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DEPARTMENT OF MATHEMATICS

Z E R O - S E T   S P A C E S  
A N D  
P S E U D O C O M P A C T N E S S

by

Christopher R.A. Gilmour

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## INTRODUCTION

In the paper 'Rings of functions determined by zero-sets' Hugh Gordon [1971] introduces the notion of a zero-set space. These spaces are shown to lie naturally between the proximity spaces and the topological spaces. Gordon defines pseudocompactness and realcompactness for zero-set spaces in a natural way and shows that the zero-set spaces are eminently suitable for the study of these concepts and of the related function rings.

M.J. Canfell [1973] shows that zero-set spaces provide a natural and very general setting for the development of dimension theory.

We are concerned in this thesis with the intrinsic properties of zero-set spaces and their relationship to other topological-type structures. Our definition of a z-space coincides with that of Gordon's zero-set space without the point separation axiom. It is of interest that Canfell also omits this axiom.

This thesis is divided into three chapters which can be summarised as follows:

In chapter 1 we give some basic results on z-spaces. We find it useful to study these from a categorical point of view, especially as an application of initiality considerations. We characterise certain special morphisms in Zero, the category of z-spaces.

In chapter 2 we elaborate on Gordon's results concerning the relationship of  $z$ -spaces to completely regular spaces, proximity spaces and uniform spaces.

Chapter 3 contains characterizations of pseudo-compactness for  $z$ -spaces.

In detail: Section 1.1 contains basic definitions and properties of  $z$ -spaces. We prove a Urysohn lemma for  $z$ -spaces.

In section 1.2 we give the analogues of  $C$ - and  $C^*$ -embeddings for  $z$ -spaces and obtain results analogous to those for completely regular spaces.

In section 1.3 we show Zero to be initiality complete and hence coinitality complete and we characterise the coinital  $z$ -structures. We also give an initiality characterization of Zeros, the category of separated  $z$ -spaces.

In section 1.4 we use the results of the previous section to characterise the epimorphisms and extreme monomorphisms, and hence the epi-reflective subcategories of Zero and Zeros.

In section 2.1 we study the relationship between a  $z$ -space and its underlying topology, and discuss the lifting of epireflectors from the category of Tychonoff spaces to Zeros.

In section 2.2 we examine the right inverses of the forgetful functors  $F' : \underline{\text{Prox}} \rightarrow \underline{\text{Zero}}$  and  $F'' : \underline{\text{Un}} \rightarrow \underline{\text{Zero}}$ , where  $\underline{\text{Prox}}$  and  $\underline{\text{Un}}$  are the categories of proximity spaces and uniform spaces respectively. We study the relationship between these right inverses and analogous functors between  $\underline{\text{Un}}$ ,  $\underline{\text{Prox}}$  and the category of completely regular spaces. In particular we obtain results about  $U^*$ , the weakest right inverse of  $F''$ .

In section 3.1 we use our results on  $z$ -complete separation to obtain an alternative to Gordon's characterization of  $\beta X$  for  $z$ -spaces. We give Gordon's definition of realcompactness for  $z$ -spaces and present his analogue of the Hewitt realcompactification.

In section 3.2 we use Gordon's definition of a pseudocompact  $z$ -space to obtain various characterizations of pseudocompactness. A summary of these characterizations appears at the end of the section.

Section 3.3 contains a brief sketch of applications of  $z$ -space theory and some open problems.

At the end of each chapter there is a short section of bibliographical notes.

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PRELIMINARIES

The term "function" shall be reserved throughout the text for a real-valued map. The functions on a set  $X$  form a ring under the usual pointwise operations and a lattice under the operations  $\wedge$  and  $\vee$  where:

$$(f \wedge g)(x) = \min\{f(x), g(x)\}$$

$$(f \vee g)(x) = \max\{f(x), g(x)\}.$$

By a ring or lattice of functions on  $X$  we shall mean a subring or sublattice of the ring of all functions on  $X$ .

We denote by  $f^+$  and  $f^-$  the functions  $f \vee 0$  and  $(-f)^+$  respectively and define  $|f| = f^+ + f^-$ .

We write  $f|_A$  for the restriction of a function to a subset  $A$  of its domain  $X$  and  $f^{-1}B$  for the preimage of a set  $B$ , i.e.  $f^{-1}B = \{x \in X : f(x) \in B\}$ . In particular,  $f^{-1}\{0\}$ , the "zero-set" of  $f$ , will be denoted by  $Z_f$ .

We denote by  $\mathbb{N}$ , the set of natural numbers  $\{1, 2, 3, \dots\}$ . The standard structures on the real line require a notation of their own.

$\mathbb{R}$  the real line with its usual topology, or regarded as a set (depending on the context).

$\mathbb{R}_{sp}$  the real line with its standard proximity structure.

$\mathbb{R}_{su}$  the real line with the standard (metric) uniform structure.

The closure of a set  $A$  in a topological space  $X$  will be denoted by  $cl_X A$  or, where no confusion arises, by  $\bar{A}$ .

We emphasise that by a completely regular space we do not necessarily mean a Hausdorff completely regular space.

The terminology on categories closely follows that of Herrlich [1968], Herrlich and Strecker [1973]. If  $\underline{C}$  is a category, then  $|\underline{C}|$  denotes the class of objects of  $\underline{C}$ . We denote the set of all morphisms  $: X \rightarrow Y$  in  $\underline{C}$  by  $\underline{C}(X, Y)$ .

Ens is the category of sets. Crg, Prox, Un denote the categories of completely regular spaces, proximity spaces and uniform spaces respectively. The corresponding subcategories of separated spaces will be denoted by Crgs, Proxs, and Uns.

All functors under consideration will be covariant. We shall use  $G$  for the forgetful functor: Zero  $\rightarrow$  Ens and  $\mathcal{G}$  and  $\mathcal{G}^*$  for those functors: Crg  $\rightarrow$  Un which induce the initial uniform structure for the continuous and the bounded continuous functions to  $\mathbb{R}_{su}$  respectively.

On page 76 there are diagrams which summarise the functors which are mentioned from section 2.1 onwards.

## CHAPTER 1 : Z-SPACES

### 1.1 BASIC CONCEPTS

In this section we introduce the concept of a z-space and give a few basic results. Our definition of the separated z-spaces coincides with that of the zero-set spaces of Gordon [1971] .

1.1.1 Definition. A z-structure on  $X \in |\text{Ens}|$  is a collection  $\underline{Z}$  of subsets of  $X$  satisfying:

- (1)  $\underline{Z}$  is closed under finite unions;  $\emptyset \in \underline{Z}$  .
- (2)  $\underline{Z}$  is closed under countable intersections;  $X \in \underline{Z}$  .
- (3) If  $A, B \in \underline{Z}$  and  $A \cap B = \emptyset$  then there are  $C, D \in \underline{Z}$  such that  $A \cap C = \emptyset = B \cap D$  and  $C \cup D = X$  .
- (4) If  $A \in \underline{Z}$  then there are  $A_1, A_2, \dots \in \underline{Z}$  such that  $X - A = \bigcup_1^{\infty} A_n$ .

$X$  with the above structure is called a z-space (with z-structure  $\underline{Z}$ ) .

If we further require:

- (5) for each pair of distinct points of  $X$  there is an  $A \in \underline{Z}$  which contains just one of the points,

then we refer to  $X$  as a separated z-space and to  $\underline{Z}$  as a separated z-structure.

1.1.2 Remarks. (a) Gordon [1971] studies only the separated z-spaces.

(b) It will sometimes be useful to denote a z-space by a single symbol  $X$ . We then denote the z-structure of  $X$  by  $z\text{-str } X$ . The members of  $z\text{-str } X$  are called the zero-sets of  $X$  and their complements with respect to  $X$  are the cozero-sets of  $X$ .

1.1.3 Examples. (a) The closed sets of  $\mathbb{R}$  clearly satisfy the axioms for a separated z-structure. We refer to this as the standard z-structure of  $\mathbb{R}$  and denote the corresponding z-space by  $\mathbb{R}_z$ .

(b) The totality of closed sets of a  $T_0$  topological space satisfy (1), (2), and (5) and if the space is normal (3) is also satisfied.

(c) The closed sets of the space  $\Gamma$  of Gillman and Jerison [1960 . Exercise 3K] satisfy (4) but not (3).

Any uniformly closed ring of bounded functions, on a set  $X$ , which contains the constants generates a z-structure on  $X$  in the following way:

1.1.4 Theorem. Let  $X \in |\text{Ens}|$ . If  $M$  is a ring of bounded functions with the above properties then

$$\underline{Z} = \{Zf : f \in M\}$$

is a z-structure on  $X$ .

Proof.  $\emptyset \in \underline{Z}$  and  $Z(fg) = Zf \cup Zg$  hence (1) of 1.1.1 is satisfied.  $X \in \underline{Z}$  and

$$Z \sum_1^{\infty} (|f_n| \wedge 2^{-n}) = \bigcap_1^{\infty} Z f_n$$

satisfying (2) .

If  $A = Zf$  ,  $B = Zg$  are such that  $A \cap B = \emptyset$  , then  $C = Z(|g| - |f|)^+$  and  $D = Z(|f| - |g|)^+$  satisfy (3) .

If  $A = Zf$  , define  $A_n = Z(|f| - \frac{1}{n})^-$  , then (4) is satisfied.

1.1.5 Corollary. If  $M$  in the above theorem separates points, then  $\underline{Z}$  is a separated  $z$ -structure on  $X$  .

1.1.6 Definition. If  $X$  and  $Y$  are  $z$ -spaces, we call a mapping  $f : X \rightarrow Y$  a  $z$ -map if  $f^{-1}A \in z\text{-str } X$  for each  $A \in z\text{-str } Y$  .

1.1.7 Remarks. (1) The definition is equivalent to the requirement that  $f^{-1}C$  is a cozero-set of  $X$  , for each cozero-set  $C$  of  $Y$  .

(2) The composition of two  $z$ -maps is a  $z$ -map. Thus we can form the category Zero (respectively Zeros), whose objects are the  $z$ -spaces (respectively the separated  $z$ -spaces) and whose morphisms are the  $z$ -maps.

(3) The closed intervals in  $\mathbb{R}$  form a countable subbase for the closed sets of  $\mathbb{R}$  . Hence by the definition of  $\mathbb{R}_z$  (1.1.3(a)) we have that  $f : X \rightarrow \mathbb{R}_z$  is a  $z$ -function precisely when  $f^{-1}A \in z\text{-str } X$  for each closed interval  $A$  of  $\mathbb{R}$  .

1.1.8 Lemma. Let  $\underline{B}$  be a collection of subsets of  $X$  closed under finite union and countable intersection. Let  $M$  be the set of all functions  $f$  such that  $f^{-1}F \in \underline{B}$  for each closed set  $F$  of  $\mathbb{R}$ . Then  $M$  is a uniformly closed ring.

Proof. For the sake of completeness we give a sketch of the proof which can be found in Gordon [1971, 2.4].

Let  $\underline{C} = \{X - A : A \in \underline{B}\}$ . Then  $\underline{C}$  is closed under finite intersections and countable unions and  $M$  is the set of functions  $f$  such that  $f^{-1}U \in \underline{C}$  for every  $U$  open in  $\mathbb{R}$ . If  $f_1, f_2 \in M$  then define  $g$  by

$$g(x) = (f_1(x), f_2(x))$$

It can be shown that the inverse image of an open set under the mappings  $k \circ g$  and  $l \circ g$ , where  $k(\alpha, \beta) = \alpha - \beta$  and  $l(\alpha, \beta) = \alpha\beta$ , lies in  $\underline{C}$ . Thus  $f_1 - f_2 = k \circ g$  and  $f_1 f_2 = l \circ g$  are in  $M$ . Thus  $M$  is a ring.

$M$  is uniformly closed. For suppose  $f_1, f_2, \dots \in M$  and let  $f_n \rightarrow f$  uniformly. It is sufficient to show that  $f^{-1}(\alpha, \beta) \in \underline{C}$  whenever  $\alpha < \beta$ . For each positive integer  $k$ , choose  $n$  such that

$$|f_{n_k}(x) - f(x)| < \frac{1}{k}$$

for all  $x \in X$ . Let

$$U_k = f_{n_k}^{-1} \left( \alpha + \frac{1}{k}, \beta - \frac{1}{k} \right)$$

Then  $U_k \in \underline{C}$  and  $U_k \subset f^{-1}(\alpha, \beta)$  for all  $k$ . But each  $x \in f^{-1}(\alpha, \beta)$  belongs to  $U_k$  for some  $k$ . Hence

$$f^{-1}(\alpha, \beta) \subset \bigcup_1^{\infty} U_k \in \underline{C}$$

1.1.9 Remark. We denote the set of all z-functions of  $X \in |\underline{\text{Zero}}|$  to  $\mathbb{R}_z$  by  $S(X)$  and the bounded z-functions to  $\mathbb{R}_z$  by  $S^*(X)$ . Then  $S(X)$  is a uniformly closed ring by 1.1.8 and hence so is  $S^*(X)$ .

If  $f \in S(X)$  then so is  $|f|$  and hence

$$f \vee g = \frac{1}{2}(f+g + |f-g|) \in S.$$

Thus  $S$  and  $S^*$  are lattices.

1.1.10 Definition. Let  $X \in |\underline{\text{Zero}}|$ . Subsets  $A$  and  $B$  of  $X$  are z-completely separated in  $X$  if there is an  $f \in S^*(X)$  such that  $0 \leq f \leq 1$  and

$$f(x) = 0 \quad \forall x \in A$$

$$f(x) = 1 \quad \forall x \in B$$

We now give a straightforward adaption of the classical Urysohn lemma to z-spaces.

1.1.11 Urysohn lemma for z-spaces. Let  $X \in |\underline{\text{Zero}}|$ . Any two disjoint zero-sets of  $X$  are z-completely separated in  $X$ .

Proof. Let  $A$  and  $B$  be disjoint zero-sets of  $X$ . Then  $X - B = G_1$  is a cozero-set. We can find cozero-sets  $G_{\frac{1}{2}}$  and  $G_{\frac{1}{2}}^*$  such that

$$A \subset G_{\frac{1}{2}} \subset X - G_{\frac{1}{2}}^* \subset G_1$$

For we have by (3) of 1.1.1 that if  $A, B \in \text{z-str } X$  are disjoint then there are zero-sets  $C, D$  with  $A \subset X - C$ ,  $B \subset X - D$

and  $(X - C) \cap (X - D) = \emptyset$ . Set

$$X - C = G_{\frac{1}{2}}, \quad X - D = G_{\frac{1}{2}}^* \quad (A)$$

Assume that we have found cozero-sets  $G_{k/2^n}$  and  $G_{k/2^n}^*$

( $k = 1, \dots, 2^n$ ) such that

$$A \subset G_{1/2^n} \subset X - G_{1/2^n}^* \subset G_{2/2^n} \subset \dots \subset X - G_{(k-1)/2^n}^* \subset G_{k/2^n} \subset \dots \subset G_1.$$

Set  $G_{2^m/2^{n+1}} = G_{m/2^n}$  and  $G_{2^m/2^{n+1}}^* = G_{m/2^n}^*$  and introduce  $G_{(2^m+1)/2^{n+1}}$

and  $G_{(2^m+1)/2^{n+1}}^*$  by requiring that

$$A \subset G_{1/2^{n+1}} \subset X - G_{1/2^{n+1}}^* \subset G_{2/2^{n+1}}$$

and that

$$X - G_{2^m/2^{n+1}}^* \subset G_{(2^m+1)/2^{n+1}} \subset X - G_{(2^m+1)/2^{n+1}}^* \subset G_{(2^m+2)/2^{n+1}}$$

That such sets exist follows by arguing as in (A) above. By induction we define  $G_r$  and  $G_r^*$  for each  $r \in \underline{R}$  where

$$\underline{R} = \left\{ \frac{m}{2^n} : 1 \leq m \leq 2^n, m, n \in \mathbb{N} \right\}$$

Then  $G_r$  and  $G_r^*$  satisfy:

$$r, s \in \underline{R}, r < s \Rightarrow A \subset G_r \subset X - G_r^* \subset G_s \subset G_1 \quad (B)$$

Define

$$f(x) = \inf \{ r : r \in \underline{R}, x \in G_r \} \quad \forall x \in G_1 = X - B$$

$$f(x) = 1 \quad \forall x \in B$$

$$\text{Then} \quad f(x) = 0 \quad \forall x \in A$$

$$f(x) = 1 \quad \forall x \in B$$

and  $0 \leq f(x) \leq 1$ , for each  $x \in X$

It only remains to show that  $f$  is a  $z$ -function. It is sufficient to show the preimages of intervals of the form  $[0, a)$  or  $(b, 1]$  are cozero-sets.

Now

$$0 \leq f(x) < a \iff x \in G_r, \text{ for some } r < a.$$

For if  $x \in G_r$ ,  $r < a$ , then  $f(x) \leq r < a$ . Conversely, if  $f(x) < a$  then there is an  $r \in \underline{\mathbb{R}}$  such that  $f(x) < r < a$ , thus  $x \in G_r$ .

$$\text{Hence } f^{-1}[0, a) = \cup \{G_r : r \in \underline{\mathbb{R}}, r < a\}$$

which is a countable union of cozero-sets and is thus a cozero-set.

We show  $f^{-1}[0, b]$  a zero-set. (Thus  $f^{-1}(b, 1]$  a cozero-set.)

Now

$$0 \leq f(x) \leq b \iff x \in G_r, \forall r > b.$$

For suppose  $x \in G_r$  for each  $r > b$ , then  $f(x) \leq r$  for each  $r > b$ . That is  $f(x) \leq b$ . On the other hand

$$\begin{aligned} f(x) \leq b &\implies f(x) < r, \quad \forall r > b \\ &\implies x \in G_r, \quad \forall r > b \end{aligned}$$

Thus  $f^{-1}[0, b] = \cap \{G_r : r > b, r \in \underline{\mathbb{R}}\}$ . But for each  $r > b$  there is an  $r' \in \underline{\mathbb{R}}$  such that  $r > r' > b$ . It follows by (B) that

$$G_r \supset X - G_{r'}^* \supset G_{r'}.$$

Hence

$$\bigcap_{r > b} G_r \supset \bigcap_{r' > b} (X - G_{r'}^*) \supset \bigcap_{r' > b} G_{r'}.$$

Thus

$$\bigcap_{r > b} G_r = \bigcap_{r' > b} (X - G_{r'}^*)$$

which is a countable intersection of zero-sets, thus a zero-set.

The following theorem is the main result of this section and justifies Gordon's terminology. Our proof follows that of Gordon [1971] but is more direct in that we use our theorem 1.1.11 instead of going via an induced proximity structure as does Gordon.

1.1.12 Theorem. Let  $X \in |\underline{\text{Zero}}|$ . Then

$$\begin{aligned} z\text{-str } X &= \{Zf : f \in S(X)\} \\ &= \{Zf : f \in S^*(X)\} . \end{aligned}$$

Proof. Let  $\underline{Z} = \{Zf : f \in S(X)\}$ . For each  $f \in S(X)$ ,  $Zf = f^{-1}\{0\}$  and  $\{0\}$  is a zero set of  $\mathbb{R}_Z$ . Thus  $Zf \in z\text{-str } X$ . Hence  $\underline{Z} \subset z\text{-str } X$ .

Let  $\underline{Z}^* = \{Zf : f \in S^*(X)\}$ . Then  $\underline{Z}^* \subset \underline{Z}$  and thus  $\underline{Z}^* \subset z\text{-str } X$ .

On the other hand, let  $A \in z\text{-str } X$ . Then  $X - A = \bigcup_1^\infty A_n$  where  $A_n \in z\text{-str } X$ , by (4) of 1.1.1. Then  $A \cap A_n = \emptyset$  for each  $n$ . Applying 1.1.11 for each  $n$ , we can find an  $f_n \in S^*(X)$  such that

$$\begin{aligned} f_n(x) &= 0 & \forall x \in A \\ &= 1 & \forall x \in A_n . \end{aligned}$$

Then  $A \subset Zf_n \subset X - A_n$ . But  $A = \bigcap_1^\infty (X - A_n)$ . Thus  $A = \bigcap_1^\infty Zf_n$ , where  $Zf_n \in \underline{Z}^*$ . However  $\underline{Z}^*$  is closed under countable intersections by 1.1.4 and 1.1.9, hence  $A \in \underline{Z}^*$ .

1.1.13 Remark. If  $f \in S(X)$  and  $f$  vanishes nowhere, then  $\frac{1}{f} \in S(X)$ . (The inverse image of each open interval in  $\mathbb{R}$  under  $\frac{1}{f}$  is the inverse image under  $f$  of some open set.)

The following theorem characterizes  $S$  in terms of  $S^*$ .

1.1.14 Theorem. Let  $X \in |\underline{\text{Zero}}|$ ,  $f: GX \rightarrow GR_Z$ . The following are equivalent:

- (a)  $f \in S(X)$
- (b)  $(f \wedge \alpha) \vee (-\alpha) \in S^*(X)$  for each  $\alpha > 0$ .
- (c) There are cozero-sets  $U_1, U_2, \dots$  and  $f_1, f_2, \dots \in S^*(X)$  such that

$$f|_{U_n} = f_n|_{U_n}$$

for each  $n$ , and

$$\bigcup_1^{\infty} U_n = X.$$

Proof. (a)  $\Rightarrow$  (b) follows immediately since  $S(X)$  is a lattice.

For (b)  $\Rightarrow$  (c), let  $U_n = f^{-1}(-n, n)$  and  $f_n = (f \wedge n) \vee (-n)$ .

(c)  $\Rightarrow$  (a) is clear, since  $f^{-1}(\alpha, \beta) = \bigcup_1^{\infty} [U_n \cap f_n^{-1}(\alpha, \beta)]$  is a cozero-set for every open interval  $(\alpha, \beta)$  of  $\mathbb{R}$ .

## 1.2 S AND S\*-EMBEDDING

Many problems of z-space theory involve the extension of z-functions. Before we give the main results about S\*-embedding and S-embedding we must introduce the concept of a z-subspace of a z-space X. To this end we need the following lemma.

1.2.1 Lemma. Let  $\underline{Z}_0$  be a collection of subsets of  $X \in |\underline{\text{Ens}}|$ . Let  $\underline{Z}$  be the collection of all countable intersections of finite unions of sets in  $\underline{Z}_0$ . Suppose that for each  $A \in \underline{Z}_0$  there is an  $f : X \rightarrow \mathbb{R}$  such that both  $Zf = A$  and  $f^{-1}F \in \underline{Z}$  for every F closed in  $\mathbb{R}$ . Then  $\underline{Z}$  is a z-structure on X.

Proof. Given any  $f : X \rightarrow \mathbb{R}$  such that  $f^{-1}F \in \underline{Z}$  for each F closed in  $\mathbb{R}$ , then the function  $|f| \wedge 1$  also has this property and  $Zf = Z(|f| \wedge 1)$ .

Let  $\underline{M}$  be the set of all bounded functions  $f : X \rightarrow \mathbb{R}$  with the above property and let  $\underline{Z}' = \{Zf : f \in \underline{M}\}$ .

Then  $\underline{Z}_0 \subset \underline{Z}'$  by hypothesis. By lemma 1.1.8  $\underline{M}$  is a uniformly closed ring and since we understand the definition of  $\underline{Z}$  to imply that  $\emptyset, X \in \underline{Z}$ ,  $\underline{M}$  contains the constants. Thus  $\underline{Z}'$  is a z-structure on X by theorem 1.1.4.

Now  $\underline{Z}' = \underline{Z}$ . For certainly  $\underline{Z}' \subset \underline{Z}$  and hence we need only show  $\underline{Z} \subset \underline{Z}'$ . But  $\underline{Z}'$  is closed under countable intersections and finite unions, being a z-structure, and  $\underline{Z}' \supset \underline{Z}_0$ . Thus  $\underline{Z}' \supset \underline{Z}$ .

1.2.2 Corollary. If in the above lemma we assume that  $\underline{Z}_0$  has the property that for each pair of distinct points of  $X$  there is an  $A \in \underline{Z}_0$  containing just one of them, then  $\underline{Z}$  is a separated  $z$ -structure on  $X$ .

Proof.  $\underline{M}$  separates points.

1.2.3 Definition. If  $X \in |\underline{\text{Zero}}|$  and  $A$  is a subset of  $X$  then

$$\underline{Z} = \{B \cap A : B \in z\text{-str } X\}$$

is a  $z$ -structure on  $A$ .  $A$  with this induced structure is a  $z$ -subspace of  $X$  and  $\underline{Z}$  is the relative  $z$ -structure on  $A$  with respect to  $X$ .

We must check that this does in fact define a  $z$ -structure on  $A$ .  $\underline{Z}$  is clearly closed under finite unions and countable intersections.

Let  $\underline{M}$  be the set of restrictions to  $A$  of  $z$ -functions on  $X$ . Then every set in  $\underline{Z}$  is the zero-set of a function in  $\underline{M}$  (applying 1.1.12 to  $X$  and restricting the function obtained).

On the other hand, the inverse image of each closed set of  $\mathbb{R}$ , under a function in  $\underline{M}$ , is in  $\underline{Z}$ . Thus applying 1.2.1 we see that  $\underline{Z}$  is a  $z$ -structure on  $A$ .

The following theorem gives a useful characterization of the  $z$ -completely separated sets of a  $z$ -space.

1.2.4 Theorem. Two sets are z-completely separated in  $X \in |\underline{\text{Zero}}|$  if and only if they are contained in disjoint zero-sets.

Proof. The "if" part of the theorem is a direct consequence of 1.1.11 .

Conversely if A and B are z-completely separated then there is an  $f \in S^*(X)$  with

$$\begin{aligned} f(x) &= 0 & \forall x \in A \\ &= 1 & \forall x \in B \end{aligned}$$

Let

$$Z_1 = f^{-1} \left[ 0, \frac{1}{3} \right], \quad Z_2 = f^{-1} \left[ \frac{2}{3}, 1 \right]$$

Then  $Z_1, Z_2 \in z\text{-str } X$  and  $A \subset Z_1, B \subset Z_2$  with  $Z_1$  disjoint from  $Z_2$  .

1.2.5 Definition. A z-subspace A of  $X \in |\underline{\text{Zero}}|$  is S-embedded in X if every function  $f \in S(A)$  can be extended to a function in  $S(X)$  .

A is  $S^*$ -embedded in X if every function  $f \in S^*(A)$  can be extended to a function in  $S^*(X)$  .

The proofs of the following two theorems are trivial adaptations of the proofs for the topological cases which can be found in Gillman and Jerison [1960, 1.17, 1.18]. It need only be pointed out that in the case of the real line  $S(\mathbb{R}_Z) = C(\mathbb{R})$ , where  $C(\mathbb{R})$  is the set of all continuous functions mapping  $\mathbb{R}$  to  $\mathbb{R}$  .

1.2.6 Urysohn's extension theorem for z-spaces.

A z-subspace  $A$  of  $X \in |\underline{\text{Zero}}|$  is  $S^*$ -embedded in  $X$  if and only if any two z-completely separated sets in  $A$  are z-completely separated in  $X$ .

1.2.7 Theorem. An  $S^*$ -embedded subset of  $X \in |\underline{\text{Zero}}|$  is  $S$ -embedded if and only if it is z-completely separated from every zero-set disjoint from it.

1.2.8 Remark. If  $A$  is a zero-set of  $X$ , then any zero-sets of the z-subspace  $A$  are zero-sets of  $X$ . Hence, applying 1.2.4 and 1.2.6, every zero-set of  $X$  is  $S^*$ -embedded in  $X$ . Furthermore, by 1.2.7 every zero-set of  $X$  is also  $S$ -embedded in  $X$ .

### 1.3 INITIALITY, COINITIALITY AND Z-SPACES

The z-spaces under consideration in this section are not necessarily separated. The definitions of initiality and the nomenclature are those of Brümmer [1971] .

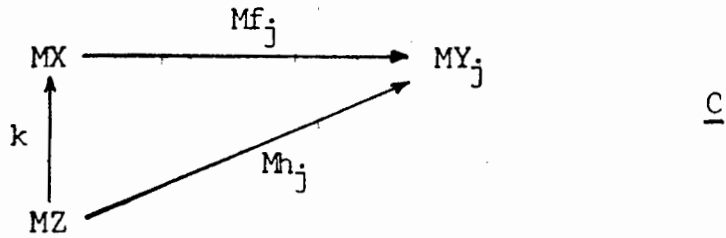
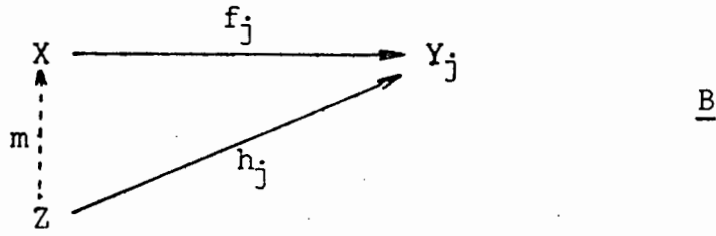
Throughout this section  $\underline{B}$  and  $\underline{C}$  will be categories with a faithful functor  $M : \underline{B} \rightarrow \underline{C}$  .

1.3.1 Definition. Let  $X \in |\underline{B}|$  . A class  $\{f_j : j \in J\}$  of  $\underline{B}$ -morphisms  $f_j : X \rightarrow Y_j$  (with common domain  $X$ ) is an M-initiality (at  $X$ ) if, whenever  $\underline{B}$ -morphisms  $h_j : Z \rightarrow Y_j$  (with the same arbitrary  $Z$  as codomain) are given together with  $k : MZ \rightarrow MX$  such that  $Mf_j \cdot k = Mh_j$  ( $j \in J$ ) , there exists  $m : Z \rightarrow X$  with  $Mm = k$  . An M-initiality  $\{f\}$  consisting of a single morphism is called simple.

1.3.2 Remarks. (a)  $M$  being faithful ensures that  $m$  is unique and that  $f_j \cdot m = h_j$  .

(b) The dual concepts are M-coinitiality (at  $X$ ) and simple M-coinitiality.

(c) We set out the diagrams for the situation of definition 1.3.1 as follows:



1.3.3 Definition. Given objects  $Y_j$  in  $\underline{B}$  and  $\underline{C}$ -morphisms  $g_j : C \rightarrow MY_j$  ( $j \in J$ ). We say that  $X \in |\underline{B}|$  is M-initial (over C) for the  $g_j$  to the  $Y_j$ , written

$$X = \text{in}_M(C, g_j, Y_j : j \in J),$$

if there exist  $f_j : X \rightarrow Y_j$  such that  $Mf_j = g_j$  and  $\{f_j : j \in J\}$  is an M-initiality at  $X$ .

1.3.4 Remarks. (1) We may express the above situation by saying that  $X$  is a solution to the M-initiality problem  $(C, g_j, Y_j : j \in J)$ .

(2) The dual notion is:  $X$  is M-cointial for the  $g_j$  from the  $Y_j$ .

(3) If the index class  $J$  is non-void then  $C$  is determined by the  $g_j$  and we may abbreviate the notation to  $(g_j, Y_j : j \in J)$ .

(4) When  $J$  is void the statement reduces to " $X$  is  $M$ -initial over  $C$ " which means: For each  $Z \in |\underline{B}|$  and each  $k : MZ \rightarrow C$ , there exists  $m : Z \rightarrow X$  with  $Mm = k$ . Thus in this case we do require that the source  $C$  be prescribed. The following proposition then remains valid for void  $M$ -initiality problems.

1.3.5 Proposition. Between any two solutions of the same initiality problem there exists a unique isomorphism  $h$  such that  $Mh = 1$ .

Proof. Follows from the uniqueness of  $m$  in 1.3.1.

1.3.6 Definition. The faithful functor  $M : \underline{B} \rightarrow \underline{C}$  is amnesic if, whenever  $h$  is an isomorphism in  $\underline{B}$  with  $Mh = 1$ , then  $h = 1$ .

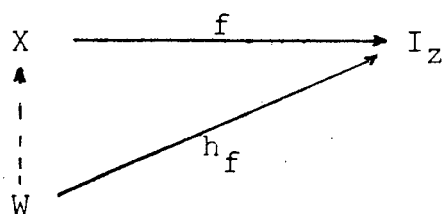
1.3.7 Remark. If  $M$  is amnesic, then by 1.3.5 each  $M$ -initiality problem has at most one solution.

The following result is useful later on

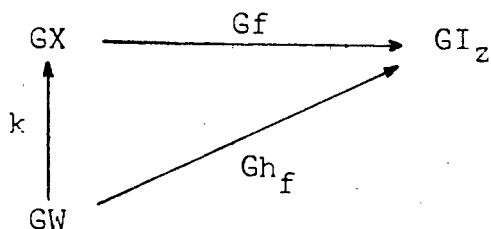
1.3.8 Proposition. Let  $X \in |\underline{\text{Zero}}|$ . Then

$$X = \text{in}_G(Gf, I_Z : f \in \underline{\text{Zero}}(X, I_Z))$$

Proof.



Zero



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Clearly we need only show that  $\underline{\text{Zero}}(X, I_Z)$  is a G-initiality at X .

Let  $h_f \in \underline{\text{Zero}}(W, I_Z)$  and  $k : GW \rightarrow GX$  be given such that  $Gf.k = Gh_f$  for each  $f \in \underline{\text{Zero}}(X, I_Z)$  .

If  $A \in z\text{-str } X$  , then  $A = Zf$  for some  $f \in \underline{\text{Zero}}(X, I_Z)$  .  
 (Certainly  $A = Zg$  for some  $g \in S(X)$  . Let  $f = (g \vee 0) \wedge 1$  .)

Now  $k^{-1}A = Z(Gf.k) = Z(Gh_f) \in z\text{-str } W$  since  $h_f \in \underline{\text{Zero}}(W, I_Z)$  .  
 Hence  $k$  is lifted to a z-map  $W \rightarrow X$  .

It is clear from the above proof that  $X = \text{in}_G(Gf, R_Z : f \in S(X))$ .  
 In the terminology of Brümmer [1971]  $I_Z$  and  $R_Z$  are G-sufficient  
objects in Zero.

1.3.9 Definition. The category B is M-initiality complete if every M-initiality problem (including void M-initiality problems) has a solution.

We now give the fundamental construction for the initiality completeness of Zero.

1.3.10 Theorem. Given a family  $\{X_\alpha\}$  of z-spaces and maps  $f_\alpha : X \rightarrow GX_\alpha$  with common domain  $X \in |\underline{\text{Ens}}|$ , let  $\underline{Z}_0$  be the collection of all sets of the form  $f_\alpha^{-1}A$  where  $A \in \text{z-str } X_\alpha$ . Let  $\underline{Z}$  be the collection of all countable intersections of finite unions of sets in  $\underline{Z}_0$ . Then

- (1)  $\underline{Z}$  is a z-structure on  $X$ , and
- (2)  $X$  with z-structure  $\underline{Z}$  is G-initial for the  $f_\alpha$  to the  $X_\alpha$ .

It follows that Zero is G-initiality complete.

Proof. (1) We show that for each  $B \in \underline{Z}_0$  there is an  $f : X \rightarrow \mathbb{R}$  with  $Zf = B$  and  $f^{-1}F \in \underline{Z}$  for each closed set  $F$  of  $\mathbb{R}$ . The result then follows from lemma 1.2.1.

If  $B \in \underline{Z}_0$  then  $B = f_\alpha^{-1}(A)$  for some  $\alpha$  and some  $A \in \text{z-str } X_\alpha$ . Then there exists  $g \in S(X_\alpha)$  with  $A = Zg$ . Let  $f = g \cdot f_\alpha$ . Then

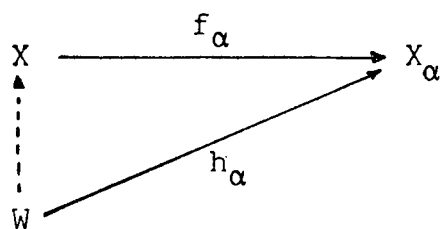
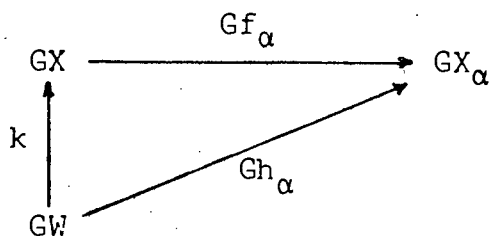
$$B = f_\alpha^{-1}A = f_\alpha^{-1}(Zg) = Z(g \cdot f_\alpha) = Zf$$

Furthermore if  $F$  is closed in  $\mathbb{R}$  then

$$f^{-1}F = (g \cdot f_\alpha)^{-1}F = f_\alpha^{-1}(g^{-1}F)$$

But  $g^{-1}F \in \text{z-str } X_\alpha$  which completes the proof of (1).

(2) Clearly  $f_\alpha : X \rightarrow X_\alpha$  is a z-map for each  $\alpha$ . It only remains to show that  $\{f_\alpha\}$  is a G-initiality at  $X$ .

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Given  $h_\alpha : W \rightarrow X_\alpha$  in Zero and  $k : GW \rightarrow GX$  such that  $Gf_\alpha \cdot k = Gh_\alpha$  for each  $\alpha$ . Let  $B \in \text{z-str } X (= \underline{Z})$ . Then  $B$  is a countable intersection of finite unions of sets of the form  $f_\alpha^{-1}A$  where  $A \in \text{z-str } X_\alpha$ . The proof is completed by observing that  $k^{-1}(Gf_\alpha)^{-1}(A)$  is a zero-set of  $W$ .

The following two results are direct consequences of 1.3.10 .

1.3.11 Corollary.     Zero is G-cointinality complete.

Proof.     The concept of M-initiality completeness is self-dual. That is, a category is M-initiality complete if and only if it is M-cointinality complete. See for example Antoine [1966] or Brümmer [1971, 1.2.2] .

1.3.12 Corollary. Zero is complete and cocomplete and  $G$  preserves limits and colimits.

Proof. See Antoine [1966] or Brümmer [1971, 1.2.6] .

Returning to the general case of a faithful functor  $M: \underline{B} \rightarrow \underline{C}$  we define:

1.3.13 Definition. If  $X, Y \in |\underline{B}|$  , then  $X \leq_M Y$  if and only if there exists a morphism  $f: Y \rightarrow X$  with  $Mf = 1$  .

We say that  $X$  is  $M$ -coarser than  $Y$  ,  $Y$  is  $M$ -finer than  $X$  .

1.3.14 Remarks. (1) Consider the forgetful functor  $G: \underline{Zero} \rightarrow \underline{Ens}$  . For  $X, Y \in |\underline{Zero}|$  , we clearly have  $X \leq_G Y$  if and only if  $z\text{-str } X \subset z\text{-str } Y$  .

(2) It follows from the proof of 1.3.10 (2) that the  $z$ -structure defined in 1.3.10 (1) is the  $G$ -coarsest  $z$ -structure on  $X$  making each  $f_\alpha$  a  $z$ -map .

(3) If  $X \leq_G Y$  and  $Y \leq_G X$  then clearly  $X = Y$  . Given any isomorphism  $f: X \rightarrow Y$  with  $Gf = 1$  , then there exists a  $g: Y \rightarrow X$  with  $fg = 1$  and  $gf = 1$  . Then  $Gg = 1$  and hence  $X = Y$  . It follows from the faithfulness of  $G$  that  $f = 1$  . Thus  $G$  is amnestic, and  $G$ -initiality problems have unique solutions.

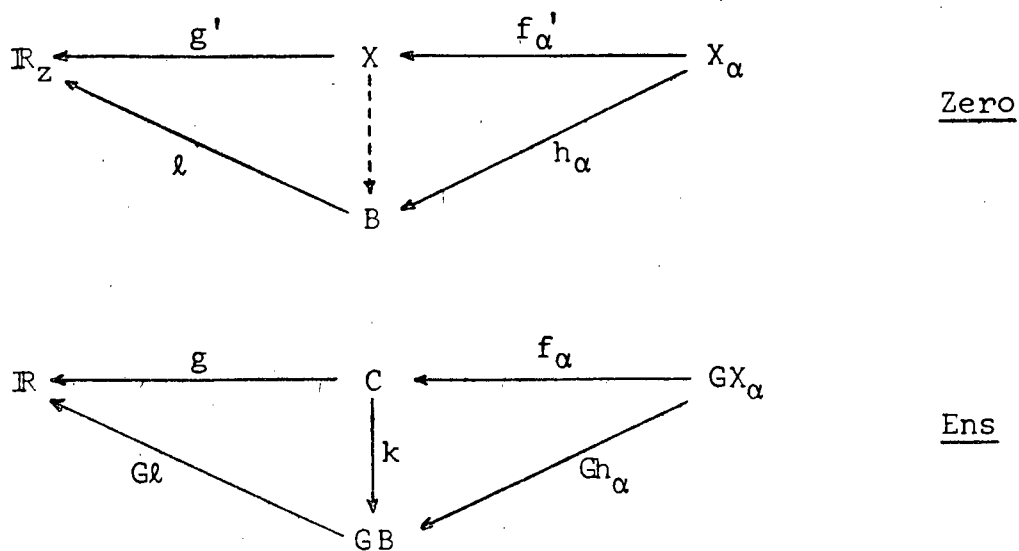
(4) Zero is complete and hence has products. The category product of  $z$ -spaces is the cartesian product with the initial  $z$ -structure for the projection maps.

(5) A z-subspace  $A$  of  $X \in |\underline{\text{Zero}}|$  has the initial z-structure for the inclusion map  $i : A \rightarrow X$ .

Given a family  $\{X_\alpha\}$  of z-spaces and maps  $f_\alpha : GX_\alpha \rightarrow X$  with common codomain  $X \in |\underline{\text{Ens}}|$ , the existence of a G-cointial z-structure on  $X$  for the  $f_\alpha$  from the  $X_\alpha$  is ensured by the cointiality completeness of Zero. This z-structure is the G-finest z-structure making each  $f_\alpha$  a z-map. The following theorem gives a useful characterization in terms of initiality.

1.3.15 Theorem. Given  $X_\alpha \in |\underline{\text{Zero}}|$  and maps  $f_\alpha : GX_\alpha \rightarrow C$  in Ens. Let the z-space  $X$  be G-initial to  $\mathbb{R}_Z$  for those functions  $g : C \rightarrow \mathbb{R}$  which satisfy  $g \cdot f_\alpha \in S(X_\alpha)$  for each  $\alpha$ . Then  $X$  is G-cointial for the  $f_\alpha$  from the  $X_\alpha$ .

Proof. Suppose  $h_\alpha \in \underline{\text{Zero}}(X_\alpha, B)$  and let  $k : C \rightarrow GB$  be such that  $k \cdot f_\alpha = Gh_\alpha$  for each  $\alpha$ .



Each  $f_\alpha$  lifts to an  $f'_\alpha : X_\alpha \rightarrow X$ .  $B$  is  $G$ -initial for all  $\ell \in S(B)$  to  $\mathbb{R}_z$  (by 1.3.8). For each  $\ell$  and  $\alpha$ ,  $(G\ell.k).f_\alpha = G\ell.Gh_\alpha \in S(X_\alpha)$ , so  $G\ell.k$  is one of the  $g$ . Hence by  $G$ -initiality of  $B$  for the  $\ell$ ,  $k$  is lifted to a  $z$ -map  $X \rightarrow B$ .

1.3.16 Remark. The zero-sets of the  $G$ -cointial  $z$ -structure on  $X$  given in the theorem above are the countable intersections of finite unions of sets  $A$  with the following property: there exists  $g : GX \rightarrow \mathbb{R}$  with  $A = Zg$  and  $g.f_\alpha \in S(X_\alpha)$  for each  $\alpha$ .

1.3.17 Definition. A morphism  $f$  in  $\underline{B}$  is an

- M-injection if  $Mf$  is an injection,
- M-surjection if  $Mf$  is a surjection,
- M-embedding if  $f$  is an  $M$ -initiality and  $M$ -injection,
- M-quotient if  $f$  is an  $M$ -cointiality and  $M$ -surjection.

1.3.18 Remark. Given a  $G$ -surjection, the existence of a quotient  $z$ -structure is ensured by 1.3.11.

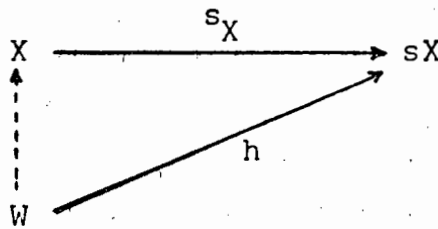
1.3.19 Definition. An object  $X$  of  $\underline{B}$  is  $M$ -separated if every simple  $M$ -initiality at  $X$  is an  $M$ -embedding.

Armed with the above definitions we give a characterization of Zeros, the category of separated  $z$ -spaces.

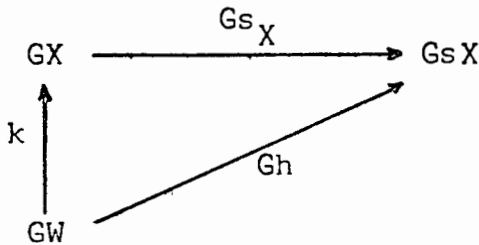
1.3.20 Proposition. The objects of the category Zeros are the  $G$ -separated objects of Zero.

Proof. Let  $X \in |\text{Zero}|$ . Define an equivalence relation on  $X$  by:  $x \equiv y$  if and only if there is no zero-set of  $X$  containing only one of the points.

Let  $sX$  be the quotient  $z$ -space obtained by this relation and let  $s_X$  be the associated quotient map. Then  $s_X$  is a  $G$ -initiality at  $X$ . For suppose  $h \in \text{Zero}(W, sX)$  and  $k : GW \rightarrow GX$  are such that  $Gs_X \cdot k = Gh$ .



Zero



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Let  $A \in z\text{-str} X$ . Then there is an  $f \in S(X)$  with  $Zf = A$ .

Let  $s_X(x) = x^*$ . Define  $f^* : GsX \rightarrow \mathbb{R}$  by  $f^*(x^*) = f(x)$ .

$f^*$  is well defined. For suppose  $f^*(x^*) \neq f^*(y^*)$ . Then

$f(x) \neq f(y)$  , and hence  $x \notin f^{-1}(f(y))$  . But  $\{f(y)\} \in z\text{-str } \mathbb{R}_z$  so that  $x^* \neq y^*$  .

Then  $f^*.s_X = f$  and hence, by definition of a quotient  $z$ -structure,  $f^*$  is a  $z$ -function.

Now

$$\begin{aligned} k^{-1}A &= k^{-1}(ZGf) = Z(Gf.k) = Z(Gf^*.Gs_X.k) \\ &= Z(Gf^*.Gh) \\ &= h^{-1}(Zf^*) \in z\text{-str } W . \end{aligned}$$

Thus  $k$  is a  $z$ -map and  $s_X$  is a  $G$ -initiality.

It follows that if  $X$  is a  $G$ -separated object, then  $s_X$  is a  $G$ -embedding and a  $G$ -surjection and hence an isomorphism. But  $sX$  is separated. For if  $x_1^* \neq x_2^*$  ,  $x_i^* \in sX$  , then there are  $x_1, x_2 \in X$  with  $s_X(x_i) = x_i^*$  . Hence there is an  $A \in z\text{-str } X$  containing  $x_1$  but not  $x_2$  , say. Now  $A = Zf$  for some  $f \in S(X)$  . Define, as above,  $f^*(x^*) = f(x)$  , where  $s_X(x) = x^*$  . Then  $Zf^*$  contains  $x_1^*$  but not  $x_2^*$  . Thus  $X$  is a separated  $z$ -space.

Conversely let  $f$  be a simple  $G$ -initiality at the separated  $z$ -space  $X$  . The zero-sets of  $X$  are preimages under  $f$  of zero-sets, hence  $f$  distinguishes points. Thus  $f$  is a  $G$ -injection and hence a  $G$ -embedding.

As the terminology on initiality has now served the purpose of clarifying concepts, we may henceforth adopt the following

CONVENTION. We omit the prefix  $M$  in terms such as  $M$ -initiality,  $M$ -injection,  $M$ -embedding etc., if  $M$  is the forgetful functor to Ens.

#### 1.4 EPIMORPHISMS AND EXTREME SUBOBJECTS

The knowledge of special morphisms such as the epimorphisms and extreme subobjects of a category proves very useful. For example, in a "nice" category, if a property is closed under the taking of products and extreme subobjects then it is epi-reflective [Herrlich 1968 , 10.2.1] .

To characterise these morphisms we use the notion of the double of a space  $Y$  along a subset  $U$  . That is we identify the coproduct of two copies of  $Y$  along  $U$  . In the case of separated  $z$ -spaces it is necessary to impose some condition on  $U$  to ensure that our double is separated. For the non-separated case no such condition is necessary.

To be precise, let  $i_1, i_2 : Y \rightarrow Y \amalg Y$  be the two inclusions into the coproduct of two copies of  $Y \in |\underline{\text{Zero}}|$  . Define an equivalence relation on  $Y \amalg Y$  by:  
 $a \equiv b$  if and only if either  $a = b$  or  $i_1^{-1}(a) = i_2^{-1}(b) \in U$  or  $i_1^{-1}(b) = i_2^{-1}(a) \in U$  . Let  $q : Y \amalg Y \rightarrow Q$  be the natural map defined onto the space  $Q$  with the quotient  $z$ -structure. Let  $r_n = q \cdot i_n$  ,  $(n = 1, 2)$  .

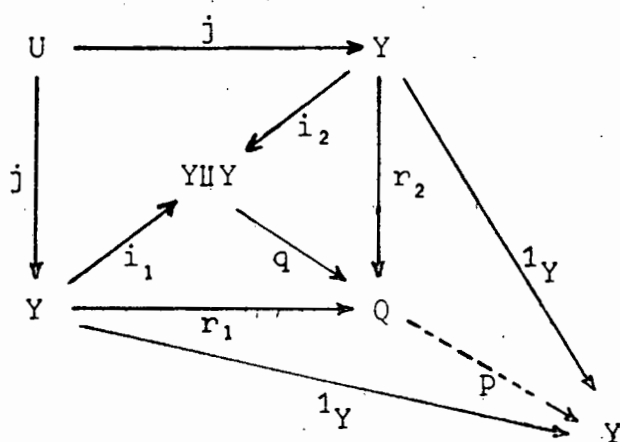
It will be convenient to consider  $Q$  as a pushout.



1.4.2 Lemma. For each (separated)  $z$ -space  $Y$  and each (zero-set) subset  $U$  of  $Y$  there is a (separated)  $z$ -space  $Q$  and  $z$ -maps  $r_1, r_2 : Y \rightarrow Q$  such that  $U = \{y \in Y : r_1(y) = r_2(y)\}$

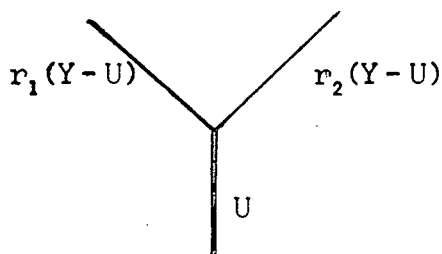
Proof. Construct the double  $Q$  as in the previous proposition. For the non-separated case  $r_1$  and  $r_2$  of 1.4.1 clearly satisfy the required conditions.

In the case where  $Y$  is separated we need to show that  $Q$  is separated. Consider the following diagram:



By the pushout property in Zero there exists a unique  $p \in \text{Zero}(Q, Y)$  with  $pr_1 = pr_2 = 1_Y$ .

It will be helpful to keep the following representation of  $Q$  in mind:



Case 1.  $x \neq y$  in  $Q$  and  $p(x) \neq p(y)$ . Then  $p(x) \neq p(y)$  in  $Y$ , which is a separated  $z$ -space, hence there is a zero-set  $A$  of  $Y$  containing only one of  $p(x)$  and  $p(y)$ . Then  $p^{-1}A$  is a zero-set of  $Q$  containing just one of  $x$  and  $y$ .

Case 2.  $x \neq y$  in  $Q$  and  $p(x) = p(y)$ . Then either  $x \in r_1(Y-U)$  and  $y \in r_2(Y-U)$  or  $y \in r_1(Y-U)$  and  $x \in r_2(Y-U)$ . Suppose the former to be true (the argument in either case is identical).

Thus  $x \in r_1(Y-U)$  and  $y \in r_2(Y-U)$ . By hypothesis  $U \in z\text{-str} Y$ , hence there exists an  $f \in S(Y)$  such that  $U = Zf$ .

Define  $g : Q \rightarrow \mathbb{R}$  by:

$$\begin{aligned} gr_1 &= f && \text{on } Y-U \\ g &= 0 && \text{otherwise.} \end{aligned}$$

Then  $Q - r_1(Y-U) = Zg$  and it will thus suffice to show that  $g \in S(Q)$ , for then  $Zg$  is a zero-set of  $Q$  containing  $y$  but not  $x$ .

We show that  $g^{-1}F \in z\text{-str} Q$  for each  $F$  closed in  $\mathbb{R}$ .

Every closed set  $F$  of  $\mathbb{R}$  is the zero-set of some continuous (and hence  $z$ -) map  $\mathbb{R} \rightarrow \mathbb{R}$ . Explicitly,  $F$  is the zero-set of the function  $k \in C(\mathbb{R}) = S(\mathbb{R})$  defined by  $k(a) = d(a, F)$  where  $d(a, F) = \inf\{|t - a| : t \in F\}$ .

Hence  $g^{-1}F = (k.g)^{-1}\{0\} = Z(k.g)$  . Thus by definition of the quotient structure on  $Q$  we need only show that  $(k.g).q \in S(Y \amalg Y)$  . By cointiality of  $Y \amalg Y$  for  $i_1, i_2$  this is equivalent to showing that  $(k.g).r_n \in S(Y)$  ( $n = 1, 2$ ) . Now  $(k.g).r_1 = k.f \in S(Y)$  and  $(k.g).r_2 = k(0)$  a constant function and hence a z-function on  $Y$  .

1.4.3 Corollary. If  $Y \in |\underline{\text{Zeros}}|$  and  $U$  is a zero-set then the diagram of proposition 1.4.1 is a pushout in Zeros .

In well-known terminology, a generator in a category  $\underline{B}$  is an object  $P$  such that for each parallel pair  $f, g : X \rightarrow Y$  in  $\underline{B}$  with  $f \neq g$  there exists an  $h : P \rightarrow X$  such that  $fh \neq gh$  .

1.4.4 Example. The one point set  $P = \{x\}$  with its unique z-structure is a generator in Zero.

1.4.5 Remark. The zero-sets of a z-space  $X$  , being closed under finite unions, form a base for some topology on  $X$  . By a closed subset of a z-space we mean closed in this topology. Thus the closure  $\bar{A}$  of a set  $A$  in this topology is given by:

$$\bar{A} = \bigcap \{B : B \in \text{z-str } X \text{ and } A \subset B\}$$

1.4.6 Theorem. Let  $f : X \rightarrow Y$  be a z-map in Zero (respectively Zeros )

- (1)  $f$  a monomorphism  $\Leftrightarrow f$  an injection
- (2)  $f$  an epimorphism  $\Leftrightarrow f$  a surjection (respectively,  $f$  dense).

Proof. (1) Suppose  $f$  is not an injection. Let  $f(x) = f(y)$  but  $x \neq y$ . Let  $P = \{p\}$  be a generator in Zero and define  $g, h \in \underline{\text{Zero}}(P, X)$  by  $g(p) = x$  and  $h(p) = y$ . Then  $fg = fh$  but  $g \neq h$  and hence  $f$  is not a monomorphism.

The converse is trivial.

(2) Non-separated case. Suppose  $f$  is not a surjection, then by 1.4.2 there is a  $z$ -space  $Q$  and  $r_1, r_2 \in \underline{\text{Zero}}(Y, Q)$  such that  $f(X) = \{y \in Y : r_1(y) = r_2(y)\}$ . That is  $r_1 f = r_2 f$  but  $r_1 \neq r_2$ . Hence  $f$  is not epic.

The converse is trivial.

Separated case. Suppose  $\overline{f(X)} = Y$  and that there exist  $g, h \in \underline{\text{Zeros}}(Y, W)$  with  $hf = gf$ .

If  $h \neq g$ , i.e.  $f$  is not epic, then there exists  $y \in Y$  with  $h(y) \neq g(y)$ . Since  $W$  is separated it follows from 1.1.1 that there are disjoint cozero-sets  $C_1$  and  $C_2$  of  $W$  with  $h(y) \in C_1$ ,  $g(y) \in C_2$ .

$h^{-1}C_1$  and  $g^{-1}C_2$  are cozero-sets and hence open sets in  $Y$  and have non-empty intersection since  $y$  is common to both. Their intersection is open in  $Y$  and since  $\overline{f(X)} = Y$  there exists an  $x \in X$  such that  $f(x) \in h^{-1}C_1 \cap g^{-1}C_2$ .

Thus  $hf(x) = gf(x) \in C_1 \cap C_2$  giving us our required contradiction.

Conversely, if  $f$  is not dense then

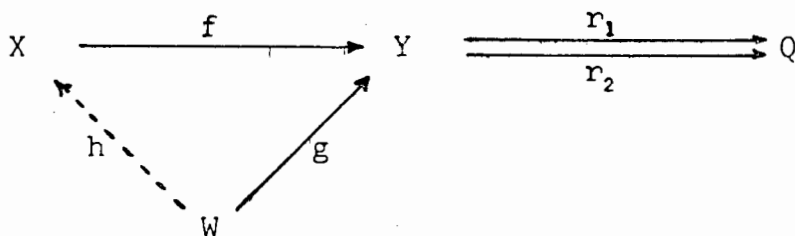
$Y \neq \overline{f(X)} = \bigcap \{A : A \in z\text{-str} Y, A \supset f(X)\}$ . Hence there exists  $A \in z\text{-str} Y$  with  $f(X) \subset A \neq Y$ . By 1.4.2 there is a  $Q \in |\underline{\text{Zeros}}|$  with  $r_1, r_2 \in \underline{\text{Zeros}}(Y, Q)$  such that  $A = \{y \in Y : r_1(y) = r_2(y)\}$ . Then  $r_1 f = r_2 f$  but  $r_1 \neq r_2$  since  $A \neq Y$ .

1.4.7 Corollary. Zero is well- and cowell-powered.

1.4.8 Theorem. Let  $X, Y \in |\underline{\text{Zero}}|$  (respectively  $|\underline{\text{Zeros}}|$ ) then for each  $z$ -map  $f : X \rightarrow Y$  the following are equivalent:

- (1)  $f$  is an embedding (respectively a closed embedding).
- (2)  $f$  is an equalizer (respectively a multiple equalizer).
- (3)  $f$  is an extreme subobject.

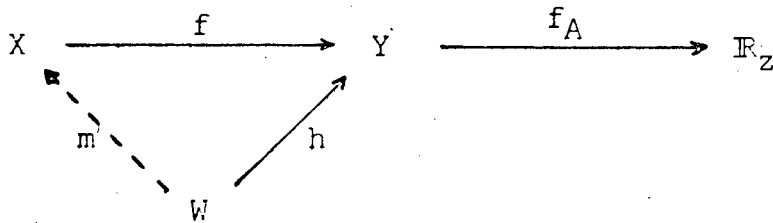
Proof. (1)  $\Rightarrow$  (2) : Non-separated case. Set  $U = f(X)$  in 1.4.2 Then there exist  $r_1, r_2 \in \underline{\text{Zero}}(Y, Q)$  with  $f(X) = \{y \in Y : r_1(y) = r_2(y)\}$ . Then  $f$  is the equalizer of  $(r_1, r_2)$ . For suppose  $r_1 g = r_2 g$ .



Then  $g(W) \subset f(X)$ . But  $f$  is an embedding hence we can define  $h \in \underline{\text{Zero}}(W, X)$  by  $fh = g$ . Uniqueness of  $h$  follows immediately, since  $f$  is an injection.

Separated case. If  $f$  is a closed embedding then  $f(X) = \bigcap \{A : A \in \underline{z}\text{-str} Y \text{ and } f(X) \subset A\}$ . Each  $A = Zf_A$  for some  $f_A \in S(Y)$ . We show that  $f$  equalizes the  $f_A$ . Denote the constant map with value 0 by  $\underline{0}$ .

Certainly  $f_A f = f_B f = \underline{0} \quad \forall A, B$ . Let  $h : W \rightarrow Y$  be given.



Suppose  $f_A h = f_B h$  for all  $A, B$ . Then  $Z(f_A h) = Z(f_B h)$  for all  $A, B$ . Thus, for fixed  $A$ ,  $h^{-1} Z f_A = \bigcap_B h^{-1} Z f_B = h^{-1} \bigcap_B Z f_B = h^{-1} [f(X)]$ . However  $\underline{0}$  is one of the  $f_A$ . Thus  $h^{-1} Z \underline{0} = h^{-1} Y = W$ . That is  $W = h^{-1} [f(X)]$ .

We can thus define  $m$  by  $m(w) = x$  where  $f(x) = h(w)$ .  $m$  is well defined since  $f$  is an injection and  $m$  lifts to a  $z$ -map since  $f$  is an initiality at  $X$ . Clearly  $m$  is unique.

(2)  $\Rightarrow$  (3): This holds true in general and is well known. For example see Herrlich [1968, 7.1.3].

(3)  $\Rightarrow$  (1): Let  $U = f(X)$  (respectively  $\overline{f(X)}$ ) with the induced  $z$ -structure, and let the inclusion map  $h : U \rightarrow Y$  be the associated embedding (respectively, closed embedding).

Define  $g : X \rightarrow U$ , by restricting the range of  $f$ . Then  $hg = f$ . But  $g$  is onto (respectively, dense) and hence, by 1.4.6, epic. Thus by hypothesis  $g$  is an isomorphism.

Hence  $f = hg$  is an embedding (respectively, a closed embedding).

1.4.9 Remark. Since Zero is complete, well- and cowell-powered, a full subcategory S is epi-reflective in Zero if and only if S is productive and hereditary [Herrlich, 1968, 10.2.1].

1.4.10 Proposition. Zeros is epi-reflective in Zero.

Proof. Each z-subspace of a separated z-space is certainly separated.

If  $\prod X_i$  denotes the product in Zero of separated z-spaces  $X_i$  and  $\pi_i$  are the associated projections to  $X_i$ , then  $\prod X_i$  is a separated z-space. For if  $x, y \in \prod X_i$  with  $x \neq y$ , then for some  $j$ ,  $\pi_j(x) \neq \pi_j(y)$ . Since  $X_j$  is separated, there exists  $A \in \text{z-str} X_j$  containing just one of  $\pi_j(x)$  and  $\pi_j(y)$ . Then  $\pi_j^{-1}A \in \text{z-str}(\prod X_i)$  contains only one of  $x$  and  $y$ .

1.4.11 Corollary. Zeros is complete.

Proof. Each reflective subcategory of a complete category is complete [Herrlich and Strecker, 1973, 36.14].

1.4.12 Remark. Zeros is also well- and cowell-powered, hence a full subcategory S of Zeros is epi-reflective in Zeros if and only if it is productive and closed hereditary.

### 1.5 NOTES

The results of section 1.1 are on the whole those of Gordon [1971] . Our approach differs from Gordon's in two ways. Firstly we need to introduce the non-separated z-spaces because the category Zeros of separated z-spaces is not initiality complete: Consider the initiality problem posed by the constant map from a two point set to a one point space.

Secondly we wanted a unified approach to z-spaces and hence we eliminated the need for introducing other structures, such as proximity structures, by our theorem 1.1.11 .

In section 1.2 , lemma 1.2.1 and the obvious definition of a z-subspace structure are due to Gordon [1971] . The remaining definitions and results on z-complete separation, S- and S\*-embedding are our own.

The definitions in section 1.3 involving arbitrary categories B and C and a faithful functor  $M: \underline{B} \rightarrow \underline{C}$  as well as proposition 1.3.5 are due to Brümmer [1971] . The remaining results of this section are own.

The notion of a double as used in section 1.4 is well-known e.g. Thornton [1971] , Herrlich [1968 , 15.2.1] . The suggestion to treat a double as a particular pushout came from my supervisor. This led to a more elegant presentation of our results in the rest of section 1.4 .

## CHAPTER 2 : THE RELATIONSHIP OF Z-SPACES TO OTHER STRUCTURES

### 2.1 TOPOLOGICAL STRUCTURES

In 1.4.5 we saw that the zero-sets of a z-space  $X$  form a base for the closed sets of some topology on  $X$ . We now study this relationship in greater detail.

In the following sections other structures will be introduced and it will be shown that Zero lies naturally, that is by means of "forgetful" functors, between Prox and Crg, the categories of proximity spaces and completely regular spaces respectively.

2.1.1 Definition. Given a z-space  $X$ , the underlying topology of  $X$  is that topology having as a base for its closed sets the zero-sets of  $X$ .

2.1.2 Remarks. (1) The cozero-sets of  $X$  form a base for the open sets of the underlying topology.

(2)  $z\text{-str } X$  is easily seen to satisfy the conditions of De Groot and Aarts [1969] for a family to be a base of closed sets for a completely regular topology. We now give a direct proof : Let  $A$  be a closed set in the underlying topology and let  $x \in X - A$ . Since  $A$  is closed,  
 $A = \bigcap \{B : B \in z\text{-str } X, B \supset A\}$ . Thus there is at least one

$B \in \mathcal{Z}\text{-str } X$  such that  $B \supset A$  and  $x \notin B$ . By 1.1.12 there exists  $f \in S^*(X)$  with  $B = Zf$ . Then  $f$  is continuous for the underlying topologies,  $f(x) \neq 0$  and  $f(A) \subset \{0\}$ . If  $X$  is a separated  $\mathcal{Z}$ -space, then the underlying topological space of  $X$  is a completely regular  $T_0$ -space and hence is a  $T_2$ -space.

(3) The underlying topology of  $X$  is delivered by the "forgetful" functor  $F : \underline{\text{Zero}} \rightarrow \underline{\text{Crg}}$ , which leaves maps unchanged. That each  $\mathcal{Z}$ -map is continuous in the underlying topologies is obvious. We shall henceforth reserve the symbol  $F$  for this functor.

2.1.3 Examples. (1) Different  $\mathcal{Z}$ -structures may have the same underlying topology. For example, the Baire sets on a metric space form a  $\mathcal{Z}$ -structure which has for its underlying topology the discrete topology, and the obvious discrete  $\mathcal{Z}$ -structure on a set also has the discrete topology,

(2) A completely regular space  $X$  which has the property that every open subspace is Lindelöf, admits a unique  $\mathcal{Z}$ -structure. For if  $X = FY$  for some  $Y \in |\underline{\text{Zero}}|$ , then each open set  $U$  of  $X$  is a union of cozero-sets of  $Y$ . Each cozero-set of  $Y$  is open in  $X$ , hence  $U$  is a countable union of cozero-sets and is thus itself a cozero-set of  $Y$ .

(3) By (2) above we have that each  $X \in |\underline{\text{Crg}}|$  which has a countable base admits a unique  $\mathcal{Z}$ -structure. Thus

$\mathbb{R}_Z$  has the unique z-structure admitted by  $\mathbb{R}$ .

Let  $H, E : \underline{A} \rightarrow \underline{B}$  be functors between categories  $\underline{A}$  and  $\underline{B}$ . If  $M : \underline{B} \rightarrow \underline{C}$  is a faithful functor then we write  $H \leq_M E$  if  $HX \leq_M EX$  for each  $X \in |\underline{A}|$ . We say that  $H$  is M-coarser than  $E$ . If further  $E \leq_M H$  and  $M$  is amnesitic, then clearly  $E = H$ .

2.1.4 Theorem. Let  $L : \underline{A} \rightarrow \underline{B}$  be a faithful functor.

If  $R$  is a right inverse of  $L$  such that  $1_A \leq_L RL$ , then

(1)  $R$  is adjoint to  $L$

(2)  $L$  preserves initiality for every faithful

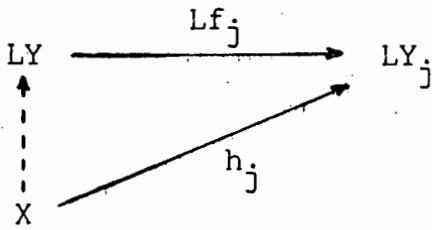
$M : \underline{B} \rightarrow \underline{C}$ .

Proof. (1) The adjunctions  $(l_B) : 1_{\underline{B}} \rightarrow LR$  and  $s : RL \rightarrow 1_{\underline{A}}$  (given by  $1_A \leq_L RL$ ) satisfy the required conditions for  $R$  to be adjoint to  $L$  (see for example Mac Lane [1971]),

(2) We must show that if  $Y = \text{in}_{ML} (C, g_j, Y_j : j \in J)$  then  $LY = \text{in}_M (C, g_j, LY_j : j \in J)$ .

$$\begin{array}{ccc}
 Y & \xrightarrow{f_j} & Y_j \\
 \uparrow m & & \uparrow n_j \\
 RX & \xrightarrow{Rh_j} & RLY_j
 \end{array}$$

Let  $Y = \text{in}_{ML} (C, g_j, Y_j : j \in J)$  and let  $MLf_j = g_j$ . Certainly  $Lf_j \in \underline{B} (LY, LY_j)$ . We need only show that  $\{Lf_j\}$  is an initiality at  $LY$ .



Given  $h_j \in \underline{B}(X, LY_j)$   
 $\underline{B}$  and  $k : MX \rightarrow C$  such that  
 $g_j \cdot k = Mh_j$  for each  $j$ .  
 As  $1 \leq_L RL$ , we have

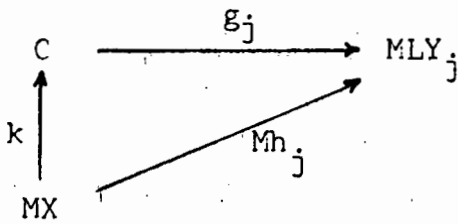
$n_j : RLY_j \rightarrow Y_j$  with  
 $Ln_j = 1$ , for each  $j$ .

Hence  $ML(n_j \cdot Rh_j) =$

$1 \cdot MLRh_j = Mh_j$ . Thus there

$\underline{C}$  exists a unique  $m \in \underline{A}(RX, Y)$   
 with  $MLm = k$ . Then

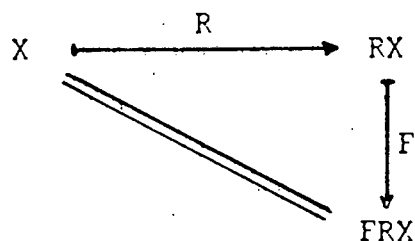
$Lm \in \underline{B}(X, LY)$ .



2.1.5 Theorem. The forgetful functor  $F : \underline{Zero} \rightarrow \underline{Crg}$  has  
 a unique right inverse  $R$ , and this  $R$  is left adjoint to  
 $F$ .

Proof. Let  $X \in |\underline{Crg}|$ . Applying 1.1.4 to  $C^*(X)$  we see  
 that the zero-sets of functions in  $C(X)$  form a  $z$ -structure  
 on  $X$ . This  $z$ -structure is delivered by a functor  
 $R : \underline{Crg} \rightarrow \underline{Zero}$  which leaves maps unchanged; we need only  
 check that  $Rf \in \underline{Zero}(RX, RY)$  for each  $f \in \underline{Crg}(X, Y)$ .

Let  $A \in z\text{-str } RY$ . Then there exists  $g \in C(Y)$  with  
 $A = Zg$ . Then  $(Rf)^{-1}A = Z(g \cdot f)$  where  $g \cdot f \in C(X)$ . Hence  
 $(Rf)^{-1}A \in z\text{-str } RX$ .



If  $A$  is closed in  $FRX$ , then it is an intersection of zero-sets of  $RX$  and hence an intersection of zero-sets of functions in  $C(X)$ . Thus  $A$  is closed in  $X$ .

Conversely, the zero-sets of functions in  $C(X)$  form a base for the closed sets of  $X$  [Gillman and Jerison, 1960, 3.2]. Hence each set closed in  $X$  is clearly closed in  $FRX$ .

Thus for every  $X \in |\underline{\text{Crg}}|$ ,  $X = FRX$ . That is  $FR = 1$ .

We now observe that:

$$RX = \text{in}_G (Mf, I_z : f \in \underline{\text{Crg}}(X, I)) \quad \dots (A)$$

where  $M : \underline{\text{Crg}} \rightarrow \underline{\text{Ens}}$  in the obvious forgetful functor.

This follows since if  $A \in \text{z-str } RX$ , then there exists  $f \in \underline{\text{Crg}}(X, I)$  with  $A = Zf = f^{-1}\{0\}$ . But

$\{0\} \in \text{z-str } I_z$  hence  $A \in \text{z-str } \text{in}_G \{Mf, I_z : f \in \underline{\text{Crg}}(X, I)\}$ .

Since for each  $f \in \underline{\text{Crg}}(X, I)$  we have  $Rf \in \underline{\text{Zero}}(RX, I_z)$ , the reverse inclusion follows immediately.

If  $L$  is any other right inverse of  $F$  then  $Lf \in \underline{\text{Zero}}(LX, LI)$  for each  $f \in \underline{\text{Crg}}(X, I)$ . But  $LI = I_z$  by 2.1.3 (3), hence  $Lf \in \underline{\text{Zero}}(LX, I_z)$ .

Thus by (A) above,  $RX \leq LX$  for each  $X \in |\underline{\text{Crg}}|$ . That is,  $R$  is the coarsest right inverse of  $F$ .

For each  $Y \in |\underline{\text{Zero}}|$ ,  $Y \leq \text{RFY}$  since:

If  $Y \in |\underline{\text{Zero}}|$  then  $Y = \text{in}_G (Gf, I_Z : f \in \underline{\text{Zero}} (Y, I_Z))$   
by 1.3.8.

If  $f \in \underline{\text{Zero}} (Y, I_Z)$  then  $\text{Rff} \in \underline{\text{Zero}} (\text{RFY}, I_Z)$  by  
again applying 2.1.3(3).

But  $\text{RFY} = \text{in}_G (Gf, I_Z : f \in \underline{\text{Zero}} (\text{RFY}, I_Z))$ ,  
hence  $Y \leq \text{RFY}$ . ..... (B)

Thus if  $L$  is any other right inverse of  $F$  and if  
 $X \in |\underline{\text{Crg}}|$  then

$$LX \leq \text{RFLX} = \text{RX}.$$

$R$  is thus the finest right inverse of  $F$  and hence the  
unique right inverse to  $F$ .

By (B),  $Y \leq \text{RFY}$  for each  $Y \in |\underline{\text{Zero}}|$ . Thus  
 $1 \leq_F \text{RF}$  so that, by 2.1.4,  $R$  is left adjoint to  $F$ .

2.1.6 Corollary. The forgetful functor  $F : \underline{\text{Zero}} \rightarrow \underline{\text{Crg}}$   
preserves initiality for the forgetful functor  
 $M : \underline{\text{Crg}} \rightarrow \underline{\text{Ens}}$ .

2.1.7 Remark. By 2.1.3 (1)  $R$  is not a left inverse to  
 $F$ . Thus  $F : \underline{\text{Zero}} \rightarrow \underline{\text{Crg}}$  is not an isomorphism. Further  
examples of completely regular spaces which admit unique  
 $z$ -structures will be given in later sections.

2.1.8 Remark. Once again it is convenient to consider a functor  $L : \underline{A} \rightarrow \underline{B}$ . Let  $r$  be a reflector in  $\underline{B}$  and denote by  $\underline{B}(r)$  the full, isomorphism closed, reflective subcategory of  $\underline{B}$  corresponding to  $r$ . Furthermore let  $L^{-1}(\underline{B}(r))$  denote the full, isomorphism closed subcategory of  $\underline{A}$  such that  $A \in |L^{-1}(\underline{B}(r))|$  if and only if  $LA \in |\underline{B}(r)|$ .

Given that  $\underline{A}$  is complete, well- and cowell-powered, then  $L^{-1}(\underline{B}(r))$  is epireflective in  $\underline{A}$  if either

- (i)  $L$  preserves products and extreme subobjects, or
- (ii)  $L$  preserves limits. [Brümmer, 1971, 1.9.2.]

If  $L^{-1}(\underline{B}(r))$  is epireflective in  $\underline{A}$  then we denote by  $r^L$  the epireflector (unique up to natural equivalence) which is such that  $L^{-1}(\underline{B}(r)) = \underline{A}(r^L)$ . We say that  $r$  can be lifted against  $L$ .

If in addition  $L$  preserves epimorphisms and has a right inverse  $R : \underline{B} \rightarrow \underline{A}$ , then there exists a natural equivalence  $r \cong Lr^LR$ .

[Brümmer, 1971, 1.9.4.]

2.1.9 Proposition. If  $r$  is an epireflector in  $\underline{Crg}$  then  $r$  can be lifted against  $F : \underline{Zero} \rightarrow \underline{Crg}$  to an epireflector  $r^L$  in  $\underline{Zero}$ . Moreover  $r \cong Fr^LR$ , where  $R$  is as in 2.1.5.

The same holds, mutatis mutandis, for the restricted

forgetful functor  $\underline{\text{Zeros}} \rightarrow \underline{\text{Crgs}}$ , where  $\underline{\text{Crgs}}$  denotes the separated subcategory of  $\underline{\text{Crg}}$ .

Proof. We need only note, firstly, that  $\underline{\text{Zero}}$  and  $\underline{\text{Zeros}}$  are complete, well- and cowell-powered (1.3.12, 1.4.7, 1.4.11, 1.4.12). Secondly,  $F$  preserves initiality (2.1.6) and hence preserves products. Thirdly we have characterized the epimorphisms and extreme monomorphisms in  $\underline{\text{Zero}}$  and  $\underline{\text{Zeros}}$  (1.4.6, 1.4.8), and it is known that the epimorphisms in  $\underline{\text{Crg}}$  and  $\underline{\text{Crgs}}$  are the surjections and the dense maps respectively, while the extreme subobjects in  $\underline{\text{Crg}}$  and  $\underline{\text{Crgs}}$  are the embeddings and the closed embeddings respectively. [Herrlich, 1968].

2.1.10 Examples. (1)  $\underline{\text{Zeros}}$  is the full, epireflective subcategory of  $\underline{\text{Zero}}$  obtained by lifting the  $T_0$  epireflector in  $\underline{\text{Top}}$ , the category of topological spaces, against the forgetful functor  $F$ .

(2) Both the compact and the realcompact completely regular spaces lift against  $F$  to full epireflective subcategories of  $\underline{\text{Zeros}}$ .

In view of 2.1.10 (2) above we now give the obvious definition for a compact  $z$ -space.

2.1.11 Definition. A  $z$ -space  $X$  is compact if its underlying topology is compact. A compact space  $Y$  is a

compactification of a  $z$ -space  $X$  if  $X$  is a dense  $z$ -subspace of  $Y$ .

2.1.12 Remarks. (1) Clearly, the  $z$ -space  $X$  is compact if and only if every covering of  $X$  by cozero-sets has a finite subcovering.

(2) By 2.1.10 (2), the compact separated  $z$ -spaces form a full epireflective subcategory of Zeros. This can be seen directly by observing that compactness is productive and closed hereditary. We denote by  $\beta X$  the compact reflection of  $X \in |\text{Zeros}|$ . As the notation implies,  $\beta X$  is the analogue of the Stone-Čech compactification of a separated completely regular space. No ambiguity will arise from this notation as the following proposition shows.

2.1.13 Proposition. Let  $Y \in |\text{Crg}|$ . If  $Y$  is compact then it admits a unique  $z$ -structure.

Proof. If  $X$  is a compact  $z$ -space, then by 1.1.8  $M = \{ff : f \in S(X)\}$  is a uniformly closed ring of continuous functions, separating points and containing the constants. Thus by the Stone-Weierstrass theorem,  $M = C(\beta X)$ .

Thus  $S(X)$  is determined by the topology, and hence  $z\text{-str } X$  is determined by the topology.

Further results concerning compactness and  $\beta X$  for  $z$ -spaces will be developed later.

## 2.2 PROXIMITY AND UNIFORM STRUCTURES

We now develop further the relationship of Zero to other categories.

Given any  $X \in |\underline{\text{Prox}}|$ , where Prox is the category of proximity spaces, it would be pleasant if the zero-sets of the proximity functions on  $X$  formed a z-structure on  $X$ . This unfortunately is not always true since the proximity functions need not form a ring (closure under addition may fail, [Fenstad, 1964]). The situation however can be remedied.

2.2.1 Theorem. Given  $X \in |\underline{\text{Prox}}|$ . Let  $\underline{Z}_0$  be the collection of all zero-sets of proximity functions  $X \rightarrow \mathbb{R}_{sp}$ . Then  $\underline{Z}$ , the family of all countable intersections of finite unions of sets in  $\underline{Z}_0$ , is a z-structure on  $X$ . Moreover  $\underline{Z}$  is delivered by a functor  $F' : \underline{\text{Prox}} \rightarrow \underline{\text{Zero}}$ .

Proof. By 1.2.1 we need only show that for each  $A \in \underline{Z}_0$  there exists  $f : X \rightarrow \mathbb{R}$  such that  $A = Zf$  and  $f^{-1}F \in \underline{Z}$  for every  $F$  closed in  $\mathbb{R}$ . It clearly suffices to show that  $f^{-1}F \in \underline{Z}_0$  for each  $f \in \underline{\text{Prox}}(X, \mathbb{R}_{sp})$ .

Let  $f \in \underline{\text{Prox}}(X, \mathbb{R}_{sp})$ . We recall that  $\mathbb{R}_{sp}$  has the standard proximity on  $\mathbb{R}$ , given by  $A \delta B$  if there exists  $\epsilon > 0$  such that  $V_\epsilon A \cap V_\epsilon B = \emptyset$  where  $V_\epsilon A = \{x : \exists y \in A, |x-y| < \epsilon\}$ .

If  $A$  is closed in  $\mathbb{R}$  then it is the zero-set of the function  $h : \mathbb{R} \rightarrow \mathbb{R}$  where  $h(a) = d(a,A)$  and  $d(a,A) = \inf \{|x-a| : x \in A\}$ .

By a straightforward application of the triangle inequality it can be shown that  $h$  is a proximity function.

Then  $A = Zh$  and  $f^1A = Z(h \cdot f)$ .

We define  $F' : \underline{\text{Prox}} \rightarrow \underline{\text{Zero}}$  to be the functor which leaves maps unchanged. To verify functoriality, consider  $f \in \underline{\text{Prox}}(X,Y)$  with  $A \in \text{z-str } F'Y$ . Then  $A$  is a countable intersection of finite unions of zero-sets of the form  $Zg$ , where  $g \in \underline{\text{Prox}}(Y, \mathbb{R}_{sp})$ . Thus  $(F'f)^1A$  is a countable intersection of finite unions of sets of the form  $Z(g \cdot F'f)$ , which are zero-sets of  $F'X$ . Hence  $f \in \underline{\text{Zero}}(F'X, F'Y)$ .

2.2.2 Corollary. If  $X \in |\underline{\text{Proxs}}|$  then  $F'X \in |\underline{\text{Zeros}}|$ .

Proof. If  $x$  and  $y$  are distinct points of  $X$  then there is a proximity function which completely separates them [Čech, 1966, 25C5].

We now define a proximity structure on a  $z$ -space  $X$ . Let us define two sets in  $X$  to be distal if they are contained in disjoint zero-sets, that is, if they are  $z$ -completely separated. That this does in fact define a proximity structure on  $X$  is easily verified by checking the axioms for a proximity structure eg. Naimpally and Warrack [1970].

The proximity structure defined above is delivered by a functor  $P : \underline{\text{Zero}} \rightarrow \underline{\text{Prox}}$  which leaves maps unchanged. (To prove functoriality of  $P$ , consider  $f \in \underline{\text{Zero}}(X, Y)$  and let  $A$  and  $B$  be distal sets in  $PY$ . Then there exists  $g \in S(Y)$  which  $z$ -completely separates  $A$  and  $B$ . Then  $g \circ f$  is a  $z$ -function which  $z$ -completely separates  $f^{-1}A$  and  $f^{-1}B$ . Thus  $Pf \in \underline{\text{Prox}}(PX, PY)$ ).

2.2.3 Proposition. Given  $X \in |\underline{\text{Zero}}|$ . Then  $f \in S(X)$  if and only if  $f \in \underline{\text{Prox}}(PX, \mathbb{R}_{sp})$ .

Proof. Let  $f : X \rightarrow \mathbb{R}_z$  be a  $z$ -map. Then  $Pf : PX \rightarrow P\mathbb{R}_z$  is a proximity map. Clearly  $P\mathbb{R}_z$  is finer than  $\mathbb{R}_{sp}$ .

Conversely let  $f \in \underline{\text{Prox}}(PX, \mathbb{R}_{sp})$ . Consider the open interval  $(\alpha, \beta)$  in  $\mathbb{R}$ . The interval  $[\alpha + \frac{1}{n}, \beta - \frac{1}{n}]$  and  $\mathbb{R} - (\alpha, \beta)$  are distal in  $\mathbb{R}_{sp}$  for each integer  $n$ , and hence their inverse images under  $f$  are distal in  $X$ . Thus for each  $n$  there are disjoint zero-sets  $A_n$  and  $A'_n$  of  $X$  with  $f^{-1}[\alpha + \frac{1}{n}, \beta - \frac{1}{n}] \subset A_n$  and  $f^{-1}(\mathbb{R} - (\alpha, \beta)) \subset A'_n$ .

Then

$$f^{-1}[\alpha + \frac{1}{n}, \beta - \frac{1}{n}] \subset A_n \subset X - A'_n \subset f^{-1}(\alpha, \beta)$$

$$\text{Thus } f^{-1}(\alpha, \beta) = \bigcup_{n=1}^{\infty} (X - A'_n)$$

which is a countable union of cozero-sets of  $X$  and is hence a cozero-set.

Let  $X \in |\underline{\text{Un}}|$ , where  $\underline{\text{Un}}$  is the category of uniform

spaces. The uniformly continuous functions generate a z-structure on  $X$  in exactly the same way as the proximity functions do for a proximity space. In detail:

2.2.4 Theorem. Given  $X \in |\underline{\text{Un}}|$ . Let  $\underline{Z}_0$  be the collection of all zero-sets of uniformly continuous functions  $X \rightarrow \mathbb{R}_{\text{su}}$ . Then  $\underline{Z}$ , the family of all countable intersections of finite unions of sets in  $\underline{Z}_0$ , is a z-structure on  $X$  which is delivered by a functor  $F'' : \underline{\text{Un}} \rightarrow \underline{\text{Zero}}$ .

Proof. Observe that the function  $h : \mathbb{R} \rightarrow \mathbb{R}$  defined in 2.2.1 by  $h(a) = d(a, A)$ , where  $A$  is any closed set in  $\mathbb{R}$ , is uniformly continuous with respect to the usual (metric) uniformity on  $\mathbb{R}$ . The proof that  $\underline{Z}$  is a z-structure follows along the lines of the proof of 2.2.1.

Once again  $F''$  is the obvious functor which leaves maps unchanged.

2.2.5 Corollary. If  $X \in |\underline{\text{Uns}}|$  then  $F''X \in |\underline{\text{Zeros}}|$ .

Proof. Distinct points of a separated uniform space can be separated by a uniformly continuous function.

2.2.6 Remark. Since the sum of two uniformly continuous functions is uniformly continuous,  $\underline{Z}_0$  of 2.2.4 is closed under finite unions. Thus in fact we need only consider the countable intersections of zero-sets of uniformly

continuous functions to generate a  $z$ -structure on a uniform space.

A uniform space  $Y$  can be thought of as a set with a family of pseudometrics defined on it, which generate the uniformity; we denote this family by  $u\text{-str } Y$ ; it consists of the pseudometrics which are uniformly continuous  $Y \times Y \rightarrow \mathbb{R}_{su}$  [Kelley, 1955].

Given  $X \in |\underline{\text{Zero}}|$  we can define a pseudometric, for each  $f \in S(X)$ , on  $X$  by  $p_f(x,y) = |f(x) - f(y)|$ . The uniform structure defined by  $\{p_f : f \in S(X)\}$  is denoted by  $UX$  and that defined by  $\{p_f : f \in S^*(X)\}$  by  $U^*X$ .

Let  $X \in |\underline{\text{Un}}|$ . It is well-known that we can define a proximity structure  $\delta$  on  $X$  by:  $A \delta B$  if and only if  $H(A) \cap B \neq \emptyset$ , where  $H(A) = \{y : (x,y) \in H, x \in A\}$ . This structure is delivered by a forgetful functor  $Q$ .

2.2.7 Theorem. Let  $X \in |\underline{\text{Zero}}|$ . Then  $PX$  is the proximity structure induced on  $X$  by both  $UX$  and  $U^*X$ .

Proof. We give a sketch of the proof given by Gordon [1971, 4.2].

If  $A$  and  $B$  are distal in  $PX$  then there exists a bounded  $z$ -map  $f$  which  $z$ -completely separates them. Then  $H = \{(x,y) : p_f(x,y) < 1\}$  is an entourage for both  $UX$  and  $U^*X$  with  $H(A) \cap B = \emptyset$ .

Conversely if  $A$  is distal from  $B$  in  $Q(UX)$  or  $Q(U^*X)$  then there is an entourage  $H$  such that  $H(A) \cap B = \emptyset$ . Then  $(x,y) \notin H$  for each  $x \in A$  and  $y \in B$ . In both cases there are  $f_1, \dots, f_n \in S(X)$  and  $\epsilon_1, \dots, \epsilon_n > 0$  such that:

$$\{(x,y) : |f_i(x) - f_i(y)| < \epsilon_i, \forall i\} \subset H,$$

by the metrization lemma.

If we let  $g_i = \left(\frac{1}{\epsilon_i}\right)f_i$  for each  $i$ , then if  $x \in A$  and  $y \in B$ ,  $|g_i(x) - g_i(y)| \geq 1$  for some  $i$ . Define  $g : X \rightarrow \mathbb{R}^n$  by  $g(x) = (g_1(x), \dots, g_n(x))$ . Then  $d(gA, gB) \geq 1$  where  $d$  is the usual Euclidean metric on  $\mathbb{R}^n$ . Define  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $f(t) = d(t, gB) \wedge 1$ . Then  $f$  is continuous and  $f.g$  is a  $z$ -map which  $z$ -completely separates  $A$  and  $B$ .

2.2.8 Theorem. Let  $X \in |\underline{\text{Zero}}|$ .  $UX$  and  $U^*X$  are delivered by functors  $U$  and  $U^*$  respectively which are right inverses of the forgetful functor  $F'' : \underline{\text{Un}} \rightarrow \underline{\text{Zero}}$ .

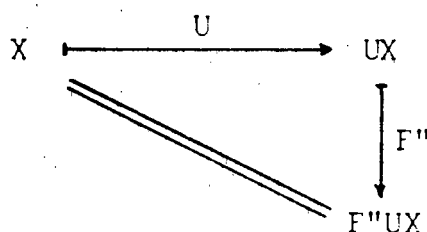
Proof.  $UX$  and  $U^*X$  are delivered by the obvious functors which for convenience we denote by  $U$  and  $U^*$  respectively. We need only check that  $U^*f \in \underline{\text{Un}}(U^*X, U^*Y)$  for each  $f \in \underline{\text{Zero}}(X, Y)$ .

$U^*Y$  is defined by pseudo-metrics of the form  $p_g$  with  $g \in S^*(Y)$ . Since  $f$  is a  $z$ -map,  $f.g \in S^*(X)$ . Thus

$$(a,b) \longmapsto p_g(fa, fb) = p_{g.f}(a,b)$$

is one of the pseudometrics defining  $U^*X$  and hence  $f$  is uniformly continuous.

To show that  $U$  and  $U^*$  are right inverses we show that the following diagram commutes for each  $X \in |\underline{\text{Zero}}|$ . (That the diagram for  $U^*$  is commutative is shown in an analogous way.)



If  $X \in |\underline{\text{Zero}}|$  let  $A \in \text{z-str } X$ . Then  $A = Zf$  for some  $f \in \mathcal{S}(X)$ . But  $f \in \underline{\text{Un}}(UX, \mathbb{R}_{\text{su}})$  by definition of  $UX$ . Thus  $A \in \text{z-str } F''UX$ .

Conversely let  $A \in \text{z-str } F''UX$ . Then  $A$  is a countable intersection of zero-sets of uniformly continuous functions from  $UX$  to  $\mathbb{R}_{\text{su}}$ . But if  $f \in \underline{\text{Un}}(UX, \mathbb{R}_{\text{su}})$  then  $f \in \underline{\text{Prox}}(QUX, \mathbb{R}_{\text{sp}}) = \underline{\text{Prox}}(PX, \mathbb{R}_{\text{sp}})$  since each uniformly continuous map is a proximity map for the underlying proximity structures and  $QUX = PX$  by 2.2.7. Thus, by applying 2.2.3,  $f \in \mathcal{S}(X)$ .

Then  $A$  is a countable intersection of zero-sets of functions in  $\mathcal{S}(X)$  and is thus a zero-set of  $X$ .

2.2.9 Corollary.  $P = QU = QU^*$ .

Proof. This follows immediately from 2.2.7.

2.2.10 Remark. Neither  $U\mathbb{R}$  nor  $U^*\mathbb{R}$  is the usual (metric) uniformity on  $\mathbb{R}$ . Since  $S(\mathbb{R}) = C(\mathbb{R})$ , the uniform structure generated by  $\{p_f : f \in S(\mathbb{R})\}$  is the initial uniform structure for  $C(\mathbb{R})$  to  $\mathbb{R}_{su}$ . Thus each  $f \in C(\mathbb{R})$  is uniformly continuous in this structure. But not all continuous functions are uniformly continuous with respect to the metric uniformity on  $\mathbb{R}$ .

Notice however that  $F''\mathbb{R}_{su} = \mathbb{R}_Z = F''U^*\mathbb{R}_Z$ .

2.2.11 Proposition. Let  $X \in |\underline{\text{Zero}}|$ . Then

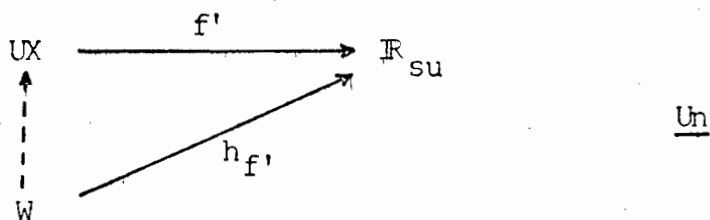
$$UX = \text{in}_{GF''} (Gf, \mathbb{R}_{su} : f \in S(X))$$

and

$$U^*X = \text{in}_{GF''} (Gf, \mathbb{R}_{su} : f \in S^*(X)).$$

Proof. Let  $f \in S(X)$  lift to  $f' \in \underline{\text{Un}}(UX, U\mathbb{R}_Z)$ . It is clear that  $U\mathbb{R}_Z$  is finer than  $\mathbb{R}_{su}$ . Thus

$f' \in \underline{\text{Un}}(UX, \mathbb{R}_{su})$  and  $GF''f' = Gf$ .



$$\begin{array}{ccc}
 GX & \xrightarrow{Gf} & \mathbb{R} \\
 \uparrow k & \nearrow GF''h_{f'} & \\
 GF''W & & 
 \end{array}$$

Ens

We need only show that  $\{f'\}$  is an initiality at  $UX$ .

Let  $h_{f'} \in \underline{\text{Un}}(UX, \mathbb{R}_{\text{su}})$  and  $k : GF''W \rightarrow GX$  be given such that  $GF''f' \cdot k = Gf \cdot k = GF''h_{f'}$  for each  $f \in S(X)$ .

Consider  $(a, b) \longmapsto p_{f'}(ka, kb) = p_{h_{f'}}(a, b)$ . Then  $h_{f'} \in \underline{\text{Un}}(W, \mathbb{R}_{\text{su}})$ , hence  $p_{h_{f'}}$  is a pseudometric in  $W$  for each  $f \in S(X)$ . Thus  $k$  lifts to a uniformly continuous map  $: W \rightarrow UX$ .

It is well known that the complete uniform spaces form an epireflective subcategory of Uns. Let  $\gamma$  denote the completeness epireflector in Uns. The following result has a well-known analogue for completely regular spaces.

2.2.12 Proposition.  $\gamma U^* \cong U^* \beta$ .

Proof.  $\beta X$  being compact is the completion of  $X$  for some uniform structure on  $X$ .  $\beta X$  admits a unique uniformity and  $U^* \beta X = \mathcal{U}^*(\beta X)$  where  $\mathcal{U}^*$  is the uniform structure defined by pseudometrics of the form  $p_f$  where  $f \in C^*(X)$ .  $\beta X$  can be considered as a separated completely regular space hence  $\mathcal{U}^*(\beta X)$  is admissible [Gillman and Jerison, 1960, 15.6].

Thus the uniformity admitted by  $\beta X$  is  $U^*\beta X$ .

But  $U^*\beta X|_X = U^*X$ , hence  $\beta X$  is the completion of  $X$  for  $U^*X$ .

2.2.13 Proposition. Let  $X \in |\underline{\text{Zero}}|$ .  $U^*X$  is the finest admissible precompact uniform structure on  $X$ .

Proof. A uniform structure is precompact if its completion is compact. By 2.2.12  $U^*X$  is certainly precompact. In fact every structure generated by a subfamily of pseudometrics of the form  $p_f$ ,  $f \in S^*(X)$  is precompact since  $U^*X$  is finer than each of these structures. Thus we need only show that every admissible precompact structure is generated by a subfamily of  $\{p_f : f \in S^*(X)\}$ .

If  $D$  is an admissible precompact structure on  $X$ , then  $\gamma D$  is compact. Being compact  $F''\gamma D$  admits a unique uniform structure, thus  $\gamma D = U^*(F''\gamma D)$ .

But  $U^*(F''\gamma D)$  is generated by pseudometrics of the form  $p_f$  where  $f \in S^*(F''\gamma D)$ .

Thus  $\gamma D|_X = D$  is generated by pseudometrics of the form  $p_g$  where  $g \in S^*(F''\gamma D)|_X \subset S^*(X)$ .

2.2.14 Proposition.  $U^*$  is the weakest right inverse of  $F''$ .

Proof. Let  $X \in |\underline{\text{Zero}}|$ . We must show that  $U^*X \leq LX$  for any right inverse  $L$  of  $F''$ .

Let  $L$  be a right inverse of  $F''$ . If  $f \in \underline{\text{Zero}}(X, I_2)$

then  $Lf \in \underline{\text{Un}}(LX, LI_Z)$ . But  $I_Z$  is compact, hence  $LI_Z = I_{\text{su}}$ . Thus  $Lf \in \underline{\text{Un}}(LX, I_{\text{su}})$ .

Hence by 2.2.11,  $U^*X \leq LX$ .

We have now developed enough machinery to study the functor  $P : \underline{\text{Zero}} \rightarrow \underline{\text{Prox}}$  in greater detail.

2.2.15 Theorem.  $P$  is the only right inverse of the forgetful functor  $F' : \underline{\text{Prox}} \rightarrow \underline{\text{Zero}}$ . Moreover  $P$  is left adjoint to  $F'$ .

Proof. That  $P$  is a right inverse of  $F'$  is a straightforward verification involving 2.2.3.

By the well-known isomorphism of  $\underline{\text{Prox}}$  to the category of precompact uniform spaces, uniqueness of  $P$  will follow if we show that there is a unique right inverse of the restriction of  $F'' : \underline{\text{Un}} \rightarrow \underline{\text{Zero}}$  to the precompact spaces. This follows immediately from 2.2.13 and 2.2.14.

By 2.1.4  $P$  is left adjoint to  $F'$  if  $1 \leq_F, PF'$ .

Let  $X \in |\underline{\text{Prox}}|$  and let  $A$  and  $B$  be distal in  $X$ . Then there exists  $f \in \underline{\text{Prox}}(X, \mathbb{R}_{\text{sp}})$  which separates  $A$  and  $B$  [Čech, 1966, 25C5]. Then  $F'f \in \underline{\text{Zero}}(F'X, \mathbb{R}_Z)$   $z$ -completely separates  $A$  and  $B$ . Thus  $A$  and  $B$  are distal in  $PF'X$ , i.e.  $X \leq_F, PF'X$  for each  $X \in |\underline{\text{Prox}}|$ .

2.2.16 Remarks. (1)  $P$  is not an isomorphism, since not

every proximity structure arises from a z-structure. For example,  $\mathbb{R}$  admits a unique z-structure but many proximity structures. In general, a proximity structure arising from a z-structure has the property that its proximity functions form a ring, but as mentioned earlier, an arbitrary proximity structure need not have this property.

(2) It should be remarked here that the usual forgetful functor  $K : \underline{\text{Prox}} \rightarrow \underline{\text{Crg}}$  has a unique right inverse  $V$  which is a left adjoint. A proof as in 2.2.15 can be given based on the result of Brümmer [1970, 1971] that the forgetful functor  $\underline{\text{Pcu}} \rightarrow \underline{\text{Crg}}$  from the category of precompact uniform spaces to the category of completely regular spaces has a unique right inverse.

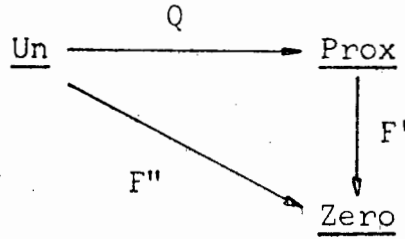
(3)  $P$  has a right adjoint  $F'$  and is an embedding. Thus  $\underline{\text{Zero}}$  is isomorphic to a full bicoreflective subcategory of  $\underline{\text{Prox}}$ .

We now show that the arrows in Figure 1 commute. This is summarised by the following theorem.

2.2.17 Theorem.

- (1)  $F'Q = F''$
- (2)  $FF' = K$
- (3)  $FF'' = T$ .

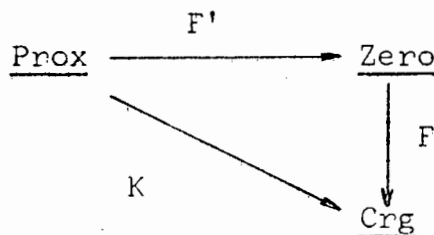
Proof. (1)



Let  $X \in |\underline{\text{Un}}|$ . Any zero-set  $A$  of  $F''X$  is a countable intersection of finite unions of zero-sets of uniformly continuous functions  $f \in \underline{\text{Un}}(X, \mathbb{R}_{\text{su}})$ . Such  $f$  is in  $\underline{\text{Prox}}(QX, \mathbb{R}_{\text{sp}})$  and hence by definition of  $F'$ ,  $A \in \text{z-str } F'QX$ .

Conversely, if  $A \in \text{z-str } F'QX$  then it is a countable intersection of finite unions of zero-sets of proximity functions  $g \in \underline{\text{Prox}}(QX, \mathbb{R}_{\text{sp}})$ . Then  $(g \vee 1) \wedge 1 = h \in \underline{\text{Prox}}(QX, \mathbb{I}_{\text{sp}})$  and  $Zg = Zh$ . Since  $\mathbb{I}_{\text{sp}}$  is compact we can apply 12.11 of Naimpally and Warrack [1970] to  $h$ . Hence  $h \in \underline{\text{Un}}(X, \mathbb{I}_{\text{su}})$ . Thus  $A \in \text{z-str } F''X$ .

(2)

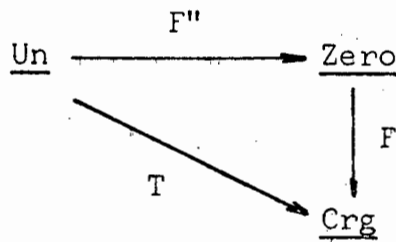


Let  $X \in |\underline{\text{Prox}}|$ . If  $A$  is closed in  $FF'X$  then it is an intersection of zero-sets of  $F'X$  and thus an intersection of finite unions of zero-sets of proximity functions  $f \in \underline{\text{Prox}}(X, \mathbb{R}_{\text{sp}})$ . Such  $f$  is in  $\underline{\text{Crg}}(KX, \mathbb{R})$ . Thus  $A$  is closed in  $KX$ .

Conversely, if  $A$  is closed in  $KX$  then each  $y \in X-A$  is distal from  $A$ . Hence we may choose  $f_y \in \underline{\text{Prox}}(X, \mathbb{R}_{sp})$  such that  $f_y(A) = 0$  and  $f_y(y) = 1$ .

Clearly  $A = \bigcap \{Zf_y : y \in X-A\}$  and since the zero-sets of proximity functions form a subbase for the closed sets of  $FF'X$ ,  $A$  is closed in  $FF'X$ .

(3)



Let  $X \in |\underline{\text{Un}}|$ . The zero-sets of uniformly continuous functions on  $X$  form a base for the closed sets of  $FF''X$ . Thus each set closed in  $FF''X$  is clearly closed in  $TX$ .

If  $A$  is closed in  $TX$  then  $A = \bigcap \{x : d[x, A] = 0, d \in \text{u-str } X\}$ . Thus  $A$  is an intersection of zero-sets of uniformly continuous functions on  $X$ . Thus  $A$  is closed in  $FF''X$ .

Further relationships between the forgetful functors and their right inverses are given in the following theorem. The arrows involved are indicated in Figure 2.

2.2.18 Theorem. (1)  $\mathcal{E}^*F = U^*$

$$(2) F'V = R$$

$$(3) PR = V$$

$$(4) U^*R = \mathcal{E}^*$$

$$(5) Q\mathcal{E}^* = V$$

$$(6) F''H = R$$

where  $H$  is any right inverse of the forgetful functor  $T : \underline{\text{Un}} \rightarrow \underline{\text{Crg}}$ . In (1), (4) and (5) the functors are restricted to the separated categories.

Proof. (1) We have that:

$$\gamma\mathcal{E}^*F \cong \mathcal{E}^*\beta F = U^*\beta \cong \gamma U^* \quad (2.2.12)$$

Thus  $\gamma\mathcal{E}^*F \cong \gamma U^*$  and  $U^* = \mathcal{E}^*F$ .

$$(2) F(F'V) = (FF')V = KV = 1 \quad (2.2.17 (2), 2.2.16 (2)).$$

But  $R$  is the unique right inverse of  $F$  (2.1.5), thus  $F'V = R$ .

(3)  $FF'.PR = F(F'P)R = F1R = 1$  (2.2.15). By uniqueness  $PR = V$ .

(4) By (1)  $\mathcal{E}^*F = U^*$ , hence  $\mathcal{E}^*FR = U^*R$  and since  $FR = 1$ ,  $\mathcal{E}^* = U^*R$ .

$$(5) Q\mathcal{E}^* = QU^*R = PR = V \quad (2.2.9).$$

(6) Let  $H$  be any right inverse of  $T = FF''$  (2.2.17 (3)).  $FF''H = TH = 1$ .

But  $R$  is the unique right inverse of  $F$ , hence  $F''H = R$ .

In particular:  $F''\mathcal{E} = R$

$$F''\mathcal{E}^* = R.$$

2.3 NOTES

The structures  $PX, U^*X, UX$  were introduced by Gordon [1971], who also showed that they are compatible with  $X$  (for a separated  $z$ -space  $X$ ). We have added the considerations about functoriality, adjointness, and uniqueness of right inverses. The obvious definition of compactness is Gordon's as well as results 2.1.12 (1), 2.1.13, 2.2.3 and 2.2.7.

With the exception of 2.1.4 (1) and 2.1.8 which are due to Brümmer [1971] the remaining results are our own. The formulation of 2.1.6 as the general result 2.1.4 (2) was suggested by my supervisor. The latter result occurs without proof in [Brümmer, 1971].

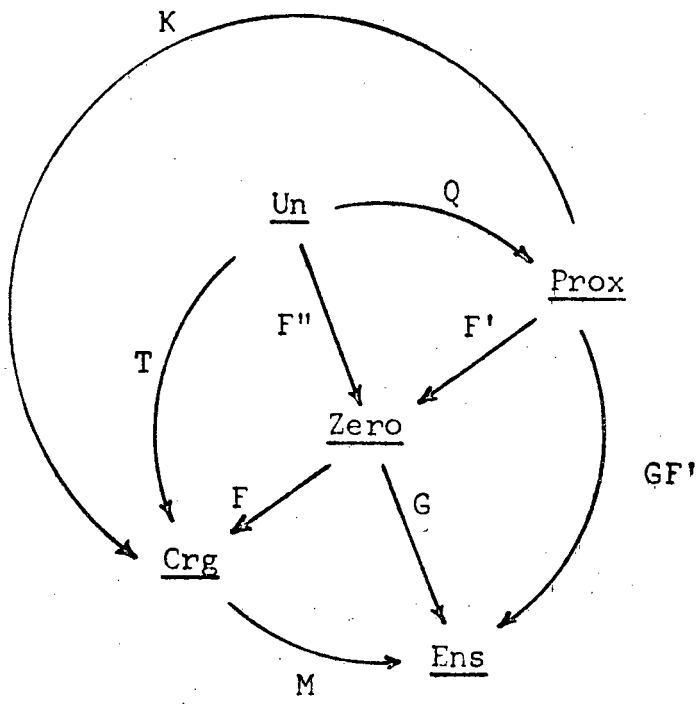


Figure 1

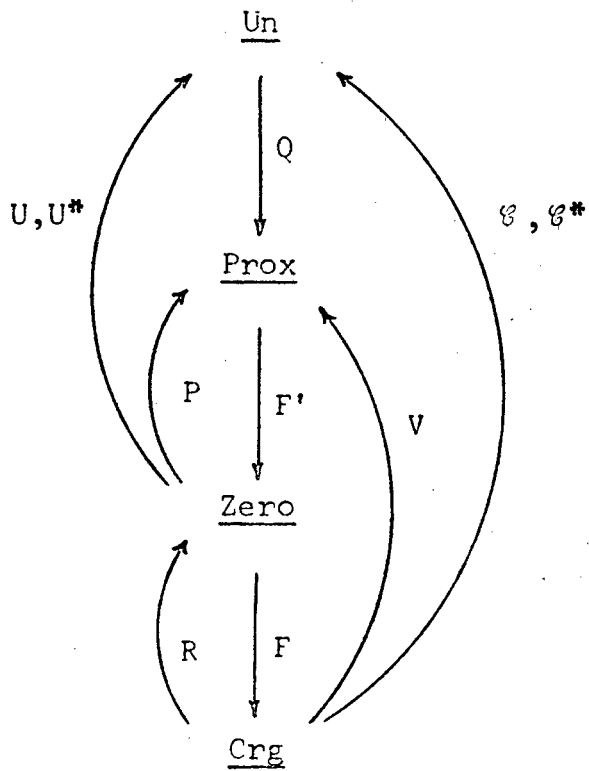


Figure 2

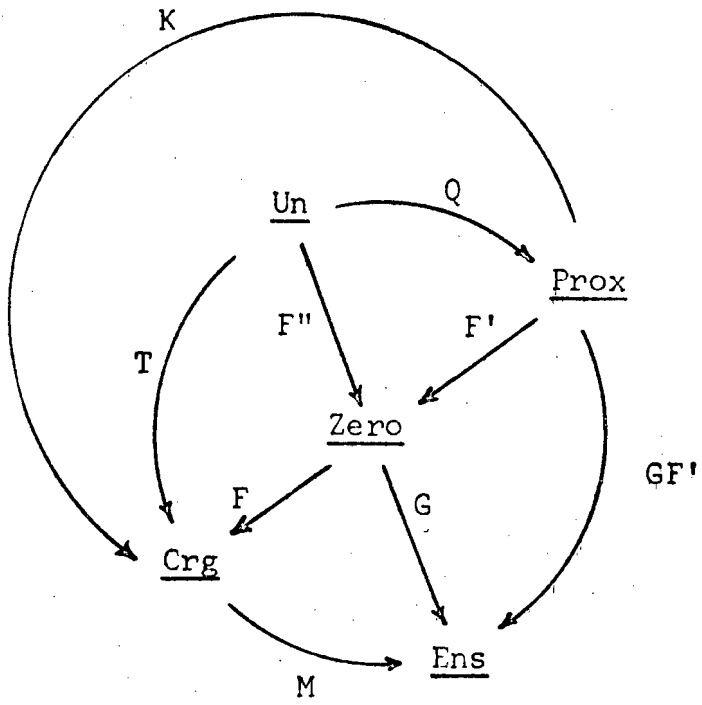


Figure 1

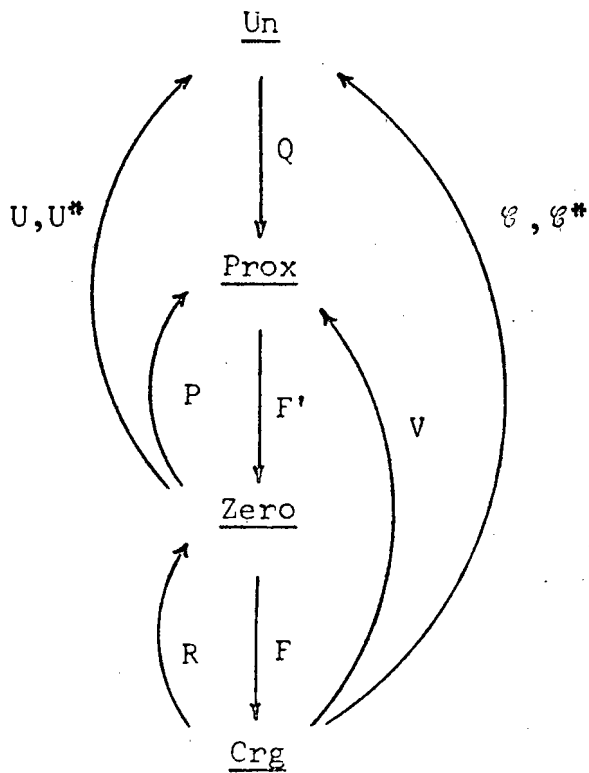


Figure 2

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We give the following counterexample;  $T$  the Tychonoff plank is pseudocompact (in Crg) and admits a unique uniform structure, hence a unique  $z$ -structure  $RT$ . Thus  $C(T) = S(RT)$  and  $RT$  is pseudocompact (in Zero). But  $T$ , and hence  $RT$ , contains a closed, countable, discrete subset  $N$  which is not pseudocompact. Also  $C(N) = S(N)$  since  $N$  has a countable base. Thus  $N$  is not pseudocompact in Zero.

Consequently, pseudocompactness is not epi-reflective in Zeros.

### 3.4 NOTES

The results and definitions of section 3.1, in one form or another, are, with the exception of 3.1.1, 3.1.6 and 3.1.7 those of Gordon [1971]. It should be noted that 3.1.1 is probably not in its most general form (cf. Gillman and Jerison [1960, 6.4]), but time and the fact that more generality was not needed dictated its form.

Results 3.2.2, 3.2.3, 3.2.4, 3.2.9 and 3.2.10 are due to Gordon [1971] and Lorch [1963]. Moreover Gordon proves the reverse implication of 3.2.10. The method we used to obtain this implication however, led to further characterizations of pseudocompactness and used machinery which had already been developed for other considerations such as the notion of  $z$ -complete separation in section 1.2, the characterization of epi-reflectors in section 1.4 as well as 2.2.13.

The remaining results of this section were, on the whole, motivated by problems from Gillman and Jerison [1960]. Problem 3.3.2(2) was posed by my supervisor.

Further properties of pseudocompactness and realcompactness for  $z$ -spaces can be found in Gordon [1971]. We note in particular that the product of pseudocompact spaces is pseudocompact. The obvious question to ask is whether or not pseudocompactness is closed hereditary in Zeros.

(3) In the study of the right inverses  $U$  and  $U^*$  of  $F'' : \underline{Un} \rightarrow \underline{Zero}$  we had to invoke the induced proximity structure  $PX$ . This would not be necessary if  $F''$  had a lari i.e. a left adjoint right inverse. Existence of a lari is equivalent in our situation to saying that  $F''$  preserves initiality [Brümmer, 1971, 1.6.9].

Thus: does  $F''$  preserve initiality (or cointiality)?

(4) Zero is embedded in Prox as a full bicoreflective subcategory (2.2.16(3)). Herrlich [1973] shows that Prox is embedded as a bireflective full subcategory  $P\text{-Near}$  of Near, the category of nearness spaces and nearness preserving maps. How is Zero embedded in Near, and what is an internal characterization of the z-spaces in Near?

(5) In 2.1.10 we observed that the realcompact completely regular spaces lift against  $F : \underline{Zero} \rightarrow \underline{Crg}$  to a full epi-reflective subcategory of Zeros. We have failed to show that a realcompact z-space has a realcompact underlying topology or vice versa, i.e. is  $FuX = uFX$ ? (where the different meanings of  $u$  should be noted).

(6) It is apparent in 2.2.18 that we have no knowledge about the relationship between  $\mathcal{C}$  and  $U$ . There is probably some connection between this problem and (5) above (see 3.1.8).

### 3.3 PROBLEMS AND APPLICATIONS

3.3.1 Applications. (1) The non-separated z-spaces provide a natural setting in which to develop dimension theory. The subspace theorem does not in general hold in arbitrary topological spaces [Gillman and Jerison, 1960, 16M9].

Canfell [1973] defines a covering dimension for z-spaces and shows that the subspace theorem, product theorem and the sum theorem hold. Moreover he deduces a subspace theorem for an arbitrary topological space  $X$  by considering the zero-sets of functions in  $C(X)$ .

(2) Gordon [1971] discusses the Baire sets and Baire functions derived from a z-space and shows that this Baire system is a z-space. Then a z-space is realcompact if and only if the derived Baire system is.

3.3.2 Problems. (1) In 1.4.8 we characterised the multiple equalizers in Zeros, the category of separated z-spaces. What are the equalizers (of pairs) in Zeros?

(2) We saw in 2.1.3(2) that a completely regular space with the property that every open subspace is Lindelöf admits a unique z-structure. The converse is false. A separated completely regular Lindelöf space is normal and, for example, the Tychonoff plank  $T$  is a space which admits a unique uniform structure (and hence a unique z-structure) but is not normal.

What is an internal characterization of those spaces which admit a unique z-structure?

3.2.15 Proposition. A z-space  $X$  is pseudocompact if and only if  $MX$  is the same for each right inverse  $M$  of  $F'' : \underline{Un} \rightarrow \underline{Zero}$ .

Proof. If  $X$  is pseudocompact then it admits a unique uniform structure.

Conversely, if each right inverse  $M$  of  $F''$  is such that  $MX$  is the same then  $UX = U^*X$  and  $X$  is pseudocompact.

3.2.16 Summary. The following conditions are equivalent for a z-space  $X$ .

- (1)  $X$  is pseudocompact
- (2)  $S(X) = S^*(X)$
- (3) Every admissible uniform structure is precompact
- (4)  $X$  admits a unique compactification
- (5)  $X$  admits a unique precompact uniform structure
- (6)  $X$  admits a unique uniform structure
- (7)  $UX = U^*X$
- (8)  $MX$  is the same for each right inverse  $M$  of  $F'' : \underline{Un} \rightarrow \underline{Zero}$ .

Hence for every admissible precompact uniform structure  $\mathfrak{U}$  on  $X$ , disjoint zero-sets of  $X$  can be separated by a uniformly continuous function  $g = f|_Y$  on  $\gamma Y|_Y = Y$  to  $\mathbb{R}_{su}$ .

Hence  $U^*X$  is the unique admissible precompact uniform structure on  $X$  (3.2.8).

Thus by 3.2.7 if  $X$  is pseudocompact then  $X$  admits a unique uniform structure.

Conversely, if  $X$  admits a unique uniform structure then this structure is  $U^*X$  and the condition of 3.2.7 is satisfied for  $X$  to be pseudocompact.

3.2.13 Remark. Each compactification of  $X$  is the completion for some precompact uniform structure on  $X$ . Thus if  $X$  admits a unique uniform structure, which is thus precompact, then  $X$  admits a unique compactification. Thus we may replace "Hence" in 3.2.12 throughout by "if and only if".

3.2.14 Proposition. A  $z$ -space  $X$  is pseudocompact if and only if  $UX = U^*X$ .

Proof. If  $X$  is pseudocompact then it is immediately apparent from the respective definitions that  $UX = U^*X$ .

Conversely, if  $X$  is not pseudocompact then  $UX \neq \beta X$ . Thus  $\gamma(UX) \neq \gamma(U^*X)$  and, since  $UX$  and  $U^*X$  have the same underlying sets,  $UX \neq U^*X$ .

3.2.10 Corollary. If a  $z$ -space  $X$  is pseudocompact then it admits a unique compactification (that is, unique up to an isomorphism leaving  $X$  pointwise fixed).

Proof. Let  $X$  be pseudocompact and let  $Y$  be any compactification of  $X$ . By 3.2.9 there exists  $X'$  isomorphic to  $\nu X$  with  $X \subset X' \subset Y$ . But  $\nu X = \beta X$ , hence  $X'$  is compact and thus closed in  $Y$ . It follows that  $X' = Y$  and  $\beta X$  is the unique compactification of  $X$ .

3.2.11 Remark. The above result, which is due to Gordon, does not hold for Tychonoff spaces. The following characterizations of pseudocompactness depend to a large extent on this result.

3.2.12 Theorem. A  $z$ -space  $X$  is pseudocompact if and only if  $X$  admits a unique uniform structure.

Proof. Let  $X$  be pseudocompact

Hence  $X$  admits a unique compactification (3.2.10)

Hence for every admissible precompact uniform structure  $Y$  on  $X$ ,  $\gamma Y = U^* \beta X$ .

Hence for every admissible precompact uniform structure  $Y$  on  $X$ , since  $X$  is  $S^*$ -embedded in  $\beta X$  (3.1.2), disjoint zero-sets of  $X$  can be separated by an  $f \in C(\beta X) = C(F''\gamma Y)$  which is uniformly continuous for  $\gamma Y$  to  $\mathbb{R}_{su}$ , since  $\gamma Y$  is compact.

separates  $A$  and  $B$ . Hence  $X$  is  $S^*$ -embedded in  $F''\gamma Y$  (1.2.6). But  $\beta X$  is the unique compactification in which  $X$  is  $S^*$ -embedded (3.1.2). Thus, since  $U^*\beta \cong \gamma U^*$ , we have that  $\gamma U^*X$  is uniformly isomorphic to  $\gamma Y$ . As  $U^*X$  and  $Y$  have the same underlying set, it follows that  $U^*X = Y$ .

It only remains to show that  $U^*X$  has the required property. Let  $A$  and  $B$  be disjoint zero-sets of  $X$ . Then  $\text{cl}_{\beta X} A \cap \text{cl}_{\beta X} B = \emptyset$  (3.1.2). Thus there exists  $f \in C(\beta X) = S(\beta X)$  which  $z$ -completely separates  $\text{cl}_{\beta X} A$  and  $\text{cl}_{\beta X} B$ . But each  $f \in C(\beta X)$  is uniformly continuous. It follows that  $A$  and  $B$  will be separated by the restriction of  $f$  to  $X$ . But  $\beta X|_X = \gamma U^*X|_X = U^*X$ . Thus  $f|_X$  is uniformly continuous for  $U^*X$  to  $\mathbb{R}_{su}$ .

3.2.9 Theorem. Let  $Y$  be a realcompact  $z$ -space and  $X \subset Y$ . Then there is an  $X' \subset Y$  which is  $uX$  in the sense that is isomorphic as a  $z$ -space to  $uX$  under an isomorphism leaving  $X$  pointwise fixed.

Proof. We give a brief sketch of the proof given by Gordon [1971, 7.9].

Let  $i: X \rightarrow Y$  be the inclusion map. By the epi-reflective property of  $u$  in Zeros we can extend  $i$  to a  $z$ -map  $i^u: uX \rightarrow Y$ . Then  $i^u$  is injective.

Set  $X' = i^u(uX)$ . The proof is completed by observing that  $i^u$  is zero-set preserving, that is if  $A \in z\text{-str } uX$  then  $i^u A \in z\text{-str } X'$ , and  $i^u$  is thus an isomorphism.

uniform structure on  $X$  (2.2.13). Thus  $UX$  is not precompact.

Conversely, let  $Y$  be an admissible uniform structure on  $X$  which is not precompact. Then there exists, for some  $d \in \mathcal{U}\text{-str}Y$ , an infinite  $d$ -discrete set  $A$  in  $Y$  [Gillman and Jerison, 1960, 15.16].

We can define an unbounded function  $g$  on  $A$  to  $\mathbb{R}$  in the obvious way. Since  $A$  is  $d$ -discrete,  $g$  is a  $z$ -function  $: A \rightarrow \mathbb{R}_z$ . But each  $d$ -discrete set is a zero-set of  $X$  and hence  $A$  is  $S$ -embedded in  $X$  (1.2.8). Hence we can extend  $g$  to a  $z$ -function on  $X$  which is unbounded on  $X$ . Thus  $S^*(X) \neq S(X)$  and  $X$  is not pseudocompact (3.2.4).

Before we give further characterizations of pseudocompactness we need the following two results.

3.2.8 Proposition.  $U^*X$  is the unique admissible precompact uniform structure on a  $z$ -space  $X$  in which any two disjoint zero-sets of  $X$  can be separated by a uniformly continuous function.

Proof. We firstly show uniqueness. Let  $Y$  be an admissible precompact uniform structure with the property that any two disjoint zero-sets of  $X$  can be separated by a function which is uniformly continuous with respect to  $Y$ .

Given disjoint zero-sets  $A$  and  $B$  of  $X$ , let  $f$  be a bounded uniformly continuous function on  $Y$  (to  $\mathbb{R}_{Su}$ ) which completely separates  $A$  and  $B$ . Then  $f$  can be extended to  $f^*$  on  $\gamma Y$ . Since  $\gamma Y$  is compact,  $f^*$  is bounded and  $z$ -completely

3.2.4 Theorem. A z-space  $X$  is pseudocompact if and only if each z-function on  $X$  is bounded.

Proof. By 1.1.13, the existence of an unbounded z-function on  $X$  is equivalent to the existence of a bounded z-function  $f$  with  $f(x) > 0$  for all  $x \in X$  but  $\inf \{f(x) : x \in X\} = 0$ . Such  $f$  extends to a z-function  $f^\beta$  on  $\beta X$  which vanishes somewhere on  $\beta X$  but nowhere on  $X$ . By 3.2.2 the proof is complete.

3.2.5 Definition. Let  $Y$  be a uniform space, and let  $d \in \text{u-str } Y$ . A subset  $A$  of  $Y$  is d-discrete if there exists  $\delta > 0$  such that  $d(x,y) \geq \delta$  for each  $x \neq y$  in  $A$ . A set  $B$  in  $Y$  is d-closed if  $d\text{-cl } B = B$  where  $d\text{-cl } B = \{x : d[A,x] = 0\}$ .

3.2.6 Remark. Let  $Y$  be any admissible uniform structure on a z-space  $X$ , i.e.  $F''Y = X$ . Since  $x \longmapsto d[A,x]$  is uniformly continuous for  $Y$  to  $\mathbb{R}_{su}$ , every d-closed set in  $Y$  is the zero-set of a z-map:  $F''Y = X \rightarrow \mathbb{R}_2$ . But each d-discrete set in  $Y$  is d-closed in  $Y$  and hence is a zero-set in  $X$ .

3.2.7 Theorem.  $X \in |\underline{\text{Zero}}|$  is pseudocompact if and only if every admissible uniform structure is precompact.

Proof. If  $X$  is not pseudocompact then  $UX \neq \beta X$  and hence  $\gamma(UX) \neq \gamma(U^*X)$ . But  $UX$  and  $U^*X$  have the same underlying set hence  $UX \neq U^*X$ . But  $U^*X$  is the finest admissible precompact

### 3.2 PSEUDOCOMPACT Z-SPACES

In this section we give Gordon's definition of pseudocompactness for a z-space and also some characterizations of pseudocompactness.

The method of study is analogous to that for completely regular spaces, though some of the results obtained are unlike those for completely regular spaces.

3.2.1 Definition. A z-space  $X$  is pseudocompact if it has no hyperreal z-ultrafilter; that is, if  $\nu X = \beta X$ .

The following two propositions are formulations, due to Gordon, of two theorems of Lorch [1963, Theorems 11 and 9].

3.2.2 Proposition. Let  $X \in |\underline{\text{Zeros}}|$  and let  $Y$  be any compactification of  $X$ . Then  $X$  is pseudocompact if and only if every non-empty zero-set of  $Y$  meets  $X$ .

3.2.3 Proposition. Let  $X \in |\underline{\text{Zeros}}|$  and let  $Y$  be any compactification of  $X$ . Then  $X$  is realcompact if and only if each  $p \in Y - X$  is contained in a zero-set of  $Y$  which does not meet  $X$ .

## CHAPTER 3 : PSEUDOCOMPACT Z-SPACES

### 3.1 COMPACT AND REALCOMPACT Z-SPACES

Throughout this chapter we will only consider the separated z-spaces unless we specifically state otherwise.

Gordon [1970] constructs the usual compactification determined by a uniformly closed ring of bounded functions, which contains the constants and separates points, on a set  $X$ . If  $S^*(X)$ , where  $X \in |\underline{\text{Zeros}}|$ , is the ring under consideration, then the compactification of  $X$  resulting is a compactification of  $X$  as a z-space and is characterised as the unique (up to an isomorphism leaving  $X$  pointwise fixed) compactification of  $X$  in which  $X$  is  $S^*$ -embedded [Gordon, 1971, 5.5]. Thus since each  $f \in S^*(X)$  is a z-map into the compact subset  $\overline{f(X)}$  of  $\mathbb{R}$  and since  $\beta X$  is epireflective in  $\underline{\text{Zeros}}$ ,  $\beta X$  (as defined in 2.1.12 (2)) is for all purposes precisely Gordon's compactification as described above.

3.1.1 Theorem. Let  $Y$  be any compactification of  $X \in |\underline{\text{Zeros}}|$ . Then the following statements are equivalent:

- (1)  $X$  is  $S^*$ -embedded in  $Y$  ;
- (2) Any two disjoint zero-sets of  $X$  have disjoint closures in  $Y$  .

Proof. (1)  $\Rightarrow$  (2): Suppose  $X$  is  $S^*$ -embedded in  $Y$  and that  $A$  and  $B$  are disjoint zero-sets of  $X$ . Then  $A$  and  $B$  are  $z$ -completely separated in  $X$  and thus, by 1.2.6, are  $z$ -completely separated in  $Y$ . Thus there exist disjoint zero-sets  $C$  and  $D$  of  $Y$  containing  $A$  and  $B$  respectively. The proof is completed by observing that  $C$  and  $D$  are closed in  $Y$ .

(2)  $\Rightarrow$  (1): If  $A$  and  $B$  are any two  $z$ -completely separated sets in  $X$  then they are contained in disjoint zero-sets  $C$  and  $D$  of  $X$ . By hypothesis  $\text{cl}_Y C \cap \text{cl}_Y D = \emptyset$ . But disjoint closed sets in a compact space are completely separated by a continuous, and hence a  $z$ -, map [Gillman and Jerison, 1960, 3.11]. Thus  $A$  and  $B$  are  $z$ -completely separated in  $Y$  and, by 1.2.6,  $X$  is  $S^*$ -embedded in  $Y$ .

3.1.2 Corollary.  $\beta X$  is the unique compactification of  $X \in |\text{Zeros}|$  satisfying either of the equivalent conditions in 3.1.1.

The notion of a  $z$ -filter is well-known in the theory of topological spaces. We now give an analogue for  $z$ -spaces.

3.1.3 Definition. Let  $X \in |\text{Zero}|$ . A non-empty subfamily  $\underline{F}$  of  $z\text{-str}X$  is called a  $z$ -filter on  $X$  provided that

- (1)  $\emptyset \notin \underline{F}$
- (2) If  $A, B \in \underline{F}$ , then  $A \cap B \in \underline{F}$
- (3) If  $A \in \underline{F}$ ,  $B \in z\text{-str}X$ , and  $B \supset A$ , then  $B \in \underline{F}$ .

A z-filter which is maximal, in the sense that it is contained in no other z-filter, is called a z-ultrafilter.

A z-ultrafilter is real if it is closed under countable intersection.

A z-ultrafilter is hyperreal if it contains a countable collection of sets whose intersection is empty. By a classical proof, a z-ultrafilter is hyperreal if and only if it is not real.

Every z-ultrafilter on  $X \in |\underline{\text{Zero}}|$  converges to some point of  $\beta X$ . Moreover each point of  $\beta X$  is the limit of a unique z-ultrafilter on  $X$  [Gordon, 1970, 3.2 and 3.3].

Let  $\nu X$  denote the z-subspace of  $\beta X$  which consists of the limit points of real z-ultrafilters on  $X$ . Each point  $x$  of  $X$  is the limit of the z-ultrafilter  $\underline{F}$  on  $X$  consisting of those zero-sets which contain  $x$ . Thus  $\underline{F}$  is real. Then  $X \subset \nu X$  and furthermore  $X$  is a dense z-subspace of  $\nu X$ . The analogue for  $\nu X$  in the completely regular spaces is the Hewitt real compactification.

3.1.4 Definition. A z-space  $X$  is realcompact if every real z-ultrafilter on  $X$  converges to a point of  $X$ .

3.1.5 Remark.  $X$  is realcompact if and only if  $\nu X = X$ .

3.1.6 Proposition. The realcompact  $z$ -spaces form a full epi-reflective subcategory of Zeros.

Proof. We show directly that the epi-reflective property holds.

Let  $X \in |\underline{\text{Zeros}}|$ , and let  $f : X \rightarrow Y$  be a  $z$ -map into a realcompact  $z$ -space  $Y$ . Then  $f$  lifts to  $Uf \in \underline{\text{Un}}(UX, UY)$  and as  $\gamma$  is an epi-reflector  $Uf$  has a unique uniformly continuous extension  $(Uf)' : \gamma UX \rightarrow \gamma UY$ . But Gordon [1971, 6.4] proved  $U\gamma Y = \gamma UY$  so that  $F''(Uf)'$  is the unique  $z$ -map  $UX \rightarrow UY = Y$ , which extends  $f$ . Denote the reflector by  $u$ . Then  $u$  is the realcompact epi-reflector in Zeros since  $X$  is dense in  $UX$  (1.4.6).

3.1.7 Corollary. Realcompactness in Zeros is productive and closed-hereditary.

3.1.8 Theorem. We have the natural equivalence

$$Uu \cong \gamma U .$$

Proof. As previously mentioned Gordon [1971, 6.4] proved  $UuX = \gamma UX$  for each  $X \in |\underline{\text{Zeros}}|$ . The naturality of  $\text{id}_X : UuX \rightarrow \gamma UX$  follows by the uniqueness of the extensions of maps from  $X$  to maps from  $uX$  and  $\gamma UX$  respectively.

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