

SOME ASPECTS OF THE MASS DEFORMED ABJM THEORY

Asadig Mohammed



A thesis submitted for the degree of Doctor of Philosophy in Applied Mathematics,
Department of Mathematics and Applied Mathematics, University of Cape Town.

Supervisor: Dr. Jeff Murugan

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Declaration

The research described in this dissertation was carried out in Department of Mathematics and Applied Mathematics, Faculty of Science, University of Cape Town. The work presented in this thesis is based on collaborations with my supervisor Dr. Jeff Murugan (Department of Mathematics and Applied Mathematics, University of Cape Town) and Prof. Houratiu Nastase (Instituto de Física Teórica of the Universidade Estadual Paulista). The results are original except where reference is made to the work of others. Below is the list of publications mainly discussed in this thesis:

1. **“Looking for a Matrix model of ABJM”**

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2. **“Towards a Realization of the Condensed-Matter/Gravity Correspondence in String Theory via Consistent Abelian Truncation”**

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3. **“Abelian-Higgs and Vortices from ABJM: towards a string realization of AdS/CMT”**

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I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other university for a degree and that it represents my own work.

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Asadig Mohammed

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Abstract

In this thesis, we discuss some aspects of the Aharony, Bergman, Jafferis & Maldacena (ABJM) theory. In particular, encouraged by the recent construction of fuzzy sphere solutions in the ABJM theory, we re-analyze the latter from the perspective of a Matrix-like model. In particular, we argue that a vortex solution exhibits properties of a supergraviton, while a kink represents a 2-brane. Other solutions are also consistent with the Matrix-type interpretation. We study vortex scattering and compare with graviton scattering in the massive ABJM background, however our results are inconclusive. We speculate on how to extend our results to construct a Matrix theory of ABJM.

We also present an embedding of the 3-dimensional relativistic Landau-Ginzburg model for condensed matter systems in an $\mathcal{N} = 6$, $U(N) \times U(N)$ Chern-Simons-matter theory (the ABJM model) by consistently truncating the latter to an abelian effective field theory encoding the collective dynamics of $\mathcal{O}(N)$ of the $\mathcal{O}(N^2)$ modes. In fact, depending on the VEV on one of the ABJM scalars, a mass deformation parameter μ and the Chern-Simons level number k , our abelianization prescription allows us to interpolate between the abelian Higgs model with its usual multi-vortex solutions and a ϕ^4 theory. We sketch a simple condensed matter model that reproduces all the salient features of the abelianization. In this context, the abelianization can be interpreted as giving a dimensional reduction from four dimensions.

Finally we present ansätze that reduce the mass-deformed ABJM model to gauged Abelian scalar theories, using the fuzzy sphere matrices G^α . One such reduction gives a Toda system, for which we find a new type of nonabelian vortex. Another gives the standard Abelian-Higgs model, thereby allowing us to embed all the usual (multi-)vortex solutions of the latter into the ABJM model. By turning off the mass deformation at the level of the reduced model, we can also continuously deform to the massive ϕ^4 theory in the massless ABJM case. In this way we can embed the Landau-Ginzburg model into the AdS/CFT correspondence as a consistent truncation of ABJM. In this context, the mass deformation parameter μ and a field VEV $\langle\phi\rangle$ act as g and g_c respectively, leading to a well-motivated AdS/CMT construction from string theory. To further this particular point, we propose a simple model for the condensed matter field theory that leads to an approximate description for the ABJM abelianization. Finally, we also find some BPS solutions to the mass-deformed ABJM model with a spacetime interpretation as an M2-brane ending on a spherical M5-brane.

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Introduction

Since its beginnings in 1997, the AdS/CFT correspondence [1] has found application in a variety of phenomena; not only in quantum gravity but also, increasingly in fields as diverse as low energy QCD and condensed matter. Its original formulation described four dimensional $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory with an $SU(N)$ gauge group in the large N limit from the perspective of a dual gravitational theory on $AdS_5 \times S^5$. As a toy model for the exploration of four dimensional QCD at strong coupling, $\mathcal{N} = 4$ SYM, with its large supersymmetry, conformal invariance, and large number of colors N (that ensures that the dual is just a gravitational theory and not a full string theory) is nearly ideal. By modifying this simple set-up, in particular by breaking supersymmetry and conformal invariance, a lot was learned about QCD itself as, for instance, in the Sakai-Sugimoto model [2]. A crucial part of this development is that the physics of gauge theories at finite temperature shows remarkable universality, which has translated into applications of $\mathcal{N} = 4$ SYM to the high temperature plasmas at RHIC and the ALICE experiment at the LHC (see for example [3] for an extensive review and references).

With the discovery of the pp-wave/BMN correspondence in 2002 [4] came the realization of the importance of operators with large R-charge to a full string theory (and not just supergravity) description of the dual. This, in turn, led to the description of spin chains from string theory [5] and, more generally, to an understanding of the integrable

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structures on both sides of the correspondence. In a sense, this was the precursor to the application of the gauge/gravity duality to condensed matter physics. More recently, based on the earlier Bagger, Lambert & Gustavsson (BLG) construction [17, 18, 19, 20] of an M2-brane action in terms of 3-algebras, Aharony, Bergman, Jafferis & Maldacena (ABJM) [21] constructed an $\mathcal{N} = 6$ supersymmetric action for the IR of N M2-branes probing a C^4/Z_k singularity, a $U(N) \times U(N)$, level- k Chern-Simons-matter gauge theory whose fields transform in the bifundamentals. Not only was this the first time that an action for an arbitrary number of M2-branes was written down but it also allowed further insight into the structure of M-theory, as well as a new and exciting example of the AdS/CFT with its gravity dual being the large k limit of $AdS_4 \times S^7/Z_k$, i.e. $AdS_4 \times CP^3$.

But while most of the work on the ABJM model concentrated either on the field theory side, or on the AdS/CFT duality, it is worth bearing in mind that the original interest in the multiple M2-brane system was, of course, the potential for a better understanding of M-theory. Indeed, if we are to use the ABJM model toward this end, the most natural possibility that comes to mind - since we are, after all, dealing with a gauge theory of $N \times N$ matrices - is a Matrix theory-type construction. In fact, very much in the spirit of M-theory, it was found in [23, 24] that the BPS fuzzy funnel solution of pure ABJM, or the vacuum fuzzy sphere solution of the massive deformation of ABJM gives rise to a $D4$ -brane on S^2 , together with the correct small fluctuations action, in the classical limit. This $D4$ -brane appears as an M5-brane on S^1/Z_k at large k , in a similar way to how $D2$ - and $D4$ -branes appear in the Matrix theory of Banks *et.al.* (BFSS) [13, 14, 15, 16].

It seems reasonable therefore to expect that a Matrix theory-type model can be constructed out of the ABJM model. Following the logic of Matrix theory, we need to find a classical solution of ABJM corresponding to a spacetime supergraviton. Since such a solution must be localized on the worldvolume, as well as in transverse space, it must be a particular type of vortex solution. However, since the ABJM action, unlike its BFSS counterpart, is conformal, we will see that a better definition of its Matrix model is given by a maximally supersymmetric deformation of ABJM [25].

The ABJM model can be considered as a prototype for strongly coupled theories in three dimensions, in particular for planar condensed matter systems. For instance, in [6, 7] it was used to study the relativistically invariant quantum critical phase and compressible Fermi surfaces, respectively. These applications of the *AdS/CFT* correspondence to condensed matter hinge on the idea that, if physics in AdS is always holographic, then we

can consider simple theories in AdS, which *should* be dual to some strongly coupled conformal field theories (see e.g. [8, 9] for a review). In an overwhelming majority of cases considered, the argument for applying the AdS/CFT duality (and trusting the answers it provided) was *universality*. In other words, the variety of theories usually considered contain a small subset of abelian operators dual to a small number of fields in AdS, usually a gauge field, some scalars and perhaps some fermions. On the other hand, the relevant condensed matter models one usually wants to describe is usually abelian to begin with. It is not entirely clear then why we can either: i) focus on a small subset of abelian operators of a large N system; or ii) consider an abelian analog of the large N system, which would not have a gravity dual. A better motivated scenario for such an “AdS/CMT” correspondence would be if, in a large N field theory with a gravity dual, we could identify a *consistent truncation* of the (in general, nonabelian) field theory to an abelian subset corresponding to the *collective dynamics* of a large number of fields, and the resulting abelian theory would be a relevant condensed matter model. It is toward this end that we explore possible abelian reductions of the ABJM model in chapters 3 and 4.

The thesis is organized as follows. In chapter 1, in the interests of being self-contained, we provide a lightning review of Supergravity, superstrings, the AdS/CFT, and the ABJM theories focusing on those elements that carry over to our case, and then in section 2.3 we will describe in detail our set-up and what we expect to find. Section 2.4 is devoted to an analysis of solutions of pure ABJM theory, and we argue that, while it is possible to identify a vortex solution with a supergraviton, such a solution is practically unfeasible due in no small part to the infinite energy of the corresponding background D2-brane. Instead in section 2.5 we will focus on the massive deformation of ABJM and identify its known solitonic spectrum (consisting of brane-filling, kink and vortex types) with branes in spacetime. In the process we identify the background in which the M2-branes corresponding to the massive ABJM move. In section 2.6 we calculate the scattering of supergravitons in this massive background, and in section 2.7 the corresponding scattering of vortices in massive ABJM. Finally, after noting a mismatch of the calculations of sections 2.6 and 2.7, in section 2.8 we speculate on the possible definition of the sought-for ABJM Matrix theory.

In chapter 3 we will take some steps towards a better understanding of AdS/CMT, by proposing a modification of the above set-up. We consider a *consistent truncation* of the 3-dimensional ABJM theory (which has a known gravity dual), a truncation that corresponds to the collective dynamics of $\mathcal{O}(N)$ fields out of the $\mathcal{O}(N^2)$ of ABJM, and gives

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an effective theory that is easily identified as the relativistic Landau-Ginzburg model. We also sketch a simple CMT model that has the same qualitative features as the ABJM abelianization, allowing us to understand better in what sense can we use ABJM for condensed matter systems. In chapter 3 we present only the main ideas, leaving the technical details to chapter 4.

In section 4.2 we explore general abelianization ansätze involving G^α , and identify two important cases of further consistent truncations for this model. In section 4.3 we study one of them, which, for BPS states, leads to a Toda system that possesses vortex-type solutions with topological charge and finite energy, but with $|\phi| \rightarrow 0$ at both $r \rightarrow 0$ and $r \rightarrow \infty$, which we describe numerically. In section 4.4 we describe a second case, more relevant for the AdS/CMT motivation above and find a reduction that, depending on certain parameters, gives us either an abelian-Higgs model, or a ϕ^4 (relativistic Landau-Ginzburg) theory. In section 4.5 we study the relevance of this reduction for condensed matter and AdS/CMT and sketch a simple condensed matter model that reproduces the general features of abelianization. In section 4.6 we study some BPS solutions suggested by the abelianizations. Finally, in section 4.7, we provide a possible spacetime interpretation for these solutions in terms of M2-branes on a background spacetime.

1.1 M-theory and 11-dimensional Supergravity

The first evidence for the existence of M-theory in eleven dimensions was provided by P.K Townsend [10][11]. The M-theory was discovered by Witten in 1995 as a theory that the strong coupling limit of type-IIA string theory and whose leading low-energy effective action is 11-dimensional supergravity [12]. M-theory is not yet fully formulated, but the evidence for its existence is very compelling. The description of M-theory in terms of an effective action is clearly not fundamental and there must be an alternative formulation of the theory. One of these alternative formulations is the Matrix theory proposed by Banks, Fischler, Shenker, and Susskind (BFSS) [13] [14][15][16]. In recent years there has been a major breakthroughs made by Bagger-Lambert-Gustavsson (BLG theory) [17][18][19][20], and Aharony-Bergman-Jafferis-Maldacena (ABJM theory) [21]. These theories give the AdS/CFT description of the quantum field theories on coincident M2-branes.

The field content of the 11-dimensional Supergravity consists of the graviton g_{MN} , a rank 3 anti-symmetric tensor field C_{MNP} and a 32 component Majorana gravitino Ψ_M^α . The graviton which is a symmetric traceless tensor of $SO(D-2)$, the little group for a massless particle. It has 44 physical degrees of freedom, the 3-form antisymmetric tensor gauge field has 84 physical degrees of freedom. Together with the graviton, this gives $44 + 84 = 128$ propagating bosonic degrees of freedom, which matches the number of propagating fermionic degrees of freedom of the gravitino. The 11-dimensional supergravity action [22] is given by

$$\begin{aligned}
 S = & \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g} \left[R - \frac{1}{2 \cdot 4!} |F_4|^2 - \frac{1}{2} \bar{\Psi}_M \Gamma^{MNP} D_N \Psi_P \right. \\
 & \left. - \frac{1}{192} \left(\bar{\Psi}_M \Gamma^{MNPQST} \Psi_T + 12 \bar{\Psi}^N \Gamma^{PQ} \Psi^S \right) F_{NPQS} \right] - \frac{1}{12\kappa_{11}^2} \int C_3 \wedge F_4 \wedge F_4 \\
 & + \text{terms quartic in } \Psi
 \end{aligned} \tag{1.1}$$

where R is the scalar curvature, $F_4 = dC_3$ is the field strength, and κ_{11} denotes the 11-dimensional gravitational coupling constant. The relation between the 11-dimensional Newton's constant G_{11} , the gravitational constant κ_{11} and the 11-dimensional Planck length ℓ_p is

$$\kappa_{11}^2 = 8\pi G_{11} = \frac{1}{4\pi} (2\pi\ell_p)^9 \tag{1.2}$$

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The last term in (1.1), which has a ChernSimons structure, is independent of the elfbein (or the metric). The first term does depend on the elfbein, but only in the metric combination

$$g_{MN} = \eta_{AB} e_M^A e_N^B \quad (1.3)$$

Supersymmetry transformations

The complete action of 11-dimensional supergravity is invariant under local supersymmetry transformations under which the fields transform according to [22]

$$\delta e_M^A = \bar{\epsilon} \Gamma^A \Psi_M \quad (1.4)$$

$$\delta A_{MNP} = -3\bar{\epsilon} \Gamma_{[MN} \Psi_{P]} \quad (1.5)$$

$$\delta \Psi_M = \nabla_M \epsilon + \frac{1}{12} \left(\frac{1}{4!} \Gamma_M F_{NPQS} \Gamma^{NPQS} - \frac{1}{2!} F_{MNPQ} \Gamma^{NPQ} \right) \epsilon \quad (1.6)$$

Here the Dirac matrices satisfy

$$\Gamma_M = e_M^A \Gamma^A, \quad \{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \quad (1.7)$$

and

$$\Gamma^{M_1 M_2 \dots M_n} = \Gamma^{[M_1} \Gamma^{M_2} \dots \Gamma^{M_n]} \quad (1.8)$$

The covariant derivative that appears in the supersymmetry transformation rule of the gravitino involves the spin connection ω and is given by

$$\nabla_M \epsilon = \psi_M \epsilon + \frac{1}{4} \omega_{MAB} \Gamma^{AB} \epsilon \quad (1.9)$$

A global supersymmetry of a given supergravity background is determined by requiring that the gravitino variation is annihilated, $\delta \Psi_M = 0$. The resulting condition on ϵ is

$$\nabla_M \epsilon + \frac{1}{12} \left(\frac{1}{4!} \Gamma_M F_{NPQS} \Gamma^{NPQS} - \frac{1}{2!} F_{MNPQ} \Gamma^{NPQ} \right) \epsilon = 0 \quad (1.10)$$

is known as the Killing spinor equation. The bosonic terms that have been included in this equation determine the possible supersymmetric solutions.

1.2 Type IIA supergravity

The action of 11-dimensional supergravity is related to the actions of the various ten-dimensional supergravity theories, which are the low-energy effective descriptions of superstring theories. M-theory compactified on a circle of radius R corresponds to type

1.2 Type IIA supergravity

IIA superstring theory in ten dimensions with coupling constant $g_s = R/\sqrt{\alpha'}$. Applying the dimensional reduction to the 11-dimensional 32-component Majorana gravitino Ψ_M we get a pair of 16-component Majorana-Weyl spinors of opposite chirality. The first ten components give the two 10-dimensional gravitinos $\psi_{\mu\alpha}^{\pm}$ and Ψ_{11} gives the two 10-dimensional dilatinos λ_{α}^{\pm} . The fermionic degrees of freedom of the $D = 10$ Type IIA theory consist of

$$\text{Type IIA}_F \quad \begin{cases} \psi_{\mu\alpha}^{\pm} & 112_F & \text{Majorana-Weyl gravitinos} \\ \lambda_{\alpha}^{\pm} & 16_F & \text{Majorana-Weyl dilatinos} \end{cases} \quad (1.11)$$

Altogether, there are **128** fermionic degrees of freedom, just as in 11 dimensions. This preservation of degrees of freedom is a general feature of dimensional reduction on circles or tori.

Upon dimensional reduction, the 11-dimensional metric G_{MN} gives rise to a 10-dimensional metric $g_{\mu\nu}$, a gauge field A_{μ} and a scalar Φ as follows

$$G_{MN} = e^{-2\Phi/3} \begin{pmatrix} g_{\mu\nu} + e^{2\Phi} A_{\mu} A_{\nu} & e^{2\Phi} A_{\mu} \\ e^{2\Phi} A_{\mu} A_{\nu} & e^{2\Phi} \end{pmatrix} \quad (1.12)$$

where all of the fields depend on the 10-dimensional space-time coordinates x^{μ} only. The exponential factors of the scalar field Φ , which turns out to be the dilaton, are introduced for later convenience. Equation (1.12) can be recast in the form

$$ds^2 = G_{MN} dx^M dx^N = e^{-2\Phi/3} g_{\mu\nu} dx^{\mu} dx^{\nu} + e^{4\Phi/3} (dx^{11} + A_{\mu} dx^{\mu})^2 \quad (1.13)$$

The three-form in eleven dimensions A gives rise to a three-form and a two-form in ten dimensions

$$A_{\mu\nu\rho}^{11} = A_{\mu\nu\rho} \quad \text{and} \quad A_{\mu\nu 11} = B_{\mu\nu} \quad (1.14)$$

with the corresponding field strengths given by

$$F_{\mu\nu\rho\lambda}^{11} = F_{\mu\nu\rho\lambda} \quad \text{and} \quad F_{\mu\nu\rho 11}^{11} = H_{\mu\nu\rho} \quad (1.15)$$

The 11-dimensional field strength is a combination of a four-form and a three-form field strength

$$F^{(4)} = e^{4\Phi/3} \tilde{F}^{(4)} + e^{\Phi/3} H^{(3)} \Gamma_{11} \quad (1.16)$$

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where

$$\begin{aligned}
 F^{(4)} &= \frac{1}{4!} F_{MNPQ} \Gamma^{MNPQ} \\
 \tilde{F}^{(4)} &= \frac{1}{4!} \tilde{F}_{MNPQ} \Gamma^{MNPQ} \\
 H^{(4)} &= \frac{1}{3!} H_{MNP} \Gamma^{MNP}
 \end{aligned} \tag{1.17}$$

Using the differential form notation, the rescaled field strength can be written as

$$\tilde{F}_4 = dA_3 + A_1 \wedge H_3 \tag{1.18}$$

And hence, the Bosonic degrees of freedom of the $D = 10$ Type IIA theory has the following field contents,

$$\text{Type IIA}_B \left\{ \begin{array}{lll}
 g_{\mu\nu} & 35_B & \text{metric-graviton} \\
 \Phi & 1_B & \text{dilaton} \\
 A_\mu & 8_B & \text{graviphoton} \\
 B_{\mu\nu} & 28_B & \text{NS-NS rank 2 antisymmetric tensor} \\
 A_{\mu\nu\rho} & 56_B & \text{antisymmetric rank 3 tensor}
 \end{array} \right. \tag{1.19}$$

The vacuum expectation value of e^Φ is the type IIA superstring coupling constant g_s and the relation between the 10-dimensional Newton's constant G_{10} , the gravitational constant κ_{10} , the string length ℓ_s and coupling constant g_s is

$$\kappa_{10}^2 = 8\pi G_{10} = \frac{1}{4\pi} (2\pi\ell_s)^8 g_s^2 \tag{1.20}$$

Since we have compactified the theory on a circle of coordinate period $2\pi R_{11}$, we can define

$$\kappa_{10}^2 = \frac{\kappa_{11}^2}{2\pi R_{11}} \tag{1.21}$$

Using $\ell_p = \ell_s g_s^{-1/3}$ and equations (1.2) and (1.20), one deduces that the radius of the circle is

$$R_{11} = \frac{1}{2\pi} \frac{\kappa_{11}^2}{\kappa_{10}^2} = g_s \ell_s \tag{1.22}$$

The action

The bosonic part of the action in the string frame for the $D = 10$ type IIA may be written as a sum of three distinct types of terms [22]:

$$S_{\text{IIA}} = S_{\text{NS}} + S_{\text{R}} + S_{\text{CS}}, \quad (1.23)$$

$$S_{\text{NS}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left(R + 4\psi_\mu \Phi \psi^\mu \Phi - \frac{1}{2} |H_3|^3 \right) \quad (1.24)$$

$$S_{\text{R}} = -\frac{1}{4\kappa^2} \int d^{10}x \sqrt{-g} \left(|F_2|^2 + |\tilde{F}_4|^2 \right) \quad (1.25)$$

$$S_{\text{CS}} = -\frac{1}{4\kappa^2} \int B_2 \wedge F_4 \wedge F_4 \quad (1.26)$$

Where we have regrouped terms according to whether the fields are in the NS-NS or R-R sector of the string theory. Note that we have defined the constant κ in terms of the 10-dimensional gravitational constant κ_{10} as

$$\kappa^2 = g_s^{-2} \kappa_{10}^2 = \frac{1}{4\pi} (2\pi\ell_s)^8 \quad (1.27)$$

1.3 Type IIB supergravity

The fermionic part of the spectrum of Type IIB theory consists of

$$\text{Type IIB}_F \quad \begin{cases} \psi_{\mu\alpha}^I & I = 1, 2 & 112_F & \text{Two left-handed Majorana-Weyl gravitinos} \\ \lambda_\alpha^I & I = 1, 2 & 16_F & \text{Two right-handed Majorana-Weyl dilatinos} \end{cases} \quad (1.28)$$

The NS-NS bosons consist of the metric $g_{\mu\nu}$, the two-form B_2 and the dilaton Φ . The R-R sector consists of form fields A_0 , A_2 and A_4 . The rank 4 antisymmetric tensor A_4 has self-dual field strength \tilde{F}_5 which imposes a significant difficulty in writing down a classical action for type IIB supergravity.

By constructing the supersymmetric equations of motion, and then writing down an action that reproduces those equations when the self-duality condition is imposed by hand. The bosonic part of the type IIB supergravity action obtained in this way takes the form [22]:

$$S_{\text{IIB}} = S_{\text{NS}} + S_{\text{R}} + S_{\text{CS}}, \quad (1.29)$$

$$S_{\text{NS}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left(R + 4\psi_\mu \Phi \psi^\mu \Phi - \frac{1}{2} |H_3|^3 \right) \quad (1.30)$$

$$S_{\text{R}} = -\frac{1}{4\kappa^2} \int d^{10}x \sqrt{-g} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right) \quad (1.31)$$

$$S_{\text{CS}} = -\frac{1}{4\kappa^2} \int A_4 \wedge H_3 \wedge F_3 \quad (1.32)$$

Introduction

where the field strengths are defined by

$$\begin{aligned} F_1 &= dA_0 \\ F_3 &= dA_2 \\ F_5 &= dA_4 \end{aligned}$$

$$\tilde{F}_3 = F_3 - A_0 H_3 \tag{1.33}$$

$$\tilde{F}_5 = F_5 - \frac{1}{2} A_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3 \tag{1.34}$$

and we have the supplementary self-duality condition

$$\tilde{F}_5 = * \tilde{F}_3 \tag{1.35}$$

1.4 Branes in Supergravity

A $(p+1)$ -form A_{p+1} naturally couples to a p -brane that is an object with $(p+1)$ -dimensional world-volume. The coupling takes the form

$$\int_V A_{p+1} \tag{1.36}$$

where V is the world volume of the p -brane. The corresponding electric charge in d -spacetime would then be

$$Q^{\text{elec}} = \int_{S_\infty^{d-2-p}} *F_{p+2} \tag{1.37}$$

Each A_{p+1} gauge field has a magnetic dual A_{d-3-p}^{magn} which is a differential form field of a rank $D - 3 - p$, whose field strength is related to that of A_{p+1} by

$$dA_{d-3-p}^{\text{magn}} = *dA_{p+1} \tag{1.38}$$

and hence the magnetically dual object to the p -brane would then be an object coupling to a $(p - 3 - p)$ -form, i.e. a $(d - 4 - p)$ -brane with magnetic charge

$$Q^{\text{magn}} = \int_{S_\infty^{p+2}} F_{p+2} \tag{1.39}$$

In the $d = 11$ supergravity we have a R-R 3-form A_3 , so we have a 2-brane, denoted M2-brane and its magnetic dual is M5-brane.

In the $d = 10$ Type IIA theory we have a R-R 1-form A_1 and a R-R 3-form A_3 , so we

1.5 Brane Solutions in Supergravity

have a D0-brane (particle) and a 2-brane which is referred to as D2-brane, plus their magnetic duals which are D6-brane and D4-brane respectively. On the other hand, the NS-NS 2-form B_2 is coupled to a 1-brane which is nothing but the fundamental string denoted F1-brane, whose magnetic dual is NS5-brane.

In the $d = 10$ Type IIB theory the R-R sector contains n -form gauge fields with $n = 0, 2, 4$. Applying the rules given above the R-R scalar should couple to a (-1)-brane, i.e. with an object which is a point-like in space-time. It is interpreted as a D-instanton, which makes sense in the Euclideanized theory. Its magnetic dual is a D7-brane. The 2-form A_2 couples electrically to a D1-brane (also called a D-string) and magnetically to a D5-brane this is clearly different from the Type IIB NS-NS 5-brane (which couples to the NS-NS 2-form B_2 and which is magnetically dual to the fundamental string F1). The 4-form A_4 couples both electrically and magnetically to a D3-brane. However, these are not distinct D-branes. Since the field strength is self-dual, $F_5 = *F_5$, the D3-brane carries a self-dual charge.

The stable D-branes (with p even in the IIA theory or odd in the IIB theory), M2-branes and M5-branes preserve half of the supersymmetry (16 supersymmetries). Therefore, they are sometimes called half-BPS D-branes. Below we present a Table of the branes occurring for various p in the $D = 11$ supergravity and in the Type IIA/B supergravities in $D = 10$.

Name	D(-1)-brane	D0-brane	D1-string	D2-brane	D3-brane	F1	M2-brane
Magnetic	D7-brane	D6-brane	D5-brane	D4-brane	D3-brane	NS5	M5-brane
$D = 11$	-	-	-	-	-	-	A_3
Type IIA	-	A_1	-	A_3	-	B_2	-
Type IIB	A_0	-	A_2	-	A_4	B_2	-

Table 1.1: Branes in various theories

1.5 Brane Solutions in Supergravity

The p -brane has a $(p + 1)$ -dimensional hyperspace, with a Poincaré invariance group $R^{p+1} \times SO(1, p)$. The presence of the extremal p -brane in the d -dimensional spacetime breaks the Lorentz symmetry

$$SO(d - 1, 1) = SO(1, p) \times SO(d - p - 1)$$

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where the group $SO(1, p)$ describes the Lorentz symmetry along the brane, and the group $SO(d-p-1)$ describes the rotational symmetry transverse to the brane. The translational symmetries along the brane R^{p+1} enlarge the Lorentz symmetry to a Poincaré symmetry. Thus the p -branes in supergravity are solutions with symmetry group

$$R^{p+1} \times SO(1, p) \times SO(d-p-1)$$

The Poincaré invariance in the $p+1$ dimensions forces the metric in the directions parallel to the brane x^μ , $\mu = 0, 1, \dots, p$ to be a rescaling of the Minkowski metric, while the rotational invariance in the transverse directions $y^u = x^{p+u}$, $u = 1, 2, \dots, d-p-1$ forces the metric on those dimensions to be a rescaling of the Euclidean metric. Denoting by r the radial coordinate in the transverse space, that is, $r = \sqrt{y^u y^u}$, it turns out that the Killing spinor equation is solved if the rescaling factor solve the $(d-p-1)$ Laplace's equation. The solution is given by

$$H_p(r) = 1 - \left(\frac{r_p}{r}\right)^{d-p-3} \quad (1.40)$$

The solutions may be expressed in terms of this single function $H_p(r)$, as

$$\text{Dp-brane:} \quad ds^2 = H_p(r)^{-1/2} dx^\mu dx_\mu + H_p(r)^{1/2} dy^u dy^u \quad (1.41)$$

$$\text{NS5-brane:} \quad ds^2 = dx^\mu dx_\mu + H_p(r)^{1/2} dy^u dy^u \quad (1.42)$$

$$\text{M2-brane:} \quad ds^2 = H_p(r)^{-2/3} dx^\mu dx_\mu + H_p(r)^{1/3} dy^u dy^u \quad (1.43)$$

$$\text{M5-brane:} \quad ds^2 = H_p(r)^{-1/3} dx^\mu dx_\mu + H_p(r)^{2/3} dy^u dy^u \quad (1.44)$$

1.6 The Maldacena AdS/CFT Correspondence

The Maldacena original formulation of the AdS/CFT Correspondence [1] describe four dimensional $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory with an $SU(N)$ gauge group in the large N limit from the perspective of a dual ten dimensional gravitational (type IIB superstring) theory on $AdS_5 \times S^5$. To illustrate the correspondence, let us consider the system of N well separated D3-branes at low energies, energies lower than $1/l_s$, then only the massless string states can be excited. The massless vector states arise from open strings starting and ending on the same brane, there are N different massless $U(1)$ states given by the diagonal strings. Since the brane breaks half of the total number of supersymmetries, these excitation modes induce a massless $U(1)^N$ gauge theory with $\mathcal{N} = 4$ supersymmetry. A massive state arises from open string with one of its endpoints is attached to a brane while the other end is attached to a different brane (the mass of

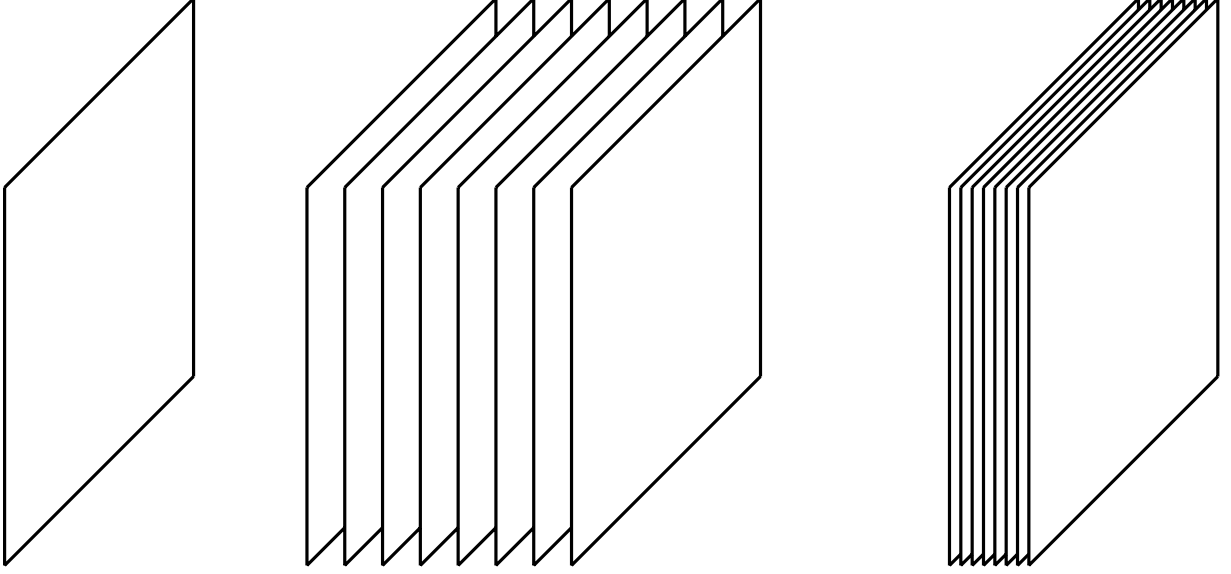


Figure 1.1: (a) single D-brane (b) 8 parallel well-separated D-branes; $U(1)^8$ gauge theory (c) coincident D-branes; $U(8)$ gauge theory

such string is proportional to the separation distance between the branes). There are $N^2 - N$ such possible states given by the off-diagonal strings. In the limit where the N D3-branes all tend to be coincident, the $N^2 - N$ extra off-diagonal string states become massless. This enhances the gauge symmetry from $U(1)^N$ to $U(N)$. The $U(1)$ subgroup of $U(N)$ corresponds to the overall collective motion of the stack of branes and may be ignored. So the gauge group is really $SU(N)$, not $U(N)$. The difference between the two groups is a subleading effect in the large N limit. The open string description reduces to $\mathcal{N} = 4$ super Yang-Mills theory, whereas the closed string description which describes the excitations of empty space (give a gravity supermultiplet in ten dimensions), reduces to string theory on $AdS_5 \times S^5$. Thus, the AdS/CFT duality arises as a consequence of the duality between open and closed strings.

On the field theory side, the world-volume theory on N D3-branes is $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory with gauge group $SU(N)$. The theory has a unique Lagrangian which can be obtained by the dimensional reduction on T^6 of the $\mathcal{N} = 1$ Supersymmetric Yang-Mills theory in ten dimensions, whose action is given by

$$S_{SYM}^{\mathcal{N}=1} = -\frac{1}{2g_{YM}^2} \int d^{10}x \text{Tr} \left(\frac{1}{2} F_{MN} F^{MN} + i \text{Tr} \bar{\psi} \Gamma^M D_M \psi \right) \quad (1.45)$$

The gauge field and the gaugino ψ (a Majorana-Weyl spinor) are written in matrix notation, and $\Gamma^M = (\Gamma^\mu, \Gamma^i)$ is the 16×16 Dirac Matrices in ten-dimensional spacetime,

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which satisfy $\text{Tr}(\Gamma^M \Gamma^N) = 16\delta^{MN}$. The supersymmetry transformations that leaves (7.1) invariant are

$$\begin{aligned}\delta_\epsilon A_M &= i\bar{\epsilon}\Gamma_M\psi \\ \delta_\epsilon\psi &= \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon\end{aligned}\tag{1.46}$$

Let us consider the Bosonic part of the action. We have the 10-dimensional gauge vector field A_M . From the 4-dimensional point of view this can be viewed as a gauge field A_μ , $\mu = 0, 1, 2, 3$ along with 6 real scalar fields $A_i = X_i$, $i = 4, \dots, 9$ of a multiplet of fields in four dimensions possessing an additional $SU(4) \sim SO(6)$ global symmetry, which is a direct consequence of the ten dimensional Lorentz invariance. From the point of view of the $\mathcal{N} = 4$ theory this $SU(4)$ global symmetry is identified with the R-symmetry group of the $\mathcal{N} = 4$ supersymmetry algebra. Thus the $\text{Tr} F^2$ term can be written as

$$\text{Tr}(F_{MN}F^{MN}) = \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + 2\text{Tr}(D_\mu X^i D^\mu X^i) - \text{Tr}\left(\sum_{i,j}[X^i, X^j]^2\right)\tag{1.47}$$

where the field strength and the covariant derivative are defined as

$$\begin{aligned}F_{\mu\nu} &= \psi_\mu A_\nu - \psi_\nu A_\mu + i[A_\mu, A_\nu] \\ D_\mu X^i &= \psi_\mu X^i + i[A_\mu, X^i]\end{aligned}\tag{1.48}$$

The fermionic part has four Weyl spinors (gluinos), which can be written in terms of the sixteen component ten-dimensional Majorana-Weyl spinor ψ as

$$\text{Tr}(\bar{\psi}\Gamma^M D_M\psi) = \text{Tr}(\bar{\psi}\Gamma^\mu D_\mu\psi) + i\text{Tr}(\bar{\psi}\Gamma^i[X_i, \psi])\tag{1.49}$$

Now the Lagrangian for the $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory in 4-dimensional spacetime is unique and given by

$$\mathcal{L}_{\text{SYM}}^{\mathcal{N}=4} = -\frac{1}{2g_{YM}^2}\text{Tr}\left\{\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + (D_\mu X^i)(D^\mu X^i) - \sum_{i<j}[X^i, X^j]^2 + i\bar{\psi}\Gamma^\mu D_\mu\psi - \bar{\psi}\Gamma^i[X_i, \psi]\right\}\tag{1.50}$$

and the supersymmetry transformations reduces to

$$\begin{aligned}\delta_\epsilon A_\mu &= i\bar{\epsilon}\Gamma_\mu\psi \\ \delta_\epsilon X^i &= i\bar{\epsilon}\Gamma^i\psi \\ \delta_\epsilon\psi &= \left(\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu} + (D_\mu X_i)\Gamma^{\mu i} + \frac{i}{2}[X_i, X_j]\Gamma^{ij}\right)\epsilon\end{aligned}\tag{1.51}$$

1.6.1 The Maldacena limit

Let us look at the picture of N D3-branes from a purely gravitational point of view. Since the number of the branes N is very large, one can allow the branes to back-react on the geometry of the bulk spacetime. The space-time metric of N coincident D3-branes and the self-dual five-form flux are given by

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{R^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2) \quad (1.52)$$

$$F_5 = (1 + *) dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d \left(1 + \frac{R^4}{r^4}\right)^{-1} \quad (1.53)$$

Here r is the radial distance away from the branes and R is the radius of the D3-brane which is by

$$R^4 = 4\pi g_s \alpha'^2 = \lambda \alpha'^2 \quad (1.54)$$

When r is very large compared to the radius of the D3-brane $r \gg R$, we recover the flat ten dimensional spacetime $\mathbb{R}^{1,9}$ and the closed strings will propagate freely. When $r < R$, the geometry is often referred to as the throat geometry. A redefinition of the coordinate $u = R^2/r$ and the large u limit, however, transform the metric into the following asymptotic form

$$ds^2 = \frac{R^2}{u^2} (du^2 + \eta_{\mu\nu} dx^\mu dx^\nu) + R^2 d\Omega_5^2 = R^2 (ds_{AdS_5}^2 + ds_{S^5}^2) \quad (1.55)$$

which corresponds to a product geometry. One component is the five-sphere S^5 and the other component represents five-dimensional anti-de-Sitter (AdS) space AdS_5 . The five-form flux integrated over the sphere S^5 gives

$$\int_{S^5} F_{(5)} = N, \quad (1.56)$$

which is the same as the charge of the N D3-branes in the gauge theory side description. Thus we have shown the original system has decoupled to the free supergravity in ten-dimensional flat region and the superstring on the near-horizon region, that is on $AdS_5 \times S^5$, where both components have identical radius R .

1.6.2 The AdS/CFT Conjecture

Maldacena conjecture states the duality between

Introduction

1. Type IIB superstring theory on $AdS_5 \times S^5$, where the 5-form has integer flux N , and the string coupling is g_s .
2. $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions, with a gauge group $SU(N)$ and Yang-Mills coupling g_{YM} in its superconformal limit.

when we identify

$$g_s = g_{\text{YM}}^2, \quad R^4 = 4\pi g_s \alpha'^2 = \lambda \alpha'^2 \quad (1.57)$$

The duality between these two theories includes a precise map between the states on the string theory side and the local gauge invariant operators on the $N = 4$ Super Yang-Mills theory side, as well as a correspondence between the correlators in both theories.

1.7 $\mathcal{N} = 6$ Superconformal Chern-Simons theory

The aim of this section is to give a brief review of the ABJM $\mathcal{N} = 6$ Superconformal Chern-Simons theory in three dimensions [21]. This theory is a supersymmetric $U(N) \times U(N)$ gauge theory with four complex scalars C^I and four Dirac fermions ψ_I , $I = 1, \dots, 4$ in the (\bar{N}, N) representation of the gauge group and their corresponding complex conjugate fields C_I^\dagger and $\psi^{I\dagger}$ in the (N, \bar{N}) representation. The gauge fields are not dynamical, and have a Chern-Simons action with opposite integer levels for the two gauge groups, k and $-k$. Its action is given by

$$S = \int d^3x (\mathcal{L}_{CS} + \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6) \quad (1.58)$$

where

$$\begin{aligned} \mathcal{L}_{CS} &= \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \\ \mathcal{L}_2 &= -\text{Tr} \left(D_\mu C_I^\dagger D^\mu C^I + i \psi^{I\dagger} \gamma^\mu D_\mu \psi_I \right) \\ \mathcal{L}_4 &= \frac{2\pi i}{k} \text{Tr} \left(C_I^\dagger C^I \psi^{J\dagger} \psi_J - \psi^{J\dagger} C^I C_I^\dagger \psi_J - 2C_I^\dagger C^I \psi^{J\dagger} \psi_J + 2\psi^{J\dagger} C^I C_I^\dagger \psi_J \right. \\ &\quad \left. + \epsilon^{IJKL} C_I^\dagger \psi_J C_K^\dagger \psi_L - \epsilon_{IJKL} C^I \psi^{J\dagger} C^K \psi^{L\dagger} \right) \\ \mathcal{L}_6 &= \frac{4\pi^2}{3k^2} \left(C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger + C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K + 4C^I C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger \right. \\ &\quad \left. - 6C^I C_J^\dagger C^J C_I^\dagger C^K C_K^\dagger \right) \end{aligned} \quad (1.59)$$

1.7 $\mathcal{N} = 6$ Superconformal Chern-Simons theory

where the gauge-covariant derivative is

$$D_\mu C^I = \psi_\mu C^I + iA_\mu C^I - iC^I \hat{A}_\mu \quad (1.60)$$

By construction, the *ABJM* action is invariant under the following $\mathcal{N} = 6$ supersymmetry variations[26][27]

$$\begin{aligned} \delta C^I &= i\omega^{IJ}\psi_J \\ \delta C_I^\dagger &= i\psi^{J\dagger}\omega_{IJ} \end{aligned} \quad (1.61)$$

$$\begin{aligned} \delta\psi_I &= -\gamma^\mu\omega_{IJ}D_\mu C^J - \frac{2\pi}{k} \left[\omega_{IJ} \left(C^K C_K^\dagger C^J - C^J C_K^\dagger C^K \right) - 2\omega_{JK} C^J C_I^\dagger C^K \right] \\ \delta\psi^{I\dagger} &= +D_\mu C_J^\dagger \gamma^\mu \omega^{IJ} - \frac{2\pi}{k} \left[\left(C_J^\dagger C^K C_K^\dagger - C_K^\dagger C^K C_J^\dagger \right) \omega_{IJ} - 2C_K^\dagger C^I C_J^\dagger \omega^{JK} \right] \end{aligned} \quad (1.62)$$

$$\begin{aligned} \delta A_\mu &= -\frac{2\pi}{k} \left(C^I \psi^{J\dagger} \gamma_\mu \omega_{IJ} + \omega^{IJ} \gamma_\mu \psi_I C_J^\dagger \right) \\ \delta \hat{A}_\mu &= +\frac{2\pi}{k} \left(\psi^{J\dagger} C^J \gamma_\mu \omega_{IJ} + \omega^{IJ} \gamma_\mu C_I^\dagger \psi_J \right) \end{aligned} \quad (1.63)$$

Classically, \mathcal{L} is scale invariant. This may be seen by assigning the standard mass-dimensions to the fields

$$[A_\mu] = [\hat{A}_\mu] = [\psi_I] = 1 \quad [C^I] = \frac{1}{2} \quad (1.64)$$

All terms in the Lagrangian are of dimension 3, from which scale invariance once follows.

1.7.1 The dual gravitational backgrounds of M-theory and type IIA string theory

The Chern-Simons-matter theories constructed above are dual to the conformal field theory living at low energies on N M2-branes probing a C^4/\mathbb{Z}_k singularity. This theory has a dual gravitational description in terms of M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$.

The space-time metric of N' coincident M2-branes [28] is given by

$$ds^2 = \left(1 + \frac{R^6}{r^6} \right)^{-2/3} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{R^6}{r^6} \right)^{1/3} (dr^2 + r^2 d\Omega_7^2) \quad (1.65)$$

Here r is the radial distance away from the branes and R is the radius of the M2-brane which is by

$$R^6 = 32\pi^2 N' \ell_p^6 \quad (1.66)$$

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A redefinition of the coordinate $u = R^3/r^2$ and the large u limit, however, transform the metric into the following asymptotic form

$$ds^2 = \frac{R^2}{4u^2}(du^2 + \eta_{\mu\nu}dx^\mu dx^\nu) + R^2 d\Omega_4^2 = R^2\left(\frac{1}{4}ds_{AdS_4}^2 + ds_{S^7}^2\right) \quad (1.67)$$

which corresponds to a product geometry. One component is the seven-sphere S^7 and the other component represents four-dimensional anti-de-Sitter (AdS) space AdS_4 . Thus we have shown that the original system has decoupled to the free supergravity in eleven-dimensional flat region and the M-theory on the near-horizon region, that is on $AdS_4 \times S^7$, where the radius of the seven-sphere is R , while that of the AdS_4 is $R/2$.

If we write the transverse space to the M2-branes using four complex coordinates z_i ($i = 1, 2, 3, 4$), then the \mathbb{Z}_k quotient acts as:

$$z_i \rightarrow e^{i\frac{2\pi}{k}} z_i \quad (1.68)$$

The gravity dual of N M2-branes in flat space is $AdS_4 \times S^7$, and we simply need to quotient this by this \mathbb{Z}_k . It is natural to use the description of S^7 as an S^1 fibration over \mathbb{CP}^3 . We can write the metric of S^7 as

$$ds_{S^7}^2 = (d\varphi' + \omega)^2 + ds_{\mathbb{CP}^3}^2 \quad (1.69)$$

where

$$\begin{aligned} ds_{\mathbb{CP}^3}^2 &= \frac{\sum_i dz_i d\bar{z}_i}{\rho^2} - \frac{|\sum_i z_i d\bar{z}_i|^2}{\rho^4}, & \rho^2 &= \sum_{i=1}^4 |z_i|^2 \\ d\varphi' + \omega &= \frac{i}{2\rho}(z_i d\bar{z}_i - \bar{z}_i dz_i) \\ d\omega &= J = id\left(\frac{z_i}{\rho}\right) d\left(\frac{\bar{z}_i}{\rho}\right) \end{aligned} \quad (1.70)$$

If we take $\varphi' = \varphi/k$, with $\varphi \sim \varphi + 2\pi$ and then perform the Hopf fibration, the metric becomes

$$ds_{S^7/\mathbb{Z}_k}^2 = \frac{1}{k^2}(d\varphi + k\omega)^2 + ds_{\mathbb{CP}^3}^2 \quad (1.71)$$

Since the radius of S^7/\mathbb{Z}_k is smaller by a factor of k than the radius of S^7 , in order to have a properly quantized flux on the quotient space we need that $N' = kN$. The radius of the circle φ is

$$\frac{R}{k} = \left(32\pi^2 \frac{N}{k^5}\right)^{1/6} \ell_p \quad (1.72)$$

1.7 $\mathcal{N} = 6$ Superconformal Chern-Simons theory

Thus, the M-theory description is valid whenever $k^5 \ll N$, and when k increases the circle becomes small and we can reduce to a weakly coupled type IIA superstring theory, viz.

$$\begin{aligned} ds_{\text{string}}^2 &= \frac{R^3}{k} \left(\frac{1}{4} ds_{\text{AdS}_4}^2 + ds_{\mathbb{CP}^3}^2 \right) \\ e^\phi &= \frac{R^3}{k^3} = 4\sqrt{2}\pi \frac{N^{1/2}}{k^{5/2}} \sim \frac{1}{N^2} \left(\frac{N}{k} \right)^{5/2} \\ F_4 &= \frac{3}{8} R^3 \hat{\epsilon}_4 \\ F_2 &= kd\omega = kJ \end{aligned} \tag{1.73}$$

This is the type IIA string theory compactified on $AdS_4 \times \mathbb{CP}^3$ with N units of F_4 flux on AdS_4 and k units of F_2 flux on the $\mathbb{CP}^1 \subset \mathbb{CP}^2$ 2-cycle.

Introduction

2

Looking for a Matrix model of ABJM

2.1 Introduction

Based on the earlier Bagger, Lambert & Gustavsson (BLG) construction [17, 18, 19, 20] of an M2-brane action in terms of 3-algebras, Aharony, Bergman, Jafferis & Maldacena (ABJM) [21] constructed an $\mathcal{N} = 6$ supersymmetric action for the IR of N M2-branes probing a C^4/Z_k singularity, a $U(N) \times U(N)$, level- k Chern-Simons-matter gauge theory whose fields transform in the bifundamentals. Not only was this the first time that an action for an arbitrary number of M2-branes was written down but it also allowed further insight into the structure of M-theory, as well as a new and exciting example of the AdS/CFT with its gravity dual being the large k limit of $AdS_4 \times S^7/Z_k$, i.e. $AdS_4 \times CP^3$.

But while most of the work on the ABJM model concentrated either on the field theory side, or on the AdS/CFT duality, it is worth bearing in mind that the original interest in the multiple M2-brane system was, of course, the potential for a better understanding of M-theory. Indeed, if we are to use the ABJM model toward this end, the most natural possibility that comes to mind - since we are, after all, dealing with a gauge theory of $N \times N$ matrices - is a Matrix theory-type construction. In fact, very much in the spirit of M-theory, it was found in [23, 24] that the BPS fuzzy funnel solution of pure ABJM,

Looking for a Matrix model of ABJM

or the vacuum fuzzy sphere solution of the massive deformation of ABJM gives rise to a $D4$ -brane on S^2 , together with the correct small fluctuations action, in the classical limit. This $D4$ -brane appears as an M5-brane on S^1/Z_k at large k , in a similar way to how $D2$ - and $D4$ -branes appear in the Matrix theory of Banks *et.al.* (BFSS) [13, 14, 15, 16].

It seems reasonable therefore to expect that a Matrix theory-type model can be constructed out of the ABJM model. Following the logic of Matrix theory, we need to find a classical solution of ABJM corresponding to a spacetime supergraviton. Since such a solution must be localized on the worldvolume, as well as in transverse space, it must be a particular type of vortex solution. However, since the ABJM action, unlike its BFSS counterpart, is conformal, we will see that a better definition of its Matrix model is given by a maximally supersymmetric deformation of ABJM [25]. Consequently, after an initial analysis of pure ABJM model. We are going to focus on the maximally supersymmetric deformation of the ABJM theory. After identifying various spacetime branes as classical solutions of the model, we compute supergraviton scattering and compare this with vortex scattering in ABJM. Unfortunately, we find a mismatch between the simplest calculations on both sides that manifests in the associated interaction potentials. We will argue that this mismatch arises because a naive application of the BFSS model does not identify the correct calculations on both sides that are supposed to match. We will then speculate on the Matrix theory rules for the correct identification of the two sides.

2.2 A short review of the BFSS matrix theory

The BFSS Matrix theory of [13] is based on a discrete light cone quantization (DLCQ) of M-theory with a compact circle. In the heuristic derivation given by Sen and Seiberg [29, 30], this light-like compactification of the given M-theory is related to a space-like compactification of a different M-theory in a decoupling limit, in which the only thing that remains is a decoupled theory of N D0-branes. Each D0-brane in this description corresponds to a single unit of momentum in the compact (11th) M-theory direction. Its action is given by

$$S = \int dt \text{Tr} \left[\frac{1}{2R} D_t \tilde{X}^i D_t \tilde{X}^i - \frac{RM_P^6}{4} [\tilde{X}_i, \tilde{X}_j]^2 - \theta^T D_t \theta - RM_P^2 \theta^T \gamma_i [\theta, \tilde{X}^i] \right], \quad (2.1)$$

where the $\tilde{X}^i = X^i/g_s^{1/3}$ are nine scalars corresponding to the nine transverse directions, $D_t = \partial_t + iA$, and A is a 0+1 dimensional $U(N)$ gauge field. In addition R is the compactification radius, M_P , the 11-dimensional Planck mass and the θ are fermionic

2.2 A short review of the BFSS matrix theory

superpartners of the \tilde{X}^i that transform as spinors under the $SO(9)$ group of transverse rotations.

The simplest classical solution of the BFSS Lagrangian

$$\tilde{X}^i(t) = (x_0^i + v^i t) \mathbb{I}_{M \times M}, \quad (2.2)$$

can be understood as $M \leq N$ D0-branes located at \vec{x}_0 and moving with velocity \vec{v} . In spacetime, this corresponds to a pointlike object with M units of momentum along the 11th direction¹ and is interpreted as a supergraviton. Other classical solutions correspond to other D -branes in spacetime - $D2$ -branes and $D4$ -branes - with different possible geometries.

Arguably, the calculation that received most attention in Matrix theory, ostensibly providing the first real test of the BFSS conjecture, is the matching of the interaction potential of two spacetime supergravitons with the corresponding interaction potential for two corresponding objects of the form (2.2) in Matrix theory. The calculation of the supergraviton interaction potential, as described in [32], is based on the observation of 't Hooft [33] that the tree level Rutherford scattering interaction potential in gravity (mediated by single graviton exchange) can be calculated by scattering two gravitational shockwaves. One of these is described by an Aichelburg-Sexl shockwave - which serves as a heavy *source graviton*, while the second is a plane wave probing it. Note that the 't Hooft calculation was in a flat space background, but this procedure was also applied successfully to the curved space case [34, 35].

The first step in determining the interaction potential in the more general case, then, is to calculate the shockwave in the given spacetime background. This plays the role of the graviton wavefunction and corresponds to adding an $h_{--}(dx^-)^2$ term to the background metric. These 'pp' shockwaves have the remarkable property (not shared by many solutions in the highly nonlinear field equations of general relativity) that the linearized solution is exact [36], i.e. h_{--} is the solution of the Poisson equation

$$\Delta_{bgr}^\perp(\vec{x})h_{--}(\vec{x}) = Q\delta^\perp(\vec{x}), \quad (2.3)$$

where the source Q depends on the momentum of the wave in the 11th dimension, $p_{11} = N_s/R$ and M_P . In the BFSS case, there are nine transverse directions (and two parallel directions - time and the 11th dimension, in which the wave propagates), so, by dimensional analysis, the solution must scale like $1/r^7$. More precisely $h_{--} = \frac{15\pi N_s}{RM_P^9 r^7} \delta(x^-)$,

¹Or, equivalently, carrying D0-brane charge M .

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and the source graviton is in a state of definite p_- so that averaging over $x^- \in (0, 2\pi R)$ gives

$$h_{--} = \frac{15N_s}{2R^2M_P^9} \frac{1}{r^7}. \quad (2.4)$$

To compute the interaction potential we then scatter a probe scalar (plane wave) off the shockwave background. Practically speaking, this requires considering the h_{--} part of the metric as an interaction and using standard quantum mechanical perturbation theory. Consequently, the free wave equation $(\square + m^2)\phi = 0$ becomes²

$$\left[\frac{\partial^2}{\partial \vec{x}^2} - \frac{\partial^2}{\partial x_0^2} + m^2 \right] \phi = \left[-\frac{h_{--}}{2} (\partial_0 - m)^2 \right] \phi \equiv V\phi \quad (2.5)$$

where $\phi = \bar{\phi} e^{ipx_{11}}$ and $p = im$. At this stage, a few points deserve some elaboration.

- Momentum in the 11th dimension acts like a mass from the 10-dimensional point of view.
- Even though gravitons move at the speed of light, we call a graviton moving entirely in the 11th dimension a $v = 0$ solution. This will be an h_{--} gravitational perturbation.
- The probe graviton has nonvanishing v (i.e., a relative velocity between the probe and the source), and so will also propagate ever so slightly in the other ten dimensions.
- For both gravitons, however, x^- plays the role of time which is as it should be seeing as how we are in a DLCQ description.

With this prescription (and the usual relativistic normalization), the probe graviton wavefunction

$$\phi(x) = \frac{1}{(2\pi)^{9/2}} \frac{1}{\sqrt{2E}} e^{i\vec{p}\vec{x} - iEx_0}, \quad (2.6)$$

where $E^2 = \vec{p}^2 + m^2$, $E = \frac{m}{\sqrt{1-v^2}}$ and $\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2}}$. The 1-loop interaction potential in momentum space is computed through the S-matrix

$$2\pi\delta(E_{in} - E_{out})V_{int}^{(1)} = S^{(1)} = \int d^{10}y \phi_{out}(y)^* V(y) \phi_{in}(y).$$

On substituting for the graviton wavefunction and the expression for $V(y)$ above,

$$S^{(1)} = \frac{i(\tilde{E}_{in} - m)^2}{(2\pi)^9 2\sqrt{E_{in}E_{out}}} \int dx_0 e^{i(E_{out} - E_{in})x_0} \int d^9x e^{i(\vec{p}_{in} - \vec{p}_{out})\vec{x}} h_{--}(\vec{x}), \quad (2.7)$$

²Throughout this thesis the D'Alembertian operator will always include the m^2 term

2.2 A short review of the BFSS matrix theory

so that after integrating, the interaction potential can be read off as

$$V_{int}^{(1)}(\vec{x}) = h_{--} \frac{(\tilde{E} - m)^2}{2\tilde{E}} \simeq m \frac{v^4}{8} h_{--} + \mathcal{O}(v^6). \quad (2.8)$$

Finally, using $m = N_p/R$ gives the full one-loop result

$$V_{int}(\vec{x}) \sim N_s N_p \frac{v^4}{r^7}. \quad (2.9)$$

It is this interaction potential - in particular the coefficient of the v^4/r^7 term - that is then matched to the one-loop Matrix-theory of BFSS [32]. To compute the interaction potential of two supergravitons

$$X^i(t) = (x_1^i + v_1^i t) \mathbb{I}_{N_1 \times N_1} + (x_2^i + v_2^i t) \mathbb{I}_{N_2 \times N_2}, \quad (2.10)$$

separated by $r = |\vec{x}_1 - \vec{x}_2|$ and with relative velocity $v = |\vec{v}_1 - \vec{v}_2|$ in Matrix theory, we note that since the two diagonal blocks are *non-overlapping* these can be interpreted as two *distinct* supergravitons with correspondingly different extent in the transverse directions. If we choose $v_1^i = -v^i$ and $v_2^i = +v^i$, we get that

$$H = \frac{1}{2R} \text{Tr} |\dot{X}^i|^2 = \frac{(N_1 + N_2) v^2}{R} \frac{1}{2} = \frac{m_1 v^2}{2} + \frac{m_2 v^2}{2} = E_1 + E_2, \quad (2.11)$$

from the Matrix action. This is the free nonrelativistic energy of the supergravitons. To find the interaction potential, we must go to one-loop, in which case one takes the one-loop fluctuation determinant around

$$X^i(t) = (b^i + v^i t) \mathbb{I}_{N \times N}, \quad (2.12)$$

where now $b^i = x_1^i - x_2^i$ and $v^i = v_1^i - v_2^i$ are the relative positions and velocities respectively. The calculation of one-loop determinants amounts to just the zero point fluctuation $\sum_n \omega_n/2$. Actually, this is the same calculation that one does to compute the quantum mass of solitons like, for example, a kink. There one calculates this sum in the background of the kink solution (see, e.g., [37]). The result is in perfect agreement with (2.9).

To summarize then, the leading order calculation of the interaction potential on the Matrix theory side matches the leading order result on the gravity side, even though the former is at *one-loop*, while the latter is *classical*. At this point, it is only natural to ask:

Is there an analogous computation of interaction potentials that can be performed in ABJM?

2.3 Set-up and expectations

We saw that the BFSS model - a (0+1)-dimensional $U(N)$ Matrix model on the worldvolume of N D0-branes - describes M-theory in discrete light cone quantization. However, since the D0-branes are momentum modes on the compact 11th dimension, this description of M-theory is not a fundamental one. Instead, as shown by Sen and Seiberg in [29, 30], it appears because of the equivalence of the original M-theory with a decoupled theory of D0-branes living in another M-theory. Any fundamental description of M-theory must involve M2-branes instead [31], but we don't know how to formulate it. In the large N limit classical D -branes appear as solutions of the classical BFSS theory. For example, D2-brane solutions in [13] were found, wrapping a fuzzy torus defined through,

$$\begin{aligned} X^8 &= R_8 P; & X^9 &= R_9 Q \\ P &= \sqrt{N} p; & Q &= \sqrt{N} q \\ [Q, P] &= 2\pi i \end{aligned} \tag{2.13}$$

where $U = e^{ip}$ and $V = e^{iq}$ are the “clock” and “shift” operators of the fuzzy torus satisfying

$$\begin{aligned} UV &= e^{\frac{2\pi i}{N}} VU \\ U^N &= V^N = 1 \end{aligned} \tag{2.14}$$

Another instructive example is Matrix theory in a pp-wave background (*i.e.* the BMN Matrix model [4]). Here, 2-branes wrapping a fuzzy S^2 are also a solution. There are two ways to think about this: either as another example of the same BFSS construction as BFSS, only in a different spacetime background, or as a massive deformation of the BFSS model since the pp-wave corresponds to a mass deformation on the brane worldvolume. The presence of the mass deformation also serves to better define the Matrix theory as it makes states discrete instead of continuous. If Matrix theory is to correctly describe M-theory (and its dimensional reduction to type IIA string theory) then it should be able to describe *all* D -branes in the theory and not just $D2$ -branes. For example, a $D4$ -brane wrapping an S^4 was found in [14], following the earlier works of [15, 16], but the solution is not without several unresolved subtleties. In general, finding the complete spectrum of D -branes from Matrix theory remains a very difficult problem. The $D2$ - and $D4$ -branes already found are reductions to ten dimensions of $M2$ - and $M5$ -branes, and while they are a minimum necessary for the spectrum of M-theory, they are by no means sufficient. Indeed, we would also need to find a $D6$ -brane, coming from an eleven

dimensional KK monopole, and a $D8$ -brane³.

Fortunately, the recent construction by Aharony *et.al.* [21] of an IR action for N $M2$ -branes at a C^4/Z_k singularity offers some much needed hope. Since this ABJM model is also a theory of $N \times N$ matrices⁴, it is natural to ask whether one can construct a new kind of Matrix theory from the ABJM model, perhaps one whose formulation is more fundamental instead of the somewhat derived one of BFSS. In support of this idea, it was shown in [23, 24] that $D4$ -branes wrapping a fuzzy S^2 appear as solutions of (a massive deformation of) the ABJM model, in much the same way as in the BFSS Matrix theory. In as much as the BFSS Matrix theory is related to the maximally supersymmetric BMN Matrix model by a massive deformation of the former, the pure ABJM model can be mass-deformed to yield a maximally supersymmetric massive ABJM model [39]. In this case the fuzzy funnel solution of the pure theory stabilizes to a fuzzy sphere in the massive one. This hints then, that the massive deformation gives a better definition of a proposed Matrix model. Nevertheless we will first begin with an analysis of the pure ABJM model and then show how we are driven to its massive deformation. Our strategy will be as follows:

- The presence of the $D4$ wrapping a fuzzy S^2 solution suggests that we start with a similar set-up to BFSS.
- The first issue to be checked is whether there is a classical solution corresponding to a supergraviton, for which we can then compute scattering.
- Thereafter, we will test whether other D -branes appear as classical solutions⁵.

We immediately notice a difference from BFSS; namely that the N $M2$ -branes of ABJM have two spatial worldvolume directions while a supergraviton needs to be pointlike in both the transverse and parallel space directions. It also carries $D0$ -brane charge $M \leq N$ so we must look for a vortex-type solution of ABJM, pointlike in the transverse space, and with some $D0$ -brane charge M . Once we find such objects, we can then try to scatter them, and match against scattering of supergravitons.

Even having identified which objects to scatter, the issue of matching superpotentials is

³The latter would appear in the massive type IIA string theory, for which the Matrix theory was constructed in [38]. Its properties remain largely unexplored.

⁴Albeit one with a $U(N) \times U(N)$ gauge group and bifundamental degrees of freedom.

⁵And reserve the right to not be surprised if, as for BFSS, it turns out to be difficult to find all of them.

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still a little murky since there is no *a priori* guide which term should correspond to which. This is not unique to our ABJM computations either. Indeed, in the BFSS model the leading supergraviton scattering is a tree-level (classical) interaction, while on the Matrix side the leading term is a one-loop interaction with the classical interaction vanishing. Moreover, on the gravity side the tree-level interactions yield an infinite series that should match various higher loop correction terms in the Matrix side but the precise matching cannot be guessed before doing at least a computation of the N_s, N_p and M_P dependences.

In our case the increased computational complexity means that even the N_s, N_p dependence is hard to obtain in the ABJM side while the M_P dependence cannot be guessed before computing the r dependence, as we discuss below. Our goal therefore will be to first compute and match the simplest, leading terms on both sides. On the gravity side, this will be a similar tree-level term as in the BFSS case, but now in the Matrix model, the leading classical interaction (for the vortex scattering) is no longer vanishing.

One other difference between the ABJM and BFSS models is that pure ABJM is a conformal theory while the BFSS Lagrangian, having M_P^3 as coupling, is not. In the BFSS interaction potential, the M_P dependence appears from two sources, one being as a coupling dependence, and the other as giving the unit of length when translating from BFSS to gravity. The latter is of the same kind as in AdS/CFT where the l_s dependence of gravity calculations also appears by introducing a unit of length in the conformal calculations of the field theory. Since pure ABJM is also conformal, only the latter type of M_P dependence would be available. So the matching is *a priori* more constrained than in the BFSS case. Of course, as we said above, the massive ABJM will be found to be more useful for defining a Matrix model. There we will see that we have one more parameter - the mass deformation μ - potentially improving the situation. We will return to these issues in some detail in section 8 where we describe the systematics of matching in BFSS and how they might apply to our case.

While on the topic of expectations, it is worth asking at this point what other D -branes we expect to find as classical solutions of the ABJM model? Certainly, as in BFSS, we would want at least an example of a $D2$ -brane and a $D4$ -brane, which would mean that $M2$ - and $M5$ -branes appear in the theory. These are already present however; for the $D2$ -branes there is at least the configuration of the ABJM 2-branes themselves⁶ while for $D4$ -branes there is (at least) the solution corresponding to a $D4$ -brane on a fuzzy

⁶See, for instance [40] for an example of how the worldvolume theory of N $D2$ -branes arises.

S^2 . Of course, this is by no means satisfactory, in part because we now have the peculiar fact that directions parallel and transverse to the ABJM worldvolume are different so we would like to see at least one example of a “shifted” brane. For example, we would expect to find *kink* solutions having a fuzzy one-dimensional transverse space and a fuzzy three-dimensional transverse space, corresponding to $D2$ - and $D4$ -branes respectively, each with only one direction parallel while transverse $D2$ - and $D4$ -branes with no parallel directions should be described as *vortex* solutions with fuzzy two-dimensional and a fuzzy four-dimensional transverse space. We will seek to find at least one example among these solutions.

2.4 Pure ABJM solutions

Solutions of the pure ABJM theory and its massive deformation appear to be related, usually in a fairly nontrivial way. For example in [23, 24], it was shown that not only does the maximally supersymmetric fuzzy sphere ground state of massive ABJM become the $\frac{1}{2}$ -BPS fuzzy funnel solution of pure ABJM, but they also enjoy many shared properties, like the same unrescaled bosonic action for fluctuations. Certainly then, we would not be surprised if the vortex and kink solitons found in both pure and massive ABJM [41, 42, 43], turn out to be similarly related. In this section we will discuss the solitons of pure ABJM, though for technical reasons we will be forced to switch to massive ABJM in the next section.

To this end, we begin by asking if we can find vortex solutions of *pure ABJM theory* that can be interpreted as supergravitons in spacetime? A good place to start answering this

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question is with the ansatz presented in [42] for a general class of vortex solutions,

$$\begin{aligned}
 Y^A &= v^A \mathbb{I}, \text{ for } A = 2, 3, 4, \\
 Y^1 &= \sum_{a=1}^{N-1} y_a e^a + y_M E^{-M}, \\
 y_M &= \frac{G(z)}{\prod_{a=1}^{N-1} y_a},
 \end{aligned} \tag{2.15}$$

$$\begin{aligned}
 \partial \bar{\partial} \ln |y_a|^2 &= 4v \left(\frac{2\pi}{k} \right)^2 \sum_{b=1}^{N-1} K_{ab} \left(|y_b|^2 - \frac{|G(z)|^2}{|c_b|^2 \prod_{c=1}^{N-1} |y_c|^2} \right), \\
 F_{12} = \hat{F}_{12} &= -\frac{s}{2} \left(\frac{2\pi v}{k} \right)^2 [Y^1, Y_1^\dagger], \\
 D_0 Y^A &= i s \frac{2\pi}{k} v^A [Y^1, Y_1^\dagger],
 \end{aligned}$$

where Y^A are the four complex $U(N)$ -valued bifundamental scalars of ABJM; A_μ and \hat{A}_μ , ($\mu = 0, 1, 2$) are worldvolume $U(N) \times U(N)$ gauge fields and e^a, h^a and E^{-M} are generators satisfying

$$(e^a)^\dagger = e^{-a}, (E^M)^\dagger = E^{-M}, [E^{-M}, e^a] = 0, [e^a, e^{-a'}] = \delta_{aa'} h^a, [h^a, e^b] = K_{ab} e^b. \tag{2.16}$$

K_{ab} is the Cartan matrix and we have defined

$$\begin{aligned}
 A &= \frac{(A_1 - iA_2)}{2}, \\
 z &= x_1 + ix_2, \\
 D_\mu Y^A &= \partial_\mu Y^A + iA_\mu Y^A - iY^A \hat{A}_\mu, \\
 A &= \sum_{a=1}^{N-1} A_a h^a, \\
 v^2 &= \sum_{A=2}^4 |v^A|^2.
 \end{aligned} \tag{2.17}$$

$G(z)$ is an arbitrary holomorphic function, c_b is a constant and x_1, x_2 are the two space worldvolume directions. The vortex solution then obtains by solving the above equations for the scalars and the gauge fields. There are two particularly simple cases:

1. By taking the trivial value for the holomorphic function, namely $G = 0$, we can obtain an approximate solution. Define $K_a = \sum_b K_{ab}$. Then $y_M = 0$ and the

equation for y_a becomes $\partial\bar{\partial}\ln|y_a|^2 = A\sum_b K_{ab}|y_b|^2$, with A a constant. Consider moreover the case when K_a is independent of a , in which case we can choose a solution with $g = |y_a|^2$ independent of a , so that $\partial\bar{\partial}\ln g = AK_ag$. Then near $z = 0$, the solution is approximately $|y^a|^2 \sim e^{AK_a|z|^2}$ or,

$$Y^1 \sim \sum_{a=1}^{N-1} e^a e^{\frac{1}{2}AK_a|z|^2} \quad (2.18)$$

We will not calculate the gauge fields, since it is not clear how to interpret this approximate solution.

2. The simplest exact solution is obtained for the simplest nontrivial G . In this case, taking

$$y^a = \frac{y^M}{c_a},$$

reduces the y^a equation to $\partial\bar{\partial}\ln|y^M|^2 = 0$ and from the y^M equation we get

$$y_M = \left(G(z) \prod_{a=1}^{N-1} c_a \right)^{1/N}, \quad (2.19)$$

Then, if we choose the simplest holomorphic function $G(z) = z - z_0$, the equation $\partial\bar{\partial}\ln|y_M|^2 = 0$ is identically satisfied and we have a complete solution. Specifically,

$$Y^1 = \left(\sum_{a=1}^{N-1} \frac{e^a}{c_a} + e^{-M} \right) \left((z - z_0) \prod_{a=1}^{N-1} c_a \right)^{1/N}, \quad (2.20)$$

$$Y^A = v^A \mathbb{I}, \quad A = 2, 3, 4.$$

For this solution can be interpreted as a supergraviton, we have to first argue that it corresponds to a *classical pointlike object* in spacetime. This is not too difficult to see. First notice that the corresponding coordinate Y^A is fixed and, according to the usual Matrix theory definitions, a VEV proportional to the identity corresponds to a fixed classical coordinate. Further, at the position of the vortex, $z = z_0$, $Y^1 = 0$ is also fixed, so that this object is extended only in time and not in any of the parallel or transverse coordinates.

Let's now calculate the gauge fields for this solution. To do so, we need to compute

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A_0, \hat{A}_0, A and \hat{A} (or equivalently, $A_1, A_2, \hat{A}_1, \hat{A}_2$). Since $F_{12} = \hat{F}_{12}$, we can choose $A = \hat{A}$. First, note that $F_{12} = \partial_1 A_2 - \partial_2 A_1$ implies that $\partial \bar{A} = \frac{1}{4}[\partial_1 A_1 + \partial_2 A_2 + i F_{12}]$. Consequently, in the Coulomb gauge, $\partial_i A_i = \partial_1 A_1 + \partial_2 A_2 = 0$, the magnetic field takes the succinct form $B = F_{12} = 4\partial \bar{A}/i$. Using the properties of the generators, $[Y^1, Y_1^\dagger] = \sum_{a=1}^{N-1} |y_a|^2 h^a$, so that substituting into (2.16) and integrating, we get

$$B = -\frac{1}{2} \left(\frac{2\pi v}{k} \right)^2 \left(\prod_{b=1}^{N-1} |c_b|^2 \right)^{1/N} \sum_{a=1}^{N-1} \frac{|z - z_0|^{2/N}}{|c_a|^2} h^a, \quad (2.21)$$

$$\frac{A_1 + iA_2}{2} \equiv A^* = - \left[\frac{Ni}{8(N+1)} \left(\frac{2\pi v}{k} \right)^2 \left(\prod_{b=1}^{N-1} |c_b|^2 \right)^{1/N} \right] \sum_{a=1}^{N-1} \frac{(z - z_0)^{\frac{N+1}{N}} (\bar{z} - \bar{z}_0)^{1/N}}{|c_a|^2} h^a,$$

On the other hand, to solve for the A_0 , we substitute the expression for the commutator $[Y^1, Y_1^\dagger]$ into

$$D_0 Y^A = iA_0 Y^A - iY^A \hat{A}_0 = i \frac{2\pi}{k} v^A [Y^1, Y_1^\dagger], \quad (2.22)$$

to obtain

$$A_0 Y^A - Y^A \hat{A}_0 = Y^A \frac{2\pi}{k} \left(\prod_{b=1}^{N-1} |c_b|^2 \right)^{1/N} \sum_{a=1}^{N-1} \frac{h^a}{|c_a|^2} |z - z_0|^{2/N}, \quad (2.23)$$

from which (after choosing $\hat{A}_0 = 0$) we can read off

$$A_0 = \frac{2\pi}{k} \left(\prod_{b=1}^{N-1} |c_b|^2 \right)^{1/N} \sum_{a=1}^{N-1} \frac{h^a}{|c_a|^2} |z - z_0|^{2/N}. \quad (2.24)$$

Note that for this vortex solution, the magnetic field $B = F_{12}$ goes to zero at $z = z_0$ (the position of the vortex), but the electric field $F_{0z} = -\partial A_0$ diverges at that same location.

This vortex solution looks like it could stand in for the spacetime supergraviton, since it is a pointlike object in spacetime, carrying $D0$ -brane charge. This last property is perhaps not obvious. However, as a massive, classical, pointlike spacetime object in ten dimensions this vortex must carry a charge corresponding to its momentum in the eleventh dimension and, as there is no other possible candidate, this must be identified with a $D0$ -brane charge. As we don't have a dual description of this solution though, it is not easy to check this assertion explicitly. We will see that in the related case of the vortex of the massive deformation of ABJM that the corresponding object does indeed carry $D0$ -brane charge.

However, it is also easy to see that the solution has infinite energy, since the energy density of the magnetic field increases away from $z = z_0$. Note also that the solution has the complex coordinate $Y^1 \propto (z - z_0)^{1/N}$ which, in the large N limit exhibits a step function-like behaviour (0 at the vortex position and 1 away from it). In other words, away from the vortex, the solution represents a two-dimensional worldvolume growing at infinity, which could be identified with a 2-brane. As similar as they sound, this solution is different from BIon branes of [44, 45], where a single coordinate $X \sim 1/r$ signals a string extending to infinity at $r = 0$, as well as the self-intersecting $M2$ -brane of [44, 46, 47], where the two complex coordinates⁷ s and t are related by $s \propto c/t$ ($c = \text{constant}$). In each of those cases, a new brane or string “grows” at the position of the singularity, and represents the spacetime intersection of branes. Charge conservation then implies that the worldvolume flux must flow through a string or brane which must either extend to infinity, as for the BIon and self-intersecting $M2$, or end on yet another brane [44, 45]. This latter solution, ending at $r = r_0$ and finite $X(r_0)$, is half of the $D \perp F1 \perp \bar{D}$ brane configuration, and provides an example of charge conservation by ending on another brane with $X(r_0)$ finite and $X'(r_0) = \infty$.

The vortex solution of pure ABJM is clearly an example of the latter. To be precise, since $Y^1(z_0) = 0$ but $Y^{1'}(z_0) = \infty$, the only way to enforce spacetime charge conservation would be to have $D0$ -branes at the endpoint. The situation is muddled however by the fact that Y^1 grows in the direction of the original 2-brane (as $z \rightarrow \infty$).

In [48], it was demonstrated that the vortex solutions of pure ABJM described here have the same supersymmetries as self-intersecting $M2$ -branes. This fact alone likely means that we can interpret the solution away from z_0 as a self-intersecting $M2$ -brane, with the caveat that $D0$ -branes be added to the $z = z_0$ point to enforce charge conservation. It seems then that the vortex solution should be thought of as a bound state of self-intersecting 2-branes and $D0$ -branes at z_0 . This interpretation matches nicely with the picture we will find in the massive ABJM case, where the vortex is a bound state of dielectric $D2$ -branes blown up into $D4$ -branes, and $D0$ -branes at the vortex position. The difference of course, is that in this case the 2-branes are infinite in extent, and have correspondingly infinite total energy.

To summarize; *even though the vortex solution looks like a supergraviton, it is hard to*

⁷One of these is a worldvolume coordinate, $t = x_1 + ix_2$ while the other, $s = X^4 + iX^5$, is transverse.

get actual physics from it. Essentially, this is because we will still need to have some appropriate regularization that to render the energy finite before we can subtract the 2-brane contribution and interpret the remainder as the supergraviton. There is no reason why this cannot be done in principle, we just have not found a simple way to do it. By contrast, it turns out to be much easier to work with the massive ABJM theory where everything is wonderfully finite. To this end then, in the next section we study the finite energy solutions of the massive ABJM theory.

2.5 ABJM and its massive deformation

2.5.1 The mass term

Since The ABJM theory a description of M2-branes in flat space, it must admit a mass-deformation that preserve its maximal supersymmetry. The pure ABJM theory preserve the $SU(4) \times U(1)$ subgroup of the R-symmetry group. The maximally supersymmetric mass-deformation breaks the $SO(8)$ group to $SO(4) \times SO(4) \times \mathbb{Z}_2$. We would expect to have set of mass deformations that the preserve the maximal supersymmetry, but all these deformations must be related by the $SO(8)$ rotations. To see this let the rotation group act on the vector $V = (v_1, \dots, v_4, w_1, \dots, w_4)$, then we can define the subgroup $SO(4) \times SO(4)$ by the $SO(8)$ rotations that do not mix v_i and w_j components of V , while the \mathbb{Z}_2 to be the transformations that swap v_i and w_i components.

On the other hand the group $SU(4)$ can be defined by the complex rotations of the complex vector $(v_1 + iv_2, v_3 + iv_4, w_1 + iw_2, w_3 + iw_4)$, while the $U(1)$ is defined to be the transformation that multiples this complex vector by an overall phase.

The group $SU(2) \times SU(2) \times U(1)$ is a common subgroup of the $SO(8)$ R-symmetry group, $SU(4) \times U(1)$ and $SO(4) \times SO(4)$. This subgroup is generated by the independent complex rotations $(v_1 + iv_2, v_3 + iv_4)$ and $(w_1 + iw_2, w_3 + iw_4)$, and the overall phase rotation. Therefore the maximally supersymmetric mass-deformation should preserve a manifest $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$ symmetry.

To preserve this $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$ symmetry without having any flat direction, the mass deformation must give equal masses to all scalar fields. Therefore we must add the following mass term

$$\mathcal{L}_{mass} = \mu^2 \text{tr} \left(C^I C_I^\dagger \right)$$

to the Lagrangian (1.58).

2.5.2 The Lagrangian

The ABJM model [21] is an $\mathcal{N} = 6$ supersymmetric $U(N) \times U(N)$ Chern-Simons gauge theory at level $(k, -k)$, with bifundamental scalars C^I and fermions ψ_I , $I = 1, \dots, 4$ in the fundamental of the $SU(4)_R$ symmetry group and gauge fields for the two groups A_μ and \hat{A}_μ . Its action is given by

$$\begin{aligned}
 S = & \int d^3x \left(\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \right. \\
 & - \text{Tr} D_\mu C_I^\dagger D^\mu C^I - i \text{Tr} \psi^{I\dagger} \gamma^\mu D_\mu \psi_I + \frac{4\pi^2}{3k^2} \text{Tr} \left(C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger \right. \\
 & + C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K + 4C^I C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger - 6C^I C_J^\dagger C^J C_I^\dagger C^K C_K^\dagger C^K \left. \right) \quad (2.25) \\
 & + \frac{2\pi i}{k} \text{Tr} \left(C_I^\dagger C^I \psi^{J\dagger} \psi_J - \psi^{J\dagger} C^I C_I^\dagger \psi_J - 2C_I^\dagger C^J \psi^{I\dagger} \psi_J + 2\psi^{J\dagger} C^I C_J^\dagger \psi_J \right. \\
 & \left. + \epsilon^{IJKL} C_I^\dagger \psi_J C_K^\dagger \psi_L - \epsilon_{IJKL} C^I \psi^{J\dagger} C^K \psi^{L\dagger} \right),
 \end{aligned}$$

where the gauge-covariant derivative is

$$D_\mu C^I = \partial_\mu C^I + iA_\mu C^I - iC_I \hat{A}_\mu. \quad (2.26)$$

The action has an $SU(4) \times U(1)$ R-symmetry associated with the $\mathcal{N} = 6$ supersymmetries. It admits a maximally supersymmetric (*i.e.*, preserving all $\mathcal{N} = 6$) massive deformation with mass parameter μ [25, 27], which breaks the R-symmetry down to $SU(2) \times SU(2) \times U(1)_A \times U(1)_B \times \mathbb{Z}_2$ by splitting the scalars as

$$C^I = (Q^\alpha, R^\alpha); \quad \alpha = 1, 2 \quad (2.27)$$

The mass deformation changes the potential to

$$V = \text{Tr} (|M^\alpha|^2 + |N^\alpha|^2), \quad (2.28)$$

where

$$\begin{aligned}
 M^\alpha &= \mu Q^\alpha + \frac{2\pi}{k} \left(2Q^{[\alpha} Q_\beta^\dagger Q^{\beta]} + R^\beta R_\beta^\dagger Q^\alpha - Q^\alpha R_\beta^\dagger R^\beta + 2Q^\beta R_\beta^\dagger R^\alpha - 2R^\alpha R_\beta^\dagger Q^\beta \right), \\
 N^\alpha &= -\mu R^\alpha + \frac{2\pi}{k} \left(2R^{[\alpha} R_\beta^\dagger R^{\beta]} + Q^\beta Q_\beta^\dagger R^\alpha - R^\alpha Q_\beta^\dagger Q^\beta + 2R^\beta Q_\beta^\dagger Q^\alpha - 2Q^\alpha Q_\beta^\dagger R^\beta \right). \quad (2.29)
 \end{aligned}$$

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This mass-deformed theory (mABJM) has ground states of the *fuzzy sphere* type given by [25, 27]

$$R^\alpha = cG^\alpha; \quad Q^\alpha = 0 \quad \text{and} \quad Q_\alpha^\dagger = cG^\alpha; \quad R^\alpha = 0 \quad (2.30)$$

where $c \equiv \sqrt{\frac{\mu k}{2\pi}}$ and the matrices G^α , $\alpha = 1, 2$, bifundamental under $U(N) \times U(N)$, satisfy (with no summation on repeated indices)

$$G^\alpha = G^\alpha G_\beta^\dagger G^\beta - G^\beta G_\beta^\dagger G^\alpha \quad (2.31)$$

This ground state corresponds to a fuzzy 2-sphere [23, 65]. An explicit solution for G^α is given by

$$\begin{aligned} (G^1)_{m,n} &= \sqrt{m-1} \delta_{m,n}, & (G^2)_{m,n} &= \sqrt{(N-m)} \delta_{m+1,n} \\ (G_1^\dagger)_{m,n} &= \sqrt{m-1} \delta_{m,n}, & (G_2^\dagger)_{m,n} &= \sqrt{(N-n)} \delta_{n+1,m}. \end{aligned} \quad (2.32)$$

We will now use these so-called GRVV matrices to posit an ansatz that effectively abelianizes the ABJM model while retaining the large N limit.

2.6 Massive ABJM solitons

Following several earlier works on massive deformations of BLG theories [18, 49, 50], a maximally supersymmetric ($\mathcal{N} = 6$) massive deformation of ABJM was recently proposed in [25]. However, it is not clear exactly what the brane interpretation of the deformed theory is. One suggestion is that before the brane backreaction, the background corresponds to the maximally supersymmetric type IIB pp-wave. In [43], a different conjecture, for the gravity dual of massive ABJM was proposed, based on a Z_k orbifold of the LLM solution corresponding to the massive deformation of N $M2$ -branes in flat space.

For our purposes, we will need to understand the massive deformed background in which the N $M2$ -branes move (*before* back-reaction) *i.e.* the analog of the C^4/Z_k orbifold for pure ABJM. Once we have computed the background dual to the deformed ABJM model, we proceed to analyze the various solutions of massive ABJM. From the point of view of the ABJM worldvolume, these may be classified as brane-filling, vortex or kink type, according to their codimension (zero, two and one respectively). We will provide a brane interpretation of these solitons.

2.6.1 Dual of massive ABJM deformation

To begin, we will compute the spacetime background corresponding to the massive deformation of ABJM. Recall that in the case of pure ABJM, the spacetime background is set up by N $M2$ -branes living at the tip of the cone on $R^{2,1} \times C^4/Z_k$. We will deal with the Z_k orbifolding at the end, but for the moment we would like to understand the background that replaces the flat eleven-dimensional $R^{2,1} \times C^4$ in the presence of the massive deformation. To this end, following the suggestion in [25] that it could be related to the IIB maximally supersymmetric pp wave, we consider the $k = 1$ case first and notice that the type IIB gravitational wave

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 + (H - 1)(dx + dt)^2 + d\vec{x}^2, \\ e^\phi &= g_s, \quad B = 0, \end{aligned} \tag{2.33}$$

is T-dual⁸ to the fundamental string (F1) solution

$$\begin{aligned} ds_{str}^2 &= H^{-1}(-dt^2 + dx^2) + d\vec{x}^2, \\ B_{tx} &= 1 - H^{-1}, \\ e^\phi &= H^{-1/2}, \end{aligned} \tag{2.34}$$

where $H = 1 + Q/r^6$ is a harmonic function of the transverse coordinates \vec{x} and the F1 solution is such that all fields are trivial at $\vec{x} \rightarrow \infty$. In turn, using

$$\begin{aligned} ds_M^2 &= e^{4\phi/3} (ds_{10}^2 + e^{2\phi} (A_\mu dx^\mu + dx_{11})^2), \\ F_{abcd}^{(M)} &= e^{4\phi/3} (F_{abcd} + 4A_{[a} H_{bcd]}), \\ F_{abc11}^{(M)} &= H_{abc}, \end{aligned} \tag{2.35}$$

⁸The easiest way to see this is to use Buscher's T-duality rules [51, 52, 53] which, in the absence of RR fields are the same for IIA \rightarrow IIB and IIB \rightarrow IIA, namely:

$$\begin{aligned} \tilde{g}_{00} &= \frac{1}{g_{00}}; \quad \tilde{g}_{0i} = \frac{B_{0i}}{g_{00}}; \quad \tilde{g}_{ij} = g_{ij} - \frac{g_{0i}g_{0j} - B_{0i}B_{0j}}{g_{00}} \\ \tilde{B}_{0i} &= \frac{g_{0i}}{g_{00}}; \quad \tilde{B}_{ij} = B_{ij} + \frac{g_{0i}B_{0j} - B_{0i}g_{0j}}{g_{00}} \\ \tilde{\phi} &= \phi - \frac{1}{2} \ln g_{00} \end{aligned}$$

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the $F1$ solution lifts to the membrane ($M2$) solution

$$\begin{aligned} ds_M^2 &= H^{-2/3} d\bar{y}^2 + H^{1/3} d\bar{y}^2, \\ F_4 &= dy^0 \wedge dy^1 \wedge dy^2 \wedge dH^{-1}, \\ H &= 1 + \frac{Q}{r^6}, \end{aligned} \tag{2.36}$$

in M -theory. Here we define $(y^0, y^1, y^2) = (t, x, y \equiv x_{11})$. On the other hand, the *maximally supersymmetric type IIB pp-wave*,

$$\begin{aligned} ds^2 &= 2dx^+ dx^- - \mu^2 \bar{x}^2 (dx^+)^2 + d\bar{x}^2, \\ F_{+1234} &= F_{+5678} = \mu, \\ e^\phi &= g_s; \quad B = 0, \end{aligned} \tag{2.37}$$

can be understood as the gravitational wave obtained in the presence of the constant flux $F_{+1234} = F_{+5678} = \mu$, which also modifies $H - 1$ from Q/r^6 to $-\mu^2 r^2$. Correspondingly, the IIA solution T-dual to this pp-wave is found by using the full IIB \rightarrow IIA Buscher rules (including those for the RR sector):

$$\begin{aligned} \tilde{A}_{ijk} &= \frac{8}{3} D_{ijk}^+ + B_{0[i} B_{jk]}^{(2)} - B_{0[i}^{(2)} B_{jk]} + B_{0[i} B_{|0|j}^{(2)} \frac{g_{k]0}}{g_{00}} - B_{0[i}^{(2)} B_{|0|j} \frac{g_{k]0}}{g_{00}}, \\ \tilde{A}_{0ij} &= \frac{2}{3} B_{ij}^{(2)} + 2 \frac{B_{0[i}^{(2)} g_{j]0}}{g_{00}}, \\ \tilde{A}_i &= -B_{0i}^{(2)} + a B_{0i}; \quad \tilde{A}_0 = 0, \end{aligned} \tag{2.38}$$

to get

$$\begin{aligned} ds_{str}^2 &= H^{-1} (-dt^2 + dx^2) + d\bar{x}^2, \\ B_{tx} &= 1 - H^{-1}, \\ e^\phi &= H^{-1/2}, \\ F_{1234} &= F_{5678} = \mu, \\ H &= 1 - \mu^2 \bar{x}^2. \end{aligned} \tag{2.39}$$

This can be interpreted as a fundamental string in the background of the constant flux $F_{1234} = F_{5678} = \mu$, which, again, modifies the harmonic function. Finally, this lifts to the M -theory solution

$$\begin{aligned} ds_M^2 &= H^{-2/3} d\bar{y}^2 + H^{1/3} d\bar{x}^2, \\ F_4 &= dy^0 \wedge dy^1 \wedge dy^2 \wedge dH^{-1} + \mu(dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8), \\ H &= 1 - \mu^2 \bar{x}^2. \end{aligned} \tag{2.40}$$

This can again be interpreted as an $M2$ -brane in the presence of constant transverse flux $F_{1234} = F_{5678} = \mu$. This flux (in particular $|F|^2$) means that H now satisfies $\partial_i^2 H = -8\mu^2$ so that, again, the harmonic function is modified. There are two points about this geometry that should be noted: before the T-dualizing, we could always rescale μ away by sending $x^+ \rightarrow x^+/\mu$ and $x^- \rightarrow \mu x^-$. After the T-duality and M-theory lift, this is no longer possible. *Now* the scale μ has physical meaning. Secondly, while there is nothing particularly interesting about the point $|\vec{x}| = 1/\mu$ in the type IIB metric, here it is potentially singular.

This solution obtained can be interpreted as an $M2$ -brane with a constant flux preserving the same manifest R -symmetry as the massive deformation of the ABJM field theory. Specifically, the $SU(4)_R$ is broken to $SU(2) \times SU(2)$, by the splitting of the scalars into the 1, 2, 3&4 and 5, 6, 7&8 directions. This gives us confidence we are on the right track. For this, and other reasons given in [25], we can say with some confidence that we have found the spacetime background corresponding to the massive deformation, at least in the $k = 1$ case. For general k , we still have to apply the \mathbb{Z}_k action (inherited from the pure ABJM case) on the transverse coordinates. Since it acts by $Y^A \rightarrow e^{2\pi i/k} Y^A$ the solution remains intact and the background is valid at any k .

There is one slight subtlety related to the amount of symmetry preserved under T-duality in the above construction. We started with maximal supersymmetry (32 supercharges, or $\mathcal{N}=16$ in three-dimensions), and ended up with a solution that *cannot* have as much since, in M-theory, the *unique* backgrounds with maximal susy are flat space, $AdS_4 \times S_7$, $AdS_7 \times S_4$ and the maximally supersymmetric eleven-dimensional pp-wave that is their Penrose limit. Since our solution is of the $M2$ -brane-type, we must have that at least the constraint $\Gamma^0 \Gamma^1 \Gamma^2 \psi = \psi$ hold. This reduces 32 supercharges to 16 (or $\mathcal{N} = 8$ in three-dimensions). In addition, the presence of the transverse flux $F_{1234} = F_{5678} = \mu$ further reduces the number of supercharges to 12 (giving $\mathcal{N} = 6$ in three-dimensions). Naively, this breaks the R -symmetry to an $SU(2) \times SU(2) \times U(1)$ which rotates the transverse coordinates Y^A . The problem is that in [25] it was argued that the massive deformation of the ABJM model retains the full $SO(6)$ R -symmetry of $N=6$ in three dimensions, so the same should apply here.

On the other hand, in [49], it was argued also that in the BLG (or, equivalently, the $N=2$ ABJM) case, the massive deformation with $M2$ on $R_t \times T^2$ is equivalent to the IIB pp-wave. There, the massive BLG model was observed to satisfy 32 supersymmetries, only

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16 of which are linearly realized, and the remaining 16 are nonlinear. The fuzzy sphere background breaks the 16 nonlinear supersymmetries, giving 16 genuine supersymmetries. These fuzzy sphere vacua correspond to $D3$ -brane giant gravitons in the IIB pp-wave background, which preserve the same 16 supersymmetries in addition to satisfying the same zero energy condition.

Returning to the case at hand, it seems that the solution to the problem of supersymmetry under T-duality is that one has to consider the full string theory (or rather Matrix theory) and not just the supergravity background. Then one needs to consider $D0$ -branes in IIB in the presence of the pp-wave. This preserves a total of 16 supercharges, instead of the full 32. After T-dualizing, in the M-theory we have an $M2$ -brane with constant transverse flux and $H = 1 - \mu^2 \vec{x}^2$, on top of the ABJM $M2$ -branes. This configuration again preserves 16 of the supercharges, so there is no problem. Of course, the above argument applies just for the $k=1$ & 2 ABJM model (that has no $3/4$ supersymmetry reduction), otherwise we need to take a Z_k quotient of the target space.

2.6.2 Brane-filling solution: the fuzzy sphere

The first type of solutions we will analyze in some detail are *brane-filling*, i.e. ground states, which can be interpreted as branes with a transverse extension. In [25] it was shown that the maximally supersymmetric ground state of the massive deformation of ABJM is given by an equation associated with a fuzzy sphere. In particular, the four complex scalars of ABJM as split as $Y^A = (Q^\alpha, R^{\dot{\alpha}})$ and the solution is given by $R^{\dot{\alpha}} = 0$ and $Q^\alpha = f G^\alpha = \sqrt{\frac{k\mu}{4\pi}}$, with

$$-G^\alpha = G^\beta G_\beta^\dagger G^\alpha - G^\alpha G_\beta^\dagger G^\beta . \quad (2.41)$$

The irreducible matrices that solve eq.(2.41) - that we will call the GRVV algebra - are

$$\begin{aligned} (\tilde{G}^1)_{m,n} &= \sqrt{m-1} \delta_{m,n} , \\ (\tilde{G}^2)_{m,n} &= \sqrt{(N-m)} \delta_{m+1,n} , \\ (\tilde{G}_1^\dagger)_{m,n} &= \sqrt{m-1} \delta_{m,n} , \\ (\tilde{G}_2^\dagger)_{m,n} &= \sqrt{(N-n)} \delta_{n+1,m} . \end{aligned} \quad (2.42)$$

Based partly on the fact that $G^\alpha G_\alpha^\dagger = N - 1$ for this irrep, this solution was conjectured to represent a fuzzy S^3 , but this was shown to not be the case in [23]. It is instead a fuzzy S^2 , as could be guessed by the fact that $G^1 = G_1^\dagger$ (so that the scalar represented by the

imaginary part of G^1 is fixed to zero). Moreover, in [24] it was shown that if we mod out by the $U(N)$ gauge transformations that leave the GRVV algebra invariant, the resulting bifundamental matrices can be viewed as fuzzy versions of Killing spinors on the S^2 . The action for fluctuations around the solution is then the supersymmetric $D4$ -brane action, compactified on S^2 . The fields can be expanded in fuzzy spherical harmonics made up of either J_i or \bar{J}_i ,

$$\begin{aligned} J_i &= (\tilde{\sigma}_i)^\alpha{}_\beta G^\beta G_\alpha^\dagger = (\tilde{\sigma}_i)^\alpha{}_\beta J^\beta{}_\alpha \equiv (\sigma_i)_\beta{}^\alpha J^\beta{}_\alpha, \\ \bar{J}_i &= (\tilde{\sigma}_i)^\alpha{}_\beta G_\alpha^\dagger G^\beta = (\tilde{\sigma}_i)^\alpha{}_\beta \bar{J}_\alpha{}^\beta \equiv (\sigma_i)_\beta{}^\alpha \bar{J}_\alpha{}^\beta, \end{aligned} \quad (2.43)$$

in the same way as the regular spherical harmonics are built from Euclidean coordinates x_i . In the classical limit, both the J_i and \bar{J}_i tend to the same x_i , up to a normalization constant. The $D4$ -brane compactified on the S^2 ground state then corresponds to a $D3$ -brane giant graviton wrapping a 3-sphere

$$|Z^1|^2 + |Z^2|^2 = R^2, \quad (2.44)$$

in the maximally supersymmetric type IIB pp-wave. Here the 4+4 coordinates transverse to the IIB pp-wave are $Z^A = (Z^\alpha, Z^{\dot{\alpha}})$ with

$$\begin{aligned} Z^1 &= X^1 + iX^2, & Z^2 &= X^3 + iX^4, \\ Z^{\dot{1}} &= X^5 + iX^6, & Z^{\dot{2}} &= X^7 + iX^8. \end{aligned} \quad (2.45)$$

We also have to mod out by \mathbb{Z}_k which acts on the transverse coordinates as $Z^A \rightarrow e^{2\pi i/k} Z^A$, and Z^A corresponds to Y^A in M-theory.

2.6.3 Vortex solutions

Moving up in co-dimension, in [41], a BPS vortex solution was found for another massive deformation of ABJM. This time the deformation is given by the superpotential term $\delta W = \mu \text{Tr}[\mathcal{Z}^\alpha \mathcal{W}_\alpha]$, and admits the supersymmetric ground state

$$Z^\alpha = W^{\dagger\alpha} = \sqrt{\frac{k\mu}{4\pi}} G^\alpha. \quad (2.46)$$

Here we have written $Y^A = (Z^\alpha, W^{\dagger\dot{\alpha}})$ instead of $(Q^\alpha, R^{\dot{\alpha}})$, since the different mass deformation implies a different kind of split. While this deformation manifestly preserves an $SU(2)_R$ symmetry and $\mathcal{N} = 2$ supersymmetry, it was demonstrated in [42] that it in

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fact preserves the full $\mathcal{N} = 6$, and it matches the GRVV deformation upon a redefinition of the fields. The BPS vortex solution is given by the ansatz

$$\begin{aligned}
 Z^\alpha &= W^{\dagger\alpha} = f(x)G^\alpha, \\
 A_\mu &= a_\mu(x)G^\alpha G^\dagger_\alpha, \\
 \hat{A}_\mu &= a_\mu(x)G^\dagger_\alpha G^\alpha, \\
 f(r, \theta) &= \sqrt{\frac{k\mu}{4\pi}}g(r)e^{in\theta}, \\
 a_i(r) &= \epsilon_{ij}\frac{\hat{x}_j}{r}[a(r) - n], \quad i = 1, 2,
 \end{aligned} \tag{2.47}$$

where $a(r)$ and $g(r)$ are real functions satisfying

$$\begin{aligned}
 a(r) &= \mp r g'(r)/g(r), \\
 (\ln g^2)'' + \frac{1}{r}(\ln g^2)' + 4\mu^2 g^2(1 - g^2) &= 0,
 \end{aligned} \tag{2.48}$$

together with the boundary conditions $a(0) - n = 0$, $ng(0) = 0$ and $g(\infty) = 1$, $a(\infty) = 0$. The vortex has magnetic flux $\Phi = -k\mu n/2$ and zero angular momentum and, at infinity it goes over to the GRVV ground state, i.e. the fuzzy S^2 . Consequently, at the position of the vortex $r = 0$, and $Z^\alpha = W^{\dagger\alpha} = 0$. It has no extension in the transverse coordinates either (the fuzzy sphere shrinks to zero there), and so can be interpreted as a point in both worldvolume and transverse space directions. This implies it can be identified with an object carrying D0-brane charge⁹ [43]. The U(1) symmetry under which the vortex is charged was identified in the dual gravity background - the Z_k reduction of the LLM solution for N $M2$'s in flat space - as $J = kQ_0 + NQ_4$ where Q_0 and Q_4 are the $D0$ -brane and $D4$ -brane charges respectively. One interpretation of this solution is as a bound state of an object with $D0$ -brane charge k , the vortex, with the $D4$ -brane wrapped on the fuzzy S^2 corresponding to the ground state. This picture was confirmed through a $D0$ -brane probe analysis with the mass of the probe at the minimum matching the soliton mass of $k\mu$.

Evidently, the vortex truly does represent a charge k $D0$ -brane object which, in the large k classical limit becomes a classical spacetime object, a supergraviton. It also follows that

⁹In [42] and [43], more general vortex solutions were given, based on a general discussion of the BPS equations of the GRVV massive deformation. Because of the identification of the massive deformations up to field redefinitions however, and the fact that vortices carry topological charge, all the vortex solutions must represent the same object. This soliton has charge k and mass $k\mu$.

the vortex solution of the massive ABJM theory is indeed a crucial ingredient in a search for a Matrix model of ABJM.

We note that if our proposal is correct, we should also find ABJM solutions corresponding to branes, at least $D2$ and $D4$, that are *completely transverse*, *i.e.* that have a worldvolume in the transverse scalar directions, but are pointlike on the worldvolume. These would be vortex solutions with fuzzy 2- and 4-dimensional spaces respectively, at the position of the vortices. We have, unfortunately, not succeeded in identifying such solutions.

2.6.4 Kink solutions

According to our proposed Matrix theory-type interpretation for ABJM, we also expect to find ABJM solutions that can be interpreted as 2-branes with one spatial worldvolume coordinate parallel to the M2-brane and one perpendicular to it. Such a solution would be a kink soliton of ABJM model, which at the position of the kink can be interpreted as a “fuzzy circle”. The problem however is two-fold: first, we need to find a kink solution of the ABJM model and second, we have to understand what exactly such a “fuzzy circle” is. Odd-dimensional fuzzy n -spheres have proven notoriously subtle to construct in the past when $n \geq 3$ [54]. The fuzzy circle, on the other hand, appears not to even have been constructed yet! We will see however that in a particular limit, the fuzzy sphere plays the role of a fuzzy circle.¹⁰

The general problem of finding a fuzzy circle can be readily described. For a fuzzy sphere, the fuzzy spherical harmonics constitute a *complete* subset of the set of $N \times N$ matrices, with a maximal angular momentum of $N^2 - 1$. This also means that the composition of spherical harmonics, $Y_k Y_l \sim \sum_n C_{kl}^n Y_n$ is still valid in the fuzzy case as long as $k + l \leq L_{max} = N^2 - 1$. For a fuzzy circle, the “fuzzy spherical harmonics” (or “fuzzy Fourier modes”) would also need to satisfy similar completeness and composition relations. These, however, appear to be quite difficult to obtain in any other way than as a limiting case of the fuzzy sphere. For instance, completeness requires that the N^2 elements of the $N \times N$ matrices need to be arranged as circle Fourier modes of increasing order n which in turn makes composition hard to accomplish.

We now turn to studying kink solutions of the massive ABJM Lagrangian. Naively, it

¹⁰A different limit was considered in [55], where a fuzzy circle arose from a “fuzzy cylinder” construction. We thank Yolanda Lozano for pointing this out to us, after the first version of our paper was posted on the electronic archives.

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might seem like a good place to start is with the construction of [41, 56] of the half supersymmetric kink

$$\begin{aligned} Z^\alpha &= W^{\dagger\alpha} = f(x_1)G^\alpha, \\ f^2(x_1) &= \frac{k}{4\pi} \frac{\mu}{1 + e^{-2\mu x_1}}. \end{aligned} \tag{2.49}$$

However, at the location of the kink, $x_1 = 0$ and $Z^\alpha = W^{\dagger\alpha} = \mu G^\alpha$, resulting in a degenerate fuzzy $S^2 \times S^2$. This is not the kind of solution we are looking for¹¹.

In search of the *fuzzy circle kink* solution, we will look for solutions with no gauge fields, and half the scalars set to zero. In the notation of GRVV [25], this corresponds to exciting only the Q^α of $Y^A = (Q^\alpha, R^{\dot{\alpha}})$. The action on such a solution would be

$$\begin{aligned} S = \int d^2x &\left[(\partial_\mu Q^\alpha)^2 + \left[\mu Q^1 + \frac{2\pi}{k} (Q^1 Q_2^\dagger Q^2 - Q^2 Q_2^\dagger Q^1) \right]^2 \right. \\ &\left. + \left[\mu Q^2 + \frac{2\pi}{k} (Q^2 Q_1^\dagger Q^1 - Q^1 Q_1^\dagger Q^2) \right]^2 \right]. \end{aligned} \tag{2.50}$$

We can readily check that the ansatz $Q^1 = f(x_1)G^1$, $Q^2 = \sqrt{\frac{-\mu k}{2\pi}}G^2$, where G^1, G^2 are matrices satisfying the GRVV fuzzy sphere algebra, results in the following quartic action for the function f ,

$$S = \frac{N}{2} \int dx_1 \left[(\partial_x f)^2 + \frac{-\mu k}{2\pi} \left(\frac{2\pi}{k} f^2 + \mu \right)^2 \right]. \tag{2.51}$$

The resulting equations of motion are solved by the usual kink solution $f(x_1) = \tanh x_1$. However, we note that *only half of the equations of motion* of the full action eq.(2.50) are solved by this ansatz. This ‘‘near miss’’ suggests another avenue: Completing the square in the action eq.(2.50) produces the BPS equations

$$\begin{aligned} \partial_s Q^1 &= \mu Q^1 + \frac{2\pi}{k} (Q^1 Q_2^\dagger Q^2 - Q^2 Q_2^\dagger Q^1), \\ \partial_s Q^2 &= \mu Q^2 + \frac{2\pi}{k} (Q^2 Q_1^\dagger Q^1 - Q^1 Q_1^\dagger Q^2), \end{aligned} \tag{2.52}$$

¹¹One could perhaps interpret this solution as a 5-brane with 4 directions transverse to the ABJM worldvolume, but this is less clear

whose solutions will be the half supersymmetric solitons of the theory. The ansatz $Q^2 = fT^2$; $Q^1 = gT^1$, where the T^i are matrices that satisfy the algebra

$$\begin{aligned} T^1 T_2^\dagger T^2 - T^2 T_2^\dagger T^1 &= -\alpha T^1, \\ T^2 T_1^\dagger T^1 - T^1 T_1^\dagger T^2 &= -T^2, \end{aligned} \tag{2.53}$$

collapses the BPS equations to two coupled differential equations for f and g whose solution is

$$\begin{aligned} f(x_1) &= \sqrt{\frac{-C_1}{1 - 2\pi\alpha e^{-2C_1 C_2/k} \exp(\frac{C_1}{\mu k} e^{2\mu x_1})}} e^{-C_1 C_2/k} \exp\left(\mu x_1 + \frac{C_1}{2\mu k} e^{2\mu x_1}\right), \\ g(x_1) &= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{-C_1}{1 - 2\pi\alpha e^{-2C_1 C_2/k} \exp(\frac{C_1}{\mu k} e^{2\mu x_1})}} e^{\mu x_1}. \end{aligned} \tag{2.54}$$

After fixing one of the integration constants as $C_1 = -k^2/2\pi$ we get

$$\begin{aligned} g &= \frac{k}{2\pi} \frac{1}{\sqrt{1 - \frac{4\pi^2\alpha}{k^2} A^2 \exp(-\frac{k}{2\pi\mu} e^{2\mu x_1})}} e^{\mu x_1}, \\ f &= \frac{A}{\sqrt{1 - \frac{4\pi^2\alpha}{k^2} A^2 \exp(-\frac{k}{2\pi\mu} e^{2\mu x_1})}} \exp\left[\mu x_1 - \frac{k}{4\pi\mu} e^{2\mu x_1}\right]. \end{aligned} \tag{2.55}$$

This solution has the following properties: As $x_1 \rightarrow \pm\infty$, $f \rightarrow 0$. Moreover, $f(x_1)$ has a maximum at

$$x_1 = \frac{1}{2\mu} \ln \left[\frac{2\pi\mu}{k} \left(L \left(-\frac{4\pi^2 A^2 \alpha}{ke} \right) + 1 \right) \right], \tag{2.56}$$

of value

$$f_{max} = A \left(\frac{2\pi\mu}{ke} \right)^{1/2} \left[\frac{L \left(-\frac{4\pi^2 A^2 \alpha}{ke} \right)}{\left(-\frac{4\pi^2 A^2 \alpha}{ke} \right)} \right]^{\frac{1}{2}}, \tag{2.57}$$

where $L(\xi)$ is a Lambert W function that satisfies the Lambert equation,

$$z(1+L) \frac{dL}{dz} = L, \quad z \neq -\frac{1}{e}. \tag{2.58}$$

This solution can be interpreted as a (non-topological) kink with regard to the field f , while nothing particularly interesting happens for the field g . As for the transverse fuzzy circle, note that the T^α can be taken to be

$$T^1 = G^1; \quad T^2 = \sqrt{\alpha} G^2, \tag{2.59}$$

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and hence define deformed fuzzy spheres. Up to now, the introduction of the parameter α seems to be superfluous. However, note that if we take the $\alpha \rightarrow 0$, then what we have is a BPS solution with $T^1 = G^1$ and $T^2 = 0$, yet satisfying the nontrivial relations¹²

$$\begin{aligned} T^1 T_2^\dagger T^2 - T^2 T_2^\dagger T^1 &= 0, \\ T^2 T_1^\dagger T^1 - T^1 T_1^\dagger T^2 &= -T^2. \end{aligned} \tag{2.60}$$

This is ample justification for our associating them with the fuzzy spherical harmonics of the fuzzy circle. In this limit, the solution becomes

$$\begin{aligned} g &= \frac{k}{2\pi} e^{\mu x_1}, \\ f &= A \exp \left[\mu x_1 - \frac{k}{4\pi\mu} e^{2\mu x_1} \right], \end{aligned} \tag{2.61}$$

and is still a kink, since f has the same asymptotics with a maximum now at

$$x_1 = \frac{1}{2\mu} \ln \left(\frac{2\pi\mu}{k} \right), \tag{2.62}$$

while the field g exhibits a simple monotonic increase. The interpretation of this result is that we have taken a fuzzy sphere and shrunk one of its directions (G^2) to zero, thus obtaining a fuzzy circle¹³.

2.7 Graviton scattering in massive ABJM background

There are two cases of interest for the scattering of gravitons in the background given in eqs.(5.8). These correspond to the two types of vortex scattering in the massive ABJM model that we want to match to. There, the two relevant cases are when the vortices are separated in the directions parallel to the ABJM worldvolume, or in the transverse (scalar) directions.

In one case the gravitons are separated only in the two directions parallel to the ABJM

¹²Of course, if $T^2 = 0$ from the start, these equations are trivial. They are only nontrivial on taking the limit of $\alpha \rightarrow 0$.

¹³As a final point, note that our choice of representation for the G 's is not unique. We could take for G^1 and G^2 either the irreducible GRVV matrices (as we have), or the reducible matrices $G^1 = \delta_{m+N/2,n}$, $G_1^\dagger = \delta_{m,n+N/2}$ and $G^2 = \delta_{mn}$ with $m, n \leq N/2$.

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worldvolume, and the 11th dimension is transverse to the worldvolume. In the second, they are separated only in the directions transverse to the ABJM worldvolume, and the 11th dimension is parallel to it. This latter case is a more direct analog of the BFSS analysis which we reviewed previously, whereas the former case is new. As in the BFSS model, we will employ 't Hooft's trick of replacing the graviton-graviton scattering with the scattering of a graviton shockwave with a probe wave, but here, it is not clear *a priori* that it will give the complete result.

As in the BFSS case, the graviton shockwaves satisfy the wavefunction equation,

$$\Delta_{bgr}^\perp h_{--} = Q\delta^\perp(\vec{x}, \vec{y}). \quad (2.63)$$

We will content ourselves with some cursory comments about the solutions of this equation here and leave a detailed analysis of the solutions for the Appendix. As earlier, we will denote the worldvolume coordinates by \vec{y} (with $|\vec{y}| \equiv y$), and transverse coordinates by \vec{x} (with, correspondingly, $|\vec{x}| \equiv r$). In the transverse separation case, we find

$$h_{--}(\vec{y} = 0, r) \sim C' \frac{Q}{r^7}, \quad (2.64)$$

as in the BFSS case, but since the result is independent of μ , we could have imposed the wrong boundary condition, so we will refrain from using this result in the following. In the case of parallel separation on the other hand, we find that

$$h_{--}(\vec{y}, r = 0) \sim C e^{-m_1|y|}. \quad (2.65)$$

The next step in this computation is to scatter a scalar probe-graviton off the source-graviton. Since it is clearer how to deal with the transverse case, we will do this first, more as a guide than anything else.

1. Transverse separation: Expanding the D'Alembertian in the probe graviton equation

$$\left[\square_{bgr}^{full} + \partial_+ h^{++} \partial_+ \right] \phi = 0, \quad (2.66)$$

gives

$$H^{-1/3} \left[H(\vec{x}) \left(\partial_y^2 - \frac{\partial^2}{\partial x_0^2} + m^2 \right) + \partial_x^2 \right] \phi = \left[-\frac{h^{++}}{2} (\partial_0 - m)^2 \right] \phi \equiv V\phi. \quad (2.67)$$

As in the, perhaps more familiar, BFSS case the momentum-space interaction potential is computed through the S-matrix

$$S^{(1)}(p_{in} - p_{out}) = \int d^{10}z \phi_{out}^{(0)}(z) * V(z) \phi_{in}^{(0)}(z), \quad (2.68)$$

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where $\phi^{(0)}$ solves the unperturbed equation

$$\left[H(\vec{x}) \left(\partial_y^2 - \frac{\partial^2}{\partial x_0^2} + m^2 \right) + \partial_{\vec{x}}^2 \right] \phi^{(0)} = 0. \quad (2.69)$$

If we factorize the plane wave $\phi^{(0)}$ as $\phi^{(0)} = \frac{e^{ip_y y - iE x_0}}{\sqrt{2E}} f(\vec{x})$, define $p_0 = \sqrt{E^2 + m^2 - p_y^2}$, $\vec{y} = \sqrt{\mu p_0} \vec{x}$ and $\tilde{E} = (p_0)^2 / (2\mu p_0)$, we find that the function f satisfies

$$\left[\partial_{\vec{y}}^2 - \vec{y}^2 + 2\tilde{E} \right] f = 0, \quad (2.70)$$

which is nothing but the equation of a d -dimensional harmonic oscillator. Therefore, if we were interested in solutions that drop off at infinity, the solution would be given in terms of Hermite polynomials as

$$\begin{aligned} f &= \prod_i f_{p_0, i}; & f_{p_0, i} &= \frac{e^{-y_i^2/2} H_{n_i}(y_i)}{\pi^{1/4} \sqrt{2^{n_i} n_i!}}, \\ \tilde{E} &= \sum_i \tilde{E}_i, \end{aligned} \quad (2.71)$$

with the quantization condition $\tilde{E}_i = n_i + 1/2$.

However, we have $|\vec{x}| \leq 1/\mu$ and, as explained in the Appendix, we need to impose Neumann boundary conditions there instead. Consequently, the ‘‘plane wave solution’’ of (2.70) is

$$\begin{aligned} f &= \prod_i f_{p_0, i}, \\ f_{p_0, i} &= e^{-y_i^2/2} \left[C_{1,i} {}_1F_1 \left(\frac{1}{4} - \frac{\tilde{E}_i}{2}, \frac{1}{2}; y_i^2 \right) + C_{2,i} y_i {}_1F_1 \left(\frac{3}{4} - \frac{\tilde{E}_i}{2}, \frac{3}{2}; y_i^2 \right) \right], \end{aligned} \quad (2.72)$$

$$\begin{aligned} \frac{(p_0)^2}{2\mu p_0} &= \tilde{E} = \sum_i \tilde{E}_i \equiv \frac{(p_{0,i})^2}{2\mu p_0}, \\ y_i &= \sqrt{\mu p_0} x_i. \end{aligned}$$

Here the relative normalization C_1/C_2 could be defined by the Neumann boundary condition at $r = 1/\mu$, but that would result in *two constants* for each dimension. We will see in what follows that another relative normalization is preferred. Note that because $p_0^2 = \sum_i (p_{0,i})^2$, \vec{p}_0 can be called the (plane wave) momentum, and we will fix the normalization by requiring that we obtain the flat space plane waves in some limit.

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Indeed, from Appendix C of [57], we know that the harmonic oscillator wavefunctions become plane waves as, for example $n \rightarrow \infty$ or $m\omega \rightarrow 0$. Explicitly,

$$\phi_n(x) = \frac{(m\omega)^{1/4}}{(\sqrt{\pi}2^n n!)^{1/2}} e^{-m\omega x^2/2} H_n(\sqrt{m\omega}x) \rightarrow \frac{1}{\sqrt{\pi}} \left(\frac{4m\omega}{n}\right)^{1/4} \cos(2\sqrt{m\omega}nx) \quad (2.73)$$

where $k_n = \sqrt{2m\omega n} = 2\pi n/L$. This limit, together with the relations

$$\begin{aligned} H_{2n}(x) &= (-1)^n \frac{(2n)!}{n!} {}_1F_1\left(-n, \frac{1}{2}; x^2\right), \\ H_{2n+1}(x) &= (-1)^n 2 \frac{(2n+1)!}{n!} x {}_1F_1\left(-n, \frac{3}{2}; x^2\right), \end{aligned} \quad (2.74)$$

generalize to the limit of our solutions, giving plane waves for large $p_{0,i}/\mu$. We fix the relative constants C_{1i}/C_{2i} such that we are able to sum $\cos(\dots) + i\sin(\dots) = e^{i(\dots)}$ after the limit. As we will argue further on, knowledge of the precise values of these constants is not necessary, only that the plane wave limit exists. The overall constant in front of the solution follows from a relativistic normalization.

We now reverse the initial logic, take the $p_{0,i}$ to be independent, and instead constrain E by the relation $E^2 + m^2 - p_y^2 = p_0^2 = \sum_i (p_{0,i})^2$. The interaction potential is then obtained from

$$\begin{aligned} S^{(1)}(E; p_{in} - p_{out}) &= \int dx_0 dy d^8 \vec{x} \overline{\phi_{out;p_y, m, \vec{p}_0}^{(0)}} \left[-\frac{h^{++}(y, \vec{x})}{2} (\partial_0 - m)^2 \right] \phi_{in;p_y, m, \vec{p}_0}^{(0)} \\ &= -\frac{2\pi \delta(E^{out} - E^{in})(E^{in} - m)^2}{4\sqrt{E^{in} E^{out}}} \int dy e^{iy(p_y^{in} - p_y^{out})} \int d^8 \vec{x} \overline{f_{\vec{p}_0^{out}}^{out}(r)} h^{++}(y, r) f_{\vec{p}_0^{in}}^{in}(r) \\ &= 2\pi \delta(E^{out} - E^{in}) V_{int}^{(1)}(E; p_{in} - p_{out}). \end{aligned} \quad (2.75)$$

with the understanding that we need to make a Fourier transform back to x space at the end. However note that since $f_{\vec{p}_0}(\vec{x}) \neq e^{i\vec{p}_0 \vec{x}}$, it is not even guaranteed that we get a function of $|\delta \vec{p}_0| = |\vec{p}_0^{in} - \vec{p}_0^{out}|$ after the \vec{x} integration. Assuming that it is nevertheless such a function, after the Fourier transform back to position space we find that

$$\begin{aligned} V_{int}^{(1)}(y, \vec{z}) &= \frac{(E - m)^2}{2E} \int d^8 \delta \vec{p}_0 e^{-i\vec{z}\delta \vec{p}_0} \int d^8 \vec{x} \overline{f_{\vec{p}_0^{out}}^{out}(\vec{x})} h^{++}(y, r) f_{\vec{p}_0^{in}}^{in}(\vec{x}) \\ &\simeq \frac{mv^4}{8} \int d^8 \delta \vec{p}_0 e^{-i\vec{z}\delta \vec{p}_0} \int d^8 \vec{x} \overline{f_{\vec{p}_0^{out}}^{out}(\vec{x})} h^{++}(y, r) f_{\vec{p}_0^{in}}^{in}(\vec{x}) \end{aligned} \quad (2.76)$$

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In particular, when $y = 0$,

$$V_{int}^{(1)}(y = 0, \vec{z}) = \frac{mv^4}{8} \int d^8 \delta \vec{p}_0 e^{-i\vec{z}\delta\vec{p}_0} \int d^8 \vec{x} \overline{f_{\vec{p}_0}^{out}(\vec{x})} h^{++}(y = 0, r) f_{\vec{p}_0}^{in}(\vec{x}) \quad (2.77)$$

While it is not clear whether the integral is dominated by the low r region, what is clear is that we cannot have a result depending on the δp_0 unless we are in the plane wave limit. This is obtained in the limit of high momentum p_{0i} . *It seems then that only if we probe with sufficiently high momentum we will have a well-defined interaction potential.*

In any case, the outcome is that we need to replace the functions $f_{\vec{p}_0}$ with plane waves and obtain, as in the BFSS flat space case,

$$V_{int}^{(1)}(y = 0, \vec{z}) = \frac{mv^4}{8} h^{++}(y = 0, r). \quad (2.78)$$

2. Parallel separation:

To write down the equation for a probe graviton, we need to first define the coordinates and metric more a little more carefully. We start with the metric

$$\begin{aligned} ds^2 &= H^{-2/3}(\vec{x})(-dt^2 + d\vec{y}^2) + H^{1/3}d\vec{x}^2, \\ d\vec{x}^2 &= dr^2 + r^2 d\Omega_7^2; \quad \vec{x} = (x^1, \dots, x^8), \end{aligned} \quad (2.79)$$

and dimensionally reduce on x_{11} . Operationally, we choose x_{11} as the fibre direction of the Hopf fibration of S^7 over CP^3 which acts on the Euclidean coordinates, Y^A , on the S^7 through the overall phase $Y^A = e^{ix_{11}} \tilde{Y}^A$. Since the coordinates satisfy $|Y^A|^2/r^2 = 1$, this means that if we write $\vec{x} = (\vec{x}', x_{11})$, then $r = |\vec{x}| = |\vec{x}'|$ and $d\vec{x}^2 = |dY^A|^2 = |d\tilde{Y}^A|^2 + dx_{11}^2$ becomes $d\vec{x}'^2 + dx_{11}^2$. Since the metric is translationally invariant in x_{11} , we can schematically rewrite

$$ds^2 \simeq H^{-2/3}(\vec{x}') (-dt^2 + H(\vec{x}') dx_{11}^2 + d\vec{y}^2) + H^{1/3}(\vec{x}') d\vec{x}'^2. \quad (2.80)$$

With this metric, the equation for the probe graviton takes a form similar to the transverse separation case,

$$H^{-1/3} \left[H(\vec{x}') \left(\partial_{\vec{y}}^2 - \frac{\partial^2}{\partial x_0^2} \right) + \Delta(\vec{x}', m^2) \right] \phi = \left[-\frac{h^{++}}{2} (\partial_0 - m)^2 \right] \phi, \equiv V \phi \quad (2.81)$$

where $\Delta(\vec{x}', m^2)$ is obtained from $\partial_{\vec{x}}^2$ by reducing on x_{11} , and setting $\phi(x_{11}, \dots) = e^{ipx_{11}} \phi(\dots)$ and $p = im$. Noting that we could rewrite $-dt^2 + H(\vec{x}') dx_{11}^2 = 2dx^+ dx^-$ only on slices

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of (nearly) constant \vec{x}' , we picked the simplest choice $\vec{x}' \simeq 0$ and fixed it so that we can form the x^+ combination for defining h^{++} without problem.

Now we can proceed as with the transverse case. Specifically, the momentum-space interaction potential obtains from

$$S^{(1)}(p_{in} - p_{out}) = \int d^{10}z \phi_{out}^{(0)}(z)^* V(z) \phi_{in}^{(0)}(z), \quad (2.82)$$

where $\phi^{(0)}$ satisfies the unperturbed equation associated to eq.(2.81). Here again, a plane wave ansatz of the form $\phi^{(0)} = \frac{e^{i\vec{p}_y \vec{y} - iE x_0}}{\sqrt{2E}} f(\vec{x})$ will solve the equation provided

$$[H(\vec{x}) (E^2 - \vec{p}_y^2) + \Delta(\vec{x}', m^2)] f(\vec{x}) = 0. \quad (2.83)$$

We again obtain

$$\begin{aligned} S^{(1)}(E; p_{in} - p_{out}) &= \int dx_0 d^2y d^7\vec{x} \overline{\phi_{out: \vec{p}_y, m, \vec{p}_0}^{(0)}} \left[-\frac{h^{++}(\vec{y}, \vec{x})}{2} (\partial_0 - m)^2 \right] \phi_{in: \vec{p}_y, m, \vec{p}_0}^{(0)} \\ &= -\frac{\delta(E^{out} - E^{in})(E^{in} - m)^2}{4\sqrt{E^{in} E^{out}}} \int d^2y e^{i\vec{y}(\vec{p}_y^{in} - \vec{p}_y^{out})} \int d^7\vec{x} \overline{f_{\vec{p}_0^{out}}^{out}} h^{++}(y, r) f_{\vec{p}_0^{in}}^{in} \end{aligned} \quad (2.84)$$

or, after manipulations similar to the transverse separation case,

$$V_{int}^{(1)}(\vec{y}, \vec{z} = 0) \simeq \frac{(E - m)^2}{4E} \left[\int d^7\delta\vec{p}_0 \int d^7\vec{x} f_{\vec{p}_0^{out}}^{out}(\vec{x})^* h^{++}(y, r) f_{\vec{p}_0^{in}}^{in}(\vec{x}) \right]. \quad (2.85)$$

We have, of course, already noted that equations (2.81) through (2.83) only make sense if $\vec{x} \simeq 0$ is implied, in order to be able to form the x^+ combination. This could be alleviated by integrating \vec{x} only in a neighbourhood of zero, in which case

$$V_{int}^{(1)}(\vec{y}, \vec{z} = 0) \simeq \frac{(E - m)^2}{4E} h^{++}(y, r = 0) \left[\int d^7\delta\vec{p}_0 \int_{|x| \simeq 0} d^7\vec{x} \overline{f_{\vec{p}_0^{out}}^{out}(\vec{x})} f_{\vec{p}_0^{in}}^{in}(\vec{x}) \right]. \quad (2.86)$$

In both cases however, we still have the same problem that we observed in the transverse separation case. Namely, in order for the momentum space result to only be a function of $\delta\vec{p}_0 = \vec{p}_0^{in} - \vec{p}_0^{out}$, we need to only consider high momentum wavefunctions, which are effectively just plane waves. In that case, the integrations disappear, irrespective of whether or not $\vec{x} \simeq 0$, and we obtain

$$V_{int}^{(1)}(\vec{y}, \vec{z} = 0) \simeq C \frac{(E - m)^2}{4E} h^{++}(y, r = 0), \quad (2.87)$$

or, since $h^{++} \propto e^{-m_1 y}$,

$$V_{int}^{(1)}(\vec{y}, \vec{z} = 0) \propto m v^4 e^{-m_1 y}. \quad (2.88)$$

So much for the graviton-graviton scattering.

2.8 Vortex scattering in massive ABJM

In this section we analyze vortex scattering in the $N = 2$ (or equivalently, the $U(2) \times U(2)$) massive ABJM model. The general N case is quite complicated, so we will not attempt it here. We will employ the vortex solution in the form of [42], as it is both guaranteed to be the most general solution¹⁴, and also because the specific form of the solution will allow the use of previous known results for the Abelian-Higgs model and its Nielsen-Olesen vortex solution. The vacuum for the massive $U(2) \times U(2)$ ABJM model is

$$Y^1 = \sqrt{\frac{k\mu}{\pi}} G^1; \quad Y^2 = \sqrt{\frac{k\mu}{\pi}} G^2; \quad Y^3 = Y^4 = 0, \quad (2.89)$$

so that, keeping $Y^3 = Y^4 = 0$ and using complex notation for $z = x^1 + ix^2$ means that in terms of

$$\mathcal{D}_z Y^\alpha = \partial Y^\alpha + i A_z Y^\alpha - i Y^\alpha \hat{A}_z; \quad \mathcal{D}_{\bar{z}} Y^\alpha = \bar{\partial} Y^\alpha + i A_{\bar{z}} Y^\alpha - i Y^\alpha \hat{A}_{\bar{z}}, \quad (2.90)$$

$$A_z = \frac{1}{2}(A_1 - iA_2); \quad A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2); \quad \hat{A}_z = \frac{1}{2}(\hat{A}_1 - i\hat{A}_2); \quad \hat{A}_{\bar{z}} = \frac{1}{2}(\hat{A}_1 + i\hat{A}_2),$$

the BPS equations read

$$\begin{aligned} \mathcal{D}_{\bar{z}} Y^1 &= 0; \quad \mathcal{D}_1 Y^2 = \mathcal{D}_2 Y^2 = 0, \\ \mathcal{D}_0 Y^1 - i(\beta_2^{21} + \mu Y^1) &= 0; \quad \mathcal{D}_0 Y^2 + i(\beta_1^{12} + \mu Y^2) = 0, \\ \frac{k}{2\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho} &= i(Y^\alpha (D^\mu Y^\alpha)^\dagger - D^\mu Y^\alpha Y_\alpha^\dagger); \quad \frac{k}{2\pi} \epsilon^{\mu\nu\rho} \hat{F}_{\nu\rho} = i((D^\mu Y^\alpha)^\dagger Y^\alpha - Y_\alpha^\dagger D^\mu Y^\alpha) \end{aligned} \quad (2.91)$$

where $\alpha = 1, 2$ and

$$\beta_\gamma^{\alpha\beta} = \frac{4\pi}{k} Y^{[\alpha} Y_\gamma^\dagger Y^{\beta]}. \quad (2.92)$$

The multi-vortex solution can then be written as

$$\begin{aligned} Y^1 &= e^{-\frac{\psi}{2}} H_0(z) \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad Y^2 = \sqrt{\frac{k\mu}{2\pi}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ A_0 &= \frac{1}{\mu} \begin{pmatrix} \partial \bar{\partial} \psi & 0 \\ 0 & 0 \end{pmatrix}; \quad \hat{A}_0 = \frac{1}{\mu} \begin{pmatrix} 0 & 0 \\ 0 & \partial \bar{\partial} \psi \end{pmatrix}; \quad A_z = \hat{A}_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{i}{2} \bar{\partial} \psi \end{pmatrix}. \end{aligned} \quad (2.93)$$

Here $H_0(z)$ is an arbitrary polynomial

$$H_0(z) = \prod_{i=1}^n (z - z_i), \quad (2.94)$$

¹⁴since it was not found using any specific ansatz, but rather by analyzing the energy functional on a case-by-case basis.

2.8 Vortex scattering in massive ABJM

and the function real $\psi(z, \bar{z})$ is determined through the equation

$$\partial\bar{\partial}\psi = \mu^2 (1 - e^{-\psi} |H_0(z)|^2) , \quad (2.95)$$

with the boundary condition at $|z| \rightarrow \infty$ requiring $\psi \rightarrow \log |H_0(z)|^2$, and where z_i are the position moduli for n vortices. Correspondingly, the energy of this solution is $nk\mu$. Note that this is the same equation governing the vortices in the Abelian-Higgs model, so we expect that the same effective action governs their scattering as well. This is indeed the case.

To obtain the effective action for vortex scattering, we let the parameters z_i become functions of t , so that the fields of the static solution become a function of $(z, \bar{z}, z_i(t), \bar{z}_i(t))$. Of course, if we do that, the equations of motion are no longer satisfied, but we can still solve them order-by-order in time derivatives $\partial_t \sim \dot{z}_i(t)$. We write

$$Y^\alpha(z, \bar{z}, t) = Y_{(0)}^\alpha(z, \bar{z}, z_i(t), \bar{z}_i(t)) + Y_{(1)}^\alpha + Y_{(2)}^\alpha + \dots \quad (2.96)$$

and similarly for A_μ, \hat{A}_μ . Because they are multiplied by the zero-th order equations of motion, the second order fields do not contribute to the effective action, so we can stop at the first order solution. At that level (on-shell),

$$L_{eff} = \int d^2x \mathcal{L}(Y^\alpha, A_\mu, \hat{A}_\mu) . \quad (2.97)$$

The first order solution is found to be

$$\begin{aligned} Y_{(1)}^\alpha = 0; \quad A_0^{(1)} = \hat{A}_0^{(1)} &= \begin{pmatrix} 0 & 0 \\ 0 & -\frac{i}{2}(\dot{z}^i \partial_i - \dot{\bar{z}}^i \bar{\partial}_i)\psi \end{pmatrix} \\ , \\ A_{\bar{z}}^{(1)} &= \begin{pmatrix} \frac{1}{2\mu} \dot{z}^i \partial_i \partial \psi & 0 \\ 0 & 0 \end{pmatrix}; \quad \hat{A}_{\bar{z}}^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2\mu} \dot{z}^i \partial_i \partial \psi \end{pmatrix} \end{aligned} \quad (2.98)$$

Finally, substituting this solution into the Lagrangian, produces an effective Lagrangian for the moduli

$$L_{eff} = \frac{k\mu}{\pi} \int d^2x \left[-\partial\bar{\partial}\psi + \frac{1}{2} \dot{z}^i \dot{\bar{z}}^j \left(\partial_i \bar{\partial}_j \psi + \frac{1}{\mu^2} (\partial_i \bar{\partial} \psi \bar{\partial}_j \partial \psi - \partial \bar{\partial} \psi \partial_i \bar{\partial}_j \psi) \right) \right] , \quad (2.99)$$

where we have used the equation of motion (2.95) and its boundary condition at infinity. This effective Lagrangian is exactly the same as the one obtained in the Abelian Higgs

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model for the Nielsen-Olesen vortex! So both the equation of motion (2.95) and the effective action for the nonabelian Chern-Simons ABJM model give the same result as the ones for the Abelian Higgs model. We can then rewrite the effective Lagrangian for large vortex separation as in the Abelian Higgs model as

$$L_{eff} \simeq -k\mu n + \sum_{i=1}^n \frac{k\mu}{2} |\dot{z}^i|^2 - k\mu q \sum_{i>j} K_0(2\mu|z^i - z^j|) |\dot{z}^i - \dot{z}^j|^2, \quad (q \simeq 1.71) \quad (2.100)$$

which means that at large separation, the interaction potential becomes

$$V_{int} \simeq -k\mu q K_0(2\mu y) v^2 \simeq -k\sqrt{\mu} q \frac{\sqrt{\pi}}{2} \frac{e^{-2\mu y}}{\sqrt{y}} v^2 \quad (2.101)$$

2.9 Towards a Matrix model of massive ABJM?

Since we have now computed the scattering of both supergravitons and vortices in ABJM, we can compare them to see if they match. Assuming that they play the role of supergravitons¹⁵, the effective potential for vortices was found to be $\propto v^2 e^{-2\mu y}$, with y the parallel separation. The v^2 dependence is a general characteristic of lowest order soliton scattering, and the $e^{-2\mu y} = e^{-m_0 y}$ arises from the presence of a lowest-mass excitation in the “broken” phase, i.e. the fuzzy sphere background at infinity. By contrast, we have found that the supergraviton result is proportional to $v^4 e^{-m_1 y}$. Here, the v^4 was due to the same calculation as in the BFSS case while $m_1 \simeq 9.1\mu$ appeared because of the necessary Neumann condition at $r = 1/\mu$ which, in effect compactifies the transverse direction, with m_1 being the lowest KK-mode. Unfortunately then, the interaction potentials for separation in the directions parallel to the ABJM worldvolume do not seem to match.

One possible reason for this mismatch - assuming, of course, that our identification of the vortices as supergravitons is at least correct - appears when we realize that the same v^4 dependence appeared on the gravity side as in the BFSS model. Unlike in BFSS where the first interaction term is a one-loop effect, however, here on the matrix side we computed a *classical effect*. Clearly then, we need to think a little harder about what terms are supposed to match on both sides, and try to find a corresponding gravity term for the vortex calculation we did, and a corresponding vortex term for the gravity calculation.

¹⁵The background of D4 on fuzzy S^2 is common to both vortices in the scattering problem and thus does not contribute to the interaction

There is potentially one other subtlety, though *a priori*, it seems to give only small corrections, namely that the gravity background we took is not complete. The pure ABJM field theory corresponds to the IR of $M2$ -branes on C^4/Z_k , but we also need to consider an appropriate vacuum. For pure ABJM, the vacua are just VEV's which correspond to a given position z^I in C^4/Z_k . On the other hand, in the massive ABJM theory the same VEVs, corresponding to positions z^I in the dual gravity background eqs.(5.8), are no longer solutions. Instead, the vacua are fuzzy spheres of radius $r = 1/\mu$, giving an $M5$ -brane wrapped on S^3/Z_k , which will backreact and modify the background, maybe smoothing out the $r = 1/\mu$ potential singularity.

Before we continue trying to fix the mismatch, it will be useful to build some intuition for the problem by looking in more detail at the matching in the BFSS model.

2.9.1 BFSS matching: details

The review of the systematics of Matrix theory matching in this section will follow closely [58]. With the rescalings $\tau = u/R$ and $X^i = y^i/M_P^3$, the bosonic part of the BFSS Matrix theory action (2.1) becomes

$$S = \frac{1}{M_P^6} \int du \operatorname{Tr} \left(\frac{1}{2} D_u y^i D_u y^i + \frac{1}{4} [y^i, y^j]^2 \right), \quad (2.102)$$

making explicit the fact that the coupling of the theory is M_P^3 (or that the loop-counting parameter is M_P^6). At L loops then, the BFSS effective action takes the schematic form

$$S_L = M_P^{6L-6} \int du f_L(y^i, D_u) = R M_P^{6L-6} \int d\tau f_L(M_P^3 X^i, R^{-1} D_\tau), \quad (2.103)$$

and, as advertised in section 3, there are two sources of M_P -dependence in the interaction potential; one coming from the coupling constant and the other from the use of M_P to build a unit of length. Further analysis, including keeping only Lorentz invariant terms, gives the result

$$S_L = \int d\tau R M_P^2 (M_P r)^{4-3L} \left(\frac{v^2}{R^2 M_P^2 (M_P r)^4} \right)^n \quad (2.104)$$

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so that, for the first few terms

$$\begin{aligned}
\mathcal{L}_0 &= \frac{c_{00}}{R} v^2 \\
\mathcal{L}_1 &= 0 + \frac{c_{11}}{M_P^9 R^3} \frac{v^4}{r^7} + \frac{c_{12}}{M_P^{15} R^5} \frac{v^6}{r^{11}} + \dots \\
\mathcal{L}_2 &= 0 + \frac{c_{21}}{M_P^{12} R^3} \frac{v^4}{r^{10}} + \frac{c_{22}}{M_P^{18} R^5} \frac{v^6}{r^{14}} + \dots \\
\mathcal{L}_3 &= 0 + \frac{c_{31}}{M_P^{15} R^3} \frac{v^4}{r^{13}} + \frac{c_{32}}{M_P^{21} R^5} \frac{v^6}{r^{17}} + \dots
\end{aligned} \tag{2.105}$$

Notice that the diagonal terms in this series that contribute to the classical general relativity scattering themselves receive contributions *from all Matrix theory loops*. The Einstein action has an infinite number of vertices, thus tree level scattering gives an infinite series, but they all come with integer powers of $M_P^9 \sim 1/G_{11} \sim 1/(\kappa_{11})^2$. On the other hand, the off-diagonal terms in the Matrix theory loop expansion must come from quantum correction to the effective action of gravity, since they have noninteger powers of κ_{11}^2 .

As an example, a source-probe approximation for the true classical general relativity interaction potential of gravitons gives

$$\mathcal{L} = \frac{N_2}{2R} v^2 + \frac{15}{16} \frac{N_1 N_2}{R^3 M_P^9} \frac{v^4}{r^7} + \frac{225}{64} \frac{N_1^2 N_2}{R^5 M_P^{18}} \frac{v^6}{r^{14}} + \mathcal{O}\left(\frac{v^8}{r^{21}}\right), \tag{2.106}$$

where the result is linear in the probe momentum N_2/R because of the approximation, and N_1/R is the source momentum. The symmetrization of the last term matches against a 2-loop Matrix theory result,

$$\frac{225}{32} \frac{1}{R^5 M_P^{18}} \frac{v^6}{r^{14}} \frac{N_1 N_2^2 + N_1^2 N_2}{2}. \tag{2.107}$$

In any case, we see that in BFSS, all the dependence on v, r, M_P and R could be determined from general principles. It seems pertinent to ask if the same could not be done for ABJM? The answer it seems is *no*, at least not for the pure ABJM theory which is conformal. There, all M_P -dependence is tied to the r -dependence so we cannot take advantage of having a dimensionful rescaled field and coupling, like y^i and M_P^3 , to use in dimensional analysis. Perhaps in the massive ABJM that we used for calculations something can be done since, for instance, the size of the fuzzy sphere vacuum $f = \sqrt{k\mu/(4\pi)}$ sets a dimensionful scale. This remains an open issue. Sadly then, it appears that the intuition of BFSS matching is not particularly applicable to our case.

2.9.2 ABJM matching and effective Lagrangian

Let's step back for a minute and take stock. Our analysis up to this point has not succeeded in producing a concrete Matrix theory for the ABJM model. However, we have learnt a significant amount about the structure of the problem that we are in a position to speculate on what the sought after Matrix theory and its matching to gravity would look like. To outline the way forward as clearly as possible, we will focus our speculations on three specific points; the matching of parameters on both sides, what calculations can be done on the ABJM gauge theory, and what on the gravity side.

- **Parameters:** The parameters available on the gravity side of the calculation (for the background dual to the massive deformation of ABJM) are: the mass deformation μ , the inverse size of the 11th direction $1/R$, the Planck scale M_P , the source and probe 11th momenta $m_s = N_s/R$ and $m_p = N_p/R$ and m_1 , the effective mass of the graviton wavefunction h_{--} in the presence of the Neumann condition at $r = 1/\mu$. In the souce-probe calculation, the source graviton had $Q \propto m_s$ and the probe graviton had mass $m = m_p$.

Moving to the massive ABJM field theory, μ is the same deformation parameter. The integer k can be used to define the $1/R$ scale with respect to some other scale (as, for instance, in the fuzzy sphere vacuum where $R = R_{sph}/k$, with R_{sph} the fuzzy sphere radius). As we noted before, there is no M_P inherent to ABJM since pure ABJM is of course conformal but M_P does appear as a unit of length in the same way as in AdS/CFT. For example, the physical radius of the spacetime sphere corresponding to the worldvolume fuzzy sphere is $R_{sph}^2 = \frac{2}{N} \text{Tr} \left(X^I X_I^\dagger \right) = 2\pi k \mu N l_P^3$. Then m_s and m_p , or rather N_s and N_p , should be obtained as sizes of the ABJM solutions corresponding to gravitons, i.e. sizes of vortices, with $N_s + N_p \leq N$. Unfortunately, for technical reasons we had to restrict ourselves to the case of $N = 2$ but we should, in principle, consider the case $1 \ll N_s, N_p \ll N$. It is not clear if it is sufficient to consider N_s vortices on top of each other (i.e. an charge N_s vortex), or if we need something more sophisticated in the case $1 \ll N_s \ll N$. Finally, we should have a corresponding object for m_1 and the graviton wavefunction h_{--} should, in principle, match on to a vortex wavefunction. One possibility for the latter would be to define it as $a(r) - n$ or $g(r) - g(\infty) = g(r) - 1$, which decays to zero with the mass of the smallest excitation of the theory, i.e. as $e^{-\mu r}$ in the case of the $N_s = 1$ vortex.

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- **Vortex calculations:** The classical scattering of vortices for parallel separation gave

$$\frac{1}{k} L_{class} = \sum_i \left(\mu + \mu \frac{v_i^2}{2} \right) - \sum_{ij} \mu q K_0(2\mu r_{ij}) v_{rel,ij}^2, \quad (2.108)$$

for $N = 2$, i.e. in the $U(2) \times U(2)$ ABJM theory. If we wanted to parallel the computation of BFSS, the next term to calculate would be at one-loop. Specifically, to obtain the interaction potential we must consider the classical solution of two vortices, one at $z = 0$ and one at $z^i = b^i + v^i t$, and then calculate the one-loop determinants $\sum_n \omega_n/2$ around it. To obtain matching with our tree-level graviton scattering result, we would need at least that the one-loop determinants give a result proportional to $m_s m_p v^4$, but even that seems very difficult to check directly, and we leave it for further work.

In the case of transverse vortex separation, we anticipate the same situation as in BFSS in that, if we take two vortices in different $SU(N)$ blocks then at the classical level they are non-interacting, even if they have non-vanishing velocities. Even though we will run into the problem of regulating their infinite energies, it is instructive to see how this works in the pure ABJM case. There, for a single vortex, $Y^A = v^A \mathbb{I}$ for $A = 2, 3, 4$. Therefore, by setting

$$Y^A(t) = (x_1^A + v_1^A t) \mathbb{I}_{N_1 \times N_1} + (x_2^A + v_2^A t) \mathbb{I}_{N_2 \times N_2}, \quad (2.109)$$

and the vortices at the same worldvolume positions in the ABJM action, we obtain the nonrelativistic mass of the supergravitons exactly as in BFSS,

$$H \sim \frac{1}{2} \text{Tr} \left| \dot{X}^i \right|^2 + \dots = E_0 + (N_1 + N_2) \frac{v^2}{2}, \quad (2.110)$$

where E_0 is the energy of the free vortices at rest and the interaction potential comes from the 1-loop fluctuation around the vortex solution with $X^i(t) = b^i + v^i t$. There are two immediate subtleties that arise: Firstly, it is not clear whether this will work for both X^i being the four fuzzy sphere directions, as well as for the four remaining transverse directions and, secondly we have no explicit realization of these vortices in the massive ABJM model. Both of these are left for future work.

- **Gravity calculations:** Even if we could match our graviton scattering result (single graviton exchange)

$$V_{int}^{(1)} = m_p \frac{v_{rel,sp}^4}{8} h^{++} \propto m_s m_p v_{rel,sp}^2 \sim \frac{N_s N_p}{R^2} v_{rel,sp}^4, \quad (2.111)$$

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with a corresponding vortex computation like the one-loop one suggested above, it remains to find a gravity calculation matching the classical vortex scattering potential. For that, we need a calculation that will give a result proportional to $v_{rel,ij}^2$. It does not necessarily need to be a classical calculation but it should certainly be something different from the leading single-graviton exchange calculated above *a la* 't Hooft, *i.e.* we want to find some term unaccounted for by that calculation.

One possibility suggests itself immediately. Since the graviton itself should correspond to a vortex, it is not unreasonable to expect that we should also be able to find an effective 3-dimensional Lagrangian for *interacting supergravitons*. Its construction, however, is plagued with subtleties. The supergraviton (a $D0$ -brane) and the other D -branes of the theory are also solutions of the 10-dimensional supergravity action, but only after introducing explicit delta function source terms in it. So we should, in principle, find a solution for interacting supergravitons and plug it back in the action, exactly as we did for the corresponding vortices. But that of course means that we now also need to do something about the extra dimensions. The most conservative route would be to explicitly perform the integration over the extra dimensions. Finally, one must still subtract the nonzero result for the background without supergravitons, corresponding to the ABJM field theory without vortices. To this end we start with

$$\mathcal{L}_{11} = \mathcal{L}_{EH} + \mathcal{L}_{matter} + \mathcal{L}_{source} - (\mathcal{L}_{EH} + \mathcal{L}_{matter})_{bgr}. \quad (2.112)$$

To arrive at the effective Lagrangian, we evaluate \mathcal{L}_{11} on its supergraviton solution, and integrate out over all the directions (parallel and transverse), obtaining ¹⁶

$$L_{eff}(\vec{y}_i(t)) = \int d^7\vec{x} d^2\vec{y} \sqrt{g} \mathcal{L}_{11}(\tilde{h}_{\mu\nu}(\vec{y}, r; \vec{y}_i(t))), \quad (2.113)$$

where

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \tilde{h}_{\mu\nu}(\vec{x}, \vec{y}; \vec{y}_i(t)) \delta(x_{11} - t), \quad (2.114)$$

and we drop the $\delta(x_{11} - t)$ factor from the metric together with the integration over x_{11} , following [32]. Note that the $\delta(x^-)$ factor in $h_{--} = \tilde{h}_{--} \delta(x^-)$ can be eliminated from L_{eff} in this way only if it is the same for all the gravitons under consideration. But the different velocities of different gravitons mean that they do not move in exactly the same direction, and the delta function cannot be eliminated! So this

¹⁶Here x^- is DLCQ time, but we rewrite in terms of usual time $t \equiv x_0$ and trivially integrating over x_{11} at fixed 11th momentum of the fields.

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approximation is exact only for the first order in the expansion in v , i.e. for the v^2 term we are interested in, but there will be extra terms starting at order v^4 . In order to calculate such v^4 effects, one must consider the full interacting problem, with differently oriented shockwaves. 't Hooft showed that in flat space this is reproduced by the source-probe calculation used by [32], but now in our curved background the calculation might be incomplete. Nevertheless, this proposal passes several simple tests:

- On the background (in the absence of a supergraviton) we have, correctly $\mathcal{L}_{11}=0$, since $\mathcal{L}_{source} = 0$ by definition.
- In the presence of a single (free) supergraviton, $\mathcal{L}_{EH} - (\mathcal{L}_{EH})_{bgr}$ is also zero, since the only nonzero equation of motion is $G_{--} - (G_{--})_{bgr} \propto \Delta h_{--}$, and $g^{--} = 0$. The only contribution to L_{eff} comes from the source term, the integral of \mathcal{L}_{source} , which is exactly equal to the energy of the free supergraviton, as it should be.
- In the case of interacting supergravitons however, we will have a nonzero interaction energy coming from $\mathcal{L}_{EH} - (\mathcal{L}_{EH})_{bgr}$. Again, this energy will only be nonzero for a relative velocity. If the gravitons are parallel, it will be as if we have a single graviton of source equal to the sum of the all sources, for which we get no interaction. Further, Galilean invariance implies that the interaction has to enter as integer powers of $(\dot{q}_i - \dot{q}_j)^2 = v_{rel,ij}^2$, so that we will only get v^2, v^4, v^6, \dots terms.
- And finally the 1-loop ABJM interaction appears from classical corrections to the $\delta(x^-)$ -factored Lagrangian as it should (although, of course, there must also be quantum gravity corrections as well, i.e. one must consider in the general the quantum effective Lagrangian instead of the classical Lagrangian).

2.9.3 M5 brane backreaction

As a final point in this section, we now sketch how to construct the solution for adding the backreacted $M5$ -brane correction to the background corresponding to massive ABJM. As we saw in Section 5, in the type IIB theory the background corresponds to a pp-wave, and the $M5$ -brane corresponds to a $D3$ -brane wrapping the S^3 defined by $r_1 = \text{fixed}$, $r_2 = 0$ inside the maximally supersymmetric pp-wave. Such a solution has been implicitly written in Fig.1f of LLM [59]. Moreover, if the M-theory dual giant graviton is to have size $R = 1/\mu$, so too must the $D3$ -giant in IIB. Our strategy will therefore be to (a) use

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the LLM prescription to find an explicit solution, (b) T-dualize this to type IIA and (c) lift to M-theory. Recall that the LLM solution for any $\frac{1}{2}$ BPS type IIB supergravity

$$\begin{aligned}
ds^2 &= -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2 \\
h^{-2} &= 2y \cosh G \\
y\partial_y V_i &= \epsilon_{ij} \partial_j z \\
y(\partial_i V_j - \partial_j V_i) &= \epsilon_{ij} \partial_y z \\
z &= \frac{1}{2} \tanh G \\
F &= dB_t \wedge (dt + V) + B_t dV + d\hat{B} \\
\tilde{F} &= d\tilde{B}_t \wedge (dt + V) + \tilde{B}_t dV + d\hat{\tilde{B}} \\
B_t &= -\frac{1}{4}y^2 e^{2G}; \quad \tilde{B}_t = -\frac{1}{4}y^2 e^{-2G} \\
d\hat{B} &= -\frac{1}{4}y^3 *_3 d\left(\frac{z+1/2}{y^2}\right); \quad d\hat{\tilde{B}} = -\frac{1}{4}y^3 *_3 d\left(\frac{z-1/2}{y^2}\right)
\end{aligned} \tag{2.115}$$

is completely determined by a single function z which is itself the solution of the *linear* equation

$$\partial_i \partial_i z + y \partial_y \left(\frac{\partial_y z}{y} \right) = 0. \tag{2.116}$$

and absence of singularities means that $z(x_1, x_2, y=0) = \pm 1/2$, and $z \leftrightarrow -z$ is particle-hole duality. Note that the equation for $\Phi = z/y^2$ is the Laplace equation in 6d flat space, with 4d spherical symmetry, with y the radial coordinate for these 4 dimensions. The solution is

$$\begin{aligned}
z(x_1, x_2, y) &= -\frac{1}{2\pi} \int_{\partial\mathcal{D}} dl n'_i \frac{x_i - x'_i}{(\vec{x} - \vec{x}')^2 + y^2} + \sigma \\
V_i(x_1, x_2, y) &= \frac{\epsilon_{ij}}{2\pi} \oint_{\partial\mathcal{D}} \frac{dx'_j}{(\vec{x} - \vec{x}')^2 + y^2}
\end{aligned} \tag{2.117}$$

where σ is a contribution from infinity if z is constant outside of a circle of very large radius (asymptotically $AdS_5 \times S_5$ geometries), with $\sigma = \pm 1/2$ if $z = \pm 1/2$ asymptotically, \mathcal{D} is a droplet, and n_i is the unit normal vector on it, pointing towards the region of $z = +1/2$.

The solution we want is a superposition of two simple solutions:

1) **The pp-wave:** This solution is obtained by having the boundary condition at $y=0$ as a half-filled plane, *i.e.* $z(x'_1, x'_2, 0) = \frac{1}{2} \text{sgn}(x'_2)$. Explicitly performing the integrals above

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gives

$$z(x_2, y) = \frac{1}{2} \frac{x_2}{\sqrt{x_2^2 + y^2}}, \quad (2.118)$$

$$V_1 = \frac{1}{2} \frac{1}{\sqrt{x_2^2 + y^2}}; \quad V_2 = 0.$$

To put the geometry into the standard pp-wave form

$$ds^2 = -2dt dx_1 - (r_1^2 + r_2^2) dt^2 + dr_1^2 + dr_2^2, \quad (2.119)$$

we defined

$$y = r_1 r_2; \quad x_2 = \frac{1}{2}(r_1^2 - r_2^2). \quad (2.120)$$

2) **The $AdS_5 \times S_5$ solution:** Now the boundary condition at $y = 0$ is a large spherical droplet, i.e. for $\tilde{z} = z - 1/2$, the 6d Laplace equation for $\tilde{\Phi} = \tilde{z}/y^2$ has source on a disk of radius r_0 . Integrating, one obtains

$$\tilde{z}(r, y; r_0) = \frac{r^2 - r_0^2 + y^2}{2\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}} - \frac{1}{2}, \quad (2.121)$$

$$-r \sin \phi V_1 + r \cos \phi V_2 = V_\phi(r, y; r_0) = -\frac{1}{2} \left(\frac{r^2 + y^2 + r_0^2}{\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}} - 1 \right).$$

Defining new coordinates by

$$y = r_0 \sinh \rho \sin \theta$$

$$r = r_0 \cosh \rho \cos \theta \quad (2.122)$$

$$\tilde{\phi} = \phi - t$$

puts the metric into the more familiar form,

$$ds^2 = r_0 \left[-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + d\theta^2 + \cos^2 \theta d\tilde{\phi}^2 + \sin^2 \theta d\tilde{\Omega}_3^2 \right], \quad (2.123)$$

with the area of the droplet in the (x_1, x_2) plane $A = 4\pi^2 l_P^4$ and where $l_P = g^{1/4} \sqrt{\alpha'}$. This is clearly proportional to N , the number of branes. Large N therefore corresponds to a large droplet. That the pp-wave is the Penrose limit of $AdS_5 \times S_5$ translates into the fact that near the surface of a large spherical droplet, locally the droplet looks like a half-filled plane.

Since the equation for the only unknown function $z(x_i, y)$ is linear, we can trivially construct a solution that on the boundary give superposition of circles (and maybe half-filled planes), by just adding up the solutions for $z(x_i, y)$. In particular, for concentric circles we get

$$\begin{aligned}\tilde{z} &= \sum_i (-1)^{i+1} \tilde{z} \left(r, y; r_0^{(i)} \right) , \\ V_\phi &= \sum_i (-1)^{i+1} V_\phi \left(r, y; r_0^{(i)} \right) .\end{aligned}\tag{2.124}$$

To obtain the solution we want - a $D3$ -brane wrapping the S^3 with $r_1 = r_{1,0}$, $r_2 = 0$ and (t, x_1) fixed inside the pp-wave - we translate into LLM language which, from (2.120), means that we need $x_2 = \frac{(r_{1,0})^2}{2}$, $y = 0$ and (t, x_1) fixed. This is a small circle at a position $x_2 = \frac{(r_{1,0})^2}{2}$ away from the $x_2 = 0$ boundary of the half-filled plane. Specifically, we are interested in the circle of size $r_{1,0} = 1/\mu$ (or, in the notation used here, with $\mu = 1$, $r_{1,0} = 1$). Thus the solution we seek is the (finite energy) solution of Fig.3f in LLM, a distance $x_2 = 1/2$ from the Fermi surface, and size given by how many branes we want wrapped, in this case one.

In terms of the function $z(x_1, x_2, y)$ this is just the sum of the pp-wave solution (2.119) and a rescaled version of (2.122), of area given by $N = 1$ (one M5-brane), where we replace r of (2.122) with $|\vec{x} - \vec{x}_0|$, and $\vec{x}_0 = (0, \frac{1}{2})$. Unfortunately, this solution looks rather complicated, but to obtain the backreacted M5-brane we still need to T-dualize it to type IIA and lift up to M-theory. This, we leave as an interesting open problem.

2.10 Appendix. Graviton wavefunction equation

In this Appendix we present some of the details of the solution of the graviton (shockwave) wavefunction equation,

$$\Delta_{bgr}^\perp h_{--} = Q\delta^\perp(\vec{x}, \vec{y}). \quad (2.125)$$

We begin by splitting the problem into two cases depending on whether the 11th dimension is parallel to the worldvolume (transverse separation) or perpendicular to the worldvolume directions (longitudinal separation).

- **Transverse separation:**

In this case, \perp denotes directions transverse to the wave. Among the $\vec{y} = t, x, y$ directions, however, only y has this property, so eq.(2.125) becomes

$$H^{-1/3}(\vec{x}) [H(\vec{x})\partial_y^2 + \partial_{\vec{x}}^2] h_{--}(y, \vec{x}) = Q\delta(y)\delta^8(\vec{x}). \quad (2.126)$$

After a Fourier transform in the worldvolume direction y ,

$$h_{--}(\vec{x}, y) = \int dp \frac{e^{ipy}}{2\pi} \phi_p(\vec{x}), \quad (2.127)$$

we get the equation

$$(1 - \mu^2 \vec{x}^2)^{-1/3} [\partial_{\vec{x}}^2 - p^2(1 - \mu^2 \vec{x}^2)] \phi_p(\vec{x}) = Q\delta^8(\vec{x}). \quad (2.128)$$

- **Parallel separation:**

Now, there are two directions, $\vec{y} = (x, y)$ that are transverse to the wave and parallel to the 3-dimensional worldvolume. Expanding the Laplacian and simplifying gives

$$(1 - \mu^2 \vec{x}^2)^{-1/3} [\partial_{\vec{x}}^2 - p^2(1 - \mu^2 \vec{x}^2)] \phi_{\vec{p}}(\vec{x}) = Q\delta^7(\vec{x}) \quad (2.129)$$

where now

$$h_{--}(\vec{x}, \vec{y}) = \int d^2\vec{p} \frac{e^{i\vec{p}\vec{y}}}{2\pi} \phi_{\vec{p}}(\vec{x}). \quad (2.130)$$

We can actually treat both cases in a unified way since, in either case, we can write the equation that needs to be solved as

$$\left[\frac{\partial}{\partial \vec{x}^2} + \mu^2 p^2 \vec{x}^2 - p^2 \right] \phi_p(\vec{x}) = -Q\delta^d(\vec{x}). \quad (2.131)$$

Two points to note about this equation are (a) that it is manifestly spherically symmetric and (b) that if we put $Q = 0$ and $p = i\tilde{p}$ we get the isotropic harmonic oscillator in d

2.10 Appendix. Graviton wavefunction equation

dimensions. To obtain a solution, we need to first impose boundary conditions, either at a finite distance or at infinity. The natural one would be to impose normalizability of the wave function at infinity, but as we will see, this is not a good condition.

To proceed, let's first solve the equation for $Q = 0$, using techniques for central interaction potential. Specifically, expand ϕ_p in spherical harmonics as

$$\phi_p(\vec{x}) = \frac{y_{p,k}(r)}{r^{n/2}} Y_k(\tilde{\Omega}), \quad (2.132)$$

where the n -sphere spherical harmonics $Y_k^{(n)}$ satisfy,

$$\square Y_k^{(n)}(\tilde{\Omega}) = -k(k+n-1)Y_k^{(n)}(\tilde{\Omega}). \quad (2.133)$$

Consequently

$$\left[\frac{d^2}{dr^2} - \frac{k(k+n-1) + n(n-2)/4}{r^2} + \mu^2 p^2 r^2 - p^2 \right] y_{p,k}(r) = 0. \quad (2.134)$$

Now note that when $p = k = 0$ we obtain $r^{n/2}$ and $r^{1-n/2}$ as solutions, since then $\phi_p(\vec{x})$ has a constant and r^{1-n} as solutions. Next, on setting $p = i\tilde{p}$, eq.(2.134) takes the form

$$y'' - \frac{a}{r^2}y - br^2y + cy = 0. \quad (2.135)$$

This we recognize as a confluent hypergeometric equation whose solutions are, of course, (a linear combination of) confluent hypergeometric functions ${}_1F_1$ and ${}_2F_1$,

$$y(r) = e^{\frac{-\sqrt{b}r^2}{2}} r^{\frac{1}{2} + \frac{\sqrt{1+4a}}{2}} \left[C_1 {}_1F_1 \left(\frac{1}{2} + \frac{\sqrt{1+4a}}{4} - \frac{c}{4\sqrt{b}}, 1 + \frac{\sqrt{1+4a}}{2}; \sqrt{b}r^2 \right) + C_2 {}_2F_1(\text{same}) \right]. \quad (2.136)$$

On substituting $a = \frac{k(k+n-1)+n(n-2)}{4}$, $b = \mu^2 \tilde{p}^2$ and $c = \tilde{p}^2$ we get that

$$\begin{aligned} \phi_p(\vec{x}) = & Y_k^{(n)}(\tilde{\Omega}) e^{-\mu \tilde{p} r^2 / 2} r^k \left[C_1 {}_1F_1 \left(-\frac{\tilde{p}}{4\mu} + \frac{k}{2} + \frac{n+1}{4}, k + \frac{n+1}{2}; \mu \tilde{p} r^2 \right) \right. \\ & \left. + C_2 {}_2F_1 \left(-\frac{\tilde{p}}{4\mu} + \frac{k}{2} + \frac{n+1}{4}, k + \frac{n+1}{2}; \mu \tilde{p} r^2 \right) \right]. \end{aligned} \quad (2.137)$$

Since we have a spherically symmetric source and we are looking for a solution that goes to zero at infinity, we need to choose a spherically symmetric solution, *i.e.* $k = 0$. Also replacing $\tilde{p} = -ip$, we finally get

$$\phi_p(\vec{x}) = e^{\frac{i\mu p r^2}{2}} \left[C_1 {}_1F_1 \left(\frac{ip}{4\mu} + \frac{n+1}{4}, \frac{n+1}{2}; -i\mu p r^2 \right) + C_2 {}_2F_1 \left(\frac{ip}{4\mu} + \frac{n+1}{4}, \frac{n+1}{2}; -i\mu p r^2 \right) \right]. \quad (2.138)$$

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Further, using the relation between ${}_2F_1$ and ${}_1F_1$,

$${}_2F_1(\alpha, \beta; z) = \frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)} {}_1F_1(\alpha, \beta; z) + \frac{\Gamma(1-\beta)}{\Gamma(\alpha)} z^{1-\beta} {}_1F_1(1+\alpha-\beta, 2-\beta; z) \quad (2.139)$$

we find that near $r = 0$

$$\phi_p(\vec{x}) = K \frac{e^{\frac{i\mu pr^2}{2}}}{r^{n-1}} {}_1F_1\left(\frac{ip}{4\mu} + \frac{3-n}{4}, \frac{3-n}{2}; -i\mu pr^2\right) \quad (2.140)$$

with the constant fixed as a function of the source Q (up to now neglected) as follows. Near $r = 0$, the full equation, with nonzero source Q becomes

$$\partial_{\vec{x}}^2 \phi_p(\vec{x}) = -Q \delta^d(\vec{x}), \quad (2.141)$$

which is solved by

$$\phi_p(\vec{x}) = \frac{Q}{(d-2)\Omega_{d-1}r^{d-2}}, \quad (2.142)$$

and so the dominating full solution near $r = 0$ must be

$$\phi_p(\vec{x}) = \frac{Q}{(n-1)\Omega_n r^{n-1}} e^{i\mu pr^2/2} {}_1F_1\left(\frac{ip}{4\mu} + \frac{3-n}{4}, \frac{3-n}{2}; -i\mu pr^2\right). \quad (2.143)$$

Notice that this solution behaves like $1/r^{n-1}$. To this we can add the subleading correction (which behaves like a constant at near zero),

$$\phi_{p,2}(\vec{x}) = \tilde{K}_p (-i\mu p)^{\frac{n-1}{2}} e^{i\mu pr^2/2} {}_1F_1\left(\frac{ip}{4\mu} + \frac{n+1}{4}, \frac{n+1}{2}; -i\mu pr^2\right), \quad (2.144)$$

where \tilde{K}_p is an arbitrary p -dependent constant and we have also kept an explicit power of p above in order to emphasize the $z = -i\mu pr^2$ dependence.

Now to impose boundary conditions. We first try normalizability at infinity and find that both solutions behave in the same way as $r \rightarrow \infty$, namely

$$\begin{aligned} \phi_p(\vec{x}) &\rightarrow \frac{Q\Gamma\left(\frac{3-n}{2}\right)}{(n-1)\Omega_{n-1}} \left\{ \frac{e^{-\frac{i\mu pr^2}{2}} (-i\mu pr^2)^{\frac{ip}{4\mu}} (-i\mu p)^{-\frac{3-n}{4}}}{\Gamma\left(\frac{ip}{4\mu} + \frac{3-n}{4}\right)} \right. \\ &\quad \left. + \frac{e^{\frac{i\mu pr^2}{2}} (i\mu pr^2)^{-\frac{ip}{4\mu}} (i\mu p)^{-\frac{3-n}{4}}}{\Gamma\left(-\frac{ip}{4\mu} + \frac{3-n}{4}\right)} \right\} \frac{1}{r^{\frac{n+1}{2}}} \propto \frac{1}{r^{\frac{n+1}{2}}}, \\ \phi_{p,2}(\vec{x}) &\rightarrow e^{\frac{i\mu pr^2}{2}} {}_1F_1\left(\frac{ip}{4\mu} + \frac{n+1}{4}, \frac{n+1}{2}; -i\mu pr^2\right) \sim (\dots) r^{\frac{ip}{2\mu} - \frac{n+1}{2}} \\ &\quad + (\dots) e^{\frac{i\mu pr^2}{2}} r^{-\frac{ip}{2\mu} - \frac{n+1}{2}} \propto \frac{1}{r^{\frac{n+1}{2}}} \end{aligned} \quad (2.145)$$

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and correctly going to zero at infinity. But this behaviour is non-normalizable, since

$$\int d^d x |\phi_p(\vec{x})|^2 \sim \int^R dr r^n \frac{1}{r^{n+1}} \sim \ln R. \quad (2.146)$$

We can extract more information from the solution by taking the $p \rightarrow \infty$ limit of the result and using the fact that the confluent hypergeometric function

$${}_1F_1(\alpha, \beta; z) \simeq \frac{\Gamma(\beta)}{\sqrt{\pi}} e^{z/2} (-\alpha z)^{-\frac{\beta}{2} + \frac{1}{4}} \cos \left[2\sqrt{-\alpha z} - \pi \left(\frac{\beta}{2} - \frac{1}{4} \right) \right], \quad (2.147)$$

as $|\alpha| \rightarrow \infty$. This, in turn means that as $p \rightarrow \infty$,

$$\phi_p(\vec{x}) \simeq \frac{\Gamma\left(\frac{3-n}{2}\right)}{\sqrt{\pi}(n-1)\Omega_{n-1}} \frac{Q}{r^{n-1}} (\pm ipr)^{\frac{n-2}{2}} \cos \left[\pm ipr + \pi \frac{n-2}{4} \right]. \quad (2.148)$$

To continue, we need to fix \tilde{K}_p by imposing appropriate boundary conditions. Again, we distinguish two cases,

Case 1. We first assume that we can put $\tilde{K}_p = 0$, thus ignoring $\phi_{p,2}(\vec{x})$. Then, in the *transverse separation* case where $n = 7$, we obtain the graviton wavefunction at $y = 0$ by a simple integration of $\phi_p(\vec{x})$ over p . Assuming also that the large p region dominates the integral, we obtain

$$\begin{aligned} \phi &\simeq C \frac{Q}{r^7} \\ C &= \lim_{n \rightarrow 7} \frac{\Gamma\left(\frac{3-n}{2}\right)}{\sqrt{\pi}(n-1)\Omega_{n-1}} (\pm i)^{\frac{n-2}{2}} \int_{-\infty}^{+\infty} dz z^{\frac{n-2}{2}} \cos \left[\pm iz + \pi \frac{n-2}{4} \right], \end{aligned} \quad (2.149)$$

which matches the flat space case (corresponding to the BFSS analysis). Also note that there are no other dimensional parameters in the Q/r^7 behaviour, it is just multiplied by a number (part of the number is an integral over the variable $z = pr$), for which we must take a limit since $\Gamma(-2) = \infty$. This independence of the result from μ is most likely the result of an incorrect initial assumption about the boundary conditions.

In the case of *parallel separation*, $n = 6$ and the Fourier transform is

$$\int \frac{d^2 p}{2\pi} e^{i\vec{p}\vec{y}} \phi_p(\vec{x}) = \int_0^\infty p dp \int_0^{2\pi} \frac{d\theta}{2\pi} e^{ipy \cos \theta} \phi_p(\vec{x}) = \int_0^\infty p dp J_0(py) \phi_p(\vec{x}) \quad (2.150)$$

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Here, by contrast, we are interested in the $r \rightarrow 0$ limit. If however, p stays finite, we find that the solution (2.143) becomes p -independent, and then the above Fourier transform gives zero (or rather, $\delta(y)$). So we must again consider the large p limit in (2.148). This region of large p can still give a nontrivial contribution to the p integral, since pr is not necessarily small. Then, also substituting $n = 6$, gives

$$\begin{aligned}\phi(y, r \rightarrow 0) &\simeq -\frac{4Q}{15\Omega_6} \frac{1}{r^3} \int_0^\infty dp p^3 J_0(py) \cos(ipr + \pi) = -\frac{4Q}{15\Omega_6} \frac{1}{r^3} \int_0^\infty dp p^3 J_0(py) \cosh(pr) \\ &= -\frac{Q}{15\Omega_6} \frac{1}{r^3} \frac{1}{r^4} {}_2F_1\left(2, 5/2; 1; -\frac{y^2}{r^2}\right).\end{aligned}\quad (2.151)$$

To understand what the behaviour of the solution near $r = 0$ is, the identities

$$\begin{aligned}\int_0^\infty dx x^{\mu-1} e^{-\alpha x} J_\nu(\beta x) &= \frac{(\beta/(2\alpha))^\nu \Gamma(\nu + \mu)}{\alpha^\mu \Gamma(\nu + 1)} {}_2F_1\left(\frac{\nu + \mu}{2}, \frac{\nu + \mu + 1}{2}; \nu + 1; -\frac{\beta^2}{\alpha^2}\right) \\ {}_2F_1(\alpha, \beta; \gamma; z) &= \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)} (-z)^{-\alpha} {}_2F_1\left(\alpha, \alpha + 1 - \gamma; \alpha + 1 - \beta; \frac{1}{z}\right) \\ &\quad + \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} (-z)^{-\beta} {}_2F_1\left(\beta, \beta + 1 - \gamma; \beta + 1 - \alpha; \frac{1}{z}\right)\end{aligned}\quad (2.152)$$

can be used to show that as $y/r \rightarrow \infty$, the hypergeometric function ${}_2F_1\left(2, 5; 1; -\frac{y^2}{r^2}\right) \sim -\frac{24}{(y/r)^{10}}$ and, subsequently $\phi(y, r \rightarrow 0) \simeq \frac{8Q}{5\Omega_6} \frac{1}{r^7} \frac{1}{(y/r)^{10}}$ near $r = 0$. Note then that we can actually set $r = 0$ directly and obtain $\phi(r = 0) = 0$, which is clearly not what we wanted, so again the initial boundary condition was incorrect.

Case 2. We now look for a more physical boundary condition, and we concentrate on the *parallel separation* case. In reality, the r space terminates at $r = 1/\mu$, so we must impose a boundary condition there. This is an apparent singularity that one should be able to continue through, thus the appropriate boundary condition at $r = 1/\mu$ is Neumann, $\phi'_p(r = 1/\mu) = 0$. The reason is that then the point $r = 1/\mu$ acts as the origin in angular coordinates, for which the above condition is the only one that makes sense (see for instance [60]). Thus imposing

$$\frac{d}{dr} \phi_p(r)|_{r=1/\mu} = 0,\quad (2.153)$$

on the function

$$\phi_p(r) = \frac{Q}{5\Omega_6 r^5} e^{\frac{i\mu pr^2}{2}} {}_1F_1\left(\frac{ip}{4\mu} - \frac{3}{4}, -\frac{3}{2}; -i\mu pr^2\right) + \tilde{K}_p(-i\mu p)^{\frac{5}{2}} e^{\frac{i\mu pr^2}{2}} {}_1F_1\left(\frac{ip}{4\mu} + \frac{7}{4}; \frac{7}{2}; -i\mu pr^2\right),\quad (2.154)$$

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and using the identity

$$\frac{d}{dz} {}_1F_1(a, b; z) = \frac{a}{b} {}_1F_1(a + 1, b + 1; z) \quad (2.155)$$

we obtain the condition

$$\frac{5\Omega_6}{Q\mu^5} \tilde{K}_p(-i\mu p)^{5/2} = \frac{{}_1F_1\left(\frac{ip}{4\mu} - \frac{3}{4}, -\frac{3}{2}; \frac{-ip}{\mu}\right) \left(1 - \frac{5\mu}{ip}\right) + \frac{4}{3} \left(\frac{ip}{4\mu} - \frac{3}{4}\right) {}_1F_1\left(\frac{ip}{4\mu} + \frac{1}{4}, -\frac{1}{2}; \frac{-ip}{\mu}\right)}{-{}_1F_1\left(\frac{ip}{4\mu} + \frac{7}{4}, \frac{7}{2}; \frac{-ip}{\mu}\right) + \frac{4}{7} \left(\frac{ip}{4\mu} + \frac{7}{4}\right) {}_1F_1\left(\frac{ip}{4\mu} + \frac{11}{4}, \frac{9}{2}; \frac{-ip}{\mu}\right)}. \quad (2.156)$$

We already saw that at $r = 0$ the Fourier transform of the first term in (2.154) vanishes, so we need to analyze only the second term. Then

$$\phi(y, r = 0) = \int \frac{d^2p}{2\pi} e^{i\vec{p}\cdot\vec{y}} \tilde{K}_p(-i\mu p)^{5/2} \quad (2.157)$$

and the right hand side of (2.156) gives in the large p limit

$$\frac{15}{16} \frac{1+i}{1-i} \left(\frac{ip}{2\mu}\right)^{-5} \quad (2.158)$$

which means that we could close the contour of integration $\int_{-\infty}^{+\infty} dp$ with a semicircle in the upper half plane (since $\tilde{K}_p(-i\mu p)^{5/2}$ goes to zero as $|p| \rightarrow \infty$), and thus if the integral above would be one-dimensional instead of two, it would be given by the residues at the poles in the upper half p plane. \tilde{K}_p has poles in the upper half plane of the complex p , given by the solutions to the equation

$${}_1F_1\left(\frac{ip}{4\mu} + \frac{7}{4}, \frac{7}{2}; \frac{-ip}{\mu}\right) = \frac{4}{7} \left(\frac{ip}{4\mu} + \frac{7}{4}\right) {}_1F_1\left(\frac{ip}{4\mu} + \frac{11}{4}, \frac{9}{2}; \frac{-ip}{\mu}\right). \quad (2.159)$$

We can easily see, however, that $p = 0$ is a solution to this equation, so half the residue at $p = 0$ will contribute to the integral, giving a constant contribution (independent of y) instead of an exponential.

The next solutions that we obtain for the poles (with `Mathematica`) are $p = 0. \pm 9.14066i\mu$, for which the residue does indeed give an exponential. So the main contribution to the integral gives a constant, plus an exponential $\sim e^{-m_1 y}$.

Here m_1 the lowest imaginary part in the upper half plane among these solutions. Note that we cannot say for certain that $m_1 = 9.14066\mu$, since `Mathematica` searches are always

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in a neighbourhood; if there is a solution of small imaginary part at large real part, we cannot say from the numerical result. Continuing the numerical search for solutions, p , of the root equation, we find $0. \pm 13.4767i\mu; 0. \pm 17.6461i\mu; 0. \pm 21.7516i\mu$, with a rather general starting point (even a *real* one). This seems to hint that all solutions are pure imaginary? If so, we would have a chance to prove that $m_1 = 9.14066\mu$.

Thus if the p integral were be one-dimensional, we would get $\phi(y \rightarrow \infty, r = 0) \sim e^{-m_1 y}$. We can then at least conclude that if we restrict the dependence to only one of the y coordinates, we indeed get $\phi(y \rightarrow \infty, r = 0) \sim e^{-m_1 y}$. A similar result is expected for a two-dimensional y , since then we would need to do the integral

$$\phi(|y|, r = 0) = \int_0^\infty p dp J_0(py) \tilde{K}_p(-i\mu p)^{5/2}, \quad (2.160)$$

which we cannot perform but the small y behaviour would likely be given by expanding $\tilde{K}_p(-i\mu p)^{5/2}$ at large p , which gives $\propto \int_0^\infty dp p^{-4} J_0(py) \propto y^3$. It is not very clear how to obtain the large y behaviour, especially since as we saw above, we are interested in a subleading term. The leading term is indeed (rather easily) obtained as a constant, since $|p| \rightarrow 0$,

$$\tilde{K}_p(-i\mu p)^{5/2} \simeq \frac{Q\mu^5}{\Omega_6} \frac{63}{2(p/\mu)^2}, \quad (2.161)$$

and

$$\phi(|y|, r = 0) \propto \int_0^\infty \frac{dp}{p} J_0(py) = \text{const.} \quad (2.162)$$

(the exact form of the constant we cannot be sure of, since the integral above is outside the range of validity for formulas we could find). It is very likely then that the same massive behaviour $e^{-m_1 y}$ persists for the first subleading term but this needs to be checked.

3

Towards a Realization of the AdS/CMT correspondence via Consistent Abelian Truncation

3.1 Introduction

The gauge/gravity duality, as manifest in the AdS/CFT correspondence [1] has evolved from its humble origins in string theory into one of the most powerful tools in the arsenal of physicists studying non-perturbative and strong-coupling phenomena in quantum field theories today. The original and, arguably, most studied of these "applied string theory" phenomena was the physics of QCD. In particular, a lot of recent work was focused on the quark-gluon plasma observed in heavy ion colliders such as the RHIC collider and the ALICE experiment at CERN (see, for example, [3] for a recent review). More recently though, the ideas of holography have found a new, lower-dimensional hunting ground in *condensed matter physics*. The pp-wave or BMN limit of AdS/CFT selects operators with large charge [4] in $\mathcal{N} = 4$ super Yang-Mills theory. The physics of such large operators is, in a very concrete sense, isomorphic to the physics of certain spin chains [5]; a realization that has led, not only to an enormous development in our understanding of integrability

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in string theory but also served as a forerunner to many of the developments in what has now become known as the “AdS/CMT” correspondence.

In these cases however, the duality was *heuristically motivated* by a relation between some system of branes and a gravitational background, in some decoupling limit. Later, it was realized that if physics in AdS is always holographic, we can consider simple theories in AdS, that should be related to some strongly coupled conformal field theory on the boundary. Thus naturally AdS/CFT came to be applied to condensed matter systems, where one encounters strongly coupled conformal field theories that cannot be dealt with in other ways (see for instance the reviews [8, 9] for an introduction and relevant references). In all these cases, however, the argument is mostly one of universality, that a variety of (large N) theories have some small set of (abelian) operators dual to some finite and small number of fields in AdS, usually a $U(1)$ gauge field, a scalar and maybe some spinors, representing a (sometimes consistent) truncation of some AdS/CFT pair. In other words, either one truncates the number of operators of the system, in which case it is not entirely clear (*a*) why one should focus on a subset, or how one understands from the point of view of a condensed matter system the focus on the few operators of the large N system; or (*b*) one thinks of an abelian condensed matter *analog* of the large N theory, in which case it is not clear why we can use just a gravity dual, as opposed to a full string theory. In either case, we find the argument less than persuasive.

In this chapter we will take some steps towards a better understanding of AdS/CMT, by proposing a modification of the above set-up. We consider instead a *consistent truncation* of the 3-dimensional ABJM theory (which has a known gravity dual), a truncation that corresponds to the collective dynamics of $\mathcal{O}(N)$ fields out of the $\mathcal{O}(N^2)$ of ABJM, and gives an effective theory that is easily identified as the relativistic Landau-Ginzburg model. We also sketch a simple CMT model that has the same qualitative features as the ABJM abelianization, allowing us to understand better in what sense can we use ABJM for condensed matter systems. Here we present only the main ideas, leaving the technical details to chapter 4.

As a point of clarity, we note that the idea of a consistent truncation in string theory is not a new one, having featured before in two primary contexts. On the gravity side of the AdS/CFT correspondence, when one is interested in a classical limit only, a consistent truncation means that we can safely drop the “nonzero modes”, as these will only appear in quantum loops. In supergravity compactifications however, a consistent truncation for a dimensional reduction means that we can drop all the nonzero (KK) modes from the low

3.2 Consistent Abelian Truncation and Condensed Matter Model

energy quantum theory as well, provided that the coupling to the nonzero modes can be made arbitrarily small, or that the masses of the nonzero modes are much larger than the mass parameters of the low energy theory. Hence quantum loops of these nonzero modes may be ignored. We will argue below that it is this latter case that arises here. To understand the collective dynamics that is crucial to our argument, consider a large number, N , of branes in some background. A classical solution that is obtained by turning on fields in all the N branes will curve the background space nontrivially corresponding, via AdS/CFT, to some finite deformation of the dual theory. On the other hand, just turning on fields on a single brane will produce a negligible deformation that will not deform the background space nor the dual. In our case it is the former situation that arises so that in this sense, the collective dynamics of $\mathcal{O}(N)$ fields really is different from the dynamics of a single field, and it is this that allows for a dual gravitational interpretation.

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Consider the following ansatz for the Chern-Simons fields and the scalar matter in the supermultiplet:

$$\begin{aligned}
A_\mu &= a_\mu^{(2)} G^1 G_1^\dagger + a_\mu^{(1)} G^2 G_2^\dagger \\
\hat{A}_\mu &= a_\mu^{(2)} G_1^\dagger G^1 + a_\mu^{(1)} G_2^\dagger G^2 \\
Q^\alpha &= \phi_\alpha G^\alpha \\
R^\alpha &= \chi_\alpha G^\alpha,
\end{aligned} \tag{3.1}$$

where, again, there is no summation over the repeated α ; $a_\mu^{(1)}$ and $a_\mu^{(2)}$ are real-valued vector fields and ϕ_α, χ_α are complex-valued scalar fields. This provides a consistent truncation of the ABJM action to

$$\begin{aligned}
S &= -\frac{N(N-1)}{2} \int d^3x \left[\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} (a_\mu^{(2)} f_{\nu\lambda}^{(1)} + a_\mu^{(1)} f_{\nu\lambda}^{(2)}) \right. \\
&\quad \left. + |D_\mu \phi_i|^2 + |D_\mu \chi_i|^2 + U(|\phi_i|, |\chi_i|) \right]
\end{aligned} \tag{3.2}$$

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where $U \equiv 2V/N(N-1)$ is a rescaling of the potential

$$\begin{aligned}
V &= \frac{2\pi^2}{k^2} N(N-1) \left[(|\phi_1|^2 + |\chi_1|^2)(|\chi_2|^2 - |\phi_2|^2 - c^2)^2 \right. \\
&\quad + (|\phi_2|^2 + |\chi_2|^2)(|\chi_1|^2 - |\phi_1|^2 - c^2)^2 \\
&\quad + 4|\phi_1|^2|\phi_2|^2(|\chi_1|^2 \\
&\quad \left. + |\chi_2|^2) + 4|\chi_1|^2|\chi_2|^2(|\phi_1|^2 + |\phi_2|^2) \right], \tag{3.3}
\end{aligned}$$

and where the, now abelian, gauge covariant derivatives are $D_\mu\phi_i = (\partial_\mu - ia_\mu^{(i)})\phi_i$ and $D_\mu\chi_i = (\partial_\mu - ia_\mu^{(i)})\chi_i$. Different choices of scalars turned on lead to different consistent truncations we collect below:

- $\chi_2 = \phi_2 = 0$: This leads to a model with two massive complex scalars with no self-interactions. This is essentially trivial and will not merit further attention.
- $\chi_1 = \phi_2 = 0$: After a minor re-labeling of $\chi_2 \rightarrow \phi_2$ this choice produces

$$\begin{aligned}
S &= -\frac{N(N-1)}{2} \int d^3x \left[\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} (a_\mu^{(2)} f_{\nu\lambda}^{(1)} + a_\mu^{(1)} f_{\nu\lambda}^{(2)}) \right. \\
&\quad \left. + |D_\mu\phi_i|^2 + U(|\phi_i|) \right], \tag{3.4} \\
V &= \frac{2\pi^2 N(N-1)}{k^2} \left[|\phi_1|^2 (|\phi_2|^2 - c^2)^2 + |\phi_2|^2 (|\phi_1|^2 + c^2)^2 \right]
\end{aligned}$$

a model which has vortex solutions with $\phi_1 = |\phi_1|e^{iN_1\alpha}$ and $\phi_2 = |\phi_2|e^{iN_2\alpha}$, where α is the polar angle on the plane, and $|\phi_{1,2}|$ go to zero at $r = 0$ and $r = \infty$.

- $\phi_1 = \phi_2 = 0$: If we also set $\chi_1 = b = \text{constant}$, and solve for the (now auxiliary) gauge field $a_\mu^{(1)}$ we obtain the action

$$S = -\frac{N(N-1)}{2} \int d^3x \left[\frac{k^2}{8\pi^2 |b|^2} (f_{\mu\nu}^{(2)})^2 + |D_\mu\chi_2|^2 + V \right] \tag{3.5}$$

where the potential is

$$\begin{aligned}
V &= \frac{2\pi^2}{k^2} N(N-1) \left[|b|^2 |\chi_2|^4 + |\chi_2|^2 ((|b|^2 - c^2)^2 - 2c^2 |b|^2) \right. \\
&\quad \left. + c^4 |b|^2 \right]. \tag{3.6}
\end{aligned}$$

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This is the most interesting case. Indeed, we see that for $||b|^2 - c^2| > \sqrt{2}|c||b|$, we obtain a regular ϕ^4 phase, whereas for $||b|^2 - c^2| < \sqrt{2}|c||b|$ we obtain the abelian-Higgs phase, i.e. this is a relativistic Landau-Ginzburg theory, with $(|b|^2 - c^2)^2 \sim g$ and $2c|b|^2 \sim g_c$. However, in order to have a consistent truncation, in the above action we need to also satisfy the “equation of motion” for the constant $|b|^2$. This is indeed the case for BPS solutions of the abelian-Higgs action.

When canonically normalizing all the fields, the quartic coupling for the *canonical* scalar $\tilde{\chi}_2$ from (3.6) becomes g^2 , with $g = 2\pi|b|/(Nk)$, and the coefficient of the middle term in (3.6) becomes $N^2g^2/2$. Then, for $|b| \sim c$, with $k \sim 1$ and N large, $g^2 \sim \mu/(N^2k) \sim \mu/N^2 \ll \mu$, and generically the mass of $|\phi|$ is $\sim \mu$. But we can tune the system to be near zero mass, so that (for $m^2 \neq 0$) we have $|m^2| \ll \mu^2$. Generic modes of the ABJM model (the “nonzero modes” dropped in our consistent truncation) have mass μ , as easily checked in (2.25), (4.6). Therefore we can drop the nonzero modes in the reduced low energy theory *even at the quantum level*, as advertised in the introduction, and consistently truncate to (4.66). Note also that substituting the reduction ansatz into the ABJM action, we find that the (sextic) potential gives a term with bilinear coupling to the nonzero modes $\delta\phi$ of the type $(\tilde{\chi}_2)^4(\delta\phi)^2/(k^2N^2) \propto 1/N^2$, the 2-fermi-2-scalar term gives a bilinear coupling $(\tilde{\chi}_2)^2\delta\bar{\psi}\delta\psi/(kN) \propto 1/N$, and mass deformation quartic in the scalars gives a bilinear coupling $(\tilde{\chi}_2)^2(\delta\phi)^2\mu/(kN) \propto \mu/N$, which is $\ll \mu$, though still $\gg \mu/N^2 \sim g^2$.

Now also note that for $|b| = c$, the potential (3.6) has the vacuum $|\phi| = |b| = c$, which is nothing but the fuzzy sphere vacuum of the massive ABJM, therefore classical solutions of the reduced theory (4.66) are some type of deformations of the fuzzy sphere. Other examples of such classical solutions will be given elsewhere [63]. These solutions, giving a collective dynamics of $\mathcal{O}(N)$ modes, then correspond to finite deformations of the gravity dual, unlike any solutions obtained by turning on a single mode. Therefore we retain the good features of the large N behaviour (classical gravity dual) with this abelian reduction, as advertised in the introduction.

At this point, one might ask whether the modes in the Landau-Ginzburg action we consider are the *only* light ones and if not, do they couple to any others? For the extra modes in (4.22) that we dropped, the answer is simple. All canonically normalized fields couple to the LG mode with coupling $\sim \mu/(N^2k) \sim g^2$. Moreover, all ϕ_i, χ_i have an explicit mass term of the order $N^2c^4/k^2 \sim \mu^2$ and there is also a contribution from the VEV $\chi_1 = b$ of $1/k^2[-4c^2b^2|\tilde{\chi}_2|^2 + b^4(|\tilde{\chi}_2|^2 + |\tilde{\phi}_2|^2)]$. Consequently in the region of parameter space where our LG mode $\tilde{\chi}_2$ can be tuned to be light all the other modes *stay heavy*.

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For generic modes outside the action (4.22) the answer is a bit more difficult. As we have already argued, since generic mass terms are of the order $m^2 \sim \mu^2 > 0$, the only thing remaining to check is if they can be (almost) cancelled by the terms coming from the Higgs VEV $\chi_1 = b$. We can obtain the total mass term by keeping only two C^I 's in the potential (2.28), and replacing the rest by $C^I = (R^1 = bG^1, R^2 = Q^1 = Q^2 = 0)$. Setting this to zero produces a very long equation for the trace of products of two C^I matrices and up to four G^1 matrices being zero, which we will not reproduce here. One solution is given by our LG light mode, $C^I = (R^2 = \chi_2 G^2, R^1 = Q^1 = Q^2 = 0)$, and amounts to an identity between the G^1 and G^2 matrices (together with the condition $|b|^2 = c^2(2 \pm \sqrt{3})$ for a massless LG field). The question is whether the solution is unique. While we don't know of a general *mathematical* proof of uniqueness, *physically* it is clear that there can't be another solution. Indeed, this solution is related to the existence of the maximally supersymmetric fuzzy sphere vacuum characterized by G^1, G^2 ; once we have G^1 turned on, there is an instability towards turning on G^2 also. Consequently, the mass of χ_2 can become negative, passing through zero. Another solution would amount to another instability towards a different vacuum with the same G^1 turned on. As there are no vacua connected in this way to the maximally supersymmetric one, this is impossible. Finally, we note that there can be other light modes in other regions of parameter space, but since all we need here is that at N large, $k \sim 1$ and the only VEV turned on is bG^1 , there are *no other light modes*.

The question at this stage is *how relevant are any of our effective field theories in the condensed matter context?* Following [66], we now outline an argument to suggest that an appropriate answer is *very*. Beginning with the Hubbard model for spinless bosons, with a ground state where each site in the lattice is populated by an equal number of bosons, one can construct a discretized field $\phi_i \sim \alpha_i a_i + \beta_i h_i^\dagger$, where a_i^\dagger creates a ‘‘particle’’ above the ground state and h_i^\dagger creates a ‘‘hole’’. One then obtains the relativistic Landau-Ginzburg action

$$S = \int d^3x (-|\partial_t \phi|^2 + v^2 |\vec{\nabla} \phi|^2 + (g - g_c) |\phi|^2 + u |\phi|^4), \quad (3.7)$$

in the continuum limit. For $g < g_c$ we have an abelian-Higgs system, i.e. a superconducting phase, while for $g > g_c$ we have an insulator phase. At $g = g_c$ (and temperature $T = 0$) the model describes a conformal field theory. The systems described by the above model have also a quantum critical phase which opens up at nonzero temperature for a T -dependent window around $g = g_c$. This quantum critical phase is strongly coupled and difficult to analyze using usual condensed matter methods and hence a good candidate for a holographic description.

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Moreover, the Hubbard model is a drastic simplification of a real condensed matter system. The model was used to describe the quantum critical phase of (bosonic) ^{87}Rb cold atoms on an optical lattice, but the description is believed to hold more generally for the quantum critical phase. For instance, high T_c superconductors have a “strange metal” phase that is believed to be of the same quantum critical phase type. In fact, a simple model for a solid with free electrons describes the qualitative features of the ABJM abelianization. One can consider an electron at site i in the model coupled to an electron at site j to form a spinless boson $\phi_{ij} = \bar{\psi}_i \psi_j$. In two spatial dimensions there are $\mathcal{O}(N^2)$ neighbours of maximum distance N away. It is not unreasonable to consider that the length between the sites has a maximum value $N \geq |\vec{i} - \vec{j}|$. We can write this field as $\phi_{i'}$, where i' is at midpoint between i and j , and a, b correspond to $1, 2, \dots, N$ in spatial directions x, y . If the normalized wavefunctions for $\phi_{i'}^{ab}$ give probabilities for existence of the pair $|\phi_{i'}^{ab}|^2$, and assuming rotational invariance so only rotationally invariant modes $\psi(a)$, thought of as eigenvalues of the matrices ϕ^{ab} , are nonzero, we can consider a decaying solution $|\psi(a)|^2 \propto N - a$. This corresponds to the ABJM matrix G^2 , with $(G^2 G_2^\dagger)_{mn} = (N - m)\delta_{mn}$. Indeed, in ABJM we have the field $\chi_2(G^2)_{mn}$, corresponding to $b_{i'} \sim \sum_a \psi(a)\phi_{i'}^{aa}$. The average distance in between sites is then

$$\langle a \rangle = \frac{\int |\psi(a)|^2 a (2\pi a da)}{\int |\psi(a)|^2 (2\pi a da)} = \frac{N}{2}, \quad (3.8)$$

consistent with the fact that there is a large distance between sites that couple, as is known to be the case.

Note that while the above fields are the only ones that are turned on, the system has, in principle, several more possibilities. For example, we can form more than one matrix scalar field, like the 4 C^I 's of ABJM, by having more electrons at each site that can couple to form spinless bosons, as well as matrix fermions, by having two electrons at site i couple among themselves and with an electron at site j . We can also construct two Chern-Simons (topological) gauge fields by a generalization of the abelian CS case (see e.g. [67]) as follows: Consider two fermions at sites i and i'' coupling to form $\phi_{i'}^{aa'}$ at their midpoint i' and two fermions at sites j and j'' coupling to form $\phi_{j'}^{bb'}$ at their midpoint j' . Then the field

$$e\vec{a}(\vec{r}_{i'}) = \vec{\nabla}_{i'} \sum_{j' \neq i'} \alpha(\vec{r}_i - \vec{r}_j), \quad (3.9)$$

where $\alpha(\vec{r}_i - \vec{r}_j)$ is the angle made by $\vec{r}_i - \vec{r}_j$ with a fixed axis, corresponds to a CS gauge field. The indices on the gauge field are the planar indices for the only variable above, $\vec{r}_{ii'} - \vec{r}_{jj'}$ (changing $\vec{r}_{ii'}$ by itself just gives a harmless overall translation), as well as the

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discrete choice for $\vec{r}_{ii'}$ to belong to i' or j' , giving two gauge fields A and \hat{A} . The scalars $\phi_{i'}^{ab}$ act as bifundamental with respect to them. It is clear then that, qualitatively at least, the model outlined above describes all the fields of ABJM, as well as the abelianization.

In some sense, the 3-dimensional Landau-Ginzburg model makes more sense as a dimensional reduction from four dimensions; the same is true of our abelianization picture. The matrix G^1 , one of the two matrices G^1, G^2 that describe the fuzzy 2-sphere, is multiplied by the constant $|b|$, so in a sense we have a “fuzzy circle” (limit of a fuzzy 2-sphere), becoming classical at large N . The physical radius of a fuzzy sphere construction was argued in the literature on brane polarizations to be (see e.g. [23])

$$R_{ph}^2 = \frac{2}{N} \text{Tr} [X^I X_I^\dagger] = \frac{2}{N} \text{Tr} [C^I C_I^\dagger] 4\pi^2 l_P^3 \quad (3.10)$$

where $l_P^3 = l_s^2 R_{11}$. Assuming the same formula holds for the “fuzzy circle” case, and that like in the pure fuzzy sphere case, the 11th direction has radius $R_{11} = R_{ph}/k$, we obtain

$$R_{ph} = (N - 1) l_s^2 \frac{4\pi^2 |b|^2}{k}. \quad (3.11)$$

The pure (massless) ABJM model corresponds to the IR limit of M2-branes on $\mathbb{R}^{2,1} \times \mathbb{C}^4/\mathbb{Z}_k$ and has as a gravity dual type IIA string theory on $AdS_4 \times \mathbb{C}P^3$. In the massive case, the spacetime for M2-brane propagation is more complicated [62], and the gravity dual even more so [43, 62], so we will not reproduce the formulas here.

4

Abelian-Higgs and Vortices from ABJM: towards a string realization of AdS/CMT

4.1 Introduction

The applications of the *AdS/CFT* correspondence to condensed matter hinge on the idea that, if physics in AdS is always holographic, then we can consider simple theories in AdS, which *should* be dual to some strongly coupled conformal field theories (see e.g. [8, 9] for a review). We will explore possible abelian reductions of the ABJM model in this chapter. Our strategy will be to look to the matrices G^α that characterize the “fuzzy funnel” BPS state of pure ABJM and the “fuzzy sphere” ground state of the massive deformation of ABJM (mABJM) since they correspond to a collective motion of $\mathcal{O}(N)$ out of $\mathcal{O}(N^2)$ degrees of freedom. They will play a central role in our abelianization ansatz. We will then argue that this ansatz furnishes a *consistent truncation* of mABJM and can be used to identify further (phenomenologically) interesting abelianizations. We then show how these find application in condensed matter physics and, finally, we will explore some BPS solutions suggested by the abelian ansätze together with their spacetime interpretation. The main ideas about the abelianization and application to AdS/CMT were outlined in chapter 3 [64], and here we present the full details.

At this point, it is worth elaborating on the idea of consistent truncations and the context in which we will use them in this chapter. To this end, let's recall previous instances where they have been used commonly. Perhaps the most frequently encountered use of consistent truncations is in the context of AdS/CFT. Here, one usually deals with a *classical theory* on the gravity side, so the existence of a consistent truncation to some reduced theory means that we can safely drop the other modes, since they will appear only in quantum loops. However, another common utilization of truncations is in supergravity compactifications. Here the relevant question is whether or not we can safely retain only the phenomenologically interesting reduced four-dimensional theory. If there exists a consistent truncation, one can check whether couplings to the “nonzero modes” can be made arbitrarily small or whether, more commonly, the masses of the nonzero modes are much larger than the mass parameters of the reduced theory. We will argue that it is the latter usage of consistent truncations that is relevant in our case and as a result, at low energies we can, with no loss of physics, drop the nonzero modes even from the quantum theory.

Another issue that warrants clarification is our use of the *collective dynamics* of $\mathcal{O}(N)$ modes. To see why this is different from the dynamics of any *single* field, consider a large number, N , of branes in some gravitational background. A classical solution obtained by turning on fields in all N branes corresponds, in the gravity dual, to a finite deformation of the background, in stark contrast to turning on fields on a *single brane* which does not produce a finite effect in the dual background.

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The ABJM model [21] is obtained as the IR limit of the theory of N coincident M2-branes moving in $\mathbb{R}^{2,1} \times \mathbb{C}^4/\mathbb{Z}_k$. It is a $\mathcal{N} = 6$ supersymmetric $U(N) \times U(N)$ Chern-Simons gauge theory at level $(k, -k)$, with bifundamental scalars C^I and fermions ψ_I , $I = 1, \dots, 4$ in the fundamental of the $SU(4)_R$ symmetry group. The gauge fields are denoted by A_μ and \hat{A}_μ . Its action is given by

$$\begin{aligned}
 S = & \int d^3x \left(\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \right. \\
 & - \text{Tr} D_\mu C_I^\dagger D^\mu C^I - i \text{Tr} \psi^{I\dagger} \gamma^\mu D_\mu \psi_I + \frac{4\pi^2}{3k^2} \text{Tr} \left(C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger \right. \\
 & \left. \left. + C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K + 4C^I C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger - 6C^I C_J^\dagger C^J C_I^\dagger C^K C_K^\dagger \right) \right)
 \end{aligned}$$

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$$\begin{aligned}
& + \frac{2\pi i}{k} \text{Tr} \left(C_I^\dagger C^I \psi^{J\dagger} \psi_J - \psi^{J\dagger} C^I C_I^\dagger \psi_J - 2C_I^\dagger C^J \psi^{I\dagger} \psi_J + 2\psi^{J\dagger} C^I C_J^\dagger \psi_J \right. \\
& \left. + \epsilon^{IJKL} C_I^\dagger \psi_J C_K^\dagger \psi_L - \epsilon_{IJKL} C^I \psi^{J\dagger} C^K \psi^{L\dagger} \right)
\end{aligned} \tag{4.1}$$

where the gauge-covariant derivative is

$$D_\mu C^I = \partial_\mu C^I + iA_\mu C^I - iC_I \hat{A}_\mu. \tag{4.2}$$

The action has a $SU(4) \times U(1)$ R-symmetry associated with the $\mathcal{N} = 6$ supersymmetries.

There is a maximally supersymmetric (*i.e.*, preserving all $\mathcal{N} = 6$) massive deformation of the model with a parameter μ [25, 27], which breaks the R-symmetry down to $SU(2) \times SU(2) \times U(1)_A \times U(1)_B \times \mathbb{Z}_2$ by splitting the scalars as

$$C^I = (Q^\alpha, R^\alpha); \quad \alpha = 1, 2 \tag{4.3}$$

The \mathbb{Z}_2 action swaps the matter fields Q^α and R^α , while the $SU(2)$ factors act individually on the doublets Q^α and R^α respectively and $U(1)_A$ symmetry rotates Q^α with a phase $+1$ and R^α with a phase -1 . The mass deformation, besides giving a mass to the fermions, changes the potential of the theory. The bosonic part of the deformed action can be written as

$$\begin{aligned}
\mathcal{L}_{\text{Bosonic}} &= \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \\
&- \text{Tr} |D^\mu Q^\alpha|^2 - \text{Tr} |D^\mu R^\alpha|^2 - V
\end{aligned} \tag{4.4}$$

where the potential is

$$V = \text{Tr} (|M^\alpha|^2 + |N^\alpha|^2), \tag{4.5}$$

and where

$$\begin{aligned}
M^\alpha &= \mu Q^\alpha + \frac{2\pi}{k} \left(2Q^{[\alpha} Q_\beta^\dagger Q^{\beta]} + R^\beta R_\beta^\dagger Q^\alpha - Q^\alpha R_\beta^\dagger R^\beta + 2Q^\beta R_\beta^\dagger R^\alpha - 2R^\alpha R_\beta^\dagger Q^\beta \right), \\
N^\alpha &= -\mu R^\alpha + \frac{2\pi}{k} \left(2R^{[\alpha} R_\beta^\dagger R^{\beta]} + Q^\beta Q_\beta^\dagger R^\alpha - R^\alpha Q_\beta^\dagger Q^\beta + 2R^\beta Q_\beta^\dagger Q^\alpha - 2Q^\alpha Q_\beta^\dagger R^\beta \right).
\end{aligned} \tag{4.6}$$

The equations of motion of the bosonic Lagrangian (4.4) are

$$\begin{aligned}
D_\mu D^\mu Q^\alpha &= \frac{\partial V}{\partial Q_\alpha^\dagger}, & D_\mu D^\mu R^\alpha &= \frac{\partial V}{\partial R_\alpha^\dagger}, \\
F_{\mu\nu} &= \frac{2\pi}{k} \epsilon_{\mu\nu\lambda} J^\lambda, & \hat{F}_{\mu\nu} &= \frac{2\pi}{k} \epsilon_{\mu\nu\lambda} \hat{J}^\lambda,
\end{aligned} \tag{4.7}$$

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where the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ and the two gauge currents J^μ and \hat{J}^μ , given by

$$J^\mu = i(Q^\alpha(D^\mu Q^\alpha)^\dagger - (D^\mu Q^\alpha)Q_\alpha^\dagger + R^\alpha(D^\mu R^\alpha)^\dagger - (D^\mu R^\alpha)R_\alpha^\dagger), \quad (4.8)$$

$$\hat{J}^\mu = -i(Q_\alpha^\dagger(D^\mu Q^\alpha) - (D^\mu Q^\alpha)^\dagger Q^\alpha + R_\alpha^\dagger(D^\mu R^\alpha) - (D^\mu R^\alpha)^\dagger R^\alpha),$$

are covariantly conserved so that $\nabla_\mu J^\mu = \nabla_\mu \hat{J}^\mu = 0$. In addition, there are two abelian currents j^μ and \hat{j}^μ corresponding to the global $U(1)_A$ and $U(1)_B$ invariances, given by

$$j^\mu = i\text{Tr} (Q^\alpha(D^\mu Q^\alpha)^\dagger - (D^\mu Q^\alpha)Q_\alpha^\dagger + R^\alpha(D^\mu R^\alpha)^\dagger - (D^\mu R^\alpha)R_\alpha^\dagger), \quad (4.9)$$

$$\hat{j}^\mu = -i\text{Tr} (Q_\alpha^\dagger(D^\mu Q^\alpha) - (D^\mu Q^\alpha)^\dagger Q^\alpha + R_\alpha^\dagger(D^\mu R^\alpha) - (D^\mu R^\alpha)^\dagger R^\alpha),$$

which are ordinarily conserved *i.e.* $\partial_\mu j^\mu = \partial_\mu \hat{j}^\mu = 0$. By choosing the gauge $A_0 = \hat{A}_0 = 0$, the energy density (Hamiltonian) is given by¹

$$H = \text{Tr} [(D_0 Q^\alpha)^\dagger (D_0 Q^\alpha) + (D_i Q^\alpha)^\dagger (D_i Q^\alpha) + (D_0 R^\alpha)^\dagger (D_0 R^\alpha) + (D_i R^\alpha)^\dagger (D_i R^\alpha) + V]. \quad (4.10)$$

Since this is a Chern-Simons theory, the equations of motion must be supplemented with the Gauss law constraints

$$F_{12} = \frac{2\pi i}{k} J^0 = \frac{2\pi i}{k} (Q^\alpha(D^0 Q^\alpha)^\dagger - (D^0 Q^\alpha)Q_\alpha^\dagger + R^\alpha(D^0 R^\alpha)^\dagger - (D^0 R^\alpha)R_\alpha^\dagger), \quad (4.11)$$

$$\hat{F}_{12} = \frac{2\pi i}{k} \hat{J}^0 = -\frac{2\pi i}{k} (Q_\alpha^\dagger(D^0 Q^\alpha) - (D^0 Q^\alpha)^\dagger Q^\alpha + R_\alpha^\dagger(D^0 R^\alpha) - (D^0 R^\alpha)^\dagger R^\alpha).$$

The gauge choice is not as restrictive as it would seem. Choosing (as we do below for our abelianization) A_0 and \hat{A}_0 different from zero produces an extra term in the Hamiltonian of the form $\epsilon^{\mu\nu\lambda} \text{Tr} [A_\mu A_\nu A_\lambda - \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda]$. In the abelian case this vanishes anyway since it is proportional to $\epsilon^{\mu\nu\lambda} a_\mu^{(i)} a_\nu^{(j)} a_\lambda^{(k)}$ and there are only two $a_\mu^{(i)}$'s. So in the abelian case, the Hamiltonian is the same even away from the gauge $A_0 = \hat{A}_0 = 0$. The mass deformed theory has ground states of the fuzzy sphere type given by

$$R^\alpha = cG^\alpha; \quad Q^\alpha = 0 \quad \text{and} \quad Q_\alpha^\dagger = cG^\alpha; \quad R^\alpha = 0 \quad (4.12)$$

¹Note that the terms $A_1 \dot{A}_2 - A_2 \dot{A}_1$ cancel from $p\dot{q} - L$, and the rest of the CS term involve A_0

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where $c \equiv \sqrt{\frac{\mu k}{2\pi}}$ and the matrices G^α , $\alpha = 1, 2$, satisfy the equations [25, 27]

$$G^\alpha = G^\alpha G_\beta^\dagger G^\beta - G^\beta G_\beta^\dagger G^\alpha. \quad (4.13)$$

It was shown in [23, 65] that this solution corresponds to a fuzzy 2-sphere.

An explicit solution of these equations is given by

$$\begin{aligned} (G^1)_{m,n} &= \sqrt{m-1} \delta_{m,n}, \\ (G^2)_{m,n} &= \sqrt{(N-m)} \delta_{m+1,n}, \\ (G_1^\dagger)_{m,n} &= \sqrt{m-1} \delta_{m,n}, \\ (G_2^\dagger)_{m,n} &= \sqrt{(N-n)} \delta_{n+1,m}. \end{aligned} \quad (4.14)$$

Clearly, these matrices satisfy $G^1 = G_1^\dagger$ also. In the case of the pure ABJM, there is a *BPS solution* of the fuzzy funnel type with c replaced by

$$c(s) = \sqrt{\frac{k}{4\pi s}}, \quad (4.15)$$

instead, where s is one of the two spatial coordinates of the ABJM model. The matrices G^α are bifundamental under $U(N) \times U(N)$, therefore $G^1 G_1^\dagger$ and $G^2 G_2^\dagger$ are in the adjoint of the first $U(N)$, and $G_1^\dagger G^1$ and $G_2^\dagger G^2$ are in the adjoint of the second.

4.2.1 An Abelianization Ansatz

Given all these properties of the G^α matrices, it is reasonable to choose the following abelianization ansatz

$$\begin{aligned} A_\mu &= a_\mu^{(2)} G^1 G_1^\dagger + a_\mu^{(1)} G^2 G_2^\dagger, \\ \hat{A}_\mu &= a_\mu^{(2)} G_1^\dagger G^1 + a_\mu^{(1)} G_2^\dagger G^2, \\ Q^\alpha &= \phi_\alpha G^\alpha, \\ R^\alpha &= \chi_\alpha G^\alpha, \end{aligned} \quad (4.16)$$

with no summation over α in the ansatz for Q^α, R^α ; $a_\mu^{(1)}$ and $a_\mu^{(2)}$ real-valued vector fields and ϕ_α, χ_α complex-valued scalar fields.

Since $G^1 G_1^\dagger$ commutes with $G^2 G_2^\dagger$ and $G_1^\dagger G^1$ commutes with $G_2^\dagger G^2$, the gauge fields $a_\mu^{(i)}$ are abelian and the field strengths decompose as

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] = f_{\mu\nu}^{(2)} G^1 G_1^\dagger + f_{\mu\nu}^{(1)} G^2 G_2^\dagger, \\ \hat{F}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + i[\hat{A}_\mu, \hat{A}_\nu] = f_{\mu\nu}^{(2)} G_1^\dagger G^1 + f_{\mu\nu}^{(1)} G_2^\dagger G^2, \end{aligned} \quad (4.17)$$

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with the *abelian* field strengths $f_{\mu\nu}^{(i)} = \partial_\mu a_\nu^{(i)} - \partial_\nu a_\mu^{(i)}$.

With this ansatz, the Chern-Simons term becomes

$$-\frac{k}{4\pi} \frac{N(N-1)}{4} \epsilon^{\mu\nu\lambda} \left(a_\mu^{(2)} f_{\nu\lambda}^{(1)} + a_\mu^{(1)} f_{\nu\lambda}^{(2)} \right), \quad (4.18)$$

while the covariant derivatives $D_\mu Q^\alpha$ and $D_\mu R^\alpha$ give rise to

$$\begin{aligned} D_\mu \phi_i &= (\partial_\mu - i a_\mu^{(i)}) \phi_i, \\ D_\mu \chi_i &= (\partial_\mu - i a_\mu^{(i)}) \chi_i, \end{aligned} \quad (4.19)$$

and the values for M^α, N^α are given by

$$\begin{aligned} M^1 &= \frac{2\pi}{k} [\phi_1 (c^2 + |\phi_2|^2 - |\chi_2|^2) - 2\chi_1 \bar{\chi}_2 \phi_2] G^1, \\ M^2 &= \frac{2\pi}{k} [\phi_2 (c^2 + |\phi_1|^2 - |\chi_1|^2) - 2\bar{\chi}_1 \chi_2 \phi_1] G^2, \\ N^1 &= \frac{2\pi}{k} [\chi_1 (-c^2 + |\chi_2|^2 - |\phi_2|^2) - 2\phi_1 \bar{\phi}_2 \chi_2] G^1, \\ N^2 &= \frac{2\pi}{k} [\chi_2 (-c^2 + |\chi_1|^2 - |\phi_1|^2) - 2\bar{\phi}_1 \phi_2 \chi_1] G^2 \end{aligned} \quad (4.20)$$

where as before, $c^2 = \mu k / (2\pi)$. Substituting into the potential gives

$$\begin{aligned} V &= \frac{2\pi^2}{k^2} N(N-1) \left[(|\phi_1|^2 + |\chi_1|^2) (|\chi_2|^2 - |\phi_2|^2 - c^2)^2 \right. \\ &\quad + (|\phi_2|^2 + |\chi_2|^2) (|\chi_1|^2 - |\phi_1|^2 - c^2)^2 \\ &\quad \left. + 4|\phi_1|^2 |\phi_2|^2 (|\chi_1|^2 + |\chi_2|^2) + 4|\chi_1|^2 |\chi_2|^2 (|\phi_1|^2 + |\phi_2|^2) \right]. \end{aligned} \quad (4.21)$$

Note that the interchange of χ with ϕ (which changes Q^α with R^α) is equivalent to a change in the sign of c^2 , *i.e.* either a change in the sign of μ , or of k . Putting everything together then gives the final *abelian* effective action

$$S = -\frac{N(N-1)}{2} \int d^3x \left[\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} (a_\mu^{(2)} f_{\nu\lambda}^{(1)} + a_\mu^{(1)} f_{\nu\lambda}^{(2)}) + |D_\mu \phi_i|^2 + |D_\mu \chi_i|^2 + U(|\phi_i|, |\chi_i|) \right], \quad (4.22)$$

with a rescaled potential $U \equiv 2V/N(N-1)$. Since the effective theory derives from a Chern-Simons theory, the equations of motion need to be supplemented with the Gauss

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law constraints which, in our ansatz, reduce to

$$\begin{aligned}
 f_{12}^{(2)} &= \frac{2\pi i}{k} [\phi_1 (D^0 \phi_1)^\dagger - (D^0 \phi_1) \phi_1^\dagger + \chi_1 (D^0 \chi_1)^\dagger - (D^0 \chi_1) \chi_1^\dagger], \\
 f_{12}^{(1)} &= \frac{2\pi i}{k} [\phi_2 (D^0 \phi_2)^\dagger - (D^0 \phi_2) \phi_2^\dagger + \chi_2 (D^0 \chi_2)^\dagger - (D^0 \chi_2) \chi_2^\dagger],
 \end{aligned}
 \tag{4.23}$$

We see, however, that these are nothing but the $a_0^{(1)}, a_0^{(2)}$ equations of motion for the action (4.22). As we will need to work away from the $a_0^{(1)} = a_0^{(2)} = 0$ gauge, we don't need to impose them.

4.2.2 Consistent Truncations

A key point to note about this abelianization ansatz is that it is a *consistent truncation* of the original ABJM theory in the sense that, using the facts that $M^\alpha \propto G^\alpha$, $N^\alpha \propto G^\alpha$, $D_\mu D^\mu (\phi_\alpha G^\alpha) = (D_\mu D^\mu \phi_\alpha) G^\alpha$ and $D_\mu D^\mu (\chi_\alpha G^\alpha) = (D_\mu D^\mu \chi_\alpha) G^\alpha$, the equations of motion that follow from the action (4.22),

$$\begin{aligned}
 \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} f_{\mu\nu}^{(1)} &= i \left[\bar{\phi}_2 D^\lambda \phi_2 - \phi_2 \overline{D^\lambda \phi_2} + \bar{\chi}_2 D^\lambda \chi_2 - \chi_2 \overline{D^\lambda \chi_2} \right], \\
 \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} f_{\mu\nu}^{(2)} &= i \left[\bar{\phi}_1 D^\lambda \phi_1 - \phi_1 \overline{D^\lambda \phi_1} + \bar{\chi}_1 D^\lambda \chi_1 - \chi_1 \overline{D^\lambda \chi_1} \right],
 \end{aligned}
 \tag{4.24}$$

and

$$\begin{aligned}
 &D_\mu D^\mu \phi_1 \\
 &= \frac{4\pi^2}{k^2} \left[(|\chi_2|^2 - |\phi_2|^2 - c^2)^2 + 2(|\phi_2|^2 + |\chi_2|^2)(|\phi_1|^2 + |\chi_1|^2 + c^2) + 4|\phi_2|^2 |\chi_2|^2 \right] \phi_1, \\
 &D_\mu D^\mu \phi_2 \\
 &= \frac{4\pi^2}{k^2} \left[(|\chi_1|^2 - |\phi_1|^2 - c^2)^2 + 2(|\phi_1|^2 + |\chi_1|^2)(|\phi_2|^2 + |\chi_2|^2 + c^2) + 4|\phi_1|^2 |\chi_1|^2 \right] \phi_2, \\
 &D_\mu D^\mu \chi_1 \\
 &= \frac{4\pi^2}{k^2} \left[(|\chi_2|^2 - |\phi_2|^2 - c^2)^2 + 2(|\phi_2|^2 + |\chi_2|^2)(|\phi_1|^2 + |\chi_1|^2 - c^2) + 4|\phi_2|^2 |\chi_2|^2 \right] \chi_1, \\
 &D_\mu D^\mu \chi_2 \\
 &= \frac{4\pi^2}{k^2} \left[(|\chi_1|^2 - |\phi_1|^2 - c^2)^2 + 2(|\phi_1|^2 + |\chi_1|^2)(|\phi_2|^2 + |\chi_2|^2 - c^2) + 4|\phi_1|^2 |\chi_1|^2 \right] \chi_2,
 \end{aligned}
 \tag{4.25}$$

satisfy the higher original ABJM equations of motion (4.7) and Gauss constraints (4.12).

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Since $\text{Tr}[G^1 G_1^\dagger] = \text{Tr}[G_1^\dagger G_1] = \text{Tr}[G^2 G_2^\dagger] = \text{Tr}[G_2^\dagger G_2] = N(N-1)/2$, the energy density (Hamiltonian) is

$$H = \frac{N(N-1)}{2} [|D_0 \phi_i|^2 + |D_0 \chi_i|^2 + |D_a \phi_i|^2 + |D_a \chi_i|^2] + V \quad (4.26)$$

where $a, b = 1, 2$. Note also that away from the gauge $A_0 = \hat{A}_0 = 0$ (which imply that $a_0^{(i)} = 0$), we would, in principle, have a term cubic in the gauge fields in the Hamiltonian. This however vanishes in the abelian case, so the above result is correct in general. This abelianization, with its four complex scalar fields, is rather general. We will study further reductions of it involving only two scalars. Looking at the scalar equations of motion above we see that putting any two of the scalars to zero is again a consistent truncation.

- A trivial choice turns out to be $\chi_2 = \phi_2 = 0$ (or equivalently $\chi_1 = \phi_1 = 0$), since in that case, the potential reduces to a simple mass term,

$$V = \frac{4\pi^2 c^4}{k^2} (|\phi_1|^2 + |\chi_1|^2) \quad (4.27)$$

while at the same time, the only $a_\mu^{(2)}$ dependence remains in the Chern-Simons term, $\sim \int \epsilon a^{(2)} f^{(1)}$, so its equation of motion is $f_{\mu\nu}^{(1)} = 0$, which means $a_\mu^{(1)}$ is also trivial (pure gauge). So we remain with two massive complex scalar fields coupled to one trivial gauge field (pure gauge, with no kinetic term), an uninteresting model.

- A much more interesting choice is $\phi_1 = \phi_2 = 0$, which will turn out to lead (with some modifications) to the Abelian-Higgs model. Since we will study this separately and extensively in section 4, we will not discuss it further here.
- Finally, setting $\chi_1 = \phi_2 = 0$, and renaming χ_2 to ϕ_2 for simplicity, we get

$$S = -\frac{N(N-1)}{2} \int d^3x \left[\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \left(a_\mu^{(2)} f_{\nu\lambda}^{(1)} + a_\mu^{(1)} f_{\nu\lambda}^{(2)} \right) + |D_\mu \phi_i|^2 + U(|\phi_i|, |\chi_i|) \right] \quad (4.28)$$

$$V = \frac{2\pi^2}{k^2} N(N-1) [|\phi_1|^2 (|\phi_2|^2 - c^2)^2 + |\phi_2|^2 (|\phi_1|^2 + c^2)^2],$$

and energy density (Hamiltonian)

$$H = \frac{N(N-1)}{2} [|D_0 \phi_i|^2 + |D_a \phi_i|^2] + V. \quad (4.29)$$

4.3 New vortex solutions for a Toda system

We should note that, until now, we have worked only with the *massive* ABJM model, but that we can analyze the *massless* (or pure) ABJM model in a straightforward way by setting $c = 0$. Since c appears only in the potential, we can check that the model with potential (4.21) is symmetric under interchange of $\phi_i \leftrightarrow \chi_i$. For the model with $\chi_1 = \phi_2 = 0$ above, for example, we obtain a purely quartic potential,

$$V = \frac{2\pi^2}{k^2} N(N-1) |\phi_1|^2 |\phi_2|^2 (|\phi_1|^2 + |\phi_2|^2). \quad (4.30)$$

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We now study BPS solutions of the effective model (4.28). Before doing so, it is worth taking a step back, and considering the more general case of the $Q^2 = R^1 = 0$ reduction, with only $Q^1 = Q$ and $R^2 = R$ nonzero, but without the abelianization ansatz. There we can ‘complete squares’ in the Hamiltonian density and write it, in complete analogy to the usual Abelian-Higgs model, as

$$\begin{aligned} \mathcal{H} = & \text{Tr} |D_0 Q - iM|^2 + \text{Tr} |D_0 R + iN|^2 + \text{Tr} |D_- Q|^2 + \text{Tr} |D_+ R|^2 \\ & + i\epsilon^{ab} \partial_a \text{Tr} \left(Q^\dagger (D_b Q) - R^\dagger (D_b R) \right) + \mu j_0, \end{aligned} \quad (4.31)$$

where $\mu j_0 = \mu k / (2\pi) \text{Tr} (F_{12})$ and $D_\pm \equiv D_1 \pm iD_2$. Just as in the Abelian-Higgs model, the term on the second line (with ϵ^{ab}) is zero on the configurations of interest, since

$$\int_V d^2x \epsilon^{ab} \partial_a (\phi^\dagger D_b \phi) = \int_{\partial V_\infty} (\phi^\dagger D_a \phi) dx_a \quad (4.32)$$

and $D_a \phi \rightarrow 0$ at $r \rightarrow \infty$ for $\phi = Q, R$ in order to have finite energy configurations. Moreover, the perfect squares on the first line are minimized by the BPS equations

$$D_- Q = 0; \quad D_+ R = 0; \quad D_0 Q = iM; \quad D_0 R = -iN, \quad (4.33)$$

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which leaves just the *topological term*, μj_0 . The BPS equations together with the Gauss law constraints are

$$\begin{aligned}
D_- Q &= 0, \\
D_+ R &= 0, \\
D_0 Q &= i\mu Q - \frac{2\pi i}{k} [QR^\dagger R - RR^\dagger Q], \\
D_0 R &= i\mu R + \frac{2\pi i}{k} [RQ^\dagger Q - QQ^\dagger R], \\
F_{12} &= -\frac{4\pi\mu}{k} (QQ^\dagger + RR^\dagger) + \frac{8\pi^2}{k^2} [QR^\dagger RQ^\dagger - RQ^\dagger QR^\dagger], \\
\hat{F}_{12} &= -\frac{4\pi\mu}{k} (Q^\dagger Q + R^\dagger R) + \frac{8\pi^2}{k^2} [R^\dagger QQ^\dagger R - Q^\dagger RR^\dagger Q],
\end{aligned} \tag{4.34}$$

where, in the Gauss law constraints, we have already substituted the BPS equations for $D_0 Q, D_0 R$. These equations are more general, and can be used in principle to find nonabelian BPS solutions. In practice, they are still too difficult to solve analytically so from now on we will go back to the abelian case $Q = \phi_1 G^1, R = \phi_2 G^2$. There, the BPS equations for $D_0 Q$ and $D_0 R$ become

$$\begin{aligned}
(\partial_0 - ia_0^{(1)})\phi_1 &= -\frac{2\pi i}{k}\phi_1 \left[|\phi_2|^2 - \frac{\mu k}{2\pi} \right], \\
(\partial_0 - ia_0^{(2)})\phi_2 &= \frac{2\pi i}{k}\phi_2 \left[|\phi_1|^2 + \frac{\mu k}{2\pi} \right].
\end{aligned} \tag{4.35}$$

For static configurations (for which $\partial_0 \phi_i = 0$) these equations can be solved for $a_0^{(i)}$ as

$$\begin{aligned}
a_0^{(1)} &= \frac{2\pi}{k} \left[|\phi_2|^2 - \frac{\mu k}{2\pi} \right], \\
a_0^{(2)} &= -\frac{2\pi}{k} \left[|\phi_1|^2 + \frac{\mu k}{2\pi} \right].
\end{aligned} \tag{4.36}$$

In other words, the $a_0^{(i)}$ are completely specified by the scalar fields ϕ_i and spatial components of the abelian gauge fields $a_a^{(i)}$, $a = 1, 2$. Consequently, the (temporal) gauge $a_0^{(i)} = 0$ would be inconsistent with the BPS equations. This is different from the Abelian-Higgs model, where the one can set both $a_0 = 0$ and $\partial_0 = 0$, reducing the system to a two dimensional one (for the spatial components). Here, this would be inconsistent with the

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Gauss law constraint which, for a Chern-Simons gauge field, relates F_{12} to terms with D_0Q and D_0R so that, if F_{12} is nonzero, so too is D_0Q and D_0R . Finally, the Gauss law constraints in the BPS case reduce to

$$\begin{aligned} f_{12}^{(1)} &= -\frac{8\pi^2}{k^2} |\phi_2|^2 \left(|\phi_1|^2 + \frac{\mu k}{2\pi} \right), \\ f_{12}^{(2)} &= \frac{8\pi^2}{k^2} |\phi_1|^2 \left(|\phi_2|^2 - \frac{\mu k}{2\pi} \right). \end{aligned} \tag{4.37}$$

In order to facilitate the rest of the analysis of the BPS system, it will prove useful to complexify the (x_1, x_2) -plane and write $z = x^1 + ix^2$. As usual, this induces a complexification of the derivatives as well as the gauge fields as,

$$\partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad a^{(i)} \equiv a_z^{(i)} = \frac{1}{2}(a_1^{(i)} - ia_2^{(i)}),$$

together with their complex conjugates. This, in turn, implies that $f_{12}^{(i)} = -2i[\partial\bar{a}^{(i)} - \bar{\partial}a^{(i)}]$, so that the BPS equations $D_-Q = D_+R = 0$ become simply

$$\partial\phi_1 - ia^{(1)}\phi_1 = 0, \quad \bar{\partial}\phi_2 - i\bar{a}^{(2)}\phi_2 = 0. \tag{4.38}$$

Equations (4.38) together with the Gauss law constraints constitute a complete set which, as we argue below, possess at least one simple set of finite energy, spatially localized solutions of the vortex type *i.e. isolated zeros of the (complex) scalar fields with nonvanishing winding number*. The analysis follows the same general logic as for the Nielsen-Olesen vortex and we start by writing

$$\phi_1 = |\phi_1|e^{i\theta_1}; \quad \phi_2 = |\phi_2|e^{i\theta_2} \tag{4.39}$$

and then (4.38) become (after taking derivatives and making the combinations $f_{12}^{(i)}$)

$$\begin{aligned} f_{12}^{(1)} &= 2\partial\bar{\partial} \ln |\phi_1|^2 - 2i(\partial\bar{\partial} - \bar{\partial}\partial)\theta_1 = \frac{1}{2}\Delta \ln |\phi_1|^2 + \epsilon^{ab}\partial_a\partial_b\theta_1, \\ -f_{12}^{(2)} &= 2\partial\bar{\partial} \ln |\phi_2|^2 + 2i(\partial\bar{\partial} - \bar{\partial}\partial)\theta_2 = \frac{1}{2}\Delta \ln |\phi_2|^2 - \epsilon^{ab}\partial_a\partial_b\theta_2. \end{aligned} \tag{4.40}$$

But since, if α is the polar angle in the 1, 2 plane, $\epsilon^{ab}\partial_a\partial_b\alpha = 2\pi\delta^2(x)$ (as can be checked by integrating over a circle of vanishingly small radius), we may take the ansatz

$$\theta_1 = -N_1\alpha, \quad \theta_2 = N_2\alpha, \tag{4.41}$$

which leads to

$$\begin{aligned} f_{12}^{(1)} &= \frac{1}{2} \Delta \ln |\phi_1|^2 - 2\pi N_1 \delta^2(x), \\ -f_{12}^{(2)} &= \frac{1}{2} \Delta \ln |\phi_2|^2 - 2\pi N_2 \delta^2(x). \end{aligned} \tag{4.42}$$

Finally, on substituting the Gauss law constraints, we obtain a continuous Toda system with delta functions sources,

$$\begin{aligned} \Delta \ln |\phi_1|^2 &= - \left(\frac{4\pi}{k} \right)^2 |\phi_2|^2 \left(|\phi_1|^2 + \frac{\mu k}{2\pi} \right) + 4\pi N_1 \delta^2(x), \\ \Delta \ln |\phi_2|^2 &= - \left(\frac{4\pi}{k} \right)^2 |\phi_1|^2 \left(|\phi_2|^2 - \frac{\mu k}{2\pi} \right) + 4\pi N_2 \delta^2(x), \end{aligned} \tag{4.43}$$

whose solutions we proceed to analyze.

4.3.1 Asymptotic Analysis of the Toda System

As in the case of the (much simpler) Nielsen-Olesen vortex, the Toda system of equations does not, as far as we are aware, exhibit any closed form analytic solution. Consequently, here too must we resort to topological, asymptotic and numerical analyses to tease out finite energy solutions from it. The argument, fortunately, goes through in much the same way as for the Nielsen-Olesen case: the topological term in the energy

$$\int_{\mathbb{R}^2} dx dy \mu j_0 = \int_{\mathbb{R}^2} dx dy \frac{\mu k}{2\pi} \text{Tr} (F_{12}) = \frac{\mu k}{2\pi} \frac{N(N-1)}{2} \int_{\mathbb{R}^2} dx dy (f_{12}^1 + f_{12}^2),$$

is quantized as usual, since for an abelian gauge field

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} F_{12} dx dy = \frac{1}{2\pi} \oint_C A_\alpha dl = \frac{1}{2\pi} \int_0^{2\pi} A_\alpha d\alpha,$$

and $D_\alpha \phi \rightarrow 0$ at $r \rightarrow \infty$, with $\phi = |\phi| e^{i\theta}$, $D_\mu = \partial_\mu - ia_\mu$ and $\partial_\mu \ln |\phi| \rightarrow 0$, means that $\partial_\alpha \theta^\infty - A_\alpha^\infty = 0$ and consequently

$$\int_{\mathbb{R}^2} F_{12} dx dy = 2\pi \tilde{N}.$$

The corresponding statement for our system (4.38) is that, at infinity

$$\partial_\alpha \theta_i^\infty = a_\alpha^{(i)\infty},$$

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which gives the energy of the BPS state as

$$E(N_1, N_2) = \mu k \frac{N(N-1)}{2} (N_1 + N_2). \quad (4.44)$$

We are now in a position to look at the Toda equations in the asymptotic regions. To obtain the $r \rightarrow 0$ behaviour, we integrate each of them over a very small disk of radius $R \rightarrow 0$, and find that

$$\int dx dy \vec{\nabla} \cdot \vec{\nabla} \ln |\phi_i| = 2\pi N_i, \quad i = 1, 2,$$

which, after using Stokes' theorem, gives

$$R \frac{d}{dr} \ln |\phi_i|_{r=R} = N_i.$$

This expression is easily integrated to show that as $r \rightarrow 0$ each of the scalars exhibits the power law behaviour,

$$|\phi_i| \sim A_i r^{N_i}. \quad (4.45)$$

This is in accordance with the usual argument says that the only possibility for the vortices with $\phi = |\phi| e^{iN\alpha}$ is to have $|\phi| \rightarrow 0$ at $r \rightarrow 0$ in order that the phase is well defined at $r = 0$. In fact we can do better and refine the conditions at $r \rightarrow 0$ by using the equations of motion away from $r = 0$. Taking as an ansatz for the scalars

$$|\phi_1|^2 = A_1 r^{2N_1} (1 + B_1 r^p), \quad (4.46)$$

$$|\phi_2|^2 = A_2 r^{2N_2} (1 + B_2 r^q),$$

and substituting into the Toda equations, we find

$$\begin{aligned} q &= 2N_1 + 2, \quad p = 2N_2 + 2, \\ B_1 &= \frac{8\pi\mu}{kp^2} A_2, \\ B_2 &= -\frac{8\pi\mu}{kq^2} A_1. \end{aligned} \quad (4.47)$$

The constants A_i are only determined from the full numerical solution.

At $r \rightarrow \infty$, we can first check that neither a constant, nor a decaying exponential, nor a power law that blows up, $\phi \sim r^p$ works for either of the two fields. This leaves a decaying power law as the only plausible behaviour for either of the two scalar fields. In

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order to use the equations above, we must consider also the first subleading terms, *i.e.* we substitute

$$\begin{aligned} |\phi_1|^2 &= \frac{\bar{A}_1}{r^m} \left(1 + \frac{\bar{B}_1}{r^p} \right), \\ |\phi_2|^2 &= \frac{\bar{A}_2}{r^n} \left(1 + \frac{\bar{B}_2}{r^q} \right), \end{aligned} \tag{4.48}$$

into the equations above, to find

$$\begin{aligned} m &= q + 2, \quad n = p + 2, \\ \bar{B}_1 &= - \left(\frac{8\pi\mu}{p^2 k} \right) \bar{A}_2, \\ \bar{B}_2 &= \left(\frac{8\pi\mu}{q^2 k} \right) \bar{A}_1. \end{aligned} \tag{4.49}$$

Since $p, q \in \mathbb{N}_*$, $m, n = 3, 4, 5, \dots$. Again, the constants \bar{A}_1, \bar{A}_2 , as well as m, n are determined from the full numerical solutions.

4.3.2 Numerical Analysis of the Toda System

To determine the various parameters of the vortex-like solutions described above, we need to solve the Toda system numerically. As in the asymptotic analysis above, our numerical solution follows the general logic of the Abelian-Higgs model. Specifically, we will use a modified *two-parameter shooting method* to numerically solve the two-point boundary value problem described by the coupled Toda equations. To facilitate the implementation of the shooting algorithm, we first rewrite the equations as a four-dimensional (non-autonomous) dynamical system. To this end, we first non-dimensionalize the system by rescaling our variables and defining

$$\begin{aligned} g &\equiv \frac{2\pi}{\mu k} |\phi_1|^2, \\ f &\equiv \frac{2\pi}{\mu k} |\phi_2|^2, \\ R &\equiv \frac{r}{2\mu}, \end{aligned} \tag{4.50}$$

with r taken to be the radial coordinate on the plane. Substituting into the system (4.44) and assuming that the solitons that we are looking for are rotationally symmetric on the

4.3 New vortex solutions for a Toda system

plane (so that the two-dimensional Laplacian is $\Delta = \frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \right)$), we find that eqs.(4.44) reduce to

$$\frac{1}{R} \frac{d}{dR} \left(\frac{R}{f} \frac{df}{dR} \right) = -g(f-1), \tag{4.51}$$

$$\frac{1}{R} \frac{d}{dR} \left(\frac{R}{g} \frac{dg}{dR} \right) = -f(g+1).$$

Finally, we reduce the order of the system by one by making the additional definitions $h \equiv \frac{df}{dR}$ and $j \equiv \frac{dg}{dR}$ so that (denoting by a ' derivatives with respect to the dimensionless radial variable R)

$$\begin{aligned} f' &= h, \\ h' &= \frac{h^2}{f} - \frac{h}{r} - gf(f-1), \\ g' &= j, \\ j' &= \frac{j^2}{g} - \frac{j}{r} - gf(g+1). \end{aligned} \tag{4.52}$$

Before directly integrating this four-dimensional non-autonomous dynamical system, it will be instructive to extract some qualitative information from it. There are two (physical) fixed points at $(f, h, g, j) = (0, 0, 0, 0)$ and $(1, 0, 0, 0)$. A linearization of the system near the former, shows that the origin is a saddle. Solutions of the kind that carry nonvanishing winding number and conform to the asymptotic boundary conditions $f(0) = g(0) = 0$ and $f(\infty) = g(\infty) = 0$ correspond to *homoclinic* orbits² that begin and end at $(0, 0, 0, 0)$ and that encircle the fixed point at $(1, 0, 0, 0)$ (see Fig.1).

Our numerical integration of the system is based on a two-parameter shooting algorithm that converts the nonlinear dynamical system above into a nonlinear parameter estimation problem. The parameters in question are precisely the undetermined constants A_1 and A_2 above and these are chosen at $R = 0$ so that the *constraint* $f(\infty) = g(\infty) = 0$ is met. In practice, the constraints at $R = \infty$ are a problem, but our asymptotic analysis above can be extended to show that solutions at $R \approx 10$ are quite safely in the far field for both f and g . Some results of our numerical integration are presented in Figures 2 and 3.

We also obtain from the numerics that the power law at infinity is $|\phi_1|^2 \propto 1/r^3$, $|\phi_2|^2 \propto 1/r^2$, i.e. $m = 3, n = 2$.

²This should be compared to the standard ANO vortices of the Abelian-Higgs model which correspond to *heteroclinic* orbits interpolating between the two fixed points of the associated dynamical system.

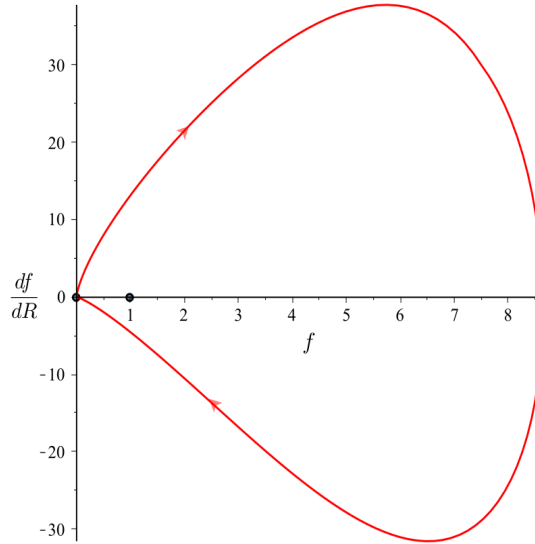


Figure 4.1: The (f, f') -subspace of the full phase space of the Toda system

As a final point, we note that in the massless ABJM case, at $\mu = 0$, these vortices vanish since their energy is proportional to μ . This agrees well with known facts about the solitonic spectrum of pure ABJM [41].

4.4 The Abelian-Higgs model from ABJM

We now look to embed the Abelian-Higgs model in ABJM, as a truncation of our general abelianization ansatz. To find the truncation we look at the multi-vortex solution we found previously in [62] for the $N = 2$ case, i.e. $U(2) \times U(2)$ ABJM. There, not only was the ansatz written in a manner similar to the multi-vortices of the conventional Abelian-Higgs model, but the action on the moduli space of vortices was also found to be the same. In retrospect, this was really a telling signal that we were actually embedding the Abelian-Higgs model into ABJM. For the reader unfamiliar with [62], we recall that the

4.4 The Abelian-Higgs model from ABJM

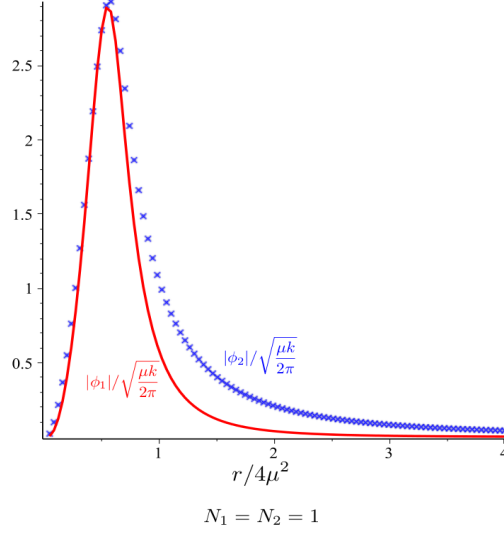


Figure 4.2: The $N_1 = N_2 = 1$ soliton profiles. Optimization of the shooting parameters yeild $A_1 = 30.00, A_2 = 30.05$

static multivortex solution there was given by

$$\begin{aligned}
 C^1 &= \sqrt{\frac{k\mu}{\pi}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \sqrt{\frac{k\mu}{\pi}} G^1, \\
 C^2 &= \sqrt{\frac{k\mu}{\pi}} e^{-\psi/2} H_0(z) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sqrt{\frac{k\mu}{\pi}} e^{-\psi/2} H_0(z) G^2; \quad C^3 = C^4 = 0, \quad (4.53) \\
 A_0 &= \frac{1}{\mu} \begin{pmatrix} \partial\bar{\partial}\psi & 0 \\ 0 & 0 \end{pmatrix}; \quad \hat{A}_0 = \frac{1}{\mu} \begin{pmatrix} 0 & 0 \\ 0 & \partial\bar{\partial}\psi \end{pmatrix}; \quad A_{\bar{z}} = \hat{A}_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{i}{2}\bar{\partial}\psi \end{pmatrix},
 \end{aligned}$$

where $H_0(z) = \prod_{i=1}^n (z - z_i)$ is an arbitrary polynomial and the real function $\psi(z)$ is determined through the equation

$$\partial\bar{\partial}\psi = \mu^2 (1 - e^{-\psi} |H_0(z)|^2) \quad (4.54)$$

with boundary conditions at $|z| \rightarrow \infty$ requiring $\psi \rightarrow \log |H_0(z)|^2$. As usual, z_i with $i = 1 \dots n$, denotes the positions of the n vortices. Treating each of these position variables

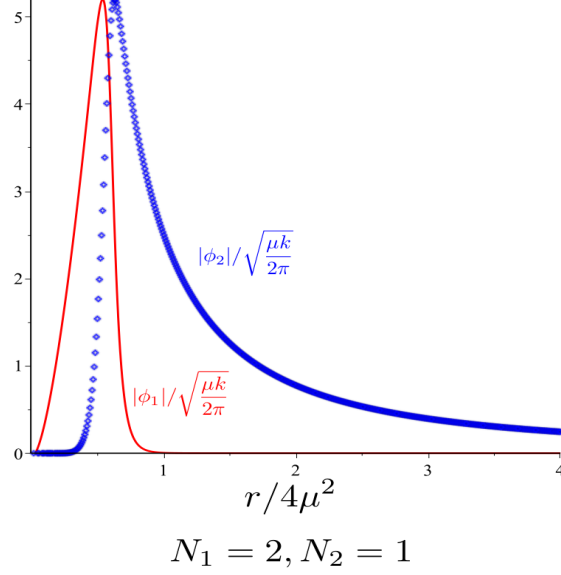


Figure 4.3: The $N_1 = 2, N_2 = 1$ soliton profiles. Optimization of the shooting parameters yield $A_1 = 50.00, A_2 = 100.00$

as (adiabatic) functions of time, $z_i(t)$, produces the first order solution

$$\begin{aligned}
 C^I &= 0, \\
 A_0^{(1)} &= \hat{A}_0^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{i}{2}(z^{\bar{i}}\partial_i - \bar{z}^{\bar{i}}\bar{\partial}_{\bar{i}})\psi \end{pmatrix}, \\
 A_{\bar{z}}^{(1)} &= \begin{pmatrix} \frac{1}{2\mu}z^{\bar{i}}\partial_i\partial\psi & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{A}_{\bar{z}}^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2\mu}z^{\bar{i}}\partial_i\partial\psi \end{pmatrix},
 \end{aligned} \tag{4.55}$$

on the moduli space of the vortices. On the other hand, when $N = 2$, our general abelianization ansatz gives

$$\begin{aligned}
 A_\mu &= a_\mu^{(1)}G^1G_1^\dagger + a_\mu^{(2)}G^2G_2^\dagger = \begin{pmatrix} a_\mu^{(1)} & 0 \\ 0 & a_\mu^{(2)} \end{pmatrix}, \\
 \hat{A}_\mu &= a_\mu^{(1)}G_1^\dagger G^1 + a_\mu^{(2)}G_2^\dagger G^2 = \begin{pmatrix} 0 & 0 \\ 0 & a_\mu^{(1)} + a_\mu^{(2)} \end{pmatrix}.
 \end{aligned} \tag{4.56}$$

Comparing with the solution above (and also denoting now the first order solution for the abelian fields with a tilde to avoid confusion with the indices (1) and (2) on the a 's) we

4.4 The Abelian-Higgs model from ABJM

find

$$\begin{aligned}
a_{\bar{z}}^{(2)} &= \frac{i}{2} \bar{\partial} \psi, & a_0^{(1)} &= \frac{1}{\mu} \partial \bar{\partial} \psi, & a_a^{(1)} &= a_0^{(2)} = 0, \\
\tilde{a}_{\bar{z}}^{(1)} &= \frac{1}{2\mu} \dot{z}^{\bar{i}} \partial_{\bar{i}} \partial \psi, \\
\tilde{a}_0^{(2)} &= -\frac{i}{2} (\dot{z}^{\bar{i}} \partial_{\bar{i}} - \dot{\bar{z}}^{\bar{i}} \bar{\partial}_{\bar{i}}) \psi.
\end{aligned} \tag{4.57}$$

Note that from (4.57), by taking complex conjugate and then sums and differences, we get

$$\begin{aligned}
\tilde{a}_1^{(1)} &= \frac{1}{2\mu} \left[(\dot{z}^{\bar{i}} \partial_{\bar{i}} + \dot{\bar{z}}^{\bar{i}} \bar{\partial}_{\bar{i}}) \partial_1 - i (\dot{z}^{\bar{i}} \partial_{\bar{i}} - \dot{\bar{z}}^{\bar{i}} \bar{\partial}_{\bar{i}}) \partial_2 \right] \psi, \\
\tilde{a}_2^{(1)} &= -\frac{i}{2\mu} \left[(\dot{z}^{\bar{i}} \partial_{\bar{i}} - \dot{\bar{z}}^{\bar{i}} \bar{\partial}_{\bar{i}}) \partial_1 - i (\dot{z}^{\bar{i}} \partial_{\bar{i}} + \dot{\bar{z}}^{\bar{i}} \bar{\partial}_{\bar{i}}) \partial_2 \right] \psi.
\end{aligned} \tag{4.58}$$

This can be written more compactly, by using the fact that ψ is a real-valued field, as

$$\tilde{a}_i^{(1)} = \frac{i}{2\mu} \epsilon_{ij} \left(\dot{z}^{\bar{k}} \partial_{\bar{k}} - \dot{\bar{z}}^{\bar{k}} \bar{\partial}_{\bar{k}} \right) \partial_j \psi. \tag{4.59}$$

In view of the above solution, and assuming that the same relation to our abelianization holds at all N , we can now identify the truncation ansatz needed to obtain the abelian-Higgs model as

$$\phi_1 = \phi_2 = 0, \chi_1 = b = \text{constant}, \tag{4.60}$$

which gives

$$\begin{aligned}
D_\mu \phi_1 &= D_\mu \phi_2 = 0, \\
D_\mu \chi_1 &= -i a_\mu^{(1)} b \\
D_\mu \chi_2 &= (D_\mu - i a_\mu^{(2)}) \chi_2
\end{aligned} \tag{4.61}$$

and the potential

$$\begin{aligned}
V &= \frac{2\pi^2}{k^2} N(N-1) [|b|^2 (|\chi_2|^2 - c^2)^2 + |\chi_2|^2 (|b|^2 - |c|^2)^2], \\
&= \frac{2\pi^2}{k^2} N(N-1) [|b|^2 |\chi_2|^4 + |\chi_2|^2 (-4|b|^2 c^2 + |b|^4 + c^4) + c^4 |b|^2].
\end{aligned} \tag{4.62}$$

We can easily arrange for the coefficient of the $|\chi|^2$ term to be negative, as is required for the mexican hat potential of the abelian-Higgs model, by choosing for instance

$$|b| = |c| \Rightarrow \mu = \frac{2\pi |b|^2}{k}. \tag{4.63}$$

Abelian-Higgs and Vortices from ABJM: towards a string realization of AdS/CMT

The action is then

$$S = -\frac{N(N-1)}{2} \int d^3x \left[\frac{k}{2\pi} \epsilon^{\mu\nu\lambda} a_\mu^{(1)} f_{\nu\lambda}^{(2)} + (a_\mu^{(1)})^2 |b|^2 + |D_\mu \chi_2|^2 + V \right], \quad (4.64)$$

with the auxiliary field $a_\mu^{(1)}$. As usual it can be eliminated through its equation of motion

$$a_\mu^{(1)} = -\frac{k}{4\pi|b|^2} \epsilon^{\mu\nu\lambda} f_{\nu\lambda}^{(2)}. \quad (4.65)$$

so that

$$S = -\frac{N(N-1)}{2} \int d^3x \left[\frac{k^2}{8\pi^2|b|^2} (f_{\mu\nu}^{(2)})^2 + |D_\mu \chi_2|^2 + V \right], \quad (4.66)$$

which is nothing but the action of the abelian-Higgs model.

Of course, we still need to check the consistency of the truncation, i.e. to check that the equations of motion of the full abelianization ansatz in section 2 are satisfied. We have fixed ϕ_1, ϕ_2 to zero and χ_1 to b , so it is these three equations of motion that we need to check. As before, the choice $\phi_1 = \phi_2 = 0$ is a consistent truncation. The equation for χ_1 reduces, in the Lorentz gauge $\partial^\mu a_\mu^{(2)} = 0$, and using (4.65), to

$$(a_\mu^{(1)})^2 b = b \frac{\partial V}{\partial |b|^2} \quad (4.67)$$

which is just the equation of motion we would obtain for the parameter b by varying in the abelian-Higgs action (4.66). We find this somewhat puzzling, since it means that the *constant parameter* $|b|$ has to be effectively treated like a field in the abelian-Higgs action, giving its own equation of motion.

It remains now to check that our multivortex solution satisfies the condition (4.65), since it certainly matched our ansatz *before* we imposed the equation of motion for $a_\mu^{(1)}$. The equations (4.65) reduce for the zeroth order solution and the first order solution respectively, to

$$\begin{aligned} a_0^{(1)} &= \frac{k}{4\pi|b|^2} \epsilon^{ij} \partial_i a_j^{(2)} \\ \tilde{a}_i^{(1)} &= -\frac{k}{4\pi|b|^2} \epsilon^{ij} \partial_j \tilde{a}_0^{(2)} \end{aligned} \quad (4.68)$$

We can check the first equation, since $a_{\bar{z}}^{(2)} = \frac{i}{2} \bar{\partial} \psi$, which written in real components reads $\tilde{a}_i^{(2)} = -\frac{1}{2} \epsilon_{ij} \partial_j \psi$. Then the zeroth order equation is satisfied if

$$\mu = \frac{2\pi|b|^2}{k} \quad (4.69)$$

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while the first order equation is satisfied when

$$\tilde{a}_i^{(1)} = \frac{i}{2\mu} \epsilon_{ij} (\dot{z}^{\bar{k}} \partial_{\bar{k}} - \dot{\bar{z}}^{\bar{k}} \bar{\partial}_{\bar{k}}) \partial_j \psi \quad (4.70)$$

as expected. The appearance of the abelian-Higgs model above is somewhat non-standard but can easily be put into canonical form by appropriately normalizing the fields as

$$a^{(2)} = \frac{2\pi b}{Nk} \tilde{a}^{(2)}, \quad \chi_2 = \frac{\tilde{\chi}_2}{N},$$

to obtain

$$S = \int d^3x \left[-\frac{1}{4} \left(\tilde{f}_{\mu\nu}^{(2)} \right)^2 - |D_\mu \tilde{\chi}_2|^2 - V \right] \quad (4.71)$$

where now $D_\mu = \partial_\mu - ig\tilde{a}_\mu^{(2)}$ and $g = \frac{2\pi|b|}{Nk}$. In terms of the canonical fields and coupling, the potential

$$V = \frac{g^2}{2} \left[|\tilde{\chi}_2|^4 + \frac{\mu^2 k^2 N^4}{4\pi^2} + |\tilde{\chi}_2|^2 N^2 \left(-\frac{4\mu k}{2\pi} + |b|^2 + \frac{\mu^2 k^2}{4\pi^2 |b|^2} \right) \right]. \quad (4.72)$$

As previously alluded to, the potential has a range of values of $|b|$ for which it is spontaneously breaking (has negative mass squared). The central value of this domain is $|b| = c$, and for this value of $|b|$, we find

$$V = \frac{g^2}{2} \left[|\tilde{\chi}_2|^2 - \frac{\mu k N^2}{2\pi} \right]. \quad (4.73)$$

Moreover, for this value of $|b|$, the equation of motion for $|b|^2$ (the extra constraint on our abelian-Higgs model), becomes

$$[|\phi_2|^2 - c^2]^2 = \frac{1}{2} \left[\frac{k}{2\pi\mu} f_{\mu\nu}^{(2)} \right]^2. \quad (4.74)$$

On the other hand, equating the kinetic (Maxwell) term for $f_{\mu\nu}^{(2)}$ with the potential term, V , gives exactly the same equation. Further, taking the square root of this equation, and imposing that $f_{0i} = 0$, we find

$$\frac{1}{Ng} f_{12}^{(2)} = \pm [|\tilde{\chi}_2| - N^2 c^2], \quad (4.75)$$

which is part of the abelian-Higgs BPS condition. In other words, the extra condition is satisfied on BPS solutions of the abelian-Higgs model with $f_{0i} = 0$, and in particular for vortices.

Abelian-Higgs and Vortices from ABJM: towards a string realization of AdS/CMT

Having established the consistent truncation to the Landau-Ginzburg model of interest, as explained in the introduction, we still need to establish the conditions under which we can decouple the nonzero modes. From the potential (4.72) we find that generically the mass term has $m \sim \mu$ (for instance for the central value $|b|^2 = \mu k / (2\pi)$), whereas it vanishes at

$$|b|^2 = \frac{\mu k}{2\pi} (2 \pm \sqrt{3}), \quad (4.76)$$

so, for values close to these, the mass of $\tilde{\chi}_2$ can be made much smaller than μ . The coupling g^2 is generically, for instance close to the central value of $|b|^2$,

$$g^2 = \frac{4\pi^2 |b|^2}{N^2 k^2} \sim \mu \frac{2\pi}{N^2 k}. \quad (4.77)$$

If we choose large N (as is required for the gravity dual) and $k \sim 1$, we see that $g^2 \ll \mu$. The constant term is large in that case, for instance for near massless $\tilde{\chi}_2$,

$$\frac{1}{2} \frac{g^2 \mu^2 k^2 N^4}{4\pi^2} \simeq \frac{\mu k N^2}{4\pi} (2 \pm \sqrt{3}). \quad (4.78)$$

Since we are in a non-gravitational theory here, this cannot be measured and, consequently, it does not matter.

It is also possible to analyze the various terms in the ABJM action to see which of them are quadratic in the nonzero modes since, these will be the terms responsible for the simplest quantum loops. We find, using the ansatz for the “zero mode” fields that give the LG action, and considering the nonvanishing $\delta\phi$ as generic (ij) modes in the $N \times N$ matrices,

$$\begin{aligned} \frac{4\pi^2}{3k^2} \text{Tr} [C^6] &\sim \frac{1}{k^2 N^2} \tilde{\chi}_2^4 (\delta\phi)^2 \propto \frac{1}{N^2}, \\ \frac{k}{4\pi} \text{Tr} [A^3] &\sim k N a (\delta\phi)^2 \propto N, \\ \text{Tr} [D_\mu C^I D^\mu C_I] &\sim (\partial_\mu \delta\phi)^2; \quad \text{Tr} [\psi^\dagger \not{D} \psi] \sim \delta\psi^\dagger \not{\partial} \delta\psi, \\ \frac{2\pi}{k} \text{Tr} [C^\dagger C \psi^\dagger \psi] &\sim \frac{1}{k N} (\tilde{\chi}_2)^2 \delta\psi^\dagger \delta\psi \propto \frac{1}{N}, \\ \frac{2\pi\mu}{k} \text{Tr} [Q^\dagger Q R^\dagger R] &\sim \frac{\mu}{k N} (\tilde{\chi}_2)^2 (\delta\phi)^2 \propto \frac{\mu}{N}. \end{aligned} \quad (4.79)$$

Evidently then, the Chern-Simons term generates a term with large coupling, and the mass term generates a term $\propto \mu/N \ll \mu$, but still $\gg \mu/N^2 \sim g^2$. These couplings cannot be made small; leading us to the situation that we advertised: the masses of the nonzero modes are much larger than the mass parameters of the reduced theory, while the couplings remain relatively large.

4.4 The Abelian-Higgs model from ABJM

Nevertheless, we still need to show that the modes of the reduced theory are the only light ones in the theory or, if they are not, that any additional light modes do not couple to ours. To this end, let's start with the other modes in (4.22). Rescaling to canonically normalized fields,

$$\chi_i = \frac{\tilde{\chi}_i}{N}; \quad \phi_i = \frac{\tilde{\phi}_i}{N}; \quad a^{(2)} = \frac{2\pi b}{Nk} \tilde{a}^{(2)}, \quad (4.80)$$

we find the following in the absence of a Higgs VEV:

- Sextic terms in the scalars go like $\sim 1/N^4 k^2 \rightarrow 0$,
- Quartic terms go like $\sim c^2/N^2 k^2 \sim \mu/(N^2 g) \sim g^2$, and
- Mass terms go like $\sim \mu^2$.

As claimed, they are generically heavy. All that remains then is to check what happens in the presence of the Higgs VEV $\chi_1 = b$. In this case we obtain the extra terms

$$\frac{1}{k^2} [-4b^2 c^2 |\tilde{\chi}_2|^2 + b^4 (|\tilde{\chi}_2|^2 + |\tilde{\phi}_2|^2)], \quad (4.81)$$

so that only the $\tilde{\chi}_2$ mode can become light; all others remain massive. What about generic modes outside the action (4.22)? We already saw that generic mass terms are of order $m^2 \sim \mu^2 > 0$, so it only remains to see that the terms coming from the Higgs VEV $\chi_1 = b$ cannot cancel them. Thus we search for solutions to the vanishing of the mass term coming from the ABJM action, where we only keep two C^I 's general in each term, and the rest we write as

$$C^I = (R^1 = bG^1, R^2 = Q^1 = Q^2 = 0). \quad (4.82)$$

Setting this mass term to zero produces a very long equation, for the trace of a sum of terms with two C^I matrices and up to four G^1 matrices being 0. One solution of this equation is given by our light mode

$$C^I = (R^2 = \chi_2 G^2, R^1 = Q^1 = Q^2 = 0); \quad b^2 = c^2 (2 \pm \sqrt{3}), \quad (4.83)$$

and is equivalent to an identity between G^1 and G^2 matrices after the ansatz has been considered. The issue is whether or not the solution is unique. While we don't know a mathematical proof of uniqueness, physically it is clear it should be so. Indeed, the solution is related to the existence of the maximally supersymmetric fuzzy sphere vacuum characterized by G^1, G^2 ; once we turn on G^1 , there is an instability towards turning on G^2 as well, apparent in the fact that the mass of χ_2 can go through zero and become negative.

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Any other solution would amount to the statement that there is another vacuum with G^1 turned on (corresponding to a different instability in the presence of $R^1 = bG^1$). As there is no other vacuum connected in this way to the maximally supersymmetric one, we conclude that there should be no other solution to the zero mass equation. Hence there are no other light modes in the presence of the Higgs VEV $\chi_1 = b$. Of course, there can be other light modes in other regions of parameter space, but all we need is that for large N , $k \sim 1$ and the only VEV turned on being $R^1 = bG^1$ we don't have other light modes, and we have argued this is indeed the case.

This completes our demonstration that (i) the abelian Higgs model can be obtained from the abelianization of the ABJM model as a quantum consistent truncation and (ii) that both the classical zeroth order and the first order (in the moduli space approximation) multivortex solutions of the latter at $N = 2$, are encoded in this model. Since this is a *bone fide* embedding of the abelian-Higgs model, we say say more even. For instance, it is natural that we obtain the same fluctuation action for vortex scattering as in the abelian Higgs case. It also means that we can now immediately write down the multivortex solution at general N , with the guarantee that we will recover the same fluctuation action for vortex scattering as in the abelian Higgs case. To be concrete, the multivortex solution at general N in ABJM

$$\begin{aligned} R^1 &= \sqrt{\frac{k\mu}{\pi}} G^1, \quad R^2 = \sqrt{\frac{k\mu}{\pi}} e^{-\psi/2} H_0(z) G^2, \quad Q^1 = Q^2 = 0, \\ A_0 &= \frac{1}{\mu} \partial \bar{\partial} \psi G^1 G_1^\dagger, \quad \hat{A}_0 = \frac{1}{\mu} \partial \bar{\partial} \psi G_1^\dagger G^1, \\ A_{\bar{z}} &= \frac{i}{2} \bar{\partial} \psi G^2 G_2^\dagger, \quad \hat{A}_{\bar{z}} = \frac{i}{2} \bar{\partial} \psi G_2^\dagger G^2, \end{aligned} \quad (4.84)$$

produces an effective Lagrangian on the moduli space

$$\begin{aligned} L_{eff} &= \frac{N(N-1)k\mu}{2} \frac{1}{\pi} \int d^2x \left[-\partial \bar{\partial} \psi + \frac{1}{2} \dot{z}^i \dot{\bar{z}}^j \left(\partial_i \bar{\partial}_j \psi + \frac{1}{\mu^2} (\partial_i \bar{\partial} \psi \bar{\partial}_j \partial \psi - \partial \bar{\partial} \psi \partial_i \bar{\partial}_j \psi) \right) \right], \\ &\simeq \frac{N(N-1)}{2} \left[-k\mu n + \sum_{i=1}^n \frac{k\mu}{2} |z^i|^2 - k\mu q \sum_{i>j} K_0(2\mu|z^i - z^j|) |\dot{z}^i - \dot{z}^j|^2 \right], \end{aligned} \quad (4.85)$$

with $q \simeq 1.71$. To close this discussion on vortices of the abelian-Higgs model and their embedding into the ABJM model, we mention briefly that in the case of the massless ABJM model, with $c = \mu = 0$, we obtain a non-symmetry breaking potential,

$$V = \frac{2\pi^2}{k^2} N(N-1) [|b|^2 |\chi_2|^4 + |b|^4 |\chi_2|^2], \quad (4.86)$$

which is just a massive gauged ϕ^4 model.

4.5 Towards a string construction of AdS/CMT

At this point, let's stop and consider what it is that we have achieved. Stripping away all the bells and whistles, essentially our truncation has produced a (2+1)-dimensional scalar field theory with potential

$$V = \frac{2\pi^2}{k^2} N(N-1) [|b|^2 |\phi|^4 + |\phi|^2 ((|b|^2 - c^2)^2 - 2c^2 |b|^2) + c^4 |b|^2] . \quad (4.87)$$

It is not too difficult to see that it is just a *Landau-Ginzburg model* in which, at fixed $|b|^2$, $c^2 \propto \mu$ acts as a coupling that takes us from a $|\phi|^4$ theory (the insulator phase) to an abelian-Higgs theory (the superconducting phase). In this sense, the parameters $|b|^2$ and c^2 control the coupling g and critical coupling g_c of the Landau-Ginzburg model. More precisely, we identify the combinations $(|b|^2 - c^2)^2$ as g and $2c^2 |b|^2$ as g_c respectively. In this light, it makes sense then to think of this abelianization as *a realization of the recently proposed AdS/CMT correspondence*. To see why our construction is markedly different from any of its pre-cursors, we recall the general ideas involved. Usually, in an AdS/CMT construction, one assumes some theory in an AdS background, usually involving gravity, a gauge field A_μ , maybe a complex (charged) scalar ϕ and some fermions ψ_i . It is then argued that this theory should be dual to some large N conformal field theory with a global current J_μ dual to the gauge field A_μ , and some other operators (in principle) dual to the other fields. It is then argued that relevant physics in AdS corresponds to some behaviour of the operators in the field theory which simulates the relevant physics, like superconductivity [61] for example, to be studied. Sometimes the AdS theory is obtained as a consistent truncation of some known AdS/CFT duality (for which there is a heuristic derivation involving a decoupling limit of some brane constructions), so that the field theory contains a small subset of operators that could possibly give the desired physics [8, 9].

However, even in these cases, it is not obvious how to directly relate the set of operators in the given CFT to the condensed matter system of interest, and usually one has to invoke some sort of universality argument. In other words, if the physics of the selected set of operators in the large N CFT describes the correct physics for the condensed matter system, then perhaps the physics is general enough to appear in many different systems, and we can try to apply our seemingly unrelated field theory to the condensed matter system of interest. While we certainly appreciate the logic of this argument, we find it less than satisfactory for a number of reasons. Primary among these is that it is not at all clear why can we choose only a very small number of operators in the large N CFT and concentrate on their physics. Secondly, if we try to write down a gravity dual of an

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abelian theory having this small number of nontrivial operators, we would fail, since the absence of the large N would mean that we could not focus on the supergravity limit in the dual.

However, we can now do better. We have found a *consistent truncation* of the large N CFT, for which there is a well-defined duality, and not just a truncation of the gravity theory. That means that this set of fields is a well defined subset at the *quantum level*³ corresponding to the collective motion of the nonabelian fields in the large N case and involving $\mathcal{O}(N)$ out of the $\mathcal{O}(N^2)$ fields of ABJM, via the nontrivial matrices G^α (which have $\mathcal{O}(N)$ nonzero elements). It is not just a simple restriction to $N = 1$ of the ABJM model, which would imply losing the supergravity limit in the dual. Therefore this abelianization still maps to a purely gravitational theory, and not a full string theory as for generic abelian theories.

We should note that the potential (4.62) for $|b| = c$ has a minimum (vacuum) at $\chi_2 = |b| = c$, which is nothing but the fuzzy sphere vacuum of the massive ABJM model, and hence classical solutions of the reduced theory (LG) can be understood as some type of deformations of the fuzzy sphere. We will see other examples of similar classical solutions in the next section. Therefore all of these solutions, representing a collective motion of $\mathcal{O}(N)$ fields, correspond to finite deformations of the gravity dual, unlike any solutions that only turn on one mode. In this sense, as already explained, the property of classical gravity dual related to large N is still preserved by our abelianization.

In our case, there already exists a well defined gravity dual of the field theory. In the case of massless ABJM, that theory corresponds to M2-branes moving in the space $\mathbb{R}^{2,1} \times \mathbb{C}^4/\mathbb{Z}_k$, and the gravity dual (*i.e.* the near-horizon limit of the backreacted background) is $AdS_4 \times \mathbb{CP}^3$. In the case of the massive ABJM, the theory corresponds to M2-branes moving in a space defined in [62, 68] with the gravity dual described in [43, 62]. Of course, we still would need to understand to what the truncation to $\langle \chi_1 \rangle = b$ and $\chi_2 \neq 0$ corresponds in this gravity dual in order to complete the picture, but we leave this for further work.

Actually, as it turns out, the theory we obtain in the abelianization is also the relevant effective theory for a CMT construction. Indeed, as reviewed for instance in [66], starting from the Hubbard model for spinless bosons hopping on a lattice of sites i with short range repulsive interactions,

$$H_b = -w \sum_{\langle ij \rangle} (b_i^\dagger b_j + b_j^\dagger b_i) + \frac{U}{2} \sum_i n_i(n_i - 1) - \mu \sum_i n_i, \quad (4.88)$$

³We can consistently put the other fields to zero even at the quantum level.

4.5 Towards a string construction of AdS/CMT

where $n_i = b_i^\dagger b_i$ and w is the hopping matrix between nearest-neighbour sites, one obtains the relativistic Landau-Ginzburg theory

$$S = \int d^3x \left(-|\partial_t \phi|^2 + v^2 |\vec{\nabla} \phi|^2 + (g - g_c) |\phi|^2 + u |\phi|^4 \right). \quad (4.89)$$

The effective field ϕ is obtained as follows. The ground state contains an equal number of bosons at each site, with the creation operators a_i^\dagger producing extra particles at each site, and creation operators h_i^\dagger that produce extra “holes” at each site; “antiparticles” in the QFT picture. Then, as is usual in field theory, $\phi_i \sim \alpha_i a_i + \beta_i h_i^\dagger$ is a discretized version of the complex field describing both particles and antiparticles, where α_i, β_i are wavefunctions for the modes.

For $g < g_c$ we have an abelian-Higgs system, i.e. a superconducting phase, while for $g > g_c$ we have an insulator phase. The marginal case $g = g_c$ is a conformal field theory. The systems described by the above model also have a quantum critical phase which opens up at nonzero temperature for a T -dependent window around $g = g_c$. This quantum critical phase is strongly coupled and very hard to describe using conventional condensed matter methods, which makes it an excellent choice for a holographic description. In [6] it was shown that by considering a gauge field in the gravity dual of ABJM and introducing a coupling for it to the Weyl curvature, $\gamma \int C_{abcd} F^{ab} F^{cd}$ one obtains a conductivity $\sigma(\omega)$ consistent with the quantum critical phase, and from which it was concluded that ABJM is a good primer for these systems, though the precise reason for the match was not obvious.

While the bosonic Hubbard model leads, in the continuum limit to the action (4.89), the model itself is a drastic simplification, of a condensed matter system. The model has been used to describe the quantum critical phase of (bosonic) ^{87}Rb cold atoms on an optical lattice, but the description is believed to hold more generally for the quantum critical phase. For instance, high T_c superconductors have a “strange metal” phase that is believed to be of the same quantum critical type. We can consider a solid with free electrons (fermions, perhaps several per atom) that could hop between fixed atoms, and unlike the simple Hubbard model, we also have in principle interactions that are not restricted to nearest neighbours. One could, for instance, generate bosons ϕ_{ij} (having the role of the bosons b_i of the Hubbard model) by coupling fermions at two sites i and j . By an abuse of notation we will call by the same ϕ_{ij} the field obtained by multiplying the corresponding “particle creation” operator with a wavefunction, and adding a corresponding “hole” part.

In fact, we can sketch a simple model for the condensed matter system above that generates the same qualitative picture as the abelianization of the ABJM model. Consider spinless bosons ϕ_{ij} generated by coupling fermions of opposite spins (Cooper pairs) at sites

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i and j with a maximum distance between sites $|i - j| \leq N$, i.e. $\bar{\psi}_i \psi_j$. The resulting ϕ_{ij} can be described by a field $\phi_{i'}^{ab}$, with $a, b = 1, \dots, N$. Since we are in two spatial dimensions, every site has $\mathcal{O}(N^2)$ neighbours a distance $\leq N$ away. Now take the point i' at which the effective field, ϕ_{ij} , lives to be midpoint of the line between i and j , and a, b to correspond to sites j in the x and y directions away from i' (so that, if i' and j are fixed, so is i). Consider that the normalized wavefunctions for the field $\phi_{i'}^{ab}$ give probabilities for existence of the pairing as $\propto |\phi_{i'}^{ab}|^2$ for a pair (ab) . In this case, any transformation on $\phi_{i'}^{ab}$ must be a unitary transformation $U^{ab, a'b'}$ inside $U(N^2)$, up to an overall factor. In particular, any *symmetry* of the system must be of this type. The symmetry of the ABJM model is $U(N) \times U(N)$, and would correspond to $U^{ab, a'b'} = f U^{aa'} V^{bb'}$.

Since the simplest type of condensed matter system is a rotationally invariant one, we should not have any angular dependence, and we should have $\phi_i^{ab} = \phi_i(\sqrt{a^2 + b^2}) = \phi_i^{ba}$. It should be then possible to diagonalize this symmetric matrix, corresponding to considering only the constant (rotationally invariant) $m = 0$ modes for the "spherical harmonics" expansion $e^{2\pi i m \theta}$ at fixed radius $r = \sqrt{a^2 + b^2}$. In this way, only $\mathcal{O}(N)$ modes, specifically those that are spherically symmetric, out of the $\mathcal{O}(N^2)$ modes in the system are turned on. These can be thought of as the eigenvalues of ϕ_i^{ab} .

Since N is the effective maximal radius for coupling of the two fermions at different sites, it makes sense for the wavefunction in the ground state to decrease from a maximum value at $a = 1$ (neighbouring sites) to zero at $a = N$ (sites at distance N). For instance, if the wavefunction $\psi(a)$ is such that $|\psi(a)|^2 \propto N - a$, then the average distance between sites is

$$\langle a \rangle = \frac{\int |\psi(a)|^2 a (2\pi a da)}{\int |\psi(a)|^2 (2\pi a da)} = \frac{N}{2}, \quad (4.90)$$

which is consistent with having a large average distance between the electrons that couple. This form of the wavefunction, $\psi(a) \propto \sqrt{N - a}$, here just a consistent choice, is exactly what we obtain in the ABJM model. Of course, in principle, if we would be able to correctly describe the interactions between various $\psi_{i'}^{ab}$, as in the ABJM model, the dynamics would select the form of $\psi(a)$. Finally, the Hubbard model field b_i must be the linear combination of the spherical modes, i.e. $b_i \sim \sum_a \psi(a) \phi_i^{aa}$.

We have already seen that to obtain the Landau-Ginzburg model from ABJM, we have only one field, χ_2 , turned on corresponding to turning on the matrix $G^2 = \sqrt{N - m} \delta_{m+1, n}$, with $(G^2 G_2^\dagger)_{mn} = (N - m) \delta_{mn}$. As in the simple model above, there are two independent rotations, in this case $U(N) \times U(N)$ rotations, acting on the indices, so the most general solution for the matrix G^2 is in fact $U(\sqrt{N - m} \delta_{mn}) V^{-1}$. We can use these to diagonalize the matrix, thus reducing the degrees of freedom turned on, from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$, as in the above condensed matter model. The ABJM field that is turned on is $\chi_2(G^2)_{mn}$,

corresponding to $b_{i'} \sim \sum_a \psi_a \phi_{i'}^{aa}$.

While this field is the only one turned on in our simple toy condensed matter model, there are, in principle, many more fields. We could, for instance, have more free electrons at each site, thus having more matrix scalars, transforming in some R-symmetry group (in ABJM we have 4 complex scalars, corresponding to $\phi_1, \phi_2, \chi_1, \chi_2$, that transform under the $SU(2) \times SU(2)$ of the mass deformed ABJM). Then, we could also have matrix fermions, corresponding for instance to two electrons at site i coupling with one electron at site j , although such modes are, of course, not turned on in the Hubbard model description. To complete the field content of the ABJM model we need also the Chern-Simons gauge fields, but since those are topological and have no dynamics, we don't need to introduce any new degrees of freedom.

Chern-Simons gauge fields are, of course, no strangers to condensed matter systems, showing up, for instance, in the fractional quantum Hall effect (see for instance the review [67]). An abelian Chern-Simons field can be obtained as follows. Consider a multi-electron wavefunction $\Psi_e(\vec{r}_1, \dots, \vec{r}_k)$ with a generic Hamiltonian

$$H_e = \sum_j \frac{|\vec{p}_j - e\vec{A}(\vec{r}_j)|^2}{2m_b} + \sum_{i < j} v(\vec{r}_i - \vec{r}_j) \quad (4.91)$$

such that $H_e \Psi_e = E \Psi_e$. We can redefine the wavefunction through the transformation

$$\Phi(\vec{r}_1, \dots, \vec{r}_k) = U \Psi_e(\vec{r}_1, \dots, \vec{r}_k) = \left[\prod_{i < j} e^{-i\tilde{\phi}\alpha(\vec{r}_i - \vec{r}_j)} \right] \Psi_e(\vec{r}_1, \dots, \vec{r}_k) \quad (4.92)$$

where $\alpha(\vec{r}_i - \vec{r}_j)$ is the angle made by $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ with a fixed axis. Since

$$U^{-1}(\vec{p}_i - e\vec{A}(\vec{r}_i))U = \vec{p}_i - e\vec{A}(\vec{r}_i) - e\vec{a}(\vec{r}), \quad (4.93)$$

where

$$e\vec{a}(\vec{r}_i) = \tilde{\phi} \sum_{j \neq i} \vec{\nabla}_i \alpha(\vec{r}_i - \vec{r}_j), \quad (4.94)$$

the new Hamiltonian reads

$$H = \sum_j \frac{|\vec{p}_j - e\vec{A}(\vec{r}_j) - e\vec{a}(\vec{r}_j)|^2}{2m_b} + \sum_{i < j} v(\vec{r}_i - \vec{r}_j) \quad (4.95)$$

so that $H\Phi = E\Phi$. Therefore after the transformation, $\vec{a}(\vec{r})$ describes a gauge field with no dynamics which, one can show is of Chern-Simons type. Such a Chern-Simons gauge

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field, coupled to the fermions and to the electromagnetic gauge field, plays a central role in the fractional quantum Hall effect, see e.g. [70].

A generalization of this construction to the nonabelian case is straightforward. If two fermions at sites i and i'' couple to form a boson $\phi_{i'}^{aa'}$, at site i' at the midpoint, and two other fermions at sites j and j'' couple to form a boson $\phi_{j'}^{bb'}$ at site j' at their midpoint, we can consider the field

$$e\vec{a}(\vec{r}_{i'}) = \vec{\nabla}_{i'} \sum_{j' \neq i'} \alpha(\vec{r}_i - \vec{r}_j), \quad (4.96)$$

where we have not yet specified the nonabelian indices on the gauge field. It is not hard to see that the only variable in this object is the vector $\vec{r}_{ii'} - \vec{r}_{jj'}$ (by changing the vector $\vec{r}_{ii'}$ we just produce a harmless global spatial translation in the value of the right hand side of (4.96)), as well as the discrete choice of $\vec{r}_{ii'}$ to belong to the fixed point i' or the summed point j' . Since the two vectors \vec{r} subtracted correspond to matrix indices (aa') and (bb') , we can think of this construction as giving us two nonabelian gauge fields \vec{a}^{ab} and $\hat{\vec{a}}^{a'b'}$, like the A and \hat{A} of ABJM. Moreover, the scalars $\phi_{i'}^{aa'}$ are bifundamental with respect to the two resulting gauge fields. There remain many open problems to understand about this model, not the least of which is the symmetry group acting on the matrix Chern-Simons fields but we leave these to the interested reader. This concludes our description of the field content of ABJM and qualitative understanding of abelianization. Suffice it to say that the ABJM abelianization gives a well motivated model of AdS/CMT.

Finally, a few comments on a four dimensional picture for the Landau-Ginzburg model (4.87). The Landau-Ginzburg model makes more sense from a theoretical viewpoint as a reduction of the corresponding four dimensional theory. But here as well, the abelianization presented has in particular an ansatz with the scalar VEV b multiplying the matrix G^1 . If we had the same VEV multiplying both G^1 and G^2 , that would lead to a description of the fuzzy two-sphere, a finite N approximation of the classical two-sphere [23, 65]. As it is, we can think of the abelianization as generating a single direction, or a "fuzzy circle", therefore the resulting Landau-Ginzburg theory must also be thought of as coming from a circle reduction of a similar theory in 4 dimensions. The physical radius obtained from a fuzzy space construction was argued to be (see for instance [23])

$$R_{ph}^2 = \frac{2}{N} \text{Tr} [X^I X_I^\dagger] = \frac{2}{N} \text{Tr} [C^I C_I^\dagger] 4\pi^2 l_P^3, \quad (4.97)$$

where $l_P^3 = l_s^2 R_{11}$. Assuming this same formula holds for the less defined "fuzzy circle" case, from $\text{Tr} [G^1 G_1^\dagger] = N(N-1)/2$, we get

$$R_{ph}^2 = (N-1)|b|^2 4\pi^2 l_s^2 R_{11}. \quad (4.98)$$

4.6 Some BPS solutions with spacetime interpretation

If, as in the pure fuzzy sphere case, the 11th direction has radius $R_{11} = R_{ph}/k$, we obtain in the “maximally Higgs” case $|b|^2 = c^2$

$$R_{ph} = (N - 1)\mu l_s^2. \quad (4.99)$$

4.6 Some BPS solutions with spacetime interpretation

We now return to the more general abelianization ansatz, and consider the system with $\phi_1 = \phi_2 = 0$ and gauge fields put to zero, but $\chi_1 \neq 0$ still a field (unlike in the abelian Higgs case previously described). This ansatz gives the reduced action

$$S = -\frac{N(N-1)}{2} \int d^3x \left[|\partial_\mu \chi_1|^2 + |\partial_\mu \chi_2|^2 + \frac{4\pi^2}{k^2} \left(|\chi_1|^2 (|\chi_2|^2 - c^2)^2 + |\chi_2|^2 (|\chi_1|^2 - c^2)^2 \right) \right] \quad (4.100)$$

which we now proceed to study.

4.6.1 Single Profile Solution

As a first pass, let's consider solutions with a single profile

$$\chi_1 = \chi_2 = f(x_1), \quad (4.101)$$

with x_1 as one of the spatial coordinates. The equation of motion for the reduced action (4.100) for this ansatz become

$$\partial_{x_1}^2 f = \frac{4\pi^2}{k^2} f \left(f^2 - \frac{\mu k}{2\pi} \right) \left(3f^2 - \frac{\mu k}{2\pi} \right), \quad (4.102)$$

from which we distill two cases:

1. Zero mass: In the massless case, $\mu = 0$, the ground state solution is simply $f(x_1) = 0$ with no other constant solutions. This is, however, not the only solution and a straightforward integration of the equation of motion yields

$$f(x_1) = \sqrt{\frac{k}{4\pi x_1}}. \quad (4.103)$$

This is a *fuzzy funnel solution* which we can check is, in fact, BPS. Indeed, the energy of solutions satisfying the above ansatz is

$$H_{\mu=0} = -N(N-1) \int dx_1 dx_2 \left[(\partial_{x_1} f)^2 + \frac{4\pi^2}{k^2} f^6 \right], \quad (4.104)$$

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which, by the usual procedure of completion of squares, can be expressed as

$$H_{\mu=0} = -N(N-1) \int dx_1 dx_2 \left(\left(\partial_{x_1} f - \frac{2\pi}{k} f^3 \right)^2 + \text{surface term} \right), \quad (4.105)$$

from which we can simply read off the BPS equation

$$\partial_{x_1} f(x_1) = \frac{2\pi}{k} f(x_1)^3. \quad (4.106)$$

It is clear that this equation is solved by the fuzzy funnel solution above.

2. Nonzero mass: In this case, the constant solutions to the equations of motion are

$$f = 0, \quad f = \sqrt{\frac{\mu k}{2\pi}}, \quad f = \sqrt{\frac{\mu k}{6\pi}}. \quad (4.107)$$

Of these, only the first two are ground states. Indeed, completing squares again, we find the BPS equation

$$\partial_{x_1} f + \frac{2\pi}{k} f \left(f^2 - \frac{\mu k}{2\pi} \right) = 0, \quad (4.108)$$

from which see that indeed $f = 0$ is a trivial ground state, while the second solution ($f^2 = \mu k/2\pi$) is again the fuzzy sphere ground state. The third solution of the equations of motion ($f^2 = \mu k/6\pi$) doesn't satisfy the BPS equation, so is a non-ground state fuzzy sphere. The BPS equation has nontrivial solutions

$$f_{\mp}(x_1) = \sqrt{\frac{\mu k/2\pi}{1 \mp e^{-2\mu x_1}}}. \quad (4.109)$$

The first solution, f_- , describes a fuzzy funnel with $x_1 \in (0, +\infty)$, so f_- varies between an infinite size at $x_1 = 0$ and the fuzzy sphere ground state at $x_1 \rightarrow +\infty$,

$$f_-(0) = +\infty, \quad f_-(+\infty) = \sqrt{\frac{\mu k}{2\pi}}. \quad (4.110)$$

The second solution, f_+ , describes a fuzzy funnel with $x_1 \in (-\infty, +\infty)$, varying in size between zero at $x_1 \rightarrow -\infty$ and the fuzzy sphere at $x_1 \rightarrow +\infty$,

$$f_+(-\infty) = 0, \quad f_+(+\infty) = \sqrt{\frac{\mu k}{2\pi}}. \quad (4.111)$$

This fuzzy funnel solution will be elaborated on in the next section, where we argue that it is a generalization of the Basu-Harvey solution that describes an M2 ending on a *spherical* M5. These solutions are plotted in figure 4. below.

4.6 Some BPS solutions with spacetime interpretation

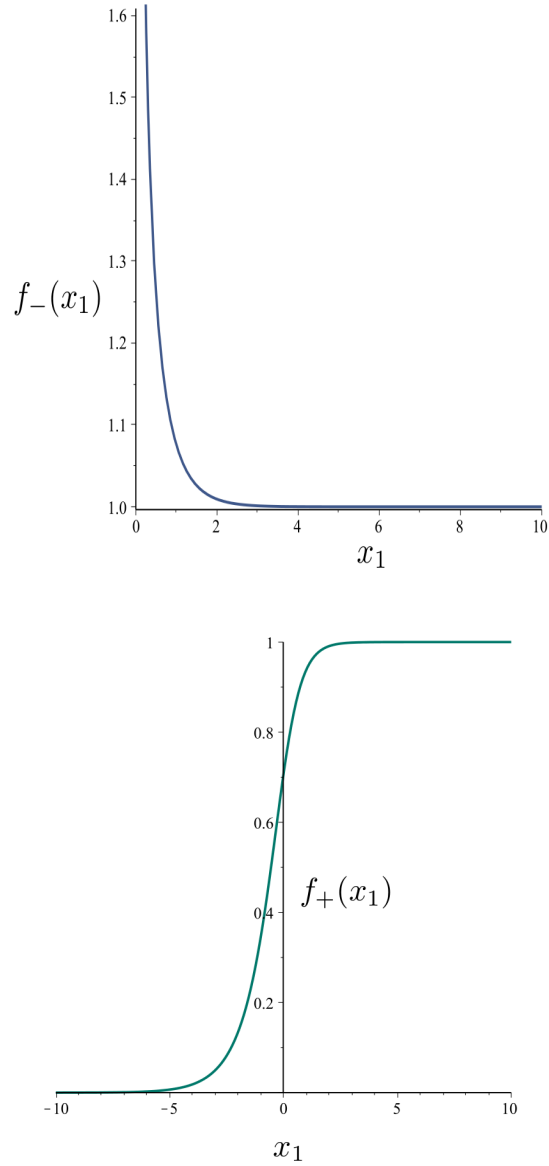


Figure 4.4: The normalized single-profile solutions $f_-(x_1)$ and $f_+(x_1)$

4.6.2 Two-Profile Solution

The above single profile solution is also fairly easily generalized to a two-profile one with

$$\chi_1 = f(x_1), \quad \chi_2 = g(x_1). \quad (4.112)$$

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With this ansatz, the equations of motion reduce to

$$\partial_{x_1}^2 f = \frac{4\pi^2}{k^2} f \left[\left(g^2 - \frac{\mu k}{2\pi} \right)^2 + 2g^2 \left(f^2 - \frac{\mu k}{2\pi} \right) \right], \quad (4.113)$$

$$\partial_{x_1}^2 g = \frac{4\pi^2}{k^2} g \left[\left(f^2 - \frac{\mu k}{2\pi} \right)^2 + 2f^2 \left(g^2 - \frac{\mu k}{2\pi} \right) \right].$$

Again, we can complete squares in the Hamiltonian and read off the BPS equations

$$\partial_{x_1} f + \frac{2\pi}{k} f \left(g^2 - \frac{\mu k}{2\pi} \right) = 0, \quad (4.114)$$

$$\partial_{x_1} g + \frac{2\pi}{k} g \left(f^2 - \frac{\mu k}{2\pi} \right) = 0.$$

As in the single profile case above, there are again two separate cases that need to be solved separately.

1. Massless case: For $\mu = 0$, we have the solutions

$$f(x_1) = \sqrt{\frac{k}{2\pi}} \frac{C e^{-x_1}}{\sqrt{1 - C^2 e^{-2x_1}}}, \quad (4.115)$$

$$g(x_1) = \sqrt{\frac{k}{2\pi}} \frac{1}{\sqrt{1 - C^2 e^{-2x_1}}},$$

that solve both the first order BPS equations of motion as well as the general second order equations. This solution blows up at $x = \log C$, and goes to a constant in g and zero in f , corresponding to a fuzzy circle. These solutions are plotted in figure 5. below.

2. Nonzero mass: For $\mu \neq 0$, we have the solutions (see figure 6)

$$f(x_1) = \sqrt{\frac{\mu k}{2\pi}} \frac{C \exp \left[\mu x_1 - \frac{1}{2} e^{2\mu x_1} \right]}{\sqrt{1 - C^2 \exp(-e^{2\mu x_1})}}, \quad (4.116)$$

$$g(x_1) = \sqrt{\frac{\mu k}{2\pi}} \frac{e^{\mu x_1}}{\sqrt{1 - C^2 \exp(-e^{2\mu x_1})}}.$$

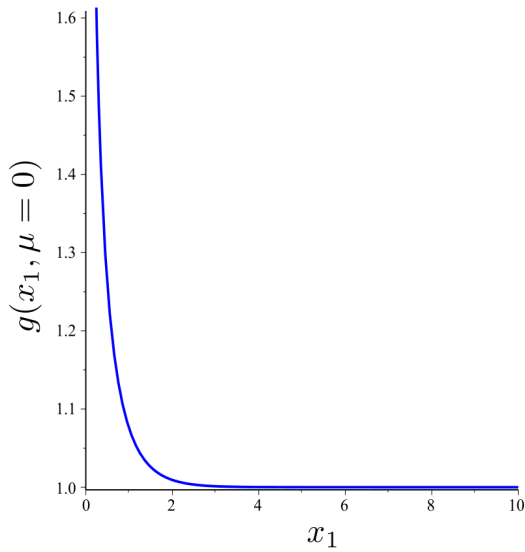
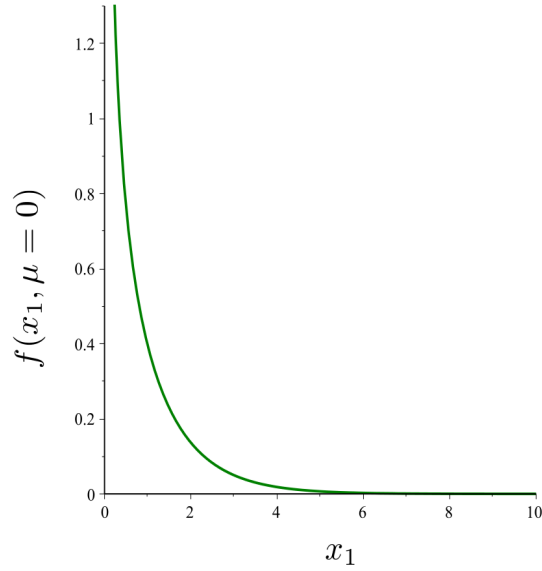


Figure 4.5: The normalized two-profile solutions $f(x_1)$ and $g(x_1)$

4.7 Funnel solutions as M2-M5 brane systems

In this section we will try to find a spacetime interpretation for the fuzzy funnel solutions in eq. (4.111). The fuzzy funnel solution (4.103), interpolating between a sphere of infinite size and a sphere of zero size, is known to have the spacetime interpretation of a flat M2-

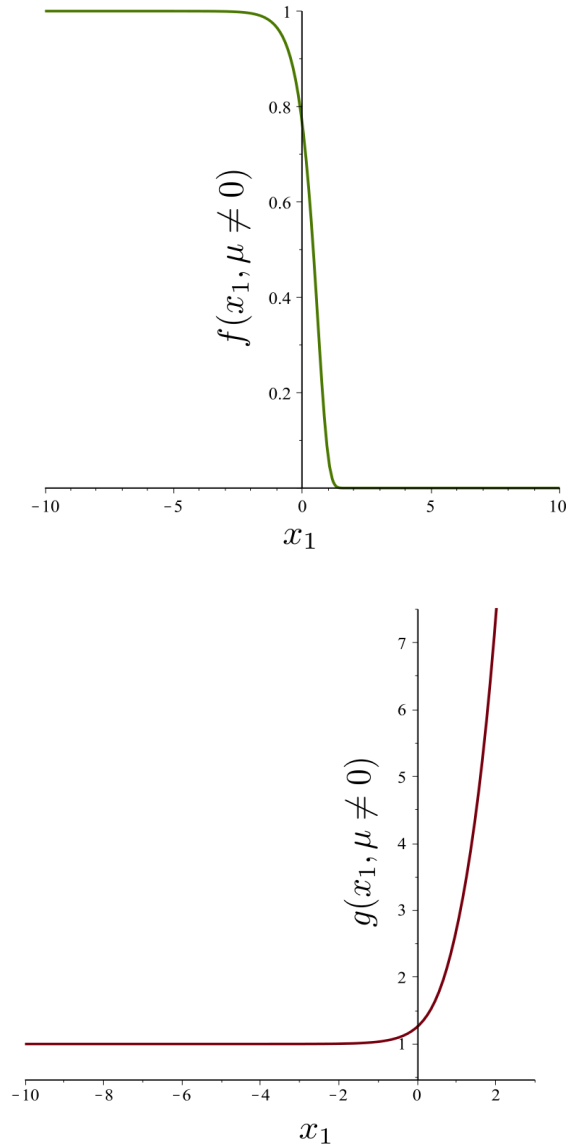


Figure 4.6: The normalized two-profile solutions $f(x_1)$ and $g(x_1)$ for $\mu \neq 0$.

brane ending on a flat M5-brane. From the point of view of the M2-brane theory given by the massless ABJM, the M5-brane appears as a spherical funnel solution, a M5-brane that grows from zero size at $x_1 = \infty$ to infinite size at $x_1 = 0$. We will review this case, reduced to string theory, i.e. a D2-brane ending on a D4-brane, later. Also, from the point of view of the M5-brane theory, we can write a BIon type solution, corresponding to

4.7 Funnel solutions as M2-M5 brane systems

an M2-brane growing out of the M5-brane (directions 0 and 5 are trivial, and in directions 1-4 the M2-brane appears as a BIon). From the point of view of the spacetime theory, we have an M2-M5 system preserving 1/4 supersymmetry in flat space, and we also have an M5-brane solution in the (backreacted) background of M2-branes. This picture matches nicely with the two worldvolume descriptions.

With this in mind, we expect that the fuzzy funnel solution of (4.111) should have a similar interpretation. The solution interpolating between zero and a fuzzy sphere vacuum was found in [56, 41], and we would guess that it can only match with a spacetime solution corresponding to an M2-brane ending on an M5-brane. We will see however that there is some ambiguity, related to the existence of two solutions, the one from zero to the fuzzy sphere, and one from the fuzzy sphere to infinity.

4.7.1 Massless case: A Fuzzy Funnel Review

The solution (4.103) corresponds in spacetime to a flat M2-brane ending on a flat M5-brane, a solution which preserves 1/4 supersymmetry as follows. In a flat background, the 11-dimensional gravitino transformation law,

$$\delta\psi_\mu = D_\mu\epsilon + \#(\Gamma^{\nu\rho\sigma\lambda}{}_\mu - 8\delta_\mu^\nu\Gamma^{\rho\sigma\lambda})F_{\nu\rho\sigma\lambda}, \quad (4.117)$$

must be set to zero in order to obtain a BPS solution. The M2-brane solution extended in the (0,1,2)-directions corresponds to a nonzero 3-form A_{012} , with a nonzero field strength component F_{012r} (here r is the radial part of all the coordinates transverse to the M2). The solution is given by a local supersymmetry parameter $\epsilon(r)$ which is a scalar function of r times a constant susy parameter $\epsilon_{(0)}$ satisfying

$$\Gamma^{012}\epsilon_{(0)} = \pm\epsilon_{(0)}. \quad (4.118)$$

The M5-brane solution extended in the (0, 1, 3, 4, 5, 6) directions similarly gives a nonzero field strength $F_{\theta_1\dots\theta_4}$, where $\theta_1, \dots, \theta_4$ are the four angles obtained for the transverse directions (2, 7, 8, 9, 10). Again, the solution for the local supersymmetry parameter $\epsilon(r)$ is a function of r times a constant susy parameter $\epsilon_{(0)}$ satisfying

$$\Gamma^{013456}\epsilon_{(0)} = \pm\epsilon_{(0)}. \quad (4.119)$$

We can then have a solution for an M2-brane ending on an M5-brane preserving 1/4 supersymmetry by imposing both conditions (which are now compatible). We can then reduce this system to 10-dimensional string theory, thereby considering a D2-brane ending on a D4-brane.

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From the point of view of the D4-brane theory in a flat background spacetime, the fuzzy funnel solution looks like a BIon-type solution. For a spacetime D2-brane in the (0,1,2)-directions, called t, x, z , and a D4-brane in the (0,1,3,4,5)-directions, with polar coordinates r, θ, ϕ for the directions (3,4,5), the worldvolume gauge field flux on the D4-brane is

$$F = (2\pi\alpha')n \sin\theta d\theta d\phi. \quad (4.120)$$

Because the solution is of the BIon type, with the D2-brane growing out of the D4-brane, we consider $z = z(r)$ on the worldvolume, leading to DBI D4-brane Lagrangian

$$\mathcal{L} = T_4 \sqrt{(1 + z'(r)^2)(r^4 + (\lambda n)^2)}, \quad (4.121)$$

where $\lambda \equiv 2\pi\alpha'$. Since \mathcal{L} is independent of z , it follows that $\partial\mathcal{L}/\partial z'$ is a constant, which we can put equal to λn , in which case we obtain

$$z' = \pm \frac{\lambda n}{r^2} \Rightarrow z = \frac{\lambda n}{r} \quad (4.122)$$

This corresponds to a funnel solution for a semi-infinite D2-brane ending on the D4-brane.

A similar story takes place for the case where there is a background created by other D2-branes (parallel with the first). It is however easier to describe what happens in the case of the type IIB solution for D3-branes ending on D5-branes (instead of D2-D4), since in that case the spacetime background is easier (there is no M theory reduction). Consider the background generated by other D3-branes, with harmonic function $f(r)$,

$$ds^2 = f(r)^{-1/2}(-dt^2 + dx^2 + dy^2 + dz^2) + f(r)^{+1/2}(dr^2 + r^2 d\Omega_2^2 + d\bar{s}^2) \quad (4.123)$$

$$C_{(4)} = (f^{-1} - 1)dt \wedge dx \wedge dy \wedge dz$$

The DBI Lagrangean for the D5-brane reads

$$\mathcal{L} = T_5 \left[\sqrt{(1 + f'(r)z'(r)^2)(r^4 + f^{-1}(r)\lambda^2 n^2)} - \lambda n(f(r)^{-1} - 1)z'(r) \right], \quad (4.124)$$

and the same calculation leads to the same solution $z(r) = \lambda n/r$, with the function $f(r)$ dropping out completely. One can also take the near-horizon limit and consider the usual scaling $r = \alpha'U$, leading to a finite funnel solution that can be interpreted from the point of view of the D3-brane theory as

$$U(z) = \frac{n}{2\pi z}. \quad (4.125)$$

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Note that if we consider a spherical D5-brane ansatz, oriented in the (t, x, y, z, Ω_2) -directions, we obtain the Lagrangian

$$\mathcal{L} = T_5 \sqrt{f^{-1}(r)(r^4 + g^{-1}(\lambda n)^2)} - \lambda n (f^{-1}(r) - 1) . \quad (4.126)$$

If we take the full harmonic function $f(r) = 1 + \frac{Q}{r^4}$, there is no fixed sphere solution with $r = R = \text{constant}$, but if we drop the 1 in f , i.e. at very large r , we obtain an identity by varying with respect to r , namely $\lambda n/Q - \lambda n/Q = 0$. Therefore in flat space, we have asymptotically a solution for very large radius sphere, but at small radius we only have the funnel solution; a fixed sphere is not a solution.

4.7.2 Massive case: Supersymmetry and a Fluctuation Solution on the M5-brane

The mass deformation changes the 11 dimensional background spacetime from flat to [62, 68]

$$ds^2 = H^{-2/3}(-dt^2 + dx_1^2 + dx_2^2) + H^{1/3}(dx_3^2 + \dots dx_{10}^2) \quad (4.127)$$

$$F_4 = 2\mu(dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 + dx^7 \wedge dx^8 \wedge dx^9 \wedge dx^{10}) + dx^0 \wedge dx^1 \wedge dx^2 \wedge dH^{-1}$$

where $H(r) = 1 - \frac{1}{4}\mu^2 r^2$. A naive guess is that the M5-brane has to live in the $(0, 1, 2, \theta, \phi, \xi)$ -directions, where θ, ϕ and ξ are the angular directions of $(3, 4, 5, 6)$, with r their radial direction, giving

$$\Gamma^{012\theta\phi\xi}\epsilon_{(0)} = \pm\epsilon_{(0)} , \quad (4.128)$$

so that the transverse M2-brane would have to be in the $(0, 1, r)$ -directions, giving

$$\Gamma^{01r}\epsilon_{(0)} = \pm\epsilon_{(0)} . \quad (4.129)$$

However we observe that the 4-form in (4.128) has nonzero F_{012r} as wanted, but since $F_{012r} = \partial_r(A_{012})$ and not $\partial_2(A_{01r})$, there must be a nontrivial Maxwell transformation that brings the gauge field A into the desired form.

How would this M2-M5-brane solution look from the point of view of the D4-brane theory (i.e., reducing to 10d string theory and focusing on the worldvolume theory)?

From the fuzzy S^2 picture in the ABJM theory an action was found for the fluctuation modes around the ground state [23]. For the scalar Φ corresponding to the fluctuations of the radius of the S^2 (transverse direction) the action reduces to just a massive mode, i.e. with potential

$$V_\Phi = \frac{1}{2} [\Phi^2 + (\nabla_{S^2}\Phi)^2] , \quad (4.130)$$

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Later, in [68] the same fluctuation action was found from the DBI action of a D4-brane in the background (4.128).

For such a potential, a solution was found in [69], and was called the BIGGon (in analogy with the BIon), representing, in spacetime, an S^3 giant graviton in type IIB on the maximally supersymmetric pp wave background with F-string spikes attached at the poles. Since the pp-wave background is T-dual to (4.128) (see, for example, [62] for the explicit construction), the same solution should apply in our case. The BIGGon was found by similarly taking a single scalar field Φ on the 3-sphere (a fluctuation of the radial coordinate), with the same potential, but on S^3 instead of S^2 , *i.e.*

$$V_\Phi = \frac{1}{2} [\Phi^2 + (\nabla_{S^3}\Phi)^2] . \quad (4.131)$$

The solution is

$$\Phi = \frac{Q}{\sin \psi} , \quad (4.132)$$

giving the full radial coordinate (background plus fluctuation)

$$X = R \left(1 + Q \frac{g_{eff}}{\sin \psi} \right) , \quad (4.133)$$

where we have taken the S^3 parametrization to be

$$\begin{aligned} X^4 &= R \cos \psi , \\ X^3 &= R \sin \psi \cos \theta , \\ X^2 &= R \sin \psi \sin \theta \sin \phi , \\ X^1 &= R \sin \psi \sin \theta \cos \phi . \end{aligned}$$

Therefore in our case we have the BIGGon solution

$$\Phi = \frac{Q}{\sin \theta} \quad (4.134)$$

where the parametrization of the S^2 is

$$\begin{aligned} X^3 &= R \sin \theta , \\ X^4 &= R \cos \theta \sin \phi , \\ X^5 &= R \cos \theta \cos \phi . \end{aligned}$$

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This solution indeed corresponds with our naive expectation of a D2-brane extending out perpendicularly from the spherical D4-brane. But the action that it extremizes corresponds to small fluctuations of the field Φ . However, in [68] it was shown how to write the full DBI action for D4-branes in the mass-deformed spacetime.

Massive case: full funnel solution

In the background (4.128) it was shown that the DBI action for the D4-brane has a fixed sphere solution, corresponding to the fuzzy sphere solution of the massive ABJM. Now we want to see if we can also find funnel solutions corresponding to the ABJM solutions (4.111) and extending the perturbative BIGGon solution above.

To this end, we consider again an M2-brane in the (0,1,2)-directions and an M5-brane in the (0,1,3,4,5,6)-directions, with (3,4,5,6) in polar coordinates r, θ, ϕ , and ξ . We reduce M-theory to type IIA on ξ and look for a D4-brane extending along 0, 1, r, θ, ϕ , with $z = z(r, \theta, \phi)$ in order to have a BIon-type solution corresponding to a perpendicular D2-brane as above.

Dimensionally reducing the background (4.128) to type IIA string theory, and writing only the terms in directions parallel to the D4-brane, we find [68]

$$\begin{aligned}
C_{(5)} &= -\frac{\mu}{2kR_*} \left(\frac{H^{-1} + 1}{2} \right) dt \wedge dx \wedge z' dr \wedge r^4 d\Omega_2 + \dots \\
C_{(3)} &= (H^{-1} - 1) dt \wedge dx \wedge z' dr + \\
B &= \frac{\mu}{2kR_*} r^4 d\Omega_2 + \\
e^\phi &= \left(\frac{r}{kR_*} \right)^{3/2} H^{1/4}
\end{aligned} \tag{4.135}$$

and a worldvolume flux $F = 2\lambda N d\Omega_2$, giving

$$\mathcal{F} = \lambda F - B = 2\lambda N - \frac{\mu}{2kR_*} r^4. \tag{4.136}$$

Substituting this ansatz into the action

$$S = T_4 \left[\int d^5 x e^{-\phi} \sqrt{-\det(g + \mathcal{F})} + \int (C^{(5)} + C^{(3)} \wedge \mathcal{F}) \right], \tag{4.137}$$

gives

$$S = T_4 \left\{ \int dt \wedge dx \wedge dr \wedge d\Omega_2 \sqrt{(1 + z'^2 H^{-1}) \left(1 + H^{-1} \frac{k^2 R_*^2}{r^6} \left(2\lambda N - \frac{\mu r^4}{2kR_*} \right)^2 \right)} \right\}$$

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$$+ \int dt \wedge dx \wedge dr \wedge d\Omega_2 \left(2\lambda N z' (H^{-1} - 1) - \frac{\mu}{2kR_*} z' \frac{H^{-1} + 1}{2} r^4 \right) \}. \quad (4.138)$$

Evidently, the Lagrangian \mathcal{L} is independent of z , which means that $\partial L/\partial z'$ is a constant, which we can put equal to $-T_4 2\lambda N$, in which case

$$z' = \pm \frac{\left[\frac{2\lambda N}{r^3} - \frac{\mu r}{2} + \frac{\mu^3 r^3}{16} \right] \sqrt{1 - \frac{\mu^2 r^2}{4}}}{\sqrt{1 - \frac{\mu^2 r^2}{4} - \frac{\mu^6 r^6}{4^4} + \frac{\mu^4 r^4}{4^2} - \frac{\mu^3 k R_* \lambda N}{8}}}. \quad (4.139)$$

Here we need to have $z' < 0$, since the equation is obtained by squaring an equation whose left hand side is linear in z' and has a positive coefficient, and whose right hand side is negative, after which we take the square root of z'^2 . Since $R_0^2 = 2\lambda N \mu k R_*$, after defining

$$x = \mu^2 r^2; \quad y = \mu z; \quad a = \mu R_0, \quad (4.140)$$

we obtain the equation

$$\frac{dy}{dx} = \pm \frac{\left(\frac{a^2}{x^2} - \frac{1}{2} + \frac{x}{4^2} \right) \sqrt{1 - \frac{x}{4}}}{\sqrt{1 - \frac{x}{4} - \frac{x^3}{4^4} + \frac{x^2}{4^2} - \frac{a^2}{4^2}}}. \quad (4.141)$$

However, as explained in [68], we are in the approximation $a = \mu R_0 \ll 1$, and the fixed sphere ground state solution is $r = R_0$, or $x = a^2 \ll 1$. That means that we can assume x small in the above equation, thus

$$\frac{dy}{dx} \simeq -\frac{a^2}{2x^2} + \frac{1}{4} \quad (4.142)$$

which gives finally

$$z \simeq \frac{R_0^2}{2\mu r^2} + \frac{\mu r^2}{4}. \quad (4.143)$$

This $z(r)$ has a minimum at

$$r_* = \left(\frac{4\lambda N k R_*}{\mu} \right)^{1/4} = \left(\frac{\sqrt{2} R_0}{\mu} \right)^{1/2} \gg R_0, \quad (4.144)$$

and at r_* , the minimum value of z is

$$z_{min} = \sqrt{\mu \lambda N k R_*} = \frac{R_0}{\sqrt{2}}. \quad (4.145)$$

Note that here R_0 is written in physical spacetime variables, $R_{ph}^2 = 8\pi^2 N l_P^3 f^2$, where $f = \sqrt{\mu k}/(2\pi)$ is the radius in the M2-brane worldvolume theory.

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To compare with the fuzzy funnel solutions (4.111), we note that z is the equivalent of the M2-brane worldvolume direction x_1 and r is the equivalent of the transverse direction $f(x_1)$. Therefore we must consider $r(z)$ but, since it has two branches, we must choose only one. The two branches are: $r(z)$ going from 0 to r_* (for z going from ∞ to z_{min}), and $r(z)$ going from r_* to infinity. That would naively match the two solutions in (4.111), except for the fact that $r_* \gg R_0$, and $r_* \rightarrow \infty$ at $\mu \rightarrow 0$ (with N very large), whereas $R_0 = 2\mu\lambda N \rightarrow 0$ as $\mu \rightarrow 0$. In fact, for $\mu \rightarrow 0$, we obtain from (4.143)

$$r(z) = \frac{R_0}{\sqrt{2\mu z}}. \tag{4.146}$$

This can be compared with the fuzzy funnel solution (4.103), written as

$$f(x_1) = \frac{f}{\sqrt{2\mu x_1}}, \tag{4.147}$$

so the spacetime solution is a *deformation* of the $\mu = 0$ case. It also matches with the first solution in (4.111) in the $\mu \rightarrow 0$ limit. However if μ is fixed, the solution becomes (4.147) in the $x_1 \rightarrow 0$ limit, corresponding to $z \rightarrow 0$, which is not even reachable by (4.143).

It is therefore unclear to us how to relate the two branches of (4.143) to the two solutions of (4.111) precisely, other than through the general qualitative behaviour. We will leave a precise understanding of the matching to future work.

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5

Conclusion

In chapter 2 we have looked for a Matrix theory-type construction from ABJM model. We have identified a BPS vortex solution of ABJM as corresponding to a spacetime supergraviton (a pointlike object carrying $D0$ -brane charge). In the case of the pure ABJM model, the identification is problematic since, in addition to the supergraviton, we also have also an infinite energy 2-brane to worry about. We find that this is resolved in the case of the massive maximally supersymmetric deformation of ABJM since, here, the solution has finite energy. We have also found a BPS kink solution of ABJM, using a certain limiting procedure. This solution can be identified with a $D2$ -brane with one direction parallel to the ABJM worldvolume, and one direction transverse. The vacuum supersymmetric fuzzy sphere solution had already been identified with the $D4$ -brane wrapped on S^2 .

In the latter half of the chapter, we identified the spacetime background corresponding to the massive ABJM deformation, and computed the leading graviton scattering interaction potential using a shockwave-probe method previously developed for the BFSS Matrix theory. We then computed the classical interaction potential of two vortices in the $N = 2$ ABJM model, and found that it differs from the supergraviton potential calculated by scattering two shockwaves *a la* 't Hooft. We then speculated on how one could fix this mismatch, noting that there are many differences with respect to the BFSS computation. Chief among these is the fact that now, the two spatial worldvolume directions are special.

Conclusion

We argued that perhaps one needs to find an effective Lagrangian for supergravitons that would match with the effective Lagrangian for vortex interaction, and that our gravity calculation could be matched by a one-loop calculation in the vortex background, similar to the BFSS case. Ultimately, though, it is clear that much more work is needed to find a concrete Matrix theory construction for ABJM.

In chapter 3, we have presented a consistent truncation of the ABJM model to a collective model of $\mathcal{O}(N)$ modes out of the $\mathcal{O}(N^2)$, reducing to an abelian Landau-Ginzburg model. We have also seen that we can map this process to a simple condensed matter model that reproduces the same general features. This provides a concrete step towards a well-defined AdS/CMT model, where there is a large N theory for the condensed matter system, with a gravity dual, and yet the relevant physics is encoded in a simple abelian model.

In chapter 4, we have studied various ansätze for abelian reductions of the ABJM model, in the general case of nonzero mass, and used them to build a better defined AdS/CMT model. We have found a general abelianization ansatz (4.22, 4.21), using the matrices G^α that describe the fuzzy funnel BPS state and fuzzy sphere ground state, and that represent a consistent truncation. A further consistent truncation led to a model with topological vortex BPS solutions, but with $|\phi| \rightarrow 0$ at both $r = 0$ and $r = \infty$ while yet another further consistent truncation led to a relativistic Landau-Ginzburg model which, depending on the parameter $c^2 = \mu k / (2\pi)$ and on the scalar vev b , extrapolates between the abelian-Higgs model, and a scalar ϕ^4 theory.

The second abelianization was used to take steps towards a better defined AdS/CMT model, since the ABJM model has a gravity dual, and the abelianization corresponds to the collective dynamics of $\mathcal{O}(N)$ out of the $\mathcal{O}(N^2)$ fields. We also sketched a simple condensed matter model for a solid with free electrons that exhibits the same general features as the abelianization and leads to a bosonic Hubbard model, which in the continuum limit gives the relativistic Landau-Ginzburg system. It will be interesting to see if we can make more the model more concrete and elaborate further on its relation to ABJM. If successful, our construction provides, in our opinion, a concrete embedding of the AdS/CMT correspondence in string theory.

In the last two sections, we studied various BPS solutions suggested by the abelianization, finding some generalizations of known solutions. We tried to find a spacetime interpreta-

tion for the BPS solutions in (4.111) as M2-M5 systems, with partial success. For small fluctuations we succeeded in matching this with the BIGGon solution for an M2 ending on a spherical M5, but for the full system we could only match only general qualitative behaviour and not the particular solution. It goes without saying that more work is needed to understand these solutions.

Conclusion

Bibliography

- [1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].
- [2] T. Sakai and S. Sugimoto, “Low energy hadron physics in holographic QCD,” *Prog. Theor. Phys.* **113**, 843 (2005) [hep-th/0412141].
- [3] S. S. Gubser, “Using string theory to study the quark-gluon plasma: Progress and perils,” *Nucl. Phys. A* **830** (2009) 657C [arXiv:0907.4808 [hep-th]].
- [4] D. Berenstein, J. Maldacena and H. Nastase, “Strings in flat space and pp waves from N=4 superYang-Mills,” *JHEP* **0204**, 013 (2002) [hep-th/0202021].
- [5] J. A. Minahan and K. Zarembo, “The Bethe ansatz for N=4 superYang-Mills,” *JHEP* **0303**, 013 (2003) [hep-th/0212208].
- [6] R. C. Myers, S. Sachdev and A. Singh, “Holographic Quantum Critical Transport without Self-Duality,” *Phys. Rev. D* **83**, 066017 (2011) [arXiv:1010.0443 [hep-th]].
- [7] L. Huijse and S. Sachdev, “Fermi surfaces and gauge-gravity duality,” *Phys. Rev. D* **84**, 026001 (2011) [arXiv:1104.5022 [hep-th]].
- [8] S. A. Hartnoll, “Lectures on holographic methods for condensed matter physics,” *Class. Quant. Grav.* **26**, 224002 (2009) [arXiv:0903.3246 [hep-th]].
- [9] C. P. Herzog, “Lectures on Holographic Superfluidity and Superconductivity,” *J. Phys. A* **42**, 343001 (2009) [arXiv:0904.1975 [hep-th]].
- [10] C. M. Hull and P. K. Townsend, “Unity of superstring dualities,” *Nucl. Phys. B* **438** (1995) 109 [hep-th/9410167].

BIBLIOGRAPHY

- [11] P. K. Townsend, “The eleven-dimensional supermembrane revisited,” *Phys. Lett. B* **350** (1995) 184 [hep-th/9501068].
- [12] E. Witten, “String theory dynamics in various dimensions,” *Nucl. Phys. B* **443** (1995) 85 [hep-th/9503124].
- [13] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, “M theory as a matrix model: A conjecture,” *Phys. Rev. D* **55** (1997) 5112–5128, arXiv:hep-th/9610043.
- [14] J. Castelino, S. Lee, and W. Taylor, “Longitudinal 5-branes as 4-spheres in matrix theory,” *Nucl. Phys. B* **526** (1998) 334–350, arXiv:hep-th/9712105.
- [15] H. Grosse, C. Klimcik, and P. Presnajder, “Finite quantum field theory in noncommutative geometry,” *Commun. Math. Phys.* **180** (1996) 429–438, arXiv:hep-th/9602115.
- [16] D. N. Kabat and W. Taylor, “Spherical membranes in matrix theory,” *Adv. Theor. Math. Phys.* **2** (1998) 181–206, arXiv:hep-th/9711078.
- [17] J. Bagger and N. Lambert, “Modeling multiple M2’s,” *Phys. Rev. D* **75** (2007) 045020, arXiv:hep-th/0611108.
- [18] J. Bagger and N. Lambert, “Comments On Multiple M2-branes,” *JHEP* **02** (2008) 105, arXiv:0712.3738 [hep-th].
- [19] J. Bagger and N. Lambert, “Gauge Symmetry and Supersymmetry of Multiple M2-Branes,” *Phys. Rev. D* **77** (2008) 065008, arXiv:0711.0955 [hep-th].
- [20] A. Gustavsson, “Algebraic structures on parallel M2-branes,” *Nucl. Phys. B* **811** (2009) 66–76, arXiv:0709.1260 [hep-th].
- [21] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” *JHEP* **10** (2008) 091, arXiv:0806.1218 [hep-th].
- [22] E. Cremmer, B. Julia and J. Scherk, “Supergravity Theory in Eleven-Dimensions,” *Phys. Lett. B* **76** (1978) 409.
- [23] H. Nastase, C. Papageorgakis, and S. Ramgoolam, “The fuzzy S^2 structure of M2-M5 systems in ABJM membrane theories,” *JHEP* **05** (2009) 123, arXiv:0903.3966 [hep-th].

- [24] H. Nastase and C. Papageorgakis, “Fuzzy Killing Spinors and Supersymmetric D4 action on the Fuzzy 2-sphere from the ABJM Model,” *JHEP* **12** (2009) 049, arXiv:0908.3263 [hep-th].
- [25] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk, and H. Verlinde, “A Massive Study of M2-brane Proposals,” *JHEP* **09** (2008) 113, arXiv:0807.1074 [hep-th].
- [26] D. Gaiotto, S. Giombi and X. Yin, “Spin Chains in N=6 Superconformal Chern-Simons-Matter Theory,” *JHEP* **04** (2009) 066, arXiv:0806.4589 [hep-th].
- [27] S. Terashima, “On M5-branes in N=6 Membrane Action,” *JHEP* **08** (2008) 080, arXiv:0807.0197 [hep-th].
- [28] M. J. Duff and K. S. Stelle, “Multimembrane solutions of D = 11 supergravity,” *Phys. Lett. B* **253** (1991) 113.
- [29] A. Sen, “D0 branes on T(n) and matrix theory,” *Adv. Theor. Math. Phys.* **2** (1998) 51–59, arXiv:hep-th/9709220.
- [30] N. Seiberg, “Why is the matrix model correct?,” *Phys. Rev. Lett.* **79** (1997) 3577–3580, arXiv:hep-th/9710009.
- [31] B. de Wit, J. Hoppe and H. Nicolai, “On the Quantum Mechanics of Supermembranes,” *Nucl. Phys. B* **305** (1988) 545.
- [32] K. Becker and M. Becker, “On graviton scattering amplitudes in M-theory,” *Phys. Rev.* **D57** (1998) 6464–6470, arXiv:hep-th/9712238.
- [33] G. 't Hooft, “Graviton Dominance in Ultrahigh-Energy Scattering,” *Phys. Lett.* **B198** (1987) 61–63.
- [34] K. Kang and H. Nastase, “Planckian scattering effects and black hole production in low M(Pl) scenarios,” *Phys. Rev.* **D71** (2005) 124035, arXiv:hep-th/0409099.
- [35] K. Kang and H. Nastase, “High energy QCD from Planckian scattering in AdS and the Froissart bound,” *Phys. Rev.* **D72** (2005) 106003, arXiv:hep-th/0410173.
- [36] T. Dray and G. 't Hooft, “The Gravitational Shock Wave of a Massless Particle,” *Nucl. Phys. B* **253** (1985) 173.

BIBLIOGRAPHY

- [37] H. Nastase, M. A. Stephanov, P. van Nieuwenhuizen, and A. Rebban, “Topological boundary conditions, the BPS bound, and elimination of ambiguities in the quantum mass of solitons,” *Nucl. Phys.* **B542** (1999) 471–514, [arXiv:hep-th/9802074](#).
- [38] D. A. Lowe, H. Nastase, and S. Ramgoolam, “Massive IIA string theory and matrix theory compactification,” *Nucl. Phys.* **B667** (2003) 55–89, [arXiv:hep-th/0303173](#).
- [39] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk, and H. Verlinde, “Supersymmetric Yang-Mills Theory From Lorentzian Three- Algebras,” *JHEP* **08** (2008) 094, [arXiv:0806.0738 \[hep-th\]](#).
- [40] S. Mukhi and C. Papageorgakis, “M2 to D2,” *JHEP* **05** (2008) 085, [arXiv:0803.3218 \[hep-th\]](#).
- [41] M. Arai, C. Montonen, and S. Sasaki, “Vortices, Q-balls and Domain Walls on Dielectric M2- branes,” [arXiv:0812.4437 \[hep-th\]](#).
- [42] C. Kim, Y. Kim, O.-K. Kwon, and H. Nakajima, “Vortex-type Half-BPS Solitons in ABJM Theory,” *Phys. Rev.* **D80** (2009) 045013, [arXiv:0905.1759 \[hep-th\]](#).
- [43] R. Auzzi and S. Prem Kumar, “Non-Abelian Vortices at Weak and Strong Coupling in Mass Deformed ABJM Theory,” [arXiv:0906.2366 \[hep-th\]](#).
- [44] C. G. Callan and J. M. Maldacena, “Brane dynamics from the Born-Infeld action,” *Nucl. Phys.* **B513** (1998) 198–212, [arXiv:hep-th/9708147](#).
- [45] P. L. H. Cook, R. de Mello Koch, and J. Murugan, “Non-Abelian Bionic brane intersections,” *Phys. Rev.* **D68** (2003) 126007, [arXiv:hep-th/0306250](#).
- [46] G. W. Gibbons, “Born-Infeld particles and Dirichlet p-branes,” *Nucl. Phys.* **B514** (1998) 603–639, [arXiv:hep-th/9709027](#).
- [47] J. P. Gauntlett, J. Gomis, and P. K. Townsend, “BPS bounds for worldvolume branes,” *JHEP* **01** (1998) 003, [arXiv:hep-th/9711205](#).
- [48] T. Fujimori, K. Iwasaki, Y. Kobayashi, and S. Sasaki, “in preparation,”.
- [49] J. Gomis, A. J. Salim, and F. Passerini, “Matrix Theory of Type IIB Plane Wave from Membranes,” *JHEP* **08** (2008) 002, [arXiv:0804.2186 \[hep-th\]](#).

-
- [50] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee, and J. Park, “N=5,6 Superconformal Chern-Simons Theories and M2-branes on Orbifolds,” *JHEP* **09** (2008) 002, [arXiv:0806.4977 \[hep-th\]](#).
- [51] T. H. Buscher, “Quantum corrections and extended supersymmetry in new sigma models,” *Phys. Lett.* **B159** (1985) 127.
- [52] T. H. Buscher, “A Symmetry of the String Background Field Equations,” *Phys. Lett.* **B194** (1987) 59.
- [53] T. H. Buscher, “Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models,” *Phys. Lett.* **B201** (1988) 466.
- [54] S. Ramgoolam, “On spherical harmonics for fuzzy spheres in diverse dimensions,” *Nucl. Phys.* **B610** (2001) 461–488, [arXiv:hep-th/0105006](#).
- [55] B. Janssen, Y. Lozano, and D. Rodriguez-Gomez, “Giant gravitons in AdS(3) x S**3 x T**4 as fuzzy cylinders,” *Nucl. Phys.* **B711** (2005) 392–406, [arXiv:hep-th/0406148](#).
- [56] K. Hanaki and H. Lin, “M2-M5 Systems in N=6 Chern-Simons Theory,” *JHEP* **09** (2008) 067, [arXiv:0807.2074 \[hep-th\]](#).
- [57] A. Jevicki and H. Nastase, “Towards S matrices on flat space and pp waves from SYM,” [arXiv:hep-th/0501013](#).
- [58] K. Becker, M. Becker, J. Polchinski, and A. A. Tseytlin, “Higher order graviton scattering in M(atrrix) theory,” *Phys. Rev.* **D56** (1997) 3174–3178, [arXiv:hep-th/9706072](#).
- [59] H. Lin, O. Lunin, and J. M. Maldacena, “Bubbling AdS space and 1/2 BPS geometries,” *JHEP* **10** (2004) 025, [arXiv:hep-th/0409174](#).
- [60] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” *Adv. Theor. Math. Phys.* **2** (1998) 505–532, [arXiv:hep-th/9803131](#).
- [61] S. S. Gubser, “Breaking an Abelian gauge symmetry near a black hole horizon,” *Phys. Rev. D* **78**, 065034 (2008) [[arXiv:0801.2977 \[hep-th\]](#)].
- [62] A. Mohammed, J. Murugan and H. Nastase, “Looking for a Matrix model of ABJM,” *Phys. Rev. D* **82**, 086004 (2010)
-

BIBLIOGRAPHY

- [63] A. Mohammed, J. Murugan and H. Nastase, “Abelian-Higgs and Vortices from ABJM: towards a string realization of AdS/CMT,” JHEP **1211**, 073 (2012) [arXiv:1206.7058 [hep-th]].
- [64] A. Mohammed, J. Murugan and H. Nastase, “Towards a Realization of the Condensed-Matter/Gravity Correspondence in String Theory via Consistent Abelian Truncation,” Phys. Rev. Lett. **109**, 181601 (2012) [arXiv:1205.5833 [hep-th]].
- [65] H. Nastase and C. Papageorgakis, “Bifundamental fuzzy 2-sphere and fuzzy Killing spinors,” SIGMA **6**, 058 (2010)
- [66] S. Sachdev, “What can gauge-gravity duality teach us about condensed matter physics?,” Ann. Rev. Condensed Matter Phys. **3**, 9 (2012)
- [67] S. H. Simon, ”The Chern-Simons Fermi Liquid Description of Fractional Quantum Hall States” chapter in ”Composite Fermions”, ed. O. Heinonen, World Scientific and arXiv:cond-mat/9812186
- [68] N. Lambert, H. Nastase and C. Papageorgakis, “5D Yang-Mills instantons from ABJM Monopoles,” Phys. Rev. D **85** (2012) 066002 [arXiv:1111.5619 [hep-th]].
- [69] D. Sadri and M. M. Sheikh-Jabbari, “Giant hedgehogs: Spikes on giant gravitons,” Nucl. Phys. B **687**, 161 (2004) [hep-th/0312155].
- [70] E. Witten, Lectures at the 2011 Swieca School in Campos de Jordão, Brazil