Flavour SU(3) symmetry breaking in Quantum Chromodynamics

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ABSTRACT

The subject of this thesis is the determination of the size of SU(3) flavour symmetry breaking in the QCD vacuum, as measured by the ratio of the strange to non-strange quark vacuum condensates $\frac{<\bar{s}s>}{<\bar{u}u>}$ (with $< \bar{u}u > \simeq < \bar{d}d >$). This is done through Laplace transform QCD sum rules for a pair of functions related to the two-point functions involving the strangeness changing vector and axial-vector current divergences. These functions are currently known in QCD perturbation theory up to four loops, and no longer involve logarithmic quark mass singularities. At the same time, there is improved information on the hadronic spectral functions, in particular regarding their threshold behaviour fixed by chiral perturbation theory. All of this allows for a more reliable determination of $\frac{<\bar{s}s>}{<\bar{u}u>}$, which turns out to be rather stable as a function of the Laplace variable, and reasonably insensitive to changes in the various parameters. The result of this determination is $\frac{<\bar{s}s>}{<\bar{u}u>} = 0.7 \pm 0.1$

CONTENTS

1 Quantum Chromodynamics

1.1 Introduction to QCD

Quantum Chromodynamics[1]-[3] is the non-abelian gauge theory describing strong interactions. The interactions are between quarks and gluons and amongst gluons themselves.
Quarks are spin 1/2 particles with fractional electric charge. The gluons are gauge fields which correspond to massless spin 1 particles with no electric charge. There are six flavours of quarks : up, down, strange, charm, bottom and top. The quarks also have three colour degrees of freedom : red, green and blue. The colour states can be identified with the three states in the fundamental representation of $SU(3)_{\text{colour}}$. The gluons are non-abelian gauge fields which carry colour charge and are responsible for mediating colour interactions between quarks and amongst themselves.
In 1964 Gell-Mann and Zweig introduced the concept of quarks [4]. This was in response to the huge volume of hadronic data discovered in the 50's and early 60's. There was a desire to understand the force that holds atomic nuclei together and to be able to calculate hadronic spectra. They proposed that hadrons can be constructed as flavor SU(3) states.
Evidence for quarks came from deep inelastic scattering experiments involving electrons and protons at SLAC in the late 60's. It was shown that electrons were scattering off pointlike constituents inside the protons. This was corroborated by subsequent neutrino experiments.
In 1972 Fritzsch, Gell-Mann and Leutwyler constructed the field theory of coloured quarks and gluons using a Yang-Mills theory based on gauged SU(3) of colour [5]. The theory was renormalizable, contained colour conservation and was gauge invariant with quark masses supplied by the same mechanism as the electro-weak theory. There are two well-known experiments which show the existence of three colour degrees of freedom. One is the formula for $\pi^0 \to 2\gamma$ decay. For $N_c = 3$ the experimental and the theoretically predicted rates agree with each other. Another is the ratio R measured at SLAC for the cross-section $R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$ in e^+e^- annihilation where once again theoretical predictions and experimental results match for $N_c = 3$.
An important feature of QCD is the phenomenon of asymptotic freedom [6]. As the distance between particles increase so the magnitude of the effective coupling constant decreases. Thus, perturbative methods can be used for small distances where the effective coupling constant is < 1.

1.2 QCD Lagrangian

The Lagrangian for free quarks fields with mass m is given by

$$L_0(x) = i\bar{q}_\alpha^A(x)\gamma^\mu\partial_\mu q_\alpha^A(x) - m\bar{q}_\alpha^A(x)\,q_\alpha^A(x) \;, \tag{1.1}$$

where $q_\alpha^A(x)$ represents the quark fields with colour components given by $\alpha =$ red, blue, green. The flavour components are A=1,2,..N_f where N_f is the number of flavours. Summation takes place over A and α.
In the above form the Lagrangian is invariant under global gauge transformations but not under local gauge transformations of the type

$$q_\alpha^A(x) \to q_\alpha^{'A}(x) = G(x)q_\alpha^A(x) \equiv \left[e^{-igT_a\theta_a}\right] q_\alpha^A(x) \qquad (a = 1, 2, ...8) \;, \tag{1.2}$$

where θ_a are eight real space-time functions, T_a are the eight generators of SU(3) and g is the coupling constant in QCD.
Under (1.2) the Lagrangian transforms as

$$L_0(x) \to L_0'(x) = i\bar{q}_\alpha^A(x)\gamma^\mu\left[G^\dagger(x)\,\partial_\mu G(x)\right]q_\alpha^A(x) - m\bar{q}_\alpha^A(x)\,q_\alpha^A(x) \;. \tag{1.3}$$

By substituting the usual derivative ∂_μ by a covariant one one is able to make the Lagrangian density invariant under local gauge transformations. The covariant derivative is given by

$$\partial_\mu \to D_\mu \equiv \partial_\mu - igT_{(a)}B_\mu^{(a)}(x) \;, \tag{1.4}$$

where $B_\mu^{(a)}(x)$ are the eight gluon fields. Redefining

$$B_\mu(x) \to igT_{(a)}B_\mu^{(a)}(x) \;, \tag{1.5}$$

we obtain

$$\partial_\mu \to D_\mu \equiv \partial_\mu - B_\mu(x) \;. \tag{1.6}$$

We see now that the new Lagrangian density will be invariant under local gauge transformations provided $q^A(x)$ and $D_\mu q^A(x)$ transform alike when acted on by $G(x)$. Thus we must have

$$\begin{aligned} D_\mu q_\alpha^A(x) &\to D_\mu' q_\alpha^{'A}(x)\,1.7 \\ &= G(x)D_\mu q^A(x) \;. \end{aligned} \tag{1.7}$$

This implies that the gluon fields transform as

$$
\begin{aligned}
B_\mu(x) &\rightarrow B'_\mu(x)\,1.8 \\
&= G(x)\,B_\mu(x)\,G^{-1}(x) + \left[\partial_\mu G(x)\right]G^{-1}(x) \ .
\end{aligned}
\tag{1.8}
$$

The kinetic energy terms for the gluons are described by adding to the Lagrangian the term

$$
-\frac{1}{4}F^{(a)}_{\mu\nu}(x)F^{\mu\nu}_{(a)}(x) \ , \tag{1.9}
$$

where

$$
\begin{aligned}
-igF_{\mu\nu}(x) &\equiv -\left[D_\mu, D_\nu\right] \\
&= -ig\left(\partial_\mu B_\nu(x) - \partial_\nu B_\mu(x)\right)_a T^a 1.10 \\
&\quad -ig^2 f_{abc} T^{(c)} B^{(a)}_\mu(x)\,B^{(b)}_\nu(x) \ .
\end{aligned}
\tag{1.10}
$$

The Lagrangian now takes the form

$$
\begin{aligned}
L_0(x) &= -\frac{1}{4}F^{(a)}_{\mu\nu}(x)F^{\mu\nu}_{(a)}(x) \\
&\quad +i\overline{q}^A_\alpha(x)\,\gamma^\mu\left(D_\mu\right)_{\alpha\beta} q^A_\beta(x)\,1.11 \\
&\quad -m\overline{q}^A_\alpha(x)\,q^A_\alpha(x) \ .
\end{aligned}
\tag{1.11}
$$

1.3 Global Symmetries of the Lagrangian

The QCD Lagrangian is invariant under the U(1) transformation

$$
q(x) \rightarrow q'(x) = \exp\left(-i\theta I\right) q(x) \ , \tag{1.12}
$$

where θ is a real constant and I the unit matrix. There is an associated conserved baryonic current

$$
J^\mu = \overline{q}(x)\,\gamma^\mu q(x) \ , \tag{1.13}
$$

and an associated charge

$$
B = \int d^3x \ J^0(t, \overrightarrow{x}) \ . \tag{1.14}
$$

The Lagrangian is also invariant under $\mathrm{U}_1(1) \otimes \mathrm{U}_2(1) \otimes ... \otimes \mathrm{U}_{N_f}$ group transformations of the type

$$q^A(x) \rightarrow q'^A(x) = \exp(-i\theta_A I)\, q_A(x) \ , \qquad A = 1, .., N_f \ , \tag{1.15}$$

where θ_A are real constants. The associated conserved vector baryonic current is

$$J^\mu = \overline{q}^A(x)\, \gamma^\mu q^A(x) \ . \tag{1.16}$$

Setting the quark masses equal to each other one can enlarge the symmetry to a global SU(N_f) symmetry.
Taking the quark masses equal to zero L_{QCD} is invariant under the transformation

$$q^A(x) \rightarrow q'^A(x) = \exp(-i\theta_A I \gamma_5)\, q_A(x) \ , \qquad A = 1, .., N_f \ , \tag{1.17}$$

with a conserved axial baryonic current

$$J_5^\mu = \overline{q}^A(x)\, \gamma^\mu \gamma_5 q^A \ . \tag{1.18}$$

The masses of the up and down quarks are both close to zero implying that SU(2) is almost an exact symmetry of L_{QCD}. By comparison the mass of the strange quark is fairly large leading to SU(3) symmetry being broken. In fact, the divergence of the flavour-changing vector current in the up- and down-quark sector is given by

$$\partial^\mu V_\mu(x) = (m_d - m_u) : \bar{d}(x) u(x) : \tag{1.19}$$

In spite of the fact that $m_d \simeq 2m_u$, SU(2) flavour symmetry is almost exact because these quark masses are very small in comparison with typical hadronic masses, as well as with the QCD scale Λ_{QCD}, to wit: $m_d \simeq 7$ MeV, $m_u \simeq 4$ MeV, at a scale of 1 GeV, and $\Lambda_{QCD} \simeq 300-400$ MeV [31]. However, in the case of the strangeness-changing vector current divergence, this is now only approximately conserved, as the strange-quark mass ($m_s \simeq 150 - 250$ MeV [31]) is not negligible in comparison with hadronic masses or Λ_{QCD}.
If the quark masses are set equal, then again L_{QCD} is invariant under the transformation

$$q(x) \rightarrow q'(x) = \exp\left(-i\theta^{(A)} T^{(A)}\right) q(x) \ , \tag{1.20}$$

where $\theta^{(A)}$ are constant parameters and $T^{(A)}$ are the generators of SU(N_f) with the corresponding conserved current

$$V_{\mu}^{(A)}(x) = \overline{q}^{i}(x)\,\gamma_{\mu} T_{ij}^{(A)} q^{j}(x) \ , \qquad (1.21)$$

and associated charge

$$Q^{(A)} = \int d^{3}x \ V_{0}(t, \overrightarrow{x}) \ . \qquad (1.22)$$

If the masses are set to zero then $L_{\rm QCD}$ is invariant under

$$q(x) \rightarrow q^{'}(x) = \exp\left(-i\theta_{5}^{(A)} T^{(A)} \gamma_{5}\right) q(x) \ , \qquad (1.23)$$

with the corresponding current

$$A_{\mu}^{(A)}(x) = \overline{q}^{i}(x)\,\gamma_{\mu}\gamma_{5} T_{ij}^{(A)} q^{j}(x) \ , \qquad (1.24)$$

and associated charge

$$Q_{5}^{(A)} = \int d^{3}x \ A_{0}(t, \overrightarrow{x}) \ . \qquad (1.25)$$

The charges

$$\begin{aligned} Q_{L}^{(A)} &= Q^{(A)} - Q_{5}^{(A)} \ , 1.26a \\ Q_{R}^{(A)} &= Q^{(A)} + Q_{5}^{(A)} \ .1.26b \end{aligned} \qquad (1.26)$$

are the generators of chiral SU$_L(N_f)\otimes$SU$_R(N_f)$ which is an exact symmetry in the massless limit. In fact, in the case of two light flavours (up and down), the divergence of the axial-vector current is

$$\partial^{\mu} A_{\mu}(x) = (m_d + m_u) : \bar{d}(x)\gamma_5 u(x) : \qquad (1.27)$$

which is almost conserved since the quark masses are negligible in comparison with hadronic and QCD scales. It should be mentioned that according to the Gell-Mann, Oakes and Renner relation [3] the light quark masses are of the same order in chiral-symmetry breaking as the pion mass, which itself is also small in comparison with the typical 1 GeV hadronic scale. The size of chiral-symmetry breaking in this case is traditionally gauged by the deviations from the Goldberger- Treiman relation [33], which are at the 1-2 % level. Including the third (strange) flavor leads to a moderate breaking of $SU(3)\otimes SU(3)$; deviations from the Goldberger-Treiman relation are now at the 20-30 % level.

1.4 Regularization and Renormalization

In QCD it is necessary to calculate the Green's functions of Feynman diagrams containing loops. This is because in perturbative theory we need to perform calculations involving higher orders in α_s. However, these loop diagrams contain ultraviolet divergences. These divergences can, however, be subtracted out through a suitable renormalization scheme. Before renormalization is performed though, the integrals need to be mathematically well-defined. The divergent integrals are made meaningful through the process of regularization.

Dimensional regularization is the traditional method used in QCD [7]-[10]. Here the 4-dimensional integral $\mathrm{d}^4\mathrm{k}$ is replaced by the D-dimensional one $\mathrm{d}^\mathrm{D}\mathrm{k}$ where D = 4 - ε. Thus, the space-time dimension D is less than four and the divergent 4-dimensional integral is replaced by a convergent D-dimensional one. In performing dimensional regularization one needs to redefine, in D dimensions, the Dirac algebra, the dimensions of the fields, coupling constants, and gauge parameters. Dimensional regularization is a purely mathematical procedure. Thus no new fields or couplings are introduced in the Lagrangian density and all symmetry properties are preserved.

After dimensional regularization has been performed and the integrals calculated one finds that the divergent contributions to the Green's functions are separated from the finite terms and are contained in terms of the form $1/\varepsilon$. The next step is to renormalize the theory so that one obtains finite Green's functions in the limit $\varepsilon \longrightarrow 0$.

The divergent terms are removed by adding counterterms to the initial QCD Lagrangian. The counterterms exactly cancel the divergent terms. Thus we have

$$L_0 \longrightarrow L_0 + L_{0C} \; . \tag{1.28}$$

In L_{0C} each term in the initial Lagrangian is multiplied by a different coefficient $\mathrm{C_i}$. The $\mathrm{C_i}$'s are a power series in α_s where

$$\alpha_s = \frac{g^2}{4\pi}\left(\nu^2\right)^{-\frac{\varepsilon}{2}} \; , \tag{1.29}$$

and ν is an arbitrary mass scale parameter.

Defining $\mathrm{Z_i} \equiv 1 - \mathrm{C_i}$ one may write the renormalized QCD Lagrangian after renormalizing the fields, coupling constants, mass and gauge parameters by multiplying them by the appropriate $\mathrm{Z_i}$'s. The different renormalization schemes involve different methods of choosing the constants $\mathrm{C_i}$ and thus $\mathrm{Z_i}$. The calculations in this thesis have been performed in the modified minimal subtraction scheme ($\overline{\mathrm{MS}}$). In this scheme the $\mathrm{Z_i}$ are chosen so as to eliminate terms of the form ($1/\varepsilon$ - $\ln 4\pi$ + γ_E) from the regularized Green's functions. Here Euler's constant $\gamma_E = 0.5772$.

1.5 Renormalization Group Equations (RGE)

Depending on the renormalization scheme we have different expressions for physical quantities. However, since these physical quantities arise from a unique Lagrangian they are thus equivalent. Thus, the physical quantities must be independant of the renormalization scheme chosen. The relation between a renormalized and bare Green's function is given by

$$\Gamma_R\left(p_1, ..., p_N; \alpha_s, a, m_i, \nu\right) = Z_R\left(\nu, \varepsilon\right) \Gamma_0\left(p_1, ..., p_N; \alpha_s^0, a^0, m_i^0, \varepsilon\right) \ , \qquad (1.30)$$

where Z(R) denotes the appropriate product of renormalization constants in the particular scheme.
Starting from Eq.(1.30) the fundamental equation of the renormalization group can be derived:

$$\left\{-\frac{\partial}{\partial t} + \beta\left(\alpha_s\right)\alpha_s\frac{\partial}{\partial \alpha_s} + \beta_a\left(\alpha_s\right)\frac{\partial}{\partial a} - \sum_A\left[1+\gamma_A\left(\alpha_s\right)\right]x_A\frac{\partial}{\partial x_A} + d_\Gamma - \gamma_\Gamma\left(\alpha_s\right)\right\} \qquad (1.31)$$
$$\cdot\Gamma_R\left(e^t p_1, ..., e^t p_N; \alpha_s, a, m_i, \nu\right) = 0 \ ,$$

where

$$\begin{aligned} x_A &= \frac{m_A}{\nu} \ , \\ \alpha_s\beta\left(\alpha_s\right) &= \nu\frac{d\alpha_s}{d\nu} \ , 1.32 \\ -\gamma_A\left(\alpha_s\right) &= \frac{\nu}{m_A}\frac{dm_A}{d\nu} \ , \\ \beta_a\left(\alpha_s\right) &= \nu\frac{da}{d\nu} \ . \end{aligned} \qquad (1.32)$$

Here d_Γ is the dimension of the Green's function. The anomalous dimension, γ_Γ, is a function of Z_i and is a function of the number of external quark, gluon and ghost lines.
Before solving for the general solution to the fundamental equation one needs to solve the differential equations for the effective coupling constant, mass and gauge parameter

$$
\begin{aligned}
\frac{d\overline{\alpha}_s(t,\alpha_s)}{dt} &= \overline{\alpha}_s\beta(\overline{\alpha}_s) \quad , \quad \overline{\alpha}_s(0,\alpha_s) = \alpha_s \; , \\
\frac{dx_j(t,\alpha_s)}{dt} &= -\left[1+\gamma_j(\overline{\alpha}_s)\right]\overline{x_j} \quad , \quad \overline{x}_j(0,\alpha_s) = x_j \; , \\
\frac{d\overline{a}(t,\alpha_s)}{dt} &= \beta(\overline{\alpha}_s) \quad , \quad \overline{a}(0,\alpha_s) = a \; .
\end{aligned}
\tag{1.33}
$$

Then we can solve for the fundamental equation as

$$
\begin{aligned}
\Gamma_R\left(e^t p_1,...,e^t p_N;\alpha_s,a,m_i,\nu\right) &= \Gamma_R\left(p_1,...,p_N;\overline{\alpha}_s,\overline{a},\overline{x}_i,\nu\right)(1.34) \\
&\cdot\exp\left\{tD-\int_0^t dt'\gamma_\Gamma\left[\overline{\alpha}_s\left(t',\alpha_s\right)\right]\right\}
\end{aligned}
$$

From the differential equations one obtains the four-loop expressions for the effective coupling constant and the quark mass for three flavours [11].

$$
\begin{aligned}
\frac{\overline{\alpha}_s(-q^2)}{\pi} &= \frac{4}{9}\frac{1}{L}-\frac{256}{729}\frac{LL}{L^2}1.35 \\
&+\left[6794-16384\left(LL-LL^2\right)\right]\frac{1}{59049}\frac{1}{L^3}+\mathcal{O}\left(\frac{1}{L^4}\right) \; , \\
\overline{m}_j\left(-q^2\right) &= \frac{\widehat{m}_j}{\left(\frac{1}{2}L\right)^{\frac{4}{9}}}\left\{1+(290-256LL)\frac{1}{729}\frac{1}{L}\right. \\
&+\left[\frac{550435}{1062882}-\frac{80}{729}\zeta(3)\right. \\
&\left.-\left(388736LL-106496LL^2\right)\frac{1}{531441}\right]\frac{1}{L^2} \\
&+\left[-\frac{126940037}{1162261467}-\frac{256}{177147}\beta_4+\frac{128}{19683}\gamma_4+\frac{7520}{531441}\zeta(3)\right. \\
&+\left(-\frac{611418176}{387420489}+\frac{112640}{531441}\zeta(3)\right)LL+\frac{335011840}{387420489}LL^2 \\
&\left.\left.-\frac{149946368}{1162261467}LL^3\right]\frac{1}{L^3}+\mathcal{O}\left(\frac{1}{L^4}\right)\right\} \; ,
\end{aligned}
\tag{1.34}
$$

where L = log ($-q^2/\Lambda^2_{QCD}$), LL = log L and

$$\beta_4 = -\frac{281198}{4608} - \frac{890}{32}\zeta(3) \ , \tag{1.36}$$

with $\gamma_4 = 88.5258$. The invariant mass, $\widehat{m}_j$, is a constant introduced through the Renormalization Group Equations, and $\zeta(z)$ is the Riemann zeta - function.

The terms of order $\mathcal{O}\left(\frac{1}{L^4}\right)$ above are known up to a constant not determined by the renormalization group [12]. This constant can be estimated e.g. using Pade' approximants. However, we have checked that our results are essentially unchanged if these terms are included.

2 Sum Rules in QCD

QCD Sum Rules [13]-[22] arise from the need to be able to obtain reliable results with respect to properties of the lowest-mass hadronic states, determining effective coupling constants or revealing information about the internal wave functions of hadrons.

As mentioned earlier QCD is an asymptotically free theory. Increasing the momentum (decreasing distances) leads to a decrease in the coupling constant. Thus, at high momentum transfers (short distances) the quarks can be considered asymptotically free. ie. the interaction between them is negligible. On the other hand at low momentum transfers (large distances) the coupling constant increases and the quarks can only exist in bound hadronic states.

Because of the small coupling constant at high momentum transfers (short distances) one is able to use perturbative theory. However, at low momentum transfer (large distances), perturbation theory breaks down as the effective coupling constant becomes extremely large. To have an accurate theory one must be able to combine both the low and high momentum regions.

QCD Sum Rules were first formulated by Shifman, Vainshtein and Zakharov [13]. In this paper they took the T product of two currents and applied the Wilson Operator Product Expansion (O.P.E) [23] to it.

They thus obtained :

$$i\int d^4x\, e^{iqx}\, T\,\{j(x)\, j(0)\} = C_I I + \sum_n C_n(q)\, O_n\;, \tag{2.1}$$

where I, O_n are local spin-zero operators and $C_I, C_n(q)$are coefficients.

The key to this OPE is that Shifman et al. gave convincing arguments in support of its validity in the presence of non-perturbative terms. The dimension of the operators concerned can be calculated using $\dim[q] = \frac{3}{2}, \dim[G_{\mu\nu}] = 2, \dim[m] = 1$. Only spin-zero operators are considered since in dealing with two-point functions in QCD sum rules only these contribute to the vacuum expectation values. In our calculations we deal only with operators of $\dim \leq 4$ since in this case operators of $\dim = 6$ are numerically negligible. The coefficients in the expansion are functions of q^{-2} and as the dimensionality of the operator increases so the coefficients contain higher powers of q^{-2}. In the OPE we assume that the short distance effects are contained in the Wilson coefficients whilst the long distance effects are contained in the vacuum condenstates. Thus, our OPE takes place with respect to the following operators:

$$\begin{aligned} I, d &= 0 \\ O_m &= m\overline{q}q \ , \ d = 4 \ 2.2 \\ O_G &= G^a_{\mu\nu} G_{a\mu\nu} \ , \ d = 4 \ . \end{aligned} \tag{2.2}$$

We see then that the purely perturbative terms are contained in the term $C_I I$ whilst the rest of the terms in the OPE are non-perturbative. Vacuum condensates involving these terms are assumed to be the contributing factors behind the resonance structure observed at low energies.

The next step in the sum rule calculation is to take a two-point function given by:

$$\Pi\left(q^2\right) = i \int d^4x e^{iq\cdot x} \left\langle 0 \left| T\left(j\left(x\right) j^\dagger\left(0\right)\right) \right| \right\rangle \ , \tag{2.3}$$

and expand it in terms of Wilson's OPE. Thus, one is able to formulate a theoretical expression for a two-point function. Equally important to the success of QCD Sum Rules is that one is also able to find an expression for the two-point function in terms of hadrons and hadronic resonances.

In formulating the general formula for the different types of sum rules we use Cauchy's theorem and obtain the dispersion relation

$$\Pi\left(Q^2\right) = \frac{1}{\pi} \int\limits_0^\infty ds \frac{\operatorname{Im}\Pi\left(s\right)}{s + Q^2} + subtraction\ terms \ , \tag{2.4}$$

where $Q^2 = -q^2 > 0$.

On the right hand side of the relation is the integral containing the hadronic representation of the two-point function , the hadronic spectral function. The subtraction terms are associated with the asymptotic behaviour of $\operatorname{Im}\Pi(s)$. These are required to render $\Pi(Q^2)$ finite in the limit s $\rightarrow \infty$.

However, whilst the above dispersion relation seem satisfying from a theoretical point of view, from a practical point of view it presents a problem. The most important contribution on the right hand side to the spectral function arises from resonances in the low-energy region. On the left hand side of the relation we consider only the first few terms in the series $\frac{1}{Q^2}$. We would like to improve the accuracy of this approximation by increasing the importance of these terms and suppressing the importance of higher orders in $\frac{1}{Q^2}$.

Thus, we require a transformation which enhances the low-energy contribution on the right and suppresses it on the left. This is achieved by taking the Laplace transform of both sides. The Laplace transform is defined by:

$$L = \lim_{Q^2\to\infty N\to\infty \frac{Q^2}{N}=M^2} \frac{(-1)^N}{(N-1)!} \left(Q^2\right)^N \frac{d^N}{(dQ^2)^N} \ . \tag{2.5}$$

Taking the Laplace transform of both sides of Eq. (2.4) we obtain

$$\hat{L}\left[\Pi\left(Q^2\right)\right] = \Pi\left(M^2\right) = \frac{1}{M^2}\int_{s_0}^{\infty} ds \; e^{-s/M^2}\frac{1}{\pi}\,\mathrm{Im}\,\Pi(s) \ . \tag{2.6}$$

The exponential term on the right hand side of Eq. (2.6) suppresses the high-energy tail of the spectral function, enhancing the better known low energy region. At the same time, the Laplace transform introduces an additional factorial suppression of the higher dimensional condensates in the OPE.

3 Sum Rules to calculate the magnitude of flavor SU(3) symmetry breaking in the vacuum

In this section we discuss the determination of the magnitude of flavour SU(3) symmetry breaking in the QCD nonperturbative vacuum. This is given in terms of the ratio of the quark condensates $\frac{\langle \overline{s}s\rangle}{\langle \overline{u}u\rangle}$. We have assumed SU(2) vacuum symmetry, ie. $\langle \overline{u}u\rangle \simeq \left\langle \overline{d}d\right\rangle$. Expressions for the condensates in terms of the renormalization-group-invariant quantities $\psi(0)$ and $\psi_5(0)$ come from the current algebra Ward identities

$$\psi(0)^s_u = -(\overline{m}_s - \overline{m}_u)\left\langle \overline{\psi}_s\psi_s - \overline{\psi}_u\psi_u\right\rangle , \tag{3.1a}$$

$$\psi_5(0)^s_u = -(\overline{m}_s + \overline{m}_u)\left\langle \overline{\psi}_s\psi_s + \overline{\psi}_u\psi_u\right\rangle . \tag{3.1b}$$

Taking $m_u \simeq 0$ and defining the ratio

$$R_{VA} \equiv \frac{\psi(0)^s_u}{\psi_5(0)^s_u} , \tag{3.2}$$

one finds the following expression for the magnitude of symmetry breaking [24]

$$\frac{\langle \overline{s}s\rangle}{\langle \overline{u}u\rangle} \simeq \frac{1+R_{VA}}{1-R_{VA}} . \tag{3.3}$$

Thus, it is necessary to find an accurate and reliable estimate of $\psi(0)^s_u$ and $\psi_5(0)^s_u$. We achieve this through Laplace-transform QCD sum rules.
Various determinations of $\frac{\langle \overline{s}s\rangle}{\langle \overline{u}u\rangle}$ were attempted in the past [24] – [25], with results in the range : $\frac{\langle \overline{s}s\rangle}{\langle \overline{u}u\rangle} \simeq 0.5 - 1$. Many of the determinations involved expression for $\psi_{(5)}(0)$ which contained unwanted logarithmic quark mass singularities. Also, the corresponding perturbative expressions were known only up to two - loops. Since then, the issue of the above mentioned singularities has been resolved [26] ,[28], and the perturbative expansion of the relevant two-point functions needed to extract $\psi_{(5)}(0)$ are now known to four - loops. At the same time, there is improved information on the hadronic spectral functions, particularly the threshold behaviour fixed by chiral perturbation

theory. All of this will allow for a more accurate determination of the ratio in Eq. 3.3.

In calculating $\psi(0)_u^s$ and $\psi_5(0)_u^s$ we make use of the two-point functions $\psi(q^2)$ and $\psi_5(q^2)$ defined by

$$\psi\left(q^2\right) = i \int d^4x \; e^{iqx} \left\langle 0 \left| T\left(\partial^\mu V_\mu(x)\, \partial^\nu V_\nu^\dagger(0)\right)\right|\right\rangle \;, \tag{3.4}$$

with V_μ being the strangeness-changing vector current and

$$\psi_5\left(q^2\right) = i \int d^4x \; e^{iqx} \left\langle 0 \left| T\left(\partial^\mu A_\mu(x)\, \partial^\nu A_\nu^\dagger(0)\right)\right|\right\rangle \;, \tag{3.5}$$

with A_μ being the strangeness-changing axial-vector current.

The functions actually used in this thesis are the functions $\phi(q^2)$ and $\xi(q^2)$ which satisfy the following dispersion relations

$$\phi\left(q^2\right) = \frac{1}{\pi}\int_{t_0}^{\infty} \frac{dt}{t}\frac{\mathrm{Im}\,\psi_5(t)}{(t+Q^2)^2}\;, \tag{3.6a}$$

$$\xi\left(q^2\right) = \frac{1}{\pi}\int_{t_0}^{\infty} \frac{dt}{t}\frac{\mathrm{Im}\;\psi(t)}{(t+Q^2)^2}\;, \tag{3.6b}$$

where

$$\phi\left(q^2\right) = \frac{\partial}{\partial q^2} D_5\left(q^2\right)\;, \tag{3.7a}$$

$$\xi\left(q^2\right) = \frac{\partial}{\partial q^2} D\left(q^2\right)\;, \tag{3.7b}$$

and the functions $D_5(q^2)$and $D(q^2)$are related to $\psi_5(q^2)$ and $\psi(q^2)$, respectively, through the Ward identities

$$q^2 D_5(q^2) = \psi_5\left(q^2\right) - \psi_5(0)\;, \tag{3.8a}$$

$$q^2 D(q^2) = \psi\left(q^2\right) - \psi(0)\;. \tag{3.8b}$$

The function $\psi_{(5)}(Q^2)$has been calculated to order α_s^3 [26] – [28]. In doing so it was assumed that $N_f = 3$ and the mass of the up quark was taken to be negligible compared to the mass of the strange quark. With $L = \ln\left(\frac{Q^2}{\nu^2}\right)$ and neglecting dimension six vacuum condensates one obtains the result

$$
\begin{aligned}
\psi_{(5)}\left(Q^2\right) \; = \; & \frac{3}{8\pi^2}m_s^2\left\{Q^2 L\left[\,1+\frac{\alpha_s}{\pi}\left(\frac{17}{3}-L\right)\right.\right. \qquad (3.9)\\
& +\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{9681}{144}-\frac{35}{2}\zeta(3)-\frac{95}{6}L+\frac{17}{12}L^2\right]\\
& +\left(\frac{\alpha_s}{\pi}\right)^3\left[\,\frac{4748953}{5184}+\frac{715}{12}\zeta(5)-\frac{91519}{216}\zeta(3)-\frac{\pi^4}{36}\right.\\
& \left.\left.\left.-\left(\frac{4781}{18}-\frac{475}{8}\zeta(3)\right)L+\frac{229}{6}L^2-\frac{221}{96}L^3\,\right]+\mathcal{O}\left(\alpha_s^4\right)\right]\right\}\\
& -\frac{3}{4\pi^2}m_s^4\left[1-L+\frac{\alpha_s}{\pi}\left(\frac{25}{3}-4\zeta(3)-\frac{16}{3}L+2L^2\right)\right]\\
& +\frac{1}{2}\frac{m_s^2}{Q^2}\left[\,\langle m_s\overline{s}s\rangle\left(1+\frac{\alpha_s}{\pi}\left(\frac{11}{3}-2L\right)\right)\right.\\
& \left.\pm 2\langle m_s\overline{u}u\rangle\left(1+\frac{\alpha_s}{\pi}\left(\frac{14}{3}-2L\right)\right)\,\right]\\
& +\frac{1}{8}\frac{m_s^2}{Q^2}\left\langle\tfrac{\alpha_s}{\pi}G^2\right\rangle+\frac{3}{16\pi^2}\frac{m_s^2}{Q^2}m_s^4\left(1+2L\right)\;.
\end{aligned}
$$

Substituting Eq. (3.9) in Eq.(3.7a) and Eq.(3.7b) we find for $\xi\ \left(Q^2\right)$and $\phi\left(Q^2\right)$

$$
\begin{aligned}
\left.\begin{array}{c}\xi\left(Q^2\right)\\ \phi\left(Q^2\right)\end{array}\right\} \; = \; & \frac{3}{8\pi^2}\frac{m_s^2}{Q^2}\left\{1+\frac{\alpha_s}{\pi}\left(\frac{17}{3}-2L\right)\right. \qquad (3.10)\\
& +\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{9631}{144}-\frac{35}{2}\zeta(3)-\frac{95}{3}L+\frac{17}{4}L^2\right]\\
& +\left(\frac{\alpha_s}{\pi}\right)^3\left[\,\frac{4748953}{5184}+\frac{715}{12}\zeta(5)-\frac{91519}{216}\zeta(3)-\frac{\pi^4}{36}\right.\\
& \left.\left.-2\left(\frac{4781}{18}-\frac{475}{8}\zeta(3)\right)L+\frac{229}{2}L^2-\frac{221}{24}L^3\,\right]+\mathcal{O}\left(\alpha_s^4\right)\right\}\\
& +\frac{3}{4\pi^2}\frac{m_s^4}{Q^4}\left[2-L+\frac{\alpha_s}{\pi}\left(\frac{41}{3}-4\zeta(3)-\frac{28}{3}L+2L^2\right)\right]\\
& -\frac{m_s^2}{Q^6}\left[\,\langle m_s\overline{s}s\rangle\left(1+\frac{\alpha_s}{\pi}\left(\frac{14}{3}-2L\right)\right)\right.\\
& \left.\pm 2\langle m_s\overline{u}u\rangle\left(1+\frac{\alpha_s}{\pi}\left(\frac{17}{3}-2L\right)\right)\,\right]\\
& -\frac{1}{4}\frac{m_s^2}{Q^6}\left\langle\tfrac{\alpha_s}{\pi}G^2\right\rangle-\frac{3}{4\pi^2}\frac{m_s^6}{Q^6}L+\frac{\psi_{(5)}(0)}{Q^4}\;.
\end{aligned}
$$

Taking the Laplace transform (see Appendix A) and using the Renormalization Group Equation improvements, i.e. setting $\nu^2 = M_L^2$ $\left(\therefore \ln\left(\frac{M_L^2}{\nu^2}\right) = 0\right)$we obtain

$$
\begin{aligned}
\left.\begin{array}{l}\xi\left(M_L^2\right)\\ \phi\left(M_L^2\right)\end{array}\right\} =\ & \frac{3}{8\pi^2}\frac{\overline{m}_s^2\left(M_L^2\right)}{M_L^2}\left\{1+\tfrac{\overline{\alpha}_s\left(M_L^2\right)}{\pi}\left(\frac{17}{3}-2\Psi\left(1\right)\right)\right. \\
& +\left(\tfrac{\overline{\alpha}_s\left(M_L^2\right)}{\pi}\right)^2\left[\frac{9631}{144}-\frac{35}{2}\zeta\left(3\right)-\frac{95}{3}\Psi\left(1\right)\right. \\
& \left.+\frac{17}{4}\left[\Psi^2\left(1\right)-\Psi'\left(1\right)\right]\right] \\
& +\left(\tfrac{\overline{\alpha}_s\left(M_L^2\right)}{\pi}\right)^3\left[\frac{4748953}{5184}+\frac{715}{12}\zeta\left(5\right)-\frac{91519}{216}\zeta\left(3\right)-\frac{\pi^4}{36}\right. \\
& -2\left(\frac{4781}{18}-\frac{475}{8}\zeta\left(3\right)\right)\Psi\left(1\right)+\frac{229}{2}\left[\Psi^2\left(1\right)-\Psi'\left(1\right)\right] \\
& \left.\left.+\frac{221}{24}\left[3\Psi\left(1\right)\Psi'\left(1\right)-\Psi^3\left(1\right)-\Psi''\left(1\right)\right]\right]\right\} \\
& +\frac{3}{4\pi^2}\frac{\overline{m}_s^4\left(M_L^2\right)}{M_L^4}\left[2-\Psi\left(2\right)\right. \\
& \left.+\tfrac{\overline{\alpha}_s\left(M_L^2\right)}{\pi}\left(\frac{41}{3}-4\zeta\left(3\right)-\frac{28}{3}\Psi\left(2\right)+2\left(\Psi^2\left(2\right)-\Psi'\left(2\right)\right)\right)\right] \\
& -\frac{1}{2}\frac{\overline{m}_s^2\left(M_L^2\right)}{M_L^6}\left[\left\langle m_s\overline{s}s\right\rangle\left(1+\tfrac{\alpha_s\left(M_L^2\right)}{\pi}\left(\frac{14}{3}-2\Psi\left(3\right)\right)\right)\right. \\
& \left.\pm2\left\langle m_s\overline{u}u\right\rangle\left(1+\tfrac{\overline{\alpha}_s\left(M_L^2\right)}{\pi}\left(\frac{17}{3}-2\Psi\left(3\right)\right)\right)\right] \\
& -\frac{1}{8}\frac{\overline{m}_s^2\left(M_L^2\right)}{M_L^6}\left\langle\tfrac{\alpha_s}{\pi}G^2\right\rangle-\frac{3}{8\pi^2}\frac{\overline{m}_s^6\left(M_L^2\right)}{M_L^6}\Psi\left(3\right)+\frac{\psi_{(5)}\left(0\right)}{M_L^4}\ ,
\end{aligned}
\tag{3.11}
$$

where $\Psi\left(1\right)=-\gamma_E$, $\gamma_E=0.5772$, $\Psi\left(z+1\right)=\Psi\left(z\right)+\frac{1}{z}$.
The hadronic spectral function is usually written as

$$
\frac{1}{\pi}\operatorname{Im}\Pi\left(s\right)|_{HAD}=\frac{1}{\pi}\operatorname{Im}\Pi\left(s\right)|_{RES}\theta\left(s_0-s\right)+\frac{1}{\pi}\operatorname{Im}\Pi\left(s\right)|_{QCD}\theta\left(s-s_0\right)\ ,
$$

where the first term describes the contribution of resonances known experimentally up to a threshold value s_0. The second term is the hadronic continuum which is usually taken to be the perturbative QCD expression.
For the resonance expression we assume dominance of the lowest-lying states. The hadronic spectral function used in the resonance term is defined by

$$\frac{1}{\pi}\,\mathrm{Im}\,\Pi\left(s\right)=\left(2\pi\right)^{3}\sum_{n}\left\langle 0\left|J_{n}\left(0\right)\right|n\right\rangle\left\langle n\left|J_{n}\left(0\right)^{\dagger}\right|0\right\rangle\delta^{4}\left(q-p_{n}\right)\ , \qquad (3.12)$$

where J is either $\partial^{\mu}V_{\mu}$ or $\partial^{\mu}A_{\mu}$ and summation is performed over the intermediate states with the correct quantum numbers.
In the case of the hadronic spectral function involving the divergences of the vector current, the appropriate intermediate states are $K^{+}\pi^{0}$ and $K^{0}\pi^{+}$. These states, however, resonate to form the $K_{0}^{*}(1430)$ and $K_{0}^{*}(1950)$ states. In turn the $K_{0}^{*}(1950)$ can also couple to the $K^{+}\eta^{'}$ intermediate state. The full expression for $\frac{1}{\pi}\,\mathrm{Im}\,\psi\left(s\right)$ can be written as [26], [28] , [29]

$$\begin{aligned}\frac{1}{\pi}\,\mathrm{Im}\,\psi\left(s\right) &= \frac{3}{32\pi^{2}}\left|d\right|^{2}\left\{\sqrt{\left(1-\tfrac{s_{+}}{s}\right)\left(1-\tfrac{s_{-}}{s}\right)}\right. \\ &\quad *\left\{BW_{1}\left(s\right)\theta\left(s_{c}-s\right)+BW_{2}\left(s\right)\theta\left(s-s_{c}\right)\right\} \\ &\quad \left.+CG\sqrt{\left(1-\tfrac{s_{+}^{'}}{s}\right)\left(1-\tfrac{s_{-}^{'}}{s}\right)}BW_{2}\left(s\right)\theta\left(s-s_{c}\right)\right\}\ ,\end{aligned} \qquad (3.13)$$

with $s_{c}=2.5\ \mathrm{GeV}^{2}$.
Here CG is the Clebsch-Gordon coefficient related to the $K_{0}^{*}(1950)$ resonance and

$$d\equiv d_{K^{0}\pi^{+}}\simeq 0.25GeV^{2}, \qquad (3.14)$$

where $d_{K^{0}\pi^{+}}$ is the form factor defined by

$$d_{K^{0}\pi^{+}}\equiv\left\langle\pi^{0}\left|i\partial^{\mu}V_{\mu}\right|K^{+}\right\rangle\ . \qquad (3.15)$$

Using isospin relations and Clebsch-Gordan coefficients we can also write $\left|d_{K^{+}\pi^{0}}\right|=\frac{1}{\sqrt{2}}d$.
The thresholds s_{+} and s_{-} are defined as $s_{\pm}=\left(M_{K}\pm\mu_{\pi}\right)^{2}$ and $s_{+}^{'}$ and $s_{-}^{'}$ as $s_{\pm}^{'}=\left(M_{K}\pm M_{\eta^{'}}\right)^{2}$.
The Breit-Wigner forms are chosen to take into account the fact that the $K\pi$ states resonate to form the $K_{0}^{*}(1430)$ and $K_{0}^{*}(1950)$ states. Below 2.5 GeV we deal with the Breit-Wigner form arising from the $K_{0}^{*}(1430)$ resonance and above 2.5 Gev we deal with the Breit-Wigner form arising from the $K_{0}^{*}(1950)$ resonance. The value 2.5 Gev is chosen to be the value at which the first resonance is equal to the second.

The hadronic spectral function containing the axial-vector divergences was constructed by Dominguez et al. [27] using the kaon-pole and it's radial excitations K(1460) and K(1830). The radial excitations can decay into the $K\pi\pi$ state as well as it's resonant sub-channels, the $K^*(892) - \pi$ and the $\rho\,(770) - K$ states. The $\rho\,(770) - K$ sub-channel turns out to be numerically negligible.
Thus, for the hadronic spectral function they obtained [27]

$$\begin{aligned}\frac{1}{\pi}\,\mathrm{Im}\,\psi_5\,(s)\,|_{HAD} &= 2f_K^2 M_K^4 \delta\left(s - M_K^2\right) \\ &\quad + \frac{1}{\pi}\,\mathrm{Im}\,\psi_5\,(s)\,|_{K\pi\pi}\left[\frac{BW_1\,(s) + \lambda BW_2\,(s)}{(1+\lambda)}\right] ,\end{aligned} \tag{3.16}$$

where through chiral symmetry normalization they found

$$\frac{1}{\pi}\,\mathrm{Im}\,\psi_5\,(s)\,|_{K\pi\pi} = \frac{M_K^4}{2f_\pi^2}\frac{3}{2^8\pi^4}\frac{I\,(s)}{s\,(M_K^2 - s)^2}\theta\left(s - M_K^2\right) , \tag{3.17}$$

and

$$\begin{aligned}I\,(s) &= \int_{M_K^2}^{s} \frac{du}{u}\left(u - M_K^2\right)(s-u) \times \{ \left(M_K^2 - s\right)\left[u - \frac{(s+M_K^2)}{2}\right] \\ &\quad - \frac{1}{8u}\left(u^2 - M_K^4\right)(s-u) + \frac{3}{4}\left(u - M_K^2\right)^2 |F_{K^*}\,(u)|^2 \} .\end{aligned} \tag{3.18}$$

Here

$$|F_{K^*}\,(u)|^2 = \frac{(M_{K^*}^2 - M_K^2)^2 + M_{K^*}^2\Gamma_{K^*}^2}{(M_{K^*}^2 - u)^2 + M_{K^*}^2\Gamma_{K^*}^2} , \tag{3.19}$$

is the contribution from the resonant sub-channel $K^*(892) - \pi$.
The parameter λ controls the importance of the second radial excitation, the K(1830), relative to the K(1460). It is taken as $\lambda = 1$ which results in a smaller contribution from the second radial excitation than the first.
Due to our approximation $m_u = 0$ the pion mass has been neglected in both spectral functions.

4 Results

The functions $\xi(Q^2)$ and $\phi(Q^2)$ satisfy the dispersion relations in Eq.(3.6), where according to duality the left hand side is computed in QCD, and the integral on the right hand side is saturated by the hadronic parametrization of the experimental data. Since $\xi(Q^2)$ and $\phi(Q^2)$ contain $\psi_{(5)}(0)$, the latter can thus be determined. The procedure is the same after performing the Laplace transforms of the QCD two-point functions and the integral of the data, viz.

$$\xi\left(M_L^2\right) = \frac{1}{M_L^4}\int_{s_+}^{\infty}\frac{ds}{s}e^{-s/M_L^2}\frac{1}{\pi}\,\mathrm{Im}\,\psi(s)\ , \tag{4.1}$$

where

$$s_+ = \left(M_K + \mu_\pi\right)^2\ , \tag{4.2}$$

and

$$\phi\left(M_L^2\right) = \frac{1}{M_L^4}\int_{s'_+}^{\infty}\frac{ds}{s}e^{-s/M_L^2}\frac{1}{\pi}\,\mathrm{Im}\,\psi_5(s)\ , \tag{4.3}$$

where

$$s'_+ = \left(M_K + 2\mu_\pi\right)^2\ , \tag{4.4}$$

with $\xi\left(M_L^2\right)$ and $\phi\left(M_L^2\right)$ given by Eq (3.11). The two equations above determine $\psi(0)$ and $\psi_5(0)$, respectively, in terms of the perturbative and non-perturbative QCD parameters in Eq.(3.11), which are known from other analyses, and in terms of the experimental data.

In both the axial-vector and the vector currents the integrals cannot be solved through analytical methods. For the vector current we use Romberg integration to evaluate the integral of the hadronic spectral function. In the case of the axial-vector current double Gaussian integration was performed.

In the determination of $\psi_{(5)}(0)$ we face a rather large parameter space, i.e. the various hadronic parameters in the spectral functions, and on the QCD side, the QCD scale $\Lambda_{\rm QCD}$, the invariant quark mass $\widehat{m}_s$, the continuum threshold and the vacuum condensates.

However, since the functions $\psi_{(5)}(0)$ determine the quark masses, the parameter space for the determination of $\psi_{(5)}(0)$ is reduced considerably in size. We shall adopt the same range of values of the hadronic and QCD parameters that have been used in [26] and [27] to determine the strange quark mass, with the exception of $\Lambda_{\rm QCD}$. The value of $\Lambda_{\rm QCD}$, extracted from experiment, has been increasing steadily over the years, from $\Lambda_{\rm QCD} \simeq 100$ MeV at the time the QCD sum rules were first proposed [13], to something as high as $\Lambda_{\rm QCD} \simeq 350 - 400$ MeV lately [31].

Such high values of $\Lambda_{\rm QCD}$ have the effect of invalidating the whole QCD sum rule program, as the perturbative contributions become huge, and the vacuum condensate terms become negligible. It has been argued in [32] that if $\Lambda_{\rm QCD} \gtrsim 330$ MeV the QCD sum rule program breaks down, and it is not even possible to extract numerical values for the condensates from experimental data. We shall then use the range $\Lambda_{\rm QCD} \simeq 300 - 320$ MeV, together with $\widehat{m}_s \simeq 150$ MeV (with errors in the 20% range), and [30]

$$\begin{aligned} \left\langle \tfrac{\alpha_s}{\pi} G^2 \right\rangle &\simeq 0.02 - 0.06 \text{ GeV}^4 \,, 4.5 \\ \langle m_s \bar{s} s \rangle &\simeq \langle m_s \bar{u} u \rangle \simeq -0.0015 \text{ GeV}^4 \,. \end{aligned} \tag{4.5}$$

Since $\frac{\langle \bar{s}s \rangle}{\langle \bar{u}u \rangle}$ is the object of this determination, we have followed an iteration procedure, i.e. we start with this ratio being unity, extract the ratio, and replace the new values, etc. The process converges very fast, as the numerical importance of the $\langle m_s \bar{s}s \rangle$ in $\xi\left(Q^2\right) \left(\phi\left(Q^2\right)\right)$ is very small (see Eqs. (3.10) - (3.11)).

Figures 1-3 show the results for $\psi(0)$, $\psi_5(0)$ and $\frac{\langle \bar{s}s \rangle}{\langle \bar{u}u \rangle}$ for $s_{0V} = s_{0A} = 6$ GeV^2, and $\Lambda_{\rm QCD} = 300$ MeV, whilst Figures 4-6 correspond to changing the continuum thresholds to $s_{0V} = s_{0A} = 8$ GeV^2. Figures 7-9 correspond to changing $\Lambda_{\rm QCD}$ to $\Lambda_{\rm QCD} = 320$ MeV.
Other intermediate values of these parameter give intermediate values for $\psi_{(5)}(0)$ and the ratio of the quark condensates. Rounding off the various results we conclude that

$$\begin{aligned} \psi(0) &\simeq -(0.0005 - 0.0010) \text{ GeV}^4 \,, 4.6 \\ \psi_{(5)}(0) &\simeq (0.0028 - 0.0035) \text{ GeV}^4, \end{aligned} \tag{4.6}$$

and

$$\frac{\langle \bar{s}s \rangle}{\langle \bar{u}u \rangle} = 0.7 \pm 0.1 \tag{4.7}$$

The result for $\psi_5(0)$ points to a large deviation from the naive PCAC (Partial Conservation of the Axial-Vector Current) prediction

$$\psi_5(0)|_{PCAC} = 2 f_K^2 \mu_K^2 \simeq 0.006 \text{ GeV}^4 \tag{4.8}$$

However, this deviation is consistent with the size of chiral SU(3)×SU(3) symmetry breaking as measured by deviations from the Goldberger - Treiman relation (GTR) [33]. In fact, one expects a correction to Eq. (4.8) of the order $\left(1 - \Delta_K\right)^2$, where $\Delta_K \simeq 0.25 - 0.30$ is the deviation from GTR.

We notice, in closing, that in spite of the various uncertainties in all the parameters, the result of Eq. (4.7) is clearly consistent with $\frac{\langle \bar{s}s \rangle}{\langle \bar{u}u \rangle} < 1$.

5 Appendix A : Laplace Transform Rules

The Laplace transform is defined as

$$\hat{L} = \lim_{N\to\infty} \frac{(-)^N}{(N-1)!} \left(Q^2\right)^N \frac{\partial^N}{\left(\partial Q^2\right)^N} \mid_{\frac{Q^2}{N} \equiv M_L^2 = fixed} \tag{A1}$$

In our Sum Rule calculations we have the expression of the form

$$\Pi\left(Q^2\right) = \frac{1}{\pi} \int ds \frac{\mathrm{Im}\,\Pi\,(s)}{s+Q^2} + polynomial \tag{A2}$$

Since

$$\hat{L}\,\{polynomial\} = 0$$

we have

$$\begin{aligned} \hat{L}\left\{\Pi\left(Q^2\right)\right\} &= \Pi\left(M_L^2\right) = \lim_{N\to\infty} \frac{(-)^N}{(N-1)!} \left(Q^2\right)^N \\ & * \frac{1}{\pi} \int ds\, \mathrm{Im}\,\Pi\,(s) \frac{\partial^N}{\left(\partial Q^2\right)^N} \left(\frac{1}{s+Q^2}\right) \end{aligned} \tag{A3}$$

Taking the N-th derivative and letting $N \to \infty$ we obtain

$$\Pi\left(M_L^2\right) = \left(\frac{1}{M_L^2}\right) \int ds\; e^{-\frac{s}{M_L^2}} \frac{1}{\pi}\, \mathrm{Im}\,\Pi\,(s) \tag{A4}$$

For the Laplace transform of the QCD derived expression $\Pi\left(Q^2\right)$ we have the following rules (with $x=\frac{-q^2}{\Lambda^2_{QCD}}$ and $y=\frac{M_L^2}{\Lambda^2_{QCD}}$):

i) $\hat{L}\left\{\frac{1}{x^{\alpha+1}(\ln x)^{\beta+1}}\right\}$

$$\begin{aligned}\hat{L}\left\{\frac{1}{x^{\alpha+1}\left(\ln x\right)^{\beta+1}}\right\} &= \frac{1}{\Gamma\left(\alpha+1\right)}\frac{1}{y^{\alpha+1}}\frac{1}{\left(\ln y\right)^{\beta+1}}\left\{1-\frac{\left(\beta+1\right)\psi\left(\alpha+1\right)}{\ln y}\right. \\ &+\frac{1}{2}\frac{\left(\beta+2\right)\left(\beta+1\right)}{\ln^2 y}\left[\psi^2\left(\alpha+1\right)-\psi'\left(\alpha+1\right)\right] \\ &+\frac{1}{3!}\frac{\left(\beta+3\right)\left(\beta+2\right)\left(\beta+1\right)}{\ln^3 y}\left[3\psi\left(\alpha+1\right)\psi'\left(\alpha+1\right)\right. \\ &\left.\left.-\psi^3\left(\alpha+1\right)-\psi''\left(\alpha+1\right)\right]+\ldots\right\}\end{aligned} \tag{A5}$$

ii) $\hat{L}\left\{\frac{\ln\ln x}{x^{\alpha+1}(\ln x)^{\beta+1}}\right\}$

$$\begin{aligned}\hat{L}\left\{\frac{\ln\ln x}{x^{\alpha+1}\left(\ln x\right)^{\beta+1}}\right\} &= \frac{1}{\Gamma\left(\alpha+1\right)}\frac{1}{y^{\alpha+1}}\frac{1}{\left(\ln y\right)^{\beta+1}}\ln\ln y\left\{1\right. \\ &-\frac{\left(\beta+1\right)\psi\left(\alpha+1\right)}{\ln y}+\frac{\left(\beta+1\right)\psi\left(\alpha+1\right)}{\left(\ln y\right)\left(\ln\ln y\right)} \\ &*\left[\psi\left(\beta+2\right)-\psi\left(\beta+1\right)\right] \\ &+\frac{1}{2}\frac{\left(\beta+2\right)\left(\beta+1\right)}{\ln^2 y}\left[\psi^2\left(\alpha+1\right)-\psi'\left(\alpha+1\right)\right] \\ &-\frac{1}{2}\frac{\left(\beta+2\right)\left(\beta+1\right)}{\left(\ln^2 y\right)\left(\ln\ln y\right)}\left[\psi^2\left(\alpha+1\right)-\psi'\left(\alpha+1\right)\right] \\ &*\left[\psi\left(\beta+3\right)-\psi\left(\beta+1\right)\right] \\ &+\frac{1}{3!}\frac{\left(\beta+3\right)\left(\beta+2\right)\left(\beta+1\right)}{\ln^3 y}\left[3\psi\left(\alpha+1\right)\psi'\left(\alpha+1\right)\right. \\ &\left.-\psi^3\left(\alpha+1\right)-\psi''\left(\alpha+1\right)\right] \\ &-\frac{1}{3!}\frac{\left(\beta+3\right)\left(\beta+2\right)\left(\beta+1\right)}{\left(\ln^3 y\right)\left(\ln\ln y\right)}\left[3\psi\left(\alpha+1\right)\psi'\left(\alpha+1\right)\right. \\ &\left.-\psi^3\left(\alpha+1\right)-\psi''\left(\alpha+1\right)\right] \\ &\left.*\left[\psi\left(\beta+4\right)-\psi\left(\beta+1\right)\right]+\ldots\right\}\end{aligned} \tag{A6}$$

iii) $\widehat{L}\left\{(\ln x)^p\right\}$ where $p = 1, 2, 3, ..$

$$\begin{aligned}
\widehat{L}\left\{\ln x\right\} &= -1 \\
\widehat{L}\left\{\ln^2 x\right\} &= -2\left(\ln y + \psi\left(1\right)\right) \\
\widehat{L}\left\{\ln^3 x\right\} &= -3\left(\ln^2 y + 2\psi\left(1\right)\ln y - \psi'\left(1\right) + \psi^2\left(1\right)\right)
\end{aligned} \tag{A7}$$

In the equations above, $\psi(z)$ is the digamma function, and $\psi^{(n)}(z)$ its derivatives, defined as

$$\psi(z) = d[ln\Gamma(z)]/dz$$

where $\Gamma(z)$ is Euler's Gamma function. The following recurrence formulas are of practical importance

$$\psi(z+1) = \psi(z) + \frac{1}{z}$$

$$\psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n n!\, z^{-n-1}$$

Some particular values needed in our calculations are

$$\psi(1) = -\gamma_E \simeq -0.5772$$

$$\psi(2) = 1 - \gamma_E$$

$$\psi(3) = \frac{3}{2} - \gamma_E$$

$$\psi'(1) = \zeta(2)$$

$$\psi''(1) = -2\zeta(3)$$

$$\psi'(2) = \zeta(2) - 1$$

where

$$\zeta(2) = \frac{\pi^2}{6}$$

$$\zeta(3) \simeq 1.20206$$

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Thanks to my friends for providing encouragement.
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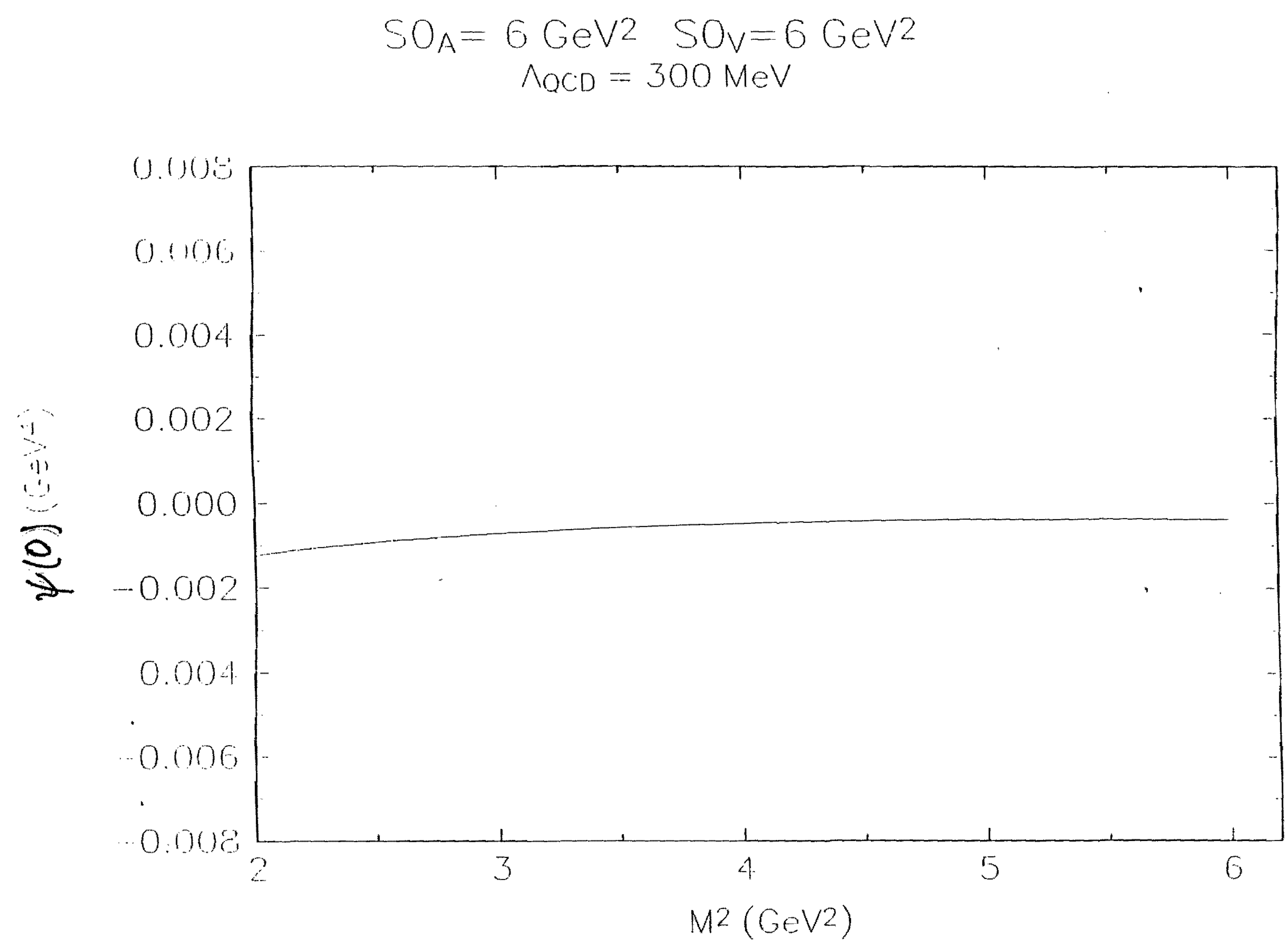
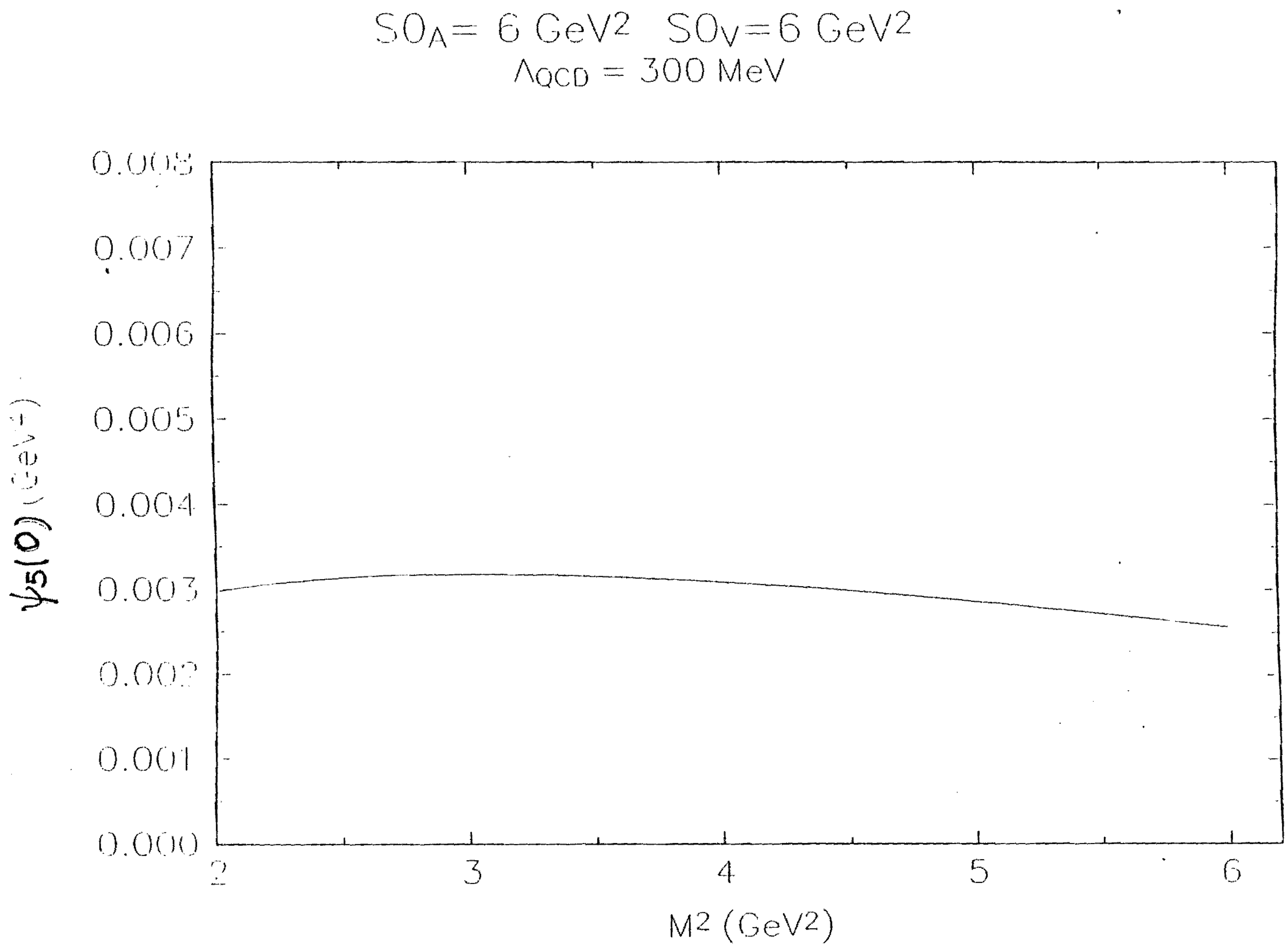
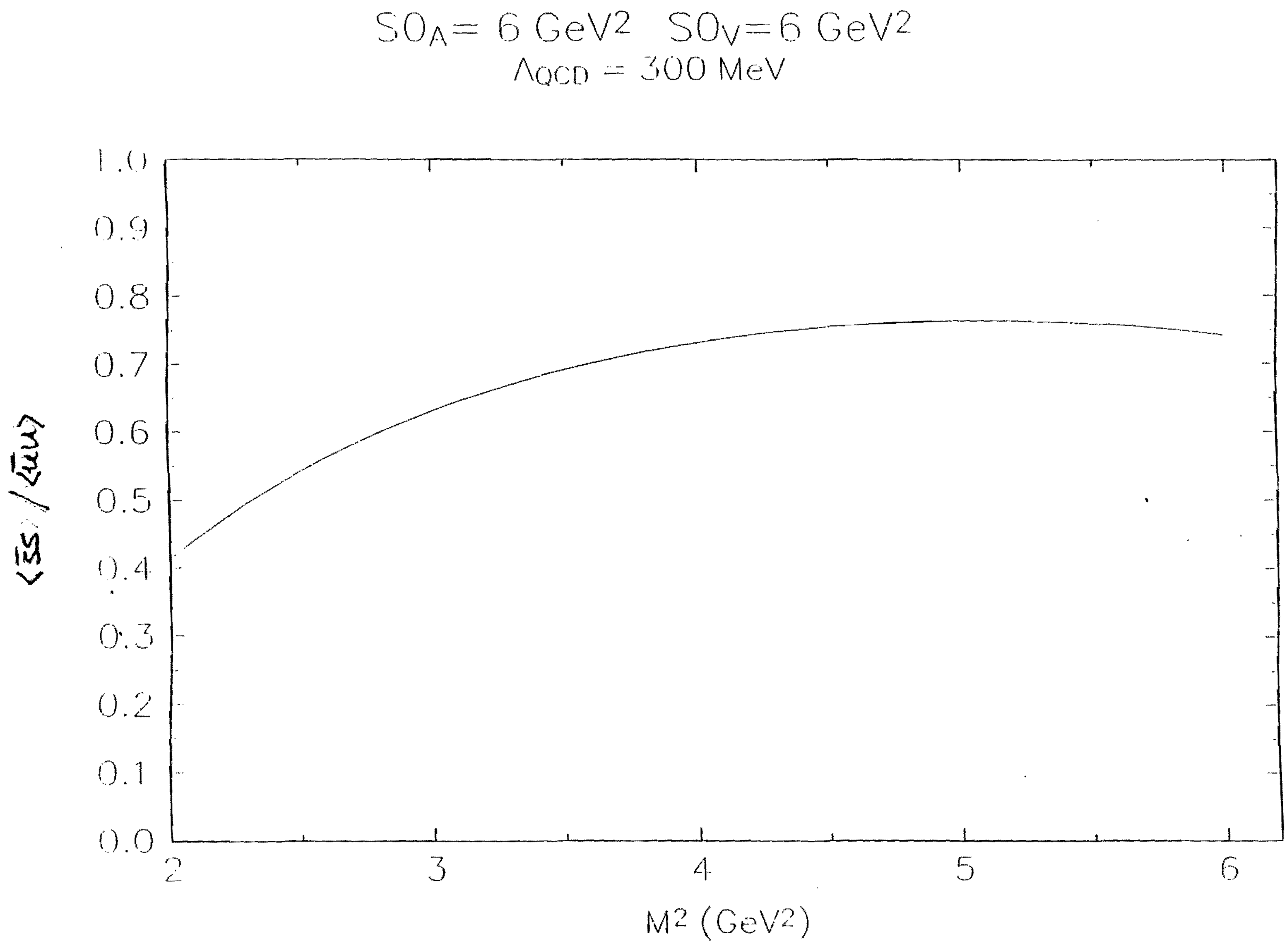


SOA= 6 GeV2 SOV=6 GeV2
ΛQCD = 300 MeV
ψ(0) (GeV4)
0.008
0.006
0.004
0.002
0.000
−0.002
0.004
−0.006
−0.008
2
3
4
5
6
M2 (GeV2)



$SO_A = 6\ GeV^2 \quad SO_V = 6\ GeV^2$
$\Lambda_{QCD} = 300$ MeV
$\psi_5(0)$ (GeV4)
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0.000
2
3
4
5
6
M^2 (GeV2)


$SO_A = 6$ GeV2 $SO_V = 6$ GeV2
$\Lambda_{QCD} = 300$ MeV
$\langle \bar{s}s \rangle / \langle \bar{u}u \rangle$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
2
3
4
5
6
M^2 (GeV^2)

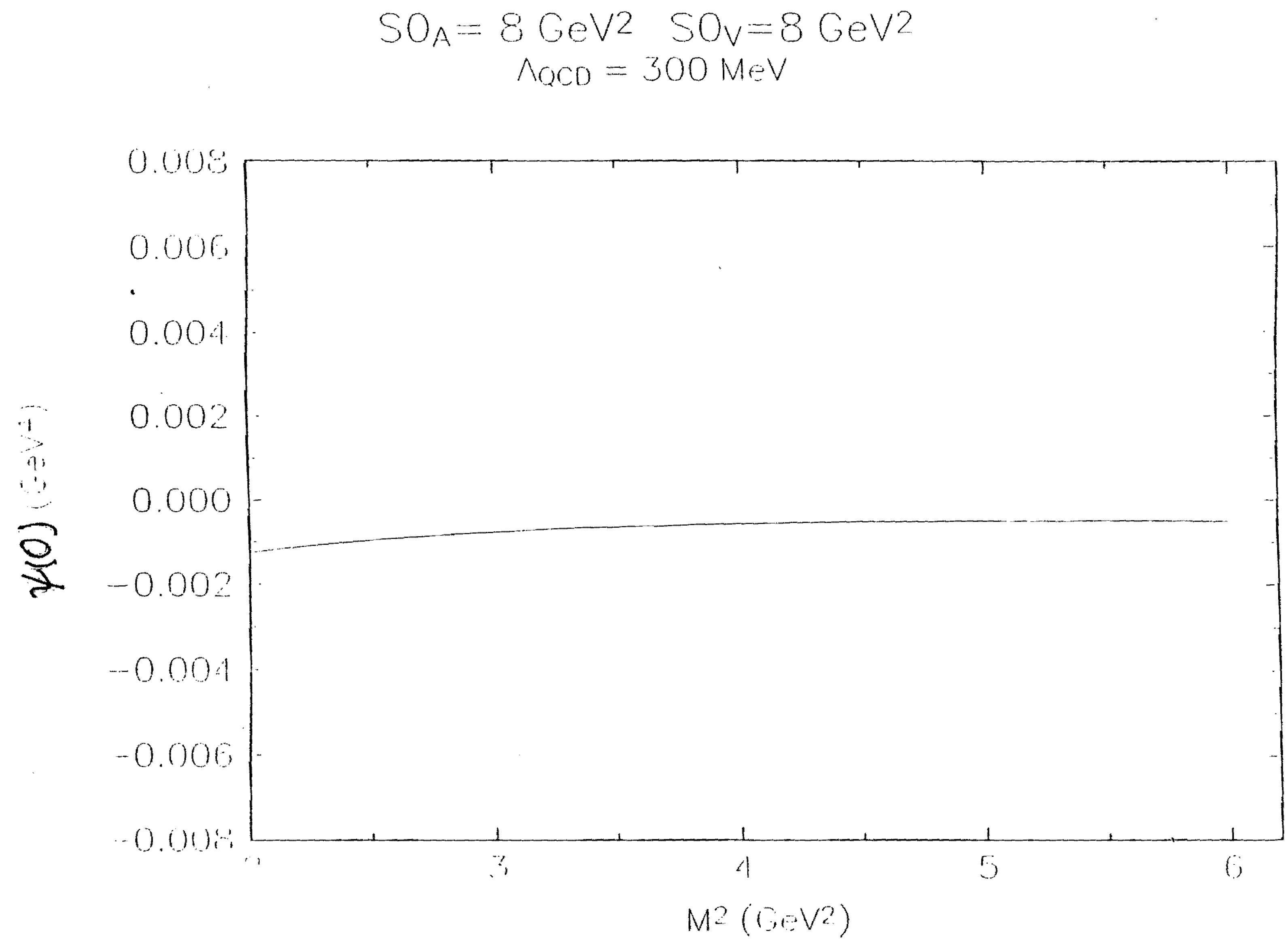
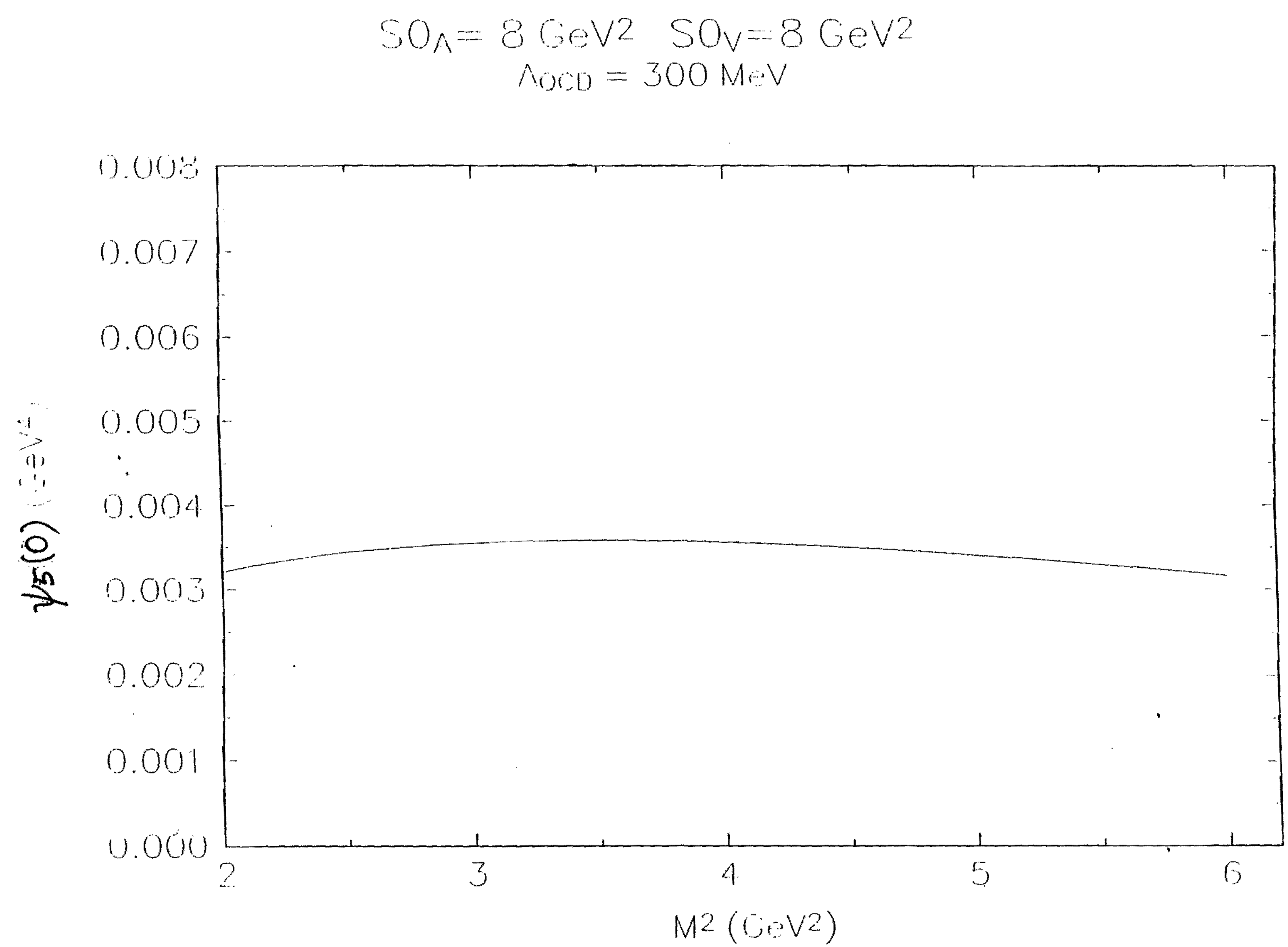
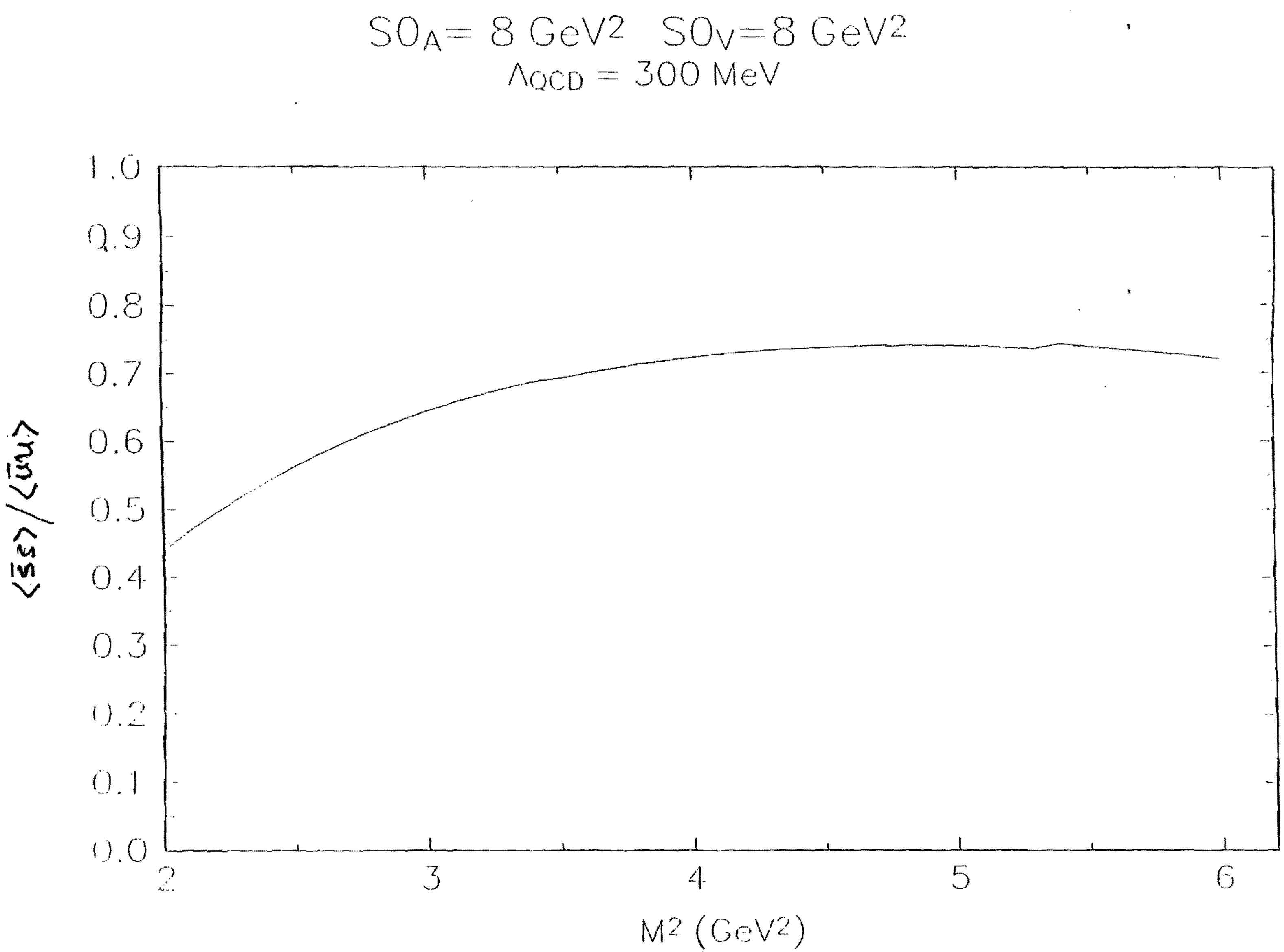

SOA= 8 GeV2 SOV=8 GeV2
ΛQCD = 300 MeV
0.008
0.006
0.004
0.002
0.000
−0.002
−0.004
−0.006
−0.008
3
4
5
6
ψ(0) (GeV4)
M2 (GeV2)

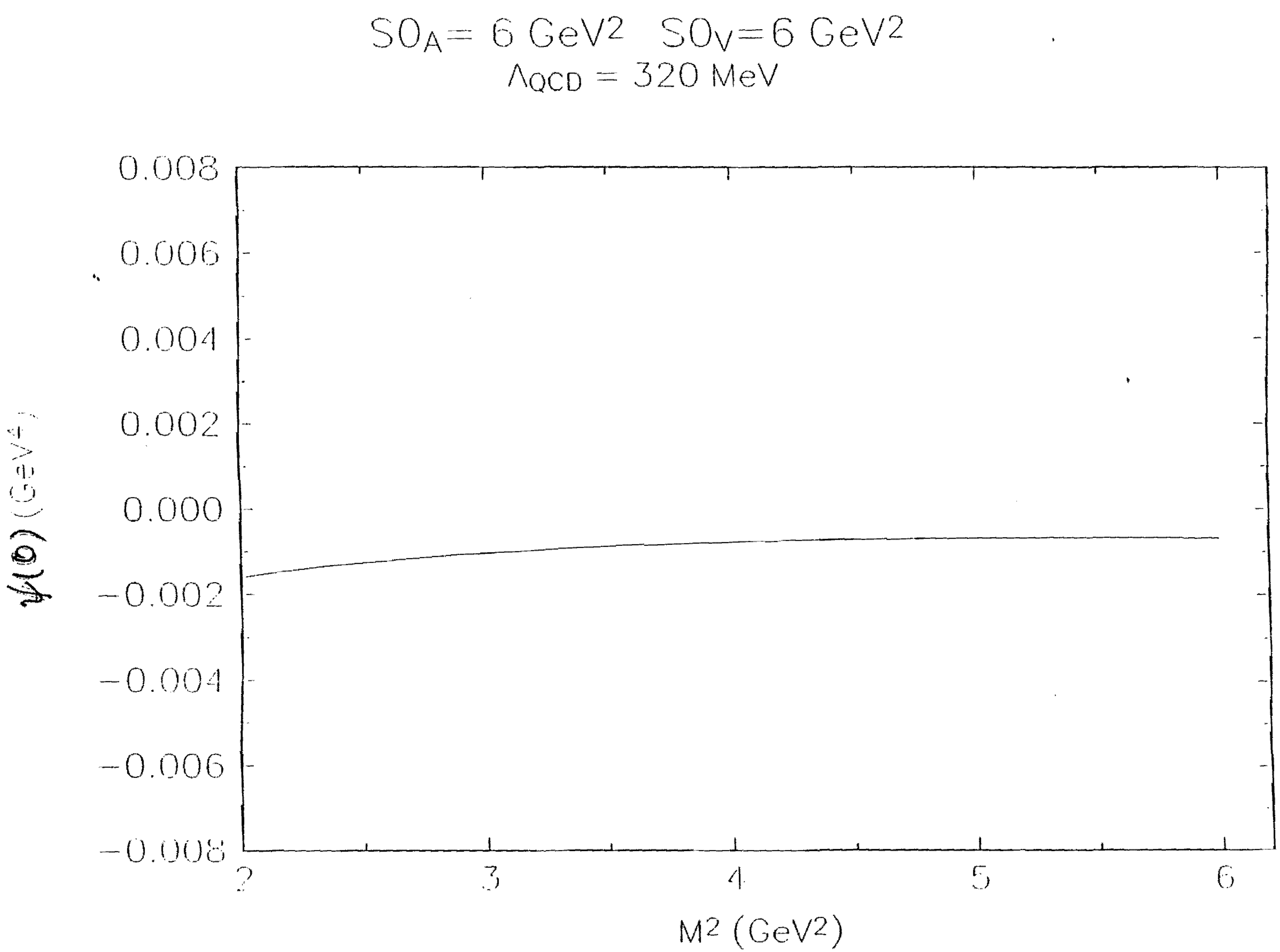
Figure 4



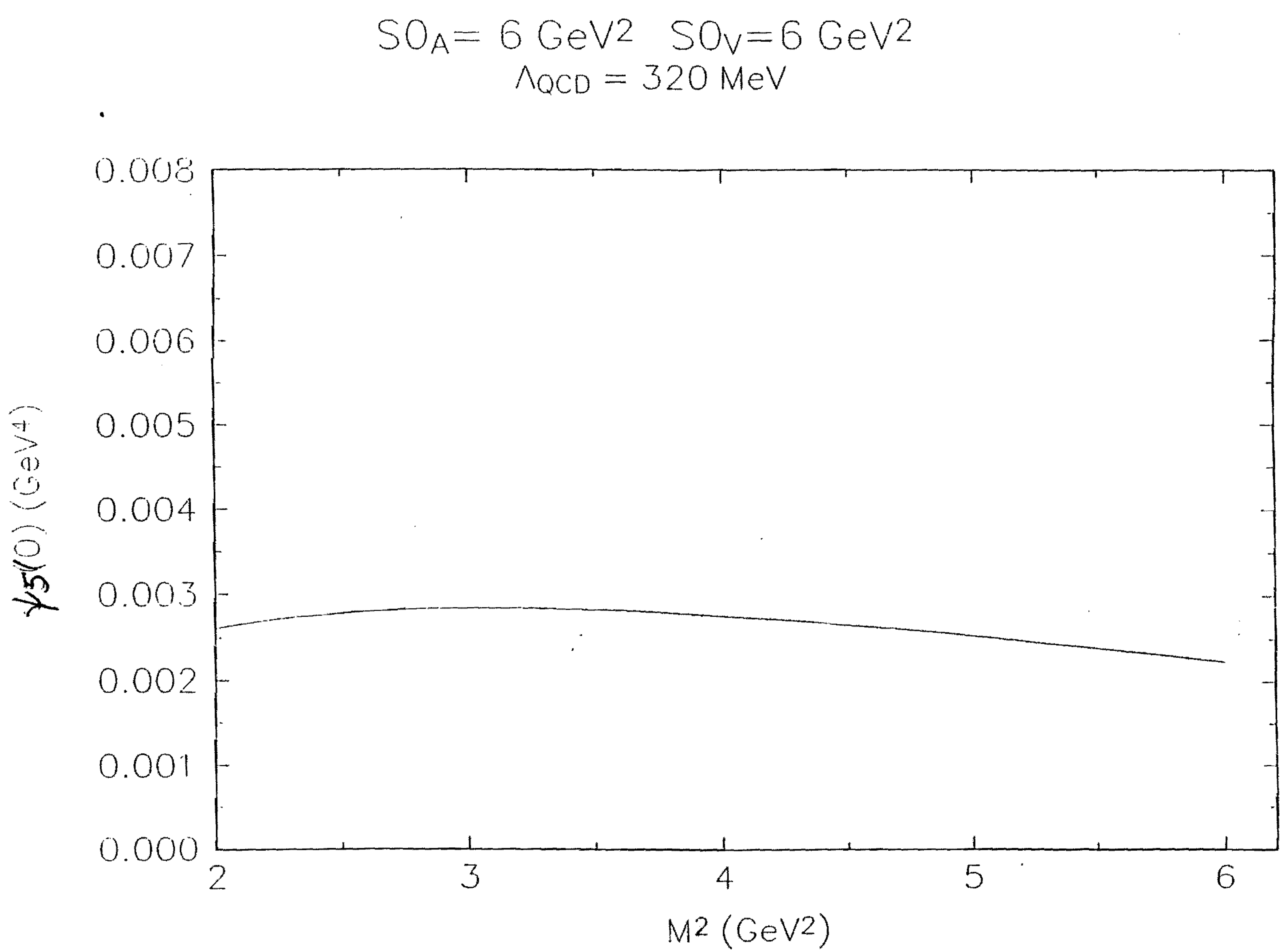
SO_A= 8 GeV2   SO_V=8 GeV2
Λ_QCD = 300 MeV
ψ5(0) (GeV4)
0.000
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
2
3
4
5
6
M2 (GeV2)



$SO_A = 8\ GeV^2 \quad SO_V = 8\ GeV^2$
$\Lambda_{QCD} = 300\ MeV$
$\langle \bar{s}s \rangle / \langle \bar{u}u \rangle$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
2
3
4
5
6
$M^2\ (GeV^2)$



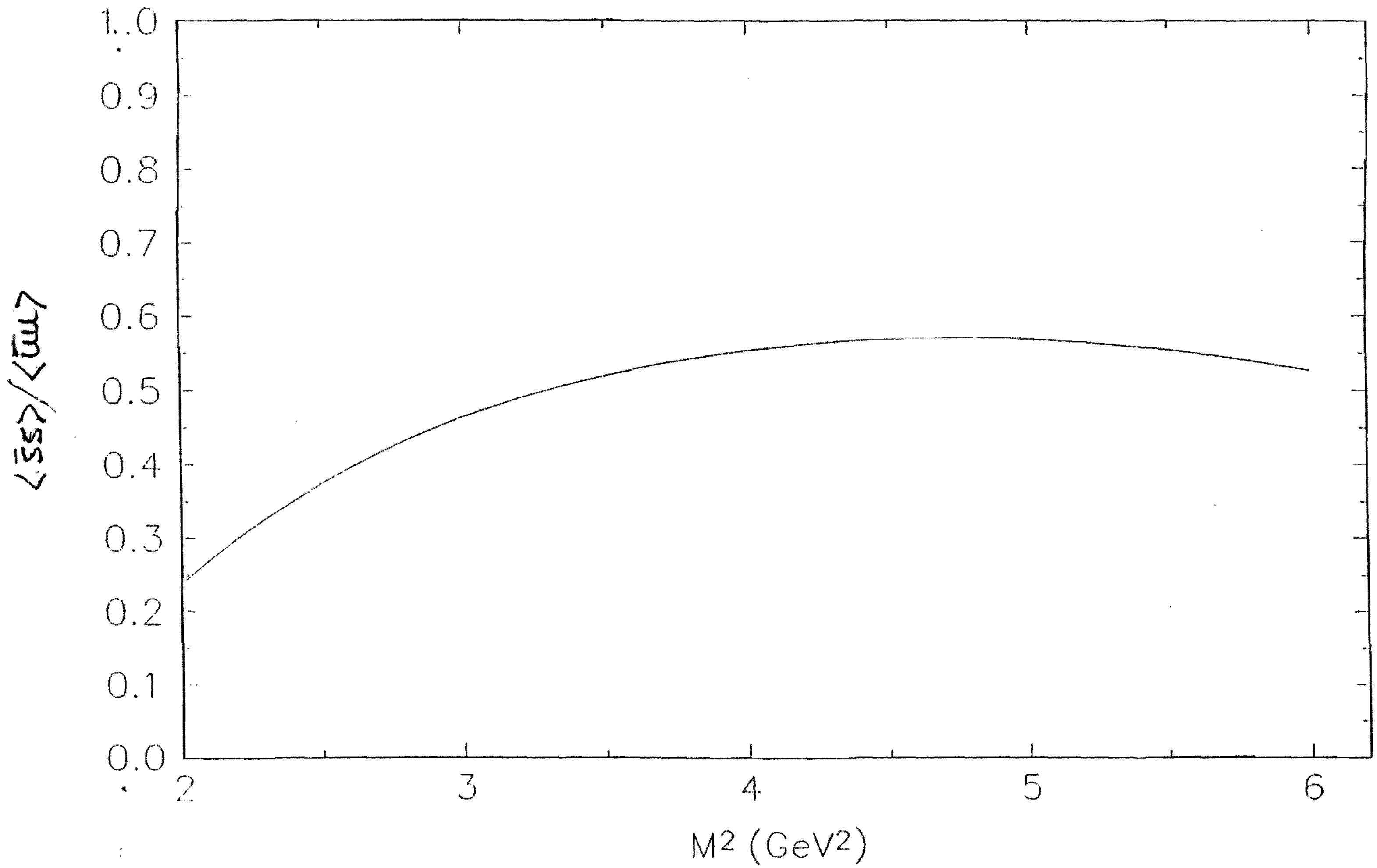
SO_A= 6 GeV2   SO_V=6 GeV2
Λ_QCD = 320 MeV
ψ(0) (GeV4)
0.008
0.006
0.004
0.002
0.000
−0.002
−0.004
−0.006
−0.008
2
3
4
5
6
M2 (GeV2)



SO_A= 6 GeV2 SO_V=6 GeV2
Λ_QCD = 320 MeV
ψ5(0) (GeV4)
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0.000
2
3
4
5
6
M2 (GeV2)

$S0_A = 6\ GeV^2 \quad S0_V = 6\ GeV^2$

$\Lambda_{QCD} = 320\ MeV$



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