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METRIZATION OF ORDERED TOPOLOGICAL SPACES

by

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CONTENTS

INTRODUCTION

In 1969, Lutzer proved that a linearly ordered topological space with a G_δ -diagonal is metrizable (32). This appears to be the first work in the field of metrization of ordered topological spaces. Very little seems to have been done in this direction. This thesis is a study of the various conditions necessary for metrizability of such spaces.

One of the earliest papers concerned with ordered topological spaces is that of Eilenberg (18). Since then, ordered spaces have been considered by various authors, but few considered the conditions under which such spaces would be metrizable. Bennet gave a characterization of metrizability for a linearly ordered topological space with a σ-point finite base.

A linearly ordered topological space is a space for which the interval topology coincides with the original topology for the space. We investigate the metrizability of linearly ordered topological space satisfying certain covering properties, countability conditions on the base, certain conditions on the diagonal and spaces which admit a symmetric. We obtain four characterizations of metrizability for linearly ordered topological space in terms of some of the above notions.

Summary

Chapter 1 is concerned with the possibility of defining a linear order on certain spaces such that the order topology coincides with the given topology. Eilenberg (18) obtained a characterization of orderability for connected spaces. This result was enlarged upon by Kok (30). Lynn gave several conditions necessary to ensure the orderability of subsets of the real line, and Rudin (45) a characterization of ordered subsets of the real line. To my knowledge, however, a complete characterization of orderability of general spaces has not been found.

In Chapter 2 we consider various conditions that need be satisfied by a linearly ordered topological space to give metrizability. We consider certain covering properties and establish two characterizations of metrizability in a linearly ordered topological space in terms of covering properties; namely that a linearly ordered topological space is metrizable if and only if it is a p-space (see Definition 2.11) with a point countable base and that a linearly ordered topological space is metrizable if and only if it is developable. Bennet (7) showed that a connected linearly ordered topological space is not only metrizable, but is homeomorphic to a subset of the real line.

We consider also certain conditions imposed on the diagonal of a space; namely spaces with G_δ -, G_δ^* -, and $\bar{G}_\delta$ -diagonals

(see Definitions 2.29 and 2.30), the latter two being introduced by Hodel (28) and Borges (10) respectively. As mentioned above, Lutzer showed that a linearly ordered topological space with a G_δ -diagonal is metrizable: we supply a proof of the converse. We show also that the G_δ , G_δ^* and $\bar{G}_\delta$ conditions are equivalent in the case of a linearly ordered topological space. To conclude Chapter 2, we show that any symmetrizable linearly ordered topological space is metrizable - a result proved by Nedev (40), but whose proof is reconstructed here as Nedev's paper was unobtainable.

In Chapter 3 we discuss various types of bases for a space. Alexandroff (2) introduced the notion of a uniform base, whose definition was strengthened by Arkhangel'skii (5) to get a strong uniform base, a condition which he showed was equivalent to the metrizability of a space. We deduce that a linearly ordered topological space is metrizable if and only if it has a uniform base. Quasi uniform bases were introduced by Lutzer (33), as a property common to linearly ordered topological space and compact Hausdorff spaces which accounts for the similarity in a number of metrization theorems for these two spaces.

For the sake of completeness, we include in Appendix I Bennet's result (7) that a connected linearly ordered topological space is homeomorphic to a subset of the real line, and in Appendix II Bing's Theorem (9) that a collectionwise normal Moore space is metrizable - a result that plays an important part in this exposition.

The notation used is standard and any undefined concepts may be found in (29) or (22). We use N to denote the natural numbers, and cl_X S to denote the closure in X of a subset S, although where it is clear from the context, the X is omitted. Unless otherwise stated, I denotes the interval topology (see Definition 1.1). We use the abbreviation "iff" for "if and only if".

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CHAPTER 1 - ORDERED TOPOLOGICAL SPACES AND ORDERABILITY

§1 ORDERED TOPOLOGICAL SPACES

We give the definitions and some preliminary results of the basic concepts we shall be concerned with.

Definitions 1.1

A linear order on a set X is a binary relation, $\leqslant$, which is reflexive, transitive, antisymmetric and decisive; i.e. the relation satisfies the properties:

(a) For $x \in X$, $x \leqslant x$

(b) For $x, y, z \in X$, $x \leqslant y$ and $y \leqslant z$ implies $x \leqslant z$.

(c) If $x \leqslant y$ and $y \leqslant x$ than $x = y$.

(d) For every $x, y \in X$, either $x \leqslant y$ or $y \leqslant x$.

As usual, $x < y$ if $x \leqslant y$ and $x \neq y$.

It can be seen that X then has a subbase consisting of all sets of the form $\{x: x < a\}$ or $\{x: x > a\}$ for some $a \in X$. This is called the order (interval) topology on a linearly ordered set.

Definition 1.2

A linearly ordered topological space (abbreviated LOTS) is a space X with a linear order such that the interval topology coincides with the original topology on X.

A topological space (X, T) is <u>orderable</u> if there is a linear order on X such that the interval topology coincides with T.

The interval topology on a subspace is weaker than the relative topology. It is a well known, yet surprising fact, that a subspace of an orderable space need not be orderable. An example is the space X obtained from the real line by deleting the interval $(0,1]$. In the interval topology every neighbourhood of 0 contains points > 1, but this is not so in the relative topology. However, the relative topology on a compact or connected subset of an orderable space coincides with the interval topology on that subspace (see § 2).

It is clear that every LOTS is a Hausdorff space, by (d) of definition 1.1.

Definition 1.3

A metric is a real valued function d on the product of a set X satisfying the following:

(a) $d(x, y) = d(y,x) \geqslant 0$ for all $x,y \in X$.

(b) $d(x,y) = 0$ iff $x = y$.

(c) $d(x,y) + d(y,z) \geqslant d(x,z)$ for all $x,y,x \in X$.

We call the set $\{y: d(x,y) < r\}$ a ball of radius r about x.

A space is metrizable if it admits a metric.

§ 2 CONDITIONS FOR ORDERABILITY

It is interesting to see under what conditions a space is orderable. It has been shown, (49), that any discrete space in orderable, as is every finite Hausdorff space; also any compact or connected subspace of an orderable space is orderable.

In 1941, Eilenberg (18) gave the characterization of Theorem 1.4 for orderability in the case of a connected space. He showed further that the order in a connected space is unique (two orders in a connected space are either identical or inverse to each other). The diagonal of a Space X is denoted by $\Delta = \{(x,x): x \in X\}$. An order is continuous if for $x < y$ there are neighbourhoods U and V of x and y respectively, such that $x < y'$ and $x' < y$ whenever $x' \in U$ and $y' \in V$.

Theorem 1.4

A connected topological space is orderable by a continuous order iff $X \times X - \Delta$ is not connected.

Proof: Suppose X is ordered by a continuous order.
Let $A(x) = \{(x,y): x < y\}$ and $B(x) = \{(x,y): y < x\}$. Since the order is linear, $X \times X - \Delta = A(x) \cup B(x)$ and $A(x) \cap B(x) = \emptyset$.
By the continuity A(x) and B(x) are open, so $X \times X - \Delta$ is not connected.

Conversely, given (x,y) in $X \times X - \Delta$, let $p(x,y) = (y,x)$. By (18, Theorem 3.1), $X \times X - \Delta$ can be decomposed into two open disjoint sets A and B such that $p(A) = B$. We define $x < y$ iff $(x,y) \in A$. It can then be seen that $<$ is a continuous linear ordering.

Proposition 1.5

Let X be a LOTS. If (Y,T) is a subspace of X, and I is the interval topology on Y, then $I < T$.

Proof: Follows from the continuity of the identity map of (Y,T) onto (Y,I). Thus if $T < I$ then (Y,T) is a LOTS.

Certain conditions for orderability of subspaces of the real line have been obtained by Lynn (35, 36). We denote the real line with its usual linear ordering by R.

Definitions 1.6

A space X is not linearly ordered at a point $x \in X$ from below (above) if x is a right hand (left hand) end point of a component of R–X, which is a half open interval. If this is not the case, X is linearly ordered at the point $x \in X$ from below (above). If X is linearly ordered at a point $x \in X$ from below and above, we say X is linearly ordered at x.

Proposition 1.7 (35)

A subset of the real line, X, is a linearly ordered space in the relative topology iff it is linearly ordered at each of its points.

Proof: Suppose X is not a linearly ordered space. By 1.5 there is and $x \in X$ which is an I-limit point of the set $S = \{y \in X: y < x\}$, but not a T-limit point. Thus x is the left hand end point of a component $C \subset R-X$ which is a half open interval, since if C were an open interval, the right hand end point of C would be in X, and hence x would not be an I-limit point of X. Thus X is not linearly ordered at x.

Conversely, suppose X is not linearly ordered at $x \in X$. Then x is an end point of a component $C \subset R-X$ which is a half open interval. Thus x is an I-limit point of the set S of points in X on the other side of C, since otherwise C would be an open interval. But x is not a T-limit point of S, thus X is not a linearly ordered space.

Corollary 1.8

(a) Any open or closed subspace of the real line is linearly orderable.

(b) Any dense subspace of the real line is linearly orderable.

Proposition 1.9 (36)

Every subset of the real line which contains no interval is linearly orderable.

The following proposition follows as a corollary to a result in (36). We omit the proof as it is rather long and involved.

<u>Proposition 1.10</u>

(a) If $X \subset R$ contains no compact open set and has only countably many components, then X is linearly orderable.

(b) If $X \subset R$ contains no isolated interval closed in R and its components are intervals, then X is linearly orderable.

(c) If X is the union of open or half open intervals, then X is linearly orderable.

Rudin, (45), has obtained a characterization (theorem 1.13) of orderability for subsets of R and necessary and sufficient conditions for orderability of a subspace of an ordered space. Closures are taken in R and Q will denote the union of all non-trivial components of a space $X \subset R$, all of whose endpoints belong to the closure of $(\bar{X}-X)$.

<u>Theorem 1.11</u>

A subspace $X \subset R$ is linearly orderable iff one of the following hold.

(a) If $(X-Q)$ is compact and $(X-Q) \cap \bar{Q} = \emptyset$ then either $Q = \emptyset$ or $(T-Q) = \emptyset$.

(b) If I is an open interval of R and p an endpoint of I and

$\{p\} \cup (I \cap (X-Q))$ is compact and $\{p\}$ is the intersection of the closures of $(I \cap Q)$ and $(I \cap (X-Q))$ then the component of X containing p, if any, is trivial.

The conditions for orderability of a subspace of an orderable space are slightly more complicated than those for R. With the necessary changes, Rudin obtained similar conditions to Theorem 1.11 above (Theorem 1.12).

Let X be a linearly orderable topological space and let $T \subset X$. If $p \in T$, let A(p) be the set of all monotonic (in the ordering) sequences of points of T approaching p and which have no subsequence of smaller cardinality approaching p. Let Q denote the set of all points q of T such that: (a) if p is the first or last (in the ordering) point of X or any point of X not in the component of T containing q, then the interval $[q, p]$ of X does not intersect T in a compact set; (b) either the component of T containing q is not trivial or there are terms of A(q) of different cardinality or there are terms A and A' of A(q) such that every subsequence of A' approaching q has a limit point not in $cl_X A$.

Theorem 1.12

If X is a linearly orderable space and $T \subset X$, then the following conditions are necessary and sufficient for T to be linearly orderable.

(a) If $(T-Q)$ is compact and does not intersect $cl_T Q$ then either $Q = \emptyset$ or $(T-Q) = \emptyset$.

(b) If I is an open interval of X and $p \in (T-I)$ and $(I \cap (T-Q)) \cup \{p\}$ is compact and non-trivial and intersects the closure of $(I \cap Q)$ in p and only in p then (i) the component of T containing p is trivial and (ii) no term of $A(p)$ is uncountable.

Kok (30), has obtained necessary and sufficient conditions (Theorem 1.13) for the orderability of a connected space, including Theorem 1.4.

A subset $A \subset X$ is a segment if A is a component of $X - \{x\}$ for some $x \in X$. A pair (A,B) of subsets of X is a separation (of $A \cup B$) if $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and both A and B are open in $A \cup B$. If $x, y, z \in X$, we say x separates y and z if there is a separation (A,B) of $X - \{x\}$ such that $y \in A$ and $z \in B$. A point $x \in X$ is an endpoint if $X - \{x\}$ is connected.

<u>Theorem 1.13</u>

A connected Hausdorff space is linearly orderable iff any of the following conditions are satisfied.

(a) $X \times X - \Delta$ is not connected.

(b) For every pair of distinct points x and y of X there is a $z \in X$ such that z separates x and y; and every point of X has at most two distinct segments.

(c) Every connected subset of X has at most two endpoints and every point of X has at most two distinct segments.

(d) For every three distinct points of X, there is one which separates the other two.

(e) For every three distinct, connected, proper subsets of X, there are two which together do not cover the space.

In (49), Venkatamaran et al. give conditions for orderability of a topological space whose underlying space is a topological group.

Finally, Ostaszewski (43), has recently given the following characterization of compact separable ordered spaces. (An ordered set is compact iff every subset has a supremum and an infimum (22)).

Theorem 1.14

Let X be a non-empty linearly ordered set. The interval topology is compact iff X is order isomorphic to a subset $Y \subset [0,1] \times \{0,1\}$ with Y ordered lexicographically (see 29), provided that $Y \cap [0,1] \times \{0\}$ is a closed subset of the interval topology of $[0,1] \times \{0\}$ and that $(t,1) \in Y \Rightarrow (t,0) \in Y$.

We give some examples of spaces which are not linearly orderable.

Examples 1.15

(a) The subspace $(0,1) \cup \{2\}$ of R is not linearly orderable, since a one-one, continuous map of the connected space $(0,1)$ into a linearly ordered space Y is order-preserving or order reversing (18). So a homeomorphism of X in Y consists of a connected open interval G and an isolated point y. But since G has neither a first point nor a last point, y is an I-limit point induced by the linear order in Y on $\{y\} \cup G$.

(b) A compact extremely disconnected space is not linearly orderable, since such a space contains a copy of βN (22) which is not orderable (49).

(c) The Sorgenfrey line (the real line with the left half open interval topology) is not linearly orderable, since it is non-metrizable since it is regular and Lindelöf and separable but does not have a countable base (29 p59), and has a G_δ -diagonal (see Theorem 2.27).

(d) An open and closed subset of a linearly orderable space need not be linearly orderable. If X is $(0,1) \cup \{2\}$ which is not orderable and Y is a countably infinite discrete space, then $X \times Y$ is orderable, and any $X \times \{y\}$ is an open and closed subset homeomorphic to X.

CHAPTER 2 - COVERING PROPERTIES AND METRIZATION

We shall be concerned in this chapter with conditions for metrizability of LOTS in terms of properties of coverings. Conditions on countability of bases has proved important in the theory of metrization. The following result of Miscenko (39) is an example: A compact Hausdorff space with a point countable base is metrizable.

Arkhangel'skii (6) has shown that a perfectly normal, collectionwise normal space with a σ-point finite base is metrizable. We shall obtain similar theorems for LOTS and investigate which can be strengthened for the case of a LOTS.

We begin by giving some properties of LOTS.

§ 1 BASIC PROPERTIES

Bing's result (9) that a paracompact Moore space is metrizable has indicated that usefulness of paracompactness in the theory of metrization. We therefore give a condition (Theorem 2.3) for paracompactness in a LOTS; A LOTS in paracompact iff every open cover of the space has an open point countable refinement.

Definitions 2.1

(a) A collection $\mathcal{B}$ of subsets of a set X is a point countable (point finite) collection if each element of X is in at most countably (finitely) many elements of $\mathcal{B}$.

$\mathcal{B}$ is a σ-point finite collection if $\mathcal{B} = \cup\{\mathcal{B}_i : i \in N\}$ where each $\mathcal{B}_i$ is a point finite collection.

A base $\mathcal{B}$ for a space X is a point countable base (σ-point finite base) if $\mathcal{B}$ is a point countable (σ-point finite) collection. For example, a metric space has a point countable base by balls of radius $\frac{1}{n}$.

(b) A collection $\mathcal{A}$ of sets is coherent if for any subcollection $\mathcal{B}$ of $\mathcal{A}$ there is an element of $\mathcal{B}$ that intersects some element of $\mathcal{A} - \mathcal{B}$.

$\mathcal{B}$ is a maximal coherent subcollection of $\mathcal{A}$ if there is no coherent subcollection $\mathcal{C}$ of $\mathcal{A}$ such that $\mathcal{B}$ is a proper subcollection of $\mathcal{C}$.

If $\mathcal{G}$ is a collection of sets we denote $\mathcal{G}^* = \cup\{G \in \mathcal{G}\}$.

(c) A subset A of a linearly ordered set is <u>convex</u> if when $a, b \in A$ then $\{x \in X: a < x < b\} \subseteq A$.

The union of any collection of convex sets with non-empty intersection is convex, so every subset of X can be uniquely expressed as the union of disjoint maximal convex sets, called convex components.

Every interval is convex but not conversely.

We can now obtain the condition for paracompactness mentioned above.

<u>Lemma 2.2</u> (7)

Let $\mathcal{G}$ be a collection of non-empty subsets of a set X. Let $\mathcal{C}$ be the family of all maximal (with respect to $\subset$) coherent subcollections of $\mathcal{G}$. Then $\mathcal{G} = \cup\mathcal{C}$ and if $\mathcal{H}$ and $\mathcal{K}$ are distinct elements of $\mathcal{C}$ then $\mathcal{H}^* \cap \mathcal{K}^* = \emptyset$.

<u>Proof:</u> For each $S \subset X$, the family $\mathcal{C}_0 = \{C \subset X : S \subset C\}$ is coherent. By Zorn's Lemma, $\mathcal{C}_0$ is contained in a maximal coherent collection. Hence $\cup\mathcal{C} = \mathcal{G}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be maximal coherent collections. Assume $\mathcal{H}^* \cap \mathcal{K}^* \neq \emptyset$. Then there is an $H \in \mathcal{H}$ and a $K \in \mathcal{K}$ such that $H \cap K \neq \emptyset$. Then $\mathcal{H} \cup \{K\}$ is coherent, contradicting the maximity of $\mathcal{H}$.

<u>Theorem 2.3</u> (7)

A LOTS is paracompact if every open cover has an open point countable refinement.

<u>Proof:</u> Let X be such a LOTS. Let $\mathcal{U}$ be an open covering of X and let $\mathcal{G}$ be an open point countable refinement of $\mathcal{U}$. We may assume the elements of $\mathcal{G}$ are convex.

Let $\mathcal{G} = \cup\{\mathcal{C}_\alpha : \alpha \in A\}$ where each $\mathcal{C}_\alpha$ is a maximal coherent subcollection of $\mathcal{G}$. By Lemma 2.2, if $\alpha \neq \beta$ then $\mathcal{C}^*_\alpha \cap \mathcal{C}^*_\beta = \emptyset$ and for each $\alpha \in A$ there is a countable subcollection $\mathcal{H} = \{H(\alpha, i) : i \in N\}$ such that $\mathcal{H}^*_\alpha = \mathcal{C}^*_\alpha$.

Thus $\mathcal{K}_i = \{H(\alpha, i) : \alpha \in A\}$ is a locally finite collection and $\mathcal{K} = \cup\{\mathcal{K}_i : i \in N\}$ is a σ-locally finite open refinement of $\mathcal{U}$. Thus X is paracompact.

<u>Corollary 2.4</u>

A LOTS with a point countable base is hereditarily paracompact.

<u>Proof:</u> Let X be a LOTS with a point countable base. Every open subspace of X is the pairwise disjoint union of its convex components.

Now every maximal convex subset A of an open set G is open, since if we take $x \in A$, there is a neighbourhood U of x such that $U \subset G$. U is convex, and $U \cap A \neq \emptyset$. Thus $U \cup A$ is convex and $U \cup A \subset A$ so U is open and hence A is open. Thus each convex component is open.

Let V be an open convex component and let T be the relative topology on V. Let I be the interval topology on V. Let $x \in W \subset V$ where W is open in T. Then W is open in X and hence there is (a,b) such that $x \in (a,b) \subset W$ and thus W is open in I. Since, by Proposition 1.5, $I < T$, the two topologies coincide.

Thus each of the convex components is a LOTS in its relative topology, by Corollary 1.8(a). Also each has a point countable base and is thus paracompact. Thus each open subspace of X is paracompact.

We shall now examine the Lindelöf property and the countable chain condition in LOTS. It is known that in metric spaces the above are equivalent to the second axiom of countability. Any space which is hereditarily Lindelöf satisfies the countable chain condition but not conversely (for example, the "tangent disc" of (48)). Lutzer and Bennet (34) have shown that for a LOTS the conditions are in fact equivalent.

Definition 2.5

A space X is Lindelöf if every open covering of X has a countable subcovering.

A space X satisfies the countable chain condition (abbreviated CCC) if any disjoint collection of open sets is countable.

A space X is collectionwise normal if every discrete collection of sets can be covered by a pairwise disjoint collection of open sets, each of which covers just one of the original sets.

Theorem 2.6 (34)

A LOTS which satisfies the CCC is hereditarily Lindelöf.

Corollary 2.7

If X is a LOTS satisfying the CCC and Y is a discrete (in the relative topology) subspace of X then Y is countable.

Theorem 2.8 (34)

A separable LOTS is hereditarily separable.

Outline of Proof: Let X be a separable LOTS and let $A \subseteq X$. Let $I(A) = \{a \in A : \{a\}$ is relatively open in $A\}$. By Corollary 2.7, I(A) is countable. Let D be a countable dense subset of X. Let $\mathcal{D} = \{(r,s) : r,s \in D,\ r < s \text{ and } A \cap (r,s) \neq \emptyset\}$. For each interval $J \in \mathcal{D}$, choose a point $a(J) \in A \cap J$. Let $D(A) = I(A) \cup \{a(J) : J \in \mathcal{D}\}$. Then D(A) is a countable dense subset of A.

It is well known that a LOTS is normal (22). However, Steen (47) has proved the following stronger result, which shows that a LOTS satisfies all the usual separation axioms. We omit the proof as it is rather lengthy.

Theorem 2.9

A LOTS is hereditarily collectionwise normal.

§2 METRIZATION OF A LOTS WITH COUNTABLE BASE

Bennet (7) has shown that a LOTS with a point countable base which satisfies the CCC satisfies the second axiom of countability. The result of Arkhangel'skii mentioned above, that a perfectly normal, collectionwise normal space with a σ-point finite base is metrizable, together with Theorem 2.9 gives the following:

<u>Theorem 2.10</u> (7)

A LOTS X with a σ-point finite base is metrizable if X satisfies either of the following.

(a) X satisfies the CCC

(b) X is perfectly normal.

<u>Remark:</u> The condition cannot be weakened to "X has a point countable base" since in (7) it is shown that if there exists a Souslin space (a non-separable LOTS satisfying the CCC) then there exists a Souslin space with a point countable base.

If we add some kind of completeness condition to the spaces above, we can obtain further metrization theorems. One such condition is that of a p-space introduced by Arkhangel'skii (4), who showed that every complete space is a p-space.

§3 p-SPACES

Definition 2.11 (4)

A space X is a p-space if there is a sequence $\{\alpha_n\}_{n=1}^{\infty}$ where each α_n is a collection of open sets in the Stone-Čech compactification of X and is a cover of X such that for each $x \in X$, $\bigcap_n st(x, \alpha_n) \subseteq X$.

$(st(x,\alpha_n) = \bigcup\{a_n : x \in a_n\}.)$.

The sequence $\{\alpha_n\}_{n=1}^{\infty}$ is called a feathering on X.

If a feathering $\{\alpha_n\}_{n=1}^{\infty}$ is such that for each $x \in X$ there is an m such that $cl_{\beta X}\, st(x,\alpha_n) \subseteq st(x,\alpha_n)$ then $\{\alpha_n\}$ is a strict feathering. A space with a strict feathering is a strict p-space.

Proposition 2.12

(a) A paracompact p-space is a strict p-space.

(b) A feathering $\{\alpha_n\}$ on X is a strict feathering iff

$\bigcap_n cl_{\beta X}\, st(x,\alpha_n) = \bigcap_n st(x,\alpha_n)$ for all $x \in X$.

(c) A locally compact T_2 space X is a p-space.

(d) A metric space or a completely regular Moore space is a strict p-space.

(e) If X has a feathering in any compactification bX then X has a feathering in its Stone-Cech compactification βX.

Proof: (a) By (12: Corollary 1.8).

(b) By (11).

(c) Since any locally compact space X is open in βX if we let $Y_n = \{X\}, \{Y_n\}_{n=1}^{\infty}$ is a feathering for X.

(d) See Definition 2.19 and Theorem 2.20.

(e) Every compactification bX is a continuous image of βX.

The following two propositions are interesting in that they give internal characterizations of p-spaces and strict p-spaces; i.e. the spaces are characterized without the use of compactifications. A sequence $\{A_n(x)\}_{n=1}^{\infty}$ of subsets of X with $x \in A(x)$ for each $n \in N$ is an x-sequence if $x_n \in A(x)$ implies that $\{x_n\}_{n=1}^{\infty}$ has a cluster point in X.

Proposition 2.13 (12)

A completely regular space X is a p-space iff there is a sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of open covers of X satisfying that condition that if $x \in X$ and $G_n \in \mathcal{G}_n$ such that $x \in G_n$, then

(a) $\bigcap_n \text{cl}G_n$ is compact

(b) $\{\bigcap_{n=1}^{k} \text{cl}G_n : k \in N\}$ is an x-sequence.

Proposition 2.14 (14)

A completely regular space X is a strict p-space iff there is a sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of open covers of X satisfying:

(a) $P_x = \bigcap_n st(x, \mathcal{G}_n)$ is a compact set for each $x \in X$.

(b) The family $\{st(x, \mathcal{G}_n) : n \in N\}$ is a neighbourhood base for the set P_x.

<u>Outline of Proof:</u> If X is a strict p-space, there is a feathering $\{\alpha_n\}$ for X in βX. We may assume α_{n+1} is a refinement of α_n. Let $P_x = \bigcap_n cl\ st(x, \alpha_n) = \bigcap_n st(x, \alpha_n)$ and $\mathcal{G}_n = \{G \cap X : G \in \alpha_n\}$. Then P_x is a compact subset of X, $P_x = \bigcap_n st(x, \mathcal{G}_n)$ and $\{st(x, \mathcal{G}_n) : n \in N\}$ is a neighbourhood base for P_x.

Conversely, suppose $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a sequence of open covers of X satisfying (a) and (b). Let α_n be the collection of all sets G' open in βX such that $G' \cap X \in \mathcal{G}_n$. Then $\{\alpha_n\}$ is a strict feathering for X.

Further, Pareek obtained another characterization of p-spaces without appealing to the compactification. If $\{F_s : s \in S\}$ is a family of subsets of a set X and $\{\alpha_n\}_{n=1}^{\infty}$ is a countable family of covers of X, then $\{F_s : s \in S\}$ has sets which are <u>base point strictly small</u> relative to $\{\alpha_n\}_{n=1}^{\infty}$ iff there exists $x_o \in X$ such that for each n, there is $s_n \in S$ and $A_n \in \alpha_n$ for which no $x_o \in A_n$ and $F_{s_n} \in A_n$.

Proposition 2.15 (44)

A completely regular space X is a p-space iff there is a sequence $\{a_n\}_{n=1}^{\infty}$ of open covers of X such that for every family of closed sets $\{F_s : s \in S\}$ which has the finite intersection property and contains sets which are base point strictly small relative to $\{a_n\}_{n=1}^{\infty}$ then $\cap (F_s : s \in S) \neq \emptyset$.

We now give an example of the type of theorem mentioned at the end of §2.

Theorem 2.16 (17)

A paracompact p-space with a point countable base is metrizable.

We can thus deduce the following characterization of metrizability for a LOTS.

Theorem 2.17

A LOTS X is metrizable iff it is a p-space with a point countable base.

Proof: Let X be a LOTS with a point countable base and which is a p-space. By Corollary 2.4 it is paracompact and thus by Theorem 2.16 is metrizable.

Conversely a metric space has a countable base by balls of radius $\frac{1}{n}$,
and by Proposition 2.12 (d) a metric space is a p-space.

Corollary 2.18

A connected LOTS with a point countable base is metrizable.

Proof: Since a connected LOTS is locally compact, it is a p-space. Then by Theorem 2.17 it will be metrizable.

Bennet (7) has shown that in fact a connected LOTS with a point countable base is homeomorphic to a connected subset of the real line. This is given in Appendix I.

§ 4 DEVELOPABLE SPACES

The concept of a developable space can be traced back to the Alexandroff-Urysohn Theorem that a T_1 space is metrizable iff it has a development $\{G_n\}$ such that for each n, the union of any two elements of G_{n+1} having a point in common is a member of G_n. It has been shown that every paracompact developable space is metrizable (9). It will be evident that the notion of a developable space plays an important role in the metrization of LOTS; in fact we will show that developability of a LOTS is equivalent to metrizability.

Definition 2.19

A space X is developable iff there is a sequence $\{G_n\}_{n=1}^{\infty}$ of open covers of X such that for $x \in X$, $\{st(x, G_n)\}_{n=1}^{\infty}$ is a local base at x. This sequence of covers is called a development. Often the extra condition is imposed that $G_{n+1} \subset G_n$, but it is unnecessary to state this as any given development can be modified to satisfy this extra condition.

Developable spaces are first countable and metric spaces are developable, since the sequence of open coverings by balls of radius $\frac{1}{n}$ is a development. However, in general, a developable space is not necessarily metrizable. Counter examples are "Cantor's Tree" and Moore's "Road Space" (48).

A regular developable space is a Moore Space. Thus in view of 2.8, a LOTS is a Moore space iff it is developable. We can now obtain the following.

Theorem 2.20

A LOTS is metrizable iff it is a developable space.

Proof: This follows from the remarks above, and Bing's result (9) that a collectionwise normal Moore space is metrizable (see Appendix II).

We now consider the relation between developable spaces and p-spaces.

Theorem 2.21 (31)

A completely regular developable space is a strict p-space.

Proof: Let bX be a compactification of X and $\{\alpha_n\}_{n=1}^{\infty}$ a development. For each n and each $G_n \in \alpha_n$, choose V_n open in bX so that $V_n \cap X = G_n$. We denote the collection of all such sets V_n by μ_n.

Let $x \in X$ and let U be any neighbourhood of X, open in bX. Since bX is regular, there is a neighbourhood W of x, open in bX, such that $cl_{\beta X} W \subset U$. Since $\{\alpha_n\}$ is a development there is an integer m such that $st(x, \alpha_m) \subset W \cap X \subset W$. But $st(x, \alpha_m) = X \cap st(x, \mu_m)$, so cl_{bX} $st(x,\alpha_n) = X \cap st(x,\mu_m)$, since X is dense in bX, and $st(x,\mu_m)$ is open in bX.

Thus $st(x,\mu_m) \subset cl_{bX}(st(x,\mu_m)) = cl_{bX}(st(x,\alpha_m)) \subset cl_{bX} W \subset U$. It than follows that $\{\mu_n\}_{n=1}^{\infty}$ is a strict feathering of X in bX, if we let $U = st(x,\alpha_n)$. In fact $\bigcap_n st(x,\mu_n) = \{x\}$ for each $x \in X$.

We shall give a characterization of developable LOTS which will be used later.

Theorem 2.22

A LOTS is developable iff it is a symmetrizable p-space.

Proof: From Theorem 2.9 and the result that a completely regular space is developable iff it is a symmetrizable p-space (12).

§ 5 THE G_δ-DIAGONAL CONDITION

The diagonal of a space is closely related to many properties of the space. We will find that the G_δ-diagonal plays a large part in the metrization of LOTS. Several results on p-spaces with a G_δ-diagonal are known. In particular, (24), a paracompact p-space with a G_δ-diagonal is metrizable. Burke (13) has shown that a p-space with a G_δ-diagonal is not necessarily developable. However, it will be seen that a LOTS with a G_δ-diagonal is developable and hence metrizable (Theorem 2.27).

Definition 2.23

A space X has a G_δ-diagonal if $\Delta = \{(x,x) : x \in X\}$ is a G_δ subset of XxX.

Ceder (15) has given the following characterization of a G_δ-diagonal in terms of covers.

Proposition 2.24

A space X has a G_δ-diagonal iff there is a sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of open covers of X such that if $x \neq y$ there is an n such that $y \notin st(x, \mathcal{G}_n)$.

Proof: Suppose $\Delta = \bigcap_n A_n$ where each A_n is open in $X \times X$. For each n, put $\mathcal{G}_n = \{G : G$ is open in X; $G \times G \subset A_n\}$. Then if $x \neq y$ there is an m such that $(x,y) \notin A_m$ and hence $y \notin st(x, \mathcal{G}_n)$.

Conversely, let $\{\mathcal{G}_n\}_{n=1}^{\infty}$ be such a sequence of covers. For each n put $A_n = st(x, \mathcal{G}_n)$. Then $\Delta = \bigcap_n A_n$.

Theorem 2.25 (32)

A LOTS with a G_δ-diagonal is symmetrizable and has a point countable base.

Proof: Let X be a LOTS with a G_δ-diagonal. Let $\Delta = \bigcap_n W(n)$ where we may suppose that $W(n+1) \subsetneq W(n)$ for each n. For each $x \in X$ and each n there is an open interval $g(n,x)$ in X such that

$(x,x) \in g(n,x) \times g(n,x) \subseteq W(n)$ and $g(n+1,x) \subseteq g(n,x)$.

This collection $\{g(n,x)\}_{n=1}^{\infty}$ is then a point countable base.

Suppose $y \in X$ and let $\langle x(n) \rangle$ be a sequence in X such that $y \in g(n,x(n))$ for all $n \geq 1$. If $z \in \bigcap_n g(n,x(n))$ then $(z,y) \in \Delta$.

Thus if $r < y < s$ there is an integer N so that if $r,s \notin g(n,x(n))$ for $n \geq N$, then $x(n) \in [r,s]$ for $n \geq N$ and thus $x(n)$ converges to y.

We can now see that this g satisfies the conditions of the following theorem of Heath (25):

A T_1 space is symmetrizable if there is a function g from ZxX to the open sets of X satisfying

(a) For each $x \in X$, $\{g(n,x)\}_{n=1}^{\infty}$ is a non-increasing sequence which forms a local base for x;

(b) If y is a point of X and x is a point sequence such that for each m, $y \in g(m,x(m))$ then x converges to y.

Corollary 2.26

A LOTS with a G_δ -diagonal is paracompact.

Proof: Follows from Theorem 2.4.

This leads to a significant characterization of metrizability in a LOTS. It is an interesting analogue to the well known result that a compact Hausdorff space is metrizable iff it has a G_δ -diagonal (17).

Theorem 2.27

A LOTS is metrizable iff it has a G_δ -diagonal.

Proof: Let X be a LOTS with a G_δ -diagonal. From Theorem 2.25 X has a point countable base, and is symmetrizable. Heath (27) has shown that a symmetrizable space with a point countable base is developable and then by Theorem 2.20 X is metrizable.

We shall obtain an alternative proof to this in Theorem 2.38 and shall obtain the converse from Remark 2.33.

Proposition 2.28

A LOTS with a G_δ -diagonal is a strict p-space.

Proof: By the result of Heath mentioned in the above proof, a LOTS with a G_δ -diagonal is developable and then by Proposition 2.12 it is a p-space.

Two other conditions on the diagonal have been considered by Hodel (28) and Borges (10) respectively.

Definition 2.29

A space X has a G^*_δ-diagonal if there is a sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of open covers of X such that $x \neq y$ implies that there is an n such that $y \notin cl_X st(x, \mathcal{G}_n)$.

Definition 2.30

A space X has a $\bar{G}_\delta$ -diagonal if there is a sequence of open covers $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of X such that

(a) $x \neq y$ implies there is an n such that $y \notin st(x, \mathcal{G}_n)$, and

(b) Given $x \in X$ and n, there is an m such that $cl_X\ st(x, \mathcal{G}_m)$
$\subseteq st(x, \mathcal{G}_n)$.

It is clear that if X has a $\bar{G}_\delta$ -diagonal then X has a G_δ^* -diagonal, and if X has a G_δ^* -diagonal then X has a G_δ -diagonal.

Proposition 2.31 (24)

A strict p-space with a G_δ -diagonal has a G_δ^* -diagonal.

Corollary 2.32

A LOTS with a G_δ -diagonal has a G_δ^* -diagonal.

Proof: By Proposition 2.28 such a space is a strict p-space, and thus by Proposition 2.31 has a G_δ^* -diagonal.

Proposition 2.33 (24)

A completely regular space is developable iff it is a strict p-space with a $\bar{G}_\delta$ -diagonal.

Remark 2.33

We can now get the converse proof required in Theorem 2.27. A metrizable LOTS is developable and thus has a $\bar{G}_\delta$ -diagonal and therefore has a G_δ -diagonal.

Proposition 2.34

A LOTS X with a G_δ -diagonal has a $\bar{G}_\delta$ -diagonal.

Proof: By the above X is developable and thus by Theorem 2.32 has a $\bar{G}_\delta$ -diagonal.

We have thus shown the following:

Proposition 2.35

In a LOTS X the following are equivalent.

(a) X has a G_δ -diagonal

(b) X has a $\bar{G}_\delta$ -diagonal

(c) X has a G^*_δ -diagonal

§6 SYMMETRIZABLE SPACES

We shall show that a symmetrizable LOTS is metrizable. This will offer an alternative proof for Theorem 2.27. The result is due to Nedev (40) but the proof is reconstructed here as Nedev's paper was not available.

Definition 2.36

A *symmetric* $d(x,y)$ on a space X is a real valued function defined on XxX such that

(a) $d(x,y) = d(y,x) \geq 0$.

(b) $d(x,y) = 0$ iff $x = y$.

(c) P is closed iff P contains every point x such that

$\{d(x,y) : y \in P\} = 0$.

A symmetric is *coherent* if for a sequence $\langle x_n \rangle \in X$ and $y \in X$ $d(x_n, y_n) \to 0$, $d(x_n, x) \to 0 \Rightarrow d(y_n, x) \to 0$.

Lemma 2.37 (50)

If a symmetric d on a space X is coherent then (X,d) is matrizable.

Theorem 2.38

A symmetrizable LOTS is metrizable.

Proof: We note first that a LOTS with a symmetric d is a first countable space since the balls of radius $\frac{1}{n}$ form a countable base at each point. It is also a Hausdorff space.

We now define an equivalent distance function ρ satisfying:

(a) $\rho(x,x) = 0$.

(b) If $x < y$ then $\rho(x,y) = \sup_{x \leq y \leq z} \{d(z,y)\}$.

(c) If $y < x$ then $\rho(x,y) = \rho(y,x)$.

This satisfies the condition $x < y < z \Rightarrow \rho(x,y) < \rho(x,z)$ since $\rho(x,y)$ is the supremum of elements of a subset of (x,z). That ρ satisfies condition (c) of Definition 2.36 can be seen as follows: If P is closed and $x \in X-P$ then there is an open ball B of radius $a > 0$ around x such that $B \subset X-P$. Thus $\rho(x,y) > a$ for every $y \in P$. Conversely, if P contains all points such that $\{\rho(x,y) : y \in P\} = 0$ then if $x \in X-P$ there is a $y \in P$ such that $\rho(x,y) = a > 0$ and a ball of radius $\frac{a}{2}$ around x will be a neighbourhood of x in X-P. So X-P is open and P is closed.

We will now show that ρ is coherent. Assume

$\rho(x,x_n) \to 0, \rho(x_n,y_n) \to 0$ and $\rho(x,y_n) \not\to 0$. Then by choosing a subsequence of y_n we get, for some $\epsilon > 0$,

$\rho(x,x_n) \to 0, \quad \rho(x_n, y_n) \to 0$ and $\rho(x,y_n) \geq \epsilon > 0$.

We may assume that $x_n > x$ for all n (or $x_n < x$ for all n in which case the proof is similar).

We can then obtain a sequence

$x_{n_1} \geq x_{n_2} \geq \ldots$ which still converges to x.

Relabel these x_n. Now consider two cases:

(a) For infinitely many n's, $y_n \leqslant x_n$.

(b) For infinitely many n's, $y_n \geqslant x_n$.

In case (a) by a similar relabelling process we can get

$\rho(x,x_n) \to 0$, $\rho(x_n,y_n) \to 0$ and $\rho(x,y_n) \geqslant \epsilon$ with

$x_1 \geqslant x_2 \geqslant \dots \geqslant x_n \geqslant \dots$ and $x_n \geqslant y_n$.

If $x \leqslant y_n \leqslant x_n$ then $\rho(x,y_n) \leqslant \rho(x,x_n)$ so $\rho(x,y_n) \to 0$, and if $y_n \leqslant x \leqslant x_n$ then $\rho(x,y_n) \leqslant \rho(x_n,y_n)$ so $\rho(x,y_n) \to 0$, which is a contradiction in either case.

In case (b) we have $y_n \geqslant x_n$ except for finitely many n. Then as above we get $x \leqslant x_n \leqslant y_n$.

Let $\epsilon > 0$ be given and let N be such that if $n \geqslant N$ then $\rho(x,x_n) < \epsilon$.

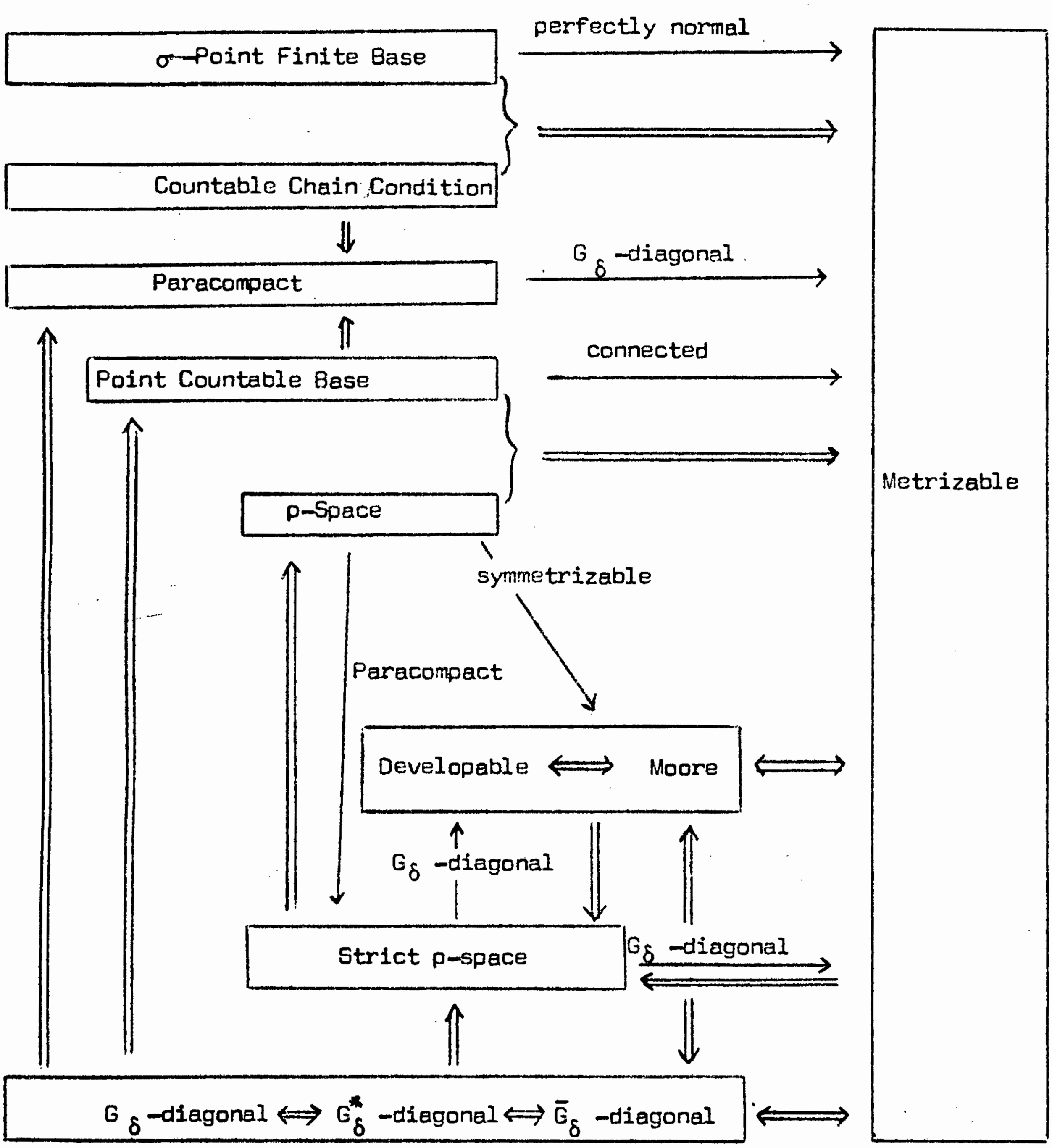
Now $\rho(x_N,x_n)$ decreases with n. Let $\alpha = \inf_n \rho(x_N,x_n)$ which is positive since $x_N \notin \mathrm{cl}\, y_n$. ($x_n \in X - \mathrm{cl}\, y_n$, a neighbourhood of x).

Thus for $m \geqslant N$, $\rho(x_m,y_m) \geqslant \rho(x_N,y_m) \geqslant \alpha > 0$, so $\rho(x_m,y_m) \not\to 0$, which gives a contradiction.

Thus ρ is coherent and by Lemma 2.37 X is metrizable.

TABLE OF RESULTS

The following is a summary of the various implications discussed for the case of a LOTS.



⟹ denotes unconditional implication for a LOTS

⟶ denotes conditional implication with a sufficient condition alongside.

CHAPTER 3 - QUASI UNIFORM BASE

The concept of a uniform base used in the metrization theory was introduced by Alexandroff (2). The definition of a uniform base was strengthened by Arkhandel'skii (5) to obtain a strong uniform base, which he showed is equivalent to metrizability. Lutzer (33) originated the concept of a quasi uniform base as a common property of compact Hausdorff spaces and LOTS which account for the similarity of a number of metrization theorems of these two types of spaces.

Definitions 3.1

A base for X is a uniform base if for each $x \in X$ and each neighbourhood U of x, only a finite number of the base sets which contain x intersect X–U. Equivalently, a base B is uniform if for all $x \in X$ any infinite subset of elements of B is a local base at x.

A base B is a strong uniform base if we replace x with a compact set K in the above definition; i.e. if for any compact subset K of X and any neighbourhood U of K, only a finite number of the base elements intersect both K and X–U.

A space is metacompact if every open cover of the space has an open point finite refinement.

A metric space has a uniform base since for each integer n the open covering by balls of radius $\frac{1}{n}$ has a locally finite subcovering. A space with a uniform base is metacompact. Alexandroff (2) has shown that a collectionwise normal space with a uniform base is metrizable and that a paracompact space with a uniform base is metrizable. Both of these follow from a result of Heath (26) that a regular space has a uniform base iff it is metacompact and developable. From the above we can obtain the following:

Proposition 3.2

A LOTS is metrizable iff it has a uniform base.

Arkhangel'skii (3) showed that a space is metrizable iff it has a strong uniform base. Each uniform base is a point countable base and Heath has shown that every symmetrizable space with a point countable base is developable.

We shall also consider bases of countable order, introduced in Arkhangel'skii in the following theorem: A paracompact Hausdorff space is metrizable iff it has a base of countable order. This has been shown to generalize the Alexandroff - Urysohn theorem mentioned in Chapter 2 §4. It turns out that the concept of a quasi uniform base is common to spaces with uniform bases, spaces with bases of countable order and LOTS.

Definition 3.3

A base of countable order is a base $\mathcal{B}$ such that if $\mathcal{b}$ is a perfectly decreasing subcollection of $\mathcal{B}$ and x is an element common to all elements of $\mathcal{b}$, then $\mathcal{b}$ is a base at x.

A space X is θ-refinable iff for every open covering $\mathcal{H}$ of X there is a countable family $\mathcal{G}$ such that each $\mathcal{F} \in \mathcal{G}$ is a collection of open sets which is a refinement of $\mathcal{H}$ covering X and for every point $x \in X$ there is an $\mathcal{F} \in \mathcal{G}$ which is finite at x.

A space is essentially T_1 if for any points x,y either $cl\{x\} = cl\{y\}$ or $cl\{x\}$ does not intersect $cl\{y\}$. This property is also called R_0 and is equivalent to the condition: if $x \in V$, open then $clx \subset V$.

Worrell and Wicke have shown that the concept of θ-refineability generalizes that of metacompactness, and also obtained the following characterization of developable spaces and metrization theorem.

Theorem 3.4 (51)

A topological space is developable iff it is essentially T_1, θ-refinable and has a base of countable order.

Theorem 3.5 (51)

A collectionwise normal T_1 space is metrizable iff it is θ-refinable and has a base of countable order.

Proof: By Theorem 3.4 the space is developable and the result follows from Bing's theorem that a collectionwise normal Moore space is metrizable. (See Appendix II)

This theorem generalizes the theorem of Arkhangel'skii mentioned previously, that a paracompact Hausdorff space is metrizable iff it has a base of countable order, since a paracompact Hausdorff space satisfies the conditions of Theorem 3.5.

Definition 3.6

A subspace Y is p-embedded in X if there is a sequence $\{\mathcal{G}(n)\}$ of covers of X by open subsets of Y such that if $x \in X$ then $\bigcap_n \{st(x, \mathcal{G}(n))\} \subseteq X$.

Thus a p-space is a space which is p-embedded in its Stone-Cech compactification.

Lutzer (33) defined the notion of a quasi uniform base as being that property responsible for the common aspect of the following two pairs of theorems.

Theorem 3.7a

A compact Hausdorff space with a G_δ-diagonal is metrizable.

Theorem 3.7b

A LOTS with a G_δ -diagonal is metrizable.

Theorem 3.8a

A paracompact space with a G_δ -diagonal which can be p-embedded in a compact Hausdorff space is metrizable.

Theorem 3.8b

A paracompact space with a G_δ -diagonal which can be p-embedded in a LOTS is metrizable.

Definition 3.9

A quasi uniform base for X is a sequence $\{\mathcal{B}(n)\}$ of bases such that a subcollection $\mathcal{F} \subset \bigcup_{n=1}^{\infty}\{\mathcal{B}(n)\}$ is a neighbourhood base at $x \in X$ if:

(a) $\mathcal{F}$ is a filterbase.

(b) $\mathcal{F} \cap \mathcal{B}(n) \neq \emptyset$ for infinitely many $n \geq 1$.

(c) $\cap \mathcal{F} = \{x\} = \cap \, cl \mathcal{F}$.

If $\mathcal{B}(n) = \mathcal{B}(1)$ for each n then X has a strong quasi uniform base.

Theorem 3.10

X has a quasi uniform base if any of the following conditions are satisfied:

(a) X is compact.

(b) X is a LOTS.

(c) X has a base of countable order.

(d) X is developable.

(e) X has a uniform base.

<u>Proof</u>: (a) From (29:5F) it can be seen that any base for a compact space is a strong quasi uniform base.

(b) The family of all open convex subsets of a LOTS form a strong quasi uniform base.

(c) Let $\mathcal{B}$ be a base of countable order and let $\mathcal{F} \in \mathcal{B}$ be a filterbase satisfying $\cap \mathcal{F} = \{x\} = \cap \text{cl} \mathcal{F}$. Then either $\mathcal{F}$ is perfectly decreasing or $\{x\} \in \mathcal{F}$; in either case $\mathcal{F}$ is a local base for x.

(d) and (e) Both follow from (c) and Heath's result mentioned after Definition 3.1 that a space with a uniform base is developable, and a developable space has a base of countable order (Theorem 3.4).

It is shown in (33) that the concepts of quasi uniform base and uniform base are inherited by open subspaces but not by closed subspaces. Also the arbitrary product of spaces with a strong quasi uniform base has the same property. It is further shown that:

Theorem 3.11

If X is p-embedded in a space Y which is either a compact Hausdorff space or a LOTS then X has a quasi uniform base.

Proof: We give an outline of the proof in the case where Y is a LOTS. Let $\{\mathcal{G}_n\}$ be a sequence of covers of X by open subsets of Y which satisfies $\bigcap_n \{ st(x, \mathcal{G}_n)\} \subset X$ for each $x \in X$. We may assume that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ for each n.

Let $\mathcal{J} = \{I : I$ is an open convex subset of $Y\}$, and let

$\mathcal{J}(n) = \{T \in \mathcal{J} : cl_Y(T) \subset G$ for some $G \in \mathcal{G}_n\}$.

Let $\mathcal{B}(n) = \{T \cap X : T \in \mathcal{J}(n)\}$. Then each $\mathcal{B}(n)$ is a base for X and $\mathcal{B}(n+1) \subset \mathcal{B}(n)$. Suppose $\mathcal{F} \subset \bigcup_{n=1}^{\infty}\{\mathcal{B}(n)\}$ is a filter base which satisfies

$\cap \mathcal{F} = \{x\} = \cap \{cl_X(F) : F \in \mathcal{F}\}$ and $\mathcal{F} \cap \mathcal{B}(n) \neq \emptyset$ for infinitely many $n \geq 1$. Then $\mathcal{F} \cap \mathcal{B}(n) \neq \emptyset$ for each n.

Let $\mathcal{E} = \{T \in \bigcup_{n=1}^{\infty}\mathcal{J}(n) : T \cap X \in \mathcal{F}\}$. Then $\mathcal{E}$ is a filterbase and $\cap \mathcal{E} = \{x\}$. Since the members of $\mathcal{E}$ are convex, $\mathcal{E}$ is a neighbourhood base for x in Y. Hence $\mathcal{F}$ is a neighbourhood base for x in X.

Lutzer's main result is that a regular space with a G_δ-diagonal and a quasi uniform base has a base of countable order. From this, Theorems 3.7 and 3.8 and various other results follow.

Theorem 3.12

A regular space with a G_δ -diagonal and a quasi uniform base has a base of countable order.

Proof: Let $\Delta = \bigcap_{n=1}^{\infty} w(n)$ where each $w(n)$ is open in XxX and $w(n+1) \subset w(n)$ for each n. Let $\{\mathcal{B}(n)\}$ be a quasi uniform base. Since each $\mathcal{B}(n)$ is a base for a regular space X, for each $x \in X$ we can choose a sequence $\{B(n,x)\}$ such that

$$(x,x) \in B(n,x) \times B(n,x) \subset clB(n,x) \times clB(n,x) \subset w(n)$$

with $B(n,x) \in \mathcal{B}(n)$, and $B(n+1,x) \subset B(n,x)$ for all $n \geqslant 1$.

Let $\mathcal{G}(n) = \{B(n,x) : x \in X\}$. Each $\mathcal{G}(n)$ is an open cover of X.

Suppose $p \in X$, $j_1 < j_2 < \ldots$ and $G_{j_k} \in \mathcal{G}(j_k)$ with $p \in G_{j_{k+1}} \subset G_{j_k}$.

Then $p \in \cap\{G_{j_k} : k \geqslant 1\} \subset \cap\{clG_{j_k} : k \geqslant 1\}$ and if $q \in \cap\{clG_{j_k} : k \geqslant 1\}$ then $(p,q) \in \cap\{w(j_k) : k \geqslant 1\} = \Delta$.

Hence $\cap\{clG_{j_k} : k \geqslant 1\} = \{p\}$. Since $G_{j_k} \in \mathcal{B}(j_k)$ it follows from Definition 3.9 that $\{G_{j_k} : k \geqslant 1\}$ is a local base for p in X.

The sequence $\{B(n,p)\}$ is a local base at p and then from a characterization of spaces with bases of countable order in (33), X has a base of countable order.

Corollary 3.13

If X is regular, the following hold:

(a) X is developable iff X is θ-refinable, has a G_δ-diagonal and has a quasi uniform base.

(b) X has a uniform base iff X is metacompact, has a G_δ-diagonal and has a quasi uniform base.

(c) X is metrizable iff X is paracompact, has a G_δ-diagonal and has a quasi uniform base.

Proof: (a) From Theorem 3.12 X has a base of countable order, and by Theorem 3.4 a regular space is developable iff it is θ-refinable and has a base of countable order.

(b) From (a), X is developable and the result follows from Heath's result mentioned in Definition 3.1 above that a space has a uniform base iff it is metacompact and developable.

(c) From (a), X is developable and the result follows from Bing's Theorem (9) that a paracompact developable space is metrizable.

Corollary 3.14

A completely regular space X is developable iff X is a θ-refinable p-space with a G_δ-diagonal.

Proof: Follows from Theorem 3.11, Theorem 3.12, Corollary 3.13.

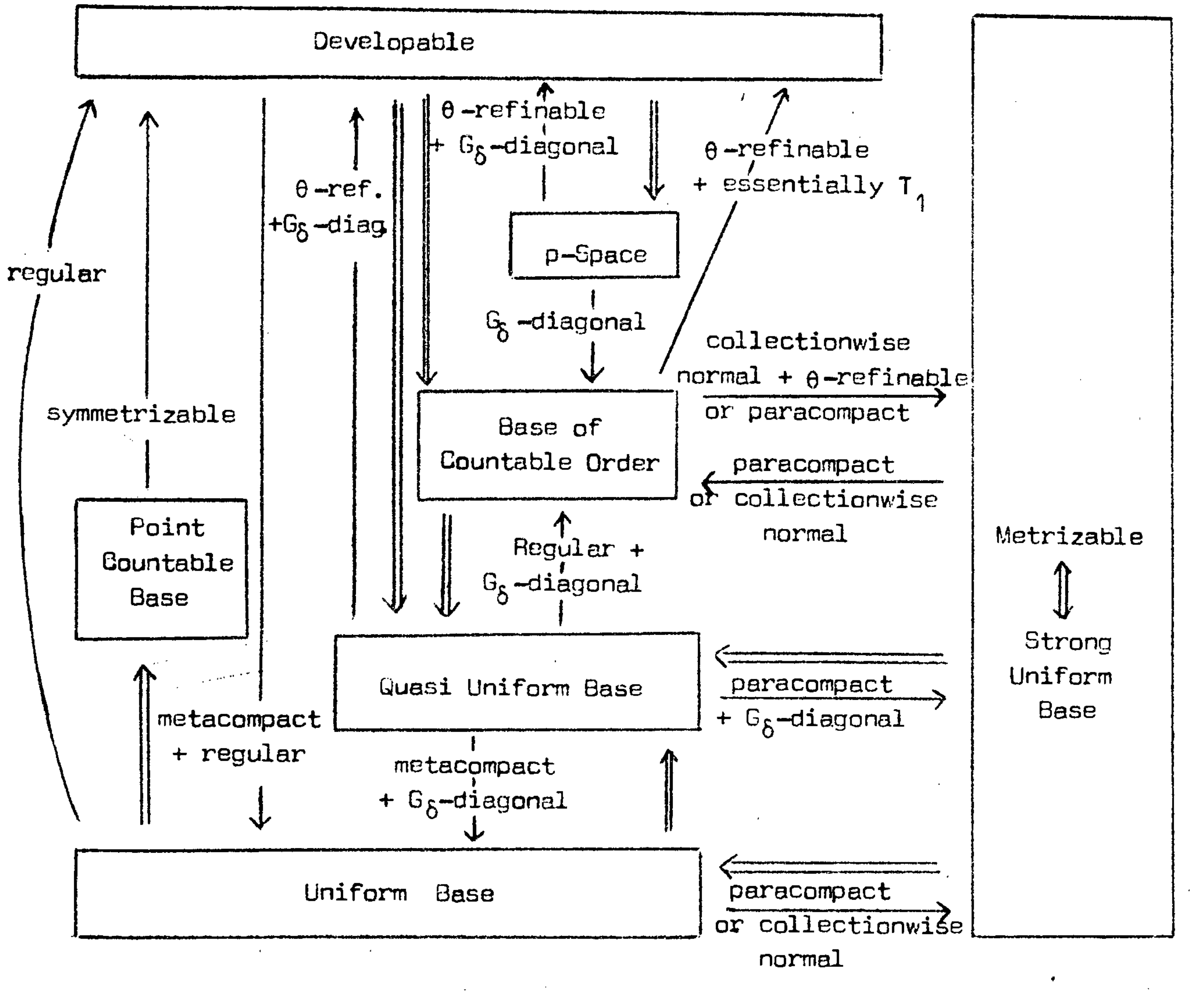
The following corollary has recently been obtained by Bennet and Berney (8).

Corollary 3.15

A p-space with a G_δ-diagonal has a base of countable order.

TABLE OF RESULTS

The following is a summary of the various implications discussed in Chapter 3.



⟹ denotes unconditional implication.

⟶ denotes conditional implication with a sufficient condition alongside.

APPENDIX I

We give the result of Bennet (7) mentioned in Corollary 2.18 that a connected LOTS is homeomorphic to a subspace of the real line. The proof involves a characterization of paracompact LOTS in terms of Q-gaps (21).

Definition I.1

An interior gap of a LOTS X is a Dedekind cut (A/B) (see 22) of X such that A has no last element and B has no first element: it may be regarded as a "virtual" element u of X that satisfies the ordering relations $a < u < b$ for $a \in A$ and $b \in B$.

If the set X has no first (last) element, we can introduce a virtual element u such that $u < x$ $(u > x)$ for all $x \in X$: u is called a left (right) end gap. The linearly ordered set of all elements and all gaps of x will be denoted by X^+.

A gap u of X is called a Q-gap from the left (right) iff there exists a regular initial ordinal ω_α and an increasing (decreasing) sequence $\{x_\beta \mid \beta < \omega_\alpha\}$ of points of X^+ such that $u = \lim_{\beta \, \omega_\alpha} x_\beta$ and if $\lambda < \omega_\alpha$ is any non-zero limit ordinal then $\lim_{\beta < \omega_\alpha} x_\beta$ is a gap of X. A gap u of X is a Q-gap if it is a Q-gap from the left and from the right.

Proposition I.2 (21)

A LOTS is paracompact iff every gap of X is a Q-gap.

It has been shown (23) that in a LOTS the concepts of paracompactness and metacompactness coincide.

Theorem I.3 (7)

A connected LOTS with a point countable base is homeomorphic to a connected subset of the real line.

Proof: Let X be a connected LOTS with a point countable base. By Corollary 2.4, X is paracompact and thus every gap is a Q-gap. Then since X is connected, the regular initial ordinal associated with any gap must be ω_o We consider three cases.

(a) If X has both endpoints, X is compact and has a point countable base. Thus by (38) X is metrizable and thus separable.

(b) If X has one endpoint, say the right end point b, then the left end point v is a virtual element of X^+ and there is a point sequence $\{x_n\}$ of elements of X which converge to v. For each $n \in N$, $[x_n, b]$ is a compact metric space and thus separable. Hence $X = \bigcup_{n=1}^{\infty} [x_n, b]$ is separable.

(c) If X has no end points, then both end points u and v are virtual elements of X^+ and the proof proceeds as above.

Then, by a characterization of the arc given in Hall and Spencer, "Elementary Topology" (1955), X is homeomorphic to a connected subset of the real line.

<u>Remark:</u> The space $\{(x,y) : 0 \leq x \leq 1, 0 < y < 1\}$ ordered lexcographically shows that "connected" cannot be replaced by "locally connected" in the above theorem, since this space is locally connected, has a point countable base but is not separable (29).

APPENDIX II

For the sake of completeness, we record Bing's Theorem (9) that a collectionwise normal Moore space is metrizable, a result that has been used repeatedly.

Definitions II.1

A space X is screenable if for every open covering $\mathfrak{F}$ there is a sequence $\{F_n\}$ of collections of pairwise disjoint open sets such that $\cup F_n$ is a refinement of $\mathfrak{F}$.

A metric space is screenable as is a paracompact space.

Proposition II.2 (9)

A normal screenable Moore space is metrizable.

Proposition II.3

For each open covering H of a developable space there is a sequence $\{X_n\}$ such that each X_n is a discrete collection of closed sets, which is a refinement of X_{n+1} and of H and $\cup X_n$ covers the space.

Proof: Let W be a well ordering of H and $\{G_n\}$ a development such that G_{n+1} refines G_n. For each $h \in H$ let $x(h_n)$ be the union of all points p such that no element of H that contains p preceeds h in W and each element of G_n containing p is a subset of h.

If X_n is the collection of all sets $x(h,n)$, X_n is a discrete collection since no element of G_n intersects two elements of X_n. If p is a point and h(p) is the first element of H and W containing p, then for some integer n, [h(p),n] contains p. Thus $\cup X_n$ covers the space.

Theorem II.4

A collectionwise normal Moore space is metrizable.

Proof: For each open cover H of X there is a sequence $\{X_n\}$ as in Proposition II.3. Since X is collectionwise normal there is a pairwise disjoint collection $\{Y_n\}$ of open sets covering $\cup X_n$ such that no element of Y_n intersects two elements of X_n but each is a subset of H.

Then $\{Y_n\}$ satisfies the condition in the definition of screenability, and the Proposition II.2 ensures that X is metrizable.

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