On Grid Diagrams, Braids and Markov Moves

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Abstract

ON GRID DIAGRAMS, BRAIDS AND MARKOV MOVES

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Grid diagrams are essential in the new combinatorial version [MOST07] of the Heegaard Floer knot homology, and proving that these homologies are actually knot and link invariants depends on knowing that two grid diagrams representing isotopic links are related by grid moves. The purpose of this paper is to prove this fact. This result has already been proved by Cromwell [Cro95] and Dynnikov [Dyn06]. We present a new proof which is built upon Markov's theorem involving moves on braid words and link isotopy.

Declaration

This thesis is the original work of the author, with the exception of sources cited in the text. The techniques in Chapters 6 and 7 are purely original as joint work with Dr David Thomas Gay.

Signed by candidate ______________________

Audry Fafa Ayivor, 5th February, 2010

Contents

1. Introduction

Grid diagrams are essential in the new combinatorial version [MOST07] of the Heegaard Floer knot homology, and proving that these homologies are actually knot and link invariants depends on knowing that two grid diagrams representing isotopic links are related by grid moves. The purpose of this paper is to prove this fact. Our proof will use Markov's theorem about braids which states that any two braid words whose closures are isotopic as links are related by Markov moves.

We begin with a discussion of links in Chapter 2, where we give definitions and state Reidemeister's Theorem. Then in Chapter 3, we discuss grid diagrams and grid moves, and we go on to give the definition of braids, discuss isotopy of braids and Markov moves in Chapter 4. In Chapter 5, we discuss how to represent braids with grid diagrams, and we produce an algorithm to turn a grid diagram into a braid. Inspired by the relationship between grid diagrams and braids, we produce a new idea in Chapter 6 - Special Grid Diagrams or SGDs - which are a unique way to represent given braids on a grid diagram. In Chapter 7, we collect our results and prove our main theorem. The proof is organised by the prism of maps in Fig 1.1 below. Here $\mathcal{G}$ is the set of all grid diagrams, $\mathcal{L}$ is the set of all links, $\mathcal{W}$ is the set of all braid words and $\sim$ is an appropriate equivalence relation on each set and the maps will be described in due course. We will show that the prism is commutative in all sections except for the upper triangular portion which is due to the fact that $\ell(G) \sim c(w(G))$, but $\ell(G)$ might not equal $c(w(G))$. The lower triangular portion of the prism forms the heart of this paper, and this will be covered in Chapter 7.

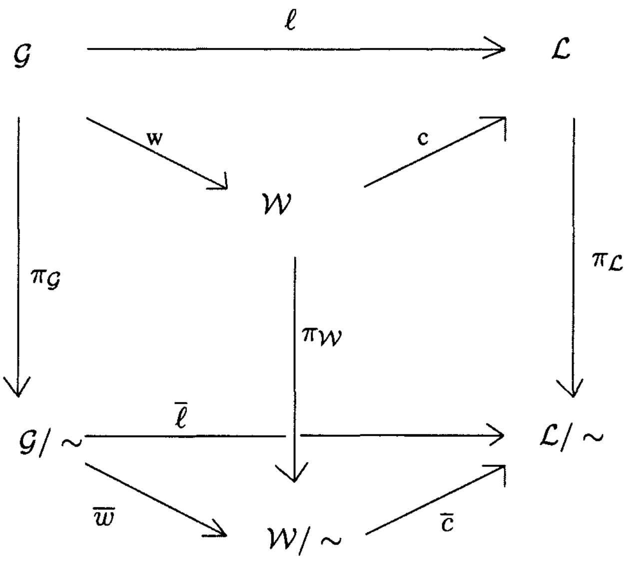


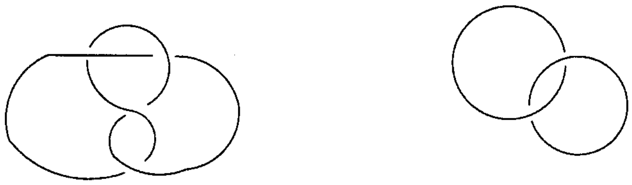
Figure 1.1: Overview: $\overline{\ell}$ is the Main theorem; $\overline{w}$ is the Sub-Theorem and $\overline{c}$ is Markov's Theorem

The maps will be described and we will prove various surjectivity, injectivity and commutativity properties of this diagram. In particular, the main result is that $\overline{\ell}$ is a bijection. The fact that $\overline{c}$ is a bijection is Markov's result. We will prove that $\overline{w}$ is a bijection and that the bottom triangle commutes. From this it follows that $\overline{\ell}$ is a bijection.

2. Links

Given two Hausdorff spaces X and Y, a mapping $f : X \to Y$ is called an embedding if $f : X \to f(X)$ is a homeomorphism. (These definitions are from [BZ85].)

Definition 2.0.1. *A* **link** *is defined as a smooth embedding in $\mathbb{R}^3$ of ℓ disjoint copies of the oriented circle. A link with ℓ=1 is a knot.*



[1] [2]

Figure 2.1: examples of links

The diagram of Fig 2.1 labelled [1] is the knot named the Figure-eight knot, while the diagram labelled [2] is a link with two components, called the Hopf link.

Definition 2.0.2 (Ambient Isotopy). *Two embeddings $f_0, f_1 : X \to Y$ are ambient isotopic if there is a level preserving isotopy*

$$H : Y \times I \to Y \times I, H(y,t) = (h_t(y), t)$$

with $f_1 = h_1 f_0$ and $h_0 = id_Y$. H is called an ambient isotopy.

Remark 2.0.3. Henceforth when we refer to isotopies of links we mean ambient isotopies.

Definition 2.0.4. *We denote the following:*

- *the set of all links by $\mathcal{L}$*
- *the equivalence relation of (ambient) isotopy by $\sim$*
- *the isotopy class determined by a particular link by $[L]$*
- *the set of isotopy classes of links by $\mathcal{L}/\sim$*
- *the quotient map from $\mathcal{L}$ to $\mathcal{L}/\sim$ by*

$$\pi_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}/\sim$$

which is obviously a surjection.

Definition 2.0.5. *A* **projection** *of a link is the 2-dimensional image of a link projected from $\mathbb{R}^3$ to $\mathbb{R}^2$ with crossings indicated.*

To motivate some ideas in the rest of the paper, we note here that:

According to history, in the 1930s, Kurt Reidemeister came up with the following theorem[Wei]:

Theorem 2.0.6. *Two given isotopic links can be related by a finite sequence of the following moves and planar isotopies:*

- **Reidemeister Move I, R1**
- **Reidemeister Move II, R2**
- **Reidemeister Move III, R3**

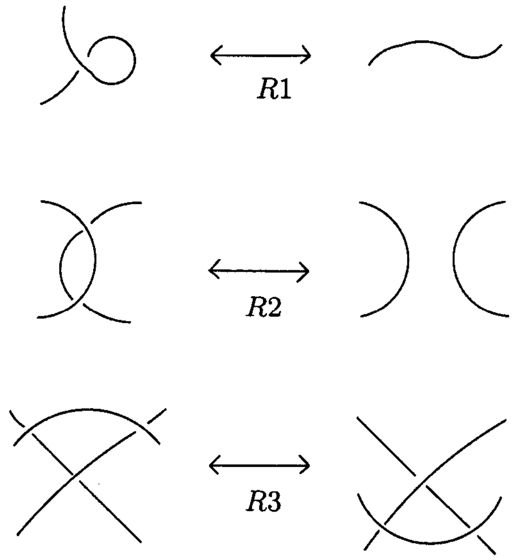


Figure 2.2: Reidemeister moves

The diagrams of Fig 2.2 illustrates the Reidemeister move types there are.

Example 2.0.7. In this example we show how a sequence of Reidemeister moves can be performed on a link projection to move it to a link projection for an isotopic link. .

We observe from the diagrams of Fig 2.3 that a succeeding diagram is obtained from a preceeding one by performing a single isotopy (Reidemeister) move.

Remark 2.0.8. Performing a finite sequence of Reidemeister moves on a link yields a class of links isotopic to the given link.

We will not use Reidemeister moves further in this paper, but rather work with other combinatorial moves on other representations of links.

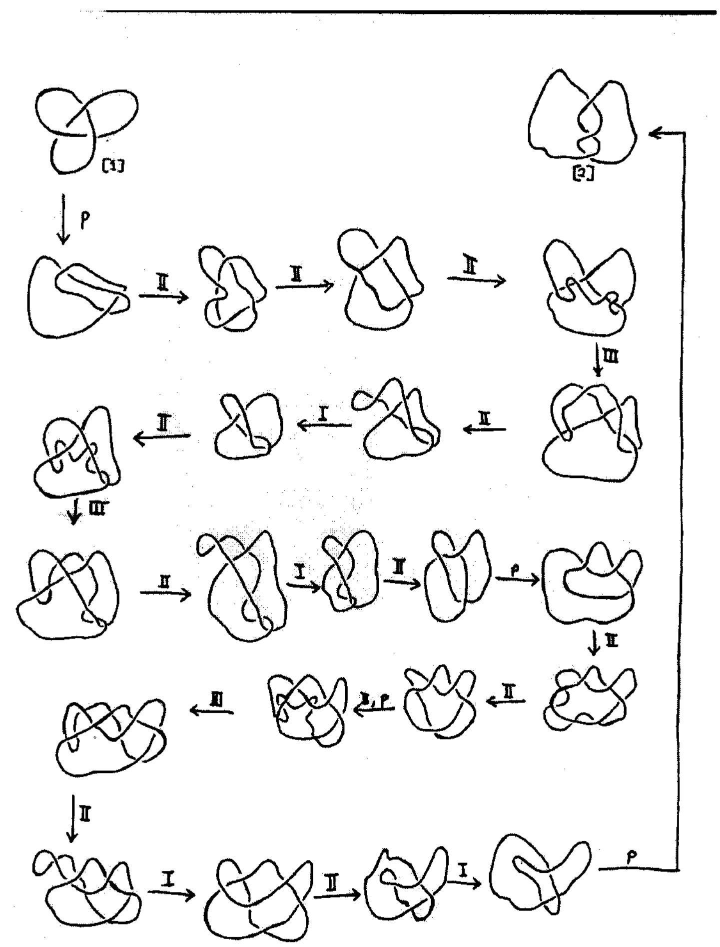


Figure 2.3: performing Reidemeister moves on a projection of the trefoil knot:the arrows show the sequence of movement starting from the link labelled [1] and ending on the link labelled [2]; the arrows labelled 'p' represent planar isotopy moves; I, II and III represent Reidemeister I, II and III moves respectively

3. Grid Diagrams

In [MOS07], the authors introduced the idea of a planar grid diagram, and how to represent a link on a grid diagram. Further, they stated a the result [Cro95][Dyn06] that grid diagrams representing the same link can be connected by a finite sequence of elementary grid moves. In this Chapter, we will discuss grid diagrams, elementary grid moves which can be performed on them and how to turn a grid diagram into a link. We will also show how to turn a link into a grid diagram.

3.1 Grid Diagram Representation of Links

Definition 3.1.1. *A* **grid diagram** *with grid number $n \in \mathbb{N}^*$ is an $n \times n$ square grid with n X's and n O's placed in distinct squares such that each row and each column contains exactly one X and one O.[NT08][MOST07]. The diagram of Fig 3.1 is an example of a grid diagram.*

O	X	
	O	X
X		O

Figure 3.1: A 3×3 grid diagram

A grid diagram can be used to either represent a link or a braid. Our focus for subsequent sections will be on achieving this.

Definition 3.1.2. *$\mathcal{G}$ is the set of all grid diagrams.*

A link diagram can be associated to any grid diagram. To achieve this, we join the O to the X in each column with an oriented vertical segment which begins from the O and ends at the X, and then in each row, by an oriented horizontal segment which begins from the X and ends at the O, and which passes over the vertical segments at each crossing.

Remark 3.1.3. In [MOST07], the convention used to turn a grid diagram into a link involves vertical strands being the overpass and the horizontal, the underpass at each crossing. Our convention used is analogous to that used in [MOST07] except for the fact that we rotate our diagram through an angle of ninety degrees to make it the same as the link constructed using the convention in [MOST07], and also, on a row, the oriented horizontal segments begin from the O and end at the X, while in a column, the oriented vertical segments begin from the O and end at the X. We explain the reason for the choice of this convention in Chapter 5.

We refer the reader to the diagram of Fig 3.2 for an illustration.

Definition 3.1.4. *We define a function* $\ell : \mathcal{G} \to \mathcal{L}$ *as follows:*

Given any $G \in \mathcal{G}$, $\ell(G)$ *is the link constructed from* G *as described in the preceeding paragraph. .*

The diagram of Fig 3.2 below illustrates this function.

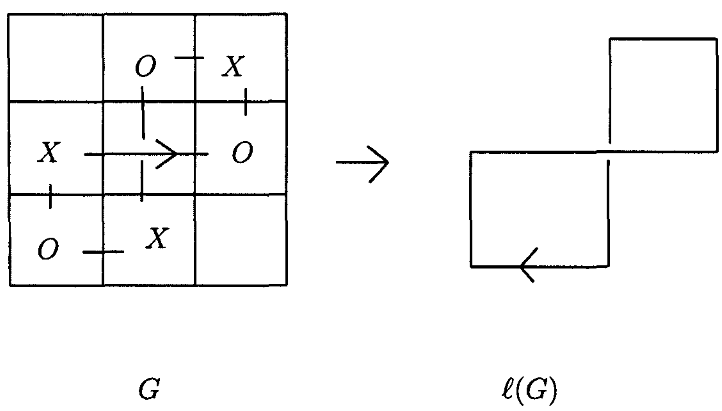


Figure 3.2: Turning a 3×3 grid diagram into a link

3.2 Turning Links into Grid Diagrams

In this section we discuss how to turn a given link diagram into a grid diagram. We put the method of achieving this result into two major parts, with the first being the procedure to place a link diagram into grid position, and the second being the procedure to place the link diagram in grid position into a grid diagram.

Definition 3.2.1. *A link diagram,* $L \in \mathbb{R}^3$ *is in* **grid position** *if it contains only vertical and horizontal oriented segments with the horizontal over vertical at all crossings and no two segments being collinear.*

Given any oriented link projection, L, we can put L in grid position as follows:

$\underline{Step1}$: Rotate near each crossing in order to arrange that, at each crossing, the overstrand is in a horizontal position, while the understrand, the vertical position.

$\underline{Step2}$: Isotope the vertical and horizontal strands of the link to turn them into oriented vertical and horizontal segments, each of which joins another segment at right angles.

$\underline{Step3}$: Isotope the oriented vertical and horizontal segments such that no two segments are colinear.

Now, we construct a grid diagram as follows:

$\underline{Step 1}$: Turn the right-angled corners of the link into X's and O's according to the orientation of the link.

$\underline{Step 2}$: Place each segment into columns and rows.

An illustration of this process is given by Figure 3.3.

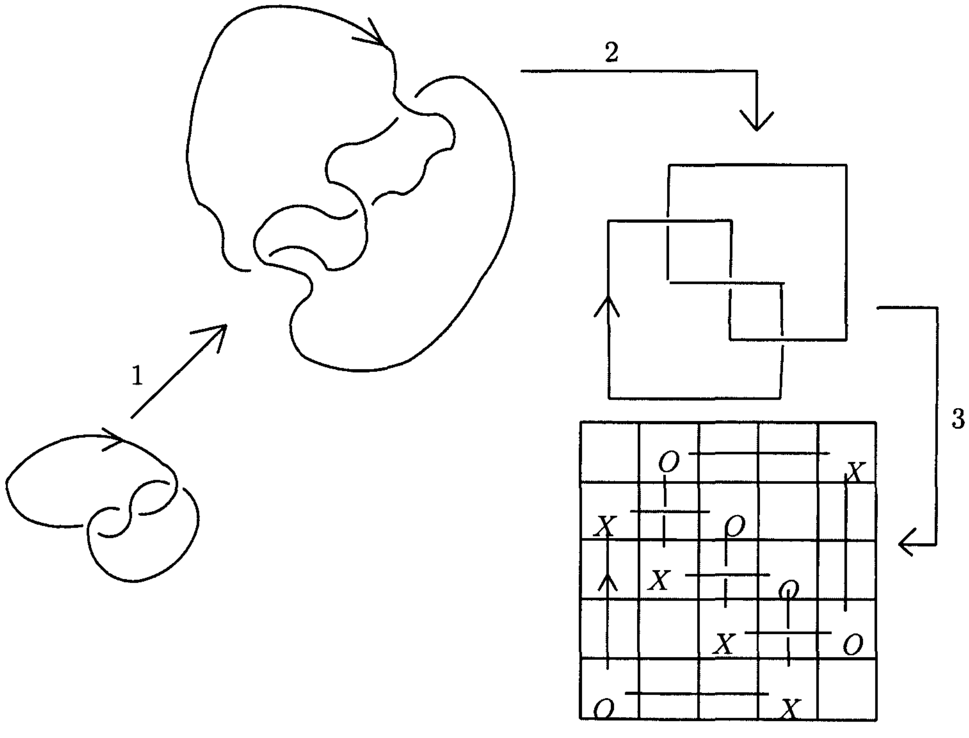


Figure 3.3: Every link up to isotopy can be represented by a grid diagram:The arrows represent the steps taken to arrive at the grid diagram; 1 represents a rotation near the crossings as stated in Step 1 above; 2 represents placing the link in grid position and 3 represents labelling the corners of the link with Xs and Os based on the orientation of the given link.

3.3 Elementary Grid Moves

These are moves performed on grid diagrams. In Chapter 2, we showed with an example how to relate any two given links in general position by Reidemeister moves. These grid moves are analogous to Reidemeister moves. These grid moves give an equivalence relation $\sim$ on $\mathcal{G}$.

Our main goal of this paper is to prove:

Theorem 3.3.1 (Main Theorem). *Any two grid diagrams which represent isotopic links can be related by a finite sequence of the following grid moves:*

1. **Cyclic permutation**. *This involves cyclically permuting the columns (resp. rows) of the grid diagram. In other words, it entails moving a column (resp. row) from the extreme left (resp. top) to the extreme right (resp. bottom) and vice versa. Refer to Fig 3.4.*

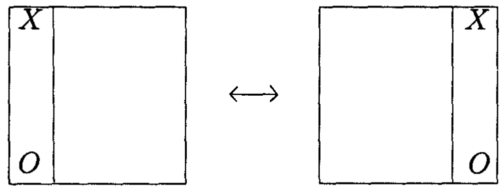


Figure 3.4: Elementary grid moves: cyclic permutation

2. **Commutation**. *Consider a pair of consecutive columns (resp. rows) in a grid diagram. Draw a straight vertical line to connect the entries starting from the entry heighest in position to that lowest in position. Label the entries $i = 1, \dots, 4$ as one would read from top to bottom. We then consider the following possible conditions:*

   - *the pair of entries of one column (resp. row) are 1 and 2, while the pair of entries for the other column are 3 and 4, or*
   - *the pair of entries of one column (resp. row) are 1 and 4, while the pair of entries for the other column (resp. row) are 2 and 3.*

   When either of these conditions is satisfied, we then switch the entries of the columns (resp. rows) as illustrated in Fig 3.5.

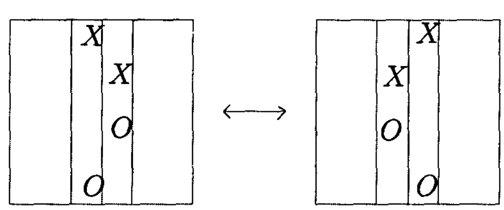


Figure 3.5: Elementary grid moves: commutation

3. **Stabilisation(Destabilisation)**. *Consider a grid diagram, G, of size n. A stabilisation (resp. destabilisation) move involves adding (resp. removing) one column and one row from G, thereby resulting in a grid diagram of size $n+1$ (resp. $n-1$). The diagrams of Fig 3.6 illustrate all the possible configurations of a stabilisation and a destabilisation move, which we have drawn without the X's and O's because it is easier to understand.*

Remark 3.3.2. All throughout this paper, we will refer to a **horizontal commutation** move as the move which involves switching the entries of two consecutive rows; a **vertical commutation** move as the move which involves switching the entries of two consecutive columns.

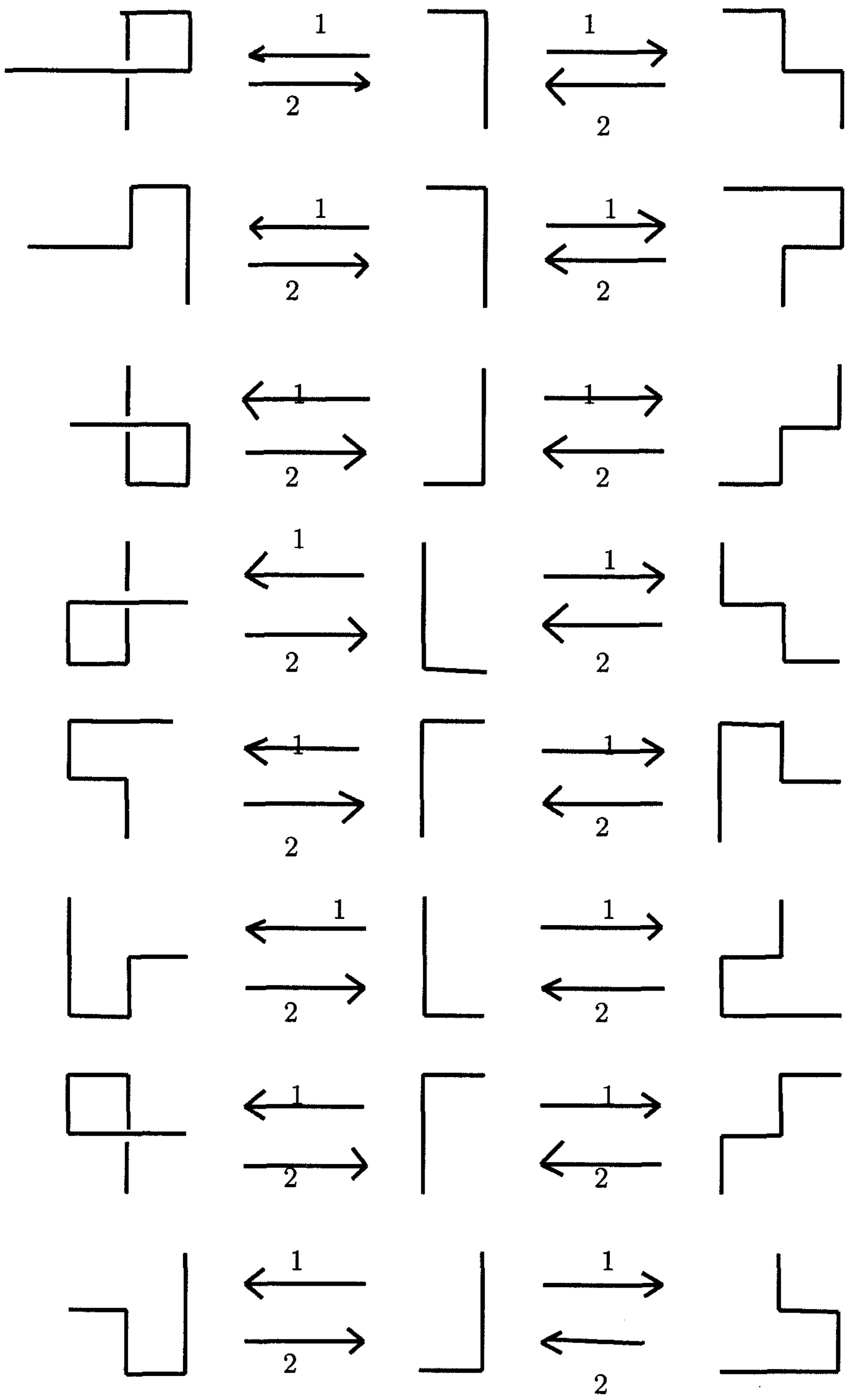


Figure 3.6: Elementary grid moves:1: stabilisation 2: destabilisation

Example 3.3.3. In this example, we begin with a given grid diagram, perform a single grid move on it, and then repeat the same for subsequent grids until we finally obtain a grid diagram which is isotopic to the original grid diagram as illustrated in Fig 3.7.

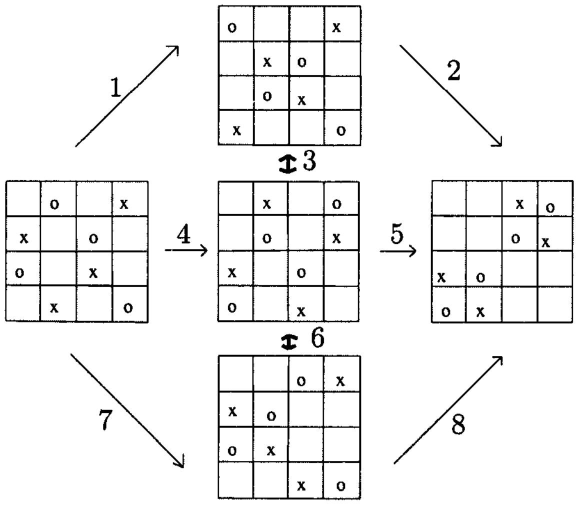


Figure 3.7: Performing grid moves on grid diagrams yields equivalent grid diagrams: The arrows with relatively thinner segments each represent a single grid move, while those with relatively thicker segments represent multiple grid moves. The arrow labelled 1 represents a commutation between columns one and two; 2 represents a cyclic permutation of row one to row 4; 3 represents multiple grid moves; 4 represents a cyclic permutation from row four to row one; 5 represents a commutation between columns two and three; 6 represents multiple grid moves; 7 represents a commutation between columns two and three ; 8 represents cyclic permutation from row four to row one.

We then give an illustration of the effects of the grid moves on the associated links as a result of performing the grid moves as in Fig 3.8.

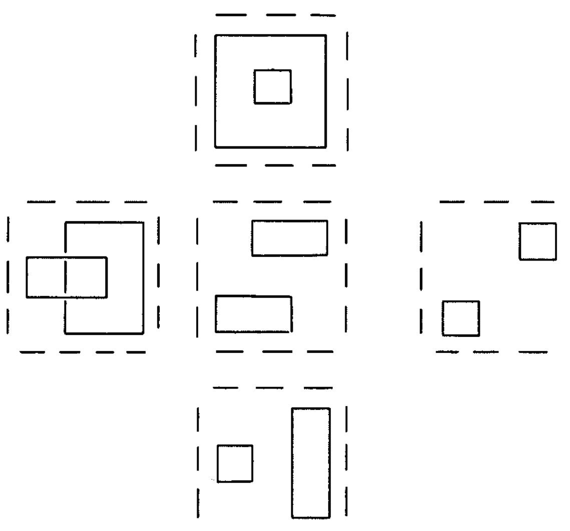


Figure 3.8: the associated links obtained from the grid diagrams of Fig 3.7

The elementary grid moves performed on grid diagrams yield an equivalence relation on the set of grid diagrams.

Definition 3.3.4. *G is equivalent to G', denoted by $G \sim G'$, if there is a finite sequence $G_i, 1 \leq i \leq q$, with*

$$G = G_1, \ldots, G_q = G'$$

and G_{i+1} is obtained from G_i by a single grid move.

Definition 3.3.5. *We denote the following:*

- *the set of all grid diagrams by*

$$\mathcal{G}$$

- *the equivalence class of grid diagrams isotopic to a given grid diagram, G by*

$$[G] = \{G' \in \mathcal{G} | G' \sim G\}$$

- *the set of all $\sim$-equivalence classes by*

$$\mathcal{G}/\!\sim = \{[G], G \in \mathcal{G}\}$$

Lemma 3.3.6. *The function $\overline{\ell} : \mathcal{G}/\!\sim \longrightarrow \mathcal{L}/\!\sim$ defined by*

$$\overline{\ell}([G]) = [\ell(G)]$$

is well-defined.

Proof. The point here is that the grid moves do not change the isotopy class of the link associated to the grid diagram. This can easily be checked by the reader by seeing that each grid move breaks into a sequence of Reidemeister moves. □

Since grid moves give an equivalence relation $\sim$ on $\mathcal{G}$, we therefore define the following projection:

Definition 3.3.7. *We define the projection $\pi_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{G}/\!\sim$ by*

$$\pi_G(G) = [G]$$

Lemma 3.3.8. *Every link diagram up to isotopy can be represented by a grid diagram i.e.*

$\pi_{\mathcal{L}} \circ \ell : \mathcal{G} \longrightarrow \mathcal{L}/\!\sim$ is surjective, and thus $\overline{\ell} : \mathcal{G}/\!\sim \longrightarrow \mathcal{L}/\!\sim$ is surjective.

Proof. We recap the following notation:

$$\begin{aligned} \mathcal{L} &= \text{set of links in } \mathbb{R}^3 \\ \mathcal{G} &= \text{set of grid diagrams} \end{aligned}$$

We consider an arbitrary link L, $L \in \mathcal{L}$. We follow the steps of Section 3.2 to turn the given link into a grid diagram. This shows that given any link $L \in \mathcal{L}$, we can find a grid diagram, $G \in \mathcal{G}$, such that $\ell(G) \sim L$.

In other words, given $[L] \in \mathcal{L}/\sim$, we use the steps of Section 3.2 to produce $G \in \mathcal{G}$ such that $\ell(G) \sim L$, and therefore $\pi_{\mathcal{L}}(\ell(G)) = [L]$. See Fig 3.9.

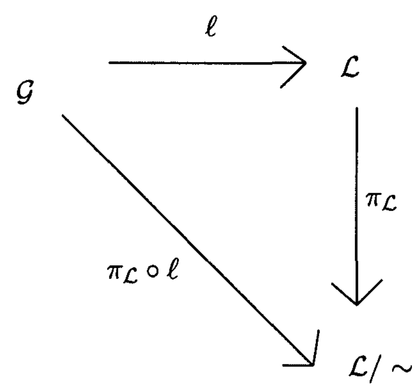


Figure 3.9: a commutative diagram

□

4. Braids

According to history, in 1925, Emil Artin, a German Mathematician, came up with a theory, which is known today as Braid theory. Braid theory has many applications and these include cryptography. In this Chapter we will discuss how to represent braids using diagrams and words, and the isotopy of braids.

4.1 Braids

In this section, we begin with a definition of a braid.

Definition 4.1.1. *An n-**braid** is a collection of n arcs (also called strands or strings), c_i, $i \leq n$, in $\mathbb{R}^3$, placed between a pair of parallel horizontal planes such that the following hold:*

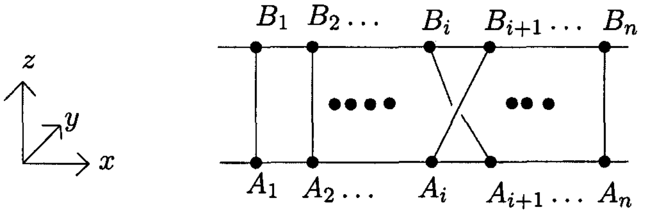


Figure 4.1: a braid

1. *each arc c_i, is located completely between the horizontal parallel planes, A= $\{z = 0\}$ and B= $\{z = 1\}$*

2. *each arc begins at exactly one point A_i on A and ends at exactly one point B_i on B ; the collection of points on each horizontal plane are as follows:*

$$A_i := \left\{\frac{1}{2}, \frac{i}{n+1}, 0\right\}$$

and

$$B_i := \left\{\frac{1}{2}, \frac{i}{n+1}, 1\right\}$$

3. *the arcs are mutually disjoint*

4. *each c_i meets B*

5. *if we take the height function, z of the embeddings, it does not have any local maxima or minima or horizontal tangencies.*

Definition 4.1.2. *A **braid diagram** is a projection onto the y-z plane of the braid with crossings indicated.*

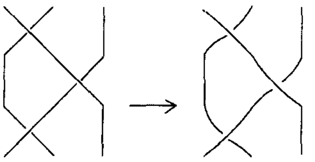


Figure 4.2: A braid with three strands

Fig 4.2 shows two isotopic braid diagrams of which the left is polygonal and the right is smooth.

Definition 4.1.3. *A* **crossing** *of a braid is a point at which an overstrand passes over an understrand of the braid.*

The diagram of Fig 4.2 has 3 crossings.

Definition 4.1.4. *The* **braid index** *of a given braid is the number of strands forming the braid.*

The diagram of Fig 4.2 is an example of a 3-braid, i.e. it has a braid index of three.

4.2 Braid words

In this section we discuss braid words. Each braid has a unique set of letters by which it can be represented. In determining this braid word, our direction of movement will be the same as that of the strands of the braid.

Definition 4.2.1. *A* **braid word** *is a pair (n, W) where $n \in \mathbb{N}^*$ is the number of strands of the braid and W, a word in the letters $\sigma_1, \ldots, \sigma_{n-1}\sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}$, called* **Artin's generators**.

Remark 4.2.2. Frequently, we will omit n from the pair, but it is implied.

The generators of a braid represent the crossings of the braid and they may either be positive i.e. σ_i, when the overstrand has a positive slope, or negative i.e. σ_i^{-1}, when the overstrand has a negative slope, as shown in the diagram of Fig 4.3.

Now, we discuss the relation between braid words and braids. In particular, we determine how to find the braid word of a braid when given an n-braid. The steps involved in this process are as follows:

1. Ensure the braid is placed between two parallel horizontal lines and no two crossings are on the same level

2. Label the strands of the braid, $i = 1, \ldots, n$

3. Determine the name of each generator by naming the crossing after the label of the string on the left hand side of the two consecutive strings which make up the crossing e.g. σ_5 for a crossing made by strings 5 and 6.

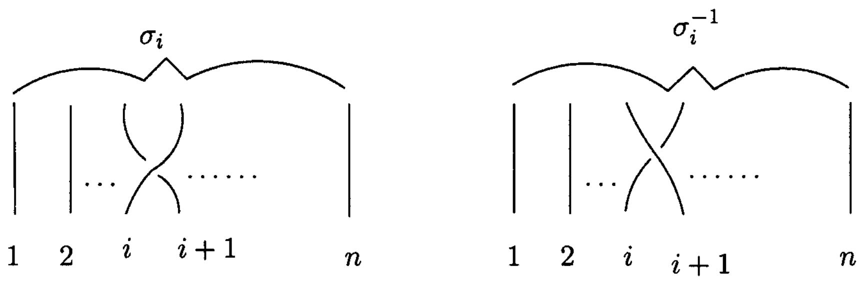


Figure 4.3: diagrams for σ_i and σ_i^{-1}

4. Determine the sign of each generator as mentioned above.
5. Write down the product of the generators determined beginning from the bottom-most generators.

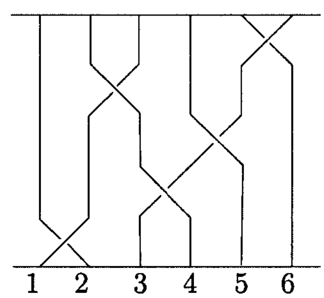


Figure 4.4: determining the braid words for a given n-braid

In the example of Fig 4.4, on reading the position and signs of the generators, beginning from the generator at the bottom, the product of the results obtained is the braid word:

$$\sigma_1\sigma_3^{-1}\sigma_4^{-1}\sigma_2^{-1}\sigma_5$$

Going the other way, given a braid word with the index number specified, we obtain the associated n-braid as follows:

1. For each letter, draw the associated elementary braid as in Fig 4.3.
2. Concatenate them, with the elementary braid corresponding to the leftmost letter at the bottom.

Definition 4.2.3. *We define the following:*

$$\mathcal{W} = \textit{set of all braid words}$$

As with links, there is a natural notion of isotopy of braids, however, we will be working with closures of braids more than with actual braids, so we do not discuss this further.

4.3 Braid Closure

In this section, we describe how to obtain the closure of a braid from a given braid. In [Dyn06], the author presented a closed braid via an oriented 'rectangular diagram' which could be converted to a braid by some deformation. However, in [Cro95], the author represented them as embeddings in sheaves of half planes which were termed the arc presentations of the link.

Definition 4.3.1. *Let b be a given braid. We define the* **closure** *of b by connecting the endpoints A_i to B_i, $1 \leq i \leq n$ in a manner which preserves order without introducing new crossings. That is, we join A_1 to B_1 etc. We denote the closure of a given n-braid, b, by $\bar{b}$.*

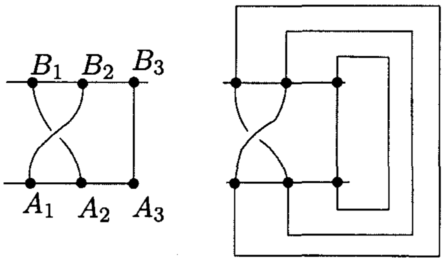


Figure 4.5: the diagrams of a braid, W $= \sigma_1$ and its closure

The resulting diagram is a link. This is illustrated in Fig 4.5.

Remark 4.3.2. Note that the number of **components** of a closed braid is the number of cycles of permutation of the braid.[BZ85]. The closure of an n-stranded braid results in a link with i components, for some $i \leq n$.[Bir74]

Example 4.3.3. For this example, we discuss the diagrams of Fig 4.5. In these diagrams, we are given a 3-braid, which closure we obtained by the method described above. We observe that the number of cycles obtained on performing the closure is two, thus the link obtained has two components.

Remark 4.3.4. The closure of a braid can still be obtained when given the braid word of the n-braid.

Definition 4.3.5. *We define a function $c : \mathcal{W} \rightarrow \mathcal{L}$ as follows:*

Given $W \in \mathcal{W}$, $c(W)$ is the closure of the braid associated to W.

Theorem 4.3.6 (Alexander, 1923). *[Bir74] Every link is isotopic to a closed braid.*

Proof. We refer the reader to [Bir74] for the proof of this theorem. □

Corollary 4.3.7. *$\pi_{\mathcal{L}} \circ c$ is surjective.*

Proof. Given $[L] \in \mathcal{L}$, let $W \in \mathcal{W}$ be the braid word for the braid whose closure is isotopic to L, and thus $c(W) \sim L$ so $\pi_{\mathcal{L}} \circ c(W) = [L]$. □

4.4 Markov Moves

In a quest to find a relationship between equivalent close braids, Markov came up with a theorem which related two close braids by a sequence of moves which do not alter the link type [Bir74]. In [Cro95], the author gave arc presentations of links and a finite sequence of isotopy preserving moves one could carry out on isotopic braids and showed how the Markov moves can be accomplished by these moves.

The following are a summary of the Markov moves [Cro95].

- **Type I Move:** A braid with a braid word $W = W_0\sigma_i \cdot \sigma_i^{-1}W_1$ can be replaced by the braid with braid word $W' = W_0W_1$; similarly a braid with braid word $W_0\sigma_i^{-1} \cdot \sigma_i W_1$ can be replaced by the braid with braid word $W' = W_0W_1$.
- **Type II Move:** If $|i-j| \geq 2$, then a braid with braid word $W = W_0\sigma_i \cdot \sigma_j W_1$ can be replaced by the braid with the braid word $W' = W_0\sigma_j \cdot \sigma_i W_1$.
- **Type III Move:** A braid with braid word $W = W_0\sigma_i \cdot \sigma_{i+1} \cdot \sigma_i W_1$ can be replaced by a braid with braid word $W' = W_0\sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} W_1$.
- **Type IV Move:** A braid with braid word W can be replaced by a braid with braid word $\sigma_i W \sigma_i^{-1}$ or $\sigma_i^{-1} W \sigma_i$
- **Type V Move:** A braid may be modified by a planar isotopy, i.e. an isotopy that does not change the braid word.

Note that moves I, II, III and IV are really moves on braid words. We illustrate these moves in the following figures:

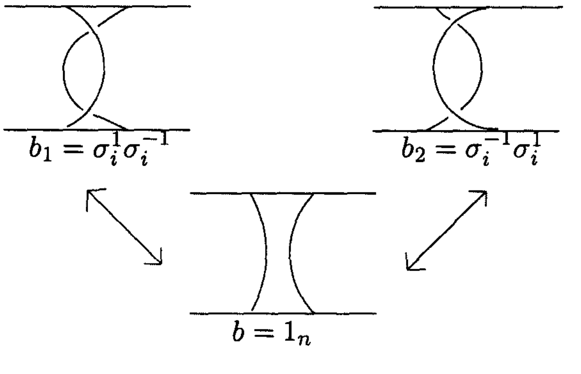


Figure 4.6: Type I Markov Move

In the Example 4.4.1, we give two braid words representing equivalent braids and show how starting from one, we could arrive at the next using braid relations inspired by the Markov moves described above.

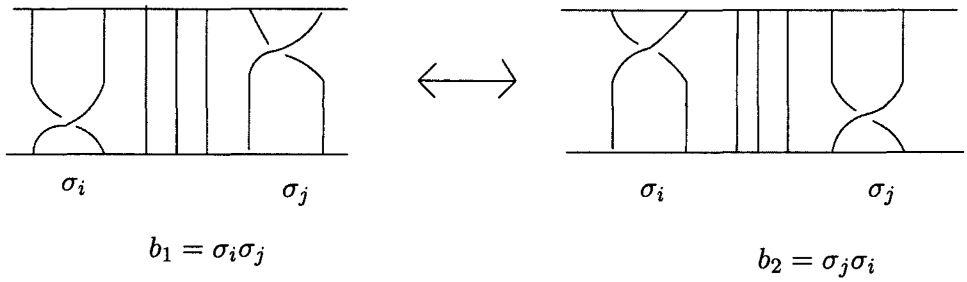


Figure 4.7: Type II Markov Move

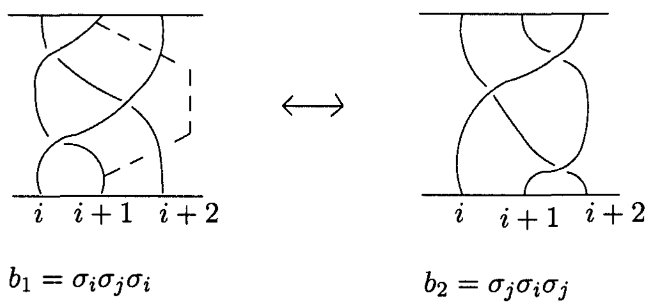


Figure 4.8: Type III Markov Move

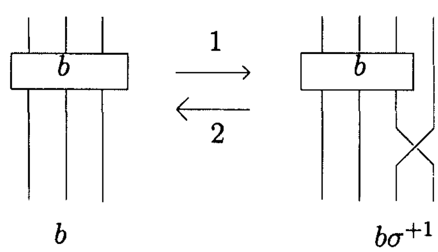


Figure 4.9: Type IV Markov Move:The arrow labelled 1 represents a stabilisation move; 2 represents a destabilisation move

Example 4.4.1. The following braid words represent the diagrams drawn in Fig 4.10.

$$\begin{aligned}
\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1 &\simeq \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_2\sigma_1 \\
&\simeq \sigma_2^{-1}\sigma_1^{-1}\sigma_1 \\
&\simeq \sigma_2^{-1}
\end{aligned}$$

Definition 4.4.2. *Two braids, b and b_1 are* **Markov equivalent** *if they can be related by a finite sequence of Markov moves.*

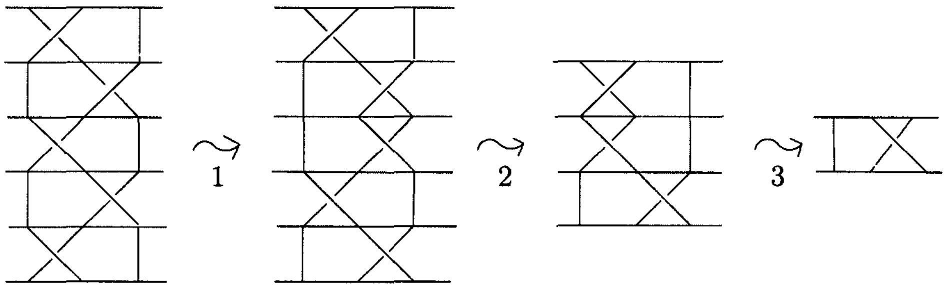


Figure 4.10: relating equivalent braid words with braid relations: the arrows represent the Markov moves performed on a braid to achieve the subsequent one; 1 represents a type 3 Markov move; 2, a type 2 Markov move; 3, a type 1 Markov move

In 1935, Markov stated the following theorem[Mar35]:

Theorem 4.4.3. *Given two braids, b and b_1, their closures, $\overline{b}$ and $\overline{b}_1$, are isotopic as links iff b and b_1 are Markov equivalent.*

Proof. We refer the reader to [Bir74], [Mar35], [BM02], [Lam93] for proof of Markov's Theorem. □

Definition 4.4.4. *Two braid words, $W, W' \in \mathcal{W}$ are equivalent, $W \sim W'$, if W and W' are related by Markov moves of type I, II, III and IV (We ignore type V because this does not change braid words).*

Definition 4.4.5. *We denote the following:*

- *the equivalence class determined by a particular braid word by $[W]$*
- *the set of equivalence classes of braid words by $\mathcal{W}/\sim$*
- *the quotient map from $\mathcal{W}$ to $\mathcal{W}/\sim$ by*

$$\pi_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}/\sim$$

which obviously is surjective.

Corollary 4.4.6. *The map $\overline{c} : \mathcal{W}/\sim \to \mathcal{L}/\sim$, defined by $\overline{c}([W]) = [c(W)]$ is well-defined and is a bijection.*

Proof. Suppose $[W] = [W']$, i.e. $W \sim W'$, then we know from Markov's theorem that $c(W) \sim c(W')$. Therefore $\overline{c}$ is well-defined.

Because $\pi_{\mathcal{L}} \circ c$ is a surjection (Corollary 4.3.7), $\overline{c}$ is also a surjection. From Markov's theorem, if $\overline{c}([W]) = \overline{c}([W'])$ i.e. $c(W) \sim c(W')$, then $W \sim W'$, so $[W] = [W']$. Therefore, $\overline{c}$ is an injection. □

[0]At this point, the author wishes to express sincere gratitute to Professor Sofia S. F. Lambropoulou, National Technical University, Athens, Greece, who gave her material on the detailed proof of Markov Theorem, which formed part of Sofia's PhD thesis.

5. Grids→Braids

5.1 Grid Representation of Braids

In Chapter 3, we discussed the use of grid diagrams to represent links. In this Chapter, we will discuss how grid diagrams can be used to represent braids using a convention for joining the Xs and Os. In Chapter 4, we represented braids using braid diagrams. However, in this Chapter, we provide another way of representing braids, which is via grid diagrams[KN09][NT08].

A grid diagram G, can be used to represent a braid. To achieve this, one should join the X to the O in each row with an oriented horizontal segment which begins at the X and ends at the O. In each column, one should join the O to the X with an oriented horizontal segment which begins at the O should the X be located above the O. If however the X is located below the O, connect the O to an oriented vertical segment which begins from the O and ends on the outside of the grid. Then connect an oriented vertical section from the outside of the grid and ends at the X. At each crossing, we keep the oriented horizontal segments as overpasses.

Remark 5.1.1. In [NT08], the convention used to represent a braid on a grid diagram involves the vertical strands being the overpass and the horizontal the underpass at each crossing. Our convention used is analogous to that used in [NT08] except for the fact that we have to rotate our diagram through an angle of ninety degrees. Our reason for choosing this convention is that we prefer to view braids vertically, unlike in [NT08], where the braids are viewed horizontally.

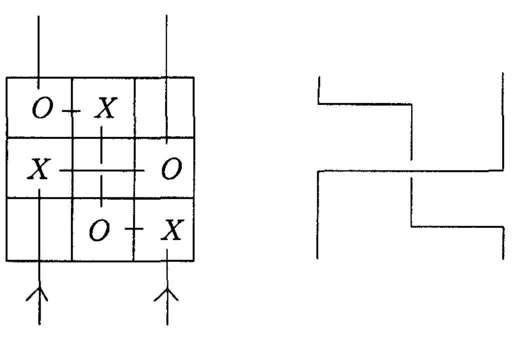


Figure 5.1: obtaining a braid from a grid diagram: the associated braid word is $W = \sigma_1$

Given a grid diagram, we can thus construct the associated braid and then read off the braid word as described above.

Definition 5.1.2. *We define the function, w which takes a grid diagram to a braid word as follows:*

$$w : \mathcal{G} \rightarrow \mathcal{W}$$

Given any $G \in \mathcal{G}$, $w(G)$ is the braid word read from the braid represented by the grid diagram, G, as discussed above.

Example 5.1.3. The braid word associated to this grid diagram is $\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3\sigma_4\sigma_3^{-1}\sigma_1\sigma_3\sigma_4^{-1}\sigma_1\sigma_2$

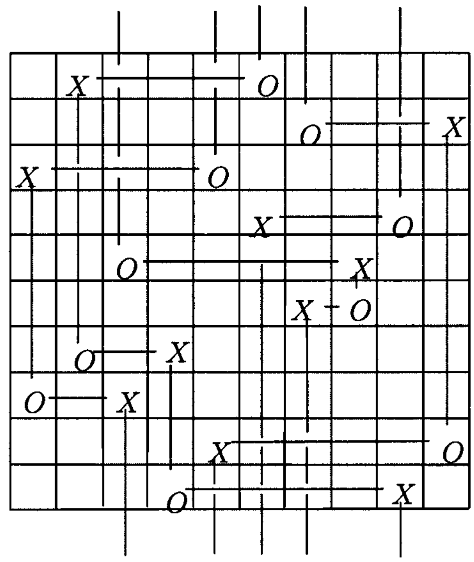


Figure 5.2: a nontrivial example of a grid representation of a braid

5.2 Commutativity of the Upper Triangle of the Prism

Now we prove that $c(w(G)) \sim \ell(G)$ i.e. the upper triangle of the prism of maps in Fig 1.1 commutes up to isotopy.

Lemma 5.2.1. $c(w(G)) \sim \ell(G)$

Proof. We want to prove $c(w(G))$ is isotopic to $\ell(G)$. We consider a given diagram, G (We consider the example in Fig 5.3). By the method described in Section 3.1, we turn G into a link. We denote this link by $\ell(G)$ (Refer to Fig 5.3). By the method described in Section 5.1, we turn G into a braid as illustrated in Figure 5.4.

We obtain the closure of this braid as described in Section 4.3. We denote this closure by $c(w(G))$ (Refer to Fig 5.4). It is clear from the example that the only difference between $\ell(G)$ and $c(w(G))$ of Fig 5.3 and Fig 5.5 respectively, is that the downward vertical strands in $\ell(G)$, which are behind all other strands have been isotoped back and out to the right, so that they end up going up and then around to the right. The reader can easily check the procedures described so far with grid diagrams of varying sizes. This shows that $c(w(G))$ is isotopic to $\ell(G)$.

□

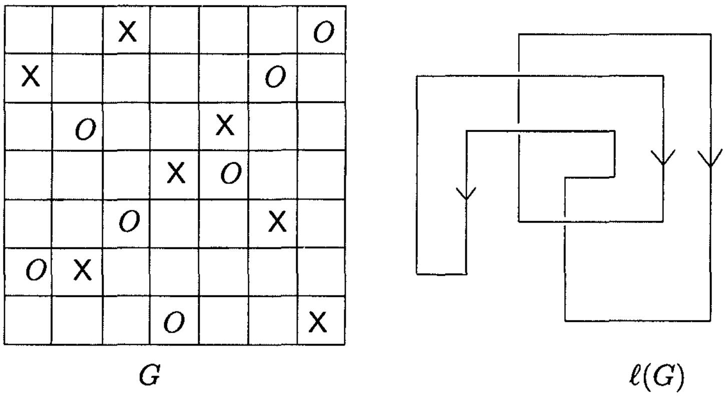


Figure 5.3: obtaining $\ell(G)$ from G

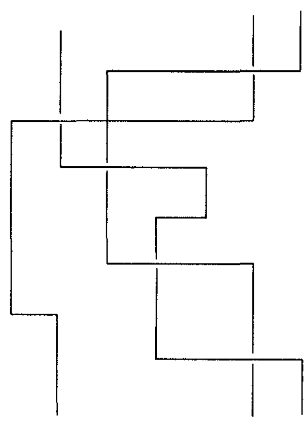


Figure 5.4: turning G into a braid

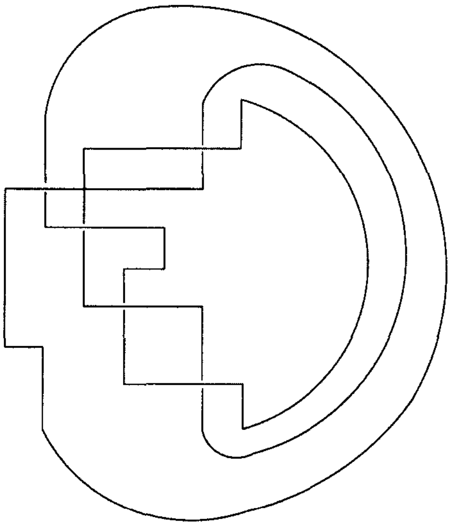


Figure 5.5: obtaining the closure

6. Special Grid Diagrams

In this chapter, we introduce the idea of special grid diagrams, as part of our contribution to existing literature. We will give an explicit method on how these grid diagrams can be constructed from given braid words. These special grid diagrams are grid diagrams associated to given braid words. We will simply use the abbreviation SGD henceforth.

6.1 Special Grid Diagrams

Given a braid word, we can find a special grid diagram associated to the braid represented by the given braid word. This will give us a function $g : \mathcal{W} \to G$, where $g(W)$ is the special grid diagram associated to the braid word, W. From Definition 5.1.2, we recall $w : \mathcal{G} \to \mathcal{W}$, where $w(G)$ is the braid word read from a braid represented by a grid diagram, G. With this, a composition of the above mentioned functions will yield the following:

$$w(g(W)) = W$$

Remark 6.1.1. Henceforth, we will refer to the Special Grid diagrams as SGDs.

6.2 Sub-Grid Representations of A Special Grid Diagram

We will describe a step by step algorithm to enable us to obtain a special grid diagram associated to a given braid word. We will begin by describing the function g, which requires pre-defining a collection of building blocks - sub-grid diagrams - which make up a special grid diagram.

Remark 6.2.1. Throughout, n is the number of strands in the braid, and not the size of the full grid.

Definition 6.2.2. *We define the following:*

- I^X is an $n \times n$ grid with X's on the diagonal; I^O is an $n \times n$ grid with O's on the diagonal.

$$I^X = \underbrace{\begin{bmatrix} X & & & \\ & X & & \\ & & \ddots & \\ & & & X \end{bmatrix}}_{n} \Bigg\} n \qquad I^O = \underbrace{\begin{bmatrix} O & & & \\ & O & & \\ & & \ddots & \\ & & & O \end{bmatrix}}_{n} \Bigg\} n$$

Figure 6.1: Sub-grids, I^X and I^O

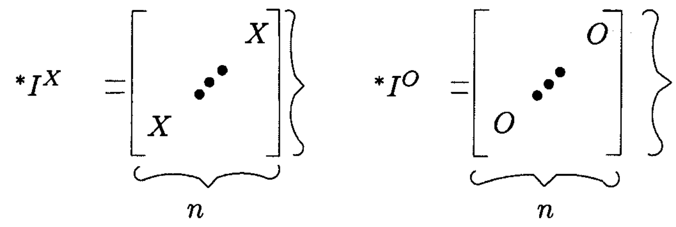


Figure 6.2: Sub-grids, $*I^X$ and $*I^O$

- $*I^X$ is an $n \times n$ grid with X's on the anti-diagonal; $*I^O$ is an $n \times n$ grid with O's on the anti-diagonal.
- J_k^X is a grid with $n+1$ columns and n rows, obtained from I^X by inserting an empty column after the k-th column; J_k^O is a grid $n+1$ columns and n rows, obtained from I^O by inserting an empty column after the k-th column.

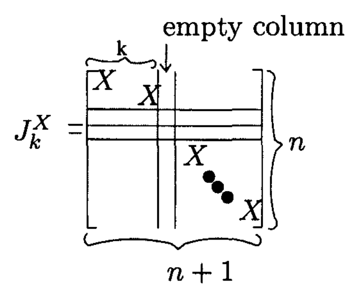


Figure 6.3: Sub-grid, J_k^X

- We construct sub-grids, C_k^{+1} and C_k^{-1}, corresponding to the positive and negative crossings of the generators of the braid word as in Fig 6.4.

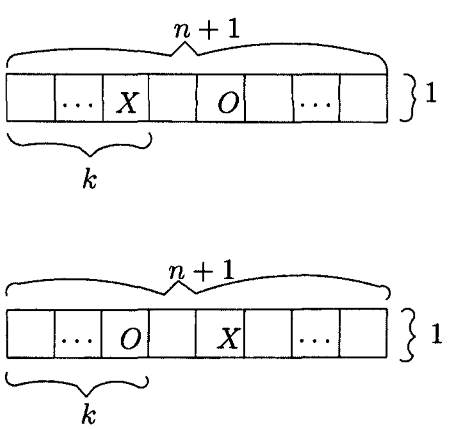


Figure 6.4: C_k^{+1} and C_k^{-1}

C_k^{+1} (resp. C_k^{-1}) is a single row of $n+1$ squares, with an X (resp. an O) in the k-th (resp. (k+2)-th) square, and an O (resp. an X) in the (k+2)-th square.

Suppose we are given a grid diagram G, we want to apply grid moves so that G looks like the outcome of the function $g : \mathcal{W} \to \mathcal{G}$ as described above, without changing any crossings. Rather than working with just the X's and O's, we work with the actual strands, since this makes it easier to see what is happening.

Definition 6.2.3. *A* **JCJ** *configuration in a grid diagram is a configuration of three sub-grids stacked vertically in one of the two cases on the right of Fig 6.5, where the version involving C_i^{+1} corresponds to the braid letter σ_i^{+1}*

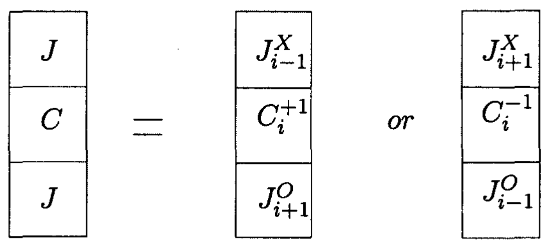


Figure 6.5: JCJ configurations

and the version involving C_i^{-1} corresponds to the braid letter σ_i^{+1}.

Remark 6.2.4. In future diagrams, we will indicate JCJ configurations as on the left of Fig 6.5 i.e. without including subscripts and superscripts.

Definition 6.2.5. *We define $g : W \to G$ as follows:*

Given a braid word $W = (n, \sigma_{i_1}^{\pm 1}, \sigma_{i_2}^{\pm 1}, \ldots, \sigma_{i_l}^{\pm 1})$ then $g(W)$ is the grid diagram drawn below:

where each I^* is $n \times n$, and J is $(n+1) \times n$, where the n is the number of strands in W, and there are no X's or O's in any unlabelled squares. Each JCJ configuration is labelled with subscripts and superscripts as in Fig 6.5, with the bottommost (which is also leftmost) JCJ corresponding to the first letter in W.

Definition 6.2.6. *A special grid diagram is a grid diagram as in Fig 6.6, i.e. a grid diagram which is an outcome of the function $g : \mathcal{W} \to \mathcal{G}$.*

Example 6.2.7. The following example illustrates that $w(g(W)) = W$ i.e. this really does produce a braid with the given braid word. Consider the braid word, $W = \sigma_2 \sigma_4^{-1} \sigma_1$ with 5 strands. In this example, we demonstrate how to construct a special grid diagram from this braid word. We refer the reader to the diagrams of Fig 6.7 and Fig 6.8.

In Fig 6.7 , we have drawn it using the sub-grid notation, while in Fig 6.8 we have actually drawn the strands.

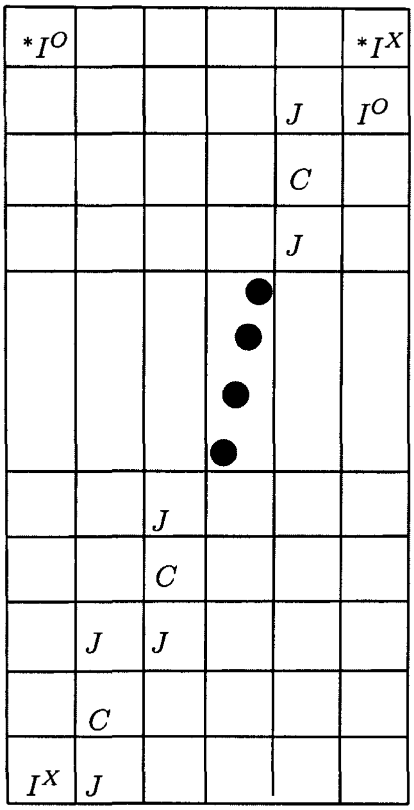


Figure 6.6: $g(W)$

6.3 Turning Grid Diagrams into Special Grid Diagrams

Given a grid diagram, we want to show that $g(W(G)) \sim G$, which is the same as showing that G can be changed into the special kind of grid diagram produced by g, using grid moves which do not change the associated braid word.

Algorithm

Now we present an algorithm to take any grid diagram and using grid moves, transform it into a special grid diagram without changing crossings and without changing the vertical ordering of the crossings. This shows that $g(w(G)) \sim G$.

We put our algorithm into 3 main steps and a number of sub-steps. It is worth noting that the construction of the special grid diagram begins from the bottom.

Step 1: Our main goal for this step is to get I^X at the bottom left, $*I^O$ at the top left, $*I^X$ at the top right and I^O below $*I^X$. We describe the step by step procedures involved in the following

$*I^O$				$*I^X$
			J_0^X	I^O
			C_1^+	
		J_5^X	J_2^O	
		C_4^-		
	J_1^X	J_3^O		
	C_2^+			
I^X	J_3^O			

Figure 6.7: example of $g(\sigma_2\sigma_4^{-1}\sigma_1)$

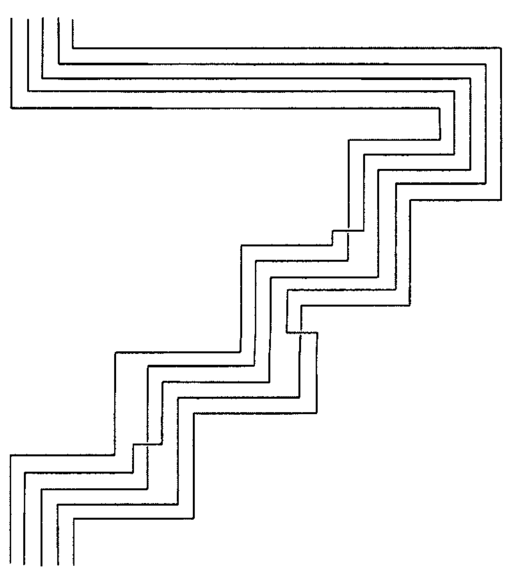


Figure 6.8: example of $g(\sigma_2\sigma_4^{-1}\sigma_1)$

sub-steps:

i) We draw a planar link projection of the braid obtained by connecting the Xs and Os using the convention stated in Section 5.1.

ii) We ensure every strand has at least one corner in it below its bottom-most c as shown in Fig 6.9 (i.e. a strand forms a right angle with itself), and at least one corner above its top-most crossing. In a situation where a strand does not have a corner above the top and/or below the bottom crossing, we perform a stabilisation move at the bottom-most corner (resp. top-most) the bottom crossing of the strand. We then perform a series of horizontal commutation moves in order to move this corner below (resp. above) the bottom (resp. top) crossing.

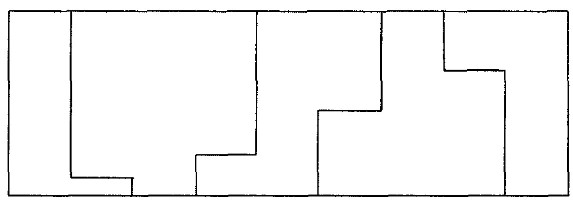


Figure 6.9: the bottom-most corners

iii) At this point the reader can check that a series of stabilisation and commutation moves at these bottom corners will result in a diagram similar to that of Fig 6.10:

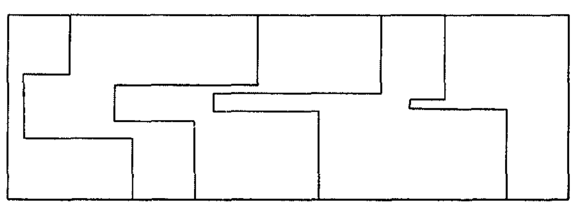


Figure 6.10: after stabilisation and commutations at the bottom-most corners

iv) Again, the reader can check that performing further commutations on the diagrams of Fig 6.10 will yield a diagram with nesting as illustrated in Fig 6.11. The nested configuration in the dashed box should be farther left than anything else in the grid diagram.

v) We perform a further series of stabilisation and vertical commutations on the bottom corners of Fig 6.11. The purpose of these commutations is to take the corners to the far right of the diagram, repeating Steps iii) and iv) but with a mirror reflection to get a configuration as in Fig 6.12.

vi) We perform a cyclic permutation, moving the rows below the portion indicated by 'p' on the diagram of Fig 6.12 to the top of the diagram and the resulting diagram is illustrated in Fig 6.13.

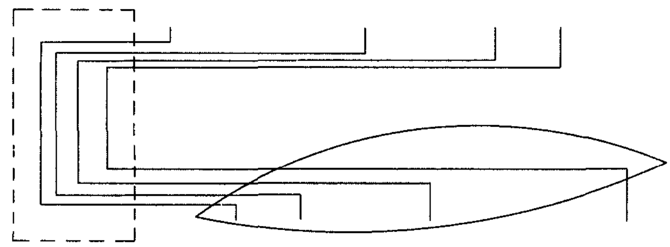


Figure 6.11: possible outcome

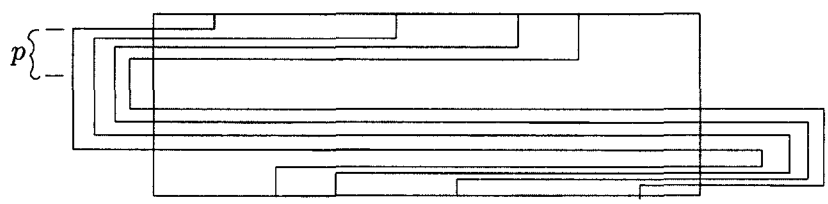


Figure 6.12: possible outcome

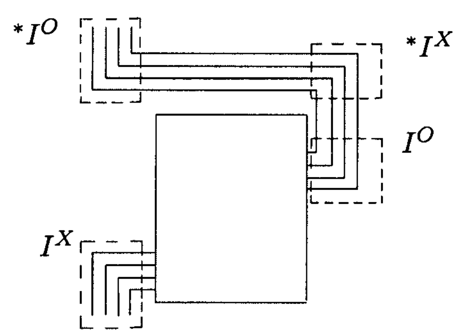


Figure 6.13: possible outcome of performing a cyclic permutation

It is clear from the diagram of Fig 6.13 that we obtain I^X at the bottom left of the SGD, $*I^O$ at the top left, $*I^X$ at the top right and then I^O below $*I^X$. At this stage we have achieved our main target for Step 1, we then proceed to Step 2, where our focus will be on the portion of Fig 6.13 inside the box, which is the rest of the grid diagram.

Step 2: Our goal for this step is to get our grid diagram to look like the diagram of Fig 6.14, where the squares labelled '?' are portions of the diagram where there are no crossings and the empty squares have no strands at all. We denote a collection of three consecutive squares labelled '?' by **'no-crossing section'**.

i) We will proceed one crossing at a time. Suppose that our configuration is as in Fig 6.15:

where no crossings in ?'s. We will show how to change to a new configuration with one more JCJ and one fewer crossing in the uncontrolled section. This is the section whereby we have no control

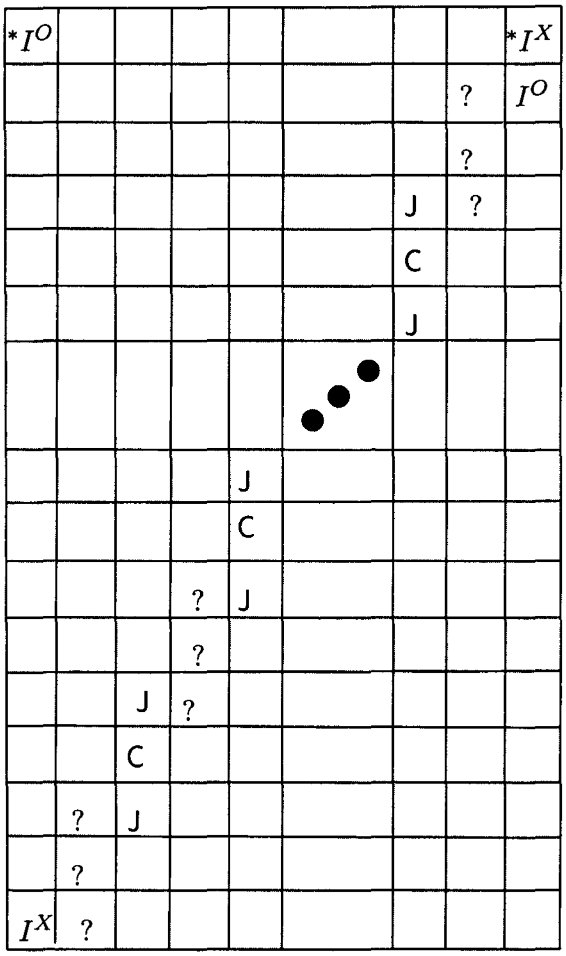


Figure 6.14: ...on the way to a SGD

over the resulting crossings added or removed from the diagram. Once there are no crossings in the uncontrolled section, then we are finished with this step.

We magnify the 'uncontrolled section' section and suppose it looks like the diagram of Fig 6.16.

ii) In the uncontrolled section, we perform a stabilisation at the closest corners above and below the lowest crossing row, and then perform a series of vertical commutation moves to put this crossing and all strands crossing this row as far to the left as possible in the uncontrolled section. The strands involved in this row will then behave as in Fig 6.17.

iii) Perform further stabilisations and commutations at the rear corners just below the crossing row as illustrated in Fig 6.18.

Note that the section in the dashed box in the last stage of Fig 6.18 is a JCJ configuration.

iv) Our uncontrolled section now looks like the diagram of Fig 6.19. The reader should note that

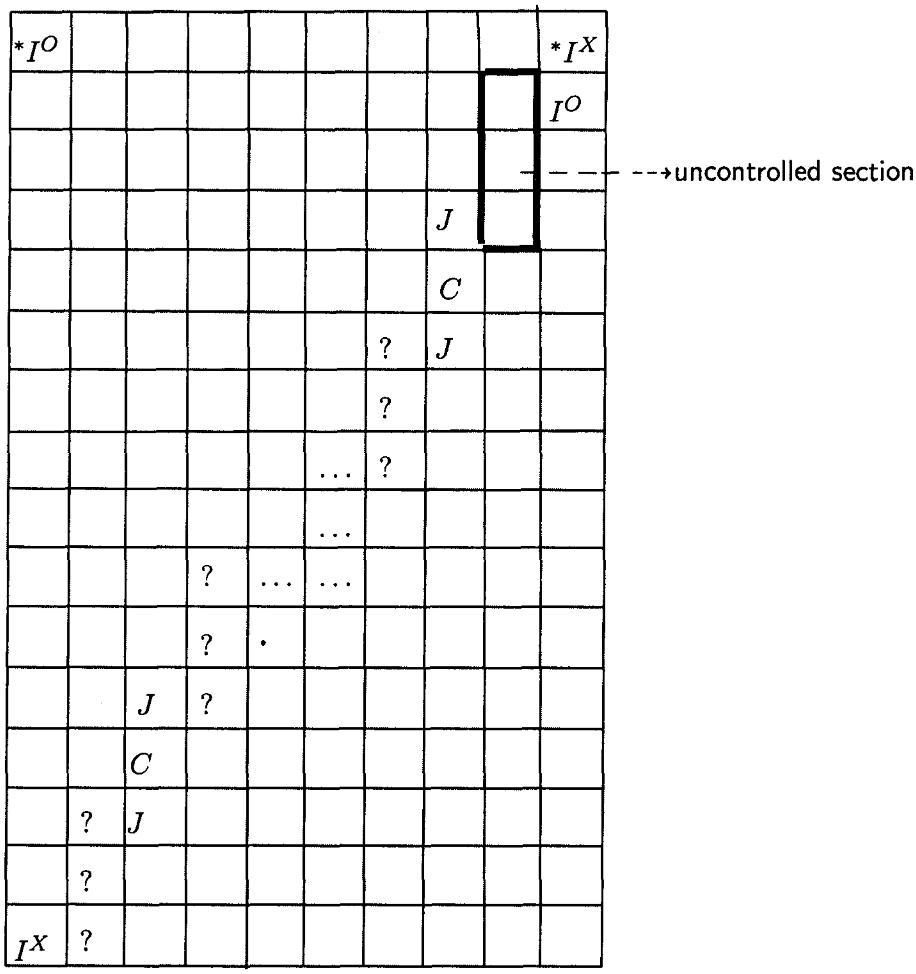


Figure 6.15: uncontrolled section

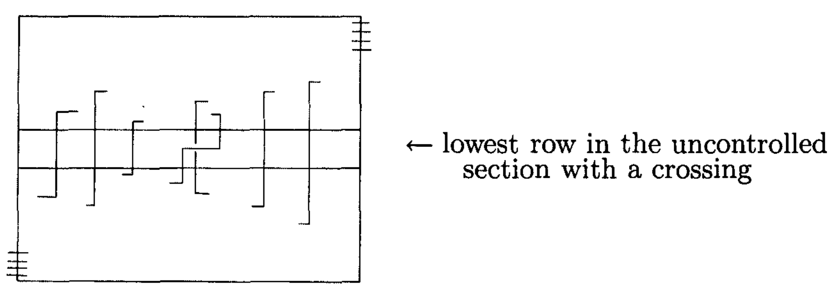


Figure 6.16: a magnified version of the no control section

we omitted the content of this section - the uncontrolled section, due to the simple fact that the resulting diagram will not end up the same for all braids, taking into consideration the braid index of the braid, and the number of unknown crossings, u, which would have been added or removed in the process. The key point is that, the smaller uncontrolled section has one fewer crossing.

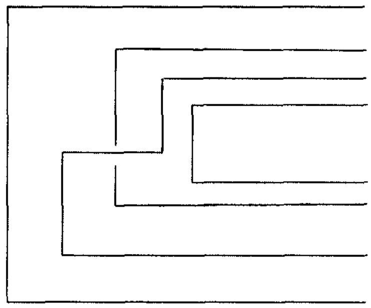


Figure 6.17: result of performing some grid moves on the no control section

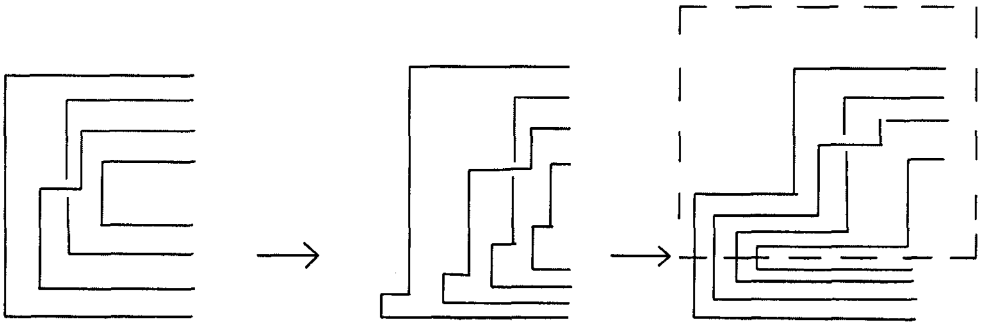


Figure 6.18: result of performing some series of commutation moves

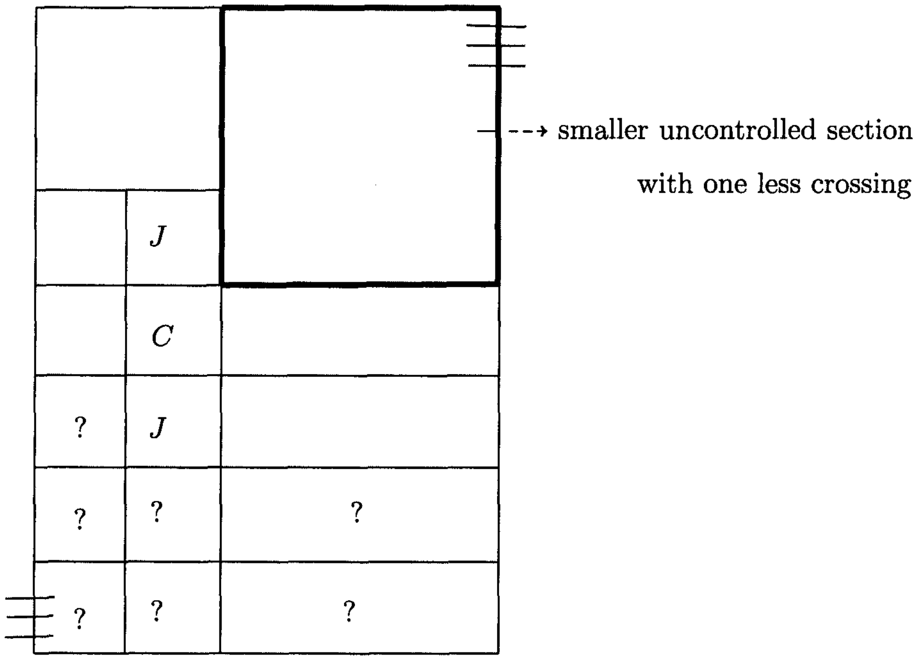


Figure 6.19: resulting 'uncontrolled section'

Since there are not crossings in this section, all vertical strands here can be commuted to the left of the JCJ configuration, to get the configuration in Fig 6.20. This completes Step 2.

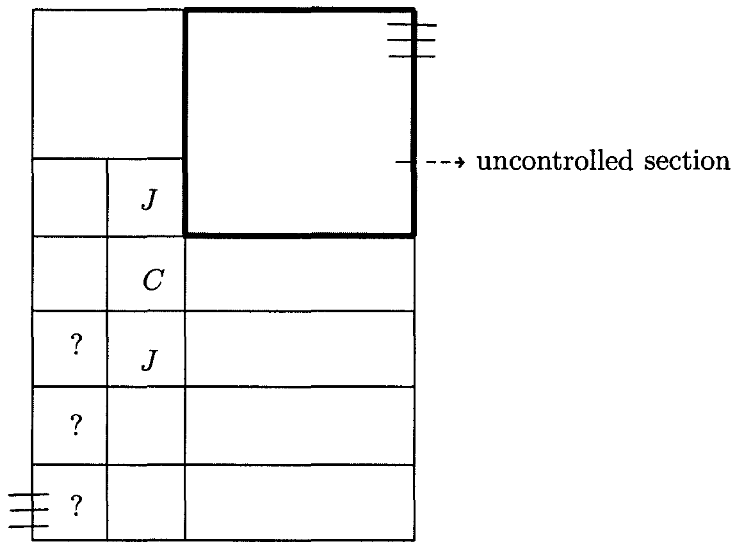


Figure 6.20: results of performing vertical commutation on all the vertical strands to cause all sections labelled '?' to move to the left of the JCJ

Step 3: The goal of this step is to remove all the columns labelled '?', i.e. the no-crossing sections of Fig 6.14. In this diagram, we demonstrate a typical example of what we might see between two consecutive JCJ configurations. It should be clear that a sequence of commutation and destabilisation moves can be used to eliminate all the vertical strands, thus placing the JCJ immediately next to each other. We illustrate this in Example 6.3.1.

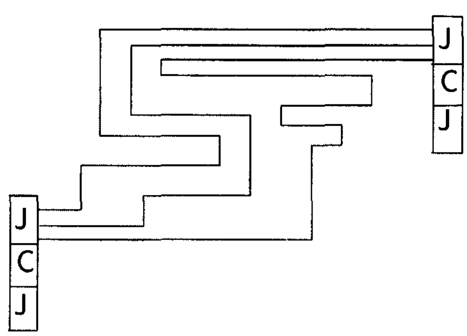


Figure 6.21: a typical no-crossing section

Example 6.3.1. In this example, we illustrate the resulting diagrams after one more set of destabilisations and commutations moves are performed in the diagrams of Fig 6.22.

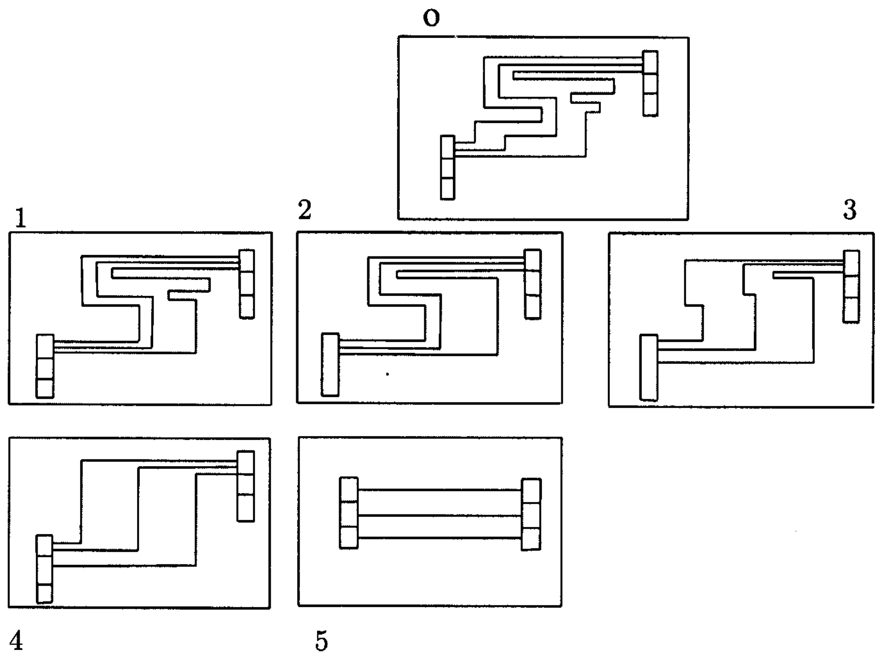


Figure 6.22: 0. original diagram; 1.results obtained from performing destabilisation moves on the original diagram; 2.results obtained from performing destabilisation moves on 1; 3. results obtained from performing commutation moves on 2; 4. results obtained from performing destabilisation moves on 3; 5. resulting diagram after one more set of destabilisations and commutations

This completes the proof of:

Lemma 6.3.2. *For any grid diagram, $G \in \mathcal{G}$, $g(w(G)) \sim G$.*

6.4 Special Grid Diagrams and Markov Moves

Lemma 6.4.1. *If W and $W^{'}$ are related by one Markov move, then $g(W) \sim g(W^{'})$*

Proof. It suffices for each Markov move to construct some grid diagrams G and $G^{'}$ with $w(G) = W$, $w(G^{'}) = W^{'}$, and show that $G \sim G^{'}$. Then by Lemma 6.3.2 we know that $G \sim g(W)$ and $G^{'} \sim g(W^{'})$.

1. Given two braid words, $W = W_0\sigma_i\sigma_i^{-1}W_1$ and $W^{'} = W_0W_1$, we construct G and $G^{'}$ as follows:

where '*' is the following diagram.

Upon performing a commutation and two destabilisation moves, the resulting diagram is that on the far right of Fig 6.25.

And we can see from Fig 6.25 that $w(G^{'}) = W^{'}$. An obvious variant converts $W_0\sigma_i^{-1}\sigma_iW_1$ into W_0W_1.

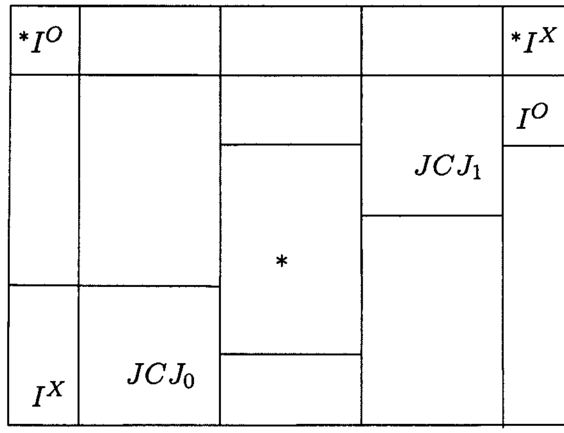


Figure 6.23: diagram of G: G^* is the same as G but with * replaced by *'

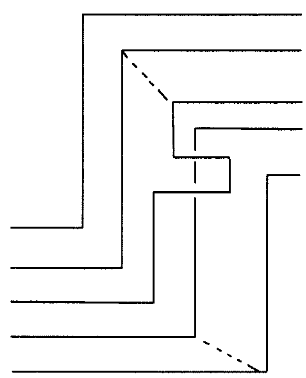


Figure 6.24: diagram of *

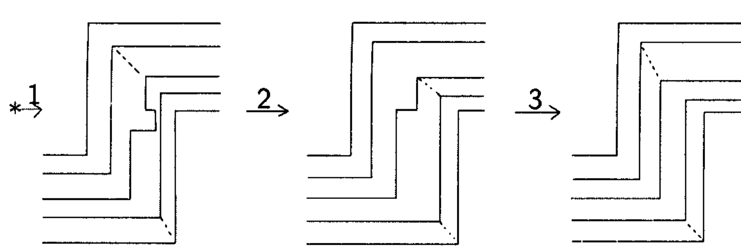


Figure 6.25: results of performing commutation and destabilisation moves on Fig 6.24; * represents the diagram of Fig 6.24; 1 represents a commutation move; 2 a destabilisation move; 3 a destabilisation move

2. Given two braid words, $W = W_0\sigma_i\sigma_j\sigma_i W_1$ and $W' = W_0\sigma_j\sigma_i\sigma_j W_1$ with $|i-j| \geq 1$, with an SGD as illustrated in the diagram of Fig 6.23, where * is the diagram of Fig 6.26, we construct G and G' as follows:

Upon performing two commutation moves on * as illustrated in Fig 6.27 below, we obtain the diagram to the far right.

Remark 6.4.2. Suppose we are given a grid diagram G, we want to apply grid moves so that G looks like the outcome of the function $g : \mathcal{W} \longrightarrow \mathcal{G}$ as described above, without changing any crossings. Rather than working with just the X's and O's, we work with the actual strands, since this makes it easier to see what is happening.

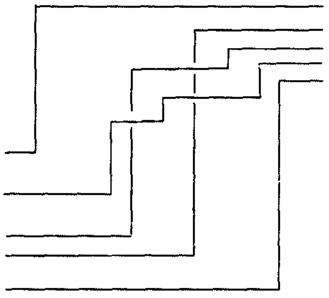


Figure 6.26: diagram of *

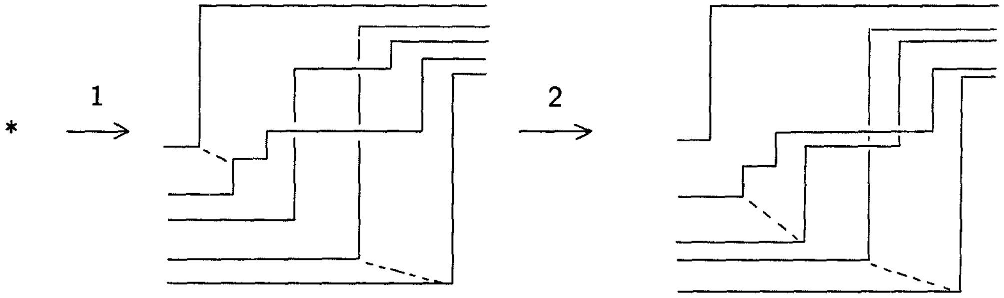


Figure 6.27: performing two commutations on diagram of *.

3. Given two braid words, $W = W_0\sigma_i\sigma_jW_1$ and $W' = W_0\sigma_j\sigma_iW_1$ with $|i - j| \geq 1$, with an SGD as illustrated in the diagram on the left of Fig 6.23, where * is the diagram of Fig 6.28, we construct G and G' as follows:

Upon performing a commutation move on *, it is immediately obvious that the resulting diagram is that on the right of Fig 6.28.

4. Given two braid words W and W', where $W' = W\sigma_n$, meaning W' has one more strand than W, begin with the grid diagram $G = g(W)$. Now perform stabilisations as in Fig 6.29, where the configuration shown is the upper right $*I^X$ and I^O of Fig 6.6. Note that the 3^{rd} diagram uses the convention for turning a grid diagram into a link not a braid, but the 4^{th} diagram uses the braid convention. We include the link diagram just to make the stabilisation clearer.

In the case where $W' = W\sigma_n^{-1}$, proceed as in Fig 6.30.

Given two braid words W and W', where $W' = \sigma_iW\sigma_i^{-1}$. Let $G = g(W)$. We therefore have the following generalised diagram as shown in Fig 6.31

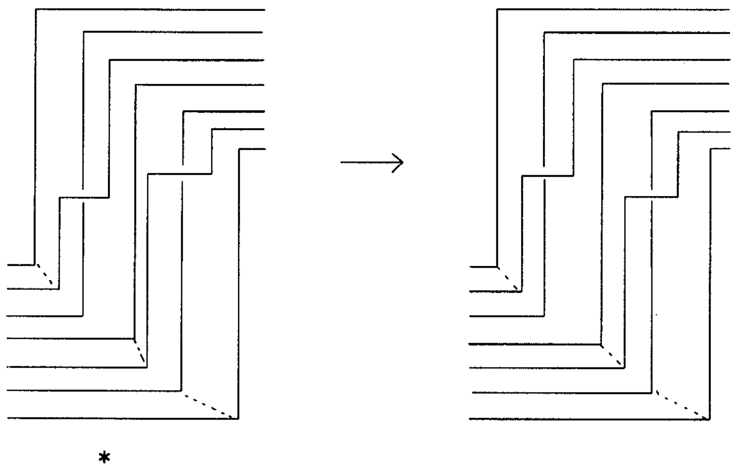


Figure 6.28: diagram of * and the resulting diagram from performing a commutation move on *

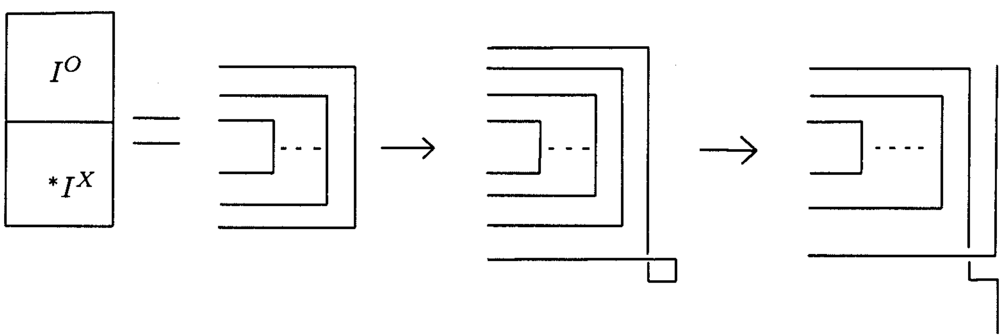


Figure 6.29: postitive stabilisation

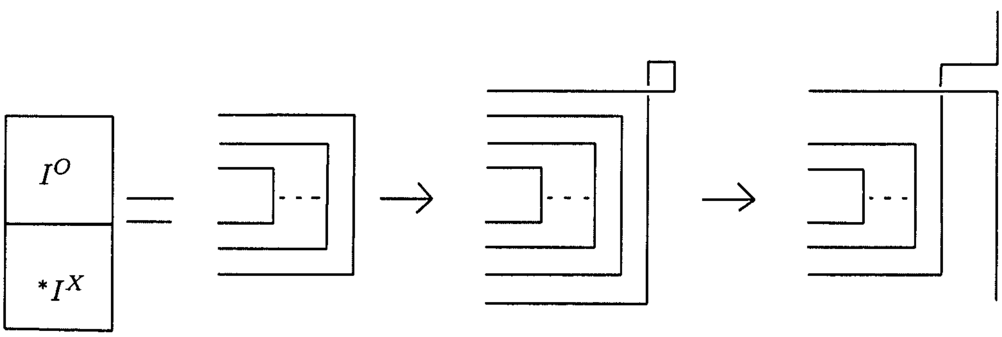


Figure 6.30: negative stabilisation

Now, focussing on the bottom-left corner of Fig 6.31, we have the following diagrams as illustrated in Fig 6.32 upon performing a serives of moves:

Finally, we performa a cyclic permutation on the bottom left portion within the oval, indicated on the last diagram of Fig 6.32, by moving it round and upwards to the top-most left section. This results in

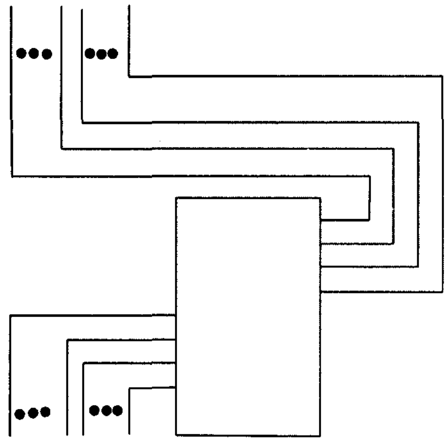


Figure 6.31: generalised diagram of G

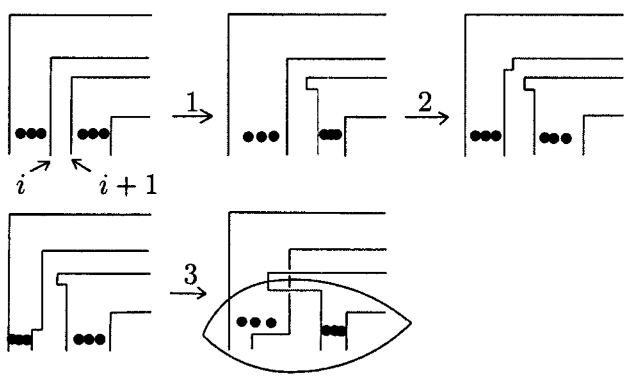


Figure 6.32: The arrows represent the sequence of moves performed on strands 'i' and 'i+1' starting from 1 to 4. 1:stabilisation move on strand i+1; 2:stabilisation move on strand i; 3:commutation move on strand i

W'. We illustrate the resulting diagram in Fig 6.33 below.

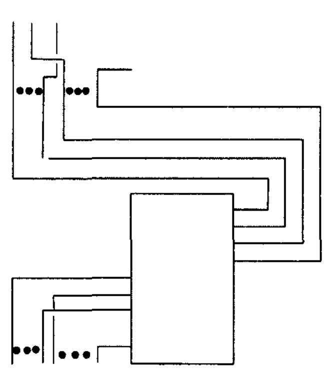


Figure 6.33: W'

□

Corollary 6.4.3. *If $W \sim W'$, then by $g(W) \sim g(W')$*

Proof. If $W \sim W'$, then there exists a sequence $W = W_0, W_1, \ldots, W_k = W'$ where each W_{i+1} is obtained from W_i by a single Markov move.

By Lemma 6.4.1, $g(W) = g(W_0) \sim g(W_1) \sim \ldots \sim g(W_k) = g(W')$ □

7. Proofs of Theorems on Prism

The goal of this Chapter is to prove our Main Theorem, which forms the heart of the paper.

Lemma 7.0.4. *The bottom triangle in the prism commutes exactly.*

Proof. We already have shown that $c(w(G)) \sim \ell(G)$ (Refer to Lemma 5.2.1) so passing to quotients we immediately have $\overline{c} \circ \overline{w} = \overline{\ell}$. □

Lemma 7.0.5. *The function $\overline{w} : \mathcal{G}/\!\sim \longrightarrow \mathcal{W}/\!\sim$ defined by*

$$\overline{w}([G]) = [w(G)]$$

is well-defined and is a bijection.

Proof. To prove this is well-defined, let us suppose $[G] = [G^{'}]$, i.e. $G \sim G^{'}$. We know that $c(w(G)) \sim \ell(G) \sim \ell(G^{'}) \sim c(w(G^{'}))$ (Refer to Lemma 5.3) Therefore by Markov's theorem, $w(G) \sim w(G^{'})$.

To prove injectivity, we suppose $\overline{w}([G]) = \overline{w}([G^{'}])$. This means

$$[w(G)] = [w(G^{'})]$$

i.e. $w(G) \sim w(G^{'})$. Therefore by Corollary 6.4.3 $g(w(G)) \sim g(w(G^{'}))$.

But by Lemma 6.3.2, $G \sim g(w(G))$, and $G^{'} \sim g(w(G^{'}))$. Therefore, $G \sim G^{'}$ and $[G] = [G^{'}]$.

To prove for surjectivity, given $[W] \in \mathcal{W}/\!\sim$, we let $G = g(W)$. Note that

$\overline{w}([G]) = [w(G)] = [w(g(W))] = [W]$

□

7.1 Proof of Main Theorem

Proof. We know $\overline{\ell}$ is well-defined and surjective (Lemma 3.3.6 and Lemma 3.3.8). We have just seen in Lemma 7.0.5 that $\overline{w}$ is well-defined and bijective and in Lemma 7.0.4 that $\overline{c} \circ \overline{w} = \ell$

Markov's theorem (Corollary 4.4.6) says that $\overline{c}$ is a bijection, so $\overline{\ell}$ is a composition of two bijections and is therefore bijective. □

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