BEYOND THE STANDARD MODEL OF COSMOLOGY: A PERTURBATIVE APPROACH

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Declaration

The work presented in this thesis is partly based on collaboration with Cyril Pitrou and J.P. Uzan (Institute of Advance Astrophysics; Paris), Kishore Ananda, C.A. Clarkson and P.K.S. Dunsby (Gravitation and Cosmology group, Mathematics and Applied Mathematics, University of Cape Town). The list below identifies sections or paragraphs which are partially based on the listed publication or article in preparation.

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   Large scale structures and evolution of density perturbations
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2. Chapter 4:
   Pure gravitational waves in Bianchi I model filled with dust
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3. Chapter 5:
   Second order gauge invariant perturbation theory
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   Locally extracting tensor modes in cosmological perturbations
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I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other university for a degree and that it represents my own work.
Bob Otieno Osano

Abstract

This thesis concerns higher order perturbations of the standard model of cosmology. The theme is addressed in two distinct research areas. The first area deals with linear perturbations of Bianchi type I model filled with dust whose flow is irrotational, and which is an analogue to second order perturbations about the standard model. We investigate both density perturbations and gravitational waves in the shear dominated and the matter dominated regimes. We find that whereas the analysis of perturbations in the matter dominated regime recovers the standard FLRW results, the analysis of perturbations in the shear dominated regime reveals that density perturbations and gravitational waves decouple only when the background shear is *locally rotational symmetric.*

The second part of this thesis is about the second-order perturbations of FLRW model. We discus a novel way of defining second-order gauge invariant variables. At second order, the various perturbations modes about FLRW present us with new challenges, ranging from gauge-problems to mode coupling. We develop a covariant method for extracting the various modes at second order. We apply the techniques developed to analyse gravitational waves generated by second-order effects during inflation. Although we find the gravitational wave spectrum to be negligible, we have identified a new contribution to the Non-Gaussianity parameter; $f_{NL}^{\mathcal{REE}}$ parameter, which needs to be taken into account when testing the consistency relation.

Dedicated to

Uel, Rani and Kendi

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Cape Town, South Africa — Osano Otieno Bob
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Notations, conventions and assumptions

The following assumptions, conventions and notations are used throughout the thesis, unless otherwise stated. We adopt the units $c = k_b = \hbar = 1 = 8\pi G$.

- t- cosmic time,
- $\cdot \equiv u^a \nabla_a$- derivative along an observer worldline,
- τ- dimensionless time,
- η- conformal time,
- $' \equiv \frac{\partial}{\partial \eta}$- derivative with respect to conformal time,
- Latin indices $(a, b, c, d\ )$ run from 0 to 3,
- Latin indices (i, j, k) run from 1 to 3,
- ∇_a- covariant derivative with respect to g_{ab} { with signature (- + + +)},
- $\tilde{\nabla}_a$- covariant derivative with respect to h_{ab},
- $_{|i} \equiv \nabla_i$ Covariant differentiation with respect to γ_{ij},
- $_{,i} \equiv \partial_i$ Partial differentiation.

Chapter 1

Introduction

This chapter deals with the current status of modern cosmology. It addresses two issues that seem to contradict; on one hand, the evidence in support of the standard model and on the other hand, observations that can not be accounted for by this model. The latter set of issues provides the motivation for investigations beyond the standard model, which is the subject of this thesis. The chapter ends with a mathematical description of the standard model of cosmology.

1.1 The best-fit Model

The current best-fit model of the Universe is the ΛCDM model. This model depicts a Universe that is spatially flat, and which is homogeneous and isotropic on large scales. The constituents of this model include ordinary matter, radiation, a cosmological constant Λ and cold dark matter (CDM). The primordial fluctuations in this model are adiabatic, nearly scale-invariant, Gaussian and random [49]. Six cosmological parameters are used to characterise the properties of this model; the density of matter ($\Omega_m h^2$), the density of baryons ($\Omega_b h^2$), the Hubble constant H_0, the amplitude of the primordial fluctuations

σ_8, the slope for scalar perturbation spectrum η_s, and the optical depth $(\tau)^1$, where h is the dimensionless Hubble parameter. These parameters are robust enough to predict the statistical properties of the cosmic microwave background (CMB) sky and the large scale distribution of matter and galaxies [48].

The primary goal of most current and future surveys is to determine these parameters, and other observational constraints, and thus place tighter bounds on this model of the Universe. Whereas the *power law* ΛCDM models fits Wilkinson Microwave Anisotropy Probe (WMAP) first year data, a stronger constraints on the cosmological parameters is obtained by combining WMAP data with data from surveys like the Sloan Digital Sky Survey (SDSS), the 2dF Galaxy RedShift Survey (2dfGRS) and a range of other experiments [46]. From WMAP third year data, $\Omega_m h^2 = 0.127^{+0.007}_{-0.013}$, $\Omega_b h^2 = 0.0223^{+0.007}_{-0.0009}$, $h = 0.73^{+0.03}_{-0.03}$, $n_s = 0.951^{+0.015}_{-0.019}$, $\tau = 0.09^{+0.03}_{-0.03}$ and $\sigma_8 = 0.74^{+0.05}_{-0.06}$ [48]. With the basic parameters fixed at the last scattering surface ($z = 1100$), the ΛCDM model is therefore able to predict both the galaxy and the gravitational lensing power spectra; it is also able to constrain the range of the Hubble constant, and the luminosity distance relationship. In addition, information about the conditions and the ionisation history of the early universe may come from the analysis of the polarization of the CMB temperature anisotropy. On large angular scales, the polarization can in principle probe the Universe at its infancy i.e., when it was about 10^{-35}*seconds* old.

1.1.1 Gravitational waves

Gravitational waves are ripples in space-time and are a generic feature of General Relativity and therefore an expanding Universe, irrespective of the mechanism through which they are produced. The existence of primordial gravitational waves was first suggested in [149] by Starobinsky in 1979.

[1] We use τ in Chapters 3 and 4 to denote dimensionless time, η without a subscript to denote conformal time and σ with letter subscripts to denote shear tensor

Experimentally, the goal is to determine the density spectrum of gravitational waves, denoted by Ω_{GW}, and which is proportional to a transfer function derived from the wave equation (2.33) (see Chapter 2). So far a number of terrestrial experiments such as LIGO [128], VIRGO [86], TAMA 300 [126], and GEO [127], have been set up to try and detect gravitational waves from a variety of sources. The most promising experiment for gravitational waves is the *Laser Interferometer Space Antenna* (LISA) [123] which is planned for launch in 2015. LISA technology is set to reach the sensitivity levels required for the detection of waves from most compact binary sources having periodic spacetime strains of $10^{-21}(Hz)^{1/2}$, where Hz is the gravitational frequency in Hertz. [148]. The detection of primordial gravitational waves is however much more challenging due to the sensitivity levels required. The planned Planck mission may have the sensitivity required to detect primordial gravitational waves (see Fig. 1.1).

The power spectrum of gravitational waves plays a crucial role, for example in cosmological studies, in that it provides a link between theory and experiment. The quantity usually determined is the ratio of the power spectrum of the tensor to that of the scalar; $r = \mathcal{P}_T/\mathcal{P}_\mathcal{R}$. Slow-roll inflation predicts $r = 16\epsilon$, where ϵ is the standard slow-roll parameter [100]. It follows that WMAP gives the upper the limit of these fundamental quantities, $r < 1.28$ (95%) which implies that $\epsilon < 0.08$, and which then provides an upper bound on the energy scale of inflation $< 3.8x10^{16}$ GeV [119]. Although the direct detection of gravitational waves remains a challenge, requiring improvement in technology, an indirect detection of gravitational wave background may come from the successful extraction of the B-Mode polarization in the CMB temperature anisotropies. This is of great interest to cosmology as a detection of polarization for $l < 100$ would lead to a better understanding of the physics of the early universe, at energy scales of $10^{15} - 10^{16}$ GeV [55]. A major challenge in extracting the B-mode polarization of CMB anisotropy is foreground contamination. The way around this is to allow the experiment to run for a longer period of time, thereby allowing for a more accurate estimate of the noise level [55]. The

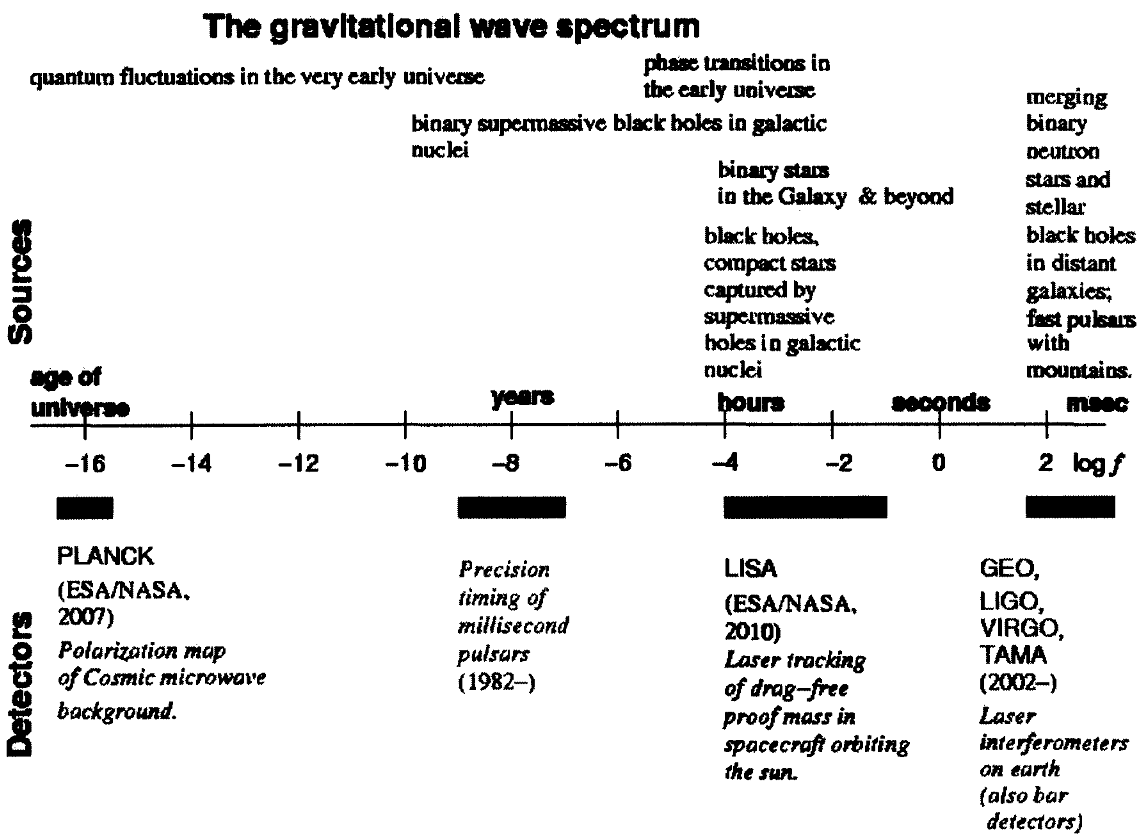

Figure 1.1: Frequency range for various experiments-courtesy [1]. The launch of Planck has since been moved to July 2008 [4]

situation should improve with the launch of Beyond Einstein Mission [3], whose primary responsibility will be to map out CMB polarization.

These experiments target stochastic gravitational wave background predicted in the inflationary paradigm, and which are in agreement with the isotropic and homogeneous models. But as we shall point out in the next section, there are several anomalies in the CMB that motivate the analysis of anisotropic models and non-linear perturbations of FLRW. In light of this, we shall consider pure gravitational waves in perturbed Bianchi I universe field with dust in Chapter 3.

1.1.2 Discrepancies

It may appear that the issue concerning a model that correctly describes the evolution of the Universe has been settled, given the review presented this far. However, continued assessments have revealed some discrepancies with our best-fit concordance model. In this model, anisotropies generated by primordial fluctuations in the inflationary field are imprinted in the CMB, which should then be statistically isotropic and Gaussian. Based on this, most statistical studies of the CMB seek to determine if there exists any violation of these properties. In some cases we may not be in a position to unequivocally determine such violation. For example, a primordial density field is theoretically a single realisation of some random process [90], and to test this idea, it is necessary to observe and average over a number of such realisations. The problem is that we only have one universe to observe and thus there will be a variance in our average. It clear that we can not do anything about this average which is referred to as the *cosmic variance.* The issues that turn to next are more disturbing.

First there was the initial detection of a curiously low quadrupole amplitude by Cosmic Background Explorer (COBE [47]), which was discussed in detail by Efstathiou in [36]. This was followed by claims that there exists a strange planarity and alignment of the quadrupole ($l = 2$) and the octopole ($l = 3$), with the possibility that this alignment extends to higher multipoles [45, 96, 130, 52, 53, 51, 50, 30, 59]. There is also a report of the detection of non-Gaussianity, seen as a very cold spot on the sky of angular scale 10 degrees. The alignment that extends between the first four multipoles ($l = 2 - 5$) has become as known the *Axis of evil* [96]; which if proved to be an intrinsic feature of the anisotropies, may break the magical spell of the concordance model. In principle, such an alignment may point to the existence of a feature in the CMB that picks a preferred direction or a form of statistical anisotropy. Although this could arise from foreground contamination, results from the *difference* maps constructed from the WMAP foreground corrected maps indicate that such contamination is unlikely to lead to the levels observed [62]. The fact that the alignment includes the quadrupole suggests that the anomaly may be an intrinsic feature of the cosmological model and therefore provides a motivation to investigate anisotropic cosmological models. For example, in [62], the authors have used what they call the best-fit Bianchi model (Bianchi type VII_h) to correct a combination of various WMAP sky maps. They find among other things, that the Bianchi corrected maps exhibit greater isotropy compared to the uncorrected maps. They also find that the general shape of the corrected spectrum is flatter than the WMAP best-fit power spectrum, which agrees with the theoretical fit made to the northern hemisphere data in [50]. This further strengthens the motivation to analyse anisotropic models.

1.1.3 Non-Gaussianity

Primordial cosmological density fluctuations are usually studied within linear perturbation theory since they are small; comparable to CMB temperature anisotropies ($\delta T/T \sim 10^{-5}$). In standard slow-roll inflation, density perturbations are Gaussian and are produced when the inflaton field φ slowly rolls down its potential $V(\varphi)$, thereby giving rise to the fluctuations in ϕ and the metric (see Kolb and Turner [95], page 272). At the end of inflation, the inflaton oscillates about the minimum of its potential and decays, which leads to the reheating of the Universe. It is then thought that the final temperature anisotropies are the result of inflation lasting for different amounts of time in the different regions of the Universe, leading to adiabatic perturbations. The single field scenario presented above is just one of the several competing inflationary scenarios. Density fluctuations could also be produced through other mechanisms, such as the inhomogeneous reheating of the Universe [139, 70, 69, 93, 121], the D-acceleration in the AdS/CFT correspondence [147] or even by ghost inflation [44], and therefore the debate over the origin of these fluctuations is far from over. The present WMAP data and data from future surveys will enable us to distinguish between the various competing inflationary paradigms.

The first and third year WMAP data sets [48] highly favour the slow-roll inflationary paradigm. It must pointed out, however, that claims such as the very cold spot in the WMAP sky data [59] merit investigating non-Gaussianity, as its presence may indicate that a mechanism other than the slow-roll inflation may have been responsible for these fluctuations. Other claims such as those of a preferred direction in the CMB temperature anisotropies do point to the possible existence of some Bianchi type phase. Nevertheless, these anomalies provide sufficient motivation to go beyond the standard model in a different way. Statistically, a pure Gaussian distribution is completely described by its two-point correlation function. Non-Gaussianity is usually parametrised by the quantity f_{NL} associated with Bardeen's gravitational potential $\Phi = \Phi_L + f_{NL}\Phi_L^2$ [119]. This equation indicates that in order to keep track of the level of non-Gausianity for the evolving Universe, one needs to perform a perturbation around the homogeneous background up to second order. Current theoretical estimates indicate that for WMAP data, the non Gaussianity should be detectable through a non-vanishing three-point function of scalar perturbations if $f_{NL} > 20$ [56]. Data from the planned Planck mission will lower this limit even further to $f_{NL} > 5$, however the critical level of $f_{NL} \sim 1$ necessary to detect the non-Gaussian contributions from second-order perturbations may still prove challenging. In Chapter 6, we shall examine a new contribution to the f_{NL} parameter given by second-order effects arising from density perturbations.

1.2 Basic concepts

Observations indicate that the universe is homogeneous and isotropic on large scales, but it is also clear that in regions nearby, the universe is highly inhomogeneous, with clumps of matter in the form of stars, galaxies and clusters of galaxies distributed across the visible universe.

In practice, the dynamics of the Universe is usually studied in two parts. The large scale behaviour of the Universe is described by a homogeneous and isotropic background, with the small-scale irregularities superimposed. For much of the evolution of the Universe, these irregularities can be considered as small perturbations of this background and are usually treated using linear perturbation theory. But as indicated in the previous section, there are a number of factors that motivate the development of *higher order perturbation theory*. This thesis is therefore an attempt to go beyond linear perturbation theory. We first consider the basic concepts upon which the rest of the thesis will be built.

1.2.1 Standard Model of Cosmology

Most well developed models of standard cosmology begin with two basic assumptions. The first assumption is that on large scales, the distribution of matter in the universe is homogeneous and isotropic (i.e. the cosmological principle) and the scales at which such smoothness is observed can be self-consistently determined by the model. It has been argued that if the expansion of the universe were anisotropic, we should be able to detect equivalent levels of anisotropies in the CMB [95].

The second assumption is that gravitational interactions determine the large scale structure of the universe and that such interactions are described by General Relativity. By assuming that matter in the universe is distributed uniformly on large scales, we can use General Relativity to compute the corresponding gravitational effects of such matter. Since gravity is a property of space-time in General Relativity, this would be equivalent to computing the dynamics of the space-time itself. The general consensus today is that the two assumptions mean the following:

1.2.2 Metric

The first assumption indicates that the large scale geometry can be described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric of the form,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\mathcal{K}r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1.1}$$

where $a(t)$ is the scale factor, while r, θ and ϕ are spherical polar co-ordinates. The constant $\mathcal{K}$ is the scalar curvature. We note that this constant takes three types of values with important implication on the geometry and the evolution of the model. When $\mathcal{K} < 0$, the curvature is said to be open and the model corresponds to a hyperbolic universe. When $\mathcal{K} = 0$, the curvature is said to be flat and the model corresponds to a universe with flat spatial sections. When $\mathcal{K} > 0$, the model corresponds to a closed universe.

A consequence of the homogeneity of the spatial hypersurfaces is that a unique time dimension is picked out. This effectively splits the 4-dimensional spacetime into three spatial and one temporal. The time coordinate will be denoted by t and is referred to as the proper time or equivalently the cosmic time. A more useful time parameter is the conformal time defined by

$$a(\eta)d\eta = dt. \tag{1.2}$$

This allows a scale factor of $a^2(\eta)$ to be taken out of the metric, leading to a line element of the form,

$$ds^2 = a^2(\eta)\left[-d\eta^2 + [\frac{dr^2}{1-\mathcal{K}r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)]\right]. \tag{1.3}$$

Conformal time is significant as it determines the particle horizon. In terms of comoving distance, this horizon is equal to the conformal time η_0 that has passed since the Big Bang times the speed of light c. In other words, the finite distance light signal can travel between the big bang and the present [131].

1.2.3 Einstein's Field Equations

The presence of matter introduces curvature in the spacetime geometry, this curvature in turn influences the motion of matter. The second assumption suggests that these relationships between matter and curvature may be given by Einstein's equations with energy momentum tensor acting as a source,

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab}, \tag{1.4}$$

where $G_{ab}, R_{ab}, R, g_{ab}$, and T_{ab} are the Einstein tensor, the Ricci tensor, the Ricci scalar, the metric tensor, the energy-momentum tensor respectively. The most general form of equation (1.4) includes the term ($-\Lambda g_{ab}$) on the rhs, where Λ is the cosmological constant. The energy momentum tensor T_{ab} represents the physical properties of the different types of matter fields.

The conservation of the energy-momentum tensor follows from equation (1.4) and the twice-contracted Bianchi identities, where

$$\nabla_b G^{ab} = 0 \Rightarrow \nabla_b T^{ab} = 0. \tag{1.5}$$

From this compact conservation equation, the separate conservation equations for the energy and the momentum are easily obtained. The energy-momentum tensor encodes the physics of the matter fields. In what follows, we examine the most commonly used form of this tensor, which turns out to be sufficient for the discussion of the standard model.

1.2.4 Energy momentum tensor

Whereas the lhs of equation (1.4) is determined by the metric tensor of spacetime given by the line element (1.3), the rhs requires matter to be of a perfect fluid form, due to the homogeneous and isotropy condition. In general, other types of sources may be important in the EFE and therefore need to be taken into account. The energy momentum tensor for a perfect fluid takes the form

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab}, \tag{1.6}$$

where u^b is the fluid 4-velocity, $\mu = T_{ab}u^a u^b$ is the energy density, $p = \frac{1}{3}T_{ab}h^{ab}$ is the isotropic pressure and g_{ab} is the metric. If we assume that p = 0, we have the simplest case: pressure-free matter (dust or Cold Dark Matter).

In general, one needs to specify the equation of state, the simplest being one that links p and μ. These not withstanding, It is usually required that one or all of the following energy conditions hold,

$$\mu > 0, \quad (\mu + p) > 0, \quad (\mu + 3p) > 0, \tag{1.7}$$

(the scalar fields in inflationary universe models, which we consider in section (1.2.6), however violate the latter condition). In addition it is usually required that the isentropic speed of sound

$$c_s^2 = \left(\frac{\partial p}{\partial \mu}\right)_{s=const}$$

obeys

$$0 \le c_s^2 \le 1 \Rightarrow 0 \le \left(\frac{\partial p}{\partial \mu}\right)_{s=const} \le 1, \tag{1.8}$$

which is required for local stability of matter (lower bound) and causality (upper bound), respectively. In these expressions and inequalities, s is a measure of the entropy.

1.2.5 Exact Solutions

Substituting (1.1) into the EFE (1.4) with perfect fluid, one finds the Friedmann and the Raychaudhuri equations,

$$\left(\frac{a'}{a}\right)^2 = \frac{\mu a^2}{3} - \mathcal{K} \tag{1.9}$$

and

$$\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = -\frac{1}{6}(\mu + 3p)a^2, \tag{1.10}$$

respectively. The immediate consequence of these two equations is the continuity equation or the energy conservation equation,

$$\mu' + 3\mathcal{H}(\mu + p) = 0, \tag{1.11}$$

where $\mathcal{H} \equiv aH$ ($H \equiv \dot{a}/a$ is the Hubble parameter). This equation can also be obtained from the energy-momentum conservation equation(1.5). In order to determine the cosmological evolution, it is easier to combine (1.9) and (1.5).

We shall assume that the equation of state (EOS) for the cosmological matter is of the form $p = w\mu$, where w is constant. This EOS includes the two types of matter that are important in cosmology, radiation given by $w = 1/3$ and dust given by $w = 0$. The integration of the energy conservation equation (1.11) gives

$$\mu \propto a^{-3(1+w)}. \tag{1.12}$$

Now substituting (1.12) into (1.9) and taking the model to be spatially flat ($\mathcal{K} = 0$), one finds that,

$$\left(\frac{a'}{a}\right)^2 = \frac{\mu_0 a^2}{3}\left(\frac{a}{a_0}\right)^{-3(1+w)}. \tag{1.13}$$

In this case, the subscript 0 stands for quantities evaluated today. It is clear that, $a'/a \propto a^{-(1+3w)/2}$ and $\mathcal{H} = 2/(3w+1)\eta$. The evolution of the scale factor in the universe dominated by non-relativistic matter; $a \propto \eta^2$, and in a universe dominated by radiation; $a \propto \eta$. We also note that the cosmological constant corresponds to an equation of state $w = -1$ (it can be shown that this corresponds to exponential growth in the proper time coordinate). We now turn to scalar fields which play an important role in the physics of the early universe.

1.2.6 Scalar fields

Let us consider a minimally coupled scalar field with Lagrangian density,

$$\mathcal{L}_\phi = -\sqrt{-g}\left[\tfrac{1}{2}\nabla_a\phi\nabla^a\phi + V(\phi)\right], \tag{1.14}$$

where $V(\phi)$ is a general (effective) potential expressing the self interaction of the scalar field. Using the conventions of [168], we note that the Hilbert-Einstein action in presence of matter is defined by,

$$\mathcal{A} = \int dx^4\sqrt{-g}\left[\tfrac{1}{16\pi G}R + \mathcal{L}_\phi\right]. \tag{1.15}$$

The equation of motion for the field ϕ following from $\mathcal{L}_\phi$ is the Klein - Gordon equation

$$\nabla_a\nabla^a\phi - V'(\phi) = 0, \tag{1.16}$$

where the prime[2] indicates a derivative with respect to ϕ. The energy - momentum tensor of ϕ takes the form

$$T_{ab} = \nabla_a\phi\nabla_b\phi - g_{ab}\left[\tfrac{1}{2}\nabla_c\phi\nabla^c\phi + V(\phi)\right]; \tag{1.17}$$

[2]Not to be confused with the use of prime elsewhere e.g. derivative with respect to conformal time (1.2).

provided $\phi_{,a} \neq 0$, equation (1.16) follows from the conservation equation

$$\nabla_b T^{ab} = 0. \tag{1.18}$$

We shall now assume that in an open region U of spacetime, the *momentum density* $\nabla^a\phi$ is *timelike*:

$$\nabla_a\phi\nabla^a\phi < 0. \tag{1.19}$$

This requirement implies two features: first, ϕ is not constant in U, and so $\{\phi = \mathit{const.}\}$ specifies well-defined surfaces in spacetime. When this is not true (i.e., ϕ is constant in U), then by (1.17),

$$\nabla_a\phi = 0 \quad \Leftrightarrow \quad T_{ab} = -g_{ab}V(\phi) \quad \Rightarrow \quad V = \text{const.}, \tag{1.20}$$

in U, [the last being necessarily true due to the conservation law (1.18))] and we have an effective cosmological constant in U rather than a dynamical scalar field. As we shall see in Chapter 6, this is a significant feature of slow-roll inflationary paradigm.

1.3 Chapter conclusion

In this chapter we reviewed the current status of cosmology, highlighting issues that the standard model fails to explain. The anomalies reviewed in this chapter provide the motivation for the work presented in the rest of this thesis.

Chapter 2

Cosmological Perturbation Theory

The main aim of developing cosmological perturbation theory is to examine the properties of primordial density fluctuations necessary to explain the observed large scale structures of the Universe, and to clarify the origin and the evolutionary behavior of such density fluctuations. The other aim, which is not any less important, is to examine the properties and evolution of gravitational waves. This chapter reviews linear perturbation theory in cosmology. The *metric* (equivalently the *co-ordinate* based approach) and the *covariant* formalisms are presented separately, followed by their comparison. Although this chapter is primarily about linear perturbation theory, it ends with a motivation for developing *higher order perturbation theory*, which is the major theme of this thesis. We begin by reviewing the idea of cosmological perturbations as given in [152].

2.1 Linear Perturbation Theory

In general, a spacetime can be regarded as a pair $(\mathcal{M}, g_{ab})$ of a smooth, four dimensional, connected, Hausdorff[1] manifold $\mathcal{M}$ endowed with a smooth Lorentz metric $g_{ab}$of signature

[1]A topological space $\mathcal{M}$ is said to be a Hausdorff space if it satisfies the Hausdorff separation axiom: Whenever x and y are two distinct points in $\mathcal{M}$, there exists disjoint open sets $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{M}$ such that $x\epsilon\mathcal{U}$ and $y\epsilon\mathcal{V}$ [79]

$(-+++)$. A perturbation of $(\mathcal{M}, g)$ can then be thought of in three different ways: (1) As the process of changing $(\mathcal{M}, g)$ to a slightly different spacetime $(\breve{\mathcal{M}}, \breve{g})$, or (2) as some measure of the difference between $(\mathcal{M}, g)$ and $(\breve{\mathcal{M}}, \breve{g})$ where $\breve{\mathcal{M}}$ is the real spacetime, or (3) a result $(\breve{\mathcal{M}}, \breve{g})$ of changing $(\mathcal{M}, g)$ slightly. The first and the third ideas may sound similar, however it should be noted that the former emphasises the *process* while the latter emphasises the *result*, a distinction worth keeping. Perturbation theory therefore deals with two spacetimes that differ slightly.

The two widely used perturbation formalisms, as already mentioned, are the *metric* approach and the *covariant* approach. The *metric* approach is the more widely used and involves perturbing the metric and the matter variables. For a long time, *gauge issues* in the metric based method hindered the development of perturbation theory as a tool for cosmological analysis. This lead to the development of two main approaches to deal with these gauge artefacts. The first method involves *fixing* the gauge, while the second involves working with gauge-invariant variables. These remedies were first suggested in Bardeen's major paper [8]. A thorough review of the metric approach can be found in [8, 118, 92].

Building on the work of Bardeen, Ellis, Bruni and coworkers, developed an alternative approach based on variables that are covariant and gauge invariant. This alternative method is referred to as the *covariant* approach. A review of this formalism can be found in [41, 43], while a comprehensive review of its applications can be found in [160]. We first examine the *metric* approach.

2.2 Metric approach

In the metric approach, fundamental quantities of the FLRW universe such as the metric (g_{ab}), are perturbed. A perturbed metric then leads to a perturbed line element, while the perturbation of the Ricci tensor and energy momentum tensors lead to perturbed Einstein

Field Equations (EFE). In order to build linear perturbation theory in this framework, one needs the complete list of linearly perturbed quantities. We note that the same scheme can be extended to higher-order perturbations (see Chapter 6).

2.2.1 Linearly perturbed metric tensor

The mathematical form of the perturbed metric can be written as follows:

$$g_{ab} = g_{ab}^{(0)} + \delta g_{ab}, \tag{2.1}$$

where the unperturbed background metric is denoted by a superscript (0). It is possible to show that the metric perturbations further split into a divergence-free and trace-free part that we will refer to as the pure *tensor*, a divergence-free part called a pure *vector* and curl-free part called a pure *scalar*. This splitting is based on the way these parts transform on the spatial hypersurfaces (for detail see [92] or [118]). In the case of the perturbed metric, we have

$$\begin{aligned}
g_{00} &= -a^2[1+2\phi], \quad g^{00} = -a^{-2}[1-2\phi], && (2.2)\\
g_{0i} &= a^2[B_{|i} - \bar{\mathcal{E}}_i], \quad g^{0i} = a^{-2}[B^{|i} - \bar{\mathcal{E}}^i], && (2.3)\\
g_{ij} &= a^2[(1-2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{(i|j)} + \bar{\mathcal{E}}_{ij}],\\
g^{ij} &= a^{-2}[(1+2\psi)\gamma^{ij} - 2E^{|ij} - 2F^{(i|j)} - \bar{\mathcal{E}}^{ij}], && (2.4)
\end{aligned}$$

where $(\phi, \psi, B\text{and } E)$ are scalars, ($\bar{\mathcal{E}}_i$ and F_i) are vectors and ($\bar{\mathcal{E}}_{ij}$) is a tensor ($\partial^i\bar{\mathcal{E}}_i = 0 = \partial^i\bar{\mathcal{E}}_{ij}$).

The linearly perturbed FLRW line element now reads

$$\begin{aligned}
ds^2 &= a^2(\eta)\{-(1+2\phi)d\eta^2 + 2(B_{|i} - \bar{\mathcal{E}}_i)d\eta dx^i\\
&+ [(1-2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{(i|j)} + \bar{\mathcal{E}}_{ij}]dx^i dx^j\}. \qquad (2.5)
\end{aligned}$$

The background metric is easily recovered by dropping all first order terms from this line element.

2.3 The connection and Ricci tensor

In a torsion-free metric, the connection coefficients are given by the Christoffel symbols

$$\Gamma^{d}{}_{bc} = \tfrac{1}{2} g^{ad} \left(g_{ab,c} + g_{ac,b} - g_{bc,a} \right), \tag{2.6}$$

from which one can obtain the individual connection coefficients. The Riemann tensor is given by,

$$R^{a}{}_{bcd} = \Gamma^{a}{}_{bd,c} - \Gamma^{a}{}_{bc,d} + \Gamma^{a}{}_{ec}\Gamma^{e}{}_{bd} - \Gamma^{a}{}_{ed}\Gamma^{e}{}_{bc}. \tag{2.7}$$

Given the Riemann tensor $R^{ab}{}_{cd}$, the Ricci tensor is obtained by simple contraction of a and c. A further contraction of b and d yields the Ricci scalar R. The components of the unperturbed Ricci tensor and Ricci scalar are,

$$R_{00} = -3\mathcal{H}', \tag{2.8}$$

$$R_{0i} = 0, \tag{2.9}$$

$$R_{ij} = (\mathcal{H}' + 2\mathcal{H}^2 + \mathcal{K})\gamma_{ij}, \tag{2.10}$$

$$R = \tfrac{6}{a^2}(\mathcal{H}' + \mathcal{H}^2 + \mathcal{K}), \tag{2.11}$$

where $\mathcal{H} = a'/a$. The linearly perturbed connection coefficients can be obtained by simplying the products of the metric and the perturbed metric, from which we obtain,

$$\delta\Gamma^{c}{}_{bd} = \tfrac{1}{2}\delta g^{ac} \left(g_{ab,d} + g_{ad,b} - g_{bd,a} \right) + \tfrac{1}{2} g^{ac} \left(\delta g_{ab,d} + \delta g_{ad,b} - \delta g_{bd,a} \right). \tag{2.12}$$

The linear perturbation of the Riemann tensor is given by,

$$\delta R^a{}_{bcd} = \delta\Gamma^a{}_{bd,c} - \delta\Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\delta\Gamma^e{}_{bd} + \delta\Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\delta\Gamma^e{}_{bc} - \delta\Gamma^a{}_{ed}\Gamma^e{}_{bc}. \quad (2.13)$$

From this expression, we calculate the linear order Ricci tensor by contracting over the first and third indices, which give,

$$\delta R_{00} = 3\mathcal{H}\phi' + \nabla^2\phi + 3(\psi'' + \mathcal{H}\psi') + \nabla^2(B' + \mathcal{H}B) - \nabla^2(E'' + \mathcal{H}E'), \quad (2.14)$$

$$\delta R_{0i} = 2\mathcal{H}\phi_{|i} + 2\psi'_{|i} + (\mathcal{H}' + 2\mathcal{H}^2)(B_{|i} - \bar{\mathcal{E}}_i) + \nabla^2(\bar{\mathcal{E}}_i + F'_i), \quad (2.15)$$

$$\begin{aligned} \delta R_{ij} &= [-\mathcal{H}\phi' - 2(\mathcal{H}' + 2\mathcal{H}^2)(\phi + \psi) - \psi'' - 5\mathcal{H}\psi' + \nabla^2\phi]\gamma_{ij} - \phi_{|ij} + \psi_{|ij} \\ &- (B'_{|ij} + 2B_{|ij}) - \mathcal{H}\nabla^2(B\gamma_{ij} + E'\gamma_{ij}) + [E'' + 2\mathcal{H}E' + 2(\mathcal{H}' + 2\mathcal{H}^2)E]_{|ij} \\ &+ \bar{\mathcal{E}}'_{(i|j)} + 2\mathcal{H}\bar{\mathcal{E}}_{(i|j)} + F''_{(i|j)} + 2\mathcal{H}F'_{(i|j)} + 2(\mathcal{H}' + 2\mathcal{H}^2)F_{(i|j)} + \tfrac{1}{2}\bar{\mathcal{E}}''_{ij} + \mathcal{H}\bar{\mathcal{E}}'_{ij} \\ &+ (\mathcal{H}' + 2\mathcal{H}^2)\bar{\mathcal{E}}_{ij} - \tfrac{1}{2}\nabla^2\bar{\mathcal{E}}_{ij}. \end{aligned} \quad (2.16)$$

Finally, the first order Ricci scalar takes the form

$$\begin{aligned} \delta R &= \tfrac{1}{a^2}[-6\mathcal{H}(3\psi' + \phi') - 6\psi'' + 2\nabla^2(2\psi - \phi) - 12\tfrac{a''}{a}\psi' \\ &- 6\mathcal{H}\nabla^2(B - E') - 2\nabla^2(B' - E'')], \end{aligned} \quad (2.17)$$

where $\nabla^2 = \nabla^i\nabla_i$.

2.4 The perturbed energy momentum tensor

In order to find the linearly perturbed EFE, we also need the linearly perturbed energy momentum tensor. This can be obtained in a manner similar to that we have used to obtain the linearly perturbed Ricci tensor. In particular,

$$T_{ab} = T^{(0)}_{ab} + \delta T_{ab}. \quad (2.18)$$

In this section we present the linearly perturbed energy momentum tensor of the imperfect fluid. The perturbed momentum tensor takes the form,

$$\delta T_{ab} = (p+\mu)(v_a\delta v_b + \delta v_a v_b) + (\delta\mu + \delta p)v_a v_b + 2(q_{(a}\delta v_{b)} + v_{(v}\delta q_{b)}) + \delta\pi_{ab}, \qquad (2.19)$$

where

$$\begin{aligned} \delta v^a &= \tfrac{1}{a}[-\phi, \delta v^i], & (2.20)\\ \delta v_a &= a[-\phi, \delta v_i + B_{,i} - \bar{\mathcal{E}}_i], & (2.21) \end{aligned}$$

from which it follows that,

$$\begin{aligned} \delta T_{00} &= a^2\delta\mu + 2a^2\phi\mu, & (2.22)\\ \delta T_{0i} &= -a\delta q_i - a^2(\mu+p)\delta v_i - a^2\mu(B_{|i} - \bar{\mathcal{E}}_{1i}), & (2.23)\\ \delta T_{ij} &= a^2\delta p\gamma_{ij} + \delta\pi_{ij} + a^2 p(-2\psi\gamma_{ij} + 2E_{|ij} + 2F_{(i|j)} + \bar{\mathcal{E}}_{ij}). & (2.24) \end{aligned}$$

where v_i is the velocity vector, q_i is the energy flux and π_{ij} is the anisotropic stress tensor, all defined on the 3-hypersurfaces.

2.4.1 Perturbed Einstein Field Equation

The general covariance of the Einstein field equation guarantees that the linearly perturbed equation takes the form,

$$\delta G_{ab} = \delta T_{ab}, \qquad (2.25)$$

where both side should ideally be expressed in terms of gauge invariant variables. Gauge-invariant formulation in the metric approach was first given by Bardeen [8] and later developed in [92,118,35]. In this approach, one considers ways of defining gauge-invariant variables based on how the various first order quantities transform. Assume, for example, that the quantities, $(\phi, \psi, E, B, \bar{\mathcal{E}}, F_i, \bar{\mathcal{E}}_{ij})$ transform to $(\tilde{\phi},\ \tilde{\psi},\ \tilde{E},\ \tilde{B},\ \tilde{\bar{\mathcal{E}}}, \tilde{F}_i,\ \tilde{\bar{\mathcal{E}}}_{ij})$, where the scalars and the vectors obey their individual transformation properties.

The perturbation of any scalar quantity f about a FLRW background transforms as

$$\delta\tilde{f} = \delta f - L^0 f_0'. \tag{2.26}$$

Therefore scalars on the hypersurfaces, such as spatial curvature, acceleration, shear and the density perturbation only depend on the choice of the temporal gauge, L^0. The spatial gauge, determined by L, will only affect the components of the 3-vectors and 3-tensors on this hypersurface.

Vector quantities that are derived from a potential, such as the velocity potential v, only depend on the shift L within the hypersurfaces and are independent of L^0. It can therefore be shown that the velocity potential transforms as,

$$\tilde{v} = v + L'. \tag{2.27}$$

The function L^i only affects the components of divergence-free 3-vectors and 3-tensors on the 3-hypersurfaces. A quantity such as the velocity perturbation v^i therefore transforms as,

$$\tilde{v}^i = v^i + L^{i\prime}. \tag{2.28}$$

There are no tensor type gauge transformations in linear perturbation theory. Because the metric and the matter perturbation defined in the previous section are not gauge invariant, one usually tries to find linear combinations that are gauge-invariant in order to proceed towards a gauge-invariant formalism. A detailed review of this can be found in [118]. In general there are several possible choices that allows for gauge-fixing. The simplest gauge-invariant linear combinations of the scalars (ϕ, ψ, E, B) were first given by

Bardeen and read,

$$\Phi = \phi + \tfrac{1}{a}\left[(B - E')a\right]', \qquad \Psi = \psi - \tfrac{a'}{a}(B - E'), \tag{2.29}$$

which turn out to coincide with the metric perturbations in the longitudinal gauge as discussed in the next section. Gauge-invariant perturbations can also be defined based on other choices of space-time time-slicing. We consider a few of these in the next section.

2.4.2 Other gauge choices

In this brief review, we shall only give a few gauge choices without showing the gauge-invariant combinations. The *synchronous* gauge choice is obtained from setting ($\tilde{\phi} = \tilde{B} = 0$). This choice, which was first used by Lifshitz [102], does not completely fix the gauge. The other common choice is the *longitudinal* (also called the conformal-Newtonian) gauge. This is arrived at by setting ($\tilde{B} = \tilde{E} = 0$). Unlike the *synchronous* choice, the longitudinal choice completely fixes the gauge. It follows that in this gauge ($\phi = \Phi, \psi = \Psi$). The *Comoving orthogonal gauge* is chosen by requiring that the 3-velocity of the fluid vanish ($\tilde{v} = 0$). The orthogonality of the constant-η hypersurfaces to the 4-velocity, then requires $\tilde{v} + \tilde{B} = 0$. One can also choose the *Uniform density* gauge. In this choice the matter content is used to pick out uniform density hypersurfaces on which perturbed quantities can be defined. In general, setting $\delta\tilde{\mu} = 0$ implies $L^0 = \mu/\mu_0'$. This has been used to define the gauge-invariant curvature perturbation ζ [29, 114, 87] on these hypersurfaces given by

$$-\zeta = \tilde{\psi} = \psi + \mathcal{H}'\delta\mu/\mu_0'. \tag{2.30}$$

The remaining degrees of freedom can be set by demanding that either $\tilde{B}, \tilde{E}$ or $\tilde{v}$ be zero.

2.4.3 Scalar perturbations

Let us consider scalar perturbations in the longitudinal gauge. In this gauge $\Phi = \phi$ and $\Psi = \psi$, in addition $\Phi = \Psi$ for perfect fluids and scalar fields. In the case of perfect fluid, substituting the individual parts of Eq. (2.2), equations (2.8) and (2.14) into the perturbed Einstein field equation given that $\delta G^a_b \equiv \delta R^a_b - \frac{1}{2}g^a_b\delta R - \frac{1}{2}R\delta g^a_b$, where $\delta R^a_b \equiv g^{ac}\delta R_{cb} + R_{cb}\delta g^{ac}$ for $a = b = 0$, yields an equation with perturbed energy density as a source. In the Newtonian limit, this equation is the usual Poisson equation for the gravitational potential induced by some energy-density perturbation. The substitution of (2.10), (2.11), and (2.16), into the perturbed EFE yields an equation with the perturbed pressure as a source. Applying the adiabatic condition $\delta p = c^2{}_s\delta\mu$, and setting $c_s^2 = w$, the two equations combine to give,

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' - c_s^2\nabla^2\Phi = 0. \tag{2.31}$$

This variable characterises the evolution of density perturbations in the metric approach, as it is related to a gauge-invariant density contrast [118]. The standard approach is then to Fourier decompose this wave equation i.e. $\Phi = \sum \Phi_k Q^{(k)}$, where Q is a scalar Fourier basis obtained by solving the Helmholtz equation.

The usual approach involves the introduction of a new variable(see [118]) that simplifies its form and then leads to desired solutions for given k. It can be shown, for example, that in the case of dust,

$$\Phi_k(\eta) = \mathcal{Y}_{1k} + \mathcal{Y}_{2k}/\eta^5,$$

where $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are constants with respect to η. The radiation case solutions are

$$\Phi_k = \left(\tfrac{\mathcal{Y}_{1k}}{(k|\eta|)^2\sqrt{3}} + \tfrac{\mathcal{Y}_{2k}}{(k|\eta|)^3\sqrt{3}}\right)\cos\left(\tfrac{k|\eta|}{\sqrt{3}}\right) + \left(\tfrac{\mathcal{Y}_{2k}}{(k|\eta|)^2\sqrt{3}} - \tfrac{\mathcal{Y}_{1k}}{(k|\eta|)^3\sqrt{3}}\right)\sin\left(\tfrac{k|\eta|}{\sqrt{3}}\right), \tag{2.32}$$

where again $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are the constants of integration.

2.4.4 Tensor perturbations

Since there exists no tensor type infinitesimal gauge transformation, $\bar{\mathcal{E}}_{ij}$ is gauge-invariant by itself. Applying the process used to isolate scalar perturbations to the case of tensor perturbations yields

$$\bar{\mathcal{E}}''_{ij} + 2\mathcal{H}\bar{\mathcal{E}}'_{ij} - \nabla^2\bar{\mathcal{E}}_{ij} = 0, \tag{2.33}$$

where equations (2.4) and (2.11) are used. Here again, one applies Fourier decomposition using a tensor basis. It can be shown that in the case of dust one obtains the solutions

$$\bar{\mathcal{E}}_k = \left(\tfrac{\mathcal{X}_{1k}}{(k|\eta|)^2} + \tfrac{\mathcal{X}_{2k}}{(k|\eta|)^3}\right)\cos(k|\eta|) + \left(\tfrac{\mathcal{X}_{2k}}{(k|\eta|)^2} - \tfrac{\mathcal{X}_{1k}}{(k|\eta|)^3}\right)\sin(k|\eta|), \tag{2.34}$$

where $\mathcal{X}_1$ and $\mathcal{X}_2$ are the constants of integration. The radiation case solutions are

$$\bar{\mathcal{E}}_k = \tfrac{\mathcal{X}_{3k}}{(k|\eta|)}\sin(k|\eta|) + \tfrac{\mathcal{X}_{4k}}{(k|\eta|)}\cos(k|\eta|), \tag{2.35}$$

where again $\mathcal{X}_1$ and $\mathcal{X}_2$ are constants. We now turn to the *covariant* formalism before comparing the two approaches at linear order. Adopting the approach usually employed in discussing perturbations in this method, we first present the full nonlinear covariant set of equations, then drop terms to obtain the linear perturbations of the standard model.

2.5 Covariant approach

In the 1+3 covariant formalism developed by Ellis and Bruni [41], kinematic quantities based on a fundamental 4-velocity, the energy momentum tensors of matter sources, the electric and the magnetic parts of the Weyl curvature tensor are used rather than the metric. The fundamental equations are the Ricci and Bianchi identities, applied to the 4-velocity vector, with the Einstein's equations included via algebraic relations between the Ricci and the energy momentum tensor.

2.5.1 Average 4-velocity of matter

In order to define a 4-velocity, we need to consider a general spacetime given by the pair $(\mathcal{M}, \mathbf{g})$. For such spacetime, it is usually assumed that there is a well-defined preferred motion of matter and hence a unique 4-velocity. At late times this is taken to be the 4-velocity defined by the vanishing of the dipole of the CMB. This is reasonable as only one 4-velocity will set this dipole to zero. This 4-velocity is given by

$$u^a = \tfrac{dx^a}{dt}, \qquad u_a u^a = -1, \tag{2.36}$$

where t is the proper time measured along the fundamental observer worldlines [43].

2.5.2 Projection and Permutation Tensors

Given u^a, there are defined unique projections tensors,

$$U^a{}_b = -u^a u_b \Rightarrow U^a{}_c U^c{}_b = U^a{}_b, U^a{}_a = 1, U_{ab} u^b = u_a, \tag{2.37}$$

$$h_{ab} = g_{ab} + u_a u_b \Rightarrow h^a{}_c h^c{}_b = h^a{}_b, h^a{}_a = 3, h_{ab} u^b = 0, \tag{2.38}$$

where the first projects parallel while the second orthogonal to the velocity vector u^a. There is also a volume element for the rest-spaces,

$$\varepsilon_{abc} = u^d \eta_{dabc} \Rightarrow \eta_{abc} = \eta_{[abc]}, \qquad \eta_{abc} u^c = 0, \tag{2.39}$$

where η_{abcd} is the 4-dimensional volume element and ε_{abc} defines a permutation tensor. We can define the covariant time derivative along the observer wordlines, where for any tensor $T^{ab}{}_{cd}$,

$$\dot{T}^{ab}_{cd} = u^e \nabla_e T^{ab}{}_{cd}, \tag{2.40}$$

and the orthogonal projected covariant derivative $\tilde{\nabla}_a$, where for any tensor $T^{ab}{}_{cd}$,

$$\tilde{\nabla}_e T^{ab}{}_{cd} = h^a{}_f h^b{}_g h^p{}_c h^q{}_d h^r{}_e \nabla_r T^{fg}{}_{pq}. \tag{2.41}$$

We next consider variables relevant to this formalism.

2.5.3 Kinematical variables

Using these projection tensors and covariant derivatives, the 4-velocity u_a may be split into its irreducible parts given by their symmetry properties [38]. This reads

$$\nabla_a u_b \;=\; -u_a A_b + \tilde{\nabla}_a u_b = \tfrac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}, \tag{2.42}$$

where $A_b = \dot{u}_b = u^c \tilde{\nabla}_c u_b$, $\Theta = \tilde{\nabla}_a u^a$, $\sigma_{ab} = \tilde{\nabla}_{\langle a} u_{b\rangle}$, $\omega_{ab} = \tilde{\nabla}_{[a} u_{b]}(\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc})$. Θ is the volume-rate of expansion of the fluid (such that $\Theta = 3H$, where H is the normal Hubble parameter in the FLRW model). The trace-free symmetric rate of shear tensor is given by σ_{ab} (such that $\sigma_{ab}u^b = 0$, $\sigma^a{}_a = 0$). The vorticity is given by the skew-symmetric tensor ω_{ab}, $(\omega_{ab}u^a = 0)$. It is important to note that $\tilde{\nabla}_a$ coincides with the 3-dimensional covariant derivatives, in the case where the vorticity vanishes.

2.5.4 Weyl Curvature Tensor

In general the Riemann tensor R_{abcd} takes the form,

$$R_{abcd} = C_{abcd} + g_{a[c}R_{b]d} - g_{b[c}R_{d]a} - \tfrac{1}{6}R g_{abcd}. \tag{2.43}$$

In this equation, $g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}$, and

$$C_{abcd} = (g_{abpq}g_{cdrs} - \eta_{abpq}\eta_{cdrs})u^p u^r E^{qs} + (\eta_{abpq}g_{cdrs} + g_{abpq}\eta_{cdrs})u^p u^r H^{qs}, \tag{2.44}$$

where C_{abcd} is the Weyl tensor. The quantities E_{ab} and H_{ab} are the *electric* and the *magnetic* parts of the Weyl tensor, alternatively given by,

$$E_{ab} = C_{acbd}u^c u^d, \qquad H_{ab} = \tfrac{1}{2}\eta_{ade}C^{de}{}_{be}u^c, \tag{2.45}$$

such that

$$E^b{}_b = 0, \quad E_{ab} = E_{(ab)}, \quad E_{ab}u^a = 0, \tag{2.46}$$

$$H^b{}_b = 0, \quad H_{ab} = H_{(ab)}, \quad H_{ab}u^a = 0. \tag{2.47}$$

These tensors represent the *free gravitational field*, enabling gravitational action at a distance, and influence the motion of matter and radiation through geodesic deviation equation for timelike and null vectors, respectively [23, 132, 133, 153, 154]. We now look at the matter tensor.

2.5.5 Matter Tensor

The matter energy-momentum tensor T_{ab} can be decomposed relative u^a as follows:

$$T_{ab} = \mu u_a u_b + q_a u_b + u_a q_b + p h_{ab} + \pi_{ab}, \tag{2.48}$$

where $\mu = (T_{ab}u^a u^b)$ is the relativistic energy density relative to u^a, $q^a = -T_{bc}u^b h^{ca}$ is the relativistic momentum density, that corresponds to the energy flux relative to u^a, $p = \frac{1}{3}(T_{ab}h^{ab})$ is isotropic pressure, and $\pi_{ab} = T_{cd}h^c{}_{\langle a}h^d{}_{b\rangle}$ is the trace-free anisotropic pressure or stress.

The physics of the situation is determined by the various components making up the energy momentum tensor. It is common to impose the perfect fluid restrictions

$$q^a = \pi_{ab} = 0, \tag{2.49}$$

which reduces the energy momentum tensor to

$$T_{ab} = \mu u_a u_b + p h_{ab}. \tag{2.50}$$

It is also standard to give the equation of state, relating pressure to the energy density (see Chapter 1).

2.5.6 Nonlinear Covariant Equations

From EFE (1.4) and its associated integrability conditions, one finds three sets of fundamental equations. The first of these sets arises from the Ricci identities for the vector field u^a, i.e

$$\nabla_a \nabla_b u^c - \nabla_b \nabla_a u^c = R_{ab}{}^c{}_d u^d. \tag{2.51}$$

The various velocity gradients can be substituted for using Eq. (2.42), while the rhs can be substituted for using the Einstein field equation. The resulting equation can then be separated into the orthogonally projected part and the parallel part. The orthogonally projected parts can be separated further into a trace, symmetric trace-free, and skew symmetric parts. These procedures yield three propagation equations and three constrain equations. The first of these is Raychaudhuri equation [138]:

$$\dot{\Theta} = \tilde{\nabla}_a A^a - \tfrac{1}{3}\Theta^2 + (A_a A^a) - \sigma^{ab}\sigma_{ab} + (\omega^{ab}\omega_{ab}) - \tfrac{1}{2}(\mu + 3p) + \Lambda, \tag{2.52}$$

which is interpreted as the equation of gravitational attraction and which shows the repulsive nature of a positive cosmological constant. The $\mu + 3p$ term counteracts the repulsive cosmological term and is interpreted to be the active gravitational mass density. The second equation is the vorticity equation given by,

$$\dot{\omega}_{\langle a \rangle} = \tfrac{1}{2}\epsilon_{abc}\tilde{\nabla}^b A^c - \tfrac{2}{3}\Theta\omega_a + \sigma^b{}_a\omega_b. \tag{2.53}$$

When the vorticity is zero, it follows from (2.53) that the curl of acceleration vanishes. Most significant however, is that the vanishing of vorticity allows us to define orthogonal hypersurfaces and leads to the spatial derivative coinciding with the 3-surface spatial gradient. Vanishing vorticity will be a recurring assumption in this thesis.

The third equation produced by the Ricci identities is the shear propagation equation which takes the form,

$$\dot{\sigma}_{\langle ab\rangle} = \tilde{\nabla}_{\langle a}A_{b\rangle} - \tfrac{2}{3}\Theta\sigma_{ab} + A_{\langle a}A_{b\rangle} - \omega_{\langle a}\omega_{b\rangle} - \sigma_{c\langle a}\sigma_{b\rangle}{}^{c} - E_{ab} + \tfrac{1}{2}\pi_{ab}. \tag{2.54}$$

Apart from the three propagation equations, the Ricci identities also produce the following constraint equations. The divergence of shear constraint is given by,

$$(C_1)_a = \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta + \epsilon_{abc}[\tilde{\nabla}^b\omega^c + 2A^b\omega^c] + q_a = 0, \tag{2.55}$$

while the vorticity divergence identity is given by,

$$(C_2) = \tilde{\nabla}_a\omega^a - A_a\omega^a = 0. \tag{2.56}$$

The Ricci identities give the a constraint equation for the magnetic part of the Weyl tensor:

$$(C_3)_{ab} = \mathsf{curl}\,(\sigma_{ab}) - H_{ab} - 2A_{\langle a}\omega_{b\rangle} - \tilde{\nabla}_{\langle a}\omega_{b\rangle} = 0. \tag{2.57}$$

This equation shows how distortion of the vorticity and the rotation of shear determines the magnetic part of the Weyl tensor. We note that this constraint has no contribution from the non-perfect fluid terms. In the absence of vorticity, the magnetic part is completely determined by the curl of the shear.

The next set of equations arise from the twice-contracted Bianchi Identities. These are the energy density propagation equation and the momentum propagation equation, given by

$$\dot{\mu} = -\tilde{\nabla}_a q^a - \Theta(\mu + p)\mu - 2A_a q^a - (\sigma_{ab}\pi^{ab}) \tag{2.58}$$

and,

$$\dot{q}_{\langle a \rangle} = -\tilde{\nabla}_a p - \tilde{\nabla}^b \pi_{ab} - \tfrac{4}{3}\theta q_a - (\mu + p)A_a - \sigma^b{}_a q_b - A^b \pi_{ab} - \epsilon_{abc}\omega^b q^c, \tag{2.59}$$

respectively. In the case of the perfect fluid, these two equations reduce to,

$$\dot{\mu} = -\Theta(\mu + p)\mu \tag{2.60}$$

and

$$0 = \tilde{\nabla}_a p + (\mu + p)A_a, \tag{2.61}$$

where $\mu + p$ is the inertial mass density. For *ordinary* matter, one requires that $\mu + p > 0$, which corresponds to the energy condition (1.7) The last set of equations come from Bianchi identities:

$$\nabla_{[a}R_{bc]de} = 0. \tag{2.62}$$

Besides the conservation of the Einstein tensor which results from the double contraction of these identities, two propagation and two constraints also arise.

The 1+3 splitting of the electric and the magnetic parts of the Weyl tensor (2.45), the Einstein field equation which includes the cosmological constant and the once-contracted Bianchi identities give the two last propagation and constraint equations needed to form a closed system of covariant equations. These are the propagation equations for the *electric* and the *magnetic* parts of the Weyl tensor given by,

$$\begin{aligned} \dot{E}_{\langle ab \rangle} &= -\tfrac{1}{2}\dot{\pi}^{ab} + \mathrm{curl}\, H_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q_{b\rangle} - \tfrac{1}{2}(\mu + p)\sigma_{ab} - \Theta(E_{ab} + \tfrac{1}{6}\pi_{ab}) \\ &\quad + 3\sigma^c_{\langle a}(E_{b\rangle c} - \tfrac{1}{6}\pi_{b\rangle c}) - A_{\langle a}q_{b\rangle} + \epsilon_{cd\langle a}[2A^c H_{b\rangle}{}^d + \omega^c(E_{b\rangle}{}^d + \tfrac{1}{2}\pi_{b\rangle}{}^d)], \end{aligned} \tag{2.63}$$

and,

$$\dot{H}_{\langle ab\rangle} = -\epsilon_{cd\langle a}[2A^c E_{b\rangle}{}^d - \tfrac{1}{2}\sigma_{b\rangle}{}^c q^d - \omega^c H^b_d] + \mathsf{curl}\,(\tfrac{1}{2}\pi_{ab} - E_{ab}) - \Theta H_{ab} + 3\sigma^c_{\langle a} H_{b\rangle c} + \tfrac{3}{2}\omega_{\langle a} q_{b\rangle}, \qquad (2.64)$$

where the $\mathsf{curl}\, E_{ab} = \epsilon_{cd\langle a}\tilde{\nabla}^c E_{b\rangle}{}^d$. The two equations (2.63 and 2.64) form the foundation of the covariant analysis of gravitational radiation, a theme we return to in Chapters (3, 5, 6).

The two constraints arising from the identities (2.62) are the divergence of the *electric* and the *magnetic* parts of the Weyl tensor. These are,

$$(C_4)_a = \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b \pi_{ab} - \tfrac{1}{3}\tilde{\nabla}_a \mu + \tfrac{1}{3}\Theta q_a - \tfrac{1}{2}\sigma^b{}_a q_b - 3\omega^b H_{ab} - \varepsilon_{abc}[\sigma^b_d H^{cd} - \tfrac{3}{2}\omega^b q^c] = 0, \qquad (2.65)$$

$$(C_5)_a = \tilde{\nabla}^b H_{ab} + (\mu + p)\omega_a + 3\omega^b(E_{ab} - \tfrac{1}{6}\pi_{ab}) + \varepsilon_{abc}\sigma^b_d(E^{cd} + \tfrac{1}{2}\pi^{cd}) = 0. \qquad (2.66)$$

Whereas the divergence of the electric part of the Weyl tensor is an analogue of the Newtonian Poisson equation [43], the magnetic part has no Newtonian analogue.

Equations (2.63-2.66) are very similar to Maxwell's equations in a curved expanding background (which is why E_{ab} and H_{ab} are referred to as the electric and the magnetic part of the Weyl tensor [43]). This important difference will be exploited when analyzing gravitational waves. It can be seen from (2.65) that the divergence of the electric part of the Weyl tensor has gradient of the energy density as a source. This represents a tidal action at a distance. The magnetic part of the Weyl tensor on the other hand has vorticity as a source. Together these equations describe the couplings that occurs between the 'free-gravitational field' represented by these parts of the Weyl tensor, and the the dynamics of the fluid flow.

2.5.7 Ricci tensor and Ricci scalar

The contraction of Riemann tensor $R^{ab}{}_{cd}$ gives the Ricci tensor $R^b{}_d$, while that of the Ricci tensor gives the Ricci scalar R. Let S_{ab} be the traceless part of Ricci tensor; $S_{ab} = R_{ab} - \frac{1}{3}g_{ab}R$. It follows that,

$$
\begin{aligned}
S_{ab} &= -\tfrac{1}{3}\theta\sigma_{ab} + \sigma_{c(a}\sigma_{b)}{}^c + \omega_a\omega_b + E_{ab} + \tfrac{1}{2}\pi_{ab}, && (2.67)\\
{}^3R &= -\tfrac{2}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - 2\omega_a\omega^a + 2\mu + 2\Lambda, && (2.68)
\end{aligned}
$$

where Λ is the cosmological constant. One can define a 3-surface by setting the vorticity to zero ($\omega^a = 0$), and therefore ${}^3S_{ab}$ corresponds to a 3- tensor and ${}^3R$ a 3-Ricci scalar of these surfaces.

2.5.8 Friedmann- Lemaître- Robertson-Walker Model

The FLRW background is characterised by energy density μ, pressure p and expansion Θ, assuming $\Lambda = 0$. The basic standard model equations can now be extracted from the covariant equations in a direct manner. Setting all other terms to zero lead to the desired background equations,

$$
\begin{aligned}
\dot{\mu} &= -\Theta(\mu + p), && (2.69)\\
\dot{\Theta} &= -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu + 3p), && (2.70)
\end{aligned}
$$

which are just equations (1.10) and (1.11) and provide the first matching between the two formalisms. The power of 1+3 formalism lies in the fact that one begins with a general set of equations suitable for studying any cosmological model containing matter with known equation of state.

In the perturbative approach, one decides before hand, the model one wishes to study. This involves determining variables that do not vanish in a chosen background, for example the background variables in the FLRW model would be (μ, Θ and p), while those of Bianchi model would be (μ, Θ, p, σ_{ab} and E_{ab}). The variables then represent the zeroth order solution of the cosmological model. The spatial derivatives of these values and all the remaining terms that appear in the covariant equation for perfect fluid vanish in the background and are considered to be first order in magnitude [41] and [72]. This guarantees that linear order variables satisfy the gauge-invariant condition [152]. By definition, all the first order quantities are small compared to the background values and are assigned perturbative order $\mathcal{O}(\epsilon)$, where ϵ is a measure of smallness.

2.5.9 Linearisation

Equations representing first order perturbations are then obtained by *linearising* the full set of covariant equations for perfect fluid about the chosen background. In this process, products of two or more first-order terms [$\mathcal{O}(\epsilon)$] are neglected. Linearization of the exact equation about FLRW background yield the required first-order propagation equations. In order to proceed, one stipulates the nature of matter. For example, if we assume that matter is of perfect fluid form and that the vorticity vanishes, we find the propagation equations.

$$\dot{\mu} = -\Theta(\mu + p), \tag{2.71}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}\mu(1 + 3p), \tag{2.72}$$

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab}, \tag{2.73}$$

$$\dot{E}_{ab} = \mathrm{curl}\, H_{ab} - \tfrac{1}{2}(\mu + p)\sigma_{ab} - \Theta E_{ab}, \tag{2.74}$$

$$\dot{H}_{ab} = -\mathrm{curl}\, E_{ab} - \Theta H_{ab}, \tag{2.75}$$

and constraints equations,

$$(C_1)_a = \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta = 0, \tag{2.76}$$
$$(C_2) = \tilde{\nabla}^a\omega_a = 0 \tag{2.77}$$
$$(C_3)_{ab} = \mathsf{curl}\,(\sigma_{ab}) - H_{ab} = 0, \tag{2.78}$$
$$(C_4)_a = \tilde{\nabla}^b E_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\mu = 0, \tag{2.79}$$
$$(C_5)_a = \tilde{\nabla}^b H_{ab} = 0. \tag{2.80}$$

These linearized equations represent the coupling of various perturbative modes. We now focus on individual perturbations as handled in the covariant approach.

2.5.10 Scalar perturbations

In the absence of vorticity, inhomogeneity in FLRW models are covariantly characterised by the variable $D_a = a\tilde{\nabla}_a\mu/\mu$ [43], a quantity that represents the usual density contrast in the comoving gauge, which features in the *metric* approach. It can be shown that this quantity obeys the second order equation,

$$D_a'' + \mathcal{H}(1-3w)D_a' - \left[\tfrac{1}{2}(1-w)(1+3w)a^2\mu + 2\mathcal{K}w\right]D_a - a^2w\tilde{\nabla}^2 D_a = 0, \tag{2.81}$$

where $w = const.$ The properties of this second order equation have been examined extensively [105, 33, 73, 72]. The importance of D_a lies in the fact that its magnitude D is both gauge invariant and corresponds to a real spatial fluctuation. The method for solving this equation is to first assume the spatial and the time dependence of the variable D_a are separable, which is achieved via a harmonic decomposition [77].

The standard procedure employs a basis $\mathcal{Q}$, which is covariantly constant, $\dot{\mathcal{Q}} = 0$ (equivalently $\mathcal{Q}' = 0$), to decompose the perturbation variable. This harmonic is defined as eigenfunctions of the covariant Laplace-Beltrami operator $\tilde{\nabla}^2\mathcal{Q} = -\kappa^2\mathcal{Q}/a^2$ [105].

It follows that, $D_a = \sum D_\kappa(\eta)Q_a^{(k)}(x)$, where x denote spatial direction. The harmonic decomposition of Eq. (2.81) has the added advantage in that it enables one to define a *long wavelength limit*, given by $k/\mathcal{H} << 1$. This allows one to effectively neglect the Laplacian term in Eq. (2.81). Consider for example the radiation case ($w = 1/3$) in a flat ($\mathcal{K} = 0$) universe, it follows that the long wave length solution is

$$D_a = d_+\eta + d_-\eta^{-1/2},$$

where d_+ and d_- are constant with respect to η. For wavelengths shorter than the Jeans length [43], the equation becomes the damped harmonic equation giving oscillations. The solutions for dust case are

$$D_a = d_+\eta^2 + d_-\eta^{-3},$$

where again d_+ and d_- are constant with respect to η. This the famous solution first discovered by Lifshitz and Khalatnikov [101], demonstrating gravitational instability.

2.5.11 Tensor perturbations

The Weyl tensors were first used to characterise gravitational radiation by Hawking [78]. A lot of work has since appeared based on this approach. Gravitational waves in FLRW models have been considered in [129, 21, 81, 137]. It has been shown that the magnetic part of the Weyl tensor obeys the second order wave equation,

$$H''_{ab} + 6\mathcal{H}H'_{ab} + 6\mathcal{H}^2(1-w)H_{ab} - a^2\tilde{\nabla}^2 H_{ab} = 0. \tag{2.82}$$

To solve this equation, one applies a tensor harmonic decomposition [78]. Note that the decomposition takes the form;

$$H_{ab} = a^{-2}\sum_\kappa \kappa^2\left(H_\kappa Q_{ab}^{(\kappa)} + \bar{H}_\kappa \bar{Q}_{ab}^{(\kappa)}\right), \tag{2.83}$$

where $\mathcal{Q}_{ab}$ is the usual tensor basis such that $\mathcal{Q}'_{ab} = 0$. It follows that the solutions to the decomposed equation are;

$$H_k = \eta^{\frac{3w-11}{2(3w+1)}} \left[c_1 J\left(\tfrac{5+3w}{2(3w+1)}, \kappa\eta\right) + c_2 Y\left(\tfrac{5+3w}{2(3w+1)}, \kappa\eta\right)\right], \tag{2.84}$$

for one parity. J and Y are Bessel functions of the first and the second kinds respectively. These recover those found in [129].

2.5.12 Consistency

A fundamental issue with the covariant equations is the consistency of constraints with the propagation equations. It follows that the full non-linear constraints equations evolve consistently with the propagation equations for the case of dust and for the case of imperfect fluid [109, 162]. Further to this, the constraints equations arising from linearization about FLRW evolve consistently [109, 162, 32]. It has been shown that there exists a linearization instability for the case of *silent* models (models with $H_{ab} = 0$). In other words consistency is satisfied at linear level but not at non-linear level for these models [76]. What is never stressed enough is the fact that consistency check should ideally be performed on two levels.

In general it is assumed that the constraints arising from linearization are satisfied at some initial time. The issue is whether or not these constraints are conserved under time evolution. We have independently confirmed the case given by [109] and have included the full calculation in the appendix (B) for completeness. In Chapters 3 and 4, we shall present new detailed consistency check for pure density perturbations and pure gravitational waves for perturbation of Bianchi type I models with irrotational dust respectively.

2.6 Comparison of covariant and Bardeen formalisms

Before going further it is instructive to compare the two formalisms and understand how they relate to each other. In this section we only make comparisons at linear order, which coincide with those of [106]. Comparison at second order will be made in Chapter 6.

2.6.1 Scalar perturbations

We have seen that the evolution of the scalars perturbations in the two appraoches take the form;

$$\Phi'' + 3\mathcal{H}(1+c_s^2)\Phi' - c_s^2\nabla^2\Phi + [2\mathcal{H}' + (1+3c_s^2)(\mathcal{H}^2 - K)]\Phi = 0, \tag{2.85}$$

and

$$D_a'' + \mathcal{H}(1-3w)D_a' - \left[\tfrac{1}{2}(1-w)(1+3w)a^2\mu + 2Kw\right]D_a - a^2w\tilde{\nabla}^2 D_a = 0, \tag{2.86}$$

respectively for the adiabatic case where $w = const$ and the vorticity is zero. We note further that $\Phi \propto \mu a^2 |D_a|$.

2.6.2 Tensor perturbations

We have also seen that gravitational waves are represented by the equations,

$$\bar{\mathcal{E}}_{ij}'' + 2\mathcal{H}\bar{\mathcal{E}}_{ij}' - \nabla^2\bar{\mathcal{E}}_{ij} = 0, \tag{2.87}$$

where $\bar{\mathcal{E}}_{ij}$ is the tensor perturbation defined in (2.4)and

$$H_{ab}'' + 6\mathcal{H}H_{ab}' + 6\mathcal{H}^2(1-w)H_{ab} - a^2\tilde{\nabla}^2 H_{ab} = 0, \tag{2.88}$$

in the two approaches respectively. At the background level the scale factors a and expansion rates H introduced in each formalism agree, so we keep the same notation. At linear order we have,

$$\sigma_{ij} = a\bar{\mathcal{E}}_{ij}^{(1)\prime}, \tag{2.89}$$

$$E_{ij} = -\tfrac{1}{2}\left(\bar{\mathcal{E}}_{ij}'' + \nabla^2\bar{\mathcal{E}}_{ij}\right), \tag{2.90}$$

$$H_{ij} = \eta_{kl\langle i}\partial^k\bar{\mathcal{E}}_{j\rangle}^{(1)\prime\, l} \equiv (\hat{\mathsf{curl}}\,\bar{\mathcal{E}}^{(1)\prime})_{ij}\,. \tag{2.91}$$

Note that η_{kli} is the completely antisymmetric tensor normalized such that $\eta_{123} = 1$, which differs from ε_{abc}. We shall return to a more detailed term by term comparison in Chapter 6. Detailed comparisons of the metric and the covariant approaches can be found in [84] and [105].

2.6.3 Beyond the Standard Model

In the introduction, we looked at several motivations for going beyond the current best-fit model of cosmology. We shall attempt to do this via perturbation theory about this model. In summary, the desire to go beyond this model are motivated by the need to determine:

- how large perturbations have to be for linear theory to breakdown,
- the accuracy of our linear order approximations through a correct determination of the next higher order correction.
- if CMB anisotropies violate the predicted statistical isotropy and Gaussianity, and
- levels of- and to verify the existence of anomalies in the WMAP data that were recently reported.

There are two main ways of going beyond the linear perturbation of the standard model. One approach is consider linear perturbations of a model which contains the standard model as a subcase, for example the anisotropic models. The second approach is to consider second-order perturbations of about the standard model. This thesis examines both approaches. As has been stated, the primary motivation for second-order theory is the need to analyse non-linear effects which may be important, for example in the inflationary paradigm. Once developed, predictions that arise from the application of the higher-order theory can then be tested against the high precision data coming from the various running or upcoming surveys.

Although the 1+3 formalism given above is elegant and easily applicable to linear perturbations about isotropic and homogeneous models, it requires further specialisation for it to be useful in inhomogeneous and/or anisotropic situations. In the next section, we review the 1+3 *orthonormal* approach and make mention the 1+1+2 approach, two approaches that will allow us to analyse perturbations beyond linear perturbations of the standard model. The 1+3 *orthonormal* frame approach is complimentary to 1+3 covariant approach presented above, but has the advantage that it is easily and more transparently applicable to modelling of dynamical process involving anisotropic and inhomogeneous situations.

2.7 General tetrad formalism

A primary consideration in any modelling procedure, is the issue of the relationship between variables in use and the existence of corresponding situations they represent. In this context, it is important then that one is able to completely determine the metric and connection from variables used in cosmological modelling. The 1+3 covariant equations are however incomplete in this respect. Besides this fact, it turns out that a tetrad or 1+1+2 formalism is what is required for a detailed analysis of anisotropic and inhomogeneous models.

Detailed reviews of the tetrad concepts can be found in [157, 80, 26, 88, 166, 42] while those of 1+1+2 are given in [25, 15]. In our case, we define and make use of orthonormal tetrad based on the normals to the surfaces of homogeneity. The 1+3 covariant equations will then appear as a subset of the resulting tetrad equations. The presentation given here follows the structure and form used by [164] and [165]. The 1+1+2 formalism will be discussed in Chapter 4. Let us begin by defining what a tetrad is. A tetrad is a set of four orthogonal unit basis vector field $\{e_a\}$ which can be written in terms of a local coordinate basis by means of tetrad components $e^{\alpha}_{a}(x^{\beta})$,[2] where

$$e_a = e^{\alpha}_{a}(x^{\beta})\frac{\partial}{\partial x^{\alpha}}, \quad e^{\alpha}_{a} = e_a(x^{\alpha}).$$

One can also define the inverse tetrad component $\mathrm{e}^{a}_{\alpha}(x^{\beta})$. Together the tetrad component and its inverse give,

$$e_a{}^{\alpha} e^{a}{}_{\beta} = \delta^{\alpha}{}_{\beta}. \tag{2.92}$$

A change from one tetrad basis to another involves tensor-like transformations, in particular $\mathbf{e}_a = \bar{\Lambda}^{a'}_{a}(x^{\alpha})\mathbf{e}_{a'}$. The inverse also transforms as follows, $\mathbf{e}_{a'} = \bar{\Lambda}^{a}_{a'}(x^{\alpha})\mathbf{e}_a$. The metric tensor components in the tetrad form are then given by,

$$g_{ab} = g_{\alpha\beta} e_a{}^{\alpha} e_b{}^{\beta} = \mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab}, \tag{2.93}$$

where η_{ab} =diag(-1,1,1,1). These components play a central role in definition of other fundamental quantities. We now turn to these quantities, beginning with commutation relations.

[2] α, β=1,2,3 are the coordinate basis

2.7.1 Commutation functions

From the commutators of the basis vectors, one can define commutation functions $\gamma^{a}{}_{bc}(x^{i})$,by

$$[\mathbf{e}_a, \mathbf{e}_b] = \gamma^{c}_{bc}\mathbf{e}_c. \tag{2.94}$$

In terms of the tetrad components,

$$\gamma^{b}{}_{bc}(x^{i}) = e^{a}{}_{i}e_{b}{}^{j}\partial_{j}e_{c}{}^{i} - e_{c}{}^{j}\partial_{j}e_{b}{}^{i} = -2e_{b}{}^{i}e_{c}{}^{j}\nabla_{[i}e^{a}{}_{j]}. \tag{2.95}$$

The connection components $\Gamma^{a}{}_{bc}$ for the tetrad are defined by the relations,

$$\nabla_{\mathbf{e}_b}\mathbf{e}_a = \Gamma^{c}{}_{ab}\mathbf{e}_c, \tag{2.96}$$

which represent the c-component of the covariant derivatives in the b-direction of the a-vector. This shows that all covariant derivatives can be written out in tetrad components in a way completely analogous to the usual tensor form. An example is

$$\nabla_{a}T_{bc} = \mathbf{e}_a(T_{bc}) - \Gamma^{d}{}_{ba}T_{dc} - \Gamma^{d}{}_{ca}T_{bd}. \tag{2.97}$$

2.7.2 Application in cosmology

For cosmological models, we choose $\mathbf{e}_0$ to be the unit tangent of the matter flow u^a. This fixing implies that the initial six-parameter freedom of using Lorentz transformations has been reduced to a three-parameter freedom of rotations of the spatial frame $\{e_i\}$. The algebraically independent frame components of the space-time connection $\Gamma^{a}{}_{bc}$ can then be split into the set,

$$\begin{aligned}
\Gamma_{i00} &= \dot{u}_i, && (2.98)\\
\Gamma_{i0j} &= \tfrac{1}{3}\Theta\delta_{ij} + \sigma_{ij} - \epsilon_{ijk}\omega^k, && (2.99)\\
\Gamma_{ij0} &= \epsilon_{ijk}\Omega^k, && (2.100)\\
\Gamma_{ijk} &= 2\varpi_{[i}\delta_{j]k} + \epsilon_{kl[i}n^l_{j]} + \tfrac{1}{2}\epsilon_{ijl}n^l{}_k\tfrac{1}{3}\Theta\delta_{ij} + \sigma_{ij} - \epsilon_{ijk}\omega^k. && (2.101)
\end{aligned}$$

Kinematical variables arise from the first two sets.

The rate of rotation of the spatial frame $\{e_i\}$ with respect to Fermi-propagated basis is expressed in the third set by Ω^i. Lastly, the quantities [3]ϖ^i and n_{ij} (symmetric) describe the 9 spatial rotation coefficients. We now list all the evolution and constraint equations in the tetrad formalism. A detailed discussion of this formalism is given by [164], [165] and [43]. From the Ricci identities we get the evolution equations for expansion, vorticity vector and the shear tensor for a perfect fluid.

$$\begin{aligned}
\mathbf{e}_0(\Theta) &= e_i(\dot{u}^i) - \tfrac{1}{3}\Theta^2 + (\dot{u}^i - 2\varpi_i)\dot{u}^i - (\sigma^i{}_j\sigma^j{}_i) + 2\omega_i\omega^i - \tfrac{1}{2}(\mu + 3p) + \Lambda, && (2.102)\\
\mathbf{e}_0(\omega) &= \tfrac{1}{2}\epsilon^{ijk}e_j(\dot{u}^k) - \tfrac{2}{3}\Theta\omega^i - \tfrac{1}{2}n^i{}_j\dot{u}^j - \tfrac{1}{2}\epsilon^{ijk}[\varpi_j\dot{u}_k - 2\omega_j\omega_k] + (\sigma^i{}_j\omega^j), && (2.103)\\
\mathbf{e}_0(\sigma^{ij}) &= \delta^{k\langle i}e_k(\dot{u}^{j\rangle}) - \tfrac{2}{3}\Theta\sigma^{ij} + (\dot{u}^{\langle i} + \varpi^{\langle i})\dot{u}^{j\rangle} - \sigma^{\langle i}{}_k\sigma^{j\rangle k} - \omega^{\langle i}\omega^{j\rangle} - E^{ij}\\
&+ \tfrac{1}{2}\pi^{ij} + \epsilon^{k\delta\langle i}[2\Omega_k\sigma^{j\rangle}{}_\delta - n^{j\rangle}{}_k\dot{u}_\delta], && (2.104)\\
\mathbf{e}_0(\mu) &= -\mathbf{e}_i(q^i) - \Theta(\mu + p) - 2(\dot{u}_i - \varpi_i)q^i - (\sigma^i{}_j\pi^j{}_i), && (2.105)\\
\mathbf{e}_0(q^i) &= -\delta^{ij}\mathbf{e}_j(p) - \mathbf{e}_j(\pi^{ij}) - \tfrac{4}{3}\Theta q^i - \sigma^i{}_jq^j - (\mu + p)\dot{u}^i - (\dot{u}_j - 3\varpi_j)\pi^{ij}\\
&- \epsilon^{ijk}[(\omega_j - \Omega_j)q_j - n_{j\delta}\pi^\delta{}_k], && (2.106)
\end{aligned}$$

[3]We have used ϖ rather than the notation a used in [164].

$$
\begin{aligned}
\mathbf{e}_0(E^{ij}) &= -\tfrac{1}{2}\pi^{ij} + \epsilon^{kl\langle i}\mathbf{e}_k(H^{j\rangle}{}_l) - \tfrac{1}{2}\delta^{k\langle i}\mathbf{e}_k(q^{j\rangle}) - \tfrac{1}{2}(\mu+p)\sigma^{ij} + \Theta(E^{ij} + \tfrac{1}{6}\pi^{ij}) \\
&+ 3\sigma^{\langle i}{}_k(E^{j\rangle k} - \tfrac{1}{6}\pi^{j\rangle k}) + \tfrac{1}{2}n^k{}_k H^{ij} - 3n^{\langle i}{}_k H^{j\rangle k} - \tfrac{1}{2}(2\dot{u}^{\langle i} + \varpi^{\langle i})q^{j\rangle} \\
&+ \epsilon^{kl\langle i}[(2\dot{u}_k - \varpi_k)H^{j\rangle}{}_l + (\omega_k + 2\Omega_k)(E^{j\rangle}{}_l + \tfrac{1}{2}\pi^{j\rangle}{}_l) + \tfrac{1}{2}n^{j\rangle}{}_k q_l],
\end{aligned}
\tag{2.107}
$$

$$
\begin{aligned}
\mathbf{e}_0(H^{ij}) &= -\epsilon^{kl\langle i}\mathbf{e}_k(E^{j\rangle}{}_l + \tfrac{1}{2}\pi^{j\rangle}{}_l) - \Theta H^{ij} + 3\sigma^{\langle i}{}_k H^{j\rangle k} + \tfrac{3}{2}\omega^{\langle i}q^{j\rangle} + \tfrac{1}{2}n^k{}_k(E^{ij} - \tfrac{1}{2}\pi^{ij}) \\
&+ 3\sigma^{\langle i}{}_k(E^{j\rangle k} - \tfrac{1}{2}\pi^{j\rangle k}) + \epsilon^{kl\langle i}[\varpi_k(E^{j}{}_l - \tfrac{1}{2}\pi^{j\rangle}{}_l) - 2\dot{u}_k E^{j\rangle}{}_l + \tfrac{1}{2}\sigma^{j\rangle}{}_k q_l + (\omega_k \\
&+ 2\Omega_k)H^{j\rangle}{}_l].
\end{aligned}
\tag{2.108}
$$

From the Ricci identities and Jacobi identities, the twice contracted Bianchi identities and the EFE 1.4 one finds the following constraint equations,

$$
\begin{aligned}
0 &= (C_1)^i = (\mathbf{e}_j - 3\varpi_j)(\sigma^{ij}) - \tfrac{2}{3}\mathbf{e}_j(\Theta) - n^i{}_j\omega^j + q^j + \epsilon^{ijk}[(\mathbf{e}_j + 2\dot{u}_j \\
&\qquad - \varpi_j)(\omega_k) - n_{jl}\sigma^l{}_k, && (2.109)\\
0 &= (C_2) = (\mathbf{e}_i - \dot{u}_i - 2\varpi_i)(\omega^i), && (2.110)\\
0 &= (C_3)^{ij} = H^{ij} + (\delta^{k\langle i}\mathbf{e}_k + 2\dot{u}^{\langle i} + \varpi^{\langle i})(\omega^{j\rangle}) - \tfrac{1}{2}n^k{}_k\sigma^{ij} + 3n^{\langle i}{}_k\sigma^{j\rangle k} \\
&\qquad - \epsilon^{kl\langle i}[(\mathbf{e}_k - \varpi_k)(\sigma^{j\rangle}{}_l) + n^{j\rangle}{}_k\omega_l], && (2.111)\\
0 &= (C_4)^i = (\mathbf{e}_j - 3\varpi_j)(E^{ij} + \tfrac{1}{2}\pi^{ij}) - \tfrac{1}{3}\delta^{ij}\mathbf{e}_j(\mu) + \tfrac{1}{3}\Theta q^i - \tfrac{1}{2}\sigma^i{}_j q^j \\
&\qquad - 3\omega_j H^{ij} - \epsilon^{ijk}[\sigma_{jl}H^l{}_k - \tfrac{3}{2}\omega_j q_k + n_{jl}(E^l{}_k + \tfrac{1}{2}\pi^l{}_k)], && (2.112)
\end{aligned}
$$

$$
\begin{aligned}
0 &= (C_5)^i = (\mathrm{e}_j - 3\varpi_j)(H^{ij}) + (\mu + p)\omega^i + 3\omega_j(E^{ij} - \tfrac{1}{6}\pi^{ij}) - \tfrac{1}{2}n^i{}_j q^j \\
&\quad + \epsilon^{ijk}[\tfrac{1}{2}(e_j - \varpi_j)(q_k)\sigma_{jl}(E^l{}_k + \tfrac{1}{2}\pi^l{}_k) - n_{jl}H^l{}_k], \qquad (2.113)\\
0 &= (C_j)^i = (\mathrm{e}_j - 2\varpi_j)(n^{ij}) + \tfrac{2}{3}\Theta\omega^i + 2\sigma^i{}_j\omega^j + \epsilon^{ijk}[\mathrm{e}_j(\varpi_k) - 2\omega_j\Omega_k], \qquad (2.114)\\
0 &= (C_G)^{ij} =^3 S^{ij} + \tfrac{1}{3}\Theta\sigma^{ij} - \sigma^{\langle i}{}_k\sigma^{j\rangle k} - \omega^{\langle i}\omega^{j\rangle} + 2\omega^{\langle i}\Omega^{j\rangle} - (E^{ij} + \tfrac{1}{2}\pi^{ij}), \qquad (2.115)\\
0 &= (C_G) =^3 R + \tfrac{2}{3}\Theta^2 - (\sigma^i{}_j\sigma^j{}_i) + 2\omega_i\omega^i - 4(\omega_i\Omega^i) - 2\mu - 2\Lambda, \qquad (2.116)
\end{aligned}
$$

where,

$$
\begin{aligned}
{}^3S_{ij} &= \mathrm{e}_{\langle i}(\varpi_{j\rangle}) + 2b_{\langle ij\rangle} - \epsilon^{kl}{}_{\langle i}(\mathrm{e}_{|k|} - 2\varpi_{|k|})(n_{j\rangle l}), \qquad (2.117)\\
{}^3R &= 2(2\mathrm{e}_i - 3\varpi_i)(\varpi^i) - \tfrac{1}{2}b^i{}_j, \qquad (2.118)\\
b &= n_{k\langle i}n^k{}_{j\rangle} - n^k{}_k n_{ij}. \qquad (2.119)
\end{aligned}
$$

To complete these equations, we also need the evolution equations for the commutation functions ϖ^i and n_{ij}. These arise from the Jacobi identities. They can be arrived at by eliminating the frame derivatives e_i of Θ, σ_{ij} and ω^i. They are,

$$
\begin{aligned}
\mathrm{e}_0(\varpi^i) &= -\tfrac{1}{3}(\Theta\delta^i{}_j - \tfrac{3}{2}\sigma^i{}_j)(\dot{u}^j + \varpi^j) + \tfrac{1}{2}n^i{}_j\omega^j - \tfrac{1}{2}q^i, \\
&- \tfrac{1}{2}\epsilon^{ijk}[(\dot{u}_j + \varpi_j)\omega_k - n_{jl}\sigma^l{}_k - (e_j + \dot{u}_j - 2\varpi_j)\Omega_k] + \tfrac{1}{2}(C_1)^i, \qquad (2.120)\\
\mathrm{e}_0(n^{ij}) &= -\tfrac{1}{3}\Theta n^{ij} - \sigma^{\langle i}{}_k n^{j\rangle k} + \tfrac{1}{2}\sigma^{ij}n^k{}_k - (\dot{u}^{\langle i} + \varpi^{\langle i})(\omega^{j\rangle} - H^{ij}) \qquad (2.121)\\
&+ (\delta^{k\langle i}\mathrm{e}_k + \dot{u}^{\langle i})(\Omega^{j\rangle}) - \tfrac{2}{3}\delta^{ij}[2(\dot{u}_k + \varpi_k)\omega^k - \sigma^k{}_l n^l{}_k + (e_k + \dot{u}_k)(\Omega^k) \\
&- \epsilon^{kl\langle i}[(\dot{u}_k + \varpi_k)\sigma^{j\rangle}{}_l - (\omega_k + 2\Omega_k)n^{j\rangle}{}_l] - \tfrac{2}{3}\delta^{ij}(C_2) + (C_3)^{ij}. \qquad (2.122)
\end{aligned}
$$

These equations generalise the 1+3 covariant equations (2.52-2.66) and provide a basis for characterising particular families of solutions. We employ these equations to complement the 1+3 equations in the analysis of perturbed Bianchi I model in the two next chapters. Tetrads have, for example, been used to study locally rotational symmetric space-times [39, 151] and Bianchi universes in [42, 166].

2.8 Chapter conclusion

This chapter reviewed the basic concepts in linear cosmological perturbation theory. The two most commonly used formalisms; *metric* and *covariant*, were presented and compared in the gauge-invariant framework. We also reviewed the 1+3 orthonormal frame approach, which generalises the 1+3 covariant equations and is suited to the analysis of anisotropic and in-homogeneous situations. With these tools, we are now able to begin analysis of spacetimes that go beyond the standard ΛCDM model.

Chapter 3

Density perturbations of Bianchi I model

In this chapter we consider density perturbations of a Bianchi type I model filled with irrotational dust. In general, scalar, vector and tensor modes are coupled in such models unlike in perturbations of FLRW models. We first examine how to characterize pure density perturbations, then we perform a detailed consistency analysis of the arising covariant equations. Although the analysis is primarily performed in the 1+3 *covariant* formalism, it turns out that the 1+ 3 *orthonormal* tetrad approach is necessary for a complete analysis. The chapter ends with a discussion of the evolution of the inhomogeneity and a presentation of exact solutions for the inhomogeneity variable D_a.

3.1 Introduction

As discussed in the introductory chapter, our understanding of the dynamics of the physical universe comes from the development of ideas which are built on the isotropic and homogeneous models (the FLRW models) and their linear perturbations.

Linear perturbations of anisotropic models were once intensively studied, because it was thought that they could help explain the origin of large scale structures by virtue of their faster growth rate [37, 16, 74, 28, 155]. However, anisotropic models have since become unattractive because they are incompatible with the *inflationary* paradigm in which a number of cosmological questions are answered. In particular, it was shown in [167] that Bianchi models with a positive cosmological constant isotropize at late times implying that inflationary era leads to the decay of anisotropies. As was pointed out in Chapter 1, current observation suggest that the CMB is statistically isotropic and Gaussian. However, some anomalies have been discovered in the analysis of both the first and the third-year WMAP data [46]. In particular, one such anomaly hints at the existence of a feature in the CMB that statistically picks out a preferred direction [45,59].

Statistical anisotropy and non-Gaussianity are closely linked, in isotropic Gaussian processes, the power C_l of each realisation is randomly distributed among all multipoles. A preferred direction is said to exist, if one can identify an axis $\mathbf{n}$ about which there is an over concentration of power, when there is an alignment between the $\mathbf{n}$ and the z-axis. In general this m-preference, which is a measure of anisotropy, does not survive realignment. It was found in [96], that although there is such an alignment for low multipoles $l = 2, .., 5$ (and to some extend a preferred Cartesian axis) this does not pick a specific m. Hence there is no preference for particular shapes, which suggests that any model with a preferred axis could be responsible. In order to pin down the model we therefore have to consider models that go beyond the standard- homogeneous and isotropic model. Models such as Bianchi I, which have different rates of expansions in the three spatial directions therefore make for good candidates for investigation. Analysis of linear density perturbations in Bianchi models may therefore prove invaluable in unravelling the so-called *axis of evil*. Previous studies of anisotropic models include [60, 159, 124, 166], however as yet no self-consistent presentation of pure density perturbation has been given using the 1+3 covariant approach.

3.2 Bianchi I model

Bianchi type I models have a metric of the form,

$$ds^2 = -dt^2 + \alpha^2(t)dx^2 + \beta^2(t)dy^2 + \gamma^2(t)dz^2, \quad u^a = \delta^a{}_0. \tag{3.1}$$

The spatial sections are mutually orthogonal and have different expansion scale factors. The effective scale factor for the model is given by the average expansion scale factor $a(t) = (\alpha\beta\gamma)^{1/3}$. It is clear that FLRW is a subset of this model, which follows when the three scale factors are equal. The spatial sections of constant time in Bianchi I models are flat and all the invariants depend only on the time coordinate. The fluid flow, which is orthogonal to the spatial surfaces, is geodesic and irrotational. These models are therefore covariantly characterized by,

$$0 = \dot{u}^a = \omega^a, \quad 0 = \tilde{\nabla}_a\mu = \tilde{\nabla}_a\Theta = \tilde{\nabla}_a p, \quad 0 =^3 R_{ab}, \tag{3.2}$$

where the variables have the definition given in Chapter 2. Equivalently, these models are characterized in the tetrad formalism (Chapter 2) by

$$\begin{aligned} 0 &= \dot{u}^j = \omega^i = \Omega^i, \quad 0 = \varpi^i = n_{ij}, \mathbf{e}_i(\Theta) = \mathbf{e}_i(\sigma_{jk}), & (3.3)\\ 0 &= e_i(\mu) = \mathbf{e}_i(p), \quad 0 =^3 R_{ij} \Rightarrow^3 R = 0 \Rightarrow^3 S_{ij} = 0, & (3.4) \end{aligned}$$

where a tetrad can be chosen as follows ($e^i{}_1 = \alpha(t)^{-1}\delta_1{}^i$,$e^i{}_2 = \beta(t)^{-1}\delta_2{}^i$ and $e^i{}_3 = \gamma(t)^{-1}\delta_3{}^i$). The substitution of these conditions into the tetrad equations (2.102-2.115) generate the desired tetrad equations describing the Bianchi I model. It is clear from equation (2.111) that $H_{ij} = 0$ and therefore $\mathbf{e}_j(E^{ij}) = 0$, given the above conditions. Similar conclusions arise in the 1+3 covariant formalism.

Bianchi I models can also be thought of as homogeneous perturbations about FLRW models. This is because in the 1+3 formalism, the concept of perturbation involves a *top-*

down approach rather than a *bottom-up* approach as is the case in the metric formalism. This is to say, that one begins with the set of full nonlinear covariant equations (2.52-2.61), which are then linearized about a background of choice. The Bianchi I background has two extra terms when compared with the FLRW background. These extra terms are the shear tensor and the electric part of the Weyl tensor. Therefore, from the full nonlinear equations, one can first linearize about a Bianchi I background, which can further be linearised about FLRW background. This makes the Bianchi I model an intermediate linearization state, between the full nonlinear and the FLRW equations. For this reason, we consider the study of linear perturbations about Bianchi I models as a first step towards a full non-linear description of perturbation about FLRW models.

3.2.1 Pressure- and vorticity- free Bianchi I model

The background variables in this model are μ, Θ, σ_{ab} and E_{ab}. We begin by applying the linearization procedure discussed in Chapter 2, to the full set of covariant equations (2.52, 2.53,2.54,2.55, 2.56, 2.57, 2.58, 2.59 and 2.61). Taking vorticity to be zero, the linearization process leads to the following propagation equations,

$$\dot{\mu} = -\Theta\mu, \tag{3.5}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} - \tfrac{1}{2}\mu, \tag{3.6}$$

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c\langle a}\sigma_{b\rangle}{}^{c} - E_{ab}, \tag{3.7}$$

$$\dot{E}_{ab} = \mathsf{curl}\, H_{ab} - \Theta E_{ab} + 3\sigma_{c\langle a}E_{b\rangle}{}^{c} - \tfrac{1}{2}\mu\sigma_{ab}, \tag{3.8}$$

$$\dot{H}_{ab} = -\mathsf{curl}\, E_{ab} - \Theta H_{ab} + 3\sigma_{c\langle a}H_{b\rangle}{}^{c}, \tag{3.9}$$

and the constraints equations

$$
\begin{aligned}
0 &= \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta, && (3.10)\\
0 &= \mathsf{curl}\,\sigma_{ab} - H_{ab}, && (3.11)\\
0 &= \tilde{\nabla}^b E_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\mu - \varepsilon_{abc}\sigma^b{}_d H^{cd}, && (3.12)\\
0 &= \tilde{\nabla}^b H_{ab} + \varepsilon_{abc}\sigma^b{}_d E^{cd}. && (3.13)
\end{aligned}
$$

Since we are interested in pure density perturbations, we need to subject the above covariant equations to further restrictions that will isolate density perturbations. In particular, we need to eliminate tensor perturbations. The process of how this is achieved is the focus of the next section.

3.3 Pure density perturbations

Because the background is anisotropic but homogeneous, all first-order gauge-invariant vector may be split into a *curl-free* and *divergence-free* part, usually referred to as scalar and vector parts respectively, which we write as

$$V_a = V_{\mathsf{S}a} + V_{\mathsf{V}a}, \qquad (3.14)$$

where $\mathsf{curl}\, V_{\mathsf{S}a} = 0$ and $\mathsf{div} V_{\mathsf{V}} = 0$. Similarly, any tensor may be invariantly split into scalar, vector and tensor parts:

$$T_{ab} = T_{\mathsf{S}ab} + T_{\mathsf{V}ab} + T_{\mathsf{T}ab}, \qquad (3.15)$$

where $\mathsf{curl}\, T_{\mathsf{S}ab} = 0$, $\mathsf{divdiv} T_{\mathsf{V}} = 0$ and $\mathsf{div} T_{\mathsf{T}a} = 0$. It follows therefore that in the above constraint equations we can separately equate scalar, vector and tensor parts and obtain equations that characterize the evolution of each type of perturbation. Perturbations about Bianchi I background include, the spatial gradients of the background variables and the magnetic part of the Weyl tensor. Apart from the magnetic part of the Weyl tensor which can be split into vector and tensor parts, all other perturbations can be split into scalar, vector and tensor parts. Now consider the case where only scalars modes are excited at first order. This is equivalent to setting $H_{ab} = 0$. It follows that pure density perturbations may be characterized by the conditions,

$$\tilde{\nabla}^b \mu \neq 0, \tilde{\nabla}^b \Theta \neq 0, \tilde{\nabla}^b \sigma^2 \neq 0, H_{ab} = 0, \tag{3.16}$$

where only the scalar part of the gradients are considered. These conditions imply that there is no information exchange via gravitational waves (characterized by H_{ab}) or via sound (with p=0). Models with these conditions are called *Silent universes* and have been analysed in [140, 104]. Now that we have a way of characterising pure density perturbations, we can proceed to determine how this characterisation affects the flow of fluid in this model. Note that these conditions generate the following set of constraint equations,

$$(C_1)_a = \tilde{\nabla}^b \sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a \Theta = 0, \tag{3.17}$$

$$(C_3)_{ab} = \mathrm{curl}\, \sigma_{ab} = 0, \tag{3.18}$$

$$(C_4)_a = \tilde{\nabla}^b E_{ab} - \tfrac{1}{3}\tilde{\nabla}_a \mu = 0, \tag{3.19}$$

$$(C_5)_a = \varepsilon_{abc}\sigma^b{}_d E^{cd} = 0, \tag{3.20}$$

$$(C_6)_{ab} = \mathrm{curl}\, E_{ab} = 0. \tag{3.21}$$

C_2 is the divergence of vorticity constraint. Let the collective notations of these equations be

$$(C_A) = \{\tilde{\nabla}^b \sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a \Theta, \mathsf{curl}\, \sigma_{ab} - H_{ab}, ..\}, \tag{3.22}$$

a notation borrowed from [109], where $A = 1...6$. Since these constraints express the nature of relationships between variables, say at some fixed time t_0, it is of interest to know if such relationships are preserved with the passage of time. Time here is measured relative to the observer worldline. Ideally, we need to find out if $\dot{C}_A = \mathcal{F}_A(C_B)$, where $\mathcal{F}_A$ do not contain time derivatives. In other words, can the propagation of each constraint be expressed in terms of the original constraints? We perform this check in the next section.

3.4 Evolving the constraints

First consider constraint (3.17). Taking the time derivative of this constraint, making use of the commutation relations (A.7, A.8, A.3), the evolution equations (4.13, 4.14), and the constraint equations (3.17, 3.18 and 3.19), we find

$$(\dot{C}_1)_a + \Theta(C_1)_a + (C_4)_a - 2\epsilon_a{}^{bc}\sigma_b{}^d(C_3)_{cd} = 0. \tag{3.23}$$

Taking the time derivative of (3.18), making substitutions using the propagation equation (4.14), the constraint equations (3.17, and 3.18), and the commutation relations (A.5 and A.11), we obtain

$$(\dot{C}_3)_{ab} + \Theta(C_3)_{ab} + \epsilon^{cd}{}_{(a}\sigma_{b)c}(C_1)_d \;\; = \;\; 0. \tag{3.24}$$

Propagating (3.19), making use of the evolution equations (4.12, 4.14 and 4.15), the constraints equations (3.17, 3.19 and 3.20), and the commutation relations (A.3 and A.7) yield,

$$(\dot{C_4})_a + \tfrac{4}{3}\Theta(C_4)_a - \tfrac{3}{2}E_a{}^b C_{1b} - \tfrac{1}{2}\sigma_a{}^b(C_4)_b + \tfrac{1}{2}\mu(C_1)_a - \tfrac{1}{2}\mathsf{curl}\,(C_5)_a \;=\; \zeta_a^{(3)}, \qquad (3.25)$$

where $\zeta_a^{(3)}$ is given below. As for constraint (3.20), we require the propagation equations (4.14, 4.15 and 3.20). These yield,

$$(\dot{C_5})_a + \tfrac{5}{3}\Theta(C_5)_a \;=\; \zeta_a^{(4)}, \qquad (3.26)$$

where $\zeta_a^{(4)}$ is given below. Propagating equation (3.21), making use of the identity (A.11), the propagation equation (4.15), and the constraint equations (3.18, 3.20 and 3.21), it can be shown that,

$$(\dot{C_6})_{ab} + \tfrac{4}{3}\Theta(C_6)_{ab} - \tfrac{3}{2}(C_1)_{c(a}E_{b)}{}^c - \tfrac{3}{2}(C_4)_{c(a}\sigma_{b)}{}^c + \tfrac{1}{2}\mu(C_3)_{ab} \;=\; \zeta_{ab}^{(5)}, \qquad (3.27)$$

where the source terms to equations (3.25-3.27) are given by,

$$\zeta^{(3)}{}_a \;=\; -\epsilon_a{}^{bc}\left(E_b{}^d(C_3)_{cd} - \sigma^d{}_b(C_6)_{cd}\right), \qquad (3.28)$$

$$\zeta^{(4)}{}_a \;=\; -\varepsilon_{abc}\sigma_e{}^{\langle b}\sigma^{d\rangle e}E^c{}_d - \varepsilon_{abc}\sigma^b{}_d\sigma_e{}^{\langle c}E^{d\rangle e}, \qquad (3.29)$$

$$\zeta^{(5)}{}_{ab} \;=\; 3\mathsf{curl}\,(\sigma^c_{\langle a}E_{b\rangle c}) - \varepsilon^{cd}{}_{(a}\left[\sigma^e{}_c\tilde{\nabla}_{|e|}E_{b)d} - \tfrac{3}{2}E_{b)d}\tilde{\nabla}_e\sigma^e{}_c - \tfrac{3}{2}\sigma_{b)d}\tilde{\nabla}_e E^e_c\right]. \qquad (3.30)$$

Using the propagation equation (4.14), it can be shown that $\zeta_a^{(4)}$ takes the form,

$$\zeta^{(4)}{}_a \;=\; \varepsilon_{abc}[\sigma^b{}_d\sigma_e{}^{\langle c}\dot{\sigma}^{d\rangle e} - \sigma_e{}^{\langle c}\sigma^{d\rangle e}\dot{\sigma}^b{}_d + \sigma^b{}_d\sigma_e{}^{\langle c}\sigma_f{}^{\langle d\rangle}\sigma^{e\rangle f} - \sigma_e{}^{\langle c}\sigma^{d\rangle e}\sigma_f{}^{\langle b}\sigma_{d\rangle}{}^f]. \qquad (3.31)$$

In general, the terms in this equation identically cancel. Again using equation (4.14), the electric part of the Weyl tensor can be eliminated from $\zeta^{(5)}_{ab}$. To show that this term vanishes we turn to tetrad formalism. In particular, we can find a set equivalent to the perturbed Bianchi I model with dust equation of state and with vanishing vorticity.

It follows from equation (3.20), that one can choose an eigenframe in which both the shear and the electric part of the Weyl tensor are diagonal. It was shown in [9] that this implies

$$0 = n_{11} = n_{22} = n_{33}, \tag{3.32}$$

which can be seen from the diagonal components of constraint (3.21) and evolution equation (4.16). It was also shown that the eigenframe $\mathbf{e}_i$ of σ_{ij} and E_{ij} are Fermi-transported a long $\mathbf{u}$, which implies,

$$\Omega^i = 0.$$

This arises from the vanishing of the off-diagonal components of the evolution equations of both the electric part of the Weyl and the shear tensors. Together with the vanishing vorticity, these conditions imply that $\mathbf{e}_i$ is spanned by four hypersurface-orthogonal basis fields. This means that we can find local co-ordinates on the spacetime with respect to which the metric tensor field $\mathbf{g}$ is diagonal [76].

We note that the evolution of constraints in the 1+3 formalism given in the previous section can be presented in an equivalent form in the tetrad formalism. However to avoid duplicity, we only consider the constraint (3.21) and the source term arising from its propagation (3.30). It follows that (3.21) is equivalent to the three tetrad equations,

$$0 = \tfrac{\sqrt{3}}{2}(\mathbf{e}_1 - \varpi_1)(E_{22} - E_{33}) - \tfrac{3\sqrt{3}}{2} n_{23}(E_{22} + E_{33}), \tag{3.33}$$

$$0 = (\mathbf{e}_2 - \varpi_2)(E_{33} + 2E_{22}) + 3n_{31}E_{22}, \tag{3.34}$$

$$0 = (\mathbf{e}_3 - \varpi_3)(2E_{33} + E_{22}) + 3n_{12}E_{33}. \tag{3.35}$$

These equations are equivalent to equations (64, 65 and 66) of [76].

The propagation of these equations yield three source terms equivalent to the single equation (3.30) and which in tetrad formalism are,

$$0 = \mathbf{e}_1(\Theta)E_- + \tfrac{1}{2}(\sigma_-)\mathbf{e}_1(\mu) - 2(\varpi_1\sigma_- + \sqrt{3}n_{23}\sigma_+)E_+ - 2(\varpi_1\sigma_+ + \tfrac{1}{\sqrt{3}}n_{23}\sigma_-)E_-, \tag{3.36}$$

$$\begin{aligned} 0 &= \mathbf{e}_2(\Theta)(E_+ - \tfrac{1}{\sqrt{3}}E_-) + (\tfrac{1}{2}\sigma_+ - \tfrac{1}{2\sqrt{3}}\sigma_-)\mathbf{e}_2(\mu) + 2(\varpi_2 - n_{31})(\sigma_+ + \tfrac{1}{\sqrt{3}}\sigma_-)E_+ \\ &+ \varpi_2(\tfrac{2}{\sqrt{3}}\sigma_+ - 2\sigma_-)E_- - n_{31}(\tfrac{2}{\sqrt{3}}\sigma_+ + \tfrac{10}{3}\sigma_-)E_-, \end{aligned} \tag{3.37}$$

$$\begin{aligned} 0 &= \mathbf{e}_3(\Theta)(E_+ + \tfrac{1}{\sqrt{3}}E_-) + (\tfrac{1}{2}\sigma_+ + \tfrac{1}{2\sqrt{3}}\sigma_-)\mathbf{e}_3(\mu) + 2(\varpi_3 + n_{12})(\sigma_+ - \tfrac{1}{\sqrt{3}}\sigma_-)E_+ \\ &- \varpi_3(\tfrac{2}{\sqrt{3}}\sigma_+ + 2\sigma_-)E_- - n_{12}(\tfrac{2}{\sqrt{3}}\sigma_+ - \tfrac{10}{3}\sigma_-)E_-, \end{aligned} \tag{3.38}$$

where we have used trace-free-adapted irreducible frame components for σ_{ab} and E_{ab} defined by

$$\sigma_+ : = -\tfrac{1}{2}\sigma_{11} = \tfrac{1}{2}(\sigma_{22} + \sigma_{33}), \quad \sigma_- := \tfrac{1}{2\sqrt{3}}(\sigma_{22} - \sigma_{33}), \tag{3.39}$$

$$E_+ : = -\tfrac{1}{2}E_{11} = \tfrac{1}{2}(E_{22} + E_{33}), \quad E_- := \tfrac{1}{2\sqrt{3}}(E_{22} - E_{33}). \tag{3.40}$$

We note that, if $\sigma- = \sigma_{22} - \sigma_{33} = 0 \Rightarrow E_- = E_{22} - E_{33} = 0$ (i.e degenerate) is identically satisfied. In this case, (3.37) and (3.38) reduce to,

$$0 = E_+\mathbf{e}_2(\Theta) + \tfrac{1}{2}\sigma_+\mathbf{e}_2(\mu) + 2(\varpi_2 - n_{31})\sigma_+E_+, \tag{3.41}$$

$$0 = E_+\mathbf{e}_3(\Theta) + \tfrac{1}{2}\sigma_+\mathbf{e}_3(\mu) + 2(\varpi_2 + n_{12})\sigma_+E_+. \tag{3.42}$$

As was shown by [76], from constraint equations (2.102-2.116), one can also derive the following equations,

$$0 = \mathbf{e}_2(\Theta) + (\varpi_2 - n_{31})\sigma_+, \tag{3.43}$$

$$0 = \mathbf{e}_3(\Theta) + (\varpi_3 + n_{12})\sigma_+, \tag{3.44}$$

$$0 = \tfrac{1}{2}\mathbf{e}_2(\mu) + (\varpi_2 - n_{31})\sigma_+, \tag{3.45}$$

$$0 = \tfrac{1}{2}\mathbf{e}_2(\mu) + (\varpi_2 + n_2)\sigma_+. \tag{3.46}$$

These equations show that (3.41) and (3.42) are identically satisfied. Putting everything together, it follows that ${\zeta^5}_{ab}$ is satisfied when the background shear is degenerate (i.e. $\sigma_{22} = \sigma_{33}$), which leads to the closure of the set of evolution equations for the constraints. It can be demonstrated that LRS yields similar results. Note that, this shows that one can consistently decouple density perturbation from gravitational waves if the background is degenerate. This was first shown by Perko et al (see page 976 in [61]).

The first presentation of a detailed analysis of constraints equations in the $'1 + 3'$ formalism was given in [109], for the case of general nonlinear perturbations in FLRW models with irrotational dust. A detailed consistency analysis for the barotropic perfect fluid case was given in [162]. Having successfully shown that the constraint equation for a characterisation of pure density perturbations evolve consistently (with respect to the propagation equations), we are now in a position to analyse the evolution of these perturbations.

3.5 Evolution of inhomogeneity

We follow the standard approach, where inhomogeneity variables are given by the comoving spatial gradients. In the case of Bianchi I model, these are the spatial gradients of the energy density, the rate of expansion and the shear-scalar. Note that these inhomogeneity variables are constrained by the spatial gradient of the Ricci scalar for the 3-surface, as can be seen from the generalised Friedmann equation (2.68), i.e.

$$a\tilde{\nabla}^3_a R = -\tfrac{4}{3}\Theta a\tilde{\nabla}_a\Theta + 2a\tilde{\nabla}_a\sigma^2 + 2a\tilde{\nabla}_a\mu, \tag{3.47}$$

where ${}^3R$ is the 3-Ricci scalar, $a = a(t)$ is the scale factor and $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$. The evolution of equation (3.47) governs the growth of inhomogeneity. We can now define variables which represent each of the terms on the right hand side of this equation. We denote the co-moving density gradient, the co-moving spatial derivatives expansion and the shear by,

$$D_a = a\tilde{\nabla}_a\mu/\mu,\ Z_a = a\tilde{\nabla}_a\Theta,\ T_a = a\tilde{\nabla}_a\sigma, \tag{3.48}$$

respectively. The gradient of the generalised Friedmann equation in terms of the new variables becomes

$$R_a = -\tfrac{4}{3}\Theta Z_a + 4\sigma T_a + 2\mu D_a, \tag{3.49}$$

where $R_a = a\tilde{\nabla}^3_a R$.

Since the 3-Ricci scalar is important, it follows that we also need to examine the 3-Ricci tensor. In particular, the traceless part of this tensor which is denoted by S_{ab}, takes the form

$${}^{(3)}S_{bc} = {}^{(3)}R_{bc} - \tfrac{1}{3}h_{bc}{}^{(3)}R = -\dot{\sigma}_{bc} - \Theta\sigma_{bc}. \tag{3.50}$$

This variable evolves as

$$\dot{S}_{bc} + \tfrac{2}{3}\Theta S_{bc} + \tfrac{1}{6}R\sigma_{bc} - S_{d(b}\sigma_{c)}{}^d = 3[\sigma^2\sigma_{bc} - \sigma^d{}_{(b}\sigma_{d)}{}^e\sigma_{de}]. \tag{3.51}$$

It is straight forward to show that the rhs vanishes if the background is LRS. Notice that if ${}^{(3)}R_{bc}$ is isotropic ${}^{(3)}R_{bc} = \frac{1}{3}h^{(3)}_{bc}R \Rightarrow S_{bc} = 0$. This has the implication that S_{bc} is first order as it vanishes in the background. Using the traceless part of the 3-Ricci tensor, we define the auxiliary variable,

$$S_a \;=\; a\tilde{\nabla}_a S = a\tilde{\nabla}_a(\sigma^{bc}S_{bc}), \tag{3.52}$$

which is also a measure of inhomogeneity. We now examine how these variables evolve.

3.5.1 Evolution of the new variables

Taking the time derivative of (3.48), using equation (A.3), we obtain the following evolution equation,

$$\dot{D}_a + \sigma_a{}^b D_b + Z_a = 0. \tag{3.53}$$

The propagation of (3.48) requires (A.3), (3.48), (3.48) and (3.48). The net result is the evolution equation,

$$\dot{Z}_a + \tfrac{2}{3}\Theta Z_a + 4\sigma T_a + \sigma_a{}^b Z_b + \tfrac{1}{2}\mu D_a = 0. \tag{3.54}$$

Propagation of (3.48) requires (3.48), (3.48), (3.48) and (3.52) and reads,

$$2\sigma\dot{T}_a + \Theta\sigma T_a + 2\sigma\sigma_a{}^b T_b + 2\sigma^2 Z_a + S_a = 0. \tag{3.55}$$

This equation involves the new variable S_a. The evolution of this variable requires (3.52) and (A.3), leading to

$$\dot{S}_a + \tfrac{5}{3}\Theta S_a + \tfrac{1}{3}R_a\sigma^2 + \sigma_a{}^b S_b + \tfrac{2\sigma}{\sqrt{3}}S_a = 0. \tag{3.56}$$

The system is closed as it stands since R_a can be written in terms of Z_a, T_a and D_a. For completeness, we note that R_a obeys the following evolution equation,

$$\dot{R}_a + \tfrac{2}{3}\Theta R_a + \sigma_a{}^b R_b + 2S_a = 0. \tag{3.57}$$

The system given by equations (3.57-3.56) can be expressed in terms of D_a, R_a and S_a. We have four equation in four unknowns. However, it turns out that we can express this system in a more compact manner without loosing any information. Notice that using equation (3.49), T_a can be eliminated from equation (3.54) leaving,

$$\dot{Z}_a + 2\Theta Z_a + R_a + \sigma_a{}^b Z_b - \tfrac{3}{2}\mu D_a = 0. \tag{3.58}$$

This leads to the elimination of (3.55) from the system. Also notice, that using the time derivative of (3.53) and equations (3.53) and (3.58), we can eliminate Z_a as well from the system. The system now takes the form,

$$\ddot{D}_a + 2\Theta\dot{D}_a + 2\sigma_a{}^b\dot{D}_b + (\Theta\sigma_a{}^b + \sigma_a{}^c\sigma_c{}^b)D_b + \tfrac{3}{2}\mu D_a - R_a = 0, \tag{3.59}$$

$$\dot{S}_a + \tfrac{5}{3}\Theta S_a + \tfrac{1}{3}R_a\sigma^2 + \sigma_a{}^b S_b + \tfrac{2\sigma}{\sqrt{3}}S_a = 0, \tag{3.60}$$

$$\dot{R}_a + \tfrac{2}{3}\Theta R_a + \sigma_a{}^b R_b + 2S_a = 0. \tag{3.61}$$

We choose a shear eigenframe of the form

$$diag\sigma_{ab} = (-\tfrac{2}{\sqrt{3}}\sigma, \tfrac{1}{\sqrt{3}}\sigma, \tfrac{1}{\sqrt{3}}\sigma), \tag{3.62}$$

where the fluid is taken to shear in the x^1direction. We can now analyse the evolution of these inhomogeneities in the three spatial directions, keeping in mind that we have a preferred direction (x^1). We first consider the coefficients for in the system given by equations (3.59-3.61) and which are made up of background variables. It is clear that these background variables are coupled via their evolution equations.

The background is flat, therefore taking the time derivative of (2.68), using (4.12-4.14) and keeping only the background terms (R is first order since the background is flat), it follows that,

$$\dot{\sigma} = -\Theta\sigma, \tag{3.63}$$

where $\sigma^2 = \sigma_{ab}\sigma^{ab}/2$. It is useful to define a dimensionless expression (i.e $\Sigma = \sigma/(H\sqrt{3})$), where H is the usual Hubble parameter. It is also convenient to define a dimensionless time variable τ, where $d\tau = Hdt$. The derivative with respect to the τ is denoted using a prime[1], while the covariant time is denoted by a dot. It can be shown that

$$\Sigma' = -\tfrac{3}{2}(1-\Sigma^2)\Sigma. \tag{3.64}$$

The fixed points are $\Sigma = 0, \pm 1$. The zero value represents a matter dominated regime that correspond to the FLRW universe, while the non-zero values represent shear dominated regimes which correspond to the Kasner universes. We can now recast the system into one with derivatives with respect to τ. It follows that we also need the following Hubble-normalised variables; $\mathcal{D}_a = D_a, \mathcal{R}_a = R_a/H^2, \mathcal{S}_a = S_a/H^3$. The system then takes the form,

$$\mathcal{D}_a'' + \mathcal{D}_a'(\tfrac{9}{2} - \tfrac{3}{2}\Sigma^2) + 2\tfrac{\sigma_a{}^b}{H}\mathcal{D}_b' + (\tfrac{9}{2} - \tfrac{9}{2}\Sigma^2)\mathcal{D}_a + (3\tfrac{\sigma_a{}^b}{H} + \tfrac{\sigma_a{}^c\sigma_c{}^b}{H^2})\mathcal{D}_b - \mathcal{R}_a = 0, \tag{3.65}$$

$$\mathcal{R}_a' - (1+3\Sigma^2)\mathcal{R}_a + \tfrac{\sigma_a{}^b}{H}\mathcal{R}_b + 2\mathcal{S}_a = 0, \tag{3.66}$$

$$\mathcal{S}_a' + (\tfrac{1}{2} + 2\Sigma - \tfrac{9}{2}\Sigma^2)\mathcal{S}_a + \tfrac{\sigma_a{}^b}{H}\mathcal{S}_b + \mathcal{R}_a\Sigma^2 = 0. \tag{3.67}$$

In order to solve the above generic system, we need to resolve the terms $\sigma_a{}^b/H$. Implicit in this analysis is the fact that the background shear is diagonal and is given by (3.62). It follows that $\mathrm{diag}(\sigma_a{}^b/H) = (-2\Sigma, \Sigma, \Sigma)$, and are determined by the values for the shear fixed points. In the next section we solve for $\mathcal{D}_1$ and $\mathcal{D}_2$ for each of the shear fixed-point.

[1]This should not be confused with the use of prime elsewhere where it represents derivative with respect to conformal time.

3.5.2 Positive shear fixed-point

When $\Sigma = +1$, the above system reduces to,

$$\mathcal{D}_a'' + 3\mathcal{D}_a' + 2\tfrac{\sigma_a{}^b}{H}\mathcal{D}_b' + (3\tfrac{\sigma_a{}^b}{H} + \tfrac{\sigma_a{}^c\sigma_c{}^b}{H^2})\mathcal{D}_b - \mathcal{R}_a = 0, \tag{3.68}$$

$$\mathcal{R}_a' - 4\mathcal{R}_a + \tfrac{\sigma_a{}^b}{H}\mathcal{R}_b + 2\mathcal{S}_a = 0, \tag{3.69}$$

$$\mathcal{S}_a' - 2\mathcal{S}_a + \tfrac{\sigma_a{}^b}{H}\mathcal{S}_b + \mathcal{R}_a = 0, \tag{3.70}$$

which can then be analysed for the various choices of the index a. We note that there are two principle directions, these are denoted by x^1 and x^2 respectively.

Solutions for D_1 and D_2 when $\Sigma = +1$

When $a = 1$, $\sigma_a{}^b/H = -2\Sigma = -2$, while $\sigma_a{}^b/H = \Sigma = 1$ when $a = 2$. In both cases $\Sigma = +1$. It can be shown easily that the system forms a fourth-order differential equation in $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. The solutions to these are,

$$D_1 = \mathcal{D}_1 = d_1^{(+)}e^{-\tau} + d_2^{(+)}e^{2\tau} + d_3^{(+)}e^{(5+\sqrt{3})\tau} + d_4^{(+)}e^{(5-\sqrt{3})\tau} \tag{3.71}$$

$$D_2 = \mathcal{D}_2 = \mathcal{A}_1^{(+)}e^{-\tau} + \mathcal{A}_2^{(+)}e^{-4\tau} + \mathcal{A}_3^{(+)}e^{(2+\sqrt{3})\tau} + \mathcal{A}_4^{(+)}e^{(2-\sqrt{3})\tau}, \tag{3.72}$$

where the coefficients $d_1^{(+)}, d_2^{(+)}, d_3^{(+)}, d_4^{(+)}, \mathcal{A}_1^{(+)}, \mathcal{A}_2^{(+)}, \mathcal{A}_3^{(+)}$ and $\mathcal{A}_4^{(+)}$ are set by initial conditions. It can be shown that $a(t) \propto t^{\frac{1}{3}}$, by solving the background evolution equations. Since $a(t) \propto e^{\tau} \propto t^{\frac{1}{3}}$,it follows that,

$$D_1 = d_1^{(+)}t^{-\frac{1}{3}} + d_2^{(+)}t^{\frac{2}{3}} + d_3^{(+)}t^{\frac{(5+\sqrt{3})}{3}} + d_4^{(+)}t^{\frac{(5-\sqrt{3})}{3}} \tag{3.73}$$

$$D_2 = \mathcal{A}_1^{(+)}t^{-\frac{1}{3}} + \mathcal{A}_2^{(+)}t^{-\frac{4}{3}} + \mathcal{A}_3^{(+)}t^{\frac{(2+\sqrt{3})}{3}} + \mathcal{A}_4^{(+)}t^{\frac{(2-\sqrt{3})}{3}}. \tag{3.74}$$

3.5.3 Negative shear fixed-point

When $\Sigma = -1$ the system reduces to,

$$\mathcal{D}_a'' + 3\mathcal{D}_a' + 2\tfrac{\sigma_a{}^b}{H}\mathcal{D}_b' + (3\tfrac{\sigma_a{}^b}{H} + \tfrac{\sigma_a{}^c\sigma_c{}^b}{H^2})\mathcal{D}_b - \mathcal{R}_a = 0, \tag{3.75}$$

$$\mathcal{R}_a' - 4\mathcal{R}_a + \tfrac{\sigma_a{}^b}{H}\mathcal{R}_b + 2\mathcal{S}_a = 0, \tag{3.76}$$

$$\mathcal{S}_a' - 6\mathcal{S}_a + \tfrac{\sigma_a{}^b}{H}\mathcal{S}_b + \mathcal{R}_a = 0. \tag{3.77}$$

The solutions to the the system in the two principle directions can be found in a process similar to the one used when $\Sigma = +1$.

Solutions for D_1 and D_2 when $\Sigma = -1$

The two principle directions are given by a=1 and a=2 respectively. Since the background shear is diagonal $\sigma_a{}^b/H = 2\Sigma = 2$ for a=1 and $\sigma_a{}^b/H = \Sigma = -1$ for a=2, given that $\Sigma = -1$. It follows that the system again forms a fourth-order differential equation in $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. The solutions to these are,

$$\mathcal{D}_1 = d_1^{(-)}e^{-2\tau} + d_2^{(-)}e^{-5\tau} + d_3^{(-)}e^{(3+\surd 3)\tau} + d_4^{(-)}e^{(3-\surd 3)\tau} \tag{3.78}$$

$$\mathcal{D}_2 = \mathcal{A}_1^{(-)}e^{\tau} + \mathcal{A}_2^{(-)}e^{-2\tau} + \mathcal{A}_3^{(-)}e^{(6+\surd 5)\tau} + \mathcal{A}_4^{(-)}e^{(6-\surd 5)\tau}. \tag{3.79}$$

As in the previous case, these solutions can be expressed in proper time as follows,

$$\mathcal{D}_1 = d_1^{(-)}t^{-\frac{2}{3}} + d_2^{(-)}t^{-\frac{5}{3}} + d_3^{(-)}t^{\frac{(3+\surd 3)}{3}} + d_4^{(-)}t^{\frac{(3-\surd 3)}{3}} \tag{3.80}$$

$$\mathcal{D}_2 = \mathcal{A}_1^{(-)}t^{\frac{1}{3}} + \mathcal{A}_2^{(-)}t^{-\frac{2}{3}} + \mathcal{A}_3^{(-)}t^{\frac{(6+\surd 5)}{3}} + \mathcal{A}_4^{(-)}t^{\frac{(6-\surd 5)}{3}}. \tag{3.81}$$

3.5.4 The zero shear fixed-point

This is the matter dominated (or the FLRW regime). In this case $\Sigma = 0$, following which the first equation in the generic system (3.65-3.66) decouples. The relevant differential equation is the second-order equation that reads,

$$\mathcal{D}_a'' + \tfrac{1}{2}\mathcal{D}_a' - \tfrac{3}{2}\mathcal{D}_a = 0,$$

and whose solutions are

$$\mathcal{D}_a = \mathcal{C}_1 e^{-\frac{3}{2}\tau} + \mathcal{C}_2 e^{\tau}. \tag{3.82}$$

These are the results for evolution of inhomogeneity in the FLRW and can be expressed in proper time as,

$$\mathcal{D}_a = \mathcal{C}_1 t^{-1} + \mathcal{C}_2 t^{\frac{2}{3}}. \tag{3.83}$$

3.6 Chapter conclusion

We have developed a framework for dealing with decoupled density perturbations in Bianchi type I model filled with irrotational dust. We found that density perturbations decouple from gravitational waves when the background is LRS, confirming the results first found by Perko et al [61]. We note that in order to examine a link between density perturbations about Bianchi I model and the existence of a preferred direction in the CMB temperature anisotropies, it will be necessary to extend this framework to a multi-fluid case along the line given by Dunsby [31] and to implement it in a CMB code such as CMBFAST [2]. Such extension will necessarily include a Boltzmann description of radiation and neutrinos. We will pursue this in future.

3.7 Conclusion

The apparent anomalies in the CMB anisotropy have raised questions about the our so-called concordance model. With improved precision in the measurements of CMB anisotropy, one would expect greater confirmation of the isotropic and homogeneous model with Gaussian distribution. As reported by the authors of [78], this is not the case. There seems to be a clear asymmetry in the CMB anisotropies and a non-Gaussian cold spot. Models with shear and vorticity may explain this asymmetry [62, 63]. Bianchi models are the obvious starting point for the search of a model that can explain the anomalies. The authors of [63] explored the issue of chance alignment by applying template-fitting methods to CMB data and compared the Bianchi model of type VII_h to WMAP first year data. Our analysis is complementary to this, in that we demonstrate that pure density perturbations about Bianchi I model filled with irrotational dust decouple and evolve in a consistent manner. Although the anisotropic Bianchi models come with directional preference and hence good candidates to consider in the search for the answer to the CMB anomalies.

Chapter 4

Gravitational waves in a perturbed Bianchi I model

This chapter examines the evolution of gravitational waves in perturbed Bianchi I models, complementing the analysis of pure density perturbations given in the previous chapter. It is known that scalar, vector and tensor modes couple in anisotropic models [61], which makes the process of isolating and analysing any one particular mode difficult. Since there exist various classes of anisotropic and homogeneous models (Bianchi Models) [166], it is a standard practice for one to choose a class best suited for the situation under investigation. A good example of factors influencing such a choice can be seen in the recent article [63], where Bianchi VII_h is used. This was motivated by the need to address the issue of existence of shear and possibly vorticity in the WMAP data [62].

Our intention is somewhat different but complementary to these studies. We examine the cosmological model rather than determine the best-fit model for data. First, because of mode-coupling, any observed feature such as polarization, intensity or spectrum may have contributions from more than a single mode. Secondly, the growth behaviour may influence such feature in different ways and these need to be understood. What we seek is a way to isolate and determine the growth pattern of a particular mode. In general, the analysis of perturbations in any anisotropic model is a complex undertaking. This is due to the mathematical complexity involved and the lack of clarity concerning the physical significance of such perturbations. It is therefore useful to begin our study with the simplest possible anisotropic model. Consequently, we investigate the evolution of pure gravitational waves in a dust filled perturbed Bianchi type I model with vanishing vorticity $w^a = 0$. We carry out our analysis in the covariant approach because the development is straightforward and it is easy to extract the meaning of the perturbation variables. The chapter first looks at a covariant characterisation of pure gravitational waves, which is then followed by a detailed analysis of the constraint equations arising in the characterisation of pure gravitational waves. The treatment is analogous to that of pure density perturbations presented in the previous chapter. This chapter ends with the solutions to the long wavelength gravitational waves in this model.

4.1 Introduction

As pointed out in the introductory chapter, studies of gravitational waves in perturbations of isotropic and homogeneous models (FLRW) are of interest because the detection of primordial gravitational waves and the analysis of their properties (e.g. polarisation, intensity, and spectrum) might reveal important information about the nature of the early universe [55]. However, the contribution to the CMB anisotropies from these waves is extremely small in comparison to the contribution from density perturbations. In order to analyse them, higher levels of precision in the collection and the analysis of data is required. But with these levels of precision comes the possibility of discovering new and startling results (see 1.1.2), such as the possible existence of a preferred direction in the CMB temperature anisotropies. These findings in turn demand the re-examination of the standard model, in particular the assumptions that go into building this model. Taking our cue from the claims in section (1.1.2), we now consider gravitational waves in a perturbed Bianchi I model.

As we saw in Chapter 2, gravitational waves propagating in the standard model are described as weak perturbations of the background metric in the *metric* approach (2.33) or by the transverse and trace-free degrees of freedom in the *electric* and the *magnetic* parts of the Weyl tensor in the *covariant* approach. The covariant approach has its origins in the paper by Hawking [78], in which both the electric and the magnetic part of the Weyl tensor, denoted by

$$E_{ab} = C_{abcd}u^c u^d, \qquad H_{ab} = \tfrac{1}{2}\epsilon_{acd}C^{cd}{}_{bf}u^f, \tag{4.1}$$

were used. A necessary condition for these fields to describe radiation is that their spatial curls and distortions should be nonzero ($\mathrm{curl}\, E_{ab} \neq 0$, $\mathrm{curl}\, H_{ab} \neq 0$, $\tilde{\nabla}_{\langle c}E_{ab\rangle} \neq 0$ and $\tilde{\nabla}_{\langle c}H_{ab\rangle} \neq 0$) [136]. In addition, the spatial divergence should vanish for a pure radiative field [20].

It is our goal to investigate if properties of linearised gravitational waves about FLRW presented above, carry through to linearisation about Bianchi I models filled with irrotational dust. In particular, we examine the divergence-free condition on the magnetic part of the Weyl tensor and the implications this has on gravitational waves in a perturbed Bianchi I model. Previous studies of perturbations in anisotropic models include [64, 61, 124, 11, 10, 85, 20, 22, 83, 82].

4.2 Pure gravitational waves

For irrotational Bianchi I model filled with dust, the background quantities are the energy density (μ), the expansion (Θ), the shear (σ_{ab}), the electric part of the Weyl tensor (E_{ab}) and their temporal derivatives. Consider an *almost* Bianchi I universe, in particular one where the magnetic part of the Weyl tensor and the spatial gradients of μ, σ_{ab} and E_{ab} are nonzero. These extra quantities are first-order perturbations about our Bianchi I model.

Now consider the case where the gradients of scalars vanish. In particular,

$$\tilde{\nabla}_a f = 0, \tag{4.2}$$

where f is any scalar such as μ, Θ or scalar product such as σ. The above condition should be understood to mean that the gradients of scalars are at-most second-order (i.e. they vanish in the background and at first order). Since density perturbations are characterised by the scalar part of $D_b = a(\tilde{\nabla}_b\mu)/\mu$ (extracted via a scalar harmonic decomposition) in covariant perturbations about FLRW background, setting $\tilde{\nabla}_a\mu = 0$ is equivalent to eliminating inhomogeneity. Condition (4.2) therefore eliminates contributions to inhomogeneity from other scalars. The magnetic part of the Weyl tensor is now a variable characterising perturbations about our Bianchi I background. We shall use this tensor to characterise gravitational waves in this model. We shall not presume that transverse degree of freedom holds for the magnetic part of the Weyl tensor, but rather determine the conditions under which this will be true.

Subjecting the covariant equations (3.10-3.13) to condition (4.2) leads to,

$$\begin{aligned}
(C^1)_a &= \tilde{\nabla}^b\sigma_{ab} = 0, && (4.3)\\
(C^2)_a &= \tilde{\nabla}_a\Theta = 0, && (4.4)\\
(C^3)_{ab} &= \mathsf{curl}\,\sigma_{ab} - H_{ab} = 0, && (4.5)\\
(C^4)_a &= \tilde{\nabla}^b E_{ab} - \varepsilon_{abc}\sigma^b{}_d H^{cd} = 0, && (4.6)\\
(C^5)_a &= \tilde{\nabla}_a\mu = 0, && (4.7)\\
(C^6)_a &= \tilde{\nabla}^b H_{ab} - \varepsilon_{abc}\sigma^b{}_d E^{cd} = 0, && (4.8)\\
(C^8)_a &= \tilde{\nabla}_b\sigma^2 = 0. && (4.9)
\end{aligned}$$

Although a Bianchi I model is flat ($\mathcal{K} = 0$), the perturbed model is not necessarily flat. This implies that the contributions from Ricci tensor and Ricci scalar, which are first order according to our classification, need to be taken into account. We find that the traceless part of Ricci tensor, denoted by S_{ab}, plays a role in the dynamics of the constraints given above. This term, whose divergence arises in the propagation of the above constraints, can be constructed from variables that already exist in our system of equations as follows,

$$S_{ab} = {}^3R_{ab} - \tfrac{1}{3}h_{ab}{}^3R = -\dot{\sigma}_{ab} - \Theta\sigma_{ab}, \tag{4.10}$$

where equation (4.14) is used. The divergence of this term has the useful property that it vanishes when condition (4.2) is used. This arises as follows, using the fact that for 3-surfaces $\tilde{\nabla}^{b3}R_{ab} = \frac{1}{2}\tilde{\nabla}_a{}^3R$ (see (A.1) for proof),

$$\tilde{\nabla}^b S_{ab} = \tilde{\nabla}^{b3}R_{ab} - \tfrac{1}{3}\tilde{\nabla}_a{}^3R = \tfrac{1}{6}\tilde{\nabla}_a{}^3R. \tag{4.11}$$

The gradient of R can be expressed in terms of gradients of scalars using the generalised Friedmann equation (2.68) and hence in terms of known constraints (4.4, 4.7 and 4.9), which are known to vanish. This has the important implication that on application of (4.2), $\tilde{\nabla}^b S_{ab} = 0$, showing that S_{ab} is divergence-free and induces tensor modes. The propagation equations for Bianchi type I with irrotational dust are,

$$\begin{aligned}
\dot{\mu} &= -\Theta\mu, && (4.12)\\
\dot{\Theta} &= -\tfrac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} - \tfrac{1}{2}\mu, && (4.13)\\
\dot{\sigma}_{ab} &= -\tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c(a}\sigma_{b)}{}^c - E_{ab}, && (4.14)\\
\dot{E}_{ab} &= \mathsf{curl}\, H_{ab} - \Theta E_{ab} + 3\sigma_{c(a}E_{b)}{}^c - \tfrac{1}{2}\mu\sigma_{ab}, && (4.15)\\
\dot{H}_{ab} &= -\mathsf{curl}\, E_{ab} - \Theta H_{ab} + 3\sigma_{c(a}H_{b)}{}^c. && (4.16)
\end{aligned}$$

We now examine how constraints equations (4.3-4.9) evolve, subject to propagation equations (4.12-4.16).

4.3 Evolving the constraints

Assume that $C_A = 0$ represents a set of solutions to the initial data and which are satisfied on an initial spatial surface at a time t_0 (denoted by $C_A|_{t_0} = 0$). We seek to determine if these constraints always remain satisfied along the dust worldlines. We do this by evolving the constraints.

Taking the time derivative of (4.3) and making substitutions using (4.14), (4.3), (4.4), (4.5), (4.6), (A.7) and (A.8) produces,

$$\dot{C}^{(1)}{}_a + \Theta C^{(1)}{}_a + \tfrac{4}{3}C^{(8)}{}_a + \tfrac{2}{3}\sigma_a{}^b C^{(2)}{}_b + C^{(4)}{}_a = 2\varepsilon_{abc}\sigma^b{}_d C^{(3)(cd)}. \tag{4.17}$$

The propagation of constraint (4.4) requires (4.4), (4.5), (4.7), (A.3) and (A.5). This yields,

$$\dot{C}^{(2)}{}_a + \Theta C^{(2)}{}_a + \tfrac{1}{2}C^{(5)}{}_a + \sigma_a{}^b C^{(2)}{}_b + 2C^{(8)}{}_a = 0. \tag{4.18}$$

Propagation of (4.19) requires (4.5), (4.14), (4.16), (4.3), (4.4), (4.5), (A.11) and yields,

$$\dot{C}^{(3)}{}_{ab} + \Theta C^{(3)}{}_{ab} + \epsilon^{cd}{}_{(a}\sigma_{b)c}[C^{(1)}{}_d - \tfrac{2}{3}C^{(2)}{}_d] = 0. \tag{4.19}$$

The next constraint requires (A.7), (4.15), (A.10) and (A.1). The propagation and substitution processes yield,

$$\begin{aligned}\dot{C}^{(4)}{}_a &+ \tfrac{4}{3}\Theta C^{(4)}{}_a - \tfrac{1}{2}\sigma_a{}^b C^{(4)}{}_b - \tfrac{3}{2}E_a{}^b C^{(1)}{}_b \\ &-\tfrac{1}{2}\mathsf{curl}\, C^{(6)}{}_a + E_a{}^b C^{(2)}{}_b + \tfrac{1}{2}\mu C^{(1)}{}_a + \tfrac{1}{2}\sigma_a{}^b C^{(5)}{}_b = 0.\end{aligned} \tag{4.20}$$

The propagation of equation (4.7) requires (A.3), (4.12), (4.4) and (4.7) and produces,

$$\dot{C}^{(5)}{}_a + \tfrac{4}{3}\Theta C^{(5)}{}_a + \sigma_a{}^b C^{(5)}{}_b + \mu C^{(2)}{}_a = 0. \tag{4.21}$$

Constraint 4.9 requires (4.14), (4.4) and (A.6),

$$\dot{C^{(8)}}_a + \tfrac{7}{3}\Theta C^{(8)}{}_a + \sigma_a{}^b C^{(5)}{}_b + 2\sigma^2 C^{(2)}{}_a = \tilde{\nabla}_a \sigma^{bc} S_{bc}. \tag{4.22}$$

We now require that the gradient of $\sigma^{bc}S_{bc}$ should vanish. We shall denote this auxiliary constraint as follows

$$C_a^{(10)} = \tilde{\nabla}_a \sigma^{bc} S_{bc}. \tag{4.23}$$

The propagation of constraint (4.23) requires (4.14), (4.15), (A.5) and (4.11)

$$\dot{C^{(10)}}_a + 2\Theta C^{(10)}{}_a + \sigma_a{}^b C^{(10)}{}_b = -2\sigma^2 \tilde{\nabla}^b S_{ab} + \tilde{\nabla}_a[\sigma^{bc}\mathsf{curl}\, H_{bc}] + \sigma^{bc}\tilde{\nabla}_a[S_{d\langle b}\sigma_{c\rangle}{}^d]. \tag{4.24}$$

Each term to the right of (4.24) can be handle separately. Notice that the first term on the rhs can be expressed as follows;

$$\tilde{\nabla}^b S_{ab} = C_a^{(9)} = \tfrac{1}{3} C^{(5)}{}_a + \tfrac{1}{3} C^{(8)}{}_a - \tfrac{2}{9}\Theta C_a^{(2)}, \tag{4.25}$$

and hence does not need to be propagated.

The second term on the rhs of (4.24) can be expressed as follows

$$\tilde{\nabla}_a(\sigma^{bc}\mathsf{curl}\, H_{bc}) \;=\; \sigma^{bc}\tilde{\nabla}_a\tilde{\nabla}_b(C_c^{(2)}) - \tilde{\nabla}^2(C_a^{(8)}) + 4\sigma^2 C_a^{(9)}, \tag{4.26}$$

where we have used (4.11), the fact that the background is flat (3-Ricci scalar is therefore first order) and the identity (A.13).

The last term on the lhs of (4.24) can also be expressed in terms of a known constraint. To do this we turn to the 1+1+2 formalism developed in [15] and [25] . Notice that what distinguishes this model from the FLRW model is the presence of shear. This is usually given by the shear tensor σ_{ab}. Consider the case where the background is *locally rotational symmetric* [43], in which case there exists a preferred direction. The way to make this information useful, is to extend the 1+3 decomposition to the 1+1+2 decomposition. Detailed description of the 1+1+2 decomposition and its application can be found in [15] and [25].

In the 1+1+2 method, a further slicing of the spatial section of the spacetime with respect to a unit vector n^a is performed. The 1+3 projection tensor $h_{ab} = g_{ab} + u_a u_b$, together with this vector generate a new projection tensor N_{ab},

$$N_a{}^b = h_a{}^b - n_a n^b = g_a{}^b + u_a u^b - n_a n^b, \tag{4.27}$$

which projects vectors and tensors orthogonal to n^a and $u^a (n^a N_{ab} = 0 = u^a N_{ab})$ onto a 2-surface $(N_a{}^a = 2)$.

The 1+3 covariant vectors and tensors quantities can now be split relative to the new unit vector n^a. Since the source involves the shear tensor multiplied by a first order quantity, the shear must take its background value for the product to remain linear. Relative to the unit vector n^a, the shear tensor decomposition takes the form,

$$\begin{aligned} \sigma_{ab} &= \tilde{\Sigma}(n_a n_b - \tfrac{1}{2}N_{ab}) + 2\tilde{\Sigma}_{(a}n_{b)} + \tilde{\Sigma}_{ab}, \\ &= \tilde{\Sigma}(3n_a n_b - h_{ab}) + 2\tilde{\Sigma}_{(a}n_{b)} + \tilde{\Sigma}_{ab}. \end{aligned} \tag{4.28}$$

The tilde on the Σ serves as a reminder that this variable is different from the Σ used in Chapters 5 and 6. The first, the second and the third terms in equation (4.28) are the scalar, the vector and the tensor parts respectively. The vector and the tensor parts have the property $\tilde{\Sigma}_a n^a = 0 = \tilde{\Sigma}_{ab} n^a$. For an LRS background, only the first term in (4.28) is non-vanishing. In the rest of the chapter, we shall require the background to be LRS (with $\sigma_2{}^2 = \sigma_3{}^3$ i.e. degenerate shear).

Using the 1+1+2 decomposition and applying the LRS conditions, it is straight forward to show that

$$\sigma^{bc}\tilde{\nabla}_a(S_{d\langle b}\sigma_{c\rangle}{}^d) = \tilde{\nabla}_a(\sigma^{bc}S_{d\langle b}\sigma_{c\rangle}{}^d) + \mathcal{O}(2) = \tfrac{9}{4}\tilde{\Sigma}^2\tilde{\nabla}_a(n^b n^c S_{bc}) + \mathcal{O}(2) = 3\tilde{\Sigma}C_a^{(10)}, \quad (4.29)$$

where the condition (4.2) has been used.

4.3.1 Divergence-free condition on the magnetic part of the Weyl tensor

The divergence-free condition on the magnetic part of the Weyl tensor leads to the separation of constraint (4.8) into two new constraints ;

$$C^{6(1)}{}_a = \tilde{\nabla}^b H_{ab} = 0, \quad (4.30)$$

$$C^{6(2)}{}_a = \varepsilon_{abc}\sigma^b{}_d E^{cd} = 0. \quad (4.31)$$

When these two new constraints are propagated, they yield,

$$-\varsigma_a = \dot{C}^{6(1)}{}_a + \tfrac{1}{3}\Theta(4C^{6(1)}{}_a - C^{6(2)}{}_a) - \sigma_a{}^b(\tfrac{1}{2}C^{6(1)}{}_b + C^{6(2)}{}_b) + \tfrac{1}{2}curlC^{(4)}{}_a, \quad (4.32)$$

$$\varsigma_a = \dot{C}^{6(2)}{}_a + \tfrac{5}{3}\Theta C^{6(2)}{}_a - \tfrac{1}{2}\sigma_a{}^b C^{6(2)}{}_b, \quad (4.33)$$

where

$$\varsigma_a = \varepsilon_{abc}\sigma^b{}_d \mathsf{curl}\, H^{cd}. \quad (4.34)$$

Equation (4.32) requires (A.7), (4.16) and (4.30). Equation (4.33) requires (A.7), (4.14) and (4.15). As in the previous consistency check, we seek to show that (4.34) either vanishes or can be expressed in terms of known constraints. Using (A.9), the source takes the form

$$\varepsilon_{abc}\sigma^b{}_d \mathsf{curl}\, H^{cd} = \sigma^{bc}\tilde{\nabla}_a H_{bc} - \sigma^{bc}\tilde{\nabla}_b H_{ac}. \quad (4.35)$$

We shall now concern ourselves with the first two terms on the right hand side.

The magnetic part of the Weyl tensor can be replaced by its 1+1+2 decomposition. Note that this decomposition takes the form,

$$H_{ab} = \mathcal{H}(n_a n_b - \tfrac{1}{2}N_{ab}) + 2\mathcal{H}_{(a}n_{b)} + \mathcal{H}_{ab}, \tag{4.36}$$

where all the terms (scalar, vector and tensor) are first order in magnitude. It is also necessary to expand the spatial derivative of the unit vector, again we only retain products that are linear in magnitude. This yields,

$$\sigma^{bc}\tilde{\nabla}_a H_{bc} = \tfrac{3}{2}\tilde{\Sigma}\tilde{\nabla}_a(\mathcal{H}), \tag{4.37}$$

since gradients of scalar are zero by virtue of (4.2), and $n^a N_{ab} = 0$. Similar analysis will show that,

$$\sigma^{bc}\tilde{\nabla}_b H_{ac} = \tfrac{1}{2}\tilde{\Sigma}C^6_a + \tfrac{1}{2}\tilde{\nabla}_b\left((n^b n_a + N_a{}^b)\tilde{\Sigma}\mathcal{H} - \mathcal{H}_a n^b\tilde{\Sigma} + 2\mathcal{H}^b n_a\tilde{\Sigma}\right). \tag{4.38}$$

These results imply,

$$\varepsilon_{abc}\sigma^b{}_d \mathsf{curl}\, H^{cd} = -\tfrac{1}{2}\sigma_a{}^b C^6{}_b - \tfrac{1}{2}\tilde{\Sigma}C^6_a + \tilde{\nabla}_a(\tilde{\Sigma}\mathcal{H}) + \tfrac{1}{2}\tilde{\nabla}^b(\tilde{\Sigma}\mathcal{H}_a n_b - \tilde{\nabla}^b(\tilde{\Sigma}\mathcal{H}_b n_a). \tag{4.39}$$

It is clear that this leads to additional constraints and so the divergence free conditions is not preserved. This implies that the wave equation for the magnetic part of the Weyl tensor is not a pure tensor wave. In order to obtain gravitational waves, we need to extract the pure tensor part of the wave equation. We shall show in the next section that for a diagonal H_{ab}, the divergence-free condition is met. Gravitational waves propagating in this background can be characterised in two different ways. The first is to use the magnetic part of the Weyl where the divergence-free condition is relaxed. In this case the issue of extracting pure tensor arises or one takes the magnetic part of the Weyl tensor to be diagonal. The second is to use the traceless part of the Ricci tensor S_{ab}, keeping in mind that the background is flat ($\mathcal{K} = 0$) and S_{ab} is first order. The issue of whether this represents genuine gravitational waves is debatable. It has been conjectured in [6] that there exists a link between the Cotton-York tensor of hypersurfaces in spacetime and the presence of gravitational waves. We note the the curl of the traceless part of 3-Ricci tensor (denoted by $\mathsf{curl}\, S_{ab}$) is proportional to the Cotton-York tensor. In this thesis we shall only consider wave equation given by the magnetic part of the Weyl tensor. The case of the traceless part of the Ricci tensor will be addressed in future.

4.3.2 Wave equation for the magnetic part of the Weyl tensor with vanishing divergence and diagonal H_{ab}

The starting point in deriving the wave equation is the evolution equation for the magnetic part of the Weyl tensor given by,

$$\dot{H}_{ab} + \Theta H_{ab} + 3\sigma_{\langle a}{}^{c} H_{b\rangle c} = -\mathsf{curl}\, E_{ab}. \tag{4.40}$$

Differentiating the above equation with respect to time, making substitutions using (4.14), (4.15) and (4.16) and the appropriate commutation relation, we find,

$$\begin{aligned} \ddot{H}_{ab} - \tilde{\nabla}^2 H_{ab} + \tfrac{7}{3}\Theta \dot{H}_{ab} + (\tfrac{2}{3}\Theta^2 + 2\sigma^2) H_{ab} - 9\Theta \sigma_{c\langle a} H_{b\rangle}{}^{c} - 9\dot{\sigma}_{c\langle a} H_{b\rangle}{}^{c} - 3\sigma_{c\langle a} \dot{H}^{c}_{b\rangle} \\ + \tfrac{3}{2}\tilde{\nabla}_{\langle a} \tilde{\nabla}^{c} H_{b\rangle c} - 6H^{c}{}_{\langle a}\sigma_{b\rangle d}\sigma_{c}{}^{d} + \sigma_{cd} H^{cd} \sigma_{ab} - \sigma_{ca} H^{cd}\sigma_{bd} + \sigma^{cd}\sigma_{c\langle a} H_{b\rangle d} \\ = -3\mathsf{curl}\,(\sigma_{c\langle a} E_{b\rangle}{}^{c}) + \sigma_{e}{}^{c}\varepsilon_{cd(a} \tilde{\nabla}^{e} E_{b)}{}^{d}. \end{aligned} \tag{4.41}$$

Notice that for $a = b$, $\tilde{\nabla}_{\langle a}\tilde{\nabla}^{c} H_{b\rangle c} \equiv 0$. This follows when one substitutes for the divergence of the magnetic part of the Weyl tensor using (4.8) and applies the LRS (equivalently the degenerate shear) condition. Similar conclusions have been arrived at by [64], who use a tetrad approach. Now note that the first term on rhs of (4.41) can be expanded as follows,

$$-3\,\mathsf{curl}\,(\sigma_{c\langle a}E_{b\rangle}{}^{c}) \equiv 3\varepsilon_{cd\langle a}\tilde{\nabla}^{e}(\sigma^{c}{}_{e}E_{b\rangle}{}^{d}), \tag{4.42}$$

where the last equivalence can be demonstrated by replacing the the electric part of the Weyl tensor using equation (4.14) and applying identity (A.5). But

$$3\varepsilon_{cd\langle a}\tilde{\nabla}^{e}(\sigma^{c}{}_{e}E_{b\rangle}{}^{d}) = 3\varepsilon_{cd\langle a}\sigma^{c}{}_{e}\tilde{\nabla}^{e}(E_{b\rangle}{}^{d}) + 3\varepsilon_{cd\langle a}E_{b\rangle}{}^{d}\tilde{\nabla}^{e}(\sigma^{c}{}_{e}), \tag{4.43}$$

where the last term vanishes given that (4.3) holds. This implies that the source reduces to

$$4\varepsilon_{cd\langle a}\sigma^{c}{}_{e}\tilde{\nabla}^{e}(E_{b\rangle}{}^{d}). \tag{4.44}$$

It can be shown that for LRS (equivalently degenerate shear) background, the rhs contributes terms equal to the product of shear and curl of the electric part of the Weyl tensor on one hand and product of the electric part of the Weyl tensor and the magnetic part of the Weyl tensor on the other. We demonstrate this for the component H_{11}. The source for this components takes the form,

$$\begin{aligned} 4\varepsilon_{cd\langle 1}\sigma^{c}{}_{e}\tilde{\nabla}^{e}E_{1\rangle}{}^{d} &= 4\left(\sigma^{2}{}_{e}\tilde{\nabla}^{e}E_{1}{}^{3} - \sigma^{3}{}_{e}\tilde{\nabla}^{e}E_{1}{}^{2}\right) \\ &= 4\sigma^{2}{}_{2}\left(\tilde{\nabla}^{2}E_{1}{}^{3} - \tilde{\nabla}^{3}E_{1}{}^{2}\right), \end{aligned} \tag{4.45}$$

where the shear is taken to be diagonal and degenerate, i.e. $\sigma_2{}^2 = \sigma_3{}^3$. Let us now examine the source term to equation (4.40), for the case H_{11}. In this case note that,

$$\mathsf{curl}\,E_{11} = \varepsilon_{cd\langle 1}\tilde{\nabla}^{c}E_{1\rangle}{}^{d} = \tilde{\nabla}^{2}E_{1}{}^{3} - \tilde{\nabla}^{3}E_{1}{}^{2}. \tag{4.46}$$

This demonstrates that in isolating the components of the H_{ab}, and applying the degenerate shear condition, one can effectively replace the source with the product of the corresponding version of $\dot{H}_{ab}$ and a diagonal component of shear. This can then be moved to the left leaving a homogeneous equation. The source term for the other components are more complex but can be analysed in a similar manner. In order to proceed, we first apply the projection tensor h^{bc}on (4.41) thereby raising one index. It turns out that it is much easier to work with one index raised. We introduce and make use of the following notation

$$H_1 = H^1{}_1,\ H_2 = H^2{}_2,\ H_3 = H^3{}_3,$$

as they lead to a neater presentation. We also transform to a dimensionless time variable τ where $d\tau = Hdt, H$ is the Hubble parameter. The derivative with respect to τ will be denoted by a prime[1], in contrast to the proper time derivative denoted by an overdot.

The long wavelength approximation of the diagonal components of the wave equation (4.41) in dimensionless time take the form,

$$H_1'' + (\tfrac{11}{2} + \Sigma - \tfrac{3}{2}\Sigma^2)H_1' + (6 + 33\Sigma - 18\Sigma^2)H_1 = 0, \tag{4.47}$$

$$H_2'' + (\tfrac{11}{2} + \Sigma - \tfrac{3}{2}\Sigma^2)H_2' + (6 + 33\Sigma - 14\Sigma^2)H_2 + 2\Sigma^2 H_1 = 0, \tag{4.48}$$

$$H_3'' + (\tfrac{11}{2} + \Sigma - \tfrac{3}{2}\Sigma^2)H_3' + (6 + 33\Sigma - 14\Sigma^2)H_3 + 2\Sigma^2 H_1 = 0, \tag{4.49}$$

where we have used expansion-normalised shear elements given by

$$\sigma_1/H = -2\Sigma, \sigma_2/H = \sigma_3/H = \Sigma,$$

which follows a choice of shear-eigen frame of the form diag $\sigma = (-\frac{2\sigma}{\sqrt{3}}, \frac{\sigma}{\sqrt{3}}, \frac{\sigma}{\sqrt{3}})$. In order to solve the above system, the values for Σ are required. These are given by the fixed

[1]The prime in chapter (2) denoted derivative with respect to conformal time. In this chapter and in chapter (3), the prime denotes dimensionless time (τ)

points for the evolution equation for Σ. It can be shown that for this background,

$$\Sigma' = -\frac{3}{2}(1-\Sigma^2)\Sigma. \tag{4.50}$$

The fixed points are $\Sigma = 0, \pm 1$. In the next section, we shall solve the system of equations given by the diagonal components of the magnetic part of the Weyl tensor, subject to these shear fixed points.

4.4 The non-zero shear subcase

The shear regime is characterised by the two nonzero shear values. We look at each of these, beginning with the case where the shear is positive.

4.4.1 Positive shear subcase

In this case the solutions to the system of diagonal elements are,

$$H_1 = \mathcal{C}_1 e^{-\frac{5}{2}\tau}, \quad H_2 = -\tfrac{1}{2}(\mathcal{C}_1 + \mathcal{C}_2)e^{-\frac{5}{2}\tau}, \quad H_3 = -\tfrac{1}{2}(\mathcal{C}_1 - \mathcal{C}_2)e^{-\frac{5}{2}\tau}, \tag{4.51}$$

where,

$$\mathcal{C}_1(\tau) = c_1 \sin(\tfrac{\sqrt{59}}{2}\tau) + c_2 \cos(\tfrac{\sqrt{59}}{2}\tau),\ \mathcal{C}_2(\tau) = c_3 \sin(\tfrac{5\sqrt{3}}{2}\tau) - c_4 \cos(\tfrac{5\sqrt{3}}{2}\tau). \tag{4.52}$$

The small c(s) are the integration constants and are determined by the initial conditions. In the shear-dominated regime, the Hubble parameter $H = e^{-3\tau} \propto t^{-1}$. We can now convert back to proper time, in which case the Hubble-normalised solutions read,

$$\mathcal{H}_1 = H_1/H^2 = \mathcal{C}_1 t^{\frac{5}{6}}, \tag{4.53}$$

$$\mathcal{H}_2 = H_2/H^2 = -\tfrac{1}{2}t^{5/6}(\mathcal{C}_1 + \mathcal{C}_2), \tag{4.54}$$

$$\mathcal{H}_3 = H_3/H^2 = -\tfrac{1}{2}t^{5/6}(\mathcal{C}_1 - \mathcal{C}_2), \tag{4.55}$$

and

$$\mathcal{C}_1(t) \propto \left[c_1 \sin\left(\tfrac{\sqrt{59}}{6}\ln(t)\right) + c_2 \cos\left(\tfrac{\sqrt{59}}{6}\ln(t)\right)\right], \tag{4.56}$$

$$\mathcal{C}_2(t) \propto \left[c_3 \sin\left(\tfrac{5\sqrt{3}}{6}\ln(t)\right) - c_4 \cos\left(\tfrac{5\sqrt{3}}{6}\ln(t)\right)\right]. \tag{4.57}$$

It follows that these coupled components oscillate with growing amplitudes. It is also clear that the components grow faster than the growing mode in the FLRW case, which implies that the presence of shear boosts the amplitude of gravitational waves. Next we consider the case where shear is negative.

4.4.2 Negative shear subcase

In this case the solutions to system (4.47-4.48) are,

$$H_1 = e^{-\frac{3+3\sqrt{21}}{2}\tau}(c_1 + c_2 e^{3\sqrt{21}\tau}), \tag{4.58}$$

$$H_2 = -\tfrac{1}{2}e^{-\frac{3+3\sqrt{21}}{2}\tau}(c_1 + c_2 e^{3\sqrt{21}\tau}) + e^{-\frac{3+\sqrt{173}}{2}\tau}(C_3 - c_4 e^{\sqrt{173}\tau}), \tag{4.59}$$

$$H_3 = -\tfrac{1}{2}e^{-\frac{3+3\sqrt{21}}{2}\tau}(c_1 + c_2 e^{3\sqrt{21}\tau}) - e^{-\frac{3+\sqrt{173}}{2}\tau}(c_3 - c_4 e^{\sqrt{173}\tau}), \tag{4.60}$$

where the coefficients are again determined by the initial conditions. As in the previous case, the Hubble normalised solutions expressed in proper time read

$$\mathcal{H}_1 = H_1/H^2 = \{c_1 t^{\frac{9-3\sqrt{21}}{6}} + c_2 t^{\frac{9+3\sqrt{21}}{6}}\}, \quad (4.61)$$
$$\mathcal{H}_2 = H_2/H^2 = -\tfrac{1}{2}\{c_1 t^{\frac{9-3\sqrt{21}}{6}} + c_2 t^{\frac{9+3\sqrt{21}}{6}}\} + c_3 t^{\frac{9-\sqrt{173}}{6}} - c_4 t^{\frac{9+\sqrt{173}}{6}}, \quad (4.62)$$
$$\mathcal{H}_3 = H_3/H^2 = -\tfrac{1}{2}\{c_1 t^{\frac{9-3\sqrt{21}}{6}} + c_2 t^{\frac{9+3\sqrt{21}}{6}}\} + c_3 t^{\frac{9-\sqrt{173}}{6}} - c_4 t^{\frac{9+\sqrt{173}}{6}}. \quad (4.63)$$

In this case we have both growing and decaying modes. Again, the presence of shear leads to an increase in the rate of growth and a decrease in the decay rate, when compared to the FLRW case.

4.5 The zero shear subcase

In this case the equations in the system (4.47-4.49) decouple, with each component evolving separately. One can apply harmonic decomposition, using the basis presented covariant section of Chapter 2. The long wave length approximation of the decomposed wave equation takes the form;

$$H_\kappa'' + \tfrac{11}{2} H_\kappa' + 6 H_\kappa = 0, \quad (4.64)$$

for one parity, with the opposite parity obeying a similar equation. The solutions to this wave equation are,

$$H_\kappa = \mathcal{C}_1 e^{-\frac{3}{2}\tau} + \mathcal{C}_2 e^{-4\tau}.$$

This can be converted to proper time. From the solutions to the evolution equations for background shear and energy density, it can be shown that the scale factor $a \propto t^{\frac{2}{3}}$, which implies that in proper time,

$$H_\kappa = \mathcal{C}_1 t^{-1} + \mathcal{C}_2 t^{-\frac{8}{3}}.$$

The dimensionless form of this solutions then read,

$$\mathcal{H}_{\kappa} = H_{\kappa}/H^2 = C_1 t + C_2 t^{-\frac{2}{3}},$$

where H is the Hubble parameter and is equal to $\frac{2}{3t}$ in this regime. These recover the standard FLRW results found in the detailed analysis given in [129]

4.6 Chapter conclusion

We have looked at the decoupling of gravitational waves from density perturbations in perturbed Bianchi type I model filled with dust. We found that the background model must be LRS for the constraint equations to propagate, after setting the gradients of scalars to zero. Whereas the issue of purely radiative gravitational field ($\mathsf{div} H_a = 0$) can easily be demonstrated for gravitational waves in perturbed FLRW, this is not the case for perturbations about Bianchi I model filled with dust. It turns out that the divergence of the magnetic part of the Weyl tensor vanishes when H_{ab} is diagonal. We have presented the solutions to the wave equation with diagonal H_{ab}, however the issue of whether or not these are genuine gravitational waves is still open. Our analysis extends those of [109] to the case of pure gravitational waves in perturbed Bianchi I model with irrotational dust and also complements the analysis by [82].

Chapter 5

Second-Order Gauge-Invariant perturbation theory

The introductory chapter to this thesis reviewed how well the current standard model fits observations. As pointed out, there exists some discrepancies (see section 1.1.2). In order to address some of these discrepancies it is necessary to refine the current standard model. A possible refinement may come from second order perturbations of this model. This chapter re-examines the covariant and gauge invariant approach to second-order perturbation theory initiated in [24] and extends it in a significant manner. We first revisit the motivations for developing techniques to study *Second-Order effects*, before presenting the formalism. The focus is on second-order tensor perturbations about FLRW with dust on the one hand and with a general barotropic equation of state on the other. Solutions to the wave equations are presented, where possible. The chapter ends with a discussion of some outstanding issues and the work in progress.

5.1 Introduction

Cosmology is now firmly data driven [160], where considerable emphasis is placed on determining the key cosmological parameters discussed in our introductory chapter. Large amounts of high quality data is now available from large scale surveys of galaxies redshifts, the measurements of the CMB temperature anisotropies and polarization, and measurements of the magnitude-redshift relation of distance supernovae, to mention a few. The precision in the measurements of the CMB temperature anisotropies in particular, is nearing the levels where it may be possible to extract imprints left by non-linear effects.

The study of small perturbations, giving rise to large-scale temperature anisotropies of the CMB is usually treated with first-order relativistic perturbation theory techniques, either in a gauge invariant manner or by specifying a suitable gauge [92, 118, 8, 34, 141]. On small and intermediate angular scales, where the description in terms of linear order perturbations theory is no longer accurate, second-order perturbation theory is necessary. A further motivation for second-order perturbation theory is that there is no way, within the linear theory, to decide when the perturbations have become too large for the theory to handle [122, 24]. For these reasons, second-order perturbation theory is beginning to receive considerable attention [161,122,24,115,7,156,68,142,67,125,113,24,97,98,89,122].

Second-order cosmological perturbations dates back to Tomita [158], where for the first time second order terms from scalar modes were examined. More recently, the authors of [68] looked at second-order perturbations of a flat dust FLRW models with a cosmological constant, while those of [142] considered the case without a cosmological constant. Second-order effects during inflation was studied in [161], allowing for the prediction of the bispectrum of perturbation from inflation. The full relativistic treatment of second order perturbation theory have also been considered in [135, 116, 141, 125]. In Chapter 6, we shall consider gravitational waves generated by second order effects during slow-roll period of inflation. Meanwhile the authors of [113, 57] have considered the evolution of curvature perturbations on super-Hubble scales after inflation which they find to be constant. The authors of [120,121] have also shown that second order effects may lead to detectable non-Gausianity in the CMB, while those of [143] have considered second order contributions to CMB polarization.

The literature cited above, with a few exceptions, develop and apply metric based formalisms in their analysis of higher-order perturbations. Our interest is to develop a covariant and gauge invariant approach to complement these efforts.

In [24], a covariant approach to nonlinear perturbation theory was initiated, and the formalism used to study second order gravitational waves sourced by first order density perturbations and second order density perturbations sourced by gravitational waves at first order. However, only the dust equation of state was considered in that work. We develop a parallel approach that represents a significant extension of the analysis in that paper and one that is able to deal with a wide range of barotropic fluids. Let us first re-examine the basics of scalar perturbations in the covariant approach.

5.2 First order dynamics

The study of linear perturbations of a FLRW background are relatively straightforward to study. Let us begin by defining the *first-order gauge-invariant* (FOGI) variables corresponding respectively to the spatial fluctuations in the energy density, expansion rate and spatial curvature for a fluid with barotropic equation of state ($p = w\mu$):

$$X_a = \widetilde{\nabla}_a \mu, \tag{5.1}$$

$$\mathcal{Z}_a = \widetilde{\nabla}_a \Theta^1, \tag{5.2}$$

[1]This differs from the notation used in (3.48), where a scale factor is involved

$$C_a = a^3\tilde{\nabla}_a\tilde{R}. \tag{5.3}$$

The quantities are FOGI because they vanish exactly in the background FLRW spacetime [152]. As we saw earlier, It turns out that a more suitable quantity for describing density fluctuations is the co-moving gradient of the energy density:

$$\mathcal{D}_a = \tfrac{a}{\mu}X_a, \tag{5.4}$$

where the ratio X_a/μ allows one to evaluate the magnitude of the energy density perturbation relative to its background value and the scale factor a guarantees that it is dimensionless and co-moving. From equation (2.52-2.66), it follows that the set of linearized equations satisfied by the FOGI variables are given by,

$$\begin{aligned}
\dot{\sigma}_{ab} &= -\tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c\langle a}\sigma_{b\rangle}{}^{c} - E_{ab} + \tilde{\nabla}_{\langle a}A_{b\rangle}, & (5.5)\\
\dot{E}_{ab} &= -\Theta E_{ab} + \mathsf{curl}\, H_{ab} - \tfrac{1}{2}\mu(1+w)\sigma_{ab} + 3\sigma_{c\langle a}E_{b\rangle}{}^{c}, & (5.6)\\
\dot{H}_{ab} &= -\Theta H_{ab} - \mathsf{curl}\, E_{ab} + 3\sigma_{c\langle a}H_{b\rangle}{}^{c} - 2\epsilon_{cd\langle a}A^{c}E_{b\rangle}{}^{d}, & (5.7)\\
\dot{X}_a &= -\tfrac{4}{3}\Theta X_a(1 - \tfrac{3w}{4}) - \mu\mathcal{Z}_a, & (5.8)\\
\dot{\mathcal{Z}}_a &= -\Theta\mathcal{Z}_a - \tfrac{1}{2}X_a(1+w), & (5.9)
\end{aligned}$$

together with the constraint equations,

$$\begin{aligned}
0 &= \mathsf{div}\sigma_a - \tfrac{2}{3}\mathcal{Z}_a, & (5.10)\\
0 &= H_{ab} - \mathsf{curl}\,\sigma_{ab}, & (5.11)\\
0 &= \mathsf{div}E_a - \tfrac{1}{3}X_a, & (5.12)\\
0 &= \mathsf{div}H_a, & (5.13)\\
0 &= \mathsf{curl}\,X_a, & (5.14)
\end{aligned}$$

where $\mathsf{div}(E_a) \equiv \tilde{\nabla}_e E_a{}^e$. Because the background is homogeneous and isotropic, each FOGI vector may be uniquely split into a *curl-free* and *divergence-free* part, usually referred to as scalar and vector parts respectively, which we write as

$$V_a = V_{\mathsf{S}a} + V_{\mathsf{V}a}, \tag{5.15}$$

where $\mathsf{curl}\, V_{\mathsf{S}a} = 0$ and $\mathsf{div} V_{\mathsf{V}} = 0$. Similarly, any tensor may be invariantly split into scalar, vector and tensor parts:

$$T_{ab} = T_{\mathsf{S}ab} + T_{\mathsf{V}ab} + T_{\mathsf{T}ab}, \tag{5.16}$$

where $\mathsf{curl}\, T_{\mathsf{S}ab} = 0$, $\mathsf{divdiv} T_{\mathsf{V}} = 0$ and $\mathsf{div} T_{\mathsf{T}a} = 0$. It follows therefore that in the above equations we can separately equate scalar, vector and tensor parts and obtain equations that independently characterize the evolution of each type of perturbation. It follows from the above discussions that pure scalar modes are characterized by the vanishing of the magnetic part of the Weyl tensor: $H_{ab} = 0$, so the above set of equations reduce to a set of two coupled differential equations for X_a and $\mathcal{Z}_a$:

$$\dot{X}_a = -\tfrac{4}{3}\Theta X_a(1 - \tfrac{3w}{4}) - \mu \mathcal{Z}_a, \tag{5.17}$$

$$\dot{\mathcal{Z}}_a = -\Theta \mathcal{Z}_a - \tfrac{1}{2}X_a(1 + w), \tag{5.18}$$

and a set of coupled evolution and constraint equations that determine the other variables

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c\langle a}\sigma_{b\rangle}{}^{c} - E_{ab} + \tilde{\nabla}_{\langle a}A_{b\rangle}, \tag{5.19}$$

$$\dot{E}_{ab} = -\Theta E_{ab} - \tfrac{1}{2}\mu(1+w)\sigma_{ab} + 3\sigma_{c\langle a}E_{b\rangle}{}^{c}, \tag{5.20}$$

$$0 = \mathsf{div}(\sigma_a) - \tfrac{2}{3}\mathcal{Z}_a, \tag{5.21}$$

$$0 = \mathsf{curl}\,(\sigma_{ab}), \tag{5.22}$$

$$0 = \mathsf{div}(E_a) - \tfrac{1}{3}X_a, \tag{5.23}$$

$$0 = \mathsf{curl}\,(X_a), \tag{5.24}$$

$$0 = \mathsf{curl}\,E_{ab}, \tag{5.25}$$

when A^c is first order. Our aim is to develop a formalism for studying second order tensor perturbations generated by first-order scalar modes. Unlike in first-order dynamics, where the different modes decouple and evolve separately, these modes couple in second-order dynamics. A mode extracting scheme is therefore necessary in situations where a specific mode needs to be analysed. In the next section, we present new covariant mode extraction procedures useful for extracting scalars, vectors and tensors. We shall then apply the tensor extractor in the rest of the chapter.

5.3 Mode extraction

We assume an FLRW geometry of curvature $\mathcal{K}$, and all relations are defined on this background. Under a perturbation at order n, all relations below hold for objects of perturbative order n; for lower perturbative order, the commutation relations below have curvature correction terms added to them. Given the usual 4-velocity u^a, we use the spatial metric $h_{ab} = g_{ab} + u_a u_b$, defined in Chapter 2. All rank-1 and -2 tensors used here are orthogonal to u^a, and rank-2 tensors are symmetric and trace-free (PSTF) which we denote using angle brackets on indices. We define the conformal spatial covariant derivative acting on scalars or spatial tensors as $\vec{\nabla}_a = a h_a{}^b \nabla_b = a\tilde{\nabla}_a$, where a is the scale factor and $\tilde{\nabla}_a$ is the spatial derivative normally used in the covariant approach. We use $\vec{\nabla}_a$ as it commutes with $u^a\nabla_a$ and is the covariant derivative normally used in the metric approach to perturbation theory. The irreducible parts of the spatial derivative of PSTF tensors are the divergence, curl, and distortion, defined as

$$\begin{aligned} \vec{\mathrm{div}}\, X_{b\ldots c} &= \vec{\nabla}^a X_{ab\ldots c}, && (5.26)\\ \vec{\mathrm{curl}}\, X_{ab\ldots c} &= \varepsilon_{de\langle a} \vec{\nabla}^d X_{b\ldots c\rangle}{}^e, && (5.27)\\ \vec{\mathrm{dis}}\, X_{ca\ldots b} &= \vec{\nabla}_{\langle c} X_{a\ldots b\rangle}. && (5.28) \end{aligned}$$

Note that the divergence decreases the rank of the tensor by one, the $\vec{\mathrm{curl}}\,(= a\,\mathrm{curl}\,)$ preserves it, while the distortion increases it by one. Keeping this in mind one can drop the indices on differential operators as long as it's explicit the valence of the PSTF tensor which is being acted on (which can sometimes be confusing). This considerably simplifies the appearance of the equations. For a rank-1 spatial tensor the following relations hold:

$$\begin{aligned} \vec{\mathrm{div}}\,\vec{\mathrm{curl}} &= 0, && (5.29)\\ \vec{\mathrm{dis}}\,\vec{\mathrm{curl}} &= 2\vec{\mathrm{curl}}\,\vec{\mathrm{dis}}, && (5.30)\\ \vec{\mathrm{curl}}^{\,2} + \vec{\nabla}^2 &= \vec{\nabla}\vec{\mathrm{div}} + 2\mathcal{K}, && (5.31)\\ \vec{\mathrm{div}}\,\vec{\mathrm{dis}} &= \tfrac{1}{2}\vec{\nabla}^2 + \tfrac{1}{6}\vec{\nabla}\vec{\mathrm{div}} + \mathcal{K}, && (5.32)\\ \vec{\nabla}^2\vec{\mathrm{div}} &= \vec{\mathrm{div}}\,\vec{\nabla}^2 - 2\mathcal{K}\vec{\mathrm{div}} && (5.33)\\ \vec{\nabla}^2\vec{\mathrm{curl}} &= \vec{\mathrm{curl}}\,\vec{\nabla}^2, && (5.34)\\ \vec{\nabla}^2\vec{\mathrm{dis}} &= \vec{\mathrm{dis}}\,\vec{\nabla}^2 + 4\mathcal{K}\vec{\mathrm{dis}}. && (5.35) \end{aligned}$$

Here we have used the notation whereby $\vec{\nabla}$ with no index represents the gradient of a scalar. Acting on any PSTF rank-2 tensor, the following commutation relations hold:

$$\begin{aligned}
\vec{\mathsf{curl}}\,\vec{\nabla}^2 &= \vec{\nabla}^2\vec{\mathsf{curl}}\,, && (5.36)\\
\vec{\mathsf{curl}}\,\vec{\mathsf{div}} &= 2\vec{\mathsf{div}}\,\vec{\mathsf{curl}}\,, && (5.37)\\
\vec{\mathsf{curl}}^{\,2} + \vec{\nabla}^2 &= \tfrac{3}{2}\vec{\mathsf{dis}}\,\vec{\mathsf{div}} + 3\mathcal{K}, && (5.38)\\
(\vec{\mathsf{dis}}\,\vec{\mathsf{div}})\vec{\mathsf{curl}} &= \vec{\mathsf{curl}}\,(\vec{\mathsf{dis}}\,\vec{\mathsf{div}}), && (5.39)\\
\vec{\nabla}^2\vec{\mathsf{div}} &= \vec{\mathsf{div}}\,\vec{\nabla}^2 - 4\mathcal{K}\vec{\mathsf{div}}\,, && (5.40)\\
\vec{\nabla}^2\vec{\mathsf{div}}\,\vec{\mathsf{div}} &= \vec{\mathsf{div}}\,\vec{\mathsf{div}}\,\vec{\nabla}^2 - 6\mathcal{K}\vec{\mathsf{div}}\,\vec{\mathsf{div}}\,, && (5.41)\\
\vec{\mathsf{div}}\,\vec{\mathsf{curl}}\,\vec{\mathsf{div}} &= 0, && (5.42)\\
\vec{\mathsf{dis}}\,\vec{\mathsf{div}} &= \tfrac{5}{2}\vec{\mathsf{div}}\,\vec{\mathsf{dis}} - \tfrac{5}{6}\vec{\nabla}^2 - 5\mathcal{K}, && (5.43)
\end{aligned}$$

where the last equation is given in [137].

5.4 Rank-1 tensors

For rank-1 tensors the scalar and vector parts correspond to the curl- and divergence-free parts respectively. Given a spatial rank-1 tensor on an FLRW background at maximum perturbative order, $X^a = S^a + V^a = \vec{\nabla}^a S + V^a$, it is easy to form quantities which are pure scalars or vectors by the following rules:

$$\begin{aligned}
\text{Scalar:}\qquad & \breve{S}_a \equiv \vec{\nabla}_a \vec{\mathsf{div}}\, X = \vec{\nabla}_a \vec{\mathsf{div}}\, S = \vec{\nabla}_a \vec{\nabla}^2 S \\
& \quad = \left(\vec{\nabla}^2 - 2\mathcal{K}\right)\vec{\nabla}_a S, \\
\text{Vector:}\qquad & \breve{V}_a \equiv \vec{\mathsf{curl}}\, X_a = \vec{\mathsf{curl}}\, V_a. && (5.44)
\end{aligned}$$

By taking spatial derivatives we have quantities which are scalars and vectors but which remain local. We can then formulate the non-local variables from the original tensor by the formal solution

$$S = \vec{\nabla}^{-2}\vec{\mathrm{div}}\,X \tag{5.45}$$

$$V_a = \left(2\mathcal{K} - \vec{\nabla}^2\right)^{-1} \vec{\mathrm{curl}}^{\,2} X_a. \tag{5.46}$$

While X^a, $\breve{S}$ and $\breve{V}^a$ have compact support, S does not [152]. Part of the conceptual utility of defining local scalar and vector quantities via equations (5.44), which preserve the rank of X^a, is that the differential operations involved commute with the Laplacian $\vec{\nabla}^2$, and time derivative. Therefore, if X^a satisfies a wave equation with source, $\mathcal{L}[X_a] = \mathcal{S}_a$, where $\mathcal{L}$ is a linear differential operator containing the Laplacian, then we may locally extract the covariant scalar and vector parts to find $\mathcal{L}[\breve{S}_a] = \vec{\nabla}_a\vec{\mathrm{div}}\,\mathcal{S}$ and $\mathcal{L}[\breve{V}_a] = \vec{\mathrm{curl}}\,\mathcal{S}_a$. One may further relate the local and non-local decompositions in Fourier space (which is inherently non-local). Defining a scalar harmonic basis in the usual way, $\vec{\nabla}^2 Q^{(S)} = -k^2 Q^{(S)}$, we find $\breve{S}^{(k)} = -k^2 S^{(k)}$. Similarly for vectors, $\vec{\nabla}^2 Q_a^{(V)} = -k^2 Q_a^{(V)}$, where we have two parities of orthogonal harmonics, $(k^2+2\mathcal{K})^{1/2}Q^{(V)} = \vec{\mathrm{curl}}\,\bar{Q}^{(V)} \Leftrightarrow (k^2+2\mathcal{K})^{1/2}\bar{Q}^{(V)} = \vec{\mathrm{curl}}\,Q^{(V)}$, in Fourier space the local extraction involves a parity switch. However, this feature can be trivially removed by defining $\breve{V}_a \equiv \vec{\mathrm{curl}}\,\vec{\mathrm{curl}}\,X_a$ instead of equation (5.46).

5.5 Rank-2 tensors

For rank-1 tensors the above considerations are trivial to find, and well known. For rank-2 tensors, on the other hand, the local decomposition into scalar, vector and tensor modes is not quite so easy. A general projected, symmetric, and trace-free rank-2 tensor has the decomposition

$$\begin{aligned} X_{ab} &= S_{ab} + V_{ab} + T_{ab} \\ &= \vec{\nabla}_{\langle a}\vec{\nabla}_{b\rangle}S + \vec{\nabla}_{\langle a}V_{b\rangle} + T_{ab}, \end{aligned} \tag{5.47}$$

where the non-local scalar part is curl-free, the vector part is solenoidal, $\vec{\mathrm{div}}\,V = 0 \Rightarrow \vec{\mathrm{div}}\,\vec{\mathrm{div}}\,V = 0$ (and $\vec{\mathrm{div}}\,V_a \neq 0$), while the tensor part is transverse, $\vec{\mathrm{div}}\,T_a = 0$. The question is, how do we form *local* scalar, vector and tensor quantities from X_{ab} and relate them to the non-local split given above? That is, what differential operations do we need to act on X_{ab} to leave only the scalar, vector or tensor part? Furthermore, let us assume that X_{ab} obeys a wave equation with source of the form

$$\mathcal{L}[X_{ab}] = \mathcal{S}_{ab}, \tag{5.48}$$

where $\mathcal{L}$ contains time derivatives and Laplacians, and any derivative operations which preserve the rank of X_{ab} – i.e., $\vec{\mathrm{curl}}$, $\vec{\mathrm{dis}}\,\vec{\mathrm{div}}$ or $\vec{\mathrm{div}}\,\vec{\mathrm{dis}}$, or combinations thereof. Ideally we would like the local extractions to be differential operators which commute with $\mathcal{L}$. To show this we need only show that the Laplacian and $\vec{\mathrm{curl}}$ commute with our extractions below: time derivatives commute trivially; $\vec{\mathrm{dis}}\,\vec{\mathrm{div}}$ commutes if $\vec{\mathrm{curl}}$ and $\vec{\nabla}^2$ does by equation (5.38); and $\vec{\mathrm{div}}\,\vec{\mathrm{dis}}$ therefore will by equation (5.43).

5.5.1 Scalars

Clearly, $\breve{S} = \vec{\mathrm{div}}\,\vec{\mathrm{div}}\,X$ is a (covariantly defined) scalar and can only depend on S, so let us define

$$\begin{aligned}
\breve{S}_{ab} &\equiv \hat{\mathscr{S}}[X_{ab}] \equiv \vec{\nabla}_{\langle a}\vec{\nabla}_{b\rangle}\vec{\mathrm{div}}\,\vec{\mathrm{div}}\,X = \vec{\nabla}_{\langle a}\vec{\nabla}_{b\rangle}\vec{\nabla}^c\vec{\nabla}^d X_{cd} \\
&= \vec{\nabla}_{\langle a}\vec{\nabla}_{b\rangle}\vec{\mathrm{div}}\,\vec{\mathrm{div}}\,S \\
&= \tfrac{2}{3}\vec{\nabla}_{\langle a}\vec{\nabla}_{b\rangle}\left(\vec{\nabla}^2 + 3\mathcal{K}\right)\vec{\nabla}^2 S.
\end{aligned} \tag{5.49}$$

The non-local scalar S is given formally from X_{ab} by

$$\begin{aligned} S &= \frac{3}{2}\left(\vec{\nabla}^2 + 3\mathcal{K}\right)^{-1}\vec{\nabla}^{-2}\vec{\mathsf{div}}\,\vec{\mathsf{div}}\,X \\ &= \frac{3}{2}\left(\vec{\nabla}^2 + 3\mathcal{K}\right)^{-1}\vec{\nabla}^{-2}\breve{S}. \end{aligned} \tag{5.50}$$

This latter relation trivially gives the relation in Fourier space by replacing $\vec{\nabla}^2 \mapsto -k^2$. Defining $\breve{S}_{ab}$ to preserve the rank of the original tensor allows us to find the wave equation it satisfies easily. First, note that any curls in $\mathcal{L}$ commute with $\hat{\mathscr{S}}$ trivially, and note that

$$\vec{\nabla}^2\left(\vec{\nabla}_{\langle a}\vec{\nabla}_{b\rangle}\vec{\mathsf{div}}\,\vec{\mathsf{div}}\right) = \left(\vec{\nabla}_{\langle a}\vec{\nabla}_{b\rangle}\vec{\mathsf{div}}\,\vec{\mathsf{div}}\right)\vec{\nabla}^2. \tag{5.51}$$

We then see that the locally defined scalar quantity $\breve{S}_{ab}$ obeys

$$\mathcal{L}[\breve{S}_{ab}] = \hat{\mathscr{S}}[S_{ab}] = \vec{\nabla}_{\langle a}\vec{\nabla}_{b\rangle}\vec{\mathsf{div}}\,\vec{\mathsf{div}}\,S. \tag{5.52}$$

5.5.2 Vectors

We begin by noting that the operator $\vec{\mathsf{curl}}\,\vec{\mathsf{div}} = 2\vec{\mathsf{div}}\,\vec{\mathsf{curl}}$ knocks out the scalar and tensor part of X_{ab}, and is solenoidal by equation (5.42) (note that it forms a rank-1 tensor from X_{ab}); thus, $\breve{V}_a = \vec{\mathsf{curl}}\,\vec{\mathsf{div}}\,X_a$ is a locally defined vector. The rank-2 extraction of vector modes may be given by taking the distortion of this operator,

$$\begin{aligned} \breve{V}_{ab} &\equiv \hat{\mathscr{V}}[X_{ab}] \equiv \vec{\mathsf{dis}}\,\vec{\mathsf{curl}}\,\vec{\mathsf{div}}\,X_{ab} \\ &= \varepsilon_{cd\langle a}\vec{\nabla}_{b\rangle}\vec{\nabla}^c\vec{\nabla}^e X_e{}^d \\ &= \vec{\mathsf{dis}}\,\vec{\mathsf{curl}}\,\vec{\mathsf{div}}\,\vec{\mathsf{dis}}\,V_{ab} && (5.53) \\ &= \tfrac{1}{2}\vec{\mathsf{dis}}\,\vec{\mathsf{curl}}\left(\vec{\nabla}^2 + 2\mathcal{K}\right)V_{ab}. && (5.54) \end{aligned}$$

The non-local vector V_a may be given formally from X_{ab} by

$$\begin{aligned} V_a &= 2\left(-\vec{\nabla}^4 + 4\mathcal{K}^2\right)^{-1} \vec{\mathsf{curl}}^{\,2}\vec{\mathsf{div}}\, X_a \\ &= 2\left(-\vec{\nabla}^4 + 4\mathcal{K}^2\right)^{-1} \vec{\mathsf{curl}}\, \check{V}_a. \end{aligned} \tag{5.55}$$

This last relation enables us to relate the local and non-local vectors in Fourier space as $\check{V}^{(k)} = \frac{1}{2}(k^2 + 2\mathcal{K})^{1/2}(-k^2 + 2\mathcal{K})\bar{V}^{(k)}$, with the same relation for the opposing parity. Again, the otherwise superfluous preservation of the rank of X_{ab} in defining the operator $\hat{\mathscr{V}}$ – the extra $\vec{\mathsf{dis}}$ term – allows $\hat{\mathscr{V}}$ and $\mathcal{L}$ to commute [to prove this for the Laplacian, we use equations (5.35), (5.34) and (5.40); for curl, we use equations (5.37) followed by (5.30)], giving our wave equation for $\check{V}_{ab}$ as

$$\mathcal{L}[\check{V}_{ab}] = \hat{\mathscr{V}}[S_{ab}] = \vec{\mathsf{dis}}\,\vec{\mathsf{curl}}\,\vec{\mathsf{div}}\, S_{ab}. \tag{5.56}$$

5.5.3 Tensors

The key difficulty here lies in finding the differential operator which when acting on $\vec{\mathsf{dis}}\, V_{ab}$ produces zero by virtue of V_a being solenoidal (while obviously leaving the transverse part of X_{ab}). It is straightforward to verify that $[\vec{\mathsf{curl}}^{\,2} + \frac{1}{2}\vec{\mathsf{dis}}\,\vec{\mathsf{div}} - \mathcal{K}]\vec{\mathsf{dis}}\, V_{ab} = 0$ for $\vec{\mathsf{div}}\, V = 0$, which follows from equations (5.30), (5.32) and (5.31). We must also take an extra $\vec{\mathsf{curl}}$ to remove the scalar part. Therefore, our local tensor extraction may be defined as

$$\begin{aligned} \check{T}_{ab} &\equiv \hat{\mathscr{T}}[X_{ab}] \equiv \left[-\vec{\nabla}^2 + 2\mathcal{K} + 2\vec{\mathsf{dis}}\,\vec{\mathsf{div}}\right] \vec{\mathsf{curl}}\, X_{ab} \\ &= \varepsilon_{cd\langle a}\left[\left(-\vec{\nabla}^2 + 2\mathcal{K}\right)\vec{\nabla}^c X_{b\rangle}{}^d + \vec{\nabla}_{b\rangle}\vec{\nabla}^e\vec{\nabla}^c X_e{}^d\right] \\ &\quad + \varepsilon_{cde}\vec{\nabla}_{\langle a}\vec{\nabla}^e\vec{\nabla}^c X_{b\rangle}{}^d \\ &= \left[-\vec{\nabla}^2 + 2\mathcal{K}\right] \vec{\mathsf{curl}}\, T_{ab}. \end{aligned} \tag{5.57}$$

Note that $\vec{\text{curl}}$ commutes with the operator in square brackets. It is relatively straightforward to verify that $\check{T}_{ab}$ is transverse, $\vec{\text{div}}\,\check{T}_a = 0$, showing that this represents a tensor mode as expected: First note that we may write

$$\hat{\mathscr{T}} = \tfrac{1}{3}\left[\vec{\nabla}^2 + 4\vec{\text{curl}}^{\,2} - 6\mathcal{K}\right]\vec{\text{curl}}\,, \tag{5.58}$$

which follows from equation (5.38), so that

$$\begin{aligned} \vec{\text{div}}\,\hat{\mathscr{T}} &= \tfrac{1}{3}\left[\vec{\nabla}^2 + \vec{\text{curl}}^{\,2} - 2\mathcal{K}\right]\vec{\text{div}}\,\vec{\text{curl}} \\ &= \tfrac{1}{3}\vec{\nabla}\,\vec{\text{div}}\,\vec{\text{div}}\,\vec{\text{curl}} = 0, \end{aligned} \tag{5.59}$$

which uses equations (5.37) and (5.40) on the first line, then equation (5.31) to get to the second, and finally equation (5.29) to show the last expression is zero. The formal, non-local, TT tensor T_{ab} is given in terms of the original tensor X_{ab} by taking a further $\vec{\text{curl}}$ to obtain

$$T_{ab} = \left(-\vec{\nabla}^2 + 3\mathcal{K}\right)^{-1}\left(-\vec{\nabla}^2 + 2\mathcal{K}\right)^{-1}\hat{\mathscr{T}}[\vec{\text{curl}}\,X_{ab}]. \tag{5.60}$$

To convert our extraction into Fourier space we define tensor harmonics as $\vec{\nabla}^2 Q^{(T)}_{ab} = -k^2 Q^{(T)}_{ab}$, where we have two parities of orthogonal harmonics, $(k^2+3\mathcal{K})^{1/2}Q^{(T)}_{ab} = \vec{\text{curl}}\,\bar{Q}^{(T)}_{ab} \Leftrightarrow (k^2+3\mathcal{K})^{1/2}\bar{Q}^{(T)}_{ab} = \vec{\text{curl}}\,Q^{(T)}_{ab}$. We therefore find

$$\check{T}^{(k)} = \left(k^2 + 3\mathcal{K}\right)^{1/2}\left(k^2 + 2\mathcal{K}\right)\bar{T}^{(k)}, \tag{5.61}$$

with the same relation for the opposite parity. In order for $\hat{\mathscr{T}}$ to be useful it will have to operate on a wave equation, equation (5.48). It is clear from the form of $\hat{\mathscr{T}}$ given in equation (5.58) that $\mathcal{L}$ and $\hat{\mathscr{T}}$ commute, so that $\check{T}_{ab}$ obeys

$$\begin{aligned} \mathcal{L}[\check{T}_{ab}] &= \hat{\mathscr{T}}[S_{ab}] = \left[-\vec{\nabla}^2 + 2\mathcal{K} + 2\vec{\text{dis}}\,\vec{\text{div}}\right]\vec{\text{curl}}\,S_{ab} \\ &= \left[\tfrac{16}{5}\vec{\text{dis}}\,\vec{\text{div}} - 3\vec{\text{div}}\,\vec{\text{dis}} + 8\mathcal{K}\right]\vec{\text{curl}}\,X_{ab}. \end{aligned} \tag{5.62}$$

We are now in a position to determine the key second-order tensor variable. Consider the situation where only scalar modes are excited at first order, a tensor such as the shear σ_{ab} will only manifest the scalar part. Let us denote this by σ_{ab_s}. Applying the tensor extractor (5.58) on σ_{ab} will therefore extract the tensor part (see equation 5.63). This part is second order given that only scalars are excited at first order. Since we do not have vectors and tensors at first order, it is sufficient to use $\vec{\mathsf{curl}}$ as a tensor extractor rather than the full $\hat{\mathscr{T}}$. This is more so because the pre-factor $(-\vec{\nabla}^2 + 2\mathcal{K})$, which appears on (5.57), is rank preserving and commutes with the time derivative. In what follows, we revert to the standard $\mathsf{curl}\,(= \vec{\mathsf{curl}}\,/a)$.

5.6 The key second order variable

As discussed above, we consider tensor modes excited by first-order scalars. Limiting our discussion to dust ($w = 0 = A^a = 0 = p = 0$), we can group variables into background, first and second-order respectively:

- background: μ, Θ.
- first-order: $X_a, \mathcal{Z}_a, \sigma_{ab}, E_{ab}$.
- second-order: $\mathsf{curl}\,\sigma_{ab}$.

This implies that at linear order, all rank-1 and rank-2 variables have only scalar contributions. But for vanishing vorticity $\mathsf{curl}\,\sigma_{ab} = H_{ab}$, which has both tensor and vector parts. $\mathsf{curl}\,\sigma_{ab}$ should therefore give a second-order tensor, in the case where vectors vanish. Let us formally define a quantity with this characteristics,

$$\Sigma_{ab} = \mathsf{curl}\,\sigma_{ab}, \tag{5.63}$$

with respect to FLRW background and where only scalar modes are excited at linear order.

It follows that equation (5.7) becomes an evolution for our second-order quantity, when the third order term is dropped:

$$\dot{\Sigma}_{ab} = -\Theta\Sigma_{ab} - \mathsf{curl}\,(E)_{ab}. \tag{5.64}$$

Taking the time derivative of (5.64), making use of the evolution equations for FOGI quantities and using the commutation relation (A.12), it follows that

$$\begin{aligned} \ddot{\Sigma}_{ab} + \mathsf{curl}\,\mathsf{curl}\,\Sigma_{ab} + \tfrac{7}{3}\Theta\dot{\Sigma}_{ab} + (\Theta^2 - \mu)\Sigma_{ab} &= \epsilon_{cd\langle a}[\tfrac{3}{2}E_{b\rangle}{}^{d}\mathsf{div}(\sigma^c) + \tfrac{3}{2}\sigma_{b\rangle}{}^{d}\tilde{\nabla}^{e}E_{e}{}^{c} \\ &\quad + \sigma^{ec}\tilde{\nabla}_{|e|}E_{b\rangle}{}^{d}] - 3\,\mathsf{curl}\,(\sigma_{c\langle a}E_{b\rangle}{}^{c}). \end{aligned} \tag{5.65}$$

There is an alternative route to obtaining this evolution equation, first note that one can derive the general wave equation for first-order shear in a straight forward way. This follows from taking the time derivative of equation (5.19), for the case of dust and using the evolution equation for σ_{ab} and E_{ab}. From these we obtain,

$$\ddot{\sigma}_{ab} + \mathsf{curl}\,\mathsf{curl}\,\sigma_{ab} + \tfrac{5}{3}\Theta\dot{\sigma}_{ab} + (\tfrac{4}{9}\Theta^2 - \tfrac{5}{6}\mu)\sigma_{ab} = \sigma_{c\langle a}\dot{\sigma}^{c}_{b\rangle} + \Theta\sigma_{c\langle a}\sigma_{b\rangle}{}^{c}. \tag{5.66}$$

Now taking the curl of this equation produces,

$$\begin{aligned} \mathsf{curl}\,\ddot{\sigma}_{ab} + \mathsf{curl}\,\mathsf{curl}\,\mathsf{curl}\,\sigma_{ab} + \tfrac{5}{3}\mathsf{curl}\,(\Theta\dot{\sigma}_{ab}) + \epsilon_{cd\langle a}\sigma_{b\rangle}{}^{d}[\tfrac{8}{9}\Theta\tilde{\nabla}^{c}\Theta - \tfrac{5}{6}\tilde{\nabla}^{c}\mu] + (\tfrac{4}{9}\Theta^2 - \tfrac{5}{6}\mu)\Sigma_{ab} \\ = \mathsf{curl}\,[\sigma_{c\langle a}\dot{\sigma}^{c}_{b\rangle} + \Theta\sigma_{c\langle a}\sigma_{b\rangle}{}^{c}]. \end{aligned} \tag{5.67}$$

We can now expand each term and re-express it in terms of Σ_{ab} where possible. Note that equation (5.64) can be written as,

$$\mathrm{curl}\,\dot{\sigma}_{ab} = \dot{\Sigma}_{ab} + \tfrac{1}{3}\Theta\Sigma_{ab} + \epsilon_{cd\langle a}\sigma^{ec}\tilde{\nabla}_{|e|}\sigma_{b\rangle}{}^{d}. \tag{5.68}$$

The time derivative of this equation takes the form,

$$\mathrm{curl}\,\ddot{\sigma}_{ab} = \ddot{\Sigma}_{ab} + 2\epsilon_{cd\langle a}\sigma^{ec}\tilde{\nabla}_{|e|}\dot{\sigma}^{d}_{b\rangle} + (\tfrac{1}{3}\dot{\Theta} + \tfrac{1}{9}\Theta^2)\Sigma_{ab} + \tfrac{2}{3}\Theta\dot{\Sigma}_{ab} + \epsilon_{cd\langle a}\dot{\sigma}^{ec}\tilde{\nabla}_{|e|}\sigma_{b\rangle}{}^{d}. \tag{5.69}$$

It can be shown that the remaining terms take the following forms:

$$\begin{aligned}
\mathrm{curl}\,\mathrm{curl}\,\Sigma_{ab} &= \mathrm{curl}\,\mathrm{curl}\,\mathrm{curl}\,\sigma_{ab}, && (5.70)\\
\tfrac{5}{3}\mathrm{curl}\,(\Theta\dot{\sigma}_{ab}) - \tfrac{5}{2}\epsilon_{cd\langle a}\dot{\sigma}^{d}_{b\rangle}\mathrm{div}(\sigma^c) &= \tfrac{5}{3}\Theta\dot{\Sigma}_{ab} + \tfrac{5}{9}\Theta^2\Sigma_{ab} + \tfrac{5}{3}\Theta\epsilon_{cd\langle a}\sigma^{ec}\tilde{\nabla}_{|e|}\sigma_{b\rangle}{}^{d}, && (5.71)\\
\epsilon_{cd\langle a}\sigma_{b\rangle}{}^{d}[\tfrac{8}{9}\Theta\tilde{\nabla}^c\Theta - \tfrac{5}{6}\tilde{\nabla}^c\mu] &= [\tfrac{9}{3}\Theta\epsilon_{cd\langle a}\sigma_{b\rangle}{}^{d}\mathrm{div}(\sigma^c) + \tfrac{5}{2}\epsilon_{cd\langle a}\sigma_{b\rangle}{}^{d}\mathrm{div}(\dot{\sigma}^c)], && (5.72)\\
\mathrm{curl}\,(\sigma_{c\langle a}\dot{\sigma}^{c}_{b\rangle}) &= -\tfrac{1}{2}\epsilon_{cd\langle a}(\tilde{\nabla}_e[\dot{\sigma}^{d}_{b\rangle}\sigma^{ce} + \sigma^{d}_{b\rangle}\dot{\sigma}^{ce}]), && (5.73)\\
\mathrm{curl}\,(\Theta\sigma_{c\langle a}\sigma_{b\rangle}{}^{c}) &= -\Theta\epsilon_{cd\langle a}\tilde{\nabla}_e[\sigma_{b\rangle}{}^{d}\sigma^{ec}],. && (5.74)
\end{aligned}$$

Substituting (5.70-5.69) into (5.67) gives,

$$\begin{aligned}
\ddot{\Sigma}_{ab} + \mathrm{curl}\,\mathrm{curl}\,\Sigma_{ab} + \tfrac{7}{3}\Theta\dot{\Sigma}_{ab} + (\Theta^2 - \mu)\Sigma_{ab} = \epsilon_{cd\langle a}[&-3\dot{\sigma}^{d}_{b\rangle}\mathrm{div}(\sigma^c) - 4\Theta\sigma_{b\rangle}{}^{d}\mathrm{div}(\sigma^c)\\
&-3\sigma_{b\rangle}{}^{d}\mathrm{div}(\dot{\sigma}^c) - \tfrac{5}{2}\sigma^{ec}\tilde{\nabla}_{|e|}\dot{\sigma}^{d}_{b\rangle} - \tfrac{3}{2}\dot{\sigma}^{ec}\tilde{\nabla}_{|e|}\sigma_{d\rangle}{}^{d} - \tfrac{8}{3}\Theta\sigma^{ec}\tilde{\nabla}_{|e|}\sigma_{b\rangle}{}^{d}].
\end{aligned} \tag{5.75}$$

This is just equation (5.65) with the source expressed in terms of shear rather than the electric part of the Weyl tensor (see 5.19).

We now seek to show that the source term to equation (5.75), which we denote by $\mathcal{F}_{ab}$, is also gauge invariant. Ideally we need to demonstrate that $\mathcal{F}_{ab} = 0$ when $\Sigma_{ab} = 0$.

From equation (5.65), we see that

$$\mathcal{F}_{ab} = -3\,\mathrm{curl}\,(\sigma_{c\langle a}E_{b\rangle}{}^{c}) + \epsilon_{cd\langle a}[\tfrac{3}{2}E_{b\rangle}{}^{d}\mathrm{div}(\sigma^{c}) + \tfrac{3}{2}\sigma_{b\rangle}{}^{d}\tilde{\nabla}^{e}E_{e}{}^{c} + \sigma^{ec}\tilde{\nabla}_{|e|}E_{b\rangle}{}^{d}]. \quad (5.76)$$

However, one can expand the first term in this source and express it in terms of Σ as follows,

$$\begin{aligned} 3\,\mathrm{curl}\,(\sigma_{c\langle a}E_{b\rangle}{}^{c}) &= \mathrm{curl}\,\dot{E}_{ab} + \Theta\,\mathrm{curl}\,E_{ab} + \tfrac{3}{2}\epsilon_{cd\langle a}E_{b\rangle}{}^{d}\mathrm{div}(\sigma^{c}) - \mathrm{curl}\,\mathrm{curl}\,\Sigma_{ab} \\ &\quad + \tfrac{1}{2}\mu\Sigma_{ab} + \tfrac{3}{2}\epsilon_{cd\langle a}\sigma_{b\rangle}{}^{d}\mathrm{div}(E^{c}). \end{aligned} \quad (5.77)$$

Upon substituting into equation (5.76) and using commutation relation (A.11), we find

$$\begin{aligned} \mathcal{F}_{ab} &= -\mathrm{curl}\,\dot{E}_{ab} - \Theta\,\mathrm{curl}\,E_{ab} + \mathrm{curl}\,\mathrm{curl}\,\Sigma_{ab} - \tfrac{1}{2}\mu\Sigma_{ab} + \epsilon_{cd\langle a}\sigma^{ec}\tilde{\nabla}_{|e|}E_{b\rangle}{}^{d} \\ &= -(\mathrm{curl}\,E_{ab})^{\cdot} - \tfrac{4}{3}\Theta\,\mathrm{curl}\,E_{ab} + \mathrm{curl}\,\mathrm{curl}\,\Sigma_{ab} - \tfrac{1}{2}\mu\Sigma_{ab}. \end{aligned} \quad (5.78)$$

Note, from equation (5.64), that $\Sigma_{ab} = 0 \Rightarrow \mathrm{curl}\,E_{ab} = 0$ and so $\mathcal{F}_{ab}$ vanishes at first order as required and implies that $\mathcal{F}_{ab}$ is also gauge-invariant. In order to the Σ_{ab} wave equation, we need to determine how $\mathcal{F}_{ab}$ evolves. This will be addressed in the next section.

5.7 Analysing the source term $\mathcal{F}_{ab}$

In order to discuss the source term, it is useful to introduce new variables to denote the first order products that make up this source [24]. These are,

$$\begin{aligned} \psi_{1ab} &= \epsilon_{cd\langle a}\mathrm{div}(\sigma^{c})\sigma_{b\rangle}{}^{d}, \qquad \psi_{2ab} = \epsilon_{cd\langle a}\mathrm{div}(\sigma^{c})\dot{\sigma}^{d}_{b\rangle}, \\ \psi_{3ab} &= \epsilon_{cd\langle a}\mathrm{div}(\dot{\sigma}^{c})\sigma_{b\rangle}{}^{d}, \qquad \psi_{4ab} = \epsilon_{cd\langle a}\mathrm{div}(\dot{\sigma}^{c})\dot{\sigma}^{d}_{b\rangle}, \end{aligned} \quad (5.79)$$

and

$$\begin{aligned}\xi_{1ab} &= \epsilon_{cd\langle a}\sigma^{ec}\tilde{\nabla}_{|e|}\sigma_{b\rangle}{}^{d}, \qquad \xi_{2ab} = \epsilon_{cd\langle a}\dot{\sigma}^{ec}\tilde{\nabla}_{|e|}\sigma_{b\rangle}{}^{d}, \\ \xi_{3ab} &= \epsilon_{cd\langle a}\sigma^{ec}\tilde{\nabla}_{|e|}\dot{\sigma}^{d}_{b\rangle}, \qquad \xi_{4ab} = \epsilon_{cd\langle a}\dot{\sigma}^{ec}\tilde{\nabla}_{|e|}\dot{\sigma}^{d}_{b\rangle}. \end{aligned} \tag{5.80}$$

It follows that the source takes a very simple form when expressed in these new variables,

$$\mathcal{F}_{ab} = -\tfrac{9}{2}\psi_{2ab} - 4\Theta\psi_{1ab} - \tfrac{3}{2}\psi_{3ab} - 4\xi_{3ab} - \tfrac{8}{3}\Theta\xi_{1ab}. \tag{5.81}$$

Taking the time derivative of each variable given in (5.79), and making use of (A.6), we find the following closed set of first order differential equations,

$$\dot{\psi}_{1ab} = \psi_{2ab} + \psi_{3ab} - \tfrac{1}{3}\Theta\psi_{1ab}, \tag{5.82}$$

$$\dot{\psi}_{2ab} = -2\Theta\psi_{2ab} + \psi_{4ab} - (\tfrac{4}{9}\Theta^2 - \tfrac{5}{6}\mu)\psi_{1ab}, \tag{5.83}$$

$$\dot{\psi}_{3ab} = -2\Theta\psi_{3ab} + \psi_{4ab} - (\tfrac{4}{9}\Theta^2 - \tfrac{5}{6}\mu)\psi_{1ab}, \tag{5.84}$$

$$\dot{\psi}_{4ab} = -(\tfrac{4}{9}\Theta^2 - \tfrac{5}{6}\mu)\psi_{2ab} - (\tfrac{4}{9}\Theta^2 - \tfrac{5}{6}\mu)\psi_{3ab} - \tfrac{11}{3}\Theta\psi_{4ab}. \tag{5.85}$$

It is clear that this set takes the form $\dot{\psi}_{ab} = \Psi\psi_{ab}$, and can be solved relatively easily. Similarly, the time derivative of (5.80) gives rise to a system of differential equations of the form; $\dot{\xi}_{ab} = \Xi\xi_{ab}$, where $\Xi = \Psi$. The substitution of the solutions to these sets of differential equations into equation (5.81) yields,

$$\mathcal{F}_{ab} = \alpha_{ab}t^{-4} + \beta_{ab}t^{-\frac{7}{3}} + \gamma_{ab}t^{-\frac{17}{3}}, \tag{5.86}$$

where (α_{ab}, β_{ab} and γ_{ab}) are the integration constants determined by the initial conditions. An alternative method for determining $\mathcal{F}_{ab}$ involves taking the time derivative of (5.81), expressing the resulting equation in terms of the original variable and isolating a new source;

$$\dot{\mathcal{F}}_{ab} + 2\Theta\mathcal{F}_{ab} = \mathcal{A}_{ab}, \tag{5.87}$$

where

$$\mathcal{A}_{ab} = 6\psi_{4ab} + 4\Theta\psi_{2ab} + 4\Theta\psi_{3ab} + \tfrac{11}{3}\Theta^2\psi_{1ab} + 4\Theta\xi_{4ab} + \tfrac{8}{3}\Theta\xi_{2ab} + \tfrac{8}{3}\Theta\xi_{3ab}. \tag{5.88}$$

Now taking the time derivative of (5.87), expressing the resulting equation in terms $\mathcal{A}_{ab}$ and isolating a new source term leads to,

$$\dot{\mathcal{A}}_{ab} + \tfrac{8}{3}\Theta\mathcal{A}_{ab} = \mathcal{B}_{ab}, \tag{5.89}$$

where

$$\begin{aligned}\mathcal{B}_{ab} \;=\; & -2\Theta\psi_{4ab} - \tfrac{4}{3}\Theta\xi_{4ab} - \tfrac{10}{3}\Theta^2\psi_{3ab} - \tfrac{10}{3}\Theta^2\psi_{2ab} - \tfrac{32}{9}\Theta^3\psi_{1ab} - \tfrac{20}{9}\Theta^2\xi_{2ab} \\ & -\tfrac{20}{9}\Theta^2\xi_{3ab} - \tfrac{64}{27}\Theta^3\xi_{1ab}.\end{aligned} \tag{5.90}$$

Finally, it follows that the evolution equation for (5.90) can be expressed in terms of $\mathcal{B}_{ab}$ and $\mathcal{A}_{ab}$, and takes the form,

$$\dot{\mathcal{B}}_{ab} + \tfrac{17}{6}\Theta\mathcal{B}_{ab} - 2\mu\mathcal{A}_{ab} = 0. \tag{5.91}$$

The solution to the coupled system of equations given by (5.87, 5.88, 5.91) reproduces (5.86). So far we have only resolved the source term to equation (5.65), we now proceed to solve the wave equation.

5.7.1 Solutions

From equation (5.86), it follows that

$$\ddot{\Sigma}_{ab} - \tilde{\nabla}^2\Sigma_{ab} + \tfrac{7}{3}\Theta\dot{\Sigma}_{ab} + (\Theta^2 - \mu)\Sigma_{ab} \;=\; \mathcal{F}_{ab} = \alpha_{ab}t^{-4} + \beta_{ab}t^{-\frac{7}{3}} + \gamma_{ab}t^{-\frac{17}{3}}. \quad (5.92)$$

Notice that $\Theta^2/3 = \rho$ if the background is flat. In fact we shall consider this case in the following analysis. We begin by performing a harmonic decomposition of the above equation. In order to achieve this, we use the 1+3 tensor harmonics introduced in Chapter (2). Details of this decomposition can be found in [77, 43, 105, 21]. Since Σ_{ab} and $\mathcal{F}_{ab}$ are function of time and space, they should ideally be written as $\Sigma_{ab}(t,x)$ and $\mathcal{F}_{ab}(t,x)$. The harmonic decomposition of these variables into temporal and spatial dependent parts take the form;

$$\Sigma_{ab}(t,x) = a^{-2}\sum_{\kappa}\kappa^2\left[\Sigma^{(\kappa)}(t)Q_{ab}(\kappa,x) + \bar{\Sigma}^{(\kappa)}(t)\bar{Q}_{ab}(\kappa,x)\right], \quad (5.93)$$

and

$$\mathcal{F}_{ab}(t,x) = a^{-2}\sum_{\kappa}\kappa^2\left[\mathcal{F}^{(\kappa)}(t)Q_{ab}(\kappa,x) + \bar{\mathcal{F}}^{(\kappa)}(t)\bar{Q}_{ab}(\kappa,x)\right]. \quad (5.94)$$

It follows that the decomposition of (5.92) reduces to,

$$\ddot{\Sigma}^{(\kappa)} + \tfrac{\kappa^2}{a^2}\Sigma^{(\kappa)} + \tfrac{7}{3}\Theta\dot{\Sigma}^{(\kappa)} + (\tfrac{2}{3}\Theta^2\Sigma^{(\kappa)} \;=\; \mathcal{F}^{(\kappa)}, \quad (5.95)$$

for one parity. The opposite parity has a similar structure. It is straight forward to show that the solutions to the homogeneous part of the wave equation (5.95) are,

$$\Sigma_1^{(\kappa)} = C_1(\kappa)t^{-\frac{11}{6}}J\left(\tfrac{5}{6},\tfrac{\kappa}{a}t\right), \qquad \Sigma_2^{(k)} = C_2(\kappa)t^{-\frac{11}{6}}Y\left(\tfrac{5}{6},\tfrac{\kappa}{a}t\right), \quad (5.96)$$

where $C_1(\kappa)$ and $C_2(\kappa)$ are the constants of integration, while J and Y are the Bessel functions of the first and the second kinds respectively. The general solutions can then be found using Green's method as follows,

$$\Sigma^{(\kappa)} = C_1(\kappa)t^{-\frac{11}{6}}J\left(\tfrac{5}{6},\tfrac{\kappa}{a}t\right)+C_2(\kappa)t^{-\frac{11}{6}}Y\left(\tfrac{5}{6},\tfrac{\kappa}{a}t\right)+\tfrac{\pi}{2}\Sigma_1^{(\kappa)}\int\Sigma_2^{(\kappa)}\mathcal{F}^{(\kappa)}dt+\tfrac{\pi}{2}\Sigma_2^{(\kappa)}\int\Sigma_1^{(\kappa)}\mathcal{F}^{(\kappa)}dt. \tag{5.97}$$

For completeness, we also present numerical solutions to the wave equation in figure (5.1) demonstrating how sensitive the wave is to the initial conditions. The results in this section recover those found in [24]. We now turn to $w \neq 0$ case, where we obtain new results. We next consider a barotropic fluid with vanishing vorticity.

5.7.2 Barotropic perfect fluid

The starting point is again the situation where only scalar perturbations are allowed at linear order, this implies that Σ_{ab} is second order. In this case the presence of pressure gives rise to an extra term in the source to the evolution equation for Σ_{ab}. In particular,

$$\dot{\Sigma}_{ab}+\Theta\Sigma_{ab}+\mathsf{curl}\,(E)_{ab}=-2\epsilon_{cd\langle a}A^cE_{b\rangle}{}^d, \tag{5.98}$$

where we have used $A^c \equiv \dot{u}^c \neq 0$ (This should not be confused with the $\mathcal{A}_{ab}$ used in (5.88)). Using the momentum conservation equation and equation (5.23), we can now express A^c in the following form,

$$A^c=-\frac{\tilde{\nabla}_a p}{(\mu+p)}=-\frac{w\tilde{\nabla}_a\mu}{\mu(1+w)}=-\frac{3w\tilde{\nabla}_eE^{ec}}{\mu(1+w)}, \tag{5.99}$$

where w is taken to be constant with respect to spatial derivatives (i.e. the spatial derivatives vanish). Note that w may still vary with time (see Chapter 6). To begin with, we consider the case where w is constant with respect to both time and space. Therefore

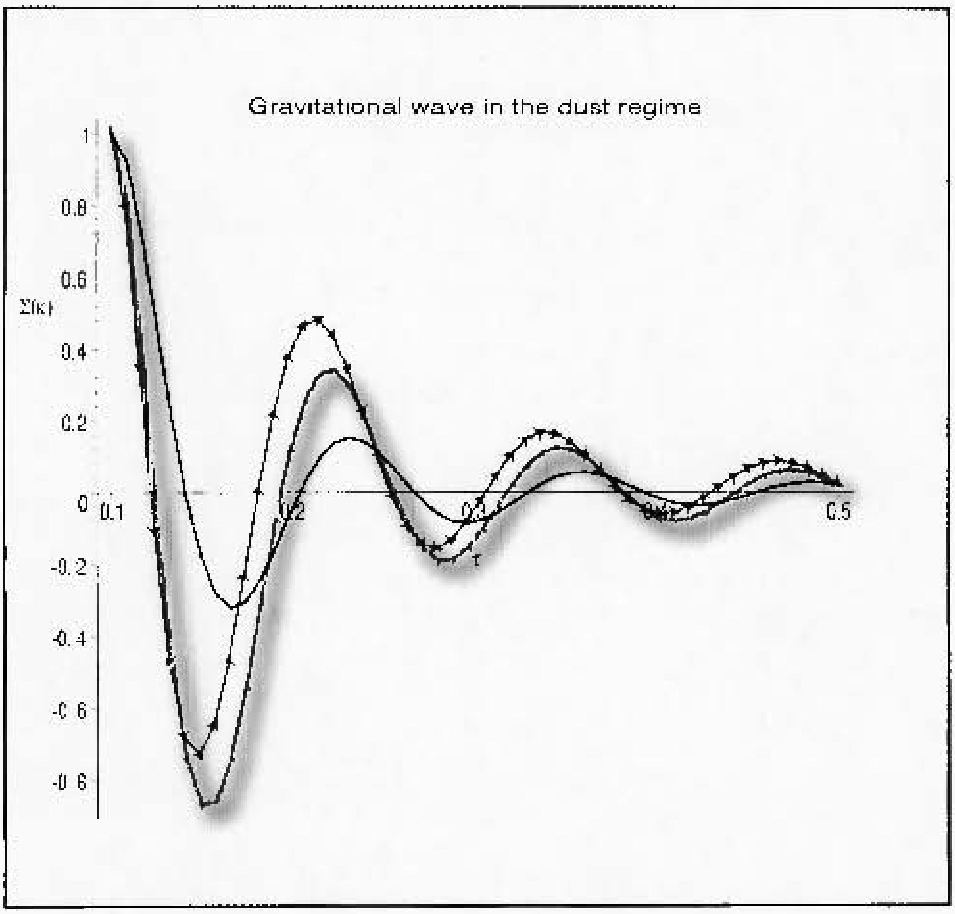

Figure 5.1: These curves represent numerical solutions to the gravitational wave equation (5.95) given different scaling of coefficients of the source term. The initial conditions are $\Sigma(\kappa) = \Sigma^{(\kappa)}/\Sigma_0^{(\kappa)} = 1, \Sigma(\kappa)' = 0,\ k/a = 25$. The red curve (with arrow head) represents the numerical solution for the homogeneous wave. The green curve (with dots) is the solution with $\alpha = 0, \beta = .08, \gamma = -0.03$. The black curve is the solution curve for GW whose source has the coefficients; $\alpha = 0.1, \beta = 0, \gamma = -0.02$.

taking the time derivative of (5.98), where w is a constant, gives rise to

$$\ddot{\Sigma}_{ab} + \mathsf{curl}\,\mathsf{curl}\,\Sigma_{ab} + \tfrac{7}{3}\Theta\dot{\Sigma}_{ab} + [\Theta^2 - \mu(1+2w)]\Sigma_{ab} = \mathcal{F}_{ab}, \tag{5.100}$$

where the source is given by,

$$\begin{aligned} \mathcal{F}_{ab} &= \epsilon_{cd\langle a}[\tfrac{9}{2}E_{b\rangle}{}^{d}\mathsf{div}(\sigma^c) + \tfrac{3}{2}(1+w)\sigma_{b\rangle}{}^{d}\tilde{\nabla}^e E_e{}^c + 4\sigma^{ec}\tilde{\nabla}_{|e|}E_{b\rangle}{}^{d}] \\ &\quad \tfrac{15w}{\mu(1+w)}\epsilon_{cd\langle a}\dot{E}_{b\rangle}\mathsf{div}(E^c) + \tfrac{(13+6w)w}{\mu(1+w)}\Theta\epsilon_{cd\langle a}E_{b\rangle}\mathsf{div}(E^c). \end{aligned} \tag{5.101}$$

In deriving this equation, we have used the commutation relation [163],

$$(\mathsf{curl}\,T_{ab})^{\cdot} = \mathsf{curl}\,\dot{T}_{ab} - \tfrac{1}{3}\Theta\mathsf{curl}\,T_{ab} - \epsilon_{cd\langle a}\sigma^{ec}\tilde{\nabla}_{|e|}T_{b\rangle}{}^{d} + \epsilon_{cd\langle a}[A^c\dot{T}^d_b + \tfrac{1}{3}\Theta A^c T_{b\rangle}{}^{d} + 3H_{c\langle a}T_{b\rangle}{}^{c}]. \tag{5.102}$$

Further simplification of the source is obtained by making use of the following relations for $\sigma_b{}^d$ and $\mathsf{div}(\sigma)^c$ given by

$$\sigma_b{}^d = -\frac{2(\dot{E}^d_b + \Theta E_b{}^d)}{\mu(1+w)}, \tag{5.103}$$

$$\mathsf{div}\sigma^c = -\frac{2(\mathsf{div}(\dot{E})^c + \Theta\mathsf{div}E^c)}{\mu(1+w)}, \tag{5.104}$$

which follows linearization of equation (5.6). The downside to using these relations is that they do not apply when $w = -1$ which would correspond to cosmological constant or dark energy dominated regimes, cases that are topical at the moment. For now, our substitutions lead to a source that is made up of products that involve only the electric part of the Weyl tensor. These are $\epsilon_{cd\langle a}E_{b\rangle}{}^{d}\mathsf{div}E^c$ and $\epsilon_{cd\langle a}E^{ec}\tilde{\nabla}_{|e|}E_{b\rangle}{}^{d}$ and their temporal derivatives. One can demonstrate that the propagation of these two terms give rise to two separate, but structurally similar, systems that have identical solutions. The source now takes the following simplified form,

$$\mathcal{F}_{ab} = \tfrac{-20+12w}{\mu(1+w)}\epsilon_{cd\langle a}\dot{E}^{d}_{b\rangle}\mathsf{div}(E^{c}) + \tfrac{-20+10w+6w^{2}}{\mu(1+w)}\Theta\epsilon_{cd\langle a}E_{b\rangle}{}^{d}\mathsf{div}(E^{c}). \tag{5.105}$$

We shall apply the techniques similar to those employed in the analysis of the dust subcase, in our analysis of the waves equation (5.100) where $w \neq 0$. In particular, we define and make use of the following new second-order variables,

$$\psi_{1ab} = \epsilon_{cd\langle a}E_{b\rangle}{}^{d}\mathsf{div}(E)^{c}, \tag{5.106}$$
$$\psi_{2ab} = \epsilon_{cd\langle a}\dot{E}^{d}_{b\rangle}\mathsf{div}(E)^{c}, \tag{5.107}$$
$$\psi_{3ab} = \epsilon_{cd\langle a}\dot{E}^{d}_{b\rangle}\mathsf{div}(\dot{E})^{c}, \tag{5.108}$$
$$\psi_{4ab} = \mu\epsilon_{cd\langle a}L_{b\rangle}{}^{d}\mathsf{div}(E)^{c}, \tag{5.109}$$
$$\psi_{5ab} = \mu\epsilon_{cd\langle a}L_{b\rangle}{}^{d}\mathsf{div}(\dot{E})^{c}, \tag{5.110}$$
$$\psi_{6ab} = \mu\epsilon_{cd\langle a}L_{b\rangle}{}^{d}\mathsf{div}(L)^{c}, \tag{5.111}$$

where $L_{ab} = \tilde{\nabla}_{\langle a}A_{b\rangle}$. Whereas the first two terms appear in the source term $\mathcal{F}_{ab}$, the remaining arise in the propagation of the source. Similarly, we define

$$\xi_{1ab} = \epsilon_{cd\langle a}E^{ec}\tilde{\nabla}_{|e|}E_{b\rangle}{}^{d}, \tag{5.112}$$
$$\xi_{2ab} = \epsilon_{cd\langle a}\dot{E}^{ec}\tilde{\nabla}_{|e|}E_{b\rangle}{}^{d}, \tag{5.113}$$
$$\xi_{3ab} = \epsilon_{cd\langle a}\dot{E}^{ec}\tilde{\nabla}_{|e|}\dot{E}^{d}_{b\rangle}, \tag{5.114}$$
$$\xi_{4ab} = \epsilon_{cd\langle a}L^{ec}\tilde{\nabla}_{|e|}E_{b\rangle}{}^{d}, \tag{5.115}$$
$$\xi_{5ab} = \epsilon_{cd\langle a}E^{ec}\tilde{\nabla}_{|e|}\dot{E}^{d}_{b\rangle}, \tag{5.116}$$
$$\xi_{6ab} = \epsilon_{cd\langle a}L^{ec}\tilde{\nabla}_{|e|}L_{b\rangle}{}^{d}, \tag{5.117}$$

where the first two terms appear in the source term $\mathcal{F}_{ab}$.

5.7.3 Analysing the source $\mathcal{F}_{ab}$ for $w \neq 0$

We now use a method similar to that employed in the analysis of the source term for the dust case, which relies on the fact the system of differential equations closes and can be solved with relative ease. The propagation of the variables (5.106-5.111) yields;

$$
\begin{aligned}
\dot{\psi}_{1ab} &= -\tfrac{1}{3}\Theta\psi_{1ab} + 2\psi_{2ab}, && (5.118)\\
\dot{\psi}_{2ab} &= -(1+\tfrac{1}{3}w)\Theta^2\psi_{1ab} - (3+w)\Theta\psi_{2ab} + \psi_{3ab} - \tfrac{1}{2}(1+w)\psi_{4ab}, && (5.119)\\
\dot{\psi}_{3ab} &= -2(1+\tfrac{1}{3}w)\Theta^2\psi_{2ab} - (\tfrac{17}{3}+2w)\Theta\psi_{3ab} - (1+w)\psi_{5ab}, && (5.120)\\
\dot{\psi}_{4ab} &= -2\Theta\psi_{4ab} + \psi_{5ab} + w\mu\epsilon_{cd\langle a}\tilde{\nabla}^2\sigma_{b\rangle}{}^d\mathsf{div}(E)^c, && (5.121)\\
\dot{\psi}_{5ab} &= -(1+\tfrac{1}{3}w)\Theta^2\psi_{4ab} - (\tfrac{14}{3}+w)\Theta\psi_{5ab} - \tfrac{1}{2}\mu(1+w)\psi_{6ab} + w\mu\epsilon_{cd\langle a}\tilde{\nabla}^2\sigma_{b\rangle}{}^d\mathsf{div}(\dot{E})^c, && \\
& && (5.122)\\
\dot{\psi}_{6ab} &= -(\tfrac{8}{3}-w)\Theta\psi_{6ab} + w\mu\epsilon_{cd\langle a}L_{b\rangle}{}^d\tilde{\nabla}^2\mathsf{div}(\sigma)^c + 2w\mu\epsilon_{cd\langle a}\tilde{\nabla}^2\sigma_{b\rangle}{}^d\mathsf{div}(L)^c. && (5.123)
\end{aligned}
$$

We now apply the standard 1+3 harmonic decomposition to system of equations. We only present the case for one parity. In order to do this, we introduce a new notation for the harmonic decomposition of our new variables. For example,

$$
\psi_{1ab} = \sum_{\kappa=\kappa'} E_1(\kappa)E_1(\kappa')\epsilon_{cd\langle a}Q_{b\rangle}{}^{(k)d}\mathsf{div}Q^{(\kappa')c} = \sum_{\kappa=\kappa'} \psi_1(\kappa,\kappa)\epsilon_{cd\langle a}Q_{b\rangle}{}^d(k)\mathsf{div}Q^c(\kappa'). \quad (5.124)
$$

All the terms in the above system can be decomposed similarly. Notice in particular that,

$$
\begin{aligned}
w\mu\epsilon_{cd\langle a}\tilde{\nabla}^2\sigma_{b\rangle}{}^d\mathsf{div}(E)^c &= \tfrac{2w}{(1+w)}\epsilon_{cd\langle a}\tilde{\nabla}^2\left(\dot{E}_b^d + \Theta E_b{}^d\right)\mathsf{div}(E)^c\\
&= \tfrac{2wk^2}{(1+w)a^2}[\psi_2(\kappa,\kappa') + \psi_1(\kappa,\kappa')]\epsilon_{cd\langle a}Q_{b\rangle}{}^{(k)d}\mathsf{div}Q^{(\kappa')c}, \quad (5.125)\\
w\mu\epsilon_{cd\langle a}\tilde{\nabla}^2\sigma_{b\rangle}{}^d\mathsf{div}(\dot{E})^c &= \tfrac{2wk^2}{(1+w)a^2}[\psi_2(\kappa,\kappa') + \psi_1(\kappa,\kappa')]\epsilon_{cd\langle a}Q_{b\rangle}{}^d(\kappa)\mathsf{div}Q^c(\kappa'),\\
& \quad (5.126)
\end{aligned}
$$

and

$$w\mu\epsilon_{cd\langle a}\tilde{\nabla}^2\sigma_{b\rangle}{}^d\mathsf{div}(L)^c+2w\mu\epsilon_{cd\langle a}L_{b\rangle}{}^d\tilde{\nabla}^2\mathsf{div}(\sigma)^c = \frac{4wk^2}{(1+w)a^2}[\psi(\kappa,\kappa')_4+2\psi(\kappa,\kappa')_5]\epsilon_{cd\langle a}Q_{b\rangle}{}^d(\kappa)\mathsf{div}Q^c(\kappa'). \tag{5.127}$$

It is clear that these terms vanish in the long wavelength limit, which leads to the closure the harmonic decomposed system. This reduces to a system of the form $\dot{\psi}_n = \Psi\psi_n$, where $n = 1..6$ and Ψ is the coupling matrix given by

$$\begin{pmatrix} -\frac{1}{3}\Theta & 2 & 0 & 0 & 0 & 0 \\ -(1+\frac{1}{3}w)\Theta^2 & -(3+w)\Theta & 1 & -\frac{1}{2}(1+w) & 0 & 0 \\ 0 & -2(1+\frac{1}{3}w)\Theta^2 & -(\frac{17}{3}+2w)\Theta & 0 & -(1+w) & 0 \\ 0 & 0 & 0 & -2\Theta & 1 & 0 \\ 0 & 0 & 0 & -(1+\frac{1}{3}w)\Theta^2 & -(\frac{14}{3}+w)\Theta & -\frac{1}{2}\mu(1+w) \\ 0 & 0 & 0 & 0 & 0 & -(\frac{8}{3}-w)\Theta \end{pmatrix}$$

The system can now be solved given that $\Theta = \frac{2}{(1+w)t}$ in the background. Solving this system, and using the solutions to reconstruct $\mathcal{F}_{ab}$ yields ,

$$\begin{aligned} \mathcal{F}_{ab} &= \alpha_{1_{ab}}t^{-\frac{4}{(1+w)}} + \alpha_{2_{ab}}t^{\frac{2(-2+w)}{(1+w)}} + \alpha_{3_{ab}}t^{\frac{2(-2+3w)}{3(1+w)}} + \alpha_{4_{ab}}t^{\frac{(-7+9w)}{3(1+w)}} + \alpha_{5_{ab}}t^{\frac{(-7+15w)}{3(1+w)}} \\ &+ \alpha_{6_{ab}}t^{-\frac{(17+3w)}{3(1+w)}} \end{aligned} \tag{5.128}$$

where

$$\alpha_{1_{ab}} = \frac{5c_{1_{ab}}}{4}(-2+w)(-248+112w+72w^2), \tag{5.129}$$

$$\alpha_{2_{ab}} = \frac{5c_{2_{ab}}}{4}(-2+w)(-253+110w+75w^2), \tag{5.130}$$

$$\alpha_{3_{ab}} = \frac{5c_{3_{ab}}}{4}(-2+w)(-253+104w+69w^2), \tag{5.131}$$

$$\alpha_{4_{ab}} = \frac{5c_{4_{ab}}}{4}(-2+w)(-248+130w+90w^2), \tag{5.132}$$

$$\alpha_{5_{ab}} = \tfrac{5c_{5_{ab}}}{4}(-2+w)(-248+118w+78w^2), \quad (5.133)$$

$$\alpha_{6_{ab}} = \tfrac{5c_{6_{ab}}}{4}(-2+w)(-258+96w+66w^2), \quad (5.134)$$

and where $c_{n_{ab}}$ are constants of integration which are determined by the initial conditions. Now applying the standard 1+3 tensor harmonic decomposition to the second-order wave equation gives,

$$\ddot{\Sigma}^{(\kappa)} + \tfrac{\kappa^2}{a^2}\Sigma^{(\kappa)} + \tfrac{7}{3}\Theta\dot{\Sigma}^{(\kappa)} + \tfrac{2}{3}\Theta^2(1-w)\Sigma^{(\kappa)} = \mathcal{F}^{(k)}. \quad (5.135)$$

for each parity. The solutions to the homogenous part are

$$\Sigma_1^{(k)} = C_1(\kappa)t^{-\frac{11-3w}{6(1+w)}} J\left(\tfrac{5+3w}{6(1+w)}, \tfrac{\kappa}{a}t\right), \quad \Sigma_2^{(\kappa)} = C_2 t^{-\frac{11-3w}{6(1+w)}} Y\left(\tfrac{5+3w}{6(1+w)}, \tfrac{\kappa}{a}t\right), \quad (5.136)$$

where again J and Y are the Bessel functions of the first and the kinds respectively. Here too, the general solutions can found using Green's method i.e.

$$\Sigma^{(k)} = \Sigma_1^{(k)} + \Sigma_2^{(k)} + \Sigma_1^{(k)} \int \tfrac{a}{\varpi k}\Sigma_2^{(k)} S^{(k)} dt + \Sigma_2^{(k)} \int \tfrac{a}{\varpi k}\Sigma_1^{(k)} S^{(k)} dt \quad (5.137)$$

where

$$\varpi = J\left(\tfrac{(5+3w)}{6(1+w)}, \tfrac{\kappa t}{a}\right) Y\left(\tfrac{(11+9w)}{6(1+w)}, \tfrac{\kappa}{a}t\right) - Y\left(\tfrac{(5+3w)}{6(1+w)}, \tfrac{\kappa}{a}t\right) J\left(\tfrac{(11+9w)}{6(1+w)}, \tfrac{\kappa}{a}t\right). \quad (5.138)$$

Figure (5.2) represents the numerical solutions for gravitational waves in the dust and radiation dominated regimes.

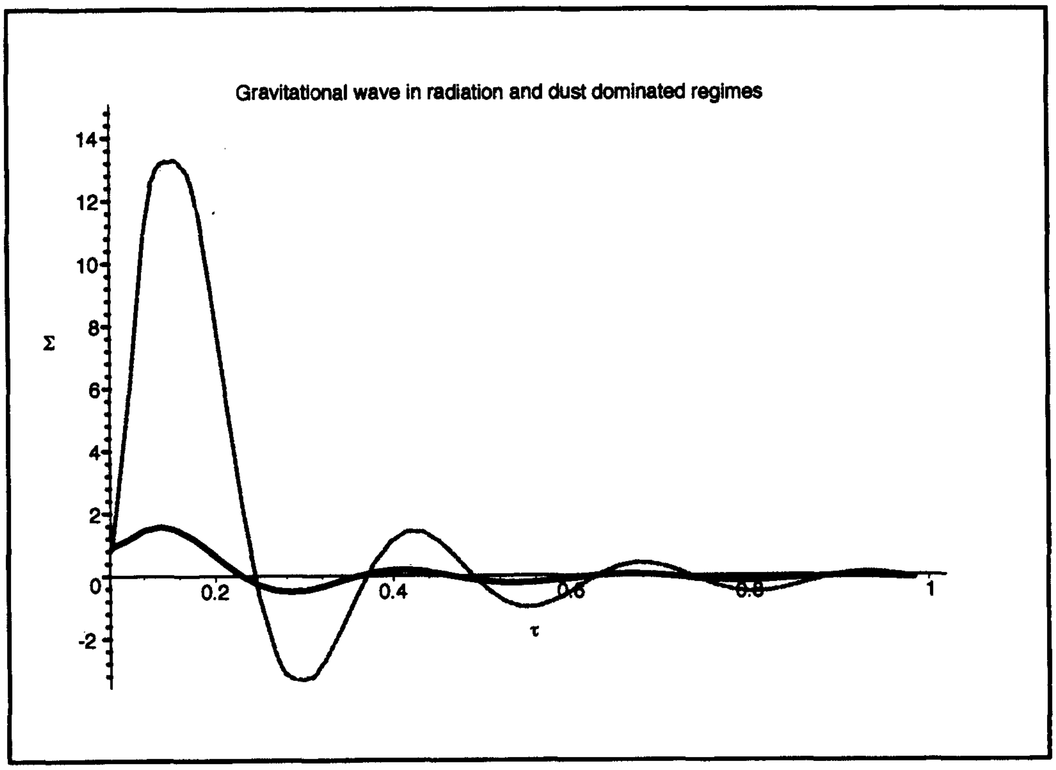

Figure 5.2: The red (dotted) line is that of gravitational wave amplitude $\Sigma = \Sigma^{(\kappa)}/\Sigma_0^{(\kappa)}$ in the radiation dominated regime while the black line that in the matter dominated regime. It is clear that the wave decays much faster in the radiation regime. The initial conditions are $\Sigma_0^{(\kappa)} = 10^{-5}$, $\Sigma^{(\kappa)\prime} = 0$, $k/a = 25$, $\alpha = \beta = \gamma = 0.1$

5.8 Chapter conclusion

Cosmology has entered a phase where observations is beginning to drive this whole field of research, thanks to the abundance of data from various experiments. Perhaps the greatest feature of this data is their ever improving precision. This in principle will allow us to determine the role played by nonlinear effects in cosmology. The greatest hindrance to the study of nonlinear relativistic effects is mathematical, for this reason perturbation theory remains the only recourse. However, nonlinear perturbation theory is made even more complicated by gauge issues and complex equations.

In this chapter, we discussed a new second-order gauge invariant formalism suited for studying models with barotropic fluids. This formalism is an extension of the dust-based formalism initiated in [24]. We found that the presence of pressure leads an even more complicated system of equations. We have shown that the system derived in our analysis generalises that given in [24].

As observed in the [24], the main difficulty encountered in the 1+3 covariant approach at second-order is linked to the gauge-issue. Whereas equation (5.105) can be written by crossing off third-order terms, they are not in general integrable. This has to do with the fact that unlike the metric approach which solves for operators at first order, the covariant approach only solves for physical variables. In order to integrate the second-order equations therefore, derivative operators must operate only on variables which vanish at first-order and in the background. These second-order gauge invariant variables are $\Sigma_{ab}, \psi_{(n)ab}, \xi_{(n)ab}$. We found Σ_{ab} by extracting the tensor part of first-order σ_{ab}. Notice that Σ_{ab} vanishes at first-order since only scalar modes are excited at this order. The $\psi_{(n)ab}$ and $\xi_{(n)ab}$, which make up the source for the Σ_{ab} wave equations, are products of first-order scalars. These SOGI variables highlight the two different ways in which second order gauge invariant variables may arise. We have presented new covariant methods for extracting the various modes and have used the tensor extraction in our subsequent analysis of second-order gravitational waves.

The greatest challenge in the above development is finding a way to deal with the source term. We have presented a new way of handling the source term which complements the approached used in [24].

Chapter 6

Gravitational waves from second-order effects during inflation

In Chapter 5, we developed a method for characterising second-order tensor modes induced by first order scalars. In this chapter, we use this method in conjunction with a metric based approach to study gravitational waves induced by second-order effects during inflation. The chapter first reviews scalar field dynamics, followed by methodical development of tools that are needed in the subsequent analysis. We then pursue a parallel approach, where the two main perturbative formalisms discussed in Chapter 2, are developed. We find that in-order to determine useful measures of gravitational wave power spectrum, we have to use the metric approach. The chapter examines the power spectrum of the wave induced by the first order scalars. We also examine a new contribution to the non-linearity parameter f_{NL}, which is a measure of non-Gaussianity.

6.1 Introduction

In this chapter we study the generation of gravitational waves during inflation arising from the non-linear coupling of scalar and tensor modes.

The generation of gravitational waves (GW) is a general prediction of an early inflationary phase [149]. Their amplitude is related to the energy scale of inflation and they are potentially detectable via observations of B-mode polarization in the cosmic microwave background (CMB) if the energy scale of inflation is larger than $\sim 3 \times 10^{15}$ GeV [107, 146, 108, 91]. Such a detection would be of primary importance to test inflationary models.

Among the generic predictions of one-field inflation [117, 150, 87, 75] are the existence of (adiabatic) scalar and tensor perturbations of quantum origin with an almost scale invariant power spectrum and Gaussian statistics. Even if non-linear effects in the evolution of perturbations are expected, a simple calculation [12], confirmed by more detailed analysis [112], shows that it is not possible to produce large non-Gaussianity within single field inflation as long as the slow-roll conditions are preserved throughout the inflationary stage. Deviations from Gaussianity can be larger in, e.g., multi-field inflation scenarios [12, 145, 13, 66, 5, 71] and are thus expected to give details on the inflationary era.

As far as scalar modes are concerned, the deviation from Gaussianity has been parameterized by a (scale-dependent) parameter, $f_{\rm NL}$. Various constraints have been set on this parameter, mainly from CMB analysis [49, 94, 56, 58] (see [119] for a review on both theoretical and observational issues). Deviation from Gaussianity in the CMB can arise from primordial non-Gaussianity, i.e. generated during inflation, post-inflation dynamics or radiation transfer [120]. It is important to understand them all in order to track down the origin of non-Gaussianity, if detected. Among the other signatures of non-linear dynamics is the fact that the Scalar-Vector and Tensor (SVT) modes of the perturbations are no longer decoupled. This implies in particular that scalar modes can generate gravity waves. Also, vector modes, that are usually washed out by the evolution, can be generated. In particular, second-order scalar perturbations in the post-inflation era will also contribute to B-mode polarization [143] or to multipole coupling in the CMB [144], and it is thus important to understand this coupling in detail.

In this chapter, we focus on the gravitational waves generated from scalar modes via second order dynamics. Second-order perturbation theory has been investigated in various works [158,68,142,67,125,113,24,97,98,89,122] and a fully gauge-invariant approach to the problem was recently given in [122]. Second-order perturbations during inflation have also been considered in [161, 112], providing the prediction of the bispectrum of perturbations from inflation.

As discussed in (2), two main formalisms have been developed to study perturbations, and hence second order effects: the $1+3$ covariant formalism [41] in which exact gauge-invariant variables describing the physics of interest are first identified and exact equations describing their time and space evolution are then derived and approximated with respect to the symmetry of the background to obtain results at the desired order, and the coordinate based approach of Bardeen in which gauge-invariants are identified by combining the metric and matter perturbations and then equations are found for them at the appropriate order of the calculation. In this chapter we carry out a detailed comparison of the two approaches up to second order, highlighting the advantages and disadvantages of each method, thus extending earlier work on the linear theory [105]. This chapter also extends the work of [97, 98], in which the relation between the two formalisms on super-Hubble scales is investigated. In particular, we show that the degree of success of one formalism over the other depends on the problem being addressed. This is the first time a complete and transparent matching of tensor perturbations in the two formalisms at first and second order is presented. We also show, using an analytical argument, that the power-spectrum of gravitational waves from second-order effects is much smaller than the first order on super-Hubble scales. This is in contrast to the fact that during the radiation era the generation of GW from primordial density fluctuations can be large enough to be detected in principle [89].

6.1.1 Kinematical quantities

Following the scalar fields dynamics given in section 1.2.6, we now give a formal description of the scalar field in terms of fluid quantities; therefore, we assign a 4-velocity vector u^a to the scalar field itself. This will allow us to define the dot derivative, i.e. the *proper* *time* derivative along the flow lines: $\dot{T}^{a\cdots b}{}_{c\cdots d} \equiv u^e\nabla_e T^{a\cdots b}{}_{c\cdots d}$. Now, given the assumption (1.19), we can choose the 4-velocity field u^a as the unique timelike vector with unit magnitude ($u^a u_a = -1$) parallel to the normals of the hypersurfaces $\{\phi = const.\}$ [111],

$$u^a \equiv -\psi^{-1}\nabla^a\phi, \tag{6.1}$$

where we have defined the field $\psi = \dot{\phi} = (-\nabla_a\phi\nabla^a\phi)^{1/2}$ to denote the magnitude of the momentum density (simply momentum from now on). In the case of more than one scalar field, this choice can still be made for each scalar field 4-velocity, but not for the 4-velocity of the total fluid.

A number of frame choices exist for the 4-velocity of the total fluid, the most common being the energy frame, where the total energy flux vanishes (see [105] for a detailed description of this case). The choice (6.1) defines u^a as the unique timelike eigenvector of the energy-momentum tensor (1.17). The quantity ψ will be positive or negative depending on the initial conditions and the potential V; in general ϕ could oscillate and change sign even in an expanding phase, and the determination of u^a by (6.1) will be ill-defined on those surfaces where $\nabla_a\phi = 0 \Rightarrow \psi = 0$ (including the surfaces of maximum expansion in an oscillating Universe). This will not cause us a problem however, as we assume the solution is differentiable and (1.19) holds almost everywhere, so determination of u^a almost everywhere by this equation will extend (by continuity) to determination of u^a everywhere in U.

The kinematical quantities associated with the "flow vector" u^a can be obtained by a standard method [40, 99]. We can define a projection tensor into the tangent 3-spaces orthogonal to the flow vector:

$$h_{ab} \equiv g_{ab} + u_a u_b \Rightarrow h^a{}_b h^b{}_c = h^a{}_c,\ h_{ab}u^b = 0; \tag{6.2}$$

with this we decompose the tensor $\nabla_b u_a$ as

$$\nabla_b u_a = \widetilde{\nabla}_b u_a - \dot{u}_a u_b, \quad \widetilde{\nabla}_b u_a = \tfrac{1}{3}\Theta h_{ab} + \sigma_{ab}, \tag{6.3}$$

where $\widetilde{\nabla}_a$ is the spatially totally projected covariant derivative operator orthogonal to u^a (e.g., $\widetilde{\nabla}_a f = h_a{}^b \nabla_b f$; see the Appendix of [106] for details), $\dot{u}_a$ is the acceleration ($\dot{u}_b u^b = 0$), and σ_{ab} the shear ($\sigma^a{}_a = \sigma_{ab} u^b = 0$). Then the expansion, shear and acceleration are given in terms of the scalar field by

$$\begin{aligned}
\Theta &= -\nabla_a(\psi^{-1}\nabla^a\phi) = -\psi^{-1}\left[V'(\phi) + \dot{\psi}\right], & (6.4)\\
\sigma_{ab} &= -\psi^{-1}\, h_{\langle a}{}^c h_{b\rangle}{}^d \nabla_c \left[\nabla_d \phi\right], & (6.5)\\
\dot{u}_a &= -\psi^{-1}\,\widetilde{\nabla}_a \psi = -\psi^{-1}(\nabla_a \psi + u_a \dot{\psi}), & (6.6)
\end{aligned}$$

where the last equality in equation (6.4) follows on using the Klein-Gordon equation (1.16). We can see from equation (6.6) that ψ is an *acceleration potential* for the fluid flow [43]. Note also that the vorticity vanishes:

$$\omega_{ab} = -h_a{}^c h_b{}^d \nabla_{[d}\left(\psi^{-1}\nabla_{c]}\phi\right) = 0, \tag{6.7}$$

an obvious result with the choice (6.1), so that $\widetilde{\nabla}_a$ is the covariant derivative operator in the 3-spaces orthogonal to u^a, i.e. in the surfaces $\{\phi = \text{const.}\}$ as discussed in Chapter 2. As usual, it is useful to introduce a scale factor a (which has dimensions of length) along each flow-line by

$$\tfrac{\dot{a}}{a} \equiv \tfrac{1}{3}\Theta = H, \tag{6.8}$$

where H is the usual Hubble parameter if the Universe is homogeneous and isotropic. Finally, it is important to stress that

$$\widetilde{\nabla}_a \phi = 0, \tag{6.9}$$

which follows from our choice of u^a via equation (6.1), a result that will be important for the choice of gauge invariant (GI) variables and for the perturbations equations.

6.1.2 Fluid description of a scalar field

It follows from our choice of the four velocity (6.1) that we can represent a minimally coupled scalar field as a perfect fluid; the energy-momentum tensor (1.17) takes the usual form for perfect fluids

$$T_{ab} = \mu u_a u_b + p h_{ab}, \tag{6.10}$$

where the energy density μ and pressure p of the scalar field "fluid" are given by

$$\begin{aligned} \mu &= \tfrac{1}{2}\psi^2 + V(\phi), & (6.11)\\ p &= \tfrac{1}{2}\psi^2 - V(\phi). & (6.12)\end{aligned}$$

If the scalar field is not minimally coupled this simple representation is no longer valid, but it is still possible to have an imperfect fluid form for the energy-momentum tensor [111].

Using the perfect fluid energy-momentum tensor (6.10) in (1.18) one obtains the energy and momentum conservation equations

$$\begin{aligned} \dot{\mu} + \psi^2\Theta &= 0, & (6.13)\\ \psi^2\dot{u}_a + \widetilde{\nabla}_a p &= 0. & (6.14)\end{aligned}$$

If we now substitute μ and p from equations (6.11) and (6.12) into equation (6.13) we obtain the 1+3 form of the Klein-Gordon equation (1.16):

$$\ddot{\phi} + \Theta\dot{\phi} + V'(\phi) = 0, \tag{6.15}$$

an exact ordinary differential equation for ϕ in any space-time with the choice (6.1) for the four-velocity. With the same substitution, equation (6.14) becomes an identity for the acceleration potential ψ. It is convenient to relate p and μ by the *index* γ defined by

$$p = (\gamma - 1)\mu \;\Leftrightarrow\; \gamma = \tfrac{p+\mu}{\mu} = \tfrac{\psi^2}{\mu}. \tag{6.16}$$

This index would be constant in the case of a simple one-component fluid, but in general will vary with time in the case of a scalar field:

$$\frac{\dot{\gamma}}{\gamma} = \Theta(\gamma - 2) - 2\frac{V'}{\psi}. \tag{6.17}$$

Finally, it is standard to *define* a speed of sound as

$$c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}} = \gamma - 1 - \frac{\dot{\gamma}}{\Theta\gamma}. \tag{6.18}$$

6.1.3 Background equations

The previous equations assume nothing on the symmetry of the spacetime. We now specify it further and assume that it is close to a flat Friedmann-Lemaître spacetime (FLRW), which we consider as our background spacetime. The homogeneity and isotropy assumptions imply that

$$\sigma^2 \equiv \tfrac{1}{2}\sigma_{ab}\sigma^{ab} = 0, \;\; \omega_{ab} = 0, \;\; \widetilde{\nabla}_a f = 0, \tag{6.19}$$

where f is any scalar quantity; in particular

$$\widetilde{\nabla}_a \mu = \widetilde{\nabla}_a p = 0 \;\; \Rightarrow \;\; \widetilde{\nabla}_a \psi = 0\,, \; \dot{u}_a = 0. \tag{6.20}$$

The background (zero - order) equations are given by [105]:

$$3\dot{H} + 3H^2 = \left[V(\phi) - \psi^2\right], \tag{6.21}$$

$$3H^2 = \left[\tfrac{1}{2}\psi^2 + V(\phi)\right] \tag{6.22}$$

$$\dot{\psi} + 3H\psi + V'(\phi) = 0 \Leftrightarrow \dot{\mu} + 3H\psi^2 = 0, \tag{6.23}$$

where all variables are a function of cosmic time t only.

6.2 Gravitational waves from density perturbations: covariant formalism

In Chapter (5), we developed in detail a covariant approach to gravitational waves induced by first order scalars. In this section, reformulate this in the context of scalar fields.

6.2.1 First order equations

We discussed the basics of scalar field in the introductory chapter to this thesis. In this we developed the perfect fluid argument. The study of linear perturbations of a FLRW background are relatively straightforward. Let us begin by defining the *first-order gauge-invariant* (FOGI) variables corresponding respectively to the spatial fluctuations in the energy density, expansion rate and spatial curvature:

$$\begin{aligned} X_a &= \widetilde{\nabla}_a \mu, \\ Z_a &= \widetilde{\nabla}_a \Theta, \\ C_a &= a^3 \widetilde{\nabla}_a \tilde{R}. \end{aligned} \tag{6.24}$$

The quantities are FOGI because they vanish exactly in the background FLRW spacetime [152, 17, 19]. It turns out that a more suitable quantity for describing density fluctuations is the co-moving gradient of the energy density:

$$\mathcal{D}_a = \tfrac{a}{\mu} X_a, \tag{6.25}$$

where the ratio X_a/μ allows one to evaluate the magnitude of the energy density perturbation relative to its background value and the scale factor a guarantees that it is dimensionless and co-moving.

These quantities exactly characterize the inhomogeneity of any fluid; however we specifically want to characterize the inhomogeneity of the scalar field: this cannot be done using the spatial gradient $\widetilde{\nabla}_a \phi$ because it identically vanishes in any space-time by virtue of our choice of 4-velocity field u^a. It follows that in our approach the inhomogeneities in the matter field are completely incorporated in the spatial variation of the momentum density: $\widetilde{\nabla}_a\psi$, so it makes sense to define the dimensionless gradient

$$\Xi_a \equiv \tfrac{a}{\psi}\widetilde{\nabla}_a\psi, \tag{6.26}$$

which is related to $\mathcal{D}_a$ by

$$\mathcal{D}_a = \tfrac{\psi^2}{\mu}\Xi_a = \gamma\Xi_a, \tag{6.27}$$

where we have used equation (6.11) and γ is given by equation (6.16); comparing equation (6.26) and equation (6.6) we see that Ξ_a is proportional to the acceleration: it is a gauge-invariant measure of the spatial variation of proper time along the flow lines of u^a between two surfaces ϕ =const. (see [40]). The set of linearized equations satisfied by the FOGI variables consists of the *evolution equations*

$$\dot{X}_a = -4HX_a - \psi^2 Z_a, \tag{6.28}$$

$$\dot{Z}_a = -3HZ_a - \tfrac{1}{2}X_a + \widetilde{\nabla}_a \mathsf{div}\dot{u}, \tag{6.29}$$

$$\dot{\sigma}_{ab} - \widetilde{\nabla}_{\langle a}\dot{u}_{b\rangle} = -2H\sigma_{ab} - E_{ab}, \tag{6.30}$$

$$\dot{E}_{ab} - \mathsf{curl}\, H_{ab} = -\tfrac{1}{2}\psi^2\sigma_{ab} - \Theta E_{ab}, \tag{6.31}$$

$$\dot{H}_{ab} + \mathsf{curl}\, E_{ab} = -3HH_{ab}; \tag{6.32}$$

and the *constraints*

$$0 = \tfrac{1}{3}X_a - \mathsf{div}E_a, \tag{6.33}$$
$$0 = \tfrac{2}{3}Z_a - \mathsf{div}\sigma_a, \tag{6.34}$$
$$0 = \mathsf{div}H_a, \tag{6.35}$$
$$0 = H_{ab} - \mathsf{curl}\,\sigma_{ab}, \tag{6.36}$$
$$0 = \mathsf{curl}\,X_a, \tag{6.37}$$
$$0 = \mathsf{curl}\,Z_a. \tag{6.38}$$

The curl operator is defined by $\mathsf{curl}\,\psi_{ab} = (\mathsf{curl}\,\psi)_{ab} = \varepsilon_{cd\langle a}\widetilde{\nabla}^c\psi^d_{b\rangle}$ where ϵ_{abc} is the completely antisymmetric tensor with respect to the spatial section defined by $\epsilon_{bcd} = \epsilon_{abcd}u^a$, ϵ_{abcd} being the volume antisymmetric tensor such that $\epsilon_{0123} = \sqrt{-g}$. The divergence div of a rank n tensor is a rank $n-1$ tensor defined by $(\mathsf{div}\psi)_{i_1\ldots i_{n-1}} \equiv \widetilde{\nabla}^{i_n}\psi_{i_1\ldots i_n}$.

Because the background is homogeneous and isotropic, each FOGI vector may be uniquely split into a *curl-free* and *divergence-free* part, usually referred to as scalar and vector parts respectively, which we write as

$$V_a = V_{\mathsf{S}a} + V_{\mathsf{V}a}, \tag{6.39}$$

where $\mathsf{curl}\,V_{\mathsf{S}a} = 0$ and $\mathsf{div}V_{\mathsf{V}} = 0$. Similarly, any tensor may be invariantly split into scalar, vector and tensor parts:

$$T_{ab} = T_{\mathsf{S}ab} + T_{\mathsf{V}ab} + T_{\mathsf{T}ab} \tag{6.40}$$

where $\mathsf{curl}\,T_{\mathsf{S}ab} = 0$, $\mathsf{divdiv}T_{\mathsf{V}} = 0$ and $\mathsf{div}T_{\mathsf{T}a} = 0$. It follows therefore that in the above equations we can separately equate scalar, vector and tensor parts and obtain equations that independently characterize the evolution of each type of perturbation. In the case of a scalar field, the vorticity is exactly zero, so there is no vector contribution to the perturbations.

Let us now concentrate on scalar perturbations at linear order. It is clear from the above discussion that pure scalar modes are characterized by the vanishing of the magnetic part of the Weyl tensor: $H_{ab} = 0$, so the above set of equations reduce to a set of two coupled differential equations for X_a and Z_a:

$$\dot{X}_a + 4HX_a = -\psi^2 Z_a \tag{6.41}$$

$$\dot{Z}_a + 3HZ_a = -\tfrac{1}{2}X_a - \psi^{-2}\tilde{\nabla}^2 X_a, \tag{6.42}$$

and a set of coupled evolution and constraint equations that determine the other variables

$$\dot{\sigma}_{ab} = -\psi^{-2}\tilde{\nabla}_{\langle a}X_{b\rangle} - 2H\sigma_{ab} - E_{ab}. \tag{6.43}$$

$$\dot{E}_{ab} = -\tfrac{1}{2}\,\psi^2\,\sigma_{ab} - 3HE_{ab}, \tag{6.44}$$

$$\mathsf{div}\sigma_a = \tfrac{2}{3}\,Z_a, \tag{6.45}$$

$$\mathsf{curl}\,\sigma_{ab} = 0, \tag{6.46}$$

$$\mathsf{div}E_a = \tfrac{1}{3}\,X_a. \tag{6.47}$$

6.2.2 Gravitational waves from density perturbations

The preceding discussion deals with first-order variables and their behavior at linear order. It is important to keep in mind that we were able to set $H_{ab} = 0$ only because pure scalar perturbations in the absence of vorticity implies that $\mathsf{curl}\,\sigma_{ab} = 0$ at first order. As was discussed in (5.6), the vanishing of the magnetic part then follows from equation (6.36). However, at second order $\mathsf{curl}\,\sigma_{ab} \neq 0$. We denote the non-vanishing contribution at second order by [24]

$$\Sigma_{ab} = \mathsf{curl}\,\sigma_{ab}.$$

The new variable is *second-order and gauge-invariant* (SOGI), as it vanishes at all lower orders [152, 152]. It should be noted that the new variable is just the magnetic part of the Weyl tensor subject to the conditions mentioned above i.e.

$$\Sigma_{ab} = H_{ab}. \tag{6.48}$$

We are interested in the properties inherited by the new variable from the magnetic part of the Weyl tensor. As demonstrated in Chapter 5, the new variable is transverse and traceless at this order and is thus a description of gravitational waves. It should be stressed that in full generality, there are tensorial modes even at first order. By assuming that there are none, we explore a particular subset in the space of solutions. From the "iterative resolution" point of view, this means that we constrain the equations in order to focus on second order GWs sourced by terms quadratic in scalar perturbations. In doing so, we isolate and switch off GW perturbations at first order.

6.2.3 Propagation equation

The propagation of the new second-order variable now needs to be investigated using a covariant set of equations that are linearized to second-order about FLRW. We make use of equations (6.13), (6.14) and the following evolution equations which are up to second order in magnitude;

$$\dot{E}_{ab} = -\Theta E_{ab} + \operatorname{curl}\Sigma_{ab} - \tfrac{1}{2}\psi^2\sigma_{ab} + 3\sigma_{c\langle a}E_{b\rangle}{}^{c}, \tag{6.49}$$

$$\dot{\Sigma}_{ab} = -\Theta\Sigma_{ab} - \operatorname{curl}E_{ab} - 2\epsilon_{cd\langle a}\dot{u}^{c}E_{b\rangle}{}^{d}, \tag{6.50}$$

together with the constraint

$$\dot{u}^a = -\psi^{-2}\tilde{\nabla}^a p = -\tfrac{3}{\psi^2}\mathrm{div}E^a. \tag{6.51}$$

Unlike at first-order, where the splitting of tensors into their scalar, vector and tensor parts is possible, at second order this can only be achieved for SOGI variables.

We may isolate the tensorial part of the equations by decoupling Σ_{ab}: since it is divergence free it is already a pure tensor mode, whereas E_{ab} is not. We saw in (5.7.2), that the wave equation for the gravitational wave contribution can be found by first taking the time derivative of (6.50) and making appropriate substitutions using the evolution equations and keeping terms up to second-order. The wave equation for Σ_{ab} then reads:

$$\ddot{\Sigma}_{ab} - \widetilde{\nabla}^2\Sigma_{ab} + 7H\dot{\Sigma}_{ab} + (12H^2 - 2\psi^2)\Sigma_{ab} = S_{ab}, \tag{6.52}$$

where the source is given by the cross-product of the electric-Weyl curvature and its divergence (or acceleration):

$$S_{ab} \quad = \quad -\left[2u^e\nabla_e + 16H - 15Hc_s^2\right]\left(\tfrac{2}{\psi^2}\epsilon_{cd\langle a}E_{b\rangle}{}^d\mathsf{div}E^c\right). \tag{6.53}$$

To obtain this, we have used the fact that with a flat background space-time

$$\mathsf{curl\,curl}\,T_{ab} \quad = \quad -\widetilde{\nabla}^2 T_{ab} + \tfrac{3}{2}\widetilde{\nabla}_{\langle a}\mathsf{div}T_{b\rangle}$$

and we have used the commutation relation

$$(\mathsf{curl}\,T_{ab})^{\cdot} \quad = \quad \mathsf{curl}\,\dot{T}_{ab} - \tfrac{1}{3}\Theta\,\mathsf{curl}\,T_{ab} - \epsilon_{cd\langle a}\sigma^{ec}D_{|e|}T_{b\rangle}{}^d + \epsilon_{cd\langle a}[\dot{u}^c\dot{T}_b^d + \tfrac{1}{3}\Theta\dot{u}^cT_{b\rangle}{}^d].$$

We have also used equations (6.17) and (6.18) to eliminate $\dot{\psi}/\psi$ from the source term. It can also be shown that S_{ab} is transverse, illustrating that equation (6.52) represents the gravitational wave contribution at second order. Note that this is a local description of gravitational waves, in contrast to the non-local extraction of tensor modes by projection in Fourier space. Since Σ_{ab} contains exactly the correct number of degrees of freedom possible in GW, any other variable we may choose to describe GW must be related by quadrature, making this a suitable master variable.

In order to express the gravitational wave equation in Fourier space, we define our normalised tensor harmonics as

$$Q^{ab} = \tfrac{\xi^{ab}}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.54}$$

where ξ^{ab} is the polarization tensor, which satisfies the (background) tensor Helmholtz equation: $\tilde{\nabla}^2 Q_{ab} = -(q^2/a^2)Q_{ab}$. As q_a is required to satisfy $q_a u^a = 0$ in the background, it can thus be identified with a 3-vector and will subsequently written in bold when necessary. We denote harmonics of the opposite polarization with an overbar. Amplitudes of Σ_{ab} may be extracted via

$$\Sigma(\mathbf{k}, t) = \int \mathrm{d}^3\mathbf{k} \left[\Sigma_{ab}(\mathbf{x}, t) Q^{*ab}(\mathbf{k}, \mathbf{x})\right], \tag{6.55}$$

with an analogous formula for the opposite parity. This implies that our original variable may be reconstructed from

$$\Sigma_{ab} = \int \mathrm{d}^3\mathbf{k} \left[\Sigma(\mathbf{k}, t) Q_{ab}(\mathbf{k}, \mathbf{x}) + \bar{\Sigma}(\mathbf{k}, t) \bar{Q}_{ab}(\mathbf{k}, \mathbf{x})\right]. \tag{6.56}$$

The same relations hold for any transverse tensor. Hence, our wave equation in Fourier space is

$$\Sigma''(\mathbf{k}, \eta) + 6\mathcal{H}\Sigma'(\mathbf{k}, \eta) + [k^2 + 12\mathcal{H}^2 - 2\psi^2]\Sigma(\mathbf{k}, \eta) = S(\mathbf{k}, \eta), \tag{6.57}$$

with an identical equation for the opposite polarization. We have converted to conformal time η, where a prime denotes derivatives with respect to η, and we have defined the conformal Hubble parameter as $\mathcal{H} = a'/a$. The source term is composed of a cross-product of the electric part of the Weyl tensor and its divergence. At first-order, the electric Weyl tensor is a pure scalar mode, and can therefore be expanded in terms of scalar harmonics. To define these, let $Q^{(s)} = e^{i\mathbf{q}\cdot\mathbf{x}}/(2\pi)^{3/2}$, be a solution to the Helmholtz equation: $\tilde{\nabla}^2 Q^{(s)} = -(q^2/a^2)Q^{(s)}$. Beginning with this basis, it is possible to derive vectorial and (PSTF) tensorial harmonics by taking successive spatial derivatives as follows:

$$Q^{(s)}_a = \tilde{\nabla}_a Q^{(s)} = i\frac{q_a}{a} Q^{(s)}, \tag{6.58}$$

$$Q^{(s)}_{ab} = \tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle} Q^{(s)} = -a^{-2}\left(q_a q_b - \tfrac{1}{3}h_{ab}q^2\right) Q^{(s)}. \tag{6.59}$$

This symmetric tensor has the additional property $q^a q^b Q^{(s)}_{ab} = -(2q^4/3a^2)Q^{(s)}$. Using this representation we can express our source in equation (6.57) in terms of a convolution in Fourier space, by expanding the electric Weyl tensor as

$$E(\mathbf{q},\eta) = \int \mathrm{d}^3\mathbf{x} E_{ab} Q^{*ab}_{(S)}(\mathbf{q},\mathbf{x}). \tag{6.60}$$

Then, the right hand side of equation (6.53) expressed in conformal time, accompanied by appropriate Fourier decomposition of each term and making use of the normalization condition for orthonormal basis, yields:

$$S(\mathbf{k},\eta) \;=\; \int \mathrm{d}^3\mathbf{q} A(\mathbf{q},\mathbf{k}) \left\{2\left[E(\mathbf{q},\eta)E(\mathbf{k}-\mathbf{q},\eta)\right]' + (16-15c_s^2)\mathcal{H}\, E(\mathbf{q},\eta)E(\mathbf{k}-\mathbf{q},\eta)\right\} \tag{6.61}$$

where

$$A(\mathbf{q},\mathbf{k}) = \tfrac{3i}{4a^3\psi^2}\epsilon_{cd(a}q_{b)}q^d(k^c-q^c)|\mathbf{k}-\mathbf{q}|^2\xi^{ab}(\mathbf{k}), \tag{6.62}$$

with a similar expression for the other polarization. In principle we can now solve for the gravitational wave contribution Σ_{ab}, and calculate the power spectrum of gravitational waves today. For this however, we need initial conditions for the electric Weyl tensor (or, alternatively Ψ_a).

6.3 Gravitational waves from density perturbations: coordinate based approach

As discussed in Chapter 2, in this formalism we consider perturbations around a FLRW universe with Euclidean spatial sections and expand the metric as

$$\mathrm{d}s^2 = a^2(\eta)\left\{-(1+2\Phi)\mathrm{d}\eta^2 + 2\omega_i \mathrm{d}x^i\mathrm{d}\eta + \left[(1-2\Psi)\delta_{ij} + h_{ij}\right]\mathrm{d}x^i\mathrm{d}x^j\right\}, \tag{6.63}$$

where η is the conformal time and a the scale factor. We perform a scalar-vector-tensor decomposition as

$$\omega_i = D_i B - \bar{B}_i, , \tag{6.64}$$

and

$$h_{ij} = \bar{\mathcal{E}}_{ij} + D_i\bar{\mathcal{E}}_j + D_j\bar{\mathcal{E}}_i + 2D_iD_j\mathcal{E}, \tag{6.65}$$

where $\bar{B}_i$, $\bar{\mathcal{E}}_i$ are transverse ($D_i\bar{\mathcal{E}}^i = D_i\bar{B}^i$=0), and $\bar{\mathcal{E}}_{ij}$ is traceless and transverse ($\bar{\mathcal{E}}^i_i = D_i\bar{\mathcal{E}}^i_j = 0$). Latin indices i, j, k... are lowered by use of the spatial metric, e.g. $B^i = \gamma^{ij}B_j$. We fix the gauge and work in the Newtonian gauge defined by $B_i = \mathcal{E} = B = 0$ so that Φ and Ψ are the two Bardeen potentials. As in the previous sections, we assume that the matter content is a scalar field ϕ that can be split into background and perturbation contributions: $\phi = \phi(\eta) + \delta\phi(\eta, \mathbf{x})$. The gauge invariant scalar field perturbation can be defined by

$$Q \equiv \delta\phi + \phi'\tfrac{\Psi}{\mathcal{H}}, \tag{6.66}$$

where $\mathcal{H} \equiv a'/a \equiv aH$. We denote the field perturbation in Newtonian gauge by χ so that $Q = \chi + (\phi'/\mathcal{H})\Psi$. Introducing

$$\varepsilon = \tfrac{3}{2}\tfrac{\phi'^2}{\mu a^2}, \tag{6.67}$$

(Note that this differs slightly from the often used version $\varepsilon = \frac{3}{2}\frac{\phi'^2}{\mu}$. The difference is due to the type of derivative used in defining the acceleration potential. We have used the covariant derivative and hence equation 6.67). The equation of state (6.16) takes the form $\gamma = w + 1 = 2\varepsilon/3$. We thus have two expansions: one concerning the perturbation of the metric and the other in the slow-roll parameter ϵ.

6.3.1 Scalar modes

Focusing on scalar modes at first order in the perturbation, it is convenient to introduce

$$v = aQ \tag{6.68}$$

and

$$z \equiv a\frac{\phi'}{\mathcal{H}}, \tag{6.69}$$

in terms of which the action (1.15) takes the form

$$S_{\rm scal} = \frac{1}{2}\int \mathrm{d}^3\mathrm{x}\mathrm{d}\eta \left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z}v^2\right], \tag{6.70}$$

when expanded to second order in the perturbations. It is the action of a canonical scalar field with effective square mass $m_v^2 = -z''/z$. v is the canonical variable that must be quantised [118]. It is decomposed as follows

$$\hat{v}(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\left[v_k(\eta)\mathrm{e}^{i\mathbf{k}.\mathbf{x}}\hat{a}_{\mathbf{k}} + v_k^*(\eta)\mathrm{e}^{-i\mathbf{k}.\mathbf{x}}\hat{a}_{\mathbf{k}}^\dagger\right]. \tag{6.71}$$

Here v_k is solution of the Klein-Gordon equation

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0 \tag{6.72}$$

and the annihilation and creation operators satisfy the commutation relation, $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta(\mathbf{k}-\mathbf{k}')$. We define the free vacuum state by the requirement $\hat{a}_{\mathbf{k}}|0\rangle = 0$ for all $\mathbf{k}$.

From the Einstein equation, one can get the expression for the Bardeen potential (recalling that $\Psi = \Phi$)

$$\Delta\Phi = \frac{1}{2}\frac{\phi'^2}{\mathcal{H}}\left(\frac{v}{z}\right)', \quad \left(\frac{a^2\Phi}{\mathcal{H}}\right)' = \frac{1}{2}zv \tag{6.73}$$

and for the curvature perturbation in comoving gauge

$$\mathcal{R} = -v/z. \tag{6.74}$$

Once the initial conditions are set, solving equation (6.72) will give the evolution of $v_k(\eta)$ during inflation, from which $\Phi_k(\eta)$ and $\mathcal{R}_k(\eta)$ can be deduced, using the previous expressions.

Defining the power spectrum as

$$\langle \mathcal{R}_{\mathbf{k}}\mathcal{R}_{\mathbf{k}'}^*\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_{\mathcal{R}}(k)\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{6.75}$$

one easily finds that

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}\left|\frac{v_k}{z}\right|^2. \tag{6.76}$$

Note also that z and ε are related by the simple relation

$$\frac{1}{\sqrt{2}}z = a\sqrt{\varepsilon}, \tag{6.77}$$

so that

$$\chi = Q - \frac{z}{a}\Phi = Q - (\sqrt{2\varepsilon})\Phi. \tag{6.78}$$

6.3.2 Gravitational waves at linear order

At first order, the tensor modes are gauge invariant and their propagation equation is given by

$$\bar{\mathcal{E}}''_{ij} + 2\mathcal{H}\bar{\mathcal{E}}'_{ij} - \tilde{\nabla}^2\bar{\mathcal{E}}_{ij} = 0 \tag{6.79}$$

since a minimally coupled scalar field has no anisotropic stress. Defining the reduced variable

$$\mu_{ij} = a\bar{\mathcal{E}}_{ij}, \tag{6.80}$$

the action (1.15) takes the form

$$S_{\text{tens}} = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\mathrm{d}\eta\left[(\mu'_{ij})^2 - (\partial_k\mu_{ij})^2 + \frac{a''}{a}(\mu_{ij})^2\right], \tag{6.81}$$

when expanded tosecond order. Developing $\bar{\mathcal{E}}_{ij}$, and similarly μ_{ij}, in Fourier space:

$$\bar{\mathcal{E}}_{ij} = \sum_{\lambda=+,\times}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\mathcal{E}_\lambda\varepsilon^\lambda_{ij}(\mathbf{k})\mathrm{e}^{\mathrm{i}\mathbf{k}.\mathbf{x}}, \tag{6.82}$$

where ε^λ_{ij} is the polarization tensor, the action (6.81) takes the form of the action for two canonical scalar fields with effective square mass $m^2_\mu = -a''/a$

$$S_{\text{tens}} = \frac{1}{2}\sum_\lambda\int \mathrm{d}^3\mathbf{x}\mathrm{d}\eta\left[(\mu'_\lambda)^2 - (\partial_k\mu_\lambda)^2 + \frac{a''}{a}\mu^2_\lambda\right]. \tag{6.83}$$

If one considers the basis ($\mathbf{e}_1$, $\mathbf{e}_2$) of the 2 dimensional space orthogonal to $\mathbf{k}$ then $\varepsilon^{\lambda}_{ij} = (e^1_i e^1_j - e^2_i e^2_j)\delta^{\lambda}_{+} + (e^1_i e^2_j + e^2_i e^1_j)\delta^{\lambda}_{\times}$.

μ_λ are the two degrees of freedom that must be quantized [118] and we expand them as

$$\begin{aligned}\hat{\mu}_{ij}(\mathbf{x},\eta) &= \sum_{\lambda}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\left[\mu_{k,\lambda}(\eta)\mathrm{e}^{i\mathbf{k}.\mathbf{x}}\hat{b}_{\mathbf{k},\lambda}\right. \\ &\qquad \left. +\mu^*_{k,\lambda}(\eta)\mathrm{e}^{-i\mathbf{k}.\mathbf{x}}\hat{b}^{\dagger}_{\mathbf{k},\lambda}\right]\varepsilon^{\lambda}_{ij}(\mathbf{k}). \end{aligned} \tag{6.84}$$

μ_k is solution of the Klein-Gordon equation

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0, \tag{6.85}$$

where we have dropped the polarization subscript. The annihilation and creation operators satisfy the commutation relations, $[\hat{b}_{\mathbf{k},\lambda}, \hat{b}^{\dagger}_{\mathbf{k}',\lambda'}] = \delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}$ and $[\hat{a}_{\mathbf{k}}, \hat{b}^{\dagger}_{\mathbf{k}',\lambda}] = 0$. We define the free vacuum state by the requirement $\hat{b}_{\mathbf{k},\lambda}|0\rangle = 0$ for all $\mathbf{k}$ and λ.

Defining the power spectrum as

$$\langle \mathcal{E}_{\mathbf{k},\lambda}\mathcal{E}^*_{\mathbf{k}',\lambda'}\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_T(k)\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}, \tag{6.86}$$

one easily finds that

$$\mathcal{P}_T(k) = \frac{k^3}{\pi^2}\left|\frac{\mu_k}{a}\right|^2, \tag{6.87}$$

where the two polarisations have the same contribution.

6.3.3 Gravitational waves from density perturbations

At second order, we split the tensor perturbation as $\bar{\mathcal{E}}_{ij} = \bar{\mathcal{E}}^{(1)}_{ij} + \bar{\mathcal{E}}^{(2)}_{ij}/2$. The evolution equations of $\bar{\mathcal{E}}^{(2)}_{ij}$ is similar to equation (6.79), but inherits a source term quadratic in the first order perturbation variables and from the transverse tracefree (TT) part of the stress-energy tensor

$$a^2\left[T^i_j\right]^{\mathrm{TT}} = \gamma^{ip}\left[\partial_j\chi\partial_p\chi\right]^{\mathrm{TT}}. \tag{6.88}$$

It follows that the propagation equation is

$$\bar{\mathcal{E}}_{ij}^{(2)\prime\prime} + 2\mathcal{H}\bar{\mathcal{E}}_{ij}^{(2)\prime} - \Delta\bar{\mathcal{E}}_{ij}^{(2)} = S_{ij}^{\rm TT}, \tag{6.89}$$

where $S_{ij}^{\rm TT}$ is a TT tensor that is quadratic in the first order perturbation variables.

Working in Fourier space, the TT part of any tensor can easily be extracted by means of the projection operator

$$\perp_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i\hat{k}_j, \tag{6.90}$$

where $\hat{k}^i = k^i/k$ (note that $\perp_{ij}(\hat{\mathbf{k}})$ is not analytic in k and is a non-local operator) from which we get

$$\begin{aligned} S_{ij}^{\rm TT}(\mathbf{k},\eta) &= \left[\perp_i^a\perp_j^b - \tfrac{1}{2}\perp_{ij}\perp^{ab}\right]S_{ab}(\mathbf{k},\eta) \\ &\equiv P_{ij}^{ab}(\mathbf{k})S_{ab}(\mathbf{k},\eta). \end{aligned} \tag{6.91}$$

The source term is now obtained as the TT-projection of the second order Einstein tensor quadratic in the first order variables and of the stress-energy tensor

$$S_{ab} = S^{(2)}_{ab,\,\rm SS} + S^{(2)}_{ab,\,\rm ST} + S^{(2)}_{ab,\,\rm TT}. \tag{6.92}$$

The three terms respectively indicate terms involving products of first order scalar quantities, first order scalar and tensor quantities and first order tensor quantities. The explicit form of the first term is

$$S_{ij}^{\rm TT} = 4\left[\partial_i\Phi\partial_j\Phi + \tfrac{1}{2}\partial_i\chi\partial_j\chi\right]^{\rm TT}. \tag{6.93}$$

The first term was considered in [18] and the second term was shown to be the dominant contribution for the production of gravitational waves during preheating [54]. In Fourier space, it is given by

$$
\begin{aligned}
S^{(2)}_{ab,\,\mathrm{SS}} &= -4\left[\int \mathrm{d}^3\mathbf{q}\, q_b q_a \Phi(\mathbf{q},\eta)\Phi(\mathbf{k}-\mathbf{q},\eta)\right.\\
&\quad \left. +\tfrac{1}{2}\int \mathrm{d}^3\mathbf{q}\, q_b q_a \chi(\mathbf{q},\eta)\chi(\mathbf{k}-\mathbf{q},\eta)\right]. \qquad (6.94)
\end{aligned}
$$

$\mu^{(2)}_{ij}(\mathbf{x},\eta)$ can be decomposed as in equation (6.82), using the same definition (6.80) at any order. The two polarizations evolve according to

$$
\mu^{(2)\prime\prime}_{\lambda} + \left(k^2 - \tfrac{a''}{a}\right)\mu^{(2)}_{\lambda} = -2aP^{ab}_{ij}S^{(2)}_{ab,\,\mathrm{SS}}\varepsilon^{ij}_{\lambda}. \qquad (6.95)
$$

Since the polarization tensor is a TT tensor, it is obvious that $P^{ab}_{ij}\varepsilon^{ij}_{\lambda} = \varepsilon^{ab}_{\lambda}$, so that

$$
\mu^{(2)\prime\prime}_{\lambda} + \left(k^2 - \tfrac{a''}{a}\right)\mu^{(2)}_{\lambda} = -4a\varepsilon^{ij}_{\lambda}\int \mathrm{d}^3\mathbf{q}\, q_i q_j \left[\Phi(\mathbf{q},\eta)\Phi(\mathbf{q}-\mathbf{k},\eta) + \tfrac{1}{2}\chi(\mathbf{q},\eta)\chi(\mathbf{q}-\mathbf{k},\eta)\right]. \qquad (6.96)
$$

From the equation (6.95), we deduce that the source term derives from an interaction Lagrangian of the form

$$
S_{\mathrm{int}} = \int \mathrm{d}\eta \mathrm{d}^3\mathbf{x} 4a\left[\partial_i\Phi\partial_j\Phi + \tfrac{1}{2}\partial_i\chi\partial_j\chi\right]\mu^{ij}. \qquad (6.97)
$$

It describes a two-scalars graviton interaction. In full generality the interaction term would also include, at lowest order, cubic terms of three scalars, two gravitons-scalar and three gravitons. They respectively correspond to second order scalar-scalar modes generated from gravitational waves and second order tensor modes. As emphasized previously, we do not consider these interactions here.

6.4 Comparison of the two formalisms

Before going further it is instructive to compare the two formalisms and understand how they relate to each other. Following the sketch of comparison given in Chapter 2, we now extend this comparison to first and second-order perturbations. Note that we go beyond [106], where a comparison of the variables was made at linear order. Here we investigate how the equations map to each other and extend the discussion to second order for the tensor sector. At the background level the scale factors a and expansion rates H introduced in each formalism agree, which explains why we made use of the same notation.

The perturbations of the metric around FLRW space-time has been split into a first-order and a second-order part according to

$$X = X^{(1)} + \tfrac{1}{2}X^{(2)} . \tag{6.98}$$

We make a similar decomposition for the quantities used in the $1+3$ covariant formalism. As long as we are interested in the gravitational wave sector, we only need to consider the four-velocity of the perfect fluid describing the matter content of the universe which we decompose as

$$u^\mu = \tfrac{1}{a}(\delta^\mu_0 + V^\mu). \tag{6.99}$$

Its spatial components are decomposed as

$$V^i = \partial^i V + \bar{V}^i , \tag{6.100}$$

$\bar{V}^i$ being the vector degree of freedom and V the scalar degree of freedom. As V^μ has only three independent degrees of freedom since u^μ satisfies $u_\mu u^\mu = -1$, its temporal component is linked to other perturbation variables. We assume that the fluid has no vorticity ($\bar{V}^i = 0$), as it is the case for the scalar fluid we have in mind and consequently we will also drop the vectorial perturbations ($\bar{\mathcal{E}}_i = 0$).

6.4.1 Matching at linear order

At first order, the spatial components of the shear, acceleration and expansion are respectively given by

$$\sigma^{(1)}_{ij} = a\left(\partial_{(i}\partial_{j)} V^{(1)} + \bar{\mathcal{E}}^{(1)\prime}_{ij}\right) , \tag{6.101}$$

$$\dot{u}_i^{(1)} = \partial_i \left(\Phi^{(1)} + \mathcal{H} V^{(1)} + V^{(1)\prime} \right) , \tag{6.102}$$

$$\delta\Theta^{(1)} = \tfrac{1}{a} \left(-3\Psi^{(1)\prime} - 3\mathcal{H}\Phi^{(1)} + \Delta V^{(1)} \right) . \tag{6.103}$$

The electric and magnetic part of the Weyl tensor take the form

$$E_{ij}^{(1)} = \partial_{\langle i}\partial_{j\rangle} \Phi^{(1)} - \tfrac{1}{2} \left(\bar{\mathcal{E}}_{ij}'' + \Delta \bar{\mathcal{E}}_{ij} \right) , \tag{6.104}$$

$$H_{ij}^{(1)} = \eta_{kl\langle i} \partial^k \bar{\mathcal{E}}_{j\rangle}^{(1)\prime\ l} \equiv (\hat{\mathsf{curl}}\, \bar{\mathcal{E}}^{(1)\prime})_{ij} . \tag{6.105}$$

Note that η_{kli} is the completely antisymmetric tensor normalized such that $\eta_{123} = 1$, which differs from ε_{abc}. We deduce from the last expression that

$$\left(\mathsf{curl}\, E^{(1)} \right)_{ij} = -\tfrac{1}{2a} \left[\left(\hat{\mathsf{curl}}\, \bar{\mathcal{E}}^{(1)\prime\prime} \right)_{ij} + \left(\hat{\mathsf{curl}}\, \Delta \bar{\mathcal{E}}^{(1)} \right)_{ij} \right] , \tag{6.106}$$

where we have used simpler notation by writing $(\hat{\mathsf{curl}}\, \bar{\mathcal{E}})_{ij}$ as $\hat{\mathsf{curl}}\, \bar{\mathcal{E}}_{ij}$. We also note that the derivative along u_μ of a tensor T of rank (n, m), vanishing in the background, takes the form

$$\dot{T}^{i_1 \dots i_n}_{\ \ j_1 \dots j_m} = \partial_t T^{i_1 \dots i_n}_{\ \ j_1 \dots j_m} + (n-m) H T^{i_1 \dots i_n}_{\ \ j_1 \dots j_m} \tag{6.107}$$

at first order, or alternatively

$$\frac{\left(a^{m-n} T^{i_1 \cdots i_n}_{\ \ j_1 \cdots j_m} \right)^{\cdot}}{a^{m-n}} = \partial_t T^{i_1 \dots i_n}_{\ \ j_1 \dots j_m} . \tag{6.108}$$

Again, recall that a dot refers to a derivative along u^μ. Indeed at first order, it reduces to a derivative with respect to the cosmic time but this does not generalise to second order.

Now, equation (6.32) can be recast a

$$a^{-2} \left(a^2 H_{ij} \right)^{\cdot} + \mathsf{curl}\, E_{ij} + H\, H_{ij} = 0 . \tag{6.109}$$

Using the expressions (6.104-6.105) for the geometric quantities, this equation takes the form

$$\hat{\mathsf{curl}}\,\left[\tfrac{1}{2a}\left(\bar{\mathcal{E}}''_{ij} + 2\mathcal{H}\bar{\mathcal{E}}'_{ij} - \Delta\bar{\mathcal{E}}_{ij}\right)\right] = 0\,. \qquad (6.110)$$

Similarly equation (6.52) can be recast as

$$\frac{(a^2 H_{ab})^{\cdot\cdot}}{a^2} + 3H\frac{(a^2 H_{ab})^{\cdot}}{a^2} + 2(H^2 + \dot{H})H_{ab} - \widetilde{\nabla}^2 H_{ab} = 0\,, \qquad (6.111)$$

so that it reduces at first order to

$$\hat{\mathsf{curl}}\,\left[\frac{1}{2a^2}\left(\bar{\mathcal{E}}^{(1)''}_{ij} + 2\mathcal{H}\bar{\mathcal{E}}^{(1)'}_{ij} - \Delta\bar{\mathcal{E}}^{(1)}_{ij}\right)'\right] = 0\,. \qquad (6.112)$$

Thus, equation (6.112) maps to equation (6.79) with the identification (6.105), if there is no vector modes. This can be understood from the fact that in the Bardeen formalism, equation (6.79) is obtained from the Einstein equation as $\hat{\mathsf{curl}}^{-1}[\hat{\mathsf{curl}}\, G_{ij}] = 0$.

In the case where there are vector modes, equation (6.105) has to be replaced by

$$H^{(1)}_{ij} = (\hat{\mathsf{curl}}\,\bar{\mathcal{E}}^{(1)'})_{ij} + \tfrac{1}{2}\eta_{kl\langle i}\partial^k\partial_{j\rangle}\bar{\mathcal{E}}^{(1)'l}$$

and H_{ab} is no longer a description of the GW, i.e. directly related to the TT part of the spacetime metric and the matching is not valid anymore.

6.4.2 Matching at second order

At second order, the matching is much more intricate mainly because the derivative along u^μ does not match with the derivative respect to cosmic time any more. Let us introduce the short hand notation

$$(X \times Y)_{ij} \equiv \eta_{kl\langle i}X^k Y_{j\rangle}{}^l \qquad (6.113)$$

for any tensors X^k and Y^{lm}. If $Y^{lm} = \partial^l\partial^m Z$, or $X^k = \partial^k W$, we also use the short-hand notation $Y = \partial\partial Z$ $X = \partial W$.

Among the terms quadratic in first-order perturbations, those involving a first-order tensorial perturbation can be omitted, as we are only interested in second-order effects sourced by scalar contributions. At second order, the geometric quantities of interest read

$$\begin{aligned} H^{(2)}_{ij} &= \left(\hat{\mathrm{curl}}\,\bar{\mathcal{E}}^{(2)\prime}\right)_{ij} - 4\left(\partial V^{(1)} \times \partial\partial\Phi^{(1)}\right)_{ij} && (6.114)\\ \left(\mathrm{curl}\,E^{(2)}\right)_{ij} &= -\tfrac{1}{2a}\left[\left(\hat{\mathrm{curl}}\,\bar{\mathcal{E}}^{(2)\prime\prime}\right)_{ij} + \left(\hat{\mathrm{curl}}\,\Delta\bar{\mathcal{E}}^{(2)}\right)_{ij}\right] - \tfrac{2}{a}\left[\left(\partial\Phi^{(1)} \times \partial\partial\Phi^{(1)}\right)_{ij}\right. \\ &\qquad \left. +\mathcal{H}\left(\partial V^{(1)} \times \partial\partial\Phi^{(1)}\right)_{ij} - \left(\partial V^{(1)} \times \partial\partial\Phi^{(1)\prime}\right)_{ij}\right]. && (6.115) \end{aligned}$$

From the latter expression, we remark that $H^{(2)}_{ij}$ has a term quadratic in first-order perturbations involving $V^{(1)}$ and $\Phi^{(1)}$. This terms arise from a difference between the two formalisms related to the fact that geometric quantities, such as H_{ij} E_{ij} etc., live on the physical space-time, whereas in perturbation theory, any perturbation variable at any order, such as $V^{(1)}$, $\mathcal{E}^{(2)}_{ij}$ etc., are fields propagating on the background space-time.

It follows that the splitting into tensor, vector and scalar modes is different. In the covariant formalism, the splitting refers to the fluid on the physical space-time, whereas in perturbation theory it refers to the co-moving fluid of the background solution. Indeed, this difference only shows up at second order as the magnetic Weyl tensor vanishes in the background. The one to one correspondence at first order between equations of both formalisms disappears, as the second order equations of the covariant formalism contain the dynamics of the first order quantities.

When keeping terms contributing to the second order, equation (6.32) has an additional source term and reads

$$\dot{H}_{ab} + \mathrm{curl}\,E_{ab} + 3HH_{ab} = -2\epsilon_{cd\langle a}\dot{u}^c E_{b\rangle}{}^d. \quad (6.116)$$

If first order tensorial perturbations are neglected then H_{ab} vanishes at first order and equation (6.107) still holds when applied to H_{ab}. Thus equation (6.116) can be recast as

$$\frac{\left(a^2 H_{ab}\right)^{\cdot}}{a^2} + \mathrm{curl}\,E_{ab} + \frac{\mathcal{H}}{a}H_{ab} = -2\epsilon_{cd\langle a}\dot{u}^c E_{b\rangle}{}^d. \quad (6.117)$$

Substituting the geometric quantities for their expressions at second order, and making use of equation (6.108) to handle the derivatives, equation (6.109) reads at second order

$$\hat{\mathsf{curl}}\left[\frac{1}{2a}\left(\bar{\mathcal{E}}^{(2)\prime\prime}_{ij}+2\mathcal{H}\bar{\mathcal{E}}^{(2)\prime}_{ij}-\Delta\bar{\mathcal{E}}^{(2)}_{ij}\right)\right]=-\frac{2}{a}\left[\left(\partial\Phi^{(1)}\times\partial\partial\Phi^{(1)}\right)_{ij}-\left(\partial V^{(1)}\times\partial\partial\Phi^{(1)\prime}\right)_{ij}\right]$$
$$-\frac{2}{a}\left[-\mathcal{H}\left(\partial V^{(1)}\times\partial\partial\Phi^{(1)}\right)_{ij}\right]. \tag{6.118}$$

Using the momentum and constraint equation (6.34) at first order

$$\Phi^{(1)\prime}+\mathcal{H}\Phi^{(1)}=\left(\mathcal{H}'-\mathcal{H}^2\right)V^{(1)} \tag{6.119}$$

and the background equation $\mathcal{H}'-\mathcal{H}^2=-\frac{1}{2}\mu(1+w)a^2$, that we deduce from the Raychauduri equation and the Gauss-Codacci equation at first order, we can link it to equation (6.89) as it then reads

$$\frac{1}{a}\hat{\mathsf{curl}}\left[\frac{1}{2}\left(\bar{\mathcal{E}}^{(2)\prime\prime}_{ij}+2\mathcal{H}\bar{\mathcal{E}}^{(2)\prime}_{ij}-\Delta\bar{\mathcal{E}}^{(2)}_{ij}\right)\right]=\frac{1}{a}\hat{\mathsf{curl}}\left[2\partial_i\Phi^{(1)}\partial_j\Phi^{(1)}+a^2(\mu+P)\partial_iV^{(1)}\partial_jV^{(1)}\right]. \tag{6.120}$$

When applied to a scalar field, this is exactly the gravitational wave propagation equation (6.89) with the source term (6.93).

6.4.3 Discussion

In conclusion, we have matched both the perturbation variables and equations at first and second order in the perturbations. This extends the work of [106] which considered the linear case, and has not been previously investigated.

Even though we restrict to the tensor sector, this comparison is instructive and illustrates the difference of approach between the two formalisms, in a clearer way than at first order. In the Bardeen approach, all perturbation variables live on the unperturbed spacetime. At each order, we write exact equations for an approximate spacetime. In particular, this implies that the time derivatives are derivative with respect to the cosmic time of the background spacetime. In the covariant approach, one derives an exact set of equations (assuming no perturbation to start with). These exact equations are then solved iteratively starting from a background solution which assumes some symmetries. The time derivative is defined in terms of the flow vector as $u^a\nabla_a$. Indeed, at first order for scalars, this derivative matches exactly with the derivative with respect to the background cosmic time. At second order, this is no longer the case. First the flow vector at first order does not coincide with its background value. This implies a (first-order) difference between the two time derivatives which must be taken into account. Then, the geometric quantities, such as H_{ij} E_{ij} etc., "live" on the physical space-time, whereas in perturbation theory, any perturbation variable at any order, such as $V^{(1)}$, $\mathcal{E}^{(2)}_{ij}$ etc., live on the background space-time. This explains why e.g. $H^{(2)}_{ij}$ has a term quadratic in first-order perturbations involving $V^{(1)}$ and $\Phi^{(1)}$.

The master variables and corresponding wave equations in both formalisms are also different in nature. In the metric approach the wave equation with source is defined non-locally in Fourier space; in the covariant approach, we are able to derive a local tensorial wave equation which, because it is divergence-free, represents the gravitational wave contribution. Of course, we can make a non-local decomposition in Fourier space as required. Furthermore, on one hand the TT part of the metric in a particular gauge is a perturbative approach used to describe GW, and this tells us the shear of spatial lengths with respect to a homogenous and isotropic background, referring implicitly to a hypothetical set of averaged observers. On the other hand, the covariant description using H_{ab} which is built out of the Weyl tensor and the comoving observer's velocity, directly describes the dynamically free part of the gravitational field [129, 136, 110] (up to second-order when rotation is zero) as seen by the true comoving observers. This is part of the dynamic spacetime curvature which directly induces the motion of test particles through the geodesic deviation equation, and it accounts for effects due to the non-homogenous comoving fluid velocity. There is one more difference between the two formalisms, concerning the initial conditions. In the Bardeen approach, as we recalled in section 6.3, there is a natural way to set up the initial conditions on sub-Hubble scales by identifying canonical variables, both for the scalar and tensor modes, and promoting them to the status of quantum operators. In the covariant formalism such variables have not been constructed in full generality (see however [134] for a proposal). Consequently this sets limitations to this formalism since it cannot account for both the evolution and the initial conditions at the same time.

6.5 The illustration: slow-roll inflation

6.5.1 Slow-roll inflation

In this section, we focus on the case of a single slow-rolling scalar field and we introduce the slow-roll parameters

$$\varepsilon = \frac{3}{2}\frac{\psi^2}{\mu a^2}, \qquad \delta = -\frac{\dot\psi}{H\psi}. \tag{6.121}$$

Using the Friedmann equations (6.21-6.22), these parameters can be expressed in terms of the Hubble parameter as

$$\varepsilon = 2\left[\frac{H'(\phi)}{H(\phi)}\right]^2, \qquad \delta = 2\frac{H''(\phi)}{H(\phi)}. \tag{6.122}$$

Interestingly equation (6.21) takes the form

$$H^2\left(1-\tfrac{1}{3}\varepsilon\right) = \tfrac{1}{3}V(\phi), \tag{6.123}$$

which implies

$$\frac{\ddot a}{a} = (1-\varepsilon)H^2. \tag{6.124}$$

The equation of state and the sound speed of the equivalent scalar field are thus given by

$$w = -1 + \tfrac{2}{3}\varepsilon, \qquad c_s^2 = -1 + \tfrac{2}{3}\delta. \tag{6.125}$$

The evolution equations for ε and δ show that $\dot{\varepsilon}$ and $\dot{\delta}$ are of order 2 in the slow-roll parameters so that at first order in the slow-roll parameters, they can be considered constant. Using the definition of the conformal time and integrating it by parts, one gets

$$a(\eta) = -\tfrac{1}{H\eta}\tfrac{1}{1-\varepsilon}, \tag{6.126}$$

assuming ε is constant, from which it follows that

$$\mathcal{H} \equiv aH = -\tfrac{1}{\eta}(1+\varepsilon) + \mathcal{O}(2), \tag{6.127}$$

where η varies between $-\infty$ and 0. This implies that

$$\tfrac{a''}{a} = \tfrac{2+3\varepsilon}{\eta^2}, \qquad \tfrac{z''}{z} = \tfrac{2+6\varepsilon-3\delta}{\eta^2}. \tag{6.128}$$

The general solution of equation (6.72) is

$$v_k = \sqrt{-\pi\eta/4}\left[c_1 H_\nu^{(1)}(-k\eta) + c_2 H_\nu^{(2)}(-k\eta)\right], \tag{6.129}$$

with $|c_1|^2 - |c_2|^2 = 1$, where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are Hankel functions of first and second kind and $\nu = 3/2 + 2\varepsilon - \delta$. Among this family of solutions, it is natural to choose the one with $c_2 = 0$ which contains only positive frequencies [118]. It follows that the solution with these initial conditions is

$$v_k(\eta) = \tfrac{\sqrt{\pi}}{2}\sqrt{-\eta}H_\nu^{(1)}(-k\eta). \tag{6.130}$$

On super-Hubble scales, $|k\eta| \ll 1$, we have

$$v_k \to 2^{\nu-3/2}\Gamma(\nu)/\Gamma(3/2)(2k)^{-1/2}(-k\eta)^{-\nu+1/2}.$$

Now, using equation (6.126) to express η and equation (6.77) to replace z in expression (6.76), we find that

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{\pi}\frac{H^2}{M_p^2\varepsilon}\left[2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]^2\left(\nu-\frac{1}{2}\right)^{-2\nu+1} \times \left(\frac{k}{aH}\right)^{-2\nu+3}, \tag{6.131}$$

where we have set $M_p^2 = G^{-1} = 8\pi$, since $8\pi G = 1$. At lowest order in the slow-roll parameter, it reduces to

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{\pi}\frac{H^2}{M_p^2\varepsilon}\left(\frac{k}{aH}\right)^{2\delta-4\varepsilon}. \tag{6.132}$$

The evolution of the gravitational waves at linear order are dictated by the same equation but with $\nu_T = 3/2+\varepsilon$, so that

$$\mu_k^{(1)}(\eta) = \frac{\sqrt{\pi}}{2}\sqrt{-\eta}H_{\nu_T}^{(1)}(-k\eta). \tag{6.133}$$

Similarly as for the scalar mode, we obtain

$$\mathcal{P}_T(k) = \frac{16}{\pi}\frac{H^2}{M_p^2}\left(\frac{k}{aH}\right)^{-2\varepsilon}. \tag{6.134}$$

6.5.2 Gravitational waves at second order

The couplings between scalar and tensor modes at second order imply that the second order variables can be expanded as

$$\mathcal{R} = \mathcal{R}^{(1)} + \frac{1}{2}\left(\mathcal{R}^{(2)}_{\mathcal{R}\mathcal{R}} + \mathcal{R}^{(2)}_{\mathcal{E}\mathcal{E}} + \mathcal{R}^{(2)}_{\mathcal{R}\mathcal{E}}\right)$$

and a similar expansion for $\mathcal{E}$, where, e.g., $\mathcal{R}^{(2)}_{\mathcal{R}\mathcal{E}}$ stands for the second order scalar modes induced by the coupling of first order scalar and tensor modes etc. The deviation from Gaussianity at the time η of the end of inflation can be characterized by a series of coefficients $f_{\rm NL}^{a,bc}$ defined for example as

$$\begin{aligned}\tfrac{1}{2}\mathcal{R}^{(2)}_{\mathcal{E}\mathcal{E}}(\mathbf{k},\eta) &= \tfrac{1}{(2\pi)^{3/2}}\int \delta^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\mathcal{E}(\mathbf{k}_1,\eta)\mathcal{E}(\mathbf{k}_2,\eta)\\ &\qquad f_{\rm NL}^{\mathcal{R},\mathcal{E}\mathcal{E}}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\eta)\mathrm{d}^3\mathbf{k}_1\mathrm{d}^3\mathbf{k}_2. \end{aligned}\tag{6.135}$$

These six coefficients appear in different combinations in the connected part of the 3-point correlation function of $\mathcal{R}$ and $\mathcal{E}$. For instance

$$\begin{aligned}\langle \mathcal{E}_{\mathbf{k}_1}\mathcal{R}_{\mathbf{k}_2}\mathcal{R}_{\mathbf{k}_3}\rangle_c &= \Big[2f_{\rm NL}^{\mathcal{E},\mathcal{R}\mathcal{R}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)P_{\mathcal{R}}(k_3)P_{\mathcal{R}}(k_2)+f_{\rm NL}^{\mathcal{R},\mathcal{E}\mathcal{R}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)P_{\mathcal{R}}(k_3)P_{\mathcal{E}}(k_1)\Big]\\ &\times\ \delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\end{aligned}\tag{6.136}$$

and $f_{\rm NL}^{\mathcal{R},\mathcal{R}\mathcal{R}}$ is the standard $f_{\rm NL}$ parameter. One can easily check that $\langle\mathcal{R}_{\mathbf{k}_1}\mathcal{R}_{\mathbf{k}_2}\mathcal{R}_{\mathbf{k}_3}\rangle_c$ involves $f_{\rm NL}^{\mathcal{R},\mathcal{R}\mathcal{R}}$, $\langle\mathcal{E}_{\mathbf{k}_1}\mathcal{E}_{\mathbf{k}_2}\mathcal{R}_{\mathbf{k}_3}\rangle_c$ involves $f_{\rm NL}^{\mathcal{E},\mathcal{E}\mathcal{R}}$ and $f_{\rm NL}^{\mathcal{R},\mathcal{E}\mathcal{E}}$, and $\langle\mathcal{E}_{\mathbf{k}_1}\mathcal{E}_{\mathbf{k}_2}\mathcal{E}_{\mathbf{k}_3}\rangle_c$ involves $f_{\rm NL}^{\mathcal{E},\mathcal{E}\mathcal{E}}$.

6.5.3 Expression for $f_{\rm NL}^{\mathcal{E},\mathcal{R}\mathcal{R}}$

From our analysis, we can give the expression of $f_{\rm NL}^{\mathcal{E},\mathcal{R}\mathcal{R}}$. Starting from the fact that $-\varepsilon\mathcal{R}=\Phi(1+\varepsilon)+\Phi'/\mathcal{H}$ and from the expression (6.78), we get that $\Phi\sim-\varepsilon\mathcal{R}-\Phi'/\mathcal{H}$ or $\Phi=-\epsilon\eta\int\frac{\mathcal{R}}{\eta^2}\mathrm{d}\eta$, and $\frac{1}{\sqrt{2}}\chi\sim-\sqrt{\varepsilon}\,[\mathcal{R}-\Phi'/\mathcal{H}]$. It follows that the source term (6.93) reduces at lowest order in the slow-roll parameter to

$$S_{ij}^{\rm TT}=4\left[\varepsilon\partial_i\mathcal{R}\partial_j\mathcal{R}\right]^{\rm TT}.$$

The interaction Lagrangian is thus given by

$$S_{\rm int}=\int \mathrm{d}\eta\mathrm{d}^3\mathbf{x}4a\,\varepsilon\,\partial_i\mathcal{R}\partial_j\mathcal{R}\,\mu^{ij},\tag{6.137}$$

which reduces to

$$S_{\rm int}=\int \mathrm{d}\eta\mathrm{d}^3\mathbf{x}\,2\partial_i v\,\partial_j v\,\bar{\mathcal{E}}^{ij}.\tag{6.138}$$

This is the same expression as obtained in [112].

In full generality, during inflation, we should use the "in-in" formalism to compute any correlation function of the interacting fields. As was shown explicitly in [65] for a self-interacting field and more generally in [169, 103], the quantum computation agrees with the classical one on super-Hubble scales at lowest order. Note however that both computations may differ (see [161] versus [112]) due to the fact that in the classical approach the change in vacuum is ignored. The difference does not affect the order of magnitude but the geometric k-dependence. In order to get an order of magnitude, we thus restrict our analysis here to the classical description. This description is also valid when considering the post-inflationary era.

In the classical approach, we can solve equation (6.96) by mean of a Green function. Since the two independent solutions of the homogeneous equation are $\sqrt{-k\eta}H_{\nu_T}^{(1/2)}(-k\eta)$, the Wronskian of which is $4i/(\pi k)$, the Green function is given by

$$\begin{aligned}\mathcal{G}(k,\eta,\eta') &= -i\tfrac{\pi}{4}\sqrt{\eta\eta'}\left[H_{\nu_T}^{(1)}(-k\eta)H_{\nu_T}^{(2)}(-k\eta')\right.\\ &\qquad \left.-H_{\nu_T}^{(1)}(-k\eta')H_{\nu_T}^{(2)}(-k\eta)\right]. \end{aligned} \tag{6.139}$$

It follows that the expression of the second order tensor perturbation is given by

$$\begin{aligned}\mu_{\mathbf{k},\lambda}^{(2)}(\eta) &= \tfrac{2}{(2\pi)^{3/2}}\int_{-\infty}^{\eta}\mathrm{d}\eta' a(\eta')\,\varepsilon\,\mathcal{G}(k,\eta,\eta')\\ &\qquad \int \mathrm{d}^3\mathbf{q}\left(q_iq_j\varepsilon_\lambda^{ij}\right)\mathcal{R}_{\mathbf{q}}(\eta')\mathcal{R}_{\mathbf{k}-\mathbf{q}}(\eta'). \end{aligned} \tag{6.140}$$

We thus obtain

$$\begin{aligned} f_{NL}^{\mathcal{E}_\lambda\mathcal{R}\mathcal{R}}(\mathbf{k},\boldsymbol{q}_1,\boldsymbol{q}_2,\eta) &= \left[\mathcal{R}_{q_1}(\eta)\mathcal{R}_{q_2}(\eta)\right]^{-1}\tfrac{\varepsilon}{a(\eta)}\int_{-\infty}^{\eta}\mathrm{d}\eta' a(\eta')\,\mathcal{G}(k,\eta,\eta')\left(q_{1i}q_{1j}\varepsilon_\lambda^{ij}(\mathbf{k})\right)\\ &\times\ \mathcal{R}_{q_1}(\eta')\mathcal{R}_{q_2}(\eta'). \end{aligned} \tag{6.141}$$

If we want to estimate equation (6.136) in the squeezed limit $k_1 \ll k_2, k_3$ the contribution coming from the term involving $f_{NL}^{\mathcal{E}_\lambda\mathcal{R}\mathcal{R}}(\mathbf{k}, \boldsymbol{q_1}, \boldsymbol{q_2}, \eta)$ can be computed by use of the super-Hubble limit of the Green function $|\mathcal{G}(k,\eta,\eta')| \simeq \frac{\sqrt{\eta\eta'}}{2\nu_T}\left[\left(\frac{\eta'}{\eta}\right)^{\nu_T} - \left(\frac{\eta'}{\eta}\right)^{-\nu_T}\right]$. This contribution will be proportional to $\frac{H^4}{M_p^4\varepsilon}k_2^{-8}\left(k_{2i}k_{2j}\varepsilon_\lambda^{ij}\right)\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)$, which is the same order of magnitude as in [112], but do not have the same geometric dependence as it goes like $k_2^{-5}k_1^{-3}$ instead.

6.5.4 Orders of magnitude

When we want to estimate $\langle\mathcal{E}^{(2)}_{\mathbf{k},\lambda}(\eta)\mathcal{E}^{(2)*}_{\mathbf{k}',\lambda'}(\eta)\rangle$, we have to evaluate the connected part of $\langle\mathcal{R}_{\mathbf{q}}(\eta')\mathcal{R}_{\mathbf{k}-\mathbf{q}}(\eta')\mathcal{R}^*_{\mathbf{p}}(\eta'')\mathcal{R}^*_{\mathbf{k}'-\mathbf{p}}(\eta'')\rangle$, where $\mathbf{q}$ and $\mathbf{p}$ are the two internal momentum and η' and η'' the two integration times. From the Wick theorem, this correlator reduces to $\mathcal{R}(q,\eta')\mathcal{R}^*(q,\eta'')\mathcal{R}(|\mathbf{k}-\mathbf{q}|,\eta')\mathcal{R}^*(|\mathbf{k}-\mathbf{q}|,\eta'')\delta(\mathbf{k}-\mathbf{k}')[\delta(\mathbf{q}-\mathbf{p})+\delta(\mathbf{k}-\mathbf{q}-\mathbf{p})]$ and because $k^i\varepsilon_{ij}=0$ the two terms give the same geometric factor. Thus, the integration on $\mathbf{p}$ is easily done and we can factorise $\delta(\mathbf{k}-\mathbf{k}')$. Now, note that the terms in the integral involve only the modulus of $\mathbf{q}$ and $\mathbf{k}-\mathbf{q}$ so that it does not depend on the angle φ of $\mathbf{q}$ in the plane orthogonal to $\mathbf{k}$. This implies that the integration of φ will act on a term of $\cos^2 2\varphi$, $\sin^2 2\varphi$ and $\cos 2\varphi \sin 2\varphi$ respectively for $++$, $\times\times$ and $+\times$ so that it gives a term $\pi\delta_{\lambda\lambda'}$. In conclusion, defining the second order power spectrum $\mathcal{P}_T^{(2)}$ by

$$\tfrac{1}{4}\langle\mathcal{E}^{(2)}_{\mathbf{k},\lambda}\mathcal{E}^{*2}_{\mathbf{k}',\lambda'}\rangle = \tfrac{2\pi^2}{k^3}\mathcal{P}_T^{(2)}(k)\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}, \tag{6.142}$$

it can be expressed as

$$\begin{aligned}\mathcal{P}_T^{(2)}(k) &= \tfrac{k^3}{(2\pi)^3\pi^2a^2}\int \mathrm{d}\eta'\mathrm{d}\eta'' a(\eta')a(\eta'')\,\varepsilon^2\,\mathcal{G}(k,\eta,\eta')\mathcal{G}^*(k,\eta,\eta'')\\ &\quad\times\int \mathrm{d}^3q\,\left(q_iq_j\varepsilon_\lambda^{ij}\right)^2\mathcal{R}(q,\eta')\mathcal{R}^*(q,\eta'')\mathcal{R}(|\mathbf{k}-\mathbf{q}|,\eta')\mathcal{R}^*(|\mathbf{k}-\mathbf{q}|,\eta'')\end{aligned}\tag{6.143}$$

Setting $\mathbf{k}\cdot\mathbf{q} = kq\mu$, this reduces to

$$\mathcal{P}_T^{(2)}(k) = \tfrac{k^3}{(2\pi)^3\pi a^2}\int q^6\mathrm{d}q\,\left(1-\mu^2\right)^2\mathrm{d}\mu\left|\int_{-\infty}^{\eta}\mathrm{d}\eta'\,a(\eta')\,\varepsilon\,\mathcal{G}(k,\eta,\eta')\mathcal{R}(q,\eta')\mathcal{R}(|\mathbf{k}-\mathbf{q}|,\eta')\right|^2, \tag{6.144}$$

after integration over φ which gives a factor $\pi\left(1-\mu^2\right)^2 q^4$.

We can now take the super-Hubble limit of this expression at lowest order in the slow-roll parameters. In order to do so, we make use of the super-Hubble limit of the Green function given above, and we perform the time integral from $1/k$ to η and keep only the leading order contribution:

$$\mathcal{P}_T^{(2)}(k) \;=\; \frac{1}{3^4 2^3 \pi^2}\left(\frac{H}{M_p}\right)^4 F(\epsilon,\delta)\left(\frac{k}{aH}\right)^{-2\epsilon}, \tag{6.145}$$

where, with the definitions $\boldsymbol{y} \equiv \mathbf{q}/k$ and $\boldsymbol{n} \equiv \mathbf{k}/k$,

$$F(\epsilon,\delta) \equiv \int \left(y\,|\boldsymbol{n}-\boldsymbol{y}|\right)^{-3-4\epsilon+2\delta} y^6 \mathrm{d}y \left(1-\mu^2\right)^2 \mathrm{d}\mu \tag{6.146}$$

is a numerical factor. In this approximation, the ratio between the second order power spectrum and the first order power spectrum at leading order in the slow-roll parameters, is given by:

$$\frac{\mathcal{P}_T^{(2)}(k)}{\mathcal{P}_T^{(1)}(k)} = \frac{1}{2^7 3^4 \pi}\left(\frac{H}{M_p}\right)^2 F(\epsilon,\delta). \tag{6.147}$$

Indeed there are ultraviolet and infrared divergences hidden in $F(\epsilon,\delta)$. We expect the infrared divergence not to be relevant for observable quantities due to finite volume effects (see for instance [14]). The ultraviolet divergence, on the other hand, has to be carefully dimensionally regularised in the context of quantum field theory (see e.g. [169, 103]).

6.6 Chapter conclusion

In this chapter we have investigated the generation of gravitational waves due to second order effects during inflation. We have considered these effects both in the covariant perturbation formalism and in the more standard metric based approach. The relation between the two formalisms at second-order has been considered and we have discussed their relative advantages. This comparison leads to a better understanding of the differences in dynamics between the two formalisms.

As an illustration, we have focused on GW generated by the coupling of first order scalar modes. To characterize this coupling we have introduced and computed the parameter $f_{\rm NL}^{\mathcal{E},\mathcal{R}\mathcal{R}}$. It enters in the expression of $\langle \mathcal{E}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \mathcal{R}_{\mathbf{k}_3} \rangle_c$ that was shown to be of order $(H/M_p)^4/\varepsilon$, as $\langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \mathcal{R}_{\mathbf{k}_3} \rangle_c$. On the other hand the power spectrum of GW remains negligible.

This shows that the contribution of $\langle \mathcal{E}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \mathcal{R}_{\mathbf{k}_3} \rangle_c$ to the CMB bispectrum is important to include in order to constrain the deviation from Gaussianity, e.g. in order to test the consistency relation [27]. In this chapter, we have considered slow-roll inflation to illustrate the formalism for studying gravitational waves resulting from quadratic effects. In principle, the formalism is applicable to a wide range of scenarios, and in particular other inflationary scenarios. We reserve the investigations into these scenarios for future work.

Chapter 7

Thesis Conclusion

In this thesis, we have examined two seemly contradictory issues; the agreement between observation and the standard model on the one hand, and discrepancies between the two on the other hand. The discrepancies highlighted in the introductory chapter underline the motivation for the study undertaken in the rest of thesis. In particular, linear perturbations about anisotropic and second-order perturbations about isotropic models have been considered.

The motivation for studying linear perturbations of anisotropic models comes from reports of discrepancies between the prediction by the standard model, and the analysis of WMAP data from the observation of the CMB. In the Standard Model picture, the anisotropies which are generated by primordial fluctuations in the inflationary field give rise to imprints on the CMB that are statistically isotropic and Gaussian. It has been claimed that there exists a strange planarity and alignment of the quadrupole $l = 2$ and the octopole $l = 3$, with the possibility that this alignment extends to higher multipoles [45, 96, 130, 52, 53, 51, 50, 30, 59]. The alignment that extends between the first four multipoles $l = 2 - 5$ has become known as the ' *Axis of evil*'.

In principle, such an alignment may point to the existence of a feature in the CMB that picks a preferred direction or some form of statistical anisotropy. Although this could arise from foreground contamination, results from the difference maps constructed from the WMAP foreground corrected maps indicate that such a contamination is unlikely to lead to the levels observed. The fact that the alignment includes the quadrupole, suggests that the anomaly may be an intrinsic feature of the cosmological model and therefore strengthens the need to investigate anisotropic cosmological models. Initial analysis, in relation to this problem, was given by [62]. The authors used what they called the best-fit Bianchi model (Bianchi type VII_h) to correct a combination of various WMAP sky maps. They found that the Bianchi corrected maps exhibit greater isotropy compared to the uncorrected maps. They also found that the general shape of the corrected spectrum is flatter than the WMAP best-fit power spectrum, which agrees with the theoretical fit made to the northern hemisphere data in [50].

Our approach is somewhat different but complementary to these studies. We examined linear perturbations about Bianchi I model rather than determine the statistical best-fit model for the available data. In Chapter 3, we considered density perturbations in Bianchi type I model filled with irrotational dust. Since the various modes are coupled in Bianchi I model, we first examined how to characterize decoupled density perturbations. We then performed a detailed consistency analysis of the arising covariant equations. Although the analysis is primarily performed in the 1+3 *covariant* formalism, it turns out that the 1+ 3 *orthonormal* approach was necessary for a conclusive result. In general, conditions used to characterise pure density perturbations lead to the modification of the original covariant equations.

We have analysed the consistency of the new set of constraint equations and found that the background shear needs to be diagonal and degenerate, for consistency to be achieved. We then presented the analysis of the growth behaviour of inhomogeneity in this model, where we considered the behaviour in the shear and matter regimes. The behaviour in the matter, as expected, recovers the FLRW results. Although Bianchi models are anisotropic and come with directional preference and hence good candidates to consider in the search for the answer to the CMB anomalies, it should be emphasied that these models are incompatible with inflationary theory. A model responsible for the preferred direction will also need to produce the acoustic peaks now measured at even smaller scales in the CMB, if its to be the accepted for our universe.

Chapter 4 is a work in progress. In this chapter we are concerned with two problems; the first first problem concerns ways of consistently setting inhomogeneity variable to zero, while the second problem concerns giving a consistent characterisation of gravitational waves. We found that one could consistently set the inhomogeneity to zero if the background shear were diagonal and degenerate. The second problem is linked to the problem of describing pure gravitational waves in Bianchi I model. We naively assumed that the tensor describing such waves should be divergence-free as is the case for gravitational waves in the FLRW models. We were unable to show that the constraint equations arising from such a condition evolve consistently, even when the background shear was degenerate. This was not surprising given that the residual term has also been found by the authors of [20]. These authors reached the same conclusion. This is not to say that there are no gravitational waves in a perturbed Bianchi I model filled with dust. The point seems to be that the transversal wave is inextricably coupled to other perturbations. Other restrictions such as insisting that the magnetic part of the Weyl tensor be diagonal, may lead to constraint equations that evolve consistently, and may also make the wave equation tractable as we have shown. The critical issue, and one that we are still trying to understand is the physical significance or lack thereof, of a wave with such restrictions. The analyses presented in this thesis extend those of [109] to the case of pure gravitational waves in perturbed Bianchi I model with irrotational dust and also complement the analysis given in [82]. In Chapter 5, we re-examined and extended the covariant and gauge invariant approach to second-order perturbation theory initiated in [24].

In [24], a covariant approach to nonlinear perturbation theory was initiated, and the formalism used to study second-order gravitational waves sourced by first-order density perturbations and second-order density perturbations sourced by gravitational waves at first order. Also developed in this thesis is a parallel approach, which not only recovers the results in that paper but one that is robust enough to deal with a wide range of barotropic fluids. We have also presented new covariant methods for extracting the various modes and have used the tensor extraction in the subsequent analysis of the second-order gravitational wave equation. The greatest challenge in the above development was finding a way of dealing with the source term. We have presented a new way of handling the source term which complements the approached used in [24]. Extending the method to situations involving pressure introduces complexities not found in the analysis of the dust subcase. The second challenge in this analysis has to do with determining the solutions. We have demonstrated that the case for fluid with barotropic equation of state is tractable, given the long wavelength approximation.

Chapter 6 illustrates the applicability of the techniques developed in Chapter 5. In this chapter we studied the generation of gravitational waves during inflation arising from the non-linear coupling of scalar and tensor modes. Part of the major result of Chapter 6 chapter is the detailed comparison of the two perturbative approaches up to second order, where we have highlighted both the advantages and disadvantages of each method, thus extending the earlier work that made a similar comparison but only up-to linear order [105]. This chapter also extends the work of [97,98], in which the relationship between variables in the two formalisms on super-Hubble scales is investigated. In particular, we have shown that the degree of success of one formalism over the other depends on the problem being addressed. This is the first time a complete and transparent matching of tensor perturbations in the two formalisms at first and at second order has been presented. We have also shown, using an analytical argument, that the power-spectrum of gravitational waves from second-order effects is much smaller than the first order on super-Hubble scales. This is in contrast to the fact that during the radiation era the generation of GW from primordial density fluctuations can, in principle, be large enough to be detected [89].

As an illustration, we focused on GW generated by the coupling of first-order scalar modes, where we have introduced and computed the parameter $f_{\rm NL}^{\mathcal{E},\mathcal{RR}}$. This parameter characterises the above coupling and enters in the expression of $\langle \mathcal{E}_{\mathbf{k}_1}\mathcal{R}_{\mathbf{k}_2}\mathcal{R}_{\mathbf{k}_3}\rangle_c$ that was shown to be of order $(H/M_p)^4/\varepsilon$, as $\langle \mathcal{R}_{\mathbf{k}_1}\mathcal{R}_{\mathbf{k}_2}\mathcal{R}_{\mathbf{k}_3}\rangle_c$. We find that the power spectrum of GW remains negligible. This shows that the contribution of $\langle \mathcal{E}_{\mathbf{k}_1}\mathcal{R}_{\mathbf{k}_2}\mathcal{R}_{\mathbf{k}_3}\rangle_c$ to the CMB bispectrum is important and should be included in order to constrain the deviation from Gaussianity, e.g. in order to test the consistency relation [27]. We only considered slow-roll inflation to illustrate applicability of the formalism in studying gravitational waves resulting from quadratic effects. In principle, the formalism is applicable to a wide range of scenarios, and in particular other inflationary scenarios. We reserve the investigations of these scenarios for future work. It was interesting to note, during the final phase of completing this thesis, that the authors of [115] have determined the bounds on tensor-to-scalar ratio for a situation involving gravitational waves sourced by scalars during inflation. This sets the bounds on equation 6.147. They find that for gravitational waves generated from scalar perturbation, an upper bound of 10^{-8} exists on the tensor-to-scalar ratio for a metric perturbation of no more than 10^{-5}.

Appendix A

Commutation Relations

The following commutation relations are derived for a scalar quantity, f and PSTF-tensor quantities, T_{ab} and V_{ab} in the case of dust with vanishing vorticity.

$$
\begin{aligned}
\varepsilon_{abc}\varepsilon^{dec} &= 2!h^d{}_{[a}h^e{}_{b]}, &\text{(A.1)}\\
\varepsilon_{abc}T^b{}_pT^p{}_qV^{cq} &= -T_{ab}\varepsilon^{bcd}T_c{}^pV_{dp}, &\text{(A.2)}\\
(\tilde{\nabla}_af)^{\cdot} &= \tilde{\nabla}_a\dot{f} - \frac{1}{3}\Theta\tilde{\nabla}_af - \sigma_a{}^b\tilde{\nabla}_bf, &\text{(A.3)}\\
\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}f &= 0, &\text{(A.4)}\\
\mathsf{curl}\,(T^2)_{ab} &= \varepsilon_{cd(a}\tilde{\nabla}^e\{T^c_{b)}T^d{}_e\}, &\text{(A.5)}\\
(\tilde{\nabla}_aT_{bc})^{\cdot} &= \tilde{\nabla}_a\dot{T}_{bc} - \frac{1}{3}\Theta\tilde{\nabla}_aT_{bc} - \sigma_a{}^d\tilde{\nabla}_dT_{bc} + 2H_a{}^d\varepsilon_{de(b}T^e_{c)}, &\text{(A.6)}\\
(\tilde{\nabla}^bT_{ab})^{\cdot} &= \tilde{\nabla}^b\dot{T}_{ab} - \frac{1}{3}\Theta\tilde{\nabla}^bT_{ab} - \sigma^{bc}\tilde{\nabla}_cT_{ab} + \varepsilon_{abc}H^b{}_dT^{bc}, &\text{(A.7)}\\
\varepsilon_{abc}T^b{}_d\,\mathsf{curl}\,V^{cd} &= T^{bc}\tilde{\nabla}_aV_{bc} - T^{bc}\tilde{\nabla}_bV_{ac} - \frac{1}{2}T_{ab}\tilde{\nabla}_cV^{bc}, &\text{(A.8)}\\
\mathsf{curl}\,(fT_{ab}) &= f\,\mathsf{curl}\,(T)_{ab} + \varepsilon_{cd(a}T^d_{b)}\tilde{\nabla}^cf, &\text{(A.9)}\\
(\tilde{\nabla}^b\mathsf{curl}\,T_{ab}) &= \frac{1}{2}\varepsilon_{abc}\tilde{\nabla}^b\tilde{\nabla}_dT^{cd} + \varepsilon_{abc}T^b{}_d(\frac{1}{3}\Theta\sigma^{cd} - E^{cd}) - \sigma_{ab}\varepsilon^{bcd}\sigma_{ce}T^ed, &\text{(A.10)}\\
(\mathsf{curl}\,T_{ab})^{\cdot} &= \mathsf{curl}\,(\dot{T})_{ab} - \frac{1}{3}\Theta\,\mathsf{curl}\,T_{ab} - \sigma_e{}^c\varepsilon_{cd(a}\tilde{\nabla}^eT_{b)}{}^d + 3H_{c\langle a}T_{b\rangle}{}^c, &\text{(A.11)}
\end{aligned}
$$

$$\text{curl curl}\,(T)_{ab} = -\tilde{\nabla}^2 T_{ab} + \frac{3}{2}\tilde{\nabla}_{\langle a}\tilde{\nabla}^c T_{b\rangle c} + (\rho - \frac{1}{3}\Theta^2)T_{ab} + 3T_{c\langle a}\{E_{b\rangle}{}^c - \frac{1}{3}\Theta\sigma_{b\rangle}{}^c\}$$
$$+\sigma_{cd}T^{cd}\sigma_{ab} - T^{cd}\sigma_{ca}\sigma_{bd} + \sigma^{cd}\sigma_{c(a}T_{b)d}. \tag{A.12}$$

Also

$$\tilde{\nabla}_a(\tilde{\nabla}^2 f) = {}^3R_a{}^b\tilde{\nabla}_b f - \tilde{\nabla}^2(\tilde{\nabla}_a f) \tag{A.13}$$

A.1 Divergence of Ricci tensor

Let us consider the simple case of dust. It can be shown that when the vorticity vanishes the Ricci tensor takes the form,

$$^3R_{ab} = -\dot{\sigma}_{ab} - \Theta\sigma_{ab} + \tfrac{1}{3}h_{ab}{}^3R, \tag{A.14}$$

which can also be written in terms of the electric part of the weyl tensor as follows,

$$^3R_{ab} = E_{ab} + \sigma_{c\langle a}\sigma_{b\rangle}{}^c - \tfrac{1}{3}\Theta\sigma_{ab} + \tfrac{1}{3}h_{ab}{}^3R. \tag{A.15}$$

The divergence of this equation leads to,

$$\tilde{\nabla}^{b3}R_{ab} = \tilde{\nabla}^b E_{ab} + \tilde{\nabla}^b(\sigma_{c\langle a}\sigma_{b\rangle}{}^c) - \tfrac{1}{3}\Theta\tilde{\nabla}^b(\sigma_{ab}) - \tfrac{1}{3}\sigma_{ab}\tilde{\nabla}^b(\Theta) + \tfrac{1}{3}\tilde{\nabla}_a^3 R. \tag{A.16}$$

Using the divergence equations for shear (2.73), the divergence equation for the electric part of the Weyl tensor (2.74) and commutation relation (A.8), its can be shown that,

$$\tilde{\nabla}^b(\sigma_{c\langle a}\sigma_{b\rangle}{}^c) = -\tilde{\nabla}^b E_{ab} + \tfrac{1}{3}\tilde{\nabla}_a\mu + \tfrac{1}{3}\tilde{\nabla}\sigma^2 + \tfrac{1}{3}\sigma_a{}^b\tilde{\nabla}_b\Theta. \tag{A.17}$$

Substituting this equation into (A.16) generates,

$$\begin{aligned}\tilde{\nabla}^{b3}R_{ab} &= \tfrac{1}{3}\tilde{\nabla}_a\mu + \tfrac{1}{3}\tilde{\nabla}\sigma^2 - \tfrac{2}{9}\Theta\tilde{\nabla}_a\Theta + \tfrac{1}{3}\tilde{\nabla}_a{}^3R,\\ &= \tfrac{1}{6}\tilde{\nabla}_a\left(2\mu + 2\sigma^2 - \tfrac{2}{3}\Theta^2\right) + \tfrac{1}{3}\tilde{\nabla}_a{}^3R,\\ &= \tfrac{1}{2}\tilde{\nabla}_a{}^3R.\end{aligned} \tag{A.18}$$

This relationship between the divergence of the Ricci tensor and spatial derivative of Ricci scalar hold for model with dust and with vanishing vorticity.

Appendix B

Propagation of constraints

In this section we present fully worked out illustration of constraint propagation for a nonlinear system with irrotational dust.

$$\begin{aligned}
C^1_a &= \tilde{\nabla}^b\sigma_{ab} - \frac{2}{3}\tilde{\nabla}_a\Theta \\
\dot{C}^1_a &= (\tilde{\nabla}^b\sigma_{ab})^{\cdot} - \frac{2}{3}(\tilde{\nabla}_a\Theta)^{\cdot}, \\
(\tilde{\nabla}^b\sigma_{ab})^{\cdot} &= \tilde{\nabla}^b\dot{\sigma}_{ab} - \frac{1}{3}\Theta\tilde{\nabla}^b\sigma_{ab} - \sigma_c{}^b\tilde{\nabla}^c\sigma_{ab} + \epsilon_{abc}H^b{}_d\sigma^{cd}, \\
(\tilde{\nabla}^b\sigma_{ab})^{\cdot} &= -\Theta\tilde{\nabla}^b\sigma_{ab} - \sigma_c{}^b\tilde{\nabla}^c\sigma_{ab} + \epsilon_{abc}H^b{}_d\sigma^{cd} - \frac{2}{3}\sigma_a{}^b\nabla_b\Theta - \tilde{\nabla}^b(\sigma_{c\langle a}\sigma_{b\rangle}{}^c) - \tilde{\nabla}^bE_{ab}.
\end{aligned} \tag{B.1}$$

The propagation of the gradient of expansion yields,

$$-\frac{2}{3}(\tilde{\nabla}_a\Theta)^{\cdot} = -\frac{2}{3}[-\Theta\tilde{\nabla}_a\Theta - \frac{1}{2}\tilde{\nabla}_a\rho - \sigma_a{}^b\nabla_b\Theta - 2\nabla_a\sigma^2] \tag{B.2}$$

The terms can now be regrouped to yield the desired result.

$$\begin{aligned}
\dot{C}^1_a &= -\Theta[\tilde{\nabla}^b\sigma_{ab} - \frac{2}{3}\tilde{\nabla}_a\Theta] - [\tilde{\nabla}^bE_{ab} - \frac{1}{3}\tilde{\nabla}_a\rho - \epsilon_{abc}\sigma^b{}_dH^{cd}] + 2\epsilon_{abc}\sigma^b{}_d[curl\sigma^{cd} - H^{cd}] \\
&= -\Theta C^1_a - C^3_a + 2\epsilon_a{}^{bc}\sigma_b{}^dC^2_{cd}.
\end{aligned} \tag{B.3}$$

As in the analysis of the previous constraint, we use the identities and the propagation of relevant terms

$$
\begin{aligned}
C^2_{ab} &= \mathrm{curl}\,\sigma_{ab} - H_{ab}, \\
\dot{C}^2_{ab} &= (\mathrm{curl}\,\sigma_{ab})^{\cdot} - \dot{H}_{ab}, \\
(\mathrm{curl}\,\sigma_{ab})^{\cdot} &= \mathrm{curl}\,\dot{\sigma}_{ab} - \frac{1}{3}\Theta\,\mathrm{curl}\,\sigma_{ab} - \sigma_e{}^c\epsilon_{cd(a}\tilde{\nabla}^e\sigma_{b)}{}^d + 3H_{c\langle a}\sigma_{b\rangle}{}^c
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
(\mathrm{curl}\,\dot{\sigma}_{ab}) &= -\frac{2}{3}\Theta\,\mathrm{curl}\,\sigma_{ab} - \frac{2}{3}\epsilon_{cd(a}\sigma_{b)}{}^d\tilde{\nabla}_c\Theta - \mathrm{curl}\,(\sigma_{c\langle a}\sigma_{b\rangle}{}^c) - \mathrm{curl}\,E_{ab}, \\
-(H_{ab})^{\cdot} &= \Theta H_{ab} - 3\sigma_{c\langle a}H^c_{b\rangle} + \mathrm{curl}\,E_{ab}
\end{aligned}
\tag{B.5}
$$

$$
\epsilon_{cd(a}\tilde{\nabla}^e\{\sigma_e{}^c\sigma_{b)}{}^d\} = \epsilon_{cd(a}\sigma_{b)}{}^d\tilde{\nabla}^e\{\sigma_e{}^c\} + \sigma_e{}^c\epsilon_{cd(a}\tilde{\nabla}^e\{\sigma_{b)}{}^d\}. \tag{B.6}
$$

The sum of (B.4), (B.5), (B.5) and (B.6) yields

$$
\dot{C}^2_{ab} = -\Theta[curl\sigma_{ab} - H_{ab}] - \epsilon^{cd}{}_{(a}\sigma_{b)}{}^c[\tilde{\nabla}^e\sigma_{de} - \frac{2}{3}\tilde{\nabla}_d\Theta] - curl(\sigma_{c\langle a}\sigma_{b\rangle}{}^c) + \epsilon_{cd(a}\tilde{\nabla}^e\{\sigma_e{}^c\sigma_{b)}{}^d\} \tag{B.7}
$$

where the two last terms identically cancel leaving,

$$
\dot{C}^2_{ab} = -\Theta C^2_{ab} - \epsilon^{cd}{}_{(a}\sigma_{b)_c}C^1_d. \tag{B.8}
$$

The complete equation therefore takes the form,

$$
\begin{aligned}
C^3_a &= \tilde{\nabla}^b E_{ab} - \frac{1}{3}\tilde{\nabla}_a\rho - \epsilon_{abc}\sigma^b{}_d H^{cd} && \text{(B.9)} \\
\dot{C}^3_a &= (\tilde{\nabla}^b E_{ab})^{\cdot} - \frac{1}{3}(\tilde{\nabla}_a\rho)^{\cdot} - \epsilon_{abc}(\sigma^b{}_d H^{cd})^{\cdot} && \text{(B.10)} \\
(\tilde{\nabla}^b E_{ab})^{\cdot} &= \tilde{\nabla}^b\dot{E}_{ab} - \frac{1}{3}\Theta\tilde{\nabla}^b E_{ab} - \sigma_c{}^b\tilde{\nabla}^c E_{ab} + \epsilon_{abc}H^b{}_d E^{cd}, && \text{(B.11)}
\end{aligned}
$$

where we have used the identity (A.8). On making substitution for the gradient of the propagation of E_{ab} , we get

$$(\tilde{\nabla}^b E_{ab})^{\cdot} = -\frac{4}{3}\Theta\tilde{\nabla}^b E_{ab} - \sigma_c{}^b\tilde{\nabla}^c E_{ab} + \epsilon_{abc}H^b{}_d E^{cd} - E_a{}^b\nabla_b\Theta + 3\tilde{\nabla}^b(\sigma_{c\langle a}E^c_{b\rangle}) + \tilde{\nabla}^b \mathsf{curl}\, H_{ab} - \frac{1}{2}\rho\tilde{\nabla}^b\sigma_{ab} - \frac{1}{2}\sigma_a{}^b\tilde{\nabla}_b\rho. \quad \text{(B.12)}$$

Using the identity (A.5) and substituting for the dot of ρ, one finds

$$-\frac{1}{3}(\tilde{\nabla}_a\rho)^{\cdot} = -\frac{1}{3}[-\frac{4}{3}\Theta\tilde{\nabla}_a\rho - \rho\tilde{\nabla}_a\Theta - \sigma_a{}^b\nabla_b\rho]. \quad \text{(B.13)}$$

The propagation equations for $\sigma^b{}_d$ and H^{cd} give

$$-\epsilon_{abc}(\sigma^b{}_d H^{cd})^{\cdot} = \frac{5}{3}\Theta\epsilon_{abc}(\sigma^b{}_d H^{cd}) + \epsilon_{abc}(\sigma_e{}^{\langle b}\sigma^{d\rangle e}H^c{}_d) + \epsilon_{abc}(E^b{}_d H^{cd}) - 3\epsilon_{abc}(\sigma^b{}_d\sigma_e{}^{\langle c}H^{d\rangle e}) + \epsilon_{abc}(\sigma^b{}_d curl E^{cd}). \quad \text{(B.14)}$$

The above expressions can now be re-expressed as follow,

$$-(E_{ab}\tilde{\nabla}^b\Theta) = -\frac{3}{2}E_a{}^b\tilde{\nabla}^c\sigma_{bc} + \frac{3}{2}E_a{}^b C^1_b \quad \text{(B.15)}$$

$$3\tilde{\nabla}^b(\sigma_{c\langle a}E^c_{b\rangle}) = \frac{3}{2}[\sigma_{ca}\tilde{\nabla}^b E_b{}^c + E_b{}^c\tilde{\nabla}^b\sigma_{ca} + E_a{}^c\tilde{\nabla}^b\sigma_{bc} + \sigma_b{}^c\tilde{\nabla}^b E_{ac}] - \tilde{\nabla}_a(\sigma_{bc}E^{bc}). \quad \text{(B.16)}$$

Using the identity (A.11) we find

$$\nabla^b(\mathsf{curl}\, H_{ab}) = \frac{1}{2}\epsilon_{abc}\tilde{\nabla}^b(\tilde{\nabla}_d H^{cd}) + \frac{1}{3}\Theta\epsilon_{abc}H^b{}_d\sigma^{cd} - \epsilon_{abc}H^b{}_d E^{cd} - \sigma_{ab}\epsilon^{bcd}\sigma_{ce}H^e{}_d. \quad \text{(B.17)}$$

From the first constraint equation, it follows that

$$-\frac{1}{3}\rho\tilde{\nabla}_a\Theta = \frac{1}{2}\rho[\tilde{\nabla}^b\sigma_{ab} - C^1_a]. \quad \text{(B.18)}$$

We need the curl of the fourth constraint. This has the form,

$$\begin{aligned}
\frac{1}{2}\mathrm{curl}\, C^4_a &= \frac{1}{2}\epsilon_{ab}{}^{c}\tilde{\nabla}^b[(\tilde{\nabla}^d H_{cd}) + \epsilon_{cef}\sigma^e{}_g E^{gf}], \\
&= \frac{1}{2}\epsilon_{ab}{}^{c}\tilde{\nabla}^b(\tilde{\nabla}^d H_{cd}) + \tilde{\nabla}^b(\sigma_{c[a}E_{b]}{}^{c}), \\
&= \frac{1}{2}\epsilon_{ab}{}^{c}\tilde{\nabla}^b(\tilde{\nabla}^d H_{cd}) + \frac{1}{2}E_b{}^{c}\tilde{\nabla}^b\sigma_{ca} + \frac{1}{2}\sigma_{ca}\tilde{\nabla}^b E_b{}^{c} - \frac{1}{2}\sigma_{bc}\tilde{\nabla}^b E^c{}_a - \frac{1}{2}E^c_a\tilde{\nabla}^b\sigma_{bc}.
\end{aligned} \tag{B.19}$$

It is straight forward to show that,

$$\begin{aligned}
\dot{C}^3_a &= -\frac{4}{3}\Theta[\tilde{\nabla}^b E_{ab} - \frac{1}{3}\tilde{\nabla}_a\rho - \epsilon_{abc}\sigma^b{}_d H^{cd}] + \frac{3}{2}E_a{}^b[\tilde{\nabla}^c\sigma_{cb} - \frac{2}{3}\tilde{\nabla}_b\Theta] - \frac{1}{2}\rho[\tilde{\nabla}^b\sigma_{ab} - \frac{2}{3}\tilde{\nabla}_b\Theta] \\
&+\frac{1}{2}curl[\tilde{\nabla}^b H_{ab} + \epsilon_{abc}\sigma^b{}_d E^{cd}] + \frac{1}{2}\sigma_a{}^b[\tilde{\nabla}^c E_{bc} - \frac{1}{3}\tilde{\nabla}_b\rho - \epsilon_{bcd}\sigma^c{}_e H^{ed}] - E_b{}^d\epsilon_a{}^{bc}[\mathrm{curl}\,\sigma_{cd} - H_{cd} \\
&+\epsilon_{abc}\sigma^b{}_d curl E^{cd} + \epsilon_{abc}E^b{}_d curl\sigma^{cd} + \frac{1}{2}E_a{}^b\tilde{\nabla}^c\sigma_{bc} + E_b{}^c\tilde{\nabla}^b\sigma_{ca} + \frac{1}{2}\sigma_a{}^c\tilde{\nabla}^b E_{bc} + \sigma_b{}^c\tilde{\nabla}^b E_{ac} \\
&-\tilde{\nabla}_a(\sigma^{bc}E_{bc}).
\end{aligned} \tag{B.20}$$

In terms of C's we find,

$$\begin{aligned}
\dot{C}^3_a &= -\frac{4}{3}\Theta C^3_a + \frac{3}{2}E_a{}^b C^1_b + \frac{1}{2}curl C^4_a + \frac{1}{2}\sigma_a{}^b C^3_b - \frac{1}{2}\rho C^1_a - E_b{}^d\epsilon_a{}^{bc}C^2_{cd} + \epsilon_{abc}\sigma^b{}_d curl E^{cd} \\
&+\epsilon_{abc}E^b{}_d curl\sigma^{cd} + \frac{1}{2}E_a{}^b\tilde{\nabla}^c\sigma_{bc} + E_b{}^c\tilde{\nabla}^b\sigma_{ca} + \frac{1}{2}\sigma_a{}^c\tilde{\nabla}^b E_{bc} + \sigma_b{}^c\tilde{\nabla}^b E_{ac} - \tilde{\nabla}_a(\sigma^{bc}E_{bc}).
\end{aligned} \tag{B.21}$$

where the three last terms are but two forms of the identity (A.9) that vanish.

We next consider the constraint given by the divergence of the magnetic part of the Weyl tensor.

$$C^4_a = \tilde{\nabla}^b H_{ab} + \epsilon_{abc}\sigma^b{}_d E^{cd} \tag{B.22}$$

$$\dot{C}^4_a = (\tilde{\nabla}^b H_{ab})^{\cdot} + \epsilon_{abc}(\sigma^b{}_d H^{cd})^{\cdot} \tag{B.23}$$

$$(\tilde{\nabla}^b H_{ab})^{\cdot} = \tilde{\nabla}^b \dot{H}_{ab} - \frac{1}{3}\Theta\tilde{\nabla}^b H_{ab} - \sigma_c{}^b\tilde{\nabla}^c H_{ab} + \epsilon_{abc}H^b{}_d H^{cd}. \tag{B.24}$$

Substituting for $\dot{H}_{ab}$ one gets

$$(\tilde{\nabla}^b H_{ab})^{\cdot} = -\frac{4}{3}\Theta\tilde{\nabla}^b H_{ab} - \sigma_c{}^b\tilde{\nabla}^c H_{ab} + \epsilon_{abc}H^b{}_d H^{cd} - H_a{}^b\tilde{\nabla}_b\Theta + 3\tilde{\nabla}^b(\sigma_{c\langle a}H^c_{b\rangle}) - \tilde{\nabla}^b curl E_{ab}. \tag{B.25}$$

On substituting for $\dot{\sigma}^b_d$ and $\dot{E}^{cd}$, we find

$$\begin{aligned}\epsilon_{abc}(\sigma^b{}_d E^{cd})^{\cdot} &= -\frac{5}{3}\Theta\epsilon_{abc}(\sigma^b{}_d E^{cd}) - \epsilon_{abc}(\sigma_e{}^{\langle b}\sigma^{d\rangle e}E^c{}_d) - \epsilon_{abc}(E^b{}_d E^{cd}) \\ &\quad +3\epsilon_{abc}(\sigma^b{}_d\sigma_e{}^{\langle c}E^{d\rangle e}) + \epsilon_{abc}(\sigma^b{}_d \mathrm{curl}\, H^{cd}) - \frac{1}{2}\rho\epsilon_{abc}(\sigma^b{}_d\sigma^{cd}).\end{aligned} \tag{B.26}$$

From (B.25) it can be shown that

$$-(H_{ab}\tilde{\nabla}^b\Theta) = -\frac{3}{2}H_a{}^b\tilde{\nabla}^c\sigma_{bc} + \frac{3}{2}H_a{}^b C^1_b \tag{B.27}$$

$$3\tilde{\nabla}^b(\sigma_{c\langle a}H^c_{b\rangle}) = \frac{3}{2}[\sigma_{ca}\tilde{\nabla}^b H_b{}^c + H_b{}^c\tilde{\nabla}^b\sigma_{ca} + H_a{}^c\tilde{\nabla}^b\sigma_{bc} + \sigma_b{}^c\tilde{\nabla}^b H_{ac}] - \tilde{\nabla}_a(\sigma_{bc}H^{bc}). \tag{B.28}$$

Using the identity that involves the gradient of the curl of E_{ab}, one finds

$$-\nabla^b(curl E_{ab}) = -\frac{1}{2}\epsilon_{abc}\tilde{\nabla}^b(\tilde{\nabla}_d E^{cd}) - \frac{1}{3}\Theta\epsilon_{abc}E^b{}_d\sigma^{cd} + \epsilon_{abc}E^b{}_d E^{cd} + \sigma_{ab}\epsilon^{bcd}\sigma_{ce}E^e{}_d. \tag{B.29}$$

We note that,

$$\begin{aligned}
\frac{1}{2}curlC^3_a &= \frac{1}{2}\epsilon_{abc}\tilde{\nabla}^b[(\tilde{\nabla}^d E^c{}_d) + \epsilon^{dec}\sigma^f{}_e H_{df}], \\
&= \frac{1}{2}\epsilon_{ab}{}^c\tilde{\nabla}^b(\tilde{\nabla}^d E_{cd}) + \tilde{\nabla}^b(\sigma_{c[a}H_{b]}{}^c), \\
&= \frac{1}{2}\left(\epsilon_{ab}{}^c\tilde{\nabla}^b(\tilde{\nabla}^d E_{cd}) + H_b{}^c\tilde{\nabla}^b\sigma_{ca} + \sigma_{ca}\tilde{\nabla}^b H_b{}^c - \sigma_{bc}\tilde{\nabla}^b H^c{}_a - H^c_a\tilde{\nabla}^b\sigma_{bc}\right),
\end{aligned} \tag{B.30}$$

where $\varepsilon_{abc}\varepsilon^{dec} = 2! h^d{}_{[a}h^e{}_{b]}$ has been used. It follows that,

$$\begin{aligned}
\dot{C}^4_a &= -\frac{4}{3}\Theta\left(\tilde{\nabla}^b H_{ab} + \epsilon_{abc}\sigma^b E^{cd}\right) + \frac{3}{2}H_a{}^b[\tilde{\nabla}^c\sigma_{cb} - \frac{2}{3}\tilde{\nabla}_b\Theta] - \frac{1}{2}\mathrm{curl}\,[\tilde{\nabla}^b E_{ab} - \frac{1}{3}\tilde{\nabla}_a\rho - \epsilon_{abc}\sigma^b{}_d H^{cd}] \\
&+\frac{1}{2}\sigma_a{}^b[\tilde{\nabla}^c H_{bc} + \epsilon_{bcd}\sigma^c{}_e E^{ed}] - H_b{}^d\epsilon_a{}^{bc}[\mathrm{curl}\,\sigma_{cd} - H_{cd}] + \epsilon_{abc}\sigma^b{}_d\mathrm{curl}\,H^{cd} + \epsilon_{abc}H^b{}_d\mathrm{curl}\,\sigma^{cd} \\
&+\frac{1}{2}H_a{}^b\tilde{\nabla}^c\sigma_{bc} + H_b{}^c\tilde{\nabla}^b\sigma_{ca} + \frac{1}{2}\sigma_a{}^c\tilde{\nabla}^b H_{bc} + \sigma_b{}^c\tilde{\nabla}^b H_{ac} - \tilde{\nabla}_a(\sigma^{bc}H_{bc})
\end{aligned} \tag{B.31}$$

and in terms of **C**'s

$$\begin{aligned}
\dot{C}^4_a &= -\frac{4}{3}\Theta C^4_a + \frac{3}{2}H_a{}^b C^1_b - \frac{1}{2}curlC^3_a + \frac{1}{2}\sigma_a{}^b C^4_b - H_b{}^d\epsilon_a{}^{bc}C^2_{cd} - \epsilon_{abc}\sigma^b{}_d\mathrm{curl}\,H^{cd} + \epsilon_{abc}H^b{}_d\mathrm{curl}\,\sigma^{cd} \\
&+\frac{1}{2}H_a{}^b\tilde{\nabla}^c\sigma_{bc} + H_b{}^c\tilde{\nabla}^b\sigma_{ca} + \frac{1}{2}\sigma_a{}^c\tilde{\nabla}^b H_{bc} + \sigma_b{}^c\tilde{\nabla}^b H_{ac} - \tilde{\nabla}_a(\sigma^{bc}H_{bc}),
\end{aligned} \tag{B.32}$$

which is just

$$\dot{C}^4_a = -\frac{4}{3}\Theta C^4_a + \frac{3}{2}H_a{}^b C^1_b - \frac{1}{2}curlC^3_a + \frac{1}{2}\sigma_a{}^b C^4_b - H_b{}^d\epsilon_a{}^{bc}C^2_{cd}. \tag{B.33}$$

This completes the consistency analysis of the constraint equations for irrotational dust. These findings recovers those given by Maartens Ref. [109].

Bibliography

[1] http://www.its.caltech.edu/ esp/lisa/.

[2] *http://cfa-www.harvard.edu/ mzaldarr/cmbfast/cmbfast.html.*

[3] *http://universe.nasa.gov/program.html.*

[4] *http://www.rssd.esa.int.*

[5] N. Afshordi and R. H. Brandenberger, *Super-hubble nonlinear perturbations during inflation*, Phys. Rev. **D63** (2001), 123505.

[6] B. K. Berger and D. M. Eardley and D. W. Olson, *Note on the spacetimes of szekeres*, Phys. Rev. **D 16** (1977), 3086.

[7] B. Osano, C. Pitrou, P. K. S. Dunsby, J-P. Uzan and C. Clarkson, *Gravitational waves generated by second order effects during inflation*, JCAP **0704** (2007), 003.

[8] J. M. Bardeen, *Gauge invariant cosmological perturbations*, Phys. Rev. **D22** (1980), 1882–1905.

[9] A. Barnes and R. Rowlingson, *Irrotational perfect fluids with a purely electric weyl tensor*, Class Quantum Grav. **6** (1989), 949.

[10] V.A. Belinskii. and D. Fargion, *Two-soliton waves in anisotropic cosmology*, Nuovo Cimento **B 59** (1980), 143.

[11] V.A. Belinskii and V. E. Zakharov, Sov. Phys.-JETP **48** (1978), 985.

[12] F. Bernardeau and J-P. Uzan, *Non-gaussianity in multi-field inflation*, Phys. Rev. **D66** (2002), 103506.

[13] ———, *Inflationary models inducing non-gaussian metric fluctuations*, Phys. Rev. **D67** (2003), 121301.

[14] ———, *Finite volume effects for non-gaussian multi-field inflationary models*, Phys. Rev. **D70** (2004), 043533.

[15] G. Betschart and C.A. Clarkson, *Scalar and electromagnetic perturbations on lrs class ii spacetimes*, Class. Quant. Grav. **21** (2004), 5587–5608.

[16] H. Bondi, *Spherically symmetrical models in general relativity*, Mon. Not. R. Astron. Soc. **410** (1947), 107.

[17] M. Bruni and S. Sonego, *Observables and gauge invariance in the theory of non-linear spacetime perturbations*, Class. Quant. Grav. **16** (1999), L29–L36.

[18] C. Carbone, C. Baccigalupi and S. Matarrese, *The stochastic gravitational wave background from cold dark matter halos*, Phys. Rev. **D73** (2006), 063503.

[19] C. F. Sopuerta, M. Bruni and L. Gualtieri, *Non-linear n-parameter spacetime perturbations: Gauge transformations through the baker campbell-hausdorff formula*, Phys. Rev. **D70** (2004), 064002.

[20] C.F. Sopuerta, R. Maartens, G.F.R. Ellis and W. Lesame, *Nonperturbative gravitomagnetic fields*, Phys. Rev. D (1999).

[21] A. Challinor, *Microwave background anisotropies from gravitational waves: The 1+3 covariant approach*, Class. Quant. Grav. **17** (2000), 871–889.

[22] H. T. Cho and A. D. Speliotopoulos, *Gravitational waves in bianchi type-i universes: The classical theory*, Phys. Rev. D **52** (1995), 5445.

[23] L. Civita, Math. Ann. **97** (1926), 291.

[24] C. A. Clarkson, *Density fluctuations and gravity waves: A covariant approach to gauge-invariant non-linear cosmological perturbation theory*, Phys. Rev. **D70** (2004), 103524.

[25] C.A. Clarkson and R. K. Barrett, *Covariant perturbations of schwarzschild black holes*, Class. Quant. Grav. **20** (2003), 3855–3884.

[26] C.B. Collins and S. W. Hawking, *Why is the universe isotropic?*, Astrophys. J. **180** (1973), 317.

[27] P. Creminelli and M. Zaldarriaga, *Cmb 3-point functions generated by nonlinearities at recombination*, Phys. Rev. **D70** (2004), 083532.

[28] D. Eardley, E. Liang and R. Sachs, *Velocity-dominated singularities in irrotational dust cosmologies.*, J. Math. Phys. **13** (1972), 99–107.

[29] N. Deruelle and V.F. Mukhanov, *Matching conditions for cosmological perturbations*, Phys. Rev **D52** (1995), 5549.

[30] E. Donoghue and J. Donoghue, *Isotropy of the early universe from cmb anisotropies*, Phys. Rev. **D71** (2005), 04002.

[31] P. K. S. Dunsby, *Gauge invariant perturbations in multicomponent fluid cosmologies*, Class. Quant. Grav. **8** (1991), 1785–1806.

[32] P. K. S. Dunsby, *Covariant perturbations of anisotropic cosmological models*, Phys. Rev.D **48** (1993), 3562.

[33] P.K.S. Dunsby, *Gauge invariant perturbations in multi-component fluid cosmologies*, Class. Quant. Grav. **8** (1991), 1785.

[34] R. Durrer, *Cosmological perturbation theory*, Astro-ph/0402129v2 (2004).

[35] R. Durrer and N. Straumann, *Some applications of the 3+1 formalism of general relativity*, Helv. Phys. Acta **353** (1988), 1027.

[36] G. Efstathiou, *A maximun likelihood analysis of the low cmb multipoles from wmap*, Mon. Not. Astron. Soc. **348** (2004), 885.

[37] A. Einstein and N. Rosen, *On gravitational waves*, J. Franklin Inst. **223** (1937), 43.

[38] J. Elhers, *Beitrage zur relativistischen mechanik kontinuierlicher medien, abhandl. math,-nat. kl.*, Akad. Wiss. Lit. Mainz **11** (1961), 793.

[39] G.F.R. Ellis, *Dynamics of pressure free matter in general relativity*, J. Math. Phys. **8** (1967), 1171.

[40] ______, *General relativity and cosmology*, Academic, New York, 1971.

[41] G.F.R. Ellis and M. Bruni, *Covariant and gauge-invariant approach to cosmological density fluctuations*, Phys. Rev. D **40** (1989), 1804.

[42] G.F.R. Ellis and M.A.H. MacCallum, *A class of homogeneous cosmological models*, Commun. Math. Phys. **12** (1969), 108.

[43] G.F.R. Ellis and H. van Elst, *Theoretical and observational cosmology*, edited by Lachieze-Rey, NATO Science series, (Kluwer Academic Publishers, Dorretcht), 1999.

[44] A-H. Nima et al, *Ghost condensation and a consistent infrared modification of gravity*, JHEP **074** (2004), 0405.

[45] A. Oliveria-Costa et al, *Significance of the largest scale cmb fluctuations in wmap*, 2004, p. 063516.

[46] C. L. Bennett et al, *First-year wilkinson microwave anisotropy (wmap) observations: Preliminary maps and basic results*, Astrophys. J. Suppl. **148** (2003), 1.

[47] C.L. Bennett et al, *Cosmic temperature fluctuations from two years of cobe dmr observations*, Astropys J. **436** (1994), 423–442.

[48] D.N. Spergel et al, *Three-year wilkinson microwave anisotropy probe (wmap) observations: Implications for cosmology*, Astrophys. J **170** (2007), 377–408.

[49] E. Komatsu et al, *First year wilkinson microwave anisotropy probe (wmap) observations: Tests of gaussianity*, Astrophys. J. Suppl. **148** (2003), 119–134.

[50] F. K. Hansen et al, *Asymmetries in the local curvature of the wmap data.*, Astrophys. J. **607** (2004), L67.

[51] F.K. Hansen et al, *Testing the cosmological principle of isotropy: local power spectrum estimates of the wmap data*, Mon. Not. Astron. Soc. (2004).

[52] H.K. Eriksen et al, *Asymmetries in the cosmic microwave background anisotropy field*, Astrophys. J. **2004** (14), 605.

[53] ______, Testing for Non-Gaussianity in the Wilkinson Microwave Anisotropy Probe Data: Minkowski Functionals and the Length of the Skeleton,astro-ph/0407271 **612** (2004), 64–80.

[54] J-F. Dufaux et al, *Theory and numerics of gravitational waves from preheating after inflation*, arXiv:0707.0875v1 (2007).

[55] L. Page et al, *Three-year wilkinson microwave anisotropy probe (wmap) observations: Polarization analysis*, Ap J. S. **148** (2003), 39.

[56] M. Liguori et al, *Testing primordial non-gaussianity in cmb anisotropies*, Phys. Rev. **D73** (2006), 043505.

[57] N. Bartolo et al, *Perturbations in cosmologies with a scalar field and a perfect fluid*, Phys. Rev **D70** (2004), 043532.

[58] P. Creminelli et al, *Limits on f_{NL} parameters from wmap 3yr data*, JCAP **0703** (2007), 005.

[59] P. Vielva et al, *Detection of non-gaussianity in the wilkinson microwave anisotropy probe first-year data using spherical wavelets*, Astrophys. J. **609** (2004), 22–34.

[60] T. E. Perko et al, *Galaxy formation in anisotropic cosmologies*, Phys. Rev. **D6** (1972), 969–83.

[61] T.E. Perko et al, *Galaxy formation in anisotropic cosmologies*, Phys. Rev.D **6** (1972), 969.

[62] T.R. Jaffe et al, *Evidence of vorticity and shear at large angular scales in the wmap data: violation of cosmological isotropy ?*

[63] ———, *Fast and efficient template fitting of deterministic anisotropic cosmological models applied to wmap data*, Astrophys. J. **643** (2006), 616–629.

[64] van den Bergh et al, *Purely radiative perfect fluids*, gr-qc/0611079v3.

[65] F. Bernardeau and T. Brunier and J-P. Uzan, *High order correlation functions for self interacting scalar field in de sitter space*, Phys. Rev. **D69** (2004), 063520.

[66] F. Bernardeau, T. Brunier and J-P. Uzan, *Models of inflation with primordial non-gaussianities*, AIP Conf. Proc. **861** (2006), 821–828.

[67] F. Finelli, G. Marozzi, G.P. Vacca and G. G. Venturi, *Second order gauge-invariant perturbations during inflation*, Phys. Rev. **D74** (2006), 083522.

[68] F.C. Mena, R. Tavakol and M. Bruni, *Second order perturbations of flat dust flrw universes with a cosmological constant*, Int. J. Mod. Phys. **A17** (2002), 4239–4244.

[69] G. Dvali, A. Gruzinov and M. Zaldarriaga, *Cosmological perturbations from inhomogeneous reheating, freeze-out, and mass domination*, Phys. Rev. **D69** (2004), 083505.

[70] ______, *New mechanism for generating density perturbations from inflation*, Phys. Rev. **D69** (2004), 023505.

[71] G. I. Rigopoulos, E. P. S. Shellard and B. J.W. van Tent, *Large non-gaussianity in multiple-field inflation*, Phys. Rev. **D73** (2006), 083522.

[72] J. Hwang G.F.R. Ellis and M. Bruni, *Covariant and gauge-independent perfect-fluid robertson-walker perturbations*, Phys. Rev. **40** (1989), 1819.

[73] M. Bruni G.F.R Ellis and J. Hwang, *Density-gradient-vorticity relation in perfect-fluid robertson-walker perturbations*, Phys. Rev. **D42** (1990), 1035–46.

[74] R. H. Gowdy, *Gravitational waves in closed universes*, Phys. Rev. Lett. **27** (1971), 827.

[75] A. Guth and S. Y. Pi, *Fluctuations in the new inflationary universe*, Phys. Rev. Lett. **49** (1982), 1110.

[76] W.M. Lesame H. van Elst, C. Uggla and G. F. R. Ellis, *Integrability of irrotational silent cosmological models*, Class. Quantum Grav. **5** (1997), 1151–1162.

[77] E. R. HARRISON, *Normal modes of vibrations of the universe*, Rev. Mod. Phys. **39** (1967), no. 4, 862–882.

[78] S. W. Hawking, *Perturbations of an expanding universe*, Astrophys. J. **145** (1966), 544.

[79] S. W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge, 1973.

[80] O. Heckmann and E. Schucking, *Gravitation*, Wiley, New York, 1962.

[81] P. Hogan and G.F.R. Ellis, *Propagation of information by electromagnetic and gravitational waves in cosmology*, Class. Quant. Grav. **14** (1997), A171.

[82] P. A. Hogan and E. M. O'shea, *Shear-free gravitational waves in an anisotropic universe*, Phys. Rev. **D66** (2002), 124016.

[83] B. L. Hu, *Gravitational waves in a bianchi type-i universe*, Phys. Rev. **D18** (1978), 969.

[84] J. Hwang and E.T. Vishniac, *Analyzing cosmological perurbations using the covariant approach*, Astrophys. J. **353** (1990), 1–20.

[85] J. Ibanez and E. Verdaguer, *Soliton collision in general relativity*, Phys. Rev. Lett. **51** (1983), 1313.

[86] IFN, *Virgo home page, http : //www.virgo.infn.it.*

[87] J. M. Bardeen and P. J. Steinhardt and M. S. Turner, *Spontaneous creation of almost scale - free density perturbations in an inflationary universe*, Phys. Rev. **D28** (1983), 679.

[88] R.T. Jantzen, *The dynamical degrees of freedom in spatially homogeneous cosmology*, Commun. Math. Phys. **64** (1979), 211.

[89] K. Ananda, and C. Clarkson and D. Wands, *The cosmological gravitational wave background from primordial density perturbations*, Phys. Rev. **D75** (2007), 123518.

[90] M. Kamionkowski and A. Leob, *Getting around cosmic variance*, Phys. Rev. **D56** (1997), 4511.

[91] L. Knox and Y-S. Song, *A limit on the detectability of the energy scale of inflation*, Phys. Rev. Lett. **89** (2002), 011303.

[92] H. Kodama and M. Sasaki, *Cosmological perturbation theory*, Prog.Theor. Phys. Supp. **78** (1984), 213–313.

[93] L. Kofman, *Probing string theory with modulated cosmological fluctuations*, astro-ph/0303614 (2004).

[94] N. Kogo and E. Komatsu, *Angular trispectrum of cmb temperature anisotropy from primordial non-gaussianity with the full radiation transfer function*, Phys. Rev. **D73** (2006), 083007.

[95] E. W. Kolb and M. S. Turner, *The early universe*, Addison-Weley Publishing Company, 1990.

[96] K. Land and J. Magueijo, *Axis of evil*, Phys.Rev.Lett. **95** (2005), 071301.

[97] D. Langlois and F. Vernizzi, *Conserved non-linear quantities in cosmology*, Phys. Rev. **D72** (2005), 103501.

[98] ———, *Evolution of non-linear cosmological perturbations*, Phys. Rev. Lett. **95** (2005), 091303.

[99] ———, *Nonlinear perturbations of cosmological scalar fields*, JCAP **0702** (2007), 017.

[100] A. R. Liddle and D. H. Lyth, *Cosmological inflation and large-scale structure*, Cambridge University Press, 2000.

[101] E. M. Lifshitz and I. M. Khalatnikov, Adv. Phys. **12** (1963), 185.

[102] E.M. Lifshitz, J. Phys. U.S.S.R **10** (1946), 116.

[103] D.H. Lyth and D. Seery, *Classicality of the primordial perturbations*, astro-ph/0607647.

[104] S. Matarrese M. Bruni and O. Pantano, *Dynamics of silent universes*, Astrophys. J. **445** (1995), 958–77.

[105] M. Bruni, and P.K.S. Dunsby and G.F.R. Ellis, *Cosmological perturbations and the physical meaning of gauge invariant variables*, Astrophys. J. **395** (1992), 34–53.

[106] M. Bruni, G. F. R. Ellis and P. K. S. Dunsby, *Gauge invariant perturbations in a scalar field dominated universe*, Class. Quant. Grav. **9** (1992), 921–946.

[107] M. Kamionkowski, A. Kosowsky and A. Stebbins, *A probe of primordial gravity waves and vorticity*, Phys. Rev. Lett. **78** (1997).

[108] M. Kesden, A. Cooray and M. Kamionkowski, *Separation of gravitational-wave and cosmic-shear contributions to cosmic microwave background polarization*, Phys. Rev. Lett. **89** (2002).

[109] R. Maartens, *Linearization instability of gravity waves?*, Phys. Rev. **D55** (1997), 463–467.

[110] R. Maartens and B. A. Bassett, *Gravitational electromagnetic duality*, Class. Quant. Grav. **15** (1998), 705.

[111] M.S. Madsen, *Scalar fields in curved space-times*, Class. Quant. Grav. **5** (1988), 627–639.

[112] J. M. Maldacena, *Non-gaussian features of primordial fluctuations in single field inflationary models*, JHEP **05** (2003), 013.

[113] K. A. Malik and D. Wands, *Evolution of second order cosmological perturbations*, Class. Quant. Grav. **21** (2004), L65–L72.

[114] J. Martin and D.J. Schwarz, *Influence of cosmological transitions on the evolution of density perturbations*, Phys. Rev **D57** (1998), 3302.

[115] P. Martineau and R. Brandenberger, *A back-reaction induced lower bound on the tensor-to-scalar ratio*, arXiv:0709.2671v3 (2007).

[116] S. Mollerach and S. Matarrese, *Cosmic microwave background anisotropies from second order gravitational perturbations*, Phys. Rev. **D56** (1997), 4494.

[117] V. F. Mukhanov and G. V. Chibisov, *Quantum fluctuation and nonsingular universe. (in russian)*, JETP Lett. **33** (1981), 532–535.

[118] V.F. Mukhanov, H.A.Feldman, and R.H. Brandenberger, *Theory of cosmological perturbations*, Physics Reports **215 (5)** (1992), 203–333.

[119] N. Bartolo, E. Komatsu, S. Matarrese and A. Riotto, *Non-gaussianity from inflation: Theory and observations*, Phys. Rept. **402** (2004), 103–266.

[120] N. Bartolo, S. Matarrese and A. Riotto, *Cmb anisotropies at second order i*, JCAP **0606** (2006), 024.

[121] ———, *Cmb anisotropies at second-order ii: Analytical approach*, JCAP **0701** (2007), 019.

[122] K. Nakamura, *Gauge-invariant formulation of the second-order cosmological perturbations*, Phys. Rev. **D74** (2006), 101301.

[123] NASA, *European lisa home page*, http://lisa.nasa.gov/SCIENCE/science-goals.html.

[124] H. Noh and J-C. Hwang, *Perturbations of an anisotropic spacetime: Ideal fluid*, Phys. Rev. **D52** (1995), 5643–52.

[125] ______, *Second-order perturbations of the friedmann world model*, Phys. Rev. **D69** (2004), 104011.

[126] National Astronomy Observatory-Tokyo, *Tama home page*, http://tamago.mtk.na.ac.jp.

[127] University of Hanover, *Geo home page*, http://www.geo600.uni-hannover.de.

[128] California Institute of Technology, *Ligo home page*, *http : //www.ligo.caltech.edu.*

[129] P. K. S. Dunsby, B. A. C. C. Bassett and G. F. R Ellis, *Covariant analysis of gravitational waves in a cosmological context*, Class. Quant. Grav. **14** (1997), 1215–1222.

[130] C. Park, *Non-gaussian signatures in the temperature fluctuation observed by thewilkinson microwave anisotropy probe*, Mon. Not. Roy. Astron. Soc.

[131] J. A. Peacock, *Cosmological physics*, Cambridge University Press, 1999.

[132] F. A. E. Pirani, Acta Phys. Polon. **15** (1956), 389.

[133] ______, *Invariant formulation of gravitational radiation theory*, Phys. Rev. **105** (1957), 1089.

[134] C. Pitrou and J-P. Uzan, *Quantization of perturbations during inflation in the 1+3 covariant formalism*, Phys. Rev. **D75** (2007), 087302.

[135] T. Pyne and S. M. Carroll, *Higher-order gravitational perturbations of the cosmic microwave background*, Phys. Rev. **D53** (996), 2920.

[136] G. F. R. Ellis R. Maartens and T. C. Siklos, Class. Quant. Grav. **14** (1997), 1927–1936.

[137] R. Maartens and G. F. R.Ellis and T.C.S. Siklos, *Local freedom in the gravitational field*, Class. Quant. Grav. **14** (1997), 1927–1936.

[138] A. Raychaudhuri, *Relativistic cosmology. i*, Phys. Rev. **98** (1955), 1123.

[139] A. Rouzbeh, *Scenarios of modulated perturbations*, Phys. Rev. **D70** (2004), 043507.

[140] O. Pantano S. Matarrese and D.Saez, *General relativistic dynamics of irrotational dust: Cosmological implications*, Phys. Rev. Lett. **72** (1994), 320–3.

[141] S. Matarrese, S. Mollerach and M. Bruni, *Relativistic second-order perturbations of the einsteinde sitter universe*, Phys. Rev. D **58** (1998), 043504.

[142] ———, *Second-order perturbations of the einstein-de sitter universe*, Phys. Rev. **D58** (1998), 043504.

[143] S. Mollerach, D. Harari and S. Matarrese, *Cmb polarization from secondary vector and tensor modes*, Phys. Rev. **D69** (2004), 063002.

[144] S. Prunet and J-P. Uzan, F. Bernardeau and T. Brunier, *Constraints on mode couplings and modulation of the cmb with wmap data*, Phys. Rev. **D71** (2005), 083508.

[145] D. S. Salopek and J. R. Bond, *Nonlinear evolution of long wavelength metric fluctuations in inflationary models*, Phys. Rev. **D42** (1990), 3936–3962.

[146] U. Seljak and M. Zaldarriaga, *A line of sight approach to cosmic microwave background anisotropies*, Astrophys. J. **469** (1996), 437–444.

[147] E. Silverstein and D. Tong, *Scalar speed limits and cosmology: Acceleration from d-cceleration*, Phys. Rev. **D70** (2004), 103505.

[148] S.Rowan and J. Hough, *Gravitational waves detection by interferometry (ground and space)*, http://www.livingreviews.org/Articles/Volumes3/2000-3hough/ **cited on 19 September 2007**.

[149] A. A. Starobinsky, *Spectrum of relict gravitational radiation and the early state of the universe*, JETP Lett. **30** (1979), 682–685.

[150] ———, *Dynamics of phase transition in the new inflationary universe scenario and generation of perturbations*, Phys. Lett. **B117** (1982), 175–178.

[151] J. M. Stewart and G. F. R. Ellis, *On solutions of einsteins equations for a fluid which exhibit local rotational symmetry*, J. Math. Phys. **9** (1968), 1072.

[152] J. M. Stewart and M. Walker, *Perturbations of spacetimes in general relativity*, Proc. Roy. Soc. Lond. **A341** (1974), 49–74.

[153] P. Szekeres, J. Maths. Phys. **6** (1965), 1387.

[154] ———, J. Maths. Phys. **7** (1966), 751.

[155] ———, J. Math. Phys. **13** (1972), 286.

[156] T. Lu and A. Kishore and C. Clarkson, *Vector perturbation*, arXiv:0709.1619 (2007).

[157] A. H. Taub, Ann. Math. **53** (1951), 472.

[158] K. Tomita, Prog. Theor. Phys. **37** (1967), 831.

[159] K. Tomita and D. Mitsue, *Gauge-invariant perturbations in anisotropic homogeneous cosmological models*, Phys. Rev. D **34** (1986), 3570–83.

[160] C. G. Tsagas, A. Challinor, and R. Maartens, *Relativistic cosmology and large-scale structure*, arXiv:0705.4397 (1997).

[161] V. Acquaviva, N. Bartolo, S. Matarrese and A. Riotto, *Second-order cosmological perturbations from inflation*, Nucl. Phys. **B667** (2003), 119–148.

[162] H. van Elst, http://www.maths.qmul.ac.uk/ hve/13coveq.pdf **private notes**.

[163] H. van Elst, private notes **http://www.maths.qmul.ac.uk/ hve/13coveq.pdf**.

[164] H. van Elst and G. F. R. Ellis, *Causal propagation of geometrical fields in relativistic cosmology*, Phys. Rev. D. **59** (1999), 024013.

[165] H. van Elst and C. Uggla, *General relativistic orthonormal frame approach*, Class. Quantum Grav. **14** (1997), 2673.

[166] J. Wainwright and G.F.R. Ellis, *Dynamical systems in cosmology*, Cambridge University Press, 1997.

[167] R.M. Wald, *Cosmological perturbation theory*, Phys. Rev. D. **28** (1983), 2118.

[168] ______, *General relativity*, University of Chicago Press, 1984.

[169] S. Weinberg, *Quantum contributions to cosmological correlations*, Phys. Rev. **D 72** (2005), 043514.