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Generalising the Concordant-Dissonant Factorisation

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Contents

1	Introduction and notation	5
1.1	Introduction	5
1.2	Notation	8
2	Factorisation Systems	11
2.1	Factorisation systems via prefactorisation systems	11
2.2	Factorisation systems via weak factorisation systems	13
2.3	Factorising sources	16
2.4	Images and subobjects	17
3	Introducing the \parallel-relation	19
3.1	Constant morphisms and the \parallel^{ob} -relation	19
3.2	The \parallel -relation	20
3.3	\parallel , \parallel^H and \perp	26
3.4	\parallel -pairs and factorisation systems	29
3.5	More on the \parallel^H -relation	31
4	\parallel-pairs on nice categories	33
4.1	Preliminary definitions	33
4.2	Characterising the \parallel -pairs on nice categories	38
4.3	Obtaining \parallel -pairs from full subcategories	43
5	Examples of \parallel-pairs	47
5.1	\parallel -pairs on topological categories	47
5.2	Examples on Top	49
5.3	Examples on SymRe	51
5.4	Examples on Prost	54
5.5	\parallel -pairs on abelian categories	56
5.6	Examples on Ab	56
5.7	Examples on Set	58
6	Generalising the concordant-dissonant factorisation	59
6.1	The original results of Collins	59
6.2	A generalisation by Preuss	60
6.3	Dispersed factorisation structures	61
6.4	γ -concordant morphisms	63
6.5	T-concordant morphisms	64

6.6	\parallel -pairs $(\mathcal{F}, \mathcal{G})$ such that $\mathcal{F} = L(\overline{\mathcal{G}})$	65
7	Generalising the submonotone-superlight factorisation	66
7.1	Submonotone-superlight in terms of component subcategories	66
7.2	The generalised component subcategories of Tiller	68
7.3	The approach of Clementino and Tholen	69
8	Bibliography	74

University of Cape Town

1 Introduction and notation

1.1 Introduction

The “concordant-dissonant factorization of an arbitrary continuous function” was introduced by Collins in [C 1971]. He defined that a continuous map is concordant iff each of its fibres is contained in a quasi-component of its domain and that a continuous map is dissonant iff each of its fibres and each quasi-component of its domain have at most one element in common. He established that each continuous map f may be expressed as a composite $f = me$ where m is dissonant and e is a concordant quotient map. Indeed he went further and established enough that it may reasonably be said that he was the first to prove that the class of all concordant quotient maps and the class of all dissonant maps form what today is called a factorisation system on the category **Top**.

In [S 1974] Strecker defined the submonotone maps and the superlight maps. The definition of a submonotone map is obtained from that of a concordant map by replacing “quasi-component” with “component” and the definition of a superlight map is obtained from that of a dissonant map by the same procedure. He proved that the class of all submonotone quotient maps and the class of all superlight maps form a factorisation system on **Top**.

There are other ways to define classes of continuous maps in terms of notions related to connectedness and thereby obtain interesting factorisations (see, for example, [CD 1977]). However we focus our attention on concordance, dissonance, submonotonicity and superlightness and on ways of generalising these notions.

To this end and as an item of independent interest, we define a relation on the class $Mor(\mathcal{C})$ of an arbitrary category \mathcal{C} . It may be thought of as a version for morphisms of a relation on $Ob(\mathcal{C})$ which Herrlich defined in [H 1968]. The symbol “ \parallel ” is sometimes used to denote Herrlich’s relation. We shall use it for our relation instead, denoting the relation on $Ob(\mathcal{C})$ by \parallel^{ob} . As a justification of this notation we offer Proposition 3.2.3 which establishes that, in the presence of a terminal object, one may regard the \parallel^{ob} -relation as being a special case of the \parallel -relation.

Let $f : X \rightarrow A$ and $g : Y \rightarrow B$ be morphisms in \mathcal{C} . We say $f \parallel g$ iff, for

any diagram in \mathcal{C} as follows

$$\begin{array}{ccccc} Z & \xrightleftharpoons[u]{v} & X & \xrightarrow{w} & Y \\ & & \downarrow f & & \downarrow g \\ & & A & & B. \end{array}$$

$wu = vw$ provided both $fu = fv$ and $gwu = gvw$. ($X \parallel^{ob} Y$ iff, for any diagram as above, $wu = vw$.)

Imitating Herrlich's approach, we define operators L and R such that, for any class \mathcal{H} of morphisms in \mathcal{C} , $L(\mathcal{H}) = \{f \text{ in } \mathcal{C} \mid f \parallel h, \forall h \in \mathcal{H}\}$ and $R(\mathcal{H}) = \{g \text{ in } \mathcal{C} \mid h \parallel g, \forall h \in \mathcal{H}\}$. We then say that a pair $(\mathcal{F}, \mathcal{G})$ is a \parallel -pair iff $\mathcal{F} = L(\mathcal{G})$ and $\mathcal{G} = R(\mathcal{F})$. \parallel^{ob} -pairs were defined analogously (using different notation) in [H 1968].

For a class of morphisms \mathcal{H} in \mathcal{C} , we use the notation $\overline{\mathcal{H}}$ to denote the class of all objects $X \in \mathcal{C}$ such that every morphism with domain X is in \mathcal{H} . We then let $L(\overline{\mathcal{H}}) = \{f \text{ in } \mathcal{C} \mid f \parallel h \text{ whenever } \text{dom}(h) \in \overline{\mathcal{H}}\}$ and $R(\overline{\mathcal{H}}) = \{g \text{ in } \mathcal{C} \mid h \parallel g \text{ whenever } \text{dom}(h) \in \overline{\mathcal{H}}\}$. We are particularly interested in \parallel -pairs $(\mathcal{F}, \mathcal{G})$ which satisfy one or both of the following properties: $\mathcal{F} = L(\overline{\mathcal{G}})$, $\mathcal{G} = R(\overline{\mathcal{F}})$. It turns out that the pair (Concordant maps, Dissonant maps) is equal to a \parallel -pair $(\mathcal{F}, \mathcal{G})$ on **Top** such that $\mathcal{F} = L(\overline{\mathcal{G}})$ and that (Submonotone maps, Superlight maps) is a \parallel -pair $(\mathcal{F}, \mathcal{G})$ on **Top** such that $\mathcal{F} = L(\overline{\mathcal{G}})$ and $\mathcal{G} = R(\overline{\mathcal{F}})$ (Examples 5.2(3) and 5.2(4)).

In Chapters 6 and 7 we consider the generalisations of concordance and dissonance and of submonotonicity and superlightness which the above observations provide. We do so by recalling the work of a number of authors who have already tackled the problem of generalising these notions. In the case of concordance and dissonance, we consider generalisations given by Preuss in [P 1979], by Herrlich, Salicrup and Vásquez in [HSV 1979], by Börgen and Tholen in [BT 1984] and by Janelidze and Tholen in [JT 1999]. In the case of submonotonicity and superlightness, we consider generalisations given by Strecker in [S 1974] and by Clementino and Tholen in [CT 1998]. We also draw on the work of Tiller in [T 1980] on generalised component subcategories.

The notion of a completely arbitrary \parallel -pair may be regarded as an extreme generalisation of concordance and dissonance. In Chapters 4 and 5 we establish that \parallel -pairs on **Top** are in one to one correspondence with families $\{\sim_X \mid X \in \mathbf{Top}\}$ of reflexive symmetric relations which are preserved by continuous maps. For example, this correspondence maps the pair (Concordant

maps, Dissonant maps) to the family $\{\sim_X \mid X \in \mathbf{Top}\}$ where, for each space X and all $x, x' \in X$, $x \sim_X x'$ iff x and x' belong to the same quasi-component of X .

It turns out that we are able to give this result in much greater generality. Doing so involves generalising the notion of a “family of reflexive symmetric relations preserved by continuous maps”. We give the relevant definitions in Section 4.1 and then establish the correspondence in Theorems 4.2.1 - 4.2.3. As a consequence we are able to characterise (in sufficiently nice categories) those classes \mathcal{F} for which there exists \mathcal{G} (and those classes \mathcal{G} for which there exists \mathcal{F}) such that $(\mathcal{F}, \mathcal{G})$ is a $\|\$ -pair (Corollaries 4.2.4 and 4.2.5).

In [AW 1975], Arhangel'skiĭ and Wiegandt characterise the full subcategories \mathcal{K} of \mathbf{Top} for which there exists \mathcal{H} (and the full subcategories \mathcal{H} for which there exists \mathcal{K}) such that $(\mathcal{K}, \mathcal{H})$ is a $\|\^{ob}$ -pair on \mathbf{Top} . Corollaries 4.2.4 and 4.2.5 are comparable to these results. Attention has also been paid to $\|\^{ob}$ -pairs on a number of other categories. For example, Dickson studied them on subcomplete abelian categories in [D 1966] and Fried and Wiegandt studied them on the category of graphs in [FW 1975].

In Chapter 5 we use the theory developed in Chapter 4 to obtain and discuss specific examples of $\|\$ -pairs on the categories \mathbf{Top} , \mathbf{SymRe} , \mathbf{Prost} and \mathbf{Ab} . We also establish the non-existence of non-trivial $\|\$ -pairs on \mathbf{Set} . We note that at least some of the theory developed in Chapter 4 is sufficiently general to be of use in both topological categories and abelian categories.

We note that the $\|\$ -relation is not the first example of its kind. In [H 2004] Holgate defines a similar notion which is also a version for morphisms of the $\|\^{ob}$ -relation. We use the symbol $\|\^H$ for his relation. In Chapter 3, once we have introduced the $\|\$ -relation and discussed some of its basic properties, we recall Holgate's definition and discuss the relationship between $\|\$ and $\|\^H$. Indeed, there is a pleasing connection between the two notions which we prove in Theorems 3.3.4 and 3.3.5. In [H 2004] Holgate established sufficient conditions for obtaining a prefactorisation system from a $\|\^H$ -pair (defined analogously to a $\|\$ -pair). Making use of this result and of a result from [FK 1972], we establish sufficient conditions for obtaining a factorisation system from a $\|\$ -pair (Theorem 3.4.10).

The idea of having a version for morphisms of the $\|\^{ob}$ -relation is also present in the work of Clementino and Tholen in [CT 1998] on generalising submonotonicity and superlightness. They use the $\|\^{ob}$ -relation in the slice categories of a category and so apply it to morphisms without having to define a new relation. We discuss this approach in Chapter 7 where we also

prove results which link it to our own.

As the title of [H 2004] suggests, obtaining a version for morphisms of the $\|^{ob}$ -relation may be used to define connectedness for morphisms. However, there are other methods. Recent work of Tran in [T 2004] provides a different approach.

In the following section we make some comments about the notation we shall be using. Chapter 2 is then devoted to recalling basic material on factorisation systems and related notions which will be of use throughout this dissertation.

1.2 Notation

We assume familiarity with the terminology of [AHS 1990]. In addition we make some comments here regarding the conventions and notation which we shall be using.

\mathcal{C} will always denote a category. “ $X \in \mathcal{C}$ ” will indicate that X is an **object of \mathcal{C}** . “ f in \mathcal{C} ” will indicate that f is a **morphism of \mathcal{C}** .

We shall refer to the following classes:

$Ob(\mathcal{C})$: the class of all \mathcal{C} -objects

$Mor(\mathcal{C})$: the class of all \mathcal{C} -morphisms

$Iso(\mathcal{C})$: the class of all isomorphisms in \mathcal{C}

$Epi(\mathcal{C})$: the class of all epimorphisms in \mathcal{C}

$RegEpi(\mathcal{C})$: the class of all regular epimorphisms in \mathcal{C}

$Mono(\mathcal{C})$: the class of all monomorphisms in \mathcal{C} .

Morphism composition will be written as fg and occasionally as $f \cdot g$ when it is thought that the \cdot might be helpful. The **identity morphism** on an object X will be denoted by 1_X (and the **identity functor** on \mathcal{C} by $1_{\mathcal{C}}$). Any **terminal object** will be denoted by 1 and the unique morphism with domain X and codomain 1 will be denoted by $!_X$. **Diagrams** will not necessarily commute. When we want it to be understood that one does commute, we shall say so or it will follow from what we have said. The **colimit** of a functor F will be denoted by $colim(F)$.

$X \times_Y X$, f_1 and f_2 are defined up to isomorphism by the requirement

that the following square be a pullback.

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{f_1} & X \\ f_2 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

(f_1, f_2) will be referred to as the **kernelpair** of f . δ_f is the unique morphism such that $f_1 \delta_f = 1_X = f_2 \delta_f$. We shall denote by \bar{f} the **coequaliser of the kernelpair** of f . When we say that f **coequalises** u and v , we mean that $fu = fv$. We do not mean that f is the coequaliser of u and v . If \mathcal{H} is a class of morphisms, we shall denote by \mathcal{H}^* the class of all regular epimorphic members of \mathcal{H} .

A **first factor** of a morphism f is a morphism k for which there exists h such that $f = hk$. An object P will be called **preterminal** iff, for each object Z , there is at most one morphism $w : Z \rightarrow P$.

\mathcal{C}^{op} shall denote the **dual category** of \mathcal{C} .

The category \mathcal{C}/X has as objects all \mathcal{C} -morphisms with codomain X . The morphisms of \mathcal{C}/X are the commutative triangles of the form

$$\begin{array}{ccc} Z & \xrightarrow{f} & W \\ & \searrow k & \swarrow h \\ & & X \end{array}$$

in \mathcal{C} . Composition in \mathcal{C}/X is obtained from composition in \mathcal{C} in the obvious way. \mathcal{C}/X will be referred to as a **slice category** of \mathcal{C} .

The category \mathcal{C}^2 has as objects all \mathcal{C} -morphisms and as morphisms all commutative squares

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ f \downarrow & & \downarrow g \\ W & \xrightarrow{v} & Z \end{array}$$

in \mathcal{C} . The above \mathcal{C}^2 -morphism will be denoted by $(u, v) : f \rightarrow g$. Composition in \mathcal{C}^2 is obtained from composition in \mathcal{C} in the obvious way.

A pair (T, η) , where $T : \mathcal{C} \rightarrow \mathcal{C}$ is a functor and $\eta : 1_{\mathcal{C}} \rightarrow T$ is a natural transformation, will be referred to as a **pointed endofunctor** on \mathcal{C} . (T, η) is an **idempotent pointed endofunctor** iff, for all $X \in \mathcal{C}$, $\eta_{T(X)} = T(\eta_X)$ and $\eta_{T(X)} \in Iso(\mathcal{C})$ (see [JT 1999]).

We shall refer to the following categories:

Ab: Objects are all abelian groups. Morphisms are all group homomorphisms between abelian groups.

SymRe: Objects are all sets equipped with a reflexive and symmetric relation. Morphisms are all functions which preserve these relations (functions which map pairs of related elements to pairs of related elements).

Prost: Objects are all preordered sets. Morphisms are all order preserving maps.

Set: Objects are all sets. Morphisms are all functions.

Top: Objects are all topological spaces. Morphisms are all continuous functions.

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2 Factorisation Systems

In this chapter we recall the definitions and some basic theory of factorisation systems and a number of related concepts. This is partly because, as indicated in the Introduction, we are particularly interested in two specific factorisation systems (and their generalisations) and partly because factorisation systems are extremely useful tools on which we shall depend when dealing with the notions of subobjects (see Section 2.4) and constant morphisms (see Section 3.1). Throughout this chapter, let \mathcal{C} be an arbitrary category.

2.1 Factorisation systems via prefactorisation systems

Factorisation systems were defined by Freyd and Kelly in [FK 1972] (where they were just called “factorizations”) as follows.

Definition 2.1.1 (FK 1972) *Let \mathcal{E} and \mathcal{M} be classes of morphisms in \mathcal{C} . $(\mathcal{E}, \mathcal{M})$ is a factorisation system on \mathcal{C} iff the following conditions are satisfied:*

- (1a) $\text{Iso}(\mathcal{C}) \subseteq \mathcal{E} \cap \mathcal{M}$
- (1b) both \mathcal{E} and \mathcal{M} are closed under composition
- (2) for every f in \mathcal{C} , there exist $e \in \mathcal{E}$ and $m \in \mathcal{M}$ such that $f = me$
- (3) for any solid arrow commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{e} & M & \xrightarrow{m} & Y \\ w \downarrow & & \downarrow k & & \downarrow h \\ X' & \xrightarrow{e'} & M' & \xrightarrow{m'} & Y' \end{array}$$

in \mathcal{C} with $e, e' \in \mathcal{E}$ and $m, m' \in \mathcal{M}$, there exists a unique k in \mathcal{C} such that the whole diagram commutes.

The following relation was also defined in [FK 1972] (where it was denoted by the symbol \downarrow , the symbol \perp being used for something else).

Definition 2.1.2 (FK 1972) *Let e and m be in \mathcal{C} . Then $e \perp m$ iff, for any w, h in \mathcal{C} such that $he = mw$, there exists a unique k in \mathcal{C} such that $w = ke$ and $mk = h$.*

$$\begin{array}{ccc} X & \xrightarrow{e} & N \\ w \downarrow & \swarrow k & \downarrow h \\ M & \xrightarrow{m} & Y \end{array}$$

It was observed in [FK 1972] that 2.1.1(3) may be replaced by the condition that $e \perp m$ for all $e \in \mathcal{E}$ and all $m \in \mathcal{M}$. It is also well known that the full strength of 2.1.1(1b) is not required. It may be replaced by the condition that both \mathcal{E} and \mathcal{M} are closed under composition with isomorphisms (see, for example, [AHS 1990]). It was observed in [FK 1972] that if $(\mathcal{E}, \mathcal{M})$ is a factorisation system on \mathcal{C} then, for each f in \mathcal{C} , there is, up to isomorphism, only one factorisation $f = me$ such that $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

Using slightly different notation, the following two definitions were given in [FK 1972].

Definition 2.1.3 (FK 1972) *Let \mathcal{H} be a class of morphisms in \mathcal{C} . $\mathcal{H}^\perp = \{f \text{ in } \mathcal{C} \mid f \perp h, \forall h \in \mathcal{H}\}$ and $\mathcal{H}_\perp = \{g \text{ in } \mathcal{C} \mid h \perp g, \forall h \in \mathcal{H}\}$.*

Definition 2.1.4 (FK 1972) *Let \mathcal{E} and \mathcal{M} be classes of morphisms in \mathcal{C} . $(\mathcal{E}, \mathcal{M})$ is a prefactorisation system on \mathcal{C} iff $\mathcal{E} = \mathcal{M}^\perp$ and $\mathcal{M} = \mathcal{E}_\perp$.*

Prefactorisation systems have a number of nice properties. The following were noted in [FK 1972], with the possible exception of 2.1.5(3) of which a slightly less general version was given. The full generality of 2.1.5(3) is not significantly harder to obtain and has been noted by later authors (see, for example, [CJKP 1997]).

Proposition 2.1.5 (FK 1972) *Let $(\mathcal{E}, \mathcal{M})$ be a prefactorisation system on \mathcal{C} . Then*

- (1) $\text{Iso}(\mathcal{C}) \subseteq \mathcal{E}$
- (2) \mathcal{E} is closed under composition
- (3) if $F, G : \mathcal{B} \rightarrow \mathcal{C}$ are functors such that $\text{colim}(F)$ and $\text{colim}(G)$ exist and $\alpha : F \rightarrow G$ is a natural transformation such that $\alpha_B \in \mathcal{E}$ for all $B \in \mathcal{B}$, then the induced \mathcal{C} -morphism with domain $\text{colim}(F)$ and codomain $\text{colim}(G)$ is a member of \mathcal{E}
- (4) \mathcal{E} is closed under the formation of pushouts
- (5) \mathcal{E} is closed under the formation of multiple pushouts
- (6) \mathcal{E} has the property that $e \in \mathcal{E}$ whenever $ek \in \mathcal{E}$ and $k \in \mathcal{E} \cup \text{Epi}(\mathcal{C})$.

The obvious fact that $(\mathcal{E}, \mathcal{M})$ is a prefactorisation system on \mathcal{C} iff $(\mathcal{M}, \mathcal{E})$ is a prefactorisation system on \mathcal{C}^{op} yields a dual result to Proposition 2.1.5.

The following gives the relationship between factorisation systems and prefactorisation systems.

Theorem 2.1.6 (FK 1972) *Let \mathcal{E} and \mathcal{M} be classes of morphisms in \mathcal{C} . Then $(\mathcal{E}, \mathcal{M})$ is a factorisation system on \mathcal{C} iff it is a prefactorisation system such that, for every f in \mathcal{C} , there exist $e \in \mathcal{E}$ and $m \in \mathcal{M}$ such that $f = me$.*

Theorem 2.1.7 gives useful sufficient conditions for a prefactorisation system to be a factorisation system. The dual result was proved, if not explicitly stated, in [FK 1972]. Freyd and Kelly assumed $(\mathcal{E}, \mathcal{M})$ to be a proper prefactorisation system (see Definition 2.4.1). However, the full strength of this assumption was not used in the relevant proofs.

Theorem 2.1.7 (FK 1972) *Let $(\mathcal{E}, \mathcal{M})$ be a prefactorisation system on \mathcal{C} such that $\mathcal{E} \subseteq \text{Epi}(\mathcal{C})$ and suppose that \mathcal{C} has pushouts and has multiple pushouts of members of \mathcal{E} . Then $(\mathcal{E}, \mathcal{M})$ is a factorisation system on \mathcal{C} .*

We draw attention to a particularly important factorisation system which exists for many well known categories. It is obvious and well known that $(\text{RegEpi}(\mathcal{C}), \text{Mono}(\mathcal{C}))$ is a factorisation system on \mathcal{C} provided every f in \mathcal{C} has a factorisation $f = me$ with $e \in \text{RegEpi}(\mathcal{C})$ and $m \in \text{Mono}(\mathcal{C})$. When this is the case we shall simply say that \mathcal{C} has (regular epi, mono)-factorisations.

We shall use the following convenient terminology.

Notation 2.1.8 *A regular (pre)factorisation system on \mathcal{C} is a (pre)factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{C} such that $\mathcal{E} \subseteq \text{RegEpi}(\mathcal{C})$. A stable (pre)factorisation system on \mathcal{C} is a (pre)factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{C} such that \mathcal{E} is pullback stable.*

2.2 Factorisation systems via weak factorisation systems

Given a factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{C} , one may choose, for each f in \mathcal{C} , an $(\mathcal{E}, \mathcal{M})$ factorisation $f = m_f e_f$. One then obtains a functor $F : \mathcal{C}^2 \rightarrow \mathcal{C}$ such that, for each commutative square $gu = vf$ in \mathcal{C} , the following diagram

$$\begin{array}{ccc}
 \bullet & \xrightarrow{u} & \bullet \\
 \downarrow f & \begin{array}{c} \nearrow e_f \\ \searrow m_f \end{array} & \downarrow g \\
 & F(f) \xrightarrow{F(u,v)} F(g) & \\
 \bullet & \xrightarrow{v} & \bullet \\
 & \begin{array}{c} \nearrow m_g \\ \searrow e_g \end{array} &
 \end{array}$$

commutes. $F(u, v)$ is necessarily the unique diagonal fill-in whose existence is guaranteed by Definition 2.1.1. The uniqueness of this fill-in guarantees the functoriality of F . Throughout this section, let $E : \mathcal{C} \rightarrow \mathcal{C}^2$ be the functor which embeds \mathcal{C} into \mathcal{C}^2 by mapping each \mathcal{C} -object X to the \mathcal{C}^2 -object 1_X and each \mathcal{C} -morphism $f : X \rightarrow Y$ to the \mathcal{C}^2 -morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ 1_X \downarrow & & \downarrow 1_Y \\ X & \xrightarrow{f} & Y. \end{array}$$

It is easy to see that $\gamma : 1_{\mathcal{C}} \rightarrow FE$, defined by $\gamma_X = e_{1_X}$ for all $X \in \mathcal{C}$, is a natural isomorphism. These observations, known to Korostenski and Tholen, suggest the approach to factorisation systems which they develop in [KT 1993].

Definition 2.2.1 (KT 1993) *A weak factorisation system F on \mathcal{C} is a functor $F : \mathcal{C}^2 \rightarrow \mathcal{C}$ such that $FE \cong 1_{\mathcal{C}}$.*

Let F be a weak factorisation system and let $\gamma : 1_{\mathcal{C}} \rightarrow FE$ be a natural isomorphism. It was observed in [KT 1993] that each \mathcal{C} -morphism $f : X \rightarrow Y$ has a factorisation as follows:

$$X \xrightarrow{\gamma_X} F(1_X) \xrightarrow{F(1_X, f)} F(f) \xrightarrow{F(f, 1_Y)} F(1_Y) \xrightarrow{\gamma_Y^{-1}} Y.$$

Definition 2.2.2 (KT 1993) *Let F be a weak factorisation system on \mathcal{C} and let $\gamma : 1_{\mathcal{C}} \rightarrow FE$ be a natural isomorphism. For any f in \mathcal{C} , $e_f = F(1_X, f) \cdot \gamma_X$ and $m_f = \gamma_Y^{-1} \cdot F(f, 1_Y)$. Also, $\mathcal{E}_F = \{f \text{ in } \mathcal{C} \mid m_f \in \text{Iso}(\mathcal{C})\}$ and $\mathcal{M}_F = \{f \text{ in } \mathcal{C} \mid e_f \in \text{Iso}(\mathcal{C})\}$.*

The pair $(\mathcal{E}_F, \mathcal{M}_F)$ is then a candidate for being a factorisation system. However certain conditions need to be satisfied. Clearly one of these is the following.

Definition 2.2.3 (KT 1993) *Let F be a weak factorisation system on \mathcal{C} . F is said to have the diagonalisation property iff, for each commutative square $gu = vf$ in \mathcal{C} , $F(u, v)$ is uniquely determined by the requirement that the first diagram in this section commutes.*

Korostenski and Tholen note that $(\mathcal{E}_F, \mathcal{M}_F)$ will be a factorisation system on \mathcal{C} provided F has the diagonalisation property and, for each f in \mathcal{C} , $e_f \in \mathcal{E}_F$ and $m_f \in \mathcal{M}_F$. They note too that, as is fairly clear from the opening remarks of this section, all factorisation systems on \mathcal{C} may be obtained in this way. It was proved in [KT 1993] that one actually gets the diagonalisation property for free provided $e_f \in \mathcal{E}_F$ and $m_f \in \mathcal{M}_F$ for all f in \mathcal{C} .

Theorem 2.2.4 (KT 1993) *Let F be a weak factorisation system on \mathcal{C} . Then $(\mathcal{E}_F, \mathcal{M}_F)$ is a factorisation system on \mathcal{C} provided $e_f \in \mathcal{E}_F$ and $m_f \in \mathcal{M}_F$ for all f in \mathcal{C} .*

Another perspective on factorisation systems is provided by considering the following intermediate definition given by Janelidze and Tholen in [JT 1999].

Definition 2.2.5 (JT 1999) *Let F be a weak factorisation system on \mathcal{C} . Define $F_r : \mathcal{C}^2 \rightarrow \mathcal{C}^2$ as follows: $F_r(f) = m_f$, for all \mathcal{C}^2 -objects f , and $F_r(u, v) = (F(u, v), v)$, for all \mathcal{C}^2 -morphisms (u, v) . Define $\eta : 1_{\mathcal{C}} \rightarrow F_r$ as follows: $\eta_f = (e_f, 1_{\text{cod}(f)})$, for all \mathcal{C}^2 -objects f . Then F is a right factorisation system on \mathcal{C} iff (F_r, η) is an idempotent pointed endofunctor on \mathcal{C} .*

Left factorisation systems were defined dually. It was noted in [JT 1999] that the factorisations obtained from a left factorisation system F are what were called locally orthogonal $\mathcal{E}_{\mathcal{F}}$ -factorisations in [MT 1982] (and that the factorisations obtained from a right factorisation system F are locally coorthogonal $\mathcal{M}_{\mathcal{F}}$ -factorisations). We recall the definition.

Definition 2.2.6 (MT 1982) *Let \mathcal{E} be a class of morphisms in \mathcal{C} such that $\text{Iso}(\mathcal{C}) \subseteq \mathcal{E}$ and \mathcal{E} is closed under composition with isomorphisms. Let f be in \mathcal{C} . A locally orthogonal \mathcal{E} -factorisation of f is a factorisation $f = me$ such that $e \in \mathcal{E}$ and such that, for any solid arrow commutative diagram*

$$\begin{array}{ccc} Z & \xrightarrow{w} & U \\ k \downarrow & & \swarrow d \quad \downarrow h \\ X & \xrightarrow{e} M \xrightarrow{m} & Y \end{array}$$

in \mathcal{C} with $w \in \mathcal{E}$, there is a unique d in \mathcal{C} such that the whole diagram commutes.

Left factorisation systems, right factorisation systems and factorisation systems are related as follows.

Theorem 2.2.7 (JT 1999) *Let F be a weak factorisation system on \mathcal{C} . The following are equivalent:*

- (1) $(\mathcal{E}_F, \mathcal{M}_F)$ is a factorisation system
- (2) F is a right factorisation system such that \mathcal{M}_F is closed under composition
- (3) F is a left factorisation system such that \mathcal{E}_F is closed under composition.

2.3 Factorising sources

It is possible and often useful to consider not just factorisations of morphisms but factorisations of sources as well (or, dually, of sinks). The following definition was given by Herrlich in [H 1974].

Definition 2.3.1 (H 1974) *Let $\mathcal{E} \subseteq \text{Epi}(\mathcal{C})$ and let \mathbf{M} be a collection of sources in \mathcal{C} such that both \mathcal{E} and \mathbf{M} are closed under composition with isomorphisms. \mathcal{C} is an $(\mathcal{E}, \mathbf{M})$ -category iff*

- (1) for each source $(f_i)_{i \in I}$ in \mathcal{C} , there exist $e \in \mathcal{E}$ and $(m_i)_{i \in I} \in \mathbf{M}$ such that $(f_i)_{i \in I} = (m_i \cdot e)_{i \in I}$
- (2) for any index class I and any family of solid arrow commutative squares

$$\begin{array}{ccc}
 X & \xrightarrow{e} & N \\
 k \downarrow & \dashrightarrow g & \downarrow h_i \\
 M & \xrightarrow{m_i} & Y
 \end{array} \quad (i \in I)$$

in \mathcal{C} with $e \in \mathcal{E}$ and $(m_i)_{i \in I} \in \mathbf{M}$, there exists a unique g such that $k = ge$ and, for all $i \in I$, $m_i \cdot g = h_i$.

It was shown in [BT 1978] that $\mathcal{E} \subseteq \text{Epi}(\mathcal{C})$ is actually a consequence of the other conditions in Definition 2.3.1. It is clear from Definition 2.1.1 and the remarks made after Definition 2.1.2 that if \mathcal{C} is an $(\mathcal{E}, \mathbf{M})$ -category and \mathcal{M} is the class of all singleton sources (otherwise known as morphisms) in \mathbf{M} then $(\mathcal{E}, \mathcal{M})$ is a factorisation system on \mathcal{C} . This means that the properties established for the class \mathcal{E} in Proposition 2.1.5 and the fact that $\mathcal{E} \cap \mathbf{M} = \text{Iso}(\mathcal{C})$ are true when \mathcal{C} is an $(\mathcal{E}, \mathbf{M})$ -category. Some of these results were also proved in [H 1974]. In addition, Herrlich observed the following.

Proposition 2.3.2 (H 1974) *Let \mathcal{C} be an $(\mathcal{E}, \mathbf{M})$ -category. Then*

- (1) *the $(\mathcal{E}, \mathbf{M})$ -factorisation of any source in \mathcal{C} is unique up to isomorphism*
- (2) *if $m : X \rightarrow Y$ and $(m_i : Y \rightarrow Z_i)_{i \in I}$ are in \mathbf{M} , then $(m_i \cdot m)_{i \in I} \in \mathbf{M}$*
- (3) *\mathbf{M} contains all sources which are limits in \mathcal{C} .*

There is also a version for sources of the locally orthogonal \mathcal{E} -factorisations mentioned in the previous section. This is defined in [T 1983].

Definition 2.3.3 (T 1983) *Let \mathcal{E} be a class of morphisms in \mathcal{C} such that $\text{Iso}(\mathcal{C}) \subseteq \mathcal{E}$ and \mathcal{E} is closed under composition with isomorphisms. Let $(f_i)_{i \in I}$ be a source in \mathcal{C} . A locally orthogonal \mathcal{E} -factorisation of $(f_i)_{i \in I}$ is a factorisation $(f_i)_{i \in I} = (m_i)_{i \in I} \cdot e$ such that $e \in \mathcal{E}$ and such that, for any family of solid arrow commutative diagrams*

$$\begin{array}{ccc}
 Z & \xrightarrow{w} & U \\
 k \downarrow & & \swarrow d \quad \downarrow h_i \\
 X & \xrightarrow{e} M \xrightarrow{m_i} & Y \quad (i \in I)
 \end{array}$$

in \mathcal{C} with $w \in \mathcal{E}$, there is a unique d in \mathcal{C} such that each whole diagram commutes.

Locally orthogonal \mathcal{E} -factorisations for sources are relevant to the generalisation of concordance and dissonance given by Börger and Tholen in [BT 1984] and discussed in Chapter 6. In [T 1983], Tholen defined an \mathcal{E} -localisation of an object $X \in \mathcal{C}$ to be a locally orthogonal \mathcal{E} -factorisation of the empty source with domain X .

2.4 Images and subobjects

Perhaps the most famous factorisation system is the pair (Surjections, Injections) on **Set**. Indeed, the theory of factorisation systems may be thought of as a generalisation of the theory of surjections and injections and can therefore be used to generalise the theory of subsets and so obtain a theory of subobjects. This was done in [FK 1972] using factorisation systems which may be thought of as having enough in common with (Surjections, Injections).

Definition 2.4.1 (FK 1972) *A factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{C} is said to be proper iff $\mathcal{E} \subseteq \text{Epi}(\mathcal{C})$ and $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$.*

The following definition differs slightly from that given in [FK 1972] in that Freyd and Kelly used different terminology and also did not define subobjects to be morphisms but rather equivalence classes of morphisms (two members of \mathcal{M} , with common codomain X , being equivalent iff they are isomorphic in the slice category \mathcal{C}/X). We follow later authors (see, for example, [CT 1998]) in avoiding the equivalence relation and defining subobjects to be morphisms, even though this means that images and pre-images are then only defined up to isomorphism.

Definition 2.4.2 (FK 1972) *Let $(\mathcal{E}, \mathcal{M})$ be a proper factorisation system on \mathcal{C} and let*

$$\begin{array}{ccc} M & \xrightarrow{e} & N \\ m \downarrow & & \downarrow n \\ X & \xrightarrow{w} & Y \end{array}$$

be a commutative square in \mathcal{C} . Then

- (1) *m is an \mathcal{M} -subobject of X iff $m \in \mathcal{M}$*
- (2) *n is an \mathcal{M} -image of m under w iff $m, n \in \mathcal{M}$ and $e \in \mathcal{E}$*
- (3) *m is an \mathcal{M} -pre-image of n under w iff $n \in \mathcal{M}$ and the square is a pullback.*

From Definition 2.1.1, it is clear that \mathcal{M} -images will always exist. \mathcal{M} -pre-images will exist whenever the necessary pullbacks exist. From the dual of Proposition 2.1.5(4) it is clear that \mathcal{M} -pre-images are indeed \mathcal{M} -subobjects. Where it is obvious which factorisation system we are using, we shall often omit the prefix “ \mathcal{M} -”. We shall also make use of the following notation.

Notation 2.4.3 *$w[m] : w[M] \rightarrow Y$ will denote an \mathcal{M} -image of $m : M \rightarrow X$ under $w : X \rightarrow Y$ and $w^{-1}[n] : w^{-1}[N] \rightarrow X$ will denote an \mathcal{M} -pre-image of $n : N \rightarrow Y$ under $w : X \rightarrow Y$.*

For each $X \in \mathcal{C}$, \mathcal{M}/X will denote the full subcategory of the slice category \mathcal{C}/X whose objects are the \mathcal{M} -subobjects of X . For $m, m' \in \mathcal{M}/X$, $m \leq m'$ iff there exists k in \mathcal{C} such that $m = m'k$.

Given a proper factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{C} and $X \in \mathcal{C}$, it is obvious and well known that \mathcal{M}/X is a preordered class.

3 Introducing the \parallel -relation

Throughout this chapter, let \mathcal{C} be an arbitrary category. We introduce a relation on the class $Mor(\mathcal{C})$ which we call the \parallel -relation. As mentioned in the Introduction, this may be regarded as a version for morphisms of a relation which was defined on $Ob(\mathcal{C})$ by Herrlich in [H 1968]. In Section 3.1 we recall Herrlich’s definitions. In Section 3.2 we define the \parallel -relation and the notion of a \parallel -pair and establish some of their basic properties. In Section 3.3 we link the \parallel -relation to two other similar relations: the \parallel^H -relation, defined (without the H) by Holgate in [H 2004], and the \perp -relation (recall Definition 2.1.2). In Section 3.4 we establish sufficient conditions for obtaining a factorisation system from a \parallel -pair and in Section 3.5 we record a technical result which will be of use in Chapter 7.

3.1 Constant morphisms and the \parallel^{ob} -relation

There are several reasonable ways to define the notion of a constant morphism (as a generalisation of the notion of a constant function). In [T 1984] Tholen assumes that every $X \in \mathcal{C}$ has an \mathcal{E} -localisation $\eta_X : X \rightarrow T(X)$ and defines $w : X \rightarrow Y$ in \mathcal{C} to be constant iff η_X is a first factor of w . In [C 1995] Clementino assumes that \mathcal{C} is equipped with a factorisation system $(\mathcal{E}, \mathcal{M})$ and a designated full subcategory of “constant objects” (satisfying certain conditions) and defines that w in \mathcal{C} is constant iff the middle object of its $(\mathcal{E}, \mathcal{M})$ -factorisation is constant. For our purposes it will be sufficient to use the following simple definition given by Herrlich in [H 1968].

Definition 3.1.1 (H 1968) *Let $w : X \rightarrow Y$ be in \mathcal{C} . Then w is constant iff $wu = wv$ for all $u, v : Z \rightarrow X$ in \mathcal{C} .*

It is obvious that constant morphisms behave as they should with regard to composition, i.e. f constant $\Rightarrow hfk$ constant, for all h, f, k in \mathcal{C} such that the composition is defined (see [H 1968]).

The following result is easy to prove and is probably well known. Relevant remarks may be found in [C 1995] and [CH 2003].

Proposition 3.1.2 *Let \mathcal{C} be a finitely complete category equipped with a proper and stable factorisation system $(\mathcal{E}, \mathcal{M})$. Let w be in \mathcal{C} and let $w = me$ be an $(\mathcal{E}, \mathcal{M})$ -factorisation. Then w is constant iff $dom(m)$ is preterminal.*

Under the conditions of Proposition 3.1.2, it is obvious that f constant $\Leftrightarrow mfe$ constant, for all $m \in \mathcal{M}$, f in \mathcal{C} and $e \in \mathcal{E}$ such that the composition is defined (see [C 1995]). The following is proved in [H 2004]. (Note that δ_Y is the unique morphism such that $\pi_1\delta_Y = 1_Y = \pi_2\delta_Y$, where π_1, π_2 are the product projections of $Y \times Y$.)

Proposition 3.1.3 (H 2004) *Let \mathcal{C} have products of pairs. Then $w : X \rightarrow Y$ in \mathcal{C} is constant iff there exists k in \mathcal{C} such that $w^2 = \delta_Y k$.*

$$\begin{array}{ccc}
 X & \xrightarrow{\delta_X} & X \times X \\
 w \downarrow & \dashrightarrow k & \downarrow w^2 \\
 Y & \xrightarrow{\delta_Y} & Y \times Y
 \end{array}$$

In [H 1968] Herrlich defined what we shall refer to as the $\|\!^{ob}$ -relation.

Definition 3.1.4 (H 1968) *Let $X, Y \in \mathcal{C}$. Then $X \|\!^{ob} Y$ iff all $w : X \rightarrow Y$ in \mathcal{C} are constant.*

He also defined two useful operators.

Definition 3.1.5 (H 1968) *Let \mathcal{A} be a full subcategory of \mathcal{C} . Then $l(\mathcal{A}) = \{X \in \mathcal{C} \mid X \|\!^{ob} Z, \forall Z \in \mathcal{A}\}$ and $r(\mathcal{A}) = \{Y \in \mathcal{C} \mid Z \|\!^{ob} Y, \forall Z \in \mathcal{A}\}$.*

Using these he defined what we shall refer to as $\|\!^{ob}$ -pairs.

Definition 3.1.6 (H 1968) *Let \mathcal{K} and \mathcal{H} be full subcategories of \mathcal{C} . Then $(\mathcal{K}, \mathcal{H})$ is a $\|\!^{ob}$ -pair iff $\mathcal{K} = l(\mathcal{H})$ and $\mathcal{H} = r(\mathcal{K})$.*

We say \mathcal{K} is the left class of a $\|\!^{ob}$ -pair iff there exists \mathcal{H} (and \mathcal{H} is the right class of a $\|\!^{ob}$ -pair iff there exists \mathcal{K}) such that $(\mathcal{K}, \mathcal{H})$ is a $\|\!^{ob}$ -pair.

3.2 The $\|\!-$ relation

We adapt Definition 3.1.1 as follows.

Definition 3.2.1 Let $f : X \rightarrow A$, $g : Y \rightarrow B$ and $w : X \rightarrow Y$ be in \mathcal{C} . Then w is (f, g) -constant iff, for all $u, v : Z \rightarrow X$ in \mathcal{C} ,

$$\begin{array}{ccccc} Z & \xrightarrow[u]{v} & X & \xrightarrow{w} & Y \\ & & \downarrow f & & \downarrow g \\ & & A & & B \end{array}$$

$wu = wv$ if both $fu = fv$ and $gwu = gwv$.

We then imitate Definition 3.1.4.

Definition 3.2.2 Let $f : X \rightarrow A$ and $g : Y \rightarrow B$ be in \mathcal{C} . Then $f \parallel g$ iff all $w : X \rightarrow Y$ in \mathcal{C} are (f, g) -constant.

Under a very mild condition on \mathcal{C} , the \parallel -relation may be thought of as an extension of the \parallel^{ob} -relation.

Proposition 3.2.3 Let \mathcal{C} have a terminal object and let $X, Y \in \mathcal{C}$. Then $X \parallel^{ob} Y$ iff $!_X \parallel !_Y$.

Proof. It is clear that $w : X \rightarrow Y$ in \mathcal{C} is constant iff w is $(!_X, !_Y)$ -constant.

We shall have much use for operators analogous to r and l .

Definition 3.2.4 Let \mathcal{H} be a class of morphisms in \mathcal{C} . Then $L(\mathcal{H}) = \{f \text{ in } \mathcal{C} \mid f \parallel h, \forall h \in \mathcal{H}\}$ and $R(\mathcal{H}) = \{g \text{ in } \mathcal{C} \mid h \parallel g, \forall h \in \mathcal{H}\}$.

We record the following basic consequences of Definition 3.2.4.

Proposition 3.2.5 Let \mathcal{H} and \mathcal{H}' be classes of morphisms in \mathcal{C} . Then

- (1) $\mathcal{H} \subseteq L(R(\mathcal{H}))$
- (2) $\mathcal{H} \subseteq R(L(\mathcal{H}))$
- (3) $\mathcal{H} \subseteq \mathcal{H}' \Rightarrow L(\mathcal{H}') \subseteq L(\mathcal{H})$
- (4) $\mathcal{H} \subseteq \mathcal{H}' \Rightarrow R(\mathcal{H}') \subseteq R(\mathcal{H})$
- (5) $L(\mathcal{H}) = L(R(L(\mathcal{H})))$
- (6) $R(\mathcal{H}) = R(L(R(\mathcal{H})))$.

We imitate Definition 3.1.6 as follows.

Definition 3.2.6 Let \mathcal{F} and \mathcal{G} be classes of morphisms in \mathcal{C} . Then $(\mathcal{F}, \mathcal{G})$ is a $\|\text{-pair}$ on \mathcal{C} iff $\mathcal{F} = L(\mathcal{G})$ and $\mathcal{G} = R(\mathcal{F})$.

We say that \mathcal{F} is the left class of a $\|\text{-pair}$ iff there exists \mathcal{G} such that $(\mathcal{F}, \mathcal{G})$ is a $\|\text{-pair}$. Dually, \mathcal{G} is the right class of a $\|\text{-pair}$ iff there exists \mathcal{F} such that $(\mathcal{F}, \mathcal{G})$ is a $\|\text{-pair}$. We record the following basic consequences of Proposition 3.2.5.

Proposition 3.2.7 Let $\mathcal{F}, \mathcal{G}, \mathcal{F}'$ and \mathcal{G}' be classes of morphisms in \mathcal{C} . Then

- (1) \mathcal{F} is the left class of a $\|\text{-pair}$ iff $\mathcal{F} = L(R(\mathcal{F}))$
- (2) \mathcal{G} is the right class of a $\|\text{-pair}$ iff $\mathcal{G} = R(L(\mathcal{G}))$
- (3) if $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ are $\|\text{-pairs}$ then
 - (a) $\mathcal{F} \subseteq \mathcal{F}' \Rightarrow \mathcal{G}' \subseteq \mathcal{G}$
 - (b) $\mathcal{G} \subseteq \mathcal{G}' \Rightarrow \mathcal{F}' \subseteq \mathcal{F}$.

We note that Propositions 3.2.5 and 3.2.7 are nothing other than basic results in the theory of Galois connections.

It turns out that an analogous result to Proposition 3.1.2 exists for the $\|\text{-relation}$. We believe that this helps to justify our approach. The proof will make use of the following lemma which is easy to prove and probably well known.

Lemma 3.2.8 Let \mathcal{C} have pullbacks and let p, q, e in \mathcal{C} be such that $p = qe$. Let e_R^2 be the unique morphism such that the following diagram

$$\begin{array}{ccccc}
 & & P & \xrightarrow{e} & Q \\
 & p_1 \nearrow & & & \nearrow q_1 \\
 P \times_R P & \xrightarrow{e_R^2} & Q \times_R Q & & \\
 & p_2 \searrow & & & \searrow q_2 \\
 & & P & \xrightarrow{e} & Q \\
 & & & & \nearrow q \\
 & & & & R
 \end{array}$$

commutes. Then there exist k, k', j in \mathcal{C} such that $e_R^2 = kj$ and such that k is obtained by pulling back e along q_1 , k' is obtained by pulling back e along q_2 and j is obtained by pulling back k' along k .

Proposition 3.2.9 *Let \mathcal{C} have pullbacks and products of pairs and be equipped with a proper and stable factorisation system $(\mathcal{E}, \mathcal{M})$. Let $f : X \rightarrow A$, $g : Y \rightarrow B$ and $w : X \rightarrow Y$ be in \mathcal{C} and form the commutative diagram*

$$\begin{array}{ccc}
 X & \xrightarrow{w} & Y \\
 \downarrow f & \searrow e & \uparrow \pi_2 \\
 & M & \\
 & \searrow m & \\
 A & \xleftarrow{\pi_1} & A \times Y & \xrightarrow{g} & B
 \end{array}$$

where π_1 and π_2 are product projections and $\langle f, w \rangle = me$ is an $(\mathcal{E}, \mathcal{M})$ -factorisation. Then w is (f, g) -constant iff $(\pi_1 m, g\pi_2 m)$ is a monosource.

Proof. (\Leftarrow) Suppose $(\pi_1 m, g\pi_2 m)$ is a monosource and let $u, v : Z \rightarrow X$ in \mathcal{C} be such that $fu = fv$ and $gwu = gwv$. So $\pi_1 me u = \pi_1 me v$ and $g\pi_2 me u = g\pi_2 me v$. It follows that $eu = ev$ and then that $wu = wv$. This establishes that w is (f, g) -constant.

(\Rightarrow) Suppose w is (f, g) -constant. Let $k = \langle \pi_1 m, g\pi_2 m \rangle : M \rightarrow A \times B$. Let $e_{(A \times B)}^2$ be the unique morphism such that $k_1 e_{(A \times B)}^2 = e \cdot (ke)_1$ and $k_2 e_{(A \times B)}^2 = e \cdot (ke)_2$. Since $e \in \mathcal{E}$ and \mathcal{E} is pullback stable and closed under composition, we have, by Lemma 3.2.8, that $e_{(A \times B)}^2 \in \mathcal{E}$. In particular, $e_{(A \times B)}^2$ is an epimorphism. From the definition of k , it is clear that $\pi_1 me \cdot (ke)_1 = \pi_1 me \cdot (ke)_2$ and $g\pi_2 me \cdot (ke)_1 = g\pi_2 me \cdot (ke)_2$. So $f \cdot (ke)_1 = f \cdot (ke)_2$ and $gw \cdot (ke)_1 = gw \cdot (ke)_2$. Therefore $w \cdot (ke)_1 = w \cdot (ke)_2$. Since m is a monomorphism and (π_1, π_2) is a monosource, it follows that $e \cdot (ke)_1 = e \cdot (ke)_2$. So $k_1 e_{(A \times B)}^2 = k_2 e_{(A \times B)}^2$. Therefore $k_1 = k_2$. It follows that k is a monomorphism which implies that $(\pi_1 m, g\pi_2 m)$ is a monosource.

If \mathcal{C} has a terminal object and $f = !_X$ and $g = !_Y$ then it is clear that $(\pi_1 m, g\pi_2 m)$ being a monosource is equivalent to M being preterminal. Since also $\pi_2 : 1 \times Y \rightarrow Y$ is an isomorphism, we may regard Proposition 3.1.2 as being a special case of Proposition 3.2.9 (provided \mathcal{C} has a terminal object).

$\|$ -pairs have a number of nice properties which we now list. There is clearly some similarity between these and the properties satisfied by prefactorisation systems (recall Proposition 2.1.5). We give results in Section 3.3 which help to explain why this is the case.

Proposition 3.2.10 Let \mathcal{F} and \mathcal{G} be classes of morphisms in \mathcal{C} and let p, f, q, g in \mathcal{C} be such that pf and qg are defined. Then

- (1) $\text{Mono}(\mathcal{C}) \subseteq L(\mathcal{G}) \cap R(\mathcal{F})$
- (2) $\text{Mono}(\mathcal{C}) = \mathcal{F} \cap \mathcal{G}$ if $(\mathcal{F}, \mathcal{G})$ is a \parallel -pair
- (3) $pf \in L(\mathcal{G}) \Rightarrow f \in L(\mathcal{G})$
- (4) $qg \in R(\mathcal{F}) \Rightarrow g \in R(\mathcal{F})$
- (5) if $f \in L(\mathcal{G})$ and p is a monomorphism then $pf \in L(\mathcal{G})$
- (6) $R(\mathcal{F})$ is closed under composition
- (7) $R(\mathcal{F})$ is closed under the formation of limits in \mathcal{C}^2 .

Proof. Let the following be a diagram in \mathcal{C} .

$$\begin{array}{ccccc}
 Z & \xrightarrow{u} & X & \xrightarrow{w} & Y \\
 & \xrightarrow{v} & & & \\
 & & f \downarrow & & \downarrow g \\
 & & A & & B \\
 & & p \downarrow & & \downarrow q \\
 & & C & & D
 \end{array}$$

Suppose f or g is a monomorphism. If $fu = fv$ and $gwu = gwv$ then $u = v$ or $wu = wv$ and so (in either case) $wu = wv$. Therefore $f \parallel g$. This proves (1).

Suppose $f \parallel f$. If $fu = fv$ then also $f1_Xu = f1_Xv$ and so $1_Xu = 1_Xv$ and so $u = v$. Therefore f is a monomorphism. Together with (1), this proves (2).

Suppose $pf \parallel qg$. If $fu = fv$ and $gwu = gwv$ then $pfu = pfv$ and $qgwu = qgwv$ and so $wu = wv$. Therefore $f \parallel g$. This proves (3) and (4).

Suppose $f \parallel g$ and p is a monomorphism. If $pfu = pfv$ and $gwu = gwv$ then also $fu = fv$ and so $wu = wv$. Therefore $pf \parallel g$. This proves (5).

Suppose $f \parallel g$ and $f \parallel q$. If $fu = fv$ and $qgwu = qgwv$ then $gwu = gwv$ (by the second assumption) and so $wu = wv$ (by the first assumption). So $f \parallel qg$. This proves (6).

Suppose g is the limit of a diagram in \mathcal{C}^2 , \mathcal{H} is the class of \mathcal{C}^2 -objects in that diagram and $f \parallel h$ for all $h \in \mathcal{H}$. Then, for each $h \in \mathcal{H}$, there are \mathcal{C} -morphisms u_h and v_h such that $hu_h = v_hg$ and such that $(u_h)_{h \in \mathcal{H}}$ is a monosource. Suppose $fu = fv$ and $gwu = gwv$. For all $h \in \mathcal{H}$, it follows that $hu_hwu = hu_hwv$ and then that $u_hwu = u_hwv$ (since $f \parallel h$). It then follows that $wu = wv$ (since $(u_h)_{h \in \mathcal{H}}$ is a monosource). Therefore $f \parallel g$. This

proves (7).

Proposition 3.2.10 says rather more about $R(\mathcal{F})$ than it does about $L(\mathcal{G})$. We correct this imbalance, to some extent, with the following.

Proposition 3.2.11 *Let \mathcal{C} have pullbacks and let \mathcal{E} be a pullback stable class of epimorphisms in \mathcal{C} . Let \mathcal{G} be a class of morphisms in \mathcal{C} and let $f : X \rightarrow A$ and $e : W \rightarrow X$ be in \mathcal{C} . If $fe \in L(\mathcal{G})$ and $e \in \mathcal{E}$ then $f \in L(\mathcal{G})$.*

Proof. Suppose $fe \in L(\mathcal{G})$ and $e \in \mathcal{E}$. Begin to form the following diagram

$$\begin{array}{ccccc}
 P & \xrightarrow{\widehat{e}_A^2} & Z & & \\
 \widehat{z} \downarrow & & \downarrow z & & \\
 W \times_A W & \xrightarrow{e_A^2} & X \times_A X & & \\
 (fe)_2 \downarrow \parallel (fe)_1 & & f_2 \downarrow \parallel f_1 & & \\
 W & \xrightarrow{e} & X & \xrightarrow{w} & Y \\
 & & f \downarrow & & \downarrow g \\
 & & A & & B
 \end{array}$$

by assuming the existence of $g \in \mathcal{G}$ and w in \mathcal{C} as drawn and of $u, v : Z \rightarrow X$ in \mathcal{C} such that $fu = fv$ and $gwu = gvw$. It follows that there exists z in \mathcal{C} such that $u = f_1z$ and $v = f_2z$. Let e_A^2 be the unique morphism such that $f_1e_A^2 = e \cdot (fe)_1$ and $f_2e_A^2 = e \cdot (fe)_2$. Complete the diagram by pulling back e_A^2 along z .

By Lemma 3.2.8 and the assumptions on \mathcal{E} , there exist $k, j \in \mathcal{E}$ such that $e_A^2 = kj$. Let \widehat{k}, z' in \mathcal{C} be such that $z\widehat{k} = kz'$ is a pullback square. Then pull back j along z' to obtain \widehat{j} . Without loss of generality we may assume $\widehat{e}_A^2 = \widehat{k}\widehat{j}$. By the assumptions on \mathcal{E} , we have that \widehat{k} and \widehat{j} are epimorphisms. Therefore \widehat{e}_A^2 is an epimorphism.

Trivially, $fe \cdot (fe)_1\widehat{z} = fe \cdot (fe)_2\widehat{z}$. Since $gwu = gvw$ we also have that $gwe \cdot (fe)_1\widehat{z} = gwe \cdot (fe)_2\widehat{z}$. Since $fe \parallel g$, it follows that $we \cdot (fe)_1\widehat{z} = we \cdot (fe)_2\widehat{z}$ and therefore that $wf_1z \cdot \widehat{e}_A^2 = wf_2z \cdot \widehat{e}_A^2$. Since \widehat{e}_A^2 is an epimorphism, it follows that $wf_1z = wf_2z$ and therefore that $wu = wv$. This establishes that $f \parallel g$. Therefore $f \in L(\mathcal{G})$.

In particular, Proposition 3.2.11 holds for any class \mathcal{E} for which there exists \mathcal{M} such that $(\mathcal{E}, \mathcal{M})$ is a proper and stable factorisation system on \mathcal{C} . In order to highlight an appealing symmetry, we record this observation together with a trivial consequence of 3.2.10(1) and 3.2.10(6).

Corollary 3.2.12 *Let \mathcal{C} have pullbacks and let $(\mathcal{E}, \mathcal{M})$ be a proper and stable factorisation system on \mathcal{C} . Let \mathcal{F} and \mathcal{G} be classes of morphisms in \mathcal{C} and let f, e, g, m in \mathcal{C} be such that fe and gm are defined. Then*

- (1) $f \in L(\mathcal{G})$ if $fe \in L(\mathcal{G})$ and $e \in \mathcal{E}$
- (2) $gm \in R(\mathcal{F})$ if $g \in R(\mathcal{F})$ and $m \in \mathcal{M}$.

We conclude this section with a trivial but useful observation. (Recall that \bar{f} denotes the coequaliser of the kernelpair of f .)

Proposition 3.2.13 *Let \mathcal{C} have kernelpairs and their coequalisers and let f, g be in \mathcal{C} . Then $f \parallel g$ iff $\bar{f} \parallel g$.*

Proof. This is a direct consequence of the fact that, for all $u, v : Z \rightarrow \text{dom}(f)$ in \mathcal{C} , $fu = fv \Leftrightarrow \bar{f}u = \bar{f}v$.

3.3 \parallel, \parallel^H and \perp

Holgate gives the following definition in [H 2004] on the assumption that \mathcal{C} has pullbacks.

Definition 3.3.1 (H 2004) *Let $f, g \in \mathcal{C}$. Then $f \parallel^H g$ iff, for any w, h in \mathcal{C} such that $gw = hf$, there exists d in \mathcal{C} such that $w_h^2 = \delta_g d$ (where w_h^2 is the unique morphism such that $g_1 w_h^2 = w f_1$ and $g_2 w_h^2 = w f_2$).*

$$\begin{array}{ccc}
 X \times_A X & \xrightarrow{w_h^2} & Y \times_B Y \\
 \delta_f \uparrow & \dashrightarrow d & \uparrow \delta_g \\
 X & \xrightarrow{w} & Y \\
 f \downarrow & & \downarrow g \\
 A & \xrightarrow{h} & B
 \end{array}$$

It is a trivial consequence of the theory developed in [H 2004] that one may equivalently define the \parallel^H -relation as follows.

Definition 3.3.2 Let $f, g \in \mathcal{C}$. Then $f \parallel^H g$ iff, for any w, h, u, v in \mathcal{C} such that $gw = hf$,

$$\begin{array}{ccccc} Z & \xrightarrow[u]{v} & X & \xrightarrow{w} & Y \\ & & \downarrow f & & \downarrow g \\ & & A & \xrightarrow{h} & B \end{array}$$

$$fu = fv \Rightarrow wu = wv.$$

Definition 3.3.2 has the advantage of defining the \parallel^H -relation in all categories (not just those with pullbacks). More importantly, it is well suited to revealing the link between the \parallel^H -relation and the \parallel -relation. For these reasons it is the definition that we shall adopt and use here. However Definition 3.3.1, being Holgate's original one, is better suited to revealing the line of thought that led to his considering it (recall Proposition 3.1.3).

The following is a trivial consequence of Definitions 3.2.2 and 3.3.2.

Proposition 3.3.3 Let f, g be in \mathcal{C} . Then $f \parallel g \Rightarrow f \parallel^H g$

However more can be said about the relationship between the two relations.

Theorem 3.3.4 Let \mathcal{C} have products of pairs and let f, g be in \mathcal{C} . Then $f \parallel g$ iff $k \parallel^H g$ for all first factors k of f .

Proof. (\Rightarrow) Suppose $f \parallel g$. By 3.2.10(3), $k \parallel g$ for all first factors k of f . By Proposition 3.3.3, it follows that $k \parallel^H g$ for all first factors k of f .

(\Leftarrow) Suppose $k \parallel^H g$ for all first factors k of f . Let w, u, v in \mathcal{C} be such that $fu = fv$ and $gwu = gwv$.

$$\begin{array}{ccccc} Z & \xrightarrow[u]{v} & X & \xrightarrow{w} & Y \\ & & \downarrow f & \searrow \langle f, gw \rangle & \downarrow g \\ & & A & & B \\ & & \leftarrow \pi_1 & A \times Y & \xrightarrow{\pi_2} \end{array}$$

It follows that $\langle f, gw \rangle u = \langle f, gw \rangle v$. Since $\langle f, gw \rangle$ is a first factor of f , it follows, by Definition 3.3.2, that $wu = wv$. Therefore $f \parallel g$.

There is, under slightly stronger assumptions on \mathcal{C} , a useful modification of this result.

Theorem 3.3.5 *Let \mathcal{C} have products of pairs, kernelpairs and coequalisers of kernelpairs. Let f, g be in \mathcal{C} . Then $f \parallel g$ iff $k \parallel^H g$ for all regular epimorphic first factors k of f .*

Proof. (\Rightarrow) The forward implication is a consequence of Theorem 3.3.4.

(\Leftarrow) Suppose $k \parallel^H g$ for all regular epimorphic first factors k of f . Let w, u, v in \mathcal{C} be such that $fu = fv$ and $gwu = gvw$. It follows that $\langle f, gw \rangle \cdot u = \langle f, gw \rangle \cdot v$. Since $\langle f, gw \rangle$ is a first factor of both f and gw , it follows, by Definition 3.3.2, that $wu = wv$. Therefore $f \parallel g$.

Holgate defines operators analogous to L and R which we shall refer to as L^H and R^H . We shall use the term \parallel^H -pair to refer to a pair $(\mathcal{F}, \mathcal{G})$ for which $\mathcal{F} = L^H(\mathcal{G})$ and $\mathcal{G} = R^H(\mathcal{F})$. It was proved in [H 2004] (under the assumption that \mathcal{C} has pullbacks) that L^H and R^H have a number of the properties listed in Proposition 3.2.10. It is also easy to see that the proof of Proposition 3.2.11 could be adapted to establish that this result is true for L^H too.

It is clear from Theorem 3.3.4 that, when comparing the notion of a \parallel -pair with that of a \parallel^H -pair, 3.2.10(3) is a particularly significant property. The following is easily deduced from Theorem 3.3.4.

Corollary 3.3.6 *Let \mathcal{C} have products of pairs and let $(\mathcal{F}, \mathcal{G})$ be a \parallel^H -pair on \mathcal{C} . Then $(\mathcal{F}, \mathcal{G})$ is a \parallel -pair iff $pf \in \mathcal{F} \Rightarrow f \in \mathcal{F}$ for all p, f in \mathcal{C} such that pf is defined.*

There is a strong connection between the \parallel^H -relation and the \perp -relation. The following was established in [H 2004].

Proposition 3.3.7 (H 2004) *Let \mathcal{C} have pullbacks and let f, g be in \mathcal{C} . Then*

- (1) $f \perp g \Rightarrow f \parallel^H g$
- (2) $f \perp g \Leftrightarrow f \parallel^H g$ provided $f \in \text{RegEpi}(\mathcal{C})$.

Combining Theorem 3.3.5 and Proposition 3.3.7 gives immediately the following useful link between the \parallel -relation and the \perp -relation.

Corollary 3.3.8 *Let \mathcal{C} have products of pairs, pullbacks and coequalisers of kernelpairs. Let f, g be in \mathcal{C} . Then $f \parallel g$ iff $k \perp g$ for all regular epimorphic first factors k of f .*

3.4 $\|\text{-}$ pairs and factorisation systems

Given a $\|\text{-}$ pair $(\mathcal{F}, \mathcal{G})$ it is sometimes the case that we obtain a factorisation system by restricting \mathcal{F} to its regular epimorphic members. In this section we discuss this phenomenon and related matters. We recall that, for a class of morphisms \mathcal{F} in \mathcal{C} , \mathcal{F}^* denotes the class of all regular epimorphic members of \mathcal{F} .

The following was established in [H 2004].

Theorem 3.4.1 (H 2004) *Suppose \mathcal{C} has pullbacks. Let $(\mathcal{E}, \mathcal{M})$ be a regular prefactorisation system on \mathcal{C} . Then there is a $\|\text{-}$ pair $(\mathcal{F}, \mathcal{G})$ on \mathcal{C} such that $\mathcal{E} = \mathcal{F}^*$ and $\mathcal{M} = \mathcal{G}$. Suppose \mathcal{C} has pullbacks and (regular epi, mono)-factorisations. Let $(\mathcal{F}, \mathcal{G})$ be a $\|\text{-}$ pair on \mathcal{C} . Then $(\mathcal{F}^*, \mathcal{G})$ is a regular prefactorisation system on \mathcal{C} provided \mathcal{F} has the property that, for all f in \mathcal{C} , $f \in \mathcal{F} \Rightarrow \bar{f} \in \mathcal{F}$ (where $f = m\bar{f}$ is a (regular epi, mono)-factorisation).*

Combining this with Theorem 2.1.7 yields the following.

Corollary 3.4.2 *Suppose \mathcal{C} has pullbacks, pushouts, cointersections of regular epimorphisms and (regular epi, mono)-factorisations. Let $(\mathcal{F}, \mathcal{G})$ be a $\|\text{-}$ pair on \mathcal{C} . Then $(\mathcal{F}^*, \mathcal{G})$ is a regular factorisation system on \mathcal{C} provided \mathcal{F} has the property that, for all f in \mathcal{C} , $f \in \mathcal{F} \Rightarrow \bar{f} \in \mathcal{F}$.*

Theorem 3.4.1 enables us, in suitable categories, to reduce the problem of finding $\|\text{-}$ pairs which are not $\|\text{-}$ pairs to the problem of finding prefactorisation systems of a particular kind.

Corollary 3.4.3 *Suppose \mathcal{C} has pullbacks and products of pairs. Let $(\mathcal{E}, \mathcal{M})$ be a regular prefactorisation system on \mathcal{C} for which there exist $e \in \mathcal{E}$ and k in \mathcal{C} such that k is a regular epimorphic first factor of e and $k \notin \mathcal{E}$. Then there is a $\|\text{-}$ pair $(\mathcal{F}, \mathcal{G})$ on \mathcal{C} such that $\mathcal{E} = \mathcal{F}^*$ and $\mathcal{M} = \mathcal{G}$ and such that $(\mathcal{F}, \mathcal{G})$ is not a $\|\text{-}$ pair.*

Proof. This is a straightforward consequence of Corollary 3.3.6 and Theorem 3.4.1.

The following notation will be extremely useful throughout the rest of this dissertation.

Definition 3.4.4 Let \mathcal{H} be a class of morphisms in \mathcal{C} . We define $\overline{\mathcal{H}}$ to be the full subcategory of \mathcal{C} whose objects are those $X \in \mathcal{C}$ such that, for all f in \mathcal{C} , $\text{dom}(f) = X \Rightarrow f \in \mathcal{H}$.

We extend the operators L and R as follows.

Definition 3.4.5 Let \mathcal{K} be a full subcategory of \mathcal{C} . Then $L(\mathcal{K}) = L(\{g \text{ in } \mathcal{C} \mid \text{dom}(g) \in \mathcal{K}\})$ and $R(\mathcal{K}) = R(\{f \text{ in } \mathcal{C} \mid \text{dom}(f) \in \mathcal{K}\})$.

There is a slight simplification of Definition 3.4.5 in the presence of a terminal object.

Proposition 3.4.6 Suppose \mathcal{C} has a terminal object. Let \mathcal{K} be a full subcategory of \mathcal{C} and let h be in \mathcal{C} . Then $h \in L(\mathcal{K})$ iff $h \parallel !_Y$ for all $Y \in \mathcal{K}$. Also, $h \in R(\mathcal{K})$ iff $!_X \parallel h$ for all $X \in \mathcal{K}$.

Proof. These are easy consequences of 3.2.10(3) and 3.2.10(4).

We record the following easy consequences of Proposition 3.2.5.

Proposition 3.4.7 Let $(\mathcal{F}, \mathcal{G})$ be a \parallel -pair on \mathcal{C} . In the following, (1) \Leftrightarrow (2) and (3) \Leftrightarrow (4):

- (1) there exists a full subcategory $\mathcal{K} \subseteq \mathcal{C}$ such that $\mathcal{F} = L(\mathcal{K})$
- (2) $\mathcal{F} = L(\overline{\mathcal{G}})$
- (3) there exists a full subcategory $\mathcal{K} \subseteq \mathcal{C}$ such that $\mathcal{G} = R(\mathcal{K})$
- (4) $\mathcal{G} = R(\overline{\mathcal{F}})$.

For any full subcategory $\mathcal{K} \subseteq \mathcal{C}$, we can obviously define $L^H(\mathcal{K})$ and $R^H(\mathcal{K})$ analogously to $L(\mathcal{K})$ and $R(\mathcal{K})$. We then obtain the following.

Proposition 3.4.8 Let \mathcal{K} be a full subcategory of \mathcal{C} . Suppose \mathcal{C} has a terminal object. Then $L(\mathcal{K}) = L^H(\mathcal{K})$. Suppose \mathcal{C} has products of pairs. Then $R(\mathcal{K}) = R^H(\mathcal{K})$.

Proof. Suppose \mathcal{C} has a terminal object. By Proposition 3.3.3, $L(\mathcal{K}) \subseteq L^H(\mathcal{K})$. Let $f : X \rightarrow A$ be in $L^H(\mathcal{K})$ and let $w : X \rightarrow Y$ in \mathcal{C} be such that $Y \in \mathcal{K}$. We obtain the following commutative square

$$\begin{array}{ccc} X & \xrightarrow{w} & Y \\ f \downarrow & & \downarrow !_Y \\ A & \xrightarrow{!_A} & 1 \end{array}$$

and it is then clear from Definition 3.3.2 that, for any $u, v : Z \rightarrow X$ in \mathcal{C} , $fu = fv \Rightarrow wu = wv$. It follows that $f \parallel !_Y$. By Proposition 3.4.6, $f \in L(\mathcal{K})$. Therefore $L^H(\mathcal{K}) \subseteq L(\mathcal{K})$.

Now suppose \mathcal{C} has products of pairs. For all f in \mathcal{C} , $\text{dom}(f) \in \mathcal{K}$ iff $\text{dom}(k) \in \mathcal{K}$ for all first factors k of f . It therefore follows from Theorem 3.3.4 that $R(\mathcal{K}) = R^H(\mathcal{K})$.

We are now ready to establish sufficient conditions for obtaining a factorisation system from a \parallel -pair.

Proposition 3.4.9 *Suppose \mathcal{C} has products of pairs and a terminal object. Let $(\mathcal{F}, \mathcal{G})$ be a \parallel -pair on \mathcal{C} such that $\mathcal{F} = L(\overline{\mathcal{G}})$. Then $(\mathcal{F}, \mathcal{G})$ is a \parallel^H -pair.*

Proof. By Proposition 3.4.8, $\mathcal{F} = L^H(\overline{\mathcal{G}})$. Since any first factor of a member of \mathcal{F} is itself a member of \mathcal{F} , it follows, by Theorem 3.3.4, that $R^H(\mathcal{F}) = R(\mathcal{F}) = \mathcal{G}$. It follows that $(\mathcal{F}, \mathcal{G})$ is a \parallel^H -pair.

Combining Proposition 3.4.9 with Corollary 3.4.2 gives the following.

Theorem 3.4.10 *Suppose \mathcal{C} has products of pairs, pullbacks, (regular epi, mono)-factorisations, pushouts, cointersections of regular epimorphisms and a terminal object. Let $(\mathcal{F}, \mathcal{G})$ be a \parallel -pair on \mathcal{C} such that $\mathcal{F} = L(\overline{\mathcal{G}})$. Then $(\mathcal{F}^*, \mathcal{G})$ is a factorisation system on \mathcal{C} .*

Proof. Let f be in \mathcal{C} . By Proposition 3.2.10(3), $f \in \mathcal{F} \Rightarrow \overline{f} \in \mathcal{F}$.

3.5 More on the \parallel^H -relation

We record here a characterisation of the \parallel^H -relation in the spirit of Propositions 3.1.2 and 3.2.9. It will be useful in Chapter 7.

Proposition 3.5.1 *Let \mathcal{C} have pullbacks and be equipped with a proper and stable factorisation system $(\mathcal{E}, \mathcal{M})$. Suppose $gw = hf$ is a commutative*

square in \mathcal{C} and form the diagram

$$\begin{array}{ccccc}
 X \times_A X & \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{array} & X & \xrightarrow{w} & Y \\
 & & \searrow e & & \nearrow \hat{h} \\
 & & M & \xrightarrow{m} & A \times_B Y \\
 & & & & \nearrow \hat{g} \\
 & & & & A & \xrightarrow{h} & B \\
 & & \downarrow f & & \downarrow g & & \\
 & & A & & B & &
 \end{array}$$

where $\hat{g}\hat{h} = h\hat{g}$ is a pullback square and m is an $(\mathcal{E}, \mathcal{M})$ -factorisation of the unique morphism such that $f = \hat{g}me$ and $w = \hat{h}me$. Then $wf_1 = wf_2$ iff $\hat{g}m$ is a monomorphism.

Proof. (\Leftarrow) If $\hat{g}m$ is a monomorphism then $ef_1 = ef_2$ and so $wf_1 = wf_2$.

(\Rightarrow) Suppose $wf_1 = wf_2$. Let e_A^2 be the unique morphism such that $(\hat{g}m)_1 e_A^2 = ef_1$ and $(\hat{g}m)_2 e_A^2 = ef_2$. By Lemma 3.2.8 and the fact that \mathcal{E} is pullback stable and closed under composition, we have that $e_A^2 \in \mathcal{E}$. In particular, e_A^2 is an epimorphism. Now $\hat{h}m(\hat{g}m)_1 e_A^2 = \hat{h}me f_1 = wf_1 = wf_2 = \hat{h}me f_2 = \hat{h}m(\hat{g}m)_2 e_A^2$. Since e_A^2 is an epimorphism, it follows that $\hat{h}m(\hat{g}m)_1 = \hat{h}m(\hat{g}m)_2$. Since (\hat{g}, \hat{h}) is a monosource and m is a monomorphism, it follows that $(\hat{g}m)_1 = (\hat{g}m)_2$ and therefore that $\hat{g}m$ is a monomorphism.

From Proposition 3.5.1 and Definition 3.3.2 we obtain the following.

Corollary 3.5.2 *Let \mathcal{C} have pullbacks and be equipped with a proper and stable factorisation system $(\mathcal{E}, \mathcal{M})$. Then $f \parallel^H g$ iff, for all diagrams as in Proposition 3.5.1, $\hat{g}m$ is a monomorphism.*

4 \parallel -pairs on nice categories

\parallel -pairs on **Top** have a neat and useful characterisation in terms of families of reflexive symmetric relations. Let $(\mathcal{F}, \mathcal{G})$ be a \parallel -pair on **Top**. For each space X we can define a relation \sim_X on X as follows: for all $x, x' \in X$, $x \sim_X x'$ iff there exists $f : X \rightarrow A$ in \mathcal{F} such that $f(x) = f(x')$. This defines a family $\{\sim_X \mid X \in \mathbf{Top}\}$ of reflexive symmetric relations with the property that, for any continuous map $w : X \rightarrow Y$ and all $x, x' \in X$, $x \sim_X x' \Rightarrow w(x) \sim_Y w(x')$. It turns out that such families of relations are in one to one correspondence with the \parallel -pairs on **Top**. It also turns out that this fact can be proved in much greater generality provided we first generalise the notion of a “family of reflexive symmetric relations preserved by continuous maps”. We give such a generalisation in Definition 4.1.10, making use of Definition 4.1.8. We use the name “family of canonical relations” which recalls the inspiration for our definition. In Section 5.1 we shall show that, when restricted to **Top** (or, in fact, any topological category), the generalisation is indeed equivalent to the original idea. In Section 4.2 we prove a number of results made possible by our generalisation. In Section 4.3 we use our theory to show how, under certain conditions, a \parallel -pair may be obtained from a full subcategory.

Throughout this chapter, \mathcal{C} is a finitely complete category which has coequalisers of kernelpairs and is equipped with a proper and stable factorisation system $(\mathcal{E}, \mathcal{M})$. Clementino and Tholen made the same basic assumptions in [CT 1998] where they noted that \mathcal{C} consequently has (regular epi, mono)-factorisations.

4.1 Preliminary definitions

We recall that the constant morphisms in \mathcal{C} are characterised, by Proposition 3.1.2, as being those f in \mathcal{C} for which the middle object of the $(\mathcal{E}, \mathcal{M})$ -factorisation of f is preterminal. We recall too that f is constant iff mfe is constant, for all $f \in \mathcal{C}$, $e \in \mathcal{E}$ and $m \in \mathcal{M}$ such that the composition is defined. The subobjects to which we shall refer are defined with respect to $(\mathcal{E}, \mathcal{M})$. That is to say, for each $X \in \mathcal{C}$, they are members of the preordered class \mathcal{M}/X (see Section 2.4).

The following definition was given in [T 1984] with the difference that Tholen did not require m to be a subobject. In our setting, where every morphism has an $(\mathcal{E}, \mathcal{M})$ -factorisation, it is clearly equivalent to consider

only $m \in \mathcal{M}/X$ and it suits our exposition to do so. (Note also that Tholen was using a different definition of “constant” which was given in terms of the class \mathcal{K} - see Section 3.1.)

Definition 4.1.1 (T 1984) A class \mathcal{K} of morphisms in \mathcal{C} is said to be fibre determined iff, for any $f \in \mathcal{K}$ and h in \mathcal{C} with common domain X , the following two conditions are equivalent:

- (1) for all $m \in \mathcal{M}/X$, fm constant $\Rightarrow hm$ constant
- (2) there exists p in \mathcal{C} such that $pf = h$.

If $\text{RegEpi}(\mathcal{C})$ is fibre determined, then the \parallel -relation has a very useful description in terms of subobjects.

Definition 4.1.2 Let $f : X \rightarrow A$ and $g : Y \rightarrow B$ be in \mathcal{C} . We say $f \parallel^C g$ iff, for all $w : X \rightarrow Y$ in \mathcal{C} and all $m \in \mathcal{M}/X$,

$$\begin{array}{ccccc} M & \xrightarrow{m} & X & \xrightarrow{w} & Y \\ & & \downarrow f & & \downarrow g \\ & & A & & B \end{array}$$

fm and gwm constant $\Rightarrow wm$ constant.

Proposition 4.1.3 Let $\text{RegEpi}(\mathcal{C})$ be fibre determined and let $f : X \rightarrow A$ and $g : Y \rightarrow B$ be in \mathcal{C} . Then $f \parallel g$ iff $f \parallel^C g$.

Proof. (\Rightarrow) Suppose $f \parallel g$. Let $w : X \rightarrow Y$ in \mathcal{C} and $m : M \rightarrow X$ in \mathcal{M}/X be such that fm and gwm are constant. Let $u, v : Z \rightarrow M$ be in \mathcal{C} . Since $fm u = fm v$ and $gwm u = gwm v$, we must have that $wmu = wmv$. This establishes that wm is constant. So $f \parallel^C g$.

(\Leftarrow) Suppose $f \parallel^C g$. Let $k : X \rightarrow C$ be a regular epimorphic first factor of f . It is clear from Definition 4.1.2 that $k \parallel^C g$. Let w, h, u, v in \mathcal{C} be such that $gw = hk$ and $ku = kv$.

$$\begin{array}{ccccc} Z & \xrightarrow[u]{v} & X & \xrightarrow{w} & Y \\ & & \downarrow k & & \downarrow g \\ & & C & \xrightarrow{h} & B \end{array}$$

It follows that, for any $m \in \mathcal{M}/X$, km constant $\Rightarrow wm$ constant. Since $\text{RegEpi}(\mathcal{C})$ is fibre determined, it follows that k is a first factor of w . So

$wu = wv$. Therefore $k \parallel^H g$. By Theorem 3.3.5, it follows that $f \parallel g$.

One of the reasons for the good behaviour of \parallel -pairs on **Top** is the abundance of quotient maps. We express categorically the aspect of this property that is of use to us.

Definition 4.1.4 \mathcal{C} is said to have enough quotients iff, for all $X \in \mathcal{C}$ and all $m \in \mathcal{M}/X$, there exists f in \mathcal{C} such that fm is constant and, for all non-constant $p \in \mathcal{M}/X$, fp constant $\Rightarrow p \leq m$.

It will be useful to have available the following terminology.

Notation 4.1.5 Let $f : X \rightarrow Y$ be in \mathcal{C} and let $m \in \mathcal{M}/X$. We say f is constant on m iff fm is constant. We say f is constant only on m iff f is as in Definition 4.1.4. We say f is monic on m iff fm is a monomorphism.

There is a characterisation of “having enough quotients” which reveals something of a duality between this notion and that of $\text{RegEpi}(\mathcal{C})$ being fibre determined.

Proposition 4.1.6 \mathcal{C} has enough quotients iff, for all $X \in \mathcal{C}$ and all $m, p \in \mathcal{M}/X$, the following two conditions are equivalent:

- (1) for all $f : X \rightarrow A$ in \mathcal{C} , fm constant $\Rightarrow fp$ constant
- (2) $p \leq m$ or p is constant.

Proof. (\Rightarrow) Suppose \mathcal{C} has enough quotients. Let f be constant only on m and suppose (1). It follows that fp is constant. By Definition 4.1.4, this implies (2). Trivially (2) \Rightarrow (1).

(\Leftarrow) Suppose (1) and (2) are equivalent for every $X \in \mathcal{C}$ and every $m, p \in \mathcal{M}/X$. For an arbitrary subobject $m : M \rightarrow X$, let f' be the coequaliser of $(m\pi_1, m\pi_2)$, where π_1 and π_2 are the product projections of $M \times M$. Suppose $p \in \mathcal{M}/X$ is such that $f'p$ is constant and p is not constant. Since any morphism which is constant on m must have f' as a first factor, it follows that condition (1) is satisfied. Therefore condition (2) is satisfied too. So $p \leq m$. Therefore \mathcal{C} has enough quotients.

Another reason for the good behaviour of \parallel -pairs on **Top** is the abundance of subspaces. We express categorically the aspect of this property that is of use to us.

Definition 4.1.7 \mathcal{C} is said to have enough subobjects iff, for all non-constant $w : X \rightarrow Y$ in \mathcal{C} , there is a non-constant $m \in \mathcal{M}/X$ such that w is monic on m .

The following definition will enable us to define the notion of a family of canonical relations in sufficient generality to be of use in a wide variety of categories. (When we say \mathcal{P} is hereditary, we mean that $m \in \mathcal{P}$ whenever $p \in \mathcal{P}$ and $m \leq p$.)

Definition 4.1.8 Let $X \in \mathcal{C}$ and let $\mathcal{P} \subseteq \mathcal{M}/X$ be hereditary. We define $c_X(\mathcal{P})$ to be the preordered class of all $m \in \mathcal{M}/X$ such that, for all $p \in \mathcal{P}$, $p \leq m \Rightarrow p$ constant.

We record the following useful basic properties.

Proposition 4.1.9 Let $X \in \mathcal{C}$ and let $\mathcal{P}, \mathcal{N} \subseteq \mathcal{M}/X$ be hereditary. Then

- (1) $c_X(\mathcal{P})$ is hereditary
- (2) $\mathcal{P} \subseteq c_X(c_X(\mathcal{P}))$
- (3) $\mathcal{P} \subseteq \mathcal{N} \Rightarrow c_X(\mathcal{N}) \subseteq c_X(\mathcal{P})$
- (4) $c_X(\mathcal{P}) = c_X(c_X(c_X(\mathcal{P})))$.

Proof. (1) is a trivial consequence of the transitivity of \leq . Let $p \in \mathcal{P}$ and let $m \in c_X(\mathcal{P})$ be such that $m \leq p$. It follows that $m \in \mathcal{P}$. Since $m \leq m$, we must then have that m is constant. It follows that $p \in c_X(c_X(\mathcal{P}))$. This proves (2). Suppose $\mathcal{P} \subseteq \mathcal{N}$ and let $m \in c_X(\mathcal{N})$. Let $p \in \mathcal{P}$ be such that $p \leq m$. Since $p \in \mathcal{N}$, it follows that p is constant. Therefore $m \in c_X(\mathcal{P})$. This proves (3). (4) is a direct consequence of (1), (2) and (3).

We are now in a position to define the central notion of this chapter.

Definition 4.1.10 For each $X \in \mathcal{C}$, let $S_X \subseteq \mathcal{M}/X$ be hereditary and such that $S_X = c_X(c_X(S_X))$. $\mathcal{S} = \{S_X | X \in \mathcal{C}\}$ is said to be a family of canonical relations on \mathcal{C} iff, for all $w : X \rightarrow Y$ in \mathcal{C} and $m \in \mathcal{M}/X$, $m \in S_X \Rightarrow w[m] \in S_Y$.

When showing that $\|$ -pairs can be obtained from families of canonical relations, we shall make use of the following notation.

Definition 4.1.11 Let $\mathcal{S} = \{S_X | X \in \mathcal{C}\}$ be a family of canonical relations on \mathcal{C} . We define $\mathcal{F}^{\mathcal{S}}$ to be the class of all $f : X \rightarrow A$ in \mathcal{C} such that, for all $m \in \mathcal{M}/X$, fm constant $\Rightarrow m \in S_X$. We define $\mathcal{G}^{\mathcal{S}}$ to be the class of all $g : Y \rightarrow B$ in \mathcal{C} such that, for all $n \in \mathcal{M}/Y$, gn constant $\Rightarrow n \in c_Y(S_Y)$.

There is an interesting characterisation of the class $\mathcal{G}^{\mathcal{S}}$ in the case where $\text{RegEpi}(\mathcal{C})$ is fibre determined.

Proposition 4.1.12 Let $\mathcal{S} = \{S_X | X \in \mathcal{C}\}$ be a family of canonical relations on \mathcal{C} . Let $g : Y \rightarrow B$ be in \mathcal{C} and consider the following:

(1) $g \in \mathcal{G}^{\mathcal{S}}$

(2) for all non-constant $n \in \mathcal{M}/Y$, gn constant $\Rightarrow n \notin S_Y$

(3) for all $n \in S_Y$, g is monic on n .

In general, (1) \Leftrightarrow (2) \Leftarrow (3). If $\text{RegEpi}(\mathcal{C})$ is fibre determined, then also (2) \Rightarrow (3).

Proof. (1) \Rightarrow (2) is easy. Suppose (2) and let $q \in \mathcal{M}/Y$ be such that gq is constant. Then there is no non-constant $n \in S_Y$ such that $n \leq q$. It follows that $q \in c_Y(S_Y)$. This proves (2) \Rightarrow (1).

Suppose (3) and let $n : N \rightarrow Y$ in S_Y be such that gn is constant. It follows that gn is a constant monomorphism. Therefore N is preterminal and so n is constant. This proves (3) \Rightarrow (2).

Let $\text{RegEpi}(\mathcal{C})$ be fibre determined and suppose (2). Let $n : N \rightarrow Y$ be in S_Y and suppose g is not monic on n . It follows that N is not preterminal and therefore that n is not constant. It also follows that the coequaliser of the kernelpair of gn is not a first factor of 1_N . Since $\text{RegEpi}(\mathcal{C})$ is fibre determined, there then exists a non-constant $p \in \mathcal{M}/N$ such that gnp is constant. Since S_Y is hereditary, we have $np \in S_Y$. Since np is non-constant, this contradicts (2). We conclude that g is monic on n . This establishes (3).

A similar result is true for $\mathcal{F}^{\mathcal{S}}$.

Proposition 4.1.13 Let $\mathcal{S} = \{S_X | X \in \mathcal{C}\}$ be a family of canonical relations on \mathcal{C} . Let $f : X \rightarrow A$ be in \mathcal{C} and consider the following:

(1) $f \in \mathcal{F}^{\mathcal{S}}$

(2) for all non-constant $m \in \mathcal{M}/X$, fm constant $\Rightarrow m \notin c_X(S_X)$

(3) for all $m \in c_X(S_X)$, f is monic on m .

In general, (1) \Leftrightarrow (2) \Leftarrow (3). If $\text{RegEpi}(\mathcal{C})$ is fibre determined, then also (2) \Rightarrow (3).

Proof. Given that $S_X = c_X(c_X(S_X))$, the reasoning is similar to that used in the proof of Proposition 4.1.12.

We conclude this section by establishing the notation which we shall use to obtain families of canonical relations from $\|\$ -pairs.

Definition 4.1.14 *Let \mathcal{F} be a class of morphisms in \mathcal{C} . For each $X \in \mathcal{C}$, we define $S_X^{\mathcal{F}} \subseteq \mathcal{M}/X$ to be the preordered class of all $m \in \mathcal{M}/X$ for which there exists $f \in \mathcal{F}$ such that fm is constant. We define $\mathcal{S}^{\mathcal{F}} = \{S_X^{\mathcal{F}} | X \in \mathcal{C}\}$.*

4.2 Characterising the $\|\$ -pairs on nice categories

In sufficiently nice categories, families of canonical relations give rise to $\|\$ -pairs.

Theorem 4.2.1 *Suppose $\text{RegEpi}(\mathcal{C})$ is fibre determined and \mathcal{C} has enough quotients. Let $\mathcal{S} = \{S_X | X \in \mathcal{C}\}$ be a family of canonical relations on \mathcal{C} . Then $(\mathcal{F}^{\mathcal{S}}, \mathcal{G}^{\mathcal{S}})$ is a $\|\$ -pair on \mathcal{C} .*

Proof. For any $f \in \mathcal{F}^{\mathcal{S}}$ and $g \in \mathcal{G}^{\mathcal{S}}$, it is easy to see that $f \|\mathcal{C} g$. By Proposition 4.1.3, it follows that $\mathcal{F}^{\mathcal{S}} \subseteq L(\mathcal{G}^{\mathcal{S}})$ and $\mathcal{G}^{\mathcal{S}} \subseteq R(\mathcal{F}^{\mathcal{S}})$.

Suppose $k : X \rightarrow A$ in \mathcal{C} is not in $\mathcal{F}^{\mathcal{S}}$. Then, by Proposition 4.1.13, there is a non-constant $m \in c_X(S_X)$ such that km is constant. Since \mathcal{C} has enough quotients, there exists g in \mathcal{C} such that g is constant only on m . Since $c_X(S_X)$ is hereditary and contains all constant members of \mathcal{M}/X , it follows that $g \in \mathcal{G}^{\mathcal{S}}$. The existence of such a non-constant m then ensures that $k \not\|\mathcal{C} g$. By Proposition 4.1.3, it follows that $k \notin L(\mathcal{G}^{\mathcal{S}})$. So $L(\mathcal{G}^{\mathcal{S}}) \subseteq \mathcal{F}^{\mathcal{S}}$.

Now suppose $h : Y \rightarrow B$ in \mathcal{C} is not in $\mathcal{G}^{\mathcal{S}}$. Then, by Proposition 4.1.12, there exists a non-constant $n \in S_Y$ such that hn is constant. Since \mathcal{C} has enough quotients, there exists f in \mathcal{C} such that f is constant only on n . Since $S_Y = c_Y(c_Y(S_Y))$ contains all constant members of \mathcal{M}/Y and is hereditary, it follows that $f \in \mathcal{F}^{\mathcal{S}}$. The existence of such a non-constant n then ensures that $f \not\|\mathcal{C} h$. By Proposition 4.1.3, it follows that $h \notin R(\mathcal{F}^{\mathcal{S}})$. So $R(\mathcal{F}^{\mathcal{S}}) \subseteq \mathcal{G}^{\mathcal{S}}$.

In even nicer categories, $\|\$ -pairs give rise to families of canonical relations.

Theorem 4.2.2 *Suppose $\text{RegEpi}(\mathcal{C})$ is fibre determined and \mathcal{C} has enough quotients and enough subobjects. Let $(\mathcal{F}, \mathcal{G})$ be a $\|\$ -pair on \mathcal{C} . Then $\mathcal{S}^{\mathcal{F}}$ is a family of canonical relations on \mathcal{C} .*

Proof. Let $w : X \rightarrow Y$ be in \mathcal{C} and let $m \in S_X^{\mathcal{F}}$. Let $e \in \mathcal{E}$ be such that $w m = w[m] \cdot e$. By Definition 4.1.14, there exists $f : X \rightarrow A$ in \mathcal{F} such that $f m$ is constant. Since \mathcal{C} has enough quotients, there exists $k : Y \rightarrow B$ in \mathcal{C} such that k is constant only on $w[m]$. Continue the construction of the following diagram

$$\begin{array}{ccccc}
 M \times_{w[M]} N & \xrightarrow{\hat{e}} & N & & \\
 \downarrow \hat{q} & & \swarrow q & & \\
 M & \xrightarrow{e} & w[M] & & \\
 \downarrow m & & \downarrow w[m] & & \\
 X & \xrightarrow{w} & Y & \xrightarrow{v} & Z \\
 \downarrow f & & \downarrow k & & \downarrow g \\
 A & & B & & C
 \end{array}$$

by assuming the existence of $g \in \mathcal{G}$ and v in \mathcal{C} as shown and of a non-constant $n \in \mathcal{M}/Y$ such that kn and gvn are constant. Since k is constant only on $w[m]$, this ensures the existence of q in \mathcal{C} such that $n = w[m] \cdot q$. Pulling back q along e then completes the construction. By Proposition 4.1.3, $k \in \mathcal{F}$ provided we can show that vn must be constant. Since $f m \hat{q}$ is constant and $g v m \hat{q} = g v n \hat{e}$ is constant (and $f \in \mathcal{F}$ and $g \in \mathcal{G}$), we have, by Proposition 4.1.3, that $v m \hat{q} = v n \hat{e}$ is constant. Since \mathcal{E} is pullback stable, we have $\hat{e} \in \mathcal{E}$. Therefore vn is constant. So $k \in \mathcal{F}$ and k is constant on $w[m]$. It follows that $w[m] \in S_Y^{\mathcal{F}}$.

From Definition 4.1.14 it is clear that $S_X^{\mathcal{F}}$ is hereditary. It remains only to prove that $S_X^{\mathcal{F}} = c_X(c_X(S_X^{\mathcal{F}}))$. By Proposition 4.1.9, $S_X^{\mathcal{F}} \subseteq c_X(c_X(S_X^{\mathcal{F}}))$. Suppose $m \in c_X(c_X(S_X^{\mathcal{F}}))$. Let $f : X \rightarrow A$ in \mathcal{C} be constant only on m and let $p \in \mathcal{M}/X$, $g \in \mathcal{G}$ and w in \mathcal{C} be such that $f p$ and $g w p$ are constant. Suppose $w p$ is not constant. It follows that p is not constant and then that $p \leq m$.

$$\begin{array}{ccccccc}
 R & \longrightarrow & N & \xrightarrow{n} & P & \longrightarrow & M \\
 & & & & \downarrow p & & \downarrow m \\
 & & & & X & \xrightarrow{w} & Y \\
 & & & \swarrow r & \downarrow f & & \downarrow g \\
 & & & & A & & B
 \end{array}$$

Since \mathcal{C} has enough subobjects, there exists a non-constant $n \in \mathcal{M}/P$ such that wp is monic on n . Suppose $r \in \mathcal{M}/X$ is non-constant and such that $r \leq pn$. If there were to exist $k \in \mathcal{F}$ such that kr is constant, then we would have a contradiction, by Proposition 4.1.3, since gwr is constant but wr is not (wr is non-constant because it is a monomorphism and its domain is not preterminal). It follows that $r \notin S_X^{\mathcal{F}}$. This establishes that $pn \in c_X(S_X^{\mathcal{F}})$. Since pn is non-constant and $pn \leq m$, this contradicts the fact that $m \in c_X(c_X(S_X^{\mathcal{F}}))$. We conclude that wp is constant. Therefore, by Proposition 4.1.3, $f \in \mathcal{F}$. By Definition 4.1.14, it follows that $m \in S_X^{\mathcal{F}}$. So $c_X(c_X(S_X^{\mathcal{F}})) \subseteq S_X^{\mathcal{F}}$.

The obvious question which now arises is answered by the following theorem.

Theorem 4.2.3 *Suppose $\text{RegEpi}(\mathcal{C})$ is fibre determined and \mathcal{C} has enough quotients and enough subobjects. Then the mappings $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{S}^{\mathcal{F}}$ and $\mathcal{S} \mapsto (\mathcal{F}^{\mathcal{S}}, \mathcal{G}^{\mathcal{S}})$ are inverse to each other and establish a bijection between the collection of all $\|\text{-}$ pairs on \mathcal{C} and the collection of all families of canonical relations on \mathcal{C} .*

Proof. Let $(\mathcal{F}, \mathcal{G})$ be a $\|\text{-}$ pair on \mathcal{C} and consider $(\mathcal{F}^{\mathcal{S}^{\mathcal{F}}}, \mathcal{G}^{\mathcal{S}^{\mathcal{F}}})$. It is clear that $\mathcal{F} \subseteq \mathcal{F}^{\mathcal{S}^{\mathcal{F}}}$. Let $g : Y \rightarrow B$ be in \mathcal{G} . Using Proposition 4.1.3, it is easy to see that gn is non-constant for any non-constant $n \in S_Y^{\mathcal{F}}$. By Proposition 4.1.12, it follows that $g \in \mathcal{G}^{\mathcal{S}^{\mathcal{F}}}$. So $\mathcal{G} \subseteq \mathcal{G}^{\mathcal{S}^{\mathcal{F}}}$. By Theorems 4.2.1 and 4.2.2, $(\mathcal{F}^{\mathcal{S}^{\mathcal{F}}}, \mathcal{G}^{\mathcal{S}^{\mathcal{F}}})$ is a $\|\text{-}$ pair. By Proposition 3.2.7(3), it follows that $(\mathcal{F}, \mathcal{G}) = (\mathcal{F}^{\mathcal{S}^{\mathcal{F}}}, \mathcal{G}^{\mathcal{S}^{\mathcal{F}}})$.

Let $\mathcal{S} = \{S_X | X \in \mathcal{C}\}$ be a family of canonical relations on \mathcal{C} and consider $\mathcal{S}^{\mathcal{F}^{\mathcal{S}}}$. Let $X \in \mathcal{C}$. It is clear that $S_X^{\mathcal{F}^{\mathcal{S}}} \subseteq S_X$. Let $m \in S_X$. Since \mathcal{C} has enough quotients, there exists f in \mathcal{C} such that f is constant only on m . Since S_X is hereditary and contains all constant members of \mathcal{M}/X , it follows that $f \in \mathcal{F}^{\mathcal{S}}$ and therefore that $m \in S_X^{\mathcal{F}^{\mathcal{S}}}$. So $S_X \subseteq S_X^{\mathcal{F}^{\mathcal{S}}}$. Therefore $\mathcal{S} = \mathcal{S}^{\mathcal{F}^{\mathcal{S}}}$.

We are now able, under the conditions of Theorem 4.2.3, to characterise those classes \mathcal{F} for which there exists \mathcal{G} such that $(\mathcal{F}, \mathcal{G})$ is a $\|\text{-}$ pair.

Corollary 4.2.4 *Suppose $\text{RegEpi}(\mathcal{C})$ is fibre determined and \mathcal{C} has enough quotients and enough subobjects. Let \mathcal{F} be a class of morphisms in \mathcal{C} . Then \mathcal{F} is the left class of a $\|\text{-}$ pair iff the following conditions are satisfied:*

(1) for any $k : X \rightarrow A$ in \mathcal{C} , $k \in \mathcal{F}$ if, for all non-constant $m \in \mathcal{M}/X$ such that km is constant, there exist $f \in \mathcal{F}$ and a non-constant $p \in \mathcal{M}/X$ such that $p \leq m$ and fp is constant

(2) for any $w : X \rightarrow Y$ in \mathcal{C} and any $m \in \mathcal{M}/X$, if there exists $f \in \mathcal{F}$ such that fm is constant then there also exists $k \in \mathcal{F}$ such that $k \cdot w[m]$ is constant.

Proof. (\Leftarrow) Suppose (1) and (2). Let $X \in \mathcal{C}$. Using the fact that \mathcal{C} has enough quotients, it follows from (1) that $S_X^{\mathcal{F}} = c_X(c_X(S_X^{\mathcal{F}}))$. Let $w : X \rightarrow Y$ be in \mathcal{C} and let $m \in S_X^{\mathcal{F}}$. It follows from (2) that $w[m] \in S_Y^{\mathcal{F}}$. So $\mathcal{S}^{\mathcal{F}}$ is a family of canonical relations. It is clear from (1) that $\mathcal{F} = \mathcal{F}^{\mathcal{S}^{\mathcal{F}}}$. By Theorem 4.2.1, \mathcal{F} is the left class of a \parallel -pair.

(\Rightarrow) Suppose \mathcal{F} is the left class of a \parallel -pair. By Theorem 4.2.3, $\mathcal{F} = \mathcal{F}^{\mathcal{S}^{\mathcal{F}}}$. It is then easy to see that we must have (1) and (2).

There is a similar characterisation of those classes \mathcal{G} for which there exists \mathcal{F} such that $(\mathcal{F}, \mathcal{G})$ is a \parallel -pair. In the following, let $\mathcal{F}^{\mathcal{S}^{\mathcal{G}}}$ be defined as in Definition 4.1.11 (with $\mathcal{S}^{\mathcal{G}}$ playing the role of \mathcal{S} even though it may not be a family of canonical relations on \mathcal{C}).

Corollary 4.2.5 *Suppose $\text{RegEpi}(\mathcal{C})$ is fibre determined and \mathcal{C} has enough quotients and enough subobjects. Let \mathcal{G} be a class of morphisms in \mathcal{C} . Then \mathcal{G} is the right class of a \parallel -pair iff the following conditions are satisfied:*

(1) for any $h : Y \rightarrow B$ in \mathcal{C} , $h \in \mathcal{G}$ if, for all non-constant $n \in \mathcal{M}/Y$ such that hn is constant, there exist $g \in \mathcal{G}$ and a non-constant $q \in \mathcal{M}/Y$ such that $q \leq n$ and gq is constant

(2) for any $w : X \rightarrow Y$ in \mathcal{C} and any $m \in \mathcal{M}/X$, if $w[m]$ is not constant and there exists $g \in \mathcal{G}$ such that $g \cdot w[m]$ is constant then there also exist $h \in \mathcal{G}$ and a non-constant $p \in \mathcal{M}/X$ such that $p \leq m$ and hp is constant.

Proof. (\Leftarrow) Suppose (1) and (2). Let $c(\mathcal{S}^{\mathcal{G}}) = \{c_X(S_X^{\mathcal{G}}) | X \in \mathcal{C}\}$ (noting that $\mathcal{S}^{\mathcal{G}}$ was defined in Definition 4.1.14). Let $w : X \rightarrow Y$ be in \mathcal{C} and let $m \in c_X(S_X^{\mathcal{G}})$. Suppose $w[m] \notin c_Y(S_Y^{\mathcal{G}})$. Then $w[m]$ is not constant and there is a non-constant $n \in S_Y^{\mathcal{G}}$ such that $n \leq w[m]$. It follows that there exist $k \in \mathcal{G}$, $q \in \mathcal{M}$ and $e \in \mathcal{E}$ such that kn is constant, $n = w[m] \cdot q$ and $wm = w[m] \cdot e$. Form the pullback square $e\hat{q} = q\hat{e}$. The relevant picture is provided by a portion of the first diagram in the proof of Theorem 4.2.2. Since \mathcal{E} is pullback stable, we have $\hat{e} \in \mathcal{E}$. Without loss of generality, we may

assume $n = w[m\hat{q}]$. It then follows from (2) that there exist $h \in \mathcal{G}$ and a non-constant $p \in \mathcal{M}/X$ such that $p \leq mq \leq m$ and hp is constant. It follows that $p \in S_X^{\mathcal{G}}$. This contradicts the fact that $m \in c_X(S_X^{\mathcal{G}})$. We conclude that $w[m] \in c_Y(S_Y^{\mathcal{G}})$.

Since $S_X^{\mathcal{G}}$ is hereditary, we have $c_X(S_X^{\mathcal{G}}) = c_X(c_X(c_X(S_X^{\mathcal{G}})))$ (Proposition 4.1.9(4)). So $c(\mathcal{S}^{\mathcal{G}})$ is a family of canonical relations on \mathcal{C} . It is clear from (1) that $\mathcal{G} = \mathcal{F}^{\mathcal{S}^{\mathcal{G}}}$. Using (1) and the fact that \mathcal{C} has enough quotients, it is easy to see that $S_Y^{\mathcal{G}} = c_Y(c_Y(S_Y^{\mathcal{G}}))$ for all $Y \in \mathcal{C}$. Therefore $\mathcal{G} = \mathcal{F}^{\mathcal{S}^{\mathcal{G}}} = \mathcal{G}^{c(\mathcal{S}^{\mathcal{G}})}$. It follows, by Theorem 4.2.1, that \mathcal{G} is the right class of a $\|\text{-}$ pair.

(\Rightarrow) Suppose \mathcal{G} is the right class of a $\|\text{-}$ pair. By Theorem 4.2.3, there exists a family of canonical relations \mathcal{S} such that $\mathcal{G} = \mathcal{G}^{\mathcal{S}}$. It is then easy to see that we must have (1) and (2).

With reference to 4.2.4(1) we introduce the following notation.

Definition 4.2.6 Let \mathcal{F} be a class of morphisms in \mathcal{C} . We define $\hat{\mathcal{F}}$ to consist of all $k : X \rightarrow A$ in \mathcal{C} such that, for all non-constant $m \in \mathcal{M}/X$ such that km is constant, there exist $f \in \mathcal{F}$ and a non-constant $p \in \mathcal{M}/X$ such that $p \leq m$ and fp is constant.

It is clear that any class of the form $\hat{\mathcal{F}}$ will satisfy 4.2.4(1) (with $\hat{\mathcal{F}}$ replacing \mathcal{F}) and that $\hat{\mathcal{F}}$ is the smallest such class which contains \mathcal{F} . It turns out, under the conditions of Corollary 4.2.4, that if \mathcal{F} satisfies 4.2.4(2) then so does $\hat{\mathcal{F}}$. We shall make use of the following lemma.

Lemma 4.2.7 Let \mathcal{C} have enough subobjects. For each $X \in \mathcal{C}$, let $S_X \subseteq \mathcal{M}/X$ be hereditary. Suppose $w[m] \in S_Y$ for all $w : X \rightarrow Y$ in \mathcal{C} and all $m \in S_X$. Then $\{c_X(c_X(S_X)) \mid X \in \mathcal{C}\}$ is a family of canonical relations on \mathcal{C} .

Proof. By Proposition 4.1.9, $c_X(c_X(S_X) = c_X(c_X(c_X(c_X(S_X))))$ for all $X \in \mathcal{C}$. Therefore we have only to prove that $w[m] \in c_Y(c_Y(S_Y))$ for all $w : X \rightarrow Y$ and all $m \in c_X(c_X(S_X))$. We begin to form the following diagram

$$\begin{array}{ccccccc}
 K & \xrightarrow{k} & P & \xrightarrow{p} & e^{-1}[Q] & \xrightarrow{e^{-1}[q]} & M & \xrightarrow{m} & X \\
 \downarrow i & & & & \downarrow d & & \downarrow e & & \downarrow w \\
 (dp)[K] & \xrightarrow{(dp)[k]} & Q & \xrightarrow{q} & w[M] & \xrightarrow{w[m]} & Y & &
 \end{array}$$

by letting $w : X \rightarrow Y$ be in \mathcal{C} , letting $m \in c_X(c_X(S_X))$ and supposing $w[m] \notin c_Y(c_Y(S_Y))$. There then exists a non-constant $q \in \mathcal{M}/w[M]$ such that $w[m] \cdot q \in c_Y(S_Y)$. It follows that Q is not preterminal and therefore that d (a member of \mathcal{E} obtained by pulling back e along q) is not constant. Since \mathcal{C} has enough subobjects, there exists a non-constant $p \in \mathcal{M}/e^{-1}[Q]$ such that dp is a monomorphism. Let $k \in \mathcal{M}/P$ be such that $m \cdot e^{-1}[q] \cdot pk \in S_X$ and let $i \in \mathcal{E}$ be such that $dpk = (dp)[k] \cdot i$. Without loss of generality, we may assume $w[m] \cdot q \cdot (dp)[k] = w[m \cdot e^{-1}[q] \cdot pk]$. It follows that $w[m] \cdot q \cdot (dp)[k] \in S_Y$. Since $w[m] \cdot q \in c_Y(S_Y)$, it follows that $w[m] \cdot q \cdot (dp)[k]$ is constant and therefore that $(dp)[k] \cdot i$ is constant (since $w[m] \cdot q \in \mathcal{M}$). Since dp is a monomorphism, this implies that k is constant and therefore that $m \cdot e^{-1}[q] \cdot pk$ is constant. This establishes that $m \cdot e^{-1}[q] \cdot p \in c_X(S_X)$. Since $m \cdot e^{-1}[q] \cdot p$ is non-constant, this contradicts the fact that $m \in c_X(c_X(S_X))$. So we conclude that $w[m] \in c_Y(c_Y(S_Y))$.

Theorem 4.2.8 *Suppose $\text{RegEpi}(\mathcal{C})$ is fibre determined and \mathcal{C} has enough quotients and enough subobjects. Let \mathcal{F} be a class of morphisms in \mathcal{C} . Then $\widehat{\mathcal{F}}$ is the left class of a $\|\text{-}$ pair provided \mathcal{F} satisfies 4.2.4(2).*

Proof. Suppose \mathcal{F} satisfies 4.2.4(2). It is clear that $\{S_X^{\mathcal{F}} | X \in \mathcal{C}\}$ then satisfies the conditions of Lemma 4.2.7. If we let $c(c(S^{\mathcal{F}})) = \{c_X(c_X(S_X^{\mathcal{F}})) | X \in \mathcal{C}\}$, it follows that $c(c(S^{\mathcal{F}}))$ is a family of canonical relations on \mathcal{C} . It is easy to see that then $\widehat{\mathcal{F}} = \mathcal{F}^{c(c(S^{\mathcal{F}}))}$. Therefore, by Theorem 4.2.1, $\widehat{\mathcal{F}}$ is the left class of a $\|\text{-}$ pair.

4.3 Obtaining $\|\text{-}$ pairs from full subcategories

In this section we consider the problem of obtaining a $\|\text{-}$ pair from a full subcategory \mathcal{K} of \mathcal{C} . This will be particularly relevant in Chapter 7, but it will also be useful when discussing examples in Chapter 5. The following is an imitation of Strecker's definition of \mathcal{A} -submonotone and \mathcal{A} -superlight (Definition 7.1.2). (When we say an "image of an object" we really mean an image of the identity morphism on an object.)

Definition 4.3.1 *Let \mathcal{K} be a full subcategory of \mathcal{C} . We define $\mathcal{F}_{\mathcal{K}}$ to consist of all $f : X \rightarrow A$ in \mathcal{C} such that, for all non-constant $m \in \mathcal{M}/X$, fm constant \Rightarrow there exists $p \in \mathcal{M}/X$ such that p is an image of an object in \mathcal{K} and $m \leq p$. We define $\mathcal{G}_{\mathcal{K}}$ to consist of all $g : Y \rightarrow B$ in \mathcal{C} such that, for all $n \in \mathcal{M}/Y$, if n is an image of an object in \mathcal{K} then g is monic on n .*

Although $(\mathcal{F}_\mathcal{K}, \mathcal{G}_\mathcal{K})$ may not itself be a $\|\text{-pair}$, we do at least obtain the following.

Theorem 4.3.2 *Suppose $\text{RegEpi}(\mathcal{C})$ is fibre determined and \mathcal{C} has enough quotients and enough subobjects. Let \mathcal{K} be a full subcategory of \mathcal{C} . Then $(\widehat{\mathcal{F}}_\mathcal{K}, \mathcal{G}_\mathcal{K})$ is a $\|\text{-pair}$.*

Proof. For each $X \in \mathcal{C}$, let S_X be the preordered class of all $m \in \mathcal{M}/X$ such that if m is not constant then there exists $p \in \mathcal{M}/X$ such that p is an image of an object in \mathcal{K} and $m \leq p$. Clearly S_X is hereditary, for all $X \in \mathcal{C}$. It is also clear that $w[m] \in S_Y$ for all $w : X \rightarrow Y$ in \mathcal{C} and all $m \in S_X$. By Lemma 4.2.7, $c(c(\mathcal{S})) = \{c_X(c_X(S_X)) \mid X \in \mathcal{C}\}$ is a family of canonical relations on \mathcal{C} . It is easy to see that $\widehat{\mathcal{F}}_\mathcal{K} = \mathcal{F}^{c(c(\mathcal{S}))}$. Let $g : Y \rightarrow B$ be in \mathcal{C} . As in the proof of Proposition 4.1.12, it is easy to see that $g \in \mathcal{G}_\mathcal{K}$ iff, for all $n \in \mathcal{M}/Y$, gn constant $\Rightarrow n \in c_Y(S_Y)$. Since $c_Y(S_Y) = c_Y(c_Y(c_Y(S_Y)))$ for all $Y \in \mathcal{C}$, it follows that $\mathcal{G}_\mathcal{K} = \mathcal{G}^{c(c(\mathcal{S}))}$. Therefore, by Theorem 4.2.1, $(\widehat{\mathcal{F}}_\mathcal{K}, \mathcal{G}_\mathcal{K})$ is a $\|\text{-pair}$.

Under the conditions of Theorem 4.3.2, $\|\text{-pairs}$ of the form $(\widehat{\mathcal{F}}_\mathcal{K}, \mathcal{G}_\mathcal{K})$ have the following characterisation.

Theorem 4.3.3 *Suppose $\text{RegEpi}(\mathcal{C})$ is fibre determined and \mathcal{C} has enough quotients and enough subobjects. Let $(\mathcal{F}, \mathcal{G})$ be a $\|\text{-pair}$ on \mathcal{C} . The following are equivalent:*

- (1) *there exists a full subcategory $\mathcal{K} \subseteq \mathcal{C}$ such that $(\mathcal{F}, \mathcal{G}) = (\widehat{\mathcal{F}}_\mathcal{K}, \mathcal{G}_\mathcal{K})$*
- (2) *$(\mathcal{F}, \mathcal{G}) = (\widehat{\mathcal{F}}_{\overline{\mathcal{F}}}, \mathcal{G}_{\overline{\mathcal{F}}})$*
- (3) *$\mathcal{G} = R(\overline{\mathcal{F}})$.*

Proof. Let \mathcal{K} be a full subcategory of \mathcal{C} and let $(\mathcal{F}', \mathcal{G}')$ be a $\|\text{-pair}$ on \mathcal{C} such that $\mathcal{K} \subseteq \overline{\mathcal{F}'}$. Using Proposition 3.2.5, it is easy to see that $L(R(\mathcal{K})) \subseteq \mathcal{F}'$.

Let $f : X \rightarrow A$ in $\mathcal{F}_\mathcal{K}$, $g : Y \rightarrow B$ in \mathcal{G}' , $m \in \mathcal{M}/X$ and $w : X \rightarrow Y$ in \mathcal{C} be such that fm and gwm are both constant. If m is constant then so is wm . Suppose m is not constant. Then there is an image $p : P \rightarrow X$ of an object $Z \in \mathcal{K}$ such that $m \leq p$. Let $e : Z \rightarrow P$ be the \mathcal{E} -morphism whose existence follows from the previous statement. We may then form the

following diagram

$$\begin{array}{ccc}
 Z \times_P M & \xrightarrow{\widehat{e}} & M \\
 \widehat{r} \downarrow & & \downarrow r \\
 Z & \xrightarrow{e} & P \\
 & & \downarrow p \\
 & & X \xrightarrow{w} Y \\
 & & \downarrow f \quad \downarrow g \\
 & & A \quad B
 \end{array}$$

by pulling back r along e . It is clear that $fpe\widehat{r}$ and $gpe\widehat{r}$ are constant and that $fpe \in \mathcal{F}'$ (since $Z \in \mathcal{K}$). It follows, by Proposition 4.1.3, that $wpe\widehat{r}$ is constant and therefore that $wm\widehat{e}$ is constant. Since $\widehat{e} \in \mathcal{E}$, it follows that wm is constant. Therefore, by Proposition 4.1.3, $f \parallel g$ and so $f \in \mathcal{F}'$. This establishes that $\mathcal{F}_{\mathcal{K}} \subseteq \mathcal{F}'$. From Corollary 4.2.4 and the comments made after Definition 4.2.6 it is then clear that $\widehat{\mathcal{F}}_{\mathcal{K}} \subseteq \mathcal{F}'$.

Given Theorem 4.3.2, it is easy to see that $(L(R(\mathcal{K}), R(\mathcal{K})))$ and $(\widehat{\mathcal{F}}_{\mathcal{K}}, \mathcal{G}_{\mathcal{K}})$ are both \parallel -pairs satisfying the assumption which we made about $(\mathcal{F}', \mathcal{G}')$. We conclude that $L(R(\mathcal{K})) = \widehat{\mathcal{F}}_{\mathcal{K}}$ and therefore that $(L(R(\mathcal{K})), R(\mathcal{K})) = (\widehat{\mathcal{F}}_{\mathcal{K}}, \mathcal{G}_{\mathcal{K}})$. This is enough to prove (2) \Leftrightarrow (3). Combined with Proposition 3.4.7, it is also enough to prove (1) \Leftrightarrow (3).

Under certain conditions, we are able to obtain \parallel -pairs from \parallel^{ob} -pairs of a particular sort.

Theorem 4.3.4 *Suppose $\text{RegEpi}(\mathcal{C})$ is fibre determined and \mathcal{C} has enough quotients. Suppose too that pushouts of members of \mathcal{M} exist and that \mathcal{M} is closed under forming pushouts. Let $(\mathcal{K}, \mathcal{H})$ be a \parallel^{ob} -pair on \mathcal{C} such that, for all $m : M \rightarrow P$ in \mathcal{M} , $P \in \mathcal{K} \Rightarrow M \in \mathcal{K}$. Then $(\mathcal{F}_{\mathcal{K}}, \mathcal{F}_{\mathcal{H}})$ is a \parallel -pair on \mathcal{C} .*

Proof. Using Proposition 4.1.3 it is easy to see that $f \parallel g$ for all $f \in \mathcal{F}_{\mathcal{K}}$ and $g \in \mathcal{F}_{\mathcal{H}}$. So $\mathcal{F}_{\mathcal{K}} \subseteq L(\mathcal{F}_{\mathcal{H}})$ and $\mathcal{F}_{\mathcal{H}} \subseteq R(\mathcal{F}_{\mathcal{K}})$. Suppose $f : X \rightarrow A$ in \mathcal{C} is not in $\mathcal{F}_{\mathcal{K}}$. Then there exists $m : M \rightarrow X$ in \mathcal{M} such that $M \notin \mathcal{K}$ and fm is constant. There then exists $N \in \mathcal{H}$ and $e : M \rightarrow N$ in \mathcal{E} such that e is not

constant. We obtain the following diagram

$$\begin{array}{ccc}
 M & \xrightarrow{e} & N \\
 m \downarrow & & \downarrow n \\
 X & \xrightarrow{w} & Y \\
 f \downarrow & & \downarrow g \\
 A & & B
 \end{array}$$

by pushing out m along e and letting g in \mathcal{C} be such that g is constant only on n (noting that $n \in \mathcal{M}/Y$ by the assumptions on \mathcal{M}). It is easy to see that $g \in \mathcal{F}_{\mathcal{H}}$ and that gwm is constant while wm is not. It follows, by Proposition 4.1.3, that $f \notin L(\mathcal{F}_{\mathcal{H}})$. Therefore $L(\mathcal{F}_{\mathcal{H}}) \subseteq \mathcal{F}_{\mathcal{K}}$.

Using the same picture, suppose instead that $g \notin \mathcal{F}_{\mathcal{H}}$. Then there exists $n \in \mathcal{M}$ such that $N \notin \mathcal{H}$ and gn is constant. There then exists $M \in \mathcal{K}$ and $e \in \mathcal{E}$ such that e is not constant. We may then choose $m = 1_M$, $w = ne$ and $f = !_X$. Then $f \in \mathcal{F}_{\mathcal{K}}$ and fm and gwm are constant while wm is not. It follows, by Proposition 4.1.3, that $g \notin R(\mathcal{F}_{\mathcal{K}})$. Therefore $R(\mathcal{F}_{\mathcal{K}}) \subseteq \mathcal{F}_{\mathcal{H}}$.

Corollary 4.3.5 *Suppose $\text{RegEpi}(\mathcal{C})$ is fibre determined and \mathcal{C} has enough quotients and enough subobjects. Suppose too that pushouts of members of \mathcal{M} exist and that \mathcal{M} is closed under forming pushouts. Let $(\mathcal{K}, \mathcal{H})$ be a $\|\text{-pair}$ on \mathcal{C} such that, for all $m : M \rightarrow P$ in \mathcal{M} , $P \in \mathcal{K} \Rightarrow M \in \mathcal{K}$. Then $(\mathcal{F}_{\mathcal{K}}, \mathcal{G}_{\mathcal{K}})$ is a $\|\text{-pair}$ on \mathcal{C} .*

Proof. By Theorem 4.3.4, $\mathcal{F}_{\mathcal{K}}$ is the left class of a $\|\text{-pair}$. By Corollary 4.2.4 and the comments made after Definition 4.2.6, it follows that $\mathcal{F}_{\mathcal{K}} = \widehat{\mathcal{F}_{\mathcal{K}}}$. The result then follows from Theorem 4.3.2.

5 Examples of $\|\text{-}$ pairs

In this section we use the theory developed in Chapters 3 and 4 to consider examples of $\|\text{-}$ pairs on topological categories, abelian categories and the category of pointed sets. In Section 5.1 we show that families of canonical relations really do generalise the notion that inspired them. In Sections 5.2, 5.3 and 5.4 we consider specific examples in the categories **Top**, **SymRe** and **Prost**. In Section 5.5 we consider abelian categories and establish that they satisfy the conditions of Theorem 4.2.1 but that they do not always have enough subobjects. In Section 5.6 we consider specific examples in the category **Ab**, including one which establishes that Theorem 4.2.2 really does need the assumption that \mathcal{C} has enough subobjects. In Section 5.7 we use our theory to show that there are no non-trivial $\|\text{-}$ pairs on **Set**. The definitions of these categories are listed in Section 1.2.

5.1 $\|\text{-}$ pairs on topological categories

Topological categories (categories which are topological over **Set**) are discussed in detail in Chapter 21 of [AHS 1990]. In this section (\mathcal{C}, U) is assumed to be a topological category. We often avoid reference to the functor U , allowing the underlying set of an object $X \in \mathcal{C}$ to be referred to as X and allowing the underlying function of a morphism f in \mathcal{C} to be referred to as f . We recall that \mathcal{C} is complete and cocomplete, that it has (regular epi, mono)-factorisations and that it is equipped with a proper and stable factorisation system $(\mathcal{E}, \mathcal{M})$, where $\mathcal{E} = \text{Epi}(\mathcal{C})$ and \mathcal{M} is the class of all embeddings in \mathcal{C} . We recall too that $\text{RegEpi}(\mathcal{C})$ is precisely the class of quotient morphisms in \mathcal{C} and that $(\mathcal{C}^{op}, U^{op})$ is topological over **Set**^{op}. The subobjects to which we shall refer are all defined with respect to \mathcal{M} .

Proposition 5.1.1 *RegEpi(\mathcal{C}) is fibre determined and \mathcal{C} has enough quotients and enough subobjects.*

Proof. This result is obviously true in the case where $\mathcal{C} = \mathbf{Set}$. Using U and U^{op} , it is then easy to lift the required properties from **Set** to \mathcal{C} (given that the regular epimorphisms in **Set** are just the surjections).

Let $m : M \rightarrow X$ be a subobject in \mathcal{C} . We shall write “ $x \in m$ ” as an abbreviation for “ x is an element of the range of the function $U(m)$ ”. It is

easy to see that m is constant iff the range of $U(m)$ is at most a singleton. We verify that families of canonical relations on \mathcal{C} are indeed equivalent to the notion that inspired them.

Definition 5.1.2 For each $X \in \mathcal{C}$, let \sim_X be a reflexive symmetric relation on X and let $\sim = \{\sim_X \mid X \in \mathcal{C}\}$ be such that, for any $w : X \rightarrow Y$ in \mathcal{C} , $x \sim_X x' \Rightarrow w(x) \sim_Y w(x')$ for all $x, x' \in X$. Then \sim is said to be a family of morphism-preserved reflexive symmetric relations (abbreviated to FPR) on \mathcal{C} .

Theorem 5.1.3 There is a bijection between the class of all FPRs on \mathcal{C} and the class of all families of canonical relations on \mathcal{C} . It is given as follows: $\sim \mapsto \mathcal{S}^\sim$ where, for each $X \in \mathcal{C}$, S_X^\sim is the preordered class of all $m \in \mathcal{M}/X$ such that $x \sim_X x' \Rightarrow m \in S_X^\sim$.

Proof. Let $\sim = \{\sim_X \mid X \in \mathcal{C}\}$ be an FPR on \mathcal{C} . We show that \mathcal{S}^\sim is a family of canonical relations. It is clear that, for any $w : X \rightarrow Y$ in \mathcal{C} and any $m \in \mathcal{M}/X$, $m \in S_X^\sim \Rightarrow w[m] \in S_Y^\sim$. Let $X \in \mathcal{C}$. Let $x, x' \in X$ and let $p \in \mathcal{M}/X$ be such that the range of the function $U(p)$ is $\{x, x'\}$. Then $x \sim_X x' \Rightarrow p \in S_X^\sim$. Therefore $c_X(S_X^\sim)$ consists precisely of all $m \in \mathcal{M}/X$ such that, for all $x, x' \in m$, $x = x'$ or $x \not\sim_X x'$. By similar reasoning we then have that $c_X(c_X(S_X^\sim)) = S_X^\sim$. So \mathcal{S}^\sim is a family of canonical relations on \mathcal{C} .

Now consider the following mapping from the class of all families of canonical relations on \mathcal{C} to the class of all FPRs on \mathcal{C} : $\mathcal{S} \mapsto \sim^\mathcal{S}$ where, for all $X \in \mathcal{C}$ and all $x, x' \in X$, $x \sim_X^\mathcal{S} x'$ iff there exists $m \in S_X$ such that $x, x' \in m$. If \mathcal{S} is a family of canonical relations, then $\sim^\mathcal{S}$ is clearly an FPR.

It is easy to see that, for any family of canonical relations \mathcal{S} on \mathcal{C} and any $X \in \mathcal{C}$, S_X is completely determined by its two-element members (those $p \in S_X$ such that the range of the function $U(p)$ has precisely two elements) and the condition that $S_X = c_X(c_X(S_X))$. It is then clear that the mappings are inverse to each other.

Combining Theorems 5.1.3 and 4.2.3 gives a one to one correspondence between the FPRs on \mathcal{C} and the $\|\$ -pairs on \mathcal{C} . In Sections 5.2, 5.3 and 5.4 it will be simpler to deal only with this correspondence and so we cut out the intermediate notion by adopting the following notation.

Notation 5.1.4 We abbreviate $\mathcal{F}^{\mathcal{S}^\sim}$ to \mathcal{F}^\sim and $\mathcal{G}^{\mathcal{S}^\sim}$ to \mathcal{G}^\sim .

The following two results are easy to establish using the proof of Theorem 5.1.3.

Proposition 5.1.5 *Let $\sim = \{\sim_X \mid X \in \mathcal{C}\}$ be an FPR on \mathcal{C} and let $f : X \rightarrow A$ and $g : Y \rightarrow B$ be in \mathcal{C} . Then $f \in \mathcal{F}^\sim$ iff, for all $x, x' \in X$, $f(x) = f(x') \Rightarrow x \sim_X x'$. Also, $g \in \mathcal{G}^\sim$ iff, for all distinct $y, y' \in Y$, $g(y) = g(y') \Rightarrow y \not\sim_Y y'$.*

Proposition 5.1.6 *Let $\sim = \{\sim_X \mid X \in \mathcal{C}\}$ be an FPR on \mathcal{C} and let \mathcal{K} be a full subcategory of \mathcal{C} . The following are equivalent:*

- (1) *for all $X \in \mathcal{C}$ and all $x, x' \in X$, $x \sim_X x'$ iff there exists $m \in \mathcal{M}/X$ such that m is an image of an object in \mathcal{K} and $x, x' \in m$*
- (2) $\mathcal{F}^\sim = \widehat{\mathcal{F}}_{\mathcal{K}}$.

With the aid of Theorem 4.3.3, Proposition 5.1.6 gives an easy way to determine, for a $\|\text{-pair}$ $(\mathcal{F}, \mathcal{G})$ on \mathcal{C} , whether or not $\mathcal{G} = R(\overline{\mathcal{F}})$. The following result gives an easy way to establish, for a $\|\text{-pair}$ $(\mathcal{F}, \mathcal{G})$ on \mathcal{C} , that $\mathcal{F} \neq L(\overline{\mathcal{G}})$.

Proposition 5.1.7 *Let \sim be an FPR on \mathcal{C} . If $\mathcal{F}^\sim = L(\overline{\mathcal{G}^\sim})$ then, for each $X \in \mathcal{C}$, \sim_X is transitive.*

Proof. Suppose $\mathcal{F}^\sim = L(\overline{\mathcal{G}^\sim})$. Let $X \in \mathcal{C}$ and $x, x', x'' \in X$ be such that $x \sim_X x'$ and $x' \sim_X x''$. Let f be constant only on p , where $p \in \mathcal{M}/X$ is such that the range of the function $U(p)$ is $\{x, x', x''\}$. Let $w : X \rightarrow Y$ in \mathcal{C} be such that $Y \in \overline{\mathcal{G}^\sim}$. Since distinct elements of Y cannot be related by \sim_Y , it follows that $w(x) = w(x') = w(x'')$. This establishes that $f \in L(\overline{\mathcal{G}^\sim}) = \mathcal{F}^\sim$. It follows that $x \sim_X x''$.

In the following three sections we consider specific examples of $\|\text{-pairs}$ on the topological categories **Top**, **SymRe** and **Prost**.

5.2 Examples on Top

(1) For each topological space X , we define a relation as follows: for all $x, x' \in X$, $x \sim_X x'$ iff there do not exist two open subsets $G_1, G_2 \subseteq X$ such that $x \in G_1 \setminus G_2$ and $x' \in G_2 \setminus G_1$. It is easy to see that $\sim = \{\sim_X \mid X \in \mathbf{Top}\}$ is an FPR on **Top**. So we obtain a $\|\text{-pair}$ $(\mathcal{F}^\sim, \mathcal{G}^\sim)$. A continuous map $g : Y \rightarrow B$ is in \mathcal{G}^\sim iff g is only constant on T_1 subspaces of Y . A continuous

map $f : X \rightarrow A$ is in \mathcal{F}^\sim iff f is only constant on subspaces of X which have no non-trivial T_1 subspaces.

$\overline{\mathcal{G}^\sim}$ is the full subcategory of **Top** whose objects are the T_1 spaces and $\overline{\mathcal{F}^\sim}$ is the full subcategory of **Top** whose objects are those spaces which have no non-trivial T_1 subspaces. Since not all \sim_X are transitive, it follows, by Proposition 5.1.7, that $\mathcal{F}^\sim \neq L(\overline{\mathcal{G}^\sim})$. For any space X and all $x, x' \in X$, $x \sim_X x'$ iff x and x' are both contained in an image of the Sierpinski space or the two-element indiscrete space. It follows, by Proposition 5.1.6, that $\mathcal{F}^\sim = \widehat{\mathcal{F}_\mathcal{K}}$ where \mathcal{K} is the full subcategory whose objects are the Sierpinski space and the two-element indiscrete space. It then follows, by Theorem 4.3.3, that $\mathcal{G}^\sim = R(\overline{\mathcal{F}^\sim})$. However, we can do better. If instead we choose $\mathcal{K} = \overline{\mathcal{F}^\sim}$ then it is clear that $\mathcal{F}^\sim = \mathcal{F}_\mathcal{K}$.

(2) For each space X , define a relation as follows: for all $x, x' \in X$, $x \sim_X x'$ iff there do not exist two disjoint open sets $G_1, G_2 \subseteq X$ such that $x \in G_1$ and $x' \in G_2$. It is easy to see that $\sim = \{\sim_X \mid X \in \mathbf{Top}\}$ is an FPR on **Top**. So we obtain a $\|\cdot\|$ -pair $(\mathcal{F}^\sim, \mathcal{G}^\sim)$.

$\overline{\mathcal{G}^\sim}$ is the full subcategory of **Top** whose objects are the T_2 spaces. Not all \sim_X are transitive and so $\mathcal{F}^\sim \neq L(\overline{\mathcal{G}^\sim})$. Let X be the set of real numbers together with an additional element $0'$ and let X have the topology generated by two copies of the usual real line topology: one as normal and one with $0'$ replacing 0 . It follows that $0 \sim_X 0'$. However there is no $Z \in \overline{\mathcal{F}^\sim}$ capable of being mapped onto a subspace of X containing both 0 and $0'$. Therefore, by Proposition 5.1.6, $\mathcal{F}^\sim \neq \widehat{\mathcal{F}_{\overline{\mathcal{F}^\sim}}}$. Therefore, by Theorem 4.3.3, $\mathcal{G}^\sim \neq R(\overline{\mathcal{F}^\sim})$.

(3) Let \mathcal{F} be the class of concordant maps and let \mathcal{G} be the class of dissonant maps (see Definition 6.1.1). Then $(\mathcal{F}, \mathcal{G}) = (\mathcal{F}^\sim, \mathcal{G}^\sim)$ where \sim_X is defined for each space X as follows: for all $x, x' \in X$, $x \sim_X x'$ iff there is a quasi-component $Q \subseteq X$ such that $x, x' \in Q$. It is easy to see that $\sim = \{\sim_X \mid X \in \mathbf{Top}\}$ is an FPR. So $(\mathcal{F}, \mathcal{G})$ is a $\|\cdot\|$ -pair on **Top**.

It is obvious and well known that $\overline{\mathcal{F}}$ is the full subcategory of **Top** whose objects are the connected spaces and that $\overline{\mathcal{G}}$ is the full subcategory of **Top** whose objects are the spaces whose quasi-components are singletons. Since $(\mathcal{F}^*, \mathcal{G})$ is a factorisation system (see Theorem 6.1.3), it follows from the discussion in Section 6.6 that $\mathcal{F} = L(\overline{\mathcal{G}})$. Since not all quasi-components are connected, it is easy to see that there is no full subcategory \mathcal{K} such that $\mathcal{F} = \mathcal{F}_\mathcal{K}$. In fact, $\mathcal{G} \neq R(\overline{\mathcal{F}})$ as shall become clear when considering the next example.

(4) Let \mathcal{F} be the class of submonotone maps and let \mathcal{G} be the class of superlight maps (see Definition 7.1.2 and the comments after it). Then $(\mathcal{F}, \mathcal{G}) = (\mathcal{F}^\sim, \mathcal{G}^\sim)$ where \sim_X is defined for each space X as follows: for all $x, x' \in X$, $x \sim_X x'$ iff there is a component $C \subseteq X$ such that $x, x' \in C$. It is easy to see that $\sim = \{\sim_X \mid X \in \mathbf{Top}\}$ is an FPR. So $(\mathcal{F}, \mathcal{G})$ is a $\|\text{-pair}$ on \mathbf{Top} .

It is obvious and well known that $\overline{\mathcal{F}}$ is the full subcategory of \mathbf{Top} whose objects are the connected spaces and that $\overline{\mathcal{G}}$ is the full subcategory of \mathbf{Top} whose objects are the spaces whose components are singletons. As in (3), we have $\mathcal{F} = L(\overline{\mathcal{G}})$ because $(\mathcal{F}^*, \mathcal{G})$ is a factorisation system (see Theorem 7.1.3). Since every component is an image of a connected space, it is clear that $\mathcal{F} = \mathcal{F}_{\overline{\mathcal{F}}}$. By Theorem 4.3.3, it follows that $\mathcal{G} = R(\overline{\mathcal{F}})$. Since the $\overline{\mathcal{F}}$ of this example and the $\overline{\mathcal{F}}$ of the previous example coincide while the two $\|\text{-pairs}$ are known to be different, the assertion made at the end of the previous example is clearly true.

5.3 Examples on \mathbf{SymRe}

It is convenient to regard the objects in \mathbf{SymRe} as graphs for which each vertex has a loop. Morphisms in \mathbf{SymRe} are then just edge-preserving maps. Let $X \in \mathbf{SymRe}$ and let $x, x' \in X$. There is an edge between x and x' iff x and x' are related in X . This edge is called a loop if $x = x'$. Let n be a non-negative integer. A path of length n between x and x' is a subset $\{x_1, \dots, x_{n+1}\} \subseteq X$ such that $x_1 = x$, $x_{n+1} = x'$ and, for $i = 1, 2, \dots, n$, there is an edge between x_i and x_{i+1} . If there is a path between x and x' , we use $d(x, x')$ to denote the length of the shortest path between x and x' . If there is no path between x and x' , we write $d(x, x') = \infty$. Let $M \subseteq X$. If $\{d(y, y') \mid y, y' \in M\}$ has a (non-infinite) maximum element then this is referred to as the diameter of M . The term “path-connected” has the obvious meaning. X is discrete iff the only edges in X are the loops. Thinking of $X \in \mathbf{SymRe}$ as a graph avoids confusion when it comes to defining \sim_X .

(1) Let n be a non-negative integer. For each $X \in \mathbf{SymRe}$, define a relation as follows: for all $x, x' \in X$, $x \sim_X x'$ iff $d(x, x') \leq n$. It is clear that $\sim = \{\sim_X \mid X \in \mathbf{SymRe}\}$ is an FPR. So we obtain a $\|\text{-pair}$ $(\mathcal{F}^\sim, \mathcal{G}^\sim)$. \mathcal{F}^\sim consists of those edge-preserving maps which are constant only on path-connected subsets with diameter not exceeding n . \mathcal{G}^\sim consists of those edge-

preserving maps which are injective on all paths of length not exceeding n .

$\overline{\mathcal{F}^\sim}$ is the full subcategory of **SymRe** whose objects are the path-connected objects with diameter not exceeding n . $\overline{\mathcal{G}^\sim}$ is the full subcategory of **SymRe** whose objects are the discrete objects. Since not all \sim_X are transitive, $\mathcal{F}^\sim \neq L(\overline{\mathcal{G}^\sim})$. It is clear that $\mathcal{F}^\sim = \mathcal{F}_{\overline{\mathcal{F}^\sim}}$. Therefore $\mathcal{G}^\sim = R(\overline{\mathcal{F}^\sim})$.

Letting n go to infinity, we obtain the following.

(2) For each $X \in \mathbf{SymRe}$, define a relation as follows: for all $x, x' \in X$, $x \sim_X x'$ iff there is a path between x and x' . It is clear that $\sim = \{\sim_X \mid X \in \mathbf{SymRe}\}$ is an FPR. So we obtain a $\|\text{-pair}$ $(\mathcal{F}^\sim, \mathcal{G}^\sim)$. \mathcal{F}^\sim consists of those edge-preserving maps which are constant only on path-connected subsets. \mathcal{G}^\sim consists of those edge-preserving maps which are injective on all paths.

$\overline{\mathcal{F}^\sim}$ is the full subcategory of **SymRe** whose objects are the path-connected objects. $\overline{\mathcal{G}^\sim}$ is the full subcategory of **SymRe** whose objects are the discrete objects. It is easy to see that $\mathcal{F}^\sim = L(\overline{\mathcal{G}^\sim})$ and that $\mathcal{F}^\sim = \mathcal{F}_{\overline{\mathcal{F}^\sim}}$. Therefore $\mathcal{G}^\sim = R(\overline{\mathcal{F}^\sim})$.

(3) For the sake of completeness, we record here the trivial example where, for each $X \in \mathbf{SymRe}$, $x \sim_X x'$ for all $x, x' \in X$. Trivially, $\sim = \{\sim_X \mid X \in \mathbf{SymRe}\}$ is an FPR. \mathcal{F}^\sim consists of all edge-preserving maps and \mathcal{G}^\sim consists of all injective edge-preserving maps. $\overline{\mathcal{F}^\sim} = \mathbf{SymRe}$ and $\overline{\mathcal{G}^\sim}$ is the full subcategory of **SymRe** whose objects are the objects which are no larger than a singleton.

It turns out that 5.3(1)-(3) is a complete list of all $\|\text{-pairs}$ on **SymRe**. The proof will make use of the following definition and lemma.

Definition 5.3.1 For any non-negative integer n , $P_n \in \mathbf{SymRe}$ has $\{0, 1, \dots, n\}$ as its underlying set and has edges between consecutive integers and between equal integers only.

Lemma 5.3.2 Let n be a non-negative integer. Let $X \in \mathbf{SymRe}$ and let $m : P_n \rightarrow X$ be an embedding such that the range of the function m is a path of minimal length between $m(0)$ and $m(n)$. Then m is a split monomorphism in **SymRe**.

Proof. We construct $f : X \rightarrow P_n$ as follows: for any $x \in X$,

- (1) $f(x) = d(x, m(0))$ if $d(x, m(0)) \leq n$
- (2) $f(x) = n$ if $d(x, m(0)) \not\leq n$.

Let $x, x' \in X$. Suppose there is an edge between x and x' . Then $d(x, m(0))$ and $d(x', m(0))$ differ by at most 1. It follows that there is an edge between $f(x)$ and $f(x')$. So f is an edge-preserving map. Since that path between $m(0)$ and $m(n)$, to which P_n is mapped by m , is of minimal length, it is clear that $fm = 1_{P_n}$.

Theorem 5.3.3 *Let $(\mathcal{F}, \mathcal{G})$ be a $\|\text{-pair}$ on \mathbf{SymRe} . Then $(\mathcal{F}, \mathcal{G})$ is one of the $\|\text{-pairs}$ considered in 5.3(1)-(3).*

Proof. Let $\sim = \{\sim_X \mid X \in \mathbf{SymRe}\}$ be an FPR on \mathbf{SymRe} . There are three possibilities:

- (1) there exists a non-negative integer n such that, for all $X \in \mathbf{SymRe}$ and all $x, x' \in X$, $x \sim_X x' \Rightarrow d(x, x') \leq n$
- (2) (1) is not true but, for all $X \in \mathbf{SymRe}$ and all $x, x' \in X$, $x \sim_X x' \Rightarrow$ there is a path between x and x'
- (3) there exist $X \in \mathbf{SymRe}$ and $x, x' \in X$ such that $x \sim_X x'$ and there is no path between x and x' .

Suppose (1). Without loss of generality, we may assume that n has been chosen to be as small as possible. It follows that there exist $X \in \mathbf{SymRe}$ and $x, x' \in X$ such that $x \sim_X x'$ and $d(x, x') = n$. By Lemma 5.3.2 there is an edge-preserving map $f : X \rightarrow P_n$ which is injective on this path. It follows that $0 \sim_{P_n} n$. It then follows that $z \sim_Z z'$ for all $Z \in \mathbf{SymRe}$ and all $z, z' \in Z$ such that $d(z, z') \leq n$. It is then clear that \sim is as described in 5.3(1).

Suppose (2). For any non-negative integer k , there must exist a larger integer n for which there exists $X \in \mathbf{SymRe}$ and $x, x' \in X$ such that $x \sim_X x'$ and $d(x, x') = n$. By the above reasoning, we may deduce that $z \sim_Z z'$ for all $Z \in \mathbf{SymRe}$ and all $z, z' \in Z$ such that $d(z, z') \leq n$. From this it is clear that \sim is as described in 5.3(2).

Clearly (3) is just 5.3(3).

Since all $\|\text{-pairs}$ on \mathbf{SymRe} are of the form $(\mathcal{F}^\sim, \mathcal{G}^\sim)$, this completes the proof.

Corollary 5.3.4 *Let $(\mathcal{F}, \mathcal{G})$ be a $\|\text{-pair}$ on \mathbf{SymRe} . Then $\mathcal{G} = R(\overline{\mathcal{F}})$ and there exists a full subcategory \mathcal{K} of \mathbf{SymRe} such that $\mathcal{F} = \mathcal{F}_{\mathcal{K}}$.*

5.4 Examples on Prost

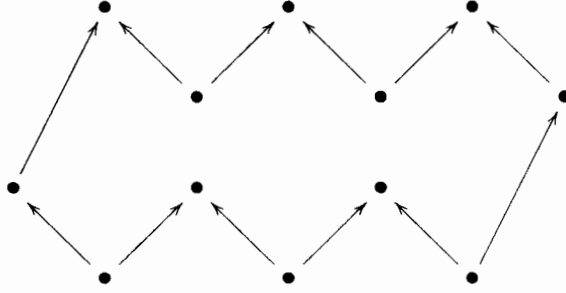
Let $X \in \mathbf{Prost}$ and let $x, x' \in X$. A path between x and x' is a finite sequence of inequalities linking x and x' . An up-first path from x to x' is a path between x and x' involving an inequality which has x on the left: “ $x \leq$ ”. A down-first path from x to x' is a path between x and x' involving an inequality which has x on the right: “ $\leq x$ ”.

(1) It is easy to see that all the examples of $\|\text{-pairs}$ on \mathbf{SymRe} have analogues on \mathbf{Prost} .

(2) Let n be a natural number. For each preordered set X we define a relation on X as follows: for all $x, x' \in X$, $x \sim_X x'$ iff there is an up-first path from x to x' and a down-first path from x to x' both of length not exceeding n . Let X be a preordered set. It is clear that \sim_X is reflexive. To see that \sim_X is symmetric, let $x, x' \in X$ and suppose there is an up-first path and a down-first path, both of length not exceeding n , from x to x' . If one of these paths may be chosen to have length less than n then clearly we could use it to obtain an up-first path from x' to x and a down-first path from x' to x both of length not exceeding n . Suppose both paths (from x to x') must have length n . Then clearly they both alternate: \leq, \geq, \leq , etc. It follows that one of them contains “ $x' \leq$ ” and the other contains $\leq x'$. Therefore they provide an up-first path and a down-first path of length n from x' to x . So $x \sim_X x' \Rightarrow x' \sim_X x$. It is clear that these relations are preserved by order-preserving maps. So $\sim = \{\sim_X \mid X \in \mathbf{Prost}\}$ is an FPR on \mathbf{Prost} and we therefore obtain a $\|\text{-pair}$ $(\mathcal{F}^\sim, \mathcal{G}^\sim)$.

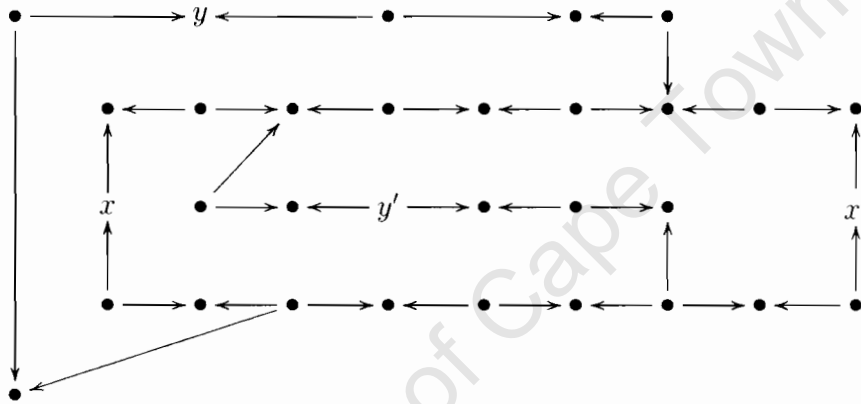
For any preordered set X and all $x, x' \in X$, it is easy to see that $x \sim_X x'$ iff x and x' are simultaneously members of an image of the preordered set generated by two paths of length n which have the same starting point and the same finishing point and for which one path is up-first and the other is down-first. For example, when $n = 6$ we mean the preordered set generated

by the following diagram:



So there exists a full subcategory $\mathcal{K} \subseteq \mathbf{Prost}$ such that $\mathcal{F}^\sim = \widehat{\mathcal{F}}_{\mathcal{K}}$. Therefore $\mathcal{G}^\sim = R(\overline{\mathcal{F}^\sim})$. Since not all \sim_X are transitive, we have that $\mathcal{F}^\sim \neq L(\overline{\mathcal{G}^\sim})$.

(3) For each preordered set X , we define a relation as follows: for all $x, x' \in X$, $x \sim_X x'$ iff X contains an image of the preordered set generated by the following diagram



with x and x' in the positions indicated. It is clear that $\sim = \{\sim_X \mid X \in \mathbf{Prost}\}$ is an FPR. So we obtain a $\|\text{-pair}$ $(\mathcal{F}^\sim, \mathcal{G}^\sim)$.

Let X be the preordered set generated by the above diagram and let x, x', y, y' be in the positions indicated. It is easy to see that $y \not\sim_X y'$. It is then clear that, although $x \sim_X x'$, there is no preordered subset $Z \subseteq X$ containing both x and x' and such that $z \sim_Z z'$ for all $z, z' \in Z$. It follows that there is no full subcategory $\mathcal{K} \subseteq \mathbf{Prost}$ such that $\mathcal{F}^\sim = \widehat{\mathcal{F}}_{\mathcal{K}}$. Therefore $\mathcal{G}^\sim \neq R(\overline{\mathcal{F}^\sim})$. Since not all \sim_X are transitive, we also have $\mathcal{F}^\sim \neq L(\overline{\mathcal{G}^\sim})$.

5.5 \parallel -pairs on abelian categories

Abelian categories are discussed in detail in Chapter VIII of [M 1971]. In this section \mathcal{C} is assumed to be an abelian category. We recall that \mathcal{C} is finitely complete and finitely cocomplete and that letting $\mathcal{E} = \text{RegEpi}(\mathcal{C})$ and $\mathcal{M} = \text{Mono}(\mathcal{C})$ gives a proper and stable factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{C} . We recall that the constant morphisms are just the zero morphisms in \mathcal{C} . We recall too that $\text{RegEpi}(\mathcal{C})$ is the class of cokernels in \mathcal{C} and that \mathcal{M} is the class of kernels in \mathcal{C} . The subobjects to which we shall refer are defined with respect to $(\mathcal{E}, \mathcal{M})$ and are therefore nothing other than the kernels in \mathcal{C} . The following is an obvious consequence of the basic theory as discussed in [M 1971].

Proposition 5.5.1 *RegEpi(\mathcal{C}) is fibre determined and \mathcal{C} has enough quotients.*

We may therefore use Theorem 4.2.1 to obtain examples of \parallel -pairs on \mathcal{C} . We do this for the special case where $\mathcal{C} = \mathbf{Ab}$, the category of abelian groups, in Section 5.6. For convenience, we shall work with subgroups (subsets which are also groups) rather than subobjects (members of \mathcal{M}). Clearly the subgroups which correspond to constant subobjects are just the trivial ones which contain no non-zero elements. It is easy to translate the definitions and results of Chapter 4 from the language of subobjects to the language of subgroups and to verify that the proofs all survive this transformation.

We observe that we may not use Theorem 4.2.2 in \mathbf{Ab} .

Proposition 5.5.2 *\mathbf{Ab} does not have enough subgroups.*

Proof. Let Z be the additive group of integers and let $q : Z \rightarrow Z/2Z$ be the standard quotient. Clearly q is non-constant and there is no non-trivial subgroup of Z on which q is monic. (All non-trivial subgroups of Z are infinite.)

5.6 Examples on \mathbf{Ab}

(1) Let P be a set of prime numbers. For each abelian group X , let S_X be the partially ordered set of all subgroups $M \subseteq X$ such that M only contains elements with order equal to a product of powers of members of P . Let X be an abelian group. It is easy to see that a subgroup $M \subseteq X$ is

in $c_X(S_X)$ iff none of its non-zero elements has order equal to a product of powers of members of P . Using similar reasoning it is then easy to see that $c_X(c_X(S_X)) = S_X$. It is then clear that $\mathcal{S} = \{S_X | X \in \mathbf{Ab}\}$ is a family of canonical relations on \mathbf{Ab} . By Theorem 4.2.1, we therefore obtain a $\|\text{-pair}$ $(\mathcal{F}^{\mathcal{S}}, \mathcal{G}^{\mathcal{S}})$.

A homomorphism $f : X \rightarrow A$ is in $\mathcal{F}^{\mathcal{S}}$ iff, for all $x \in X$, $f(x) = 0 \Rightarrow$ the order of x is a product of powers of members of P . A homomorphism $g : Y \rightarrow B$ is in $\mathcal{G}^{\mathcal{S}}$ iff, for all non-zero $y \in Y$, $g(y) = 0 \Rightarrow$ the order of y is not a product of powers of members of P (so y could have infinite order). $\overline{\mathcal{F}^{\mathcal{S}}}$ is contained in the class of all torsion abelian groups and $\overline{\mathcal{G}^{\mathcal{S}}}$ contains the class of all torsion free abelian groups.

We give an example to show that not all $\|\text{-pairs}$ in \mathbf{Ab} are obtained from families of canonical relations.

(2) Let \mathcal{F} consist of all homomorphisms $f : X \rightarrow A$ such that $f(x) = 0 \Rightarrow x$ is a square. (By a square we mean an element x of an abelian group X for which there exists $k \in X$ such that $k + k = x$.) Let \mathcal{G} consist of all homomorphisms $g : Y \rightarrow B$ such that $g(y) = 0 \Rightarrow y$ is not a non-zero square. It is clear that a homomorphism will always map a square to a square. It follows, using Proposition 4.1.3, that $f \|\ g$ for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$. So $\mathcal{F} \subseteq \mathcal{L}(\mathcal{G})$ and $\mathcal{G} \subseteq \mathcal{R}(\mathcal{F})$. Suppose $g : Y \rightarrow B$ is homomorphism which is not in \mathcal{G} . Then $g(y) = 0$ for some non-zero square y . Clearly the subgroup generated by y consists only of squares. Therefore its cokernel $k : Y \rightarrow C$ is in \mathcal{F} . By Proposition 4.1.3, it follows that $k \not\|\ g$ and therefore that $g \notin \mathcal{R}(\mathcal{F})$. So $\mathcal{R}(\mathcal{F}) \subseteq \mathcal{G}$. Now suppose $f : X \rightarrow A$ is not in \mathcal{F} . So $f(x) = 0$ for some non-square $x \in X$. It follows that x has even or infinite order. Let $q : X \rightarrow Y$ be the cokernel whose kernel is generated by the element $x + x$. It follows that $q(x)$ is not a square and that $q(x) + q(x) = 0$. Let $g : Y \rightarrow B$ be the cokernel whose kernel is the subgroup $\{0, q(x)\}$. It follows that $g \in \mathcal{G}$. Since f and gq both kill x but q does not, it follows by Proposition 4.1.3 that $f \notin \mathcal{L}(\mathcal{G})$. So $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{F}$. So $(\mathcal{F}, \mathcal{G})$ is a $\|\text{-pair}$ on \mathbf{Ab} .

We now show that $\mathcal{S}^{\mathcal{F}}$ is not a family of canonical relations on \mathbf{Ab} . Let Z denote the additive group of integers. Then $S_Z^{\mathcal{F}}$ contains all subgroups of Z other than Z itself. It follows $c_Z(S_Z)$ contains only the trivial subgroup $\{0\}$. Consequently $c_Z(c_Z(S_Z))$ contains all subgroups of Z (including Z itself). Therefore $S_Z \neq c_Z(c_Z(S_Z))$.

5.7 Examples on Set

Theorem 5.7.1 *There are no non-trivial \parallel -pairs on Set.*

Proof. Trivially, **Set** is a topological category. So a pair $(\mathcal{F}, \mathcal{G})$ of classes of functions is a \parallel -pair on **Set** iff there exists an FPR \sim on **Set** such that $(\mathcal{F}, \mathcal{G}) = (\mathcal{F}^\sim, \mathcal{G}^\sim)$.

Let $\sim = \{\sim_X \mid X \in \mathbf{Set}\}$ be an FPR on **Set**. Suppose we have a set X and distinct elements $x, x' \in X$ such that $x \sim_X x'$. Then there is a function $w : X \rightarrow \{0, 1\}$ such that $w(x) = 0$ and $w(x') = 1$. It follows that $0 \sim_{\{0,1\}} 1$. It then follows that $z \sim_Z z'$ for all sets Z and all $z, z' \in Z$. So \mathcal{F}^\sim contains all functions and \mathcal{G}^\sim contains only the injections.

Suppose instead that $x \sim_X x' \Rightarrow x = x'$ for all sets X and all $x, x' \in X$. Then \mathcal{G}^\sim contains all functions and \mathcal{F}^\sim contains only the injections.

We have considered the only two possibilities for \sim and in each case the \parallel -pair $(\mathcal{F}^\sim, \mathcal{G}^\sim)$ is trivial.

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6 Generalising the concordant-dissonant factorisation

In this chapter we discuss a number of generalisations of the notions of concordance and dissonance which Collins introduced in [C 1971] and to which we have already referred in the Introduction and Example 5.2(3). In Section 6.6 we consider the generalisation provided by the notion of a $\|\text{-pair}$ $(\mathcal{F}, \mathcal{G})$ such that $\mathcal{F} = L(\overline{\mathcal{G}})$.

6.1 The original results of Collins

Collins defined the concordant maps and the dissonant maps as follows. (Recall that the quasi-component of $x \in X$ is the intersection of all the clopen subsets of X which contain x .)

Definition 6.1.1 (C 1971) *A continuous map $f : X \rightarrow A$ is concordant iff, for each $a \in A$, $f^{-1}(\{a\})$ is contained in a quasi-component in X . A continuous map $g : Y \rightarrow B$ is dissonant iff, for each $b \in B$ and each quasi-component Q in Y , $g^{-1}(\{b\}) \cap Q$ is at most a singleton.*

Collins proved the following characterisation of the concordant quotient maps. It will be of particular interest when considering generalisations.

Proposition 6.1.2 (C 1971) *Let (Q, η) be the idempotent pointed endofunctor on \mathbf{Top} such that, for each X , $\eta_X : X \rightarrow Q(X)$ is the quotient map induced by the partition of X into quasi-components. Then a quotient map $f : X \rightarrow A$ is concordant iff $Q(f)$ is a homeomorphism.*

Although he did not use precisely Definition 2.1.1, Collins did prove all but trivial aspects of the following result. (Recall that \mathcal{F}^* is the class of all regular epimorphic members of \mathcal{F} and that quotient maps and regular epimorphisms coincide in \mathbf{Top} .)

Theorem 6.1.3 (C 1971) *Let \mathcal{F} be the class of all concordant maps and let \mathcal{G} be the class of all dissonant maps. Then $(\mathcal{F}^*, \mathcal{G})$ is a factorisation system on \mathbf{Top} .*

6.2 A generalisation by Preuss

In [CD 1977], Collins and Dyckhoff note that the concordant quotient maps are precisely the $\{2\}$ -extendable quotient maps (where 2 is the two-element discrete space). We recall the definition.

Definition 6.2.1 *Let \mathcal{C} be a category and let \mathcal{K} be a full subcategory of \mathcal{C} . A morphism $f : X \rightarrow A$ in \mathcal{C} is \mathcal{K} -extendable iff, for each solid arrow \mathcal{C} -diagram as follows*

$$\begin{array}{ccc} X & \xrightarrow{w} & Y \\ f \downarrow & \nearrow k & \\ A & & \end{array}$$

with $Y \in \mathcal{K}$, there exists k in \mathcal{C} such that the triangle commutes.

In [P 1979] Preuss assumes \mathcal{C} to be complete, cocomplete, wellpowered and cowellpowered and considers the \mathcal{P} -extendable extremal epimorphisms for an arbitrary full subcategory $\mathcal{P} \subseteq \mathcal{C}$. He defines analogues of the dissonant maps as follows.

Definition 6.2.2 (P 1979) *Let \mathcal{P} be a full subcategory of \mathcal{C} and let $\mathcal{E}^*(\mathcal{P})$ be the class of all \mathcal{P} -extendable extremal epimorphisms in \mathcal{C} . Then a morphism in \mathcal{C} is said to be relative \mathcal{P} -light iff it is a member of $(\mathcal{E}^*(\mathcal{P}))_{\perp}$.*

Preuss established that his generalisation yields a factorisation system.

Theorem 6.2.3 (P 1979) *Let \mathcal{P} be a full subcategory of \mathcal{C} and let $\mathcal{E}^*(\mathcal{P})$ be the class of all \mathcal{P} -extendable extremal epimorphisms in \mathcal{C} . Then $(\mathcal{E}^*(\mathcal{P}), (\mathcal{E}^*(\mathcal{P}))_{\perp})$ is a factorisation system on \mathcal{C} .*

For the rest of this section \mathcal{C} is assumed to be a topological category. Preuss had defined, in [P 1977], a generalised notion of quasi-component by means of the following.

Definition 6.2.4 (P 1977) *Let \mathcal{D} be a full subcategory of \mathcal{C} , let $X \in \mathcal{C}$ and let $M \subseteq X$. Then M is \mathcal{D} -connected with respect to X iff w is constant on M for all $w : X \rightarrow Y$ with $Y \in \mathcal{D}$. $C_{\text{rel}}(\mathcal{D})$ is defined to be the class of all pairs (X, M) such that M is \mathcal{D} -connected with respect to X .*

He used the term “relative connectedness” for a class \mathcal{K} of pairs (X, M) for which there exists $\mathcal{D} \subseteq \mathcal{C}$ such that $\mathcal{K} = C_{rel}(\mathcal{D})$. He characterised relative connectednesses, in [P 1977], in a way which enabled him to deduce that each $X \in \mathcal{C}$ may be partitioned into maximal M such that $(X, M) \in \mathcal{K}$. He called these \mathcal{K} -quasicomponents. He noted that the familiar quasi-components of a topological space (the ones in terms of which Collins’s concordant maps and dissonant maps are defined) are the special case obtained when $\mathcal{C} = \mathbf{Top}$, $\mathcal{D} = 2$ and $\mathcal{K} = C_{rel}(\mathcal{D})$.

In [P 1979] he related \mathcal{P} -extendable extremal epimorphisms and \mathcal{P} -quasicomponents as follows.

Theorem 6.2.5 (P 1979) *Let \mathcal{P} be a full subcategory of \mathcal{C} and let $f : X \rightarrow A$ be an extremal epimorphism in \mathcal{C} . The following are equivalent:*

- (1) *f is \mathcal{P} -extendable*
- (2) *for all $a \in A$, $f^{-1}(\{a\})$ is contained in a \mathcal{P} -quasicomponent of X*
- (3) *$Q(f)$ is an isomorphism, where (Q, η) is an idempotent pointed endofunctor corresponding to the extremal epireflective hull of \mathcal{P} .*

This established that much of the flavour of Collins’s original concordant quotient maps is retained by the generalisation. Similarly, the relative \mathcal{P} -light morphisms were shown to be a good generalisation of the dissonant maps.

Theorem 6.2.6 (P 1979) *Let \mathcal{P} be a full subcategory of \mathcal{C} . A morphism $g : Y \rightarrow B$ in \mathcal{C} is relative \mathcal{P} -light iff, for all $b \in B$ and all \mathcal{P} -quasicomponents $Q \subseteq Y$, $g^{-1}(\{b\}) \cap Q$ is at most a singleton.*

6.3 Dispersed factorisation structures

In [HSV 1979], Herrlich, Salicrup and Vásquez generalise further by assuming \mathcal{C} to be an $(\mathcal{E}, \mathbf{M})$ -category and considering, for an arbitrary full subcategory \mathcal{A} of \mathcal{C} , the class of all \mathcal{A} -extendable morphisms in \mathcal{C} which are also members of \mathcal{E} .

Definition 6.3.1 (HSV 1979) *Let \mathcal{A} be a full subcategory of \mathcal{C} , let $f : X \rightarrow A$ be a morphism in \mathcal{C} and let $(g_i : Y \rightarrow B_i)_{i \in I}$ be a source in \mathcal{C} . f is said to be \mathcal{A} -concentrated iff both $f \in \mathcal{E}$ and f is \mathcal{A} -extendable. $(g_i)_{i \in I}$ is said to be \mathcal{A} -dispersed iff \mathbf{M} contains the source which is obtained by enlarging $(g_i)_{i \in I}$ to include all $h : Y \rightarrow C$ in \mathcal{C} such that $C \in \mathcal{A}$.*

Letting $C(\mathcal{A})$ denote the class of all \mathcal{A} -concentrated morphisms in \mathcal{C} and letting $D(\mathcal{A})$ denote the collection of all \mathcal{A} -dispersed sources in \mathcal{C} , we have the following.

Theorem 6.3.2 (HSV 1979) \mathcal{C} is a $(C(\mathcal{A}), D(\mathcal{A}))$ -category.

Factorisation structures of the form $(C(\mathcal{A}), D(\mathcal{A}))$ were referred to as dispersed factorisation structures and were characterised as follows.

Theorem 6.3.3 (HSV 1979) Let \mathcal{C} be a (C, D) -category. The following are equivalent:

- (1) (C, D) is dispersed
- (2) $C \subseteq \mathcal{E}$ and, for all k and f in \mathcal{C} such that kf is defined, $f \in C$ if both $kf \in C$ and $f \in \mathcal{E}$.

Although dispersed factorisation structures are more general than the notions which were discussed in Section 6.2, something of Theorem 6.2.5 remains. The following is a trivial combination of two results given in [HSV 1979].

Theorem 6.3.4 (HSV 1979) Let \mathcal{A} be a full subcategory of \mathcal{C} and let $f : X \rightarrow A$ be in \mathcal{E} . The following are equivalent:

- (1) f is \mathcal{A} -concentrated
- (2) there exists k in \mathcal{C} such that $kf = \eta_X$, where (Q, η) is an idempotent pointed endofunctor corresponding to the \mathcal{E} -reflective hull of \mathcal{A} .

The comparison between Theorem 6.3.4 and Theorem 6.2.5 is aided by the trivial observation that, because $\mathcal{E} \subseteq \text{Epi}(\mathcal{C})$, 6.3.4(2) is equivalent to $Q(f)$ being an isomorphism. \mathcal{A} -dispersed sources were characterised along similar lines.

Theorem 6.3.5 (HSV 1979) Let \mathcal{A} be a full subcategory of \mathcal{C} and let $(g_i : Y \rightarrow B_i)_{i \in I}$ be a source in \mathcal{C} . The following are equivalent:

- (1) $(g_i)_{i \in I}$ is \mathcal{A} -dispersed
- (2) the source obtained by adding η_Y to $(g_i)_{i \in I}$ is in \mathbf{M} , where (Q, η) is an idempotent pointed endofunctor corresponding to the \mathcal{E} -reflective hull of \mathcal{A} .

In [HSV 1979], in the case where \mathcal{E} is the class of all extremal epimorphism in \mathcal{C} and \mathbf{M} is the collection of all monosources in \mathcal{C} , the \mathcal{A} -concentrated morphisms were called \mathcal{A} -concordant and the \mathcal{A} -dispersed sources were called \mathcal{A} -dissonant.

6.4 γ -concordant morphisms

In this section, let \mathcal{C} be an arbitrary category. In [BT 1984], Börger and Tholen generalise even further, giving definitions in terms of \mathcal{E} -prereflections. These were defined in [T 1984].

Definition 6.4.1 (T 1984) *Let \mathcal{E} be a class of morphisms in \mathcal{C} and let (R, γ) be a pointed endofunctor on \mathcal{C} . Then γ is an \mathcal{E} -prereflection iff, for each $X \in \mathcal{C}$, γ_X is an epimorphic member of \mathcal{E} .*

For the rest of this section, let \mathcal{E} be a fixed class of \mathcal{C} -morphisms which is assumed to be closed under composition and to contain all isomorphisms in \mathcal{C} . Assume too that each $X \in \mathcal{C}$ has an epimorphic \mathcal{E} -localisation.

Definition 6.4.2 (BT 1984) *Let (R, γ) be a pointed endofunctor on \mathcal{C} such that γ is an \mathcal{E} -prereflection on \mathcal{C} . A morphism $f : X \rightarrow A$ in \mathcal{E} is γ -concordant iff, for any solid arrow \mathcal{C} -diagram as follows,*

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ h \downarrow & & \downarrow k \\ Z & \xrightarrow{\gamma_Z} & R(Z) \end{array}$$

there exists a unique k in \mathcal{C} such that the square commutes. A source $(g_i : Y \rightarrow B_i)_{i \in I}$ in \mathcal{C} is γ -dissonant iff $f \perp (g_i)_{i \in I}$ for every γ -concordant f .

(Note that the \perp -relation between a morphism and a source is defined analogously to the \perp -relation between a morphism and a morphism.) Let the class of γ -concordant morphisms be denoted by $con(\gamma)$ and the collection of γ -dissonant sources by $diss(\gamma)$. It is worth noting the following observation for comparison with Theorems 6.2.5 and 6.3.4.

Proposition 6.4.3 (BT 1984) *Let $f : X \rightarrow A$ be an epimorphic member of \mathcal{E} . The following are equivalent:*

- (1) $f \in con(\gamma)$
- (2) there exists k in \mathcal{C} such that $kf = \gamma_X$.

The following factorisation result was obtained.

Theorem 6.4.4 (BT 1984) *If \mathcal{C} is \mathcal{E} -cocomplete (that is pushouts of members of \mathcal{E} exist and are in \mathcal{E} and multiple pushouts of members of \mathcal{E} exist and are in \mathcal{E}), then every source in \mathcal{C} has a locally orthogonal $con(\gamma)$ -factorisation.*

6.5 T-concordant morphisms

More recently, in [JT 1999], Janelidze and Tholen give a generalisation in terms of a weak factorisation system obtained from a pointed endofunctor. For this section, \mathcal{C} is assumed to be finitely complete.

Definition 6.5.1 (JT 1999) Let (T, η) be a pointed endofunctor on \mathcal{C} and, for each $f : X \rightarrow Y$ in \mathcal{C} , let $\tilde{T}(f)$ be defined such that the square

$$\begin{array}{ccc} \tilde{T}(f) & \xrightarrow{\widehat{\eta}_Y} & T(X) \\ \widehat{T}(f) \downarrow & & \downarrow T(f) \\ Y & \xrightarrow{\eta_Y} & T(Y) \end{array}$$

is a pullback. For each \mathcal{C}^2 -morphism

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{v} & W \end{array},$$

define $\tilde{T}(u, v)$ such that $\widehat{\eta}_W \cdot \tilde{T}(u, v) = T(u) \cdot \widehat{\eta}_Y$ and $\widehat{T}(g) \cdot \tilde{T}(u, v) = v \cdot \widehat{T}(f)$.

The pullback ensures that \tilde{T} is defined unambiguously on \mathcal{C}^2 -morphisms (once a choice has been made for the value of \widehat{T} on \mathcal{C}^2 -objects). It was noted in [JT 1999] that \tilde{T} is a weak factorisation system on \mathcal{C} . $\text{Conc}(T)$ and $\text{Diss}(T)$ were then defined as follows.

Definition 6.5.2 (JT 1999) Let (T, η) be a pointed endofunctor on \mathcal{C} such that $T(1) \cong 1$. Then $\text{Conc}(T) = \mathcal{E}_{\tilde{T}} \cap \text{RegEpi}(\mathcal{C})$. For each $f : X \rightarrow Y$ in \mathcal{C} , let $e_f : X \rightarrow \tilde{T}(f)$ be the unique morphism such that $f = \widehat{T}(f) \cdot e_f$ and $\widehat{\eta}_Y \cdot e_f = \eta_X$. Then $\text{Diss}(T) = \{f \text{ in } \mathcal{C} \mid e_f \in \text{Mono}(\mathcal{C})\}$.

The following factorisation theorem was obtained.

Theorem 6.5.3 (JT 1999) Let (T, η) be a pointed endofunctor on \mathcal{C} such that $T(1) \cong 1$. Suppose $(\mathcal{E}_{\tilde{T}}, \mathcal{M}_{\tilde{T}})$ is a factorisation system on \mathcal{C} and that \mathcal{C} has (regular epi, mono)-factorisations. Then $(\text{Conc}(T), \text{Diss}(T))$ is a factorisation system on \mathcal{C} .

Under the conditions of Theorem 6.5.3, the following characterisation of $\text{Conc}(T)$ was also provided.

Proposition 6.5.4 (JT 1999) *Let (T, η) be a pointed endofunctor on \mathcal{C} such that $T(1) \cong 1$. Suppose $(\mathcal{E}_{\bar{T}}, \mathcal{M}_{\bar{T}})$ is a factorisation system on \mathcal{C} and that \mathcal{C} has (regular epi, mono)-factorisations. Then, for any regular epimorphism $f : X \rightarrow Y$ in \mathcal{C} , $f \in \text{Conc}(T)$ iff there exists k in \mathcal{C} such that $kf = \eta_X$.*

6.6 $\|\text{-pairs } (\mathcal{F}, \mathcal{G}) \text{ such that } \mathcal{F} = L(\bar{\mathcal{G}})$

A recurring theme in the generalisations that have been considered in this chapter is that the class of morphisms which plays the role of Collins's concordant quotient maps is (or may be) defined in terms of a pointed endofunctor (T, η) and a class of morphisms \mathcal{E} . Let us restrict to the case where $\mathcal{E} = \text{RegEpi}(\mathcal{C})$ and where \mathcal{C} is finitely complete and has (regular epi, mono)-factorisations. It is well known that extremal epimorphisms and regular epimorphisms coincide under these conditions. All of the factorisation systems obtained in the previous sections of this chapter are then regular factorisation systems $(\mathcal{H}, \mathcal{G})$ on \mathcal{C} such that, for any $f : X \rightarrow A$ in $\text{RegEpi}(\mathcal{C})$ and $k : A \rightarrow C$ in \mathcal{C} , $kf \in \mathcal{H} \Rightarrow f \in \mathcal{H}$. From Theorems 3.4.1 and 3.3.5 it is easy to see that any such factorisation system on \mathcal{C} is of the form $(\mathcal{F}^*, \mathcal{G})$ where $(\mathcal{F}, \mathcal{G})$ is a $\|\text{-pair}$.

Let $(\mathcal{F}, \mathcal{G})$ be a $\|\text{-pair}$ on \mathcal{C} such that $(\mathcal{F}^*, \mathcal{G})$ is a factorisation system. It is obvious from Definition 2.1.1 and well known that one obtains an idempotent pointed endofunctor (T, η) on \mathcal{C} such that, for each $X \in \mathcal{C}$, η_X is the \mathcal{F}^* -part of the $(\mathcal{F}^*, \mathcal{G})$ -factorisation of $!_X$. It is then clear that, for all $f : X \rightarrow A$ in $\text{RegEpi}(\mathcal{C})$, $f \in \mathcal{F}^* \Leftrightarrow$ there exists k in \mathcal{C} such that $kf = \eta_X$. Let \mathcal{P} be the full subcategory of \mathcal{C} whose objects are all $X \in \mathcal{C}$ such that $!_X \in \mathcal{G}$. It is then well known and obvious that \mathcal{F}^* is precisely the class of all \mathcal{P} -extendable regular epimorphisms. Using Corollary 3.3.8 and Proposition 3.4.6, it is then easy to see that $\mathcal{F} = L(\mathcal{P})$. It follows that $\mathcal{F} = L(\bar{\mathcal{G}})$.

We conclude that the notion of a $\|\text{-pair } (\mathcal{F}, \mathcal{G})$ such that $\mathcal{F} = L(\bar{\mathcal{G}})$ fits in well with the existing generalisations of concordance and dissonance. We note too that it has the potentially valuable quality of being defined in all categories. It may be interesting to investigate its properties in unusual categories.

7 Generalising the submonotone-superlight factorisation

In this chapter we look at generalising the submonotone maps and superlight maps defined by Strecker in [S 1974] and already discussed in the Introduction and Example 5.2(4). We begin by considering a generalisation provided by Strecker himself in [S 1974]. We relate it to our theory with the aid of some definitions and observations given by Tiller in [T 1980]. We then focus our attention on work done by Clementino and Tholen in [CT 1998]. We conclude by suggesting that a good generalisation of submonotonicity and superlightness is provided by the $\|\text{-}$ pairs $(\mathcal{F}, \mathcal{G})$ such that $\mathcal{F} = L(\overline{\mathcal{G}})$ and $\mathcal{G} = R(\overline{\mathcal{F}})$.

7.1 Submonotone-superlight in terms of component subcategories

We begin by recalling Strecker's original definitions which present submonotone maps and superlight maps immediately in the company of something more general.

Definition 7.1.1 (S 1974) *A full subcategory \mathcal{A} of \mathbf{Top} is a component subcategory iff it satisfies the following conditions:*

- (1) *every topological space has a partition into maximal subspaces belonging to \mathcal{A} (known as \mathcal{A} -components)*
- (2) *if $X \in \mathcal{A}$ and $e : X \rightarrow Y$ is a continuous surjection then $Y \in \mathcal{A}$*
- (3) *\mathcal{A} contains no discrete spaces other than singletons.*

Definition 7.1.2 (S 1974) *Let \mathcal{A} be a component subcategory of \mathbf{Top} . A continuous map $f : X \rightarrow A$ is said to be \mathcal{A} -submonotone iff, for each $a \in A$, $f^{-1}(\{a\})$ is contained in an \mathcal{A} -component of X . A continuous map $g : Y \rightarrow B$ is said to be \mathcal{A} -superlight iff, for each $b \in B$ and each \mathcal{A} -component C of Y , $g^{-1}(\{b\}) \cap C$ is at most a singleton.*

We recall Definition 4.3.1 and observe that the class of \mathcal{A} -submonotone maps is precisely $\mathcal{F}_{\mathcal{A}}$ and the class of \mathcal{A} -superlight maps is precisely $\mathcal{G}_{\mathcal{A}}$. Strecker highlighted the case where \mathcal{A} is the full subcategory of \mathbf{Top} whose objects are the connected spaces. He omitted the prefix “ \mathcal{A} –” in this special case and proved the following.

Theorem 7.1.3 (S 1974) *Let \mathcal{F} be the class of all submonotone maps and let \mathcal{G} be the class of all superlight maps. Then $(\mathcal{F}^*, \mathcal{G})$ is a factorisation system on \mathbf{Top} .*

It is clear (see [T 1980]) that 7.1.1(3) is, in the presence of the other conditions of Definition 7.1.1, equivalent to saying that \mathcal{A} does not contain all topological spaces. Using this fact we establish the following characterisation of pairs of the form $(\mathcal{F}_{\mathcal{A}}, \mathcal{G}_{\mathcal{A}})$, where \mathcal{A} is a component subcategory of \mathbf{Top} . The proof makes use of Theorem 7.2.3 and some additional observations from [T 1980]. We supply these in Section 7.2.

Theorem 7.1.4 *Let $(\mathcal{F}, \mathcal{G})$ be a pair of classes of continuous maps. The following are equivalent:*

(1) *there exists a component subcategory \mathcal{A} of \mathbf{Top} such that $\mathcal{F} = \mathcal{F}_{\mathcal{A}}$ and $\mathcal{G} = \mathcal{G}_{\mathcal{A}}$*

(2) *$(\mathcal{F}, \mathcal{G})$ is a $\|\text{-}$ pair with the following properties:*

(a) $\mathcal{G} = R(\overline{\mathcal{F}})$

(b) *there is a pointed endofunctor (T, η) on \mathbf{Top} such that, for all spaces X , $\eta_X \in \mathcal{F}^*$ and, for all continuous maps k with domain X , $k \in \mathcal{F}^* \Rightarrow k$ is a first factor of η_X*

(c) \mathcal{F} *does not contain all continuous maps.*

Proof. (\Rightarrow) Suppose (1). From Definition 7.1.1 it is easy to see that $\mathcal{F}_{\mathcal{A}} = \widehat{\mathcal{F}_{\mathcal{A}}}$. It follows, by Theorem 4.3.2, that $(\mathcal{F}, \mathcal{G})$ is a $\|\text{-}$ pair and, by Theorem 4.3.3, that $\mathcal{G} = R(\overline{\mathcal{F}})$. For each space X , let η_X be the quotient map induced by the partition of X into \mathcal{A} -components. It is clear that $\eta_X \in \mathcal{F}^*$ and that all members of \mathcal{F}^* with domain X are first-factors of η_X . Let $w : X \rightarrow Y$ be a continuous map. From Definition 7.1.1 it follows that $\eta_Y w$ is constant on any subspace on which η_X is constant. Therefore, since $\text{RegEpi}(\mathbf{Top})$ is fibre determined and η_X is a regular epimorphism, there exists a unique continuous map $T(w)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{w} & Y \\ \eta_X \downarrow & & \downarrow \eta_Y \\ T(X) & \xrightarrow{T(w)} & T(Y) \end{array}$$

commutes. This establishes that we have a pointed endofunctor (T, η) which satisfies (2b). Since \mathcal{A} does not contain all topological spaces, it follows that \mathcal{F} cannot contain all continuous maps.

(\Leftarrow) Suppose (2). Let $\mathcal{A} = \overline{\mathcal{F}}$. It follows, by Theorem 4.3.3, that $\mathcal{F} = \widehat{\mathcal{F}}_{\mathcal{A}}$ and $\mathcal{G} = \mathcal{G}_{\mathcal{A}}$. Since \mathcal{F} does not contain all continuous maps, it follows that \mathcal{A} cannot contain all topological spaces. Therefore, by Theorem 7.2.3 and the comments that precede it, \mathcal{A} is a component subcategory of **Top**. It is then easy to see that $\widehat{\mathcal{F}}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}}$. This establishes (1).

7.2 The generalised component subcategories of Tiller

In this section, let \mathcal{C} be an arbitrary category. A generalisation of Strecker's component subcategories was given by Tiller in [T 1980]. We use the phrase "generalised component subcategory" to distinguish this notion from that of Definition 7.1.1.

Definition 7.2.1 (T 1980) *A sink $(u_i : Z_i \rightarrow X)_{i \in I}$ in \mathcal{C} is said to be chained iff, for any $w : X \rightarrow Y$ in \mathcal{C} , w is constant provided wu_i is constant for all $i \in I$.*

Definition 7.2.2 (T 1980) *A full subcategory \mathcal{A} of \mathcal{C} is said to be a generalised component subcategory of \mathcal{C} iff, for all $X \in \mathcal{C}$, $X \in \mathcal{A}$ provided there is a chained sink $(u_i : Z_i \rightarrow X)_{i \in I}$ in \mathcal{C} with $Z_i \in \mathcal{A}$ for all $i \in I$.*

Tiller observed that, when $\mathcal{C} = \mathbf{Top}$, generalised component subcategories coincide with component subcategories except for the fact that he allows **Top** to be a generalised component subcategory of itself. Under mild conditions on \mathcal{C} we are able to obtain generalised component subcategories from $\|\$ -pairs of a special type (those that satisfy 7.1.4(2b)).

Theorem 7.2.3 *Suppose \mathcal{C} has a terminal object, kernelpairs and coequalisers of kernelpairs. Let $(\mathcal{F}, \mathcal{G})$ be a $\|\$ -pair on \mathcal{C} for which there is a pointed endofunctor (T, η) such that, for all X , $\eta_X \in \mathcal{F}^*$ and, for all k with domain X , $k \in \mathcal{F}^* \Rightarrow k$ is a first-factor of η_X . Then $\overline{\mathcal{F}}$ is a generalised component subcategory of \mathcal{C} .*

Proof. Let $(u_i : Z_i \rightarrow X)_{i \in I}$ be a chained sink in \mathcal{C} such that $Z_i \in \overline{\mathcal{F}}$ for all $i \in I$. For each $i \in I$, since the coequaliser of the kernelpair of $!_{Z_i}$ is in \mathcal{F}^* , we must have that η_{Z_i} is constant. Since (T, η) is a pointed endofunctor, it follows that $\eta_X u_i$ is constant for all $i \in I$ and therefore that η_X is constant. It is then easy to see that all regular epimorphisms with domain X are in \mathcal{F}^* . Since \mathcal{C} has kernelpairs and their coequalisers, we then have, by Proposition 3.2.13, that all morphisms with domain X are in \mathcal{F} . So $X \in \overline{\mathcal{F}}$. Therefore $\overline{\mathcal{F}}$ is a generalised component subcategory of \mathcal{C} .

7.3 The approach of Clementino and Tholen

In this section \mathcal{C} is assumed to be finitely complete, to have coequalisers of kernelpairs and to be equipped with a proper and stable factorisation system $(\mathcal{E}, \mathcal{M})$. These assumptions were made in [CT 1998] where it was observed that \mathcal{C} consequently has (regular epi, mono)-factorisations. For any full subcategory \mathcal{A} of \mathcal{C} , we shall denote by \mathcal{A}/B the full subcategory of the slice category \mathcal{C}/B whose objects are the \mathcal{C} -morphisms $f : A \rightarrow B$ such that $A \in \mathcal{A}$. When using the operators r and l (recall Definition 3.1.5) in the slice category \mathcal{C}/B , we shall refer to them as r_B and l_B . This will help to keep track of which slice category we are working in.

We note that the term “concordant” is used in [CT 1998] to mean submonotone and “dissonant” is used to mean superlight. We shall continue to use “submonotone” and “superlight”. However, we shall avoid renaming the classes $Conc(\mathcal{A})$ and $Diss(\mathcal{A})$ which Clementino and Tholen define in [CT 1998].

We note also that Clementino and Tholen define the notion of a constant morphism in the equivalent way provided by Proposition 3.1.2. We shall continue to think in terms of Definition 3.1.1.

Definition 7.3.1 (CT 1998) *Let \mathcal{A} be a full subcategory of \mathcal{C} . Then $Diss(\mathcal{A}) = \bigcup\{r_B(\mathcal{A}/B) \mid B \in \mathcal{C}\}$ and $Conc(\mathcal{A}) = \bigcup\{l_B(r_B(\mathcal{A}/B)) \mid B \in \mathcal{C}\}$.*

The following useful characterisation was given.

Proposition 7.3.2 (CT 1998) *Let the following be a commutative \mathcal{C} -diagram in which $e \in \mathcal{E}$ and $m \in \mathcal{M}$.*

$$\begin{array}{ccccc} X & \xrightarrow{e} & P & \xrightarrow{m} & Y \\ & \searrow & & \swarrow & \\ & & & & B \\ & f & & g & \end{array}$$

$Diss(\mathcal{A})$ is the class of all g such that $X \in \mathcal{A} \Rightarrow gm$ is a monomorphism. $Conc(\mathcal{A})$ is the class of all f such that $g \in Diss(\mathcal{A}) \Rightarrow gm$ is a monomorphism.

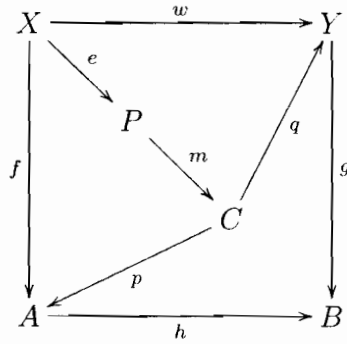
It was observed in [CT 1998] that $gm : P \rightarrow B$ is a monomorphism iff gm is preterminal in the category \mathcal{C}/X . We draw attention to the obvious similarity between Proposition 7.3.2 and Corollary 3.5.2. Indeed, there is a strong connection between the $(Conc(\mathcal{A}), Diss(\mathcal{A}))$ -pairs and the $\|H$ -pairs which we shall prove with the aid of the following lemma.

Lemma 7.3.3 (CT 1998) *Let \mathcal{A} be a full subcategory of \mathcal{C} . Then $Diss(\mathcal{A})$ is pullback stable and contains all monomorphisms in \mathcal{C} .*

Theorem 7.3.4 *Let \mathcal{A} be a full subcategory of \mathcal{C} . Then $(Conc(\mathcal{A}), Diss(\mathcal{A})) = (L^H(R^H(\mathcal{A})), R^H(\mathcal{A}))$.*

Proof. Let \mathcal{A} be a full subcategory of \mathcal{C} . Using Proposition 7.3.2 and Corollary 3.5.2, it is easy to see that $R^H(\mathcal{A}) \subseteq Diss(\mathcal{A})$. Using Proposition 7.3.2 and Definition 3.3.2, it is easy to see that $Diss(\mathcal{A}) \subseteq R^H(\mathcal{A})$.

Suppose $f \in Conc(\mathcal{A})$. Let $g \in R^H(\mathcal{A}) = Diss(\mathcal{A})$ and suppose we have the following commutative \mathcal{C} -diagram



where $e \in \mathcal{E}$, $m \in \mathcal{M}$ and the square $gq = hp$ is a pullback. By Lemma 7.3.3, $p \in Diss(\mathcal{A})$. By Proposition 7.3.2, it follows that pm is a monomorphism. By Corollary 3.5.2, it follows that $f \parallel^H g$. Therefore $f \in L^H(R^H(\mathcal{A}))$. Now suppose $f \in L^H(R^H(\mathcal{A}))$. Let $g \in Diss(\mathcal{A}) = R^H(\mathcal{A})$. Let $e \in \mathcal{E}$ and $m \in \mathcal{M}$ be such that $f = gme$. By Corollary 3.5.2, it follows that gm is a monomorphism. By Proposition 7.3.2, it follows that $f \in Conc(\mathcal{A})$. Therefore $Conc(\mathcal{A}) = L^H(R^H(\mathcal{A}))$.

From Theorem 7.3.4 and Proposition 3.4.8 we obtain the following.

Corollary 7.3.5 *For any full subcategory \mathcal{A} of \mathcal{C} , $Diss(\mathcal{A})$ is the right class of a \parallel -pair.*

The following establishes a property which $(Conc(\mathcal{A}), Diss(\mathcal{A}))$ -pairs and \parallel -pairs have in common.

Proposition 7.3.6 *Let \mathcal{A} be a full subcategory of \mathcal{C} . Then $f \in \text{Conc}(\mathcal{A})$ iff $\bar{f} \in \text{Conc}(\mathcal{A})$ (where \bar{f} is the regular epimorphic part of the (regular epi, mono)-factorisation of f).*

Proof. Let the following be a commutative \mathcal{C} -diagram such that m is a monomorphism.

$$\begin{array}{ccccc}
 & & Z & \xrightarrow{v} & Y \\
 & w \nearrow & \downarrow u & & \downarrow g \\
 X & \xrightarrow{\bar{f}} & M & \xrightarrow{p} & N \\
 & f \searrow & \downarrow m & \nearrow n & \\
 & & A & &
 \end{array}$$

(\Rightarrow) Suppose $f \in \text{Conc}(\mathcal{A})$ and $u \in \text{Diss}(\mathcal{A})$. By Proposition 3.2.10 and Corollary 7.3.5, $mu \in \text{Diss}(\mathcal{A})$. It follows, using Proposition 7.3.2, that w must coequalise the kernelpair of f and therefore that w coequalises the kernelpair of \bar{f} . It follows that the triangle $\bar{f} = uw$ is a constant morphism in the category \mathcal{C}/M . By Definition 7.3.1, it follows that $\bar{f} \in \text{Conc}(\mathcal{A})$.

(\Leftarrow) Suppose instead that $\bar{f} \in \text{Conc}(\mathcal{A})$. Further suppose that ng is a (regular epi, mono)-factorisation and that $ng \in \text{Diss}(\mathcal{A})$. Suppose too that vw is an arbitrary morphism such that $f = ngvw$ and that the square $gv = pu$ is a pullback. This construction depends on the existence of p which is guaranteed by the fact that $\bar{f} \in \text{RegEpi}(\mathcal{C})$ and $n \in \text{Mono}(\mathcal{C})$. By Proposition 3.2.10 and Corollary 7.3.5, $g \in \text{Diss}(\mathcal{A})$. Then, by Lemma 7.3.3, $u \in \text{Diss}(\mathcal{A})$. It follows, using Proposition 7.3.2, that w must coequalise the kernelpair of \bar{f} . Therefore vw coequalises the kernelpair of f . It follows that the triangle $f = ngvw$ is a constant morphism in the category \mathcal{C}/A . By Definition 7.3.1, it follows that $f \in \text{Conc}(\mathcal{A})$.

Proposition 7.3.6 has the following trivial consequence.

Corollary 7.3.7 *Let $(\mathcal{F}, \mathcal{G})$ be a \parallel -pair on \mathcal{C} . For any full subcategory \mathcal{A} of \mathcal{C} , $\mathcal{F} = \text{Conc}(\mathcal{A})$ iff $\mathcal{F}^* = \text{Conc}^*(\mathcal{A})$.*

From Theorems 7.3.4 and 3.3.5 we obtain the following.

Corollary 7.3.8 *Let \mathcal{A} be a full subcategory of \mathcal{C} . Then $(\text{Conc}(\mathcal{A}), \text{Diss}(\mathcal{A}))$ is a \parallel -pair iff every regular epimorphic first factor of any member of $\text{Conc}(\mathcal{A})$ is itself a member of $\text{Conc}(\mathcal{A})$.*

From Propositions 3.4.7, 3.4.8 and 3.4.9 and Theorem 7.3.4 we obtain the following.

Corollary 7.3.9 *Let $(\mathcal{F}, \mathcal{G})$ be a \parallel -pair on \mathcal{C} such that $\mathcal{F} = L(\overline{\mathcal{G}})$ and $\mathcal{G} = R(\overline{\mathcal{F}})$. Then $(\mathcal{F}, \mathcal{G}) = (\text{Conc}(\overline{\mathcal{F}}), \text{Diss}(\overline{\mathcal{F}}))$.*

From Proposition 7.3.6 and Theorems 3.4.1 and 7.3.4 we obtain the following (where $\text{Conc}^*(\mathcal{A})$ is the class of all regular epimorphic members of $\text{Conc}(\mathcal{A})$).

Corollary 7.3.10 *For any full subcategory \mathcal{A} of \mathcal{C} , $(\text{Conc}^*(\mathcal{A}), \text{Diss}(\mathcal{A}))$ is a prefactorisation system on \mathcal{C} .*

Half of Corollary 7.3.10 was established in [CT 1998]. From Corollary 7.3.10 and Theorem 2.1.7 we obtain the following.

Theorem 7.3.11 *Let \mathcal{C} have pushouts and have cointersections of regular epimorphisms and let \mathcal{A} be a full subcategory of \mathcal{C} . Then $(\text{Conc}^*(\mathcal{A}), \text{Diss}(\mathcal{A}))$ is a factorisation system.*

Recall that a morphism f is the generalised coequaliser of a family of pairs of morphisms $\{(u_i, v_i) | i \in I\}$ provided $fu_i = fv_i$, for all $i \in I$, and whenever there exists g such that $gu_i = gv_i$, for all $i \in I$, then there is a unique k such that $g = kf$. Recall too that a full subcategory \mathcal{A} is closed under \mathcal{E} -images if $Y \in \mathcal{A}$ whenever $e : X \rightarrow Y$ is in \mathcal{E} and $X \in \mathcal{A}$. Clementino and Tholen obtain factorisations by means of the following definition.

Definition 7.3.12 (CT 1998) *Suppose \mathcal{C} has generalised coequalisers and let \mathcal{A} be a full subcategory of \mathcal{C} which is closed under \mathcal{E} -images. Let f be in \mathcal{C} and let p_f be the generalised coequaliser of all pairs $(a \cdot (fa)_1, a \cdot (fa)_2)$ such that $a : A \rightarrow X$ is in \mathcal{C} and $A \in \mathcal{A}$. Let d_f be the unique morphism such that $f = d_f \cdot p_f$.*

$$\begin{array}{ccccc}
 A \times_Y A & \xrightarrow[(fa)_2]{(fa)_1} & A & \xrightarrow{a} & X \\
 & & & & \downarrow f \\
 & & & & Y \\
 & & & & \nearrow d_f \\
 & & & & P_f \\
 & & & & \nwarrow p_f
 \end{array}$$

f is said to be strongly \mathcal{A} -concordant iff d_f is an isomorphism. $SConc(\mathcal{A})$ is the class of all strongly \mathcal{A} -concordant morphisms.

It was observed in [CT 1998] that $f \in Diss(\mathcal{A})$ iff p_f is an isomorphism. The following factorisation result was established.

Theorem 7.3.13 (CT 1998) *Let \mathcal{C} have generalised coequalisers and let \mathcal{A} be a full subcategory of \mathcal{C} such that \mathcal{A} is closed under \mathcal{E} -images. If $SConc(\mathcal{A}) = Conc^*(\mathcal{A})$ then $(Conc^*(\mathcal{A}), Diss(\mathcal{A}))$ is a factorisation system on \mathcal{C} .*

Intuitively, one may think of a strongly \mathcal{A} -concordant morphism as being one which is constant on images of objects in \mathcal{A} and which kills as little else as possible. Arguably this is the essential quality of (\mathcal{A} -)submonotonicity. Therefore, if the factorisation system $(Conc^*(\mathcal{A}), Diss(\mathcal{A}))$ is to be considered a good generalisation of the submonotone-superlight factorisation, it seems reasonable to want $SConc(\mathcal{A})$ and $Conc^*(\mathcal{A})$ to coincide.

It is our opinion that $Conc(\mathcal{A})$ and $Diss(\mathcal{A})$ provide pleasing and interesting generalisations of the class of submonotone maps and the class of superlight maps. We suggest, however, that it might be good to impose the condition that every first factor of any member of $Conc(\mathcal{A})$ is itself a member of $Conc(\mathcal{A})$. This is certainly satisfied by the class of submonotone maps. Assume $(Conc(\mathcal{A}), Diss(\mathcal{A}))$ satisfies this condition. By Corollary 7.3.8, it follows that $(Conc(\mathcal{A}), Diss(\mathcal{A}))$ is a $\|\$ -pair. By Theorem 7.3.4 and Propositions 3.4.7 and 3.4.8, it follows that $Diss(\mathcal{A}) = R(\overline{Conc(\mathcal{A})})$. If in addition $(Conc^*(\mathcal{A}), Diss(\mathcal{A}))$ is a factorisation system then, by the comments in Section 6.6, $Conc(\mathcal{A}) = L(\overline{Diss(\mathcal{A})})$. Conversely, we have Corollary 7.3.9. We suggest that the notion of a $\|\$ -pair $(\mathcal{F}, \mathcal{G})$ such that $\mathcal{F} = L(\overline{\mathcal{G}})$ and $\mathcal{G} = R(\overline{\mathcal{F}})$ is both a simple and a good generalisation of the pair (Submonotone maps, Superlight maps). It may be interesting to investigate its properties in unusual categories.

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