THE MAPPING OF THE PRIMARY AND SECONDARY HARMONICS OF GYRO-MOTIONS USING MANUAL TIMING

G.J. VAN RIJSEWIJK

SUBMITTED TO THE UNIVERSITY OF CAPE TOWN

IN FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN ENGINEERING

CAPE TOWN

FEBRUARY 1992

DECLARATION BY CANDIDATE :

I hereby declare that this thesis is my own work and that it has not been submitted for a degree at any other university.

Signed

G.J. van Rijsewijk

Cape Town

10th February 1992

To my parents.

ACKNOWLEDGEMENTS

First, I want to thank my supervisor, Professor H.S. Williams, for always setting a very high standard in his work, and for requiring this same quality from me. He taught me to be more critical of my own writing and without his constant drive and new ideas it would have been impossible for me to complete this dissertation. He is especially thanked for devising the manual timing method for mapping the primary and secondary harmonics of the gyro-motion. I am also grateful to him for knowing where to find Category 1 T2/GAK1 configurations for my use. Associate Professor C.L. Merry assisted me in the final stages of this dissertation and his support is acknowledged.

I am deeply grateful for the assistance I received from all those who booked the observations for me, but in particular I want to thank Lauren Krumm who was willing to come to UCT in the early hours of the morning on many occasions. She supported me with the bulk of the observations and never complained for having to book non-stop for two hours at a time. On occasion, she was helped by Steven Pinker, Jenny D'Arcy-Evans and Albert Geiger. Professor Williams is also thanked for taking the first six hours of observations to provide the comparison standard for the work.

The willingness of the Department of Mining Engineering, University of the Witwatersrand, Johannesburg, the South African Navy from its naval dockyard in Simonstown, and the Chief Directorate of Surveys and Land Information, Mowbray for kindly making their instruments available is acknowledged.

Miss Bekker, the librarian of the Chief Directorate of Surveys and Land Information, Mowbray, was always extremely helpful in finding the right text among the many volumes of books on the shelves. A special thank you goes to Miss Bekker. The staff of the Computer Centre of the University of Cape Town are also acknowledged with appreciation for all the assistance given to me on the many occasions when I needed help. In particular, I want to thank Roger Haylett for providing me with the guidelines to document the software included in the appendices. Also, Jacqui King is kindly thanked for always being willing to assist me with the printing of the final text on the laser printer. Her profound knowledge of Chiwriter and the many intricacies that went along with it is appreciated.

Jenny D'Arcy-Evans kindly lent me her personal computer to type most of the thesis. My grateful thanks are extended to her for this great kindness. Colin Martin and Sue Binedell proofread the final draft and they are thanked for their time and effort.

Last, but by no means least, I thank my father, other family members, friends and colleagues for their affection and continuous encouragement, and regret that they cannot be acknowledged individually. My very grateful thanks go to my mother who died suddenly shortly before the completion of this work. While she was alive, she always encouraged and supported me in my studies. She will be remembered always.

SYNOPSIS

The suspended spinning-mass gyroscope (SMG) has been used for nearly 30 years to provide astronomic azimuth in a wide range of survey applications, such as underground mining, tunnelling and geodetic networks. The best accuracies which have been reported in the past are of the order of $1''.5$ to $3''.0$ for a single azimuth determination. All these results were obtained in the non-tracking mode and used the same fundamental mathematical model, namely the lightly-damped simple harmonic motion. The various observation models and observation reduction models have been used in conjunction with both electronically and manually-timed gyrotheodolites. The primary aim of these different observation methods had been to improve the accuracy of the gyro-azimuth. Forcing and multi-periodicity of the gyro-motion was first observed during the 1960s but it was not until the early 1980s that the first attempt was made to map the higher-order harmonics extant in the motion.

In the present study multi-period modelling of the gyro-motion from manually-timed observations is examined by making use of least squares spectral analysis. The influence such higher-order harmonics have on the accuracy of the gyro-azimuth is also investigated.

Chapter 1 deals with the historical development of the gyro-theodolite and a comparison of the accuracy of some of the instruments and observation methods is made. A brief overview of the work which led to the first mapping of a four-periodic oscillation from electronically-registered times is given.

Chapter 2 contains a summary of the various methods available for the analysis of time series, and a description of the theoretical model which represents the motion of the suspended gyroscope is given. The method of spectral analysis by least squares used in the present study is described in detail.

Chapter 3 provides a study of simulated gyro-motions and gives a comprehensive interpretation of the results from each of the manually-timed gyro-theodolite configurations used.

In Chapter 4 an investigation is made to explain some of the properties of the least squares power spectra from the different spin-ups. A sequential adjustment procedure is set up and the results thus derived are compared with results from smaller subsets. This is followed by an attempt to ascertain whether correlation exists between some of the parameters of the bi-periodic models found.

Chapter 5 contains the conclusions. The experiments show that the first two harmonics of a gyro-oscillation can be mapped from manually-timed observations. It has also been shown that the motion of the SMG can be amply described by a lightly-damped simple harmonic without loss of accuracy.

Detailed results from the single and two-periodic models are tabulated in Appendix A, while Appendices B-G contain additional information related to the simulated and manually-timed gyro-motions. Appendices H, J and K contain the listings of programs SPECTRAL, SYNTH2 AND SYNTH4, respectively, and include a summary of the computation time needed for various applications.

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CHAPTER 1 INTRODUCTION

1.1 HISTORICAL BACKGROUND

1.1.1 EARLY HISTORY

The basic principles which describe the motion of a spinning-mass gyroscope were first investigated in the middle of the last century by the French physicist Léon Foucault, famous for his pendulum experiments at Paris and at the cathedrals of Rheims and Amiens, which were aimed at proving that the earth rotated about its axis (Lauf 1963). He is also credited as being the originator of the term "gyroscope" which was used to describe the instrument he had invented; this was a fast-spinning wheel mounted in gimbals and suspended by a thin thread. It had only two degrees of freedom as the wheel was constrained by gravity to rotate only about the spin axis and the suspension thread, or vertical. The initial momentum of the spinning wheel was induced mechanically by means of an auxiliary geared machine (Bennett 1970), but because Foucault had no suitable source of power (electricity was still something for the future in his day) he was unable to maintain the angular velocity long enough to demonstrate the phenomenon of gyroscopic precession about the vertical. The electric motor was first applied to the gyroscope in 1878 and the first accurate gyrocompasses appeared at the turn of the century (Bennett

1970).

1.1.2 THE MERIDIAN INDICATOR

The mass of the gyroscope on its suspension tape stabilizes its motion and enables it to be used as a north-seeking device. The gyroscope precesses about the meridian through an angle of decreasing amplitude, the very light damping of the oscillations being mainly due to the resistance of the air in the gyro-case and to frictional forces. By contrast, the gyrocompass, unlike the suspended gyroscope, is a gyro-rotor suspended in gimbals with a mass fixed to the lower part of the gyro-frame. This unbalancing of the gyroscope creates a pendulous effect and constrains its motion to two degrees of freedom like the suspended gyroscope. However, the gyrocompass oscillatory motion has heavier damping to keep its heading as close to true north as possible. The great advantage of this type of compass is that it is immune to magnetic effects, electric fields or other interferences. This immunity to interference was of considerable importance to the shipbuilding industry when it switched from wooden to steel-hulled vessels which, together with the introduction of electrical machinery, made magnetic compasses unreliable. The first gyro-systems for navigational purposes were developed by Anschütz (Germany) in 1908, Sperry (USA) in 1911 and Brown (England) in 1916. A floated gyroscope designed by Dr. Anschütz for use in submarines and first produced in 1926 was subsequently modified for underground work by

Dr. Jungwirth (an assistant of Professor Rellensmann) at the Mining University of Clausthal in 1948. This rather cumbersome instrument, called the meridian indicator, came equipped with a compressed air generator to supply the electric current for the gyroscope and had a combined mass of about 500 kg. The MW2 meridian indicator was brought to South Africa by Jungwirth in 1951 and later made available to the University of the Witwatersrand where it was used on behalf of the Chamber of Mines by Lauf until 1960 for the transfer of bearings into many mines in southern Africa (Lauf 1963; Bennett 1970; Williams 1991a). The position of the axis of the spinning gyro-rotor could be determined relative to the outer case and the angular values of the reversal points of the precessional motion were then used to determine the null-position of the oscillations. This was done using the Schuler mean (Schuler 1932) with the observed extremal positions of the oscillation. The time taken for a single azimuth determination was about five hours and accuracies of no better than 60 arcseconds were obtained (Lauf 1963).

1.1.3 THE FENNEL KT1 GYROTHEODOLITE

During this period research into improving the gyroscope as a direction-finding instrument and making the instrument lighter and smaller was being carried out at Clausthal by J.I. McLelland in co-operation with Professor Rellensmann which resulted in the construction of the Fennel KT1 gyrotheodolite in 1959. This was a suspended spinning-mass gyroscope fitted underneath a single-second theodolite and was extensively used on various mines in southern Africa from 1961 onwards. The electric power was supplied by a transistorised converter (Bennett 1970). Lauf described the motion of the suspended gyroscope in terms of Newtonian principles in some detail in 1963 and showed that it followed that of a conical pendulum with light damping, in both the spinning and non-spinning mode (Lauf 1963). Like its predecessor, the MW2, the KT1 could only be used in the tracking mode and despite a considerable improvement in compactness the KT1 still had a mass of about 50 kg and took about two hours for a single bearing determination. Accuracies of about 5 arcseconds were reported by Lauf.

1.1.4 THE WILD GAK1 GYRO-ATTACHMENT

The Fennel KT1 gyrotheodolite was superseded in 1965 by the Wild GAK1 gyro-attachment, jointly developed by Wild-Heerbrugg (Switzerland) and Professor Rellensmann (Strasser and Schwendener, 1966). This instrument was initially intended for mining purposes only and was designed to be mounted on top of a Wild T16 or T1A theodolite (reading to approximately 5 arcseconds) and later on other conventional models by means of a bridge as well. One of the important features of the GAK1 was the inclusion of a graduated auxiliary scale in the field of view of the collimator telescope which enabled the observer to time the fiducial mark as it passed a particular scale division. For the first time gyro-observations could be taken in a non-tracking mode, as described by the transit method (Schwendener, 1964 ; Strasser and Schwendener, 1966) in which the horizontal circle of the theodolite remained clamped while the gyro-mark was timed across the zero graduation line on the auxiliary scale and the amplitude read directly off the scale. The position of the null-line of the oscillations could then be determined. According to the manufacturers of the GAK1 roughly comparable accuracies for the tracking and non-tracking methods could be expected in less than 30 minutes.

1.1.5 FORCING OF THE GYRO-MOTION

In 1966, Williams used a modified Wild T2 theodolite (with a 6° extended horizontal tangent screw motion) and a GAK1 gyro-attachment and was the first to report on the presence of forcing, disturbing or secondary harmonics in the gyro-motion (Williams and Belling 1967a). The amplitude of these harmonics was found to be as large as 50 arcseconds. This effect (named SHAR - Secondary HARmonic) was also manifested in limited tests carried out on both a Wild T16/GAK1 arrangement and a Fennel KT1 (single second) gyrotheodolite. An experiment involving protracted oscillation is also reported by Chrzanowski in which he observed longer spin-ups lasting 12-25 hours (Chrzanowski 1969) using a T16/GAK1 configuration. Apart from revealing a cyclic change with a period of about 24 hours, he mentions a "short period" of $1\frac{1}{2}$ to 3 hours with an amplitude of 20 to 80 arcseconds, which is in agreement with the empirical results obtained in the 1967 study by Williams. The secondary and tertiary harmonics obtained by Chrzanowski appeared to explain the discrepancies in azimuth determinations noticed in earlier experiments carried out by him at the University of New Brunswick, Canada. He proposed that for accurate work the mean of two sets of observations separated by half a wavelength (that is, about 12 hours) could remove the cyclic errors inherent in the motion.

Subsequently, a number of people have attempted to explain the phenomena observed during the 1960s. Halmos used MOM Gi-B1, Gi-B2, Gi-C and Gi-D instruments to observe series of 130-160 successive turning points each lasting 12-14 hours (Halmos 1971b; Grafarend and Halmos 1974) to which he applied classical harmonical analysis to differentiate between periodic and aperiodic signals. The observations were taken under laboratory conditions. From the results obtained he concluded that the regular pattern observed in the null-position over a period of time were largely due to internal thermal effects, and are different for each instrument (Halmos 1971b, Figures 1, 2 & 5). He repudiated the presence of any harmonic changes in the mean. The following year Grafarend derived a theoretical frequency spectrum from Eulerian equations, in which the nutational frequency has a period of only 0.85 msec, compared to 420 sec for the fundamental frequency (Grafarend 1972a). He assumed a mean latitude of 45° to obtain those quantities. Based on this small value for the second frequency, he predicted that such a superimposed oscillation would quickly dampen out and concluded that a long periodicity of the order suggested by Williams and Chrzanowski could not be derived from theory. Grafarend also observed long series with MW3, MW4a and GAK1 instruments consisting of about 70-90 turning points for the WBK instruments taken in the tracking mode, and about 90 turning points in the clamped mode for the WILD gyro-attachment (Grafarend 1972a, Table 3). On speculation that the SHAR-effect is stochastical in nature, he used the method of generalized harmonic analysis (Lee 1960) to distinguish between periodic functions, transient functions and random processes. In this method, the theory of correlation is applied to signal detection and statistical filtering and prediction, and allowance is made for the statistical description of a random process to its spectrum. The formulae used by Grafarend are applicable to the analysis of observations taken at equidistant points in time, that is, separated by a constant time-interval Δt. The highest frequency which can thus be determined is the so-called Nyquist frequency of $\frac{1}{2\Delta t}$ Hz, or a period of $2\Delta t$ seconds.

Like Halmos, Grafarend rejected the SHAR-effect and suggested a number of random influences on the motion instead (Halmos 1971b; Grafarend 1972a, 1972b; Grafarend and Halmos 1974). He further reported that the Fox-Schuler mean, when applied to the observations taken with the MW3 and MW4 instruments, did not completely filter out the first harmonic because of the heavy damping in the motion (Grafarend 1972a). This incomplete filtering was not noticed when the Thomas-mean (see below) or Thomas' so-called "exact mean" (Thomas 1967, formula 2) was used to centralize the observations. General harmonic analysis has also been applied to the long series observed by Halmos in 1971, using the Fox-Schuler mean for the lightly-damped oscillations (Grafarend and Halmos 1974). From their analyses they similarly concluded that the observations contained no secondary harmonic.

1.1.6 THE NULL-POSITION FROM TRACKING METHODS

The null-position, θ_0, of gyro-oscillations was usually obtained with the Fox-Schuler mean, developed independently by Fox (1924) and Schuler (1932). Other related methods were established by Thomas (1965), Lauf (1967) and Williams and Belling (1967b), who first offered general formulae for finding the null-position from n turning points. Thomas' expression to calculate the mean contains the coefficients of the binomial expansion and is rigorous for 4 turning points. A least squares solution for the adjustment of any number of turning points was published by Bennett in September 1968 (Bennett 1968). The L2-norm solution for the general case had been proved by Williams in April 1968 and used in lectures to students, but was never published (Williams 1991c).

1.1.7 THE MODIFIED TRANSIT METHOD

Notwithstanding the availability of the more convenient transit method (Strasser and Schwendener 1966), it was not until about 1970 that there was a shift to more general use of the GAK1 in the clamped mode. In a bid to acquire a greater number of observations during each spin-up of the gyroscope by timing the gyro-mark across more than one graduation line of the auxiliary scale, the so-called modified transit method was developed (Bennett 1969; Williams 1970; Halmos 1971a; Grafarend and Rymarzyk 1971). The various formulae were essentially similar in purpose. The formula offered by Halmos is identical to the one given by Bennett while, as opposed to Bennett, the fundamental term of the formula derived by Williams includes the light damping of the oscillation. Williams concluded in 1981 that the influence of damping was negligible and contributed very little to improving performance accuracy of the GAK1 (Williams 1981b). This new method of 1969 of sampling more transits events than had Schwendener in the transit method did not improve accuracy. Also, the increased amount of calculating generated by the modified transit method did not justify the effort required for its application.

1.1.8 EARLY EXPERIMENTS WITH SEMI-AUTOMATIC INSTRUMENTS

L.F. Gregerson, Chief of Research and Geodesy of the Geodetic Survey of Canada, was first to experiment with electronic timing of the transit events of the gyro-mark. The aim of the work done between 1969 and 1972 by the Survey of Canada was to replace the traditional astronomical method of determining orientation by gyro-azimuths Gregerson 1973). The instrumentation developed by the Geodetic Survey of Canada comprised a modified Wild GAK1 mounted on a Kern DKM3 geodetic theodolite (Gregerson 1974). A diaphragm with three slits replaced the conventional auxiliary scale and the electronic timing system had a resolution of 1 millisecond (Gregerson 1974; Gregerson et al. 1974; Gregerson 1977). In February 1972 this modified instrument was used to determine twelve gyro-azimuths on a reference line (Gregerson et al. 1974) and the results were compared with those obtained on another line using a manually operated MOM Gi-B2. The standard deviation produced by the electronic instrument was 0.76 arcsecond compared to 4.45 arcsecond with the standard MOM Gi-B2 instrument, using a variant of the transit method. The instrument was also tested in high latitudes, where it achieved lower accuracies (Gregerson et al. 1974). Based on the results from these experiments, Gregerson stated that accuracies of no better than 16-40 arcseconds could be obtained with a standard gyrotheodolite and concluded that an improvement in accuracy could only be possible if an automatic instrument with an electronic timing device was used (Gregerson et al. 1974; Gregerson 1974).

The Geodetic Survey of Canada started experimenting in 1970 to establish which instrument was most suitable for modification.

In 1972 the Canadian firm Tellurometer-Plessy, in cooperation with the Hungarian Optical Works MOM (Magyar Optikai Mürek), modified its Gi-B1 to include electronic registration and a new mathematical model was developed to include "gyroscopic drift, band torque drift and hysterises" (Gregerson 1977). This instrument, the Gi-B1A, was marketed as the "GYMO", and in 1974 it was used to determine gyro-azimuths in latitudes up to 80°N (Gregerson 1977, Table 1) with an accuracy of less than 6 arcseconds.

The improved accuracies achieved by Gregerson et al. in Canada were correctly attributed to more precise timing of the transit events, but their work failed to show that (Williams 1981a; 1986)

- accuracies better than the 16 to 40 arcseconds quoted by the manufacturers could be obtained with a manual instrument and a simple reduction model.
- gyrotheodolites operating in different environments (for example, on a tripod in a haulage underground) will produce different accuracies.
- the performance of a gyroscope diminishes over a period of time.
- different accuracies can be produced by gyrotheodolites of identical manufacture.

Apart from Gregerson's efforts, it was natural that subsequent research in gyro-technology concentrated much of its attention on automatic gyrotheodolites with electronic timing devices. Independent of Gregerson et al., Halmos also carried out experiments with semi-automatic MOM Gi-B1 and Gi-B2 instruments, having essentially the same electronic timing system as was developed by Gregerson. He used two photodiodes to capture the transit events accurate to 0.02 second (Halmos 1971a). After the gyro-rotor was spun up, the instrument was released and the observations were automatically recorded. When a voltage regulator was used to ensure a stable current supply, gyro-azimuths could reportedly be determined with an accuracy of 1-2 arcseconds for the MOM Gi-B2 instrument. 8-10 sets of observations were used. Slightly lower accuracies were reported for the Gi-B1 (Halmos 1971a). In a later publication, in a description of the measuring system for the MOM Gi-B2 instrument, Halmos quotes an attainable accuracy of 3 arcseconds after observing for two to three cycles of oscillation (Halmos 1977). In this updated version the null-line of the oscillations could be determined from a combination of the transit times and manually recorded turning points, or from transit times only. Similar results are obtained with the successor of the Gi-B1, the Gi-B11 (Lászió 1985). The method described by Halmos utilises either two or three photodiodes (Halmos 1977,1980), the angular separation between the diodes being 600 arcseconds (20 scale divisions) or 300 arcseconds (10 scale divisions), respectively. If time-observations were taken at all three photocells the transit time differences could be used in one of the hybrid forms of the modified transit method, such as the TEP/FOP methods developed by Grafarend and Rymarzyk (1971). The central photodiode was later abandoned (Halmos 1980). A reduction model applicable to MOM-instruments equipped with two diodes and used by MOM for their Gi-B21 model has been derived by Halmos (1980).

1.1.9 THE ROYAL SCHOOL OF MINES GAK1 MODIFICATION

A somewhat different approach was used by the Royal School of Mines, Imperial College, University of London in co-operation with Wild U.K. (Smith 1977) to improve the accuracy of the estimated turning points read off the auxiliary scale of a standard GAK1. As originally suggested by Dr Thomas, a micrometer device was fitted to the GAK1 container to measure the fractional part between the auxiliary scale division just before the turning point and the turning point itself. One turn of the micrometer drum covered one full scale division. The modified gyro-attachment was mounted on a T16 theodolite and tested in 1975 at the Royal School of Mines, London and in Cornwall (Smith, 1977). The amplitudes could be estimated to within $\frac{1}{100}$ of a scale division and to an accuracy of 0.02 divisions. During trials carried out in Cornwall a set of 39 observations produced a standard deviation of a "single azimuth determination" of 7.65 arcseconds, using the amplitude method.

1.1.10 BASIC PRINCIPLES OF MAMET

More than twenty years ago it was established that the turning points of the oscillations could be more accurately determined from the transit events (Williams 1970) than was possible by simple estimation from reading the extremal position off the scale. He stated that

> "Normally, when applying the transit method in practice, the initial half-amplitude 'a' is determined from direct auxiliary scale readings by substituting these, viz. a_l and a_r, in the formula $a = \frac{1}{2}(a_l+a_r)$. It would be reasonable to assume that each 'a' reading could be read to 0.2 of a scale division and that this estimate adequately represents the standard error of a_l and a_r, σ. Each reading of the scale is assumed independent and consequently $\sigma_a^2 = \frac{\sigma}{2}$, σ_a being the standard error of the half-amplitude. Therefore, $\sigma_a = 0.14$ div.
>
> If transit times of the gyro-mark across the auxiliary scale graduations +n and -n are observed it is more accurate to determine 'a' from these using (11).
>
> [equation (11) is
>
> $$n = a.\sin\left[\frac{\pi(\psi_n + \psi_{-n})}{T_t}\right]\cos\left[\frac{\pi(\psi_n - \psi_{-n})}{T_t}\right]$$
>
> in which ψ_n, ψ_{-n} are the time intervals between passages of the gyro-mark across the null-line and auxiliary

scale graduations n and -n respectively. The other symbols have the usual meanings.]*

...Differentiating (11) with respect to T_t, ψ_n, ψ_{-n} and 'a', recognising that n is a constant, we obtain (after simplification, solving explicitly for σ_a) ...

$$\sigma_a = \left(\frac{\pi a^2}{nT_t}\right)\left(\frac{3}{8}\right)^{\frac{1}{2}}\left\{\cos^2\left(\frac{2\pi\psi_n}{T_t}\right) + \cos^2\left(\frac{2\pi\psi_{-n}}{T_t}\right)\right\}^{\frac{1}{2}}\sigma_t \qquad \text{(17c)}$$

...For (the case when the angle between the zero graduation line of the auxiliary scale and the null-line, θ_0)* = 15.000 (T_t= 372.000 sec, ψ_n= 40.003 sec, ψ_{-n}= 22.796 sec, n = 5, a = 10.00 div. and σ_t= 0.2 sec), σ_a= 0.025 div. Thus, by determining the initial half-amplitude from the observed transit times a theoretical better result for 'a', than is possible from pure scale readings, is possible."

The above quotation contains the basic principles of what eventually became called the Modified Amplitude METhod (MAMET) which Williams developed circa 1975. This method (Williams 1978a, 1978b, 1979, 1981b, 1986) is really a refinement of the basic amplitude method rejected by Schwendener in favour of his transit method in 1964. See Williams (1978a) for a detailed mathematical proof of MAMET.

* The underlined portion in brackets is here added to the original text for explanatory reasons.

1.1.11 ACCURACY OF THE MODIFIED AMPLITUDE METHOD

Generally, transit time measurements taken at or near the null-line of the gyro-oscillations (that is, at the zero graduation line) are accurate to within 0.1 to 0.2 seconds, using a hand-held electronic stopwatch. Error in timing using the transit method, or its derivations, has a direct or even enhanced effect on the derived gyro-azimuth. Consequently, electronic timing improves the accuracy of both the transit measurement and the azimuth. With MAMET the effect of (manual) timing errors has been significantly reduced. Williams has shown that it is possible to use a standard GAK1 and a hand-held electronic stopwatch in conjunction with MAMET to obtain accuracies comparable with gyrotheodolites equipped with electronic timing devices (Williams 1979). In essence the gyro-mark is timed as it crosses the same auxiliary scale division just before and after a turning point, and the distance from the scale graduation line to the turning point can then be established in units of scale divisions. The stopwatch therefore replaces the function of the micrometer introduced at the Royal School of Mines in London (Smith 1977; Williams 1978b). In principle, the RSM micrometer solution would appear to be attractive since it offers the possibility of measuring the amplitude direct from the scale without recourse to time. In practice however, this method is deficient because the operator is forced to have "hands-on" contact with the instrument while its gyro-rotor is spinning. For best azimuth accuracy, such contact between the operator and the gyrotheodolite is unacceptable since it will disturb the motion of the gyroscope. By comparison, the MAMET procedure requires no contact between the observer and the GAK1, for example. During tests in 1978 under laboratory conditions, this chronometric procedure was used to reduce 82 sets of observations and produced an average standard deviation for a single gyro-azimuth determination, σ_0, of 1.75 arcseconds. The standard deviation of the arithmetic mean, σ_m, was 0.79 arcsecond. Schwendener's transit method, using transit times taken during the same spin-up of the gyro-rotor, was used for comparison and produced a mean σ_0 of 4.56 arcseconds from 80 determinations, with a σ_m of 2.07 arcseconds (Williams 1979).

1.1.12 IRREGULARITIES IN THE MOTION OF THE GYRO-ROTOR

Williams' experience with the GAK1 gyro-attachment in conjunction with T16, T1A and T2 theodolites over a period of some 15 years led him to conclude that the accuracy that could be expected with a gyrotheodolite configuration is dependent on the performance of the gyro-rotor. This is an electronic motor and therefore has a finite life (Williams 1978b). Williams suggested three distinct categories in which the behaviour of gyro-rotors could be classified, depending on the condition and/or age of the gyro-rotor :

- Category 1 - new and/or stable. Gyrotheodolite can be used for accurate azimuth determinations ($\sigma_0 \leq 5$ arcseconds)

- Category 2 - appearance of a systematic effect, for instance SHAR. Gyrotheodolite will take longer to produce the same order of accuracy as a category 1 instrument.
- Category 3 - erratic performance, that is, approaching failure. These instruments are unreliable and should not be used.

The gyrotheodolite used in the 1978 experiments was a Category 1 instrument. Also, experience in the Orange-Fish Tunnel (OFT) in the late 1960's and early 1970's (Williams 1978b, Table 2) with a GAK1 displaying a "marked SHAR effect" (Category 2, therefore) shows good results. However, these were obtained with great difficulty. Similar accuracies could have been achieved in a much shorter time had a Category 1 instrument been available (Williams 1981a, 1991c). The gyro-rotor of the GAK1 used in the OFT was replaced shortly afterwards. Another example of the use of a fairly new Category 2 instrument is the gyro-attachment used in the tests at the University of the Witwatersrand when the SHAR phenomenon was first studied (Williams and Belling 1967a). For an example of the deteriorating performance of a Category 3 instrument Table 6 of Williams (1978b) should be referred to. In that table, anomalous jumps in the amplitude during oscillation are patently obvious. This category of gyroscope is unreliable for accurate work. Electronic timing of the gyro-mark transit event does not compensate for a faulty rotor.

1.1.13 MAMET -VERSUS- TIME-CRITICAL METHODS

The MAMET procedure gives best results when the scale divisions nearest to the turning points are used. This ensures that the time-difference $\Delta\tau$ calculated from the successive transit events just prior to, and after, reversal is kept as small as possible. If this requirement is not observed, the fundamental equation of MAMET (Williams 1981b, equation 25) becomes sensitive to errors made in timing the transit event. The non time-criticality of MAMET will then be lost. If this condition is not met then, after some manipulation, the formula derived by Halmos in 1980 may be obtained from equation 25 (compare Halmos 1977, 1980). Halmos' method, as well as, for example, Schwendener's transit method, may therefore be described as belonging to the time-critical group of methods. The theoretical accuracies of these three methods have been compared (Williams 1981b, Table 4). There can be expected to be an improvement in accuracy if electronic timing is used, regardless of which reduction model is utilised, but such an improvement will be more noticeable for those methods which are sensitive to small errors in time-observations.

1.1.14 MULTI-PERIODICITY IN THE GYRO-MOTION

Experiments have shown that the motion of the suspended gyroscope is more complex than the simple harmonic pattern described by Lauf (1963). Higher order harmonics were first mentioned in 1967 by Williams and Belling when they coined the name "SHAR" to describe the effect which they observed during prolonged oscillation. Later work by Chrzanowski (1969) supported this suggestion. Notwithstanding the findings reported by Grafarend (1972a) and Halmos (1971b), a superficial inspection of the graphs showing the temporal changes in the Fox-Schuler mean (Halmos 1971b, figures 1, 2 & 5, especially Numbers 9-12) gives a clear indication that a similar cyclic effect as that observed by Williams and Chrzanowski is present in the motion. Partly in order to gain a better understanding of systematic effects, the least squares collocation technique, which separates the error model into "noise" and "signal", has been applied to six long series of turning points, including two observed with a MOM Gi-B1 instrument (Becker, Groten and Hein 1981). The parameters of superimposed harmonics contained in the motion can be determined by including these higher-order terms in the "signal" (a random component in the observables) of the observation equation, provided that provisional values for the amplitude and frequency of such harmonics are known. Becker et al. pointed out that the disturbances in the oscillation varied with time. This could have been a result of the insufficient sampling of the oscillation. They further suggested that the collocation method would be useful if "systematic or quasi-systematic errors" were assumed to be present.

1.1.15 LEAST SQUARES SPECTRAL ANALYSIS OF THE GYRO-MOTION

Jeudy extended the theory of motion of the suspended gyroscope (Jeudy 1981, 1982) and concluded that a total of five harmonics exist in a typical gyro-oscillation. In an attempt to measure these higher order harmonics more accurately, he applied least squares spectral analysis to 25 sequential transit times (Jeudy and Gagnon 1981) which were spread over four cycles, or almost 30 minutes. This method allowed for the evaluation of a n^{th}-order harmonic after subtracting the (n-1) harmonics from the observed signal, that is, the spectral value is a function of the variance factor determined from the residuals of the approximate n-periodic model. The transit times were recorded electronically to 0.001 second accuracy, using a Wild GAK1 instrument equipped with three photodiodes. The times were assumed to be error-free in the spectral analysis, and the angular values for the slits in front of the photodiodes were treated as quasi-observations. Their experimental values appeared to substantiate Jeudy's extended theory of motion for the gyroscope. They reported that the period of the second harmonic was roughly half that of the primary oscillation, whilst its initial half-amplitude was about 71 arcseconds compared to about 11534 arcseconds for the first harmonic. They also found tertiary and quaternary harmonics based on statistical considerations. The periods of these lower order harmonics were found to be 0.021 sec and 0.496 sec, respectively. This is much smaller than the smallest transit time sampling interval of about 24 seconds. The coarse sampling interval used by Jeudy is commented on critically (Williams 1991a) as not satisfying the Nyquist criterion, which broadly requires that for the determination of any period at least three sampled events must be captured within the time-interval containing the period to be determined.

Recent work at UCT has confirmed Jeudy's numerical results. In addition, it has been verified that the power spectrum after allowing for bi-periodicity reveals many peaks throughout the entire spectrum which are higher than a critical value for a preselected significance level (Steeves 1981) and any one of these peaks could be chosen to represent the approximate frequency of the third harmonic, in many cases resulting in a lower σ_0 than 1.071 arcseconds found by Jeudy.

Jeudy's work with the GAK1 attachment has since been extended to enable the full oscillation to be sampled. The eyepiece of the GAK1 was replaced by an array of 512 charged coupled device (CCD) photodiodes (Jeudy, Jobin and Fournier 1988), each photodiode being 25μm wide. Sampling of the oscillation at a rate of 30 Hz enables the gyro-azimuth to be obtained after one cycle, having applied "a correction for damping". By repeating the process every 10 minutes, the estimate of the azimuth can be updated. Experiments with this system under laboratory conditions have indicated that the standard deviation of the gyro-azimuth is about 16 arcsecs after one cycle, reducing to about 5 arcsecs after 40-50 minutes (3-4 cycles) of observing time (Jeudy, et al. 1988, Table 1). No improvement in accuracy could be achieved with additional observations.

1.1.16 COMPARISON OF DIFFERENT INSTRUMENTS

As has already been mentioned, accuracy comparisons of different instruments (MOM Gi-B2, WILD GAK1) under geodetic conditions were made in Canada during the early 1970s. This work was extended by Caspary and Heister (1981) using a MOM GiB1, Wild GAK1, MK10-2 and a Gyromat instrument to observe gyro-azimuths at all stations of the INNTAL test network maintained by the military university in Munich (Universität der Bundeswehr München). The latter two are semi-automatic instruments. Their aim first was to establish how accurate an azimuth could be determined with the different instruments and secondly, how accurate a geodetic network could be orientated with each instrument. The observed gyro-azimuths were compared to astronomical azimuths which had previously been established to an accuracy of 0.6 arcsecond. A number of different network adjustments, using the gyro-azimuths obtained with the various instruments, were also compared.

Varying observation techniques were employed with the four different gyrotheodolites in the INNTAL network. The MK10-2 was developed by the BodenseeWerk Gerätetechnik for military purposes and is an earlier version of the WILD GG1 automatic gyrotheodolite. The MK10-2 is a tracking instrument which, like the GiB2 (Gregerson 1970), uses servo-motors to align the rotation axis of the instrument to north, minimising torque present in the tape. This zero torque position of the tape is then included as a correction in the measuring process. Finally, the direction of the gyro-axis is read off via an auto-collimation system (Caspary and Heister 1981). The observing procedure takes a few minutes. For the INNTAL network observations, the MK10-2 was set up eccentrically since the horizontal circle readings of the attached theodolite could not be read off to geodetic accuracy. In the case of the Gyromat, the instrument is pre-oriented and the null-line of the oscillation, as well as the tape zero position, are displayed after a few minutes. A standard WILD GAK1 instrument was used for which Caspary and Heister reported an accuracy of only 15.8 arcseconds, which is in conflict with accuracies obtained by other writers. Lauf (1963) and Williams (1979) established the accuracy for the better GAK1 instruments at about 4 arcseconds. In light of their better results, it is possible that the instrument used by Caspary and Heister for their comparison was approaching the failure stage.

1.1.17 THE GYROMAT

The early work associated with gyro-azimuth determinations focused on the understanding of the motion of the gyroscope, and evolved contemporaneously with developments in the construction of the gyroscope (Lauf 1963, 1967; Williams and Belling 1967a; Bennett 1970; Williams 1970; Grafarend and Rymarzyk 1971; Halmos 1971a, 1971b; Gregerson 1977). The meridian indicator was succeeded by the Fennel KT1, the first gyro-theodolite, which in turn was followed by the GAK1-type of gyro-attachment. Interest then changed to achieving automatic azimuth. This appears to be a fourth generation development of the gyro-theodolite of which the Gyromat (Eichholz and Schäfler 1981; Heister and Schödlbauer 1990) is probably the most successful example to date. This instrument was developed largely as a result of the work of Dr. Stier, Dr. Jungwirth's successor at Clausthal, and is a semi-automatic variation of the manually-operated MW77 gyro-theodolite (Stier 1980). Improvements in the manufacture of the Gyromat has reduced the effects of some possible error sources (Becker, Groten and Hein 1981). The moment of gravity was increased to shorten the period of the oscillation (171 sec for a latitude of 51° compared with 386 sec at 34° for the GAK1) and a much lower angular velocity (3600 rpm compared with 22000 rpm for the GAK1) was chosen with the intention of extending the high-accuracy service life of the gyro-rotor (Eichholz and Schäfler 1981). Other features include high torsion rigidity of the gyro-suspension tape, improved mounting of the gyro-rotor bearings and a smaller amplitude range to protect the suspension tape (Williams 1991b). Continuous sampling of the oscillation by a photodiode enables the null-position to be established after one cycle has been completed, applying corrections for "geographic latitude, the tape zero position and certain temperature effects" (Caspary and Heister 1983). The instrument does not need to be pre-oriented and, after a 20 second warm-up period, the actual measurement is started, the oscillation being sampled at a rate of 40-50 Hz. The gyro-azimuth is displayed digitally. The Gyromat was recently used to provide orientation for the Channel tunnel (Chunnel) project between France and Great Britain.*

The manufacturers of the Gyromat, the Westfälische Berggewerkschaftskasse (WBK), specify an attainable accuracy of less than 1 mgon (about 3"), which has been confirmed by field tests (Caspary and Heister 1981; Heister and Schödlbauer 1990). The gyro-azimuths determined in the INNTAL network were accurate to about 2.9 arcseconds, using astronomical results as a reference. The Gyromat has its most appeal in that only a few additional computations are necessary once the offset of the null-line of the oscillations has been displayed.

1.2 THE AIM OF THE PRESENT STUDY

Due to a lack of electronic timing facilities for gyrotheodolites at the University of Cape Town when the writer commenced his work the research described in this dissertation relating to gyro-motion studies has been restricted to manual timing.

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* On 1 December 1990 the two tunnels, started from opposite sides in France and Britain, met with a lateral offset of 0.35m .

The purpose of the work described in the chapters that follow has been to study the form of realisation of the motion of the spinning gyroscope, and to examine the forcing of its motion. An attempt is made to model the causes of forcing by using multi-periodicity in conjunction with least squares spectral analysis. An investigation of the effects such distorting harmonics have on azimuth accuracy is also made.

As an introduction to this study simulated gyro-oscillations were generated by synthesizing a number of multi-period harmonics which represent real gyro-oscillations fairly closely. The times at which this simulated motion crossed the θ-axis at certain pre-selected θ-values were calculated and stored in the form of a time series. From these discrete data points the various harmonics were recovered successfully by least squares spectral analysis. This recovery is an indicator of the reliability of the different software developed. This enabled a better understanding of what could be expected from the spectral analysis of a real gyro-motion.

It is envisaged that the software described in Appendix H will ultimately be used in conjunction with a real-time GAK1 gyro-modification being developed at the Department of Electrical and Electronic Engineering, University of Cape Town. Consequently, sequential determinations of the null-position of the oscillations were also studied and the pattern so obtained was compared with similar experiences with the Fox-Schuler mean.

CHAPTER 2 DESCRIPTION OF THE MATHEMATICAL MODEL

2.1 NUMERICAL AND STATISTICAL TECHNIQUES APPLICABLE TO TIME-VARYING FUNCTIONS

A number of methods exist which permit the solution of a systematic signal contained in the observations. The most commonly used technique applied to time series involves the removal of a "systematic noise function" after which Fourier analysis is applied to the residuals to establish the spectrum (Taylor and Hamilton 1972). This method requires that the data are equally spaced. However, severe distortion occurs in the spectrum in the vicinity of the noise function removed earlier. An alternative method which can be used is least squares collocation, which allows for the solution of a "signal" besides the vector of parameters (Balmino 1978). The measurement is assumed to consist of a systematic part and a random part. Another technique, developed concurrently with the method of collocation, is called spectral analysis by least squares (Wells, Vaníček and Pagiatakis 1985; Jeudy and Gagnon 1981). This technique was first described by Vaníček (1969,1971) and does not require a constant time interval between observations. It has been demonstrated that Vaníček's method does not shift a peak in the spectrum situated in the vicinity of the "systematic noise function" (Taylor and Hamilton 1972). It is this latter method of least squares spectral analysis which we will use in this dissertation for the analysis of manually timed gyro-oscillations.

2.2 THE FUNDAMENTAL EQUATION OF GYROSCOPIC MOTION

Lauf (1963) has shown that the motion of the gyroscope can be expressed by the Newtonian differential equation,

$$\left(I_z + \frac{J^2}{M}\right) \frac{\partial^2 \theta}{\partial t^2} + p \frac{\partial \theta}{\partial t} + \left(J\Omega\cos\phi\right)\theta = 0 \tag{1}$$

This is a linear equation of second order with constant coefficients, in which

I_z = moment of inertia of the gyroscope about the vertical axis

J = angular momentum of the spinning wheel

M = gravity couple of the gyroscope

p = a constant

Ω = angular velocity of the earth

ϕ = latitude

$\frac{\partial \theta}{\partial t}$ = angular velocity of the gyroscope at instant t

$\frac{\partial^2 \theta}{\partial t^2}$ = angular acceleration of the gyroscope at instant t

The solution of this equation is a lightly damped simple harmonic motion,

$$\theta = Be^{-ft}\cos(\omega t - \gamma) \tag{2}$$

in which θ = precession angle about the meridian

B = initial half-amplitude of the oscillation

f = damping constant

ω = angular frequency (radians/second)

γ = arbitrary phase angle (radians)

t = time

and the period of the oscillation is given by $T = \frac{2\pi}{\omega}$ sec. If the oscillations are observed on a graduated auxiliary scale then a constant can be included to denote the equilibrium position of the oscillations,

$$\theta = \theta_0 + Be^{-ft}\cos(\omega t - \gamma) \qquad (3)$$

The properties of this lightly damped simple harmonic motion have been described in detail by Lauf (1963).

2.3 EXTENDED EQUATIONS OF GYROSCOPIC MOTION

Grafarend, in his description of the behaviour of the gyroscope, obtained two differential equations of the second order from Eulerian kinematical equations (Grafarend 1971, 1972a, 1972b). These two equations contained terms for both the horizontal and vertical motion of the gyroscope. The final solution for the precession angle of the gyroscope, allowing for two periodicities, had the form

$$\theta = \theta_0 + B_1 e^{-f_1 t} \sin(\omega_1 t) + B_2 e^{-f_2 t} \sin(\omega_2 t) \tag{4}$$

The theory of gyroscopic motion was extended by Jeudy who, from eight differential equations of the second order in eight unknowns (three coordinates for the lower end of the suspension tape, three Eulerian angles, the tension of the tape and the angular velocity of the gyroscope), derived a theoretical frequency spectrum containing five periodicities (Jeudy 1981, 1982) to describe the motion of the suspended gyroscope, that is,

$$\theta = \theta_0 + \sum_{i=1}^{N} B_i e^{-f_i t} \sin(\omega_i t - \gamma_i) \tag{5}$$

where $N = 1,2,\ldots,5$, the number of periodicities in the model. According to Jeudy's theory, the period of the second harmonic is half that of the first harmonic (the fundamental period) and the damping coefficient of the second frequency is twice the damping coefficient of the first frequency. Following his theoretical five-periodic model, multi-periodicity of the gyro-motion was tested by Jeudy and Gagnon (1981) using a single spin-up of a GAK1 in a least squares spectral analysis (Vaníček 1971; Wells and Vaníček 1978). In their emperical study it was confirmed that $T_1 = 2T_2$, whilst the ratio $f_1:f_2$ was also found to be in accordance with Jeudy's theory.

2.4 SPECTRAL ANALYSIS BY LEAST SQUARES

2.4.1 DESCRIPTION OF THE METHOD

In the present dissertation it is attempted to map the harmonic components contained in a gyroscopic motion from manually recorded observations as used by Williams in 1982 (Williams 1989, 1991a, 1991b), the multi-period harmonics being recovered by sampling across the auxiliary scale. Analysis of the motion is achieved by computing least squares power spectra from the recorded data points. This technique of spectral analysis as applied to gyroscopy was first demonstrated by Jeudy and Gagnon in 1981 when they recovered four harmonics from 25 transit times. The different spectra in the analysis give an indication of the contribution a particular frequency makes to the observation model. In other words, the first spectrum (that of the observed signal) shows the position of the fundamental frequency, the second spectrum (after allowing for single periodicity) indicates the frequency of the second harmonic, and so forth. After each spectrum has been evaluated, a least squares model is computed to allow for the newly-identified harmonic.

As was mentioned earlier, classical spectral analysis is usually based on a Fourier transformation of the covariance function (Lee 1960; Taylor and Hamilton 1972). This method breaks down when applied to non-stationary time series, that is if long linear or periodic trends are present in the motion (Wells, Vaníček and Pagiatakis 1985). This presence causes the power spectra to become disturbed. Vaníček developed a method dubbed "successive spectral analysis" in which the maximum possible contribution of a frequency ω to $F^T F$, the norm of a function F, is evaluated (Vaníček 1969, 1970; Wells, Vaníček and Pagiatakis 1985). This method lends itself to discrete time series containing unknown periodicities or for example linear, quadratic or exponential trends, all of which can be expressed as functions of time. In addition, the time series could be related to some arbitrary datum. A typical example of such a time series is one obtained from discrete data points observed on a trigonometrical function described by equation (5), in which the amplitudes θ_0 and B_i, the frequencies ω_i, the phases γ_i and the decay of the oscillations $e^{-f_i t}$ are unknown. The time series consists of two vectors, t_i (the observed times) and θ_i (the observed signal in terms of scale divisions), each containing n elements. Unlike Fourier analysis, Vaníček's method can be applied to both equally-spaced and unequally-spaced data. The term "time series" signifies data stored in chronological order. Strictly speaking, least squares spectral analysis does not require the observations to be ordered in any specific way.

2.4.2 SPECTRUM OF THE OBSERVED TIME SERIES

The parameters in (5) are solved in a stepwise fashion. Initially, the only systematic variation whose functional form is known, but not its magnitude, is θ_0 . Such constituents contained in the motion are called "systematic noise" (Vaníček 1970; Steeves 1981). To calculate the least squares spectrum of the observed signal, a simple mathematical model like (3) is assumed in which θ_0 is the datum bias. A single trigonometrical term is used where ω_1 is the unknown periodic signal, called the "systematic signal". Apart from θ_0, no other systematic noise functions are removed from the time series since we have no knowledge of any trends in the function at this stage, besides the obvious main oscillation which we are attempting to map. The exponential function in (3) is ignored to simplify the computations, giving the approximate mathematical model,

$$\theta = \theta_0 + B_1 \cos(\omega_1 t - \gamma_1) \tag{6}$$

This can also be written as

$$\theta = \theta_0 + B_1 (\cos \gamma_1 \cos \omega_1 t + \sin \gamma_1 \sin \omega_1 t)$$

or

$$\theta = \theta_0 + a_1 \cos(\omega_1 t) + b_1 \sin(\omega_1 t) \tag{7}$$

in which

$$\gamma_1 = \tan^{-1}\left(b_1/a_1\right) \tag{8}$$

and

$$B_1 = \sqrt{a_1^{\,2} + b_1^{\,2}} \tag{9}$$

The spectrum is then calculated in which j spectral values representing j different frequencies are plotted. An appropriate frequency increment is selected to ensure that the spectrum is sensitive enough to register the frequency under investigation.

Furthermore, the increment has to be such that when all the spectral values have been plotted on paper, a continuous curve can be drawn through all the points. Here, reference should be made to the computer output of program SPECTRAL in Appendix H where this point is illustrated. For our experiments, we took note of the fact that the period of the fundamental oscillation of the GAK1 in the clamped mode ranges from approximately 6 minutes (ω = 0.0175 rad/sec) at the equator to about $9\frac{1}{2}$ minutes (ω = 0.0110 rad/sec) in latitude 75° (Strasser and Schwendener 1966). A frequency interval of 0.0001 rad/sec between successive spectral values in the least squares spectrum will therefore be adequate to capture the main oscillation frequency. An initial and final frequency of 0.01 and 0.02 rad/sec, respectively, would define the total range of the primary frequency and these would specify 100 spectral values in the spectrum of the observed signal.

The unknowns θ_0, a_1 and b_1 for each frequency ω_j in (7) are yielded by the solution of the normal equations

$$\underline{X} = (\underline{A}^T\underline{A})^{-1}\underline{A}^T\underline{f} \tag{10}$$

The normal equations themselves are obtained from the canonical form $\underline{V} = \underline{AX} - \underline{f}$, in which

$$\underline{A} = \begin{pmatrix} 1 & \cos(\omega_1 t_1) & \sin(\omega_1 t_1) \\ 1 & \cos(\omega_1 t_2) & \sin(\omega_1 t_2) \\ \vdots & \vdots & \vdots \\ 1 & \cos(\omega_1 t_n) & \sin(\omega_1 t_n) \end{pmatrix} \tag{11}$$

and $\underline{f}$, the vector of absolute terms, contains the observed scale readings. Observational errors in the times are neglected in the spectrum of the observed signal, and it is assumed that no correlation exists between the observations. Unit weight is used therefore. Compare against formula 3 of Jeudy, et al. (1981) where it is shown how the normal equations are formed. The residuals which correspond to each spectral frequency need not be evaluated in the parametric adjustment, since the normalised spectral value can be calculated from (Wells, Vaníček and Pagiatakis 1985)

$$s(\omega) = \frac{\underline{f}^T \underline{AX}}{\underline{f}^T \underline{f}} \quad , \quad \text{where} \quad 0 \le s(\omega) \le 1 \tag{12}$$

In effect, the larger the spectral value, the smaller the difference between $\underline{AX}$ and $\underline{f}$ and hence the residuals obtained from $\underline{V} = \underline{AX} - \underline{f}$. This means that the largest spectral value corresponds to a frequency which yields the smallest residuals, that is, the smallest variance factor. This best-fitting frequency is then selected as the approximate value for ω_1, $(\omega_1)_0$, where $()_0$ is used to denote approximation. Steeves has shown that, assuming that the observed time series contains statistically independant variables, a critical value at a given significance level α can be computed from a knowledge of the degrees of freedom in the least squares solution (Steeves 1981; Wells, Vaníček and Pagiatakis 1985). The 99.5% critical value is given by

$$CR = 1 - (\alpha)^{2(df)^{-1}} \tag{13}$$

in which α = 0.005 and df are the degrees of freedom. It follows that any spectral value lower than the critical value, CR, is assumed to have been caused by random affects only and its spectral frequency is rejected on the grounds of being statistically insignificant.

2.4.3 THE SOLUTION OF THE SINGLE PERIODIC MODEL

The parameters in the single periodic model

$$\theta = \theta_0 + B_1 e^{-f_1 t} \cos(\omega_1 t - \gamma_1) \tag{14}$$

are now solved by least squares. In their experiments, Jeudy and Gagnon recorded the times to msec accuracy. The method developed by Vaníček which they utilised is applicable to time-series such as tidal gauge records where errors in time are negligible. Jeudy et al. were justified in assuming that the observations could be regarded as error-free (Jeudy and Gagnon 1981). This assumption is certainly valid for the relatively low frequencies of the first two harmonics which they found.

The angular values of the slots situated in front of the light-sensitive photodiodes were treated as quasi-observations. In contrast, the time-observations for our experiments were recorded with an electronic stopwatch and timing errors must be taken into account. Hence, like Breach (1985), we have to deviate from Jeudy and Gagnon in the determination of our parameters. Implicitly, it can be assumed that there is no significant error in the graduation lines of the auxiliary scale, making t the only observable quantity. Linearisation of (14) gives

$$\theta = (\theta)_0 + \frac{\partial\theta}{\partial\theta_0}d\theta_0 + \frac{\partial\theta}{\partial f_1}df_1 + \frac{\partial\theta}{\partial\omega_1}d\omega_1 + \frac{\partial\theta}{\partial B_1}dB_1 + \frac{\partial\theta}{\partial\gamma_1}d\gamma_1 + \frac{\partial\theta}{\partial t}dt \qquad (15)$$

in which

$$\left.\begin{aligned}
\frac{\partial\theta}{\partial\theta_0} &= 1 \\
\frac{\partial\theta}{\partial f_1} &= -B_1 t e^{-f_1 t}\cos(\omega_1 t-\gamma_1) \\
\frac{\partial\theta}{\partial\omega_1} &= -B_1 t e^{-f_1 t}\sin(\omega_1 t-\gamma_1) \\
\frac{\partial\theta}{\partial B_1} &= e^{-f_1 t}\cos(\omega_1 t-\gamma_1) \\
\frac{\partial\theta}{\partial\gamma_1} &= B_1 e^{-f_1 t}\sin(\omega_1 t-\gamma_1) \\
\frac{\partial\theta}{\partial t} &= \sum_{i=1}^{N} -B_i e^{-f_i t}\Big(f_i\cos(\omega_i t-\gamma_i) + \omega_i\sin(\omega_i t-\gamma_i)\Big)
\end{aligned}\right\} \qquad (16)$$

where N is the number of harmonics solved for in the model. For the single periodic model, N = 1. Equation (15) can be written in the form of the general adjustment case,

$$\underline{AX} + \underline{BV} = \underline{f} \tag{17}$$

in which the design matrix has the form

$$\underline{A} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_0} & \left(\frac{\partial\theta}{\partial f_1}\right)_1 & \left(\frac{\partial\theta}{\partial\omega_1}\right)_1 & \left(\frac{\partial\theta}{\partial B_1}\right)_1 & \left(\frac{\partial\theta}{\partial\gamma_1}\right)_1 \\ \frac{\partial\theta}{\partial\theta_0} & \left(\frac{\partial\theta}{\partial f_1}\right)_2 & \left(\frac{\partial\theta}{\partial\omega_1}\right)_2 & \left(\frac{\partial\theta}{\partial B_1}\right)_2 & \left(\frac{\partial\theta}{\partial\gamma_1}\right)_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial\theta}{\partial\theta_0} & \left(\frac{\partial\theta}{\partial f_1}\right)_n & \left(\frac{\partial\theta}{\partial\omega_1}\right)_n & \left(\frac{\partial\theta}{\partial B_1}\right)_n & \left(\frac{\partial\theta}{\partial\gamma_1}\right)_n \end{pmatrix} \tag{18}$$

the misclosure vector is

$$\underline{f} = \begin{pmatrix} \theta_1 - \left(\theta_0 + \sum_{i=1}^{N} B_i e^{f_i t_1} \cos(\omega_i t_1 - \gamma_i)\right)_0 \\ \theta_2 - \left(\theta_0 + \sum_{i=1}^{N} B_i e^{f_i t_2} \cos(\omega_i t_2 - \gamma_i)\right)_0 \\ \vdots \\ \theta_n - \left(\theta_0 + \sum_{i=1}^{N} B_i e^{f_i t_n} \cos(\omega_i t_n - \gamma_i)\right)_0 \end{pmatrix} \tag{19}$$

$\underline{X}$ contains the corrections to the parameters, $\underline{B}$ is a diagonal matrix of order n (the number of observations) with $\frac{\partial\theta}{\partial t_i}$ in the i^{th} position and $\underline{V}$ is the vector of residuals. N has the same meaning as before. To calculate the elements in (18) and (19) an initial value of zero for f_1 has to be adopted. Prior to the determination of the single periodic model, the approximate frequency $(\omega_1)_0$ selected from the spectrum is used to evaluate the coefficients θ_0, a_i and b_i, where i = 1,2,...,N (N = 1, the number of harmonics in a single periodic model) in a parametric adjustment. A value for $(B_i)_0$, the initial half-amplitude for the i^{th} harmonic, is then derived from (9) and $(\gamma_i)_0$ is obtained by putting

$$\Gamma_i = \tan^{-1}\left(\frac{|b_i|}{|a_i|}\right) \tag{20}$$

and by using the following sign convention :

b_i	a_i	γ_i
+	+	$\pi - \Gamma$
+	-	$\pi - \Gamma$
-	-	$\pi + \Gamma$
-	+	$2\pi - \Gamma$

TABLE 1 : RELATIONSHIP BETWEEN a, b AND γ

The corrections to the unknowns are calculated from

$$\underline{X} = \left[\underline{A}^T \left(\underline{BW}^{-1}\underline{B}^T\right)^{-1} \underline{A}\right]^{-1} \underline{A}^T \left(\underline{BW}^{-1}\underline{B}^T\right)^{-1} \underline{f} \tag{21}$$

and the values for the parameters in (14) are updated. This iteration process is repeated until $|\underline{X}| < 1E^{-10}$. Once this threshold value has been reached, the correlates are then obtained from

$$\underline{K} = -\left(\underline{BW}^{-1}\underline{B}^T\right)^{-1} \left(\underline{AX}-\underline{f}\right) \tag{22}$$

and since $\underline{X}$ contains very small numbers in an iterative solution, (22) can be approximated by

$$\underline{K} = \left(\underline{BW}^{-1}\underline{B}^T\right)^{-1} \underline{f} \tag{23}$$

Finally, the residuals of the least squares model are determined by

$$\underline{V} = \underline{W}^{-1}\underline{B}^T\underline{K} \tag{24}$$

and the *a posteriori* standard deviation of the time-observations is given by

$$\hat{\sigma}_0 = \left(\frac{V^T WV}{n-m} \right)^{\frac{1}{2}} \tag{25}$$

in which n are the number of observations, m are the number of unknowns and n-m are the degrees of freedom in the solution.

2.4.3.1 CHOICE OF WEIGHT MODELS

Two different weight models were tried in (17) and (21). The first model was a simple weight matrix with unit weight. The second model was obtained from (2) by putting

$$\theta \simeq B\cos(\omega t-\gamma) = B\cos\Phi \tag{26}$$

Linearisation of (26) and propagation of error gives

$$\sigma_\theta^2 = (\cos\Phi)^2\sigma_B^2 + (B\sin\Phi)^2\sigma_\Phi^2 \tag{27}$$

If the assumption is made that B is a constant, (27) can be simplified to

$$\sigma_\theta^2 \propto (\sin\Phi)^2\sigma_\Phi^2 \quad , \text{ where } \sigma_\Phi^2 = \omega^2\sigma_t^2 + \sigma_\gamma^2 = \text{a constant}$$

$$\therefore \quad \sigma_\theta^2 \propto \sin^2\Phi \tag{28}$$

and from (28) it follows that each observation i can be given a variance of $W_i \propto \sin^2(\omega_1 t_i - \gamma_1)$ on the main diagonal of $\underline{W}$. It is assumed that no correlation exists between the observations. Since both $\underline{B}$ and $\underline{W}$ are diagonal matrices, it is implicit that the matrix containing the quasi-weights, $\left(\underline{BW}^{-1}\underline{B}^T\right)^{-1}$, is also a diagonal matrix. This is true for both weight models which were tried in the spectral analysis. The quasi-weights used in (21) and (24) are determined from Table 2.

OBSERVATION$_i$	UNIT WEIGHT	APPROX. WEIGHT
QUASIWEIGHT	$\dfrac{1}{\left(\dfrac{\partial\theta}{\partial t_i}\right)^2}$	$\dfrac{\sin^2(\omega_1 t_i - \gamma_1)}{\left(\dfrac{\partial\theta}{\partial t_i}\right)^2}$

TABLE 2 : FORM OF THE QUASI-WEIGHTS

The results in Appendix A show that the determination of θ_0, the null-position of the oscillation, is affected minimally by the choice of weight models, the largest difference being $0\overset{''}{.}54$ (observation set 6, Tables A2 and A11) against the smallest difference of $0\overset{''}{.}06$ for observation set 21 (Tables A5 and A14). Intuitively it can be said that the second weight model is probably more correct, since the gyromark is moving slower as it moves further away from the zero graduation line and the corresponding errors in timing tend to become more noticeable.

2.4.4 SPECTRUM AFTER ALLOWING FOR SINGLE PERIODICITY

Having solved for the first harmonic, the second harmonic (now the "signal") has to be represented in a least squares spectrum. This spectrum can not be affected by the removal of the noise constituents that have been solved for in the single periodic model. The general form of the "systematic noise" function, that is, the datum and the damped harmonic, is known (from the solution of the single periodic model), but not its magnitude, θ_0 and B_1. The usual technique of calculating the spectrum of a time series with noise consists of the preliminary removal of the noise, having determined its magnitudes by least squares, and then by carrying out a Fourier analysis of the residuals (Taylor and Hamilton 1972). In our case, having removed the systematic noise, namely the datum bias and the damped trigonometrical function, as effectively as possible from the time series, least squares estimates were made of the "signal", a_2 and b_2, as well as for improved values for the magnitude of the systematic noise. This was done for each spectral frequency, ω_2. The functional relationship

$$\theta = \theta_0 + B_1 e^{-f_1 t} \cos(\omega_1 t - \gamma_1) + a_2 \cos(\omega_2 t) + b_2 \sin(\omega_2 t) \qquad (29)$$

was used to describe the bi-periodical model in the spectrum after allowing for single periodicity. Here, and in all spectra hereafter, the general adjustment case (equation 17) is used. An appropiate weight model is used, as before, from Table 2. The four parameters (θ_0, B_1, a_2 and b_2) are solved from (21), in which the design matrix now has the form

$$\underline{A} = \begin{pmatrix} 1 & e^{-f_1 t_1}\cos(\omega_1 t_1 - \gamma_1) & \cos(\omega_2 t_1) & \sin(\omega_2 t_1) \\ 1 & e^{-f_1 t_2}\cos(\omega_1 t_2 - \gamma_1) & \cos(\omega_2 t_2) & \sin(\omega_2 t_2) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & e^{-f_1 t_2}\cos(\omega_1 t_2 - \gamma_1) & \cos(\omega_2 t_2) & \sin(\omega_2 t_2) \end{pmatrix} \qquad (30)$$

while the misclosure vector $\underline{f}$ and the $\frac{\partial\theta}{\partial t}$ terms in $\underline{B}$ are obtained from (19) and (16), respectively. $N = 2$ in the bi-periodical model. However, since we have no knowledge of either B_2, f_2 or γ_2 at this stage of the analysis, the summation in effect applies only up to $i = 1$ in both $\underline{f}$ and $\underline{B}$. These two matrices are therefore identical to their counterparts in the final iteration of the single periodic model in (21). The residuals corresponding to each spectral value are then calculated from

$$\underline{R} = \underline{W}^{-1}\underline{B}^{T}\left(\underline{BW}^{-1}\underline{B}^{T}\right)^{-1}\left(\underline{AX} - \underline{f}\right) \tag{31}$$

and the normalised spectral value $s(\omega)$ is

$$s(\omega) = 1 - \frac{\underline{R}^{T}\underline{WR}}{\underline{V}^{T}\underline{WV}} \tag{32}$$

in which $\underline{V}$ is the residual vector from the single periodic model.

2.4.5 THE SOLUTION OF THE BI-PERIODIC MODEL

Once again, a frequency is selected from the spectrum. The two phase angles γ_1 and γ_2 are evaluated from (20) and table 1, and the bi-periodic model is solved from

$$\theta = \theta_0 + \sum_{i=1}^{N=2} B_i e^{-f_i t} \sin\left(\omega_i t - \gamma_i\right) \tag{33}$$

augmenting $\underline{A}$ (equation 18) by four columns (for f_2, ω_2, B_2 and γ_2) and putting $N = 2$ in both (16) and (19). The various bi-periodical models so recovered are tabulated in Appendix A.

Purely as an experiment, the influence of a linear drift as suggested by Gregerson (1970), Halmos (1971b), Breach (1985) et al. was also investigated in the present study, but such a drift was found to be insignificant. Furthermore, incorporation of a linear drift made no difference to the various frequencies of the single and two-periodic models shown in Appendix A. This expanded mathematical model also did not alter the fact that more than one peak occurs in the spectra after allowing for single periodicity (see Figure 3, p 78).

Rayleigh's criterion (Godin 1972) states that two close frequencies can only be resolved if the time series is sufficiently long. Stated more explicitly, the product of the frequency difference and the length of the time series, L, should be greater than one, i.e.

$$\frac{L}{2\pi}\,|\omega_1-\omega_2| \geq 1 \qquad (33b)$$

where L is in seconds. As will be seen later, Rayleigh's criterion is not always satisfied in the analyses of the gyro-motions observed in the present study. From (33b) it follows that theoretically, no solution for ω_2 is possible if

$$\frac{T_1 L}{L + T_1} \leq T_2 \leq \frac{T_1 L}{L - T_1} \qquad (33c)$$

e.g. in the case of the simulated motions studied in Chapter 4 (see Table 12) the period of the second harmonic can not be situated in the range 378.947sec $\leq T_2 \leq$ 423.529sec, given that L = 2 hours. It is not clear why two peaks appear in the least squares spectra after removal of the fundamental harmonic, cf. Appendices B and G. According to Rayleigh, Figures G1-G8 should not yield a solution for the second harmonic. However, the two peaks that appear in the analysis of both simulated and observed gyro-data are removed when a two-periodic model is solved. See also the sample output in Appendix H, pp 199-201. It appears therefore that both peaks in the spectra after allowing for single periodicity are caused by the second harmonic.

2.4.6 THE SOLUTION OF HIGHER-ORDER HARMONICS

Having solved for the new systematic noise function (the second harmonic), this is now also removed from the observations before the spectrum which identifies the third harmonic component in the gyroscopic oscillation can be calculated. Following the method described in section 2.4.4, one systematic noise function, $e^{-f_2 t} \cos(\omega_2 t - \gamma_2)$, is now added to $\underline{A}$ in (30). There are now three noise functions, and their magnitudes, that is θ_0 , B_1 and B_2 , must be evaluated for each spectral frequency ω_3 .

As before, the signal a_3 and b_3 is also obtained and each spectral value is computed using (31) and (32), in which $\underline{V}$ is the residual vector from the bi-periodical model. The tri-periodic model is computed just like described in section 2.4.5, and by putting $N = 3$.

This process is repeated if necessary until all the harmonics have been established. This will be the case if no more spectral values of statistical significance can be determined.

CHAPTER 3 RESULTS FROM THE UCT EXPERIMENTS

3.1 SIMULATION OF A GYRO-MOTION

As an introduction to the present study, the software developed (program SPECTRAL, Appendix H) was tested using data from a simulated gyro-oscillation. Two programs were written to generate the datapoints, namely SYNTH2 and SYNTH4. The listings of these two programs are included in Appendices J and K, respectively. SYNTH2 enables one or two harmonics to be included in the mathematical model expressed by (5), while SYNTH4 allows for the synthesis of three or four harmonics. The harmonics which were used are shown in Table 3. The parameters of the first two harmonics closely represent those encountered in a real gyroscopic motion (compare the results from observation set 6, Appendix A, Table A7). Harmonics 3 and 4 are approximately the same as found by Jeudy and Gagnon (1981, Table 5). The ratios between the initial half-amplitudes of the different harmonics in Table 3 is similar to the respective ratios from their spectral analysis.

The reason for having two different sets of software stems from the fact that the very high frequencies of the third and fourth harmonics, but especially the third harmonic, cause the synthesized motion to oscillate rapidly in a very short period of time. See Figure 1 where this effect has been shown graph-ically. This complex oscillation does not occur when the first two harmonics only are synthesized, producing a smooth oscillation. Therefore, different algorithms have to be used to determine the exact time when the synthesized motion reaches a certain value.

HARMONIC	1	2	3	4
ω (rad/sec)	0.01626475	0.01550282	285.599332	12.6422240
T (sec)	386.307	405.293	0.022	0.497
B (scl div)	9.5952	0.0704	0.0027	0.00006
f (sec^{-1})	1.4802e-6	7.0589e-4	1.1624e-4	1.7354e-3
γ (rad)	1.5881914 ($90^\circ 59' 48''$)	0.0189611 ($01^\circ 05' 11''$)	1.0878686 ($62^\circ 19' 49''$)	4.8489901 ($277^\circ 49' 36''$)
θ_0 (scl div)	0.1021 ($76.''01$)			

TABLE 3 : THE HARMONICS USED IN A SIMULATED GYRO-MOTION

The aim of the simulation software was to establish these exact times when certain values for θ in (5) would be reached. These θ-values represent fictitious scale divisions of a gyroscope. In essence, the software calculates a value for θ at equally-spaced time intervals $\Delta T = 2$ seconds until the desired "scale division" has been surpassed. The time-interval is then halved and the next synthesized value is calculated $\Delta t = \frac{\Delta T}{2}$ seconds backward in time. If this new value then falls on the other side of the desired θ-value, the increment in time is halved again and a value $\frac{\Delta T}{4}$ seconds forward in time is calculated. This process is repeated until satisfactory convergence has taken place, that is when $\Delta t < 10^{-6}$ seconds. The time-increment is then reset to the original value of ΔT and the next scale division is approached. The software also enables the exact point of turning to be established, using roughly the same technique. The algorithms which were set up were previously described in an undergraduate thesis which led up to this work (Potts 1984; van Rijsewijk 1988).

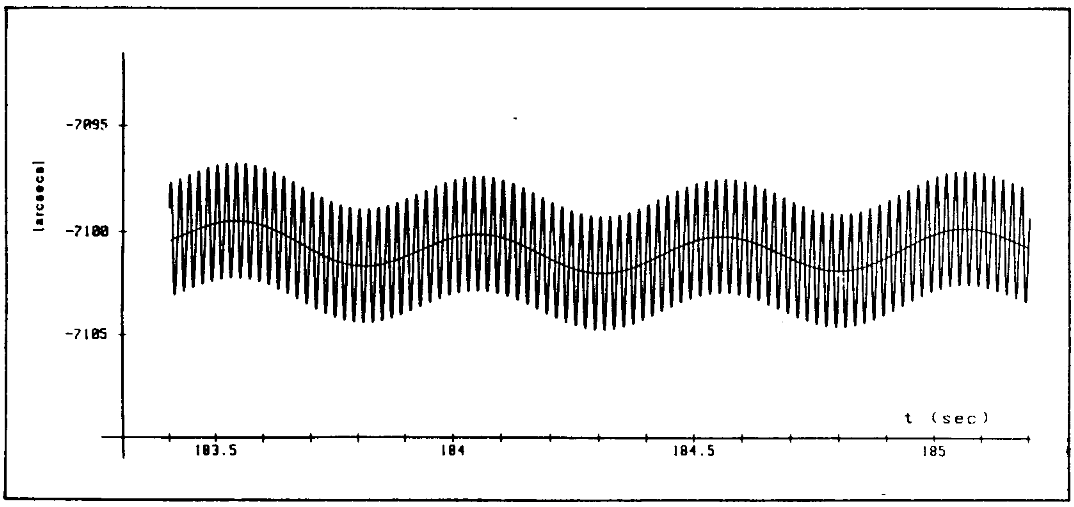

FIGURE 1 : A SIMULATED FOUR-HARMONIC GYRO-MOTION IN THE VICINITY OF A TURNING POINT

When the third harmonic (Table 3, column 4) is also included in the synthesized motion simulated by SYNTH4, the direction of the oscillation changes approximately every 0.011 second, and if the motion occurs in the vicinity of one of the "scale divisions", more than one transit of this particular θ-value takes place. This is always an odd number, and the median value is chosen to represent the time at which this θ-value is reached. To speed up the computations, only the first two harmonics are synthesized initially to provide an approximate time of crossing. Once this has been established, the other harmonics are included, the time is decreased by 0.08 second, and 20 turning points which are about 0.011 second apart are calculated for the four-periodic simulated motion. This ensures that all possible transits are included. These turning points are stored in an array, and each time of turning is later used as an initial value to establish the time of transit between that and the next turning point, provided that these successive turning points are situated on either side of the θ-value in question.

It was intended to produce a time series consisting of values which would represent an actual gyro-oscillation as closely as possible. Using the values in Table 3, the program was terminated when a time of 7200 seconds, or two hours, was reached. This amounts to a total of 19 cycles, or 571 simulated values representing the transit times of a gyromark across the various auxiliary scale divisions (0 to ±7 inclusive) These times were accurate to the nearest 10^{-6} seconds.

We then proceeded to simulate four data sets from a single, bi-, tri- and four-periodic motion, each consisting of 571 pairs (θ-value and corresponding time of crossing). These fictitious transit times were then processed using SPECTRAL, assigning unit weight to each "observation".

3.1.1 A SINGLE HARMONIC MOTION

The numerical values of the first harmonic in Table 3 were used in program SYNTH2 to simulate the transit times of a simple harmonic motion. After a least squares spectral analysis only one sharp peak occurred in the spectrum, namely at 0.0163 radians per second (Figure F1, Appendix F). Incidentally, it was also established from the four simulated sets that the spectra after allowing for n-periodicity (n = 0, 1, 2 or 3), derived from a p-periodical model, where $(n+1) \le p \le 4$, was identical for each p. The values θ_0, f_1, ω_1, B_1 and γ_1 which were obtained after a least squares curve fit are shown in Table 4, and are identical to the original values of the single periodic model simulated by SYNTH2. The *a posteriori* reference standard deviation of the least squares model was 0.0000 second.

SINGLE PERIODIC MODEL		
ω_1 (rad/sec)	0.016264746	*1.56e-13*
T_1 (sec)	386.307	*3.71e-9*
B_1 (scalediv)	9.5952	*1.09e-8*
f_1 (sec^{-1})	1.4802e-6	*2.67e-13*
γ_1 (rad)	1.5881914 (90°59′48″)	*6.62e-10*
θ_0 (scalediv)	0.1021 (76.″01)	*2.73e-9*

TABLE 4 : SINGLE PERIODIC MODEL FROM A SIMULATED SINGLE HARMONIC MOTION (std deviation in italics)

3.1.2 A BI-HARMONIC MOTION

Apart from simulating a single periodic model, program SYNTH2 was also used to synthesize the first two harmonics of Table 3. As was the case with the simulated simple harmonic motion, the first spectrum of the analysis, with no noise removed, revealed one peak at ω_1 = 0.0163 radians/second (compare Figure F1). A

HARMONICS IN MODEL	1		2	
ω_1 (rad/sec)	0.016264194	*2.81e-8*	0.01626475	*3.78e-13*
T_1 (sec)	386.320	*6.68e-4*	386.307	*8.97e-9*
B_1 (scalediv)	9.6175	*1.98e-3*	9.5952	*2.32e-8*
f_1 (sec^{-1})	1.9098e-6	*4.82e-8*	1.4802e-6	*4.93e-13*
γ_1 (rad)	1.5855246 (90°50′38″)	*1.19e-4*	1.5881914 (90°59′48″)	*1.99e-9*
ω_2 (rad/sec)			0.01550282	*5.06e-10*
T_2 (sec)			405.293	*1.32e-5*
B_2 (scalediv)			0.0704	*3.35e-8*
f_2 (sec^{-1})			7.0589e-4	*5.42e-10*
γ_2 (rad)			0.0189622 (01°05′11″)	*6.86e-7*
θ_0 (scalediv)	0.1020 (75.95″)	*4.93e-4*	0.1021 (76.01″)	*2.77e-9*

TABLE 5 : SINGLE AND BI-PERIODIC MODELS FROM A SIMULATED BI-HARMONIC MOTION (std deviations in italics)

single periodic model was then solved (Table 5, column 2), resulting in a $\hat{\sigma}_0$ *a posteriori* of 0.0878 seconds. Since we know that the second harmonic was specified to have a value of about 0.0155 radians/second (see Table 3, column 3), it was expected that the power spectrum after allowing for single periodicity would indicate a peak at approximately this frequency. However, the spectrum reveals two peaks, one at about 0.0153 radians/second and the other at about 0.0172 rads/sec (figure F2). The frequency corresponding to the highest peak in the spectrum was selected as $(\omega_2)_0$ in (33), resulting in the two-periodic model shown in the 3rd column of Table 5. As before, the recovered parameters were identical to their original values. The standard deviation of the final model was 0.0000 second.

SINGLE PERIODIC MODEL		
ω_1 (rad/sec)	0.016264194	*2.81e-8*
T_1 (sec)	386.320	*6.68e-4*
B_1 (scalediv)	9.6175	*1.98e-3*
f_1 (sec^{-1})	1.9106e-6	*4.81e-8*
γ_1 (rad)	1.5855259 (90°50′38″)	*1.19e-4*
θ_0 (scalediv)	0.1020 (75.″93)	*4.92e-4*

TABLE 6 : SINGLE PERIODIC MODEL FROM A SIMULATED TRI-HARMONIC MOTION (std deviations in italics)

3.1.3 A TRI-HARMONIC MOTION

Harmonics 1, 2 and 3 of Table 3 were synthesized using program SYNTH4. In the analysis of this time series, the first two spectra were exactly the same as shown in Figures F1 and F2 of Appendix F. The *a posteriori* reference standard deviations of the single and bi-periodic models were 0.0876 and 0.0056 seconds, respectively. In contrast with the spectrum to identify the third harmonic from Jeudy's 25 transit times (Jeudy and

HARMONICS IN MODEL	2		3	
ω_1 (rad/sec)	0.016264753	*4.18E-9*	0.01626475	*6.23e-13*
T_1 (sec)	386.307	*9.93e-5*	386.307	*1.48e-8*
B_1 (scalediv)	9.5958	*2.58e-4*	9.5952	*3.59e-8*
f_1 (sec^{-1})	1.4930e-6	*5.50E-9*	1.4802e-6	*6.71e-13*
γ_1 (rad)	1.5882379 ($90^\circ 59' 58''$)	*2.19e-5*	1.5881914 ($90^\circ 59' 48''$)	*3.54e-9*
ω_2 (rad/sec)	0.015507854	*5.95e-6*	0.01550282	*7.56e-10*
T_2 (sec)	405.161	*1.55*	405.293	*1.98e-5*
B_2 (scalediv)	0.0711	*3.82e-4*	0.0704	*6.07e-8*
f_2 (sec^{-1})	7.2203e-4	*6.39e-6*	7.0589e-4	*8.08e-10*
γ_2 (rad)	0.0084646 ($00^\circ 29' 06''$)	*7.83e-3*	0.0189582 ($01^\circ 05' 10''$)	*1.38e-6*
ω_3 (rad/sec)			285.5993321	*2.01e-9*
T_3 (sec)			0.022	*1.55e-13*
B_3 (scalediv)			0.0027	*1.75e-8*
f_3 (sec^{-1})			1.1624e-4	*1.76e-9*
γ_3 (rad)			1.0878733 ($62^\circ 19' 50''$)	*9.44e-6*
θ_0 (scalediv)	0.1021 ($76.''00$)	*3.14e-5*	0.1021 ($76.''01$)	*3.40e-9*

TABLE 7 : BI- AND TRI-PERIODIC MODELS FROM A SIMULATED TRI-HARMONIC MOTION (std deviations in italics)

Gagnon 1981), only one significant peak appeared in our spectrum at a frequency of 285.5993 radians/second, as expected. This spectrum is shown in figure F3, Appendix F. The reason for no other peaks being registered in the spectrum is possibly due to the larger number of observations that were used in this simulation study. The one, two and three periodic models recovered are shown in Tables 6 and 7.

HARMONICS IN MODEL	1		2	
ω_1 (rad/sec)	0.016264194	*2.81e-8*	0.01626475	*4.18e-9*
T_1 (sec)	386.320	*6.67e-4*	386.307	*9.93e-5*
B_1 (scalediv)	9.6175	*1.98e-3*	9.5958	*2.58e-4*
f_1 (sec^{-1})	1.9106e-6	*4.81e-8*	1.4929e-6	*5.50e-9*
γ_1 (rad)	1.5855259 (90°50′38″)	*1.19e-4*	1.5882378 (90°59′58″)	*2.19e-5*
ω_2 (rad/sec)			0.01550784	*5.95e-6*
T_2 (sec)			405.162	*1.55*
B_2 (scalediv)			0.0711	*3.83e-4*
f_2 (sec^{-1})			7.2200e-4	*6.39e-6*
γ_2 (rad)			0.0084733 (00°29′08″)	*7.83e-3*
θ_0 (scalediv)	0.1020 (75.93″)	*4.92e-4*	0.1021 (76.00″)	*3.14e-5*

<u>TABLE 8</u> : SINGLE AND BI-PERIODIC MODELS FROM A SIMULATED FOUR-HARMONIC MOTION (std deviations in italics)

HARMONICS IN MODEL	3		4	
ω_1 (rad/sec)	0.016264746	*3.41e-11*	0.01626475	*6.27e-13*
T_1 (sec)	386.307	*8.10e-7*	386.307	*1.49e-8*
B_1 (scalediv)	9.5952	*1.98e-6*	9.5952	*3.63e-8*
f_1 (sec^{-1})	1.4802e-6	*3.70e-11*	1.4802e-6	*6.77e-13*
γ_1 (rad)	1.5881919 (90°59′48″)	*1.94e-7*	1.5881914 (90°59′48″)	*3.56e-9*
ω_2 (rad/sec)	0.015502881	*4.16e-8*	0.01550282	*7.81e-10*
T_2 (sec)	405.291	*1.09e-3*	405.293	*2.04e-5*
B_2 (scalediv)	0.0704	*3.34e-6*	0.0704	*6.29e-8*
f_2 (sec^{-1})	7.0605e-4	*4.42e-8*	7.0589e-4	*8.18e-10*
γ_2 (rad)	0.0189498 (01°05′09″)	*7.58e-5*	0.0189608 (01°05′11″)	*1.41e-6*
ω_3 (rad/sec)	285.5993321	*1.10e-7*	285.5993321	*2.02e-9*
T_3 (sec)	0.022	*8.49e-12*	0.022	*1.55e-13*
B_3 (scalediv)	0.0027	*9.41e-7*	0.0027	*1.77e-8*
f_3 (sec^{-1})	1.1603e-4	*9.56e-8*	1.1624e-4	*1.78e-9*
γ_3 (rad)	1.0879223 (62°20′00″)	*5.17e-4*	1.0878526 (62°19′46″)	*9.45e-6*
ω_4 (rad/sec)			12.6422219	*2.20e-6*
T_4 (sec)			0.497	*8.63e-8*
B_4 (scalediv)			0.00006	*6.34e-8*
f_4 (sec^{-1})			1.7344e-3	*2.11e-6*
γ_4 (rad)			4.8496212 (277°51′46″)	*1.18e-3*
θ_0 (scalediv)	0.1021 (76.01″)	*1.87e-7*	0.1021 (76.01″)	*3.44e-9*

TABLE 9 : TRI- AND FOUR-PERIODIC MODELS FROM A SIMULATED FOUR-HARMONIC MOTION (std deviations in italics)

3.1.4 A FOUR-HARMONIC MOTION

Here, all four harmonics shown in Table 3 were synthesized and a time series was produced from program SYNTH4. The first three harmonics were recovered in the same manner as described in the previous section, with reference standard deviations of 0.0876 sec, 0.0056 sec and 0.0000 sec for the single, two- and three-periodic models respectively (Tables 8 and 9). Notwithstanding the small $\hat{\sigma}_0$ for this latter model, it was known that a fourth harmonic with a frequency of about 12.64 radians/second was also contained in the motion, and a least squares spectrum after allowing for tri-periodicity was computed. Once again a sharp peak appeared in the spectrum at its expected location, namely at ω = 12.6422 radians/second, albeit not a very well-defined peak (Figure F4). It should be noted that the vertical scale in Figure F4 has been exaggerated by a factor of 10 to enable the different peaks to be clearly distinguished. The sharp, yet flattish peak at the abovementioned frequency, with a spectral value of only 0.07, is probably that small because of the extremely small value for B_4, the amplitude of the 4^{th} harmonic. Nevertheless, selecting this frequency as $(\omega_4)_0$ enabled the final four-periodic model to be recovered to the values shown in Table 9. By comparing these results with the original values of Table 3, it will be seen that the four harmonics were again recovered by the spectral analysis.

The successful mapping of the synthesized harmonics confirmed the reliability of program SPECTRAL, regardless of the number of harmonics extant in a multi-period oscillation and provided more understanding of what to expect from real gyro-motions. An inspection of the results in Tables 4-9 reveals that the null-position of the oscillation can be recovered with sufficient accuracy from a simple harmonic motion in all the cases cited in Sections 3.1. The largest discrepancy of 0.″08 from the true value of 76.″01 (Tables 6 and 8) is negligible.

3.2 COMPUTER TIME UTILISED IN THE SPECTRAL ANALYSIS

The first two harmonics can be recovered in a relatively short period of time. The calculation of both the single and bi-periodic models, including the number of iterations needed and the determination of each spectrum, takes about 67 seconds of CPU (Central Processing Unit) time, allowing for 100 spectral values in each spectrum, about 6 iterations for the simple harmonic model and 20 iterations for the two-periodic model. In comparison, the determination of the third harmonic of Table 3 required a vast amount of computer processing time. It was neccessary to ensure that the third harmonic, with its known frequency, could be pinned down with absolute certainty. All possible frequencies had to be studied in the spectrum to enable an unambigious choice of ω_3 to be made. This required the use of a frequency increment of 0.0002 radians/second to ensure registration of the relevant peak in the spectrum, computed between 0 and 320 radians/second and amounting to a total of 1600000 spectral values. The total CPU time needed to complete this task on the VAX mainframe at UCT was 13 days, 12 hours, and 29 minutes. or about 0.74 second per spectral value.

The same rigour was applied to establish the spectrum after allowing for tri-periodicity. 75000 spectral values were computed between 0 and 15.0000 radians/second, and this experiment took 19 hours, 36 minutes and 50.86 seconds of CPU time. Once again, the fourth harmonic was positively identified.

For a full listing of CPU time required on the VAX 6000-330 mainframe computer installed at the University of Cape Town, refer to the title pages of programs SPECTRAL, SYNTH2 and SYNTH4 in Appendices H, J and K, respectively.

3.3 CONFIRMATION OF JEUDY'S RESULTS

Software was also developed to verify the results obtained from the time series observed and analysed by Jeudy and Gagnon in 1981. The software was identical to theirs in that two values (μ and ν) replace θ_0 in (5) and residuals were calculated in terms of scale divisions. Our numerical values of the four harmonics were in exact agreement with their values. However, it is quite clear that their sampling interval is too sparse to adequately map the third and fourth harmonics. In this study of Jeudy's data 390000 spectral values were computed at every

0.001 radians/sec after allowing for bi-periodicity. The highest peak of statistical significance (Steeves 1981) in a section of 15000 consecutive spectral values was determined and a tri-periodic model solved for. Out of these 26 models, half produced a lower $\hat{\sigma}_0$ than the $1''.071$ for ω_3 = 288.639 radians per second quoted in Table 4 of their 1981 paper. Our best-fitting model was obtained for $(\omega_3)_0$ = 70.432 radians/sec which resulted in a reference standard deviation of $0''.423$, even lower than the standard error for their four-periodic model.

3.4 SPECTRAL ANALYSIS OF MANUALLY-TIMED OBSERVATIONS

3.4.1 INSTRUMENTS USED

The University of Cape Town possesses (1991) no standard gyro-theodolite equipment with electronic timing. The consequence of this was that we were forced to rely on manual timing methodology. In order to test the proposition that the first two harmonics in a real gyro-motion can be mapped from manual timing, different gyro-systems equipped with a graduated auxiliary scale were used in conjunction with an electronic stop-watch. The early experiments (sets 1,2, and 3) are based on observations taken by Professor H.S. Williams on the WILD T2/GKK3/GAK1:196368/20847/19135 configuration. This empirical work was later extended to other configurations, namely

i) WILD T2/GKK1/GAK1:86278/2978/3140 (DSM)

ii) WILD T2/GKK1/GAK1:86278/2970/3139 (UCT)

iii) SOKKISHA TM1A/GP1/INV:4177/84274/84274 (S.A. Navy)

iv) SOKKISHA TM1A/GP1/INV:4071/81271/84274 (S.A. Navy)

v) WILD T2/GKK3/GAK1:131641E/20847/2877 (Wits)

where DSM is the Chief Directorate of Surveys and Land Information (formerly, the Directorate of Surveys and Mapping), Mowbray, UCT is the Department of Surveying and Geodetic Engineering at the University of Cape Town, S.A. Navy is the South African naval dockyard in Simonstown and Wits is the University of the Witwatersrand, Johannesburg.

For the purpose of our investigation, a total of 23 sets last ing two hours each were observed, of which sets 16, 18 and 20 were rejected due to power problems. The instrument was set up on a concrete pillar inside an air-conditioned laboratory and pre-orientated, allowing the instrument to warm up for at least half an hour before observations commenced. Sets 1-6 were taken with the same configuration as used in the 1978 tests at UCT (Williams 1979), namely T2/GKK3/GAK1:196368/20847/19135. Sets 7-13 were taken with an instrument from the DSM, using inverter GKK3:20847 for the last four sets in that group. Sets 14 and 15 were taken with the instrument belonging to UCT, once again using two different inverters. Sets 16-18 were observed with Sokkisha instruments from the S.A. Navy, while sets 19-23 were taken with T2/GAK1:131641E/2877 belonging to Wits University, using the same GKK3:20847 inverter as in sets 1-6 and 10-13. This latter T2/GAK1 configuration is identical to that

INSTRUMENT	AGE
GAK1 19135	New when used in 1/1978 at UCT
GAK1 3140	Circa 1966
GAK1 3139	Circa 1966
GP1 84274	Circa 1984
GAK1 2877	Circa 1968. Rotor replaced in 1973

TABLE 10 : APPROXIMATE AGES OF INSTRUMENTS USED

SET	DATE	THEODOLITE	INVERTER	GYRO
1	30-10-1989	T2 196368	GKK3 20847	GAK1 19135
2	07-12-1989	T2 196368	GKK3 20847	GAK1 19135
3	15-01-1990	T2 196368	GKK3 20847	GAK1 19135
4	06-11-1990	T2 196368	GKK3 20847	GAK1 19135
5	25-01-1991	T2 196368	GKK3 20847	GAK1 19135
6	07-02-1991	T2 196368	GKK3 20847	GAK1 19135
7	07-03-1991	T2 86278	GKK1 2978	GAK1 3140
8	13-03-1991	T2 86278	GKK1 2978	GAK1 3140
9	15-03-1991	T2 86278	GKK1 2978	GAK1 3140
10	20-03-1991	T2 86278	GKK3 20847	GAK1 3140
11	26-03-1991	T2 86278	GKK3 20847	GAK1 3140
12	27-03-1991	T2 86278	GKK3 20847	GAK1 3140
13	28-03-1991	T2 86278	GKK3 20847	GAK1 3140
14	03-04-1991	T2 86278	GKK1 2970	GAK1 3139
15	08-04-1991	T2 86278	GKK3 20847	GAK1 3139
17	06-05-1991	TM1A 4177	84274	GP1 84274
19	21-05-1991	T2 131641E	GKK3 20847	GAK1 2877
21	28-05-1991	T2 131641E	GKK3 20847	GAK1 2877
22	29-05-1991	T2 131641E	GKK3 20847	GAK1 2877
23	05-06-1991	T2 131641E	GKK3 20847	GAK1 2877

TABLE 11 : WILD T2/GKK1/GAK1, T2/GKK3/GAK1 AND SOKKISHA TM1A/GP1 CONFIGURATIONS USED IN THE PRESENT STUDY

used to demonstrate the MAMET procedure in 1977 (Williams 1978a).

The various gyro-attachments used are tabulated in Table 10 above, giving a brief historical background of each instrument, while a summary of all the gyrotheodolite configurations is given in Table 11.

3.4.2 METHOD OF OBSERVATION USED

For the UCT experiments, the gyro-oscillations were observed over 19 cycles or for about 2 hours in the latitude of Cape Town ($\phi \simeq 34^{\circ}S$). Subsets of 10 and 15 cycles, respectively, were also analysed but it was found that from such a reduced number of observations the peaks in the least squares spectra were not well-defined and the second harmonic could not be identified with absolute certainty. Hence, 19 cycles seems to be the minimum number that need be observed when isolating different harmonics from stopwatch-timed observations at the relevant graduation lines. The amplitude of the oscillations was set by the operator to about 9 or 10 scale divisions and the times were recorded when the gyro-mark crossed the graduation lines -7 through +7 inclusive, or 30 times per cycle. This amounted to 571 observations after 19 cycles, the initial and final observations each taken on the zero graduation line.

3.4.3 SINGLE HARMONIC MODELLING OF THE GYRO-MOTION

The power spectra indicating the main frequency of the various instruments used had the same shape for all the observation sets, with only one peak appearing at a frequency of about 0.016 radians/second. This value changed slightly for some of the spin-ups, depending on which instrument was used. See output of program SPECTRAL in Appendix H for a typical power spectrum with no systematic noise removed. After 5 or 6 iterations the single periodic models summarised in Appendix A were obtained from (21). Tables A1-A5 show the results using the quasi-weights in column 3 of Table 2, while Tables A10-A14 refer to the results from $\underline{W} = \underline{I}$ in (21)-(25). From these models it can be seen that the standard deviation of the null-position of the oscillation, σ_{θ_0}, was 0.0008 scale divisions, or $0\overset{''}{.}6$, on average. This is considerably better than the standard deviation of 0.004 scale divisions obtained by Breach (1985) for the single periodic model, after incorporation of a linear term in the observation equation to "take account of the drifting of the tape".

For the UCT experiments, the angular equivalent of one scale division on the GAK1:19135 has been calibrated to be approximately $12\overset{'}{.}408$ for latitude $\phi = 33^\circ 57' S$, and this value has been used throughout for the conversion of θ_0 from scale units to arcseconds, as indicated in the second last row of each of the tables in Appendix A.

3.4.4 TWO-PERIODIC MODELLING OF THE GYRO-MOTION

Spectra from observational data, allowing for single periodicity, are shown in Appendix B. In each of these spectra, the effect of periodic constituents revealed in the spectrum of the pure observed signal and accomodated in the single periodic model have been suppressed and the remaining peaks (those signifying the second harmonic) are enhanced, as expected. As was pointed out by Vanícek (1971), the appearance of several peaks in the spectrum could be due to mutual interference of the periodic terms contained in a multiperiod gyro-oscillation described by (5).

Observation sets 1-6, 10-13 and 19-21 all produced good results after adoption of the frequency corresponding to the highest peak in their associated spectra as the value for $(\omega_2)_0$. Although Breach (1985) does not give any indication of the magnitude of the second harmonic, he does mention that "including terms in the observation equation to take account of Jeudy's second oscillation has no effect on either the determined value of the midpoint of swing or on its standard error." Breach's statement is in agreement with our results. If the equilibrium position of the oscillation observed by Jeudy and Gagnon is given by $m = \frac{1}{2}(\mu + \nu)$, following their notation (Jeudy and Gagnon 1981, p 643), then m has the values $10.''25$, $65.''74$, $68.''58$ and $68.''64$ for the single, two-, three- and four-periodic models, respectively. The unacceptable change between the first two values is most likely due to the low number of degrees of freedom available to them, and not because inclusion of higher-order harmonics changes the determination of m.

3.4.5 RESULTS FROM THE VARIOUS INSTRUMENTAL CONFIGURATIONS

The following results relate to the approximate weight model, $weight_j \propto \sin^2(\omega_1 t_j - \gamma_1)$, previously described in section 2.4.3.1. For a full list of the results, refer to Appendix A.

3.4.5.1 OBSERVATION SETS 1 TO 3

The first three sets were observed by Professor Williams with T2/GKK3/GAK1:196368/20847/19135. The first period ranged from 385.991 seconds to 386.071 seconds, a variation of only 0.08 second, with a standard deviation $\sigma_T = \left(\frac{T^2}{2\pi}\right)\sigma_\omega$ ranging from 0.00091 sec to 0.00115 sec. Some variation was displayed in f_1,the damping factor of the first harmonic, with the damping of the first set being nearly twice that of the third set. Their corresponding standard deviations σ_f had a range of about 3e-8 sec^{-1}. Overall, the average magnitude of f_1 of about 2.4e-6 sec^{-1} is not too far removed from the 4.1e-6 sec^{-1} obtained from electronically-timed observations (Jeudy and Gagnon 1981).

The manually-recorded observations resulted in a similar pattern of dual peaks in the spectra after allowing for single periodicity (Figure 2) as did the spectra from simulated data. Also compare Figures B1-B3 with Figure F2 and Figures G1-G18. The bi-periodic models reveal a surprisingly consistent relationship between the magnitudes of the first and second periodicities, the ratios $\frac{T_1}{T_2}$ for the three sets being 0.96, 1.02 and 0.95 respectively. As was also observed by Jeudy and

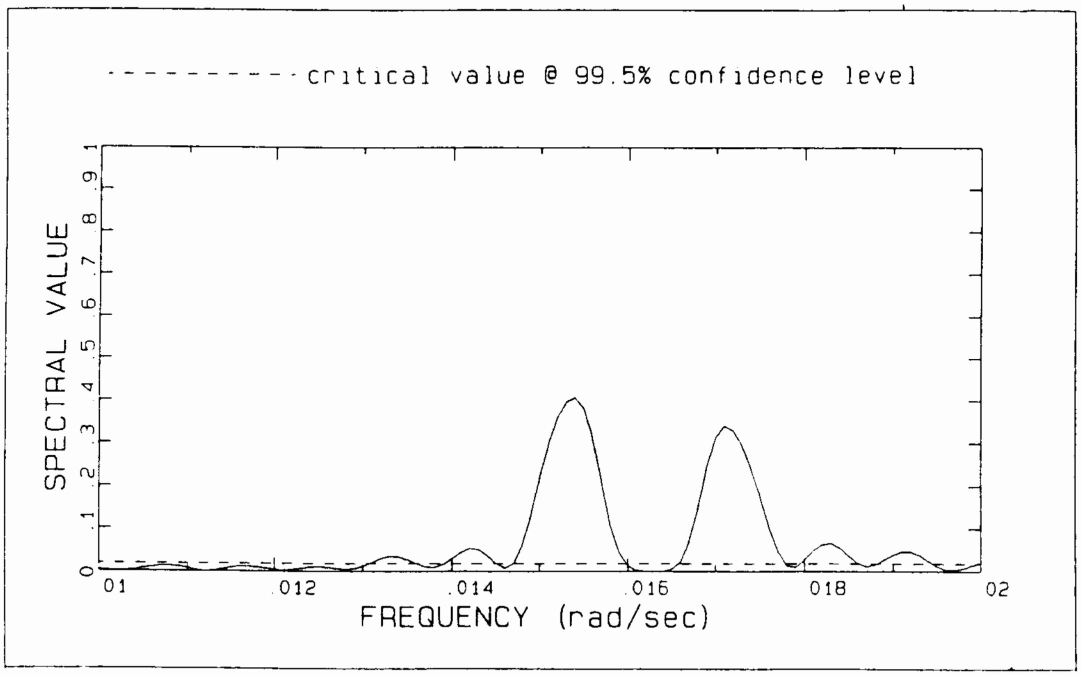

FIGURE 2 : POWER SPECTRUM AFTER ALLOWING FOR SINGLE PERIODICITY (OBSERVATION SET 1)

Gagnon (1981), the inclusion of a second harmonic in the model did not significantly alter the values of T_1 established in the single-harmonic model, the largest difference being 0.01 sec. The initial half-amplitude of the second harmonic was found to be about $\frac{1}{200}$ of B_1 , and ranged from 0.0364 to 0.1020 scale divisions. The second harmonic has a much heavier damping than reported by Jeudy and Gagnon (1981), up to 550 times that of the first harmonic. As can be expected from the more complex mathematical model, the residuals were smaller than yielded by the simple harmonic model, seen from the smaller $\hat{\sigma}_0$.

However, it is significant that the null-position could be established with equal precision in both mathematical models. θ_0 from the single periodic models for the three sets were $58\overset{''}{.}90$, $65\overset{''}{.}40$ and $67\overset{''}{.}71$ respectively, while their counterparts in the bi-periodic models were $58\overset{''}{.}98$, $65\overset{''}{.}28$ and $67\overset{''}{.}77$.

3.4.5.2 OBSERVATION SETS 4 TO 6

These three sets, as well as all subsequent sets, were observed by the writer. Sets 4 to 6 were observed on the same instrument used in sets 1 to 3. As far as the single periodic models are concerned, T_1 did not appear to be quite as stable as in the first three sets, ranging from 386.031 to 386.320 seconds. This variation could possibly be due to the relative inexperience of the observer at this early stage of our study. The standard deviations associated with T_1 were similar in magnitude to those from sets 1 to 3. It appears that the damping factor of the first harmonic had settled down to a much more consistent value than was observed previously, with a variation of only 2.8e-7 sec^{-1}. This is $\frac{1}{5}$ of the variation in sets 1 to 3. The negative phase angle of set 4 is due to the gyromark moving from the zero graduation line towards the -1 line in the first quarter cycle, instead of towards the +1 line as was the case with all the other sets.

Similar to the first group of observation sets, the only parameter relating to the first harmonic which changed somewhat in the bi-periodic model was B_1. The same relationship was found to exist between T_1 and T_2 as revealed in the first three sets, the respective ratios $\frac{T_1}{T_2}$ for sets 4-6 being 1.07, 0.89 and 0.95. The ratio $\frac{B_1}{B_2}$ was more unstable than before, especially for observation set 5, where B_1 is almost 800 times as large as B_2. This could explain why 67 iterations were needed before convergence took place in the bi-periodic model. In contrast, the damping factor f_2 is far more consistent than in sets 1 to 3. The value for f_2 in set 6 is 4 times as large as f_2 in set 5, while f_2 in set 3 is 90 times as large as f_2 in set 1. The average *a posteriori* standard deviation of the two-periodical model is about 10%, or 0.01 second, smaller than in the first three sets, possibly because of different observers or else because the gyro-rotor was running more evenly now after a lenghty period of non-use.

3.4.5.3 OBSERVATION SETS 7 TO 9

These three sets, which were observed with WILD T2/GKK1/GAK1: 86278/2978/3140, produced a sudden variation in θ_0 of more than 50 arcseconds between sets. In addition, the fundamental frequency changed by as much as 0.7 seconds. The *a posteriori* standard deviation for the single periodic models were much larger than in the previous six sets, with a $\hat{\sigma}_0$ of 0.6951 seconds for set 9 being totally unacceptable. Based on experience gained with the previous six sets of observation, it was felt that such an ill-fitting model could not have been a result of bad observing.

Apart from the poor results from the single periodic models, the bi-periodic models as identified from Figures B7-B9 also did not have the same consistency as before. Despite the fact that it is not immediately obvious from these three figures that unreliable results were obtained, a further investigation via a power spectrum after allowing for bi-periodicity showed that such allowance had left the shape of each spectrum virtually unaltered. In all three sets concerned, it was found that a third harmonic could be solved as well. The tri-periodic models each had closely-related frequencies for the three harmonics, but set 9 showed a positive damping for the third harmonic. After some consideration it was concluded that the Nickel-Cadmium batteries of the GKK1:2978 inverter had become unreliable due to lack of use over a long period of time, resulting in an unstable supply of current to the gyro-rotor. This caused the frequency of the oscillation to vary, and hence curves with varying frequency could be fitted to at least a part of the observations, creating the false impression that a third harmonic existed in the motion. On replacement of the GKK1 with the GKK3:20847 inverter used in sets 1 to 6, good results were obtained with this new configuration for sets 10-13.

3.4.5.4 OBSERVATION SETS 10 TO 13

The WILD T2/GKK3/GAK1:86278/20847/3140 configuration produced good results. The single harmonic models showed a range of 0.113 seconds in the determination of T_1, which is considerably smaller than was achieved with the GKK1 inverter in the preceding four sets. The value for f_1 ranged from 1.7572e-6 sec^{-1} to 1.9509e-6 sec^{-1}, roughly the same as was determined from sets 7 to 9, but the standard deviation for the model had decreased from an average of 0.343 second to an average of 0.087 seconds with the GKK3.

In the bi-periodic models that were solved, it was once again revealed that the frequencies of the first two harmonics in a multi-period gyro-oscillation have similar magnitudes. In these four sets, the ratios $\frac{T_1}{T_2}$ were 0.99, 1.00, 1.01 and 1.06 respectively. The two periodicities from set 11 differ by only 0.57 second. As with all the other sets so far discussed, the initial half-amplitude of the second harmonic differ considerably from one another. Observation set 13 produced the smallest B_2 of 0.0099 scale division, which is about $\frac{1}{1000}$ the value of B_1 for that set, while set 11 had the largest B_2 of 0.1018 scale divisions, or about $\frac{1}{100}$ B_1. The determination of θ_0 hardly changed from the mono-periodical models, and $\hat{\sigma}_0$ for the four sets were the smallest yet, ranging from 0.0656 second to 0.0772 second. Ihe least squares spectra for the identification of any possible higher-order harmonics in the motion did not show any significant spectral values, signifying that replacing the inverter had been the right decision. Similar consistencies in the determination of the damping factor of the second harmonic as those in sets 4-6 were achieved, with values of 1.1729e-3, 4.377e-4, 6.9703e-4 and 1.4384e-4 sec^{-1} respectively. residuals of the bi-periodic models of these four sets were sufficiently small, and no more peaks were indicated in the spectra after allowing for bi-periodicity. The *a posteriori* standard deviation of the bi-periodic models was lower than those obtained from the first six sets, and θ_0 had a range of about 18 arcseconds.

3.4.5.5 OBSERVATION SETS 14 TO 15

During the observation of set 14, it was noticeable that the gyro-mark suddenly picked up a visible tremor after about 24 minutes of observing, at the same time causing the amplitude of the oscillation to actually increase by nearly 0.6 of a scale division. A subsequent inspection of the periodicity calculated from the transit times across the null-line shows that the periodicity had also increased by about 2.5 seconds. On processing the observations it was found that the damping factor, f_1 , had the wrong sign and that the residuals from the single periodic model were unacceptably high (Table A4, Appendix A). In addition, the standard deviation of the primary period was 10 times as large as its average standard deviation from the earlier sets. Calculation of a power spectrum after allowing for single periodicity resulted in very high sidelobes (Figure B14).

Replacing the GKK1/2978 with the reliable GKK3/20847 did not improve the results. In fact, worse results were obtained. The observed amplitude gradually increased by about 0.2 scale divisions for the first 40 minutes, stabilised for the next 50 minutes, and then suddenly decreased by half a division over the next 20 minutes. The observed periodicity from the transit times had a range of more than six seconds. The single periodic models for the two sets showed that T_1 differed by nearly 0.7 seconds from that calculated from set 14, and the standard deviation was even larger than before, namely 3.9139 seconds. Also, it took nearly twice the number of iterations to establish the least squares model. Figure B15, like Figure B8, reveals oddly-shaped sidelobes.

Quite clearly the gyro-rotor was performing erratically, and had deteriorated to a Category 3 instrument. A similar anomalous pattern has been noted on other instruments in the past (Williams 1978b, Table 6). The observations from either set 14 or 15 did not yield a bi-periodic model.

3.4.5.6 OBSERVATION SETS 16 TO 18

The Sokkisha instruments were found to be of poor quality. The image of the gyromark on both the GP1:84274 and GP1:81271 could not be brought to coincide with the image of the graduated scale with the result that there was severe parallax in the collimator. The optics could no longer be used. Errors of about 0.2 to 0.3 of a scale division were introduced when estimating the relative position of the gyro-mark on the auxiliary scale, depending on the location of the observer's eye at the time. Problems were also encountered with the batteries in that they tended to lose their charge at a faster rate than normal. Set 16, observed with TM1A/GP1/INV:4177/84274/84274, had to be discarded after having observed for $1\frac{1}{2}$ hours when it was noticed that the turning point suddenly increased by nearly 1 division. This was caused by the loss in momentum of the gyroscope, in turn caused by the low voltage in the battery. Enough power was available to observe set 17, using the same configuration, but unusually large residuals were found for virtually all the observations. Nevertheless, the spectrum after allowing for single periodicity still showed the same pattern as recorded in the other sets (see Figure B17). However, the large errors in the observations, evident from a $\hat{\sigma}_0$ of 0.2023 second, did not permit a bi-periodic model to be found. Set 18 was observed on the other Sokkisha gyro-instrument made available to us, namely TM1A/GP1/INV:4071/81271/84274, but had to be aborted after about 30 minutes when it became obvious that the errors in timing were induced by the parallax present in the instrument.

3.4.5.7 OBSERVATION SETS 19 TO 23

These four sets (set 20 was aborted due to loss of power) produced the most consistent results (Tables A5 and A9). Although $\hat{\sigma}_0$ of the single periodic models tend to be a little larger than obtained with some of the other configurations, these values dropped significantly after allowance was made for the second harmonic. It should be noted that the initial half-amplitudes of the second harmonic, ranging from 0.1423 to 0.2358 scale divisions, were much larger than the average B_2 of 0.06 scale divisions found with the other configurations. This is probably why the single periodic models could not be properly fitted to the observations. There also seems to be a direct relationship between the amplitude of the peaks in Figures B17-B20 and the magnitude of B_2. In the bi-periodic models (Table A9) the range in T_2 is only 10 seconds, compared to 30 seconds for sets 1-3, 71 seconds for sets 3-6 and 26 seconds for sets 10-13. T_1 varies by only 0.116 seconds in the four sets and the damping factors for the two harmonics also remain most constant, especially f_2 which ranged from 2.4932e-4 to 5.5764e-4 sec^{-1}. The *a posteriori* standard deviation of the four two-periodical models are similar in magnitude.

CHAPTER 4 ANALYSIS OF RESULTS

4.1 RELATIONSHIP BETWEEN ω_1, ω_2 AND THE SHAPE OF THE SPECTRUM AFTER ALLOWING FOR SINGLE PERIODICITY

4.1.1 ANALYSIS OF SIMULATED DATA

The UCT experiments with three different GAK1 instruments have shown that the magnitude of the second period is remarkably similar to that of the first period (Appendices A and B). Furthermore, it was learnt from the experience that in some cases it was irrelevant which one of the two peaks in the spectra, allowing for single periodicity (see Appendix B, Figures B1-B20), was selected to represent the approximate frequency of the second harmonic. The final two-periodical models derived at were exactly the same, with the exception of γ_2 which was sometimes shifted by 180°, forcing B_2 to change sign in the least squares adjustment. In an endeavour to explain this, and other phenomena, a number of data-sets (571 observations each) from simulated two-periodic gyro-motions (Table 12) were generated using the program SYNTH2 (appendix K). Different ratios $\frac{T_2}{T_1}$ were used to investigate its effect on the shape of the spectra after allowance for single periodicity. For the purpose of this study, the value of x (Table 12, column 3) was systematically changed to allow for the ratios shown in Figures G1-G18. In each case the simulated transit times generated by SYNTH2 from the two-period oscillation were processed by least squares spectral analysis (program SPECTRAL). After a solution was obtained for the first harmonic, the spectra shown in Appendix G were calculated.

HARMONIC	1	2
T (sec)	400	x
B (scl div)	10.5	0.1
f (sec^{-1})	1.5e-6	7.0e-4
γ (degrees)	85	125
θ_0 (scl div)	0.1	

TABLE 12 : VALUES USED TO DEMONSTRATE DISPLACEMENT OF PEAKS IN THE SPECTRUM

As indicated in Figure 3, the vertical dashed line shows the position of ω_1 = 0.0157 radians/sec (or T_1 = 400 sec) in the spectrum. The spectral value at this position was always nearly zero. Taylor and Hamilton (1972) mention that "Vaníček's method has the property of preserving the position (though not the amplitude) of any single peak of the spectrum no matter how close in frequency to the interfering noise function". We have found this not to be so. Taylor et al. investigated some of the properties of Vaníček's method using simple spectra "dealing with fairly high frequencies". Presumably the frequencies mentioned are those relating to electromagnetic waves. The spectra found in the present study of simulated data were similar in shape to the plot from the variance estimate $\hat{V}(f)$ (Taylor and Hamilton 1972, Figure 1) and in each case

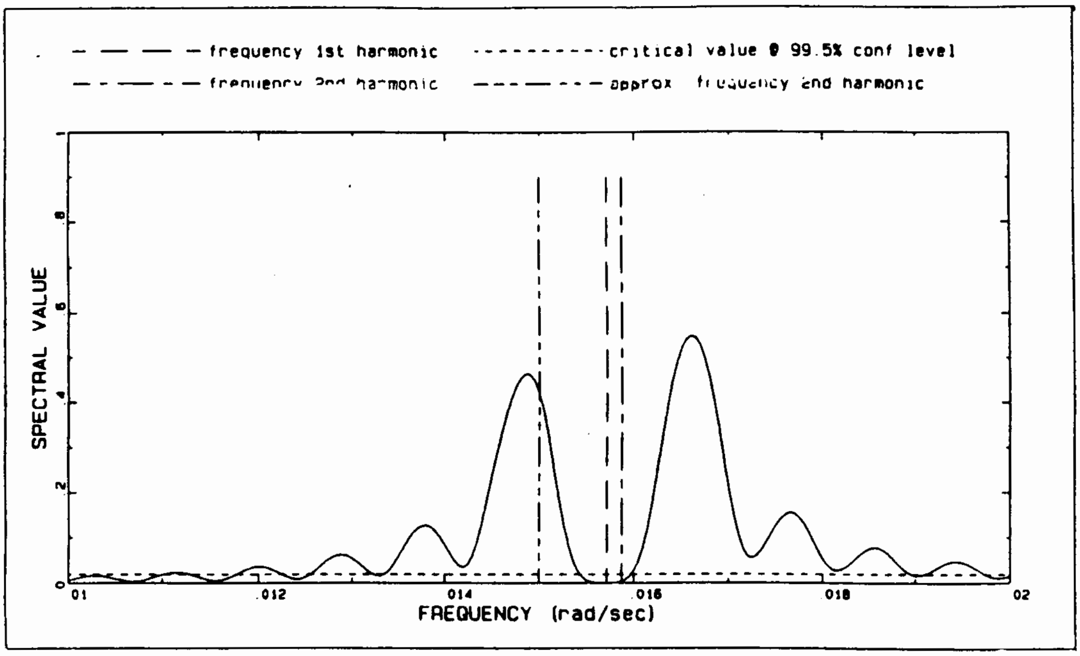

FIGURE 3 : TYPICAL POWER SPECTRUM AFTER ALLOWING FOR SINGLE PERIODICITY

study both harmonics were recovered exactly by SPECTRAL. It must be pointed out that successful convergence of the two-periodical models only occurred when the approximate frequency $(\omega_2)_0$ was taken at the position indicated by the dash-dot-dot line on Figures G1-G20, and not necessarily corresponding with the highest peak in the spectrum. The dash-dot-dash line indicates the true position of ω_2, that is, the frequency corresponding to x . From the figures shown in Appendix G it emerges that :

- two well-defined peaks are situated more or less symmetrically about the position of the peak in the first spectrum.
- there is a direct relationship between the shape of the curve and the ratio between T_1 and T_2. The peaks become higher and more distinct and more sidelobes appear in the spectrum when

T_2 approaches T_1 .

- the higher of the two peaks is always situated on the same side of ω_1 as the true ω_2 is, but the position of the peak has been shifted if $T_2 \geq 0.94T_1$ (Figures G1-G11).
- for $T_2 \leq 0.93T_1$ the approximate value for ω_2 to ensure conversion coincides roughly with the true ω_2 as well as with the highest spectral value in the spectrum (Figures G12-G18).
- for $0.96T_1 \leq T_2 \leq 0.94T_1$ the approximate ω_2 sometimes coincides with the highest spectral value, and the peak has been shifted from its true position (Figures G9-G11).
- for $T_2 \geq 0.97T_1$ the approximate ω_2 tends to be situated at or near the frequency corresponding to the largest spectral value of the lower peak (Figures G1-G8).
- for $T_2 \geq 0.995T_1$, B_2 could become negative in the final two-periodical model and the phase angle be displaced by 180° (Figures G1-G5)

The results summarised above give no clear-cut indication of what to expect when $T_1 \simeq T_2$, and much has still to be learnt in this regard. Nevertheless, the analysis was successful from the point of view that the numerical quantities of Table 15 could be recovered without any difficulty, once again underscoring the reliability of program SPECTRAL.

4.1.2 ANALYSIS OF OBSERVED GYRO-DATA

The power spectra from the manually-timed observations, allowing for single periodicity (Appendix B), reveal a similar pattern of twin peaks appearing in the spectrum, although their amplitudes are generally not as high as in the simulation studies. This reduction in amplitude could be partly due to the random errors present in the time-observations. However, based on the findings from study of simulated data in the previous section, it can be inferred that the characteristic pattern present in the analysis of the real gyro-oscillations (Figures B1-B20) is caused by the similarity in magnitude of the two frequencies ω_1 and ω_2. A third harmonic could not be identified from the observation sets in Table 11. In passing, it is interesting to note that no peak appeared at $\omega_2 = 2\omega_1$ in any of the spectra allowing for ω_1, as suggested by Jeudy (1981,1982) and later established from a single spin-up of a GAK1 (see Sections 1.1.15 and 3.3).

The experiments with a variety of gyro-theodolites have shown that it is possible to map the second harmonic in a gyro-oscillation from manual time-observations. The frequency of the second harmonic can be positively identified from a least squares spectral analysis. With the exception of the WILD GAK1:3139 and the SOKKISHA GP1:84274 and GP1:81271 gyro-attachments, the remaining three WILD instruments proved to be stable and reliable, consistently producing good results.

4.2 SEQUENTIAL ADJUSTMENT

In a semi-automatic gyro-system like that developed by Jeudy, Jobin and Fournier (1988) which incorporated a WILD GAK1 gyro-attachment, a built-in feature to update the azimuth is essential. The Department of Electrical and Electronic Engineering at the University of Cape Town, in conjunction with Professor H.S. Williams, has developed a system which utilises 35 photo-cells to capture the image of the scale and gyro-mark of the GAK1. The position of the mark is then determined by measuring the output level of the cells by means of an analogue-to-digital converter (Williams 1991b). The UCT system has got the advantage that the mark can be tracked in real time and, after inclusion of the observed directions and the instrument calibration constant, the gyro-azimuth and its standard deviation can be digitally displayed. The gyrotheodolite system therefore operates in near real-time. The GAK1 gyro-attachment is widely used in South Africa and by using this real-time conversion option the GAK1 could be upgraded with relative ease. This system could probably become commercially viable in the near future.

Unfortunately, the UCT system never got beyond the test-rig stage and hence it was not possible to investigate its full potential. Nevertheless, the software to be used in conjunction with this real-time gyro modification makes allowance for a sequential determination of those parameters describing a single harmonic motion. A description of the sequential adjustment routine (Mikhail 1976) follows below, after which the technique used in program SPECTRAL is discussed.

4.2.1 SEQUENTIAL PROCESSING FOR A FIXED NUMBER OF PARAMETERS

Let us assume we have obtained a solution for the vector of unknowns describing the single periodical model in (21), using p observations from the first cycle of the oscillation. Since the gyromark is continually being tracked, the n additional observations contained within the next quarter cycle, say, must be added to the least squares adjustment to yield an updated approximation for the unknowns. If we put

$$\underline{N} = \underline{A}^T(\underline{BW}^{-1}\underline{B}^T)^{-1}\underline{A} = \underline{A}^T\underline{Q}^{-1}\underline{A} \quad (34)$$

and

$$\underline{F} = \underline{A}^T(\underline{BW}^{-1}\underline{B}^T)^{-1}\underline{f} = \underline{A}^T\underline{Q}^{-1}\underline{f} \quad (35)$$

in (21), then by using recursive partitioning $\underline{N}$ and $\underline{F}$ can be written in terms of their predecessors,

$$\underline{N}_i = \underline{N}_{i-1} + \underline{A}_i{}^T\underline{Q}_i{}^{-1}\underline{A}_i \quad (36)$$

and

$$\underline{F}_i = \underline{F}_{i-1} + \underline{A}_i{}^T\underline{Q}_i{}^{-1}\underline{f}_i \quad (37)$$

in which $\underline{Q}_i^{-1}$ contains the quasiweights of the observations for the combined adjustment case. The sequential estimating procedure for the parameters can then be written as

$$\underline{X}_{i-1} = \underline{N}_{i-1}\ \underline{F}_{i-1} \tag{38}$$

$$\underline{X}_i = \underline{N}_i\ \underline{F}_i \quad , \text{ etc...} \tag{39}$$

It can be shown that the recursive formula for a sequential adjustment with a fixed number of parameters (five in the case of a single periodic model) is given by

$$\underline{X}_i = \underline{X}_{i-1} + \underline{N}_{i-1}^{-1}\ \underline{A}_i^T\ (\ \underline{Q}_i^{-1} + \underline{A}_i\ \underline{N}_{i-1}^{-1}\ \underline{A}_i^T\ )^{-1}\ (\ \underline{A}_i \underline{X}_{i-1} - \underline{f}_i\ ) \tag{40}$$

for the p+n observations (Mikhail 1976). Also compare Williams (1971) where similar formulae are given. A closer inspection of (40) will reveal that that this expression requires the inversion of a n×n matrix. However, since there are only 5 unknowns in the single periodic model, and the sampling rate of the gyro-oscillation will be high, it is more computer efficient to calculate the 5×5 covariance matrix $\underline{N}_i$ directly from (36). The contribution made by each observation equation to the normal equations $\underline{A}_i^T \underline{Q}_i^{-1}\ \underline{A}_i$ is evaluated and added to $\underline{N}_{i-1}$ (see program SPECTRAL, subroutine SEQADJUST, lines 530-570), the latter matrix having been stored in memory after the previous adjustment.

4.2.2 SEQUENTIAL ALGORITHM USED IN THE CURRENT SOFTWARE

The 20 observation sets from Table 11 were analysed using the method suggested in the previous section. It was decided to add one cycle of observations at a time in order to investigate the changes in the determination of θ_0, if any. In addition, the least squares solutions for n cycles (n = 1, 2,...,19) were derived from batch processing to enable the reliability of the sequential routine to be determined. The results were identical in each case. Similar to the batch processing mode, an iterative procedure had to be used for each sequential determination. A detailed analysis of the results showed that $\underline{N}_i$ in (36) changed significantly in the first iteration only, and that any further changes to $\underline{N}_i$ in subsequent iterations are negligible. However, this was only the case if at least three cycles of observations were included in the adjustment. The fluctuation in $\underline{A}_i^T \underline{Q}_i^{-1} \underline{A}_i$ between iterations for a low number of observations could be due to the relatively large errors in the manually-recorded times. Subsequently, the software was modified such that $\underline{N}_i$ was computed in the first iteration only, while a new $\underline{F}_i$ was computed for each iteration by evaluating $\underline{f}_i$ only and using the same coefficient matrix to calculate $\underline{N}_i$. This approach lowered the CPU time needed for each iteration by nearly 80% without the sacrificing of accuracy.

The drift in θ_0 which takes place over a time period of two hours can be seen from the results of the sequential adjustment. The various graphs for the 20 observation sets are given in appendix E, Figures E1-E20. An example of this drift is indicated in Figure 4 below. It is further apparent from virtually all the results shown that one cycle of manually-timed observations across the graduation lines used in the present study is inadequate to yield a good estimate for θ_0. Observations never commenced before the spinning gyro-rotor had warmed up for at least half an hour, and hence it can be said with certainty that the odd kicks in the beginning of the sequential adjustments (especially sets 3, 5, 13 and 23) are not caused by an insufficient warm-up period.

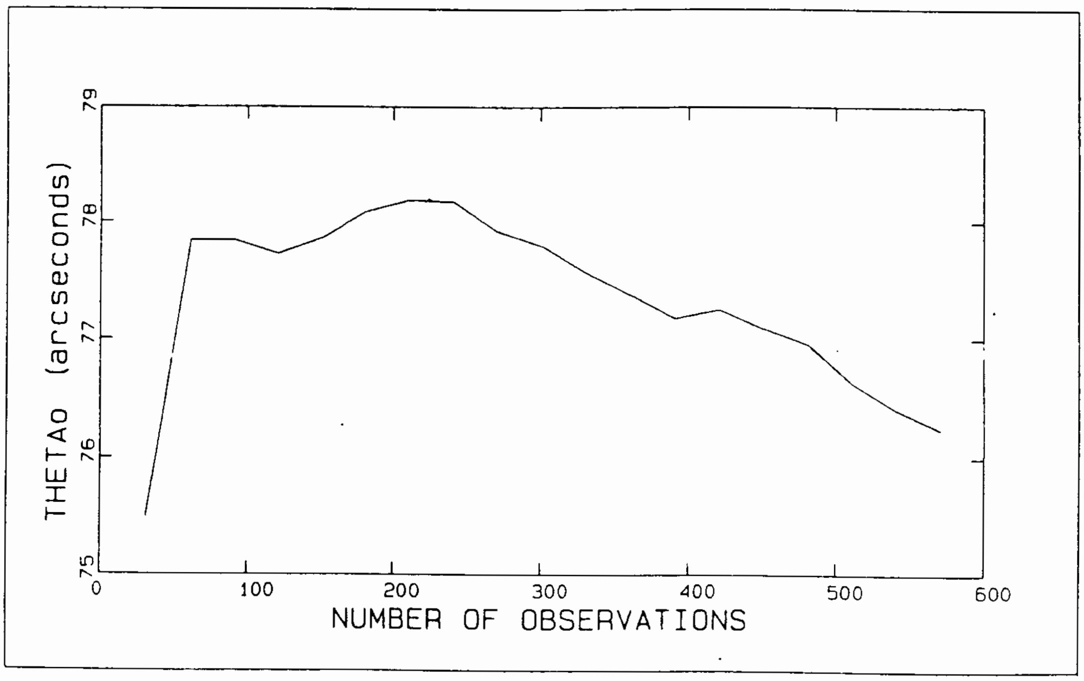

FIGURE 4 : NULL-POSITION FROM A SEQUENTIAL ADJUSTMENT (OBSERVATION SET 6)

As was expected, the graphs relating to those sets deemed unsatisfactory, namely sets 7-9, 14-15 and 17, displayed some eccentric solutions for θ_0. Generally, sets 7-9 have an upward trend but after replacement with a reliable GKK3 inverter (number 20847), the drift in the null-position changed direction, as indicated in Figures E10-E13. Set 14, which was taken with a Category 3 instrument, drifted by nearly 13 arcsecs over the first 5 cycles while the 15^{th} set (Figure E15) clearly demonstrates that this category instrument is unsuitable for accurate work. Concluding the unreliable instruments, the sequential adjustment of the observations taken with the Sokkisha instruments show a deceptive near-linear trend in the null-position. The large residuals from the adjustment prove that the parallax present in the optical system of this instrument rendered it unserviceable (see also section 3.4.5.6).

4.3 NULL-POSITION FROM SUBSETS OF 46 OBSERVATIONS, OR $1\frac{1}{2}$ CYCLE

Here, the observations were divided into subsets of $1\frac{1}{2}$ cycles and θ_0 solved from a single periodical model using all 46 observations contained therein. The subsets were stepped forward by $\frac{1}{2}$ cycle, or 15 observations, each time until the last subset contained the observations 526-571. Classical reduction models, for example, the transit method or the TIMET group of methods (Williams 1986), rely on a minimum of three discrete points on the oscillation curve for a unique solution of θ_0 (Williams 1986). If more information is available, such as additional transit times, the accuracy of θ_0 can be increased from a combination of reduction methods. This has been demonstrated with the Improved TRANsit method (ITRAN) in which "a" , the initial half-amplitude in the conventional transit method (θ_0 = CaΔt) has been approximated by the MAMET procedure (Williams 1986; Strasser and Schwendener 1966). The current method under discussion, therefore, could be described as a form of the ITRAN method.

All the results are shown in Appendix C, and generally are variations of the drift pattern indicated in Figure 5. A comparison between Figures C1-C20 and their counterparts from the sequential adjustment in Appendix E clearly indicates the occurrence of drift in the null-position of each observation set. The erratic behaviour of GAK1:3139 is further manifested in Figures C14 and C15. What also emerged from this particular group of experiments is that the arithmetic mean of θ_0 from the 36 subsets compares favourably with the simultaneous least squares adjustment of all 571 observations.

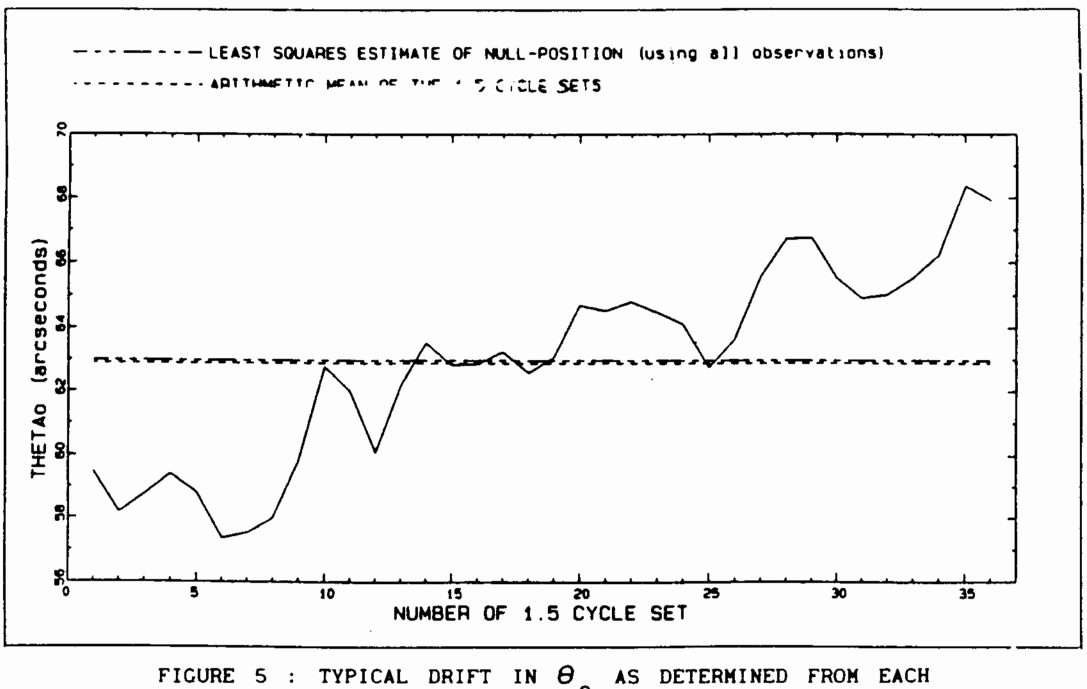

FIGURE 5 : TYPICAL DRIFT IN θ_0 AS DETERMINED FROM EACH 1.5 CYCLE

Simulated motions generated by the software included in Appendices J and K were also analysed by this ITRAN-like procedure. Figure D1 illustrates that a synthesized motion consisting of harmonics 1 and 2 in Table 3 shows a negligible range of less than 0.35 arcsecond in θ_0. The oscillation produced by this graph is virtually symmetrical about the arithmetic mean of θ_0, although no explanation could be found for the "large" variation in θ_0 in the first six or seven subsets. In Figure D2 a more realistic picture of the variation in the null-position is presented by a four-period oscillation, although still not quite as large a variation as detected in the manually-timed two-hour observations (Figures C1-C20). Apparently the simulated motions were insensitive to pick up a more noticeable drift in θ_0, as found in the real gyro-oscillations. It is possible that the two-hour sets were insufficient to map a long periodicity of say, 24 hours (Williams and Belling 1967a; Chrzanowski 1969). Further research in attempting to map such low-frequency components in multi-period gyro-oscillations would be impractical, even if a suitable power supply were available.

4.4 INTERDEPENDENCE BETWEEN THE FIRST TWO HARMONICS OF A MULTI-PERIOD GYRO-MOTION

In the 16 observation sets which enabled a two-periodic model to be determined (Appendix A, Tables A6-A9) it was noticed that f_2, the damping of the second harmonic, was always much larger than f_1. On review of section 3.4.5.3, it is probably adviseable not to include the two-periodical models from sets 7 and 8 in any further analysis, and interest will now be focussed on observation sets 1-6, 10-13 and 19, 21-23, or 14 sets in total. A wide variety of relationships between f_1 anf f_2 exists but, on average, the UCT experiments show that $f_2 \simeq 250f_1$. This factor is almost 50 times larger than claimed by Jeudy and Gagnon (1981) from their spectral analysis of 25 transit times and substantially different from Jeudy's theoretical ratio of $f_2 = 2f_1$. Our experiments have also shown that the amplitude of the second harmonic is not a fixed value, nor does the number of iterations in (21) needed to determine a two-periodic motion

N	f_1 (sec^{-1})	f_2	B_1 (scale div.)	B_2	$\frac{f_1}{f_2}$	$\frac{B_1}{B_2}$	It.	ΔT (sec)
1	1.8e-6	9.1e-6	10.6	0.04	0.199	290.1	18	14.7
2	4.0e-6	2.2e-4	10.9	0.10	0.018	106.5	12	9.2
3	1.5e-6	8.4e-4	10.1	0.06	0.002	172.1	17	20.8
4	2.3e-6	5.2e-4	11.0	0.06	0.004	199.1	26	23.8
5	2.1e-6	1.6e-4	10.7	0.01	0.013	761.1	67	46.6
6	1.5e-6	6.8e-4	9.6	0.07	0.002	142.8	26	20.2
10	1.9e-6	1.2e-3	11.2	0.10	0.002	114.2	22	5.5
11	1.6e-6	4.4e-4	10.4	0.10	0.004	101.9	63	0.6
12	1.7e-6	7.0e-4	10.8	0.06	0.002	167.9	20	5.2
13	1.9e-6	1.4e-4	10.3	0.01	0.013	1036.5	43	20.8
19	7.1e-7	2.6e-4	9.6	0.24	0.003	40.7	15	4.0
21	4.1e-6	2.5e-4	10.1	0.14	0.016	70.8	22	5.6
22	2.2e-6	5.6e-4	10.6	0.21	0.004	49.5	20	2.3
23	2.4e-6	5.1e-4	9.2	0.21	0.005	44.5	17	0.4

TABLE 13 : SUMMARY OF RESULTS FROM THE UCT EXPERIMENTS

remain constant. The different ratios $\frac{f_2}{f_1}$ found from the 14 two-periodical models derived from the approximate weight model $WEIGHT_i \propto SIN^2(\omega_1 t_i - \gamma_1)$, as well as other associated information, is shown in Table 13. "It." is the number of iterations, $\Delta T = |T_1 - T_2|$ and N is the number of the observation set. If we now group the entries in Table 13 according to the three good configurations that were used (see Table 11), namely sets 1-6, 10-13 and 19-23 (excluding 22), and we put $W = \frac{f_1}{f_2}$, $X = \frac{B_1}{B_2}$, Y = the number of iterations and Z = ΔT, we can attempt to establish if any interdependance exists between X and Y, X and Z, or Y and Z. Based on the assumption that such a relationship between X,Y and Z is linear, the linear correlation coefficient r between e.g. X and Y can be computed from

$$r_{XY} = \frac{\sum xy}{\sqrt{\sum x^2 \sum y^2}} \qquad (41)$$

in which $x = X - \bar{X}$ and $y = Y - \bar{Y}$. The results are summarised in table 14 :

SETS	correlation coefficient					
	r_{WX}	r_{WY}	r_{WZ}	r_{XY}	r_{XZ}	r_{YZ}
1-6	0.05	-0.22	-0.30	0.94	0.90	0.97
10-13	0.98	0.34	0.91	0.16	0.97	-0.06
19-23	0.97	0.79	0.66	0.90	0.65	0.41

TABLE 14 : RELATIONSHIP BETWEEN f, B AND T OF THE TWO HARMONICS IN A 2-PERIODICAL MODEL, AND THE NUMBER OF ITERATIONS

Cognisance should be taken of the fact that the WILD GAK1:19135 gyro-attachment used for the first six observation sets had not been used at all for about three years prior to the commencement of the current study, and in all likelihood regular use of the gyro-rotor resulted in stabilisation of its performance. Hence, if we ignore observation sets 1 and 2 the row for sets 3-6 in Table 14 reads like 0.98, 0.98, 0.99, 0.98, 1.00 and 0.98. It would appear therefore that, based solely on the results given for the various correlation coefficients, a direct relationship exists between W and X, X and Z and W and Z and possibly between W and Y as well as between X and Y. No correlation seems to exist between Y and Z. In summary, we have that

- if the ratio $\frac{B_1}{B_2}$ measured between two spin-ups increases, then the second harmonic of the latter oscillation is likely to have lighter damping.
- Increasing similarity in magnitude of ω_2 and ω_1 will result in a larger B_2 relative to B_1, as well as in a heavier-damped second harmonic.
- The number of iterations needed to determine the two-periodical model could be directly related to the ratios $\frac{f_1}{f_2}$ and $\frac{B_1}{B_2}$.

One other important piece of information inherent to any spin-up of a gravitationally-suspended gyroscope is the amount of time the instrument is allowed to warm up before the observations are commenced. Unfortunately, no record was kept of this period of time which varied from about half an hour to 45-50 minutes. Given sufficient time, the second harmonic will disappear entirely (the decay $e^{-f_2 t}$ of observation set 10, for example, is less than 0.001 after 100 minutes have elapsed. From the UCT experiments, the average time for this to happen is about 5 hours, but often far less) but this aspect was not studied in the present study due to its impracticality.

CHAPTER 5 CONCLUSIONS

The observations taken with three different gyro-attachments have proven beyond a doubt that the first and second harmonics of a multi-period gyro-oscillation can be positively identified from a set of manually-timed observations. A $\hat{\sigma}_0$ of about 0.075 seconds from the two-period models indicates the optimum accuracy that can be achieved from observations with a hand-held stopwatch. The periodicity of the second harmonic is different from what was theorized by Grafarend (1972a) or Jeudy (Jeudy and Gagnon 1981). Instead, it was found to be surprisingly similar to the periodicity of the fundamental harmonic. The results further show that the frequency of the second harmonic can not be pinned down to a definite value, unlike the fundamental oscillation which is a function of the latitude concerned. From the final results in Appendix A, it is shown that T_2 has a range of about 70 seconds in sets 1-6, 26 seconds in sets 10-13 and 10 seconds in sets 19-23. These erratic variations indicate that the secondary, or forcing, harmonics are in all likelihood operator-influenced and unique to each spin-up (Williams 1991b). It is not known what the exact causes are of the fluctuation in the second periodicity.

Some of the most accurate gyro-azimuths have been established with manual timing in conjunction with the MAMET procedure (Williams 1979). If manually-timed observations are to be used in the reduction model then, from a practical point of view, the non time-critical MAMET is to be preferred (Williams 1970, 1979, 1986). For a comparison of the unit forms of Halmos' 1980 formula, Schwendener's transit method and MAMET, using various values for σ_t, refer to Williams (1981b), Table 4.

The excessive computer time needed to find the third harmonic (see section 3.2) underscores the fact that least squares spectral analysis of the high-frequency harmonics (see Figure 1) can never be a viable proposition. This will be the case even if electronic timing is used to determine the position of the gyromark of the GAK1. Besides, the present study has proved that θ_0 changes by less than 0.15 arcsecond after inclusion of the second harmonic in the mathematical model. A negligible change in θ_0 was also pointed out by Breach (1985). Apart from the time factor, the intense concentration needed on the part of the observer to accumulate 19 cycles worth of observations is unjustified.

It is perhaps fortuitous that electronic timing facilities relating to gyroscopic research have not been available in South Africa up to the present time (1991), otherwise the same conclusions would have been made as Halmos (1977,1980) and Gregerson et al. (1974), who reported that improved results were due solely as a result of more accurate timing. It must be pointed out that high accuracy is not a function of electronic timing, but is rather produced by the quality of the gyro-rotor (Williams 1978b). Furthermore, it is absolutely essential to allow the gyro-rotor to warm up in order to reach a stable operating temperature before the observations are taken. For geodetic accuracies to be achieved it would be adviseable to make gyro-azimuth determinations at both the high and low range of the curves showing the cyclic variations of θ_0 during protracted oscillations (Williams and Belling 1967a; Chrzanowski 1969). There is no guarantee of getting good results from a set of, say, four turning points. Similarly, taking samples on one oscillation only (Eichholz and Schäfler 1981) is not adequate to map the motion of the gyroscope. A more complex mathematical model to include a term for linear drift (Gregerson 1970; Breach 1985) is also not recommended.

The damping of the first period was found to have an average value of 2.1e-6 sec^{-1} ± 8.8e-8 sec^{-1}, as determined from a single periodic model. Jeudy and Gagnon (1981) fixed this value as 4.7e-6 sec^{-1} ± 1.3e-6 sec^{-1}, while Williams (1981b, page 9) postulates that the damping factor α' (= f_1T) is 0.0005, or f_1 is 1.4e-6 sec^{-1} for a period T of 370 sec. The implication from the present study of manually-timed observations is that the damping can be established with a similar accuracy than possible with electronic timing. In passing, it is of interest that the damped motion can be observed in the third decimal of the calculated reversal points from MAMET (Williams 1978a, Table 3a).

Finally then, while least squares spectral analysis and the identification of higher-order harmonics is important for the determination of the statistical model, it does not result in a significant improvement in accuracy for θ_0, the equilibrium position of the oscillations. It is therefore sufficient to use a simple harmonic model to describe the motion of a spinning-mass gyroscope. If an instrument such as the GAK1 is used in conjunction with MAMET accuracies comparable with gyrotheodolites equipped with electronic timing can be obtained (Williams 1979).

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APPENDIX A

SINGLE PERIODIC MODEL

$$\theta = \theta_0 + B_1 e^{-f_1 t} \cos(\omega_1 t - \gamma_1)$$

$$weight_j \propto \sin^2(\omega_1 t_j - \gamma_1)$$

SET	1	2	3	4
iteration	6	6	5	6
ω_1 (rad/sec)	0.016276623 *4.86e-8*	0.016278077 *4.73e-8*	0.016274668 *3.82e-8*	0.016276379 *3.98e-8*
T_1 (sec)	386.025 *1.15e-3*	385.991 *1.12e-3*	386.071 *9.07e-4*	386.031 *9.45e-4*
B_1 (scl div)	10.6103 *4.88e-3*	10.7776 *4.93e-3*	10.1092 *3.46e-3*	10.9729 *4.31e-3*
f_1 (sec^{-1})	3.0954e-6 *1.08e-7*	2.4683e-6 *1.07e-7*	1.6989e-6 *8.06e-8*	2.0296e-6 *9.25e-8*
γ_1 (rad)	1.5790870 *2.06e-4* ($90^\circ 28' 30''$)	1.5824746 *2.00e-4* ($90^\circ 40' 09''$)	1.5820008 *1.62e-4* ($90^\circ 38' 31''$)	-1.5729074 *1.69e-4* ($-90^\circ 07' 15''$)
θ_0 (scl div)	0.0791 *9.86e-4* ($58''.90$)	0.0878 *9.82e-4* ($65''.40$)	0.0910 *7.37e-4* ($67''.71$)	0.0786 *8.46e-4* ($58''.53$)
$\hat{\sigma}_0$ (sec)	0.1379	0.1349	0.1076	0.1139

TABLE A1 : SINGLE PERIODIC MODELS FROM OBS. SETS 1-4, USING AN APPROXIMATE WEIGHT MODEL (std dev. in italics)

SINGLE PERIODIC MODEL

$$\theta = \theta_0 + B_1 e^{-f_1 t} \cos(\omega_1 t - \gamma_1)$$

$$\text{weight}_j \propto \sin^2(\omega_1 t_j - \gamma_1)$$

SET	5	6	7	8
iteration	6	6	6	6
ω_1 (rad/sec)	0.016265958 *3.23e-8*	0.016264180 *4.34e-8*	0.016352001 *5.57e-8*	0.016344654 *6.30e-8*
T_1 (sec)	386.278 *7.67e-4*	386.320 *1.03e-3*	384.246 *1.31e-3*	384.418 *1.48e-3*
B_1 (scl div)	10.7364 *3.34e-3*	9.6169 *3.51e-3*	10.0967 *5.01e-3*	10.4426 *6.11e-3*
f_1 (sec^{-1})	2.1817e-6 *7.31e-8*	1.9025e-6 *8.58e-8*	1.7929e-6 *1.17e-7*	1.7927e-6 *1.38e-7*
γ_1 (rad)	1.5802474 *1.37e-4* (90°32′29″)	1.5855156 *1.84e-4* (90°50′36″)	1.6015333 *2.35e-4* (91°45′40″)	1.5965078 *2.66e-4* (91°28′23″)
θ_0 (scl div)	0.1017 *6.69e-4* (75.″72)	0.1024 *7.85e-4* (76.″24)	0.2055 *1.07e-3* (153.″02)	0.1354 *1.26e-3* (100.″79)
$\hat{\sigma}_0$ (sec)	0.0921	0.1207	0.1552	0.1771

TABLE A2 : SINGLE PERIODIC MODELS FROM OBS. SETS 5-8, USING AN APPROXIMATE WEIGHT MODEL (std dev. in italics)

SINGLE PERIODIC MODEL

$$\theta = \theta_0 + B_1 e^{-f_1 t} \cos(\omega_1 t - \gamma_1)$$

$$\text{weight}_j \propto \sin^2(\omega_1 t_j - \gamma_1)$$

SET	9	10	11	12
iteration	7	6	6	6
ω_1 (rad/sec)	0.016375619 *2.54e-7*	0.016431713 *3.74e-8*	0.016427225 *3.12e-8*	0.016432086 *2.83e-8*
T_1 (sec)	383.691 *5.95e-3*	382.382 *8.70e-4*	382.486 *7.27e-4*	382.373 *6.59e-4*
B_1 (scl div)	9.5213 *2.00e-2*	11.2177 *4.21e-3*	10.3785 *2.97e-3*	10.7606 *2.91e-3*
f_1 (sec^{-1})	1.7005e-6 *4.96e-7*	1.9509e-6 *8.91e-8*	1.7701e-6 *6.79e-8*	1.7572e-6 *6.43e-8*
γ_1 (rad)	1.6198178 *1.07e-3* (92°48′31″)	1.5934380 *1.57e-4* (91°17′50″)	1.5917379 *1.31e-4* (91°12′00″)	1.5881728 *1.19e-4* (90°59′44″)
θ_0 (scl div)	0.2003 *4.52e-3* (149″.11)	0.1923 *8.08e-4* (143″.17)	0.1867 *6.16e-4* (139″.02)	0.1771 *5.83e-4* (131″.82)
$\hat{\sigma}_0$ (sec)	0.6951	0.1053	0.0867	0.0792

TABLE A3 : SINGLE PERIODIC MODELS FROM OBS. SETS 9-12, USING AN APPROXIMATE WEIGHT MODEL (std dev. in italics)

SINGLE PERIODIC MODEL

$$\theta = \theta_0 + B_1 e^{-f_1 t} \cos(\omega_1 t - \gamma_1)$$

$$\text{weight}_j \propto \sin^2(\omega_1 t_j - \gamma_1)$$

SET	13	14	15	17
iteration	5	8	11	5
ω_1 (rad/sec)	0.016428222 *2.70e-8*	0.016523988 *4.30e-7*	0.016553199 *1.59e-6*	0.016987529 *8.19e-8*
T_1 (sec)	382.463 *6.28e-4*	380.246 *9.89e-3*	379.575 *3.65e-2*	369.870 *1.78e-3*
B_1 (scl div)	10.2562 *2.50e-3*	9.1980 *3.11e-2*	7.9473 *7.70e-2*	8.6860 *4.99e-3*
f_1 (sec^{-1})	1.7776e-6 *5.78e-8*	-2.2976e-6 *8.12e-7*	1.4250e-5 *2.27e-6*	3.9892e-6 *1.41e-7*
γ_1 (rad)	1.5928175 *1.13e-4* ($91^\circ 15' 42''$)	1.5906953 *1.79e-3* ($91^\circ 08' 24''$)	1.6607698 *6.55e-3* ($95^\circ 09' 18''$)	1.5678359 *3.32e-4* ($89^\circ 49' 49''$)
θ_0 (scl div)	0.2009 *5.25e-4* ($149.''57$)	0.0012 *7.36e-3* ($0.''90$)	0.0118 *2.05e-2* ($8.''77$)	-0.0886 *1.23e-3* ($-65.''94$)
$\hat{\sigma}_0$ (sec)	0.0747	1.1474	3.9139	0.2023

TABLE A4 : SINGLE PERIODIC MODELS FROM OBS. SETS 13-15 AND 17, USING AN APPROXIMATE WEIGHT MODEL (std dev. in italics)

SINGLE PERIODIC MODEL

$$\theta = \theta_0 + B_1 e^{-f_1 t} \cos(\omega_1 t - \gamma_1)$$

$$\text{weight}_j \propto \sin^2(\omega_1 t_j - \gamma_1)$$

SET	19	21	22	23
iteration	6	6	6	6
ω_1 (rad/sec)	0.016570567 *4.71e-8*	0.016567704 *4.11e-8*	0.016566636 *5.31e-8*	0.016565613 *5.66e-8*
T_1 (sec)	379.177 *1.08e-3*	379.243 *9.41e-4*	379.267 *1.21e-3*	379.291 *1.30e-3*
B_1 (scl div)	9.7590 *3.86e-3*	9.9582 *3.53e-3*	10.5911 *5.22e-3*	9.1840 *4.05e-3*
f_1 (sec^{-1})	2.4636e-6 *9.47e-8*	2.4740e-6 *8.48e-8*	2.5055e-6 *1.18e-7*	2.3773e-6 *1.05e-7*
γ_1 (rad)	1.5860389 *1.96e-4* (90°52′24″)	1.5853245 *1.71e-4* (90°49′57″)	1.5875516 *2.21e-4* (90°57′36″)	1.5859411 *2.35e-4* (90°52′04″)
θ_0 (scl div)	0.0923 *8.50e-4* (68″.74)	0.1030 *7.61e-4* (76″.70)	0.1103 *1.06e-3* (82″.13)	0.0846 *9.47e-4* (62″.98)
$\hat{\sigma}_0$ (sec)	0.1266	0.1112	0.1454	0.1498

TABLE A5 : SINGLE PERIODIC MODELS FROM OBS. SETS 19 AND 21-23, USING AN APPROXIMATE WEIGHT MODEL (std dev. in italics)

TWO-PERIODIC MODEL

$$\theta = \theta_0 + \sum_{i=1}^{2} B_i e^{-f_i t} \cos(\omega_i t - \gamma_i)$$

$$\text{weight}_j \propto \sin^2(\omega_1 t_j - \gamma_1)$$

SET	1	2	3	4
iteration	18	12	17	26
ω_1 (rad/sec)	0.016276544 *2.94e-7*	0.016278489 *5.42e-7*	0.016275115 *7.59e-8*	0.016276628 *6.69e-8*
T_1 (sec)	386.027 *6.96e-3*	385.981 *1.29e-2*	386.061 *1.80e-3*	386.025 *1.59e-3*
B_1 (scl div)	10.5613 *1.18e-2*	10.8674 *3.88e-2*	10.0994 *5.79e-3*	10.9899 *4.73e-3*
f_1 (sec^{-1})	1.8193e-6 *2.39e-7*	3.9903e-6 *4.45e-7*	1.5223e-6 *1.17e-7*	2.3223e-6 *9.63e-8*
γ_1 (rad)	1.5775407 *1.26e-3* (90°23′11″)	1.5819727 *3.02e-3* (90°38′25″)	1.5842340 *4.04e-4* (90°46′12″)	-1.5717461 *3.31e-4* (-90°03′16″)
ω_2 (rad/sec)	0.015679764 *7.44e-5*	0.016674814 *1.27e-4*	0.015443883 *1.55e-4*	0.017346241 *8.31e-5*
T_2 (sec)	400.719 *1.90*	376.807 *2.88*	406.840 *4.09*	362.222 *1.74*
B_2 (scl div)	0.0364 *1.13e-2*	0.1020 *4.60e-2*	0.0587 *8.16e-3*	0.0552 *4.75e-3*
f_2 (sec^{-1})	9.1497e-6 *7.34e-5*	2.2075e-4 *8.81e-5*	8.4448e-4 *1.64e-4*	5.2361e-4 *7.40e-5*
γ_2 (rad)	1.0308561 *0.359* (59°03′49″)	-1.1660455 *0.318* (-66°48′34″)	-0.3085554 *0.181* (-17°40′44″)	3.0868891 *0.175* (176°51′57″)
θ_0 (scl div)	0.0792 *7.15e-4* (58.″98)	0.0877 *7.45e-4* (65.″28)	0.0910 *6.26e-4* (67.″77)	0.0788 *6.40e-4* (58.″65)
$\hat{\sigma}_0$ (sec)	0.0998	0.1022	0.0912	0.0862

TABLE A6 : TWO-PERIODIC MODELS FROM OBS. SETS 1-4, USING AN APPROXIMATE WEIGHT MODEL (std dev. in italics)

TWO-PERIODIC MODEL

$$\theta = \theta_0 + \sum_{i=1}^{2} B_i e^{-f_i t} \cos(\omega_i t - \gamma_i)$$

$$\text{weight}_j \propto \sin^2(\omega_1 t_j - \gamma_1)$$

SET	5	6	7	8
iteration	67	26	29	24
ω_1 (rad/sec)	0.016266011 *3.32e-8*	0.016264695 *8.76e-8*	0.016353401 *1.02e-7*	0.016345326 *5.17e-8*
T_1 (sec)	386.277 *7.88e-4*	386.308 *2.08e-3*	384.213 *2.39e-3*	384.403 *1.22e-3*
B_1 (scl div)	10.7313 *3.22e-3*	9.5942 *5.61e-3*	10.0802 *8.64e-3*	10.4276 *4.89e-3*
f_1 (sec^{-1})	2.0707e-6 *7.01e-8*	1.4582e-6 *1.20e-7*	1.5052e-6 *1.55e-7*	1.5005e-6 *1.03e-7*
γ_1 (rad)	1.5804822 *1.44e-4* (90°33′18″)	1.5879771 *4.69e-4* (90°59′04″)	1.6089226 *6.19e-4* (92°11′04″)	1.5998314 *2.41e-4* (91°39′49″)
ω_2 (rad/sec)	0.014515413 *6.02e-5*	0.015455322 *9.95e-5*	0.016062639 *1.09e-4*	0.014635931 *1.78e-4*
T_2 (sec)	432.863 *1.80*	406.539 *2.62*	391.168 *2.66*	429.299 *5.22*
B_2 (scl div)	0.0141 *2.34e-3*	0.0672 *6.99e-3*	0.1660 *7.04e-3*	0.1934 *1.15e-2*
f_2 (sec^{-1})	1.6289e-4 *5.60e-5*	6.7805e-4 *1.08e-4*	9.2565e-4 *6.26e-5*	2.2358e-3 *1.97e-4*
γ_2 (rad)	-0.4234966 *0.189* (-24°15′52″)	0.0123013 *0.156* (00°42′17″)	-0.0505050 *0.064* (-02°53′37″)	-0.4016771 *0.078* (-23°00′52″)
θ_0 (scl div)	0.1018 *6.21e-4* (75.″76)	0.1025 *6.02e-4* (76.″29)	0.2054 *5.58e-4* (152.″95)	0.1354 *7.67e-4* (100.″82)
$\hat{\sigma}_0$ (sec)	0.0855	0.0924	0.0809	0.1074

TABLE A7 : TWO-PERIODIC MODELS FROM OBS. SETS 5-8, USING AN APPROXIMATE WEIGHT MODEL (std dev. in italics)

TWO-PERIODIC MODEL

$$\theta = \theta_0 + \sum_{i=1}^{2} B_i e^{-f_i t} \cos(\omega_i t - \gamma_i)$$

$$\text{weight}_j \propto \sin^2(\omega_1 t_j - \gamma_1)$$

SET	10	11	12	13
iteration	22	63	20	43
ω_1 (rad/sec)	0.016432362 *7.24e-8*	0.016428343 *2.60e-7*	0.016432693 *1.38e-7*	0.016428177 *9.21e-8*
T_1 (sec)	382.367 *1.68e-3*	382.460 *6.05e-3*	382.359 *3.21e-3*	382.464 *2.14e-3*
B_1 (scl div)	11.2114 *8.47e-3*	10.3684 *4.69e-2*	10.7594 *1.29e-2*	10.2615 *3.13e-3*
f_1 (sec^{-1})	1.8378e-6 *1.42e-7*	1.6148e-6 *5.50e-7*	1.6920e-6 *2.00e-7*	1.8953e-6 *7.50e-8*
γ_1 (rad)	1.5968058 *4.08e-4* (91°29′25″)	1.5987430 *2.17e-3* (91°36′04″)	1.5914987 *9.60e-4* (91°11′10″)	1.5926507 *3.90e-4* (91°15′08″)
ω_2 (rad/sec)	0.016199611 *2.66e-4*	0.016404021 *2.23e-4*	0.016660040 *2.72e-4*	0.017372693 *1.38e-4*
T_2 (sec)	387.860 *6.36*	383.027 *5.20*	377.141 *6.15*	361.670 *2.88*
B_2 (scl div)	0.0982 *5.59e-3*	0.1018 *2.07e-2*	0.0641 *1.20e-2*	0.0099 *3.15e-3*
f_2 (sec^{-1})	1.1729e-3 *1.28e-4*	4.3770e-4 *1.02e-4*	6.9703e-4 *1.28e-4*	1.4384e-4 *1.29e-4*
γ_2 (rad)	-0.0891214 *0.116* (-05°06′23″)	0.1078514 *0.433* (06°10′46″)	0.1971296 *0.179* (11°17′41″)	-0.4539655 *0.517* (-26°00′37″)
θ_0 (scl div)	0.1922 *5.94e-4* (143.″10)	0.1867 *4.96e-4* (138.″98)	0.1770 *4.84e-4* (131.″74)	0.2009 *4.97e-4* (149.″53)
$\hat{\sigma}_0$ (sec)	0.0772	0.0697	0.0656	0.0708

TABLE A8 : TWO-PERIODIC MODELS FROM OBS. SETS 10-13, USING AN APPROXIMATE WEIGHT MODEL (std dev. in italics)

TWO-PERIODIC MODEL

$$\theta = \theta_0 + \sum_{i=1}^{2} B_i e^{-f_i t} \cos(\omega_i t - \gamma_i)$$

$$\text{weight}_j \propto \sin^2(\omega_1 t_j - \gamma_1)$$

SET	19	21	22	23
iteration	15	22	20	17
ω_1 (rad/sec)	0.016573225 *9.52e-7*	0.016568921 *8.20e-7*	0.016568866 *1.97e-7*	0.016568135 *2.29e-7*
T_1 (sec)	379.117 *2.18e-2*	379.215 *1.88e-2*	379.216 *4.50e-3*	379.233 *5.25e-3*
B_1 (scl div)	9.6085 *5.50e-2*	10.0746 *4.69e-2*	10.5717 *2.55e-2*	9.1914 *2.99e-2*
f_1 (sec^{-1})	7.1090e-7 *7.00e-7*	4.0768e-6 *4.67e-7*	2.2397e-6 *3.46e-7*	2.4494e-6 *4.31e-7*
γ_1 (rad)	1.6006252 *9.76e-3* (91°42′33″)	1.5893093 *6.42e-3* (91°03′39″)	1.6004195 *1.51e-3* (91°41′50″)	1.6009743 *1.73e-3* (91°43′45″)
ω_2 (rad/sec)	0.016400530 *9.92e-5*	0.016824251 *1.32e-4*	0.016467235 *9.36e-5*	0.016584065 *9.06e-5*
T_2 (sec)	383.109 *2.32*	373.460 *2.93*	381.557 *2.17*	378.869 *2.07*
B_2 (scl div)	0.2358 *8.21e-2*	0.1423 *6.71e-2*	0.2135 *1.53e-2*	0.2066 *1.46e-2*
f_2 (sec^{-1})	2.5886e-4 *5.49e-5*	2.4932e-4 *7.42e-5*	5.5764e-4 *4.48e-5*	5.1270e-4 *4.81e-5*
γ_2 (rad)	0.6631252 *0.310* (37°59′39″)	-0.9742460 *0.347* (-55°49′13″)	0.0261967 *0.109* (01°30′03″)	-0.0434258 *0.133* (-02°29′17″)
θ_0 (scl div)	0.0923 *5.16e-4* (68.″71)	0.1029 *5.38e-4* (76.″60)	0.1102 *5.07e-4* (82.″04)	0.0845 *4.92e-4* (62.″87)
$\hat{\sigma}_0$ (sec)	0.0766	0.0785	0.0695	0.0777

TABLE A9 : TWO-PERIODIC MODELS FROM OBS. SETS 19 AND 21-23, USING AN APPROXIMATE WEIGHT MODEL (std dev. in italics)

SINGLE PERIODIC MODEL

$$\theta = \theta_0 + B_1 e^{-f_1 t} \cos(\omega_1 t - \gamma_1)$$

$$weight_j = 1$$

SET	1	2	3	4
iteration	6	6	6	6
ω_1 (rad/sec)	0.016276626 *4.85e-8*	0.016278065 *4.77e-8*	0.016274669 *3.88e-8*	0.016276386 *4.01e-8*
T_1 (sec)	386.025 *1.15e-3*	385.991 *1.13e-3*	386.071 *9.19e-4*	386.031 *9.51e-4*
B_1 (scl div)	10.6106 *4.40e-3*	10.7761 *4.50e-3*	10.1094 *3.11e-3*	10.9731 *3.96e-3*
f_1 (sec^{-1})	3.0958e-6 *9.69e-8*	2.4330e-6 *9.79e-8*	1.6974e-6 *7.23e-8*	2.0277e-6 *8.48e-8*
γ_1 (rad)	1.5790964 *2.05e-4* ($90^\circ 28' 32''$)	1.5824590 *2.02e-4* ($90^\circ 40' 06''$)	1.5820206 *1.64e-4* ($90^\circ 38' 35''$)	-1.5728884 *1.70e-4* ($-90^\circ 07' 12''$)
θ_0 (scl div)	0.0788 *9.65e-4* ($58.''70$)	0.0877 *9.72e-4* ($65.''29$)	0.0907 *7.29e-4* ($67.''54$)	0.0784 *8.38e-4* ($58.''38$)
$\hat{\sigma}_0$ (sec)	0.1510	0.1485	0.1207	0.1248

TABLE A10 : SINGLE PERIODIC MODELS FROM OBS. SETS 1-4, USING UNIT WEIGHT (std dev. in italics)

SINGLE PERIODIC MODEL

$$\theta = \theta_0 + B_1 e^{-f_1 t} \cos(\omega_1 t - \gamma_1)$$

$$\text{weight}_j = 1$$

SET	5	6	7	8
iteration	6	6	6	6
ω_1 (rad/sec)	0.016265953 *3.25e-8*	0.016264192 *4.43e-8*	0.016352004 *5.58e-8*	0.016344659 *6.31e-8*
T_1 (sec)	386.278 *7.72e-4*	386.320 *1.05e-3*	384.246 *1.31e-3*	384.418 *1.48e-3*
B_1 (scl div)	10.7362 *3.04e-3*	9.6183 *3.11e-3*	10.0951 *4.44e-3*	10.4418 *5.49e-3*
f_1 (sec^{-1})	2.1816e-6 *6.64e-8*	1.9258e-6 *7.58e-8*	1.7617e-6 *1.04e-7*	1.7755e-6 *1.24e-7*
γ_1 (rad)	1.5802142 *1.38e-4* (90°32′23″)	1.5855166 *1.88e-4* (90°50′36″)	1.6015436 *2.35e-4* (91°45′42″)	1.5965652 *2.66e-4* (91°28′35″)
θ_0 (scl div)	0.1013 *6.60e-4* (75.″41)	0.1017 *7.76e-4* (75.″70)	0.2052 *1.04e-3* (152.″78)	0.1355 *1.23e-3* (100.″86)
$\hat{\sigma}_0$ (sec)	0.1013	0.1381	0.1721	0.1949

TABLE A11 : SINGLE PERIODIC MODELS FROM OBS. SETS 5-8, USING UNIT WEIGHT (std dev. in italics)

SINGLE PERIODIC MODEL

$$\theta = \theta_0 + B_1 e^{-f_1 t} \cos(\omega_1 t - \gamma_1)$$

$$weight_j = 1$$

SET	9	10	11	12
iteration	7	6	6	6
ω_1 (rad/sec)	0.016375649 *2.53e-7*	0.016431709 *3.74e-8*	0.016427224 *3.15e-8*	0.016432078 *2.86e-8*
T_1 (sec)	383.691 *5.94e-3*	382.382 *8.71e-4*	382.486 *7.33e-4*	382.373 *6.66e-4*
B_1 (scl div)	9.5165 *1.72e-2*	11.2167 *3.87e-3*	10.3781 *2.68e-3*	10.7606 *2.67e-3*
f_1 (sec^{-1})	1.6003e-6 *4.26e-7*	1.9343e-6 *8.16e-8*	1.7591e-6 *6.11e-8*	1.7594e-6 *5.88e-8*
γ_1 (rad)	1.6199531 *1.07e-3* (92°48′59″)	1.5934502 *1.57e-4* (91°17′53″)	1.5917435 *1.32e-4* (91°12′01″)	1.5881443 *1.20e-4* (90°59′38″)
θ_0 (scl div)	0.1998 *4.36e-3* (148.″73)	0.1921 *7.98e-4* (143.″01)	0.1863 *6.08e-4* (138.″66)	0.1766 *5.79e-4* (131.″51)
$\hat{\sigma}_0$ (sec)	0.7793	0.1144	0.0962	0.0874

TABLE A12 : SINGLE PERIODIC MODELS FROM OBS. SETS 9-12, USING UNIT WEIGHT (std dev. in italics)

SINGLE PERIODIC MODEL

$$\theta = \theta_0 + B_1 e^{-f_1 t} \cos(\omega_1 t - \gamma_1)$$

$$weight_j = 1$$

SET	13	14	15	17
iteration	5	8	12	5
ω_1 (rad/sec)	0.016428225 *2.73e-8*	0.016523887 *4.39e-7*	0.016553781 *1.65e-6*	0.016987509 *8.68e-8*
T_1 (sec)	382.463 *6.35e-4*	380.249 *1.01e-2*	379.562 *3.79e-2*	369.871 *1.89e-3*
B_1 (scl div)	10.2560 *2.25e-3*	9.1819 *2.68e-2*	7.9454 *4.97e-2*	8.6837 *4.26e-3*
f_1 (sec^{-1})	1.7626e-6 *5.19e-8*	-2.6172e-6 *7.02e-7*	1.6094e-5 *1.15e-6*	3.9690e-6 *1.19e-7*
γ_1 (rad)	1.5928210 *1.14e-4* ($91^\circ 15' 43''$)	1.5900177 *1.83e-3* ($91^\circ 06' 05''$)	1.6611980 *6.86e-3* ($95^\circ 10' 47''$)	1.5677773 *3.52e-4* ($89^\circ 49' 37''$)
θ_0 (scl div)	0.2006 *5.18e-4* ($149''.38$)	0.0005 *7.23e-3* ($0''.34$)	0.0123 *1.50e-2* ($9''.17$)	-0.0884 *1.22e-3* ($-65''.84$)
$\hat{\sigma}_0$ (sec)	0.0833	1.3249	4.9325	0.2481

TABLE A13 : SINGLE PERIODIC MODELS FROM OBS. SETS 13-15 AND 17, USING UNIT WEIGHT (std dev. in italics)

SINGLE PERIODIC MODEL

$$\theta = \theta_0 + B_1 e^{-f_1 t} \cos(\omega_1 t - \gamma_1)$$

$$\text{weight}_j = 1$$

SET	19	21	22	23
iteration	6	6	6	6
ω_1 (rad/sec)	0.016570553 *4.76e-8*	0.016567702 *4.14e-8*	0.016566634 *5.32e-8*	0.016565619 *5.76e-8*
T_1 (sec)	379.178 *1.09e-3*	379.243 *9.47e-4*	379.267 *1.22e-3*	379.291 *1.32e-3*
B_1 (scl div)	9.7597 *3.41e-3*	9.9582 *3.13e-3*	10.5910 *4.72e-3*	9.1844 *3.48e-3*
f_1 (sec^{-1})	2.4649e-6 *8.34e-8*	2.4546e-6 *7.49e-8*	2.4963e-6 *1.06e-7*	2.3696e-6 *9.03e-8*
γ_1 (rad)	1.5860091 *1.98e-4* ($90^\circ 52' 18''$)	1.5852884 *1.72e-4* ($90^\circ 49' 49''$)	1.5875335 *2.21e-4* ($90^\circ 57' 32''$)	1.5859941 *2.40e-4* ($90^\circ 52' 15''$)
θ_0 (scl div)	0.0921 *8.33e-4* ($68''.60$)	0.1029 *7.45e-4* ($76''.64$)	0.1101 *1.04e-3* ($81''.93$)	0.0843 *9.22e-4* ($62''.75$)
$\hat{\sigma}_0$ (sec)	0.1429	0.1243	0.1599	0.1731

TABLE A14 : SINGLE PERIODIC MODELS FROM OBS. SETS 19 AND 21-23, USING UNIT WEIGHT (std dev. in italics)

TWO-PERIODIC MODEL

$$\theta = \theta_0 + \sum_{i=1}^{2} B_i e^{-f_i t} \cos(\omega_i t - \gamma_i)$$

$$\text{weight}_j = 1$$

SET	1	2	3	4
iteration	16	11	15	25
ω_1 (rad/sec)	0.016276645 *2.86e-7*	0.016278483 *4.62e-7*	0.016275138 *7.55e-8*	0.016276662 *6.30e-8*
T_1 (sec)	386.025 *6.77e-3*	385.981 *1.10e-2*	386.060 *1.79e-3*	386.024 *1.49e-3*
B_1 (scl div)	10.5585 *1.23e-2*	10.8602 *3.24e-2*	10.1011 *5.46e-3*	10.9900 *4.52e-3*
f_1 (sec^{-1})	1.8047e-6 *2.44e-7*	3.8759e-6 *3.94e-7*	1.5516e-6 *1.09e-7*	2.3167e-6 *9.08e-8*
γ_1 (rad)	1.5778113 *1.30e-3* (90°24′07″)	1.5822396 *2.55e-3* (90°39′20″)	1.5843838 *4.01e-4* (90°46′43″)	-1.5715920 *3.12e-4* (-90°02′44″)
ω_2 (rad/sec)	0.015694977 *7.53e-5*	0.016696363 *1.18e-4*	0.015482390 *1.61e-4*	0.017344318 *8.38e-5*
T_2 (sec)	400.331 *1.92*	376.321 *2.66*	405.828 *4.23*	362.262 *1.75*
B_2 (scl div)	0.0405 *1.22e-2*	0.0958 *3.84e-2*	0.0620 *8.04e-3*	0.0574 *4.92e-3*
f_2 (sec^{-1})	3.4559e-5 *7.24e-5*	2.2426e-4 *8.38e-5*	8.9110e-4 *1.61e-4*	5.6235e-4 *7.68e-5*
γ_2 (rad)	1.0046131 *0.346* (57°33′36″)	-1.1023045 *0.296* (-63°09′27″)	-0.3316021 *0.161* (-18°59′58″)	3.1186233 *0.163* (178°41′02″)
θ_0 (scl div)	0.0790 *7.07e-4* (58.″79)	0.0875 *7.36e-4* (65.″17)	0.0908 *6.19e-4* (67.″62)	0.0786 *6.34e-4* (58.″51)
$\hat{\sigma}_0$ (sec)	0.1105	0.1123	0.1023	0.0945

TABLE A15 : TWO-PERIODIC MODELS FROM OBS. SETS 1-4, USING UNIT WEIGHT (std dev. in italics)

TWO-PERIODIC MODEL

$$\theta = \theta_0 + \sum^{2} B_i e^{-f_i t} \cos(\omega_i t - \gamma_i)$$

$$weight_j = 1$$

SET	5	6	7	8
iteration	54	28	22	27
ω_1 (rad/sec)	0.016266002 *3.33e-8*	0.016264792 *9.44e-8*	0.016353403 *1.01e-7*	0.016345345 *5.37e-8*
T_1 (sec)	386.277 *7.91e-4*	386.306 *2.24e-3*	384.213 *2.37e-3*	384.402 *1.26e-3*
B_1 (scl div)	10.7314 *2.95e-3*	9.5962 *5.88e-3*	10.0783 *8.00e-3*	10.4308 *4.62e-3*
f_1 (sec^{-1})	2.0765e-6 *6.41e-8*	1.5050e-6 *1.20e-7*	1.4732e-6 *1.42e-7*	1.5607e-6 *9.68e-8*
γ_1 (rad)	1.5804358 *1.44e-4* (90°33′08″)	1.5884518 *5.16e-4* (91°00′42″)	1.6089319 *6.10e-4* (92°11′06″)	1.5999753 *2.51e-4* (91°40′19″)
ω_2 (rad/sec)	0.014505451 *5.94e-5*	0.015560590 *1.13e-4*	0.016061070 *1.00e-4*	0.014837724 *2.04e-4*
T_2 (sec)	433.160 *1.77*	403.788 *2.93*	391.206 *2.44*	423.460 *5.82*
B_2 (scl div)	0.0134 *2.29e-3*	0.0743 *7.57e-3*	0.1657 *6.84e-3*	0.2004 *1.18e-2*
f_2 (sec^{-1})	1.4982e-4 *5.56e-5*	7.3570e-4 *1.11e-4*	9.1600e-4 *6.29e-5*	2.4065e-3 *2.02e-4*
γ_2 (rad)	-0.4492876 *0.190* (-25°44′32″)	0.0260425 *0.141* (01°29′32″)	-0.0377041 *0.059* (-02°09′37″)	-0.4047549 *0.073* (-23°11′27″)
θ_0 (scl div)	0.1014 *6.14e-4* (75.″46)	0.1018 *6.00e-4* (75.″76)	0.2052 *5.53e-4* (152.″76)	0.1356 *7.72e-4* (100.″92)
$\hat{\sigma}_0$ (sec)	0.0942	0.1067	0.0910	0.1212

TABLE A16 : TWO-PERIODIC MODELS FROM OBS. SETS 5-8, USING UNIT WEIGHT (std dev. in italics)

TWO-PERIODIC MODEL

$$\theta = \theta_0 + \sum_{i=1}^{2} B_i e^{-f_i t} \cos(\omega_i t - \gamma_i)$$

$$\text{weight}_j = 1$$

SET	10	11	12	13
iteration	22	41	18	35
ω_1 (rad/sec)	0.016432360 *7.06e-8*	0.016428141 *2.83e-7*	0.016432647 *1.22e-7*	0.016428180 *9.22e-8*
T_1 (sec)	382.367 *1.64e-3*	382.465 *6.58e-3*	382.360 *2.85e-3*	382.464 *2.15e-3*
B_1 (scl div)	11.2127 *8.40e-3*	10.4001 *2.35e-2*	10.7603 *1.06e-2*	10.2624 *3.26e-3*
f_1 (sec^{-1})	1.8592e-6 *1.38e-7*	2.0121e-6 *3.30e-7*	1.7145e-6 *1.68e-7*	1.9082e-6 *7.73e-8*
γ_1 (rad)	1.5968421 *3.94e-4* (91°29′32″)	1.5969725 *2.36e-3* (91°29′59″)	1.5912200 *8.18e-4* (91°10′13″)	1.5926255 *4.05e-4* (91°15′03″)
ω_2 (rad/sec)	0.016272467 *2.62e-4*	0.016605119 *2.03e-4*	0.016686975 *2.60e-4*	0.017324515 *1.38e-4*
T_2 (sec)	386.124 *6.22*	378.388 *4.63*	376.532 *5.88*	362.676 *2.90*
B_2 (scl div)	0.0984 *5.16e-3*	0.0858 *2.21e-2*	0.0618 *1.02e-2*	0.0112 *3.66e-3*
f_2 (sec^{-1})	1.1696e-3 *1.29e-4*	4.7129e-4 *1.04e-4*	7.3706e-4 *1.31e-4*	1.7692e-4 *1.33e-4*
γ_2 (rad)	-0.0891806 *0.106* (-05°06′35″)	-0.2154185 *0.290* (-12°20′33″)	0.1945781 *0.162* (11°08′55″)	-0.5585455 *0.485* (-32°00′08″)
θ_0 (scl div)	0.1920 *5.93e-4* (142.″94)	0.1862 *4.92e-4* (138.″60)	0.1765 *4.84e-4* (131.″43)	0.2006 *4.90e-4* (149.″34)
$\hat{\sigma}_0$ (sec)	0.0847	0.0777	0.0729	0.0788

TABLE A17 : TWO-PERIODIC MODELS FROM OBS. SETS 10-13, USING UNIT WEIGHT (std dev. in italics)

TWO-PERIODIC MODEL

$$\theta = \theta_0 + \sum_{i=1}^{2} B_i e^{-f_i t} \cos(\omega_i t - \gamma_i)$$

$$\text{weight}_j = 1$$

SET	19	21	22	23
iteration	22	19	19	12
ω_1 (rad/sec)	0.016573006 *8.37e-7*	0.016568836 *6.97e-7*	0.016568831 *1.88e-7*	0.016568078 *2.28e-7*
T_1 (sec)	379.122 *1.92e-2*	379.217 *1.60e-2*	379.217 *4.31e-3*	379.234 *5.21e-3*
B_1 (scl div)	9.6038 *5.20e-2*	10.0713 *4.23e-2*	10.5723 *2.26e-2*	9.2026 *2.41e-2*
f_1 (sec^{-1})	5.7062e-7 *6.09e-7*	4.0613e-6 *4.35e-7*	2.2445e-6 *3.08e-7*	2.6145e-6 *3.53e-7*
γ_1 (rad)	1.5981834 *7.93e-3* (91°34′09″)	1.5887274 *5.28e-3* (91°01′39″)	1.6001654 *1.42e-3* (91°40′58″)	1.6004801 *1.71e-3* (91°42′03″)
ω_2 (rad/sec)	0.016374482 *8.74e-5*	0.016837345 *1.18e-4*	0.016463699 *8.61e-5*	0.016614734 *7.88e-5*
T_2 (sec)	383.718 *2.05*	373.170 *2.60*	381.639 *1.99*	378.169 *1.79*
B_2 (scl div)	0.2207 *7.23e-2*	0.1364 *5.72e-2*	0.2117 *1.43e-2*	0.2043 *1.42e-2*
f_2 (sec^{-1})	2.5026e-4 *5.55e-5*	2.4969e-4 *7.25e-5*	5.6612e-4 *4.54e-5*	5.2719e-4 *4.95e-5*
γ_2 (rad)	0.7433516 *0.273* (42°35′27″)	-0.9912937 *0.312* (-56°47′49″)	0.0246853 *0.098* (01°24′52″)	-0.0917468 *0.108* (-05°15′24″)
θ_0 (scl div)	0.0921 *5.11e-4* (68″.59)	0.1028 *5.27e-4* (76″.55)	0.1100 *5.04e-4* (81″.88)	0.0842 *4.88e-4* (62″.66)
$\hat{\sigma}_0$ (sec)	0.0875	0.0878	0.0771	0.0913

TABLE A18 : TWO-PERIODIC MODELS FROM OBS. SETS 19 AND 21-23, USING UNIT WEIGHT (std dev. in italics)

APPENDIX B

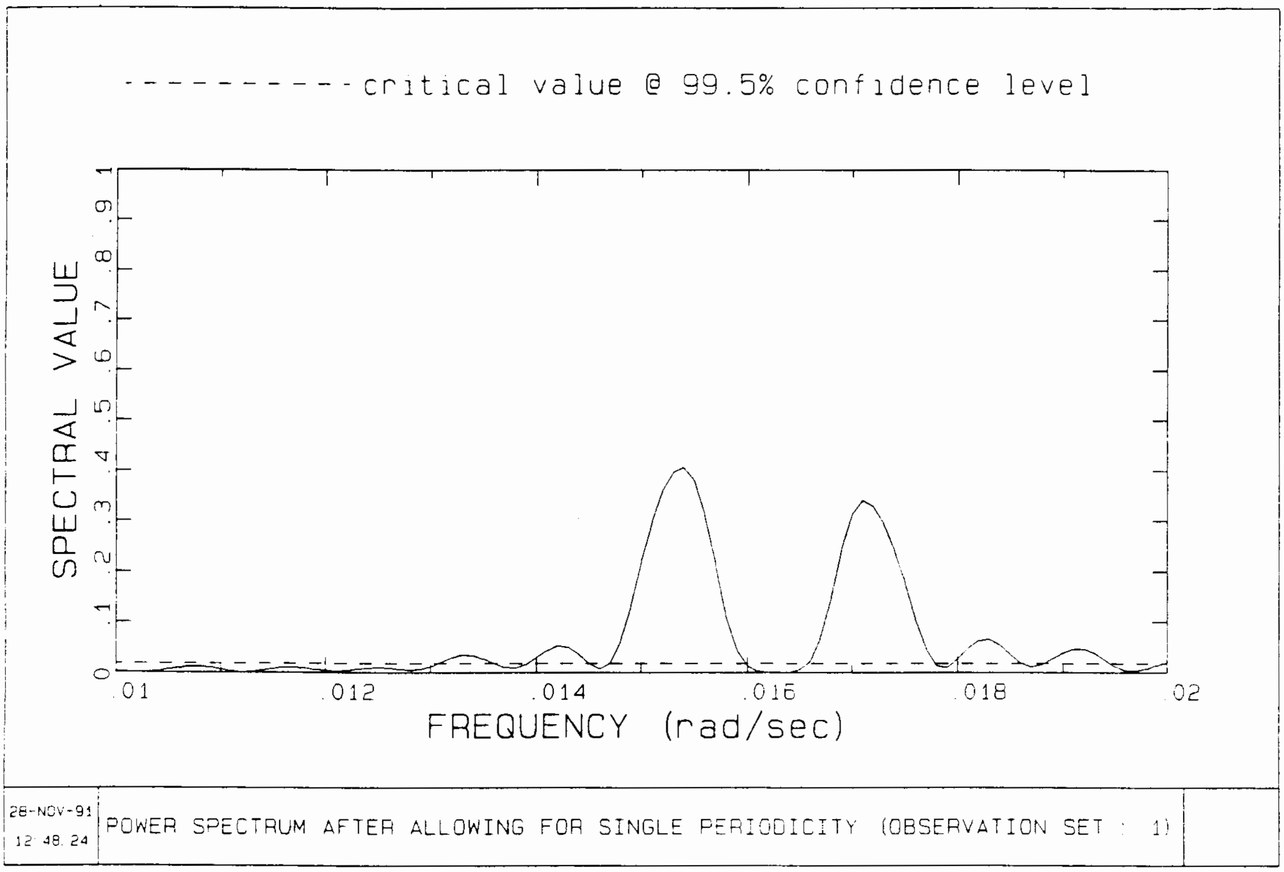


FIGURE B1

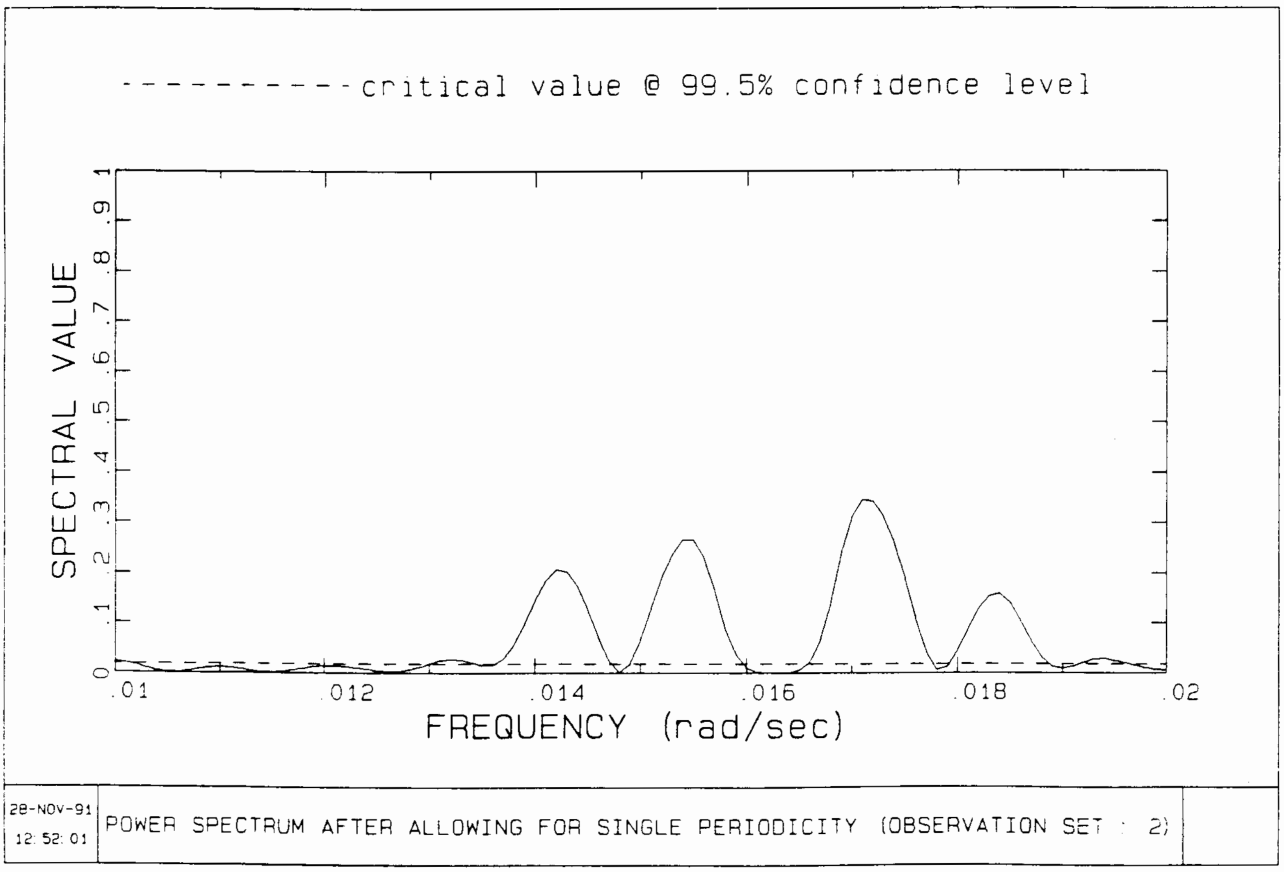


FIGURE B2

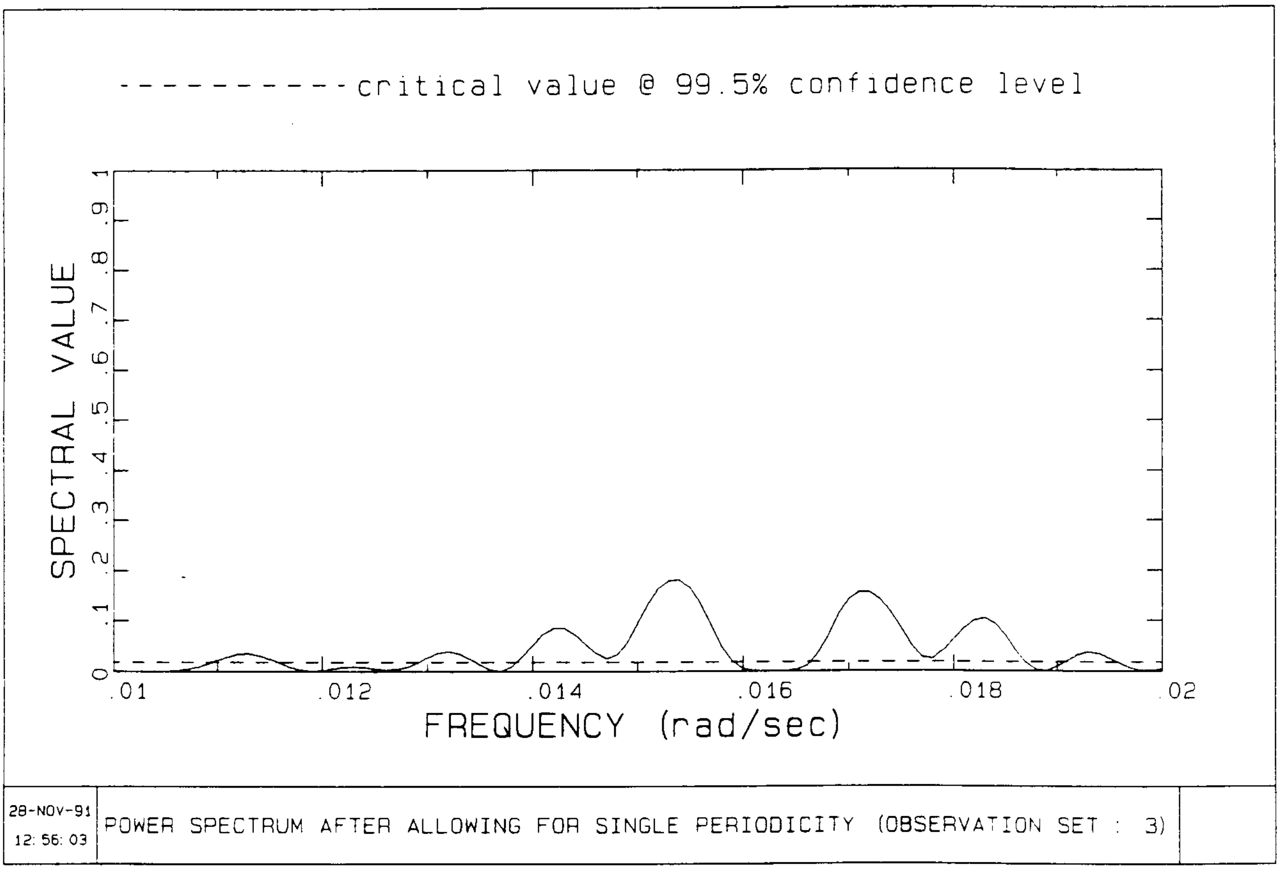


FIGURE B3

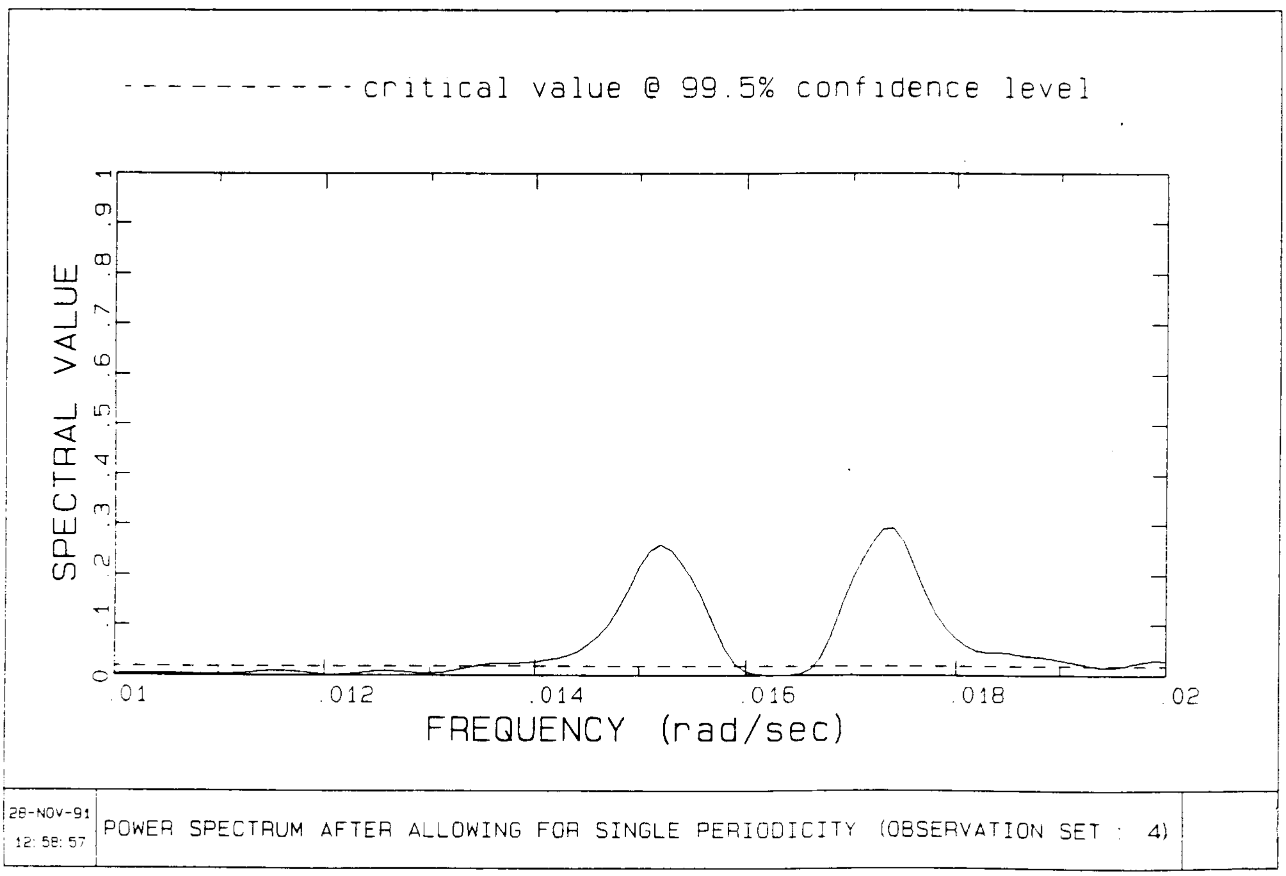


FIGURE B4

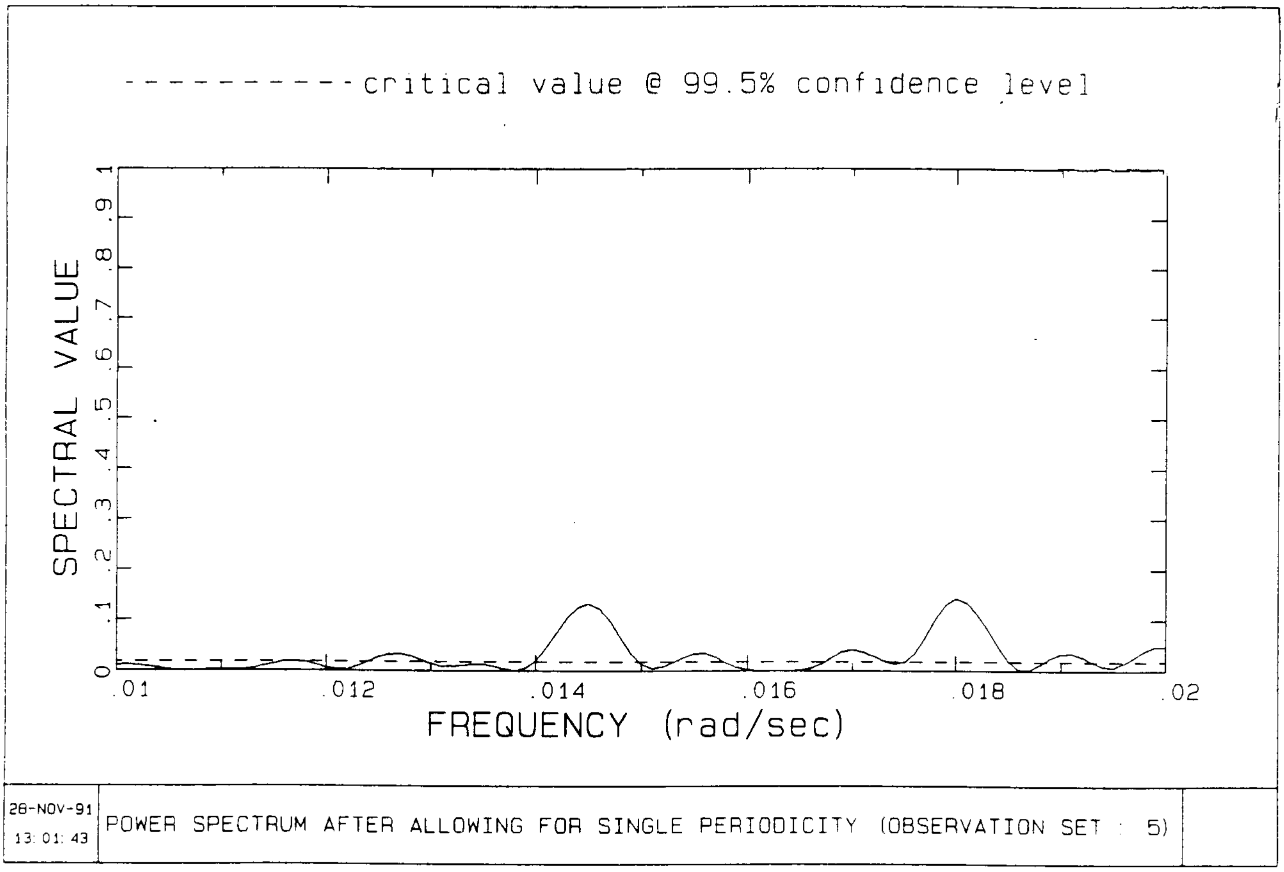


FIGURE B5

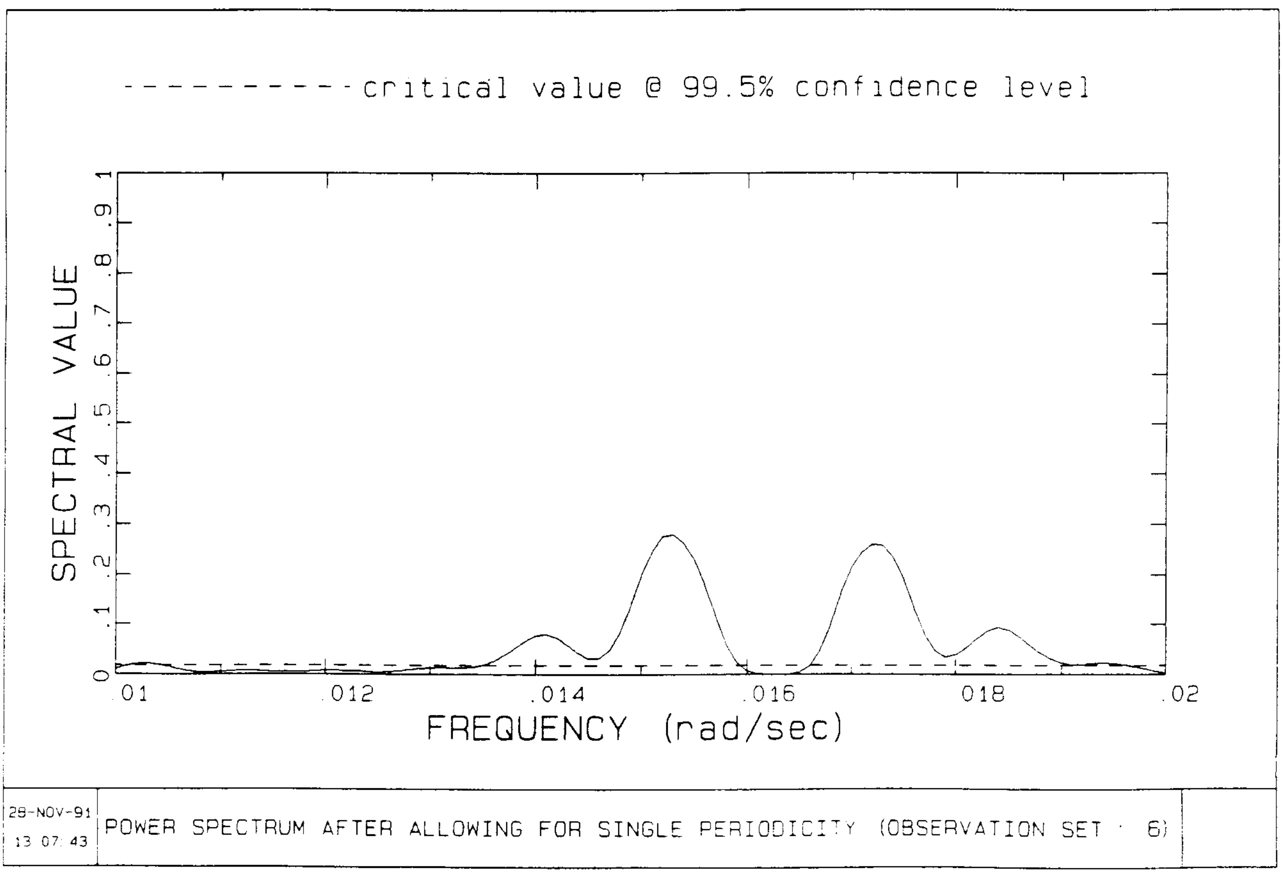


FIGURE B6

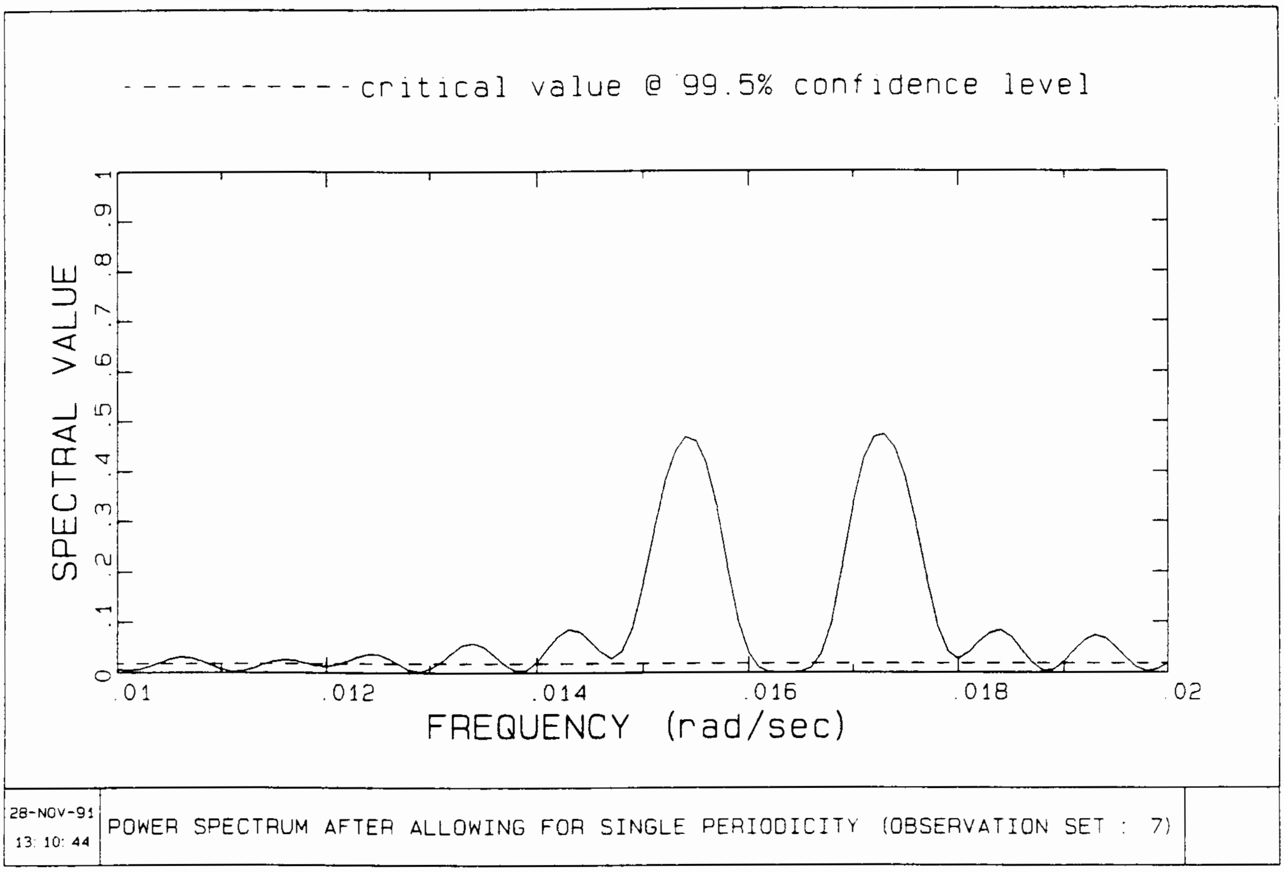


FIGURE B7

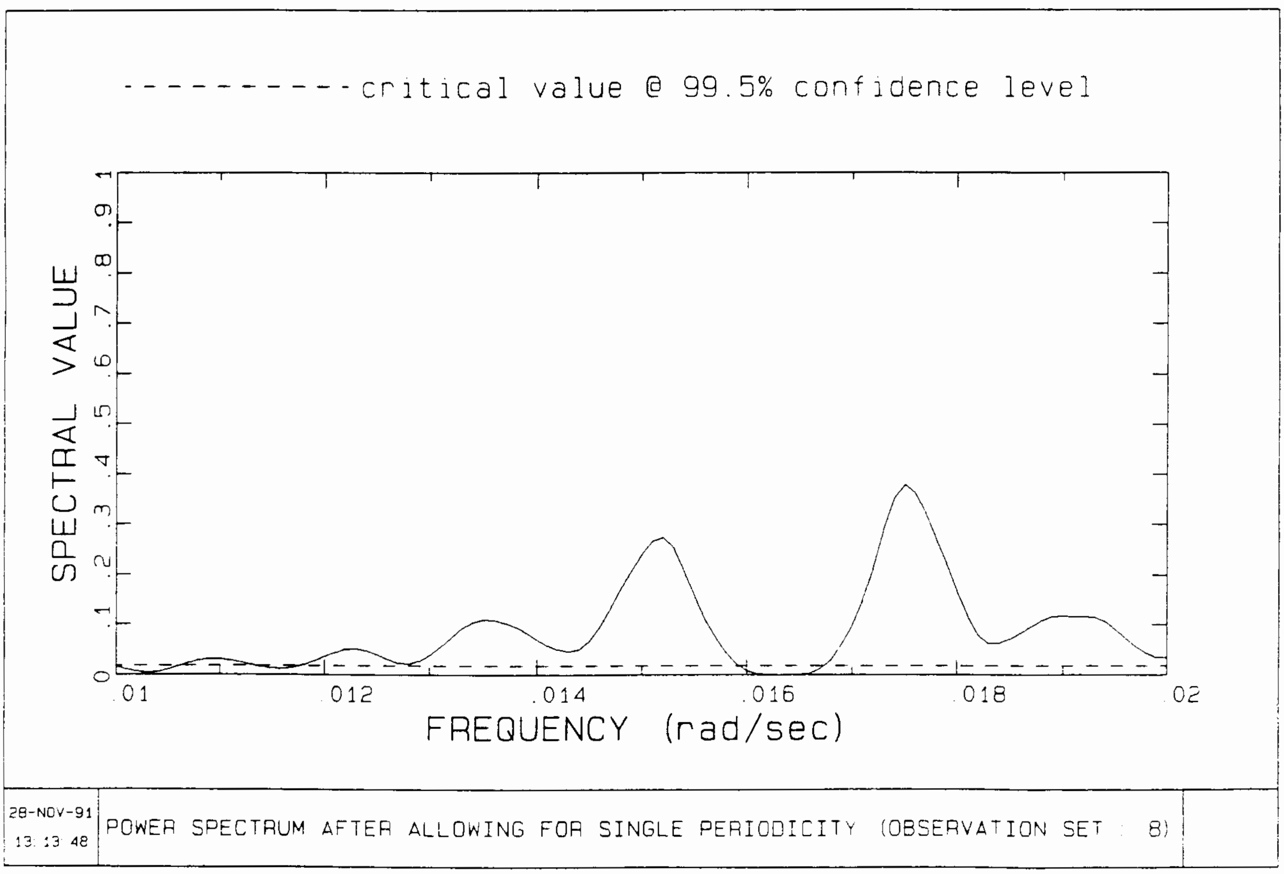


FIGURE B8

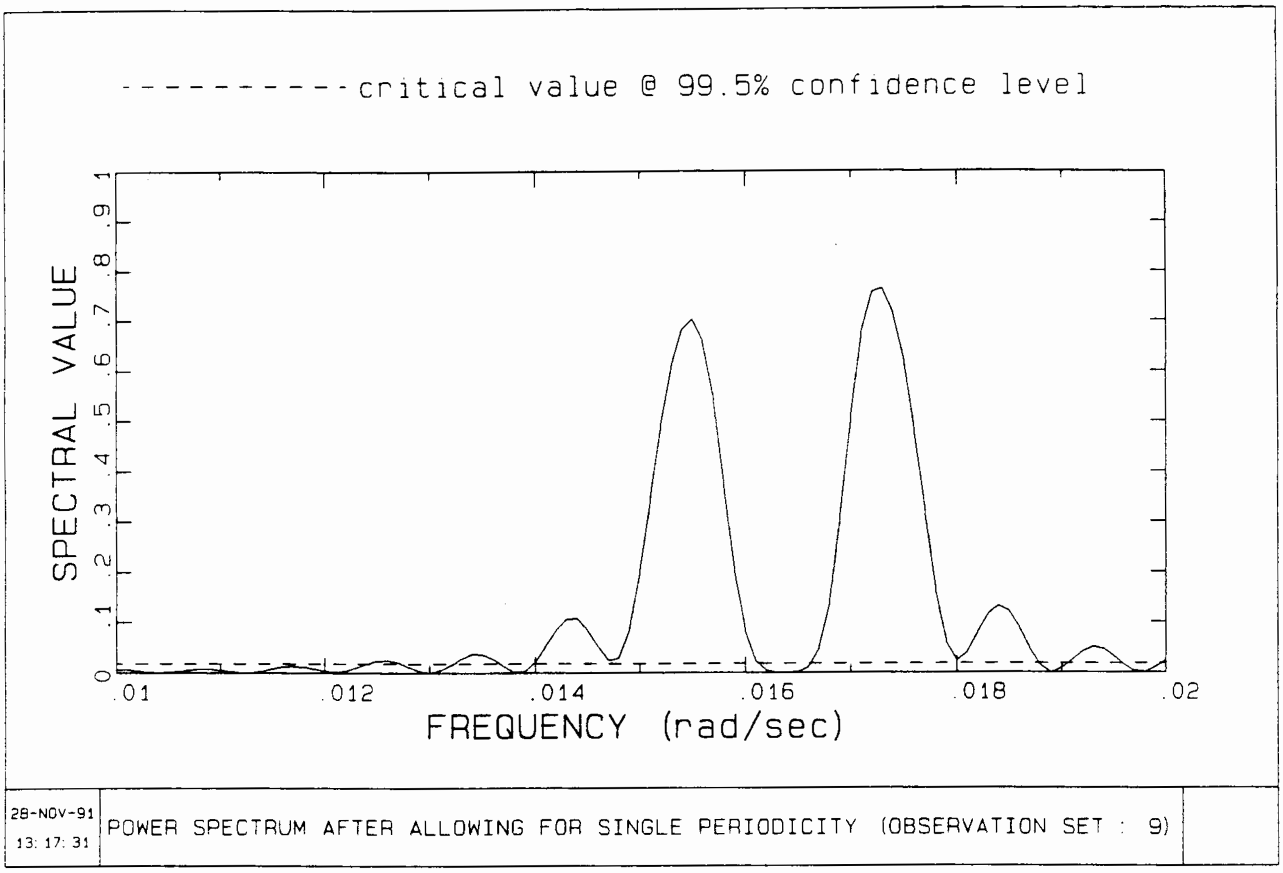


FIGURE B9

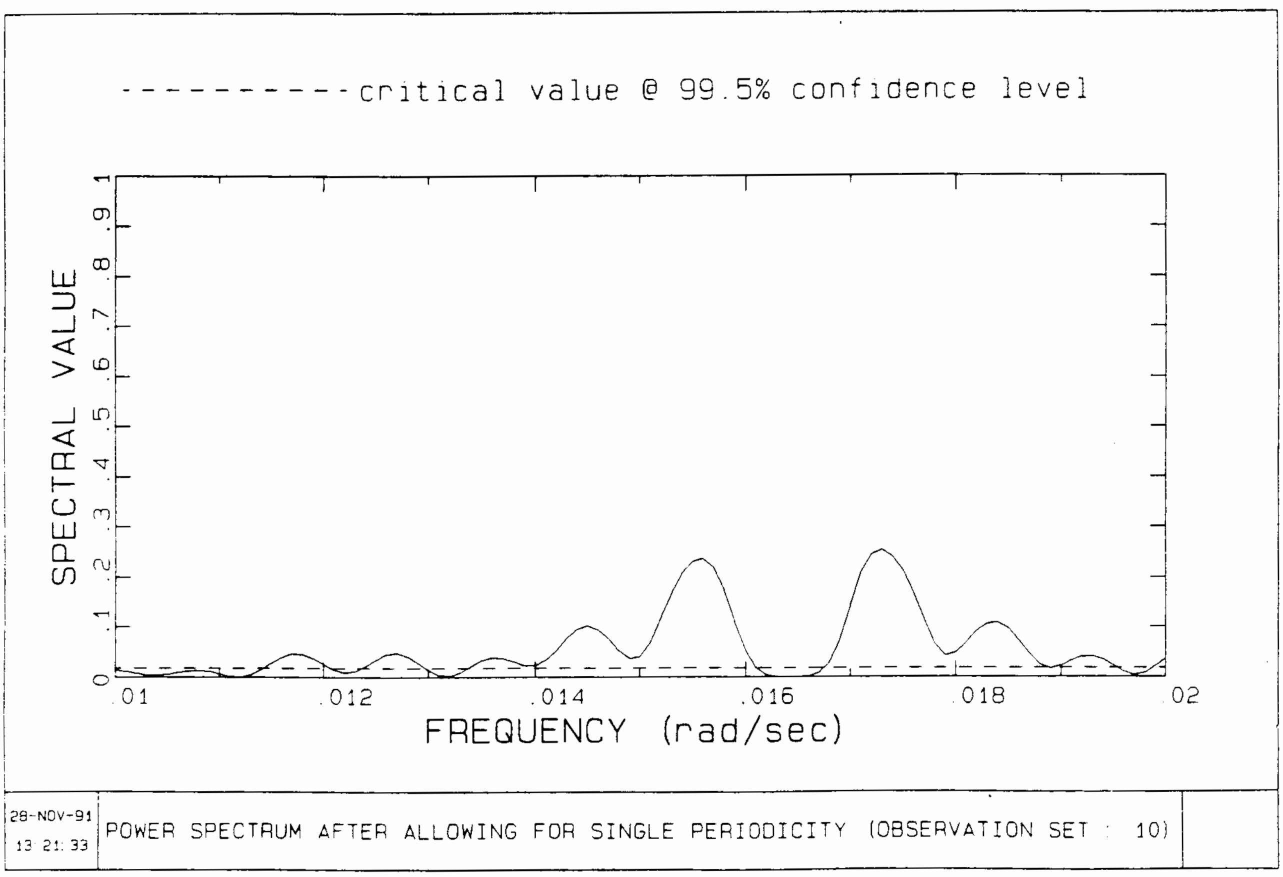


FIGURE B10

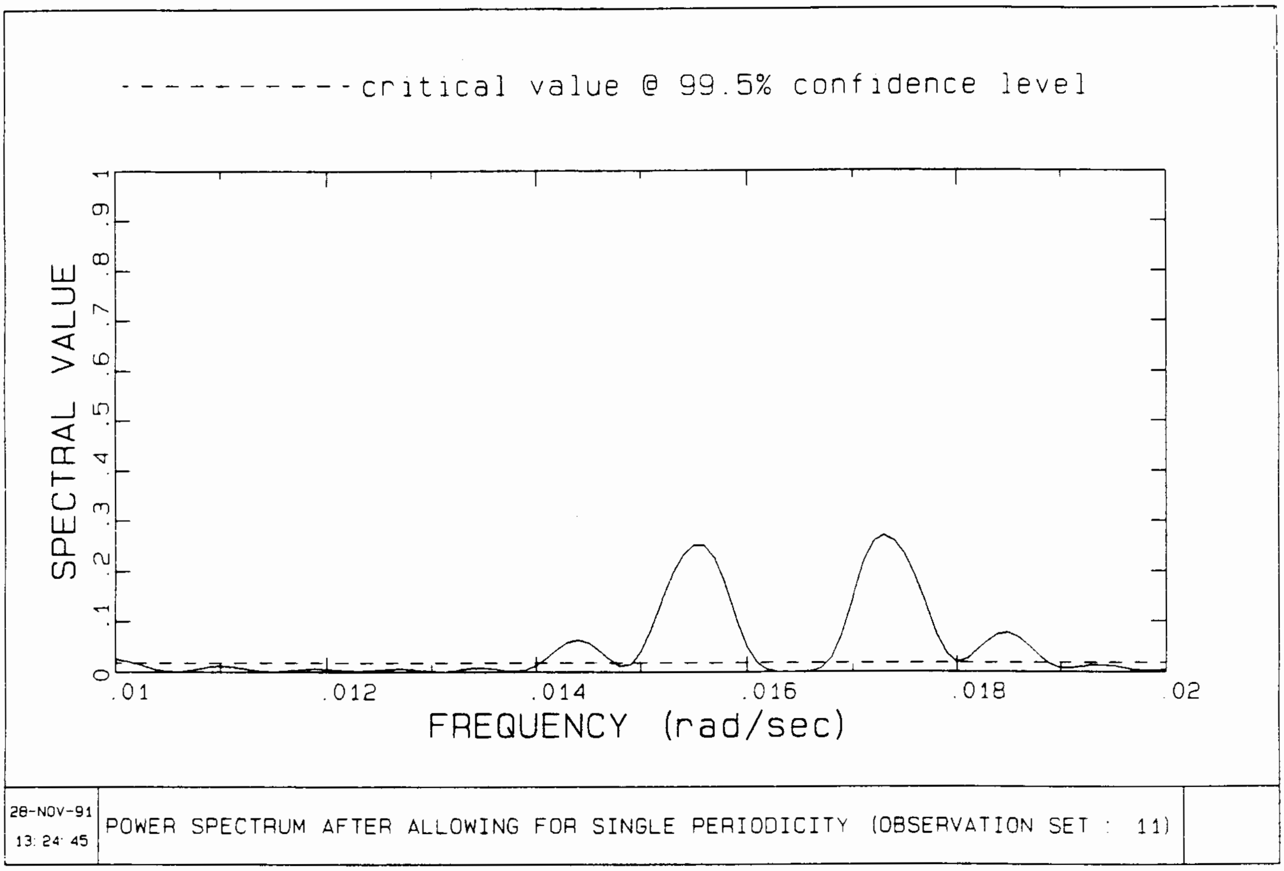


FIGURE B11

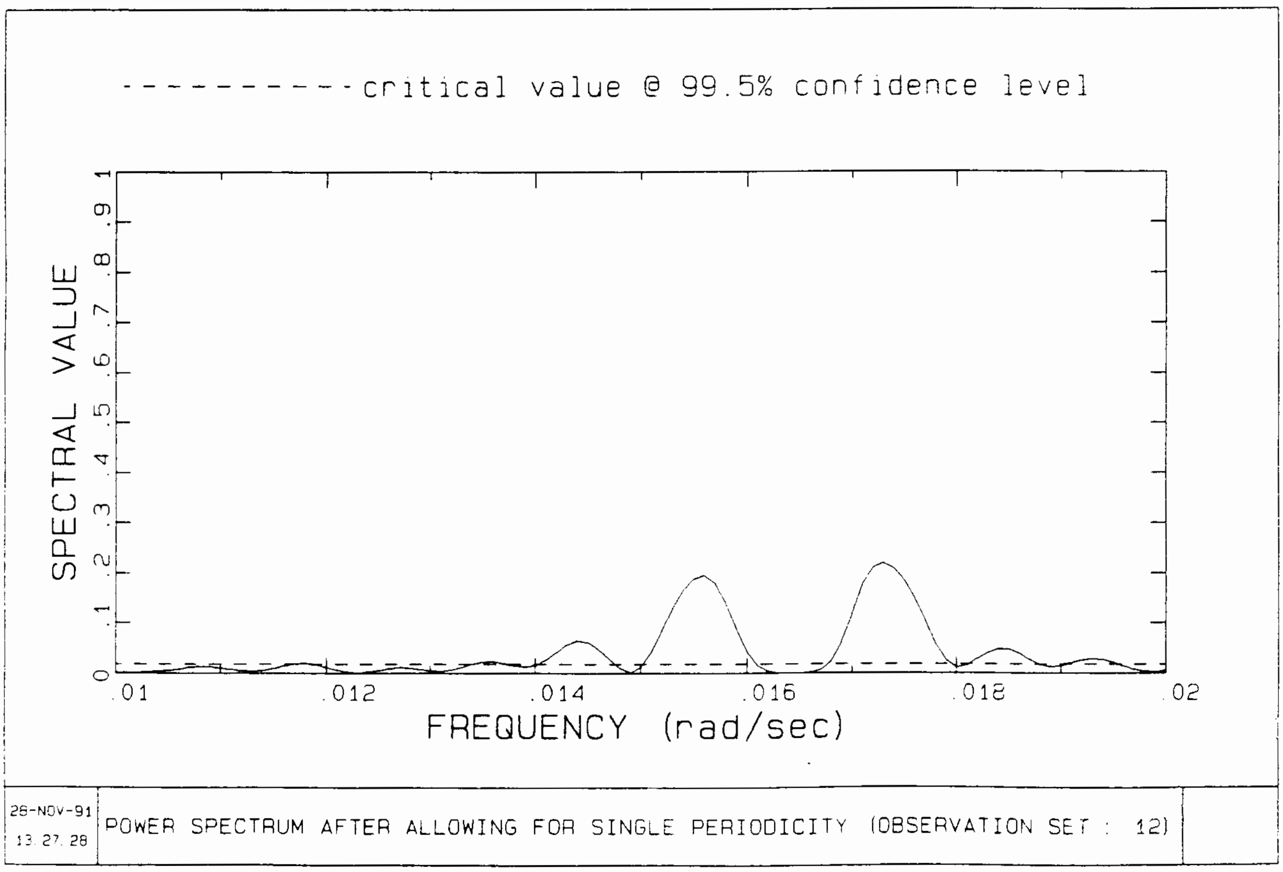


FIGURE B12

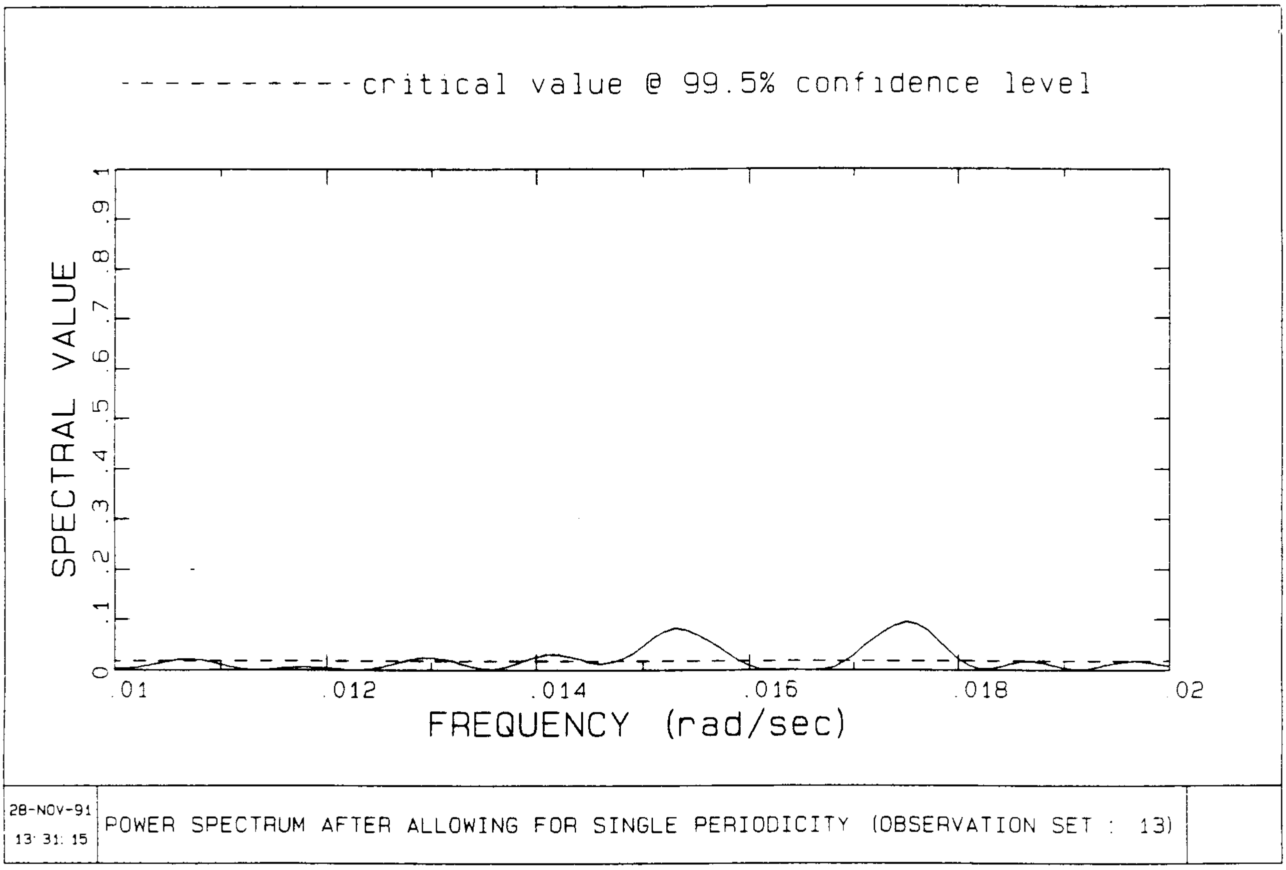


FIGURE B13

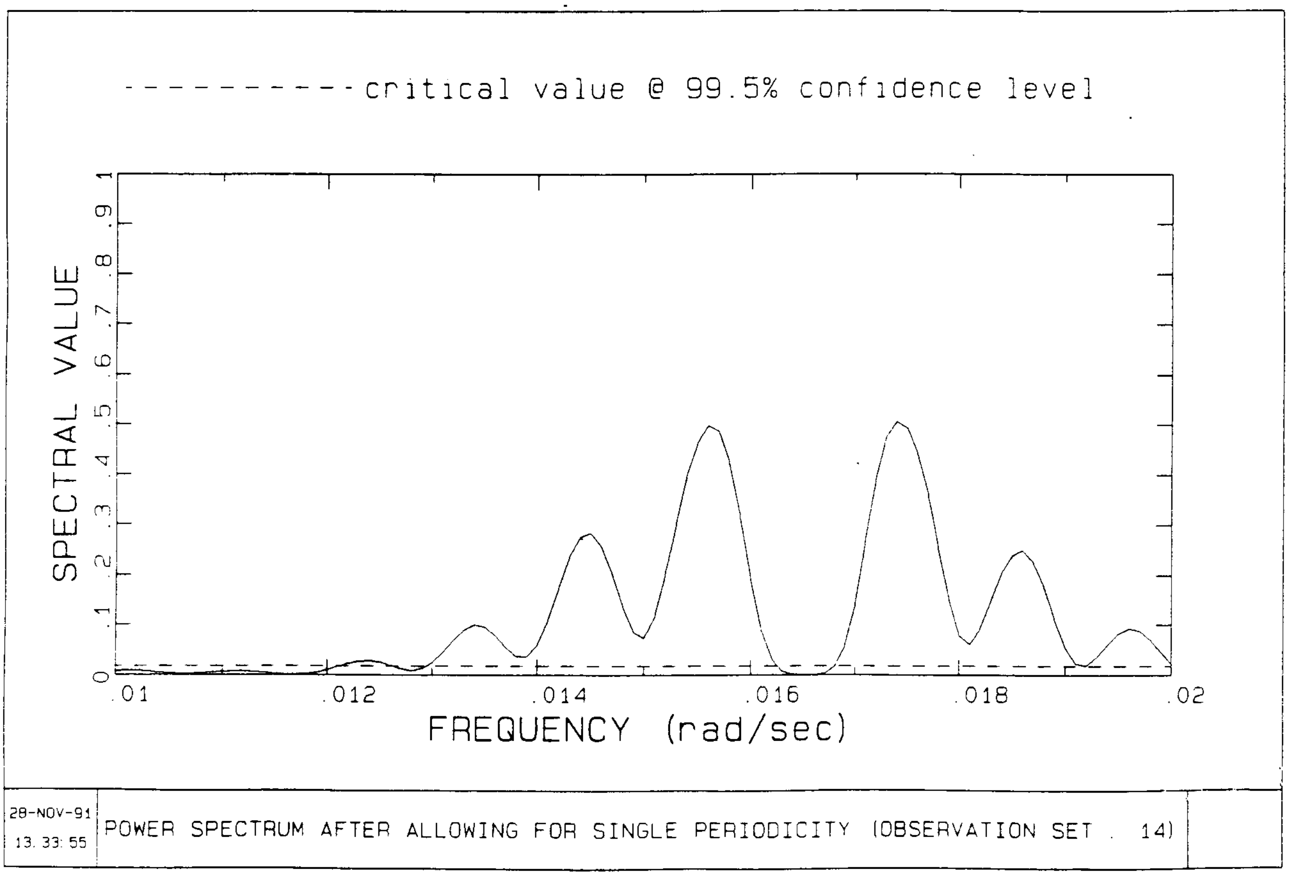


FIGURE B14

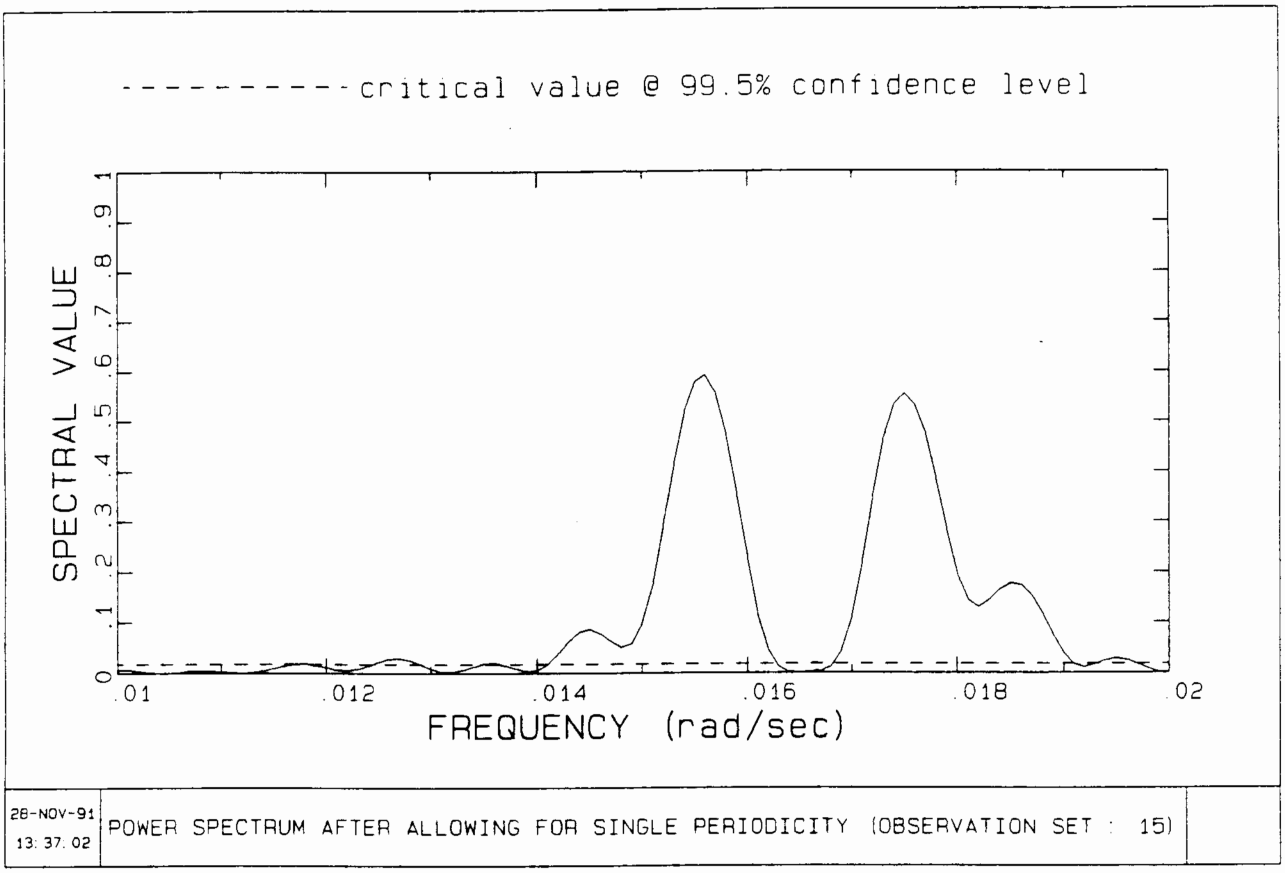


FIGURE B15

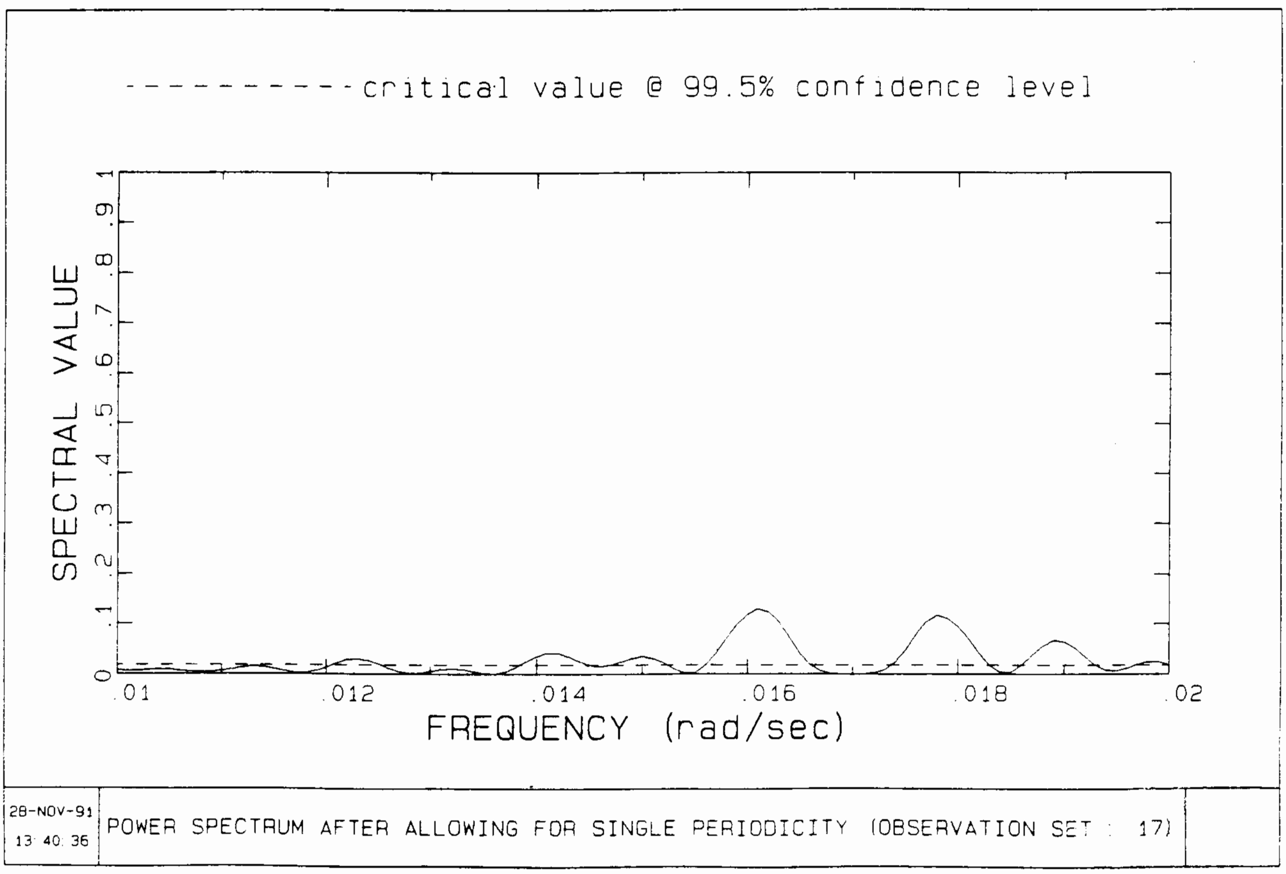


FIGURE B16

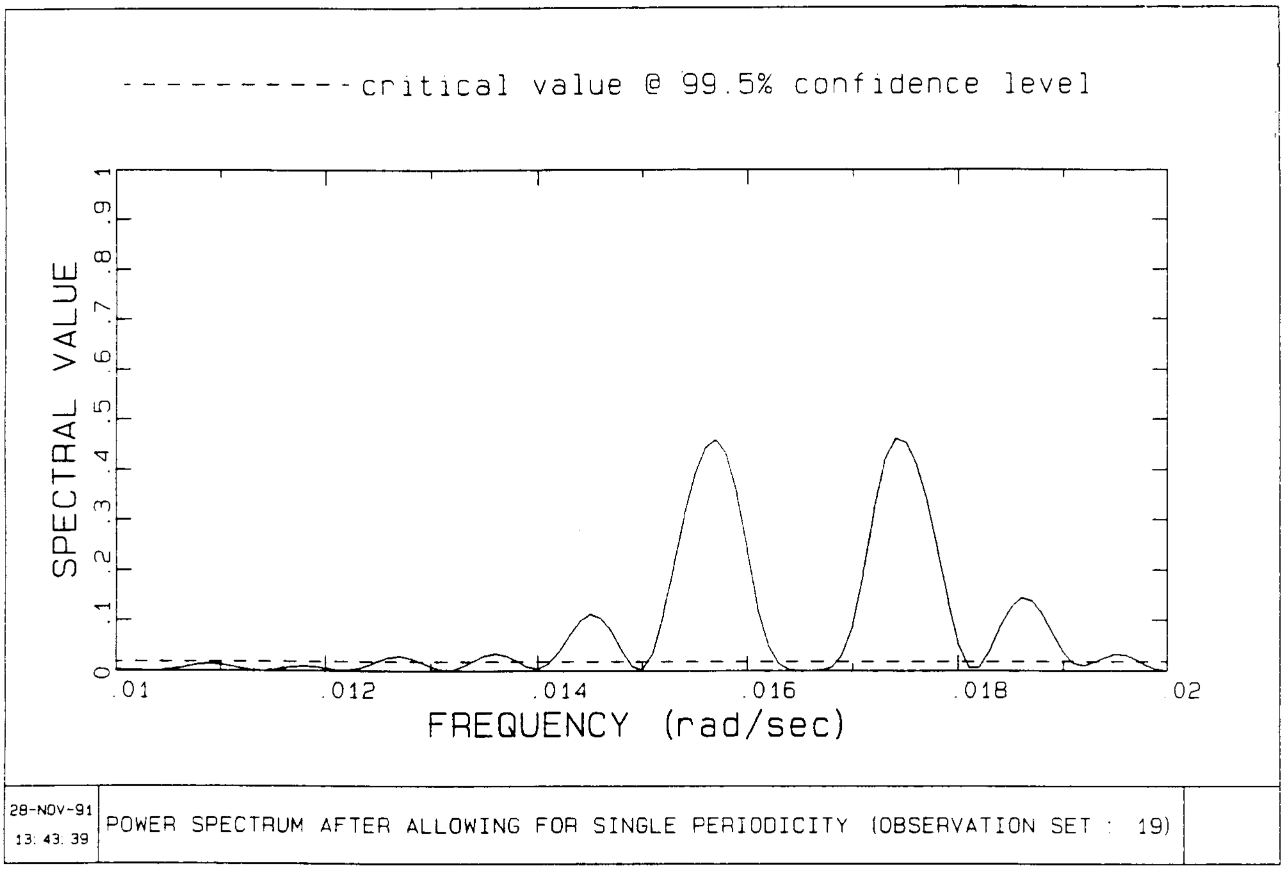


FIGURE B17

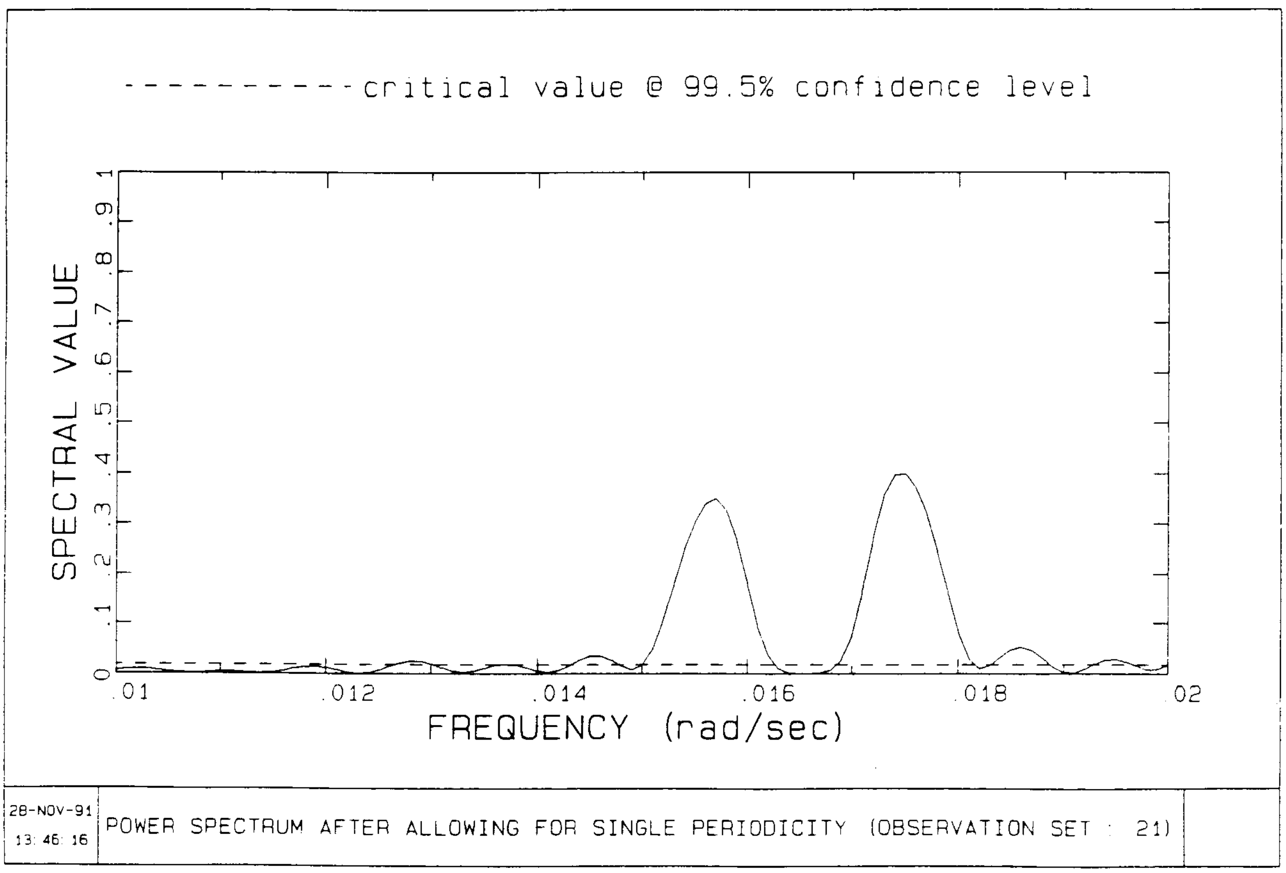


FIGURE B18

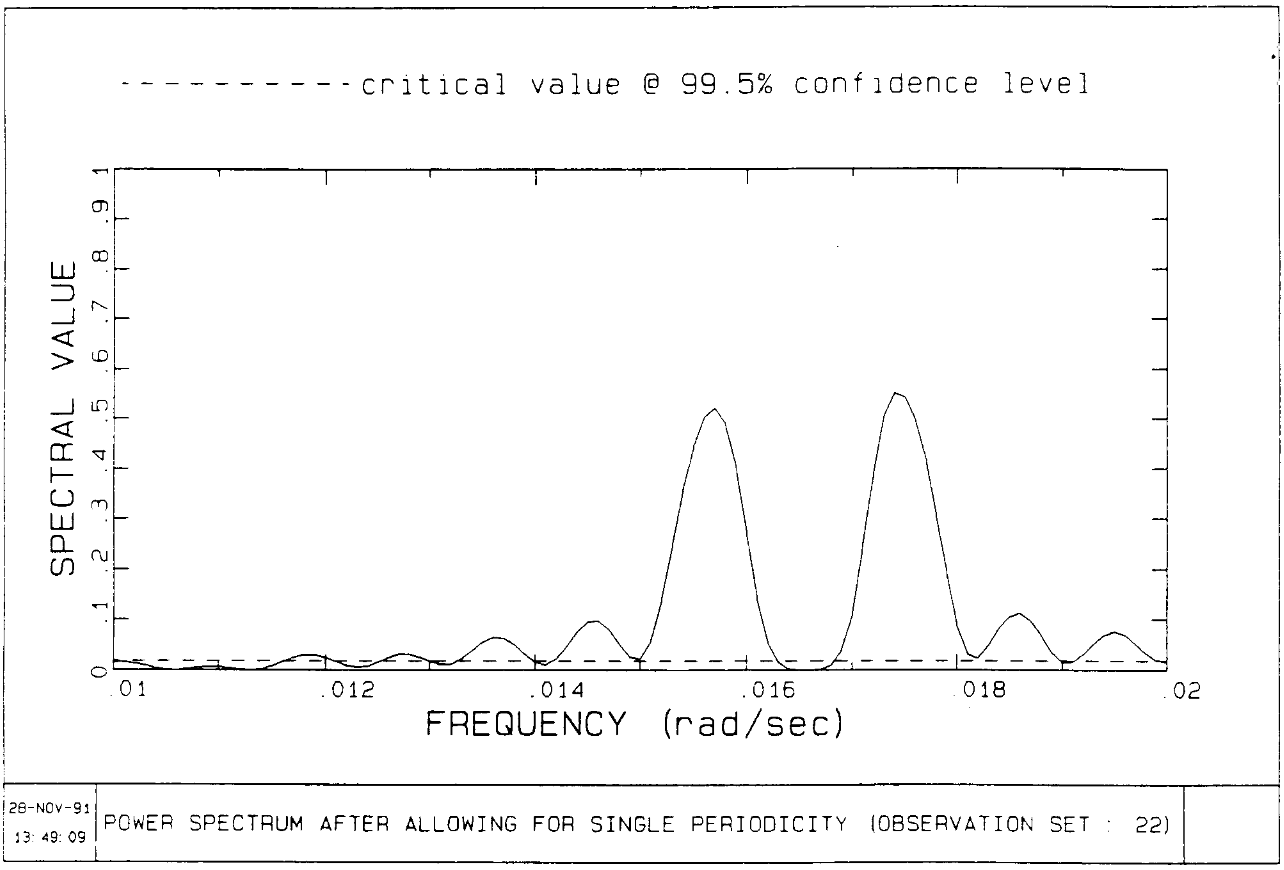


FIGURE B19

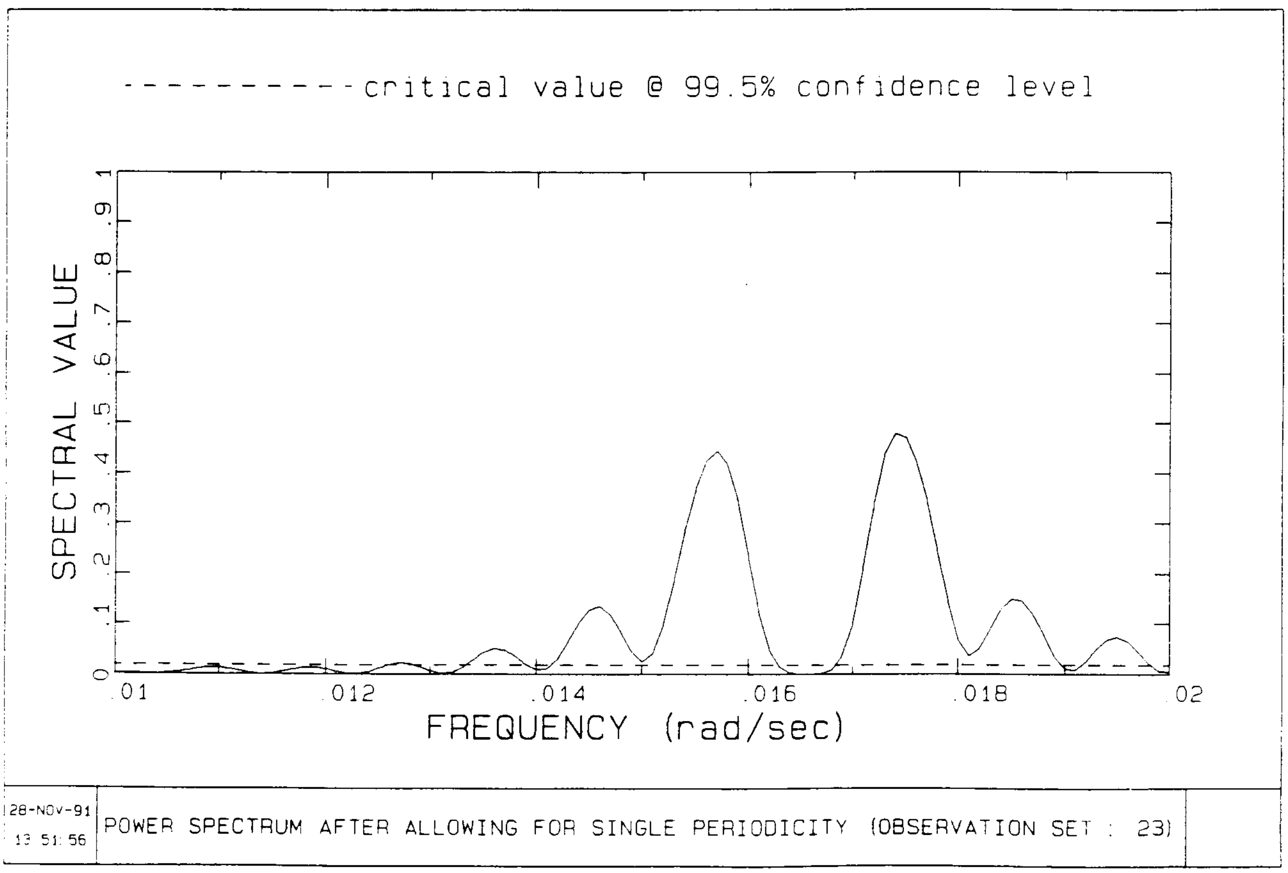


FIGURE B20

APPENDIX C

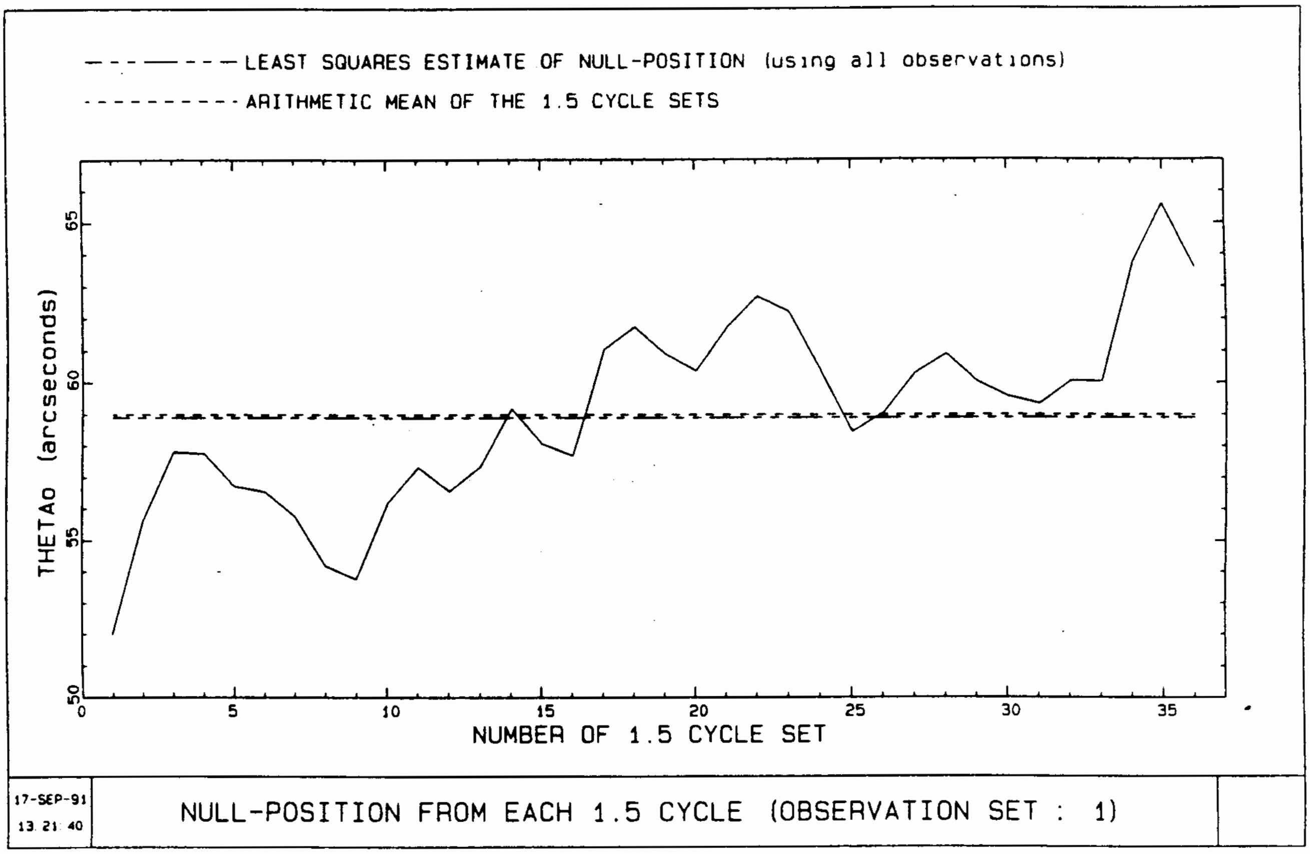


FIGURE C1

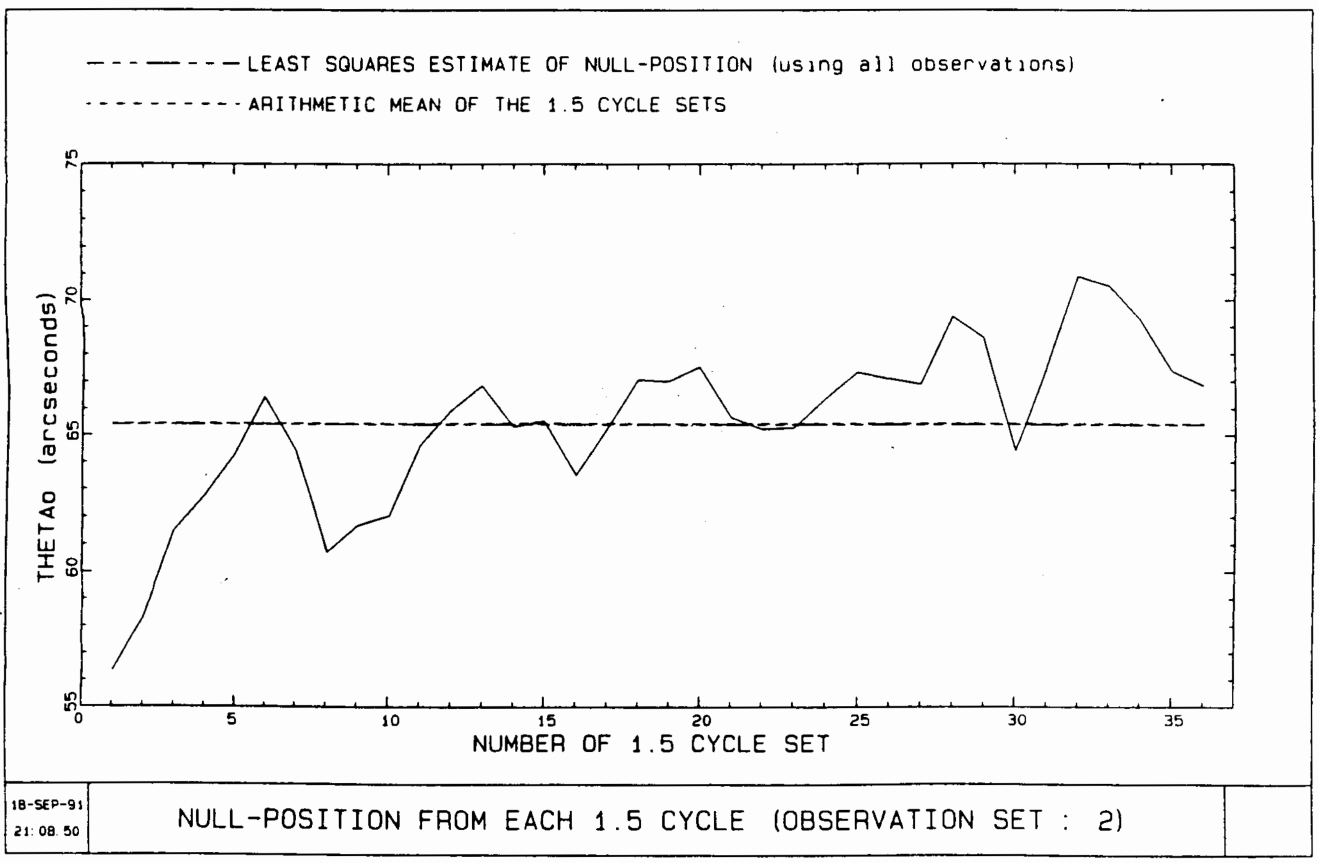


FIGURE C2

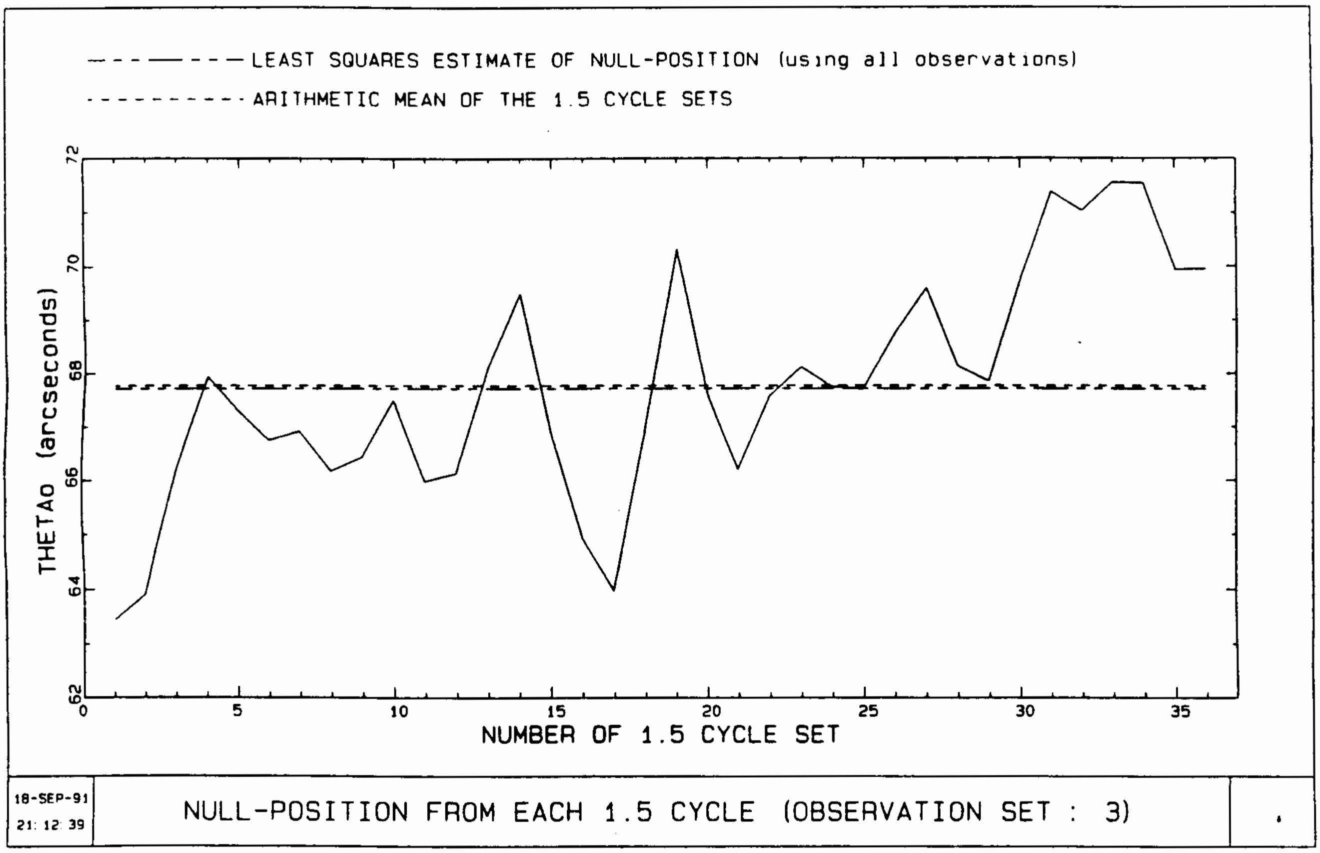


FIGURE C3

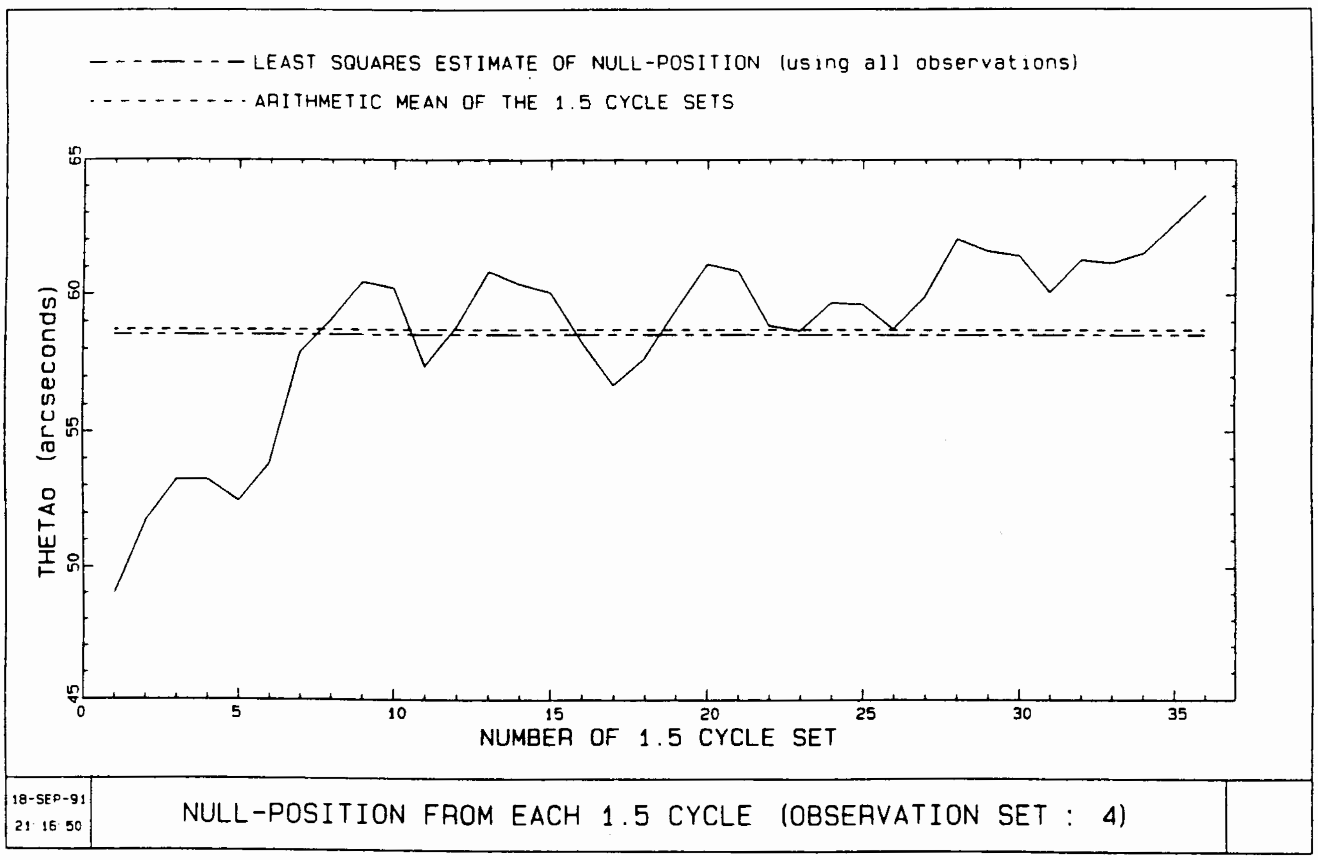


FIGURE C4

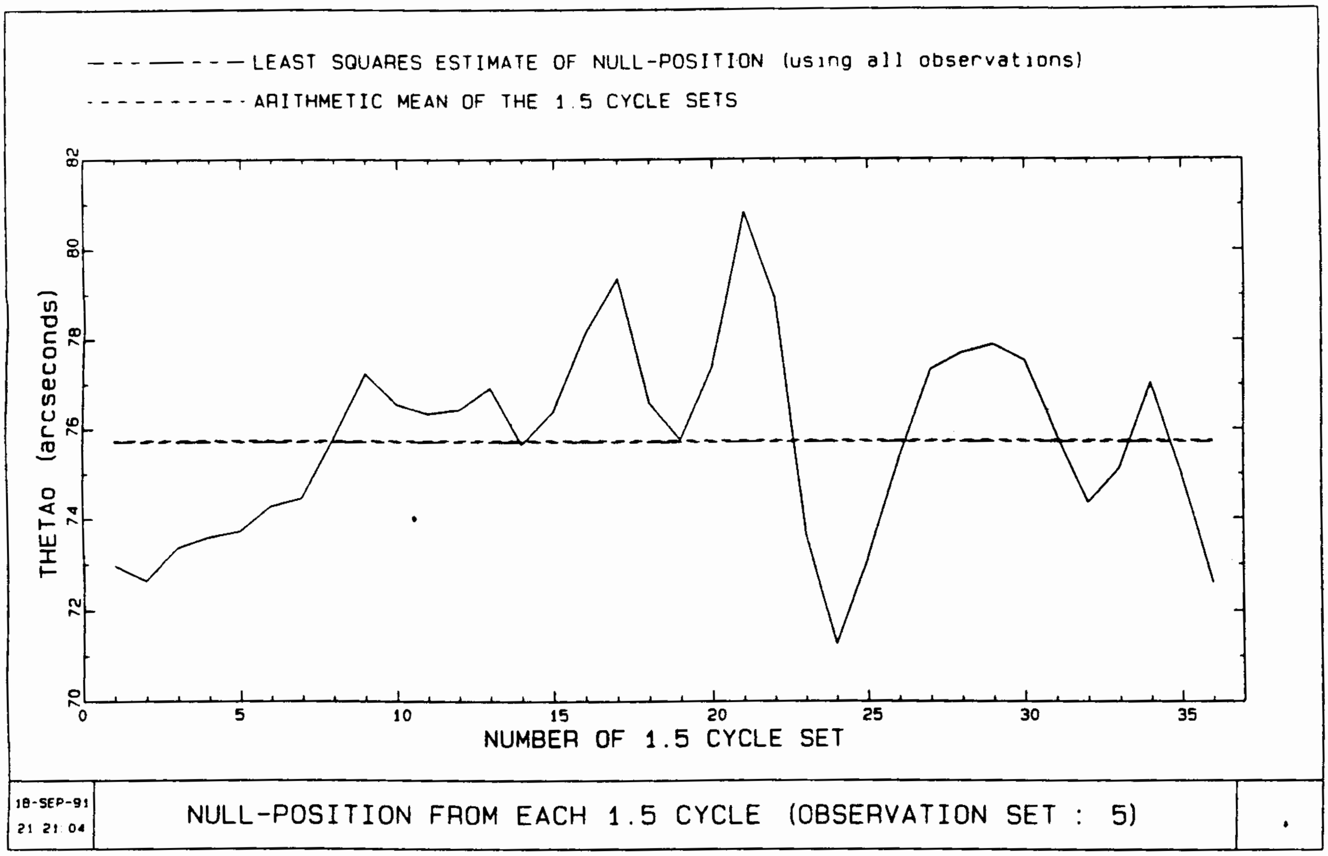


FIGURE C5

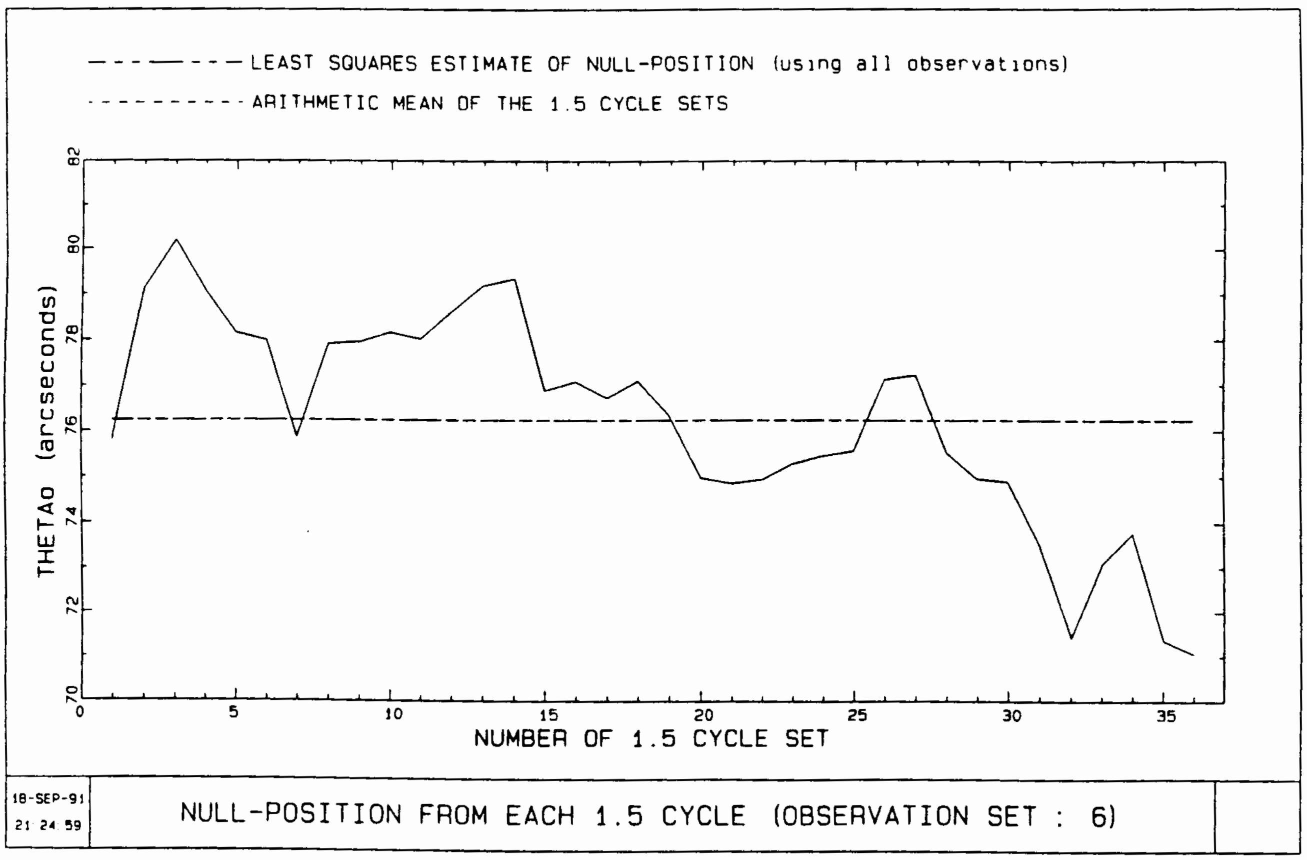


FIGURE C6

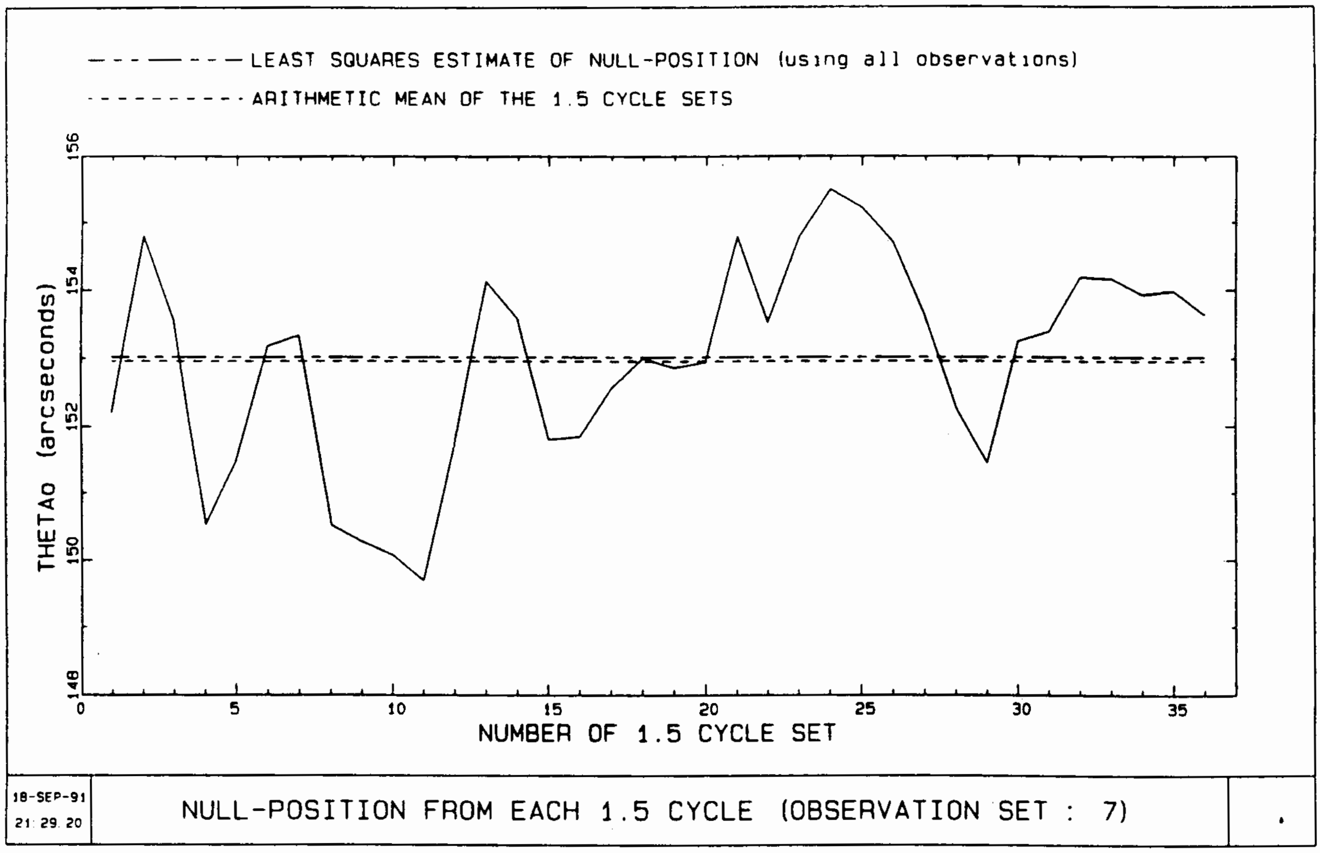


FIGURE C7

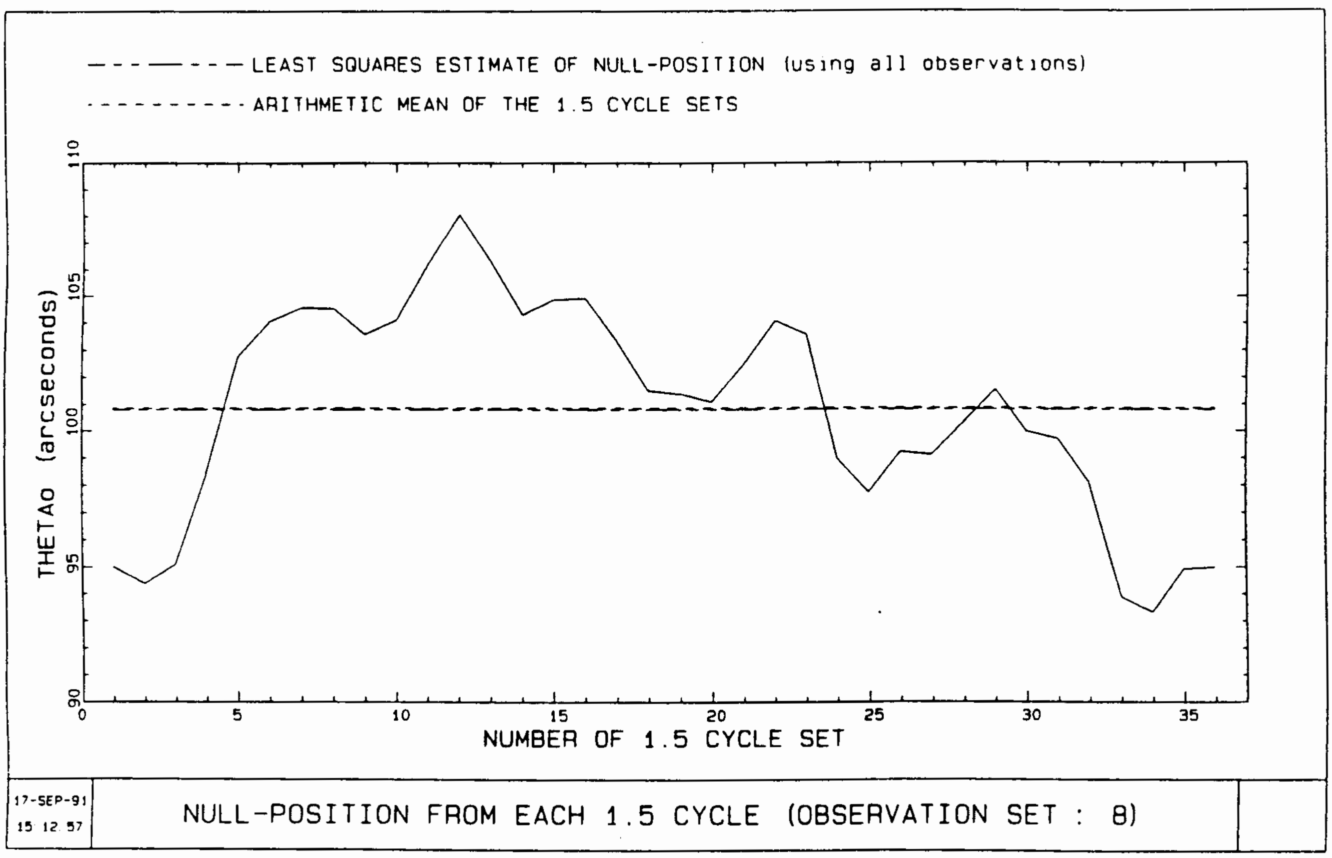


FIGURE C8

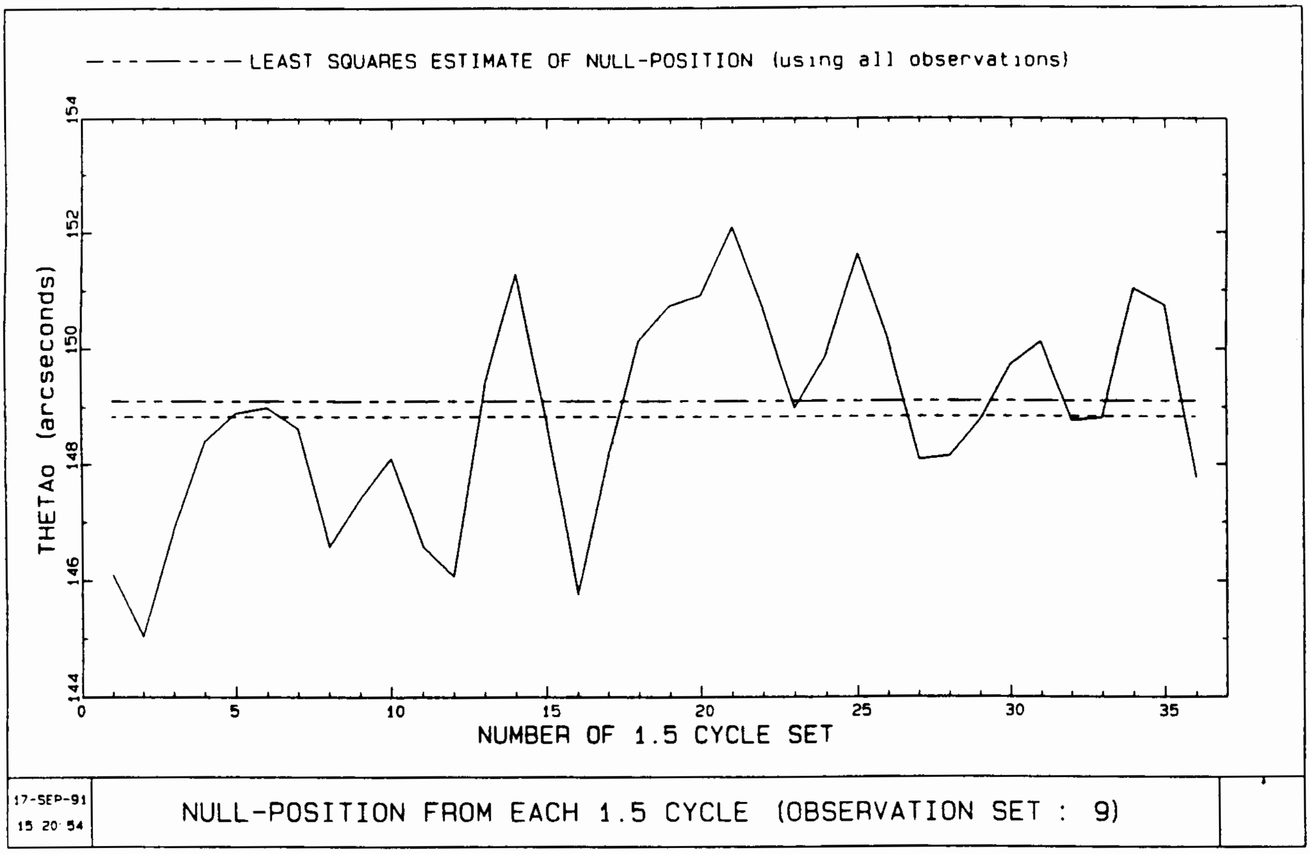


FIGURE C9

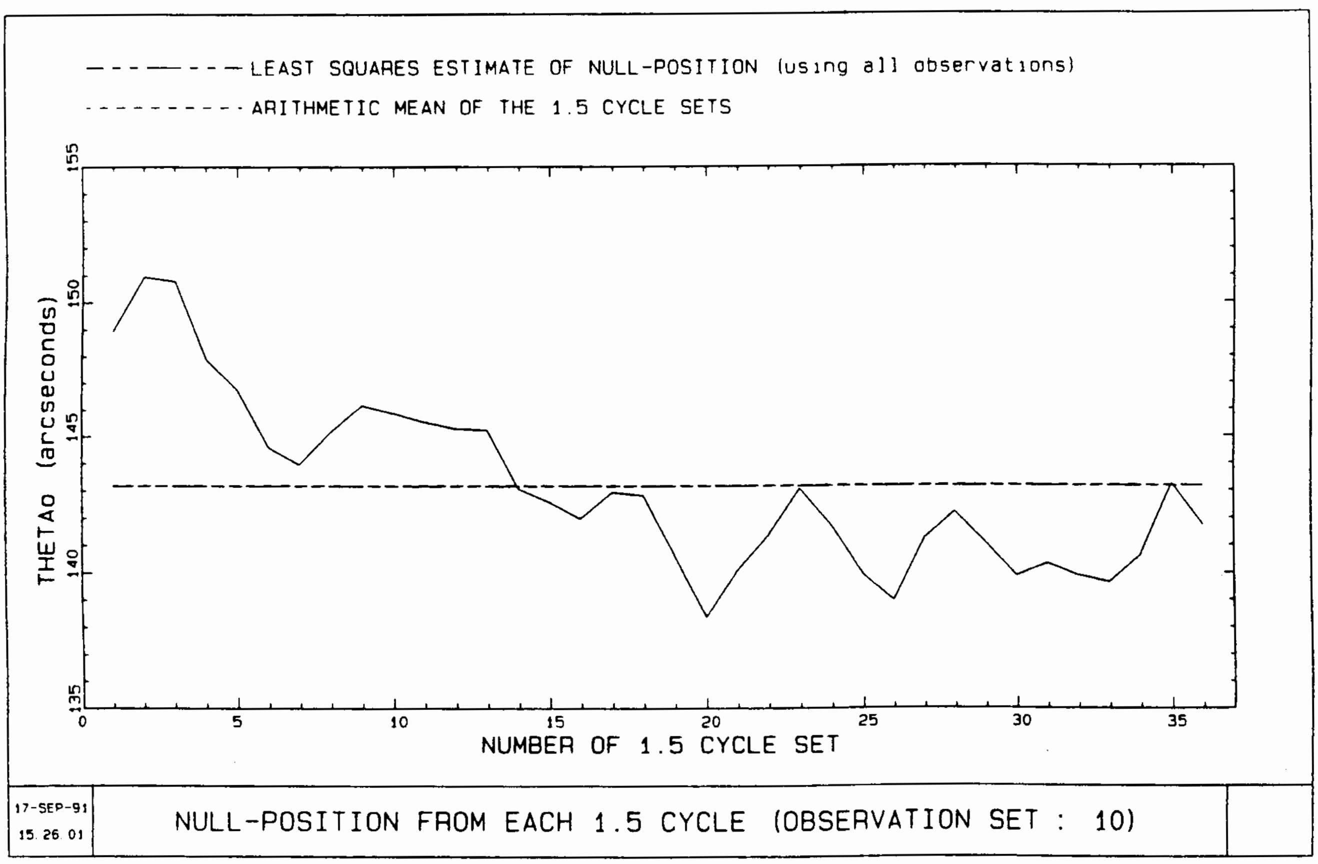


FIGURE C10

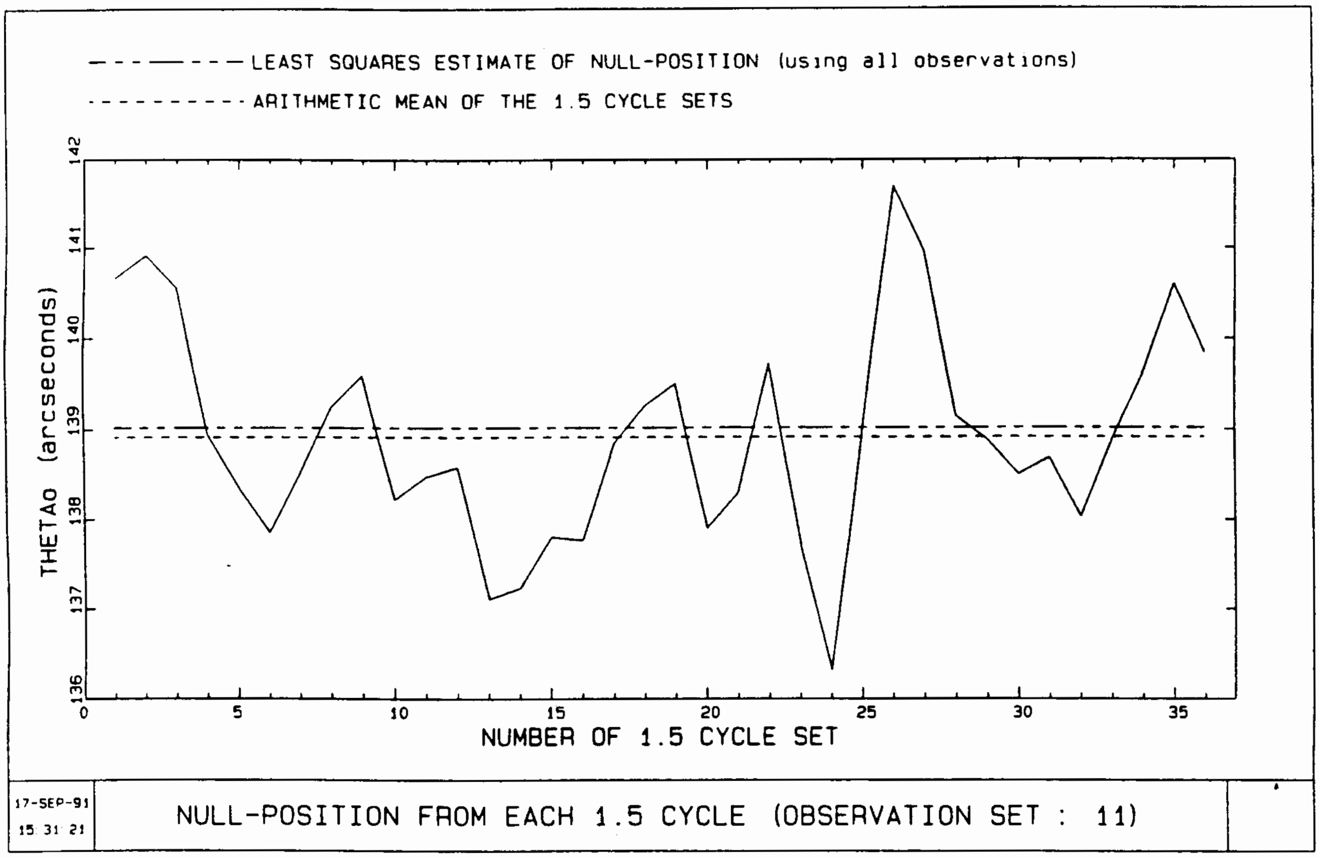


FIGURE C11

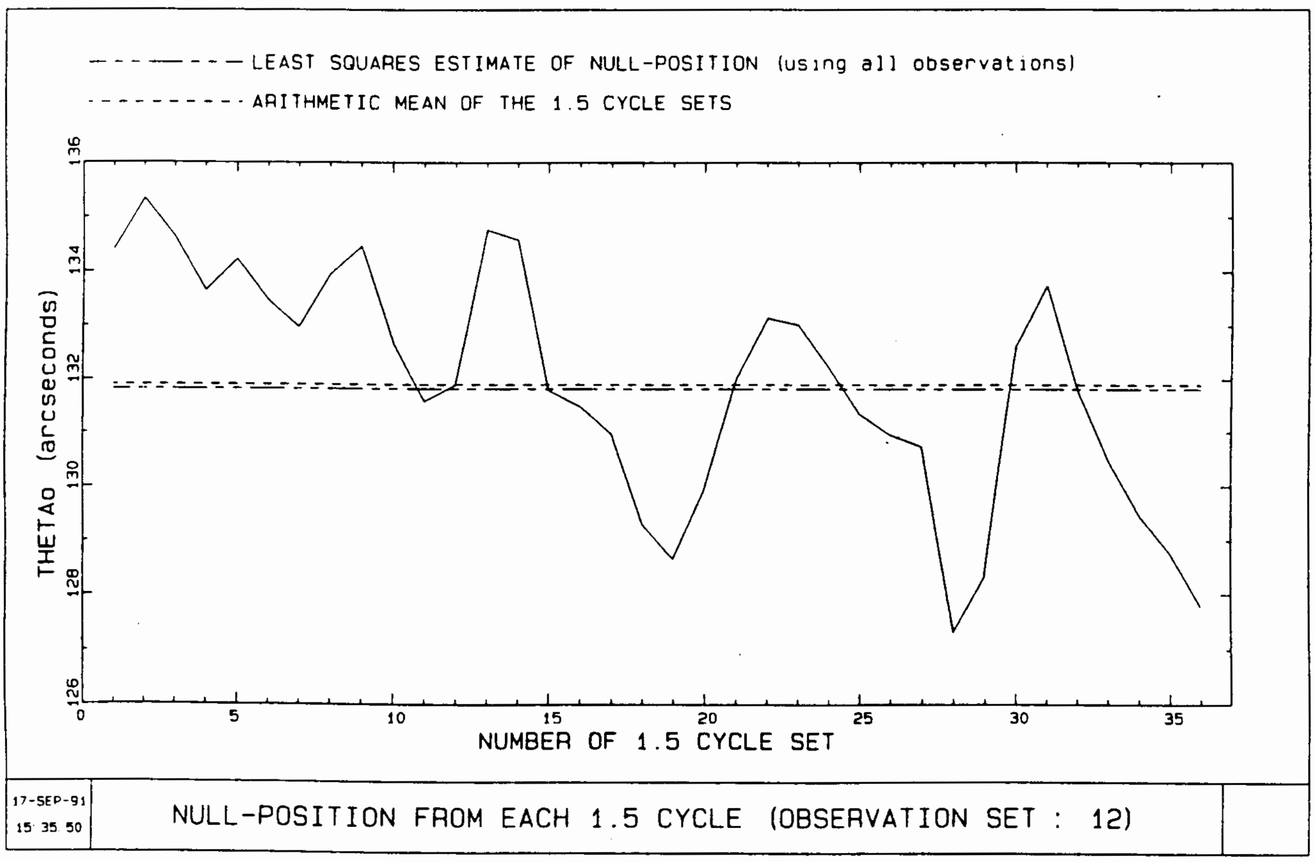
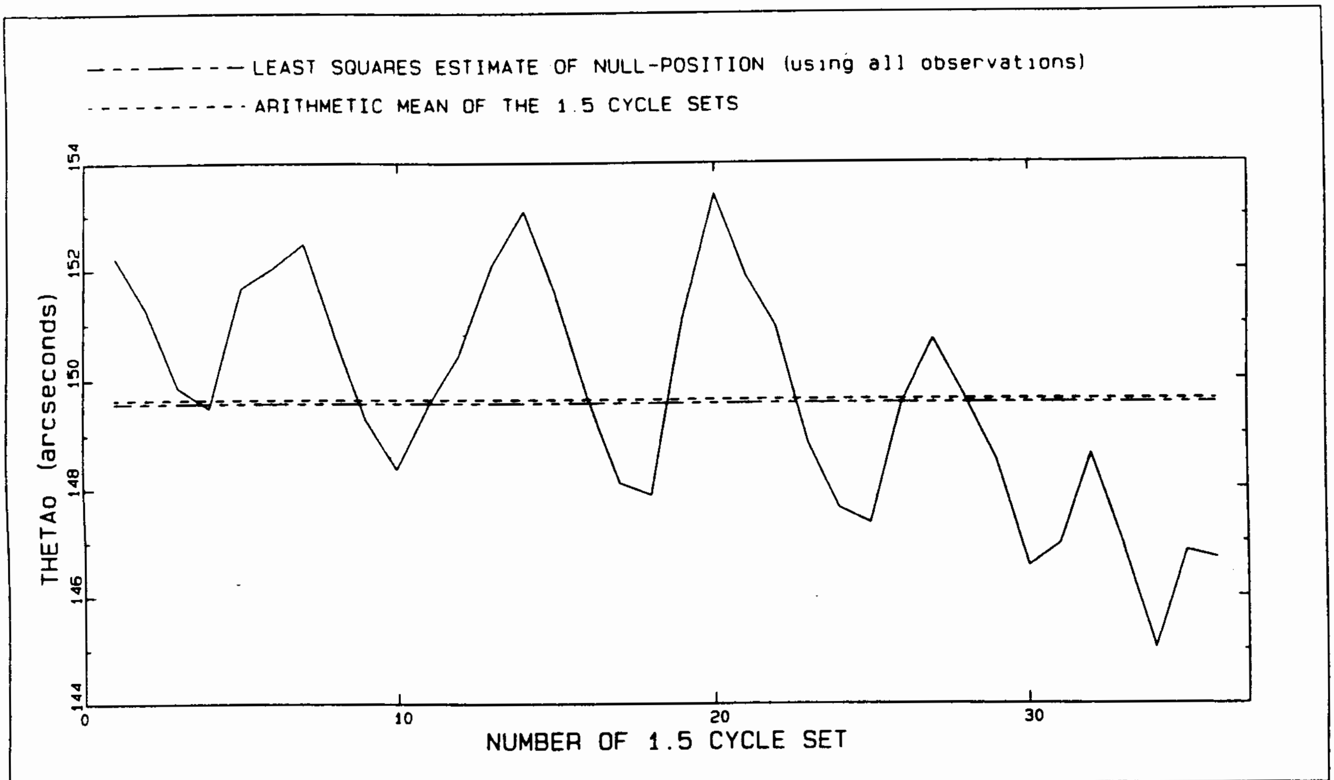


FIGURE C12



17-SEP-91 15:42:57 NULL-POSITION FROM EACH 1.5 CYCLE (OBSERVATION SET : 13)

FIGURE C13

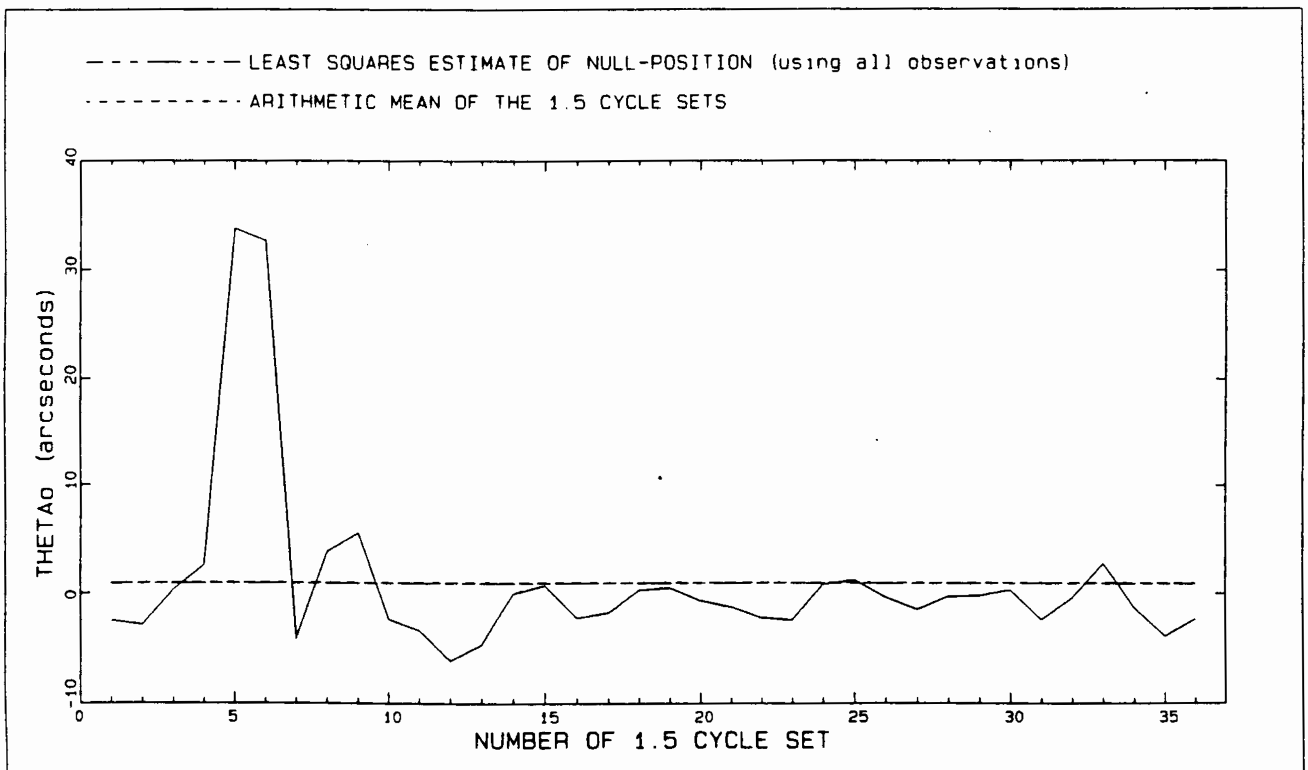


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FIGURE C14

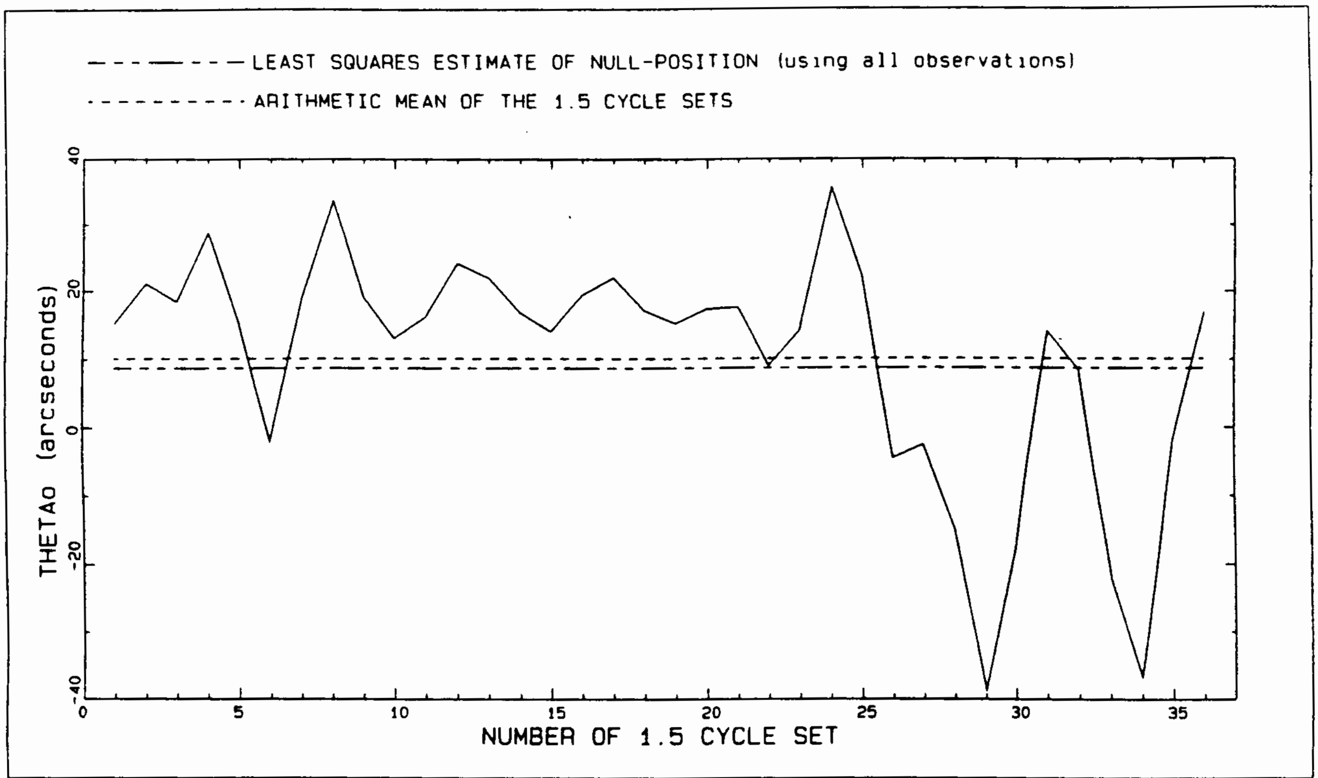

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FIGURE C15

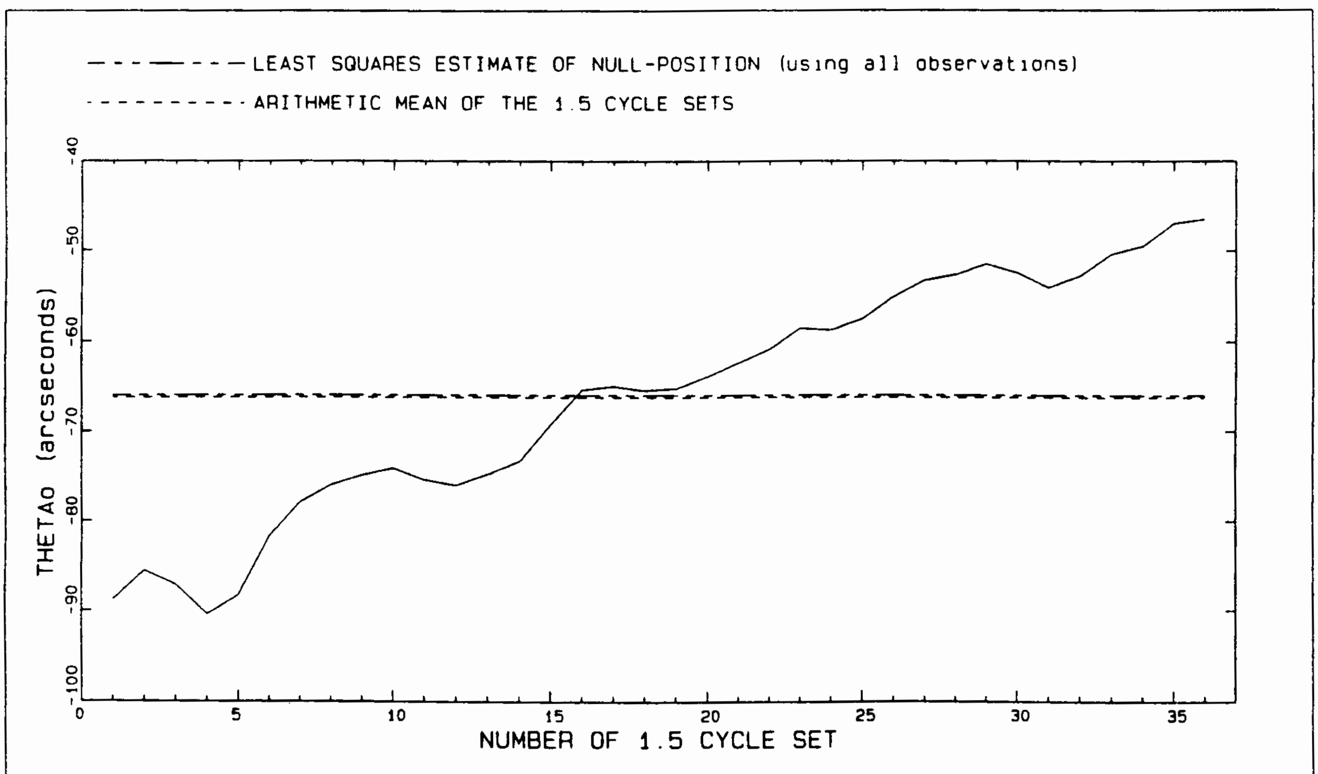

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FIGURE C16

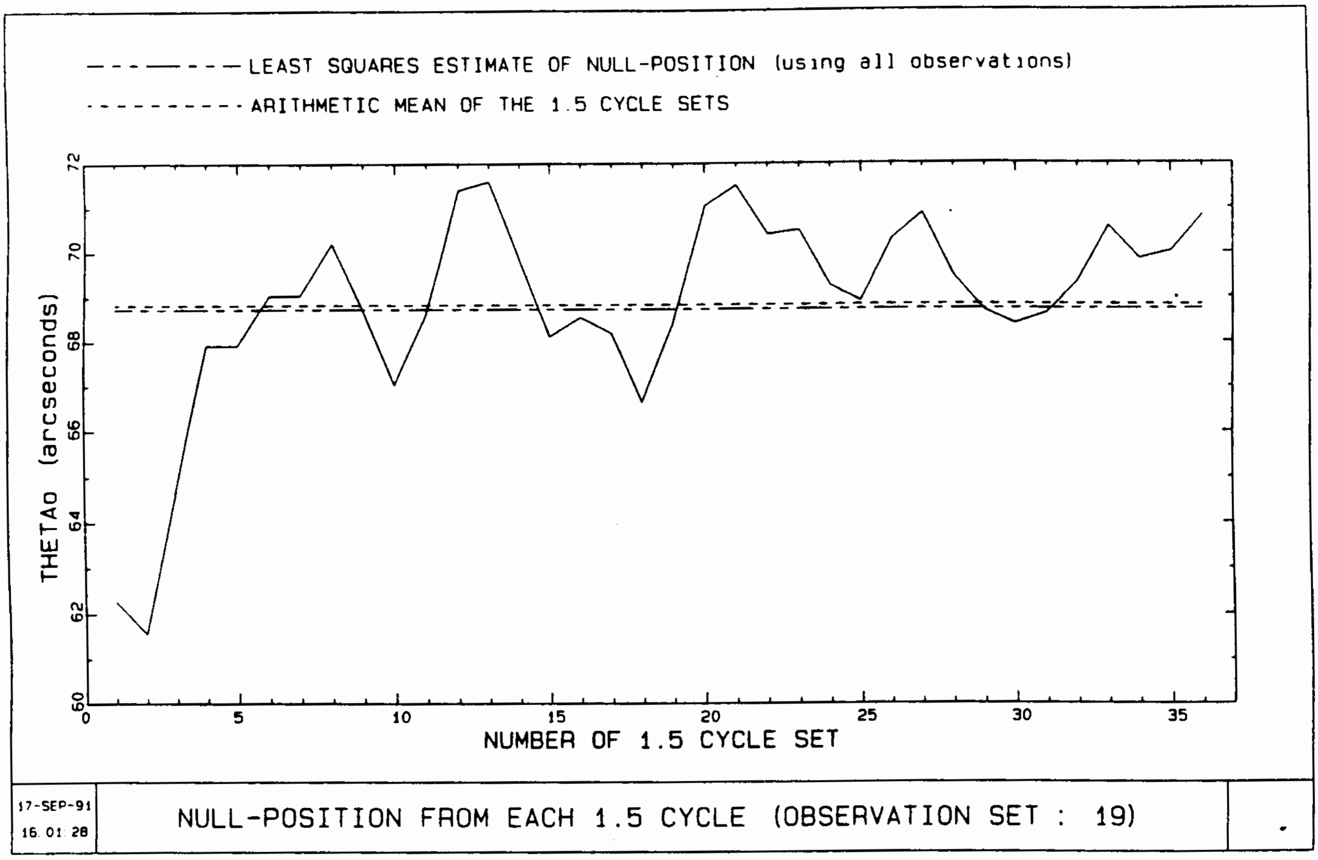


FIGURE C17

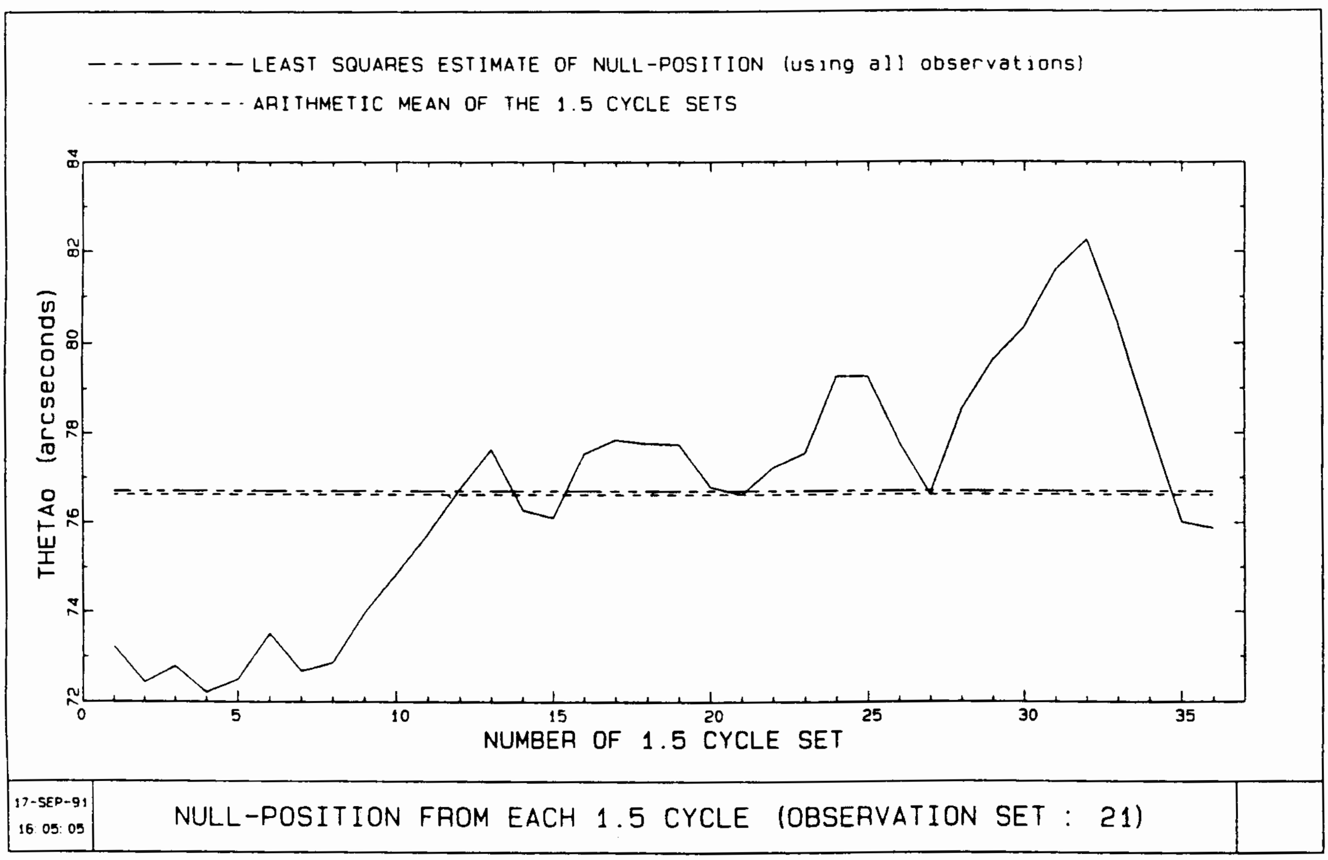


FIGURE C18

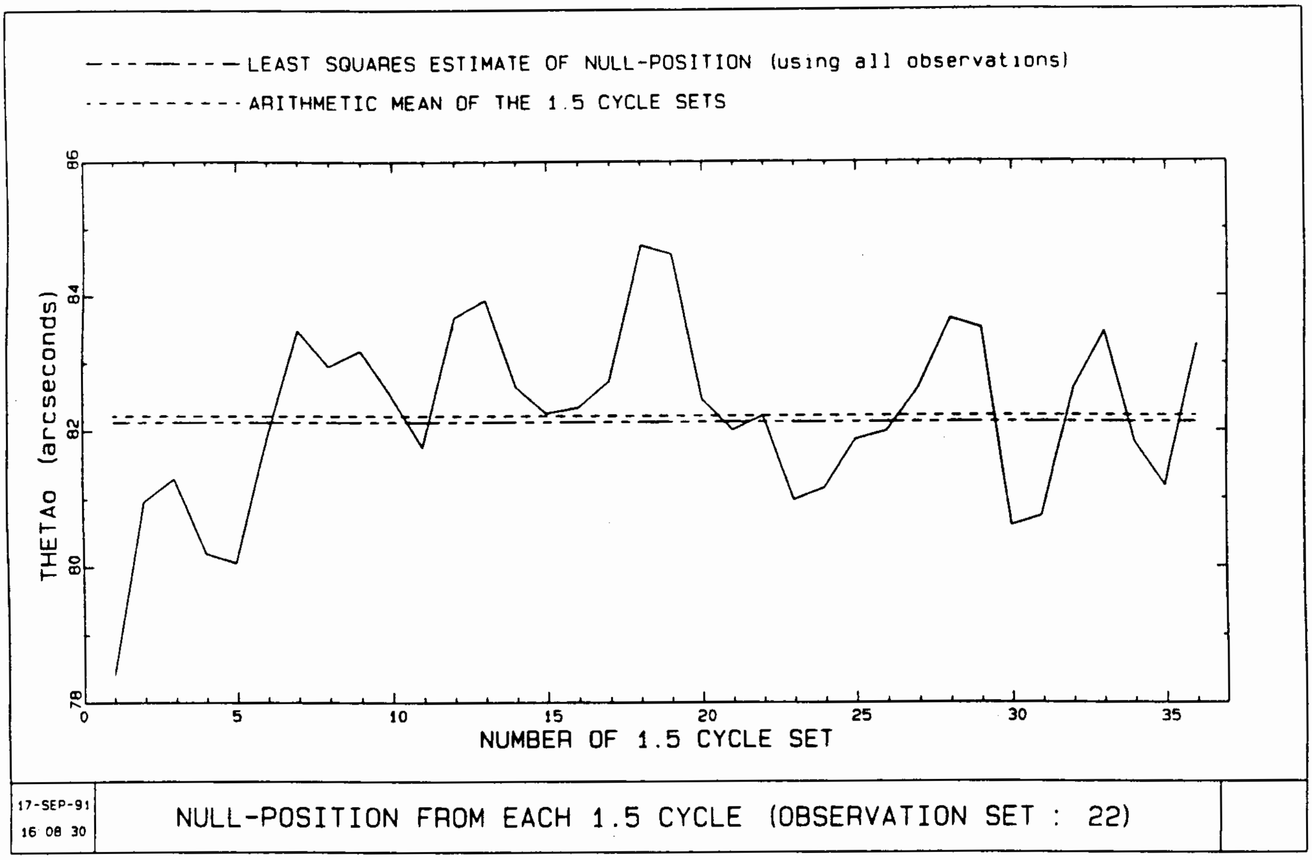


FIGURE C19

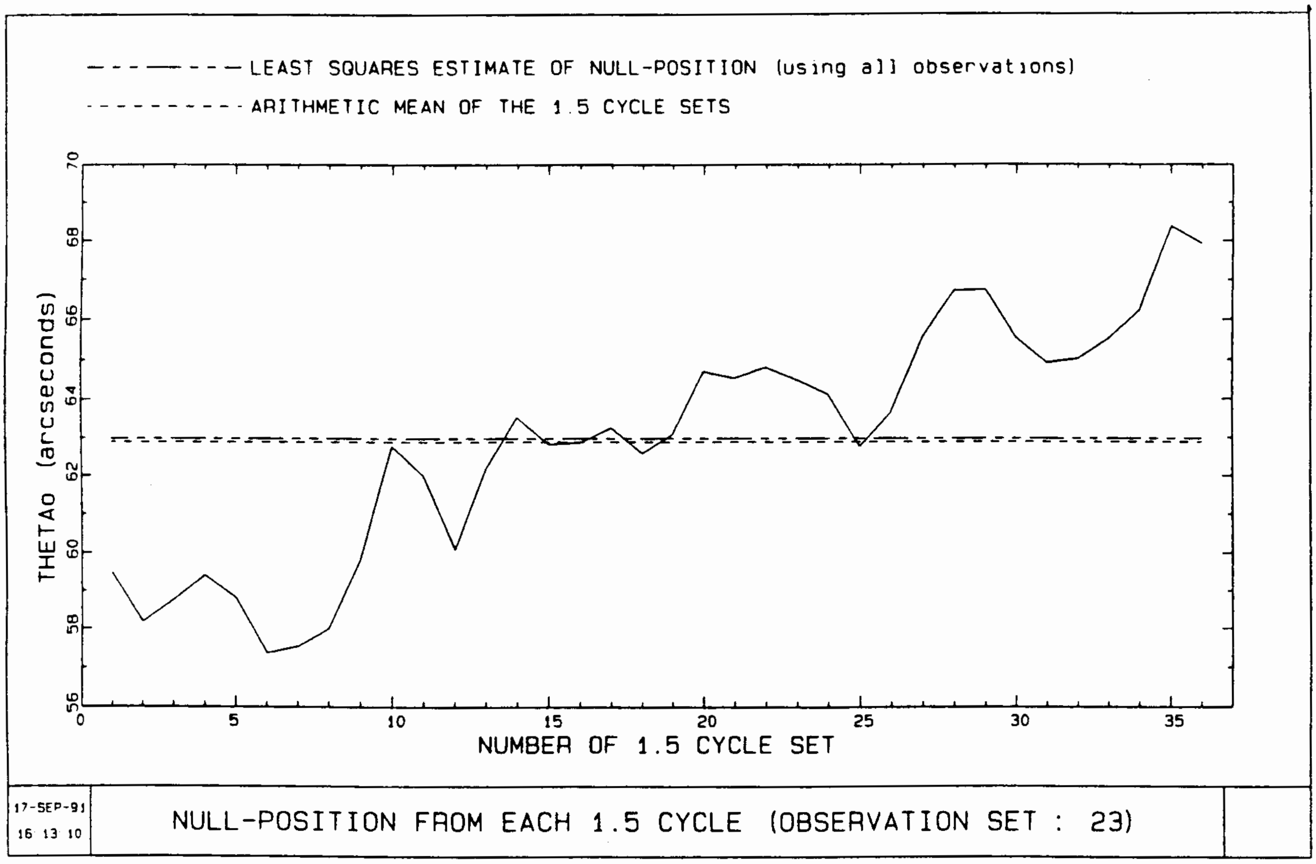


FIGURE C20

APPENDIX D

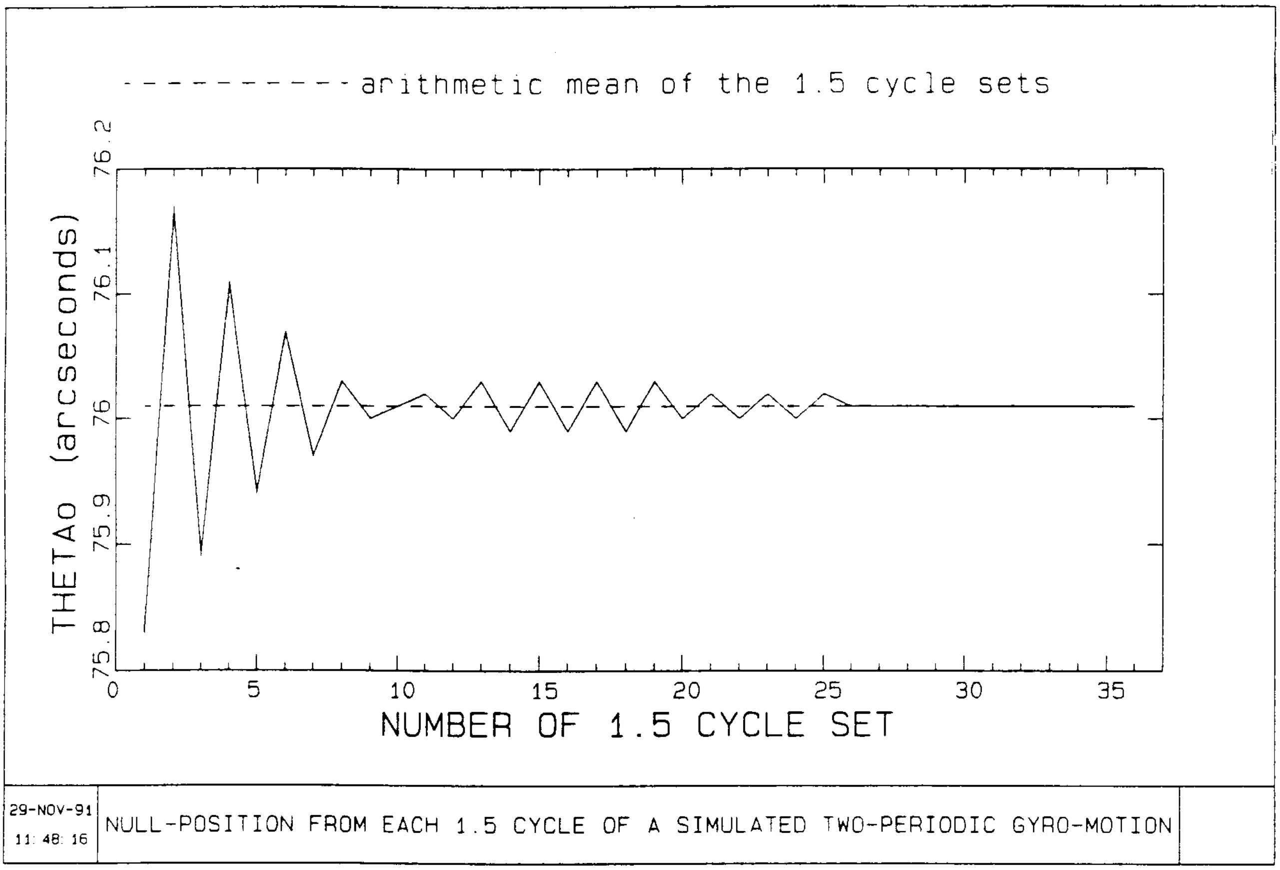


FIGURE D1

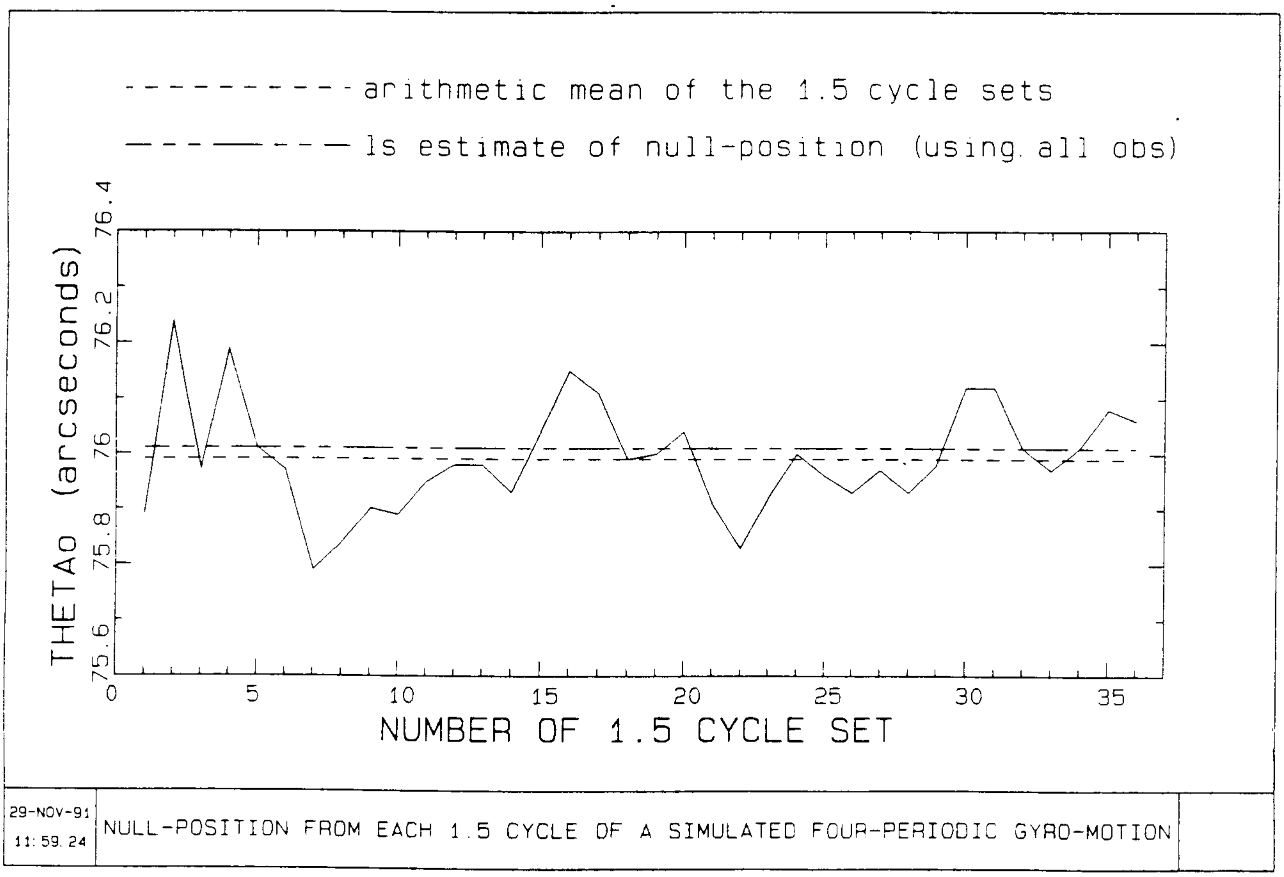


FIGURE D2

APPENDIX E

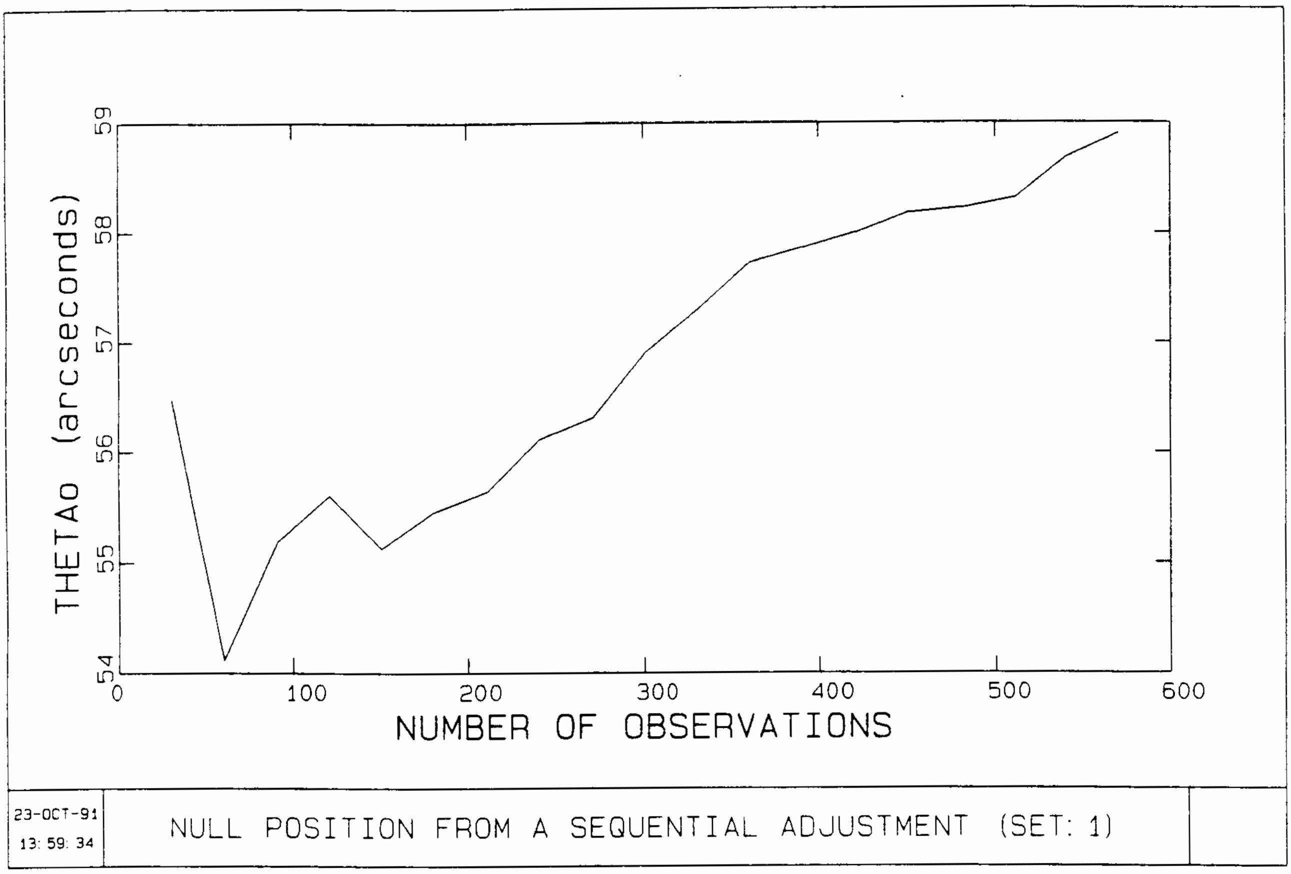


FIGURE E1

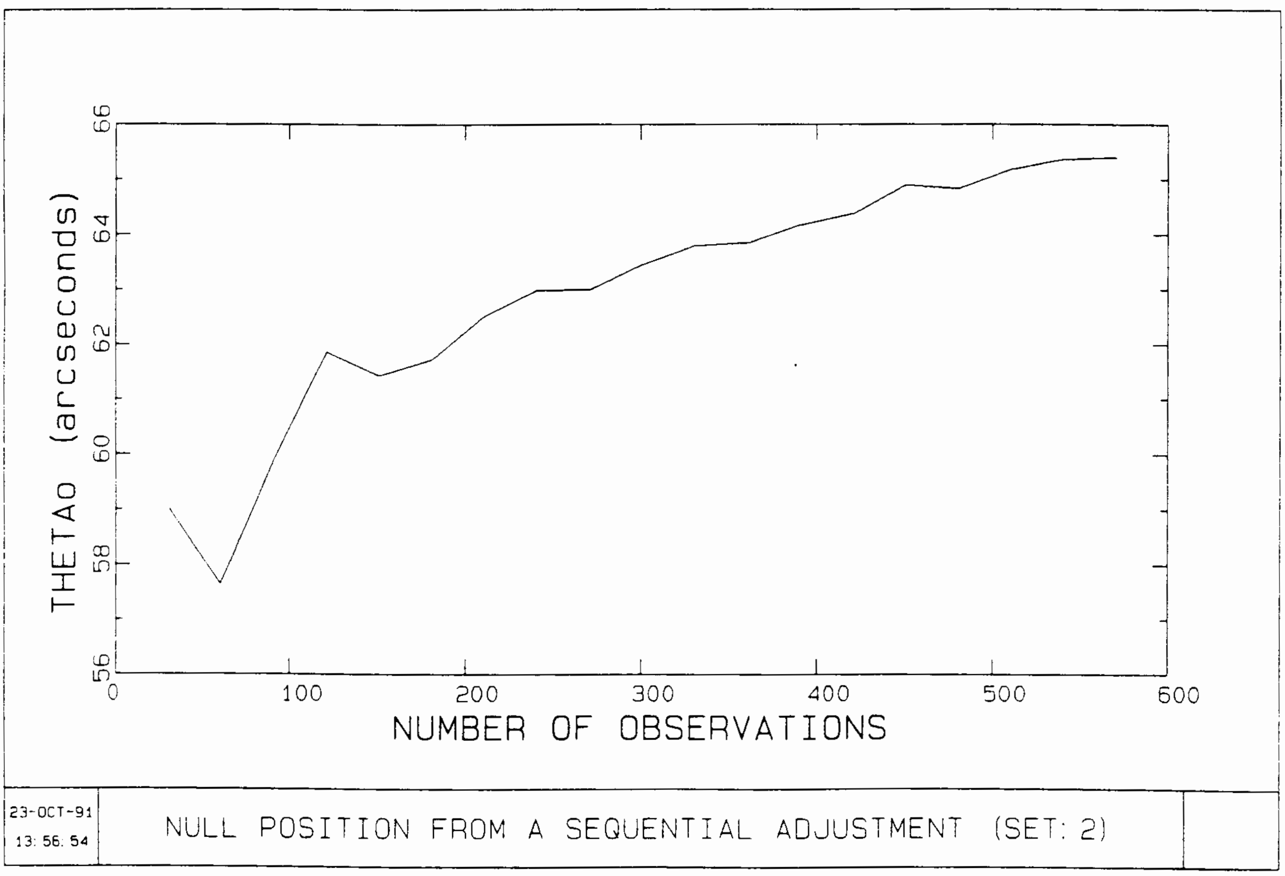


FIGURE E2

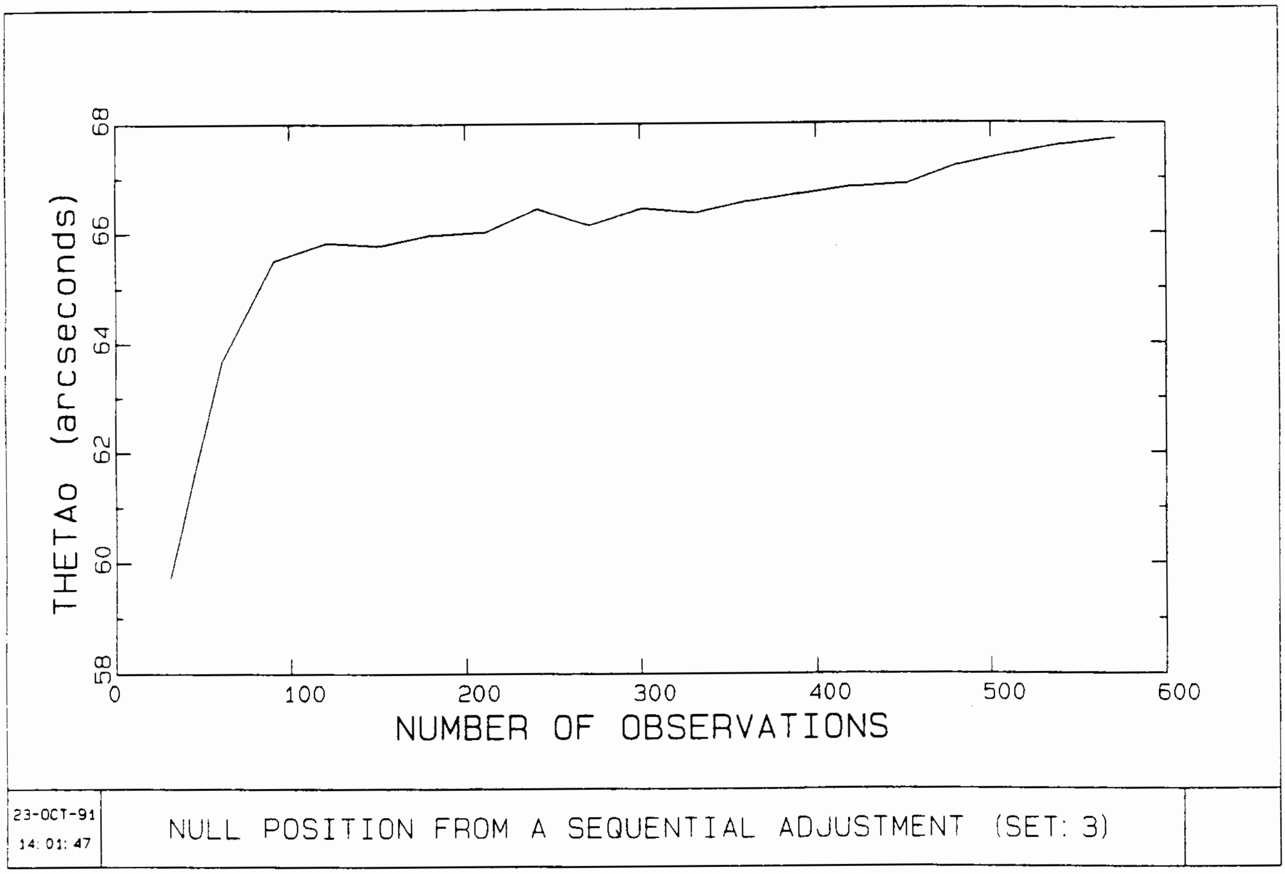


FIGURE E3

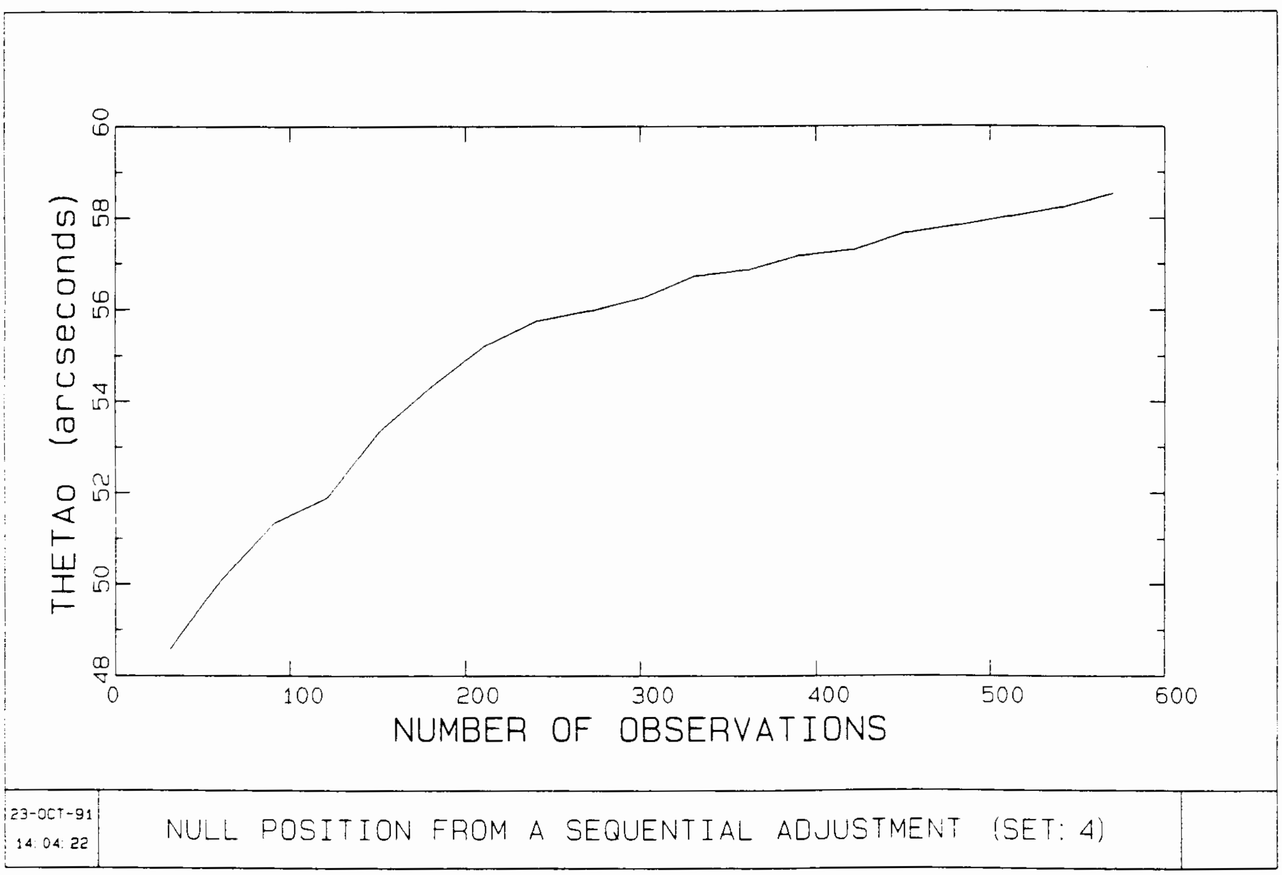
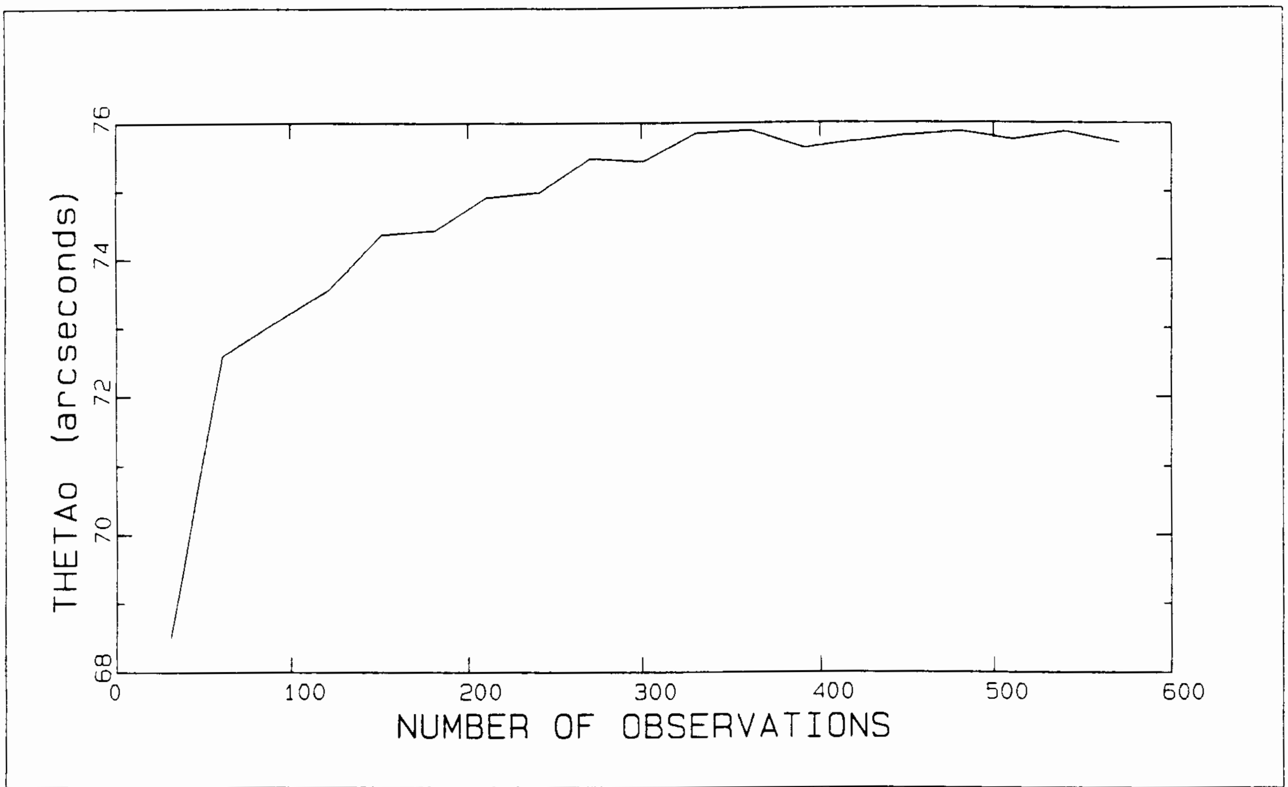


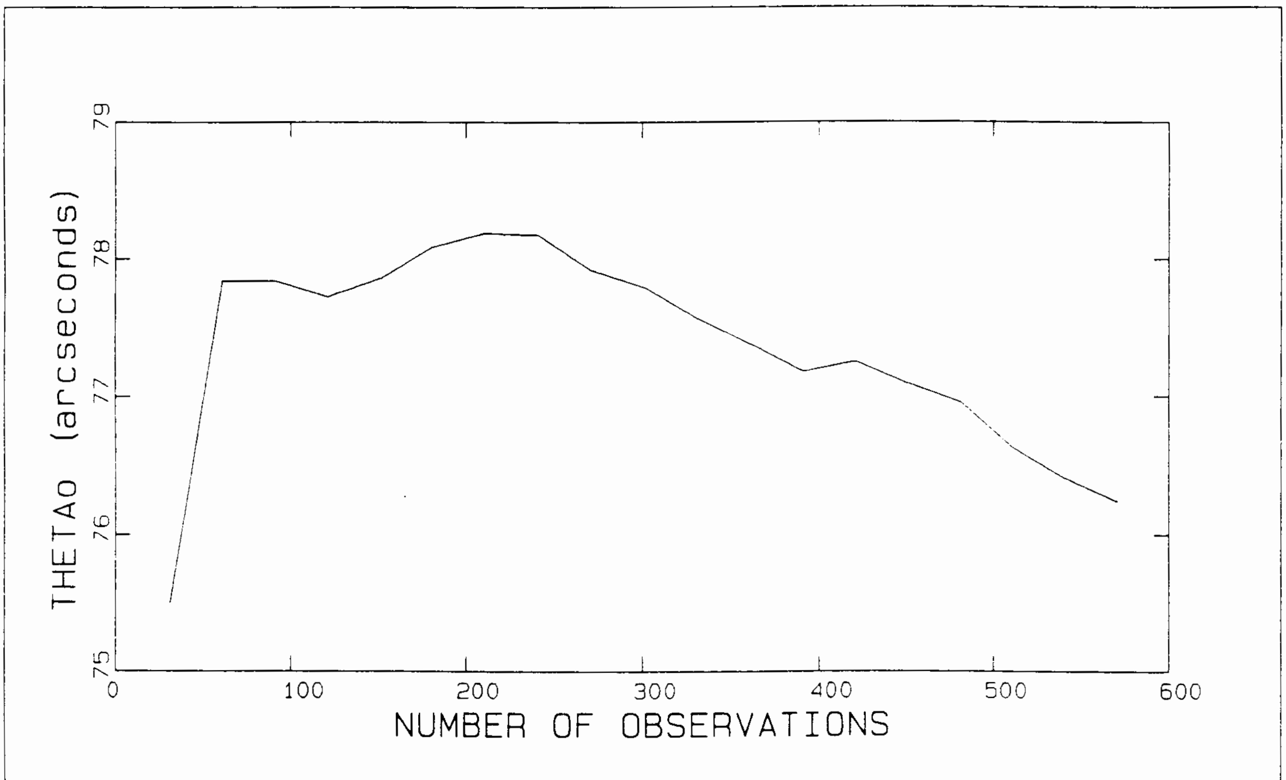
FIGURE E4



23-OCT-91 14:06:20

NULL POSITION FROM A SEQUENTIAL ADJUSTMENT (SET: 5)

FIGURE E5



23-OCT-91 14 09:10

NULL POSITION FROM A SEQUENTIAL ADJUSTMENT (SET: 6)

FIGURE E6

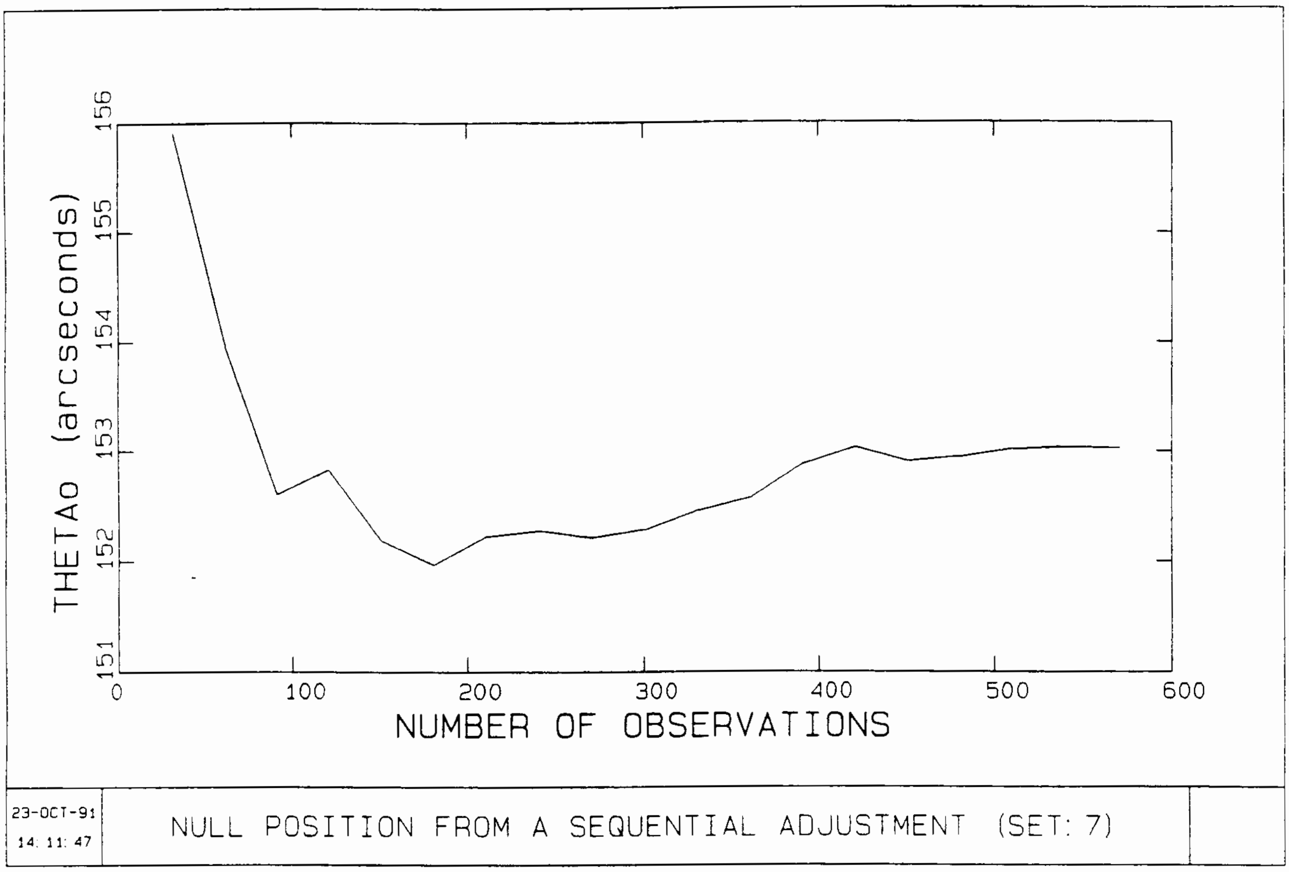


FIGURE E7

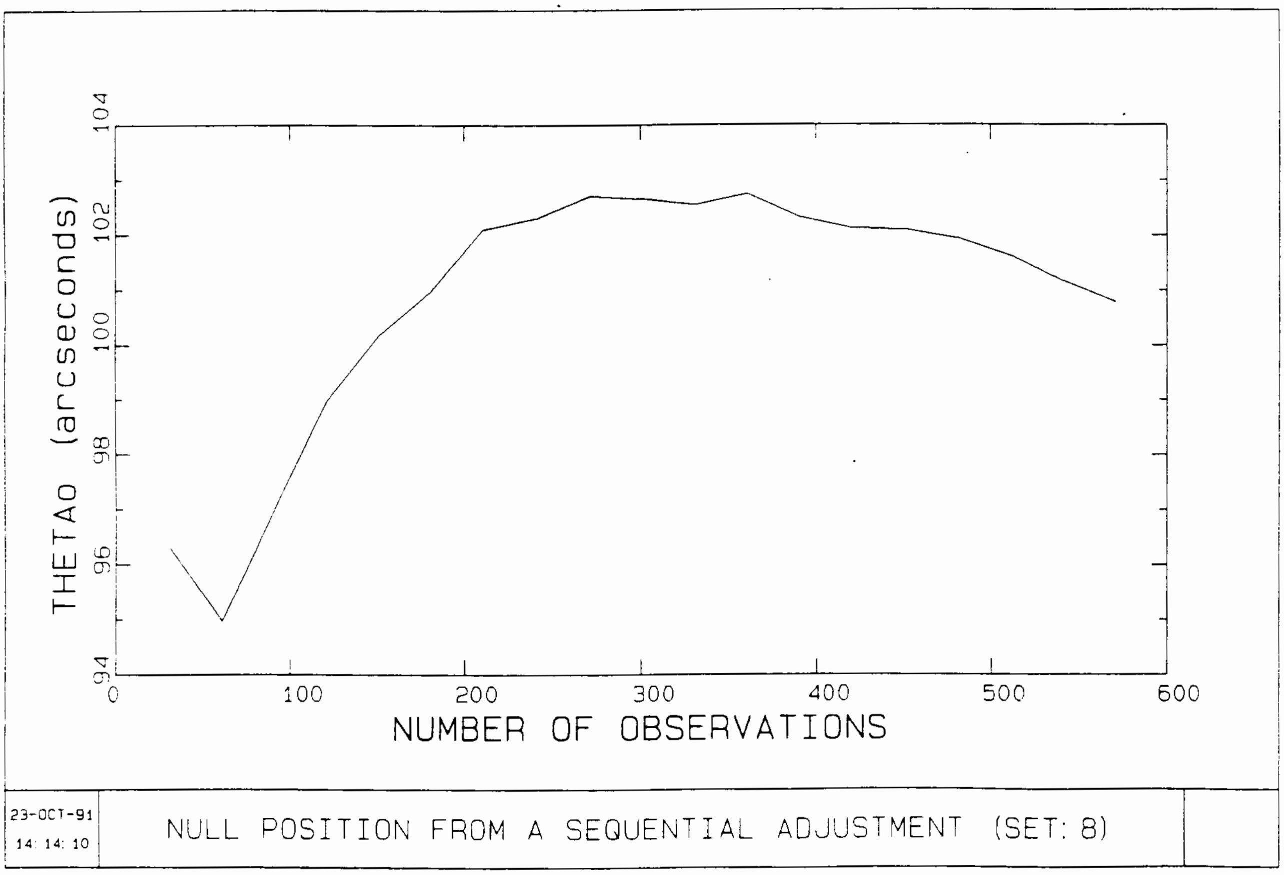


FIGURE E8

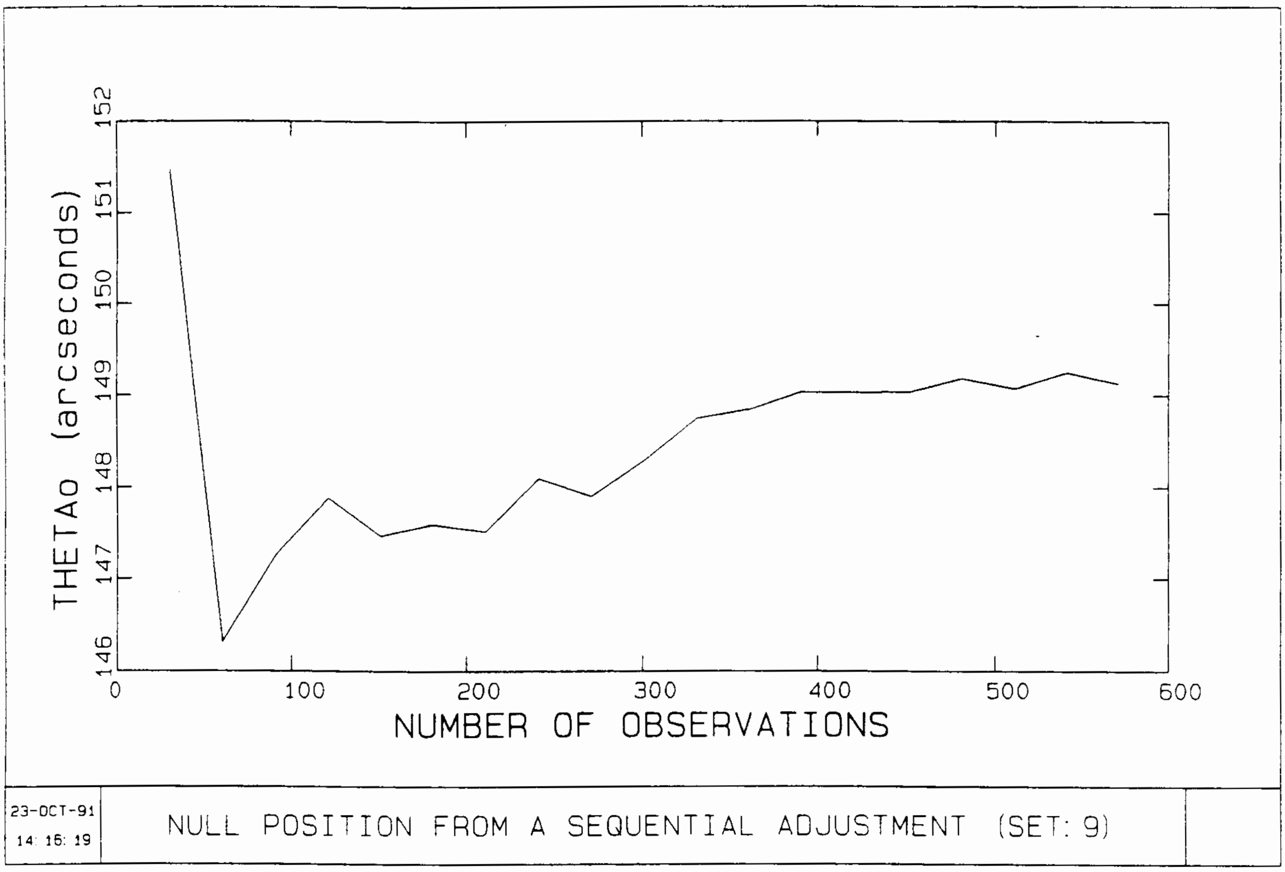


FIGURE E9

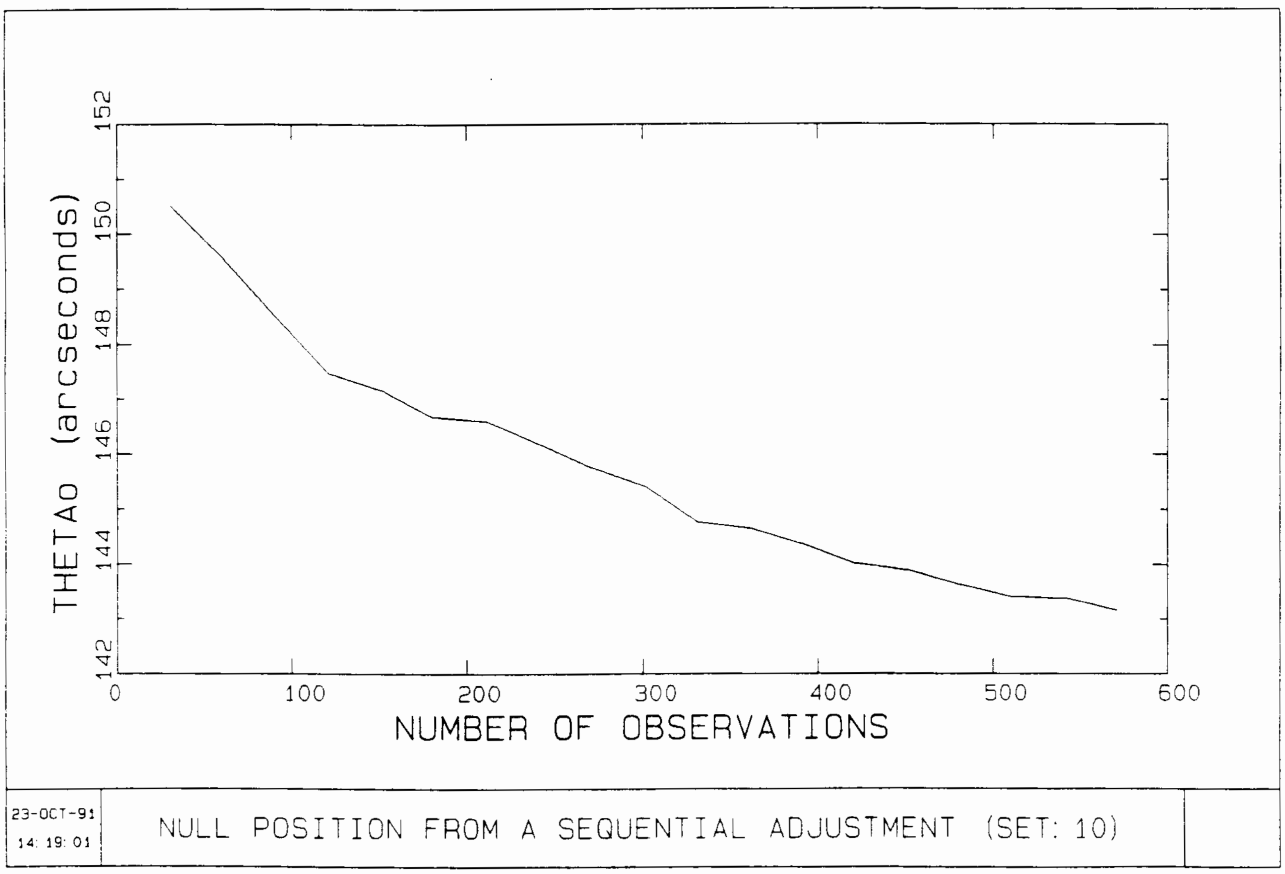


FIGURE E10

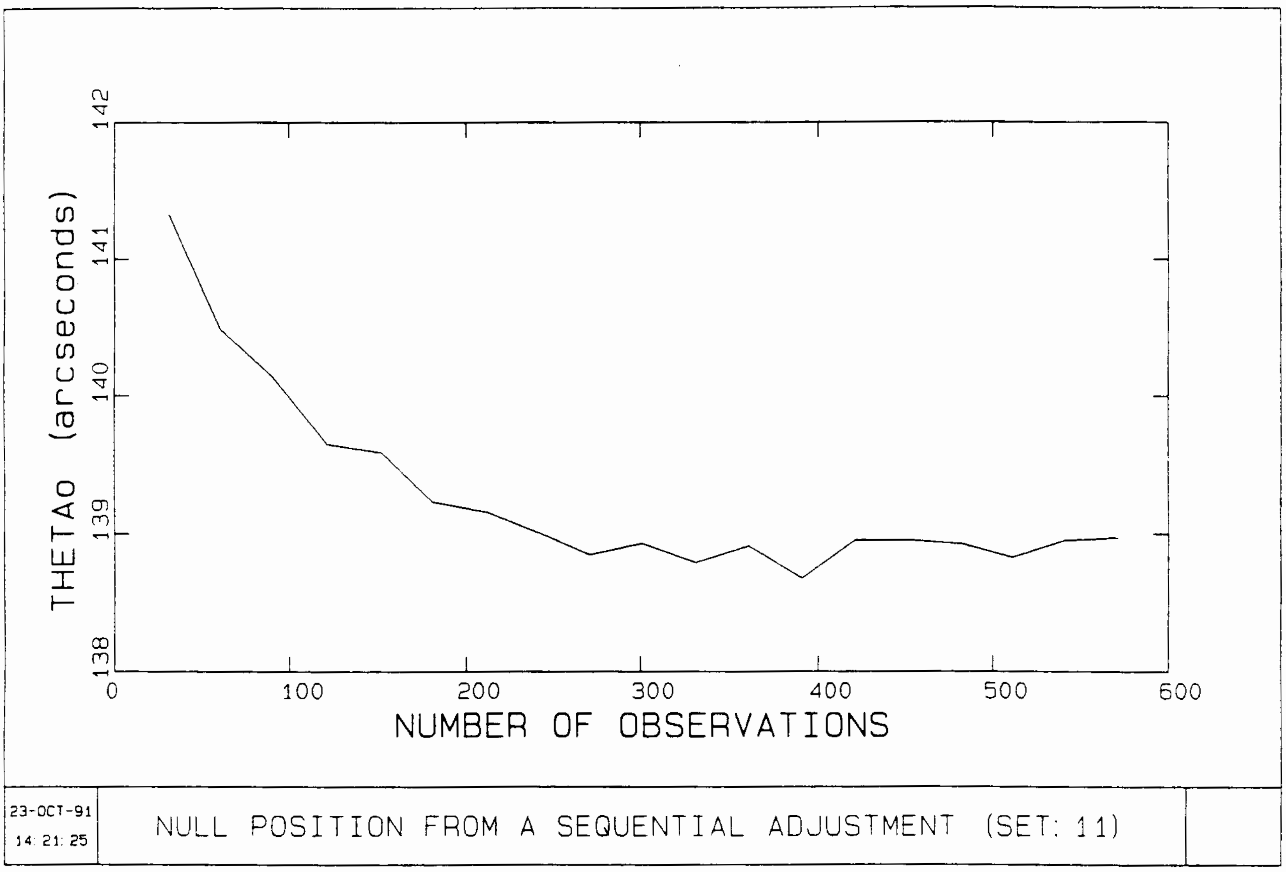


FIGURE E11

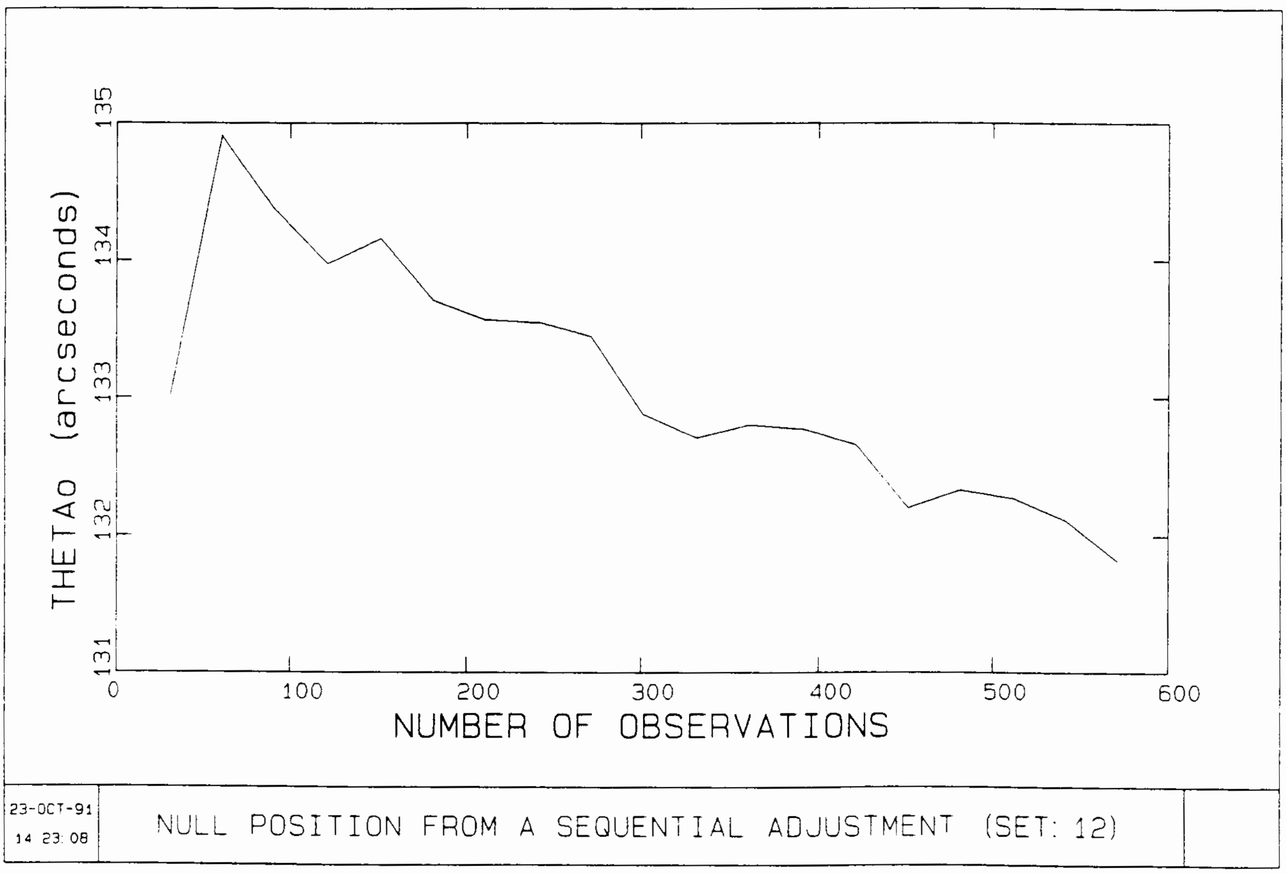


FIGURE E12

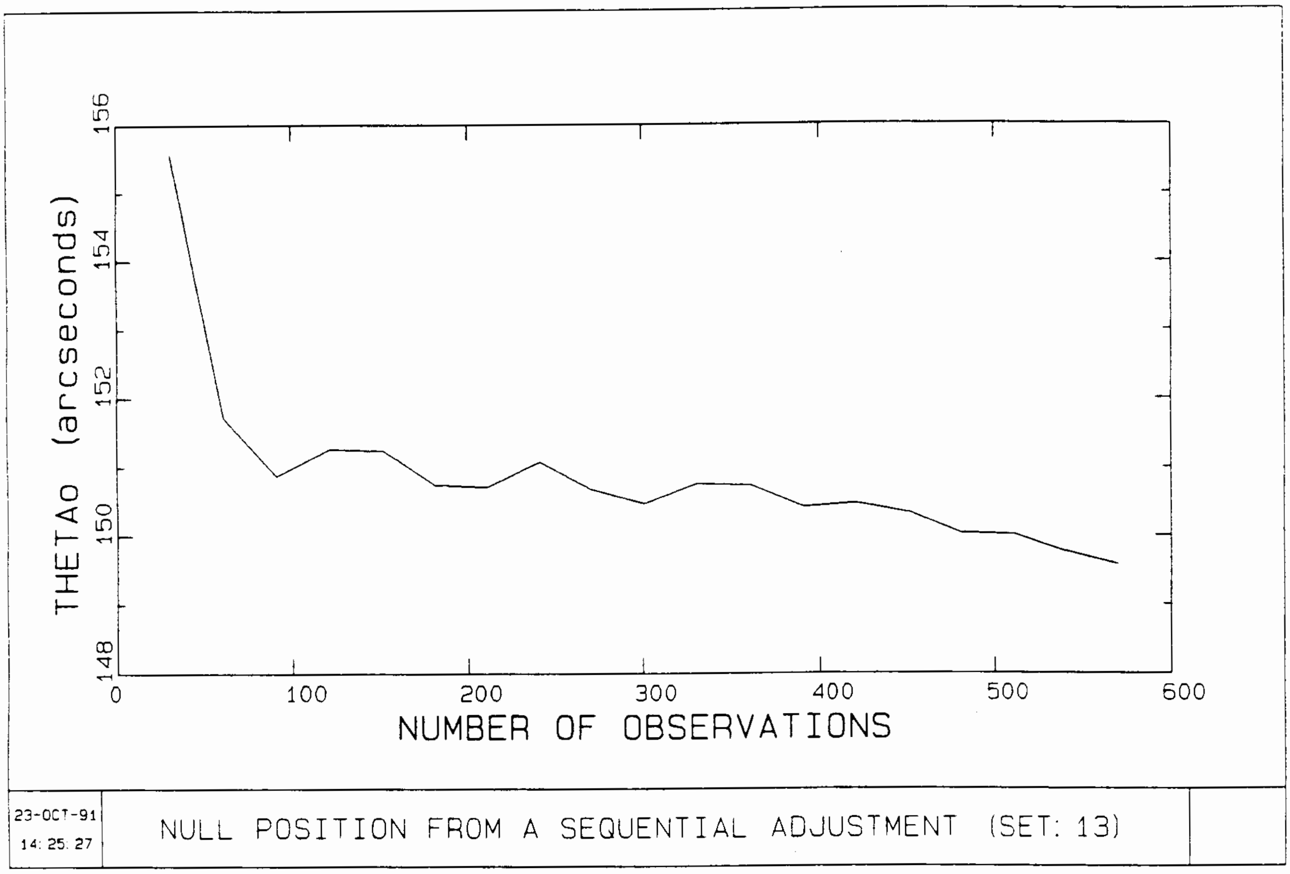


FIGURE E13

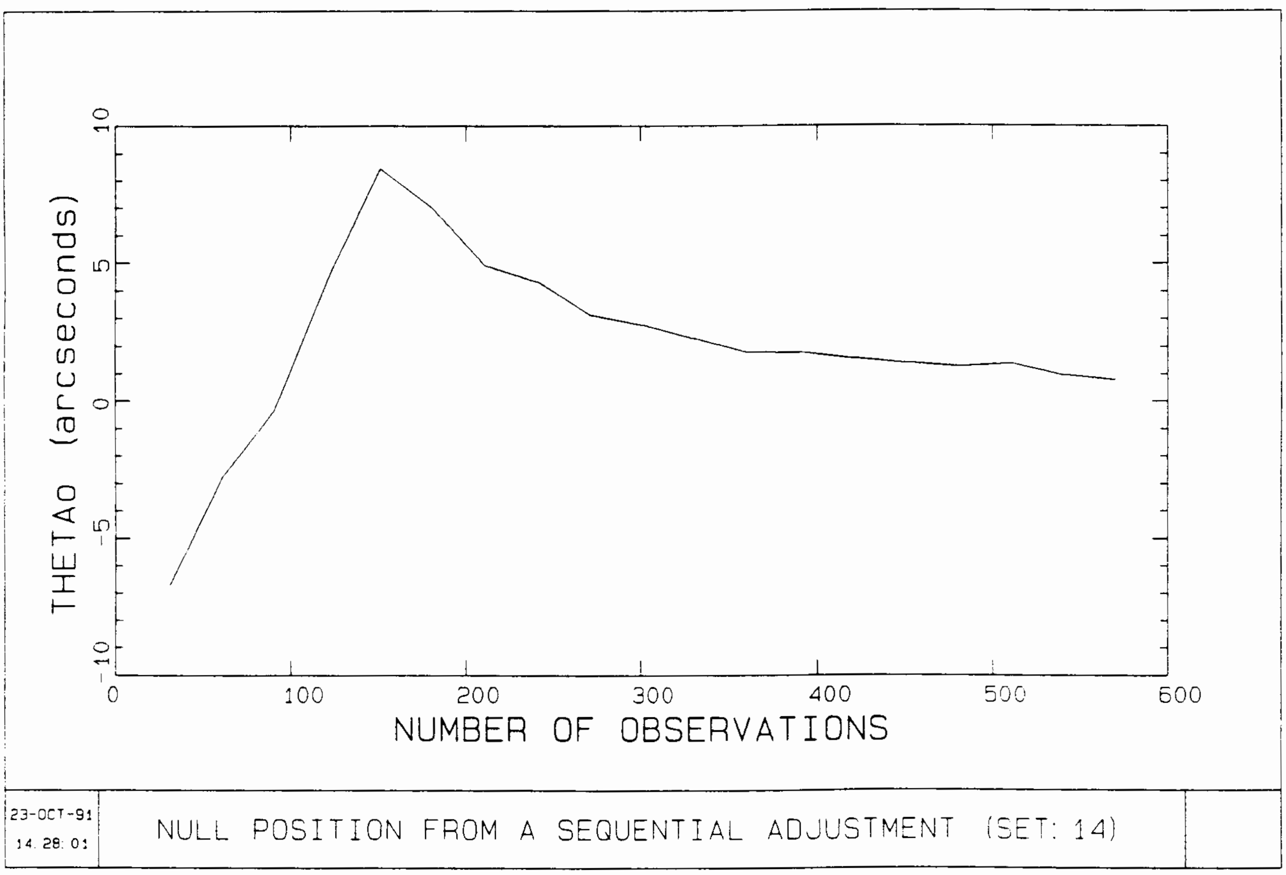
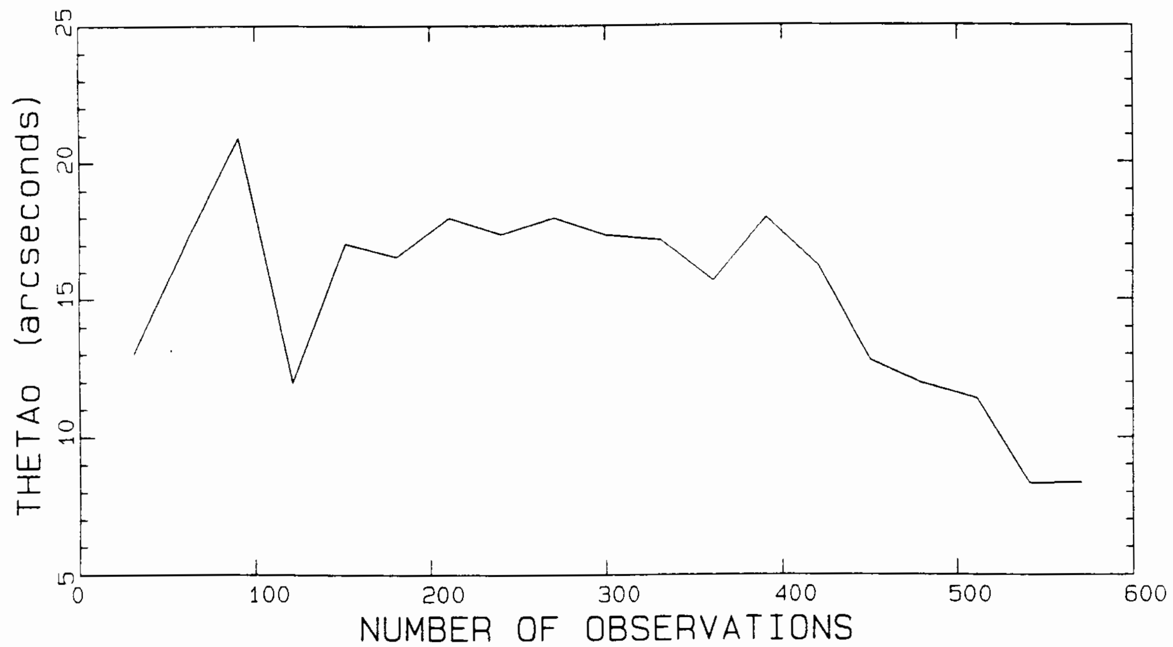
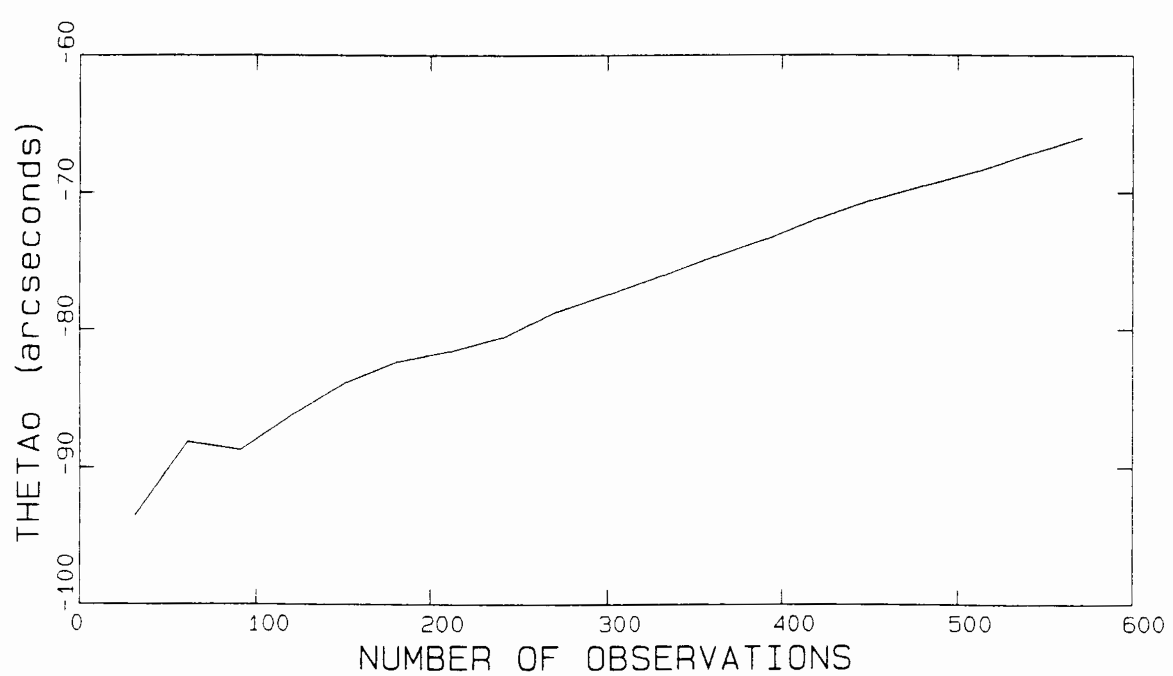


FIGURE E14



23-OCT-91 14:29:49	NULL POSITION FROM A SEQUENTIAL ADJUSTMENT (SET: 15)	

FIGURE E15



23-OCT-91 14:31:37	NULL POSITION FROM A SEQUENTIAL ADJUSTMENT (SET: 17)	

FIGURE E16

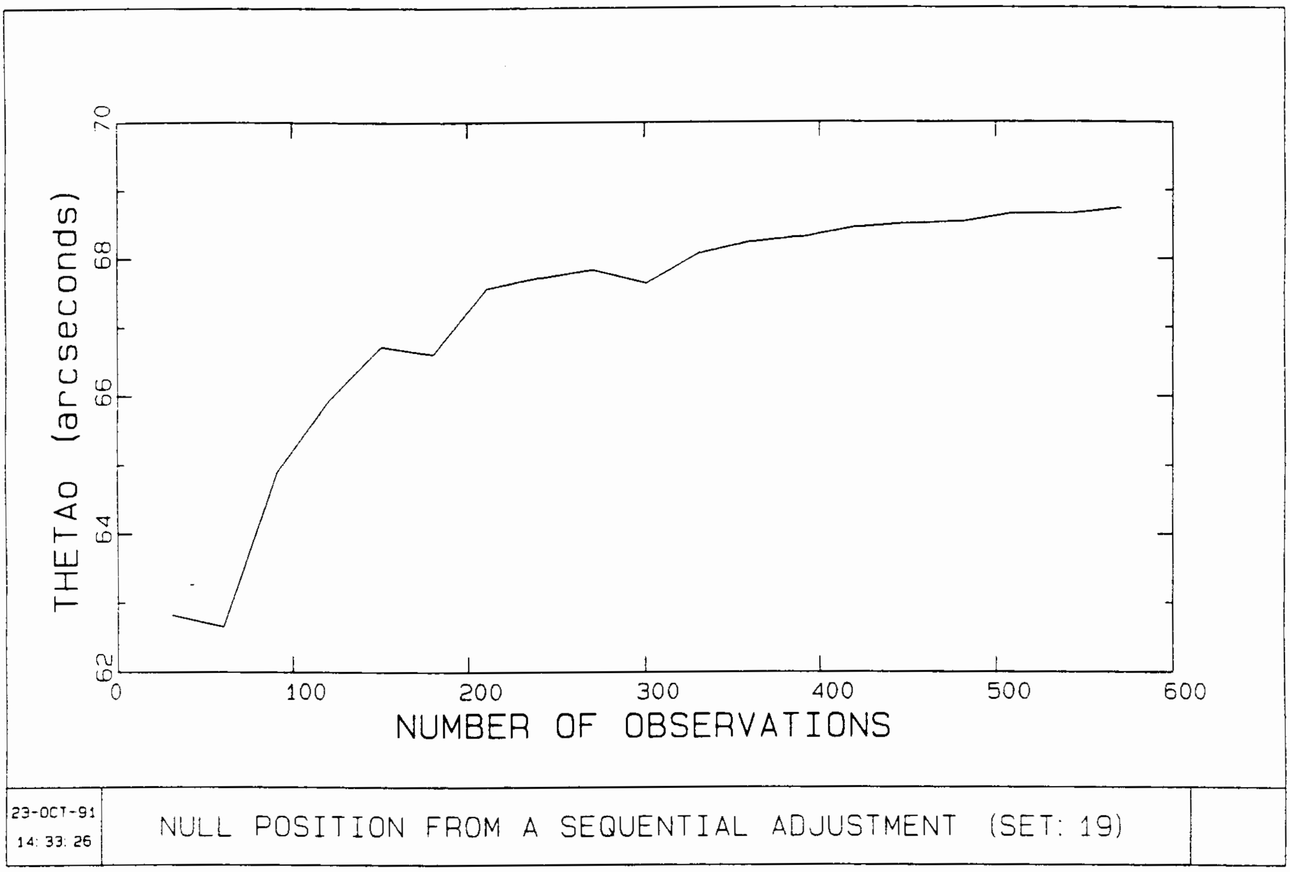


FIGURE E17

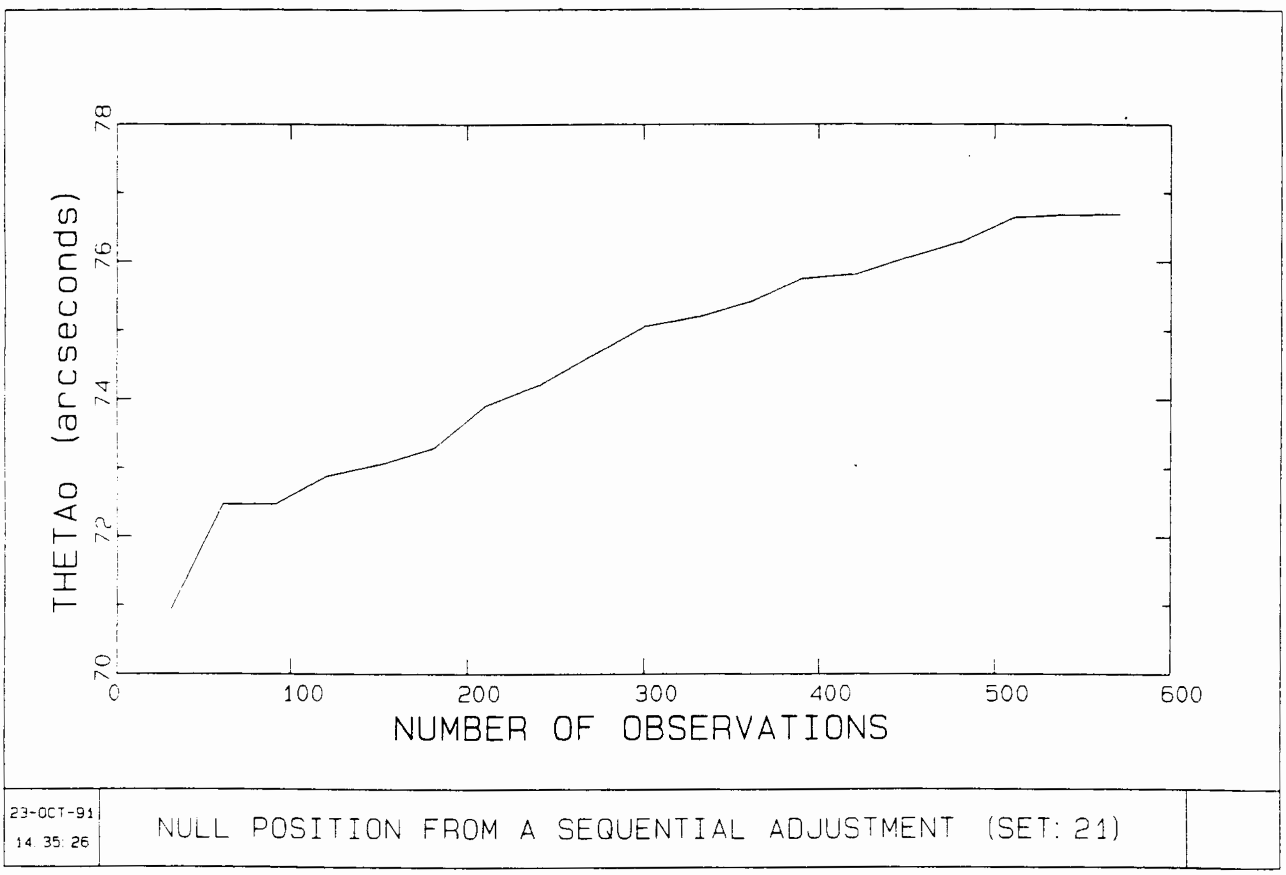


FIGURE E18

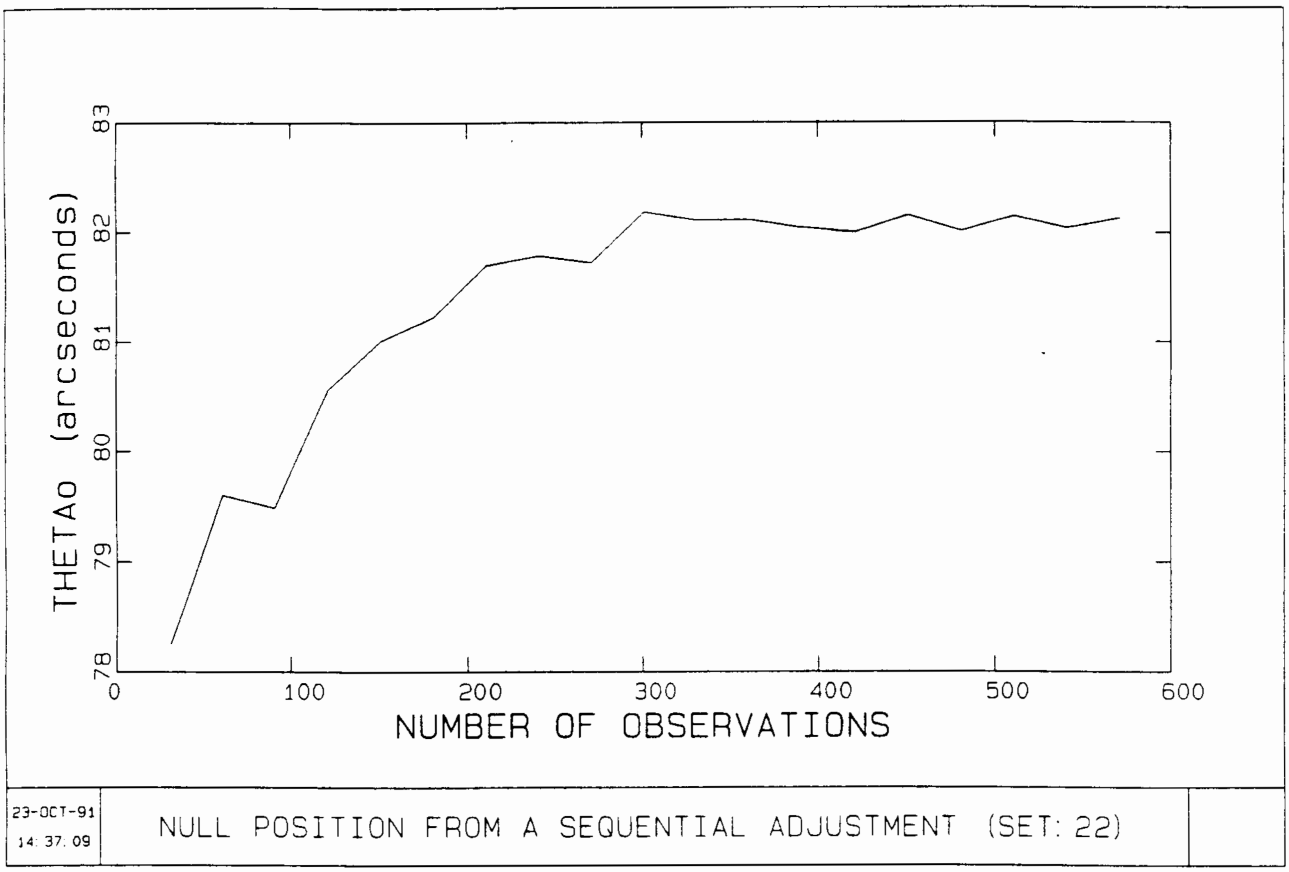


FIGURE E19

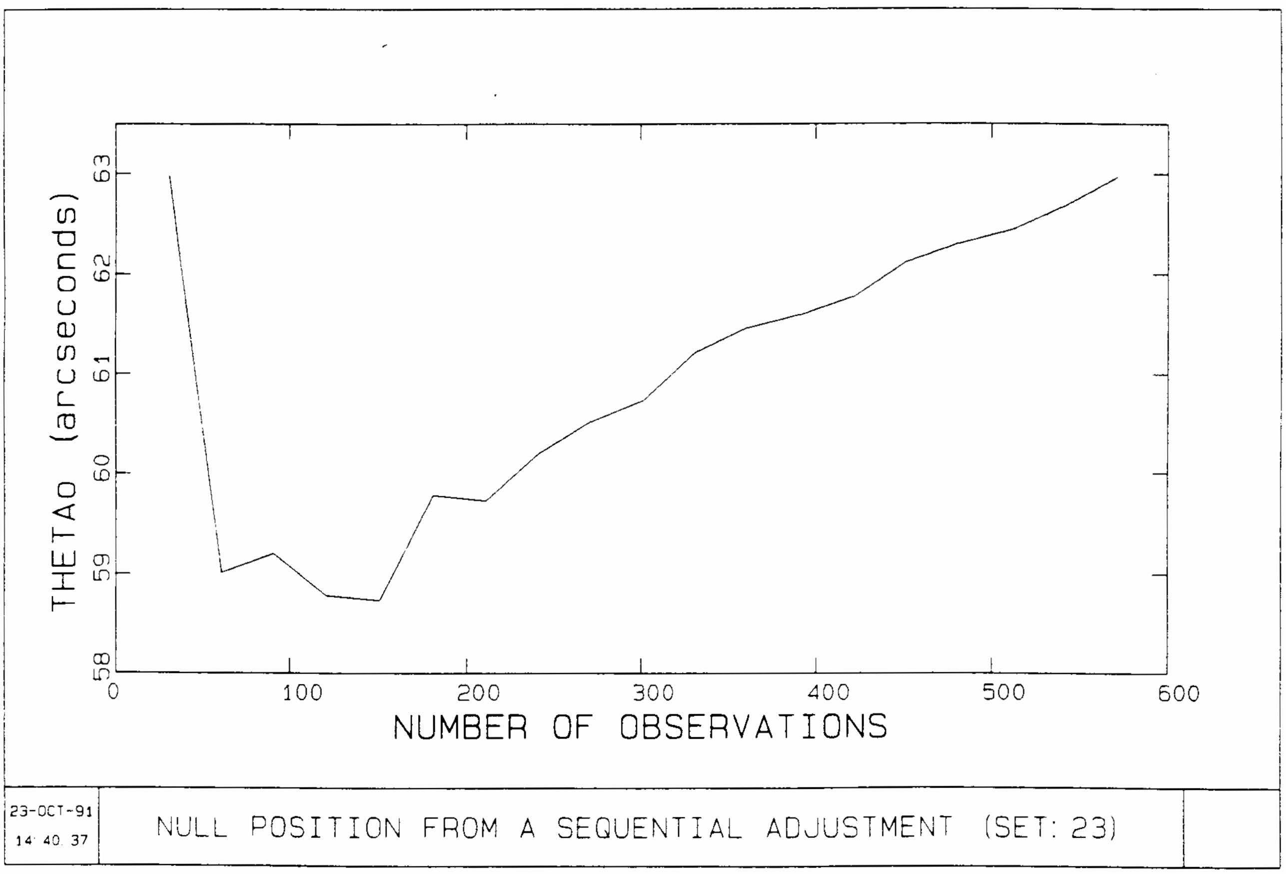


FIGURE E20

APPENDIX F

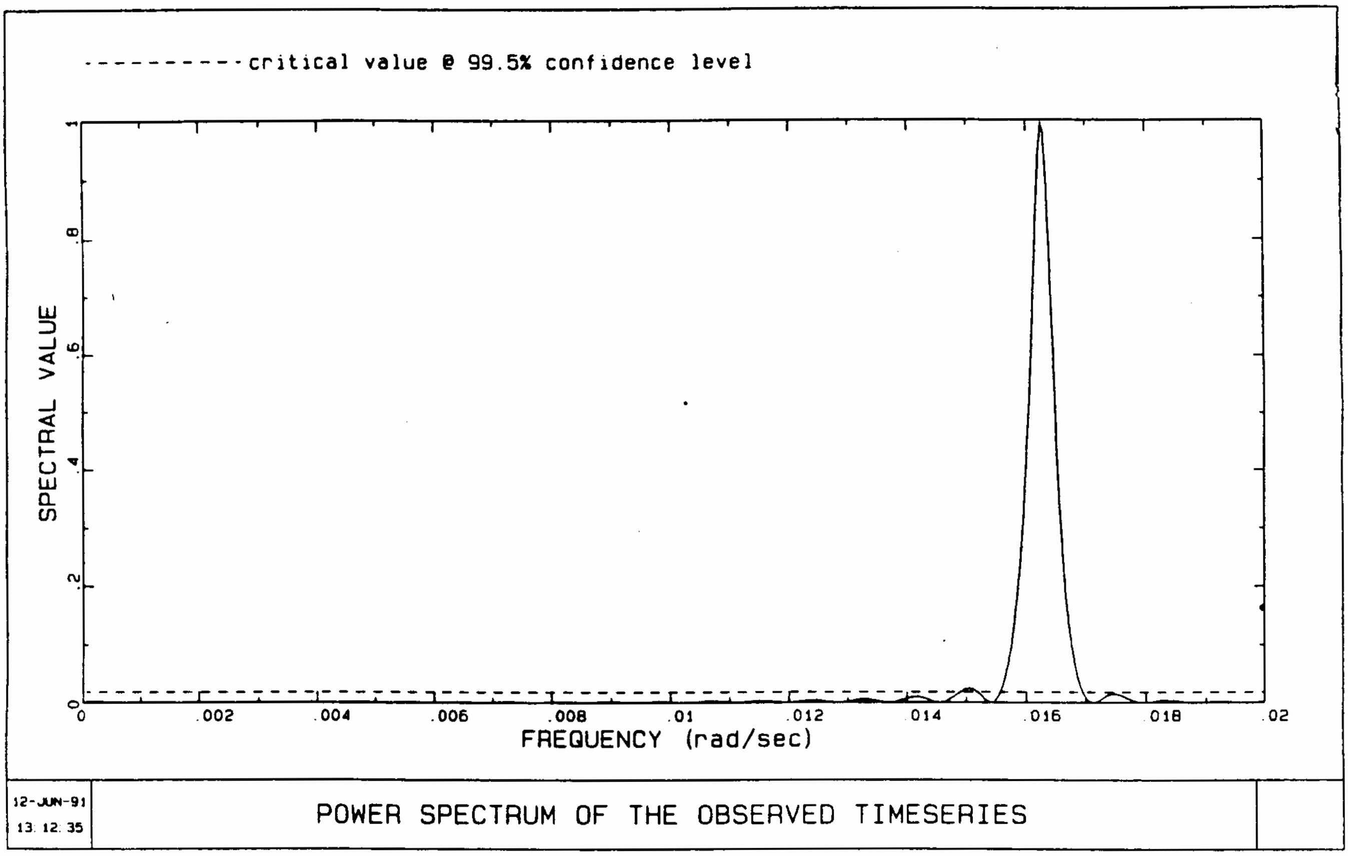


FIGURE F1 : Analysis of a simulated four-periodic gyro-motion

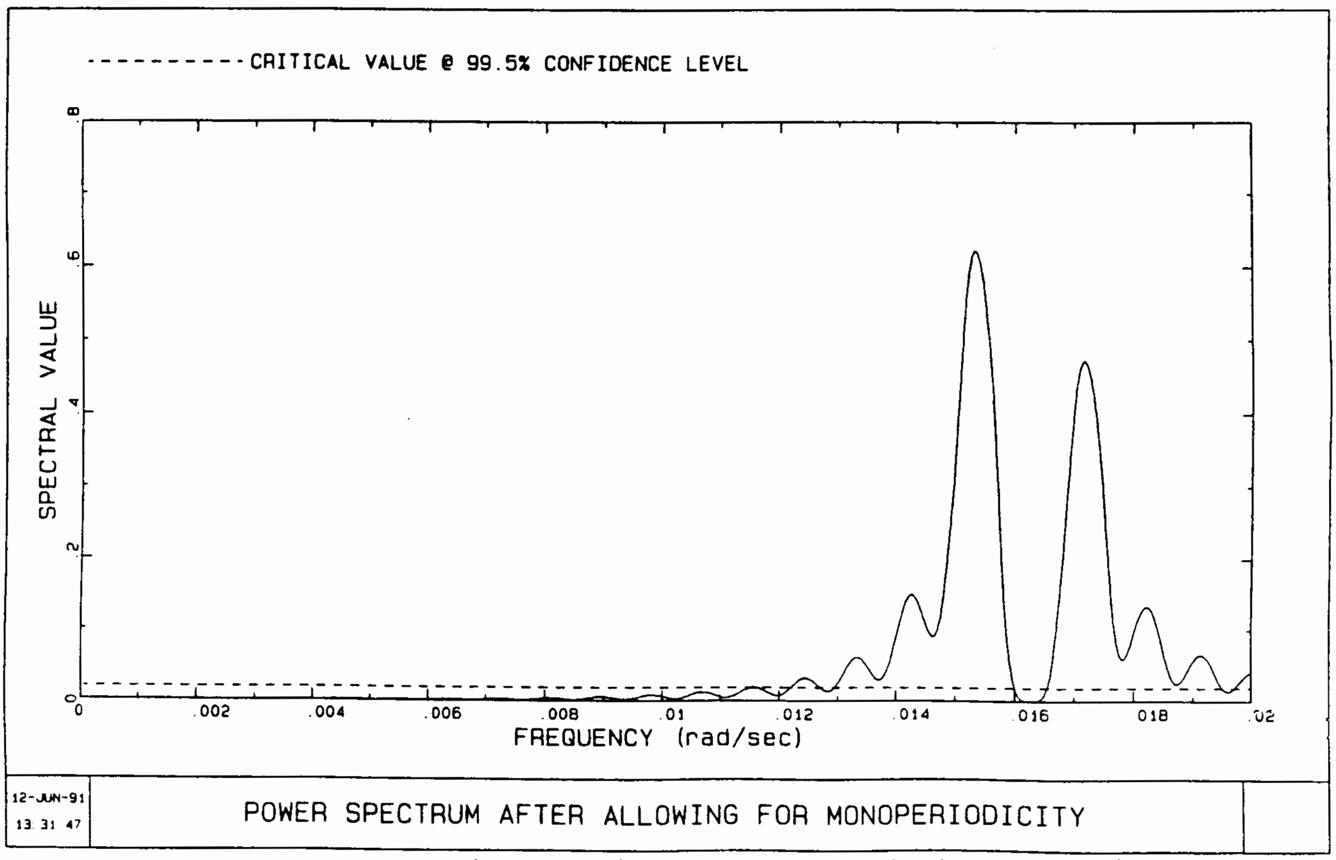


FIGURE F2 : Analysis of a simulated four-periodic gyro-motion

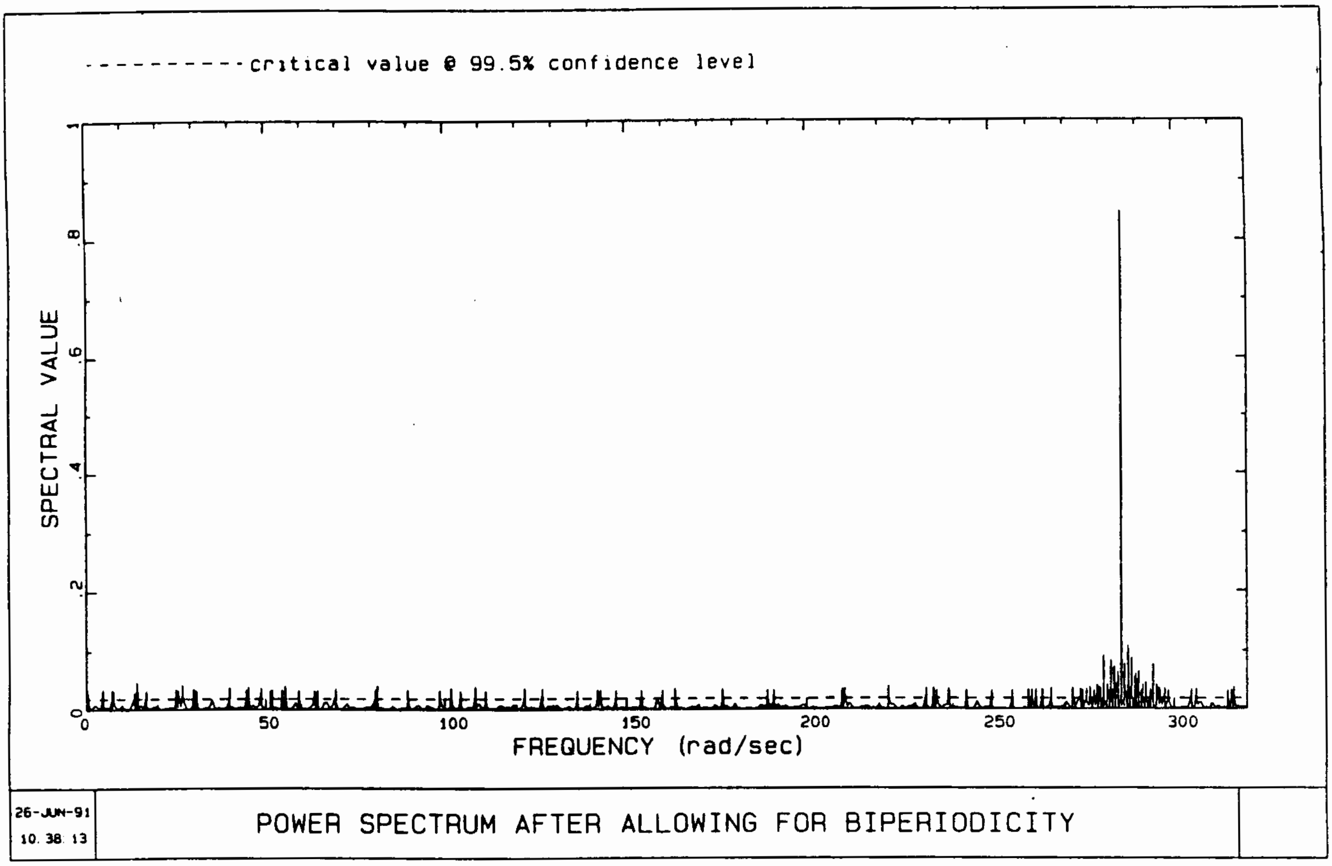


FIGURE F3 : Analysis of a simulated four-periodic gyro-motion

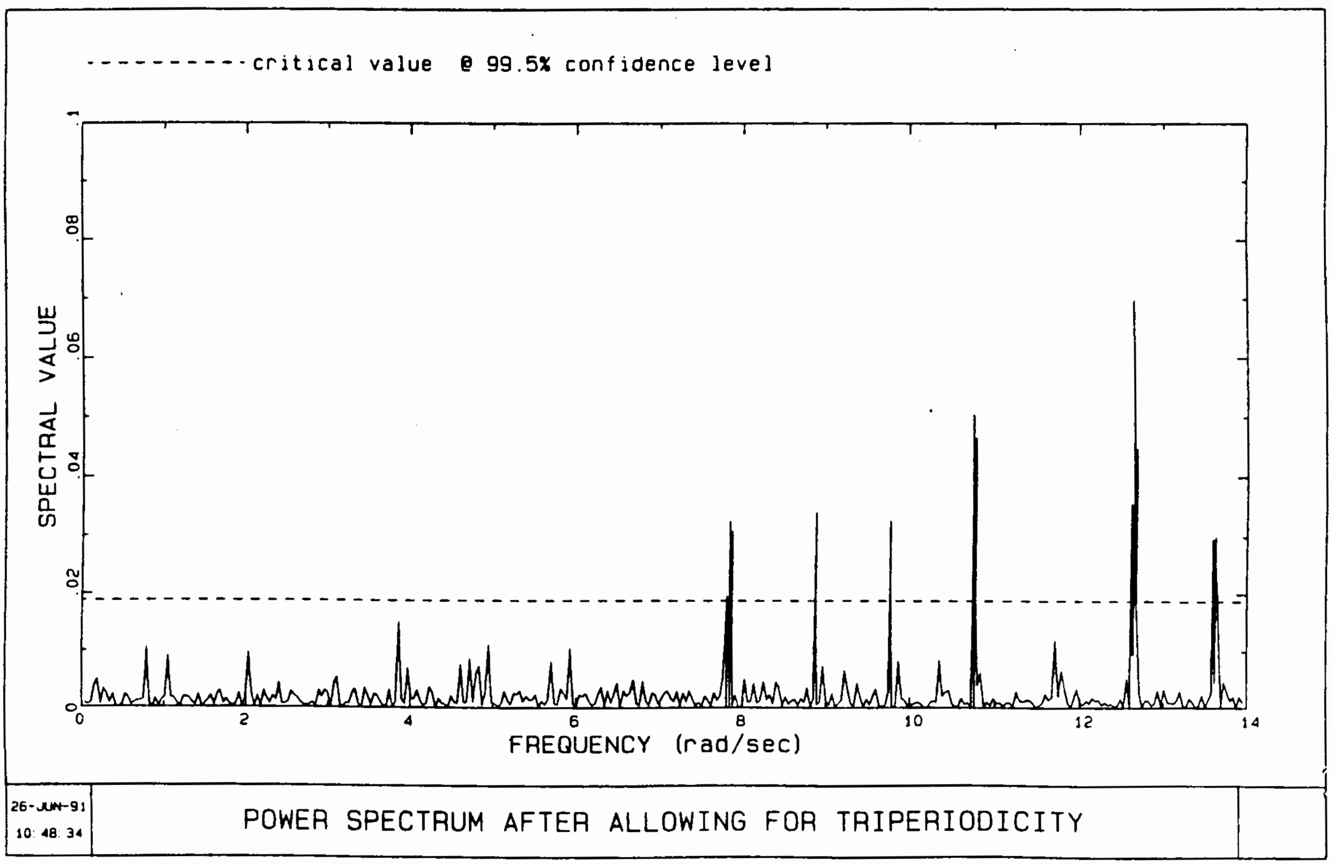


FIGURE F4 : Analysis of a simulated four-periodic gyro-motion

APPENDIX G

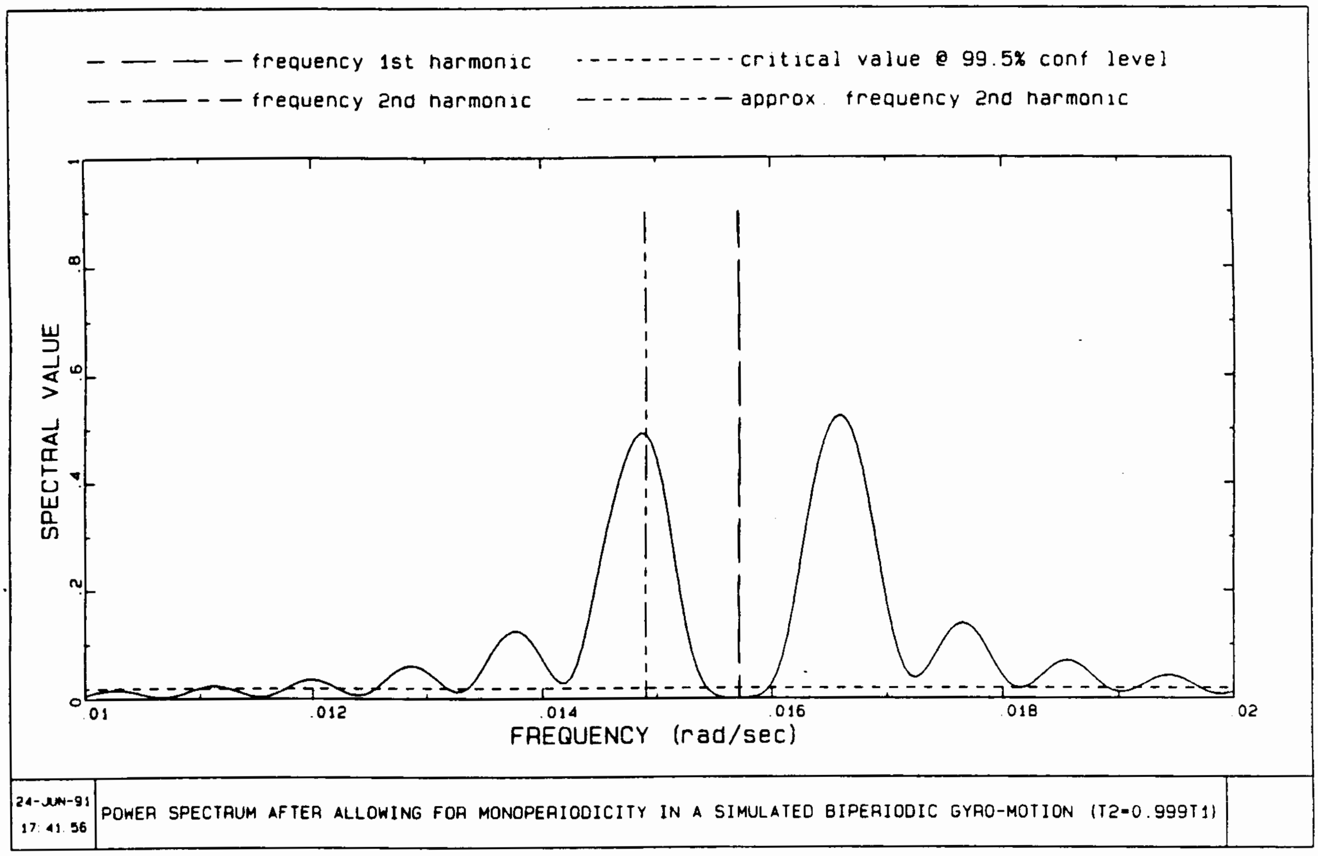


FIGURE G1

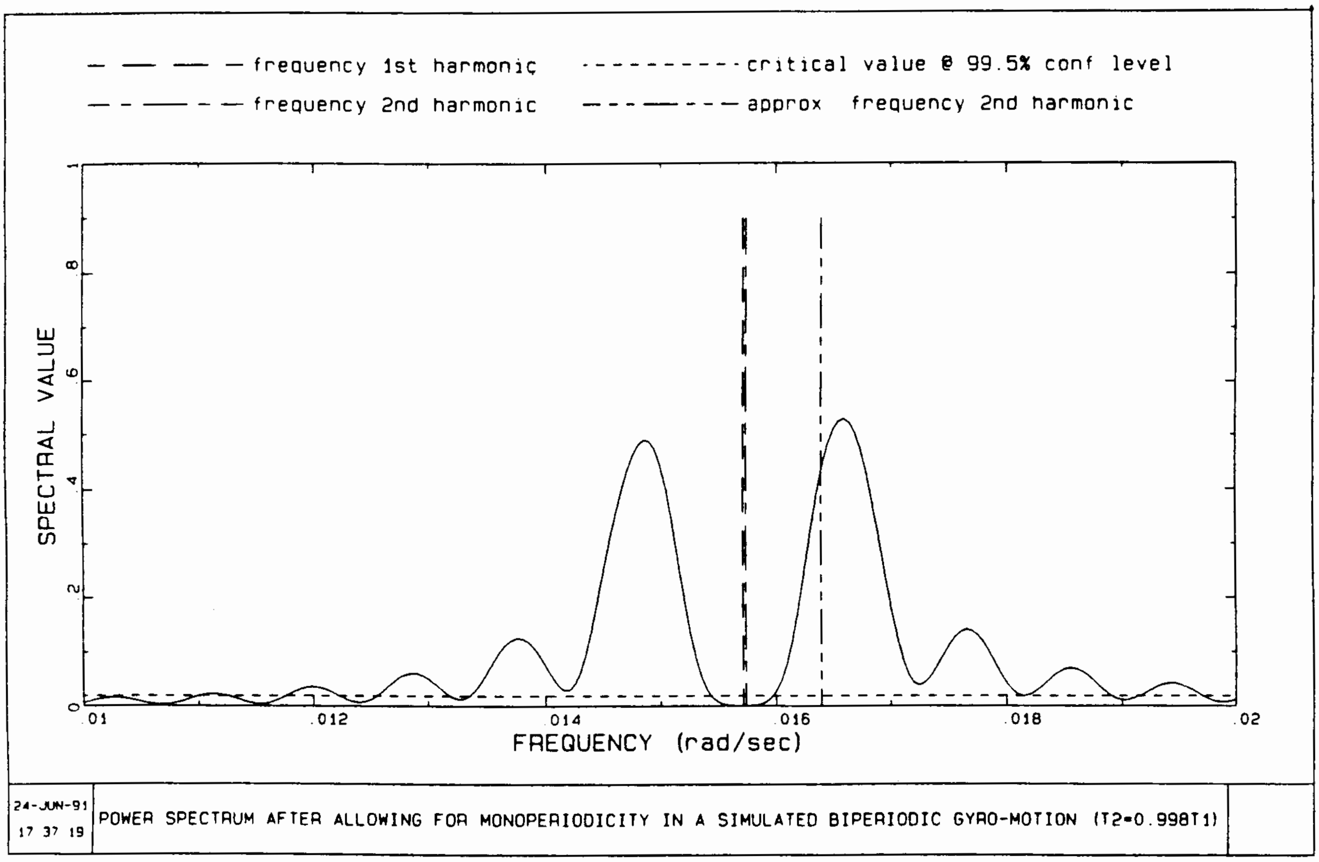


FIGURE G2

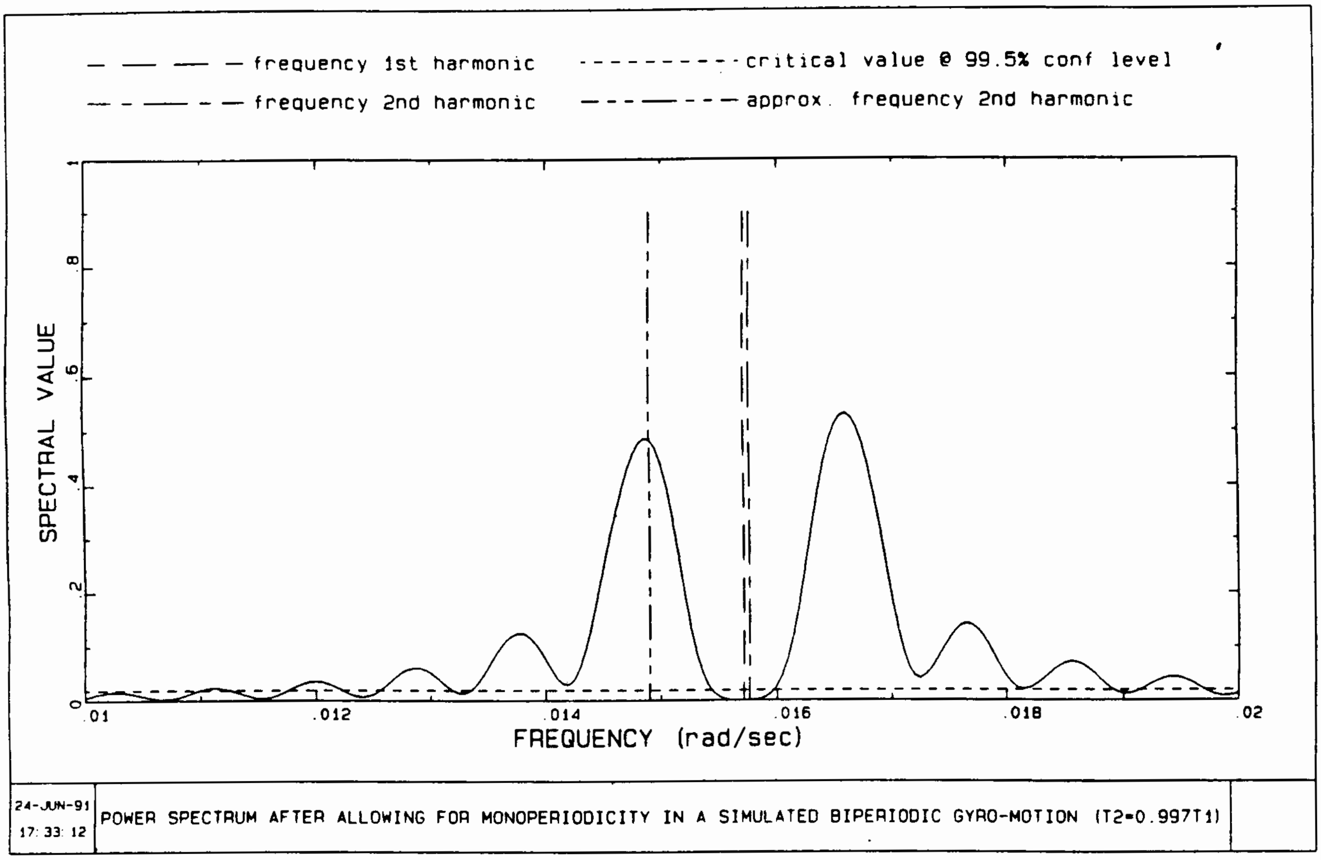


FIGURE G3

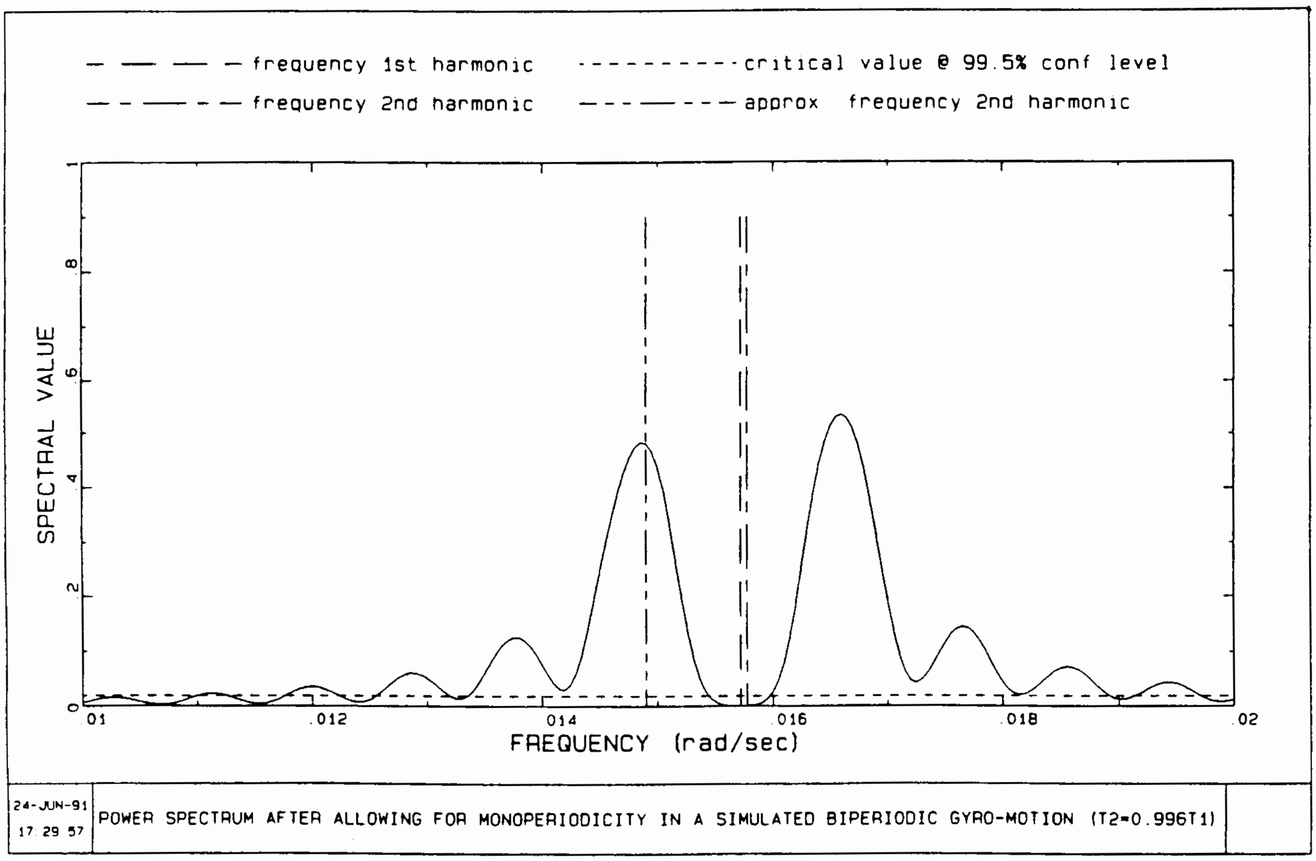
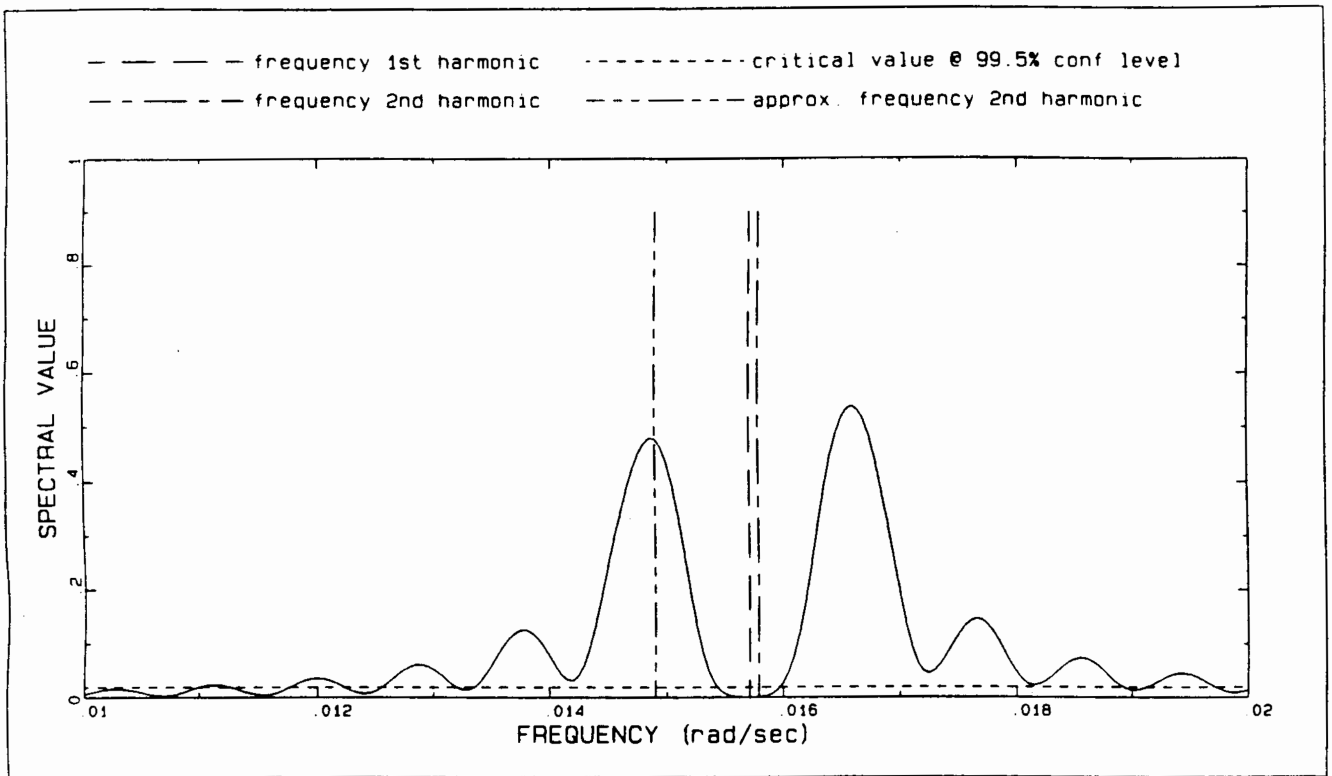
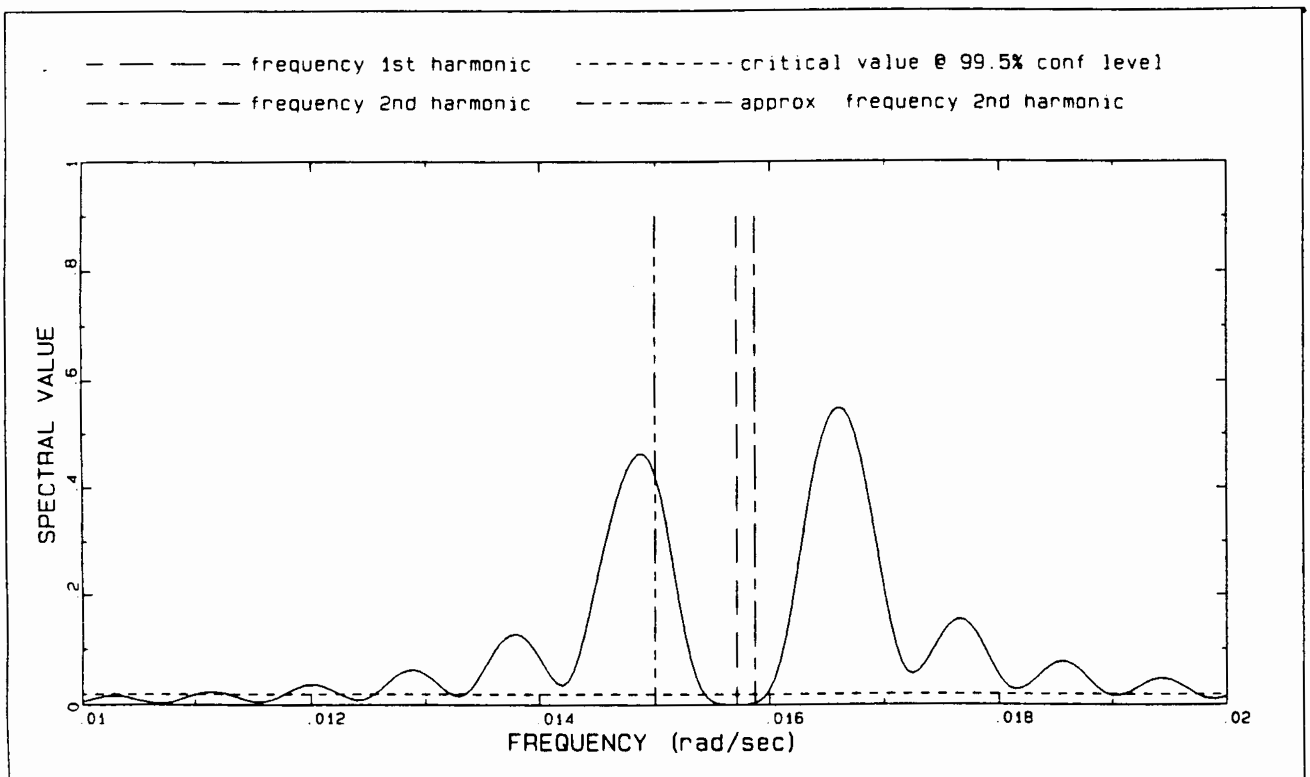


FIGURE G4



POWER SPECTRUM AFTER ALLOWING FOR MONOPERIODICITY IN A SIMULATED BIPERIODIC GYROMOTION (T2=0.995T1)

FIGURE G5



POWER SPECTRUM AFTER ALLOWING FOR MONOPERIODICITY IN A SIMULATED BIPERIODIC GYRO-MOTION (T2=0.99T1)

FIGURE G6

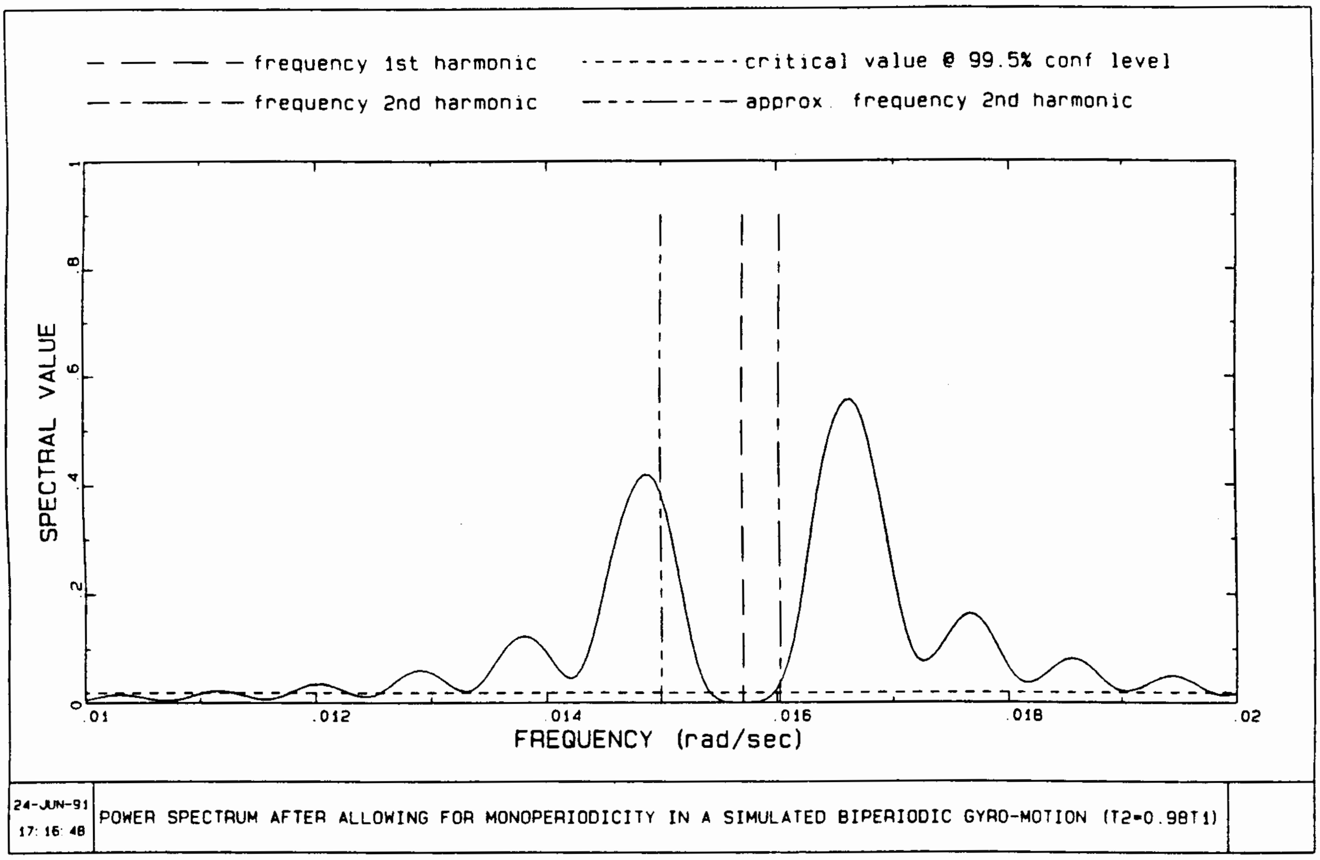


FIGURE G7

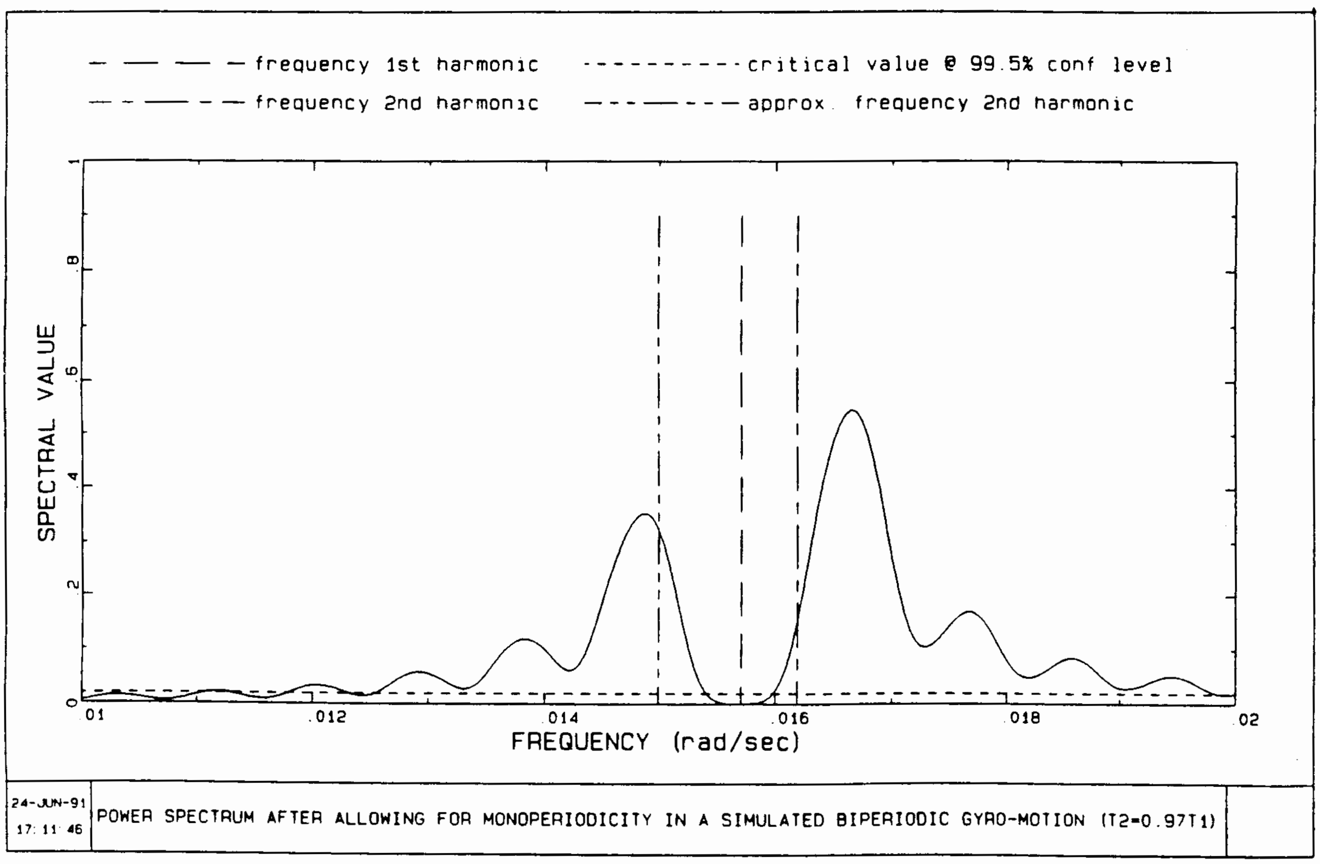
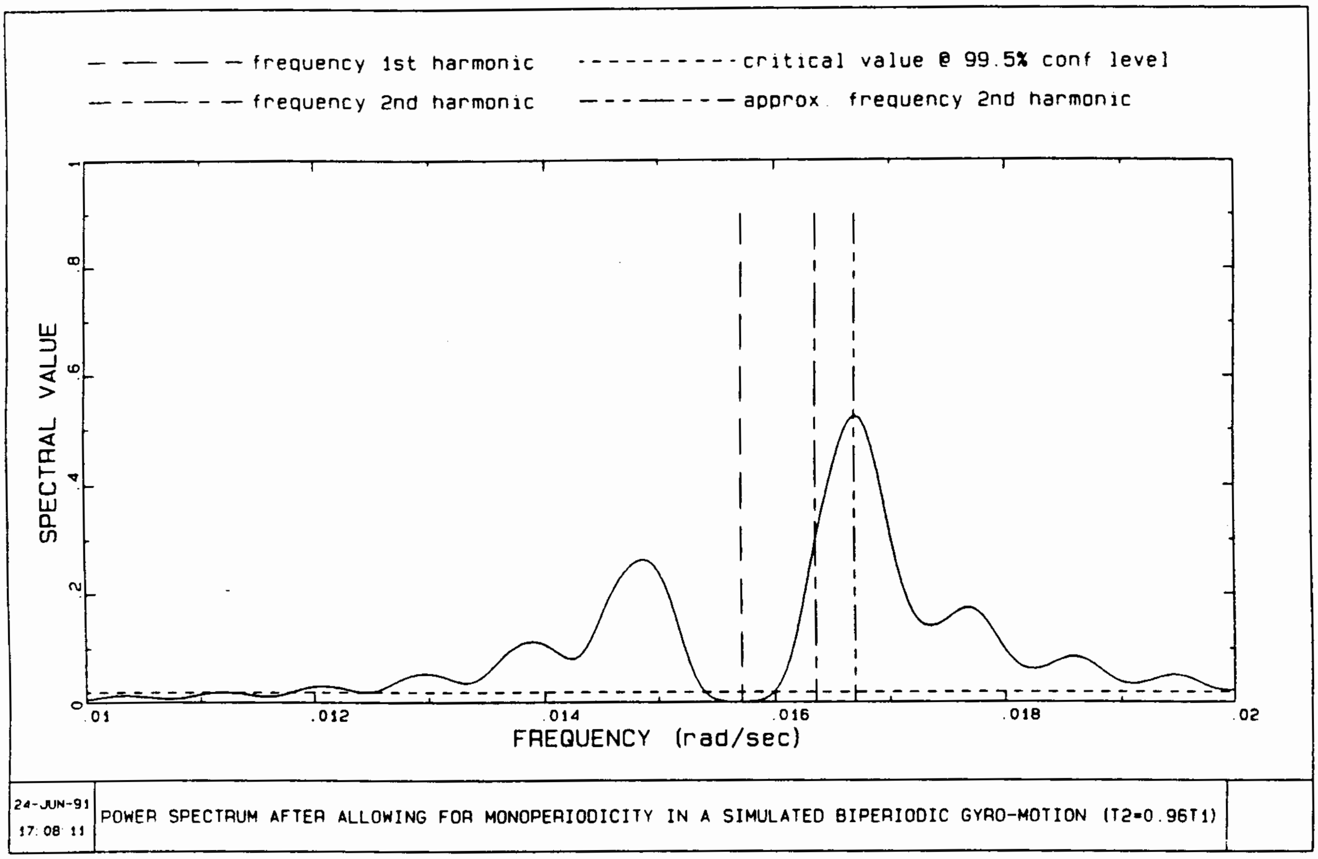
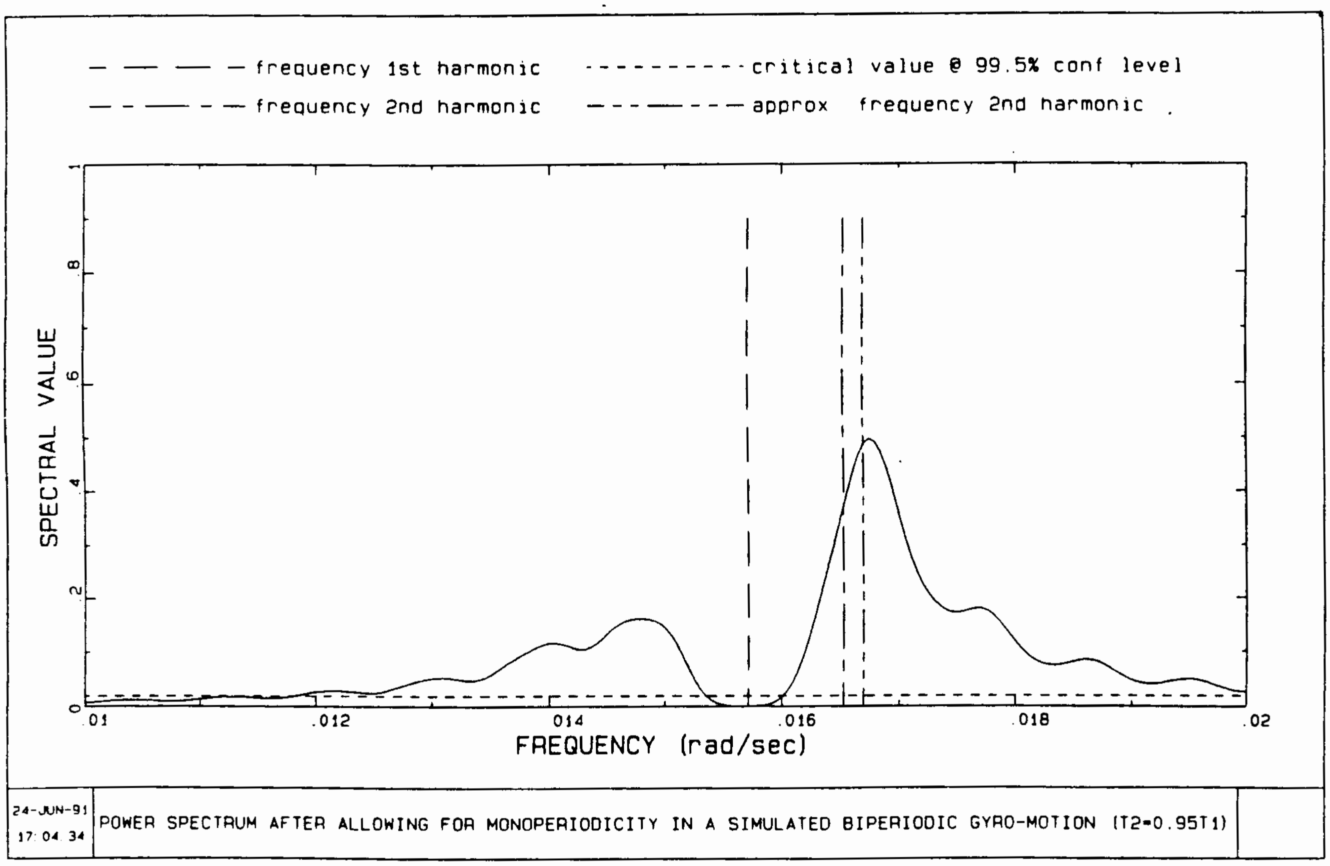


FIGURE G8



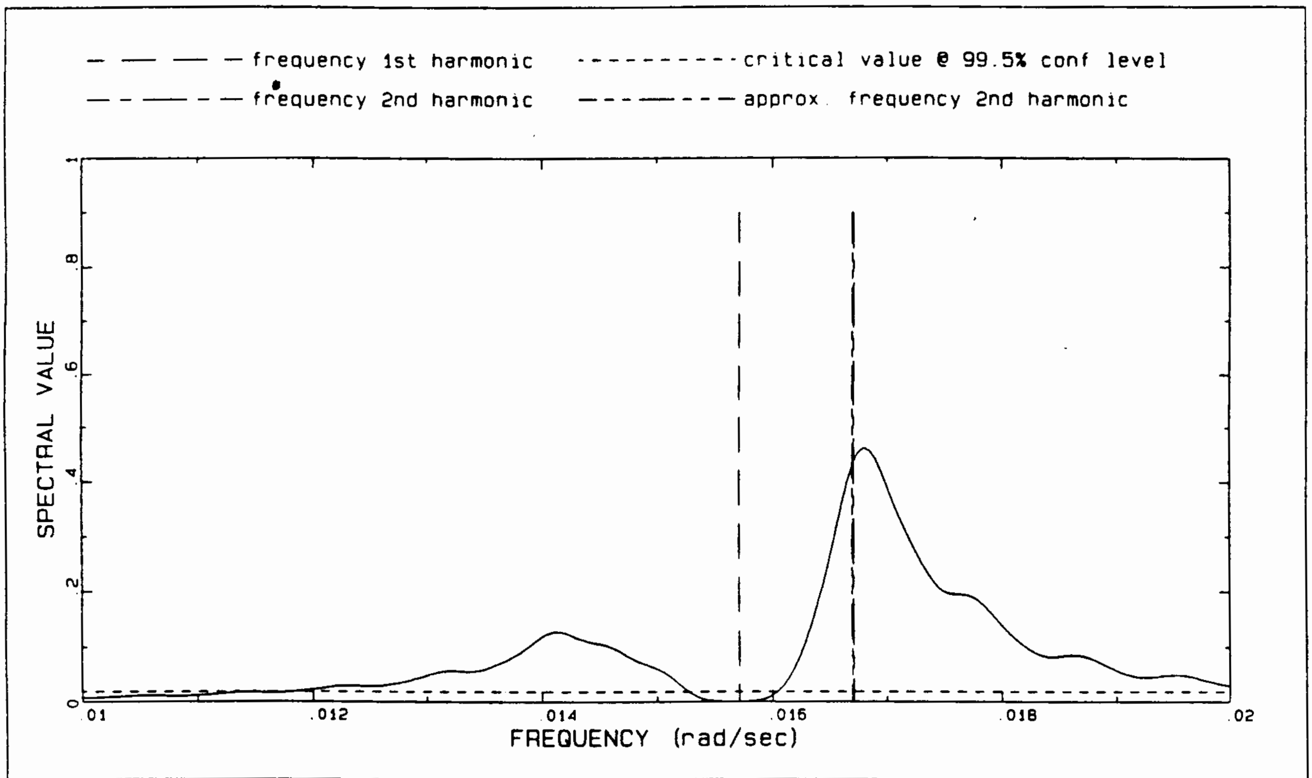
POWER SPECTRUM AFTER ALLOWING FOR MONOPERIODICITY IN A SIMULATED BIPERIODIC GYRO-MOTION (T2=0.96T1)

FIGURE G9



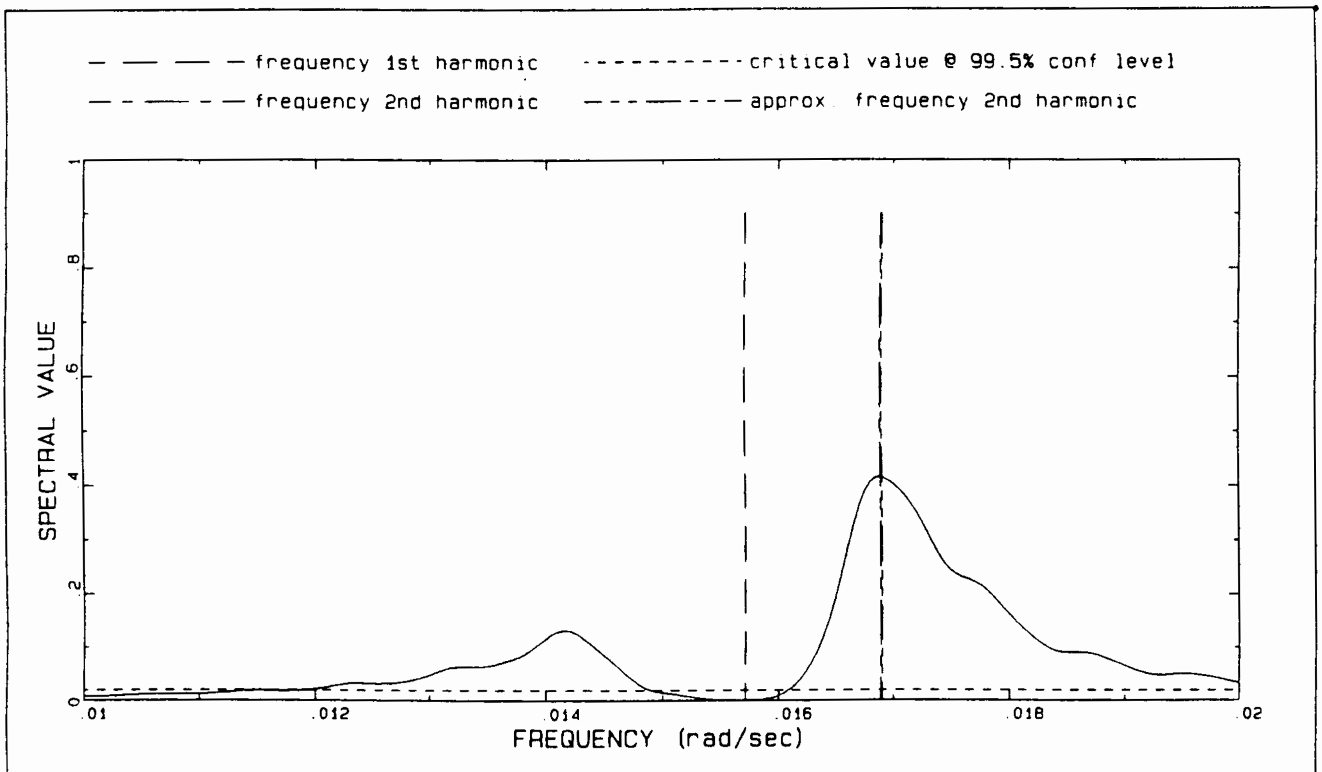
POWER SPECTRUM AFTER ALLOWING FOR MONOPERIODICITY IN A SIMULATED BIPERIODIC GYRO-MOTION (T2=0.95T1)

FIGURE G10



24-JUN-91 17:00:00 POWER SPECTRUM AFTER ALLOWING FOR MONOPERIODICITY IN A SIMULATED BIPERIODIC GYRO-MOTION (T2=0.94T1)

FIGURE G11



24-JUN-91 16:55:57 POWER SPECTRUM AFTER ALLOWING FOR MONOPERIODICITY IN A SIMULATED BIPERIODIC GYRO-MOTION (T2=0.93T1)

FIGURE G12

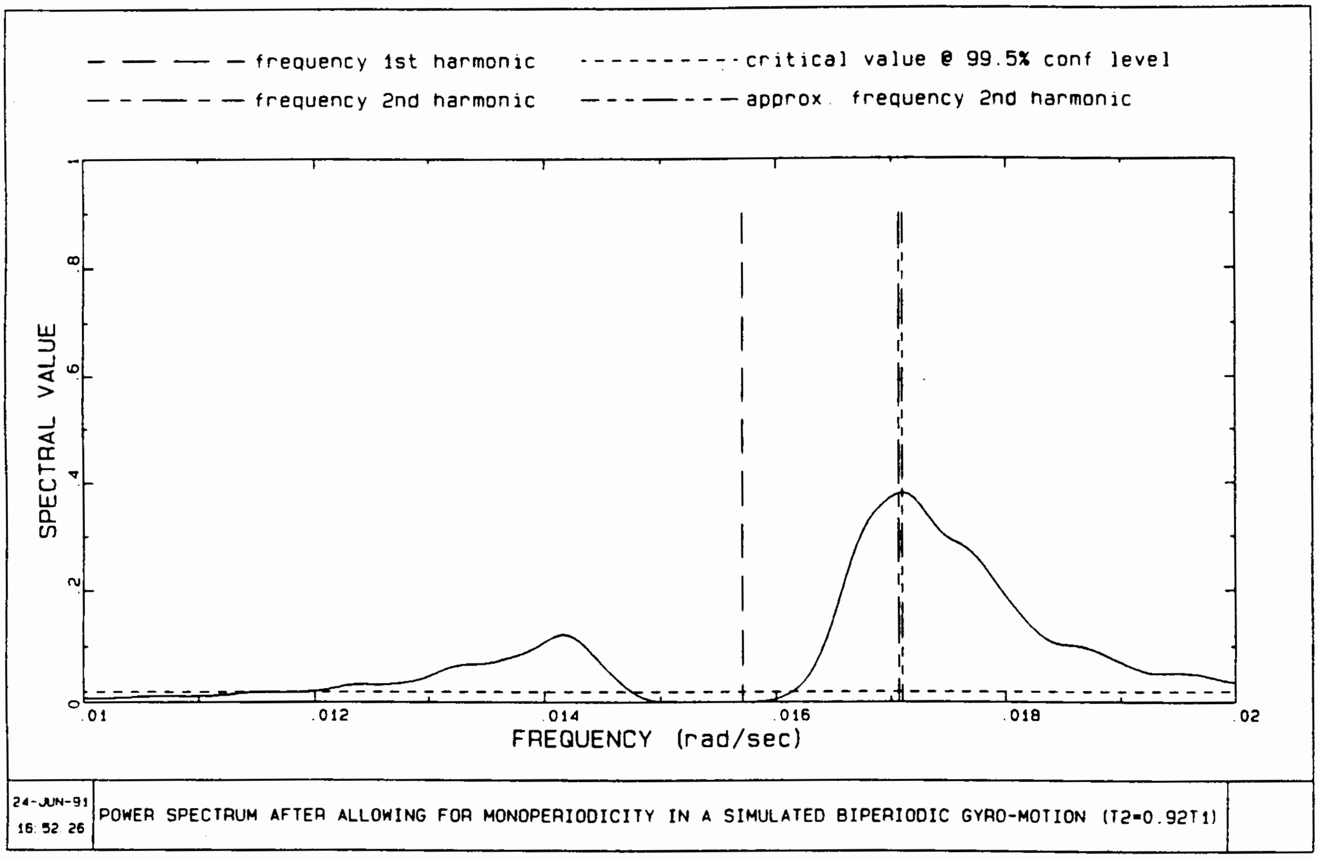


FIGURE G13

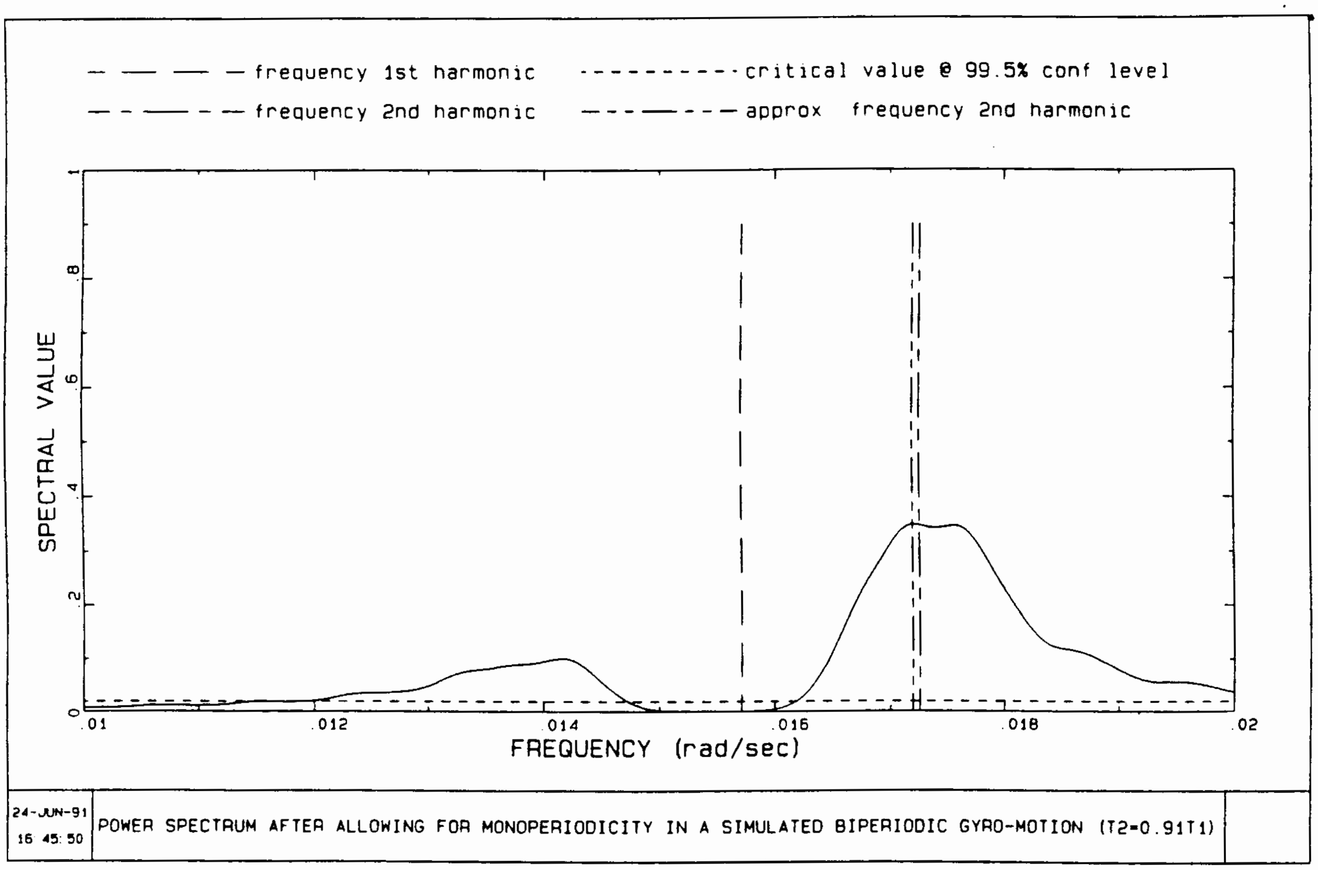
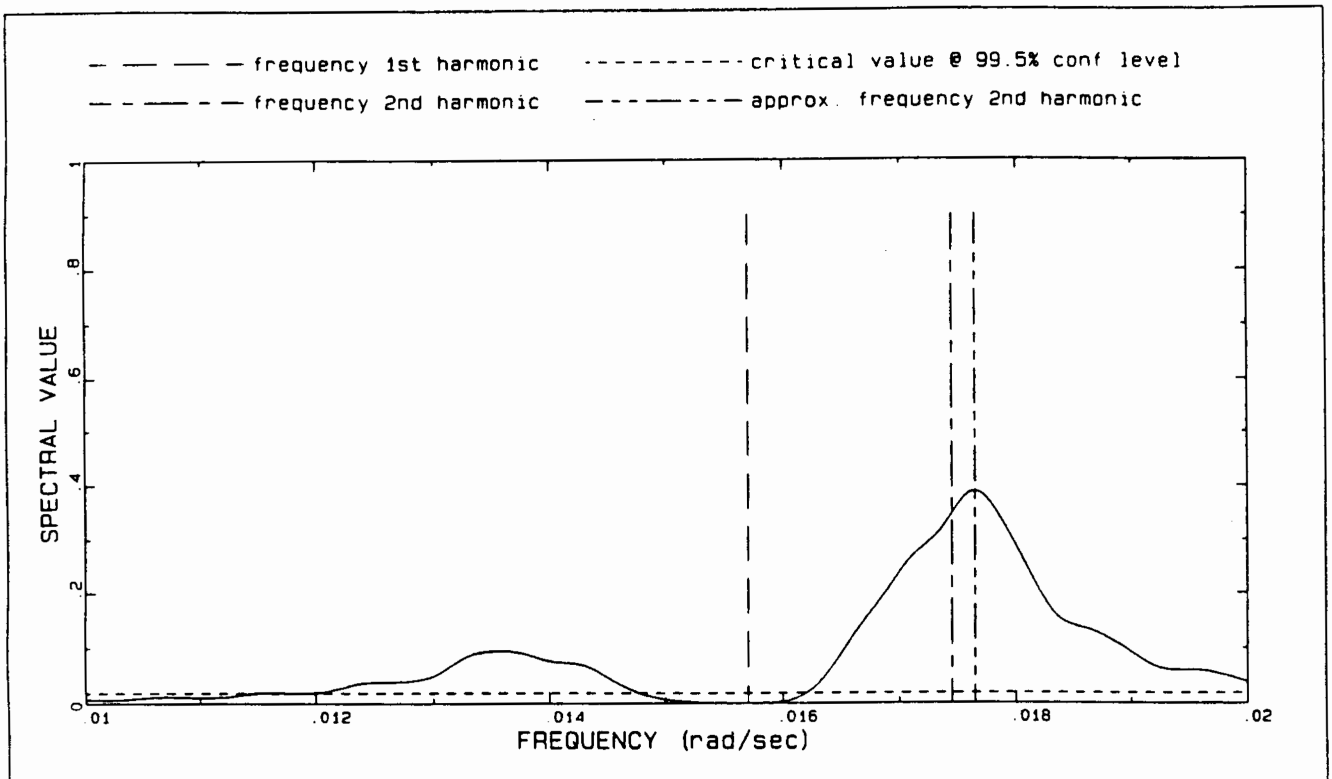
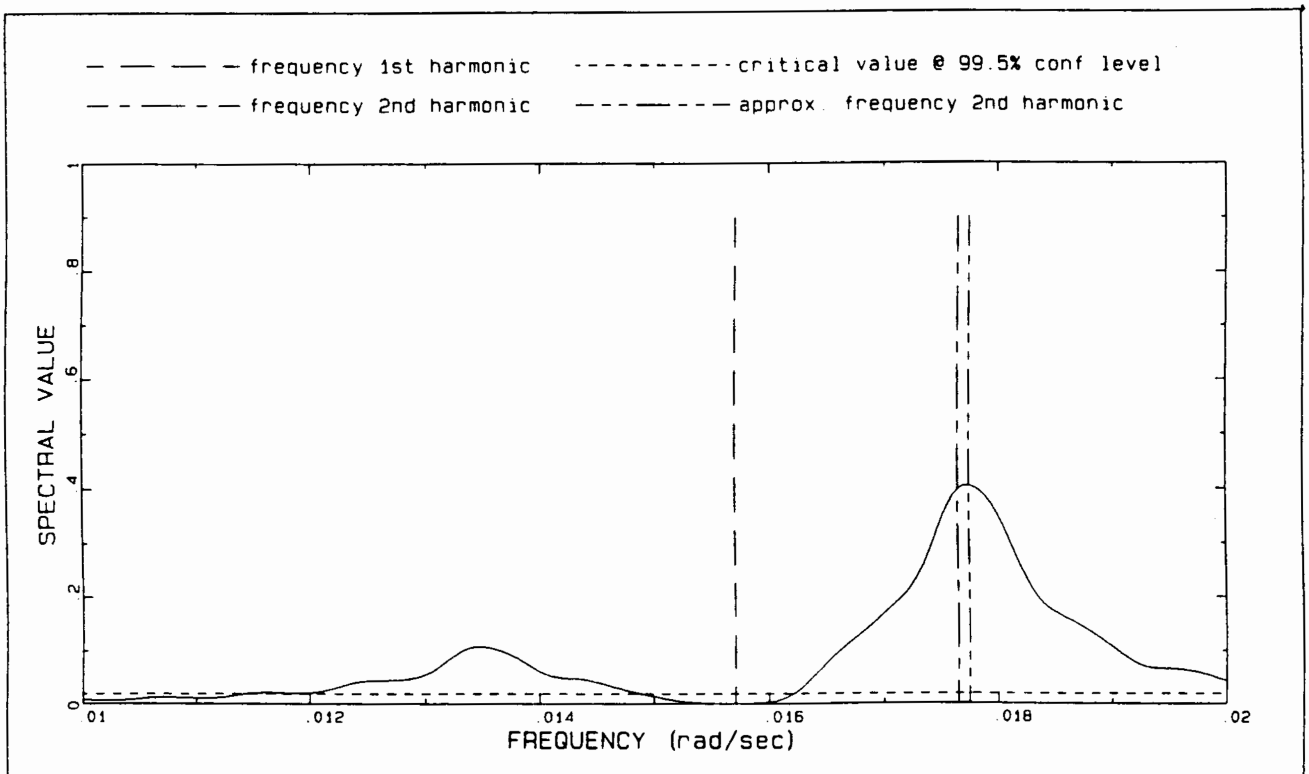


FIGURE G14



24-JUN-91 16:41:32 POWER SPECTRUM AFTER ALLOWING FOR MONOPERIODICITY IN A SIMULATED BIPERIODIC GYRO-MOTION (T2=0.90T1)

FIGURE G15



24-JUN-91 16:35:47 POWER SPECTRUM AFTER ALLOWING FOR MONOPERIODICITY IN A SIMULATED BIPERIODIC GYRO-MOTION (T2=0.89T1)

FIGURE G16

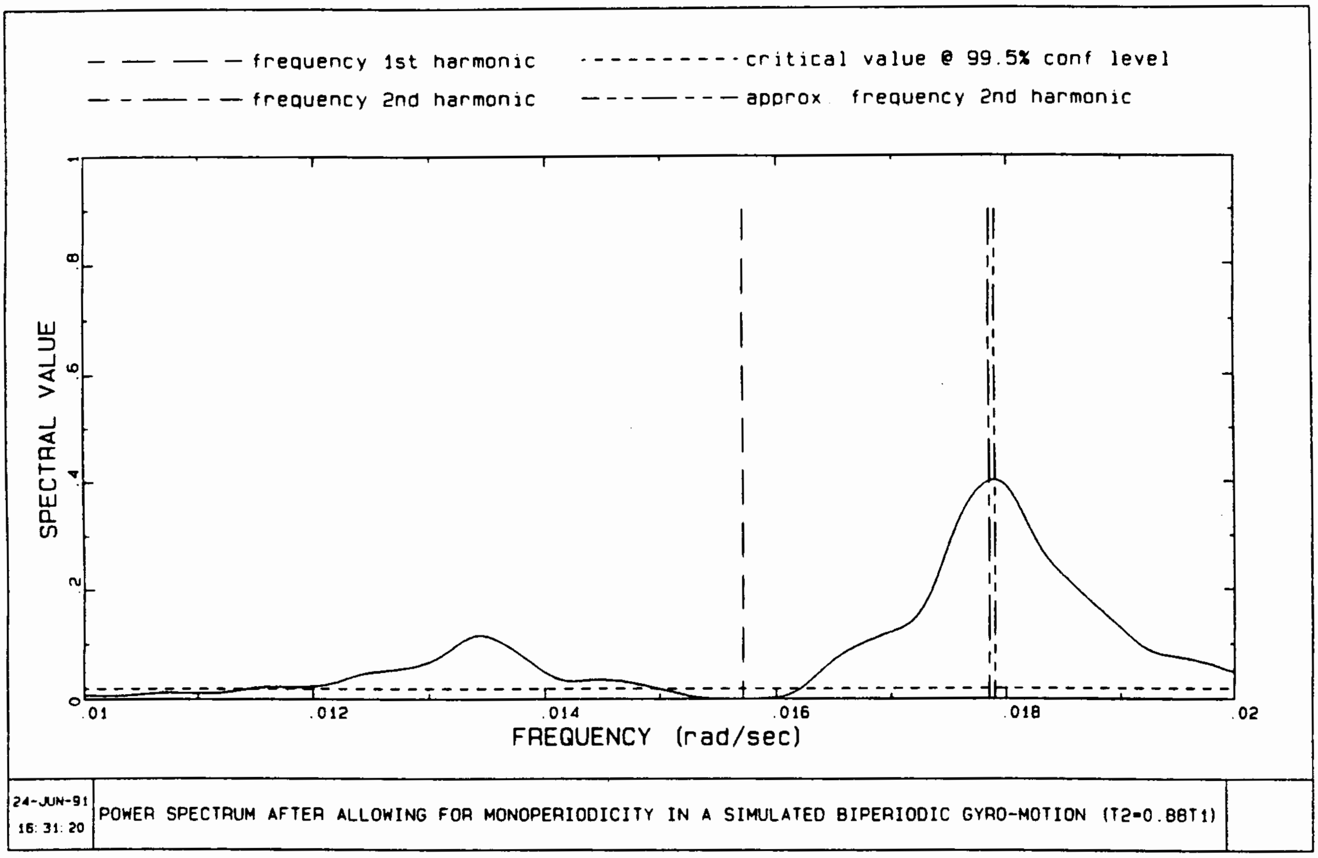


FIGURE G17

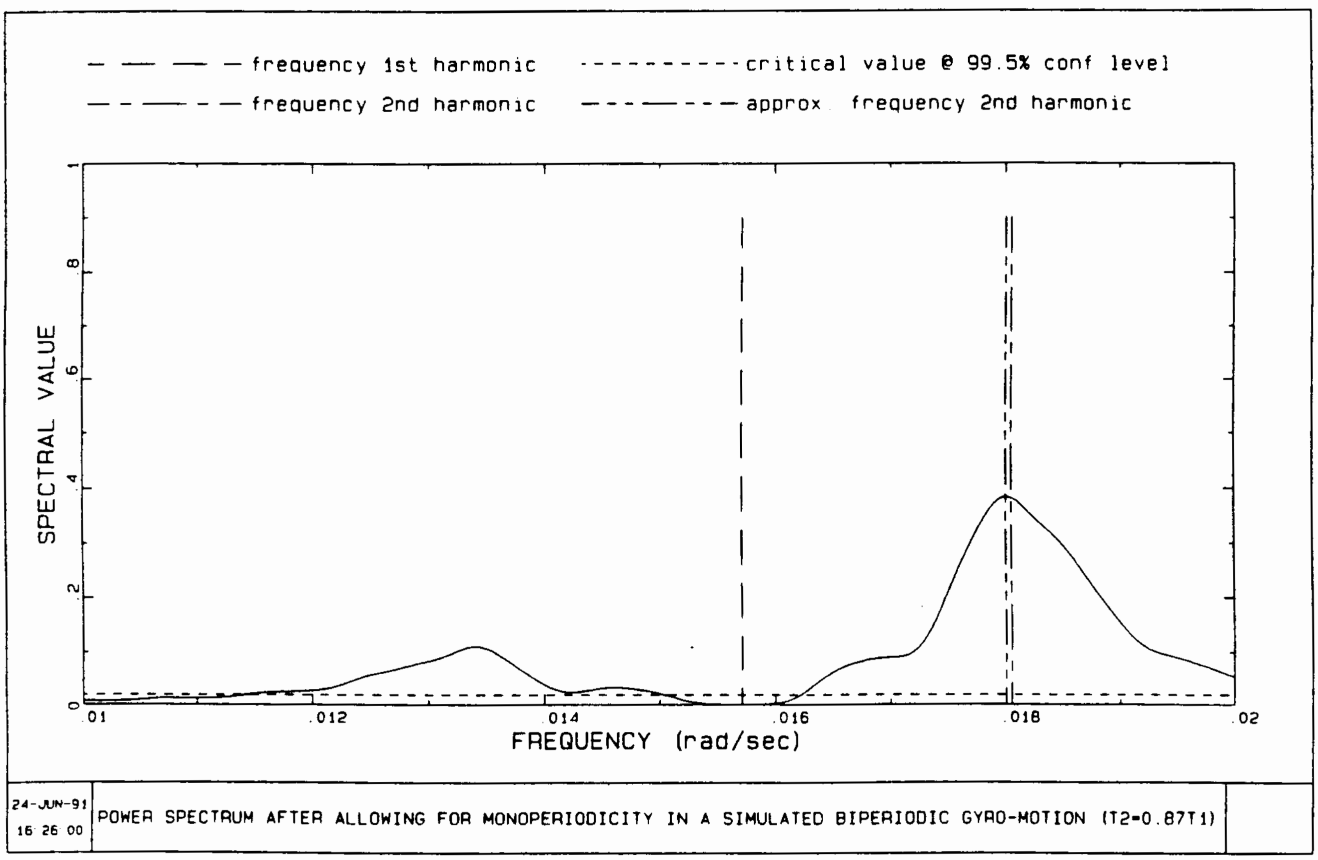


FIGURE G18

APPENDIX H

<u>PROGRAM</u>	:	SPECTRAL
<u>TYPE</u>	:	MAIN PROGRAM WITH SUBROUTINES
<u>LANGUAGE</u>	:	FORTRAN 5.5 (NB - NOT STANDARD FORTRAN)
<u>INSTALLATION</u>	:	VAX 6000-330, WITH 48 Mb RAM AND 3.6 Gb DISK STORAGE
<u>RUNNING SPEED</u>	:	TO CALCULATE 100 SPECTRAL VALUES IN EACH SPECTRUM, AND USING 571 OBSERVATIONS, THE CPU TIME TAKEN WAS : SPECTRUM 1 (no allowances made) = 10.01 sec SPECTRUM 2 (allowing for mono-periodicity) = 49.50 sec SPECTRUM 3 (allowing for bi-periodicity) = 67.64 sec SPECTRUM 4 (allowing for tri-periodicity) = 87.66 sec TO DETERMINE THE n-PERIODICAL MODEL, THE CPU TIME TAKEN FOR EACH ITERATION WAS : 1-PERIODICAL MODEL = 0.29 sec 2-PERIODICAL MODEL = 0.74 sec 3-PERIODICAL MODEL = 1.37 sec 4-PERIODICAL MODEL = 2.15 sec

```
      PROGRAM SPECTRAL
C     *-------------------------------------------------------------------*
C     *                                                                   *
C     * FUNCTION:                                                         *
C     * ========                                                          *
C     * THE VARIOUS HARMONICS CONTAINED IN A MULTIPERIOD OSCILLATION SUCH AS *
C     * A GYROSCOPIC MOTION ARE MAPPED BY USING LEAST SQUARES SPECTRAL ANA-  *
C     * LYSIS.                                                            *
C     * A SEQUENTIAL ADJUSTMENT IS ALSO POSSIBLE, WHERE THE NUMBER OF OBSER- *
C     * VATIONS ARE INCREASED IN STEPS OF 30 (i.e. THE NUMBER OF OBS/CYCLE)  *
C     *                                                                   *
C     * INPUT:                                                            *
C     * =====                                                             *
C     *                                                                   *
C     * A TIME SERIES CONSISTING OF 571 TRANSIT TIMES (REPRESENTING 19     *
C     * CYCLES) AND THEIR ASSOCIATED AUXILIARY GRADUATION LINES (0 TO +/- 7 *
C     * INCLUSIVE)                                                        *
C     *                                                                   *
C     * OUTPUT:                                                           *
C     * ======                                                            *
C     *                                                                   *
C     * FIRST, A SPECTRUM FROM THE OBSERVED SIGNAL IS PLOTTED, FROM WHICH  *
C     * THE FREQUENCY CORRESPONDING TO THE HIGHEST PEAK IS CHOSEN TO REPRE- *
C     * SENT THE FREQUENCY OF THE OSCILLATION. A LEAST SQUARES MONOPERIODIC *
C     * MODEL IS THEN COMPUTED, AFTER WHICH THE KNOWN CONSTITUENTS ARE REMO- *
C     * VED FROM THE OBSERVATIONS BEFORE THE BIPERIODICAL MODEL CAN BE ESTAB-*
C     * LISHED.                                                           *
C     * THE SPECTRUM AFTER ALLOWING FOR MONOPERIODICITY IS COMPUTED NEXT TO *
C     * IDENTIFY THE SECOND HARMONIC. THE BIPERIODIC MODEL IS THEN COMPUTED. *
C     * THIS PROCESS IS REPEATED FOR ANY POSSIBLE HIGHER-ORDER HARMONICS,   *
C     * UNTIL NO MORE PEAKS APPEAR IN THE SPECTRUM.                        *
C     * NOTE- A '6' IN A READ OR WRITE STATEMENT DENOTES INPUT FROM THE KEY- *
C     *       BOARD OR OUTPUT TO THE SCREEN RESPECTIVELY                  *
C     *                                                                   *
C     *                                                                   *
C     *              written by Gert-Jan van Rijsewijk                    *
C     *                  University of Cape Town                          *
C     *                      October 1991                                 *
C     *                                                                   *
C     *                                                                   *
C     *-------------------------------------------------------------------*
C      F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      INTEGER          NOBS            ! P Total number of observations in 19
      PARAMETER        (NOBS=571)      !   cycles
      DOUBLE PRECISION A(NOBS,20)      ! L Design matrix
      DOUBLE PRECISION A1
      DOUBLE PRECISION A2
      DOUBLE PRECISION ATWA(20,20)     ! L The normal equations
      DOUBLE PRECISION ATWF(20)        ! L The normal equations
      DOUBLE PRECISION B(4)            ! L Initial half-amplitude (scale div)
      DOUBLE PRECISION CA              ! L Coefficient of sine-function (used
                                       !   to determine the phase angle
      DOUBLE PRECISION CB              ! L Coefficient of cosine-function (used
                                       !   to determine the phase angle
      DOUBLE PRECISION COL(20)         ! L Column vectors in A(,)
      DOUBLE PRECISION CRITVAL         ! L Critical value at 99.5% confidence
```

```
                                        !   level
      EXTERNAL          DEGTODMS        ! S Converts decimal degrees to degrees,
                                        !   minutes and seconds
      DOUBLE PRECISION DF               ! L Degrees of freedom
      INTEGER           DGR             ! L Integer degrees in phase angle
      DOUBLE PRECISION DSDT(NOBS)       ! L The partial derivative of the obser-
                                        !   vations with respect to time. DSDT(i)
                                        !   is the i-th element in the diagonal
                                        !   B-matrix, used to determine QUNOCOR()
      DOUBLE PRECISION F(NOBS)          ! L Misclosure vector
      DOUBLE PRECISION FLAT(4)          ! L Damping factor (1/sec)
      DOUBLE PRECISION FREQ(4)          ! L Frequency (rad/sec)
      DOUBLE PRECISION FRINCREM         ! L Frequency increment used in the spec-
                                        !   trum
      DOUBLE PRECISION GAMMA(4)         ! L Phase angle (radians)
      DOUBLE PRECISION GMRAD            ! L Phase angle in radians (in first
                                        !   quadrant)
      EXTERNAL          HARMLIST        ! S Prints the results of the NPER harmo-
                                        !   nics to the output file in subdirec-
                                        !   tory ANSWERS
      EXTERNAL          INVERT          ! S Inverts matrix ATWA, dimension NUN
      CHARACTER*1       IPLT(0:56)      ! L Vector used to plot the spectrum. It
                                        !   contains a '*' to indicate the spec-
                                        !   tral value, and a ':' to indicate
                                        !   CRITVAL
      DOUBLE PRECISION LASTFR           ! L Last frequency in the spectrum
      INTEGER           LASTOB          ! L The total number of observations in
                                        !   the observation file
      CHARACTER*40      LCRIT           ! L File in subdirectory GRAPH called
                                        !   'CRIT', from which CRITVAL can be
                                        !   plotted if neccessary
      CHARACTER*40      LPLOT           ! L Output file in subdirectory GRAPH
                                        !   containing 2 vectors: i) all the
                                        !   frequencies in the spectrum and ii)
                                        !   their corresponding spectral values.
                                        !   These points can also be plotted
      INTEGER           MIN             ! L Integer minutes in phase angle
      INTEGER           NITER           ! L Number of iterations
      INTEGER           NLASTPER        ! L Number of harmonics to be mapped
      INTEGER           NO(NOBS)        ! L Vector of sequentially-numbered ob-
                                        !   servations in OBSSET
      INTEGER           NPER            ! L Counter from 1 to NLASTPER, i.e. it
                                        !   is the number of periodicities to be
                                        !   solved in the model
      INTEGER           NR              ! L The number of an observation
      INTEGER           NUN             ! L Number of unknowns
      CHARACTER*40      OBSSET          ! L Name of observation file in subdirec-
                                        !   tory OBSFILES
      DOUBLE PRECISION OM               ! L First frequency in the spectrum, in-
                                        !   cremented by FRINCREM until LASTFR
                                        !   has been reached
      DOUBLE PRECISION PERIOD           ! L Period of the oscillation (in sec)
      DOUBLE PRECISION PI               ! L Number of radians in 180 degrees
      DOUBLE PRECISION QUNOCOR(NOBS)! L Quasiweights, assuming no correlation
                                        !   between observations, i.e. the matrix
                                        !   is diagonal
      CHARACTER*1       RESL            ! L =Y/y if list of residuals required
```

```
                                        !    (also writes all large residuals
                                        !     > 0.2 sec to the screen, and puts
                                        !     an asterisk next to residuals
                                        !     > 0.3 sec)
                                        !   =N/n otherwise
      CHARACTER*40      RESULTS         ! L Name of output file
      DOUBLE PRECISION RTWR             ! L The variance factor for the (n+1) pe-
                                        !   riodical model. (used to calculate
                                        !   SPVAL if systematic noise is present)
      DOUBLE PRECISION S(NOBS)          ! L Vector of scale readings in OBSSET
      DOUBLE PRECISION SCALEDIV         ! L Units of scale division for an obser-
                                        !   vation
      INTEGER           SEC             ! L Integer seconds in phase angle
      CHARACTER*1       SEQ             ! L =Y/y if sequential adjustment
                                        !   =N/n otherwise
      EXTERNAL          SEQADJUST       ! S Starts off by solving a monoperiodic
                                        !   model from the first cycle (31 obser-
                                        !   vations) only, then adds a cycle at a
                                        !   time and adjusts the observations
                                        !   sequentially until all 571 observa-
                                        !   tions are included
      INTEGER           SEQL            ! L The number of the last observation in
                                        !   the last cycle
      EXTERNAL          SETZERO         ! S Sets all the elements in the matrices
                                        !   ATWA(,) and ATWF() to zero
      DOUBLE PRECISION SGMA             ! L A posteriori std deviation of FREQ()
      EXTERNAL          SOLSPVAL        ! S Calculates RtWR needed in SPVAL
      EXTERNAL          SOLVEX          ! S Solves the normal equations
      EXTERNAL          SOLVMODEL       ! S Solves the parameters for the n-
                                        !   periodical model
      INTEGER           SPCGRAPH        ! L The number of periods in the spectrum
                                        !   output to subdirectory GRAPH for sub-
                                        !   sequent plotting on A4 paper
      DOUBLE PRECISION SPVAL            ! L Normalised spectral value for fre-
                                        !   quency OM
      DOUBLE PRECISION STAX             ! L Product of S() transpose, A(,) and
                                        !   X()
      DOUBLE PRECISION STDEV            ! L A posteriori standard deviation
      DOUBLE PRECISION STS              ! L The sum of the squares of S()
      DOUBLE PRECISION T(NOBS)          ! L Vector of times in OBSSET
      DOUBLE PRECISION THETAo           ! L Nullposition of the oscillation
      DOUBLE PRECISION TIME             ! L The time for an observation
      INTEGER           TOTSPVLS        ! L Number of spectral values in spectrum
      DOUBLE PRECISION V(NOBS)          ! L Vector of residuals
      DOUBLE PRECISION VTWV             ! L The variance factor from the n perio-
                                        !   dical model
      INTEGER           WTCHOICE        ! L = 1 if unit weights are used
                                        !   = 2 if an approximate weightmodel is
                                        !       used
      DOUBLE PRECISION WTNOCOR(NOBS)    ! L Weightmatrix, assuming no correlation
                                        !   between observations, i.e. the matrix
                                        !   is diagonal
      DOUBLE PRECISION X(20)            ! L The vector of unknowns

1000  FORMAT( /, 3X, 'NAME OF OBSERVATION FILE ? ', $ )
1010  FORMAT( A40 )
1020  FORMAT( 3X, 'NAME OF OUTPUT FILE ? ', $ )
```

```
1030  FORMAT( I5, 1X, F13.6, 2X, F13.6 )
1040  FORMAT( /, 3X, 'NUMBER OF OBSERVATIONS IN FILE : ', I5 )
1050  FORMAT( 3X, 'NUMBER OF OBSERVATIONS USED    : ', I5 )
1060  FORMAT( //////, 3X, 'CHANGE "NOBS" IN PROGRAM !!!!', ////// )
1070  FORMAT( /, 3X, 'No. OF PERIODICITIES TO BE SOLVED = ', $ )
1080  FORMAT( I1 )
1090  FORMAT( /, 3X, 'WTMODEL TO BE USED : 1 = UNIT', /, 24X,
     &         '2 = [ SIN( WT - G ) ] ^2  ', $ )
1095  FORMAT( /, 3X, 'WHICH SPECTRUM MUST BE PLOTTED ON A4 ?'
     &         ' (enter 0,1,2,3, or 4) ', $ )
1100  FORMAT( 3X, 'IS A SEQUENTIAL ADJUSTMENT REQUIRED ? (Y/N) ', $ )
1110  FORMAT( A1 )
1120  FORMAT( /, 2X, 'ENTER APPROX. FREQ FOR FIRST CYCLE :  ', $ )
1130  FORMAT( F9.4 )
1140  FORMAT( /, 1X, 'NAME OF OBSERVATION FILE : ', A40 )
1150  FORMAT( 1X, 'PERIODS ALLOWED FOR IN THIS SPECTRUM : ', I1 )
1160  FORMAT( 3X, 'ENTER FIRST FREQ. IN SPECTRUM ', I1, ' : ', $ )
1170  FORMAT( 3X, 'ENTER LAST  FREQ. IN SPECTRUM ', I1, ' : ', $ )
1180  FORMAT( 3X, 'ENTER FREQ. INCREMENT IN THIS SPECTRUM : ', $ )
1190  FORMAT( 1X, 'CRITICAL VALUE @ 99.5% CONFIDENCE LEVEL = ',F7.4, / )
1200  FORMAT( 1X, F10.5, 1X, F7.4 )
1210  FORMAT( 1X, F10.5, 1X, F7.4, 1X, '|', 57A1, '|' )
1220  FORMAT( /, 2X, 'ENTER FREQUENCY TO BE USED FOR LS-FIT : ', $ )
1230  FORMAT( 3X, '(frequency', I1, ')o = ', F10.5 )
1240  FORMAT( 23X, 'Iterations completed : ', I3 )
1250  FORMAT( //, 1X, I1, '-PERIODIC MODEL :' )
1260  FORMAT( 1X, '=================' )
1270  FORMAT( 1X, 'No. OF ITERATIONS   :', I5 )
1420  FORMAT( 3X, 'IS A LIST OF RESIDUALS REQUIRED ? (Y/N) ', $ )
1430  FORMAT( /, 20X, 'the residuals (in seconds) are :', / )
1440  FORMAT( 10X, I5, 5X, 'residual = ', F10.5,' sec', 1X, '@ ', F8.1 )
1450  FORMAT( 1X, 'Large residual: ', I5, 5X, F10.5, ' sec', 2X, F13.5 )
1460  FORMAT( 1X, 'Large residual: ', I5, 4X, '*', F10.5, ' sec', 2X,
     &         F13.5 )

C     *** Set a value for PI and initialise the necessary functions      ***

      PI = 4D0 * ATAN( 1D0 )
      SEQL = 0
      StS = 0D0
      DO 100 I = 0,56
        IPLT( I ) = ' '
100   CONTINUE
      COL(1) = 1D0
      THETAo = 0D0
      I = 0

C     *** Enter the names of the input- and output- files                ***

      WRITE( 6, 1000 )
      READ ( 6, 1010 ) OBSSET
      WRITE( 6, 1020 )
      READ ( 6, 1010 ) RESULTS
      OBSSET = '[.OBSFILES]'//OBSSET

C     *** Go to subdirectory OBSFILES, read the values NR, TIME and      ***
C     *** SCALEDIV line by line from the appropiate file, and store these ***
```

```
C     *** values in the arrays NO(), T() and S() respectively            ***

      OPEN( UNIT = 10, FILE = OBSSET, STATUS = 'OLD' )
      LASTOB = 0
110   READ( 10, 1030, END = 120 ) NR, TIME, SCALEDIV
        LASTOB = LASTOB + 1
        GOTO 110
120   CONTINUE
      REWIND 10
130   READ( 10, 1030, END = 140 ) NR, TIME, SCALEDIV
        I = I + 1
        NO(I) = NR
        T(I) = TIME
        S(I) = SCALEDIV
        StS = StS + S(I)**2
        GOTO 130
140   CONTINUE
      CLOSE( 10 )

C     *** LASTOB, the number of observations in the file, has to be     ***
C     *** exactly the same as NOBS, otherwise the program will abort     ***

      WRITE( 6, 1040 ) LASTOB
      WRITE( 6, 1050 ) NOBS
      IF( NOBS .GT. LASTOB ) THEN
      WRITE( 6, 1060 )
      GOTO 320
      ENDIF

C     *** OM and SPVAL are stored in a file LPLOT in subdirectory GRAPH, ***
C     *** from which the spectrum can later be plotted. Another file     ***
C     *** LCRIT in the same subdirectory will enable a line to be plotted ***
C     *** on the spectrum to signify the statistical importance of the   ***
C     *** peaks in the spectrum. All the results will be sent to a file  ***
C     *** in subdirectory ANSWERS                                        ***

      LPLOT = '[.GRAPH]'//RESULTS
      LCRIT = '[.GRAPH]'//'CRIT'
      RESULTS = '[.ANSWERS]'//RESULTS
      OPEN( UNIT = 1, FILE = RESULTS , STATUS = 'NEW' )
      OPEN( UNIT = 2, FILE = LPLOT   , STATUS = 'NEW' )
      OPEN( UNIT = 4, FILE = LCRIT   , STATUS = 'NEW' )

C     *** Specify the number of periodicities to be solved and which     ***
C     *** weight model will be used                                      ***

      WRITE( 6, 1070 )
      READ ( 6, 1080 ) NLASTPER
      WRITE( 6, 1090 )
      READ ( 6, 1080 ) WTCHOICE
      WRITE( 6, 1095 )
      READ ( 6, 1080 ) SPCGRAPH

C     *** Solve for one harmonic at a time                               ***

      DO 300 NPER = 1, NLASTPER
      NUN = NPER + 2
```

```
C     *** The unknowns in X for the various spectra are :                 ***
C
C         | NPER | NUN |      UNKNOWNS          |
C         |------|-----|-----------------------|
C         |   1  |  3  | THETAo,a,b            |
C         |   2  |  4  | THETAo,B1,a,b         |
C         |   3  |  5  | THETAo,B1,B2,a,b      |
C         |   4  |  6  | THETAo,B1,B2,B3,a,b   |
C         |______|_____|_______________________|

        FLAT( NPER ) = 0D0
        FREQ( NPER ) = 0D0

C     *** Enter OM, LASTFR and FRINCREM so that TOTSPVLS can be estab-    ***
C     *** lished                                                          ***

      WRITE( 1, 1140 ) OBSSET
      WRITE( 1, 1150 ) NPER - 1
        WRITE( 6, 1160 ) NPER
        READ ( 6, 1130 ) OM
        WRITE( 6, 1170 ) NPER
        READ ( 6, 1130 ) LASTFR
        WRITE( 6, 1180 )
        READ ( 6, 1130 ) FRINCREM
        TOTSPVLS = INT( ( LASTFR - OM ) / FRINCREM )

C     *** Calculate the critical value to be used in the spectrum         ***

        DF = NOBS - NUN
        CRITVAL = 1D0 - 0.005D0**( 2D0 / DF )
        WRITE( 1, 1190 ) CRITVAL
        WRITE( 6, 1190 ) CRITVAL
        IF( NPER .EQ. SPCGRAPH ) WRITE( 4, 1200 ) OM, CRITVAL
        IF( NPER .EQ. SPCGRAPH ) WRITE( 4, 1200 ) LASTFR, CRITVAL

C     *** Calculate the spectrum to map the NPER-periodic model           ***

        DO 180 I = 1, TOTSPVLS + 1
          IF( I .GT. 1 ) OM = OM + FRINCREM
          CALL SETZERO( ATWA, ATWF )

C     *** For the spectrum of the observed signal a simple sinusoidal     ***
C     *** function can be used                                            ***

          IF( NPER .EQ. 1 ) THEN
            DO 170 J = 1, NOBS
              COL(2) = COS ( OM * T(J) )
              COL(3) = SIN ( OM * T(J) )
              DO 160 K = 1, NUN
                DO 150 L = 1, NUN
                  ATWA(K,L) = ATWA(K,L) + COL(K) * COL(L)
150             CONTINUE
              ATWF(K) = ATWF(K) + COL(K) * S(J)
160           CONTINUE
170         CONTINUE
            CALL SOLVEX( ATWA, ATWF, NUN, X )
```

```
          StAX = ATWF(1) * X(1) + ATWF(2) * X(2) + ATWF(3) * X(3)
          SPVAL = StAX / StS

C     *** If some constituents have already been identified, a general    ***
C     *** adjustment has to be carried out to calculate the spectral      ***
C     *** value                                                           ***

        ELSE IF( NPER .GT. 1 ) THEN
          CALL SOLSPVAL( B, COL, DSDT, F, FLAT, FREQ, GAMMA, NOBS,
     &                   NPER, NUN, OM, QUNOCOR, RtWR, S, T, THETAo,
     &                   WTCHOICE, WTNOCOR )
          SPVAL = 1D0 - RtWR / VtWV
        ENDIF

C     *** Store the spectral values in the appropiate output file         ***

        IPLT( NINT ( 56 * CRITVAL ) ) = ':'
        IPLT( NINT ( 56 * SPVAL ) ) = '*'
        WRITE( 1, 1210 ) OM, SPVAL, ( IPLT(J), J = 0, 56 )
        WRITE( 6, 1210 ) OM, SPVAL, ( IPLT(J), J = 0, 56 )
        IF( NPER .EQ. SPCGRAPH ) WRITE( 2, 1200 ) OM, SPVAL
        IPLT( NINT ( 56 * SPVAL ) ) = ' '
180   CONTINUE
      WRITE( 1, * )
      IPLT( NINT ( 56 * CRITVAL ) ) = ' '

C     *** Select a spectral frequency from the spectrum which corresponds ***
C     *** to the maximum SPVAL, or which lies close to it (the spectrum   ***
C     *** could be distorted due to mutual interference of peaks which    ***
C     *** lie close to one another)                                       ***

      WRITE( 6, 1220 )
      READ ( 6, 1130 ) FREQ(NPER)
      WRITE( 1, 1230 ) NPER, FREQ(NPER)
      NUN = NUN - 1 + NPER

C     *** The unknowns in X for the determination of the provisional      ***
C     *** coefficients, and the provisional phase angle(s) are :          ***
C
C       | NPER | NUN  |            UNKNOWNS                |
C       |------|------|------------------------------------|
C       |   1  |  3   | THETAo,a1,b1                       |
C       |   2  |  5   | THETAo,a1,b1,a2,b2                 |
C       |   3  |  7   | THETAo,a1,b1,a2,b2,a3,b3           |
C       |   4  |  9   | THETAo,a1,b1,a2,b2,a3,b3,a4,b4     |
C       |______|______|____________________________________|

190   CALL SETZERO( ATWA, ATWF )
      DO 230 I = 1, NOBS
        DO 200 J = 1, NPER
          COL( 2*J   ) = EXP(-FLAT(J) * T(I) ) * COS( FREQ(J) * T(I) )
          COL( 2*J+1 ) = EXP(-FLAT(J) * T(I) ) * SIN( FREQ(J) * T(I) )
200     CONTINUE
        DO 220 K = 1, NUN
          DO 210 M = K, NUN
            ATWA(K,M) = ATWA(K,M) + COL(K) * COL(M)
            ATWA(M,K) = ATWA(K,M)
```

```
210         CONTINUE
            ATWF(K) = ATWF(K) + COL(K) * S(I)
220       CONTINUE
230     CONTINUE
        CALL SOLVEX( ATWA, ATWF, NUN, X )

C     *** Calculate THETAo, the initial half-amplitude and the phase      ***
C     *** angle for each harmonic                                         ***

        THETAo = X(1)
        DO 240 I = 1, NPER
          CA =  X( 2*I   )
          CB =  X( 2*I+1 )
          B(I) = SQRT( CA**2 + CB**2 )
          GMRAD = ATAN( ABS( CB ) / ABS( CA ) )
          IF( CB .GT. 0 .AND. CA .GT. 0 ) GAMMA(I) = GMRAD
          IF( CB .GT. 0 .AND. CA .LT. 0 ) GAMMA(I) = PI - GMRAD
          IF( CB .LT. 0 .AND. CA .LT. 0 ) GAMMA(I) = PI + GMRAD
          IF( CB .LT. 0 .AND. CA .GT. 0 ) GAMMA(I) = 2D0 * PI - GMRAD
          IF( GAMMA(I) .GT. 2D0 * PI ) GAMMA(I) = GAMMA(I) - 2D0 * PI
240     CONTINUE
        NUN = NUN + 2 * NPER

C     *** The unknowns in X for the determination of the parameters in    ***
C     *** the NPER-periodic model :                                       ***
C
C        | NPER  | NUN  |             UNKNOWNS                 |
C        | ------|----- |------------------------------------  |
C        |   1   |  5   |  THETAo,FLAT1,FREQ1,B1,GAMMA1        |
C        |   2   |  9   |  THETAo,(FLAT1,FREQ1,B1,GAMMA1)x2    |
C        |   3   |  13  |  THETAo,(FLAT1,FREQ1,B1,GAMMA1)x3    |
C        |   4   |  17  |  THETAo,(FLAT1,FREQ1,B1,GAMMA1)x4    |
C        |_______|______|______________________________________|

C     *** If a sequential adjustment is required, update the parameters   ***
C     *** for the monoperiodic model obtained from the first cycle by ad- ***
C     *** ding a cycle at a time. Once all the observations are included, ***
C     *** calculate the next spectrum if a multi-period model is wanted   ***

        IF( NPER .EQ. 1 ) THEN
          WRITE( 6, 1100 )
          READ ( 6, 1110 ) SEQ
          IF( SEQ .EQ. 'y' .OR. SEQ .EQ. 'Y' ) THEN
            CALL SEQADJUST( A, ATWA, ATWF, B, DSDT, F, FLAT, FREQ,
     &                      GAMMA, NOBS, 1, QUNOCOR, S, T, THETAo,
     &                      V, VtWV, WTCHOICE, WTNOCOR, X )
            SEQL = 0
            GOTO 300
          ENDIF
        ENDIF

        NITER = 0

C     *** Determine the NPER-periodic model                               ***

250     CALL SOLVMODEL( A, ATWA, ATWF, B, DSDT, F, FLAT, FREQ, GAMMA,
     &                  NPER, NOBS, NUN, QUNOCOR, S, SEQL, T, THETAo,
```

```
    &                    WTCHOICE, WTNOCOR, X )
255    THETAo = THETAo + X(1)
       DO 260 I = 1, NPER
         FLAT(I) = FLAT(I) + X( 4*I-2 )
         FREQ(I) = FREQ(I) + X( 4*I-1 )
         B(I) = B(I) + X( 4*I )
         GAMMA(I) = GAMMA(I) + X( 4*I+1 )
260    CONTINUE

C     *** Write the number of iterations completed to the screen and con- ***
C     *** tinue iterating if |x(i)| > 1d-10                                ***

       NITER = NITER + 1
     WRITE( 6, 1240 ) NITER
       DO 270 I = 1, NUN
         IF( ABS( X(I) ) .GT. 1D-10 ) THEN

C     *** If less then five iterations have been completed, re-compute   ***
C     *** both ATWA and ATWF in the next iteration                        ***

           IF( NITER .LT. 5 ) THEN
             GOTO 250
           ELSE

C     *** If 5 or more iterations have been completed, re-compute ATWF    ***
C     *** only in the next iteration                                      ***

           CALL INVERT( ATWA, 20, NUN )
             DO 264 K = 1, NOBS
               DO 262 J = 1, NPER
                 IF( J .EQ. 1 ) F(K) = S(K) - THETAo
                 A1 = EXP( -FLAT(J) * T(K) )
                 A2 = COS( FREQ(J) * T(K) - GAMMA(J) )
                 F(K) = F(K) - B(J) * A1 * A2
262            CONTINUE
264          CONTINUE
             DO 268 J = 1, NUN
               ATWF(J) = 0D0
               DO 266 K = 1, NOBS
                 ATWF(J) = ATWF(J) + A(K,J) * QUNOCOR(K) * F(K)
266            CONTINUE
268          CONTINUE
             CALL SOLVEX( ATWA, ATWF, NUN, X )
             GOTO 255
           ENDIF
         ENDIF
270    CONTINUE

C     *** Evaluate the a posteriori standard deviation for the NPER-      ***
C     *** periodical model                                                ***

       VtWV = 0D0
       DO 280 I = 1, NOBS
         V(I) = F(I) / DSDT(I)
         VtWV = VtWV + V(I)**2 / WTNOCOR(I)
280    CONTINUE
       STDEV = SQRT( VtWV / ( NOBS - NUN ) )
```

```
C     *** Write the results to the output file, including the number of    ***
C     *** harmonics, the number of iterations completed, the number of     ***
C     *** observations used, which weightmodel was used, the standard de-  ***
C     *** viations of the unknowns and the angular equivalent of THETAo    ***
C     *** for the GAK1 at a latitude of Cape Town                          ***

        WRITE( 1, 1250 ) NPER
        WRITE( 1, 1260 )
        WRITE( 1, 1270 ) NITER
        NITER = 0
        CALL HARMLIST( ATWA, B, FLAT, FREQ, GAMMA, NPER, NOBS, STDEV,
     &                 THETAo, WTCHOICE )
300   CONTINUE

C     *** After the NLASTPER-periodic model has been solved, choose        ***
C     *** whether a list of residuals is needed                            ***

      WRITE( 6, 1420 )
      READ ( 6, 1110 ) RESL
      IF( RESL .EQ. 'n' .OR. RESL .EQ. 'N' ) GOTO 320
      WRITE( 1, 1430 )
      DO 310 I = 1, NOBS
        WRITE( 1, 1440 ) NO(I), V(I), S(I)
        IF( ABS( V(I) ) .GT. 0.2D0 .AND. ABS( V(I) ) .LT. 0.3D0 )
     &  WRITE( 6, 1450 ) NO(I), V(I), T(I)
        IF( ABS( V(I) ) .GT. 0.3D0 ) WRITE( 6, 1460 ) NO(I), V(I), T(I)
310   CONTINUE
320   END
```

```
      SUBROUTINE DEGTODMS( DEG, MD, MM, MS )
C     *-----------------------------------------------------------------------*
C     *                                                                       *
C     * FUNCTION:                                                             *
C     * ========                                                              *
C     * CONVERTS AN ANGLE FROM DECIMAL DEGREES TO DEGREES, MINUTES AND        *
C     * SECONDS                                                               *
C     *                                                                       *
C     *-----------------------------------------------------------------------*
C      F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      DOUBLE PRECISION    DEG           ! I  Angle in decimal degrees
      DOUBLE PRECISION    DMM           ! L  Decimal minutes in DEG
      DOUBLE PRECISION    DMS           ! L  Decimal seconds in DEG
      INTEGER             MD            ! O  Number of whole degrees in DEG
      INTEGER             MM            ! O  Number of whole minutes in DMM
      INTEGER             MS            ! O  Number of whole seconds in DMS

      MD  = INT( DEG )
      DMM = ABS( DEG - MD ) * 60D0
      MM  = INT( DMM )
      DMS = ( DMM - MM ) * 60D0
      MS  = NINT( DMS )
      RETURN
      END
```

```
      SUBROUTINE HARMLIST( ATWA, B, FLAT, FREQ, GAMMA, NPER, SEQL,
     &                     STDEV, THETAo, WTCHOICE )
C     *------------------------------------------------------------------------*
C     *                                                                        *
C     * FUNCTION:                                                              *
C     * ========                                                               *
C     * WRITES THE RESULTS OF THE NPER-PERIODICAL MODEL, USING SEQL OBSERVA-  *
C     * TIONS, TO THE OUTPUT FILE. (HARMLIST = "LIST OF HARMONIC(S)")          *
C     *                                                                        *
C     *------------------------------------------------------------------------*
C      F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      DOUBLE PRECISION ATWA(20,20)  ! I The normal equations
      DOUBLE PRECISION B(4)         ! I Initial half-amplitude (scale div)
      EXTERNAL         DEGTODMS     ! S Converts decimal degrees to degrees,
                                    !   minutes and seconds
      INTEGER          DGR          ! L Integer degrees in phase angle
      DOUBLE PRECISION FLAT(4)      ! I Damping factor (1/sec)
      DOUBLE PRECISION FREQ(4)      ! I Frequency (rad/sec)
      DOUBLE PRECISION GAMMA(4)     ! I Phase angle (radians)
      INTEGER          MIN          ! L Integer minutes in phase angle
      INTEGER          NPER         ! I The number of harmonics in the model
      DOUBLE PRECISION PERIOD       ! L Period of the oscillation (in sec)
      DOUBLE PRECISION PI           ! L Number of radians in 180 degrees
      INTEGER          SEC          ! L Integer seconds in phase angle
      INTEGER          SEQL         ! I The number of observations used
      DOUBLE PRECISION SGMA         ! L A posteriori std deviation of FREQ(1)
      DOUBLE PRECISION STDEV        ! I A posteriori standard deviation
      DOUBLE PRECISION THETAo       ! I Nullposition of the oscillation
      INTEGER          WTCHOICE     ! I = 1 if unit weights are used
                                    !   = 2 if an approximate weightmodel is
                                    !       used

1020  FORMAT( 1X, 'No. OF OBSERVATIONS :', I5 )
1030  FORMAT( 1X, 'WTMODEL USED : UNIT WEIGHT' )
1040  FORMAT( 1X, 'WTMODEL USED : WEIGHT = [ SIN( WT - G ) ] ^2' )
1050  FORMAT( //, 44X, 'Standard deviation', / )
1060  FORMAT( 4X, 'frequency', I1, ' (rad/sec) ', F14.9, 10X, E9.3 )
1070  FORMAT( 7X, 'period', I1, ' (sec) ', 8X, F8.3, 12X, E9.3 )
1080  FORMAT( 9X, 'f', I1, ' (1/sec)', 7X, E13.5, 10X, E9.3 )
1090  FORMAT( 9X, 'B', I1, ' (scale div.)', 2X, F11.4, 12X, E9.3 )
1100  FORMAT( 8X, 'gamma', I1, ' (radians)', F14.7, 11X, E9.3 )
1110  FORMAT( 15X, '(DMS)', 8X, I4, '.', I2, '.', I2 )
1120  FORMAT( /, 5X, 'THETAo (scale div.)', 3X, F9.4, 13X, E9.3 )
1130  FORMAT( 5X, 'THETAo (arcsecs) = ', F7.2, ' for latitude 33 57.' )
1140  FORMAT( /, 5X, 'A posteriori standard deviation =', F8.4, ' sec.')

      PI = 4D0 * ATAN( 1D0 )
      WRITE( 1, 1020 ) SEQL
      IF( WTCHOICE .EQ. 1 ) WRITE( 1, 1030 )
      IF( WTCHOICE .EQ. 2 ) WRITE( 1, 1040 )
      WRITE( 1, 1050 )
      DO 500 I = 1, NPER
        SGMA = STDEV * SQRT( ATWA( 4*I-1, 4*I-1 ) )
        WRITE( 1, 1060 ) I, FREQ(I), SGMA
        PERIOD = 2D0 * PI / FREQ(I)
        WRITE( 1, 1070 ) I, PERIOD, SGMA * PERIOD ** 2 / ( 2D0 * PI )
```

```
        WRITE( 1, 1080 ) I, FLAT(I),
     &                  STDEV * SQRT( ATWA( 4*I-2, 4*I-2 ) )
        WRITE( 1, 1090 ) I, B(I), STDEV * SQRT( ATWA( 4*I, 4*I ) )
        IF( GAMMA(I) .GT. 2D0 * PI ) GAMMA(I) = GAMMA(I) - 2D0 * PI
        WRITE( 1, 1100 ) I, GAMMA(I),
     &                  STDEV * SQRT( ATWA( 4*I+1, 4*I+1 ) )
        CALL DEGTODMS( GAMMA(I) * 180D0 / PI, DGR, MIN, SEC )
        WRITE( 1, 1110 ) DGR, MIN, SEC
500   CONTINUE
      WRITE( 1, 1120 ) THETAo, STDEV * SQRT( ATWA(1,1) )
      WRITE( 1, 1130 ) THETAo * 744.48D0
      WRITE( 1, 1140 ) STDEV
      RETURN
      END
```

```
      SUBROUTINE INVERT( A, NUN, N )
C     *------------------------------------------------------------------------*
C     *                                                                        *
C     * FUNCTION:                                                              *
C     * ========                                                               *
C     * INVERTS MATRIX A IN PLACE., USING THE CHOLESKI MATRIX INVERSION ROU-  *
C     * TINE.                                                                  *
C     * NUN = ROW DIMENSION OF MATRIX A, i.e. THE SIZE SPECIFIED IN THE BE-   *
C     *       GINNING OF THE MAIN PROGRAM                                      *
C     *   N = DIMENSION OF MATRIX TO BE INVERTED                               *
C     *                                                                        *
C     *------------------------------------------------------------------------*

      IMPLICIT REAL*8( A-H, O-Z )
      DIMENSION A( NUN, N )
      A(1,1) = SQRT( A(1,1) )
      DO 500 I = 2, N
500   A(I,1) = A(I,1) / A(1,1)
      DO 540 J = 2, N
        SUM = 0D0
        J1 = J - 1
        DO 510 K = 1, J1
510     SUM = SUM + A(J,K) * A(J,K)
        A(J,J) = SQRT( A(J,J) - SUM )
        IF( J .EQ. N ) GOTO 540
        J2 = J + 1
        DO 530 I = J2, N
          SUM = 0D0
          DO 520 K = 1, J1
520       SUM = SUM + A(I,K) * A(J,K)
530       A(I,J) = ( A(I,J) - SUM ) / A(J,J)
540   CONTINUE
      DO 550 I = 1, N
550   A(I,I) = 1D0 / A(I,I)
      N1 = N - 1
      DO 570 J = 1, N1
      J2 = J + 1
      DO 570 I = J2, N
      SUM = 0D0
      I1 = I - 1
      DO 560 K = J, I1
560   SUM = SUM + A(I,K) * A(K,J)
570   A(I,J) = -A(I,I) * SUM
      DO 620 J = 1, N
      IF( J .EQ. 1 ) GOTO 590
      J1 = J - 1
      DO 580 I = 1 ,J1
580   A(I,J) = A(J,I)
590   DO 610 I = J, N
      SUM = 0D0
      DO 600 K = I, N
600   SUM = SUM + A(K,I) * A(K,J)
610   A(I,J) = SUM
620   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SEQADJUST( A, ATWA, ATWF, B, DSDT, F, FLAT, FREQ,
     &                      GAMMA, NOBS, NPER, QUNOCOR, S, T, THETAo,
     &                      V, VtWV, WTCHOICE, WTNOCOR, X )
C     *----------------------------------------------------------------------*
C     *                                                                      *
C     * FUNCTION:                                                            *
C     * ========                                                             *
C     * AFTER THE 5 PARAMETERS OF THE MONOPERIODIC MODEL HAVE BEEN SOLVED    *
C     * FROM THE FIRST CYCLE (31 OBSERVATIONS), THE 30 OBSERVATIONS FROM     *
C     * EACH OF THE FOLLOWING CYCLES ARE ADDED SEQUENTIALLY. A CORRECTION TO *
C     * THE X-VECTOR IS EVALUATED AND ADDED TO THE PREVIOUS VECTOR TO OBTAIN *
C     * AN IMPROVED VALUE FOR X.                                             *
C     *                                                                      *
C     *----------------------------------------------------------------------*
C      F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      INTEGER          NOBS          ! I Total number of observations in 19
                                     !   cycles
      DOUBLE PRECISION A(NOBS,20)    ! I Design matrix
      DOUBLE PRECISION A1            ! L Intermediate value to calculate A(,),
                                     !   DSDT() and F()
      DOUBLE PRECISION A2            ! L Intermediate value to calculate A(,),
                                     !   DSDT() and F()
      DOUBLE PRECISION A3            ! L Intermediate value to calculate A(,)
                                     !   and DSDT()
      DOUBLE PRECISION ATWA(20,20)   ! I The normal equations
      DOUBLE PRECISION ATWF(20)      ! I The normal equations
      DOUBLE PRECISION B(4)          ! U Initial half-amplitude (scale div)
      INTEGER          CYCLE         ! L Number of cycles in the adjustment
      DOUBLE PRECISION DSDT(NOBS)    ! I The partial derivative of the obser-
                                     !   vations with respect to time. DSDT(i)
                                     !   is the i-th element in the diagonal
                                     !   B-matrix, used to determine QUNOCOR()
      DOUBLE PRECISION F(NOBS)       ! I Misclosure vector
      DOUBLE PRECISION FLAT(4)       ! U Damping factor (1/sec)
      DOUBLE PRECISION FREQ(4)       ! U Frequency (rad/sec)
      DOUBLE PRECISION GAMMA(4)      ! U Phase angle (radians)
      EXTERNAL         HARMLIST      ! S Prints the results of the NPER harmo-
                                     !   nics to the output file in subdirec-
                                     !   tory ANSWERS
      EXTERNAL         INVERT        ! S Inverts matrix ATWA, dimension NUN
      INTEGER          NPER          ! I The number of harmonics in the model,
                                     !   i.e. 1
      DOUBLE PRECISION QUNOCOR(NOBS)! I Quasiweights, assuming no correlation
                                     !   between observations, i.e. the matrix
                                     !   is diagonal
      DOUBLE PRECISION S(NOBS)       ! I Vector of scale readings in OBSSET
      INTEGER          SEQF          ! L The number of the first observation
                                     !   in the last cycle
      INTEGER          SEQL          ! L The number of the last observation in
                                     !   the last cycle
      EXTERNAL         SETZERO       ! S Sets all the elements in the matrices
                                     !   ATWA(,) and ATWF() to zero
      EXTERNAL         SOLVEX        ! S Solves the normal equations
      EXTERNAL         SOLVMODEL     ! S Solves the parameters for the mono-
                                     !   periodical model
      DOUBLE PRECISION STDEV         ! L A posteriori standard deviation
```

```
      DOUBLE PRECISION T(NOBS)        ! I Vector of times in OBSSET
      DOUBLE PRECISION THETAo         ! U Nullposition of the oscillation
      DOUBLE PRECISION V(NOBS)        ! I Vector of residuals
      DOUBLE PRECISION VTWV           ! O The variance factor for the monope-
                                      !   riodical model
      INTEGER          WTCHOICE       ! I = 1 if unit weights are used
                                      !   = 2 if an approximate weightmodel is
                                      !       used
      DOUBLE PRECISION WTNOCOR(NOBS)! I Weightmatrix, assuming no correlation
                                      !   between observations, i.e. the matrix
                                      !   is diagonal
      DOUBLE PRECISION X(20)          ! U The vector of unknowns

1000  FORMAT( //, 1X, 'SEQUENTIAL ADJUSTMENT :' )
1010  FORMAT( 1X, '======================' )
1020  FORMAT( 23X, 'Cycles in model : ', I3 )

      CYCLE = 0

C     *** For the first two cycles, ATWA and ATWF are fully re-computed    ***
C     *** in each iteration. When convergence has taken place, the a pos- ***
C     *** teriori standard deviation is computed                           ***

      DO 530 CYCLE = 1, 2
        IF( CYCLE .EQ. 1 ) SEQL = 31
        IF( CYCLE .GT. 1 ) SEQL = SEQL + 30
500     CALL SOLVMODEL( A, ATWA, ATWF, B, DSDT, F, FLAT, FREQ, GAMMA, 1,
     &                  NOBS, 5, QUNOCOR, S, SEQL, T, THETAo, WTCHOICE,
     &                  WTNOCOR, X )
        THETAo   = THETAo   + X(1)
        FLAT(1)  = FLAT(1)  + X(2)
        FREQ(1)  = FREQ(1)  + X(3)
        B(1)     = B(1)     + X(4)
        GAMMA(1) = GAMMA(1) + X(5)
        DO 510 I = 1, 5
          IF( ABS( X(I) ) .GT. 1D-10 ) GOTO 500
510     CONTINUE
        VtWV = 0D0
        DO 520 I = 1, SEQL
          V(I) = F(I) / DSDT(I)
          VtWV = VtWV + V(I)**2 / WTNOCOR(I)
520     CONTINUE
        STDEV = SQRT( VtWV / ( SEQL - 5 ) )
        WRITE( 1, 1000 )
        WRITE( 1, 1010 )
      WRITE( 6, 1020 ) CYCLE
        CALL HARMLIST( ATWA, B, FLAT, FREQ, GAMMA, NPER, SEQL, STDEV,
     &                 THETAo, WTCHOICE )
530   CONTINUE

C     *** If 3 or more cycles are included in the calculations, then the  ***
C     *** ATWA-matrix from the previous (cycle-1)-model is retained and   ***
C     *** only the contribution this last cycle makes to ATWA is computed ***
C     *** and a new value for ATWA obtained. ATWF is re-computed in its   ***
C     *** entirety. For the next iterations, only ATWF is calculated      ***

      DO 640 CYCLE = 3, 19
```

```
      CALL INVERT( ATWA, 20, 5 )
      SEQF = SEQL + 1
      SEQL = SEQF + 29
      DO 540  I = SEQF, SEQL
      A(I,1) = 1D0
        A1 = EXP( -FLAT(1) * T(I) )
        A2 = COS( FREQ(1) * T(I) - GAMMA(1) )
        A3 = SIN( FREQ(1) * T(I) - GAMMA(1) )
        A(I,2)  = -B(1) * T(I) * A1 * A2
        A(I,3)  = -B(1) * T(I) * A1 * A3
        A(I,4)  =  A1 * A2
        A(I,5)  =  B(1) * A1 * A3
        DSDT(I) = -B(1) * A1 * ( FLAT(1) * A2 + FREQ(1) * A3 )
540    CONTINUE
      DO 570 I = 1, 5
        DO 560 J = I, 5
          DO 550 K = SEQF,SEQL
            IF( WTCHOICE .EQ. 1 ) THEN
              WTNOCOR(K) = 1D0
            ELSE IF( WTCHOICE .EQ. 2 ) THEN
              WTNOCOR(K) = 1D0 / (SIN(FREQ(1) * T(K) - GAMMA(1) ))**2
            ENDIF
            QUNOCOR(K) = 1D0 / ( WTNOCOR(K) * DSDT(K) ** 2 )
            ATWA(I,J) = ATWA(I,J) + A(K,I) * QUNOCOR(K) * A(K,J)
550       CONTINUE
          ATWA(J,I) = ATWA(I,J)
560     CONTINUE
570   CONTINUE
580   DO 590 I = 1, SEQL
        A1 = EXP( -FLAT(1) * T(I) )
        A2 = COS( FREQ(1) * T(I) - GAMMA(1) )
        F(I) = S(I) - THETAo - B(1) * A1 * A2
590   CONTINUE
      DO 610 J = 1, 5
        ATWF(J) = 0D0
        DO 600 K = 1, SEQL
          ATWF(J) = ATWF(J) + A(K,J) * QUNOCOR(K) * F(K)
600     CONTINUE
610   CONTINUE
      CALL SOLVEX( ATWA, ATWF, 5, X )
      THETAo   = THETAo   + X(1)
      FLAT(1)  = FLAT(1)  + X(2)
      FREQ(1)  = FREQ(1)  + X(3)
      B(1)     = B(1)     + X(4)
      GAMMA(1) = GAMMA(1) + X(5)
      DO 620 I = 1, 5
        IF( ABS( X(I) ) .GT. 1D-10 ) THEN
          CALL INVERT( ATWA, 20, 5 )
          GOTO 580
        ENDIF
620   CONTINUE
      VtWV = 0D0
      DO 630 I = 1, SEQL
        V(I) = F(I) / DSDT(I)
        VtWV = VtWV + V(I)**2 / WTNOCOR(I)
630   CONTINUE
      STDEV = SQRT( VtWV / ( SEQL - 5 ) )
```

```
C     *** Write the results to the output file, including the number of   ***
C     *** observations used, which weightmodel was used, the standard de- ***
C     *** viations of the unknowns and the angular equivalent of THETAo   ***
C     *** for the GAK1 at a latitude of Cape Town.                        ***
C     *** Write the number of cycles completed to the screen              ***

        WRITE( 1, 1000 )
        WRITE( 1, 1010 )
        WRITE( 6, 1020 ) CYCLE
        CALL HARMLIST( ATWA, B, FLAT, FREQ, GAMMA, NPER, SEQL, STDEV,
     &                 THETAo, WTCHOICE )
640   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SETZERO (ATWA,ATWF)
C    *----------------------------------------------------------------------*
C    *                                                                      *
C    * FUNCTION:                                                            *
C    * ========                                                             *
C    * SETS ALL THE ELEMENTS IN MATRIX ATWA AND VECTOR ATWF TO ZERO         *
C    *                                                                      *
C    *----------------------------------------------------------------------*
C     F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      DOUBLE PRECISION    ATWA(20,20)    ! I/O The normal equations
      DOUBLE PRECISION    ATWF(20)       ! I/O The normal equations
      DO 710 I = 1, 20
        DO 700 J = 1, 20
          ATWA(I,J) = 0D0
700     CONTINUE
        ATWF(I) = 0D0
710   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SOLSPVAL( B, COL, DSDT, F, FLAT, FREQ, GAMMA, NOBS,
     &                     NPER, NUN, OM, QUNOCOR, RtWR, S, T, THETAo,
     &                     WTCHOICE, WTNOCOR )
C    *-------------------------------------------------------------------*
C    *                                                                   *
C    * FUNCTION:                                                         *
C    * ========                                                          *
C    * EVALUATES A SPECTRAL VALUE FOR FREQUENCY OM IN THE LEAST SQUARES  *
C    * SPECTRUM, TAKING SYSTEMATIC NOISE INTO ACCOUNT                    *
C    *                                                                   *
C    *-------------------------------------------------------------------*
C     F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      INTEGER          NOBS            ! I Total number of observations in 19
                                       !   cycles
      DOUBLE PRECISION A1              ! L Exponential function
      DOUBLE PRECISION ATWA(20,20)     ! L The normal equations
      DOUBLE PRECISION ATWF(20)        ! L The normal equations
      DOUBLE PRECISION B(4)            ! I Initial half-amplitude (scale div)
      DOUBLE PRECISION COL(20)         ! U Column vectors in A(,)
      DOUBLE PRECISION DSDT(NOBS)      ! I The partial derivative of the obser-
                                       !   vations with respect to time. DSDT(i)
                                       !   is the i-th element in the diagonal
                                       !   B-matrix, used to determine QUNOCOR()
      DOUBLE PRECISION F(NOBS)         ! I Misclosure vector
      DOUBLE PRECISION FLAT(4)         ! I Damping factor (1/sec)
      DOUBLE PRECISION FREQ(4)         ! I Frequency (rad/sec)
      DOUBLE PRECISION GAMMA(4)        ! I Phase angle (radians)
      DOUBLE PRECISION HK              ! L Product of matrix A and vector X,
                                       !   evaluated one observation equation at
                                       !   a time
      INTEGER          NPER            ! I Counter from 1 to NLASTPER, i.e. it
                                       !   is the number of periodicities to be
                                       !   solved in the model
      INTEGER          NUN             ! I Number of unknowns
      DOUBLE PRECISION OM              ! I First frequency in the spectrum, in-
                                       !   cremented by FRINCREM until LASTFR
                                       !   has been reached
      DOUBLE PRECISION QUNOCOR(NOBS)! I Quasiweights, assuming no correlation
                                       !   between observations, i.e. the matrix
                                       !   is diagonal
      DOUBLE PRECISION RTWR            ! O The variance factor for the (n+1) pe-
                                       !   riodical model. (used to calculate
                                       !   SPVAL if systematic noise is present)
      DOUBLE PRECISION S(NOBS)         ! I Vector of scale readings in OBSSET
      EXTERNAL         SETZERO         ! S Sets all the elements in the matrices
                                       !   ATWA(,) and ATWF() to zero
      EXTERNAL         SOLVEX          ! S Solves the normal equations
      DOUBLE PRECISION T(NOBS)         ! I Vector of times in OBSSET
      DOUBLE PRECISION THETAo          ! I Nullposition of the oscillation
      DOUBLE PRECISION VI              ! L The residual for observation I
      INTEGER          WTCHOICE        ! I = 1 if unit weights are used
                                       !   = 2 if an approximate weightmodel is
                                       !       used
      DOUBLE PRECISION WTNOCOR(NOBS)! I Weightmatrix, assuming no correlation
                                       !   between observations, i.e. the matrix
                                       !   is diagonal
```

```
      DOUBLE PRECISION X(20)          ! L The vector of unknowns

      CALL SETZERO( ATWA, ATWF )
      DO 530 I = 1, NOBS
        DO 500 J =1, NPER - 1
          IF( J .EQ. 1 )  F(I) = S(I) - THETAo
          A1 = EXP( -FLAT(J) * T(I) )
          COL( J+1 ) = A1 * COS( FREQ(J) * T(I) - GAMMA(J) )
          F(I) = F(I) - B(J) * COL( J+1 )
500     CONTINUE
        COL( NUN-1 ) = COS( OM * T(I) )
        COL( NUN   ) = SIN( OM * T(I) )
        IF( WTCHOICE .EQ. 1 ) THEN
          WTNOCOR(I) = 1D0
        ELSE IF( WTCHOICE .EQ. 2 ) THEN
          WTNOCOR(I) = 1D0 / ( SIN( FREQ(1) * T(I) - GAMMA(1) ) ) ** 2
        ENDIF
        QUNOCOR(I) = 1D0 / ( WTNOCOR(I) * DSDT(I) ** 2 )
        DO 520 K = 1, NUN
          DO 510 M = K, NUN
            ATWA(K,M) = ATWA(K,M) + COL(K) * COL(M) * QUNOCOR(I)
            ATWA(M,K) = ATWA(K,M)
510       CONTINUE
          ATWF(K) = ATWF(K) + COL(K) * F(I) * QUNOCOR(I)
520     CONTINUE
530   CONTINUE
      CALL SOLVEX( ATWA, ATWF, NUN, X )
      RtWR = 0D0
      DO 560 I = 1, NOBS
      COL( NUN-1 ) = COS( OM * T(I) )
      COL( NUN   ) = SIN( OM * T(I) )
      DO 540 J = 1, NPER - 1
          A1 = EXP( -FLAT(J) * T(I) )
          COL( J+1 ) = A1 * COS( FREQ(J) * T(I) - GAMMA(J) )
540     CONTINUE
        HK = 0D0
        DO 550 J = 1, NUN
        HK = HK + COL(J) * X(J)
550     CONTINUE
        VI = ( F(I) - HK ) / DSDT(I)
        RtWR = RtWR + VI**2 / WTNOCOR(I)
560   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SOLVEX( ATWA, ATWF, NUN, X )
C    *-------------------------------------------------------------------*
C    *                                                                   *
C    * FUNCTION:                                                         *
C    * ========                                                          *
C    * DETERMINES THE SOLUTION TO THE NORMAL EQUATIONS                   *
C    *                                                                   *
C    *-------------------------------------------------------------------*
C     F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      DOUBLE PRECISION ATWA(20,20)  ! I The normal equations
      DOUBLE PRECISION ATWF(20)     ! I The normal equations
      EXTERNAL         INVERT       ! S Inverts matrix ATWA, dimension NUN
      INTEGER          NUN          ! I Number of unknowns
      DOUBLE PRECISION X(20)        ! O The vector of unknowns

      CALL INVERT( ATWA, 20, NUN )
      DO 580 J = 1, NUN
        X(J) = 0D0
        DO 570 I = 1, NUN
          X(J) = X(J) + ATWA(J,I) * ATWF(I)
570     CONTINUE
580   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SOLVMODEL( A, ATWA, ATWF, B, DSDT, F, FLAT, FREQ,
     &                      GAMMA, NPER, NOBS, NUN, QUNOCOR, S, SEQL, T,
     &                      THETAo, WTCHOICE, WTNOCOR, X )
C     *-----------------------------------------------------------------------*
C     *                                                                       *
C     * FUNCTION:                                                             *
C     * ========                                                              *
C     * SOLVES THE PARAMETERS IN THE NPER-PERIODICAL MODEL                    *
C     *                                                                       *
C     *-----------------------------------------------------------------------*
C      F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      INTEGER          NOBS             ! I Total number of observations in 19
                                        !   cycles
      DOUBLE PRECISION A(NOBS,20)       ! I Design matrix
      DOUBLE PRECISION A1               ! L Intermediate value to calculate A(,),
                                        !   DSDT() and F()
      DOUBLE PRECISION A2               ! L Intermediate value to calculate A(,),
                                        !   DSDT() and F()
      DOUBLE PRECISION A3               ! L Intermediate value to calculate A(,)
                                        !   and DSDT()
      DOUBLE PRECISION ATWA(20,20)      ! I The normal equations
      DOUBLE PRECISION ATWF(20)         ! I The normal equations
      DOUBLE PRECISION B(4)             ! U Initial half-amplitude (scale div)
      DOUBLE PRECISION DSDT(NOBS)       ! I The partial derivative of the obser-
                                        !   vations with respect to time. DSDT(i)
                                        !   is the i-th element in the diagonal
                                        !   B-matrix, used to determine QUNOCOR()
      DOUBLE PRECISION F(NOBS)          ! U Misclosure vector
      DOUBLE PRECISION FLAT(4)          ! U Damping factor (1/sec)
      DOUBLE PRECISION FREQ(4)          ! U Frequency (rad/sec)
      DOUBLE PRECISION GAMMA(4)         ! U Phase angle (radians)
      INTEGER          NPER             ! I Counter from 1 to NLASTPER, i.e. it
                                        !   is the number of periodicities to be
                                        !   solved in the model
      INTEGER          NUN              ! I Number of unknowns
      INTEGER          OBSM             ! L Number of observations used in the
                                        !   model
      DOUBLE PRECISION QUNOCOR(NOBS)    ! U Quasiweights, assuming no correlation
                                        !   between observations, i.e. the matrix
                                        !   is diagonal
      DOUBLE PRECISION S(NOBS)          ! I Vector of scale readings in OBSSET
      INTEGER          SEQL             ! I The number of observations used
      EXTERNAL         SETZERO          ! S Sets all the elements in the matrices
                                        !   ATWA(,) and ATWF() to zero
      EXTERNAL         SOLVEX           ! S Solves the normal equations
      DOUBLE PRECISION T(NOBS)          ! I Vector of times in OBSSET
      DOUBLE PRECISION THETAo           ! U Nullposition of the oscillation
      INTEGER          WTCHOICE         ! I = 1 if unit weights are used
                                        !   = 2 if an approximate weightmodel is
                                        !       used
      DOUBLE PRECISION WTNOCOR(NOBS)    ! U Weightmatrix, assuming no correlation
                                        !   between observations, i.e. the matrix
                                        !   is diagonal
      DOUBLE PRECISION X(20)            ! O The vector of unknowns

      CALL SETZERO( ATWA, ATWF )
```

```
      IF( SEQL .EQ. 0 ) OBSM = NOBS
      IF( SEQL .NE. 0 ) OBSM = SEQL
      DO 600 I = 1, OBSM
        A(I,1) = 1D0
        DSDT(I) = 0D0
        DO 590 J = 1, NPER
          IF( J .EQ. 1 ) F(I) = S(I) - THETAo
          A1 = EXP( -FLAT(J) * T(I) )
          A2 = COS( FREQ(J) * T(I) - GAMMA(J) )
          A3 = SIN( FREQ(J) * T(I) - GAMMA(J) )
          A( I, 4*J-2 ) = -B(J) * T(I) * A1 * A2
          A( I, 4*J-1 ) = -B(J) * T(I) * A1 * A3
          A( I, 4*J   ) =  A1 * A2
          A( I, 4*J+1 ) =  B(J) * A1 * A3
          DSDT(I) = DSDT(I) - B(J) * A1 * ( FLAT(J) * A2 + FREQ(J) * A3)
          F(I) = F(I) - B(J) * A1 * A2
590     CONTINUE
600   CONTINUE
      DO 630 I = 1, NUN
        DO 620 J = I, NUN
          ATWF(J) = 0D0
        DO 610 K = 1, OBSM
            IF( WTCHOICE .EQ. 1 ) THEN
              WTNOCOR(K) = 1D0
            ELSE IF( WTCHOICE .EQ. 2 ) THEN
              WTNOCOR(K) = 1D0 / ( SIN(FREQ(1) * T(K) - GAMMA(1) ) )**2
            ENDIF
            QUNOCOR(K) = 1D0 / ( WTNOCOR(K) * DSDT(K) ** 2 )
            ATWA(I,J) = ATWA(I,J) + A(K,I) * QUNOCOR(K) * A(K,J)
            ATWF(J)   = ATWF(J)   + A(K,J) * QUNOCOR(K) * F(K)
610       CONTINUE
          ATWA(J,I) = ATWA(I,J)
620     CONTINUE
630   CONTINUE
      CALL SOLVEX( ATWA, ATWF, NUN, X )
      RETURN
      END
```

```
NAME OF OBSERVATION FILE : [.OBSFILES]L1
PERIODS ALLOWED FOR IN THIS SPECTRUM : 0
CRITICAL VALUE @ 99.5% CONFIDENCE LEVEL =  0.0185

    0.01000  0.0014 |*:
    0.01010  0.0008 |*:
    0.01020  0.0002 |*:
    0.01030  0.0000 |*:
    0.01040  0.0004 |*:
    0.01050  0.0011 |*:
    0.01060  0.0019 |*:
    0.01070  0.0023 |*:
    0.01080  0.0021 |*:
    0.01090  0.0014 |*:
    0.01100  0.0006 |*:
    0.01110  0.0000 |*:
    0.01120  0.0001 |*:
    0.01130  0.0008 |*:
    0.01140  0.0018 |*:
    0.01150  0.0026 |*:
    0.01160  0.0028 |*:
    0.01170  0.0022 |*:
    0.01180  0.0013 |*:
    0.01190  0.0003 |*:
    0.01200  0.0000 |*:
    0.01210  0.0005 |*:
    0.01220  0.0016 |*:
    0.01230  0.0028 |*:
    0.01240  0.0036 |*:
    0.01250  0.0034 |*:
    0.01260  0.0025 |*:
    0.01270  0.0011 |*:
    0.01280  0.0001 |*:
    0.01290  0.0001 |*:
    0.01300  0.0012 |*:
    0.01310  0.0029 |*:
    0.01320  0.0045 |*:
    0.01330  0.0052 |*:
    0.01340  0.0045 |*:
    0.01350  0.0028 |*:
    0.01360  0.0009 |*:
    0.01370  0.0000 |*:
    0.01380  0.0008 |*:
    0.01390  0.0031 |*:
    0.01400  0.0060 |*:
    0.01410  0.0083 |*:
    0.01420  0.0087 |*:
    0.01430  0.0069 |*:
    0.01440  0.0035 |*:
    0.01450  0.0006 |*:
    0.01460  0.0002 |*:
    0.01470  0.0032 |*:
    0.01480  0.0089 |*:
    0.01490  0.0155 | *
    0.01500  0.0205 | *
```

```
0.01510  0.0211 | *
0.01520  0.0155 | *
0.01530  0.0064 |*:
0.01540  0.0002 |*:
0.01550  0.0040 |*:
0.01560  0.0216 | *
0.01570  0.0562 | : *
0.01580  0.1129 | :    *
0.01590  0.2038 | :         *
0.01600  0.3521 | :                  *
0.01610  0.5889 | :                               *
0.01620  0.8862 | :                                                *
0.01630  0.9879 | :                                                     *
0.01640  0.7357 | :                                       *
0.01650  0.4342 | :                      *
0.01660  0.2389 | :           *
0.01670  0.1274 | :     *
0.01680  0.0638 | :  *
0.01690  0.0274 | :*
0.01700  0.0080 |*:
0.01710  0.0004 |*:
0.01720  0.0014 |*:
0.01730  0.0067 |*:
0.01740  0.0112 | *
0.01750  0.0120 | *
0.01760  0.0097 | *
0.01770  0.0061 |*:
0.01780  0.0028 |*:
0.01790  0.0006 |*:
0.01800  0.0000 |*:
0.01810  0.0007 |*:
0.01820  0.0020 |*:
0.01830  0.0030 |*:
0.01840  0.0031 |*:
0.01850  0.0024 |*:
0.01860  0.0014 |*:
0.01870  0.0005 |*:
0.01880  0.0001 |*:
0.01890  0.0001 |*:
0.01900  0.0005 |*:
0.01910  0.0009 |*:
0.01920  0.0011 |*:
0.01930  0.0011 |*:
0.01940  0.0008 |*:
0.01950  0.0004 |*:
0.01960  0.0001 |*:
0.01970  0.0000 |*:
0.01980  0.0001 |*:
0.01990  0.0003 |*:
0.02000  0.0004 |*:
```

(frequency1)o =   0.01630

```
1-PERIODIC MODEL :
==================
No. OF ITERATIONS    :    5
No. OF OBSERVATIONS :  571
WTMODEL USED : WEIGHT = [ SIN( WT - G ) ] ^2


                                                      Standard deviation

   frequency1 (rad/sec)     0.016276623            0.486E-07
      period1 (sec)           386.025              0.115E-02
        f1 (1/sec)          0.30954E-05            0.108E-06
        B1 (scale div.)       10.6103              0.488E-02
     gamma1 (radians)        1.5790871             0.206E-03
              (DMS)           90.28.30

    THETAo (scale div.)       0.0791               0.986E-03
    THETAo (arcsecs) =    58.90 for latitude 33 57.

    A posteriori standard deviation =  0.1379 sec.

NAME OF OBSERVATION FILE : [.OBSFILES]L1
PERIODS ALLOWED FOR IN THIS SPECTRUM : 1
CRITICAL VALUE @ 99.5% CONFIDENCE LEVEL =  0.0185

   0.01250  0.0098 | *
   0.01260  0.0087 |*:
   0.01270  0.0065 |*:
   0.01280  0.0050 |*:
   0.01290  0.0063 |*:
   0.01300  0.0117 | *
   0.01310  0.0202 | *
   0.01320  0.0290 | :*
   0.01330  0.0342 | :*
   0.01340  0.0335 | :*
   0.01350  0.0270 | :*
   0.01360  0.0178 | *
   0.01370  0.0102 | *
   0.01380  0.0085 |*:
   0.01390  0.0153 | *
   0.01400  0.0291 | :*
   0.01410  0.0439 | :*
   0.01420  0.0524 | : *
   0.01430  0.0497 | : *
   0.01440  0.0363 | :*
   0.01450  0.0184 | *
   0.01460  0.0077 |*:
   0.01470  0.0181 | *
   0.01480  0.0604 | : *
   0.01490  0.1326 | :     *
   0.01500  0.2186 | :          *
   0.01510  0.2989 | :               *
   0.01520  0.3603 | :                  *
   0.01530  0.3972 | :                    *
   0.01540  0.4058 | :                     *
   0.01550  0.3800 | :                   *
   0.01560  0.3135 | :                *
```

```
0.01570  0.2138 | :          *
0.01580  0.1132 | :    *
0.01590  0.0447 | : *
0.01600  0.0124 | *
0.01610  0.0020 |*:
0.01620  0.0001 |*:
0.01630  0.0000 |*:
0.01640  0.0004 |*:
0.01650  0.0048 |*:
0.01660  0.0223 | *
0.01670  0.0671 | :  *
0.01680  0.1472 | :      *
0.01690  0.2425 | :            *
0.01700  0.3131 | :                *
0.01710  0.3399 | :                 *
0.01720  0.3299 | :                *
0.01730  0.2949 | :               *
0.01740  0.2421 | :            *
```

(frequency2)o =   0.01540

```
2-PERIODIC MODEL :
==================
No. OF ITERATIONS   :   19
No. OF OBSERVATIONS :  571
WTMODEL USED : WEIGHT = [ SIN( WT - G ) ] ^2
```

		Standard deviation
frequency1 (rad/sec)	0.016276537	0.281E-06
period1 (sec)	386.027	0.666E-02
f1 (1/sec)	0.18618E-05	0.227E-06
B1 (scale div.)	10.5632	0.108E-01
gamma1 (radians)	1.5776291	0.108E-02
(DMS)	90.23.29	
frequency2 (rad/sec)	0.015665298	0.809E-04
period2 (sec)	401.089	0.207E+01
f2 (1/sec)	0.60017E-05	0.811E-04
B2 (scale div.)	0.0349	0.924E-02
gamma2 (radians)	0.9861817	0.423E+00
(DMS)	56.30.15	
THETAo (scale div.)	0.0792	0.715E-03

THETAo (arcsecs) =   58.98 for latitude 33 57.

A posteriori standard deviation = 0.0999 sec.

NAME OF OBSERVATION FILE : [.OBSFILES]L1
PERIODS ALLOWED FOR IN THIS SPECTRUM : 2
CRITICAL VALUE @ 99.5% CONFIDENCE LEVEL = 0.0185

```
0.01000  0.0012 |*:
0.01010  0.0010 |*:
0.01020  0.0007 |*:
0.01030  0.0005 |*:
```

```
0.01040  0.0006 |*:
0.01050  0.0011 |*:
0.01060  0.0020 |*:
0.01070  0.0032 |*:
0.01080  0.0043 |*:
0.01090  0.0049 |*:
0.01100  0.0048 |*:
0.01110  0.0039 |*:
0.01120  0.0024 |*:
0.01130  0.0010 |*:
0.01140  0.0002 |*:
0.01150  0.0004 |*:
0.01160  0.0017 |*:
0.01170  0.0038 |*:
0.01180  0.0058 |*:
0.01190  0.0072 |*:
0.01200  0.0073 |*:
0.01210  0.0061 |*:
0.01220  0.0041 |*:
0.01230  0.0019 |*:
0.01240  0.0005 |*:
0.01250  0.0005 |*:
0.01260  0.0021 |*:
0.01270  0.0049 |*:
0.01280  0.0078 |*:
0.01290  0.0098   *
0.01300  0.0103   *
0.01310  0.0092   *
0.01320  0.0070 |*:
0.01330  0.0046 |*:
0.01340  0.0034 |*:
0.01350  0.0044 |*:
0.01360  0.0081 |*:
0.01370  0.0143   *
0.01380  0.0219   *
0.01390  0.0294   :*
0.01400  0.0355   :*
0.01410  0.0394   :*
0.01420  0.0403   :*
0.01430  0.0379   :*
0.01440  0.0325   :*
0.01450  0.0254   *
0.01460  0.0182   *
0.01470  0.0120   *
0.01480  0.0073 |*:
0.01490  0.0042 |*:
0.01500  0.0021 |*:
0.01510  0.0009 |*:
0.01520  0.0003 |*:
0.01530  0.0001 |*:
0.01540  0.0000 |*:
0.01550  0.0000 |*:
0.01560  0.0000 |*:
0.01570  0.0000 |*:
0.01580  0.0000 |*:
0.01590  0.0000 |*:
0.01600  0.0000 |*:
```

```
0.01610   0.0000  |*:
0.01620   0.0000  |*:
0.01630   0.0000  |*:
0.01640   0.0000  |*:
0.01650   0.0000  |*:
0.01660   0.0000  |*:
0.01670   0.0000  |*:
0.01680   0.0000  |*:
0.01690   0.0000  |*:
0.01700   0.0001  |*:
0.01710   0.0001  |*:
0.01720   0.0002  |*:
0.01730   0.0006  |*:
0.01740   0.0015  |*:
0.01750   0.0031  |*:
0.01760   0.0057  |*:
0.01770   0.0094  | *
0.01780   0.0145  | *
0.01790   0.0211  | *
0.01800   0.0291  | :*
0.01810   0.0371  | :*
0.01820   0.0429  | :*
0.01830   0.0449  | :  *
0.01840   0.0433  | :*
0.01850   0.0391  | :*
0.01860   0.0336  | :*
0.01870   0.0273  | :*
0.01880   0.0206  | *
0.01890   0.0144  | *
0.01900   0.0095  | *
0.01910   0.0066  |*:
0.01920   0.0054  |*:
0.01930   0.0052  |*:
0.01940   0.0052  |*:
0.01950   0.0050  |*:
0.01960   0.0044  |*:
0.01970   0.0035  |*:
0.01980   0.0022  |*:
0.01990   0.0011  |*:
0.02000   0.0004  |*:
```

APPENDIX J

<u>PROGRAM</u>	:	SYNTH2
<u>TYPE</u>	:	MAIN PROGRAM WITH SUBROUTINES / FUNCTIONS
<u>LANGUAGE</u>	:	FORTRAN 5.5 (NB - NOT STANDARD FORTRAN)
<u>INSTALLATION</u>	:	VAX 6000-330, WITH 48 Mb RAM AND 3.6 Gb DISK STORAGE
<u>RUNNING SPEED</u>	:	CPU TIME TO GENERATE 761 VALUES REPRESENTING 19 CYCLES (i.e. $19\times38+1=723$ TRANSIT TIMES AND $19\times2=38$ TURNING POINTS) IS 9.63 sec

```
      PROGRAM SYNTH2
C     *-----------------------------------------------------------------------*
C     *                                                                       *
C     * FUNCTION:                                                             *
C     * ========                                                              *
C     * THIS PROGRAM SIMULATES A GYROMOTION AND COMPUTES THE TIME AND THETA-  *
C     * VALUE AT EACH TURNING POINT, AS WELL AS THE TRANSIT TIMES AT WHICH    *
C     * THE SYNTHESIZED MOTION CROSSES VARIOUS AUXILLIARY SCALE DIVISIONS.    *
C     * TWO HARMONICS HAVING PERIODS OF THE SAME ORDER OF MAGNITUDE SUCH      *
C     * THAT THEIR SUM DOES NOT PRODUCE A NOTICEABLE TREMOR ON THE OSCILLA-   *
C     * TION, ARE COMBINED.                                                   *
C     *                                                                       *
C     * OUTPUT:                                                               *
C     * ========                                                              *
C     * TRANSIT TIMES OF THE ULTIMATE AND PENULTIMATE AUXILLIARY SCALE DIVI-  *
C     * SIONS AND THE TIME AND THETA-VALUE AT EACH TURNING POINT ARE STORED   *
C     * IN A DATAFILE CALLED TPSYNTH2:DAT, WHILE ALL OTHER TRANSIT TIMES ARE *
C     * STORED IN AUXSYNTH2.DAT.                                              *
C     * FORMAT( I5, 1X, F13.6, 2X, F13.6 ) IS USED THROUGHOUT.                *
C     *                                                                       *
C     *                                                                       *
C     *                  written by Gert-Jan van Rijsewijk                    *
C     *                       University of Cape Town                         *
C     *                              May 1991                                 *
C     *                                                                       *
C     *                                                                       *
C     *-----------------------------------------------------------------------*
C      I=input,O=output,U=update,L=local,S=subroutine,F=function,P=parameter

      INTEGER              ATOT         ! P Total number of auxilliary scale
      PARAMETER           (ATOT=19)     !   graduation lines
      DOUBLE PRECISION     AUX(ATOT)    ! L Theta-values representing the aux-
                                        !   illiary scale divisions of a gyro-
                                        !   theodolite

      DATA AUX/ 9D0,   8D0,   7D0,   6D0,   5D0,   4D0,   3D0,
     &          2D0,   1D0,   0D0,  -1D0,  -2D0,  -3D0,  -4D0,
     &         -5D0,  -6D0,  -7D0,  -8D0,  -9D0 /

      EXTERNAL             AUXSCLE      ! S Evaluates the time and theta-value
                                        !   at an auxilliary scale division
      INTEGER              AUXUP        ! L = 1 if moving from aux(i) to
                                        !       aux(i+1)
                                        !   = 0 if moving from aux(i) to
                                        !       aux(i-1)
      DOUBLE PRECISION     START1       ! L Initial value for theta
      DOUBLE PRECISION     START2       ! L Theta-value 5E-5 sec after START1
      DOUBLE PRECISION     DFR          ! L Difference between START2 and the
                                        !   auxilliary scale div. nearest to it
      DOUBLE PRECISION     DIFF(ATOT)   ! L Array containing the difference
                                        !   between the starting value of the
                                        !   motion and the various aux()
      EXTERNAL             DMSTODEG     ! F Converts degrees, minutes and
                                        !   seconds to decimal degrees
      EXTERNAL             GYROMTN      ! S Evaluates theta-value for a given
                                        !   time
      INTEGER              NEAR         ! L The i-th entry in aux(), i.e. the
```

```
                                         !   next scale division to be crossed
      INTEGER              NOBS          ! L Number of the observation
      DOUBLE PRECISION     SMDIFF        ! L Smallest value in diff()
      DOUBLE PRECISION     START         ! L Initial value for theta
      DOUBLE PRECISION     THETA         ! L Result of the synthesized harmonics
      DOUBLE PRECISION     TIME          ! O Time at which the two harmonics
                                         !   are synthesized
      EXTERNAL             TURNPTS       ! S Evaluates the time and theta-value
                                         !   at a turning point


      OPEN( UNIT = 1, FILE = 'AUXSYNTH2', STATUS = 'NEW' )
      OPEN( UNIT = 2, FILE = 'TPSYNTH2' , STATUS = 'NEW' )

C     *** Initialise and set the time such that a transit at or near time ***
C     *** = 0 can also be evaluated                                        ***

      TIME = -5D0
      NOBS = 0

C     *** Calculate two values for theta 5E-5 seconds apart                ***

      CALL GYROMTN( START1, TIME )
      CALL GYROMTN( START2, TIME + 5E-5 )

C     *** Determine which auxilliary graduation line lies closest to the   ***
C     *** starting value for theta                                         ***

      SMDIFF = 1E10
      DO 100 I = 1, ATOT
        DIFF(I) = ABS( START1 - AUX(I) )
        IF( DIFF(I) .LT. SMDIFF ) THEN
          NEAR = I
          SMDIFF = DIFF(I)
        ENDIF
100   CONTINUE

C     *** Determine the direction in which the synthesized motion is mo-   ***
C     *** ving, and the auxilliary graduation line which will be crossed   ***
C     *** next                                                             ***

      DFR = ABS( START2 - AUX( NEAR ) )
      IF( DFR .GT. DIFF( NEAR ) ) THEN         ! Theta moving away from nearest
                                               ! scale division to START1
        IF( START2 .GT. START1 ) THEN          ! Theta moving in a +ve direc-
                                               ! tion, i.e. from aux(i) to
                                               ! aux(i-1)
          IF( NEAR .NE. 1 ) THEN               ! Not approaching a turning
                                               ! point
            NEAR = NEAR - 1
            AUXUP = 0
          GOTO 110
          ELSE                                 ! Approaching a turning point
          GOTO 120
          ENDIF
        ELSE IF( START2 .LT. START1 ) THEN     ! Theta moving in a -ve direc-
                                               ! tion, i.e. from aux(i) to
```

```
                                           ! aux(i+1)
      IF( NEAR .NE. ATOT ) THEN            ! Not approaching a turning
                                           ! point
        NEAR = NEAR + 1
        AUXUP = 1
        GOTO 110
      ELSE                                 ! Approaching a turning point
        GOTO 120
      ENDIF
    ENDIF
  ELSE IF( DFR .LT. DIFF( NEAR ) ) THEN        ! Theta moving towards nea-
                                               ! rest scale div.to START1
    IF( START2 .GT. START1 ) AUXUP = 0         ! Theta moving in a +ve
                                               ! direction
    IF( START2 .LT. START1 ) AUXUP = 1         ! Theta moving in a -ve
                                               ! direction
  GOTO 110
  ENDIF
110 CALL AUXSCLE( ATOT, AUX, AUXUP, NEAR, NOBS, THETA, TIME )
120 CALL TURNPTS( ATOT, AUX, AUXUP, NEAR, NOBS, THETA, TIME )
  END
```

```
      SUBROUTINE AUXSCLE( ATOT, AUX, AUXUP, NEAR, NOBS, THETA, TIME )
C     *-----------------------------------------------------------------------*
C     *                                                                       *
C     * FUNCTION:                                                             *
C     * ========                                                              *
C     * CALCULATES THE TIME AT WHICH THE SYNTHESIZED MOTION CROSSES A PRE-   *
C     * SELECTED AUXILLIARY SCALE DIVISION.                                   *
C     *                                                                       *
C     *-----------------------------------------------------------------------*
C     I=input,O=output,U=update,L=local,S=subroutine,F=function,P=parameter

      INTEGER              ANUM       ! L Number of theta-values calculated
      PARAMETER           (ANUM=30)   !   since transit of the last gradua-
                                      !   tion line
      INTEGER              ATOT       ! I Total number of auxilliary scale
                                      !   graduation lines
      DOUBLE PRECISION     AUX(ATOT)  ! I Theta-values representing the
                                      !   auxilliary scale divisions of a
                                      !   gyrotheodolite
      INTEGER              AUXUP      ! U =1 if moving from aux(i) to
                                      !      aux(i+1)
                                      !   =0 if moving from aux(i) to
                                      !      aux(i-1)
      DOUBLE PRECISION     CNT        ! L COUNT expressed as a real number
                                      !   (for the purpose of calculating the
                                      !   next timeincrement)
      INTEGER              COUNT      ! L Counter used to make the increment
                                      !   in time 1/2 the previous increment.
                                      !   After 25 times the increment in
                                      !   time is smaller than 1E-8, and the
                                      !   motion proceeds to the next scale
                                      !   division.
      EXTERNAL             GYROMTN    ! S Evaluates theta-value for a given
                                      !   time
      INTEGER              NEAR       ! I The i-th entry in aux(), i.e. the
                                      !   next scale div. to be crossed
      INTEGER              NOBS       ! I Number of the observation
      INTEGER              NUM        ! L Counter in th()
      DOUBLE PRECISION     TFORW      ! L =1  if next theta-value has to be
                                      !       backward in time from THL
                                      !   =-1 if next theta-value has to be
                                      !       forward in time from THF
      DOUBLE PRECISION     TH(ANUM)   ! L Theta-values from GYROMTN are sto-
                                      !   red here, until the motion has gone
                                      !   past the scale division. Only the
                                      !   last two values in th() are then
                                      !   used to determine the final time
      DOUBLE PRECISION     THF        ! L Theta-value just before aux()
      DOUBLE PRECISION     THL        ! L Theta-value just after  aux()
      DOUBLE PRECISION     THETA      ! L Result of the synthesized harmonics
      DOUBLE PRECISION     TIME       ! L Time at which the two harmonics
                                      !   are synthesized
      DOUBLE PRECISION     TSTEP      ! P Timestep chosen such that at
                                      !   least 3 values for theta can be
                                      !   evaluated between aux.scale values

 1000 FORMAT( I5, 1X, F13.6, 2X, F13.6 )
```

```
540   NUM = 0
      TSTEP = 2D0
550   TIME = TIME + TSTEP
      NUM = NUM + 1
      CALL GYROMTN( THETA, TIME )
      TH( NUM ) = THETA

C     *** If theta is moving in a -ve direction and aux(i) has not yet    ***
C     *** been crossed then calculate another value closer to aux(i)     ***

      IF( AUXUP .EQ. 1 .AND. AUX( NEAR ) .LT. THETA ) GOTO 550

C     *** If theta is moving in a +ve direction and aux(i) has not yet    ***
C     *** been crossed the calculate another value closer to aux(i)       ***

      IF( AUXUP .EQ. 0 .AND. AUX( NEAR ) .GT. THETA ) GOTO 550

C     *** Initialise and set the theta-value just before aux(i) to thf,   ***
C     *** and the one just after aux(i) to thl                            ***

      COUNT = 0
      TFORW = 1D0
      THF = TH( NUM-1 )
      THL = TH( NUM )
560   COUNT = COUNT + 1
      CNT = REAL( COUNT )
      TIME = TIME - TFORW * TSTEP / 2D0**CNT
      CALL GYROMTN( THETA, TIME )

C     *** If thf lies closer to aux(i) then thl does then                 ***

      IF( ABS( AUX( NEAR ) - THL ) .GT. ABS( AUX( NEAR )-THF ) )
     &THEN

C     *** If the latest value for theta lies on the same side of aux(i)   ***
C     *** as thf then let this theta become the new thf and calculate     ***
C     *** another theta forward from thf and halfway in time between thf  ***
C     *** and thl                                                         ***

        IF( ( AUX( NEAR ) - THETA ) * ( AUX( NEAR ) - THF )
     &  .GT. 0D0 ) THEN
          THF = THETA
        TFORW = -1D0

C     *** Else if the latest value for theta lies on the same side of     ***
C     *** aux(i) as thl then let this theta become the new thl and calcu- ***
C     *** late another theta backward from thl and halfway in time be-    ***
C     *** tween thf and thl                                               ***

        ELSE
          THL = THETA
          TFORW = 1D0
        ENDIF

C     *** else if thl lies closer to aux(i) then thf does then            ***

      ELSE
```

```
C     *** If the latest value for theta lies on the same side of aux(i)    ***
C     *** as thl then let this theta become the new thl and calculate      ***
C     *** another theta backward from thl and halfway in time between thf ***
C     *** and thl                                                          ***

        IF( ( AUX( NEAR ) - THETA ) * ( AUX( NEAR ) - THL )
     &  .GT. 0D0 ) THEN
        THL = THETA
        TFORW = 1D0

C     *** Else if the latest value for theta lies on the same side of      ***
C     *** aux(i) as thf then let this theta become the new thf and calcu- ***
C     *** late another theta forward from thf and halfway in time be-      ***
C     *** tween thf and thl                                                ***

        ELSE
          THF = THETA
          TFORW = -1D0
        ENDIF
      ENDIF

C     *** If value for theta not near enough to aux(i) then repeat the     ***
C     *** above process                                                    ***

      IF( ABS( THETA - AUX( NEAR ) ) .GT. 1E-7 ) GOTO 560

C     *** Give this 'observation' a number and write answer to the appro- ***
C     *** piate file                                                       ***

      NOBS = NOBS + 1
      IF( NEAR .GT. 2 .AND. NEAR .LT. ATOT-1   ) THEN
        WRITE( 1, 1000 ) NOBS, TIME, THETA
      ELSE
        IF( AUXUP .EQ. 0 .AND. NEAR .EQ. 2 )      WRITE( 2, *  )
        IF( AUXUP .EQ. 1 .AND. NEAR .EQ. ATOT-1 ) WRITE( 2, *  )
        WRITE( 2, 1000 ) NOBS, TIME, THETA
      ENDIF

C     *** If the time elapsed since the starting time exceeds 2 hours and ***
C     *** a full number of cycles have been completed then terminate the  ***
C     *** program                                                          ***

      IF( TIME .GT. 7200D0 .AND. AUX( NEAR ) .EQ. 0D0
     &    .AND. AUXUP .EQ. 0) GOTO 570

C     *** If the next value to be calculated is at a turning point then    ***
C     *** go back to subroutine turnpts                                    ***

      IF( AUXUP .EQ. 1 .AND. NEAR .EQ. ATOT ) RETURN
      IF( AUXUP .EQ. 0 .AND. NEAR .EQ. 1 )    RETURN

C     *** Else move on to the next auxilliary scale division               ***

      IF( AUXUP .EQ. 1) NEAR = NEAR + 1
      IF( AUXUP .EQ. 0) NEAR = NEAR - 1
      GOTO 540
```

```
570   END
```

```
      DOUBLE PRECISION FUNCTION DMSTODEG( DMS )
C    *-----------------------------------------------------------------------*
C    *                                                                       *
C    * FUNCTION:                                                             *
C    * ========                                                              *
C    * CONVERTS AN ANGLE FROM DEGREES, MINUTES AND SECONDS TO DECIMAL        *
C    * DEGREES.                                                              *
C    *                                                                       *
C    *-----------------------------------------------------------------------*
C     I=input,O=output,U=update,L=local,S=subroutine,F=function,P=parameter

      DOUBLE PRECISION    DGR          ! L   Number of degrees in DMS
      DOUBLE PRECISION    DMS          ! I/O Input  angle in degrees, min, sec
                                       !     Output angle in decimal degrees
      DOUBLE PRECISION    MIN          ! L   Number of minutes in DMS
      DOUBLE PRECISION    MINSEC       ! L   Number, in which the integer part
                                       !     contains the minutes, and the
                                       !     fractional part the seconds
      DOUBLE PRECISION    SECOND       ! L   Number of seconds in DMS

      DGR = INT( DMS )
      MINSEC = ( DMS - DGR ) * 100D0
      MIN = INT( MINSEC )
      SECOND = ( MINSEC - MIN ) * 100D0
      DMSTODEG = ( MIN + SECOND / 60D0 ) / 60D0 + DGR
      RETURN
      END
```

```
      SUBROUTINE GYROMTN( THETA, TIME )
C    *-----------------------------------------------------------------------*
C    *                                                                       *
C    * FUNCTION:                                                             *
C    * ========                                                              *
C    * SYNTHESIZES THE HARMONICS AT A GIVEN INSTANT OF TIME AND OUTPUTS THE  *
C    * THETA-VALUE IN UNITS OF SCALE DIVISIONS.                              *
C    *                                                                       *
C    *-----------------------------------------------------------------------*
C     I=input,O=output,U=update,L=local,S=subroutine,F=function,P=parameter

      INTEGER             AHAR         ! P Total number of harmonics in the
      PARAMETER          (AHAR=2)      !   motion
      DOUBLE PRECISION    BETA(AHAR)   ! P Initial half-amplitude (scale div.)
      DOUBLE PRECISION    COMP         ! L Sum of all harmonics, excl. thetao
      DOUBLE PRECISION    DMSTODEG
      EXTERNAL            DMSTODEG     ! F Converts degrees, minutes and
                                       !   seconds to decimal degrees
      DOUBLE PRECISION    FLAT(AHAR)   ! P Damping factor  (1/sec)
      DOUBLE PRECISION    GAMMA(AHAR)  ! P Phase angle in degrees,min and sec
      DOUBLE PRECISION    GMMA         ! L Phase angle (radians)
      INTEGER             NOBS         ! L Number of the observation
      DOUBLE PRECISION    OM           ! L Frequency of harmonic (rad/sec)
      DOUBLE PRECISION    PERIOD(AHAR)! P Period of the harmonic (sec)
      DOUBLE PRECISION    PI           ! L Number of radians in 180 degrees
      DOUBLE PRECISION    THETA        ! O Result of the synthesized harmonics
      DOUBLE PRECISION    THETAO       ! P Nullposition of the oscillation
      DOUBLE PRECISION    TIME         ! I Time at which the AHAR harmonics
                                       !   are synthesized

      THETAO = 0.1021D0
      DATA FLAT   / -0.14802E-5, -0.70589E-3 /
      DATA PERIOD / 386.307D0  , 405.293D0   /
      DATA GAMMA  / 90.5948D0  , 1.0511D0    /
      DATA BETA   / 9.5952D0   , 0.0704D0    /
      PI = 4D0*ATAN(1D0)
      COMP = 0D0
      DO 500 I = 1, AHAR
        GMMA = DMSTODEG( GAMMA(I) ) * PI / 180D0
        OM = 2D0 * PI / PERIOD(I)
        COMP = COMP + BETA(I) * EXP( FLAT(I) * TIME )
     &                         * COS( OM * TIME - GMMA )
500   CONTINUE
      THETA = THETAO + COMP
      RETURN
      END
```

```
      SUBROUTINE TURNPTS( ATOT, AUX, AUXUP, NEAR, NOBS, THETA, TIME)
C     *----------------------------------------------------------------------*
C     *                                                                      *
C     * FUNCTION:                                                            *
C     * ========                                                             *
C     * CALCULATES THE TIME AT WHICH THE SYNTHESIZED MOTION REACHES A TURN-   *
C     * ING POINT, AS WELL AS  ITS ANGULAR VALUE IN UNITS OF SCALE DIVISIONS.*
C     *                                                                      *
C     *----------------------------------------------------------------------*
C      I=input,O=output,U=update,L=local,S=subroutine,F=function,P=parameter

      INTEGER              ANUM          ! L Number of theta-values calculated
      PARAMETER           (ANUM=30)      !   since transit of the last gradua-
                                         !   tion line
      INTEGER              ATOT          ! I Total number of auxilliary scale
                                         !   graduation lines
      DOUBLE PRECISION     AUX(ATOT)     ! I Theta-values representing the
                                         !   auxilliary scale divisions of a
                                         !   gyrotheodolite
      EXTERNAL             AUXSCLE       ! S Evaluates the time and theta-
                                         !   value at an auxilliary scale div.
      INTEGER              AUXUP         ! U =1 if moving from aux(i) to
                                         !      aux(i+1)
                                         !   =0 if moving from aux(i) to
                                         !      aux(i-1)
      DOUBLE PRECISION     CNT           ! L COUNT expressed as a real number
                                         !   (for the purpose of calculating the
                                         !   next timeincrement)
      INTEGER              COUNT         ! L Counter used to make the increment
                                         !   in time 1/2 the previous increment.
                                         !   After 25 times the increment in
                                         !   time is smaller than 1E-7, and the
                                         !   motion proceeds to the next scale
                                         !   division.
      EXTERNAL             GYROMTN       ! S Evaluates theta-value for a given
                                         !   time
      INTEGER              NEAR          ! I The i-th entry in aux(), i.e. the
                                         !   next scale division to be crossed
      INTEGER              NOBS          ! I Number of the observation
      INTEGER              NUM           ! L Counter in th()
      DOUBLE PRECISION     TH(ANUM)      ! L Theta-values from GYROMTN are
                                         !   stored in here, until the turning
                                         !   point has been passed. Only the
                                         !   last three values in th() are then
                                         !   used to determine the final value
                                         !   for theta and the corresponding
                                         !   time
      DOUBLE PRECISION     THETA         ! L Result of the synthesized harmonics
      DOUBLE PRECISION     THF           ! L First value of the last 3 theta-
                                         !   values determined
      DOUBLE PRECISION     THL           ! L Last value of the last 3 theta-
                                         !   values determined
      DOUBLE PRECISION     THM           ! L Middle value of the last 3 theta-
                                         !   values determined
      DOUBLE PRECISION     TIME          ! L Time at which the two harmonics
                                         !   are synthesized
      DOUBLE PRECISION     TMEINCR       ! L Timeincrement to calculate the next
```

```
                                        !   time for GYROMTN
      DOUBLE PRECISION    TSTEP         ! P Timestep chosen such that at
                                        !   least 3 values for theta can be
                                        !   evaluated between aux.scale values

1000  FORMAT( I5, 1X, F13.6, 2X, F13.6 )
510   NUM = 0
      TSTEP = 2D0
      COUNT = 0
520   TIME = TIME + TSTEP
      CALL GYROMTN( THETA, TIME )
      NUM = NUM + 1
      TH( NUM ) = THETA
      IF( NUM .LT. 3 ) GOTO 520

C     *** If theta is still moving towards the turning point or if, at     ***
C     *** most, one theta-value has been calculated beyond the turning     ***
C     *** point but which still lies further away from the null-line than ***
C     *** th(num-1) does, then determine another theta                     ***

      IF( ABS( TH( NUM ) ) .GT. ABS( TH( NUM-1 ) ) ) GOTO 520

      THF = TH( NUM-2 )
      THM = TH( NUM-1 )
      THL = TH( NUM )
      DO 530 COUNT = 1, 25
        CNT = REAL( COUNT )

C     *** If the turning point lies between thf and thm and this is so     ***
C     *** for the first time then calculate another value for theta half- ***
C     *** way in time between thf and thm, else move back in time from     ***
C     *** thm by an increment which is half the previous increment. Let    ***
C     *** this theta become a new thm, and the previous thm becomes thl.   ***

        IF ( ABS( THF-THM ) .LT. ABS( THM-THL ) ) THEN
          IF( COUNT .EQ. 1 ) TMEINCR = 1.5D0 * TSTEP
          IF( COUNT .NE. 1 ) TMEINCR = TSTEP / 2D0**CNT
          THL = THM

C     *** Else if the turning point lies between thm and thl and this is  ***
C     *** so for the first time then calculate another value for theta    ***
C     *** halfway in time between thm and thl, else move forward in time   ***
C     *** from thm by an increment which is half the previous increment.   ***
C     *** Let this theta become a new thm, and the previous thm becomes    ***
C     *** thf.                                                             ***

        ELSE
          IF( COUNT .EQ. 1 ) TMEINCR = 0.5D0 * TSTEP
          IF( COUNT .NE. 1 ) TMEINCR = -TSTEP / 2D0**CNT
          THF = THM
        ENDIF

        TIME = TIME - TMEINCR
        CALL GYROMTN( THETA, TIME )
        THM = THETA
530   CONTINUE
```

```
C     *** Give this 'observation' a number and write answer to the appro- ***
C     *** piate file                                                      ***

      NOBS = NOBS + 1
      WRITE( 2, 1000 ) NOBS, TIME, THETA

C     *** If 2 hours have not yet expired, calculate another ATOT transit ***
C     *** times of the various scale divisions until the next turning     ***
C     *** point has been reached                                          ***

      IF( NEAR .EQ. 1 ) AUXUP = 1
      IF( NEAR .NE. 1 ) AUXUP = 0
      CALL AUXSCLE( ATOT, AUX, AUXUP, NEAR, NOBS, THETA, TIME )
      IF( TIME .LT. 7300D0 ) GOTO 510
      RETURN
      END
```

APPENDIX K

<u>PROGRAM</u>	:	SYNTH4
<u>TYPE</u>	:	MAIN PROGRAM WITH SUBROUTINES / FUNCTIONS
<u>LANGUAGE</u>	:	FORTRAN 5.5 (NB - NOT STANDARD FORTRAN)
<u>INSTALLATION</u>	:	VAX 6000-330, WITH 48 Mb RAM AND 3.6 Gb DISK STORAGE
<u>RUNNING SPEED</u>	:	CPU TIME TO GENERATE 761 VALUES REPRESENTING 19 CYCLES (i.e. $19\times38+1=723$ TRANSIT TIMES AND $19\times2=38$ TURNING POINTS) IS 598.63 sec

```
      PROGRAM SYNTH4
C     *-----------------------------------------------------------------*
C     *                                                                 *
C     * FUNCTION:                                                       *
C     * ========                                                        *
C     * THE MOTION OF A GYROSCOPE IS SIMULATED BY SYNTHESIZING FOUR HARMO-   *
C     * NICS. THE TIME AND THETA-VALUE AT EACH TURNING POINT, AS WELL AS THE *
C     * TIMES AT VARIOUS PRE-SELECTED AUXILLIARY SCALE DIVISIONS ARE CALCU-   *
C     * LATED.                                                          *
C     * THE HIGHER ORDER HARMONICS (THE 3rd HARMONIC IN PARTICULAR) PRODUCE  *
C     * A TREMOR ON THE OSCILLATION AND THE SYNTHESIZED MOTION MOVES BACK-    *
C     * WARDS AND FORWARDS ACROSS A SCALE DIVISION A NUMBER OF TIMES IN THE  *
C     * SPACE OF ABOUT 0.1 SECOND. THIS WILL ALWAYS BE AN ODD NUMBER AND THE *
C     * MEDIAN VALUE IS CHOSEN AS THE TIME OF TRANSIT FOR THAT SCALE DIVI-   *
C     * SION.                                                           *
C     * SIMILARLY AT A TURNING POINT, A LARGE NUMBER OF TURNING POINTS FOR   *
C     * THE 3rd HARMONIC OCCUR (APPROXIMATELY 100 TURNING POINTS PER SECOND).*
C     * AFTER A TIME OF TRANSIT AT THE ULTIMATE GRADUATION LINE HAS BEEN DE- *
C     * TERMINED A RELATIVELY LARGE STEP IN TIME (2 SEC) AND ONLY THE FIRST  *
C     * TWO HARMONICS ARE SYNTHESIZED TO GET A ROUGH VALUE FOR THE TURNING   *
C     * POINT. 100 SMALL TURNING POINTS ARE THEN CALCULATED ON EITHER SIDE   *
C     * AND THE ONE HAVING THE LARGEST ABSOLUTE VALUE FOR THETA IS THEN      *
C     * ADOPTED AS THE FINAL TURNING POINT.                             *
C     *                                                                 *
C     * OUTPUT:                                                         *
C     * ========                                                        *
C     * TRANSIT TIMES OF THE ULTIMATE AND PENULTIMATE AUXILLIARY SCALE DIVI- *
C     * SIONS AND THE TIME AND THETA-VALUE AT EACH TURNING POINT ARE STORED  *
C     * IN A DATAFILE CALLED TPSYNTH4.DAT, WHILE ALL OTHER TRANSIT TIMES ARE *
C     * STORED IN AUXSYNTH4.DAT                                         *
C     *                                                                 *
C     *                                                                 *
C     *              written by Gert-Jan van Rijsewijk                  *
C     *                  University of Cape Town                        *
C     *                         May 1991                                *
C     *                                                                 *
C     *                                                                 *
C     *-----------------------------------------------------------------*
C     F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      INTEGER             ATOT        ! P  Total number of auxilliary scale
      PARAMETER          (ATOT=19)    !    graduation lines
      DOUBLE PRECISION    AUX(ATOT)   ! L  Theta-values representing the aux-
                                      !    illiary scale divisions of a gyro-
                                      !    theodolite

      DATA AUX/  9D0,  8D0,  7D0,  6D0,  5D0,  4D0,  3D0,
     &           2D0,  1D0,  0D0, -1D0, -2D0, -3D0, -4D0,
     &          -5D0, -6D0, -7D0, -8D0, -9D0 /

      EXTERNAL            AUXSCLE     ! S  Calculates 20 turning points of
                                      !    the fully synthesized motion at
                                      !    a scale division
      INTEGER             AUXUP       ! L  = 1 if moving from aux(i) to
                                      !        aux(i+1)
                                      !    = 0 if moving from aux(i) to
                                      !        aux(i-1)
```

```
      EXTERNAL            COMPTRN     ! S  Calculates 100 turning points of
                                      !    the 3rd harmonic on either side of
                                      !    an approximate value obtained from
                                      !    synthesizing the first 2 harmonics
                                      !    only
      DOUBLE PRECISION    DFR         ! L  Difference between START2 and the
                                      !    auxilliary scale division nearest
                                      !    to it
      DOUBLE PRECISION    DIFF(ATOT)  ! L  Array containing the difference
                                      !    between START1 and the various
                                      !    aux()
      EXTERNAL            GYROMTN     ! S  Synthesizes the harmonics and re-
                                      !    turns a value for theta
      INTEGER             NEAR        ! L  The i-th value in aux(), i.e. the
                                      !    next scale division to be crossed
      INTEGER             NOBS        ! L  Number of the observation
      INTEGER             NSYN        ! L  Number of harmonics synthesized
      DOUBLE PRECISION    SMDIFF      ! L  Smallest value in diff()
      DOUBLE PRECISION    SMTHTA(100) ! L  Array containing the theta-values
                                      !    for the small turning points cal-
                                      !    culated in AUXSCLE and COMPTRN
      DOUBLE PRECISION    SMTIME(100) ! L  Times corresponding to smthta()
      DOUBLE PRECISION    START1      ! L  Initial value for theta
      DOUBLE PRECISION    START2      ! L  Theta-value 1E-2 sec after START1
      DOUBLE PRECISION    THETA       ! L  The result from GYROMTN
      DOUBLE PRECISION    TIME        ! P/U Time at which the harmonics are
                                      !    synthesized

      OPEN( UNIT = 1, FILE = 'AUXSYNTH4', STATUS = 'NEW' )
      OPEN( UNIT = 2, FILE = 'TPSYNTH4' , STATUS = 'NEW' )

C     *** Initialise and set the time such that a transit at or near time ***
C     *** = 0 can also be evaluated                                       ***

      TIME = -5D0
      NOBS = 0

C     *** Synthesize harmonics 1 and 2 and calculate two values 0.01 sec  ***
C     *** apart.                                                          ***

      NSYN = 2
      CALL GYROMTN( START1, TIME, NSYN )
      CALL GYROMTN( START2, TIME + 1E-2, NSYN )

C     *** Determine which auxilliary graduation line lies closest to the  ***
C     *** first theta-value, i.e. START1                                  ***

      SMDIFF = 1E10
      DO 100 I = 1, ATOT
        DIFF(I) = ABS( START1 - AUX(I) )
        IF( DIFF(I) .LT. SMDIFF ) THEN
          NEAR = I
          SMDIFF = DIFF(I)
        ENDIF
100   CONTINUE

C     *** Determine the direction in which the synthesized motion is mo-  ***
```

```
C     *** ving, and the auxilliary graduation line which will be crossed  ***
C     *** next                                                            ***

      DFR = ABS( START2 - AUX( NEAR ) )
      TIME = -5D0
      IF( DFR .GT. DIFF( NEAR ) ) THEN          ! Theta moving away from near-
                                                ! est scale division to START1
        IF( START2 .GT. START1 ) THEN           ! Theta moving in a +ve direc-
                                                ! ion, i.e. from aux(i) to
                                                ! aux(i-1)
          IF( NEAR .NE. 1 ) THEN                ! Not approaching a turning
                                                ! point
            NEAR = NEAR - 1
            AUXUP = 0
            GOTO 110
          ELSE
            GOTO 120
          ENDIF
        ELSE IF( START2 .LT. START1 ) THEN      ! Theta moving in a -ve direc-
                                                ! ion, i.e. from aux(i) to
                                                ! aux(i+1)
          IF( NEAR .NE. ATOT ) THEN             ! Not approaching a turning
                                                ! point
            NEAR = NEAR + 1
            AUXUP = 1
            GOTO 110
          ELSE
            GOTO 120
          ENDIF
        ENDIF
      ELSE IF( DFR .LT. DIFF( NEAR ) ) THEN     ! Theta moving towards nearest
                                                ! scale division to START1
        IF( START2 .GT. START1 ) AUXUP = 0      ! Theta moving in a +ve dirn
        IF( START2 .LT. START1 ) AUXUP = 1      ! Theta moving in a -ve dirn
        GOTO 110
      ENDIF
110   CALL AUXSCLE( ATOT, AUX, AUXUP, NEAR, NOBS, SMTIME, SMTHTA, THETA,
     &              TIME )
120   CALL COMPTRN( ATOT, AUX, AUXUP, NEAR, NOBS, SMTIME, SMTHTA, THETA,
     &              TIME )
      END
```

```
      SUBROUTINE AUXSCLE( ATOT, AUX, AUXUP, NEAR, NOBS, SMTIME, SMTHTA,
     &                    THETA, TIME )
C     *----------------------------------------------------------------------*
C     *                                                                      *
C     * FUNCTION:                                                            *
C     * ========                                                             *
C     * CALCULATES ALL THE TIMES AT WHICH THE FULLY SYNTHESIZED MOTION       *
C     * CROSSES AN AUXILLIARY SCALE DIVISION, AND THEN WRITE THE MEDIAN TIME *
C     * AND THE CORRESPONDING THETA-VALUE TO THE APPROPIATE FILE             *
C     *                                                                      *
C     *----------------------------------------------------------------------*
C     F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      INTEGER             ATOT         ! I  Total number of auxilliary scale
                                       !    graduation lines
      DOUBLE PRECISION    AUX(ATOT)    ! I  Theta-values representing the aux-
                                       !    illiary scale divisions of a gyro-
                                       !    theodolite
      INTEGER             AUXUP        ! I  = 1 if moving from aux(i) to
                                       !        aux(i+1)
                                       !    = 0 if moving from aux(i) to
                                       !        aux(i-1)
      EXTERNAL            CONV         ! S  Calculates the exact time of tran-
                                       !    sit for NSYN harmonics
      DOUBLE PRECISION    CROSS(30)    ! L  Array containing all the transit
                                       !    times at aux(), taking all the
                                       !    harmonics into account
      INTEGER             CROSSAUX     ! L  The i-th value in CROSS
      DOUBLE PRECISION    D1           ! L  Difference i.t.o. scale divisions
                                       !    between aux() and the small turn-
                                       !    ing point immediately prior to it
      DOUBLE PRECISION    D2           ! L  Difference i.t.o. scale divisions
                                       !    between aux() and the small turn-
                                       !    ing point immediately after it
      EXTERNAL            GYROMTN      ! S  Synthesizes the harmonics and re-
                                       !    turns a value for theta
      INTEGER             LCOUNT       ! L  The number of times the increment
                                       !    in time is halved before being
                                       !    smaller than 1E-7 seconds
      INTEGER             NEAR         ! U  The i-th value in aux(), i.e. the
                                       !    next scale division to be crossed
      INTEGER             NOBS         ! L  Number of the observation
      INTEGER             NSYN         ! L  Number of harmonics synthesized
      INTEGER             NUM          ! L  Counter in th()
      INTEGER             ONAUX        ! P  =1 if SMALLTRN called from AUXSCLE
                                       !    =0 if SMALLTRN called from COMPTRN
      EXTERNAL            SMALLTRN     ! S  Calculates a given number of small
                                       !    turning points near an auxilliary
                                       !    scale division or in the vicinity
                                       !    of a turning point of the full
                                       !    motion
      DOUBLE PRECISION    SMTHTA(100)  ! I  Array containing the theta-values
                                       !    for the small turning points cal-
                                       !    culated in AUXSCLE and COMPTRN
      DOUBLE PRECISION    SMTIME(100)  ! I  Times corresponding to smthta()
      DOUBLE PRECISION    TH(300)      ! L  Theta-values from GYROMTN are sto-
                                       !    red here until the motion has pas-
```

```
                                          !    sed the scale division. Only the
                                          !    last 2 values in th() are then
                                          !    used to determine the final time
                                          !    of transit
      DOUBLE PRECISION    THETA           ! O  Result of the synthesized harmo-
                                          !    nics
      DOUBLE PRECISION    THF             ! L  Theta-value just before aux()
      DOUBLE PRECISION    THL             ! L  Theta-value just after  aux()
      DOUBLE PRECISION    TIME            ! O  Time at which harmonics are syn-
                                          !    thesized
      DOUBLE PRECISION    TSTEP           ! P  Timestep chosen such that at
                                          !    least 3 values for theta can be
                                          !    evaluated between aux. scale div.
                                          !    or between small turning points
                                          !    calculated in SMALLTRN
      INTEGER             UPPER           ! L   = 1 if moving in a +ve direction
                                          !     between the small turning points
                                          ! L   = 0 if moving in a -ve direction
                                          !     between the small turning points

1000  FORMAT( I5, 1X, F13.6, 2X, F13.6 )
      ONAUX = 1
500   NUM = 0
      TSTEP = 2D0

C     *** Determine an approximate time of transit by synthesizing the     ***
C     *** first two harmonics only                                         ***

      NSYN = 2
510   TIME = TIME + TSTEP
      NUM = NUM + 1
      CALL GYROMTN( THETA, TIME, NSYN )
      TH( NUM ) = THETA

C     *** If theta is moving in a -ve direction and aux(i) has not yet     ***
C     *** been crossed then calculate another value closer to aux(i)       ***

      IF( AUXUP .EQ. 1 .AND. AUX( NEAR ) .LT. THETA ) GOTO 510

C     *** If theta is moving in a +ve direction and aux(i) has not yet     ***
C     *** been crossed then calculate another value closer to aux(i)       ***

      IF( AUXUP .EQ. 0 .AND. AUX( NEAR ) .GT. THETA ) GOTO 510

C     *** Set the theta-value just before aux(i) to thf, and the one just ***
C     *** after aux(i) to thl                                              ***

      THF = TH( NUM-1 )
      THL = TH( NUM )

C     *** Determine lcount and evaluate the transit time                   ***

      LCOUNT = NINT( LOG( TSTEP / 1E-7 ) / LOG( 2D0 ) )
      CALL CONV( ATOT, AUX, LCOUNT, NEAR, NSYN, THF, THL, TIME, TSTEP )

C     *** Move back 0.08 sec from this approximate time of transit and     ***
C     *** calculate 20 small turning points from there. The graduation     ***
```

```
C     *** line will be crossed more then once                             ***

      TIME = TIME - 0.08D0
      CALL SMALLTRN( MAXTH, MAXTI, ONAUX, SMTIME, SMTHTA, THETA, TIME )
      NSYN = 4
      CROSSAUX = 0
      DO 530 I = 1, 19
        D1 = AUX( NEAR ) - SMTHTA(I)
        D2 = AUX( NEAR ) - SMTHTA(I+1)

C     *** If 2 successive turning points lie on either side of the grad.  ***
C     *** line then calculate a transit time                              ***

        IF( ( D1 * D2 ) .LT. 0D0 ) THEN
          CROSSAUX = CROSSAUX + 1
          TSTEP   = 0.3D0 * ABS( D1 )
          UPPER = 0
          IF( SMTHTA(I) .LT. SMTHTA(I+1) ) UPPER = 1
          NUM = 0

C     *** Start from the turning point with the smaller time and move    ***
C     *** forward using tstep until aux(i) has been crossed              ***

          TIME = SMTIME(I)
520       TIME = TIME + TSTEP
          NUM = NUM + 1
          CALL GYROMTN( THETA, TIME, NSYN )
          TH( NUM ) = THETA
          IF( ( UPPER .EQ. 0 .AND. AUX( NEAR ) .LT. THETA ) .OR. (
     &    UPPER .EQ. 1 .AND. AUX( NEAR ) .GT. THETA ) ) GOTO 520
          THF = TH( NUM-1 )
          THL = TH( NUM )

C     *** Determine lcount and evaluate the transit time between these 2  ***
C     *** small turning points                                            ***

          LCOUNT = NINT( LOG( TSTEP / 1E-7 ) / LOG( 2D0 ) )
          CALL CONV( ATOT, AUX, LCOUNT, NEAR, NSYN, THF, THL, TIME,
     &               TSTEP )
          CROSS( CROSSAUX ) = TIME
        ENDIF
530   CONTINUE

C     *** Take the median time of all the transit times just calculated   ***
C     *** and check if the synthesized motion returns a value for theta   ***
C     *** which corresponds to aux(i)                                     ***

      TIME = CROSS( CROSSAUX / 2 + 1 )
      CALL GYROMTN( THETA, TIME, NSYN )

C     *** Give this 'observation' a number and write the answer to the    ***
C     *** appropiate file                                                 ***

      NOBS = NOBS + 1
      IF( NEAR .GT. 2 .AND. NEAR .LT. ATOT-1 ) THEN
        WRITE( 1, 1000 ) NOBS, TIME, THETA
      ELSE
```

```
         IF( AUXUP .EQ. 0 .AND. NEAR .EQ. 2 )      WRITE( 2, * )
         IF( AUXUP .EQ. 1 .AND. NEAR .EQ. ATOT-1 ) WRITE( 2, * )
         WRITE( 2, 1000 ) NOBS, TIME, THETA
       ENDIF

C      *** If the time elapsed since the starting time exceeds 2 hours and ***
C      *** a full number of cycles have been completed then terminate the  ***
C      *** program                                                         ***

       IF( TIME .GT. 7200D0 .AND. AUX( NEAR ) .EQ. 0D0 .AND. AUXUP .EQ.
      &0 ) GOTO 535

C      *** If the next 'observation' occurs at a major turning point then  ***
C      *** go back to subroutine comptrn                                   ***

       IF( AUXUP .EQ. 1 .AND. NEAR .EQ. ATOT ) RETURN
       IF( AUXUP .EQ. 0 .AND. NEAR .EQ. 1 )    RETURN

C      *** Else move on to the next auxilliary scale division              ***

       IF( AUXUP .EQ. 1 ) NEAR = NEAR + 1
       IF( AUXUP .EQ. 0 ) NEAR = NEAR - 1
       GOTO 500
535    END
```

```
      SUBROUTINE COMPTRN( ATOT, AUX, AUXUP, NEAR, NOBS, SMTIME, SMTHTA,
     &                    THETA, TIME )
C     *-----------------------------------------------------------------*
C     *                                                                 *
C     * FUNCTION:                                                       *
C     * ========                                                        *
C     * CALCULATES AN APPROXIMATE TURNING POINT USING THE FIRST TWO HARMO-  *
C     * NICS ONLY AND THEN CALCULATES 100 SMALL TURNING POINTS OF THE FULLY *
C     * SYNTHESIZED MOTION ON EITHER SIDE OF THIS APPROXIMATE VALUE.     *
C     * THE MAXIMUM THETA-VALUE AND ITS CORRESPONDING TIME ARE THEN WRITTEN *
C     * TO THE APPROPITE FILE.                                          *
C     *                                                                 *
C     *-----------------------------------------------------------------*
C      F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      INTEGER             ATOT         ! I  Total number of auxilliary scale
                                       !    graduation lines
      DOUBLE PRECISION    AUX(ATOT)    ! I  Theta-values representing the aux-
                                       !    illiary scale divisions of a gyro-
                                       !    theodolite
      EXTERNAL            AUXSCLE      ! S  Calculates 20 turning points of
                                       !    the fully synthesized motion at
                                       !    a scale division
      INTEGER             AUXUP        ! U  = 1 if moving from aux(i) to
                                       !        aux(i+1)
                                       !    = 0 if moving from aux(i) to
                                       !        aux(i-1)
      DOUBLE PRECISION    CNT          ! L  COUNT expressed as a real number
                                       !    for the purpose of calculating
                                       !    the next TIME
      INTEGER             COUNT        ! L  Counter used to halve the incre-
                                       !    ment in time for each iteration.
                                       !    After LCOUNT times an approximate
                                       !    turning point is adopted
      EXTERNAL            GYROMTN      ! S  Synthesizes the harmonics for a
                                       !    given time
      INTEGER             LCOUNT       ! L  The number of times by which the
                                       !    timeincrement is halved to reach
                                       !    a value of 1E-7 seconds
      DOUBLE PRECISION    MAXTH        ! L/O The max theta-value for NTURN
                                       !    turning points on either side of
                                       !    TGOTHERE in subroutine SMALLTRN
      DOUBLE PRECISION    MAXTI        ! L/O Time associated with MAXTH
      INTEGER             NEAR         ! I  The i-th value in aux(), i.e. the
                                       !    scale division closest to the
                                       !    turning point
      INTEGER             NOBS         ! U  Number of the observation
      INTEGER             NSYN         ! L  Number of harmonics synthesized
      INTEGER             NUM          ! L  Counter in th()
      INTEGER             ONAUX        ! P  =1 if SMALLTRN called from AUXSCLE
                                       !    =0 if SMALLTRN called from COMPTRN
      EXTERNAL            SMALLTRN     ! S  Calculates a given number of small
                                       !    turning points near an auxilliary
                                       !    scale division or in the vicinity
                                       !    of a turning point of the full
                                       !    motion
      DOUBLE PRECISION    SMTHTA(100)  ! I  Array containing the theta-values
```

```
                                   !     for the small turning points cal-
                                   !     culated in AUXSCLE and COMPTRN
      DOUBLE PRECISION    SMTIME(100) ! I  Times corresponding to smthta()
      DOUBLE PRECISION    TH(300)     ! L  Theta-values from GYROMTN are sto-
                                   !     red here until the turning point
                                   !     has been passed. Only the last 3
                                   !     values in th() are then used to
                                   !     determine the approximate turning
                                   !     point, using the first 2 harmo-
                                   !     nics only
      DOUBLE PRECISION    THETA       ! L  The result from GYROMTN
      DOUBLE PRECISION    THF         ! L  First value of the last three
                                   !     theta-values determined.
      DOUBLE PRECISION    THL         ! L  Last value of the last three
                                   !     theta-values determined.
      DOUBLE PRECISION    THM         ! L  Middle value of the last three
                                   !     theta-values determined.
      DOUBLE PRECISION    TIME        ! I/O Time at which the harmonics are
                                   !     synthesized
      DOUBLE PRECISION    TSTEP       ! P  Timestep chosen such that at
                                   !     least 3 values for theta can be
                                   !     evaluated between the last aux-
                                   !     illiary scale division and the
                                   !     maximum turning point

1000  FORMAT( I5, 1X, F13.6, 2X, F13.6 )
540   TSTEP = 2D0
      ONAUX = 0
      NUM = 0

C     *** Determine an approximate turning point by synthesizing the     ***
C     *** first two harmonics only                                       ***

      NSYN = 2
550   TIME = TIME + TSTEP
      CALL GYROMTN( THETA, TIME, NSYN )
      NUM = NUM + 1
      TH( NUM ) = THETA

C     *** Ensure that at least 3 theta-value have been determined.       ***
C     *** If theta is still moving towards the turning point or if, at   ***
C     *** most, one theta-value has been calculated beyond the turning   ***
C     *** point but which still lies further away from the null-line than ***
C     *** th(num-1) does, determine another theta                        ***

      IF( NUM .LT. 3 .OR. ABS( TH(NUM) ) .GT. ABS( TH(NUM-1) )) GOTO 550
      THF = TH( NUM-2 )
      THM = TH( NUM-1 )
      THL = TH( NUM )
      LCOUNT = NINT( LOG( TSTEP / 1E-7 ) / LOG( 2D0 ) )
      DO 560 COUNT = 1, LCOUNT
        CNT = REAL( COUNT )

C     *** If the turning point lies between thf and thm and count=1 then ***
C     *** calculate another value for theta halfway in time between thf  ***
C     *** and thm, else move back in time from thm by an increment which ***
C     *** is half the previous increment. Let this theta become a new thm ***
```

```
C     *** and the old thm becomes thl                                       ***

        IF( ABS( THF - THM ) .LT. ABS( THM - THL ) ) THEN
          IF( COUNT .EQ. 1 ) TIME = TIME - 1.5D0 * TSTEP
          IF( COUNT .NE. 1 ) TIME = TIME - TSTEP / 2D0**CNT
          THL = THM

C     *** Else if the turning point lies between thm and thl and count=1   ***
C     *** then calculate another value for theta halfway in time between   ***
C     *** thm and thl, else move forward in time from thm by an increment ***
C     *** which is half the previous increment. Let this theta become a    ***
C     *** new thm and the old thm becomes thf                               ***

        ELSE
          IF( COUNT .EQ. 1 ) TIME = TIME - 0.5D0 * TSTEP
          IF( COUNT .NE. 1 ) TIME = TIME + TSTEP / 2D0**CNT
          THF = THM
        ENDIF
        CALL GYROMTN( THETA, TIME, NSYN )
        THM = THETA
560   CONTINUE

C     *** Calculate the maximum theta-value of 200 small turning points    ***
C     *** symmetrical about this approximate turning point                  ***

      CALL SMALLTRN( MAXTH, MAXTI, ONAUX, SMTIME, SMTHTA, THETA, TIME )

C     *** Give this 'observation' a number and write the answer to the     ***
C     *** appropiate file                                                   ***

      NOBS = NOBS + 1
      WRITE( 2, 1000 ) NOBS, MAXTI, MAXTH

C     *** If 2 hours have not yet expired, calculate another atot transit ***
C     *** times of the various scale divisions until the next turning      ***
C     *** point has been reached                                            ***

      IF( TIME .LT. 7300D0 ) THEN
        IF( NEAR .EQ. 1 ) AUXUP = 1
        IF( NEAR .NE. 1 ) AUXUP = 0
        CALL AUXSCLE( ATOT, AUX, AUXUP, NEAR, NOBS, SMTIME, SMTHTA,
     &                THETA, TIME )
        GOTO 540
      ENDIF
      RETURN
      END
```

```
      SUBROUTINE CONV( ATOT, AUX, LCOUNT, NEAR, NSYN, THF, THL, TIME,
     &                 TSTEP )
C    *-----------------------------------------------------------------------*
C    *                                                                       *
C    * FUNCTION:                                                             *
C    * ========                                                              *
C    * CALCULATES THE EXACT TIME OF TRANSIT AT AN AUXILLIARY SCALE DIVISION *
C    * FOR A GIVEN NUMBER OF HARMONICS.                                      *
C    *                                                                       *
C    *-----------------------------------------------------------------------*
C     F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      INTEGER              ATOT          ! I  Total number of auxilliary scale
                                         !    graduation lines
      DOUBLE PRECISION     AUX(ATOT)     ! I  Theta-values representing the aux-
                                         !    illiary scale divisions of a gyro-
                                         !    theodolite
      DOUBLE PRECISION     CNT           ! L  COUNT expressed as a real number
                                         !    for the purpose of calculating
                                         !    the next TIME
      INTEGER              COUNT         ! L  Counter used to halve the incre-
                                         !    ment in time for each iteration.
                                         !    After LCOUNT times an approximate
                                         !    turning point is adopted
      EXTERNAL             GYROMTN       ! S  Synthesizes the harmonics for a
                                         !    given time
      INTEGER              LCOUNT        ! L  The number of times by which the
                                         !    timeincrement is halved to reach
                                         !    a value of 1E-7 seconds
      INTEGER              NEAR          ! I  The i-th value in aux(), i.e. the
                                         !    next scale division to be crossed
      INTEGER              NSYN          ! I  Number of harmonics synthesized
      DOUBLE PRECISION     TFORW         ! L  Determines whether next theta-
                                         !    value lies forward or backward
                                         !    from the latest value
      DOUBLE PRECISION     THETA         ! L  The result from GYROMTN
      DOUBLE PRECISION     THF           ! U  Theta-value just before aux()
      DOUBLE PRECISION     THL           ! U  Theta-value just after  aux()
      DOUBLE PRECISION     TIME          ! I/U Time at which harmonics are syn-
                                         !    thesized
      DOUBLE PRECISION     TSTEP         ! I  Timestep chosen such that at
                                         !    least 3 values for theta can be
                                         !    evaluated before reaching time
                                         !    of transit

      TFORW = 1D0
      DO 570 COUNT = 1, LCOUNT
        CNT = REAL( COUNT )
        TIME = TIME - TFORW * TSTEP / 2D0**CNT
        CALL GYROMTN( THETA, TIME, NSYN )

C     *** If thf lies closer to aux(i) then thl does then                  ***

        IF( ABS( THL - AUX( NEAR ) ) .GT. ABS( AUX( NEAR ) - THF ) )
     &  THEN

C     *** If the latest value for theta lies on the same side of aux(i)   ***
```

```
C     *** as thf then let this theta become the new thf and calculate     ***
C     *** another theta forward from thf and halfway in time between thf  ***
C     *** and thl                                                         ***

          IF( ( AUX( NEAR ) - THETA ) * ( AUX( NEAR ) - THF) .GT. 0D0 )
     &    THEN
            THF = THETA
            TFORW = -1D0

C     *** Else if the latest value for theta lies on the same side of     ***
C     *** aux(i) as thl then let this theta become the new thl and calcu- ***
C     *** late another theta backward from thl and halfway in time be-    ***
C     *** tween thf and thl                                               ***

          ELSE
            THL = THETA
            TFORW = 1D0
          ENDIF

C     *** Else if thl lies closer to aux(i) then thf does then            ***

        ELSE

C     *** If the latest value for theta lies on the same side of aux(i)   ***
C     *** as thl then let this theta become the new thl and calculate     ***
C     *** another theta backward from thl and halfway in time between thf ***
C     *** and thl                                                         ***

          IF( ( AUX( NEAR ) - THETA ) * ( AUX( NEAR ) - THL ) .GT. 0D0 )
     &    THEN
            THL = THETA
            TFORW = 1D0

C     *** Else if the latest value for theta lies on the same side of     ***
C     *** aux(i) as thf then let this theta become the new thf and calcu- ***
C     *** late another theta forward from thf and halfway in time be-     ***
C     *** tween thf and thl                                               ***

          ELSE
            THF = THETA
            TFORW = -1D0
          ENDIF
        ENDIF
570   CONTINUE
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION DMSTODEG( DMS )
C     *----------------------------------------------------------------------*
C     *                                                                      *
C     * FUNCTION:                                                            *
C     * ========                                                             *
C     * CONVERTS AN ANGLE FROM DEGREES, MIN AND SEC TO DECIMAL DEGREES.      *
C     *                                                                      *
C     *----------------------------------------------------------------------*
C      F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      DOUBLE PRECISION    DGR          ! L   Number of degrees in DMS
      DOUBLE PRECISION    DMS          ! I/O Input  angle in degrees, min, sec
                                       !     Output angle in decimal degrees
      DOUBLE PRECISION    MIN          ! L   Number of minutes in DMS
      DOUBLE PRECISION    MINSEC       ! L   Number, in which the integer part
                                       !     contains the minutes, and the
                                       !     fractional part the seconds
      DOUBLE PRECISION    SECOND       ! L   Number of seconds in DMS

      DGR = INT( DMS )
      MINSEC = ( DMS - DGR ) * 100D0
      MIN = INT( MINSEC )
      SECOND = ( MINSEC - MIN ) * 100D0
      DMSTODEG = ( MIN + SECOND / 60D0 ) / 60D0 + DGR
      RETURN
      END
```

```
      SUBROUTINE GYROMTN( THETA, TIME, NSYN )
C     *-----------------------------------------------------------------*
C     *                                                                 *
C     * FUNCTION:                                                       *
C     * ========                                                        *
C     * SYNTHESIZES THE HARMONICS AT A GIVEN INSTANT OF TIME AND OUTPUTS THE *
C     * THETA-VALUE IN UNITS OF SCALE DIVISIONS.                        *
C     *                                                                 *
C     *-----------------------------------------------------------------*
C      F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      INTEGER             AHAR        ! P Total number of harmonics in the
      PARAMETER          (AHAR=4)     !   motion
      DOUBLE PRECISION    BETA(AHAR)  ! P Initial half-amplitude (scale div.)
      DOUBLE PRECISION    COMP        ! L Sum of all harmonics, excl. theta0
      DOUBLE PRECISION    DMSTODEG
      EXTERNAL            DMSTODEG    ! F Converts degrees, min and sec to
                                      !   decimal degrees
      DOUBLE PRECISION    FLAT(AHAR)  ! P Damping factor (1/sec)
      DOUBLE PRECISION    GAMMA(AHAR) ! P Phase angle in degrees, min and sec
      DOUBLE PRECISION    GMMA        ! L Phase angle in radians
      INTEGER             NSYN        ! I Number of harmonics synthesized
      DOUBLE PRECISION    OM          ! L Frequency of harmonic (rad/sec)
      DOUBLE PRECISION    PERIOD(AHAR)! P Period of the harmonic (sec)
      DOUBLE PRECISION    PI          ! L Number of radians in 180 degrees
      DOUBLE PRECISION    THETA       ! O Result of the synthesized harmonics
      DOUBLE PRECISION    THETAO      ! P Nullposition of the oscillation
      DOUBLE PRECISION    TIME        ! I Time at which the AHAR harmonics
                                      !   are synthesized


      THETAO = 0.1021DO
      DATA BETA   / 9.5952DO   , 0.0704DO   , 0.0027DO   , 0.00006DO/
      DATA FLAT   / -0.14802E-5, -0.70589E-3, -0.11624E-3, -0.17354E-2 /
      DATA GAMMA  / 90.5948DO  , 1.0511DO   , 62.1949DO  , 277.4936DO/
      DATA PERIOD / 386.307DO  , 405.293DO  , 0.022DO    , 0.497DO/
      PI = 4DO * ATAN(1DO)
      COMP = ODO
      DO 580 I = 1, NSYN
        GMMA = DMSTODEG( GAMMA(I) ) * PI / 180DO
        OM = 2DO * PI / PERIOD(I)
        COMP = COMP + BETA(I) * EXP( FLAT(I) * TIME )
     &                        * COS( OM * TIME - GMMA )
580   CONTINUE
      THETA = THETAO + COMP
      RETURN
      END
```

```
      SUBROUTINE SMALLTRN( MAXTH, MAXTI, ONAUX, SMTIME, SMTHTA, THETA,
     &                     TIME )
C    *-------------------------------------------------------------------*
C    *                                                                   *
C    * FUNCTION:                                                         *
C    * ========                                                          *
C    * CALCULATES A GIVEN NUMBER OF SMALL TURNING POINTS AND             *
C    *    1. STORES THESE VALUES IN SMTIME() AND SMTHTA() AND RETURNS TO *
C    *       AUXSCLE                                                     *
C    * OR 2. STORES THE MAXIMUM ABSOLUTE THETA-VALUE AND ITS CORRESPONDING *
C    *       TIME IN MAXTH AND MAXTI AND RETURNS TO COMPTRN              *
C    *                                                                   *
C    *-------------------------------------------------------------------*
C     F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      INTEGER             ANUM          ! L  Number of theta-values since the
      PARAMETER          (ANUM=10)      !    previous small turning point
      DOUBLE PRECISION    CNT           ! L  COUNT expressed as a real number
                                        !    for the purpose of calculating
                                        !    the next TIME
      INTEGER             COUNT         ! L  Counter used to halve the incre-
                                        !    ment in time for each iteration.
                                        !    After LCOUNT times an approximate
                                        !    turning point is adopted
      EXTERNAL            GYROMTN       ! S  Synthesizes the harmonics for a
                                        !    given time
      INTEGER             LCOUNT        ! L  The number of times by which the
                                        !    timeincrement is halved to reach
                                        !    a value of 1E-10 seconds
      DOUBLE PRECISION    MAXTH         ! L/O The max theta-value for NTURN
                                        !    turning points on either side of
                                        !    TGOTHERE
      DOUBLE PRECISION    MAXTI         ! L/O Time associated with MAXTH
      INTEGER             NSYN          ! L  Number of harmonics synthesized
      INTEGER             NTURN         ! I  Number of small turning points
                                        !    to be evaluated
      INTEGER             NUM           ! L  Counter in th()
      INTEGER             ONAUX         ! I  =1 if SMALLTRN called from AUXSCLE
                                        !    =0 if SMALLTRN called from COMPTRN
      DOUBLE PRECISION    SMTHTA(100)   ! L/O Array containing the theta-values
                                        !    for the small turning points
      DOUBLE PRECISION    SMTIME(100)   ! L/O Times corresponding to smthta()
      DOUBLE PRECISION    TDIFF         ! L  2*NTURN turning points are cal-
                                        !    culated between TGOTHERE +/- TDIFF
      DOUBLE PRECISION    TGOTHERE      ! L  The approximate time for the
                                        !    turning point (obtained from
                                        !    COMPTRN, using 2 harmonics).
      DOUBLE PRECISION    TH(ANUM)      ! L  Stores the theta-values from
                                        !    GYROMTN, until the turning point
                                        !    has been passed. Only THF, THM
                                        !    and THL are then used to deter-
                                        !    mine the final value for theta
                                        !    and the corresponding time.
      DOUBLE PRECISION    THETA         ! L  The result from GYROMTN
      DOUBLE PRECISION    THF           ! L  First value of the last three
                                        !    theta-values determined.
      DOUBLE PRECISION    THL           ! L  Last value of the last three
```

```
                                           !    theta-values determined.
      DOUBLE PRECISION    THM              ! L  Middle value of the last three
                                           !    theta-values determined.
      DOUBLE PRECISION    TIME             ! I/O Time at which harmonics are syn-
                                           !    thesized
      DOUBLE PRECISION    TSTEP            ! P  Timestep chosen such that at
                                           !    least 3 values for theta can be
                                           !    evaluated between turning points
      EXTERNAL            UPDOWN           ! S  Determines a value for UPPER
      INTEGER             UPPER            ! L  = 1 if moving in a +ve direction
                                           !    between the small turning points
                                           !    = 0 if moving in a -ve direction
                                           !    between the small turning points

      MAXTH = 1E-10
      TGOTHERE = TIME
      TDIFF = 0D0
      IF( ONAUX .EQ. 1 ) NTURN = 20
      IF( ONAUX .EQ. 0 ) NTURN = 100
590   TIME = TGOTHERE -TDIFF
      CALL UPDOWN( TIME, UPPER )
      NSYN = 4
      TSTEP = 0.003D0
      DO 620 I = 1, NTURN
        NUM = 0
        COUNT = 0
600     TIME = TIME + TSTEP
        CALL GYROMTN( THETA, TIME, NSYN )
        NUM = NUM + 1
        TH( NUM ) = THETA
        IF( NUM .LT. 3 ) GOTO 600
        IF( ( THETA .GT. 0 .AND. UPPER .EQ. 1 .OR. THETA .LT. 0 .AND.
     &  UPPER .EQ. 0 ) .AND. ABS( TH( NUM ) ) .GT. ABS( TH( NUM-1 ) ) )
     &  GOTO 600
        IF( ( THETA .LT. 0 .AND. UPPER .EQ. 1 .OR. THETA .GT. 0 .AND.
     &  UPPER .EQ. 0 ) .AND. ABS( TH( NUM ) ) .LT. ABS( TH( NUM-1 ) ) )
     &  GOTO 600
        THF = TH( NUM-2 )
        THM = TH( NUM-1 )
        THL = TH( NUM )
        LCOUNT = NINT( LOG( TSTEP / 1E-10 ) / LOG( 2D0 ) )
        DO 610 COUNT = 1, LCOUNT


          CNT = REAL( COUNT )

C     *** If the turning point lies between thf and thm and count=1 then  ***
C     *** calculate another value for theta halfway in time between thf   ***
C     *** and thm, else move back in time from thm by an increment which  ***
C     *** is half the previous increment. Let this theta become a new thm ***
C     *** and the old thm becomes thl                                     ***

          IF( ABS( THF - THM ) .LT. ABS( THM - THL ) ) THEN
            IF( COUNT .EQ. 1 ) TIME = TIME - 1.5D0 * TSTEP
            IF( COUNT .NE. 1 ) TIME = TIME - TSTEP / 2D0**CNT
            THL = THM
```

```
C     *** Else if the turning point lies between thm and thl and count=1  ***
C     *** then calculate another value for theta halfway in time between  ***
C     *** thm and thl, else move forward in time from thm by an increment ***
C     *** which is half the previous increment. Let this theta become a   ***
C     *** new thm and the old thm becomes thf                             ***

          ELSE
            IF( COUNT .EQ. 1 ) TIME = TIME - 0.5D0 * TSTEP
            IF( COUNT .NE. 1 ) TIME = TIME + TSTEP / 2D0**CNT
            THF = THM
          ENDIF
          CALL GYROMTN( THETA, TIME, NSYN )
          THM = THETA
610     CONTINUE

C     *** Store the time and theta-value for this turning point           ***

        SMTIME(I) = TIME
        SMTHTA(I) = THETA

C     *** If this subroutine is called from comptrn and this turning      ***
C     *** point has a larger absolute value than the previous maxth then  ***
C     *** make this the new maxth                                         ***

        IF( NTURN .EQ. 100 .AND. ABS( SMTHTA(I) ) .GT. ABS( MAXTH ) )
     &  THEN
          MAXTH = SMTHTA(I)
          MAXTI = SMTIME(I)
        ENDIF

C     *** The motion now changes direction to calculate the next turning  ***
C     *** point                                                           ***

        UPPER = ABS( UPPER - 1 )
620   CONTINUE

C     *** If this subroutine is called from comptrn then calculate        ***
C     *** another nturn turning points on the other side of tgothere      ***

      IF( ONAUX .EQ. 0 ) THEN
        TDIFF = TIME - TGOTHERE
        ONAUX = 1
        GOTO 590
      ENDIF
      RETURN
      END
```

```
      SUBROUTINE UPDOWN( TIME, UPPER )
C    *------------------------------------------------------------------*
C    *                                                                  *
C    * FUNCTION:                                                        *
C    * ========                                                         *
C    * EVALUATES TEN THETA-VALUES 0.001 SECONDS APART, USING ALL FOUR    *
C    * HARMONICS, AND THEN SETS THE TIME SUCH THAT A TURNING POINT HAS JUST *
C    * BEEN PASSED. A VALUE FOR 'UPPER' IS THEN RETURNED.               *
C    *                                                                  *
C    *------------------------------------------------------------------*
C     F=function,I=input,L=local,O=output,P=parameter,S=subroutine,U=update

      EXTERNAL            GYROMTN     ! S   Synthesizes the harmonics for a
                                      !     given time
      INTEGER             NSYN        ! L   Number of harmonics synthesized
      INTEGER             SUM         ! L   Total of the last four values in
                                      !     ud()
      DOUBLE PRECISION    START1      ! L   Theta-value at a given time
      DOUBLE PRECISION    START2      ! L   Theta-value 1E-5 seconds after
                                      !     START1
      DOUBLE PRECISION    TIME        ! I/O Time at which the four harmonics
                                      !     are synthesized
      INTEGER             UD(10)      ! L   Array consisting of 0's and 1's,
                                      !     which are the values for UPPER
                                      !     corresponding to UDTIME(10)
      DOUBLE PRECISION    UDTIME(10)  ! L   Array containing the 10 times
      INTEGER             UPPER       ! O   = 1 if moving in a +ve direction
                                      !     between the small turning points
                                      !     = 0 if moving in a -ve direction
                                      !     between the small turning points

      NSYN = 4
      DO 630 I = 1, 10
        UD(I) = 0
        TIME = TIME + 0.001D0
        UDTIME(I) = TIME
        CALL GYROMTN( START1, TIME, NSYN )
        CALL GYROMTN( START2, TIME + 1E-5, NSYN )
        IF( START2 .GT. START1 ) UD(I) = 1
        IF( I .GT. 4 ) THEN
          SUM = UD(I) + UD(I-1) + UD(I-2) + UD(I-3)
          IF( SUM .EQ. 0 .OR. SUM .EQ. 4 ) THEN
            TIME = UDTIME(I-3)
            RETURN
          ENDIF
        ENDIF
630   CONTINUE
      END
```