



A study of Grade 8 and 9 learner thinking about linear equations, from a commognitive perspective

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COMPULSORY DECLARATION

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I dedicate this work to different people on different levels ...

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Abstract

The problem of poor learner performance in school mathematics in South Africa is persistent. Many studies have pointed to learner difficulties with algebra and their inadequate access to mathematical properties as a problem-solving resource. This small-scale qualitative study focuses on how, in an interview, fifteen Grade 8 and 9 learners at two South African schools think about linear equations. Sfard's theory of commognition, and particularly her concepts of ritualized and explorative discourse, are used as a framework to analyse how learners' words, gestures, narratives and routines intersect to build a picture of the mathematical objects they perceive.

Many national and international research studies focusing on functions and linear equations from a cognitivist perspective, suggest that the reason for poor performance can be ascribed to a lack of relational understanding. Using a discursive rather than a cognitive lens, the study concludes that learners' discourse is ritualistic and that learners favour working with whole numbers, even when the context is negative integers or algebraic terms. Furthermore, they do not make a link between the solution of the equation and the function. As a result they have limited flexibility to adapt their routines.

The findings are that the learners in the study do not access the mathematical resources specified for grade 8 and 9 learners in the South African curriculum. The commognitive framework yields a particularly detailed account of ritualistic learner thinking and raises awareness that high scores on written assessment tasks is a limited measure of learner understanding. Although the study is restricted to one group of learners and only describes their discourse at one moment in their learning, the findings contribute to our understanding of the bigger problem of poor learner performance in South Africa.

Key words: Commognition, linear equation, discourse, ritualistic, explorative.

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Chapter 1: Motivation for and background to the study

1.1. Introduction

This study focuses on how, in an interview, fifteen Grade 8 and 9 learners at two South African schools think about their solutions to a set of linear equations. I use Sfard's (2008) theory of commognition, which views thinking as a form of communication, to describe what learners write, say and do in these interviews. In particular, her concepts of ritualized and explorative mathematical discourse form the framework for describing in detail and commenting on how learners solve the equations.

In this chapter I establish the focus of the study and locate it in the general problem of poor school mathematics in South Africa, my own work in teacher professional development, and in the research literature on linear equations in the school context.

In mapping out my study, I begin by presenting the problem of poor learner performance internationally, in South Africa and in the context of my work. Thereafter I present the curriculum expectations of learners at Grade 8 and 9 level regarding linear equations, and some initial insights from research. Finally, I present the details of my study and briefly introduce my chosen theory.

1.2. The problem of poor learner performance in school mathematics

Problems with learner performance in school mathematics are widely recognised across socio-economic and geographical contexts (Adler, 2011; de Lima & Tall, 2008; Essien & Setati, 2006; Linchevski & Livneh, 1999), and research has shown that there are “common idiosyncracies” in how learners respond to mathematics problems, which are hard to explain (Hall, 2002; Seng, 2010; Sfard & Cobb, 2014, p. 10).

The problem in South Africa is proving particularly stubborn, and after 21 years of democracy and despite extensive changes to the structure of the curriculum, the situation still seems critical. Spaul (2013, p. 4), citing the 2011 Trends in International Mathematics and Science Study (TIMSS), concluded that except for a small privileged minority, South African learners are functionally innumerate, and that “the average South African Grade Nine child performs between two and three grade levels lower than the average Grade Eight child from other middle-income countries”. My

research is done in the context of such poor-performing schools. In the school district of the schools attended by the learners in this study, Grade 9 learner performance on the Annual National Assessment¹ declined from a pass rate of 15,9% to 12,1% between 2013 and 2014 (Department of Basic Education, 2014).

Researchers in South Africa have focused on aspects of the problem at different levels, including teachers' practice and teacher development (Adler, 2010; Heyd-Metzuyanim & Graven, 2015) and school and classroom management (Fleisch, 2008; Taylor, 2009). From a socio-economic-political perspective researchers have looked at factors responsible for differentiated performance among schools in different quintiles (Setati, 2003; Spaull, 2013; Taylor, 2009).

Internationally, research has focused on how learners reason (Brodie, 2009), learner cognition (Fillooy & Rojano, 1989; Hoch & Dreyfus, 2004; Linchevski & Livneh, 1999) and learner errors (Gcasamba, 2014; Seng, 2010). My study is located in this body of research, where I focus on what manifests at the level of learner discourse. I use Sfard's (2008) commognition theory to investigate learners' thinking as they solve a set of linear equations. I motivate these choices briefly in the sections that follow, and in detail in Chapters 2 and 3.

1.3. Experience of the problem in my own practice in teacher professional development

In my 13 years of work in the field of teacher professional development at a South African university, I have been involved in various projects that aim to improve the mathematics performance at working class schools. The project for which I did the interviews used in my study, was with Grade 7 – 9 teachers and learners at a group of working class schools in close proximity to one another. At this level of schooling there is a strong focus on algebra and although learners have been solving linear equations at lower grades, the introduction of the syntax of algebra in my experience presents problems, such as the situations below.

¹ A national standardised assessment administered in all government schools in South Africa.

1.3.1. My encounter with a set of hard-to-understand Grade 9 learner solutions to linear equations

In a Grade 9 assessment task, learners were asked to solve the equation ' $4x + 3 = 5x + 2$ '. One correct representation of the solution to the equation is:

$$\begin{aligned}4x + 3 &= 5x - 2 \\ \therefore 4x - 5x &= -2 - 3 \\ \therefore -x &= -5 \\ \therefore x &= 5\end{aligned}$$

Figure 1: Correct representation of the solution to the equation $4x + 3 = 5x - 2$

Below is a selection of incorrect solutions to the same equation:

a. $4x + 3 = 5x - 2$ $\therefore x = 4 + 3 = 7$ $\therefore x = 7$	b. $4x + 3 = 5x - 2$ $9x = -2 + 3 = 1$ $x = 1$
c. $4x + 3 = 5x - 2$ $= 4x + 3 = 5 - 2$ $= 7 = 3$	d. $4x + 3 = 5x - 2$ $= 5x - 3 + 4 - 2$ $= 2x - 5x = 3x$

Figure 2: Selection of learner solutions to the equation $4x + 3 = 5x - 2$

In a discussion about these results, neither I nor their teachers could explain why learners would produce such written responses. I reflected on the fact that the correct solution, which would be awarded full marks, would only indicate that the learners knew *how* to solve the equation because they could apply the rules for the procedure. Conventional written testing methods do not make it possible to determine whether the learner understands *when* to use a particular strategy or *why* that strategy is appropriate. Is the learner able to justify the transformation from one step to the next or merely following the teacher's rules?

These unanswered questions prompted me to investigate how I could explain why learners would produce solutions like the ones in Figure 2. It was in this context that my observation of a classroom encounter at another school drew my attention to the way classroom communication could create

barriers for learners. In the next section I briefly describe what I observed as typical of the teaching that the learners in my study would have experienced, and also the curriculum that teachers draw on.

1.3.2. Classroom talk

I observed a lesson where the teacher² was teaching Grade 8 learners how to substitute values for variables in expressions. The teacher had chosen integers for substitution, which resulted in the lesson devolving into one on integer arithmetic. Two incidents in the lesson resonated with Sfard's (2008) observation that the quandaries in mathematics might be a product of the way we speak – the way we communicate with others. Firstly, when the teacher asked learners what happens when you square a negative number, they responded that it becomes a positive number. He accepted their answer. Next, he reminded them to always use the sign that is on the left-hand side of the number. From these interactions, learners could accept that the sign is separate from the digit, and that numbers can change their value. I interrupted at a suitable point, to ask the learners whether '2' and '-2' were the same number. Most said that it was and I realized that these learners did not view '-2' (or any negative integer) as having its own value.

Later the teacher gave learners examples of different types of expressions in which to substitute values. These included monomials, polynomials and exponential expressions, for which he showed them different routines they should learn. There was no discussion on the structure of the different expressions or how to use one routine flexibly.

Sfard (2008) argues that imitation of the teacher's procedure is a necessary initial activity in learning school mathematics. However, imitation is not the aim of school mathematics; ultimately learners should gain enough understanding of the content to be able to justify their solutions and apply the procedures in unfamiliar contexts. This is unlikely to be an outcome of the pedagogy I observed in the classroom. However, because lesson delivery is aligned to the curriculum, I feel it is necessary for the reader to appreciate that choice of pedagogy is influenced by guidelines in the curriculum.

² This teacher has given permission for use of this information.

1.4. The curriculum that teachers draw on

In South Africa, the explicit conventions of presenting mathematical ideas formally at school level are shaped by the National Curriculum Statement: Curriculum & Assessment Policy Statement (CAPS) (Department of Basic Education, 2011), the current implemented version of the curriculum for Senior Phase. Thus in this section I briefly describe this curriculum as it pertains to algebra, which is the focus of my study. Specifically the document states that at Grade 7 – 9 level (Department of Basic Education, 2011, p. 21):

In Patterns, Functions and Algebra, learners' conceptual development progresses from:

- *A view of Mathematics as memorized facts and separate topics to seeing Mathematics as interrelated concepts and ideas represented in a variety of equivalent forms ...*

CAPS outlines the “Concepts and Skills” for teaching and learning about linear equations. In Grade 8 learners such as those in my study should (Department of Basic Education, 2011, p. 91):

- *Use additive and multiplicative inverses*
- *Use laws of exponents*
- *Use substitution in equations to generate tables of ordered pairs*

In Grade 9 learners revisit linear equations. The types of equations are extended to include quadratic equations and others, but the “skills” remain the same (Department of Basic Education, 2011, p. 132).

The “concepts and skills” are explicated for teachers under the heading “Some Clarification Notes or Teaching Guidelines”. The example below shows how Grade 8 learners should apply their “skills” (Department of Basic Education, 2011, p. 94):

d) Solve x if $3x + 1 = 7$

To solve the equation requires two steps:

Add -1 to both sides of the equation:

$$3x + 1 - 1 = 7 - 1, \text{ therefore } 3x = 6$$

Then divide both sides of the equation by 3

$$\frac{3x}{3} = \frac{6}{3}, \text{ therefore } x = 2$$

The examples provided in the document all indicate that learners should know *how* to solve equations. Although there is reference to the need to link algebraic manipulations to a table of ordered pairs, there is no direct reference to the structure of the equation or of links between the linear equation and the notion of *function*.

Table 1 shows how, based on the CAPS guidelines, the solution of a linear equation – which is a focus of my study – would look:

Table 1: Solution to a linear equation, using CAPS guidelines

Statement	Reasoning that underpins the mathematical action	Example from my study
$ax + b = c$	Add the additive inverse of $+b$ to the expression on each side of the equation to isolate ax on the LHS of the equation	$2x + 7 = 13$
$ax + b - b = c - b$	Simplify each expression	$2x + 7 - 7 = 13 - 7$
$ax = c - b$	To determine the value of x , multiply the expression on each side of the equation by the multiplicative inverse of a .	$2x = 6$
$\frac{1}{a} \times ax = \frac{c - b}{a}$	Simplify each expression	$\frac{1}{2} \times 2x = \frac{1}{2} \times 6$
$x = \frac{c - b}{a}$		$x = 3$

Whilst the curriculum dictates the specific “skills” learners should be taught, the socio-economic environment of the community the school serves influences learning and impacts on the social practices of the classroom and on assessment (Morgan, 1998, p. 79). The implication is that learners’ experience in the classroom in working class schools might differ from learners’ experience in schools serving middle class communities. In my field of work I have access to the classroom as a learning space. The difficulties that learners experience with algebra seem to span the socio-economic divide. Although learners in better-resourced schools perform better on formal assessment tasks, the difficulties they experience when faced with non-routine (unfamiliar) tasks

compare with those experienced by better-performing learners at poorly-resourced schools. This suggests that the *how* might be communicated differently in different school contexts, but the *when* could be neglected across schools.

1.5. Other research on how learners solve linear equations

Before I state the particular focus of my study, I note briefly that much research has been done in South Africa and internationally regarding learners solving linear equations, as well as other related topics in algebra. For example, researchers have studied learners' sense of the structure of the equation, common errors in algebra and learner difficulties with navigating the shift from arithmetic to algebra (Essien & Setati, 2006; Filloy & Rojano, 1989; Hoch & Dreyfus, 2006; Linchevski & Livneh, 1999). Other researchers have shown how mathematics classroom discourse could be a possible lever for modifying learners' ways of mathematizing (Clark, 2014; Gcasamba, 2014; Heyd-Metzuyanin & Graven, 2015; Nachlieli & Tabach, 2012; Tabach & Nachlieli, 2015). I present an overview of this literature in Chapter 2.

Insights gained from a review of the literature have contributed to shaping my own research focus, which is learner discourse in the context of linear equations. I see the potential for improving learners' mathematics by changing interpersonal communication between teachers and learners in mathematics classes. The first step in this regard, is developing a means for describing learner discourse in order to better understand how they think about linear equations. For this purpose I have chosen to use Sfard's (2008) theory of commognition. In the next section I provide a brief overview of the theory. In Chapter 3 I describe the theory in detail and motivate why I have chosen it.

1.6. Commognition as the chosen theory for the study

The term "commognition" stems from the notion that cognitive processes and interpersonal communication processes are different manifestations of the same thing (Sfard, 2008, p. 83). Within the commognitive framework there are deep connections between the terms 'communication' and 'discourse'. By using discourse as the unit of analysis, Sfard is able to conceptualise mathematics learning as participation in mathematical discourse and her theory of commognition provides a framework for measuring whether a learner's discourse is endorsable by the community. This is

useful for my study, which aims to analyse the discourse of learners in order to gain access to their thinking about linear equations.

Sfard (2008, p. 250) refers to the practice of generating solutions to problems through imitation as “ritualized” mathematical discourse. She refers to the practice of deriving new solutions by applying mathematical principles – whether to solve routine or non-routine problems – as “explorative” mathematical discourse (2008, p. 244). By establishing the features of ritualistic and explorative discourse, the theory enables me to develop an analytic framework for classifying learners’ discourse as either ritualistic or explorative, and to determine whether any learners are in a transition phase between the two discourses. A detailed exposition of my analytic framework is presented in Chapter 4.

1.7. My study

In my work with teachers I observe what learners do and say in their classes. My interest in this respect when solving linear equations led to the development of a short course for teachers with a focus on learner thinking. Thus I conducted interviews with learners about a selection from the linear equations they had solved in their assessment task. What emerged in the interviews led me to ask a number of questions about learner thinking that required more intensive research, hence this study using Sfard’s (2008) theory.

Using the theory of commognition to describe learner thinking means I will investigate learners’ communication about solving equations in order to describe their routines for solving a linear equation. As indicated earlier, my analysis will explicate their discourse using Sfard’s concepts of ritual and exploration.

Drawing on my personal experience of working with learners and teachers and with the curriculum, I hypothesise that at Grade 8 and 9 level learners’ routines are, in Sfard’s terms, ritualistic. However, my study aims to identify nuances in their rituals, and in particular to answer the following questions:

1. How can a learner’s discourse when solving linear equations be described using Sfard’s notions of exploration and ritual?

2. Do all learners who have ritualized discourse, display the same characteristics in their talk, gestures and writing? If not, what are these differences?
3. Are there any learners whose discourse is explorative, or are in a transition phase between ritualistic and explorative discourse?

Finding answers to these questions will help me to understand how to intervene in situations where there is seemingly no potential for a transition towards explorative discourse. However, given the limitations of the scope of this study, I only consider learners' discourse at one moment in their learning – at the point of the interview. This means that I cannot make more general claims about their discourse.

1.7.1. Rationale of the study

In my work in teacher professional development it is not unusual to encounter hard-to-understand learner solutions to linear equations like the ones shown in Figure 2. Since teachers have no way to explain why learners produce such solutions, I believe there is value in a study that could ultimately contribute to helping teachers to better understand learner thinking. Conducting the interviews in preparation for a short course had been a first step, but because of what the interviews revealed about the way the learners think about linear equations, I decided – with the necessary permissions – to use them to develop the research texts for this study. By developing a detailed description of learner discourse I can better understand their solutions and their difficulties. This insight will improve my own communication with teachers about their learners.

1.8. Conclusion

In this chapter I have described the general problem of poor school mathematics performance in South Africa, with a particular focus on the problems I and others (nationally and internationally) see learners encounter when solving linear equations. I presented my research focus on the problem, which is a description of learner discourse as a lever for changing mathematics classroom practice. In the chapter that follows I shall discuss research outputs linked to the solution of linear equations, including research using Sfard's commognition theory. Thereafter I describe the theory and present my analytical framework. Lastly, I present my findings and evaluate them in relation to other studies.

Chapter 2: Literature Review

2.1. Introduction

The focus of this study is how Grade 8 and 9 learners think about linear equations. In this chapter I locate my study in the mathematics education research literature by providing a critical literature review of other empirical studies regarding engagement with algebra, particularly linear equations in the context of school mathematics. The review discusses results, as well as the theories and methodologies for these studies. The dichotomy between theories in two research paradigms is presented in order to prepare the reader for the choice of theory that underpins the construction of an analytical framework for my study.

2.2. Two research paradigms in mathematics education

Sfard (2015, p. 129) describes the field of research on learning as a “theoretical battlefield” because while there is consensus that learning implies change, there is no consensus on what it is that changes. This lack of consensus is caused by the “two colliding requirements” of research, namely the need for scientific rigor and the fact that “human ways of acting may be scientifically intractable” (2015, p. 130). This, she argues, has given rise to two research paradigms – the “acquisitionist” paradigm (2015, p. 130), which embraces the cognitive theory of learning and the “participationist” paradigm (2015, p. 130) which subscribes to a theory that learning is aligned to communication. Whilst theories from the acquisitionist paradigm dominate in earlier studies, there has been a shift to participationism in the past 25 years (Lerman, 2000). Earlier studies in my review subscribe to cognition theory, while most of the later studies are located in theories aligned to discursive approaches.

2.3. Measuring the acquisition of knowledge

2.3.1. Theoretical approaches

Most cognitive theorists in mathematics education are aligned to Piaget’s theory of learning and developed “homegrown theories of mathematical thinking” based on his theory of human development. Among these, the “process-object” theories gained much support from mathematics researchers (Sfard & Cobb, 2014, p. 7). Proponents theorise that the attainment of true understanding of a mathematical concept followed a hierarchically-structured trajectory where the

student's skill shifted from performing processes upon a mathematical object, to developing a conceptual perspective on the object. These theories include Sfard's (1991, p. 1) theory of "reification", Gray and Tall's (1994, p. 6) theory of the "procept", and Dubinsky and McDonald's (2001, p. 2) APOS theory. Other theories distinguish between operational and relational understanding (Skemp, 1976) and structure sense (Linchevski & Livneh, 1999).

The term 'structure sense' as coined initially by Linchevski and Livneh (1999), and expanded upon by Hoch & Dreyfus (2004), links to relational understanding. Hoch & Dreyfus developed a definition of structure sense as it pertains to algebra, the focus of my study:

"Structure sense, as it applies to high school algebra, can be described as a collection of abilities. These abilities include the ability to [...] recognize mutual connections between structures, [...] and recognize which manipulations it is useful to perform" (Hoch & Dreyfus, 2004, p. 50).

This aligns with Skemp's (1976, pp. 14–15) distinction between operational understanding, where the learner depends on outside guidance to reach the solution (goal) and is not aware of "the overall relationship between successive stages"; and relational understanding, where the learner has a conceptual structure (schema) that allows for independent decisions about how to solve mathematical problems. By implication, operational understanding refers to knowing procedures (algorithms) for solving equations and relational understanding refers to knowing the properties that define the structure of an equation.

2.3.2. Studies relating to algebra and linear equations

The concept of structure sense has dominated studies on how learners relate to algebra, including linear equations. These studies have influenced my perspective on the various factors that impact on how learners think about linear equations, from the structure of the equation (Fillooy & Rojano, 1989; Hoch & Dreyfus, 2006; Linchevski & Livneh, 1999) to the significance of the equal sign (Asquith, Stephens, Knuth, & Alibali, 2007; Essien & Setati, 2006). Some studies have influenced the methodology of my study (de Lima & Tall, 2008; Gripper, 2011a; Herscovics & Linchevski, 1992).

Linchevski and Livneh (1999) showed that learner difficulties in algebra stem from not having a structural sense of the number system. They posited that when learners performed arithmetic operations with numbers, they could achieve success by working solely operationally. However, if

learners were not encouraged to use the properties and conventions of the number system to achieve operational proficiency, they would not have access to those same properties and conventions when performing arithmetic operations with algebraic terms. Thus they would not have access to the means for gaining relational understanding in algebra (Linchevski & Livneh, 1999).

Based on a questionnaire designed to measure structure sense and procedural knowledge with a group of Grade 10 learners, Hoch & Dreyfus (2006) found that the majority of learners in their study did not use structure sense as a resource when solving algebraic problems. However, those that used structure sense made fewer errors. Whilst these findings were not unexpected, the researchers were surprised that there was no correlation between learners' structure sense and their manipulation skills. This difficulty of defining the attributes of mathematical understanding is an example of Sfard's (2008) quandary of understanding mentioned in Chapter 1.

Studies by Asquith et al. (2007) and Essien and Setati (2006) both looked at how learners related to the equal sign in the early high school years, and how lack of relational understanding negatively influenced their flexibility. Asquith et al. saw a correlation between learners' relational understanding of the equal sign, and their ability to solve equations (Asquith et al., 2007, p. 308). Essien and Setati (2006) concluded that the dominant interpretation of the equal sign among senior phase learners was still operational, which they interpreted as a barrier to the manipulation of more sophisticated algebraic equations.

Another frequently referenced study on linear equations was Filloy and Rojano's (1989) study with 12 - 13-year-olds in Mexico. The study showed that learners responded differently to linear equations where there was only one term on the left-hand side containing the variable, as opposed to equations where the variable occurred on both sides. They referred to this phenomenon as the "didactic cut" (1989, p. 20). They posited that solving equations containing one variable on the left-hand side was an arithmetic operation, whereas equations with the variable on both sides represented the learner's "evolutionary line of thought from arithmetic to algebra" (1989, p. 20). This, they said, mirrors the development in the history of algebra towards the possibility of operating on an unknown entity. In this experimental study they provided a *mathematical* explanation for the differences they saw when learners solved linear equations that differed in structure.

Herscovics & Linchevski (1994) conducted an experimental research study with Grade 7 learners to investigate their strategies for solving linear equations. They critiqued Filloy and Rojano's (1989)

study from the perspective that it did not provide enough permutations of the linear equation to provide adequate data to describe what learners did. While their findings supported Filloy and Rojano's that there was a distinction in the way learners operated when the variable appeared multiple times in an equation, they differed in the sense that Filloy and Rojano saw the distinction when the variable appeared on both sides of the equal sign, whereas Herscovics and Linchevski saw the distinction regardless of where the variable was situated. Both agreed that the distinction marked a shift from arithmetic to algebra. Herscovics & Linchevski (1994, p. 63) referred to this as the "cognitive gap" and provided an explanation for learners' cognitive difficulties based on the fact that when the variable appeared only once, the equation could be solved arithmetically, but when it appeared multiple times, the same procedure would not necessarily work for the learner and they needed algebraic strategies

In a later study Herscovics and Linchevski (1996) conducted a research experiment with six Grade 7 learners over a period of three lessons, to teach them to use the cancellation principle to simplify equations. Learners had to decompose individual terms to transform the equation to a form where one side mirrored the other. I included this strategy in the design of my interviews. Where learners had very successfully solved the equation ' $2x + 7 = 13$ ', I then presented them with the equation ' $2x + 7 = 6 + 7$ ' to determine whether they could use their structure sense (as used in my study – horizontal equivalence) to determine whether ' x ' would have the same value.

The cancellation theory also featured in a study by Gripper (2011b), which formed part of a set of studies with Grade 10 learners in South Africa (Davis & Gripper, 2012; Gripper, 2011a, 2011b). Gripper (2011a) selected the most competent learners at three schools on the premise that what these learners constituted as mathematics would reflect what their teachers presented as such in the classroom. I adopted this methodological strategy when selecting learners for my study. The design of Gripper's (2011b) pilot study made it possible to determine whether learners could use structure sense rather than algorithms to solve equations, which mirrored the strategy of Herscovics and Linchevski (1992).

Two other studies that critiqued Filloy and Rojano's (1989) concept of didactic cut were the de Lima & Tall (2008) study with 15-year-old Brazilian learners and the Andrews (2011) study with one teacher each from Finland, Flanders and Hungary. Both studies used the concepts of conceptual and structural embodiment to view how learners solved linear equations. Andrews' teachers used the

balance scale as a form of embodiment for inducting learners to the notion of the equation as an entity that constitutes equivalence. de Lima & Tall's learners shifted symbols in a manner they described as human embodiment, whereby symbols were picked up and moved around.

In their study de Lima & Tall (2008) asked learners to solve the equation ' $4m = 2m$ ', an equation also used in my study. They noted that only 10% could solve the equation, and that no learner considered a possible value for ' m ' that would make the equation true. They posit that the problem lay in the embodiment of ' m '. If ' m ' were a physical object like a melon, ' $4m = 2m$ ' would not have been possible. Barahmand and Shahvarani (2014) conducted a similar study to de Lima and Tall's (2008), with Iranian learners. They found that 43% of the learners obtained the correct solution using the same algorithm. They concluded that the learners solved equations based on their arithmetic knowledge, and that "many of their errors in understanding equations were related to the concept of variables and algebraic expressions" (Barahmand & Shahvarani, 2014, p. 181). This resonates to some extent with Herscovics & Linchevski's (1994) findings, and also with those of Gcasamba (2014), which I discuss in Section 2.4.

2.3.3. Studies relating to learner misconceptions

A number of studies of how learners solve linear equations, have focused on learner errors and have explained these errors using the concept of *misconceptions*. For example the concept *function*, which has no real-world counterpart, is misconceived in similar ways across different contexts and mirrors the difficulties faced by mathematicians through history (Gcasamba, 2014, pp. 16–17). Particularly, studies by Hall (2002) and Seng (2010) show that learners across geographic and socio-economic contexts present the same misconceptions. Errors include the "change side – change sign" technique (Kieran, as cited by Hall, 2002, p. 12) and erroneous application of the distributive law (Seng, 2010, p. 151). Sfard (2008) presents this as a quandary, since the same misconceptions appear in classrooms across the world, even when curricula and teaching methodologies differ. An advantage of determining common misconceptions encourages the development of teaching strategies to counter them, although these strategies could favour an operational rather than structural orientation to the content. Another advantage of identifying common misconceptions is that it has led to the development of an often-referenced list of the "most common idiosyncrasies of the learner's mathematical thinking" and has given researchers insight into how learners think and that "learning is not a linear process" (Sfard & Cobb, 2014, p. 10).

2.3.4. Reflections on studies of learner thinking about linear equations within the acquisitionist paradigm

The studies reviewed thus far were all conducted from a cognitivist perspective. They have contributed to how the mathematics education community understands what is measured in order to determine changes in learner conceptions, either through knowledge being transmitted by teachers and texts, or constructed by learners. In particular, it has informed my own understanding of how learners think about linear equations, especially the difficulties learners experience when they are challenged to utilize the structure of the equation rather than an operational approach for solving equations. Some studies I have reviewed influenced my criteria for the selection of learners for my study and also in the choice of equations for my interviews. However, these studies do not explain *why* learners think as they do and what has influenced their thinking when solving linear equations. In Sfard's (2008, p. 53) terms, they do not resolve the "quandary of understanding" related to this topic.

2.4. Filling the gap left by the quandaries

It has become clear that looking at learning as the acquisition of an existing body of knowledge leaves some unanswered questions, which Sfard (2008) refers to as quandaries. These issues give voice to researchers whose theories subscribe to the participationist approach and who regard learning mathematics as "the process of becoming a participant" in mathematics as a distinct form of human activity (Sfard & Cobb, 2014, p. 4). While a number of theories can be subsumed under this paradigm, I subscribe in this study to a particular version, which is Sfard's (2008) theory of commognition. As noted in briefly Chapter 1 this theory sees thinking as communication, and learning as coming to participate in mathematical discourse.

Stahl (2008) in his review of Sfard's (2008) book, demonstrates the value of her theory for addressing the quandary of understanding. Particularly, he mentions how the theory helped him as a researcher, to investigate learners who knew that they had solved an equation correctly but could not explain why (Stahl, 2008). This resonates with my focus, because the learners in my study all know *how* to solve equations, but not necessarily *why* their solutions are endorsed. Thus Sfard's theory gives me access to how learners think about the mathematics, as well as the factors that influence

their thinking. This knowledge has implications for my professional development work with teachers. In the section that follows, I review research outputs that use this theory.

2.4.1. Studies using commognition as a theory

There is a growing body of researchers who have used commognition to study various aspects of mathematics amongst learners. To date there have been two special issues (Tabach & Nachlieli, 2016; Thurston, 2012) for such research outputs. In the editorial of the latest special issue, Presmeg (2015, p. 1) comments that “the analysis of how commognitive theory is used in the papers suggests that it is broad enough to be a useful theoretical lens in diverse settings ... (but) that there is much more potential for use of the theory than has been realized”. In these and other journals, research outputs using commognition have included the shift from arithmetic to algebra (Caspi & Sfard, 2012), functions (Clark, 2014; Gcasamba, 2014; Nachlieli & Tabach, 2012; Tabach & Nachlieli, 2011) and geometry (Sinclair & Moss, 2012). Furthermore, Viirman (2011), Berger and Bowie (2012) and Bogdanova (2012) all used the theory to study the discourse of teachers or lecturers or to develop courses for teachers. I focus here on researchers that have used this theory to study learners’ thinking about mathematics.

A number of researchers use commognition to study change in learner discourse over time (Heyd-Metzuyanim, 2015; Nachlieli & Tabach, 2012). Given the limitations of this study, I use the theory to describe learners’ discourse at a particular moment in their learning. Furthermore, my literature review did not reveal any studies using commognition, which focused exclusively on linear equations. Given that a linear equation is, in Sfard’s (2008, p. 224) terms, “an endorsable narrative” about functions, and that the CAPS curriculum (Department of Basic Education, 2011) is explicit about the links between equations and other representations of functions, I focus on studies pertaining to functions. The studies I cite use the theory and methodology in a way that informed the development of my own methodology.

Some studies have combined commognition with other theories. Gcasamba (2014) combined commognition with the cognition theory of error analysis to research why learners perform badly on questions relating to ‘function’. Unlike my qualitative study, she used both quantitative and qualitative data. Using error analysis enabled her to identify the *what*. In other words, she could identify error-types, some of which resonate with the idiosyncrasies seen in international studies such as Seng’s (2010). Using commognition enabled Gcasamba (2014) to access the *why* and *how*,

because she could describe what learners were thinking when solving the problems on function. By analyzing learners' word use, visual mediators, narratives and routines she came to the conclusion that learners had not developed to the extent that they were using mathematical discourse appropriately at all times. The resultant disconnect between visual mediators and their word use led to routines that caused them to produce errors. Furthermore, their word choice indicated an "action-oriented" approach to the problems, which indicated that they had not objectified 'function' – they could see the process but not the object (Gcasamba, 2014, p. 8).

Tabach and Nachlieli (2011, p. 2529) combined commognition with systemic functional linguistics (SFL) to develop a framework for studying prospective teachers' learning. Within the paradigm of SFL, they differentiated between different categories of classroom discourse – "the mathematical, the social and the organizational". Drawing on Sfard's (2008) commognition theory, they broadened their framework to include gestures and visual mediators. Their findings were that there is a difference between the way the teacher and the learners use words, visual mediators, routines and narratives. The learners resorted to colloquial word use that led to an imprecise expression of ideas. As a result of the course, the teacher's pedagogy was adapted to encourage learners to become more active participants and so grow their mathematical discourse. I am interested in the way Tabach and Nachlieli (2011) combined SFL with commognition. As described in Chapter 4, in my study I use some of the features of language to investigate learners' word use.

In a later study, Tabach and Nachlieli (2012) focus on the autopoietic nature of the mathematical object 'function'. Specifically, the study looks at how the notion of 'function' can be nurtured through learning about equations. They discuss learner difficulties with the discursive construction of the function and acknowledge that learners rarely see the familial link between the algebraic formulation and the graph. Their findings are that even when learners are exposed to the word, they do not incorporate it into their discourse but stay at the level of the component discourses on equations and graphs, and focus on the operational rather than the structural. They observe that "having recourse to well-established routines proved [...] an effective survival technique for those immersed in other people's discourse" (Nachlieli & Tabach, 2012, p. 23). This resonates with Gcasamba's (2014) findings, because her study also shows that learners were using relevant words, but had not objectified 'function'.

Both Gcasamba (2014) and Tabach and Nachlieli (2012) show that learning begins with an action-oriented approach to function as an object. For my study, I would look for evidence in their discourse of either the action-oriented perspective on linear equations, or the shift towards a more objectified perspective.

A study that provided useful methodological insights was Sfard's (personal communication³, January 21, 2014) South African classroom study, regarding a lesson on inequalities. In the study Sfard first describes the teacher's discourse, then analyzes the learners' discourse. The study shows how, when the teacher's discourse does not communicate the mathematics effectively, it impacts on how learners repeat the story about the mathematics. While the topic of inequalities is not the focus of my study, Sfard's methodology involved a careful recording of learners' words and actions. The latter gave an indication of the visual mediators. Using their written solutions, their verbal explanations and visual mediators, Sfard could give a detailed description of their discourse by looking at their levels of objectification, the extent to which their narratives were endorsed, as well as the goal of their activity. In my study, I use the same tools to develop my descriptions of learner discourse.

2.5. Conclusion

Given my interest in learner thinking about linear equations, I have reviewed studies that look at this phenomenon from different perspectives using theories in both the acquisitionist and participationist paradigms. While the cognitive theorists focus on learners' mental structures when thinking about linear equations, researchers who subscribe to the commognitive theory describe learning in terms of change in learners' discourse by focusing on different features of the discourse.

As indicated earlier, I have chosen to locate my study within the participationist paradigm, using Sfard's (2008) commognition theory. In the next chapter I present a detailed discussion of the features of the theory.

³ From the book "Research for Educational Change: Transforming researchers' insights into improvement in mathematics teaching and learning", edited by Adler and Sfard (2016), currently in press.

Chapter 3: Theory

3.1. Introduction

To investigate how learners think about linear equations at one moment in their learning, I aim to analyse their discourse, for which I need access to what they say and do. I reviewed different ways of studying learner thinking about linear equations in Chapter 2, and choose to adopt the view that learning can be viewed as a change in discourse. Therefore I use Sfard's (2008) commognition theory that presents thinking as communication, as an appropriate means of developing analytic tools for what learners communicate through their discourse.

Crotty's (1998, p. 2) framework, which I have used in my research design, poses a four-step process, namely epistemology, theoretical framework, methods and methodology. In Chapter 2 I discuss participationism as the theory of learning that underpins commognition. Because Sfard and Cobb (2014) say that knowledge is discursive, the epistemology of commognition is social constructivist. In this chapter I present the theoretical framework for my study and propose that this theoretical perspective suits the methodology and methods I present in Chapter 4.

The theory of commognition provides a lens for analyzing the stories learners tell about the linear equation. This is achieved by looking at different elements of the discourse from the perspective of what is or is not endorsable by the mathematics discourse community. The particular concepts from Sfard's (2008) theory that I use are the notions of mathematical object, mathematical discourse (keywords, mediators, narratives and routines), and explorative and ritualized routines. In this chapter I discuss the theory with a particular focus on its use for viewing thinking about linear equations. Thereafter I motivate for its use in my study.

3.2. The theory of commognition and its underpinnings

As noted earlier, Sfard's (2008) theory can be located in the participationist paradigm that how we think is influenced by the community we find ourselves in, and the community's stance on any particular activity or subject. Sfard acknowledges the influences of Wittgenstein (1961, as cited by Sfard, 2008) and Vygotsky (1987, as cited by Sfard, 2008) on the development of her ideas. Wittgenstein's notion of a disobjectified discourse on thinking influenced her quest to develop a theory of disobjectified discourse on mathematical thinking. Vygotsky's theory foregrounds the

“inherently social nature of human practices” (Sfard, 2008, p. 77) and influenced her position that thinking mathematically means communicating with oneself or others in a particular way that is acceptable to the mathematics community. Her theory of thinking conceptualises mathematics as a well-defined form of communication, and mathematics learning as participating in a discourse.

Sfard’s (2008) term ‘commognition’ stems from the notion that cognitive processes and interpersonal communication processes are different manifestations of the same thing. The word ‘commognition’ is a combination of ‘communication’ and ‘cognition’, which symbolizes the unanimity of these processes. Within the commognitive framework there are deep connections between the terms ‘communication’ and ‘discourse’. Sfard notes that there are different types of communication, each with specific objects, mediators and rules. She calls each a ‘discourse’.

The word ‘discourse’ implies the use of words and symbols in a way that is generally endorsed by members of a community. She further explains that mathematics is one such type of communication; mathematical discourse communicates mathematical ideas that are ratified by the body of theorems, proofs and laws that govern mathematics. Members of the mathematics community are unified by participation in mathematical discourse through what is mediated by mathematical keywords and symbols. Sfard argues that the discourse of school mathematics is less rigorous than that of professional mathematicians. However, the features that unify them as ‘mathematical discourses’ are their word use, visual mediators, narratives and routines (2008, p. 133).

Sfard (2008) argues that unlike the objects of many school subjects, mathematical objects are abstract and not accessible to our senses. Learners therefore construct these objects through their discourse and we speak of discursively constructed objects. Thus mathematics is described as an autopoietic system, because it is “a system that contains the objects of talk along with the talk itself” (2008, p. 129). It is this feature, Sfard notes that makes mathematics difficult to learn.

In my study, learners work with mathematical objects like integers, rational numbers, variables and algebraic terms and functions as they are related in the linear equation. The proxies of these mathematical objects, the symbols and icons are merely representations of the objects, which are discursive constructs. Sfard and Cobb (2014) view learning as coming to participate in this special discourse. The study looks at how learners participate in the discourse of school mathematics as it applies to linear equations because having insight into how they think, will benefit my work with

teachers to improve their learning. Before I discuss Sfard's (2008) conception of mathematical objects, I shall discuss the features of the discourse itself.

3.3. Features of mathematical discourse

Sfard's (2008) unit of analysis is discourse, of which the four characteristics are keywords, visual mediators, narratives and routines. I explain these next, with reference to linear equations.

3.3.1. Words, visual mediators and narratives

According to Sfard (2008), the tools of the discourse are keywords and visual mediators. In the context of the study, keywords are words that signify quantity like numbers, variables and function. Visual mediators are discursive prompts like symbolic artifacts, including numerals, tables, algebraic expressions, equations and graphs created to communicate relationships and operations with mathematical objects. For example, when a learner says "two x plus seven equals thirteen", she is using the keywords 'two x ', 'seven', 'equals' and 'thirteen'. The visual mediators are ' $2x$ ', '7', '=' and '13' respectively. Alternatively, she could be talking about the linear equation ' $2x + 7 = 13$ ', which communicates a relationship about the linear function, ' $f(x) = mx + c$ ' – which in this case is the mathematical object. In my study I will be looking at the words and visual mediators learners use and whether they are used appropriately in the context of mathematical discourse.

According to Sfard (2008), words and visual mediators are used to produce narratives. A narrative is any text, spoken or written, that is "framed as a description of objects, of relations between objects, or processes with or by objects" (Sfard, 2008, p. 300). Thus narratives are a description of what is done with mathematical entities. Sfard introduces the terminology 'endorsed narrative' (2008, p. 134) to indicate that the narrative is true. Narratives are constructed by learners using words and visual mediators, but they can also be constructed by researchers as interpretations of what they see in the learner discourse. The latter pertain to how researchers interpret learners' descriptions about, and justification for their procedures, based on their use of words and visual mediators.

To illustrate how Sfard conceptualizes words and visual mediators in the construction of narratives, I cite an incident from Sfard (2008) where two students, Ari and Gur are given a table and have to determine $g(6)$:

Table 2: Ari and Gur's source to determine $g(6)$ (Sfard, 2008, p. 18)

X	$g(x)$
0	-5
1	0
2	5
3	10
4	15
5	20

Pointing to the right-hand column (see Table 2), Ari states that the intercept is '-5' and later says "...the intercept is the zero (points to the 0 in the left column)" (Sfard, 2008, p. 152). Two keywords are 'intercept' and 'zero', while '0' in the left-hand column is a visual mediator. Sfard presents Ari's procedure as (Sfard, 2008, p. 153):

- 1. Find the zero in the left column of the table*
- 2. In the right column of the table, find the number b corresponding to that zero*

She has utilized Ari's words and visual mediators to construct a narrative about his procedure.

She employs a similar strategy in a study where she analyses a teacher's (Mr P's) discourse to determine whether it counts as explorative discourse. She further posits that the only legitimate statement she can make about his mathematizing is how it appeared to be. Thus she submits that in building her argument she relies on that which is *perceptible* (personal communication⁴, January 21, 2014).

Adler and Venkat (2014) also consider learner narratives in their research on how teachers respond to learners in a lesson. In my study I also construct narratives based on what is perceptible from

⁴ From the book "Research for Educational Change: Transforming researchers' insights into improvement in mathematics teaching and learning", edited by Adler and Sfard (2016), currently in press.

learners' words and visual mediators. This is Nadia's solution to the equation $7(x - 3) = 7 - 2(3x + 1)$:

Line 1	$7(x-3) = 7-2(3x+1)$
Line 2	$7x-21 = 7-6x-2$
Line 3	$7x+6x = 7-2+21$
Line 4	$13x = 26$
Line 5	$= 26 \div 13$
Line 6	$= 2$

Figure 3: Vignette 4 - Nadia's solution to ' $7(x - 3) = 7 - 2(3x + 1)$ '.

Nadia's narrative about her action in Line 1 is:

“... negative two times positive three ex is negative six ex because two times three is six”
(Nadia – Vin 4, line 8a).

‘Negative two’ and ‘positive three ex’ are keywords that signify quantity, and ‘times’ is a keyword signifying a mathematical operation. Besides the visual mediators ‘-2’ and ‘3x’, the brackets are a visual mediator that signal multiplication between the two numbers. From her statement I constructed the narrative: “ $-2 \times (+3) = -6$ because $2 \times 3 = 6$ ”. I describe my methods for doing this in Chapter 4.

From the foregoing I conclude that narratives are constructed by learners using words and visual mediators, but they can also be constructed by researchers as interpretations of what they see in the learner discourse. In my study I construct narratives based on my interpretation of what learners say about objects and about actions with and by objects. Further details will be provided in Chapter 4.

3.3.2. Routines

3.3.2.1. Rules that define routines:

Sfard (2008) describes the relationship between narratives and routines:

Specifically, mathematical regularities can be noticed whether one is watching the use of mathematical words and mediators or following the process of creating and substantiating narratives about numbers or geometrical shapes. (2008: 134).

The ‘regularities’ Sfard refers to are the routines. She describes routines as “the anatomy of mathematizing” (2008, p. 220). A routine may be a procedure; it may also be a practice such as generalizing, justifying or endorsing (or rejecting) mathematical narratives. Thus, routines are rule-based. A commonly referenced routine that learners follow when solving linear equations, is the “change sides, change signs” routine (Kieran, as cited by Hall, 2002, p. 12). This is a rule that most of the learners in my study reference, and which they justify by citing the teacher’s authority.

Sfard (2008, p. 201) identifies two categories of rules that define routines: “object-level rules” and “meta-level rules” (metarules). Whereas object-level rules depict regularities in the behavior of discursive objects, metarules reflect the structured, regular nature of discursants’ actions and define the patterns of their activities (Sfard, 2008). For example, the ‘change sides, change signs’ routine is a metarule that regulates learners’ response to linear equations where the position of algebraic terms is reorganised. Learners in my study explain ‘the sign changes when you take it over’. I will show that there is no object-level rule guiding their routine because there is no mathematical property that endorses their narrative.

According to Sfard (2008), one set of metarules pertains to the *how* of a routine. The other pertains to the *when*. Both sets can either determine or constrain the course of action. Sfard suggests that in school mathematics the focus is on the *how* of the routine, which could constrain learners’ mathematical development.

3.3.2.2 Ritualized and explorative routines

Sfard (2008) identifies three types of routines – deeds, rituals and explorations. Deeds involve change to physical objects and are not relevant to my study, which involves engagement with abstract (mathematical) objects. The routines of significance for my study are rituals and explorations. I discuss these briefly here and revisit the detail after explaining Sfard’s notion of mathematical object.

According to Sfard (2008), rituals are characterized by strict rules that are determined by an authority (the teacher). The discourse of rituals is limited to justifying how to do something, but not when to do so or why it works. The fact that mathematics is an autopoietic system, as indicated in Section 3.2, makes the discourse difficult to learn. Thus learners first imitate others, which makes rituals an acceptable interim phase in the learning process (Sfard, 2008). Ideally, the learner will

gradually gain an understanding of the ‘how’ and ‘when’, which is one of the markers of the transition from the discourse of rituals to explorative discourse, which I turn to next.

Explorations are the most sophisticated form of routine resulting in the production of narratives about mathematical objects that are endorsable in terms of mathematical axioms, definitions and theorems. From the perspective of school mathematics there are three processes that could count as exploratory routines. These are construction, which results in new endorsable narratives, substantiation, which endorses previously constructed narratives, and recall, where previously endorsed narratives are reconstructed (Sfard, 2008). I present detail of the features of ritualistic and explorative discourse in section 3.5. First, I describe the discursive construction of mathematical objects within the context of school mathematics, with a particular focus on the linear equation as a discursive construct.

3.4 Sfard’s concept of a mathematical object

As noted in Chapter 2, Linchevski & Livneh (1999), Essien & Setati (2006), Hoch & Dreyfus (2004), and Davis and Gripper (2012) in one way or another, and using different terminology, all refer to the need for learners to develop a sense of the structure of the mathematical objects they work with. Although this is not a focus of my study, it should be noted that there are wide debates in mathematics and mathematics education about the nature of mathematical objects. In Sfard’s (2008) theory, objects such as numbers, variables and functions are constructed through discourse, and she presents the realization tree as a visual representation of the structure that emerges through the discourse. Figure 4 shows a realization tree for the solution to the linear equation ‘ $2x + 7 = 13$ ’.

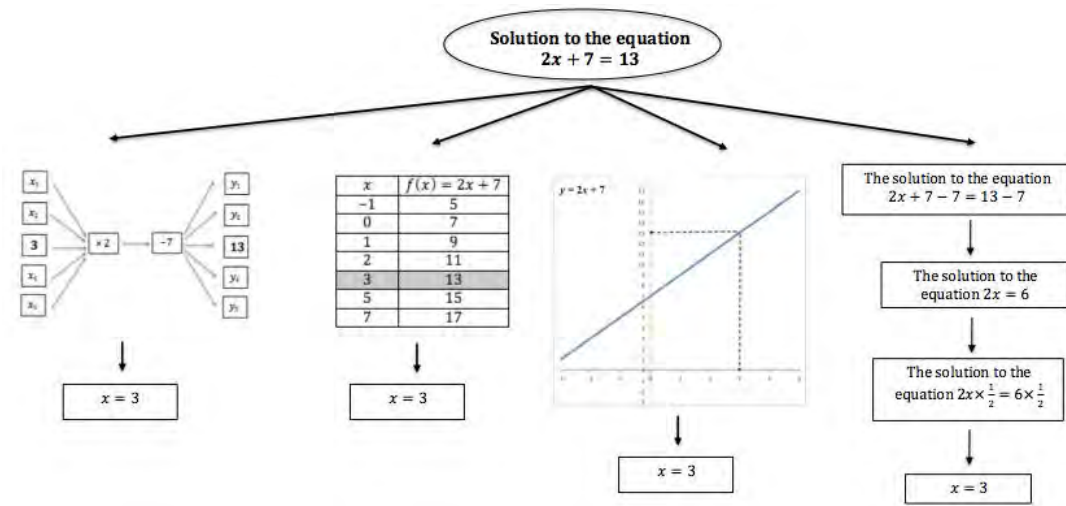


Figure 4: Realization tree for the solution to the linear equation $2x+7=13$

A key feature of Sfards' (2008) notion of a mathematical object is the relationship between signifier and realization. A signifier associates meaning between one object and another. In the realization tree in Figure 3.3, the equation ' $2x + 7 = 13$ ' is a signifier associated with the realization of the equation ' $2x + 7 - 7 = 13 - 7$ '. Thus ' $2x + 7 = 13$ ' and ' $2x + 7 - 7 = 13 - 7$ ' are a signifier-realization pair. There are four nodes on the fourth branch of the realization tree; each node represents a realization of the signifier in the node above it. In summary, the realization tree is "a hierarchically organized set of all the realisations of a given signifier", together with their realizations (Sfard, 2008, p. 301).

Figure 4 shows that the original signifier could produce different realizations, each of which would be a separate branch on the tree and would generate its own set of nodes (2008: 165). In this realization tree there are four realizing procedures for the signifier, each appearing as a separate branch. One branch of the realization tree shows that the solution of the equation is the x -coordinate of a point on the graph ' $y = 2x + 7$ '. That realization then becomes a signifier for the realization that for the output $y = 13, x = 3$. Each of the other branches signifies realizations leading to the same final output. The algebraic solution appears as the fourth branch on the tree. I now discuss this branch in more detail.

Sfard (2008) posits that the endorsement of each realization is communicated by the narrative – a story that justifies the realization by showing its relationship to the original signifier. In my example in Figure 4, the first realization, ' $2x = 13 - 7$ ', is justified by applying the additive inverse of '+7' to

the expression on either side of the equal sign. This justification would be part of the narrative, which is endorsable because equivalence is preserved. Table 3 below shows the iterative relationship between signifiers and realizations for the solution of the equation.

Table 3: Signifier-realization-narrative framework for the solution of '2x + 7 = 13'

Signifier	Realization	Realizing procedure
The solution to the equation $2x + 7 = 13$	$2x = 13 - 7$	Apply the additive inverse of (+7) to both sides of the equation
The solution to the equation $2x = 13 - 7$	$2x = 6$	Add the like terms
The solution to $2x = 6$	$x = 3$	Apply the multiplicative inverse of 2 to both sides of the equation.

Sfard (2008, p. 166) notes that the realization tree is a “personal construct” because it represents the learner’s discourse. My study considers the level to which learners have discursively constructed the solution to the linear function. As part of my analysis I construct realization trees as visual representations of the discursive objects they construct.

3.5. More about routines in school mathematics

Having defined Sfards’ notion of a mathematical object, I return to my discussion of routines in school mathematics. As noted, Sfard (2008) posits that the aim of school mathematics is to tell stories (i.e., produce narratives) about mathematical objects. Because of their autopoietic nature, mathematical objects are difficult to communicate about. Thus, the routines that teachers most often impose on learners are focused on the *how* rather than the *when* of the routine. Ultimately this defines the type of routine learners engage in – an issue that is central to my study.

Sfard (personal communication, January 21, 2014, p. 3) in a letter directed at teachers, says that school mathematics should empower learners to “be the masters of their mathematical activity [and to] decide when and how to use mathematics for their own needs”. This statement defines explorative routines. The opposite would be ritualistic routines, where learners are trapped in a cycle of performing mathematical rituals whose sole aim is to adhere to social norms (personal communication, January 21, 2014). To acquire explorative routines, learners need to have a strong

sense of mathematical objects so that these become the source of their narratives. Whilst learners familiarize themselves with new mathematics by imitating the teacher's procedure, imitation is not the aim of school mathematics; ultimately learners should have access to explorative discourse to be able to justify their solutions and apply the procedures in unfamiliar contexts

3.6. Features for classifying routines as either rituals or explorations

Sfard (2008) identifies a set of features for classifying ritualistic and explorative routines. I refer to these features in my discussion. A key feature of explorative discourse is talk about mathematical objects, which renders the discourse objectified. Objectified discourse is characterized by endorsed narratives about mathematical objects (numbers, variables and functions). Within an equation, this implies the use of words and phrases in one line of an equation that signify numbers and algebraic terms as mathematical objects. Discourse about equivalence between the left-hand side and right-hand side of an equation (horizontal equivalence) and discourse about one equation as a signifier that realizes an equivalent narrative (vertical equivalence) are also features of objectified discourse (Sfard, 2008).

In ritualistic discourse keywords would be used in a “phrase-driven” (2008, p. 243) way. This implies that the learner would talk about numbers, variables and other mathematical objects as disobjectified entities, either by separating integers and algebraic terms into parts – which renders them digits, letters and operatory signs, or spatially arranging them in the equation. In summary, the way learners use words and visual mediators is different for ritualistic and explorative discourse. When the learner talks about and acts on entities in a disobjectified way, it is an indicator of ritualistic discourse as opposed to when learners' word use and mediator use signify numbers and algebraic terms as mathematical objects in their own right.

Another key feature that is closely aligned to objectified discourse, is the extent to which learners produce endorsed narratives (Sfard, 2008). Before discussing this it is necessary to clarify the interrelationship between the source of the narrative, the narrative itself and routines. Narratives are the stories about entities and their relations, and processes with these entities. They provide the *why* of the routine. The source of these narratives could be a previously endorsed narrative or some other authority like the teacher. Sfard (2014) states that “only those narratives are endorsed that can be

logically deduced from stories already endorsed” whereas the source of other narratives would be “recourse to memory or to authority” (personal communication⁵, January 21, 2014).

In explorative discourse endorsed narratives are produced about the properties of objects, as well as relationships between objects and actions with objects. Furthermore, these endorsed narratives would be the source of the narrative for the realizations of signifiers (Sfard, 2008). For example, if the equation $2x = 6$ signifies the realization $x = 3$, the learner might explain the use of inverse operations because ‘2’ is the multiplier of ‘ x ’, which signifies that both ‘ x ’ and ‘6’ should be divided by ‘2’. In this case the property that horizontal equivalence should be preserved in an equation, is the source of the narrative. Alternatively, in ritualistic discourse learners use visual appearance and spatial arrangement as the source of the narrative, or refer to the authority of another person, like the teacher or the interviewer.

The goal of the activity (the closing condition) indicates why the routine is performed. In ritualistic discourse the goal of the activity is to form a social bond with others, which implies following the same procedure as others (Sfard, 2008). The goal is also to find a solution, which is synonymous with finding an ‘answer’. Thus the goal when solving a linear equation is seen as getting to a solution in the form $x = [\text{some number}]$. In explorative discourse, however, the goal is to produce an endorsed narrative. Sfard (2008, p. 155) describes a learner’s (Jas’s) response to finding a solution to the equation $7x + 4 = 5x + 8$. His narrative clearly describes how he visualizes the y-intercept and the linear path of each function to an approximate point of intersection. The closing condition of his story is the production of an endorsed narrative.

By whom and for whom a routine is performed also indicates the nature of the routine. A feature of ritualistic discourse is when the learner follows “other performers’ rules” (Sfard, 2008, p. 242) and the procedure is scaffolded for the learner. In explorative discourse the learner demonstrates a sense of internal persuasion and does not require scaffolding. These features relate closely to the learner’s level of flexibility and, by extension, correctibility. A learner who performs the routine with and for others has a lower level of flexibility to change the *how* of the routine than the learner whose performance is independent of others. Thus, to correct a situation would imply repeating the procedure entirely – often with the same outcome (Sfard, 2008).

Sfard (2008) indicates that often a learner’s routine contains features of both ritualistic and explorative discourse because the transformation is a gradual process. It could be that a person’s

discourse with respect to some mathematical objects is more explorative than with others. For example, a learner who is internally persuaded when talking about the sum of two positive integers might not demonstrate the same level of independence when talking about the sum of two negative integers. I will show in Chapter 5 that learners often apply a metarule that splits integers into signs and digits and performs separate operations on them.

Sfard (2008, p. 249) states that learners display “thoughtful imitation” when their discourse is in the transition phase between ritualization and explorative discourse. This implies that learners in the transition phase would not merely imitate, but would try to make sense of their routines. The time taken for the transition from ritualistic to explorative discourse differs from learner to learner because gaining access to a discourse is gradual (Sfard, 2008). Preferably one would describe shifts in discourse, but given the restrictions of my study I describe learners’ discourse at one particular moment in their learning. This rich description allows me to look for differences in their discourse and to determine whether any have explorative discourse or are in transition towards it.

3.7. Conclusion

I identified in Chapter 1 that I am interested in learner thinking about linear equations. In Sfard’s (2008) theory this means investigating their discourse. In order to describe their discourse as ritualized, explorative or in transition from one to the other, I will look in detail at the features of their mathematical discourse (keywords, visual mediators, routines, narratives). This will allow me to investigate the goals of their activity, their level of objectification, whether the narratives are endorsed, what their levels of flexibility and correctibility are, and whether they have internal persuasion. Ultimately my analysis will facilitate a comprehensive description of their discourse. In the next chapter I explain how I operationalize the theory.

Chapter 4: Methodology

4.1. Introduction

In Chapter 3 I presented Sfard's theory of commognition as a productive theoretical perspective for viewing my empirical problem. According to the commognition theory, to investigate learners' thinking I have to look at their discourse – what they say and do as they solve a set of linear equations. In my study I aim to describe learners' discourse as ritualized, explorative or in transition from ritualised to explorative, using Sfard's (2008) features of the discourse. In this chapter I complete Crotty's (1998) four-step process by responding to the remaining steps in the process, namely the methods and methodology for my study.

Before proceeding, my choice of theory makes two preliminary points necessary. The communication of the learners in my study is not a “proxy for discourse- independent objects” (Sfard, 2008, p. 278). I can only analyse what is perceptible through speech and gestures. Thus I work with verbatim utterances and interactions in a quest to preserve the “fidelity” (2008, p. 277) of the learners' responses when developing my data. While I adopt this sensitivity in the analysis, I acknowledge that ‘data’ in this study is itself discursive, and that my choices and discourse would necessarily influence the research text (Setati, 2003). In recognition of this I follow Le Roux's (2011, p. 3) use of “research text” for the data in this study.

My research texts comprise the videoed interviews, transcripts, and analyses. As I show in this chapter, the interviews and the resultant videos are the initial research texts because the questions I posed to learners influenced the data I could access. I then transcribed the interviews and operationalized Sfard's (2008) theory by identifying the tools of the discourse, namely words, visual mediators, narratives and routines. I refer to this as my Level 1 analysis, which for each learner comprises a spreadsheet containing seven sheets. I then use these tools to determine the nature of learners' discourse. I refer to this as my Level 2 analysis.

In this chapter I describe the production of the research texts. First I describe my methods for the selection of learners and how the interviews were conducted, and then I describe and justify my chosen methodology and illustrate how I use my analytical tools. I conclude by discussing issues of quality and ethics. I use double quotation marks when I quote learners and single quotation marks to refer to entities.

4.2. Method for producing research texts

As noted in Chapter 1, the interviews I use in my study were originally conducted for use in a short course with teachers. In this section I motivate for, and describe their particular use in the study.

4.2.1. Selection of learners

Given my research focus and chosen theory, the interviews needed to facilitate maximum access to learners' discourse about linear equations. Since accessing their understanding of the *when* and *why* is more challenging, my original choice of learners who knew *how* to solve linear equations was appropriate for this study. Learners were selected based on good performance on a written assessment task about the solution of linear equations. This strategy for selection was partially motivated by Gripper (Gripper, 2011a, p. 83) who, for his study, selected the "most competent learners" on the premise that what these learners constituted as mathematics would reflect what their teachers presented as such in the classroom.

In preparation for the short course, I originally conducted 22 interviews at two schools that service working class communities. For this study, I developed my research texts from the interviews of the eleven Grade 8 learners and four Grade 9 learners who agreed to participate in the study. For two of the Grade 9 learners English is not their mother-tongue, although it is their medium of instruction.

4.2.2. The design of the interviews

Prior to the interviews, I accessed the tasks and learners' marked scripts. From the tasks I selected the equations on which the interviews would focus, as indicated in Table 4. The choice of the equations was linked to earlier research findings. I have called each content section a vignette.

Table 4: Vignettes / Equations for the interview

Description	Grade 8 task	Grade 9 task
Vignette 1:	a. Learners' explanation of the instruction. "Solve the following equations" b. Learners explanation of the meaning of the equal sign	
Vignettes 2, 3 & 4 – equations from assessment tasks ⁵		
Equation 1: $ax + b = c$ Only one term contains the variable	Vignette 2: $2x + 7 = 13$	Vignette 3: $2x + 7 = 13$
Equation 2: $ax + b = cx + d$ The variable appears more than once (e.g. Filloy & Rojano, 1989; Herscovics & Linchevski, 1994)	Vignette 3: $2x + 8 = -2x - 3$	Vignette 2: $6x - 12 = 2x + 4$
Equation 3: $a(x + b) = c(x + d)$ Inclusion of brackets; learners first have to transform the equation to the form $ax + b = cx + d$.	Vignette 4: $7(x - 3) = 7 - 2(3x + 1)$	Vignette 4: $2(4x - 5) - (3x + 6) = -2(x + 3)$
Vignette 5:	a. Learner's response to whether $13 = 2x + 7$ is the same as $2x + 7 = 13$ b. Determine the value of $2x$ in the equation $2x + 7 = 6 + 7$	
Vignette 6:	Learner's response to whether substituting their solution for Equation 3 into the original equation, would yield the same value on left-hand side and right-hand side	
Vignette 7: Unseen equation, no constant term (de Lima & Tall, 2008; Barahmand & Shahvarani; 2014).	Learner's solution to $4m = 2m$ or equivalent	

⁵ For Vignette 2, I focused on an equation from the assessment. The Grade 9's did not have an equation of the form $ax + b = c$, so I included to the equation $2x + 7 = 13$ in Vignette 3.

After a few learners had been interviewed, one learner confirmed that they had gone back to discuss ‘ $4m = 2m$ ’ with their teacher. I then changed the question to read either ‘ $4x = 2x$ ’ or ‘ $2x = -3x$ ’. Subsequently it seemed that learners saw this as a different problem, and showed no influence of their teacher’s prompt.

In preparation for each interview, I captured the cover page of each learner’s task in order to match the learner’s identity with the solution set (Figure A). I took images of the relevant solutions (Figure B) and I transcribed each learner’s written solution onto folio paper (Figure C) to avoid any influence of by the teacher’s marking. An example of a full set of these artefacts is provided for below, in Figure 5.

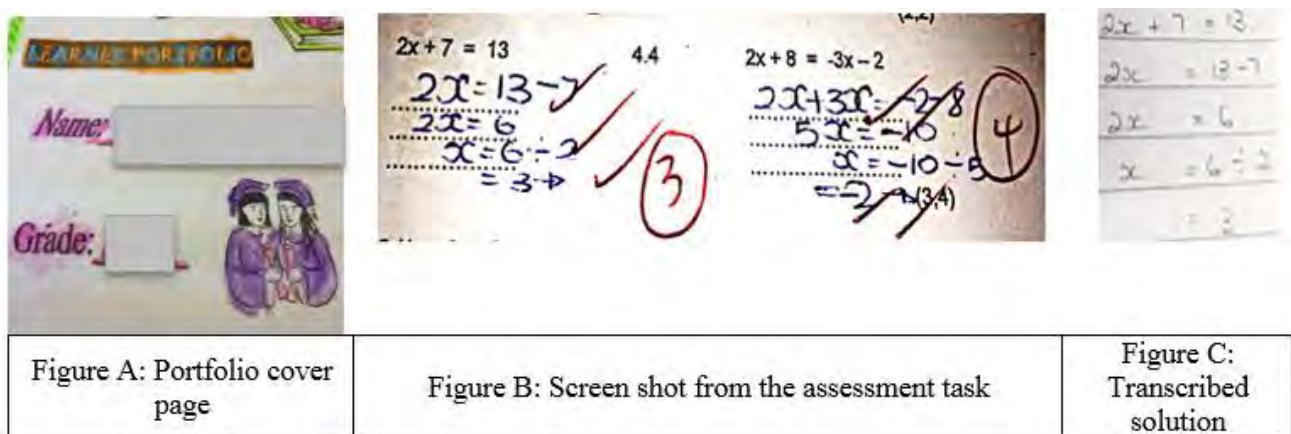


Figure 5: A set of learner artefacts from the assessment task.

I stored all learners’ documents in secure individual electronic folders, labeled using pseudonyms to protect their and their school’s identity. These pseudonyms were used as reference in all subsequent research texts.

4.2.3. The interview process

While talking about their methods, I encouraged interviewees to point at entities in their transcribed written solutions in order to identify their visual mediators. I video-recorded each interview, focusing on the solution rather than on the learner. The video recording provided a “re-presentation of data” (Setati, 2003, p. 294) consisting of what learners said and wrote, as well as their gestures. I used a casual tone to reduce learners’ levels of intimidation, sometimes adopting some of their terms and phrases even though these do not conform to literate mathematical discourse. I have taken this into account in my analysis.

As interviewer, my intention was to investigate learners' communication, and at times it was necessary to provide prompts and scaffolding to better understand their communication. Transcript 4.1 illustrates my use in lines 13 and 15 of the student Nadia's initial word "jump" in line 12c and how I prompted her to address errors. The transcript shows how I adopted Nadia's terminology to put her at ease.

In line 15 Nadia's realization from the equation ' $13 = 2x + 7$ ' was ' $-13 = -2x - 7$ '. Nadia had changed '+7' in the first equation to '-7' in the second equation. I prompted her to communicate the reasons for her action in line 16:

Transcript 4.1 – Nadia, Vignette 5

- 12c Nadia: *It's actually negative thirteen 'cause it jumps, right?*
- 13 Interviewer: *Did two ex not jump?*
- 14 Nadia: *Sorry (Changes '2x' to '- 2x')*
- 15 Interviewer: *Why did seven become minus seven? Did that also jump?*
- 16 Nadia: *Sorry I'm getting confused (Changes '- 7' to '+7')*

4.2.4. Interview transcription

The organization of the interview transcript was informed by how learners use words and visual mediators in their talk, writing and gestures. I transcribed each interview from the video soundtrack, and captured screenshots of written work from the video. For each interview my transcription and accompanying analysis was done on a spreadsheet (Appendices 1 and 2). The transcript was separated into vignettes (see Table 4 above), each on a separate sheet within the spreadsheet.

Within a vignette, each speech turn was numbered. What was said and done were organized into adjacent columns, to identify the visual mediators that accompanied the learner's word use (see Figure 6 above). To allow me to focus in detail on words and visual mediators, speech turns with more than one related visual mediator were sub-divided, with each sub-division numbered separately.

Speech Turn		Said	Done
1	Interviewer:	Mm, okay. How did you do that one?	Points to $2x + 8 = -3x - 2$
2a	Nadia:	Okay. Must I explain it?	Points to $2x$ in first line and $+3x$ in second line
2b		Okay, two ex plus three ex	
2c		because the letters goes there	Points to $-3x$ in first line
2d		and the numbers goes there.	
3	Interviewer:	Mmm	
4a	Nadia:	Negative two because it stays and it didn't jump.	
4b		Positive ..	Points to $+8$ in first line and $+8$ in second line
4c		Doesn't it change to negative because it jumps, Miss?	Points to equal sign in line 2
5	Interviewer:	You can change it.	

Figure 6: Extract from Nadia, Vignette 2 – ‘Said’ and ‘Done’ columns

4.3. Methodology

In this section I present my analytic framework, which was developed in interaction between my research texts and the theory presented in Chapter 3. Within the constraints of this dissertation, I illustrate my use of the framework.

4.3.1. Level 1 analysis

Level 1 is my operationalization of Sfard’s tools of the discourse – words, visual mediators, narratives and routines – to “re-present” (Setati, 2003, p. 294) learners’ discourse about linear equations. Although I can only make a judgment about a learner’s discourse by considering the interview as a whole, my analysis began by analyzing the discourse in each vignette separately. Figure 7 below shows the column headings for each vignette’s sheet in the spreadsheet.

Speech Turn	Speaker	Transcript		Tools		Notes	Narratives (stories)		Source of the narrative	Notes	Notes (Routine)	Routine		
		Said	Done	Words	Visual Mediators		Description of entities and relations between entities	Activities with / by entities				Realisation Routine	Links to Words, Mediators, Narratives	Realisation tree

Figure 7: Column headings used in the spreadsheet

For each vignette the transcript was combed and the relevant words and visual mediators were used to develop the narratives, which are the learner’s story about the objects and their relations with other objects (Sfard, 2008). From the narratives I constructed descriptions of the learner’s routines and produced realization trees as representations of their discursively constructed objects. Because the learners do not necessarily produce endorsed narratives about mathematical objects, I avoided using the term ‘mathematical objects’ and rather referred to the learners’ objects as entities.

Sfard (2008, p. 161) says that “trying to characterize mathematical discourses according to their external features rather than their objects is the researcher’s way out of the entanglement” caused by the fact that mathematics is an autopoietic system and that there is no clear divide between mathematical discourse and its objects. Morgan (1998, p. 76) has developed a system for “describing the forms of language” within text. Within the tools, narratives and routines I have identified a set of linguistic and gestural features that are the external features relevant to my study, and draw on both Sfard (2008) and Morgan (1998) for this purpose. I list these features together with their function in the rest of **section 4.3.1**.

4.3.1.1. Words and visual mediators

Figure 8 shows how I extracted words and identified the related visual mediators. The latter are underlined to identify them in their context. It also includes the column where I noted the linguistic features, which I discuss next.

Speech Turn	Words	Visual Mediators	Notes
1		$2x + 8 = -3x - 2$	
2a			
2b	two ex plus three ex	$2x + 8 = -3x - 2$	Expression. Keywords linked with relational verb
2c	the letters		reference to visual mediator; wrong keyword (variables)
	there		adv of place
2d	the numbers		reference to visual mediator; wrong keyword (constants)
	there		adv of place
3			
4a	Negative two	$2x + 8 = -3x - 2$	keyword - number word
	stays		verb - material process
	didn't jump		verb - material process
4b	positive		keyword - operator sign
	change	$2x + 8 = -3x - 2$	verb - material process
	negative		keyword - operator sign
4c	jumps		verb - material process
5			

Figure 8: Extract from Nadia, Vignette 2 – ‘Words’, ‘Visual Mediators’ and ‘Notes’ columns

4.3.1.1.1. Linguistic features of words

- a. Words used to name entities. These nouns and noun phrases indicate the kinds of entities being acted upon (Morgan, 1998), and may or may not be keywords (Sfard, 2008). I include number words, algebraic terms and terminology, expressions and equations.
- b. I identify three features of verbs, namely voice, process and modality. Active voice describes actions with entities. This presents “mathematical facts and [relations as being] dependent upon human action” (Morgan, 1998, p. 83). Passive voice describes a relationship between (mathematical) objects, which are both subject and object of the sentence. The passive voice is a means of establishing the “impersonal discursive form” (Sfard, 2008, p. 50). In my study, use of

the passive voice mostly shows a relationship between numbers or algebraic terms. This sentence construction could indicate a phrase-driven expression, such as ‘six is divided by two’.

- i. Process relates the types of processes and participants involved. Using Morgan (1998, p. 80), I identify “material process” verbs as indication of action on entities, and relational process verbs as indication of a system of relationships between objects.
 - ii. “Modal auxiliary verbs” indicate the degree of likelihood or authority the speaker attaches to the utterance (Morgan, 1998, p. 87). I identify verbs with high modality as an indication of obedience to authority. Examples of high modality auxiliary verbs are ‘must’ and ‘have to’.
- c. The prefacing of an entity with an article (‘the’, ‘a’) represents that entity as separate from another. In their talk, I identify when learners preface keywords with ‘the’ or ‘a’ because this effects a separation of the entity from the rest of the expression, rendering it a disobjectified entity.
 - d. Adverb of place indicates the position of an entity. Learners regularly refer to sides of the equation and other positions of entities.
 - e. Adverb of time indicates time in a routine. Adverbs such as ‘first’, ‘next’ and ‘then’ construct a story about the action (Morgan, 1998).
 - f. Personal pronoun ‘I’ or ‘you’ as subject of sentence links with the notion of action, as the subject is the doer of the action. Thus it indicates a story about action rather than an object (Sfard, 2008).

4.3.1.2. Visual mediators

To identify visual mediators I identified the symbols, terms and expressions that the learner pointed at or named, as in ‘this plus sign’. When the learner referred to a term using an article, as when ‘ $2x$ ’ was referenced as ‘the two’ and ‘the x ’, I concluded that ‘2’ and ‘ x ’ were visual mediators that operated in a disobjectified way. For example, in Figure 6 Nadia pointed at ‘ $2x$ ’ [see line 2b, Column D] so I classified it as a visual mediator. Figure 8 shows the underlined ‘ $2x$ ’ in the “visual mediator” column [see line 2b, Column G].

4.3.1.3. Narratives

As was explained in Chapter 3, narratives are based on learners' words and visual mediators. I identified two categories of narratives – narratives about description of entities and relations between entities, and narratives about activities with or by entities. For example, in Figure 6 Nadia explains that she groups '2x' and '3x' "because the letters goes there and the numbers goes there" [lines 2c-d, Column C]. In Figure 9, I constructed the narrative as a description of the relations between entities [see line 2c Column J]. Where appropriate, the source of the narrative is included. For this narrative the source is 'spatial arrangement'. The 'Notes' column indicates whether the narrative is endorsed or not, and could include reasons for this.

A	J	K	L	M
Speech Turn	Narratives (stories about entities)		Source of the narrative	Notes
	Description of entities and relations between entities	Activities with / by entities		
1				
2a				
2b				
2c	In an equation, algebraic terms go on the LHS and numbers go on the RHS		spatial arrangement	Not endorsed
2d				
3				
4a		"-2" does not "jump" over "=". The sign does not change.		Not endorsed
4b				
4c		"+8" "jumps over" "=". The sign changes.	reference - interviewer	Not endorsed
5				

Figure 9: Extract from Nadia, Vignette 2 - 'Narratives', 'Source of the narrative' and 'Notes' columns

4.3.1.4. Routines and realisation trees

In the ‘Routines’ column I developed a list of the learner’s sequential steps, as is shown in Figure 10 below. I considered all aspects of their discourse, namely their speech, gestures and written work, and indicated the links to the tools and narratives. The example in Figure 10 below relates to Nadia’s routine for solving $2x + 8 = -3x - 2$.

Realisation Routine	Links to Words, Mediators, Narratives
1-a Scan the equation. Determine how terms should be rearranged.	1-a
1-b Write "2x" on LHS of line 2	2b; 1-b
1-c Write "+3x" on LHS of line 2	2b; 1-c
1-d Write "-2" on RHS of line 2	4a; 1-d
Write "+8" on RHS of line 2 (Realisation 1a)	4b, (Realisation 1a)
Correction	
Write '-8' on RHS of line 2 (Realisation 1b)	4c, 6a; (Realisation 1b)
2-a Add "2x" and "3x" on LHS	6b; 2-a
2-b Write "="	6c; 2-b
Add "-2" and "-8" on RHS (Realisation 2)	6e, 10; (Realisation 2)
3-a Write "="	3-a
Write "-10 ÷ 5" on RHS of line 4 (Realisation 3)	18; (Realisation 3)
4-a Write "5x" on LHS of line 5	22; 4-a
Divide "-10" by "5" (Realisation 4)	18; (Realisation 4)

Figure 10: Nadia’s routine for solving $2x + 8 = -3x - 2$

From the routines in Vignettes 2, 3, 4 and 7 I developed realization trees. Figure 11 below shows the realization tree that visually represents Nadia’s discursive construction of the solution of the equation as a realization of a narrative about the functions $f(x) = 2x + 8$ and $g(x) = -3x - 2$, alongside the routine.

In the ‘Realization Routine’ column, each action between one equation and its realization is noted. I call these actions sub-nodes, with each node numbered as a realization. The sub-nodes carry the number of its related realization. At each sub-node and realization I have circled the visual mediator. In Figure 11 sub-nodes 1-a, 1-b, 1-c and 1-d come before Realization 1 (R1). This means that there are four ‘steps’ before the last one, which makes up R1. Between the ‘Realization Routine’ column and the realization tree, I note the links to other columns and to the realization tree. Where the learner corrected an error, as is done at R1, the correction is represented separately.

In the next section I show how I identified discourse as ritualized or explorative in my Level 2 analysis.

4.3.2. Level 2 analysis

The Level 2 analysis uses the perceptible features of the learners’ discourse about linear equations recorded at Level 1, to classify their routines using Sfard’s features of ritualized and explorative discourse. I use evidence from all the vignettes for this purpose, but as noted in Chapter 3, my analysis has limitations because it is a snapshot of the learners’ discourse based on one moment in their learning.

In Section 3.6 I discussed Sfard’s criteria for classifying discourse as either ritualistic or explorative. Using these criteria I developed a set of questions, which I answer using evidence from the Level 1 analysis. The questions shown in the Level 2 table are:

1. What is the degree of objectification of the learner’s discourse?
2. To what extent are the learner’s narratives endorsed?
3. What is the closing condition (goal)?
4. For whom is the routine performed?
5. By whom is the routine performed?
6. What is the learner’s level of flexibility?
7. What is the learner’s level correctibility?

I illustrate how I answer Question 1.1 in this section. The full analytic table can be found in Appendix 3.

4.3.2.1. Level 2 – Question 1

What is the degree of objectification of the learner’s discourse?

Question 1.1 considers the degree of objectification of the learner’s discourse by focusing on word and mediator use in one line of an equation. For each sub-question there are indicators that I access from the research text at Level 1, as is seen in Figure 11. Because of space constraints I only present and discuss the indicators for sub-question 1.1 in detail here.

Table 5: Sub-questions and indicators for Question 1.1 in Level 2

Sub-questions	Indicators	Source at Level 1
1.1.1 Does the learner’s word use and mediator use signify numbers and algebraic terms as mathematical objects in their own right?	1.1.1 The use of words and phrases in one line of an equation signify numbers and algebraic terms as mathematical objects.	1.1.1.1 <u>Tools - Words</u> Keywords with relational process verbs
		1.1.1.2 <u>Tools - Words</u> Keywords as subject and object, with the verb in the passive voice
1.1.2 Does the learner talk about and act on disobjectified mediators within a function?	1.1.2.1 Phrase-driven use of keywords	1.1.2.1 <u>Tools – Words</u> Keywords linked with constant phrases e.g. naming equations
	1.1.2.2 Reference to visual mediators as disobjectified entities	1.1.2.2 <u>Tools – Words</u> Keywords prefaced by articles, pronouns or wrong use of keywords
	1.1.2.3 Reference to actions with disobjectified entities (digits, operatory signs, algebraic symbols and pronouns representing them).	1.1.2.3 <u>Tools - Words</u> Digits, algebraic symbols, operatory signs and pronouns, mostly in conjunction with material process verbs
	1.1.2.4 The naming and spatial arrangement of symbols, digits and letters.	1.1.2.4 <u>Tools - Words</u> Adverbs of place

Sub-question 1.1.1 relates to explorative discourse at the level of words and phrases in one line of the equation. The indicators for objectified discourse are keywords with relational process verbs, as well as use of the passive voice with keywords as subject and object.

Sub-question 1.1.2 relates to ritualised discourse at the level of words and phrases in one line of the equation. Reference to and action with, and the spatial arrangement of disobjectified entities are seen as indicators for ritualistic discourse. Active voice and material process verbs are linguistic features that indicate action with disobjectified entities; adverbs of place indicate spatial arrangement and prefacing keywords with articles indicate that the learner is using these as visual mediators disobjectively (see indicators 1.1.2.1 – 1.1.2.4).

Because the degree of objectification of the discourse is only one of a set of criteria for explorative and ritualistic discourse, I cannot make any judgments based on one vignette or one criterion. I conduct a similar analysis of the other questions, which can be accessed in the Level 2 tables in Appendices 4 and 5. Once I have considered all seven questions and their related sub-questions, I can draw conclusions about the learner's discourse at the time of the interview.

4.4. Ethical Considerations

The interviews used in my study were conducted under the auspices of the project I worked in. Originally, permission for the use of the interviews in the development of a short course for teachers was covered by a Memorandum of Understanding between the organization responsible for the project and the participating schools. In the short course teachers engaged with extracts from interview transcriptions and video extracts. The anonymity of learners and schools was preserved and no identity can be deduced from the materials. I had built up a relationship of trust with teachers and learners through the project and the purpose of the interviews was explained to the principal, the teachers and their learners who gave verbal consent to be interviewed.

When I considered using the interviews for my study I formally obtained written permission from all stakeholders i.e. the learners, their teachers, their principal, their parents and the provincial education department. Separate consent forms were prepared for all stakeholders (See Appendix 6).

4.4.1. Informed consent

I met with the learners and their teachers to explain my research intentions. The consent forms, which they were asked to complete within one week, outlined my intentions. The provincial education department, all teachers and principals consented to my use of the interviews. In this study I have only used the interviews of the 15 learners who, with their parents, gave consent.

4.4.2. Anonymity and confidentiality

I undertook to protect learners' anonymity. Learners' faces do not appear in the videos at any time. Furthermore, I labeled interview transcripts with pseudonyms, which were then used in all research texts and in the study itself. I stored each learner's documents in secure individual electronic folders.

4.4.3. Avoiding harm

Although I have attended to the anonymity and confidentiality of learners and schools, it is unavoidable that some responses might be recognised. This is typical of small-scale studies where the researcher is involved in the context; however, I have taken steps to ensure that no harm is done. On account of the theoretical approach, the study subscribes to a non-deficit view of individual schools and learners and all data is used to understand prevalent problems with a view to improving teaching and learning.

4.5. Quality of the study

I use Maxwell's (1992) perspective on validity and generalizability to discuss the quality of my research study. Maxwell posits that in a qualitative study, validity implies the validity of the relationship between the account of what happened (in this case my research text) and the real-life situation (in this case, the interview).

I focus on two categories of validity as identified by Maxwell, namely "descriptive validity" (1992, p. 285) and "theoretical validity" (1992, p. 291). Descriptive validity refers to the factual accuracy of the transcripts based on the videotapes. All speech and gestures have been accurately recorded at a level of detail appropriate for the research focus. During the analysis process I refined the transcripts, revisited the recordings and engaged with my supervisor as necessary. My Level 1 analytic table is a re-presentation of the discourse, but in every line of the research text the learner's

discourse stands alongside my interpretation thereof. I have been mindful at all times that interpretation is necessarily subjective. Thus in Section 4.3.1 I have defined each category of interpretation as accurately as possible, based on the theory, to minimise my subjectivity.

Theoretical validity refers to whether my chosen theory is suited to my study (Maxwell, 1992). I developed my analytic framework using Sfard's (2008) theory of commognition. The theory provides the tools and indicators for developing an analytic framework suited to using the features of the discourse as I have done. As shown in Section 2.4.1, the validity of the theory for looking at learner discourse in the context of mathematics is borne out by the growing group of researchers who have used the theory to study various aspects of mathematics amongst learners.

Because this is a small-scale qualitative study, the findings are not generalizable to any other group of learners or topic in mathematics. However, the analytic framework allows for the provision of a rich description of the learners' discourse, and the reader can decide how it may or may not apply to his/her context.

4.6. Conclusion

In this chapter I have shown how I developed my analytic framework. I started by developing transcripts from the interviews conducted with the 15 learners. Thereafter I constructed my Level 1 framework using learners' words, visual mediators, narratives and routines. From the Level 1 analysis I developed my Level 2 analysis in which I answered a set of questions pertaining to the features of ritualistic and explorative questions. In the next chapter I show how I operationalized the framework in my analysis of learner discourse.

Chapter 5: Analysis

5.1. Introduction

My research aims to present a detailed description of the discourse of a group of Grade 8 and 9 learners about linear equations. In Chapter 4 I described how I use Sfard's (2008) concepts of 'exploration' and 'ritual' to classify learner discourse. In this chapter I present the analysis of the discourse.

I first provide an overview of the discourse of all learners in the study, followed by a detailed descriptions of two learners, Thomas and Emily, which give the reader deeper insight into how I used my analytic framework to describe their discourse. Thomas and Emily's discourses are representative of the range in the discourse in my study. Both Level 1 and Level 2 analyses of Thomas and Emily can be found in Appendices 4 and 5.

5.2. A visual tool to see learner differences

As a prelude to the analysis, I present learners' scores in Tables 6 and 7, from the written assessment tasks set and marked by their respective teachers. Vignette 2 in Table 6 and Vignette 3 in Table 7 reflect the marks obtained for the equation ' $2x + 7 = 13$ '. All learners solved this equation where the variable appears only once. In the other equations, where the variable appears more than once, some learners made errors. There were more errors where the equation included terms in brackets. This observation resonates with the findings of Filloy & Rojano (1989) and Linchevski and Livneh (1999) respectively, that the different formats represent a didactic cut /cognitive gap in the skills needed to solve the equations.

Table 6: Grade 8 scores on equations from their written assessment task

	GRADE 8 LEARNERS										
	Sheena	Joshua	William	Nadia	Shakira	Fatima	Erin	Carla	Gadija	Alison	Thomas
Vin 2: $2x + 7 = 13$	3/3	3/3	3/3	1/3	3/3	3/3	3/3	3/3	3/3	3/3	3/3
Vin 3: $2x + 8 = -3x - 2$	4/4	4/4	4/4	0/4	4/4	4/4	3/4	4/4	4/4	4/4	4/4
Vin 4: $7(x - 3) = 7 - 2(3x + 1)$	5/5	2/5	5/5	5/5	1/5	1/5	5/5	3/5	5/5	0/5	5/5
Percentage scored	100%	75%	100%	50%	67%	67%	92%	83%	100%	58%	100%

Table 7: Grade 9 scores on equations from their written assessment task

	GRADE 9 LEARNERS			
	Kabelo	Emily	Tumisho	Zahir
Vin 2: $6x - 12 = 2x + 4$	3/3	3/3	3/3	3/3
Vin 3: $2x + 7 = 13$	5/5	5/5	5/5	5/5
Vin 4: $2(4x-5) - (3x+6) = -2(x+3)$	5/5	1/5	4/5	5/5
Percentage scored	100%	69%	92%	100%

The learner scores might lead one to think that most of the learners work in an explorative way. However, my analysis using Sfard's tools of commognition shows that all learners in my study have ritualized discourse. I developed Table 8 to help me organize my thinking and to visualize the

discursive differences between these learners. This table represents no more than my initial sense of the nuances in each learner's discourse. I used a scale of 1 to 4 to describe the different components of a learners' discourse, based on the following descriptors:

- 1 *Strongly ritualistic*
- 2 *Ritualistic discourse, with some indicators of explorative discourse*
- 3 *Even combination of ritualistic and explorative discourse*
- 4 *Strongly explorative*

I only have access to learners' levels of correctibility where there are mistakes in their solutions. Therefore, where there are no errors, I indicate with a question mark that I could not determine their levels of correctibility.

Table 8: Features of the discourse, using a scale of 1 – 4

	GRADE 8											GRADE 9			
	Sheena	Joshua	William	Nadia	Shakira	Fatima	Erin	Carla	Gadija	Alison	Thomas	Kabelo	Emily	Tumisho	Zahir
1. The degree of objectification of the discourse															
At the level of words and visual mediators	2	2	2	2	1	2	2	1	1	1	2	2	2	2	2
At the level of horizontal equivalence	2	2	2	1	2	2	2	1	1	1	2	2	2	1	1
At the level of vertical equivalence	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2. Extent to which narratives are endorsed															
At the level of objects	1	2	2	1	1	1	2	1	1	1	2	2	2	2	1
At the level of actions	1	1	1	1	1	1	1	1	1	1	1	2	2	1	1
At the level of source of the narrative	1	1	1	1	1	2	1	1	1	1	1	2	2	1	1
3. Closing condition (goal)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4. For whom is the routine performed	2	1	2	1	1	1	2	1	1	1	1	2	2	1	1
5. By whom is the routine performed	1	1	2	1	1	1	1	1	1	1	1	1	2	1	1
6. Learner's level of flexibility	1	1	1	1	1	2	2	1	1	1	1	2	1	1	2
7. Learner's level of correctibility	1	1	?	1-2	1	1	?	2	?	1	1	2	1-2	?	?

Based on Table 8, there is little difference between the Grade 8 and Grade 9 learners' discourse. In the following section I describe the differences between each feature of learner discourse, that is, what '1' or '2' might mean in the context of ritualized discourse. The value of this distinction lies in being able to see that the ritualized discourse of all learners is not the same. Despite the similarity in their written solutions, there are nuances in the ways they view the linear equation as a realization of a function.

5.3. Synopsis of learner discourse when solving linear equations

Based on the evidence from the research texts, no learner uses explorative discourse when talking about their solutions to the linear equations. However, there are subtle differences in their discourse. In the rest of this section I provide a description of what this ritualized discourse looks like for all learners, using the Level 2 indicators as indicated in Table 8. I draw on evidence from all vignettes to support my argument. Thereafter I provide a detailed description of Thomas and Emily's discourse.

5.3.1. Degree of objectification

5.3.1.1. Words and visual mediators

In my study there is no evidence that learners use keywords objectively to signify objects such as integers in their own right. Rather, all learners use words disobjectively, but display different characteristics. I distinguish between those learners who do not use keywords, use them disobjectively or replace them with colloquial words, and learners who use keywords in a phrase-driven way. By way of example, I compare Carla's explanation with Nadia's below:

Carla: *"I moved this ... negative three ex over but made it a ... a positive number ... Because of the ... the negative and positive ... the opposite"* [Carla, Vignette 3, 2b-4].

Nadia: *"Negative two times positive three ex is negative six ex because two times three is six"* [Nadia, Vignette 4, 8a].

Carla's use of the pronoun "this" and her action on ' $-3x$ ' render it a disobjectified entity. She also uses the colloquial word "opposite" instead of talking about inverse operations as the curriculum specifies. On the other hand, Nadia's narrative suggests objectified talk about integers because she

does not split the objects into separate entities, and the relationship between them is endorsed. Nadia's narrative could be regarded as phrase-driven, because she uses entire phrases that are endorsed.

There is a difference between the way learners talk about positive integers (whole numbers) and the way they talk about negative integers, variables and algebraic terms. All learners talk about positive integers objectively, but only some talk about negative integers and algebraic terms as mathematical objects, as Nadia's narrative illustrates. By comparison, Shakira's discourse about the sum ' $-2 - 8$ ' is disobjectified:

"And negative plus negative equals negative, two plus eight equals ten." [Shakira, Vignette 3, 10b].

Unlike Nadia, Shakira has split the objects (integers) into signs and digits, and operates on these disobjectified entities separately. While all learners talk about and perform actions on disobjectified entities sometimes, there are learners who do this more often than others.

5.3.1.2. Horizontal and vertical equivalence

Although no learners talk about equations objectively, there are distinguishable differences in the way they use horizontal equivalence. No learner uses horizontal or vertical equivalence as a signifier or source of the narrative except when prompted to do so. Therefore I argue that nobody uses equivalence exploratively. Shakira's narrative, which is typical of all learners, shows how her use of adverbs of place, active voice and material process verbs work together to interrupt equivalence:

"And then you have two ex I took the ex down ... and I ... move that (points to '2' of '2x' on left-hand side) over to there (points to right-hand side of next line)" [Shakira, Vignette 2, 14a-d].

In this excerpt the adverb of place "down" indicates 'movement' of "the ex", and her gesture shows where she 'moved' '2' to the right-hand side of the next line. All her actions are unendorsable and interrupt vertical and horizontal equivalence respectively.

I draw further evidence for learners' discourse about equivalence, from Vignettes 1 and 5. In Vignette 1, regarding the meaning of the equal sign, some learners make reference to equivalence in

the structure of the equation, while others do not consider the notion of equivalence. To illustrate the difference, I compare three learners' explanations.

Fatima: *"Um ... the constants on the one side and the variables on the other side"* [Fatima, Vignette 1, 4].

Erin: *"If you calculate something, that (points to equal sign) will indicate that you get your answer"* [Erin, Vignette 1, 6]

Nadia: *"It means that (points to left-hand side) is equal to that (points to right-hand side)"* [Nadia, Vignette 2, 10a-b].

For Fatima the equal sign determines how entities are spatially organized, for Erin it is an indication of a calculation. Only Nadia mentions equivalence – albeit in a phrase-driven way.

In Vignette 5, the nature of the questions and visual mediators prompt learners to use horizontal equivalence as a source of their narrative. Some learners use horizontal equivalence to show that '2x' equals '6' in the equation ' $2x + 7 = 6 + 7$ ' and to see that ' $2x + 7 = 13$ ' and ' $13 = 2x + 7$ ' are equivalent equations. Here William explains that '2x' equals '6' in the equation ' $2x + 7 = 6 + 7$ ':

".. that six represents that two ex, so... it equals two times three" [William, Vignette 5, 6-8].

Others, such as Zahir, do not talk about equivalence and sometimes revert to the routine for finding the solution – another feature of ritualistic discourse.

"If I carry the seven over ... and then the difference between seven and ... negative seven is nought" [Zahir, Vignette 5, 14; 18a-c].

My source of evidence for vertical equivalence was Vignette 6, where learners were asked about the significance of substituting the solution into the original equation. Because no learner was familiar with the notion of substitution, it was not productive to analyse their responses. No learner makes endorsable reference to vertical equivalence in the interview.

5.3.2. Endorsed narratives

Since no learner uses words and visual mediators exploratively, it is not surprising that they do not produce endorsed narratives about mathematical objects. There are two features of narratives that are common to all the learners. Everyone performs actions with disobjectified entities, and everyone produces endorsed narratives about operations with positive integers (whole number arithmetic). Often learners perform whole number arithmetic instead of arithmetic with negative integers and algebraic term, as Zahir's routine shows. His narrative about his realization from the visual mediator ' $6x - 2x$ ' in the equation ' $6x - 2x = 12 + 4$ ', is:

"Then I minus um ... two from ... from six, and it becomes four" [Zahir, Vignette 2, 10a].

Although Zahir writes the solution as ' $4x$ ', he only talks about the whole numbers.

As indicated, all learners produce unendorsed narratives about actions with entities, using spatial arrangement and visual appearance as the source of his narrative. In the example that follows, Erin uses spatial arrangement on the signifier ' $2x = 6$ ' to realize ' $x = \frac{6}{2}$ '. She performs separate actions on ' 2 ' and ' x ':

"You put the ex at the bottom 6 and bring over the two because ... you must get a equal sign to ex" [Erin, Vignette 2, 12b-14].

Other features of narratives are not seen in all learners' discourse. Only some learners produce endorsed narratives about arithmetic operations with negative integers and algebraic terms, and about the structure (horizontal and vertical equivalence) of the equation. Also, only some learners use mathematical properties of objects as the source of their narrative. Others only use non-mathematical sources.

⁶ Reference to the next line in her written solution.

Some learners structure their thinking by talking about the relationship between objects in a phrase-driven way as Sheena's narrative shows:

"Negative two times three will give you negative six" [Sheena, Vignette 4, 10b].

Some learners produce endorsed narratives about the nature of integers, variables and equations. An example is Tumisho's narrative about the equation ' $2(4x - 5) - (3x + 6) = -2(x + 3)$ '. He points to the negative sign in the term ' $-(3x + 6)$ ' and says:

"... There is a one there that you can't see" [Tumisho, Vignette 4, 12-14].

Nobody uses endorsed narratives about objects as the source of their realizations, unless prompted to do so. In Tumisho's narrative about the "one that you can't see", he is responding to my prompt about an arithmetic error. Yet, this description of the signifier ' $-(3x + 6)$ ' still leads to the unendorsed realization that ' $-3x + 6$ '.

My study shows that learners whose discourse is more objectified, produce more endorsed narratives. However, because of the overwhelming presence of unendorsable actions with disobjectified entities and operations with negative integers and algebraic terms, learners' discourse is mostly ritualistic.

5.3.3. Closing condition

No learner sees the closing condition as being to produce an endorsed narrative about the original equation. For all learners the closing condition, which is obtained by rigidly following a routine, is that the final realization should have the appearance ' $x = [\text{some number}]$ '. This is particularly evident in Vignette 7, where the equation is non-routine. As indicated in Section 4.2.2, I sometimes varied the form of the equation ' $4m = 2m$ '. Thus, I asked Nadia to solve the equation ' $2x = -3x$ '. Her initial response is that the answer is ' $-1x$ '. She explains:

"I think three minus two is one, right? ... Because the signs are different." [Nadia, Vignette 7, 8-10].

But later she changes her mind and adds:

“I don’t think you should put x there, because you’re trying to solve x .” [Nadia, Vignette 7, 14b].

Nadia’s narrative illustrates how the need to have an integer as a solution made her discard the variable. There are various similar narratives about the closing condition for the equation in Vignette 7.

5.3.4. For whom the routine is performed

My analysis suggests that all learners solve the equation for, and with others. The use of high modality verbs indicates that they follow the rules of others. Sheena’s statement, which contains both the adverb of time “always” and the high modality auxiliary verb “must”, is typical of learners’ talk, which shows that visual appearance is the source of the narrative about their actions:

“The last sum 7 must always end with the variable” [Sheena, Vignette 2, 10b-c]

There is almost no reference to the teacher as the authoritative source of their narrative, except for Zahir who, when faced with the equation ‘ $4m = 2m$ ’ explains:

“My ma’am never gave me a sum like this before. There must be a constant or something” [Zahir, Vignette 7, 30b-32].

Many learners see me, the interviewer, as a source of authority for solving the equation ‘ $4m = 2m$ ’. When I provide scaffolding by asking whether ‘ m ’ could be zero, many assume that the solution is zero. However, when I prompt them to endorse the narrative, besides Emily who solved the equation without prompting, only Tumisho can do so.

The only difference between learners’ routines as described thus far, is that some learners can defend their realizations in an endorsable way when prompted to do so. Thus, although they are doing the routine for others, their own voice comes through because they use the properties of number to justify their realizations – with prompts. No learner shows internal persuasion about the

⁷ Reference to the final line in her written solution.

routine they follow. Thus I would not classify anyone's behavior as "thoughtful imitation" (Sfard, 2008, p. 249).

5.3.5. By whom the routine is performed

There is strong evidence that learners follow the metarules of others and also rely on visual mediators and spatial organization. This means that their routines are not performed independently but by using other people's rules or non-mathematical sources of the narrative like spatial arrangement and visual arrangement. Some learners, when asked to justify a narrative, refer to the rule overtly as I will show in my discussion on Thomas. Others show obedience to a rule through the use of high modality verbs as Sheena has done in this example:

"The negative twenty one must change to a positive twenty one" [Sheena, Vignette 4, 20d].

Thus, their routines are performed with others.

5.3.6. Level of flexibility

My evidence for learners' level of flexibility should have been sourced from Vignettes 5 and 6. However, as indicated in Section 5.3.1.2, the research text from Vignette 6 was not productive. I have used learners' responses to alternate representations of the equation ' $2x + 7 = 13$ ' in Vignette 5 to determine their flexibility to use horizontal equivalence as a source of the narrative. They are asked to make deductions that are couched in the equivalence between the two sides of the equation ' $2x + 7 = 6 + 7$ ' and in the equivalence between the equations ' $2x + 7 = 13$ ' and ' $13 = 2x + 7$ '. Here, the context and my questions provide prompts. I see a distinction between learners who, like William, use the prompts productively, and learners like Zahir who are less flexible and have to start the routine from the beginning, as shown in Section 5.3.1.2.

5.3.7. Level of correctibility

I only have access to the levels of correctibility of learners who made mistakes in their solutions. I see two types of errors – arithmetic errors and errors relating to the structure of the equation. In the test, teachers only penalized learners for arithmetic errors. Nadia, Carla, Kabelo and Emily all corrected arithmetic errors, although Nadia and Emily missed some errors, as I show in Section 5.5.5.

Both Nadia and Alison have errors relating to horizontal equivalence that they cannot correct, even with prompting. As an example, I present Alison's solution to the equation ' $2x + 8 = -3x - 2$ ', for which she obtained full marks in the assessment task:

$$\begin{aligned}2x + 8 &= -3x - 2 \\2x + 3x &= -2 - 8 \\5x &= -10 \\10/5 &= -2\end{aligned}$$

Figure 12: Alison's solution to $2x + 8 = -3x - 2$

Alison's realization from the signifier ' $5x = -10$ ' is ' $10/5 = -2$ '. When I ask her what the value of x is, she responds:

"Where's x Miss? I think the x fell away." [Alison, Vignette 3, 14-16].

Thus far I have described the learners' discourse using the indicators from my Level 2 analysis. Taken together, the evidence shows that all learners' discourse is ritualistic. However, there are subtle differences between them. Some learners only talk about positive integers objectively; they produce unendorsed narratives about negative integers and algebraic terms; they do not use the properties of number or the structure of the equation as the source of their narrative; their closing condition is the appearance of the solution; they perform their routine for and with others; they have low levels of flexibility and correctibility. Other learners talk about all numbers in a phrase-driven way; they produce more endorsed narratives about numbers and the structure of the equation; their closing condition is the same, but they show some internal persuasion about the narrative when prompted to do so, and they have higher levels of flexibility.

To strengthen my argument I present detailed analyses for two learners, Thomas and Emily. Of the two, Thomas is typical of those learners whose discourse is highly ritualistic. Emily's discourse is also ritualistic but there are some signs of objectified discourse. She produces endorsed narratives about negative integers and algebraic terms and her level of flexibility is higher. Thus in Sections 5.4 and 5.5, I demonstrate how my analytic framework gives me the tools to drill down in more detail and uncover the subtle differences between learners whose discourse would all be classified ritualistic.

5.4. Detailed discussion on Thomas's discourse

5.4.1. Summary of Thomas's discourse

Based on my description of the general features of the discourse of learners whose talk is strongly ritualistic, I describe Thomas's level of objectification as low. Most of his endorsed narratives pertain to whole numbers and whole number arithmetic. There are some endorsed narratives about the structure and properties of numbers and horizontal equivalence, but he does not use these as the source of his narrative in any vignette. The closing condition of his routine is to obtain an answer in the form "variable = [some number]". He performs the task with and for others because the source of his narrative is mostly non-mathematical – visual appearance, spatial arrangement and the rules of others. Furthermore, he has a low level of flexibility and correctibility. Although he responds positively to prompts to correct unendorsed narratives in Vignettes 3 and 4, his obedience to a rigid routine inhibits him from using his endorsed narratives about numbers or to use vertical equivalence as a resource to correct unendorsed narratives.

In the description that follows, I provide evidence to support this argument. Because of the constraints of the required length of this study I focus on Vignette 2 as an example of Thomas's routine for the solution of all equations in his assessment task. Thereafter I move to Vignette 5, which is a continuation of the conversation in Vignette 2. Because he made an arithmetic error in Vignette 4, I show my analysis of his levels of correctibility based on the part of his solution that needed correction. Finally I describe his responses in Vignette 7, where he solved a non-routine equation.

5.4.2. Vignette 2: Thomas's routine for solving $2x + 7 = 13$

Thomas's routine for solving $2x + 7 = 13$ is typical of most learners in my study. He begins by spatially arranging entities – variables on the left-hand side and constants on the right-hand side. Thereafter he adds like terms on either side and divides both sides by the coefficient of 'x' to reach a solution in the form 'x = [some number]'. His solution for the equation and the accompanying realization tree, are shown in Figures 13a and 13b respectively. I shall base my discussion on these artefacts.

Figure 13a

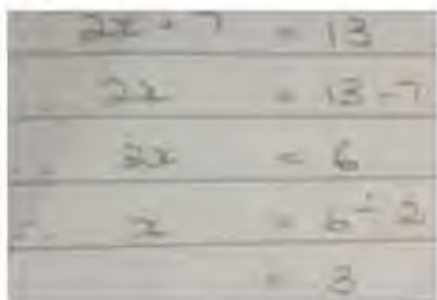


Figure 13b

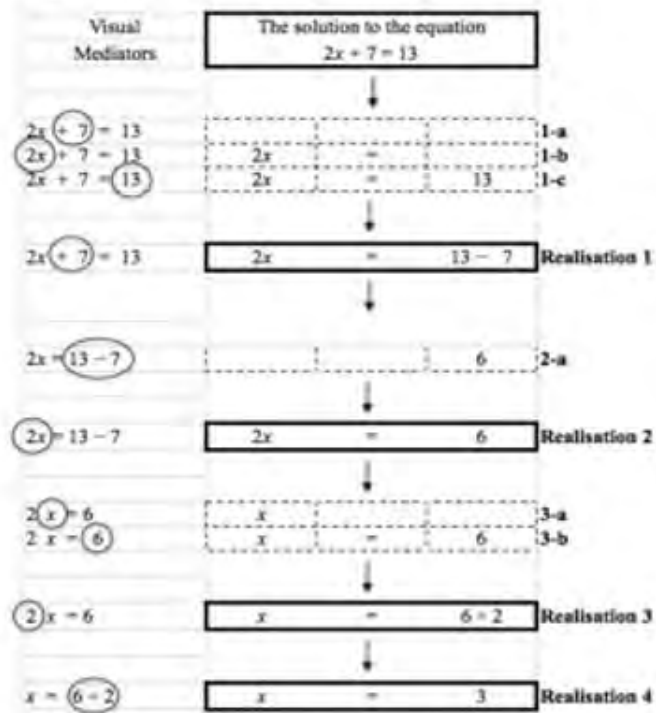


Figure 13: Vignette 2 - Thomas's solution and realization tree for ' $2x + 7 = 13$ '

There are four component actions between the original equation and Realization 1 (R1). Thomas begins by scanning the equation [1-a] and explains that '+7' should be moved "because variables on the one side and constants on the other side" [10]. He uses the phrase "bring down" [12, 14a] to describe the action with '2x' and '13' in nodes 1-b and 1-c, and "bring over" [14b] for the action with '+7'. The adverbs of place "down" and "over" indicate the direction in which entities are moved and the material process verb "bring" indicates that he moves disobjectified entities to particular places on the page. He then changes '+7' to '-7' [R1], explaining that "if you take a positive and bring it over ... then it automatically change to a negative" [6a-b]. When asked why '+7' becomes '-7', he explains that "it's just a rule" [8].

In his explanation Thomas speaks of "a positive" and "a negative". By prefacing the keywords with an article, he seemingly separates the sign from the digit and operates on the sign only. His reference to "a rule" indicates that the source of his narrative is the authority of someone or some text. Even his choice of the word "automatically" suggests that there is no endorsed narrative underpinning his action. When I ask him why he moved '+7' to the right-hand side, his response confirms that his

routine is dominated by spatial arrangement of terms: “because variables on the one side and constants on the other side” [10].

Realization 2 (R2) in Figure 13b is achieved through the addition of like terms on the right-hand side, followed by further spatial arrangement, as Transcript 5.1 shows.

Transcript 5.1 – Vignette 2

18b Thomas: *Then you bring the ex down*

18c *and then you put the six down*

18d *... the two is a times, Miss because ... it has a variable and then you bring it over then ... automatically it becomes division.. divide*

19a Interviewer: *No wait, now I don't understand ...*

20 Thomas: *No, because ... if you bring a minus over it becomes a positive, if you bring a positive over it becomes a negative and ... if you bring multiplication over it becomes division and if you bring division over it becomes multiplication.*

Lines 18b and 18c of Transcript 5.1 pertain to sub-nodes 3-a and 3-b respectively in Figure 13b. He separates ‘ $2x$ ’ and works with the letter and digit separately. He brings down “the ex” [3-a] and “the six” [3-b]. He writes ‘2’ as a divisor on the right-hand side [R3] because “it becomes division” [20].

A number of features of Thomas’s discourse in this explanation suggest that he is working with disobjectified signifiers. He regularly prefaces keywords with definite and indefinite articles as in “the two” and “a variable” [18d] followed by “a minus” and “a positive” [20]. The article identifies the visual mediator and signals the separation of the operatory sign from digits and letters for addition and subtraction, and digits from letters for multiplication and division. For example, he separates “the two” [18d] from ‘ $2x$ ’ as a disobjectified signifier [3-a] and writes it as a divisor on the right-hand side, as illustrated in R3 in Figure 13b. His choice of the word “automatically” [18d] confirms that there is no sense of needing to endorse the action mathematically, which reinforces his belief that there is a “rule” [8] that dictates the action.

Other indicators of his disobjectified discourse are his use of the active voice (“you bring” [20]) as he describes actions with entities using the verbs “bring” and “become” [18d, 20]. The sense of movement is supported by the adverb of place, “over” [20]. At the level of positive integers there is some evidence of objectification in Thomas’s discourse because he produces endorsed narratives about operations – albeit that his talk suggests a phrase-driven use of keywords – when he says, “Thirteen plus negative seven is six” [16] and “six divided by two is equal to three” [22b].

Next, I look at evidence in Vignette 5, which links to the equation I have discussed in Vignette 2.

5.4.3. Vignette 5: Thomas’s routine for solving ‘ $13 = 2x + 7$ ’ and ‘ $2x + 7 = 6 + 7$ ’

Thomas’s comparison of ‘ $2x + 7 = 13$ ’ and ‘ $13 = 2x + 17$ ’ indicates that he has a low level of flexibility. His response when I ask him whether the two equations are equivalent, is to want to repeat the routine:

“... if you do it, the same rules apply, Miss. You’ll bring the two ex over ... can I first do this?” [6a-b].

His request in 6a-b to “first do this” indicates that he needs to apply “the same rules” to endorse the equivalence between the equations, instead of accessing horizontal equivalence as a source of the narrative. I allow him to complete the routine, in which he makes an error in line 2 (Figure 14).

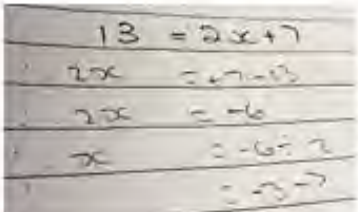
	Line 1	$13 = 2x + 7$
	Line 2	$\therefore 2x = +7 - 13$
	Line 3	$\therefore 2x = -6$
	Line 4	$\therefore x = -6 \div 2$
	Line 5	$\therefore = -3 \rightarrow$

Figure 14: Vignette 5 – Thomas’s solution for ‘ $13 = 2x + 7$ ’

Line 2 indicates that Thomas has moved ‘13’ and ‘2x’ but that ‘2x’ does not ‘become’ ‘-2x’. Even with scaffolding he remains convinced that his routine in line 2 is correct, leading to his conclusion that the two equations are not equivalent. It is possible that the visual appearance ‘ $2x = \dots$ ’ in both equations was more compelling than the application of the unendorsed “rule” that he should change signs when moving entities horizontally. However, I did not interrogate this in the interview.

I also ask him whether ' $2x + 7 = 13$ ' is the same as ' $2x + 7 = 6 + 7$ '. He agrees that they are the same. When I ask for a reason, he attempts to follow the same routine, but I stop him and push him for a verbal explanation. His response, shown below, confirms his need to complete the routine rather than use horizontal equivalence as a source of the narrative:

"Because six plus seven is thirteen, and if you bring the seven over it becomes negative seven the same as here, Miss." (Points to '-7' in line 2 of the solution to ' $2x + 7 = 13$ ') [Thomas, Vignette 5, 22a-b]

Even with prompting he does not see the symmetry between ' $2x + 7$ ' on the left-hand side and ' $6 + 7$ ' on the right-hand side.

5.4.5. Vignette 4: Thomas's routine for solving ' $7(x - 3) = 7 - 2(3x + 1)$ '

In Vignette 4 Thomas continues to use the routine followed in Vignettes 2 and 3. I present his written solution and the realization tree in Figures 15a and 15b respectively, but only describe in detail those features that are different or additional to the features in the previous vignettes.

Figure 15a

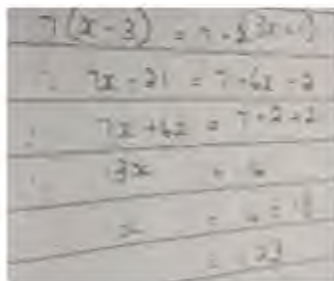


Figure 15b

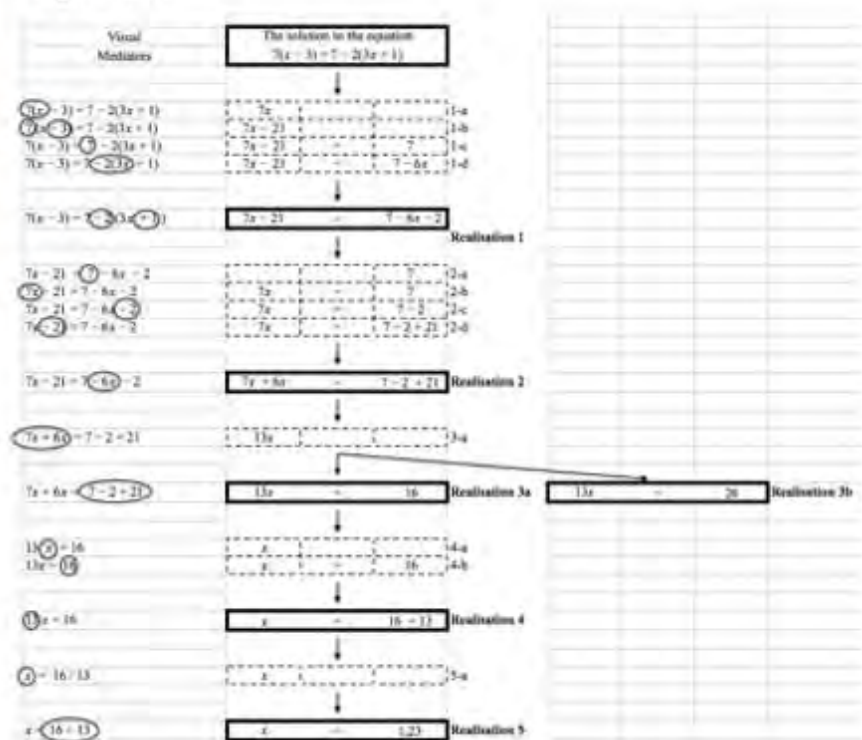


Figure 15: Vignette 4 - Thomas's solution and realization tree for ' $7(x-3) = 7 - 2(3x+1)$ '

The realization tree shows that Thomas’s calculations that led to R1 are endorsed. He then performs the component actions for R2 in a particular order, as indicated by his use of adverbs of time. Figure 15b shows that he first acts upon ‘7’ [2-a] and ‘7x’ [2-b]. He then wants to move ‘-21’ to the right-hand side, but corrects himself because he first has to act upon ‘-2’:

“... and then this negative twenty one .. carry it over it becomes ... no, Miss I must first bring that negative two down” [Thomas, Vignette 4, 6c]

Because he prefaces keywords with demonstrative pronouns, it indicates that he sees ‘-21’ and -2’ as disobjectified entities. His actions are processual, as indicated by the active voice and material process verbs”bring” and”carry”. When adding the integers, Thomas makes a calculation error [R3a]. To check his level of correctibility, I prompt him and he corrects the error [R3b]. I do not pursue him to recalculate the solution in the interview, as the realization tree shows.

I have shown that Thomas’s discourse when describing how he solves the equations in the assessment task is mostly disobjectified, with a high level of unendorsed narratives about spatial arrangement and visual appearance. I have also shown that his routine is consistent enough to conclude that he follows a rigid routine. There is some indication of his levels of correctibility, with prompts, in Vignette 4. his response to the equation ‘ $4m = 2m$ ’ assesses the flexibility of his routine.

5.4.6. Vignette 7: Thomas’s routine for solving ‘ $4m = 2m$ ’

The solution and realisation tree for the solution of ‘ $4m = 2m$ ’ in Vignette 7 are presented in Figures 16a and 16b respectively.

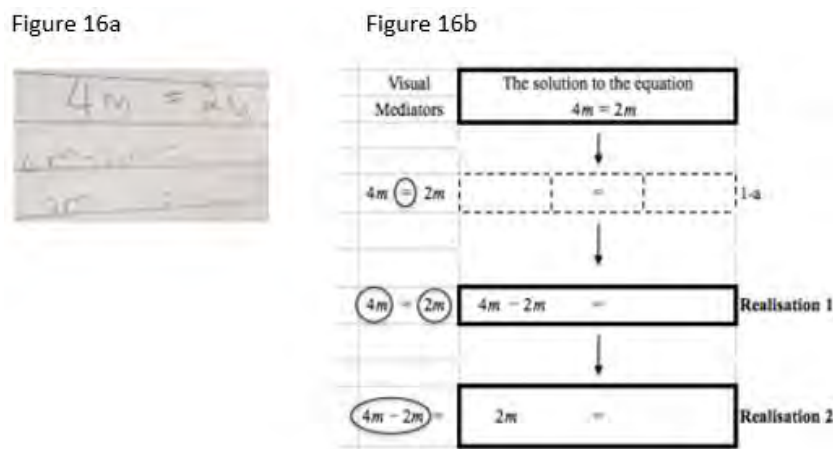


Figure 16: Vignette 7 - Thomas's solution and realization tree for ‘ $4m = 2m$ ’

Figure 16a shows that Thomas does not solve the equation; he only completes the left-hand side. Although he was silent, based on my observations as he wrote down his solution, I am able to represent the sequence of his actions in the realization tree. His discourse, including gestures, confirms that he uses the same routine as in the previous vignettes. He starts by writing the equal sign [1-a]. This suggests that it is a visual mediator for spatially arranging the other entities. He then moves ‘ $2m$ ’ to the left-hand side as ‘ $-2m$ ’. After simplifying the left-hand side of the equation he pauses, not sure how to proceed. He offers a solution:

Transcript 5.2 – Vignette 7

- 2 Thomas: *I think it's two em, Miss.*
3 Interviewer: *So what's em?*
4 Thomas: *What's em? Em is now in place of the ex, Miss.*
5 Interviewer: *Mm. Now what's the value?*
6 Thomas: *The value of em is two*

The entity ‘ $2m$ ’ [R2] has two signifiers for Thomas. He knows that the variable ‘ m ’ has the role of ‘ x ’ in the previous equations. It seems that, since the digit ‘2’ is the only visible number, he decides that it is the solution to the equation. Because the visual appearance of the equation is unfamiliar to him, he cannot produce endorsed narratives about its structure. This results in the production of an unendorsed narrative about ‘ $2m$ ’, which ignores the operation couched in ‘ $2m$ ’.

I ask whether he would get the same answer on both left- and right-hand side if he substituted ‘2’ for ‘ m ’ in the equation. His response is that he would not get the same answer, although he does not offer a reason. When I ask whether ‘ m ’ could be zero, his response is immediate:

“No, Miss ‘cause ... everything times zero is equal to nought, Miss.” [12]

I perceive that ‘everything times zero’ implies that he recognises that ‘ $2m$ ’ means ‘ 2×0 ’, but that he does not realize that the blank space on the right-hand side is also zero. I then scaffold the situation, to see whether he will use horizontal equivalence as a source for his narrative:

Transcript 5.3 – Vignette 7

- 13 Interviewer: *So what would ... four em be if em was zero? (Points to $4m$ on the left-hand side)*
- 14 Thomas: *... It will be nought, Miss.*
- 15 Interviewer: *And two em? (Points to $2m$ on right-hand side)*
- 16 Thomas: *Also nought, Miss.*
- 17 Interviewer: *So can em be zero?*
- 18 Thomas: *No, Miss.*

The fact that Thomas does not link the endorsed narratives ' $4m = 0$ ' and ' $2m = 0$ ' with the value of ' m ', shows that he does not see the horizontal equivalence between the left-hand side and right-hand side of the original equation. As a result, he has no resource for solving the equation. Although he produces an endorsed narrative pertaining to the identity element of zero that "everything times zero is equal to nought" [12], his dependence on the visual appearance of the right-hand side exposes his inflexibility to deviate from his routine. This confirms that his routine is performed with others and that his conviction of the narrative depends on "rules", spatial arrangement and the visual appearance of entities. Taken together, the evidence shows that Thomas's discourse is ritualistic at this moment in his learning.

5.5. Detailed discussion: Emily

I have chosen Emily, a Grade 9 learner, for my detailed discussion, because she represents learners whose discourse shows a difference from the way learners like Thomas talk, regarding linear equations. Furthermore, she is the only learner who solved the equation $4m = 2m$ correctly.

5.5.1. Summary of Emily's discourse

Based on the nuances in the discourse of some learners that I refer to in Section 5.3, Emily's discourse is more objectified than Thomas's because she uses keywords appropriately. However, because she acts on entities in a disobjectified way, I would argue that she does not use the keywords in an explorative way, but that her discourse is mostly phrase-driven.

She produces endorsed narratives about positive and negative integers, algebraic terms and the structure of equations. She talks about horizontal equivalence in equations, but does not use it as the

source of her narrative, even when prompted to do so. She does not talk about or use vertical equivalence either. Like Thomas, the closing condition of her routine is visual appearance of the answer – “variable = [some number]”. Knowing how to rigidly apply the routine, together with endorsed narratives about the properties of number, allow Emily to solve the equation $4m = 2m$. Although she performs the routine for, and with others, Emily defends her realizations in an endorsable way when prompted to do so. Thus, her discourse is sometimes internally persuasive because at times she uses the properties of number as the source of her narrative. However, she generally depends on support from non-mathematical mediators in her routine. Emily’s flexibility is low despite her more objectified discourse. There is not enough evidence to say that Emily is in transition to explorative discourse, because she does not have the characteristics of “thoughtful imitation” (Sfard, 2008, p. 249).

As with Thomas, I describe her routine for Vignette 3 as representative of her routine for solving all the equations in her assessment task. I then show how the conversation continues with the comparison of different forms of the equation ‘ $2x + 7 = 13$ ’ in Vignette 5, followed by some discussion of Vignette 4, where she corrects an error. Finally, I look at her solution of ‘ $4m = 2m$ ’ in Vignette 7.

5.5.3. Vignette 3: Emily’s routine for solving ‘ $2x + 7 = 13$ ’

Emily’s description of her solution to this equation provides evidence of her reference to disobjectified signifiers and unendorsed actions with entities. Her solution and the realization tree for the equation are shown in Figures 17a and 17b respectively.

Figure 17a

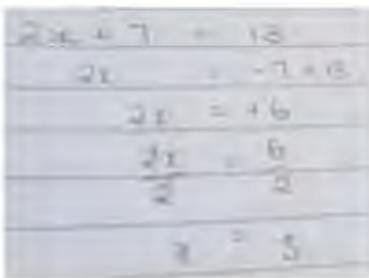


Figure 17b

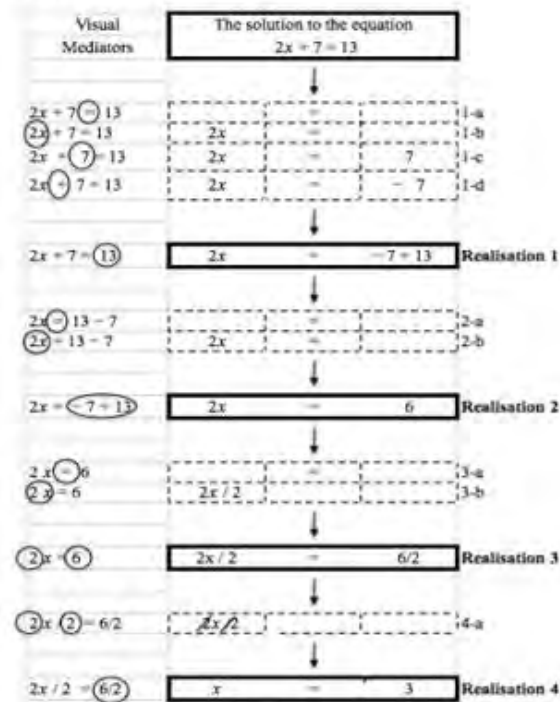


Figure 17: Vignette 3 - Emily's solution and realization tree for ' $2x + 7 = 13$ '

The realization tree in Figure 17b shows her routine, which is similar to her routine for all equations. The realization tree shows that in the sub-nodes leading to R1 she rearranges entities on either side of the equal sign and applies the unendorsed rule that the sign of entities changes when there are horizontal shifts. R1 serves as signifier for her to simplify the expression on the right-hand side [R2]. She then divides the terms on either side by the coefficient of 'x' [3-b, R3] and performs arithmetic operations to realize the solution in the form ' $x = [\text{some number}]$ ' [R4].

For Emily as a Grade 9 learner, this equation was not part of her assessment task, so she sees it for the first time. She starts off by saying "I have to get the value of ex ..." [4], which is the goal of her activity – to find a solution in the form ' $x = [\text{some number}]$ '. She then describes how she obtained Realization 1:

"I put all the like terms on one side and the unlike terms on the other side. ... Then I put the seven on the right hand side. When ... I moved the seven... to this side, the sign changes to a negative." [Emily, Vignette 3, 6a-b, 8a-b].

Here Emily speaks in the active voice, describing her actions with entities “the seven” [6b, 8a-b] and “the sign” [8b]. Her use of the definite article to preface each of these entities indicates that she acts upon them as separate disobjectified signifiers. Her consistent reference to “side” indicates the significant role of spatial arrangement in her routine. This is supported by evidence in the realization tree in Figure 17b, where sub-nodes 1-a, 2-a and 3-a show that for each line of the solution, Emily first writes the equal sign, confirming its role as visual mediator in the process of spatial arrangement.

Her description of actions with ‘+7’ in the extract shows that she acts upon ‘7’ and ‘+’separately [8b]. Her use of material process verbs such as “put”, “took” and “moved” to describe her action and her use of adverbs of place such as “one side” and “the other side” indicate how she moves “the seven” to a particular place on the page. These actions are reflected in sub-nodes 1-c and 1-d of the realization tree.

Transcript 5.4 contains Emily’s explanation of how R1 in Figure 17b signifies R2:

Transcript 5.4 – Vignette 3

- 12a *Emily: Then it’s gonna be two ex, then I’m gonna equal to that, then I sub- ..
then I’m gonna subtract.*
- 12b *I’m gonna take positive seven away from thirteen which is gonna
give me .. (inaudible) have a six left.*
- 12c *Then I brought that down. It was two ex over .. ‘cause two can .. it’s
only two that can go into two and into six without giving a*
- 14a *remainder. ...
then I divided this two ex by two, and then what I do on this side I
have to do on this side*

Her routine is typified by actions with entities including vertical movement, as indicated by the adverb of place “down” [12c]. She uses the active voice, with “I” as the subject of the sentence and doer of the material actions and speaks of entities in a disobjectified way, as is suggested by her use of demonstrative pronouns in “this two ex” [14a] and “that” [12a and 12c]. This use of pronouns creates the impression that these are separate entities, which is similar to the effect of her use of articles as in “a six” [12b].

In Transcript 5.5 Emily justifies her actions with a mathematically endorsed narrative about the properties of positive integers. She explains: “It’s only two that can go into two and into six without giving a remainder” [12c]. However, this is not a useful resource when solving linear equations and causes a problem in Vignette 4, as I show. Another endorsed narrative relates to horizontal equivalence and the need to perform the same operation on both sides [14a]. Despite the endorsed narratives, her discourse has elements of disobjectification, as illustrated in Transcript 5.5:

Transcript 5.5 – Vignette 3

14b Emily: *So then I said six over... then I said six divided by two as well.*

Then I... so... I scr... how do I say now again? Then I simplified

14c *this, I got... I just got the ex.*

14d *I scratched out the... I cancelled out the twos, which gave me a ex.*

Then I had to say six divided by two, which gave me three.

When working with ‘ $2x$ ’, she prefaces ‘2’ and ‘ x ’ with the definite article, rendering “the two” and “the ex” separate entities that she operates on in a disobjectified way [14c]. She makes no reference to the identity element that leaves ‘1’ as the coefficient of ‘ x ’. Rather she eliminates both twos by physically ‘scratching’ them out [sub-node 4-a]. In contrast, when working with the positive integers ‘6’ and ‘2’ she refers to the endorsed narrative pertaining to the arithmetic relation between them [14b and 14d].

5.5.4. Vignette 5: Emily’s routine for solving $2x + 7 = 6 + 7$

After Emily has solved the equation ‘ $2x + 7 = 13$ ’ I ask whether ‘ $2x + 7 = 6 + 7$ ’ is the same as the previous equation. She immediately says “yes” [2], but the evidence suggests that Emily would only be convinced of the horizontal equivalence between left- and right-hand side if she were to complete the routine. The appearance of ‘+7’ on both sides of the equation confuses her:

“... Seven plus six is also thirteen, Miss, but ... if you do the sum then this is gonna go on this side (points to ‘+7’ on left-hand side), then this is not gonna be the same (points towards right-hand side)” [Emily, Vignette 5, 6a-b].

This statement indicates that Emily’s sense of horizontal equivalence is not fully objectified. She uses it in some instances but not in this instance, which prevents her from seeing the symmetry

between left- and right-hand sides. Later, when I point to '2x' on the left-hand side and '6' on the right-hand side simultaneously and ask whether they are equal, she cannot see the equivalence either. She responds to both situations by indicating that she would have to perform the routine to determine equivalence. Although Emily has already determined that 'x' was '3' in the earlier equation, she says that '2x' would only equal '6' if she substituted 'x = 3' into '2x':

“[Points to “x”] If I had to substitute the three in there, Miss, then it will be the same” [10b].

Her explanation suggests that 'x' could also have a different value, in which case '2x' would not equal '6'.

Further evidence of Emily's lack of horizontal equivalence emerges when I present her with the equation in the form '13 = 2x + 7'. She explains that the equation would be “the same” [16] as '2x + 7 = 13' but that “the only thing that's gonna change is the signs” [22]. Thus 'x' would equal '-3'. The three situations I have just described show that Emily's flexibility is fragile. Her safety net remains the repetition of the routine. This same careful execution of her routine leads to the successful solution of the equation '4m = 2m', which I discuss in the next section.

5.5.5. Vignette 4: Emily's routine for solving $2(4x - 5) - (3x + 6) = -2(x + 3)$

In solving the equation, Emily follows the same routine as described for Vignette 3, but her written response includes an error. In this section I only focus on how she addresses the error, which illustrates her level of correctability. Her original solution contains some unendorsed narratives about arithmetic relations that she corrects during the interview. For that purpose I present both the original and corrected version of her solution:

$2(4x - 5) - (3x + 6) = -2(x + 3)$	$2(4x - 5) - (3x + 6) = -2(x + 3)$
$8x - 10 - 3x + 6 = -2x - 6$	$8x - 10 - 3x - 6 = -2x - 6$
$8x - 3x + 6x = 40 - 6 - 6$	$8x - 3x + 2x = 40 + 6 - 6$
$\frac{11x}{11} = \frac{-2}{11}$	$\frac{+7x}{7} = \frac{+10}{7}$
$x = \frac{-2}{11} \text{ or } 0,18$	$x = 1,4$

Figure 18: Vignette 4 - Emily's correction of ' $2(4x - 5) - (3x + 6) = -2(x + 3)$ '

These unendorsed narratives are illustrated in the realization tree in Figure 5.19, where the visual mediators are circled. In sub-node 1-a the visual mediators are ‘2’ and ‘4’, although the realization is ‘8x’ and in 1-c the visual mediators are ‘2’ and ‘5’ although the realization is ‘-10’.

Emily’s narrative for simplifying ‘ $-(3x + 6)$ ’ is also unendorsed. She copies down the negative sign, then points to the bracket and says

“you gonna leave that the same way” [6]. Thus ‘ $-(3x + 6)$ ’ becomes ‘ $-3x + 6$ ’.

When I question her, she explains her simplification of the left-hand side again, starting with ‘ $2(4x - 5)$ ’, then repeating the fact that “I just left this” i.e. ‘ $(3x + 6)$ ’ “the same” [10]. Finally I ask whether there is a number in front of the bracket. She does not immediately see what I am inferring and shows some confusion, as illustrated in Transcript 5.6:

Transcript 5.6 – Vignette 4

34 *Emily:* ... before the bracket there’s always a one [Writes “1” after “-”]
 36 Then I said one times three will give me minus three ex
 38 And then I timesed the one by that as well Miss, [Points to “+6”]
 which gave me six
 39 *Interviewer:* Why’s it positive?
 40 *Emily:* [Silence. Points from “1” to “+6”]

The transcript shows that Emily does not see ‘-1’ as a term, but sees ‘-’ and ‘1’ as separate disobjectified entities, because she speaks of “a one” [34] and “the one” [38]. As a result she reproduces the unendorsed narratives, namely ‘ $1 \times 3 = -3$ ’ and ‘ $1 \times 6 = 6$ ’ in lines 36 and 38 respectively. When I prompt her that there is possibly a problem, her initial response is to return to the beginning and check the calculations pertaining to ‘ $2(4x - 5)$ ’ even though there is no problem. This confirms her low level of flexibility, because she is not able to correct the immediate problem in isolation of the rest of the calculations on the left-hand side. In fairness, though, she was probably intimidated by my questions and might have shown more flexibility in a different situation. After my prompt in line 41 she corrects her error. Sub-node 1-dii in Figure 5.18 illustrates how she corrects the error so that ‘ $-(3x + 6)$ ’ becomes ‘ $-3x - 6$ ’.

From Transcript 5.6 it appears that she does not have a high level of correctability, because she does not immediately respond to my scaffolding. However, the written error in ' $2(4x - 5) - (3x + 6) = -2(x + 3)$ ' is the only opportunity to engage with Emily on an error in her written solution, and is thus the only opportunity to comment on her level of correctability.

As indicated in the discussion of Vignette 3, Emily's justification for division by the coefficient of ' x ' because it is a common factor, would cause problems later. In Vignette 4 she tries to apply the same narrative. This causes confusion because, as she explains, she has to "look for a number that will go into the seven and the ten" [78e]. She cannot find a common factor of '7' and '10'. Ultimately, with scaffolding she realizes that she should divide both terms by the coefficient of ' x ' (see R4 in the realization tree).

5.5.6. Vignette 7: Emily's routine for solving $4m = 2m$

Emily's inflexible routine might have disadvantaged her in vignette 5, but it works in her favour when solving ' $4m = 2m$ '. Figures 20a and 20b show her written solution and realization tree respectively:

Figure 20a

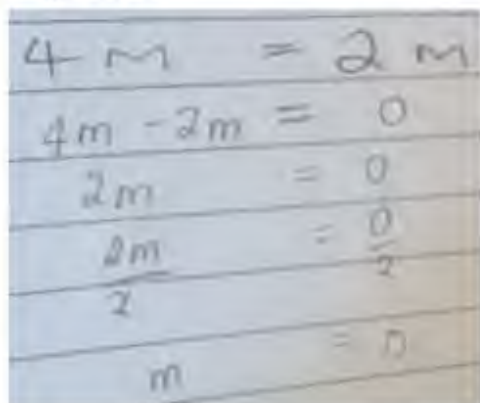


Figure 20b

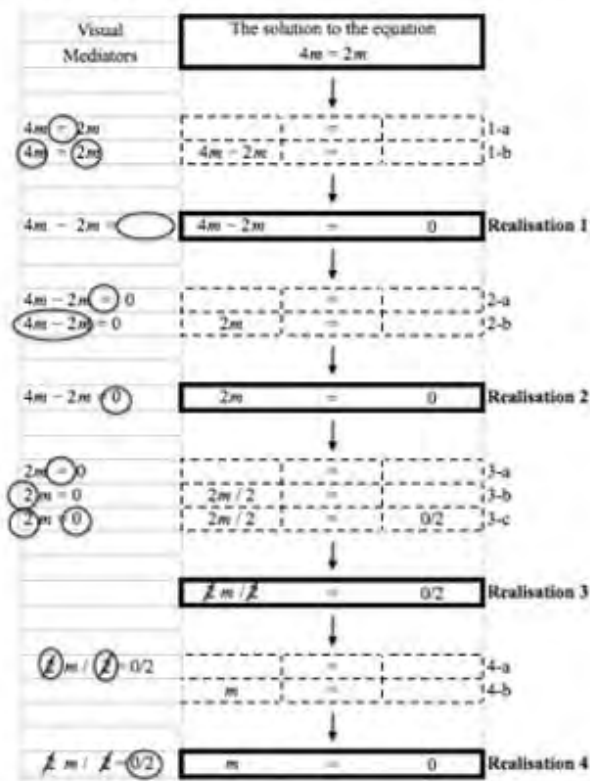


Figure 20: Vignette 7 - Emily's solution and realization tree for '4m = 2m'

As in her other solutions, the equal sign is the first visual mediator for the development of each realization [1-a, 2-a, 3-a, 4-a]. The original equation signifies the spatial arrangement that sees the horizontal shift of '2m' to the left-hand side, with the application of the unendorsed rule that '2m' becomes '-2m' [1-b]. R1 signifies the simplification of the expression on the left-hand side [2-b]. At this point Emily's access to the endorsed narratives about the properties of zero give her an advantage that no other learner has or uses. She recognizes that the right-hand side would equal zero after shifting '2m' to left-hand side, as indicated in Transcript 5.7:

Transcript 5.7 – Vignette 7

- 2c Emily: Must this stay a zero this side?
- 3 Interviewer: What do you think?
- 4 Emily: 'Cause there's nothing 'cause it have to be this side then there's nothing there
- 5 Interviewer: Mm

6f Emily: *[Uses calculator to find the value of “0/2”. Not happy with the answer, she then enters $0 \div 2$]. Oh, any number that divides by zero stays zero.*

Transcript 5.7 contains two endorsed narratives about zero in lines 4 and 6f respectively. Firstly, because there is “nothing” on the right-hand side, the value is zero. Secondly, zero divided by any number stays zero. Although she says that a number divided by zero stays zero, she actually divides ‘0’ by ‘2’. Thus, by strictly following the routine, Emily solves the equation ‘ $4m = 2m$ ’.

5.6. Conclusion

I started this chapter by presenting learners’ scores on their assessment tasks, and providing Table 8 which reflects my initial sense of all the learners’ discourse. After presenting a summary of my analysis of the discourse of all the learners, I presented detailed analyses of Thomas and Emily’ discourse, to give the reader insight into how I drilled down using the research text from my Level 2 analysis.

In the next chapter I use these results to answer my research questions and relate these answers to the empirical research reviewed in Chapter 2. I also make recommendations for practice and further research.

Chapter 6: Discussion and Conclusion

6.1. Introduction

This research study has been a journey of discovery, on both a personal and professional level. When I started the journey I was convinced that the body of mathematical knowledge that exists ‘out there’ would be accessible to all learners if only they were taught properly. I believed that if the schema of knowledge was very clear in the teacher’s mind, what was said to learners would be accurate and unambiguous. Thus in the development of teacher professional interventions as part of my work in the field of teacher development, a key focus was ensuring that teachers had this clarity, and that I could model pedagogic strategies that would ensure that their message was indeed accurate and unambiguous.

In my professional experience I have observed that, as Spaul (2013) indicates, poor school mathematics performance is not confined to schools that service working class communities. However, this does not mean that all working class schools produce poor performance. In a major project in a group of schools in very close proximity to one another, I experienced the conundrum that some schools produced much better mathematics results than others. I decided to interview learners that produce these good results. At that point I was convinced that I would have a story to tell teachers about the impact of teaching practices on learner performance.

The interviews marked a crossroads in my journey. The topic was linear equations, and learner responses were so different to what I had expected, I had to search for a way to make sense of the fact that this group of learners’ written solutions to a set of linear equations, if classified according to the scale of achievement for the National Curriculum Statement, would be placed between “meritorious achievement” and “outstanding achievement” (Department of Basic Education, 2011, p. 158). Yet, the interviews showed that they could not coherently explain why any of their solutions were sensible, using mathematically endorsable properties.

And so was born this study where I set out to use research to answer the conundrum that learners who achieve good results in formal assessments cannot explain why their solutions are correct. In my search for a theory to view this problem, I identified Sfard’s (2008) theory of commognition as suitable because, as indicated in Section 3.6, it would afford me a way to show teachers why learners produce solutions as the ones presented in Section 1.3.1. Although this study has some limitations,

as discussed in Section 6.3, looking at learners' talk, gestures and writing using Sfard's tools of words, visual mediators, narratives and routines, proved a productive way to find the answers I sought.

In this concluding chapter, I present the answers to my research questions, relate this to what others have said about the problem, acknowledge the limitations, and end with some recommendations for practice and further research. .

6.2. Description of learner discourse when solving linear equations

In this section I answer the questions that directed my study.

1. How can a learner's discourse when solving linear equations be described using Sfard's notions of exploration and ritual?

All learners in my study – Grades 8 and 9– present with ritualistic discourse. The only mathematical objects learners regularly produce endorsed narratives about are positive integers. Mostly they produce unendorsed narratives about disobjectified entities, including actions with these entities. For all learners the closing condition is the appearance of the solution. They perform their routines for and with others using the rules of others, visual appearance and spatial organization as the source of their narrative. In addition, learners have little or no flexibility. Based on my limited evidence, there is a reasonable level of correctibility regarding arithmetic operations, but not regarding the structure of their solutions.

From this finding I conclude that no learner in my study meets the requirements of the curriculum as stated in Chapter 1. No learner talks about the properties of number or axioms (like inverse operations) as a source of the narrative for solving linear equations. Even endorsed narratives about integers and algebraic terms are limited, as these mostly relate to positive integers. Moreover, the learners' realization trees only have one branch, which implies that they do not see the equation as a narrative of the function.

These findings resonate with the literature reviewed in Chapter 2. Using a discursive rather than cognitive theory, my analysis confirms the view of Linchevski and Livneh (1999) and Hoch and Dreyfus (2004) that learners who do not have what they call structure sense, cannot use the properties of number or function to solve linear equations, but rely on other people's routines. The

findings of my study also concur with Hoch and Dreyfus's (2006) claim that there is no correlation between learners' structure sense and their manipulation skills. For in my study, learners score good marks in their task but do not use endorsed narratives about mathematical objects as the source of their narrative.

In my study learners describe how they move disobjectified entities in the equation. de Lima & Tall (2008, p. 4) use the concept of "embodiment" to describe how learners move terms when solving linear equations. From a commognitive perspective I use learners' choice of words to show how they spatially arrange disobjectified entities as part of their routine, to similar effect. The focus on the notion of disobjectification in my study, however, adds the dimension that learners are not working with mathematical objects,

2. Do all learners who have ritualized discourse, display the same characteristics in their talk, gestures and writing? If not, what are these differences?

All learners in my study have ritualized discourse, but their discourse does not necessarily have identical features. There are differences in the way learners' use keywords, how they respond to prompts and their level of flexibility.

While some learners use keywords in a phrase-driven way, others use colloquial words or talk about disobjectified entities. Some learners produce endorsed narratives about positive and negative integers, algebraic terms and the structure of equations, others only produce endorsed narratives about positive integers. When prompted, some learners, use endorsed narratives as a source of the narrative for the solution to an equation, others do not. All learners perform their routines for and with others but, with prompting, only some learners show internal persuasion and defend their realisations with endorsed narratives. Furthermore, some learners have the flexibility to see the structure of the equation as a signifier for their realization, whereas others rely solely on the routine.

While my findings resonate with the findings of other studies conducted from a cognitive perspective, the answers to my research question offer a particularly detailed account of learner thinking that has been classified as ritualized. Solving linear equations in an explorative way implies more than whether a learner 'has' or does not 'have' structure sense (Hoch & Dreyfus, 2006) or relational understanding (Skemp, 1976). I argue that my analytic tools, using Sfard's (2008) theory

of commognition, have allowed me to drill down and identify differences between learners that other studies have not shown.

Regarding learners' use of the equal sign, Essien and Setati (2006) found that the dominant interpretation of the equal sign among senior phase learners was operational. Using the tools of Sfard's (2008) commognitive theory, I show that some learners see the equal sign relationally, but do not use this property as a source of the narrative to see that the expressions in the equation are equal and interchangeable. As a result, no learner uses horizontal equivalence to conclude that ' x ' has the same value in ' $2x + 7 = 13$ ' and ' $13 = 2x + 7$ '. Thus, while I would agree with Essien and Setati (2006) that there are barriers to the manipulation of more sophisticated algebraic equations when learners do not have relational understanding, I would argue that some learners could be classified as having relational understanding, when actually their talk about the equal sign is phrase-driven.

3. Are there any learners whose discourse is explorative, or are in a transition phase between ritualistic and explorative discourse?

No learner in my study is in this transition stage, because no learner shows that they are "thoughtful imitators" – that they think about their routines and try to make sense of things for themselves. However, some learners respond positively to prompts, and then try to make sense of their routines using endorsed narratives about mathematical objects. The nuances that emerge between those who respond to prompts and those who don't, are a possible lever for shifting learners towards explorative discourse.

My study raises awareness that learners who score well on written assessment tasks about linear equations, do not necessarily know why they perform routines, or when to adapt the routine. This means that learners do not meet the curriculum requirements. These interviews were conducted in the eighth month of the school year, by which time learners should, especially at Grade 9 level, have progressed from "a view of Mathematics as memorized facts and separate topics to seeing Mathematics as interrelated concepts and ideas represented in a variety of equivalent forms" (Department of Basic Education, 2011, p. 21).

In trying to explain this result, I note that the curriculum encourages teaching in an operational way. For example, in Section 3.3.2 the Grade 8 "Concepts and Skills / Clarification Notes and Teaching Guidelines" section of the curriculum document, teachers are given examples of linear equations of

different complexity levels with the “steps” for solving each (Department of Basic Education, 2011, p. 94). Therefore, the curriculum could encourage learners to work in a ritualistic way, because the implication is that learners would depend on the visual appearance of the problem for the selection of the routine. This observation has also been made by Gcasamba (2014).

In trying to explain this result, I also note what Sfard (2008; personal communication, January 21, 2014), and others who have used her theory (Nachlieli & Tabach, 2012) have to say about the role of the teacher in leading learners to explorative discourse. It is possible, therefore, that the discourse of the learners in my study has been constrained by the teacher talk, as I have described briefly in section 1.3.2. This is a topic for further research, as discussed in section 6.5.

6.3. Limitations of my study

Sfard compares the reconstruction of learner thinking that her theory facilitates, to the work of an archaeologist (Sfard, 2008: 276). And, like an archaeologist, I cannot make claims that my interpretation of learners’ thinking is entirely accurate. Although I have argued that Sfard’s theory allows me to drill down and generate rich data about all features of learner discourse, my findings pertain to one topic, linear equations, for one moment in their learning. The way I posed questions to learners could have influenced their responses, as could their perception of my position of authority.

My findings are limited further by the fact that I have no indication of what the teachers were saying or doing. Because there are no significant differences between the Grade 8 and 9 learners’ discourse, it would have added value to the study if I could have compared the discourse of the teachers in their separate classrooms.

6.4. Recommendations for practice

Sfard (2008) argues that learners learn by imitation. Some studies have pointed to the impact of teachers’ discourse on learners’ discourse (Heyd-Metzuyanim & Graven, 2015; Tabach & Nachlieli, 2011; Sfard, personal communication, January 21, 2014). In the light of the example of classroom discourse I presented in Section 1.3.2, the findings of this study suggest that classroom discourse should be a focus of teacher professional development. The nuances in learner discourse that my study shows, could become points of engagement in a teacher professional development course that focuses on levers for shifting learner discourse from the ritualistic to the explorative.

6.5. Recommendations for further study

Analysis of learner errors remains a very popular way of evaluating learner performance and such analyses form the backbone of research reports on learner performance in national assessment tasks. Geasamba (2014), in her study on learner errors used commognition to determine why learners made errors as they did. Her conclusion was that learners had not objectified the function and were acting in a processual manner. In my study I did not have sufficient evidence to use learner errors to explore the question of their levels of correctibility. She has shown that commognition puts a different theoretical perspective on learner errors. I see potential for structuring an engagement with learners specifically regarding their errors, using my analytic framework.

I also see relevance in a longitudinal study to measure shifts in learner discourse, as Tabach and Nachlieli (2012) have done. If there were a parallel intervention for teachers, one could measure shifts in discourse as a result of a teacher professional development intervention that includes course work and classroom support. I note that my analytic framework would be suitable for analysing both learner and teacher discourse.

6.7. Conclusion

Using Sfard's (2008) commognition theory, I developed an analytic framework that facilitates a detailed analysis of Grade 8 and 9 learner discourse about their solutions to a set of linear equations. The findings of the study show that the only objectified discourse all learners have, is that pertaining to whole number arithmetic and that learners are not meeting the specifications in the curriculum as presented in Section 1.4 of this study. The differences in these learners' written and spoken discourse indicate that even though they perform well on written assessment tasks, they might not have explorative discourse about the topic. This vindicates Spaul's (2013) statement that learners are "functionally innumerate" and that Grade 9 learners perform two grades below Grade 8 learners from other countries. Thus, these findings could contribute to an explanation of the current crisis in school mathematics in the country.

Furthermore, ritualised discourse is not the same for all learners. The nuances in their discourse revealed by the analysis could become suitable levers for shifting learners towards explorative discourse. These findings, limited as they are, contribute to our understanding of the bigger problem

of poor learner performance in international and national assessment tasks. Reflecting on her the theory of commognition, Sfard (2008, pp. 277–278) states that “whatever is said, although uttered by a specific individual, is the work of many”. Thus, this study gives insight into the discourse community of the classrooms of these learners. Seen this way, the potential for changing classroom discourse by analysing the discourse of learners and teachers becomes a tangible prospect. The analytic framework developed for this study is suitable for the purpose.

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Appendix Table

Appendix 1: Vignette 1 – 7 (excel spread sheets on CD)	Thomas Level 1 research text
Appendix 2: Vignette 1 – 7 (excel spread sheets on CD)	Emily Level 2 research text
Appendix 3	Level 2 analytic framework
Appendix 4	Thomas Level 2 research text
Appendix 5	Emily Level 2 research text
Appendix 6	Information letters

Appendix 3: Level 2 analytic framework

Questions	Sub-questions	Indicators	Source at Level 1	Evidence	
1. What is the degree of objectification of the learner's discourse	1.1.1 Does the learner's word use and mediator use signify numbers and algebraic terms as mathematical objects in their own right?	1.1.1 The use of words and phrases in one line of an equation signify numbers and algebraic terms as mathematical objects.	1.1.1.1 <u>Tools - Words</u> Keywords with relational process verbs		
			1.1.1.2 <u>Tools - Words</u> Keywords as subject and object, with the verb in the passive voice		
			1.1.2.1 <u>Tools - Words</u> Keywords linked with constant phrases e.g. naming equations		
	1.1.2 Does the learner talk about and act on extra-discursive mediators within a function?	1.1.2.1 Phrase-driven use of keywords	1.1.2.2 Reference to visual mediators as disobjectified entities	1.1.2.2 <u>Tools - Words</u> Keywords prefaced by articles, pronouns or wrong use of keywords	
			1.1.2.3 Reference to actions with disobjectified entities (digits, operatory signs, algebraic symbols and pronouns representing them).	1.1.2.3 <u>Tools - Words</u> Digits, algebraic symbols, operatory signs and pronouns, mostly in conjunction with material process verbs	
			1.1.2.4 The naming and spatial arrangement of symbols, digits and letters.	1.1.2.4 <u>Tools - Words</u> Adverbs of place	
	1.2.1 Does the learner's word use and mediator use support the horizontal equivalence between functions?	1.2.1 Discourse about equivalence or symmetry between the LHS and RHS of an equation in one line	1.2.1 Narratives – Notes Relations between entities: endorsed narratives about equivalence between functions on LHS and RHS of an equation	1.2.1 Narratives – Notes Relations between entities: endorsed narratives about equivalence between functions on LHS and RHS of an equation	
		1.2.2 Does the learner's word use and mediator	1.2.2 Discourse about entities that change appearance or purpose,	1.2.2.1 <u>Tools - Words</u> Adverbs of place that refer to	

	use interrupt the horizontal equivalence between functions?	accompanied by reference to a shift to the opposite side of the equation.	horizontal shifts (over, this side, that side)	1.1.2.2 <u>Narratives – Actions with entities</u> Unendorsed narratives about actions with or by entities	1.1.2.2
				1.3.1.1 <u>Narratives – Notes</u> Endorsed narratives about the relationship between equations in consecutive lines	
	1.3.1 Does the learner’s word use and visual mediators support the vertical equivalence between narratives? 1.3.2 Does the learner’s word use and visual mediators interrupt the vertical equivalence between narratives?	1.3.1 Discourse about one equation as a signifier that realizes an equivalent narrative.	1.3.1.1 <u>Narratives – Notes</u> Acceptance of incorrect arithmetic operations		
		1.3.2.1 Simplification of expressions with no obvious reference to consecutive equations as equivalent narratives. 1.3.2.2 Discourse about shifts of entities to consecutive lines	1.3.2.1 <u>Narratives – Notes</u> Acceptance of incorrect arithmetic operations		
	1.4.1 Does the learner see that the original equation as a signifier has more than one realisation for the function? 1.4.2 Does the learner see that the original equation as a signifier has only one realisation for the function?	1.4.1 Solution of equations using more than one representation of function i.e. a graph, table or algebraic equation	1.4.1 <u>Realisation trees</u> Vignettes 2, 3, 4, 7 More than one branch		
		1.4.2 Solution of equations using only one representation of function i.e. the algebraic equation	1.4.2 <u>Realisation trees</u> Vignettes 2, 3, 4, 7 One branch	Vin 2 Vin 3 Vin 4 Vin 7	

2. To what extent are the learner’s narratives endorsed?	2.1.1 Does the learner recall and reconstruct previously endorsed narratives?	2.1.1.1 Production of endorsed narratives about the structure and properties of integers, variables and equations	2.1.1.1 <u>Narratives - Description of entities</u> Endorsed narratives describing integers, variables and equations	
		2.1.1.2 Production of endorsed narratives about relations between	2.1.1.2.1 <u>Narratives - Relations between entities</u>	

		mathematical objects	Endorsed narratives about arithmetic operations with integers and variables. 2.1.1.2.2 <u>Narratives - Relations between entities</u> Endorsed narratives about equivalence between functions on LHS and RHS of an equation 2.1.1.2.3 <u>Narratives - Relations between entities</u> Endorsed narratives about the relationship between equations in consecutive lines 2.1.1.3 <u>Narratives – Activities with or by entities</u> Actions with or by mathematical objects 2.1.1.4 <u>Narratives - Source of the narrative</u> Properties of number 2.1.2.1 <u>Narratives - Description of entities</u> Unendorsed narratives about numbers, variables and equations 2.1.2.2.1 <u>Narratives - Relations between entities</u> Unendorsed narratives about operations with numerals and letters. 2.1.2.2.2 <u>Narratives - Relations between entities</u> Unendorsed narratives about rules for operations that apply to entities including operatory signs, letters and numerals. 2.1.2.2.3 <u>Narratives - Relations between entities</u> Unendorsed narratives about expressions on LHS and RHS of an equation	1.2.1.1
		2.1.1.3 Production of endorsed narratives about actions with or by mathematical objects 2.1.1.4 Application of rules to mathematical objects		
		2.1.2.1 Production of unendorsed narratives about entities		
2.1.2 Does the learner produce narratives that are not endorsable?		2.1.2.2 Production of unendorsed narratives about relations between entities, including the application of rules to disobjectified signifiers		

				narrative Properties of number	
	4.2 Is the learner's discourse for others?	4.2 Conviction of the narrative depends upon reference to an outside authority, or on spatial arrangement and visual appearance of entities	4.2.1 Narratives - Source of the narrative Teacher, peers, interviewer	4.2.1 Narratives - Source of the narrative	
			4.2.2 Narratives - Source of the narrative	4.2.2 Narratives - Source of the narrative	
				Spatial arrangement and visual appearance	

5. By whom is the routine performed?	5.1 Does the learner work independently?	5.1 No scaffolding required	5.1 Routine - Notes Mathematical justification for routine	5.1 Routine - Notes		
	5.2 Does the learner depend on scaffolding?	5.2.1 Scaffolding required, as well as prompting from the interviewer	5.2.1.1 Routines - Notes Positive response to prompt	5.2.1.1 Routines - Notes		
		5.2.2 Dependence on visual mediators as prompts for the next step in the procedure	5.2.2.1 Routines - Notes	5.2.2.1 Routines - Notes Entities and visual appearance as prompts	5.2.2.1 Routines - Notes	
			5.2.2.2 Wrong / no response to prompt	5.2.2.2 Routines - Notes Mention of absent mediators	5.2.2.2 Routines - Notes	3.2.2.2
	5.2.3 Reference to the involvement of others in the completion of the task	5.2.3 Routines - Notes Source of the narrative	5.2.3 Routines - Notes Teacher, interviewer, peers	5.2.3 Routines - Notes		

6. What is the learner's level of flexibility?	6.1 Does the learner make permissible variations to the procedure?	6.1.1 Discourse about equivalence or symmetry between different representations of the same problem	6.1.1 Narratives – Description of entities (Vin 5 & 6 only) Relations between entities: endorsed narratives about horizontal equivalence between different representations of an equation	6.1.1 Narratives – Description of entities (Vin 5 & 6 only)	
		6.1.2 Modifies the routine by introducing an endorsable sub-routine suited to the task	6.1.2.1 Routine / realisation trees More than one branch / branch other than algebraic manipulation	6.1.2.1 Routine / realisation trees	

			6.1.2.2 <u>Routine – notes</u> Positive response to scaffolding and prompts	5.2.1.2
		6.2.1 Rigid sequence of steps and spatial arrangement of entities in particular places on the page	6.2.1.1 <u>Tools - Words</u> Adverbs of time	3.2.2.1.2
		6.2.2 Obedience to a set routine	6.2.1.2 <u>Tools - Words</u> Adverbs of place.	1.3.1.1
	6.2 Does the learner follow the routine in a strictly defined and rigid way?		6.2.2.1 <u>Tools - Words</u> High modality verbs	
		6.2.3 Use of entities as prompts	6.2.2.2 <u>Routine – notes</u> Wrong / no response to scaffolding and prompts	3.2.2.2
		6.2.4 Repetition of the same routine and acceptance of the solution if the closing narrative is in the form 'x = ...'	6.2.3 <u>Routine - Notes</u> Entities as prompts.	5.2.2
		6.2.5 Difficulty with completion of the routine if the requisite mediators are absent	6.2.4 <u>Realisation trees</u> Unendorsable revision of a step in the procedure	
			6.2.5 <u>Narratives - Notes</u> Mention of absent mediators	

7. What is the learner's level of correctibility?	7.1 Does the learner recognise unendorsed narratives, and correct them?	7.1.1 Correction of incorrect statements about mathematical objects, with or without prompting 7.1.2 Uses vertical equivalence as a resource to correct unendorsed narratives	7.1.1 <u>Routines - Notes</u> Correction of error 7.1.2.1 <u>Narratives – Notes</u> Vertical equivalence relationship 7.1.2.2 <u>Realisation trees</u> Revision of a node that produces an endorsable link between signifier and realisation	1.3.1.2
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					1.3.2.1.1
				7.2.1.1 <u>Routines – Notes</u> Acceptance of incorrect arithmetic operations	
		7.2.1 Does not amend unendorsed narratives, even with prompting		7.2.1.2 <u>Routines - Notes</u> Amendment to error produces another error	
	7.2 Does the learner fail to recognise unendorsed narratives, or correct them?			7.2.1.3 <u>Routines - Notes</u> No / wrong response to prompt	3.2.2.2
		7.2.2 Does not use vertical equivalence as a resource to correct unendorsed narratives		7.2.2.1 <u>Narratives – Notes</u> No vertical equivalence relationship	
				7.2.2.2 <u>Realisation trees</u> Nodes do not illustrate an endorsable link between signifier and realisation	

Appendix 4: Thomas Level 2 research text

Based on my seven criteria I classify Thomas's discourse as ritualistic. His discourse is mostly disobjectified, with some exceptions. He uses phrases about the relationship between mathematical objects when he identifies equations or when he describes operations like division. He also acknowledges relationships of horizontal equivalence when prompted to do so, and recognises the equivalence between the equations in vignettes 2 and 5. However, he more regularly talks about and acts on disobjectified entities, which is identified by the way he prefaces keywords with articles and by his actions with disobjectified entities including operator signs. Another indicator of his spatial arrangement of entities is the many adverbs of place. These also contradict his narratives about horizontal equivalence and confirm his lack of vertical equivalence because they denote action with entities, rather than a structural orientation towards the equation.

Although Thomas produces a few endorsed narratives about the structure of numbers and equations, and many about arithmetic operations, his unendorsed narratives about actions with entities dominate his discourse. Other related unendorsed narratives pertain to the source of the narrative, which is mostly spatial arrangement and visual appearance. Both unendorsed sources underpin his routine, which aims to produce a solution that has the visual appearance " $x = \dots$ ". Further support that this is his goal lies in his regular use of adverbs of time that indicate a specific order of operations, and his failure to see the relationship between the original equation and the closing narrative, as is evidenced in Vignettes 3 and 5. The evidence provided supports my impression that Thomas's routine is performed mainly for others and is not internally persuasive. The evidence also indicates that he cannot work independently of visual appearance and spatial arrangement as sources of his narrative, which renders his routine inflexible. Although he responds positively to prompts to correct unendorsed narratives in Vignettes 3 and 4, his obedience to a rigid routine inhibits him from being persuaded by endorsed narratives about numbers to make permissible variations to the routine or to use vertical equivalence as a resource to correct unendorsed narratives.

Questions	Sub-questions (Close to Sfard, not specific to linear equations)	Indicators (Specific to the linear equations in your study, motivation not needed)	Source at Level 1 (Columns and rows to look at in spreadsheet)	Evidence
1. What is the degree of objectification of the learner's discourse	1.1.1 Does the learner's word use and mediator use signify numbers and algebraic terms as mathematical objects in	1.1.1 The use of words and phrases in one line of an equation signify numbers and algebraic terms as mathematical objects.	1.1.1.1 Tools - Words Keywords with relational process verbs	Vin 2 (16, 22b) Vin 3 (2h, 2i, 2k) Vin 4 (2a, 4a-b, 12a, 14d) Vin 5 (22a) Vin 7 (6, 12)

	their own right?		1.1.1.2 <u>Tools - Words</u> Keywords as subject and object, with the verb in the passive voice	Vin 3 (2k) Vin 4 (14d)
1.1.2 Does the learner talk about and act on extra-discursive mediators within a function?	1.1.2.1 Reference to visual mediators as disobjectified entities	1.1.2.1 Reference to actions with disobjectified entities (digits, operator signs, algebraic symbols and pronouns representing them).	1.1.2.1 <u>Tools – Words</u> Keywords prefaced by articles, pronouns or wrong use of keywords	Vin 1 (10) Vin 2 (6a-b, 12, 14a-b, 18a-d, 20, 22a) Vin 3 (2a, 2d-f, 2j, 8) Vin 4 (2a-c, 4a, 6a-c, 8a-c, 14a-c) Vin 5 (6a-b, 10, 14d, 22a) Vin 7 (4)
	1.1.2.2 Discourse about entities that change appearance or purpose, accompanied by reference to a shift to the opposite side of the equation.	1.1.2.2 Reference to actions with disobjectified entities (digits, operator signs, algebraic symbols and pronouns representing them).	1.1.2.2 <u>Tools - Words</u> Digits, algebraic symbols, operator signs and pronouns, mostly in conjunction with material process verbs	Vin 2 (6a-b, 12, 14a-b, 18a-d, 20, 22a) Vin 3 (2a, 2d-f, 2j, 8) Vin 4 (2a, 4a, 6a-c, 8a, 8c, 14b-c) Vin 5 (6a, 10, 22a)
1.2.1 Does the learner’s word use and mediator use support the horizontal equivalence between functions?	1.2.1 Discourse about equivalence or symmetry between the LHS and RHS of an equation in one line	1.1.3.1 <u>Tools - Words</u> Adverbs of place 1.1.2.3 The naming and spatial arrangement of symbols, digits and letters.	1.1.3.1 <u>Tools - Words</u> Adverbs of place 1.1.3.2 Realisation trees Operator signs and digits as visual mediators	Vin 1 (10, 14) Vin 2 (6b, 10, 12, 14a-b, 18a-d, 20, 22a) Vin 3 (2a, 2c-f, 2j, 8) Vin 4 (4a, 6a-c, 8a, 8c, 14b-c) Vin 5 (6a-b, 10, 22a-b) Vin 2 (R1, 3-a, 3-b, R3) Vin 3 (1-c, 1-d, 1-e, R1) Vin 4 (2d, R2) Vin 5 (2, 20)
1.2.2 Does the learner’s word use and mediator use interrupt the horizontal equivalence	1.2.2 Discourse about entities that change appearance or purpose, accompanied by reference to a shift to the opposite side of the equation.	1.2.1.1 Discourse about equivalence or symmetry between the LHS and RHS of an equation in one line	1.2.1.1 <u>Narratives – Notes</u> Relations between entities: endorsed narratives about equivalence between functions on LHS and RHS of an equation 1.2.1.2 <u>Narratives – description of entities</u> Endorsed narratives about the structure of the equation	Vin 5 (2, 20) Vin 7 (10)
			1.2.2.1 <u>Tools - Words</u> Adverbs of place that refer to horizontal shifts (over, this side, that side)	Vin 1 (10, 14) Vin 2 (6b, 10, 14b, 18d, 20, 22a) Vin 3 (2a, 2c, 2e-f) Vin 4 (6c, 8a, 8c, 14b-c)

	between functions?		1.2.2.2 <u>Narratives – Actions with entities</u> Unendorsed narratives about actions with or by entities 1.2.2.3 <u>Realisation trees</u> Operator signs and digits as visual mediators	Vin 2 (6a-b, 12, 14a-b, 18a-d, 20, 22a) Vin 3 (2a, 2d-f, 2j, 8) Vin 4 (2a, 4a, 6a-c, 8a, 8c, 14b-c) Vin 5 (6a, 10, 22a) Vin 2 (R1, 3-a, 3-b, R3) Vin 3 (1-c, 1-d, 1-e, R1) Vin 4 (2d, R2)
	1.3.1 Does the learner's word use and visual mediators support the vertical equivalence between narratives?	1.3.1 Uses vertical equivalence as a resource to correct unendorsed narratives	1.3.1.1 <u>Narratives – Notes</u> Endorsed narratives about the relationship between equations in consecutive lines 1.3.1.2 <u>Realisation trees</u> Revision of a node that produces an endorsable link between signifier and realisation	Vin 4 (R3)
	1.3.2 Does the learner's word use and visual mediators interrupt the vertical equivalence between narratives?	1.3.2.1 Does not use vertical equivalence as a resource to develop an endorsed solution	1.3.2.1.1 <u>Narratives – Notes</u> Acceptance of incorrect arithmetic operations 1.3.2.1.2 <u>Realisation trees</u> Nodes do not illustrate an endorsable link between signifier and realisation	Vin 2 (R1, R3) Vin 3 (1-c, 1-d, 1-e, R1) Vin 4 (2-d, R2, 4-a, R4)
		1.3.2.2 Discourse about shifts of entities to consecutive lines	1.3.2.2 <u>Tools - Words</u> Adverbs of place that refer to vertical shifts (down)	Vin 2 (12, 14a, 18a-c) Vin 3 (2a, 2d, 2j) Vin 4 (4a, 6a-c)
	1.4.1 Does the learner see that the original equation as a signifier has more than one realisation for the function?	1.4.1 Solution of equations using more than one representation of function i.e. a graph, table or algebraic equation	1.4.1 <u>Realisation trees</u> Vignettes 2, 3, 4, 7 More than one branch	
	1.4.2 Does the learner see that the original equation as a signifier has only one realisation for the function?	1.4.2 Solution of equations using only one representation of function i.e. the algebraic equation	1.4.2 <u>Realisation trees</u> Vignettes 2, 3, 4, 7 One branch	Vin 2 Vin 3 Vin 4 Vin 7

Thomas's discourse is mostly disobjectified. There are some instances where he uses keywords with relational process verbs, but these are limited to stating equations (see Vin 3 – 2h-i), except for Vignette 7 where he signifies zero as a mathematical object (see Vin 7 – 12). He almost never speaks in the passive voice. The only exception is when he makes a statement about one term divided by another (see Vin 3 – 2k). Mostly, he talks about and acts on extra-discursive mediators. These include keywords prefaced by articles (see Vin 1 – 10), the wrong use of keywords or the use of pronouns (see Vin 2 – 6a-b) and actions with disobjectified entities (see Vin 2 – 18a-d). There is much dependence on visual appearance and spatial arrangement and actions are often accompanied by adverbs of place (see Vin 3 – 2c-f) that indicate where entities are moved to. The realisation trees illustrate actions with operator signs and digits (see Vin 3 – 1-c, 1-d, 1-e, R1). Although he acknowledges the equivalence between the two sides of equations when asked, he does not ever refer to horizontal equivalence without scaffolding (see Vin 5 – 20). His regular movement of entities from one side of the equation to the other, which is accompanied by application of the unendorsed rule that the sign changes, interrupts horizontal equivalence. These actions are identified by adverbs of place that refer to horizontal shifts (see Vin 3 – 2a) and unendorsed narratives about actions with entities (see Vin 3 – 2d-f). There is no evidence that Thomas considers the relationship between equations in consecutive lines, although his realisation trees illustrate that sometimes nodes do not show an endorsable link between signifier and realisation (see Vin 3 – 1-c, 1-d, 1-e, R1). His use of adverbs of place to denote vertical shifts (see Vin 2 – 12) also suggests that he does not use vertical equivalence as a resource when solving equations. Sfar's conception of a mathematical object implies that someone who has an objectified view of the equation as a realisation of the function, would have multiple branches on their realisation tree. Thomas, like all the other learners, only has one branch on each of his trees. This is expected, as learners were oriented towards solving the equations algebraically in the assessment task.

<p>2. To what extent are the learner's narratives endorsed?</p>	<p>2.1.1 Does the learner recall and reconstruct previously endorsed narratives?</p>	<p>2.1.1.1 Production of endorsed narratives about the structure and properties of integers, variables and equations</p> <p>2.1.1.2 Production of endorsed narratives about relations between mathematical objects</p>	<p>2.1.1.1 <u>Narratives - Description of entities</u> Endorsed narratives describing integers, variables and equations</p> <p>2.1.1.2.1 <u>Narratives - Relations between entities</u> Endorsed narratives about arithmetic operations with integers and variables.</p> <p>2.1.1.2.3 <u>Narratives - Relations between entities</u> Endorsed narratives about equivalence between functions on LHS and RHS of an equation</p> <p>2.1.1.2.4 <u>Narratives - Relations between</u></p>	<p>Vin 2 (18d) Vin 3 (8, 10b) Vin 4 (2b-c) Vin 5 (2, 20)</p> <p>Vin 2 (16, 22b) Vin 3 (2h-1, 2k) Vin 4 (2a, 2d, 14a-b, 12a, 18) Vin 5 (14b-c, 22a) Vin 7 (12, 14, 16) Vin 7 (10)</p>
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			<p>entities</p> <p>Endorsed narratives about the relationship between equations in consecutive lines</p> <p>2.1.1.3 <u>Narratives – Activities with or by entities</u></p> <p>Actions with or by mathematical objects</p> <p>2.1.1.4 <u>Narratives - Source of the narrative</u></p> <p>Properties of number</p> <p>2.1.2.1 <u>Narratives - Description of entities</u></p> <p>Unendorsed narratives about numbers, variables and equations</p> <p>2.1.2.2.1 <u>Narratives - Relations between entities</u></p> <p>Unendorsed narratives about operations with numerals and letters.</p> <p>2.1.2.2.2 <u>Narratives - Relations between entities</u></p> <p>Unendorsed narratives about rules for operations that apply to entities including operatory signs, letters and numerals.</p> <p>2.1.2.2.3 <u>Narratives - Relations between entities</u></p> <p>Unendorsed narratives about expressions on LHS and RHS of an equation</p> <p>2.1.2.2.4 <u>Narratives - Relations between entities</u></p> <p>Unendorsed / no narratives about relations between consecutive equations in the solution</p> <p>2.1.2.3 <u>Narratives – Activities with or by entities</u></p> <p>Actions with or by disobjectified</p>	<p>Vin 2 (16)</p> <p>Vin 3 (2g)</p> <p>Vin 4 (2a)</p> <p>Vin 7 (12)</p> <p>Vin 1 (14)</p> <p>Vin 7 (2, 18)</p> <p>Vin 4 (14a, 14d)</p> <p>Vin 2 (6b, 18d, 20)</p> <p>Vin 3 (2e-f, 2j)</p> <p>Vin 4 (8b-c, 14c)</p> <p>Vin 5 (10, 22a)</p> <p>Vin 5 (14d)</p> <p>Vin 7 (18)</p> <p>Vin 2 (6a-b, 12, 14a-b, 18a-d, 20, 22a)</p> <p>Vin 3 (2a, 2d-f, 2j, 8)</p> <p>Vin 4 (2a, 4a, 6a-c, 8a, 8c, 14b-c)</p>
	<p>2.1.1.3 Production of endorsed narratives about actions with or by mathematical objects</p> <p>2.1.1.4 Application of rules to mathematical objects</p> <p>2.1.2.1 Production of unendorsed narratives about entities</p>			
2.1.2 Does the learner produce narratives that are not endorsable?	<p>2.1.2.2 Production of unendorsed narratives about relations between entities, including the application of rules to disobjectified signifiers</p> <p>2.1.2.3 Production of unendorsed narratives that indicate actions with or by entities, and spatial</p>			

	organization of mathematical objects	signifiers	Vin 5 (6a, 10, 22a)
	2.1.2.4 Reference to non-mathematical resources	2.1.2.4 <u>Narratives - Source of the narrative</u> Person, spatial arrangement or visual appearance	Vin 1 (10) Vin 2 (10, 18d) Vin 3 (2a, 8) Vin 7 (2)

Thomas produces a mix of endorsed and unendorsed narratives. There are endorsed narratives about the structure of numbers and expressions (see Vin 3 – 8), although most of his endorsed narratives are about arithmetic operations with integers and variables (see Vin 3 – 2h-l). He sometimes refers to the properties of number (see Vin 3 – 10b) including the rule for the signs in integer operations with reference to integers (see Vin 4 – 2b-c), but most of the time his reference to the rule pertains to operator signs as disobjectified signifiers (see Vin 2 – 20). He produces few unendorsed narratives about entities (see Vin 1 – 14) and there is only one instance where he made a calculation error (see Vin 4 – 14a). However, as indicated earlier, he performs many unendorsed actions with disobjectified signifiers (see Vin 2 – 6a-b), and the source of the narrative for these actions is mostly spatial arrangement and visual appearance (see Vin 2 – 10).

3. What is the closing condition (goal)?	3.1.1 Does the learner view the goal as producing an endorsed narrative about mathematical objects?	3.1.1 Endorsement of the relationship between the closing narrative and original equation	3.1.1 <u>Narratives - Relations between entities</u> Equivalent narratives about consecutive equations in the solution	Vin 1 (10, 14) Vin 2 (6b, 10, 12, 14a-b, 18a-d, 20, 22a) Vin 3 (2a, 2c-f, 2j) Vin 4 (4a, 6a-c, 8a, 8c, 14b-c)
	3.2.2 Does the learner view the goal as getting to the line $x = \dots$?	3.2.1 Production of narratives that describe actions with entities in order to realize a solution $x = \dots$	3.2.1.1 <u>Narratives – Activities with or by entities</u> Horizontal and vertical shifts of disobjectified entities 3.2.2.1.2 <u>Tools - Words</u> Adverbs of time 3.2.2.1.3 <u>Tools - Words</u> Adverbs of place	Vin 2 (6b, 14a-b, 16, 18a-d, 22b) Vin 3 (2a, 2c-g, 2j, 2k) Vin 4 (2b-c, 4a-b, 6a-c, 8a-c, 12b-c, 14a-d) Vin 5 (28) Vin 7 (4) Vin 1 (10, 14) Vin 2 (6b, 10, 12, 14a-b, 18a-d, 20, 22a) Vin 3 (2a, 2c-f, 2j, 8) Vin 4 (4a, 6a-c, 8a, 8c, 14b-c) Vin 5 (6a-b, 10, 22a-b)
			3.2.2.1.4 <u>Narratives - Source of the</u>	Vin 1 (10)

			narrative Spatial arrangement	Vin 2 (10) Vin 3 (2a)
			3.2.2.1.5 Realisation trees Spatial arrangement of disobjectified signifiers (visual mediators)	Vin 2 (R1, 3-a, 3-b, R3) Vin 3 (1-c, 1-d, 1-e, R1) Vin 4 (2d, R2)
			3.2.2.1.6 Routine - Notes Absent mediators	
		3.2.2.2 Failure to correct unendorsed narratives, because the procedure was completed	3.2.2.2 Routines - Notes No / wrong response to scaffolding and prompts	Vin 3 (6, 8) Vin 5 (12, 40) Vin 7 (12, 18)

There is no evidence that Thomas's goal is to produce an endorsed narrative about a mathematical object. His routine is characterized by narratives that describe actions in order to realize a solution in the form ' $x = \dots$ '. Evidence for this is his systematic horizontal and vertical shifting of disobjectified entities (see Vin 2 (18a-d) and dependence on spatial arrangement as the source of the narrative as illustrated in the realisation trees (see Vin 2 – 3-a, 3-b, R3). A clear indication that Thomas does not see the relationship between the closing narrative and the original equation is his response in vignette 3, when he does not realize that the solution is the value of ' x ', even with prompting (see Vin 3 – 6, 8).

4. For whom is the routine performed?	4.1.1 Is the learner's discourse internally persuasive?	4.1.1 Conviction of the narrative is not dependent upon others	4.1.1.1 Narratives - Source of the narrative Self	Vin 7 (2)
	4.1.2 Is the learner's discourse for others?	4.1.2 Conviction of the narrative depends upon reference to an outside authority, or on spatial arrangement and visual appearance of entities	4.1.1.2 Narratives - Source of the narrative Properties of number 4.1.2.1 Narratives - Source of the narrative Teacher, peers, interviewer 4.1.2.2 Narratives - Source of the narrative Spatial arrangement and visual appearance	Vin 7 (12) Vin 3 (8) Vin 1 (10) Vin 2 (10, 18d) Vin 3 (2a)

Thomas's routine is performed mainly for others and is not internally persuasive. Although he uses the properties of number when solving the equation in vignette 7 (see Vin 7 – 12), his routine is characterized by dependence upon reference to outside authority (see Vin 3 (8), spatial

arrangement (see Vin 2 (10) and visual appearance (see Vin 2 – 18d). Since the solution for ‘ x ’ is usually a value other than zero, he cannot accept that the solution in vignette 7 could be zero.

5. By whom is the routine performed?	5.1 Does the learner work independently?	5.1.1 No scaffolding required	5.1.1.1 Routine - Notes Mathematical justification for routine	Vin 7 (12)
	5.2 Does the learner depend on scaffolding?	5.2.1 Scaffolding required, as well as prompting from the interviewer	5.2.1.2 Routines - Notes Positive response to prompt	Vin 3 (10b) Vin 4 (16b)
		5.2.2 Dependence on visual mediators as prompts for the next step in the procedure	5.2.1.3 Routines - Notes Wrong / no response to prompt	Vin 3 (7) Vin 5 (12, 40) Vin 7 (12, 18)
			5.2.2.1 Routines - Notes Entities and visual appearance as prompts	Vin 2 (18d)
	5.2.3 Reference to the involvement of others in the completion of the task	5.2.2.2 Routines - Notes Mention of absent mediators		
			5.2.3 Narratives - Source of the narrative Teacher, interviewer, peers	Vin 3 (8)

Thomas does not work independently. In vignettes 2, 3, 4 and 7 he depends on prompts to complete his routine. In vignette 2 he depends on visual appearance (see Vin 2 – 18d), in vignette 3 he needs scaffolding to relate the solution to the original equation (see Vin 3 – 7, 10) and in vignette 5 even prompting did not persuade him to correct an error (see Vin 5 – 12).

6. What is the learner's level of flexibility?	6.1 Does the learner make permissible variations to the procedure?	6.1.1 Discourse about equivalence or symmetry between different representations of the same problem	6.1 Narratives – Description of entities (Vin 5 & 6 only) Relations between entities: endorsed narratives about horizontal equivalence between different representations of an equation	Vin 5 (2, 20)
		6.1.2 Modifies the routine by introducing an endorsable sub-routine suited to the task	6.1.2.1 Routine / realisation trees More than one branch / branch other than algebraic manipulation 6.1.2.2 Routine – notes Positive response to scaffolding and prompts	Vin 3 (10b) Vin 4 (16b)

	6.2.1 Rigid sequence of steps and spatial arrangement of entities in particular places on the page	6.2.1.1 <u>Tools - Words</u> Adverbs of time 6.2.1.2 <u>Tools - Words</u> Adverbs of place.	Vin 2 (6b, 14a-b, 16, 18a-d, 22b) Vin 3 (2a, 2c-g, 2j, 2k) Vin 4 (2b-c, 4a-b, 6a-c, 8a-c, 12b-c, 14a-d) Vin 5 (28) Vin 7 (4) Vin 1 (10, 14) Vin 2 (6b, 10, 12, 14a-b, 18a-d, 20, 22a) Vin 3 (2a, 2c-f, 2j, 8) Vin 4 (4a, 6a-c, 8a, 8c, 14b-c) Vin 5 (6a-b, 10, 22a-b)
6.2 Does the learner follow the routine in a strictly defined and rigid way?	6.2.2 Obedience to a set routine	6.2.2.1 <u>Tools - Words</u> High modality verbs 6.2.2.2 <u>Routine – notes</u> Wrong / no response to scaffolding and prompts 6.2.3 <u>Routine - Notes</u> Entities as prompts. 6.2.4 <u>Realisation trees</u> Unendorsable revision of a step in the procedure 6.2.5 <u>Narratives - Notes</u> Mention of absent mediators	Vin 1 (14) Vin 4 (2a, 6c) Vin 3 (7) Vin 5 (12, 15, 40) Vin 7 (12, 18)
	6.2.3 Use of entities as prompts	6.2.5 Narratives - Notes Mention of absent mediators	
	6.2.4 Repetition of the same routine and acceptance of the solution if the closing narrative is in the form 'x = ...'	6.2.5 Difficulty with completion of the routine if the requisite mediators are absent	

There is very little flexibility in Thomas's routine. Although he responds positively to prompts in vignettes 3 and 4 to correct unendorsed narratives (see Vin 3 – 10b) or calculations (see Vin 4 – 16b), he follows a rigid, strictly defined routine. This is characterized by the systematic spatial organization of entities in a rigid sequence of steps, as indicated by the adverbs of place and time (see Vin 4 – 6a-c), as well as his occasional use of the high modality form of the verb (see Vin 4 – 2a). It is significant that in vignette 5 his obedience to the set routine means that he does not respond positively to my scaffolding, and is not internally persuaded that if there is horizontal equivalence between the two forms of the equation, the solution should be the same (see Vin 5 – 12, 16).

7. What is the learner's level	7.1 Does the learner recognise unendorsed	7.1.1 Correction of incorrect statements about mathematical	7.1.1 <u>Routines - Notes</u> Correction of error Vin 4 (18)
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of correctibility?	narratives, and correct them?	objects, with or without prompting		
		7.1.2 Uses vertical equivalence as a resource to correct unendorsed narratives	7.1.2.1 <u>Narratives – Notes</u> Vertical equivalence relationship 7.1.2.2 <u>Realisation trees</u> Revision of a node that produces an endorsable link between signifier and realisation 7.1.2.3 <u>Realisation trees</u> Revision of a node that produces an endorsable link between signifier and realisation	Vin 4 (R3)
	7.2 Does the learner fail to recognise unendorsed narratives, or correct them?	7.2.1 Does not amend unendorsed narratives, even with prompting	7.2.1.1 <u>Routines – Notes</u> Acceptance of incorrect arithmetic operations 7.2.1.2 <u>Routines - Notes</u> Amendment to error produces another error 7.2.1.3 <u>Routines - Notes</u> No / wrong response to prompt	Vin 3 (7) Vin 5 (12, 40) Vin 7 (12, 18)
		7.2.2 Does not use vertical equivalence as a resource to correct unendorsed narratives	7.2.2.1 <u>Narratives – Notes</u> No vertical equivalence relationship 7.2.2.2 <u>Realisation trees</u> Nodes do not illustrate an endorsable link between signifier and realisation	Vin 2 (R1, R3) Vin 3 (1-c, 1-d, 1-e, R1) Vin 4 (2-d, R2, 4-a, R4)

Thomas has a low level of correctibility. In vignette 4 he corrects an arithmetic error (see Vin 4 – 18 and R3). However, in vignette 5 he cannot correct his error because the routine had been completed to his satisfaction (see Vin 5 – 12, 16) even though he had earlier agreed that there was horizontal equivalence in the two representations of the equation. Similarly, in vignette 7 he could not accept that ‘x’ was zero (see Vin 7 – 18). He did not use vertical equivalence as a resource to correct unendorsed narratives, which resulted in unendorsable links between signifier and realisation (see Vin 4 – 2d).

Appendix 5: Emily Level 2 research text

Emily frequently produces endorsed narratives about horizontal equivalence when explaining her routine. Furthermore, the source of her narrative is frequently the properties of number. From this evidence I conclude that she has been encouraged to justify her routine mathematically. However, in solving equations her discourse is mostly ritualistic. There are instances when she talks about integers, variables and the equation in an objectified way, but generally the degree of objectification of her discourse is low. She produces a mix of endorsed and unendorsed narratives, but the fact that the source of her narratives is mostly spatial arrangement and visual appearance support the argument that her discourse is generally disobjectified. Furthermore, her goal is to follow a rigid routine to achieve a solution in the form ‘ $x = \dots$ ’ and she has a low level of flexibility. Although her discourse is sometimes internally persuasive and she can work independently at times, she depends on support from non-mathematical mediators in her routine. These traits in her discourse are typical of what Sfard describes as the transitory phase learners might pass through on the way to explorative discourse (2008: 249-250). Detailed evidence for this argument is provided below, with further evidence in the table that follows.

1. The degree of objectification of the discourse

1.1 Objectification at the level of words and visual mediators

There is no evidence that Emily’s word use and mediator use signify numbers and algebraic terms as mathematical objects in their own right. All evidence indicates a high level of disobjectification in her discourse. She refers to and acts upon visual mediators as disobjectified signifiers (see Vin 3 – 6a-b), prefaces keywords and pronouns with articles (see Vin 2 – 2a-c) and describes shifts in their movement with adverbs of place (see Vin 3 – 6a-b). This tendency is also reflected in the realisation trees (see Vin 4 – 2-b, 2-c, R2).

1.2 Objectification at the level of horizontal equivalence

Emily thinks about horizontal equivalence and often references it in stating a property (see Vin 3 – 14a) or talking about the structure of the equation (see Vin 1 – 2). However, her general discourse, typified by action on disobjectified signifiers, includes many adverbs of place that reference movement across the equal sign. These shifts sometimes lead to a change in appearance (see Vin 3 – 8b). Therefore, although she is aware of horizontal equivalence, my perception is that her word and mediator use do not support horizontal equivalence.

1.3 Objectification at the level of vertical equivalence

There is no narrative??? to suggest that Emily considers the vertical equivalence between consecutive equations. There is no conclusive evidence that she accesses vertical equivalence as a resource to correct errors. Action with entities also extends to vertical shifts, as indicated by adverbs of place (see Vin 3 – 10a).

2. Extent to which her narratives are endorsed

Contrary to the fact that Emily's discourse has such a high level of disobjectification, she has produced many endorsed narratives about the properties and structure of mathematical objects. This includes descriptions of algebraic terms (see Vin 6 – 14), the structure of complex terms (see Vin 4 – 34) and equations (see Vin 1 – 10), arithmetic operations (see Vin 4 – 50a-b) and reference to the properties of number as source of her narratives (see Vin 2 – 12). However, she also produces unendorsed narratives about numbers (see Vin 2 – 2b) and acts on and applies rules to disobjectified signifiers (see Vin 2 – 2a and Vin 3 – 8b). Furthermore, her routine depends on non-mathematical resources like visual appearance and spatial arrangement (see Vin 3 – 6a).

3. Her closing condition

Since she never endorses the relationship between the closing narrative and original equation I conclude that she does not think about producing an endorsed narrative about mathematical objects, but that her goal is to get to the line $x = \dots$. Her routine includes many practices that suggest a mechanical set of operations. These include spatial arrangement (see Vin 4 – 18, 20) and adverbs of time, which support the sequential pattern of her steps (see Vin 3 – 14a-b). The horizontal and vertical shifts of disobjectified entities is illustrated in the realisations trees (see Vin 4 – 2-b, 2-c, R2).

4. For whom the routine is performed

At times Emily's discourse is internally persuasive (see Vin 5 – 22) and she is able to defend her opinion by referencing properties of number (see Vin 7 – 6f). However, there is strong evidence that she depends on visual appearance and spatial arrangement in her routine (see Vin 5 – 6a). She also sometimes refers to the interviewer for guidance (see Vin 6 – 18). I conclude that her discourse is only partially internally persuasive, and that her routine is performed partially for others.

5. By whom the routine is performed

There is evidence that Emily can work independently and justify her routine (see Vin 6 – 24). At the same time, though, she depends on visual mediators as prompts (see Vin 4 – 4) and benefits from prompting by the interviewer (see Vin 4 – 84/86). This implies that her routine is performed with support from others.

6. Her level of flexibility

Emily does not have a high level of flexibility. Although she recognises the horizontal equivalence between different representations of the same problem (see Vin 5 – 10b), she does not use this as a resource for her argument in vignette 5, even with prompting (see Vin 5 – 12). Instead she remains obedient to her set routine. Evidence for this from her discourse is her use of the high modality form of the verb (see Vin 1 – 4 and Vin 5 – 4), adverbs of time (see Vin 4 – 50a-b) and adverbs of place (see Vin 5 – 6a).

7. Her level of correctness

Emily locally replaces part of the solution in vignette 4, in order to correct an error (see Vin 4 – 64, 76a). She had initially accepted the unendorsed narrative (see Vin 4 – 10) but later responded positively to prompts. There is no other evidence to comment on her level of correctness.

Questions	Sub-questions (Close to Sfard, not specific to linear equations)	Indicators (Specific to the linear equations in your study, motivation not needed)	Source at Level 1 (Columns and rows to look at in spreadsheet)	Evidence
1. What is the degree of objectification of the learner's discourse	1.1.1 Does the learner's word use and mediator use signify numbers and algebraic terms as mathematical objects in their own right?	1.1.1 The use of words and phrases in one line of an equation signify numbers and algebraic terms as mathematical objects.	1.1.1.1 Tools - Words Keywords with relational process verbs	Vin 5 (24) - not endorsed
	1.1.2 Does the learner talk about and act on extra-discursive mediators within a function?	1.1.2.1 Reference to visual mediators as disobjectified entities	1.1.1.2 Tools - Words Keywords as subject and object, with the verb in the passive voice	Vin 1 (4, 10) Vin 2 (2a-d, 4, 10, 12, 14a-c, 16) Vin 3 (6a-b, 8a-b, 10a-b, 12b-c, 14a-c) Vin 4 (2a-b, 6, 8, 10, 14, 18, 20, 22, 26, 32, 34, 38, 44, 48a-b, 50a, 52, 60c-e, 62a-c, 64, 68, 70, 74, 76b, 76d-e, 80, 82, 88) vin 5 (4, 6a-b, 10b, 14b, 22) Vin 6 (4, 6, 18, 20, 24, 26, 30a-b)

				Vin 7 (2c)
	1.1.2.2 Reference to actions with disobjectified entities (digits, operator signs, algebraic symbols and pronouns representing them).		1.1.2.2 <u>Tools - Words</u> Digits, algebraic symbols, operator signs and pronouns, mostly in conjunction with material process verbs	Vin 1 (4) Vin 2 (2a, 4, 10) Vin 3 (6a-b, 8b, 10a, 12c, 14c) Vin 4 (12, 26, 48a, 54, 60c-d, 62c, 64, 70, 88) Vin 5 (6a, 18)
	1.1.2.3 The naming and spatial arrangement of symbols, digits and letters.		1.1.3.1 <u>Tools - Words</u> Adverbs of place	Vin 1 (4, 10) Vin 2 (2c-d, 4) Vin 3 (4, 6a-b, 8a-b, 10a-c, 12c, 14a) Vin 4 (12, 18, 20, 26, 34, 48a-b, 54, 60c, 60e, 70, 76b, 78c, 86, 90) Vin 5 (6a, 10b, 18, 22) Vin 6 (94, 26) Vin 7 (2c, 4)
			1.1.3.2 <u>Realisation trees</u> Operator signs and digits as visual mediators	Vin 2 (1-c, R1) Vin 3 (1-b) Vin 4 (2-b, 2-c, R2)
1.2.1 Does the learner's word use and mediator use support the horizontal equivalence between functions?	1.2.1 Discourse about equivalence or symmetry between the LHS and RHS of an equation in one line		1.2.1.1 <u>Narratives – Notes</u> Relations between entities: endorsed narratives about equivalence between functions on LHS and RHS of an equation	Vin 3 (4, 14a) Vin 4 (86) Vin 6 (24, 30a)
			1.2.1.2 <u>Narratives – description of entities</u> Endorsed narratives about the structure of the equation	Vin 1 (2, 10) Vin 3 (4, 14a)
1.2.2 Does the learner's word use and mediator use interrupt the horizontal equivalence between functions?	1.2.2 Discourse about entities that change appearance or purpose, accompanied by reference to a shift to the opposite side of the equation.		1.2.2.1 <u>Tools - Words</u> Adverbs of place that refer to horizontal shifts (over, this side, that side)	Vin 1 (4, 10) Vin 2 (2c, 4) Vin 3 (4, 6a-b, 8a-b, 10c, 14a) Vin 4 (12, 18, 20, 26, 48a-b, 54, 60e, 86) Vin 5 (6a, 18, 22) Vin 6 (26) Vin 7 (2c, 4)
			1.2.2.2 <u>Narratives – Actions with entities</u> Unendorsed narratives about actions with or by entities	Vin 1 (4) Vin 2 (2a, 4, 10) Vin 3 (6a-b, 8b, 10a, 12c, 14c) Vin 4 (12, 26, 48a, 54, 60c-d, 62c, 64, 70, 88)

				1.2.2.3 <u>Realisation trees</u> Operatory signs and digits as visual mediators	Vin 5 (6a, 18) Vin 2 (1-c, R1) Vin 3 (1-b) Vin 4 (2-b, 2-c, R2)
1.3.1 Does the learner's word use and visual mediators support the vertical equivalence between narratives?	1.3.1 Uses vertical equivalence as a resource to correct unendorsed narratives	1.3.1.1 <u>Narratives – Notes</u> Endorsed narratives about the relationship between equations in consecutive lines	1.3.1.2 <u>Realisation trees</u> Revision of a node that produces an endorsable link between signifier and realisation		Vin 4 (1-dii)
1.3.2 Does the learner's word use and visual mediators interrupt the vertical equivalence between narratives?	1.3.2.1 Does not use vertical equivalence as a resource to develop an endorsed solution	1.3.2.1.1 <u>Narratives – Notes</u> Acceptance of incorrect arithmetic operations	1.3.2.1.2 <u>Realisation trees</u> Nodes do not illustrate an endorsable link between signifier and realisation		Vin 4 (10)
	1.3.2.2 Discourse about shifts of entities to consecutive lines	1.3.2.2 <u>Tools - Words</u> Adverbs of place that refer to vertical shifts (down)			Vin 3 (10a, 12c)
1.4.1 Does the learner see that the original equation as a signifier has more than one realisation for the function?	1.4.1 Solution of equations using more than one representation of function i.e. a graph, table or algebraic equation	1.4.1 <u>Realisation trees</u> Vignettes 2, 3, 4, 7 More than one branch			
1.4.2 Does the learner see that the original equation as a signifier has only one realisation for the function?	1.4.2 Solution of equations using only one representation of function i.e. the algebraic equation	1.4.2 <u>Realisation trees</u> Vignettes 2, 3, 4, 7 One branch			Vin 2 Vin 3 Vin 4 Vin 7

2. To what extent are the	2.1.1 Does the learner recall and reconstruct	2.1.1.1 Production of endorsed narratives about the structure and	2.1.1.1 <u>Narratives - Description of entities</u>	Vin 1 (2, 10) Vin 2 (12, 16)
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learner's narratives endorsed?	previously endorsed narratives?	properties of integers, variables and equations	Endorsed narratives describing integers, variables and equations	Vin 3 (4, 12c) Vin 4 (34) Vin 5 (22) Vin 6 (2, 18) Vin 7 (4, 6f)
		2.1.1.2 Production of endorsed narratives about relations between mathematical objects	2.1.1.2.1 <u>Narratives - Relations between entities</u> Endorsed narratives about arithmetic operations with integers and variables.	Vin 2 (6, 8, 10, 14b-c) Vin 3 (12b, 14c-d) Vin 4 (4, 24, 36, 46, 50a-b, 78b, 88, 92) Vin 6 (20, 22a)
			2.1.1.2.3 <u>Narratives - Relations between entities</u> Endorsed narratives about equivalence between functions on LHS and RHS of an equation	Vin 3 (14a) Vin 4 (86) Vin 6 (24, 30a)
			2.1.1.2.4 <u>Narratives - Relations between entities</u> Endorsed narratives about the relationship between equations in consecutive lines	
		2.1.1.3 Production of endorsed narratives about actions with or by mathematical objects	2.1.1.3 <u>Narratives – Activities with or by entities</u> Actions with or by mathematical objects	Vin 3 (12a, 14a) Vin 4 (2a, 32, 86)
		2.1.1.4 Application of rules to mathematical objects	2.1.1.4 <u>Narratives - Source of the narrative</u> Properties of number	Vin 2 (12, 16) Vin 3 (12c, 14a) Vin 4 (86) Vin 5 (4) Vin 6 (24) Vin 7 (6f)
		2.1.2.1 Production of unendorsed narratives about entities	2.1.2.1 <u>Narratives - Description of entities</u> Unendorsed narratives about numbers, variables and equations	Vin 2 (2c) Vin 3 (10b) Vin 4 (6, 10, 14, 18, 20, 48b) Vin 5 (6b, 16, 22, 24) Vin 6 (14, 16)
	2.1.2 Does the learner produce narratives that are not endorsable?	2.1.2.2 Production of unendorsed narratives about relations between entities, including the application of rules to disobjectified signifiers	2.1.2.2.1 <u>Narratives - Relations between entities</u> Unendorsed narratives about operations with numerals and letters.	Vin 4 (38, 72, 74)

			<p>2.1.2.2.2 <u>Narratives - Relations between entities</u> Unendorsed narratives about rules for operations that apply to entities including operator signs, letters and digits.</p> <p>2.1.2.2.3 <u>Narratives - Relations between entities</u> Unendorsed narratives about expressions on LHS and RHS of an equation</p> <p>2.1.2.2.4 <u>Narratives - Relations between entities</u> Unendorsed / no narratives about relations between consecutive equations in the solution</p>	<p>Vin 2 (2c) Vin 3 (8b, 10b) Vin 4 (60c, 62c)</p> <p>Vin 6 (10)</p>
		<p>2.1.2.3 Production of unendorsed narratives that indicate actions with or by entities, and spatial organization of mathematical objects</p> <p>2.1.2.4 Reference to non-mathematical resources</p>	<p>2.1.2.3 <u>Narratives – Activities with or by entities</u> Actions with or by disobjectified signifiers</p> <p>2.1.2.4 <u>Narratives - Source of the narrative</u> Person, spatial arrangement or visual appearance</p>	<p>Vin 1 (4) Vin 2 (2a-b, 4, 10) Vin 3 (6a-b, 8b, 10a, 12c, 14c) Vin 4 12, 26, 48a, 54, 60c-d, 62c, 64, 70, 88) Vin 5 (6a, 18)</p> <p>Vin 1 (4) Vin 2 (2a-b) Vin 3 (6a) Vin 4 (12, 18, 20, 48a) Vin 5 (6a, 8a, 16) Vin 6 (18) Vin 7 (2c, 4)</p>

<p>3. What is the closing condition (goal)?</p>	<p>3.1.1 Does the learner view the goal as producing an endorsed narrative about mathematical objects?</p>	<p>3.1.1 Endorsement of the relationship between the closing narrative and original equation</p>	<p>3.1.1 <u>Narratives - Relations between entities</u> Equivalent narratives about consecutive equations in the solution</p>	
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	<p>3.2.2 Does the learner view the goal as getting to the line $x = \dots$?</p>	<p>3.2.2.1 Production of narratives that describe actions with entities in order to realize a solution $x = \dots$</p>	<p>3.2.1.1 <u>Narratives – Activities with of by entities</u> Horizontal and vertical shifts of disobjectified entities</p> <p>3.2.2.1.2 <u>Tools - Words</u> Adverbs of time</p> <p>3.2.2.1.3 <u>Tools - Words</u> Adverbs of place</p> <p>3.2.2.1.4 <u>Narratives - Source of the narrative</u> Spatial arrangement</p> <p>3.2.2.1.5 <u>Realisation trees</u> Spatial arrangement of disobjectified signifiers (visual mediators)</p> <p>3.2.2.1.6 <u>Routine - Notes</u> Absent mediators</p>	<p>Vin 1 (4, 10) Vin 2 (2c, 4) Vin 3 (4, 6a-b, 8a-b, 10a, 10c, 12c, 14a) Vin 4 (12, 18, 20, 26, 48a-b, 54, 60c, 86) Vin 5 (6a, 18, 22) Vin 6 (26) Vin 7 (2c, 4)</p> <p>Vin 1 (6) Vin 2 (6, 8, 10, 14a-c, 16) Vin 3 (8a, 10a, 12a, 12c, 14a-b, 14d) Vin 4 (2b, 4, 6, 12, 24, 26, 34, 36, 38, 48a-b, 50a-b, 52, 60c, 62a-b, 70, 72, 76b, 78e-f, 82) Vin 5 910b) Vin 6 (14, 16, 20, 26)</p> <p>Vin 1 (4, 10) Vin 2 (2c-d, 4) Vin 3 (4, 6a-b, 8a-b, 10a-c, 12c, 14a) Vin 4 (12, 18, 20, 26, 34, 48a-b, 54, 60c, 60e, 70, 76b, 78c, 86, 90) Vin 5 (6a, 10b, 18, 22) Vin 6 94, 26) Vin 7 (2c, 4)</p> <p>Vin 1 (4) Vin 2 (2a-b) Vin 3 (6a) Vin 4 (12, 18, 20, 48a) Vin 5 (6a)</p> <p>Vin 2 (1-c, R1) Vin 3 (1-c, 1-d) Vin 4 (2-b, 2-c, 2-d)</p> <p>Vin 4 (8, 40) Vin 5 (12)</p>
	<p>3.2.2 Failure to correct unendorsed narratives, because the procedure was completed</p>	<p>3.2.2.2 <u>Routines - Notes</u> No / wrong response to scaffolding and prompts</p>		

4. For whom is the routine performed?	4.1.1 Is the learner's discourse internally persuasive?	4.1.1 Conviction of the narrative is not dependent upon others	4.1.1.1 <u>Narratives - Source of the narrative</u> Self	Vin 5 (22)
	4.1.2 Is the learner's discourse for others?	4.1.2 Conviction of the narrative depends upon reference to an outside authority, or on spatial arrangement and visual appearance of entities	4.1.2.1 <u>Narratives - Source of the narrative</u> Teacher, peers, interviewer 4.1.2.2 <u>Narratives - Source of the narrative</u> Spatial arrangement and visual appearance	Vin 2 (12, 16) Vin 3 (12c, 14a) Vin 4 (86) Vin 5 (4) Vin 6 (24) Vin 7 (6f) Vin 5 (8a) Vin 6 (18) Vin 7 (2c) Vin 1 (4) Vin 2 (2a-b) Vin 3 (6a) Vin 4 (12, 18, 20, 48a) Vin 5 (6a, 16) Vin 7 (4)

5. By whom is the routine performed?	5.1 Does the learner work independently?	5.1.1 No scaffolding required	5.1.1.1 <u>Routine - Notes</u> Mathematical justification for routine	Vin 6 (24) Vin 7 (6f)
		5.2.1 Scaffolding required, as well as prompting from the interviewer	5.2.1.2 <u>Routines - Notes</u> Positive response to prompt	Vin 4 (32, 34, 42, 56, 70, 78a, 84/86) Vin 5 (10b) Vin 6 (30a)
		5.2.2 Dependence on visual mediators as prompts for the next step in the procedure	5.2.1.3 <u>Routines - Notes</u> Wrong / no response to prompt 5.2.2.1 <u>Routines - Notes</u> Entities and visual appearance as prompts 5.2.2.2 <u>Routines - Notes</u> Mention of absent mediators	Vin 4 (8, 40) Vin 5 (12) Vin 2 (12a) Vin 5 (16) Vin 7 (2a, 4)
	5.2.3 Reference to the involvement of others in the completion of the task	5.2.3 <u>Narratives - Source of the narrative</u> Teacher, interviewer, peers	Vin 5 (8a) Vin 6 (18) Vin 7 (2c)	

<p>6. What is the learner's level of flexibility?</p>	<p>6.1 Does the learner make permissible variations to the procedure?</p>	<p>6.1.1 Discourse about equivalence or symmetry between different representations of the same problem</p> <p>6.1.2 Modifies the routine by introducing an endorsable sub-routine suited to the task</p>	<p>6.1.1 <u>Narratives – Description of entities (Vin 5 & 6 only)</u> Relations between entities: endorsed narratives about horizontal equivalence between different representations of an equation</p> <p>6.1.2.1 <u>Routine / realisation trees</u> More than one branch / branch other than algebraic manipulation</p> <p>6.1.2.2 <u>Routine – notes</u> Positive response to scaffolding and prompts</p> <p>6.2.1.1 <u>Tools - Words</u> Adverbs of time</p> <p>6.2.1.2 <u>Tools - Words</u> Adverbs of place.</p> <p>6.2.2.1 <u>Tools - Words</u> High modality verbs</p> <p>6.2.2.2 <u>Routine – notes</u> Wrong / no response to scaffolding and prompts</p> <p>6.2.3 <u>Routine - Notes</u></p>	<p>Vin 5 (4, 10b)</p> <p>Vin 4 (32, 34, 42, 56, 70, 78a, 84/86) Vin 5 (10b) Vin 6 (30a)</p> <p>Vin 1 (6) Vin 2 (6, 8, 10, 14a-c, 16) Vin 3 (8a, 10a, 12a, 12c, 14a-b, 14d) Vin 4 (2b, 4, 6, 12, 24, 26, 34, 36, 38, 48a-b, 50a-b, 52, 60c, 62a-b, 70, 72, 76b, 78e-f, 82) Vin 5 (10b) Vin 6 (14, 16, 20, 26)</p> <p>Vin 1 (4, 10) Vin 2 (2c-d, 4) Vin 3 (4, 6a-b, 8a-b, 10a-c, 12c, 14a) Vin 4 (12, 18, 20, 26, 34, 48a-b, 54, 60c, 60e, 70, 76b, 78c, 86, 90) Vin 5 (6a, 10b, 18, 22) Vin 6 (4, 26) Vin 7 (2c, 4)</p> <p>Vin 1 (2, 4, 10) Vin 2 (2a-b, 2d, 4) Vin 3 (4) Vin 4 (8) Vin 5 (4) Vin 7 (2c) Vin 4 (8, 40) Vin 5 (12)</p> <p>Vin 2 (12a)</p>
<p>6.2 Does the learner follow the routine in a strictly defined and rigid way?</p>	<p>6.2.1 Rigid sequence of steps and spatial arrangement of entities in particular places on the page</p> <p>6.2.2 Obedience to a set routine</p>	<p>6.2.3 Use of entities as prompts</p>		

			Entities as prompts.	Vin 7 (2a)
		6.2.4 Repetition of the same routine and acceptance of the solution if the closing narrative is in the form 'x = ...'	6.2.4 Realisation trees Unendorsable revision of a step in the procedure	
		6.2.5 Difficulty with completion of the routine if the requisite mediators are absent	6.2.5 Narratives - Notes Mention of absent mediators	

7. What is the learner's level of correctibility?	7.1 Does the learner recognise unendorsed narratives, and correct them?	7.1.1 Correction of incorrect statements about mathematical objects, with or without prompting	7.1.1 <u>Routines - Notes</u> Correction of error	Vin 4 (60b, 64, 76a)
		7.1.2 Uses vertical equivalence as a resource to correct unendorsed narratives	7.1.2.1 <u>Narratives – Notes</u> Vertical equivalence relationship 7.1.2.2 <u>Realisation trees</u> Revision of a node that produces an endorsable link between signifier and realisation	Vin 4 (1-dii)
	7.2 Does the learner fail to recognise unendorsed narratives, or correct them?	7.2.1 Does not amend unendorsed narratives, even with prompting	7.2.1.1 <u>Routines – Notes</u> Acceptance of incorrect arithmetic operations 7.2.1.2 <u>Routines - Notes</u> Amendment to error produces another error	Vin 4 (10)
		7.2.2 Does not use vertical equivalence as a resource to correct unendorsed narratives	7.2.1.3 <u>Routines - Notes</u> No / wrong response to prompt 7.2.2.1 <u>Narratives – Notes</u> No vertical equivalence relationship 7.2.2.2 <u>Realisation trees</u> Nodes do not illustrate an endorsable link between signifier and realisation	Vin 4 (8, 40) Vin 5 (12)

Appendix 6: Information letters

Appendix 6.1: Parent information letter

UNIVERSITY OF CAPE TOWN



[Date]

School of Education

[Address]

Dear Parent/Guardian
[Name of school]

MASTERS DEGREE RESEARCH

You might be aware that [name of school] is participating in a three-year project with UCT. The [name of] Project works with teachers to improve teaching and learning in Grades 7, 8 and 9 mathematics. In July 2012 I conducted interviews with the top-performing mathematics learners at [name of school], of which your child was one. The purpose of the interview was to determine the strategies used by the best-performing learners, to solve linear equations. The interviews were used in the development of a short course for teachers in the project.

Learners were given an undertaking of anonymity and that their identity would be protected. In this respect, learners' faces do not appear at any time, with the camera only focussing on a sheet of paper on which both the interviewer and the child wrote. Permission for the collection of the data was covered by the Memorandum of Understanding between the school and the [name of organisation] was concluded with each school before commencement of the project, and initially the only purpose of the data was the short course.

I am presently registered at the University of Cape Town for the M.Ed degree in Mathematics Education. The data from the interviews is of a high quality and I hereby request that I may use the interview conducted with your child, as part of the data for my Master's research. The research project seeks to describe the mathematical resources learners use to solve linear equations. The study subscribes to a non-deficit view of schools and learners and all data will be used to understand prevalent problems with a view to improving teaching and learning. The results of the study will be published in the thesis and other possible publications emanating from the thesis. Learners will be anonymous in the write-up of the research and all writing will be informed by my commitment to do no harm to learners or schools.

In accordance with UCT policy, permission will also be sought from the principal and the [provincial department of education].

I give assurance

- (a) that the interview was conducted with integrity and with the utmost respect for the dignity of every person involved
- (b) that the anonymity of your child has been protected. At no point is the learner's name mentioned and the camera did not focus on the face at any point.

I undertake

- (a) not to use the outcomes of the tests to make any inferences about the academic abilities of any of the learners
- (b) not to use the outcomes of the tests to make any inferences about [name of school], as an institution of teaching and learning

Please note that you may withdraw from the research process at any time, without fear of retribution.

Please contact me (contact details above) or my supervisor, Dr Kate Le Roux, if you require additional information regarding my research.

I trust that you will accede to my request and, in this event, extend my sincere thanks to you and your son/daughter for agreeing to be a research participant.

Kindly please complete and return the attached consent form, acknowledging receipt of the information letter and giving your son/daughter permission to be a research participant.

Thank you.

Yours sincerely

Anthea Roberts
Student number: TYLANT004

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Appendix 6.2: Learner information letter

UNIVERSITY OF CAPE TOWN



[Date]

School of Education

[address]

Dear Learner
[name of school]

MASTERS DEGREE RESEARCH

You might be aware that [name of school] is participating in a three-year project with the University of Cape Town (UCT). The [name of project] works with teachers to improve teaching and learning in Grades 7, 8 and 9 mathematics. In July 2012 I conducted interviews with the top-performing mathematics learners, of which you were one. In the interview you explained how you solved a set of linear equations in an assessment task. As I explained at the time, the purpose of the interview was to determine how the best-performing learners understood the mathematics. The interviews were used to develop a very successful short course for teachers in the project.

The data from your interview is of a high quality and I would like to use the interview as part of the data for my research towards my Masters degree in Mathematics Education. The research project seeks to describe the mathematical resources learners use to solve linear equations.

In accordance with UCT policy, permission will also be sought from the principal, your Grade 8 Mathematics teacher, your parents or guardians and the [provincial education department].

I assure you

- (a) that the interview was conducted with integrity and with the utmost respect for the dignity of every person involved
- (b) that you cannot be identified in the interview. At no point is your name mentioned and the camera did not focus on your face at any point.
- (c) Whether or not you give permission for the interview to be used, your mathematics results at school will not be affected in any way.

I undertake

- (a) not to use the outcomes of the tests to make any inferences about your academic ability
- (b) not to use the outcomes of the tests to make any inferences about [name of school], as an institution of teaching and learning
- (c) that the results of the study will be published in the thesis and other possible publications emanating from the thesis. You will be anonymous in the write-up of the research and all writing will be informed by my commitment to do no harm to learners or schools.

I trust that you will accede to my request and, in this event, extend my sincere thanks for agreeing to be a research participant. However, you reserve the right not to

participate in this research. Should you agree to participate, kindly please complete and return the attached consent form, acknowledging receipt of the information letter and consenting to be a research participant.

Please note that you may withdraw from the research process at any time, without fear of retribution.

Please contact me (contact details above) or my supervisor, Dr Kate Le Roux, on [phone] if you require additional information regarding my research.

Thank you.

Yours sincerely

Anthea Roberts
Student number: TYLANT004

Appendix 6.3: Teacher information letter

UNIVERSITY OF CAPE TOWN



School of Education

[ADDRESS]

[DATE]

Dear [TEACHER]

MASTERS DEGREE RESEARCH

I, Anthea Roberts, am presently registered at the University of Cape Town for the M.Ed degree in Mathematics Education. My earlier exploratory conversation with you refers.

In July 2012 I conducted interviews with the top-performing Grade 8 mathematics learners in one class at [name of school]. The purpose of the interview was to determine the strategies used by the best-performing learners, to solve linear equations. The interviews were used in the development of a short course for teachers in the project. Each interview was videotaped. A learner's identity was completely concealed, with the camera only focussing on a sheet of paper on which both the interviewer and the learner wrote.

The data from the interviews is of a high quality and I now formally request permission to use the interviews as data for my research project. The research project seeks to describe the mathematical resources learners use to solve linear equations. In accordance with UCT policy, permission will also be sought from the parents, learners, Grade 8 teacher and the [provincial education department].

I give assurance

- (a) that the interviews were conducted with integrity and with the utmost respect for the dignity of every person involved
- (b) that the anonymity of learners has been protected. At no point are names mentioned and the camera did not focus on any faces.

I undertake

- (a) not to use the outcomes of the tests to make any inferences about the academic abilities of any of the learners
- (b) not to use the outcomes of the tests to make any inferences about [name of school], as an institution of teaching and learning
- (c) only I know the identity of the interviewees and schools. This identity will never be revealed to my supervisor or any other party. Furthermore, the interview records will only be viewed and discussed by myself and my research supervisor. Although there is a high level of anonymity and confidentiality around identities (schools and pupils) it is unavoidable that some responses might be recognised. This is typical of small-scale studies where the researcher is involved in the context; however, no harm will be done.

- (d) that the study subscribes to a non-deficit view of schools and learners and all data is used to understand prevalent problems with a view to improving teaching and learning. The results of the study will be published in the thesis and other publications emanating from the thesis. All writing will be informed by my commitment to do no harm to learners or schools.
- (e) to give the school a report (verbally and in writing) of my preliminary findings
- (f) to inform and request the consent of the selected learners and their parents or guardians, as well as the [provincial education department].

Please contact me (contact details above) or my supervisor, Dr Kate Le Roux, on [phone] if you require additional information regarding my research.

I trust that my request will be given your approval.

Thank you.

Yours sincerely

Anthea Roberts
Student number: TYLANT004