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Renormalization of Cavity Field Theories

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Abstract

A major obstacle to calculating Feynman diagrams in field theories, confined to a cavity, has always been the divergent loop diagrams. So far, only the quantum chromodynamic and electrodynamic self-energies of a $1s_{1/2}$ quark, confined to a static spherical cavity, have been accurately calculated. These quantities are of immediate interest in the M.I.T. bag model. The existing methods to calculate loop diagrams are based on the multiple reflection scheme, in which the zero reflection term is separated out analytically, and evaluated separately. Thus far, there are some indications that this method is unsuitable for the quadratically divergent one loop vacuum polarization.

In this thesis we firstly develop a set of Fourier transforms, appropriate to a discussion of renormalization in a cavity. Using these, we renormalize the cavity propagators to one loop for scalar, Dirac, and gauge fields. We then introduce a new computational method to subtract out the divergences, based on dimensional regularization. Using this method, we present results for various loop diagrams. The scalar ϕ^4 theory is used as a pedagogical example. We then present the quark self-energy for several low lying cavity modes. Finally we tackle the long standing and hitherto unresolved question of the vacuum polarization. For this we give a detailed discussion of surface divergences, and present results for scalar quantum electrodynamics. We make a suggestion for the implementation of the running coupling constant in the cavity.

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Chapter 1

Introduction

The study of finite or infinite systems, subject to boundary conditions, is an important part of physics. In classical electrodynamics, boundary conditions represent an approximate description of interfaces between macroscopic media of different properties.

Since the advent of the M.I.T. bag model [24], there has been considerable interest in the Casimir effect [40]. If a quantum field is subjected to boundary conditions, the presence of discretely spaced eigenmodes causes a finite change in the vacuum energy. Calculations of the Casimir effect are difficult, mainly because they involve strongly divergent expressions, however the problem has been tackled for many different fields in cavities of different geometries. For recent work on the non-trivial problem of quarks confined to spheres see [41,42,43]. Insofar as the Casimir effect describes measurable phenomena, the boundary conditions are an approximation for macroscopic classical objects, such as a gold foil.

Subsequent to the M.I.T. bag model there has also been interest shown in the problem of *interacting* quantum fields subjected to boundary conditions, or cavity field theories.

1.1 Cavity Field Theory

Broadly speaking, there are two possible reasons for studying cavity field theories.

- The results of Feynman diagrams calculated in cavity quantum chromodynamics and cavity electrodynamics find immediate application in the static sphere approximation of the M.I.T. bag model, and other models of hadrons.

- The study of cavity field theories may be compared to other theories, such as quantum electrodynamics in $1+1$ dimensions, or lattice gauge theories. While they do not purport to describe nature exactly, they may extend our theoretical understanding of theories that do, or they may provide an approximate description. Cavity field theory has yet to prove itself in this role.

Of course, while one may make a distinction as to *why* one studies cavity field theory, the calculations are the same. The early authors simply calculated the necessary graph, often in quite a crude semiclassical formalism, with time ordered perturbation theory. Later papers introduced cavity Feynman rules [4], and dealt more thoroughly with questions of gauge invariance, and quantization [2].

When one compares the predictions of cavity QCD in the framework of the M.I.T. bag model with experimental data, one has to introduce several additional assumptions or approximations. Firstly, the linear and quadratic boundary conditions introduce the phenomenon of confinement 'by hand', whereas it is believed to be already contained in QCD. Thus the possibility of other confinement mechanisms has been discarded, as well as the possibility of deformation [1], and surface dynamics.

Secondly, as translational invariance is broken in the cavity theory, we are still faced with the intimidating center-of-mass problem. Thirdly, although not essential, we usually take perturbation theory for granted, and regard confinement as having been implemented via the boundary conditions. For this reason the MIT bag model will always be a substitute for a non-perturbative effect that is 'non-understood'. Despite considerable effort, and some improvement in the model predictions, the MIT bag model remains at the 10% level in terms of its predictive ability.

In connection with the second possibility we simply note that the theory will be infra-red finite, due to the presence of a natural low momentum cut-off, namely the cavity radius. Due to the property of asymptotic freedom it seems reasonable to believe that for a sufficiently small R, perturbation theory will be 'valid', or at least free from the problems that crop up due to infra-red divergences.

Subsequent to the MIT bag model there have been other bag models developed, in particular 'soft' bag models in which the fields are coupled to fields outside the bag in a non-renormalizable way. For this reason if we want to calculate loop diagrams, we must restrict our attention to theories like cavity field theories.

1.1.1 Tree diagrams

A number of authors have developed cavity QCD in previous years. T. D. Lee [7] discussed the theory in the Coulomb gauge, and Close and Horgan [8], as well as Baacke, Igarashi, and Kasperidus [19], looked in detail at interactions to order α_S. By now, tree diagrams are straightforward to calculate, and ref [2] contains an exhaustive list of all the tree graphs to order α_S. In addition a recent work by Marbach [3] discussed the question of unitarity.

Using the hyperfine splittings of one gluon exchange, Carlson, Hansson and Peterson [10], calculated meson, baryon and glueball masses, in an 'improved' bag model. Gluonia have also been discussed by Barnes, Close and Monaghan [12] and Hess and Viollier [11]. Excited states have been calculated by DeGrand and Jaffe [13].

A more recent step has been the evaluation of order α_S corrections to other baryon properties, such as magnetic moments, charge radii and weak decays [14,15,16]. Selected finite graphs to order α_S^2, like two gluon exchange and two gluon annihilation, have also been calculated [28,29]. The latter diagram splits the mass of the π- and η-mesons.

1.1.2 Divergent Graphs

Considerable time elapsed between the first attempt at a loop diagram [32], and the first accurate calculation of the quark self-energy in a spherical cavity 9 years later [5]. Loop diagrams contain divergences, and analytic calculations are not easily performed. Hansson and Jaffe developed the Multiple Reflection Expansion (MRE) for cavity field theories [4], with the problem of divergent loop diagrams primarily in mind.

Of the diverging one-loop diagrams in a sphere, only the chromodynamic and electrodynamic self-energies of a quark in the $1s_{1/2}$ cavity mode of a spherical cavity have been calculated so far. An attempt has been made to calculate the photon self-energy by Peterson, Hansson and Johnson, [22]. They calculate the self-energy of a photon in scalar QED in a cube. In a cube one may construct the propagator with the method of images, thus analytic calculation is possible, although rather tedious. They present an approximate result for a single cavity mode. This surprising lack of progress is partly due to the fact that in the cavity theory the integrals associated with Feynman diagrams are much more difficult than their free space counterparts.

Integrations over momenta become sums over angular momentum and radial quantum numbers. This makes both loop and higher order diagrams more difficult to calculate. The loop diagrams are particularly difficult because the divergences must be removed analytically.

The knowledge of the chromodynamic self-energy of a quark in a cavity is of great importance in hadron spectroscopy. In fact we need the quark self-energy in order to predict the masses of the baryons relative to the mesons correctly in the framework of the MIT bag model.

A related problem is that of the electrodynamic self-energy of a quark in a cavity, which is an important ingredient in calculating the neutron-proton mass difference [31,6]. It should be noted that the difference in the mass of the up and down quarks, while coincidentally also of magnitude α/R, seems to dominate the splitting. Thus this quantity, while known from experiment to great accuracy, has yet to be explained by theory.

The first attempt to calculate a cavity loop diagram was by Chodos and Thorn [32]. They tackled the problem of the electrodynamic self-energy of a quark confined to an infinite slab using analytical methods. In a subsequent paper [33] they computed the spherical case numerically.

Since the self-energy of a massless quark in the cavity is finite, some other authors also attempted a direct numerical calculation. In this manner, Chin, Kerman and Yang [17] arrived at a value for the $1s_{1/2}$ state of $\Sigma = 0.40\alpha_s/R$. However their paper contains an error in equation (4.20). Breit [18] attempted a similar calculation with the results $\Sigma = 0.25\alpha_s/R$, [18]. These results are expressed according to our convention, which is to use the Feynman gauge Coulomb interaction. To compare with other conventions, see the discussion at the end of section 4.2.4. It has been suggested by Goldhaber et al [6], that this lack of agreement is caused by the fact that the finite self-energy is the sum of a conditionally convergent series.

Other authors have separated out the divergence analytically, and applied a gauge invariant regularization procedure, to get their result. Baacke, Igarashi and Kasperidus [19] reported $\Sigma = 0.85\alpha_s/R$. In order to deal with this problem, Hansson and Jaffe introduced the multiple reflection expansion (MRE) [4] to cavity QCD. In this scheme the propagators, and thereby the self-energy, is expanded into terms containing zero, one or more reflections. In this manner they calculated the self-energy of a quark in the $1s_{1/2}$-state of a cavity, $\Sigma = 0.910 \pm 0.001\alpha_s/R$. [5,6]. Marbach and Zimak [21] have repeated their calculation independently and report $\Sigma = 0.91 \pm 0.01\alpha_s/R$. However this method involves considerable analytical and numerical work, which seems to have deterred anyone from extending the work to other states.

The problem of the electromagnetic self-energy shift has also been tackled by Mohr and Sapirstein [31], and Goldhaber, Hansson and Jaffe [6]. Our results are published in [23].

1.1.3 The Multiple Reflection Expansion

We discuss briefly and schematically the argument behind use of the Multiple Reflection Expansion. The self-energy of a quark in the cavity arises due to an integral over a product of a quark and a gluon propagator,

$$\Sigma \sim \int S\, D \tag{1.1}$$

The propagators may be seperated into a term involving no reflections, e.g. S^0, and a term involving one or more reflections, e.g. $\tilde{S}$.

$$\begin{aligned} S &= S^0 + \tilde{S}, && (1.2)\\ D &= D^0 + \tilde{D}. && (1.3) \end{aligned}$$

Thus we may write the self-energy as due to a free space part, and a part due to reflections

$$\Sigma = \Sigma^0 + \tilde{\Sigma} = \int S^0 D^0 + \int (S^0 \tilde{D} + \tilde{S} D^0 + \tilde{S}\tilde{D}) \tag{1.4}$$

The result for the Σ^0 part is, in the case of a massless quark, not divergent, and does not need to be renormalized. However the actual expression is divergent, and only becomes well defined after being regularized in a proper regularization scheme, Pauli-Villars or dimensional regularization being the most common. This is why a direct numerical calculation of the self-energy will not guarantee a correct result. The series form would be a conditionally convergent series, and we investigate this possibility in section 4.2.5. This presumably explains the variety of published results.

The usual method for resolving this problem is to separate out the Σ^0 part, regularize it analytically until a finite expression is achieved, and then compute it numerically. The $\tilde{\Sigma}$ piece has been shown to be finite [4], and is evaluated numerically. In fact it is a significantly more difficult calculation than Σ^0. Authors using this approach generally present results in agreement. While there is a little more to the MRE than what we have presented here, for convenience we refer to the procedure as the MRE approach.

In summary then there are three approaches to the problem

- Direct analytic calculation, possible for slab and cube geometries.
- Direct numerical calculation, probably wrong.
- The Multiple Reflection Expansion Approach.

It should be noted that one of the notable omissions from the list of calculated diagrams is that of the vacuum polarization. In this case the reflection part still contains divergences, and in order to use the MRE pieces containing one reflection must be seperated out. This would be a formidable task, to say the least. This is why the only result thus far, has been for a cube.

1.1.4 Dimensional Regularization in the Cavity

In this section we schematically outline the technique that we intend to use to compute divergent cavity quantities. In the framework of the Dimensional Regularization scheme space time is generalized to $D = 4-2\varepsilon$ dimensions. Quantities that are divergent in $D = 4$ dimensions become finite in $D \neq 4$ dimensions, and the divergence is manifested by a pole in ε.

If we consider a divergent quantity like the free space self-energy Σ^0, then we may write it as

$$\Sigma^0(\varepsilon) = S(\varepsilon) + F. \tag{1.5}$$

$S(\varepsilon)$ will be proportional to $1/\varepsilon$ + const ,F is finite and does not depend on ε. We ignore terms of order ε, because we anticipate taking the limit $D \to 4$, i.e. $\varepsilon \to 0$. The quantity of physical interest is F, the nontrivial finite part. The divergent term $S(\varepsilon)$, plus some constant, is usually absorbed into a counterterm. In this discussion we ignore the constant whose precise form is dictated by the renormalization scheme chosen. In summary the dynamics are contained in the term F. It may be evaluated as

$$F = \lim_{\varepsilon \to 0}(\Sigma^0(\varepsilon) - S(\varepsilon)), \tag{1.6}$$

where the limit $\varepsilon \to 0$ is the final step in the calculation, because the divergent quantities $\Sigma^0(\varepsilon)$ and $S(\varepsilon)$ do not exist in this limit. We wish to tackle the ultraviolet divergences, and in dimensional regularization ultraviolet divergences usually manifest themselves as poles in the gamma function. The gamma function can be represented by an integration over a parametric variable, which we refer to as 'z'. In a similar way we can derive a form in which the divergent quantity in question is expressed as an integral over some parameter 'z', (called the 'z-form').

$$\Sigma^0(\varepsilon) = \int_0^\infty dz\, \Sigma^0(\varepsilon; z), \tag{1.7}$$

$$S(\varepsilon) = \int_0^\infty dz\, S(\varepsilon; z). \tag{1.8}$$

These forms may be chosen in a special way so that we may exchange the limit over ε and the integral over z, to get

$$F = \lim_{\varepsilon \to 0}\left[\int dz\, \Sigma^0(\varepsilon; z) - \int dz\, S(\varepsilon; z)\right], \tag{1.9}$$

$$= \int dz\, \left[\Sigma^0(0; z) - S(0; z)\right], \tag{1.10}$$

$$= \int dz\, F(z). \tag{1.11}$$

A concrete example of how we would make such a subtraction is given in section 4.1.3. The important point is that the function $\Sigma^0(\varepsilon; z)$ has a non-integrable divergence for $z \to 0$. We choose $S(\varepsilon; z)$ to be a simple function which makes $F(z)$ integrable as $z \to 0$.

At this point we recognize a useful feature; to calculate F we only need to know $\Sigma^0(\varepsilon; z)$ and $S(\varepsilon; z)$ for $\varepsilon = 0$. This is achieved by the special choice of $S(\varepsilon; z)$. We see that some of the information contained in $\Sigma^0(\varepsilon; z)$ is redundant, the only need we have for it when

$\varepsilon \neq 0$ is to choose an appropriate form of $S(\varepsilon; z)$. We have now come full circle, because we can use the ingredients of $S(\varepsilon)$ and F to recover $\Sigma^0(\varepsilon)$ by using equation (1.5).

How does this help us in the cavity? Here we have some corresponding cavity self-energy Σ^C. Usually we can write

$$\Sigma^C = \Sigma^0 + \tilde{\Sigma}, \tag{1.12}$$

where $\tilde{\Sigma}$ is the part due to the presence of the boundary. We assume that this part is finite. Suppose for the moment that we know this quantity in D dimensions, we could then write

$$\Sigma^C(\varepsilon) = \Sigma^0(\varepsilon) + \tilde{\Sigma} = S(\varepsilon) + F + \tilde{\Sigma}. \tag{1.13}$$

Naturally a finite term like $\tilde{\Sigma}$ is independent of ε to lowest order. Now with $\Sigma^C(0; z)$ we can calculate the finite part using

$$F + \tilde{\Sigma} = \int dz \left[\Sigma^C(0; z) - S(0; z)\right] = \int dz\, F'(z). \tag{1.14}$$

Can we define $\Sigma^C(\varepsilon)$ or $\Sigma^C(\varepsilon; z)$? To define a D dimensional cavity theory in principle is easy. We could suppose that a spherical cavity becomes a $D-1$ dimensional sphere and the wave functions are known (see Appendix C.4), (but other shapes would be difficult), or we could change the time axis to $D-3$ dimensions. It is however not clear whether the asymmetrical treatment of space and time would not perhaps violate Lorentz invariance.

Whether or not we can *define* $\Sigma^C(\varepsilon)$ or $\Sigma^C(\varepsilon; z)$, it would be exceedingly difficult to *compute* them. An analytical approach would probably be necessary, because there is a divergence present. However it does prove possible to derive and compute $\Sigma^C(0; z)$. We can do it in such a way that the behaviour in the limit $z \to 0$ is the same as in the free space case. In other words, we do it in such a way that the subtraction factor is the same, which allows us to use (1.14). Then, equation (1.13) may be used to construct $\Sigma^C(\varepsilon)$. Equation (1.14) is suitable for a computer calculation, although we cannot compute $\Sigma^C(0; z)$ for arbitrary small z. Even if we could there would be a large subtraction error. Thus it is necessary to extrapolate the curve $F'(z)$ to get an accurate result.

We cannot prove that $\Sigma^C(\varepsilon; z)$ is such that the identical subtraction factor should be used. This remains at the level of conjecture. However we can motivate for the correctness of the procedure. The point is that $S(\varepsilon)$ contains a trivial momentum dependence in momentum space. It is designed to subtract out a term of the form $\delta(x, y) \times 1/\varepsilon$ (or derivatives thereof) from the self-energy expressed in configuration space. In others words the divergent part is not of a form which one would expect to be affected by the presence of the boundary. Thus far we have suppressed the space dependence of our quantities. Including it we can write

$$\Sigma^0(\varepsilon; x, y) = \Sigma^0(\varepsilon; x, y) - S(\varepsilon; x, y), \qquad x \neq y, \tag{1.15}$$

and would expect, (although it isn't defined)

$$\Sigma^C(\varepsilon; x, y) = \Sigma^C(\varepsilon; x, y) - S(\varepsilon; x, y), \qquad x \neq y. \tag{1.16}$$

Viewed in this way, we would naturally expect $S(\varepsilon)$ to be the same for the free space and cavity theories.

For $S(0;z)$ to be the same we have to generate $\Sigma^0(0;z)$ and $\Sigma^C(0;z)$ using the same procedure. This we may view from another perspective. The free space quantity is obtained from an integral, and the cavity quantity from a sum. For large momentum, which corresponds to small z, the sum becomes an Euler-Maclaurin series approximation of the integral. To leading order such quantities will be the same. This point is referred to in section 2.1.1.

While the foregoing argument may be plausible, a rigorous proof would rely on computation of $\Sigma^C(\varepsilon;z)$ which we cannot do. Notwithstanding this, we can however make an even stronger conjecture. Based on the previous argument we have no grounds to make a statement about subleading orders. However we may make the *additional conjecture* that the first non-zero subleading order in z is due to the boundary in the way that we would naively guess. This may be computed for a plane boundary. Thus in the case of the scalar tadpole in ϕ^4 field theory, the leading divergence in $\Sigma^C(0;z)$ is order z^{-2}. We predict the surface divergence to be order $z^{-3/2}$ for Neumann boundary conditions and $z^{-1/2}$ for Dirichlet boundary conditions. Within our errors we find agreement in the magnitude and sign of the coefficients of these subleading terms. For the Dirichlet case we have predicted the *3rd* subleading order in the relevant expansion parameter $z^{1/2}$. This is discussed in section 3.2.3.

The only other assumption which should be mentioned is the assumption that the surface part is finite. This has been shown for the quark [4]. The case of the vacuum polarization has some special features. In the MRE it is seen that both the one and two reflection pieces are superficially divergent. There is also a problem with the cavity Fourier transform of $\partial^2\delta(x,y)$ at the boundary, mentioned in [4]. In section 5.2 we discuss this problem in detail. We show that the reflection part is finite except for a piece that conveniently solves the Fourier transform problem, thus we may apply the method outlined above.

1.1.5 Purpose of this thesis

Broadly speaking we do four things in this thesis.

- We develop a set of Fourier transforms that provide a concise way of expressing field theories in a cavity. They are convenient for considering renormalization, and enable the easy extension of concepts such as virtuality and transversality to the cavity. As an example, we derive the gauge propagator in an arbitrary gauge.
- We renormalize the cavity propagators in a way appropriate to the cavity, for the self-energy of scalars and Dirac fields.
- We introduce a new regularization scheme that enables a practical evaluation of these quantities.
- We present results for ϕ^4 scalar field theory, the massless chromodynamic quark self-energy, and the self-energy of a photon in massless scalar QED. All results are for spheres, and we present results for several low lying cavity modes in each case.

The second point needs some comment. To evaluate some quantity, e.g. an energy shift, in the cavity we need a starting point. In most work on this subject the starting point is taken to be the Gell-Mann and Low theorem [25,26,27,2]. In [4,31] a similar expression is used based on the S matrix. This is simply a more symmetric form of the Gell-Mann and Low expression. We develop a different approach, based more closely on the standard procedures of renormalizing propagators to one loop in free space. In this formalism the placing of the counterterms we believe is more transparent, and more information may be extracted. The expressions that we finally evaluate agree, in the appropriate limit, with what would be expected from the Gell-Mann and Low approach.

The third point is the major contribution of the thesis. The method has a general applicability, and relies on a qualitatively different approach to previous methods. It is primarily an efficient calculational scheme, since the likelihood of analytic evaluation is remote. Insofar as we evaluate finite quantities that result from the subtraction of infinities, it is an unusual computer application.

The method can be used to calculate any logarithmically or linearly divergent loop integral in cavity field theory, as long as there are no divergences caused by the boundary. We have also had some success with a quadratically divergent diagram, in which there *is* a surface divergence.

Using this method we have calculated the self-energies of a few of the low-lying excited states of quarks in a spherical cavity. Only for the $1s_{1/2}$ state do results exist in the literature, and we agree, within the estimated errors, with Goldhaber, Jaffe and Hansson [6], as well as Baacke, Igarashi and Kasperidus [19]. Our method is computationally much simpler than existing methods, and considerably more accurate. After this we present results for the self-energies of some low lying photon cavity modes in scalar QED. We can also make a suggestion for implementing the running coupling constant in a cavity.

The thesis is laid out as follows: in Chapter 2, we develop Fourier transforms appropriate to the cavity. These can be used to derive convenient forms for the propagators. In Chapters 3, 4, and 5 we attack the problems of scalar self-energy, quark self-energy and vacuum polarization respectively. The Chapter on the scalar self-energy is particularly easy, and is mainly pedagogical. Chapter 4 is rather detailed, and in Chapter 5 we assume familiarity with the two previous chapters in the interests of brevity. Chapter 6 concludes.

Appendix A contains our conventions, and information relating to the details of the cavity modes. In Appendix B we compute the vertex integrals. Appendix C contains useful mathematics, and Appendix D discusses peculiarities of the Coulomb interaction in a cavity.

As a general principle, we err on the side of including too much calculational detail, and for this we apologize in advance.

Chapter 2

Fourier Transforms and Propagators

In this chapter we introduce Fourier transforms appropriate to the cavity, and use these to derive cavity propagators. While we introduce a particularly elegant notation, this is not the sole objective of the chapter.

The notation presented in this chapter proves convenient for a discussion of renormalization in a cavity field theory, and highlights the differences between the cavity theory, and free space.

In particular, it is important to realize that there *is* a difference, and that this difference is intimately tied up with the concept of 'virtuality'. A particle that is on-shell in the cavity is off-shell in free space, and vice versa.

Before proceeding, we review the role of the plane wave Fourier transform in free space. In free space one can easily Fourier transform from configuration space to momentum space, using a unitary transformation matrix which is given by a plane wave.

The propagator is seldom written down in configuration space, because, in this representation, it has a complicated form involving modified Bessel functions. In the momentum space representation, however, quantities like the propagator take on a particularly simple form. For this reason, we usually rely on the momentum space form, when we renormalize. The subtraction points in renormalization are specified in terms of momentum squared, or 'virtuality'.

This much is usually taken for granted. However when dealing with quantities in a cavity the 'natural' representation in which to renormalize is the 'cavity mode' or 'cavity momentum' representation. We shall show that in order to renormalize in a cavity one may follow a closely analagous process to that of free space, provided that one uses this representation.

Another advantage of using the Fourier set is that we may make full use of the Feynman rules. Using the Feynman rules, the interaction may be written down in configuration space, followed by Fourier transforms as appropriate.

2.1 Scalar Fields

For scalar fields we choose the Fourier set as

$$\phi(q;x) = (2\pi)^{-1/2}\phi(p;\vec{r})e^{-i\omega t}. \tag{2.1}$$

$\phi(q;x)$, a function of $x = \{t, \vec{r}\}$, is labelled by q. Here $\phi(p, \vec{r})$ denotes the scalar cavity modes, which are solutions of the time-independent Klein-Gordon equation, subject to the boundary conditions chosen (see Appendix A.2). ω denotes a continuous energy parameter which is not related to the energy eigenvalue of the cavity mode. q is shorthand for the labels

$$q = \{\omega, p\} = \{\omega, n, l, m\}. \tag{2.2}$$

This set of functions has orthonormality and completeness relations,

$$\int d^4x\, \phi^*(q;x)\phi(q';x) = \delta(q,q'), \tag{2.3}$$

$$\sum_q \phi(q;x)\phi^*(q;x') = \delta(x,x'). \tag{2.4}$$

Here we have introduced the shorthand notations

$$\delta(q,q') = \delta(\omega,\omega')\delta_{nn'}\delta_{ll'}\delta_{mm'}, \tag{2.5}$$

$$\sum_q = \int_{-\infty}^{\infty} d\omega \sum_{nlm}. \tag{2.6}$$

With this Fourier set (2.1) we can expand any scalar function of space time

$$f(x) = \sum_q c_q \phi(q;x), \tag{2.7}$$

with the Fourier coefficients given by the overlap integral

$$c_q = \int d^4x\, \phi^*(q;x)f(x). \tag{2.8}$$

We note that if the function $f(x)$ is a continuous function that obeys the same boundary condition as the Fourier set, then the cavity mode expansion converges absolutely. If it does not obey the boundary condition then it converges only in the least squares sense [49], in other words there may be some Gibbs phenomenon at the boundary.

The functions used as a Fourier set do *not* satisfy the Klein-Gordon equation, but rather

$$(\Box + m^2)\phi(q;x) = (-\omega^2 + k_p^2 + m^2)\phi(q;x). \tag{2.9}$$

This suggests that the cavity analogue of 4 momentum squared or virtuality should be defined as

$$q^2 = \omega^2 - k_p^2. \tag{2.10}$$

An on-shell cavity mode has $q^2 = m^2$. It may seem strange that the angular momentum does not appear explicitly in a quantity that should correspond to 3-momentum squared, namely k_p^2. However the angular momentum, l, makes an appearance in the spherical Bessel differential equation, and is implicitly included in k_p^2.

What is the free space virtuality of the particle? To find out we would have to Fourier transform $\phi(p;\vec{r})$ using the plane wave basis,

$$\phi(p;\vec{r}) = \int d\vec{k}\, e^{i\vec{k}.\vec{r}}\phi(p;\vec{k}), \tag{2.11}$$

with the inverse transform

$$\phi(p;\vec{k}) = \int d\vec{r}\, e^{-i\vec{k}.\vec{r}}\phi(p;\vec{r})\beta(r). \tag{2.12}$$

$\beta(r) = 1 - \theta(r - R)$ is zero outside the cavity. We regard the function $\phi(p;\vec{r})$ and it's derivatives to be continuous, and explicitly cut off the wave function by putting in the step function. Most of the time such a distinction is unnecessary, and it is inserted explicitly because it can help resolve ambiguities, as will be seen later. The free space virtuality may be defined as the expectation value of the quantity $s^2 = \omega^2 - k^2$, where s is the 4-momentum, with components ω, and planar three momentum $\vec{k}$. We see from the above that a cavity mode has a range of values of planar 3-momentum, but it has only one cavity eigenmomentum k_p (see equation (A.22)). Hence we see that the free space virtuality of a cavity mode is only defined in the sense of an expectation value, whereas q^2 is defined in the sense of an eigenvalue. In contrast a plane wave has good free space virtuality, but is not a cavity eigenmode. Why can one not find states which simultaneously have good q^2 and good s^2, when they both correspond to eigenstates of the operator $(\Box + m^2)$? The reason is that equation (2.9) is true for all $\vec{r}$ in the cavity, but it is not true on the surface, where there is either a discontinuity in the wavefunction, or its derivative (see section 3.2.4). This is not to say that the plane wave Fourier transform above has no use (see Appendix A.5).

The Feynman propagator in the cavity must satisfy

$$(\Box + m^2)\Delta(x,y) = -\delta(x,y), \tag{2.13}$$

subject to the chosen boundary condition. By applying the wave equation (2.9), and the completeness relation (2.4), we see that the propagator is given by

$$\Delta(x,y) = \sum_q \frac{\phi(q;x)\phi^*(q;y)}{q^2 - m^2 + i0}. \tag{2.14}$$

If we apply the Fourier transform

$$\Delta(q,q') = \int d^4x d^4y\, \phi^*(q;x)\Delta(x,y)\phi(q';y), \tag{2.15}$$

we get the following form for the propagator in q space,

$$\Delta(q,q') = \frac{\delta(q,q')}{q^2 - m^2 + i0}. \tag{2.16}$$

The propagator is diagonal in all quantum labels.

2.1.1 The Scalar Propagator in Free Space

In this section we briefly consider the scalar propagator in free space, and compare it with the cavity propagator. The expression for the cavity propagator (2.14) may be written out in full as

$$\Delta(x,x') = \left[\int \frac{d\omega}{2\pi}\sum_{nlm} N_{nl}^2\right] \frac{1}{\omega^2 - k_{nl}^2 - m^2 + i0}$$
$$[j_l(k_{nl}r)Y_{lm}(\hat{r})e^{-i\omega t}][j_l(k_{nl}r')Y_{lm}^*(\hat{r}')e^{i\omega t'}]. \quad (2.17)$$

In free space we usually write the propagator as

$$\Delta^0(x,x') = \int \frac{d^4s}{(2\pi)^4} e^{-is(x-x')} \frac{1}{s^2 - m^2 + i0}. \quad (2.18)$$

If we note the Rayleigh relation (C.39)

$$e^{i\vec{k}.\vec{r}} = 4\pi \sum_{lm} i^l j_l(kr) Y_{lm}^*(\hat{k}) Y_{lm}(\hat{r}), \quad (2.19)$$

then, with $s^\mu = \{\omega, \vec{k}\}$, we may re-express the free propagator as

$$\Delta^0(x,x') = \left[\int \frac{d\omega}{2\pi}\sum_{lm} \frac{2}{\pi}\int dk\, k^2\right] \frac{1}{\omega^2 - k^2 - m^2 + i0}$$
$$[j_l(kr)Y_{lm}(\hat{r})e^{-i\omega t}][j_l(kr')Y_{lm}^*(\hat{r}')e^{i\omega t'}]. \quad (2.20)$$

This illustrates conveniently two points. Firstly, for large k the roots of the Bessel functions become evenly spaced, and the sum over n may be shown to be equal to the integral over k to leading order in k. (Provided, of course, that one does not choose positions actually on the surface.) This is physically expected, since the short distance behaviour is dominated by the free space part.

Secondly, since a D-dimensional form of the Raleigh relation exists (C.40), we may, at least in principle, formulate a D-dimensional free propagator in a spherical basis, or one confined to a spherical cavity. See discussion in C.4.

2.1.2 The Scalar Propagator in the MRE

Since we will frequently refer to the Multiple Reflection Expansion (MRE), we include a short derivation of the scalar propagator for massless particles in this scheme. A fuller version, as well as the Dirac and gauge fields, is treated in [4]. In this scheme we use a mixed representation for the propagator, namely $\Delta(\omega, \vec{r}, \vec{r}')$, related to the usual propagator in the usual way,

$$\Delta(x,x') = \int d\omega\, e^{-i\omega(t-t')}\Delta(\omega, \vec{r}, \vec{r}'), \quad (2.21)$$

and then perform a partial wave expansion,

$$\Delta(\omega, \vec{r}, \vec{r}') = \sum_{lm} \Delta_l(\omega, r, r') Y_{lm}(\hat{r}) Y_{lm}^*(\hat{r}'). \quad (2.22)$$

Now, the partial wave propagator must satisfy the equation

$$\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{l(l+1)}{r^2} + \omega^2\right]\Delta_l(\omega, r, r') = \frac{\delta(r, r')}{rr'}, \tag{2.23}$$

and the usual boundary condition. The free space solution of this equation is given by

$$\Delta_l^0(\omega, r, r') = \begin{cases} -i\omega j_l(\omega r_<)h_l^1(\omega r_>) & Im\,\omega > 0 \\ i\omega j_l(\omega r_<)h_l^2(\omega r_>) & Im\,\omega < 0 \end{cases}, \tag{2.24}$$

where h^1 and h^2 are the spherical Hankel functions of the first and second kind. If we add a piece, $\tilde{\Delta}$, that satisfies the homogenous version of equation (2.23), then we may have a solution which obeys the boundary conditions, namely the cavity propagator,

$$\Delta_l(\omega, r, r') = \Delta_l^0(\omega, r, r') + \tilde{\Delta}_l(\omega, r, r'), \tag{2.25}$$

and the boundary part is given by

$$\tilde{\Delta}_l(\omega, r, r') = i\omega a_l\, j_l(\omega r) j_l(\omega r'). \tag{2.26}$$

The coefficients a_l, may be chosen to satisfy Dirichlet or Neumann boundary conditions,

$$a_l^D = h_l^1(x)/j_l(x), \tag{2.27}$$

$$a_l^N = h'^1_l(x)/j'_l(x), \tag{2.28}$$

for $Im\,\omega > 0$, where $x = \omega R$. For $Im\,\omega < 0$, h^1 is replaced by $-h^2$. The free propagator has a cut on the real ω axis. The Feynman prescription determines that the cut be displaced downwards for the right-hand cut (particles), and upwards for the left-hand cut (antiparticles). The cavity propagator has discrete poles at the eigenenergies rather than a cut. The scheme may be elaborated in order to separate out 0,1,2, ..., reflections, rather than incorporating all reflections into one term, $\tilde{\Delta}$, as we do [4].

What are the uses of the MRE? The most important point is that each reflection that the propagator undergoes softens the divergence. Thus one may isolate the singularity, and use the usual free space methods to regularize.

One should also compare this form of the propagator with the cavity mode expansion form (2.17). In the cavity mode expansion we have a sum over n, the radial quantum number, of terms which are separable in r and r'. In the MRE form, this sum is performed, to give a non-separable quantity with a discontinuity (non-separable because it depends on $r_<$ and $r_>$). In the MRE form, it is usual to do the ω integral numerically in the complex plane, whereas in the cavity mode expansion it is a trivial analytic calculation.

As usual, it is best to have both forms, since each have their own peculiar advantages.

2.2 Dirac Fields

For Dirac fields the appropriate Fourier set is chosen as

$$\psi(q; x) = (2\pi)^{-1/2} u(p; \vec{r}) e^{-i\omega t}. \tag{2.29}$$

ψ, a function of $x = \{t, \vec{r}\}$, is labelled by q. Here $u(p; \vec{r})$ denotes the well-known quark cavity modes, which are solutions of the time-independent Dirac equation, subject to boundary conditions of the M.I.T. bag model (see Appendix A.3). ω denotes a continuous energy parameter which is not related to the energy eigenvalue of the quark. q is shorthand for the labels

$$q = \{\omega, p\} = \{\omega, \nu, \kappa, \mu\}. \tag{2.30}$$

ν, κ, and μ are respectively the radial, Dirac, and magnetic quantum numbers of the cavity mode.

This set of functions has orthonormality and completeness properties,

$$\sum_{\alpha} \int d^4x \; \psi_\alpha^*(q; x)\psi_\alpha(q'; x) = \delta(q, q'), \tag{2.31}$$

$$\sum_{q} \psi_\alpha(q; x)\psi_\beta^*(q; x') = \delta^4(x, x')\delta_{\alpha\beta}. \tag{2.32}$$

Here we have shown explicitly the Dirac indices α and β, and also introduced the shorthand notations

$$\delta(q, q') = \delta(\omega, \omega')\delta_{\nu\nu'}\delta_{\kappa\kappa'}\delta_{\mu\mu'}, \tag{2.33}$$

and

$$\sum_{q} = \sum_{\nu\kappa\mu} \int_{-\infty}^{+\infty} d\omega. \tag{2.34}$$

Now we can expand any spinor function of space-time in terms of the Fourier set (2.29) as

$$f_\alpha(x) = \sum_{q} c_q \psi_\alpha(q; x). \tag{2.35}$$

The Fourier coefficients are given by the overlap integral

$$c_q = \int d^4x \; \bar{\psi}(q; x)\gamma^0 f(x). \tag{2.36}$$

The adjoint spinor can be expanded as

$$\bar{f}(x) = \sum c_q^* \bar{\psi}(q; x), \tag{2.37}$$

with Fourier coefficients

$$c_q^* = \int d^4x \; \bar{f}(x)\gamma^0 \psi(q; x). \tag{2.38}$$

The functions used as a Fourier set do *not* satisfy the Dirac equation, but rather

$$(i\partial\!\!\!/ - m)\psi(q; x) = (\omega - \varepsilon_p)\gamma^0\psi(q; x). \tag{2.39}$$

This equation is true for all x inside the cavity, but is not true on the surface, where the wavefunction cuts off abruptly. The Feynman propagator in the cavity must satisfy

$$(i\partial\!\!\!/_x - m)S(x, x') = \delta^4(x, x'), \tag{2.40}$$

$$S(x, x')(-i \overleftarrow{\partial\!\!\!/}_{x'} - m) = \delta^4(x, x'), \tag{2.41}$$

subject to the boundary condition

$$(i\vec{\gamma}.\hat{r} + 1)S(x,x')|_{\vec{r}\in S} = 0. \tag{2.42}$$

By applying equations (2.39) and (2.32), we see that the propagator may be expressed as

$$S(x,y) = \sum_q \psi(q;x)\bar{\psi}(q;y)/(\omega - \varepsilon_p \pm i0). \tag{2.43}$$

The $\pm i0$, or Feynman prescription, indicates that right hand poles are displaced downwards, and left hand poles upwards, as is usual. We note two points: firstly S is formed from $\psi\bar{\psi}$, as might be expected from the definition $\langle T[\psi\bar{\psi}]\rangle$, and secondly the denominator is a number, as opposed to the free space denominator $\not{p} - m$, which is a spinor matrix. This is because the Fourier set includes the spinor indices, whereas the usual free space Fourier set, e^{ipx}, does not. If we now apply the Fourier transform

$$S(q,q') = \int d^4x d^4y\, \bar{\psi}(q;x)\gamma^0 S(x,y)\gamma^0\psi(q';y), \tag{2.44}$$

we get the form for the quark propagator in q-space,

$$S(q,q') = \delta(q,q')\frac{1}{(\omega - \varepsilon_p \pm i0)} = \delta(q,q')\frac{\omega + \varepsilon_p}{\omega^2 - \varepsilon_p^2 + i0}. \tag{2.45}$$

This propagator is diagonal in all quantum labels.

2.3 Vector Fields

In a similar manner we can derive the Feynman propagator of a vector field that is confined to a static and spherical cavity. Once again we introduce ω, the labels $q = \{\omega, p\}$ and $p = \{N, J, M\}$, and the set of Fourier functions

$$A^\mu(\Sigma, q; x) = (2\pi)^{-1/2} a^\mu(\Sigma, p; \vec{r})e^{-i\omega t}. \tag{2.46}$$

The cavity modes of a vector field denoted $a^\mu(\Sigma, p; \vec{r})$, with polarization Σ, are solutions of the time-independent d'Alembert equation, subject to the boundary conditions of the M.I.T. bag model (see Appendix A.4). The orthogonality and completeness relations are now

$$\int d^4x\; g^{\mu\nu} A^*_\mu(\Sigma, q; x)A_\nu(\Sigma', q'; x) = g^{\Sigma\Sigma'}\delta(q,q'), \tag{2.47}$$

$$\sum_{\Sigma q} g^{\Sigma\Sigma} A^\mu(\Sigma, q; x)A^{\nu*}(\Sigma, q; x') = g^{\mu\nu}\delta(x,x'). \tag{2.48}$$

Any vector function of space-time can be expanded in terms of the Fourier set,

$$V^\mu(x) = \sum_{\Sigma q} c_{\Sigma q} A^\mu(\Sigma, q; x), \tag{2.49}$$

where the Fourier coefficients are given by the overlap integral

$$c_{\Sigma q} = \int d^4x g_{\mu\nu} A^{\mu*}(\Sigma, q; x)V^\nu(x). \tag{2.50}$$

The Feynman propagator (in the Feynman gauge) will be defined by the equation

$$\Box D_{\mu\nu}(x,y) = g_{\mu\nu}\delta(x,y), \tag{2.51}$$

and will obey the M.I.T. boundary conditions (see Appendix A.4). We note that

$$\Box A^{\mu}(\Sigma,q;x) = -(\omega^2 - \Omega^2_{\Sigma p})A^{\mu}(\Sigma,q;x), \tag{2.52}$$

and once again this equation is true for all x inside the cavity, but is not true on the surface, where the wavefunction cuts off abruptly. We see by substitution that the Feynman propagator in the Feynman gauge is given by

$$D^{\mu\nu}(x,y) = -\sum_{\Sigma q} g^{\Sigma\Sigma}\frac{A^{\mu}(\Sigma,q;x)A^{\nu *}(\Sigma,q;y)}{(\omega^2 - \Omega^2_{\Sigma p} + i0)}. \tag{2.53}$$

Applying the Fourier transform (2.50) to the propagator, we get

$$D^{\Sigma\Sigma'}(q,q') = -\frac{g^{\Sigma\Sigma'}\delta(q,q')}{\omega^2 - \Omega^2_{\Sigma p} + i0}, \tag{2.54}$$

which is diagonal in all quantum labels.

At this stage we observe that the appropriate definition of virtuality for a vector field in the cavity is

$$q^2 = \omega^2 - \Omega^2_{\Sigma p}. \tag{2.55}$$

2.3.1 Transversality

We can immediately use the notation we have developed to show how certain concepts in free space field theory should be implemented in a cavity theory. The concept we have in mind is that of transversality in gauge field theories, connected to gauge invariance. We discuss this, and then proceed to develop the Feynman propagator for a vector field in an arbitrary gauge.

We begin by considering an arbitrary scalar function $\Lambda(x)$, defined in a cavity. It can be expressed in the Fourier set of the scalar mode of the vector field,

$$\Lambda(x) = \sum_q c(q)A^0(S,q;x). \tag{2.56}$$

We note that the Scalar mode is related to the Longitudinal mode by

$$\vec{A}(L,p;\vec{r}) = \frac{-i}{\Omega_{Sp}}\vec{\nabla}A^0(S,p;\vec{r}), \tag{2.57}$$

$$A^0(S,p;\vec{r}) = \frac{-i}{\Omega_{Sp}}\vec{\nabla}\cdot\vec{A}(L,p;\vec{r}), \tag{2.58}$$

$$\Omega_{Sp} = \Omega_{Lp}. \tag{2.59}$$

We would like to find an equation involving the cavity modes that corresponds to the free space relation containing the four momentum s^{μ},

$$i\partial^{\mu}e^{-isx} = s^{\mu}e^{-isx}. \tag{2.60}$$

Applying ∂^μ to Λ we get

$$i\partial^\mu \Lambda(x) = \sum_{q\Sigma} q^\Sigma c(q) A^\mu(\Sigma, q; x). \tag{2.61}$$

Usually we choose the z axis parallel to $\vec{k}$, for $s^\mu = \{\omega, \vec{k}\}$. Then we define

$$q^\Sigma = \{\omega, \Omega_p, 0, 0\}, \tag{2.62}$$

in analogy to the usual

$$s^\mu = \{\omega, 0, 0, s_z\}. \tag{2.63}$$

Here the four components of q^Σ correspond to S, L, M, E in that order. We will also need a 'contravariant' version, if we consider

$$V^\mu(x) = \sum_{q\Sigma} c^\Sigma(q) A^\mu(\Sigma, q; x), \tag{2.64}$$

where we write the Fourier coefficient in a suggestive way. If we apply the four derivative, we get

$$i\partial_\mu V^\mu(x) = \sum_{q\Sigma} q_\Sigma c^\Sigma(q) A^0(\Sigma, q; x), \tag{2.65}$$

where now we have

$$q_\Sigma = \{\omega, -\Omega_p, 0, 0\}. \tag{2.66}$$

Thus the quantity q_Σ is the generalization of transversality that will be needed in the cavity, for example we consider a conserved current, and its cavity generalization,

$$\partial_\mu j^\mu(x) = 0 \;\rightarrow\; \sum_\Sigma q_\Sigma j^\Sigma(q) = 0. \tag{2.67}$$

2.3.2 The Vector Propagator in an Arbitrary Gauge

The first task we turn to, armed with this observation, is the calculation of the vector propagator in an arbitrary gauge. For the most general case we assume that the vector field has a mass, i.e. it is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu - \frac{\lambda}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu). \tag{2.68}$$

For $\lambda \neq 0$, the limit $\mu \rightarrow 0$ exists. The propagator must satisfy the equation

$$\left[(\Box + \mu^2)g^\mu_{\ \alpha} - (1-\lambda)\partial^\mu\partial_\alpha\right] D^{\alpha\nu}(x,y) = g^{\mu\nu}\delta(x,y). \tag{2.69}$$

The propagator in free space is given by

$$D^{\mu\nu}(s) = \frac{-g^{\mu\nu} + s^\mu s^\nu/\mu^2}{s^2 - \mu^2 + i0} - \frac{s^\mu s^\nu/\mu^2}{s^2 - m^2 + i0}, \tag{2.70}$$

$$= \frac{-g^{\mu\nu}}{s^2 - \mu^2 + i0} - \frac{1-\lambda}{\lambda}\frac{s^\mu s^\nu}{(s^2 - \mu^2 + i0)(s^2 - m^2 + i0)}, \tag{2.71}$$

where $m^2 = \mu^2/\lambda$. In the limit $\mu \to 0$ we get simply

$$D^{\mu\nu}(s) = -\left[\frac{g^{\mu\nu}}{s^2+i0} + \frac{1-\lambda}{\lambda}\frac{s^\mu s^\nu}{(s^2+i0)^2}\right]. \tag{2.72}$$

We now calculate this propagator in the cavity, assuming that the boundary conditions (A.41) remain the same. By substitution one may confirm that

$$D^{\mu\nu}(x,y) = \sum_{q\Sigma\Sigma'} A^\mu(\Sigma,q;x)A^\nu(\Sigma',q;y)\left[\frac{-g^{\Sigma\Sigma'}}{q^2+i0} - \frac{1-\lambda}{\lambda}\frac{q^\Sigma q^{\Sigma'}}{(q^2+i0)^2}\right], \tag{2.73}$$

satisfies the defining equation. The similarity is rather striking! We have used the identities

$$i\partial_\mu \sum_\Sigma q^\Sigma A^\mu(\Sigma,q;x) = q^2 A^0(S,q;x), \tag{2.74}$$

$$i\partial_\mu A^\mu(\Sigma,q;x) = q_\Sigma A^0(S,q;x), \tag{2.75}$$

$$i\partial^\mu A^0(S,q;x) = \sum_\Sigma q^\Sigma A^\mu(\Sigma,q;x). \tag{2.76}$$

Chapter 3

Scalar Field Theory

We wish to develop a scheme that will be able to renormalize any field theory confined to a cavity. ϕ^4 scalar field theory, whilst not actually corresponding to any particle currently found in nature, is the simplest renormalizable field theory. For this reason it is usually used as a 'toy' theory for pedagogical purposes. We firstly review the free space theory. In section 3.1.1 we renormalize the propagator, and then regularize the order λ (tadpole) diagram in section 3.1.2.

We then turn to the case of the cavity theory, and examine how to renormalize the propagator, followed by introduction of the regualization technique. There are some problems associated with the boundary, which crop up in particular for the case of Neumann boundary conditions. Finally we compute the self-energies, and present the results.

The corresponding calculation in the MRE approach would be relatively simple. We present this case rather to introduce the method, and gain some understanding of problems encountered due to presence of the surface.

3.1 Free ϕ^4 Theory

We immediately introduce the dimensional regularization scheme [35], a brief discussion and some standard integrals are given in Appendix C.4. The Lagrangean for massive ϕ^4 field theory in $D = 4 - 2\varepsilon$ dimensions is

$$\mathcal{L} = \tfrac{1}{2}\left[(\partial_\mu\phi)(\partial^\mu\phi) - m_b^2\phi^2\right] - \frac{\lambda_b\mu^{4-D}}{4!}\phi^4. \tag{3.1}$$

λ_b is a dimensionless (bare) coupling constant, and μ an arbitrary mass scale. m_b is the bare mass. The action is given by

$$S = \int d^Dx\,\mathcal{L}. \tag{3.2}$$

3.1.1 Renormalization

The dressed two point function is given by

$$G(s^2) = \frac{i}{s^2 - m_b^2} + \frac{i}{s^2 - m_b^2}[-i\Sigma^0(s^2)]\frac{i}{s^2 - m_b^2} + \cdots, \tag{3.3}$$

$$= \frac{i}{s^2 - m_b^2 - \Sigma^0(s^2)}, \tag{3.4}$$

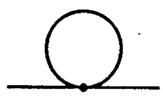

Figure 3.1: Order λ_b self-energy

where $\Sigma^0(s^2)$ is the irreducible self-energy insertion. $s^\mu = \{\omega, \vec{k}\}$ is the 4-momentum. The superscript 0 reminds us that it is a free space quantity. The self-energy is at most quadratically divergent, so all derivatives after the first, (with respect to s^2), will be finite. In an on-shell renormalization scheme, we usually define δm^2 as

$$\delta m^2 = \Sigma^0(m^2), \tag{3.5}$$

where m is the physical mass,

$$m^2 = m_b^2 + \delta m^2. \tag{3.6}$$

and the wave-function renormalization constant, Z_ϕ, is defined as

$$[1 - Z_\phi^{-1}] = \frac{\partial}{\partial s^2}\Sigma^0(s^2)|_{s^2=m^2}. \tag{3.7}$$

Finally the renormalized self-energy, Σ_R^0, is defined by

$$\Sigma^0(s^2) = \delta m^2 + (s^2 - m^2)[1 - Z_\phi^{-1}] + Z_\phi^{-1}\Sigma_R^0(s^2). \tag{3.8}$$

If we insert this in equation (3.4), we get

$$G(s^2) = \frac{iZ_\phi}{s^2 - m^2 - \Sigma_R^0(s^2)}. \tag{3.9}$$

Finally, by defining the renormalized wave function $\phi_R = Z_\phi^{-1/2}\phi$, we get the renormalized two point function

$$G_R(s^2) = \frac{i}{s^2 - m^2 - \Sigma_R^0(s^2)}. \tag{3.10}$$

3.1.2 Regularization

The order λ_b self-energy (the tadpole diagram), shown in figure 3.1, is given by

$$-i\Sigma^0(s^2) = \tfrac{1}{2}\lambda_b\mu^{4-D}\int\frac{d^Dl}{(2\pi)^D}\frac{1}{l^2 - m_b^2 + i0}. \tag{3.11}$$

We note that the right-hand side has no dependence on s^2, i.e. it is a delta function in configuration space and has no dynamical content. If we regularize this in the usual way we get (C.50),

$$\Sigma^0(s^2) = \delta m^2 = -\lambda_b m_b^2\frac{1}{32\pi^2}\left[\frac{1}{\varepsilon} - \gamma + 1 - \log\frac{m_b^2}{4\pi\mu^2}\right]. \tag{3.12}$$

Although this is a standard result of dimensional regularization, we shall derive it explicitly, as a first step towards understanding how dimensional regularization may be implemented in a cavity. We begin by Wick rotating equation (3.11), followed by elevating the denominator according to

$$\frac{1}{\bar{l}^2+\bar{m}^2} = \int dz\, e^{-\bar{l}^2 z - \bar{m}^2 z}, \tag{3.13}$$

where we have defined the dimensionless mass $\bar{m} = m/\mu$, and momentum $\bar{l} = l/\mu$. Using the Gaussian integral (C.12), we can perform the integral over D-momentum l to get

$$\begin{aligned} \Sigma^0(s^2) &= \tfrac{1}{2}\lambda_b \mu^2 \int_0^\infty dz \left(\frac{1}{4\pi z}\right)^{D/2} e^{-\bar{m}^2 z}, & (3.14)\\ &= \int_0^\infty dz\, \Sigma^0(s^2; z). & (3.15) \end{aligned}$$

This equation we will refer to as the 'z-form'. To proceed, we perform the z integral, getting a Gamma function (see Appendix C.1),

$$\begin{aligned} \Sigma^0(s^2) &= \tfrac{1}{2}\lambda_b \mu^2 \frac{1}{(4\pi)^{D/2}} (\bar{m}^2)^{\frac{D}{2}-1} \Gamma(1-D/2), & (3.16)\\ &= \lambda_b m_b^2 \frac{1}{32\pi^2} \left(\frac{m_b^2}{4\pi\mu^2}\right)^{-\varepsilon} \Gamma(-1+\varepsilon). & (3.17) \end{aligned}$$

Equation (3.12) follows straightforwardly.

3.2 Cavity ϕ^4 Theory

We now consider the same field theory confined to a static spherical cavity, subject to some boundary condition. The Lagrangean, and the configuration space Feynman rules will be the same as in free space.

3.2.1 Renormalization

We may Fourier transform the equation for the two point function (3.3) from (free) momentum space to configuration space. The result is valid for the cavity,

$$G(x,x') = i\Delta(x,x') + \int dy\, dy'\, i\Delta(x,y)[-i\Sigma(y,y')]i\Delta(y',x') + \cdots. \tag{3.18}$$

We transform each function according to

$$G(q,q') = \int dx\, dx' \phi^*(q;x) G(x,x') \phi(q',x'), \tag{3.19}$$

inserting the completeness relation (2.4) where appropriate. The two point function is once again a geometric series,

$$G(q,q') = \frac{i\delta(q,q')}{q^2-m_b^2} + \frac{i}{q^2-m_b^2}[-i\Sigma(q,q')]\frac{i}{q'^2-m_b^2} + \cdots \tag{3.20}$$

For the moment we divide up the self energy into a free space part, and a part due to the boundary,

$$\Sigma(x,y) = \Sigma^0(x,y) + \tilde{\Sigma}(x,y). \tag{3.21}$$

We assume that the boundary term has no additional singularities, and recall the usual form for Σ^0 in planar momentum space

$$\Sigma^0(s^2) = \delta m_b^2 + (s^2 - m^2)[1 - Z_\phi^{-1}] + Z_\phi^{-1}\Sigma_R^0(s^2). \tag{3.22}$$

We may transform this into configuration space, and then into cavity mode space,

$$\Sigma^0(q,q') = \delta m_b^2 \delta(q,q') + (q^2 - m^2)[1 - Z_\phi^{-1}]\delta(q,q') + Z_\phi^{-1}\Sigma_R^0(q,q') \tag{3.23}$$

We have relied on the identity

$$< s^2 - m^2 >=< q^2 - m^2 > \tag{3.24}$$

which is discussed in section (3.2.4). We may now add $\tilde{\Sigma}$ to both sides to get a cavity analogue of the equation defining Σ_R^0, which supplies a definition for the renormalized cavity self-energy, Σ_R,

$$\Sigma(q,q') = \delta m_b^2 \delta(q,q') + (q^2 - m^2)[1 - Z_\phi^{-1}]\delta(q,q') + Z_\phi^{-1}\Sigma_R(q,q') \tag{3.25}$$

We should note that the renormalization point of Z_ϕ is specified by the plane wave prescription, $s^2 = m^2$. Due to time translation invariance all quantities must obey

$$G(q,q') = G(\omega,p,p')\delta(\omega,\omega') \tag{3.26}$$

The geometric series for the two point function may now be summed,

$$G(\omega,p,p') = \frac{iZ_\phi}{\omega^2 - k_p^2 - m^2 - \Sigma_R(\omega,p,p')}, \tag{3.27}$$

yielding a renormalized cavity dressed Green's function.

3.2.2 Regularization

We need a mathematical device that will allow us to tame the singularities. The 'elevation' of denominators is a common trick in both Pauli-Villars and dimensional regularization schemes. It permits messy momentum integrals to be done using standard Gaussian integrals (see Appendix C.2), and replaces these with a single *analytic* variable z. There is also a clear relation between small z and large momentum. In the dimensional regularization scheme the ultraviolet singularities manifest themselves as divergences as $z \to 0$. Our method is based on dimensional regularization, we regularize by varying the analytic variable D, the number of dimensions.

We now consider the process of elevating the denominators in the Feynman integral. In section 3.1.2 we divided out the mass scale from l and m, and elevated the dimensionless quantity

$$\frac{\mu^2}{l^2 + m^2} = \int_0^\infty dz \, \exp\left\{-\frac{zl^2 + zm^2}{\mu^2}\right\}. \tag{3.28}$$

Clearly we can perform an analogous procedure for the cavity propagator, however the cavity result must be in the same units. Cavity quantities have some dimension, for example q^2 has dimensions of mass squared. In order to show the units explicitly, we introduce the dimensionless quantity $\bar{q}$

$$q^2 = \frac{\bar{q}^2}{R^2}. \tag{3.29}$$

Now we must elevate the corresponding dimensionless quantity

$$\frac{R^2\mu^2}{\bar{q}^2 + \bar{m}^2} = \int_0^\infty dz \exp\left\{-\frac{z\bar{q}^2 + z\bar{m}^2}{\mu^2 R^2}\right\}, \tag{3.30}$$

It is natural to choose $\mu R = 1$, so that we may always omit the scale without ambiguity. (However this means that we should use another variable, say ν, to denote the renormalization point, and remember to include a $\log(\nu/\mu)$ where appropriate.)

In this way, instead of the usual procedure where we elevate the planar 4-momentum l^2, we have shown how to elevate the 'cavity 4-momentum' $q^2 = \omega^2 - k_p^2$. While the procedure is analogous, it is clearly mathematically distinct. However we know, from arguments similar to those of section 2.1.1, that at the very least the leading order behaviour of some complicated expression as a function of $z \to 0$ will be the same. Here we use our experience of the multiple reflection expansion, to see that the most singular behaviour always arises in the free space part.

How do we implement this idea? We tackle the simplest possible diagram, that of figure 3.1. Clearly $Z_\phi = 1$, and if we take the limit $m \to 0$ then $\delta m = 0$. (But see the discussion on massless tadpoles in C.4.) What we need is a 'z-form' for the cavity self-energy, which is given by the Feynman rules as

$$-i\Sigma(x,x') = \delta(x,x')\frac{1}{2}\left[-i\lambda_b\mu^{4-D}\right] i\Delta(x,x). \tag{3.31}$$

The delta function is inserted to remind us that the self-energy is usually dependent on two variables, the half comes from the topology of the diagram, then the D-dimensional vertex in brackets, followed by the propagator. Now, transforming to cavity mode space

$$-i\Sigma(q,q') = \tfrac{1}{2}\lambda_b\mu^{4-D}\int d^4x\, d^4x'\, \phi^*(q;x)[\Delta(x,x)\delta(x,x')]\phi(q';x'). \tag{3.32}$$

We now insert the cavity mode expansion for Δ, and absorb the spatial integral into a vertex factor Q (see Appendix B). Since we will only need the result in $D = 4$ dimensions, we drop the mass scale $\mu^{2\epsilon}$.

$$-i\Sigma(q,q') = \tfrac{1}{2}\lambda_b\int d^4x\, \phi^*(q;x)\phi(q';x)\sum_{q''}\frac{|\phi(q'';x)|^2}{q''^2 - m^2 + i0}, \tag{3.33}$$

$$= \tfrac{1}{2}\delta(\omega,\omega')\lambda_b\int\frac{d\omega''}{2\pi}\sum_{n''l''}\frac{Q(p,p',p'')}{\omega''^2 - k_{p''}^2 - m^2}. \tag{3.34}$$

After elevating the denominator as in (3.30), and a Wick rotation, we can perform the ω integral to get

$$\Sigma(\omega,p,p') = \int dz\, \tfrac{1}{2}\lambda_b\frac{1}{\sqrt{4\pi}}\sum_{n''l''}Q(p,p',p'')z^{-1/2}e^{-k_{p''}^2 z - m^2 z}, \tag{3.35}$$

$$= \int dz\, \Sigma(\omega,p,p';z). \tag{3.36}$$

This serves as the definition of the z-form of the cavity self-energy. It is perhaps not obvious from this definition of $\Sigma(\omega, p, p'; z)$ that it should, to leading order be the same as its free space counterpart,

$$\Sigma(\omega, p, p'; z) \sim \Sigma^0(\omega, p, p'; z). \tag{3.37}$$

From equation (3.14) we see that the integral over z in free space is divergent at the origin due to the leading $z^{-D/2}$ behaviour. By looking closely at the sum for the 'z-form' of the cavity self-energy, and the integral for the 'z-form' of the free space self-energy, we see that the sum is an Euler-Maclaurin approximation of the integral. This enables one to see that the leading order in $z \to 0$ will be the same. In any case, as we shall see later, it can be checked on the computer.

From the definition of the renormalized cavity self-energy (3.25), and the definitions (3.5) and (3.7) for δm and Z_ϕ, (we note that $Z_\phi = 1$) we get the simple form for the renormalized self-energy in which we simply subtract out the divergent free space part,

$$\Sigma_R(\omega, p, p') = \Sigma(\omega, p, p') - \Sigma^0(\omega, p, p'). \tag{3.38}$$

Implicitly we have some regularization scheme in mind here, since the right hand side contains two divergent quantities. This is explained in the introduction. Thus we finally have the expression, ready for numerical calculation in $D = 4$ dimensions,

$$\Sigma_R(\omega, p, p') = \int dz\, [\Sigma(\omega, p, p'; z) - \Sigma^0(\omega, p, p'; z)]. \tag{3.39}$$

The two terms each contain $z^{-D/2}$ divergences which cancel. If the difference contains no other non-integrable divergence we may proceed to compute the finite quantity $\Sigma_R(\omega, p, p')$ in $D = 4$ dimensions.

Is $\tilde{\Sigma}$ finite? If so, we expect $\Sigma_R(\omega, p, p')$ to be finite. Can we predict the presence of non-integrable subleading terms in z? Is $< s^2 >=< q^2 >$? This is necessary to define the renormalized cavity self-energy. These questions will be dealt with in the next two sections.

The reader may protest that this looks remarkably similar to the MRE. We have simply subtracted out the free space part. In a sense, while we have not used that particular form of the propagator, this is true. The distinction will become more clear in an example where the free space part has some dynamical content, and is not simply a delta function.

3.2.3 The Boundary : $\tilde{\Delta}$

The next task is to find out whether the boundary term is finite. We calculate a 'z-form' of the boundary piece, B(z). If it is integrable, we conclude that the reflection contribution is finite.

The 'z-form' of the boundary form has a second use. In the introduction we refered to a second conjecture. Schematically it is that to leading order in small z

$$\Sigma^C(z) - \Sigma^0(z) \sim B(z), \tag{3.40}$$

(provided that $B(z)$ is divergent for $z \to 0$, whether or not it is integrable.) The first conjecture was that the leading order behaviour of the cavity self-energy for $z \to 0$, is given by the free space part. This is intuitively appealing. From the MRE we know that

$$\Sigma^C = \Sigma^0 + \tilde{\Sigma}. \tag{3.41}$$

However from the expression for the 'z-form' of the cavity self energy it does not follow that

$$\Sigma^C(z) = \Sigma^0(z) + \tilde{\Sigma}(z). \tag{3.42}$$

because is no separate definition of $\tilde{\Sigma}(z)$. We can however calculate the quantity $B(z)$, which is an guess as to what signal the presence of the surface might give.

Small z corresponds to high momentum, or short distance, so it is not unreasonable to expect the sub-leading z behaviour to be caused by the presence of the boundary. Can we calculate what it should be?

The behaviour of propagators at a curved boundary is rather complicated. In the MRE a closed form for 1,2,..., reflections has been derived, and it is quite possible that this could be extended to D dimensions. However, the analytic work at present seems to be rather intractable, and even the computer time needed for such calculations can be considerable.

Much insight can, however, be gained by studying simpler problems. For this reason we examine the simplest possible system with a boundary condition, scalar field theory in a half-space. We study infinite D dimensional space, with a $D-1$ dimensional plane at $x_1 = 0$. A position is given by $x = \{x_0, x_1, ..., x_D\}$, and we may apply either the Dirichlet or Neumann boundary condition,

$$\phi(x)|_{x_1=0} = 0, \tag{3.43}$$

$$\frac{\partial}{\partial x_1}\phi(x)|_{x_1=0} = 0. \tag{3.44}$$

The same boundary conditions would apply to the propagators. The half space propagator will be the sum of a direct and reflected part,

$$\Delta^h(x,y) = \Delta^0(x,y) + \tilde{\Delta}(x,y). \tag{3.45}$$

For this simple system we may write down the reflection part immediately using the method of images [50],

$$\Delta^h_D(x,y) = \Delta^0(x,y) - \Delta^0(x,y_\perp), \tag{3.46}$$

$$\Delta^h_N(x,y) = \Delta^0(x,y) + \Delta^0(x,y_\perp). \tag{3.47}$$

Here we use the subscripts D, N to denote the boundary condition choice, and the image point for the D dimensional vector $y = \{y_0, y_1, y_2, \ldots, y_{D-1}\}$ is given by $y_\perp = \{y_0, -y_1, y_2, \ldots, y_{D-1}\}$.

We wish to study the effect of the image part on the self energy, similar to equation (3.32), but with a specific interest in the reflection part of the propagator. We ignore the mass scale, (it is the same as Σ), and specialize to the massless case. We will need to know

$$I = \int_0^\infty dx_1\, g(x_1) i\Delta^0(x, x_\perp), \tag{3.48}$$

where the function g contains the external wave functions, and the propagator from the image point is given by

$$i\Delta^0(x, x_\perp) = i\int \frac{d^D s}{(2\pi)^D}\frac{1}{s^2 + i0}\exp\{i2s_1x_1\} \tag{3.49}$$

After a Wick rotation, and elevating the denominator,

$$i\Delta^0(x, x_\perp) = \int_0^\infty dz \int \frac{d^Ds}{(2\pi)^D} \exp\{-s^2 z + i2s_1x_1\}, \tag{3.50}$$

where we now understand s^2 to be the Euclidean D-momentum, not the Minkowski D-momentum. Using the standard integral (C.14),

$$i\Delta^0(x, x_\perp) = \int_0^\infty dz \left(\frac{1}{4\pi z}\right)^{D/2} \exp\left\{-\frac{1}{4}\frac{(2x_1)^2}{z}\right\}. \tag{3.51}$$

In other words, after elevating the denominator in the usual way, we get a standard form for the δ distribution listed in Appendix C.2. For small z, near a boundary, this distribution obeys an identity (C.17), and we may evaluate I as

$$I = \int dz \left(\frac{1}{4\pi z}\right)^{(D-1)/2} \left[\frac{1}{4}g(0) + \frac{1}{4}\sqrt{\frac{z}{\pi}}g'(0) + \frac{z}{16}g''(0) + \cdots\right]. \tag{3.52}$$

In the limit of short distance, or $z \to 0$, the leading order behaviour of the surface in z will be supplied by assuming it behaves like a flat surface of area $A = 4\pi$. The final ingredient will be $g(0)$ or $g''(0)$. In the case of Neumann boundary conditions

$$g(0) = \frac{1}{4\pi} N_p j_l(k_p) \; N_{p'} j_{l'}(k_{p'}), \tag{3.53}$$

and for Dirichlet boundary conditions, where $g(0)$ and $g'(0)$ are zero, the leading term will be

$$g''(0) = \frac{1}{4\pi}2 \; N_p k_p j_l'(k_p) \; N_{p'} k_{p'} j_{l'}'(k_{p'}). \tag{3.54}$$

The factor of $1/4\pi$ comes from the Y_{lm}'s. Thus the leading boundary behaviour in the cavity in $D = 4$ dimensions will be given by the function $B(z)$ given by

$$B_N(z) = z^{-3/2}\left(\frac{1}{4\pi}\right)^{3/2} 4\pi\frac{1}{4}g(0), \tag{3.55}$$

and

$$B_D(z) = -z^{-1/2}\left(\frac{1}{4\pi}\right)^{3/2} 4\pi\frac{1}{16}g''(0). \tag{3.56}$$

Note the change in sign of the Dirichlet case which comes from (3.46). Now we are able to establish the validity of the assumption made previously, namely whether the integral over z for Σ_R in (3.39) is finite. In the case of Dirichlet boundary conditions it is, whereas for Neumann boundary conditions it is not. For Neumann boundary conditions we get a $z^{-3/2}$ divergence that is not integrable.

3.2.4 The Boundary : $< s^2 - m^2 >$

It has become apparent that we need the expectation value

$$< s^2 - m^2 >= \int ds\, \phi^*(q'; s)(s^2 - m^2)\phi(q; s). \tag{3.57}$$

Here we have already assumed a planar momentum representation for the cavity modes (see Appendix A.5), in addition to the usual configuration space representation. Thus we have three different representations for any quantity, $q = \{\omega, p\}$, $s = \{\omega, \vec{k}\}$ or $x = \{t, \vec{r}\}$. The above equation may be translated into configuration space,

$$
\begin{aligned}
< s^2 - m^2 > &= < -\Box - m^2 >, && (3.58)\\
&= \int dx\, \phi^*(q'; x)(-\Box - m^2)\phi(q; x). && (3.59)
\end{aligned}
$$

Now we know that inside the cavity (2.9) holds

$$
(\Box + m^2)\phi(q; x) = (-q^2 + m^2)\phi(q; x), \tag{3.60}
$$

so that one might assume that

$$
< q'| - \Box - m^2|q >= (q^2 - m^2)\delta(q, q'), \tag{3.61}
$$

but on the boundary either the function or its derivative are discontinuous. When we then apply the spatial derivative to the discontinuity, we get a delta function, and this could give a finite or even an infinite result. We note that the time part gives no problems, so to examine the question more closely, we consider only the spatial part, which we call A. We start from the momentum space definition, and calculate the quantity A defined by,

$$
A = \int d\vec{k}\, \phi^*(p', \vec{k})(-k^2 + k_p^2)\phi(p, \vec{k}). \tag{3.62}
$$

If we Fourier transform strictly, by substituting in (2.12), we get

$$
A = \int d\vec{r}\, d\vec{r}'\, \phi^*(p', \vec{r}')\beta(r')\delta(\vec{r}', \vec{r})(\nabla_r^2 + k_p^2)\phi(p, \vec{r})\beta(r). \tag{3.63}
$$

We note that a more casual derivation might not reveal the presence of *two* step functions $\beta(r)$, and how the derivative acts on them. We separate $\phi(\vec{r}) = R(r)Y_{lm}(\hat{r})$, and we can insert the Laplacean in sphericals

$$
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\hat{l}^2}{r^2}, \tag{3.64}
$$

and perform the angular integration, to get

$$
\begin{aligned}
A &= \delta_{m'm}\delta_{l'l} \int_0^\infty r^2 dr\, [R_{p'}(r)\beta(r)] \times \\
&\quad \left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k_p^2 - \frac{l(l+1)}{r^2} \right\} [R_p(r)\beta(r)]. && (3.65)
\end{aligned}
$$

The step function, $\beta(r)$, and its derivatives, are given by

$$
\begin{aligned}
\beta(r) &= 1 - \theta(r - R), && (3.66)\\
\beta'(r) &= -\delta(r - R), && (3.67)\\
\beta''(r) &= -\delta'(r - R). && (3.68)
\end{aligned}
$$

The radius is set to one. Using the derivatives of the step function, and the wave equation, (3.65) becomes

$$\begin{aligned} A &= \delta_{m'm}\delta_{l'l}\int dr\, r^2\, [R_{p'}(r)\beta(r)] \times \\ & [-\frac{2}{R}R_p(r)\delta(r-R) - 2R'_p(r)\delta(r-R) - R_p(r)\delta'(r-R)]. \end{aligned} \tag{3.69}$$

The dangerous part of this equation is the term containing $\delta'\beta$. Some useful δ function identities are listed in Appendix C.2.1. In particular we note that $\int \delta'\theta x^2 = 0$ and $\int \delta'\theta = \infty$, so we can now see that under Dirichlet boundary conditions this quantity is zero, while under Neumann boundary conditions the quantity is either divergent or undefined. Thus we may conclude that

$$< s^2 - m^2 > = < q^2 - m^2 >, \tag{3.70}$$

only in the case of Dirichlet boundary conditions. While it is reassuring to know that all our assumptions necessary for renormalization of the Dirichlet field theory are satisfied, it seems clear that the Neumann field theory is problematic. We do not know of any fundamental reason why this should be so, but the Neumann theory does have some peculiarities (see Appendix D).

In Appendix A.6 we derive the corresponding identity for quarks.

3.2.5 Computation of $\Sigma(z)$

A vital ingredient of our calculation is an accurate result for $\Sigma(\omega, p, p'; z)$, as given by equation (3.35). Clearly we need the eigenenergies and wavefunctions. We generate spherical Bessel functions from either a series or recursion relation. The result is good to 10^{-14} for most arguments, except for a crossover regime that increases in size as the index l increases. This regime is always well below the first root. With a rootfinder we can find the eigenvalues, to a similar accuracy. With a numerical integration routine based on Gaussian quadrature we can calculate the vertex integrals.

As a test of the wavefunctions, normalization constants, and the numerical integration, we can evaluate the orthogonality of normalized wavefunctions. The results are accurate to 10^{-14}.

We can now proceed to calculate $\Sigma(z)$. The error comes mainly from truncating the infinite series. A notable virtue of our scheme is that the series is suppressed by the exponential term $\exp\{-k_p^2 z\}$. In practice we calculate and store all eigenenergies and vertices below some cutoff energy E_{max}, typically chosen between 50/R and 100/R. The units will always be $1/R$, so we will often omit them, i.e. by energy 50 we will mean 50/R (see Appendix A.1). For $E_{max} = 50$ we need the eigenenergies up a maximum of $l = 42$ and $n = 15$. The vertex integrals are of order 1 or smaller, so the error may be estimated from $\exp\{-k_p^2 z\}$. One should keep an eye on this assumption, it is less accurate in the vacuum polarization case. The error clearly increases for small z, so that if, for example, we want 8 figure accuracy, the smallest z that we may expect to be accurate will be z_{min}

$$\exp\{-E_{max}^2 z_{min}\} = 10^{-8}, \tag{3.71}$$

or more usefully,

$$z_{min}(E_{max}) = \frac{20}{E_{max}^2}. \tag{3.72}$$

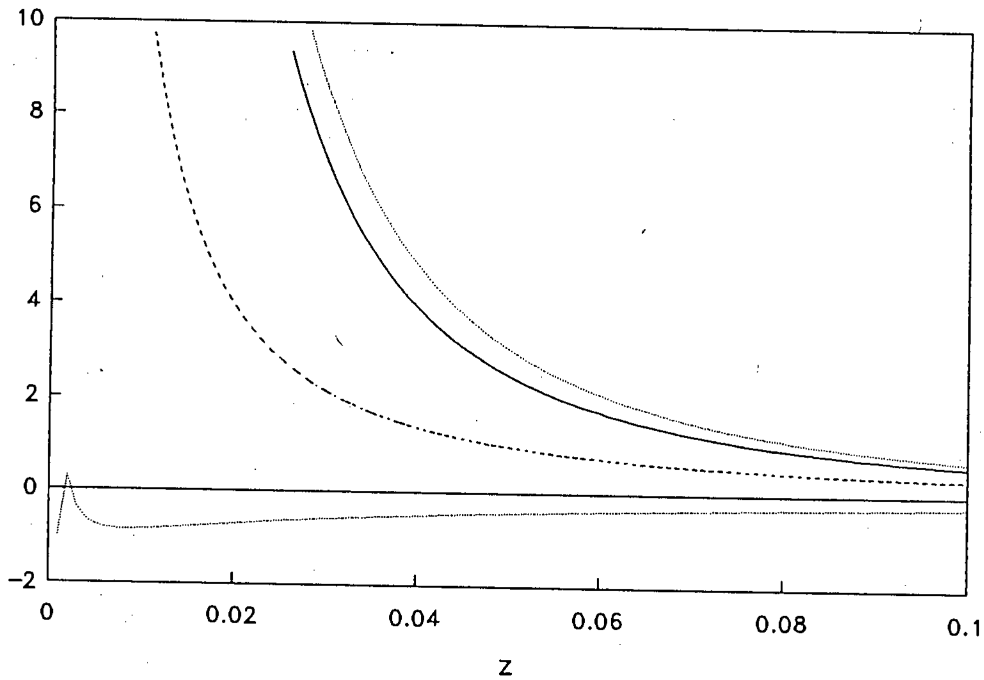

Figure 3.2: Neumann boundary conditions : 1s

If we consider the numerical error we see that $\Sigma(z)$ really depends on our choice E_{max}, so we can write $\Sigma(z; E_{max})$. We can estimate the error in $\Sigma(z; E_{max})$, (caused by the truncation), as

$$e(z; E_{max}) = \Sigma(z; E_{max}) - \Sigma(z; E_{max} - \pi). \tag{3.73}$$

We choose π as it is approximately the interlevel spacing between eigenenergies. In practice we find that $e(z; E_{max})$ blows up sharply (i.e. exponentially) at about $z = z_{min}(E_{max})$. It is a matter of great diagnostic importance that this is the case, it means that we can easily distinguish between at least this source of numeric error, and coding 'bugs', surface singularities, etc. Short distance singularities due to the surface or volume, always show up as powers in $z^{1/2}$. This may be seen by general consideration of the integrals that arise in the calculation of (3.52).

3.2.6 Results

We now turn to the question of a practical evaluation of the self energy. We start by considering the Neumann case. As we have already seen, this is always the problematic case. Firstly, the assumption $< s^2 - m^2 >=< q^2 - m^2 >$ is invalid. Secondly the boundary z dependence is $z^{-3/2}$, so the integral over z with the free space part subtracted out is not finite.

What does $\Sigma(\omega, p, p'; z)$ actually look like? For Neumann boundary conditions, with $E_{max} = 80$, and the cavity mode $p = p' = 1s$, with eigenenergy 4.4934, we show some relevant quantities as a function of z in figure 3.2.

The bold line is the free space contribution, $\Sigma^0(z)$, and the dotted line above it is the cavity contribution, $\Sigma(z)$, both with a z^{-2} divergence. The dashed line shows the boundary contribution, $B_N(z)$, with a $z^{-3/2}$ divergence. If we subtract the free space and leading boundary contribution from the cavity part we see the remainder. Clearly these

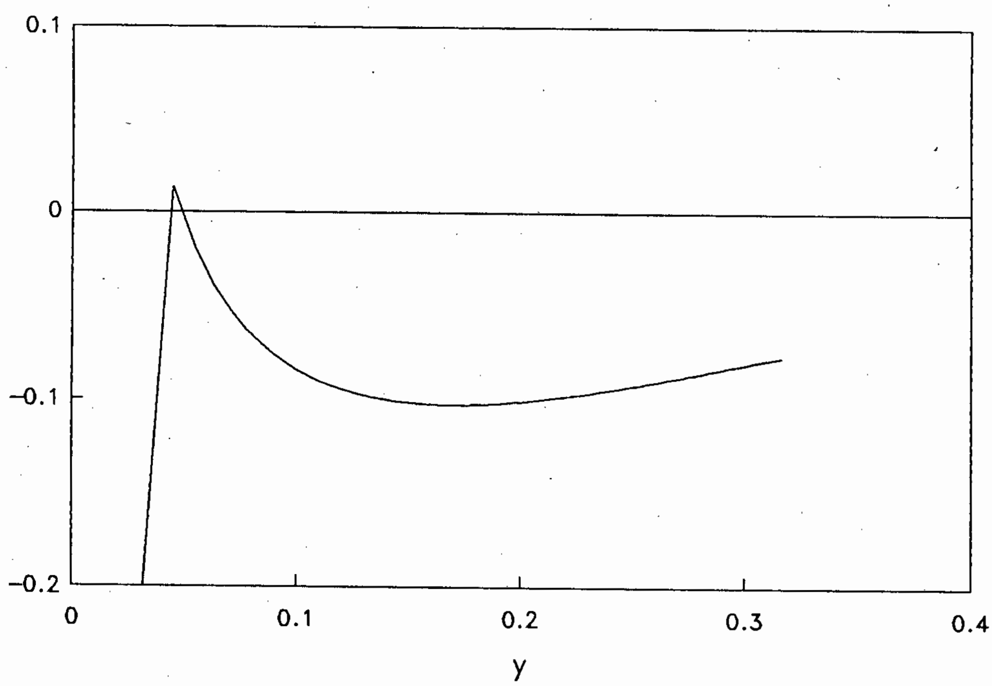

Figure 3.3: Neumann boundary conditions : 1s

two contributions dominate, but we are particularly interested in the next order behaviour. Is the remaining behaviour integrable? Clearly the danger comes from a z^{-1} term. We let

$$z = y^2, \tag{3.74}$$

and plot $y(\Sigma_R(z) - B(z))$ vs y in figure 3.3. A z^{-1} divergence would now be a y^{-1} divergence, whereas a possible $z^{-1/2}$ divergence would be y^0.

As an aside, we may mention that we could associate units with y, by considering $\exp\{-k^2y^2\}$. y would have units of R, so we could associate a function that depends on y as being related to behaviour of the theory at length scale yR.

The figure shows that there is a z^{-1} divergence present. The $z^{-3/2}$ divergence is not integrable, but we could interpret it as $\Gamma(-1/2)$, and therefore finite. The z^{-1} divergence is clearly a new infinite quantity, and we cannot offer any interpretation of it. The figure also shows the error quite well, at $y \sim 0.05$ we see the error cut in abruptly. This corresponds to $z \sim 0.0025$, which is to be compared with $z_{min} = 0.0031$.

In the next figure, 3.4, we consider the Dirichlet case. Once again we consider the $1s$ state, with eigenenergy π. Once again the bold line shows the free space part $\Sigma^0(z)$, and the dotted line slightly below shows the cavity self-energy, $\Sigma(z)$. The boundary part, $B_D(z)$, shown by the dashed line, diverges much less strongly, $B_D(z) \sim z^{-1/2}$, and the difference is well under control, $\Sigma_R(z) - B(z) \sim z^0$.

If we dont know the function down to zero, it may be asked how can we exclude the possibility of some weak $\log z$ or $1/z$ divergence. Clearly one can never exclude this possibility beneath the range of one's numerical error. However for a reasonable strength (~ 1), a good feel is obtained from a suitably blown up plot of $y^n(\Sigma(y) - \Sigma^0(z) - B(z))$ where n is some appropriate power. This should qualify the statement that $\Sigma_R(z) - B(z) \sim z^0$.

To conclude the calculation we need to integrate $\Sigma_R(z)$ to get Σ_R. It is not possible to calculate the function all the way down to $z = 0$, but the function can easily be

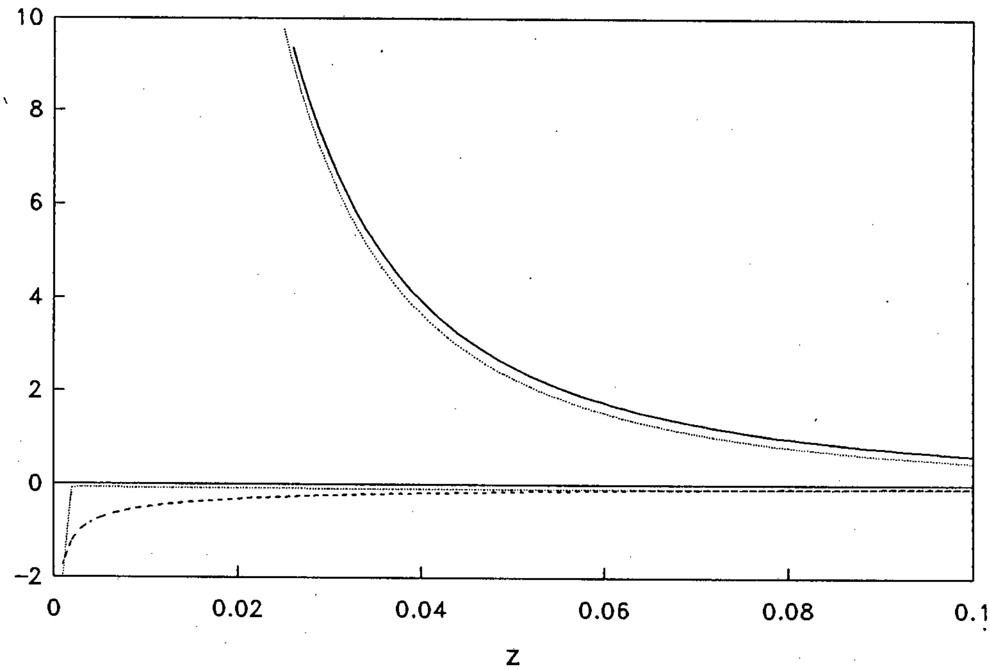

Figure 3.4: Dirichlet boundary conditions : 1s

Level	Energy	Self-energy
1s	3.1416	-0.07223
2s	6.2832	-0.13767
1p	4.4934	-0.10685
1d	5.7635	-0.14254

Table 3.1: Scalar self-energies : Dirichlet case

extrapolated, in order to complete the integral. Finally in table 3.1 we present a few results. The self-energies are rather small relative to the eigenenergies, and show that the self-energy *lowers* the energy of the mode.

Chapter 4

The Quark Self-Energy

In this chapter we tackle the problem of the quark self-energy. We begin by considering the free space case, renormalization of the one loop propagator, and regularizing the order α_s diagram. We then consider a subtraction that allows us to reduce the finite part of the self-energy to a form appropriate for numerical computation on a computer.

Next we renormalize the cavity quark propagator, regularize and present results. Finally we briefly examine the series form of the self-energy, and its convergence.

4.1 The Free Quark Self-Energy

4.1.1 Renormalization

The two point function, $G(\not{p})$, is given by the sum over the irreducible self-energy insertions

$$G(\not{p}) = \frac{i}{\not{p} - m_b} + \frac{i}{\not{p} - m_b}[-i\Sigma^0(\not{p})]\frac{i}{\not{p} - m_b} + \cdots \tag{4.1}$$

$$= \frac{i}{\not{p} - m_b - \Sigma(\not{p})}. \tag{4.2}$$

The mass appearing in the propagator is the bare mass m_b. In an on-shell renormalization scheme, we usually consider the Taylor series of the self-energy, expanded around $\not{p} = m$, where m is the physical mass,

$$\Sigma^0(\not{p}) = \Sigma^0(m) + (\not{p} - m)\frac{\partial}{\partial \not{p}}\Sigma^0(\not{p})|_{\not{p}=m} + \text{higher order terms.} \tag{4.3}$$

The first two terms contain divergent parts, and the remainder is finite. This is used to define the quantities δm, Z_2^{-1} and Σ_R^0,

$$\Sigma^0(\not{p}) = \delta m - (Z_2^{-1} - 1)(\not{p} - m) + Z_2^{-1}\Sigma_R^0(\not{p}). \tag{4.4}$$

With these definitions and the relationship between the bare and renormalized mass, m_b and m,

$$m = m_b + \delta m, \tag{4.5}$$

the free space two point function becomes,

$$\frac{1}{\not{p} - m_b - \Sigma^0(\not{p})} = \frac{Z_2}{\not{p} - m - \Sigma_R^0(\not{p})}. \tag{4.6}$$

4.1.2 Regularization

We now consider the explicit form for the quark self-energy in free space. From the D-dimensional Feynman rules in configuration space we have the irreducible free space self-energy insertion Σ^0,

$$-i\Sigma^0(x,y) = C(-ig\mu^\varepsilon\gamma_\alpha)iS^B(x,y)(-ig\mu^\varepsilon\gamma_\beta)iD^{\alpha\beta}(x,y). \tag{4.7}$$

The factor $C = 4/3$ takes account of the colour matrices. In momentum space this becomes

$$-i\Sigma^0(\not{s}) = C\int\frac{d^Dl}{(2\pi)^D}(-ig\mu^\varepsilon\gamma_\alpha)iS^B(\not{l}+\not{s})(-ig\mu^\varepsilon\gamma_\beta)iD^{\alpha\beta}(l), \tag{4.8}$$

or, in the Feynman gauge,

$$\Sigma^0(\not{s}) = Cg^2\mu^{2\varepsilon}\int\frac{d^Dl}{(2\pi)^D}\,\frac{\gamma_\alpha(\not{s}+\not{l}+m)\gamma^\alpha}{(l)^2[(l+s)^2-m^2]}. \tag{4.9}$$

We specialize to the case of massless quarks, $m_b = 0$ and $m = 0$. From the Feynman integrals in section C.4.1 we get the result

$$\Sigma^0(\not{s}) = -C\,\not{s}\frac{\alpha_s}{4\pi}\left[\frac{1}{\varepsilon}-\gamma+1-\log\left(\frac{-s^2}{4\pi\mu^2}\right)\right]. \tag{4.10}$$

4.1.3 A Subtraction

So far we have examined standard results. In this section we examine the method which leads to this result in more detail, and develop a subtraction that may be transferred to the cavity theory. In dimensional regularization, the singularities always arise as poles in ε, where space-time is $D = 4-2\varepsilon$ dimensional. More specifically, we encounter the gamma function,

$$\Gamma(w+1) = \int_0^\infty dz\, e^{-z}z^w. \tag{4.11}$$

The pole in linearly or logarithmically divergent integrals usually arises as $\Gamma(\varepsilon)$. We now introduce a trick to evaluate integrals of the above kind. To illustrate the trick, we consider as an example the integral

$$X = \int_0^\infty dzX(z) = \int_0^\infty dz\, z^{-1+\varepsilon}e^{-az} = a^{-\varepsilon}\Gamma(\varepsilon) = \frac{1}{\varepsilon}-\gamma-\log a+O(\varepsilon). \tag{4.12}$$

This is a typical expression encountered in dimensional regularization. Usually the physics is contained in the '$\log a$' part. We can separate $X(z)$ into two parts such that

$$X(z) = S(z)+F(z), \tag{4.13}$$

and S(z) and F(z) are given by

$$S(z) = z^{-1+\varepsilon}e^{-z}, \tag{4.14}$$

$$F(z) = z^{-1+\varepsilon}(e^{-az}-e^{-z}). \tag{4.15}$$

If we now perform the integral

$$S = \int_0^\infty S(z) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon), \tag{4.16}$$

and

$$F = \int_0^\infty F(z) = -\log a + O(\varepsilon). \tag{4.17}$$

The notation is intended to suggest that S contains the singularity and F contains the finite (and physically meaningful) part. S depends on the dimension D or ε and is singular when $D = 4$. The interesting thing is that F is well-defined in $D = 4$ dimensions and is *cut-off independent*. From here on we ignore terms order ε, since we will be interested in the limit $\varepsilon \to 0$. An important caveat to this method, is that we should take care that S is chosen correctly, *including* multiplicative factors $(1-\varepsilon)$, or x^ε, otherwise we may loose a finite contribution. If we do not take sufficient care with this our result may be wrong. This is a feature of dimensional regularization more generally. The free space dimensional regularization scheme is not unique, since we may always choose to multiply by a function that is one for integer dimensions D. However this produces finite contributions that are absorbed by counterterms, and don't affect the final result.

How then do we manipulate our integrals into a form suitable for this kind of subtraction? To demonstrate, we consider a typical divergent integral which crops up in loop diagrams. In this section we do a more explicit derivation of a particular Feynman integral; this and other integrals are listed with a short derivation in Appendix C.4.1. We proceed using standard methods of dimensional regularization [35,34]. Space-time is $D = 4 - 2\varepsilon$ dimensional, and q and p are momenta in free space. μ is a mass scale included to make A dimensionless. Consider the expression

$$A(p) = \mu^{2\varepsilon} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 (q-p)^2}. \tag{4.18}$$

We convert to Euclidean space, and then elevate the propagator denominators to exponential factors

$$\frac{\mu^2}{q^2} = \int_0^\infty dt_1 e^{-q^2 t_1/\mu^2}. \tag{4.19}$$

$$\frac{\mu^2}{(q-p)^2} = \int_0^\infty dt_2 e^{-(q-p)^2 t_2/\mu^2}. \tag{4.20}$$

After changing the order of integration the integral over q is finite, and may be done analytically, by a shift of variables. We thus arrive at

$$A(p) = i \int dt_1 dt_2 [4\pi(t_1+t_2)]^{-D/2} \exp\left\{-\frac{p^2}{\mu^2}\frac{t_1 t_2}{t_1+t_2}\right\}. \tag{4.21}$$

With a change of variables $t_1 = zt$ and $t_2 = z(1-t)$ we get

$$A(p) = \frac{i}{(4\pi)^{D/2}} \int_0^\infty dz \int_0^1 dt\, z^{1-D/2} \exp\left\{-\frac{p^2}{\mu^2} z t(1-t)\right\}. \tag{4.22}$$

We now define $A(p,z)$ by the integrand of

$$A(p) = \int_0^\infty dz A(p,z). \tag{4.23}$$

When we have a quantity in this form, namely an integral over the variable z, we may do a similar subtraction to that of equation (4.13). The important point is that the ultraviolet divergences show up as divergences as $z \to 0$. $A(p,z)$ diverges at the origin and the leading order is a gamma function term $\Gamma(\varepsilon) = 1/\varepsilon - \gamma$. The final result in Minkowski space is

$$A(p) = \frac{i}{(4\pi)^2}\left[\frac{1}{\varepsilon} - \gamma + 2 - \log\left(\frac{-p^2}{4\pi\mu^2}\right)\right]. \tag{4.24}$$

We also note that the singular part does not depend on the momentum. In other words it is a delta function in configuration space.

One may now straightforwardly generate the 'z-form' of $\Sigma(\not p)$ as we have done for $A(p)$. The 'z-forms' of the standard Feynman integrals that we will need are listed in Appendix C.4.1, and we get in Euclidean space

$$\Sigma^0(z,\not p) = -C\frac{\alpha_s}{4\pi}\not p\left(\frac{1}{4\pi}\right)^{-\varepsilon} 2(1-\varepsilon)\int_0^1 dt\, tz^{-1+\varepsilon}\exp\left\{-\frac{p^2}{\mu^2}zt(1-t)\right\}. \tag{4.25}$$

We then split

$$\Sigma^0(z,\not p) = S(z,\not p) + F(z,\not p). \tag{4.26}$$

By inspection the singular part is given by

$$S(\not p,z) = -C\frac{\alpha_s}{4\pi}\not p\left(\frac{1}{4\pi}\right)^{-\varepsilon}(1-\varepsilon)z^{-1+\varepsilon}e^{-z}, \tag{4.27}$$

which yields the final result

$$S(\not p) = -C\frac{\alpha_s}{4\pi}\not p\left[\frac{1}{\varepsilon} - \gamma - 1 - \log\frac{1}{4\pi}\right], \tag{4.28}$$

To evaluate the finite part we use

$$F(\not p,z) = \Sigma^0(\not p,z) - S(\not p,z), \tag{4.29}$$

and

$$F(\not p) = \int_0^\infty F(\not p,z)dz. \tag{4.30}$$

and we may now work in $D = 4$ dimensions. The singular part in $D = 4$ dimensions is

$$S(\not p,z) = -C\frac{\alpha_s}{4\pi}\not p\frac{e^{-z}}{z}. \tag{4.31}$$

This yields the correct result for the finite part of the quark self-energy

$$F(\not p) = -C\frac{\alpha_s}{4\pi}\not p\left[2 + \log\left(\frac{-p^2}{\mu^2}\right)\right], \tag{4.32}$$

and from (4.4) we get to lowest order in α_s

$$\Sigma_R^0(\not p) = F(\not p) + 2C\frac{\alpha_s}{4\pi}\not p. \tag{4.33}$$

4.2 The Cavity Quark Self-Energy

4.2.1 Renormalization

In this section we develop a formalism with which we may renormalize the quark propagator in the cavity. The question is, can an analagous procedure to that of section 4.1.1 be performed in the cavity? The cavity self-energy consists of a direct part, and a part in which either the quark or the gluon undergo at least one reflection. This multiple reflection scheme (MRS) was introduced by Hansson and Jaffe in ref. [4] where they show that the part containing the reflections is finite and absolutely convergent. We may write this as

$$\Sigma^C = \Sigma^0 + \tilde{\Sigma}, \tag{4.34}$$

where the finite term $\tilde{\Sigma}$ contains all the reflections.

We now proceed to renormalize the quark propagator in the cavity. Our starting point is the configuration space Feynman rules. Iterating the irreducible self-energy $\Sigma^C(x,y)$, the dressed quark propagator G can be expanded in terms of the bare propagator S^B,

$$G(x,y) = iS^B(x,y) + \int d^4x'd^4y' iS^B(x,x')[-i\Sigma^C(x',y')]iS^B(y',y) + \cdots \tag{4.35}$$

The bare propagator satisfies an equation containing the bare mass m_b,

$$(i\partial\!\!\!/_x - m_b + i0)S^B(x,x') = \delta^4(x,x'). \tag{4.36}$$

If we Fourier transform equation (4.35) using

$$S(q,q') = \int d^4xd^4y\, \bar{\psi}(q;x)\gamma^0 S(x,y)\gamma^0\psi(q';y), \tag{4.37}$$

$$\Sigma^C(q,q') = \int d^4xd^4y\, \bar{\psi}(q;x)\Sigma^C(x,y)\psi(q';y), \tag{4.38}$$

and the orthonormality relation,

$$\delta(q,q') = \int d^4xd^4y\, \bar{\psi}(q;x)\delta(x,y)\gamma^0\psi(q';y), \tag{4.39}$$

we arrive at an expansion for the dressed propagator in 'q-space',

$$-iG(q,q') = S^B(q,q') + \sum_{q_1q_2} S^B(q,q_1)\Sigma^C(q_1,q_2)S^B(q_2,q') + \cdots \tag{4.40}$$

We wish to renormalize the mass. We firstly note that our Fourier transformation equation (4.37) for S^B contains two masses, the bare propagator contains the bare mass m_b, whereas the Fourier set incorporates cavity modes that we choose to correspond to the physical mass m. What does S^B look like in this set? It must obey the equation (4.36). If we transform both sides of the equation by the usual Fourier transform (4.39), and insert the standard set of complete states (2.32) in front of S we arrive at

$$\sum_{q'} O(q,q')S^B(q',q'') = \delta(q,q''), \tag{4.41}$$

where $O(q,q')$ is given by

$$O(q,q') = \int d^4x\, \bar{\psi}(q;x)(i\partial\!\!\!/ - m_b)\psi(q';x). \tag{4.42}$$

Due to time translation invariance G, S^B, O and Σ all have delta functions in the ω variable. If X stands for one of these functions, we will often refer to

$$X(q,q') = \delta(\omega,\omega')X(\omega,p,p'). \tag{4.43}$$

If we remove the delta functions in (4.42), we can write the matrix equation

$$\sum_{p'}(\omega - \varepsilon(p,p'))S^B(\omega,p',p'') = \delta(p,p''). \tag{4.44}$$

We can now write the propagator as

$$S^B(\omega,p,p') = [\omega - \varepsilon(p,p')]^{-1}, \tag{4.45}$$

where we imply matrix inversion in the discrete labels p, p', and the operator $O(q,q')$ has been simplified by defining a quantity like an energy, but which is not diagonal,

$$\varepsilon(p,p') = \int d\vec{r}\, \bar{u}(p;\vec{r})[-i\vec{\gamma}.\vec{\nabla}_r + m_b]u(p';\vec{r}). \tag{4.46}$$

We note the presence of two different masses, one (m) attached to the cavity mode, and the other (m_b) appearing in the bare propagator. If we insert this form for S^B into (4.40) we get

$$\begin{aligned} -iG(\omega,p,p') &= [\omega - \varepsilon(p,p')]^{-1} + \\ &\quad [\omega - \varepsilon(p,p_1)]^{-1}\Sigma^C(\omega,p_1,p_2)[\omega - \varepsilon(p_2,p')]^{-1} + \cdots \end{aligned} \tag{4.47}$$

This sum can be performed yielding

$$-iG(\omega,p,p') = [\omega - \varepsilon(p,p') - \Sigma^C(\omega,p,p')]^{-1}. \tag{4.48}$$

We now consider the equation that defines the renormalized self-energy, (4.4), in more detail. In order to convert into a cavity form, we need an identity that was shown in [4]. The identity is the Dirac analogue of $< s^2 - m^2 > = < q^2 - m^2 >$ for scalar fields, and we give a derivation in Appendix A.6. It does not follow automatically from the wave equation (2.39), because this equation is not satisfied on the surface. The identity is

$$\int dx\, \bar{\psi}(q';x)(i\partial\!\!\!/_x - m)\psi(q;x) = (\omega - \varepsilon_p)\delta(q',q). \tag{4.49}$$

We can transform equation (4.4) into configuration space, and then into cavity Fourier space, yielding

$$\Sigma^0(\omega,p,p') = \delta m(p,p') - (Z_2^{-1} - 1)(\omega - \varepsilon_p)\delta(p,p') + Z_2^{-1}\Sigma^0_R(\omega,p,p'). \tag{4.50}$$

By adding the finite quantity $\tilde{\Sigma}$ to both sides we can define Σ^C_R,

$$\Sigma^C(\omega,p,p') = \delta m(p,p') - (Z_2^{-1} - 1)(\omega - \varepsilon_p)\delta(p,p') + Z_2^{-1}\Sigma^C_R(\omega,p,p'). \tag{4.51}$$

Since $m = m_b + \delta m$, we note that

$$\varepsilon_p \delta(p,p') = \varepsilon(p,p') + \delta m(p,p'), \tag{4.52}$$

where ε_p is the eigenenergy of a quark of mass m.

Finally we obtain the dressed cavity propagator

$$-iG(\omega,p,p') = Z_2[\omega - \varepsilon_p \delta(p,p') - \Sigma_R^C(\omega,p,p')]^{-1}, \tag{4.53}$$

The denominator of this expression is a finite matrix. We note here that if the renormalization scheme were off mass shell $\varepsilon(p,p')$ would not be diagonal.

How do we extract useful information from this dressed propagator? The energies of the particles are given by the poles of the propagator, i.e. the zeros of the matrix in the denominator. Schematically we would then have to diagonalize the following matrix

$$\begin{pmatrix} \varepsilon_1 + \Sigma_{11} - \omega & \Sigma_{12} & \Sigma_{13} & \\ \Sigma_{21} & \varepsilon_2 + \Sigma_{22} - \omega & \Sigma_{23} & \cdots \\ \Sigma_{31} & \Sigma_{32} & \varepsilon_3 + \Sigma_{33} - \omega & \\ & \vdots & & \end{pmatrix}, \tag{4.54}$$

where Σ_{ij} is shorthand for $\Sigma_R^C(\omega, p_i, p_j)$. We note that the matrix elements depend on ω. At this point it is not surprising to see that if we ignore all the off-diagonal terms and take $\omega = \varepsilon_p$ in Σ_R^C, the energy shift reduces to the usual perturbation theory result.

The existence of *off diagonal* self-energy terms in (4.54) may be surprising. It means that a $1s_{1/2}$ quark may self interact and become a $2s_{1/2}$ quark. This violates no conservation laws, although in one of the two states the quark will be off-shell. The implications of this would be interesting to investigate.

4.2.2 The Cavity Subtraction

We need to make the subtraction in the cavity representation, so we Fourier transform $S(\not p; z)$,

$$S(\not p; z) \to S(x,y;z) \to S(\omega,p,p';z). \tag{4.55}$$

Using once again using (4.49) we get

$$S(\omega,p,p';z) = -C\frac{\alpha_s}{4\pi}(\omega - \varepsilon_p)\delta(p,p')\frac{e^{-z}}{z}, \tag{4.56}$$

which gives

$$S(\omega,p,p') = -C\frac{\alpha_s}{4\pi}(\omega - \varepsilon_p)\delta(p,p')\,[\frac{1}{\varepsilon} - \gamma - 1]. \tag{4.57}$$

By comparison with 4.51, we see that for the massless case, we have

$$-(Z_2^{-1} - 1)(\omega - \varepsilon_p) = -C\frac{\alpha_s}{4\pi}(\omega - \varepsilon_p)\delta(p,p')\,[\frac{1}{\varepsilon} - \gamma + 1], \tag{4.58}$$

and, recalling equation (4.33), we get

$$\Sigma_R^C(\omega,p,p') = 2C\frac{\alpha_s}{4\pi}(\omega - \varepsilon_p) + \int_0^\infty dz\,[\Sigma^C(\omega,p,p';z) - S(\omega,p,p';z)]. \tag{4.59}$$

The latter equation may be used for a numerical calculation.

4.2.3 Calculation

The necessary ingredient now is simply $\Sigma^C(z)$. From (4.38) and (4.43) we obtain

$$\delta(\omega,\omega')\Sigma^C(\omega,p,p';z) = \int d^4x d^4x' \; \bar{\psi}(q;x)\Sigma^C(x,x';z)\psi(q';x'), \tag{4.60}$$

where $\Sigma^C(x,y)$, the irreducible self-energy insertion, is given by the Feynman rules in configuration space. The expression is identical to (4.7), except that we insert the cavity propagators (2.43) and (2.53). Using the concise notation for the spatial overlap integrals of (B.10), we arrive at

$$\begin{aligned} \Sigma^C(\omega,p,p') &= ig^2 C \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \sum_{p_1 p_2 \Sigma_2} g^{\Sigma_2\Sigma_2} Q^{\Sigma_2 p_2}_{pp_1} \tilde{Q}^{\Sigma_2 p_2}_{p_1 p'} \times \\ & \frac{\omega - \omega_2 + \varepsilon_{p_1}}{[(\omega-\omega_2)^2 - \varepsilon^2_{p_1} + i0]} \; \frac{1}{[(\omega_2)^2 - \Omega^2_{p_2} + i0]}. \end{aligned} \tag{4.61}$$

Σ_2 and p_2 label the intermediate gluon, and p_1 the intermediate quark. If we simply go ahead and do the ω_2 integral, we get a series form for the self-energy,

$$\Sigma^C(\omega,p,p') = \alpha_s C \sum_{p_1 p_2 \Sigma_2} M(p_1,p_2,\Sigma_2) \frac{1}{2\Omega_{p_2}[\omega - sgn(p_1)\Omega_{p_2} - \varepsilon_{p_1}]}. \tag{4.62}$$

Clearly we may evaluate this quantity for both on- and off-shell values of the parameter ω, and the reader will notice the poles in ω. At the poles the Feynman prescription becomes important and the denominator should be replaced by

$$\begin{aligned} \frac{1}{[\omega - sgn(p_1)\Omega_{p_2} - \varepsilon_{p_1}]} &\rightarrow \frac{1}{[\omega - sgn(p_1)\Omega_{p_2} - \varepsilon_{p_1} + i\varepsilon]} = \\ P.V. \frac{1}{[\omega - sgn(p_1)\Omega_{p_2} - \varepsilon_{p_1}]} &+ i\pi\delta(\omega - sgn(p_1)\Omega_{p_2} - \varepsilon_{p_1}). \end{aligned} \tag{4.63}$$

We recall that the free space self-energy contains a $\log(-s^2)$ term, which has an imaginary part for $s^2 > 0$. This corresponds to a probability of decay into a quark and gluon. Here we see that the cavity version has a similar imaginary piece. We note that technically an onshell quark can never decay, since the eigenenergies will never satisfy the delta function, however physically there will be some width, and the likelihood of decay will depend on the density of delta functions in that energy region. All this is as expected, since we expect to recover the free space theory in the limit $R \rightarrow \infty$.

The factor M contains the vertex integrals

$$M(p_1,p_2,\Sigma_2) = \sum_{\mu_2 M_2} 4\pi g^{\Sigma_2\Sigma_2} Q^{\Sigma_2 p_2}_{pp_1} \tilde{Q}^{\Sigma_2 p_2}_{p_1 p'}. \tag{4.64}$$

In section 4.2.5 we will discuss the series in a little more detail, however what we need for our purposes is a 'z-form'. The denominators can be elevated in a similar way to (4.19). We thus arrive at

$$\begin{aligned} \Sigma^C(\omega,p,p') &= \int_0^{\infty} dz \sum_{p_1 p_2 \Sigma_2} \alpha_s C\, M(p_1,p_2,\Sigma_2) K(p_1,p_2,\Sigma_2;z), \qquad (4.65) \\ &= \int_0^{\infty} dz\, \Sigma^C(\omega,p,p';z) \qquad (4.66) \end{aligned}$$

a form that is suitable for numerical evaluation on a computer. The factor M is fortunately z independent, and may be calculated once and stored. The vertex integrals are discussed in more detail in section B.3, and an explicit form for M is given. After elevating the denominators, we can perform the $d\omega_2$ integration, and the resulting factor K is z dependent, and depends only on the energies,

$$K(p_1, p_2, \Sigma_2; z) = i^2 z^{1/2} \int_0^1 dt \frac{1}{\sqrt{4\pi}} (\omega t + \varepsilon_{p_1}) e^{z[\omega^2 t(1-t) - \varepsilon_{p_1}^2 (1-t) - \Omega_{\Sigma_2 p_2}^2 t]}. \tag{4.67}$$

As can be seen the above expression is problematic for $|\omega| > (\Omega_{\Sigma_2 p_2} + |\varepsilon_{p_1}|)$. If this is the case it blows up for large z. This corresponds to states into which it is possible for the quark with off-shell energy ω to decay. We simply exclude this finite set from (4.65) and use the form (4.62), adding the two together afterwards. This does no damage to the $z \to 0$ behaviour that we expect, since each term goes as $z^{1/2}$. Only the infinite sum for $\Sigma(z)$ has z^{-1} behaviour, like the S singular part which must be subtracted.

4.2.4 Results

Before quoting the results we want to make a few comments about the computation of Σ^C, with particular reference to accuracy. We need firstly the eigenenergies ε_p. With accurate numerical spherical Bessel functions, these can be calculated to an accuracy of one part in 10^{14}. We need the radial part of the vertex functions, Q, which can also be numerically integrated to a similar accuracy. (This can be verified by checking their normalization.) The radial vertex integrals are calculated once and stored.

The expression for Σ^C is an infinite series, and truncating this series is the major source of error. Fortunately it is suppressed by the exponential factor K. The series is cut off when either ε or Ω exceeds some energy E_{max}. An estimate of the error can be made by comparing the result for Σ^C for some E_{max} and $E_{max} + \pi$. π is chosen because it is approximately the spacing between energy levels in the cavity. The error is vanishingly small for $z \sim 1$ and blows up in the region of some z_{min} given by

$$z_{min} E_{max}^2 \sim 1. \tag{4.68}$$

A major advantage of this method are the built in checks. The vertex factors obey a sum rule (see Appendix B.3.1). The sum rule converges rather slowly, so it functions more as a qualitative check, rather than an accuracy check. One may numerically integrate K from 0 to ∞ to get the denominator term in equation 4.62. The calculation of K involves Dawson or Erf functions, and particular care must be taken with over/underflow problems or subtraction errors, but with suitable care we get accuracy of 10^{-14}, as confirmed by the z integral. The leading part of Σ^C is clearly given by $S(z)$. Together these form an almost infallible check.

While the integral of Σ_R^C is finite, the function $\Sigma_R^C(z)$ diverges like $z^{-1/2}$ for small z. It is convenient to rather express it using a change of variable, $z = y^2$,

$$\int_0^\infty dz \Sigma(z) = 2 \int_0^\infty dy\, y\Sigma(y). \tag{4.69}$$

The function $y\Sigma_R^C(y)$ has the attractive feature that it is regular at $y = 0$. Because of (4.68) we do not have access to $\Sigma^C(y)$ below a certain y. To compute (4.69) we fit a

Table 4.1: Self-energies for low-lying states

Cavity mode	Eigenenergy	Self-Energy
$1s_{1/2}$	2.04279	0.91191 ± 0.00000
$2s_{1/2}$	5.39602	1.90261 ± 0.00010
$1p_{1/2}$	3.81154	1.50305 ± 0.00002
$1p_{3/2}$	3.20392	1.75380 ± 0.00000
$1d_{3/2}$	5.12311	2.03445 ± 0.00010
$1d_{5/2}$	4.32730	2.36113 ± 0.00002

polynomial in y through points sufficiently above $y^2_{min} = z_{min}$ to be accurate, and use this to estimate the 'missing' integral. This crude method works remarkably well. The rest is integrated directly, and the error, which comes mainly from the 'missing' part, may be estimated by changing the order of the polynomial. In this fashion we can get up to 6 figure accuracy, for $E_{max} \sim 40$. The computing time for so large a set of vertex integrals is non-negligable, $\sim$ one hour on a modern minicomputer.

Hitherto the most reliable figure for the quark self-energy has been due to Goldhaber, Hansson and Jaffe [5,6]. Their result for the self-energy of a massless quark in the lowest cavity mode is

$$\Sigma^C_R(\omega = \varepsilon_p, p = p' = 1s_{1/2}) = 0.910 \pm 0.001\ \alpha_s/R. \tag{4.70}$$

We have inserted explicitly the cavity radius which carries dimensions of inverse energy. The analytical work involved seems to have deterred them from extending their calculation to any of the excited states. We present our results in Table 4.1. The value for the $1s_{1/2}$ state is slightly outside the errors quoted by Goldhaber, but has a much higher accuracy.

Before closing this section we should make a few comments on conventions. The values listed in table 4.1 are calculated in the Feynman gauge. There is an ambiguity in the Coulomb part of the cavity propagator that is discussed in detail in Appendix D. The self-energy calculated by workers in the Coulomb gauge is different due to this ambiguity. (Thus far cavity calculations have only been done in these two gauges.)

A convenient quantity which is independent of the scheme used is half the spin independent energy shift of a colour singlet, and we choose a $q\bar{q}$ colour singlet made up of states in the same cavity mode. In other words we drop the hyperfine splitting, or one (physical) gluon exchange diagram. The Coulomb energies of the $q\bar{q}$ pair we denote by E_c Then the invariant quantity will be $\Delta E(p)$ where

$$\Delta E(p) = \Sigma(p) + E_c(p)/2. \tag{4.71}$$

If the hyperfine splitting is given by E_{hf}, then including the dimension, the energy of the $q\bar{q}$ pair to order α_s will be

$$E(p) = 2\varepsilon_p/R + 2\alpha_s\Delta E(p)/R + \alpha E_{hf}(p)/R. \tag{4.72}$$

For an example of these different conventions, in the Feynman gauge as we have seen we get for the self-energy $\Sigma(1s_{1/2}) = 0.9119$. Thus we get $\Delta E(p) = 0.9054$ as given by Goldhaber et al [6], and using the table in Appendix D, can then convert to the Coulomb gauge, with the result now in the convention of Baacke [43], $\Sigma(1s_{1/2}) = 1.7576$.

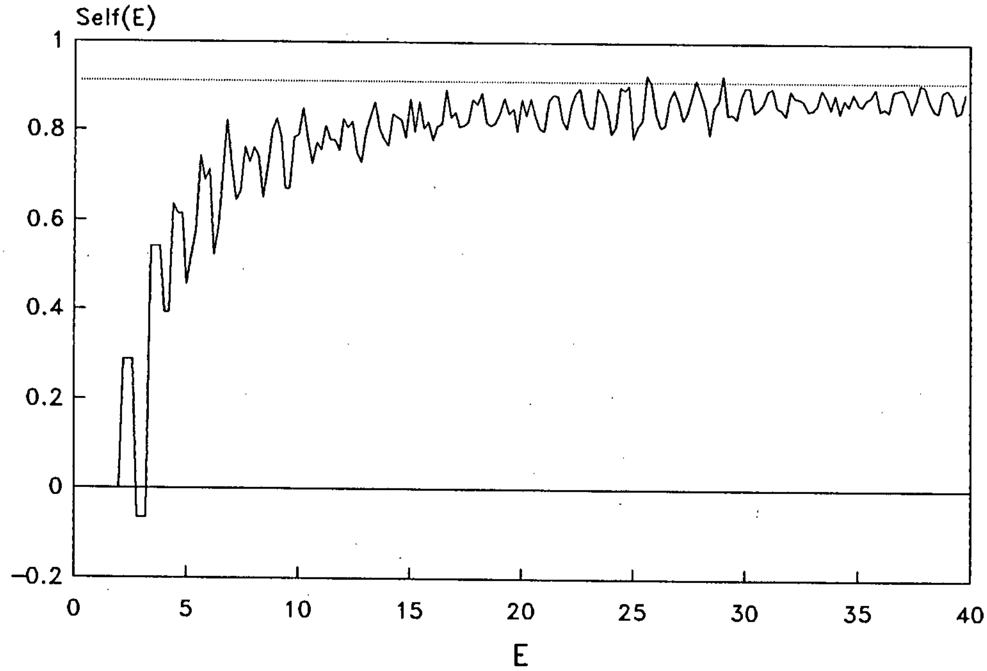

Figure 4.1: Self(E) vs E

4.2.5 Properties of the series

To deepen our understanding of the problem, and to relate our method to existing literature, we include a section on the properties of the series expression for the self-energy. It is an infinite sum over four separate indices, the quark radial and Dirac quantum numbers (from $-\infty$ to ∞), and the gluon radial and angular momentum quantum numbers (from 1 and 0 to ∞ respectively), and a finite sum over the gluon polarizations and the magnetic quantum numbers. This quantity is supposed to be finite, and originally some authors tried simply to evaluate it as it stands [17]. Since it is not rapidly convergent, in fact it oscillates, they applied some sophisticated convergence accelerating algorithms. Later, it was claimed [6] that the series is only conditionally convergent, and that it has no meaningful result, without proper regularization. We find in favour of this claim.

We have examined these possibilities in some detail. We note firstly that the self-energy, as given by equation (4.62), is the sum of terms $T(p_1, p_2, \Sigma_2)$,

$$\Sigma^C = \sum_{p_1 p_2 \Sigma_2} T(p_1, p_2, \Sigma_2). \tag{4.73}$$

We use this to define the function

$$Self(E) = \sum_{\epsilon_{p_1} \epsilon_{p_2} < E} T(p_1, p_2, \Sigma_2), \tag{4.74}$$

in which the series is simply truncated at some energy E. The value of this function is shown in figure 4.1. The function is sampled at intervals of 0.2, and we can see that it is rather rapidly and irregularly oscillating, and it is not clear whether it converges to the correct value.

The series is composed of terms that oscillate in sign. Is it conditionally convergent? To answer this we plot a new function, the sum of the absolute value of the terms in the

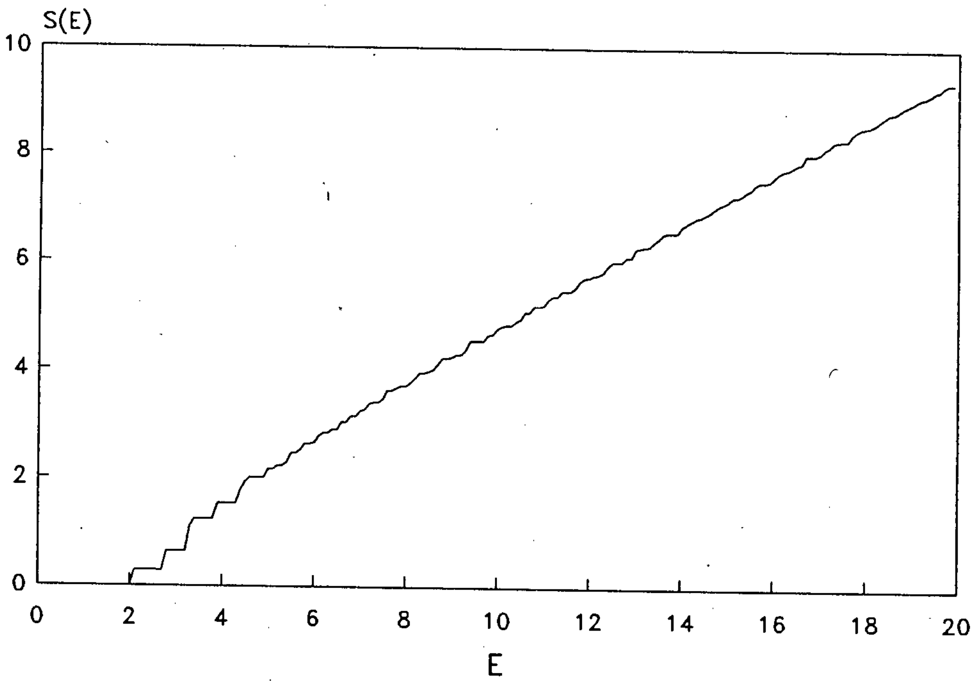

Figure 4.2: Sum(E) vs E

series, in figure 4.2.

$$Sum(E) = \sum_{\varepsilon_{p_1}\varepsilon_{p_2}<E} |T(p_1, p_2, \Sigma_2)|. \tag{4.75}$$

We note that the free space integral for the self-energy diverges linearly, i.e.

$$\Sigma \sim \int^{\Lambda} d^4q \, \frac{q}{q^4} \sim \int^{\Lambda} dq \, 1 \sim \Lambda \tag{4.76}$$

Thus we see that our graph is exactly as expected.

A conditionally convergent series may have a different limit if it is summed in a different order. Is this the case for our example? Originally Chin [17] tried to accelerate the convergence of the sum over the quantum numbers of the intermediate quark. Suppose we define the quantities

$$T(p_1) = \sum_{p_2\Sigma_2} T(p_1, p_2, \Sigma_2), \tag{4.77}$$

$$T(p_2) = \sum_{p_1\Sigma_2} T(p_1, p_2, \Sigma_2). \tag{4.78}$$

In other words we sum over all possible cavity modes of one particle, while fixing the quantum numbers of the other. The sum over the other particles angular momentum is finite, and the sum over the other particles radial quantum number is fairly rapidly convergent, and may be truncated after 4 steps either way, to give 4 figure accuracy. The magnetic quantum numbers of both particles are summed.

Finally we can sum over a finite set of the remaining quantum numbers

$$I_q(N) = \sum_{|\nu|,|\kappa|\leq N} T(p_1) \tag{4.79}$$

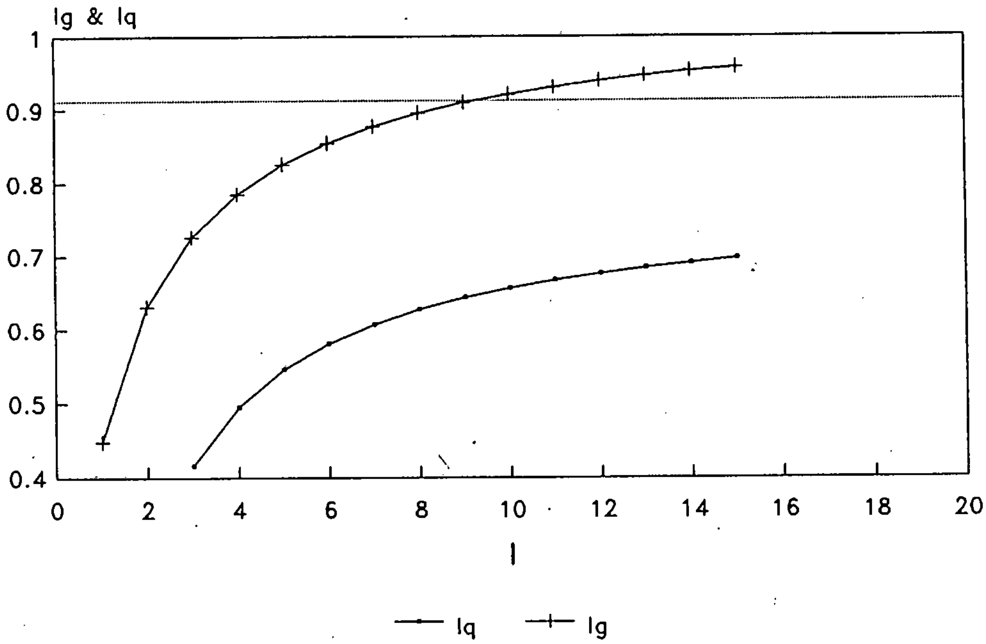

Figure 4.3: I_q and I_g vs N

$$I_g(N) = \sum_{N,J \leq N} T(p_2) \tag{4.80}$$

We show these two quantities in Figure 4.3.

From the graph it is not clear whether these curves are in fact convergent. The analytic version of the series is not very transparent, and the computing time increases as something like N^3. It would be difficult to say with certainty that the series converges, there could be a small logarithmic component. However it seems likely that it converges like $1/N$, as can be seen from a plot of I_q vs $1/N$ in figure 4.4.

In any case, from Figure 4.3 it is immediately clear that I_q and I_g do not converge to the same result, nor do they converge to the correct value, indicated by a dotted line.

From this we see that the series is, at best, conditionally convergent and it has no unique limit. While I_q and I_g are clearly hopeless, it is not quite so clear that $Self(E)$ does not converge to the correct result, however it is clearly not practical to use it as it stands.

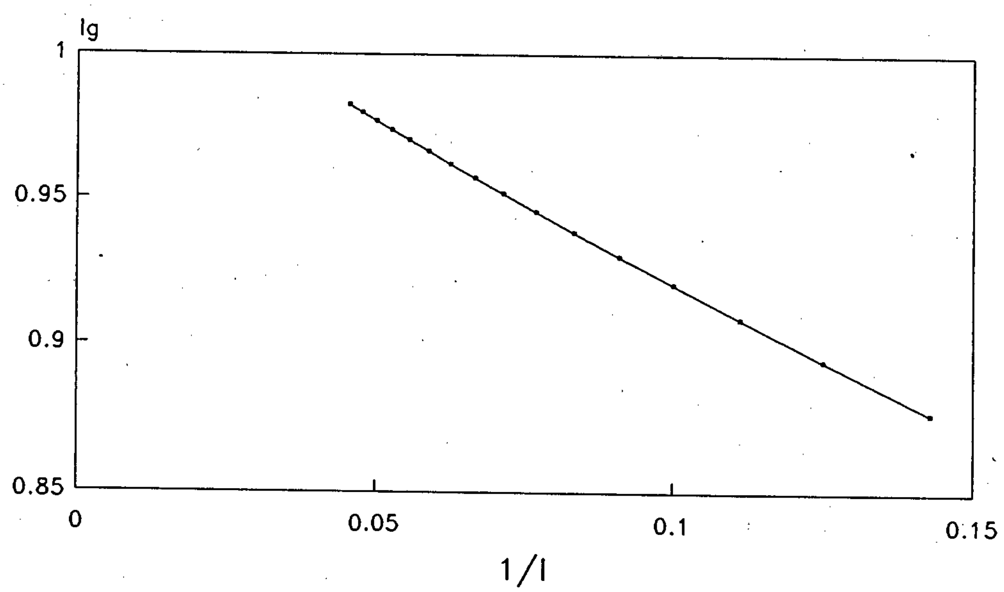

Figure 4.4: I_q vs $1/N$

Chapter 5

Vacuum Polarization

In this section we show how the methods we have developed may be applied to the problem of the vacuum polarization, or the self- energy of gauge fields. We shall specialize to the case of scalar quantum electrodynamics, as this is the simplest theory. Firstly, we briefly discuss the free space problem, to establish notation and a conceptual framework. The Vacuum Polarization has a quadratic divergence, and this usually leads to $\Gamma(-1+\varepsilon)$ poles. This means that the integrals involved need to be defined by an analytic continuation, and we discuss this point in the light of the anticipated application, a numerical subtraction.

The quadratic divergence also means that the surface behaviour will no longer be the simple finite behaviour previously encountered in the quark case. In fact, this seems to be the main reason that the MRE approach has not yet been applied to this problem. While not impossible, the degree of difficulty in applying the MRE to a sphere goes up an order of magnitude [21]. As we have mentioned in the introduction the cube can be done analytically, but the amount of work is non-trivial.

The surface behaviour was discussed in some detail in chapter 3, partly as a warm up for this problem. In order to understand the surface problem we study in section 5.2 a simple system containing a surface, the half space. The advantage is that we can evaluate all the integrals analytically, and we see that, due largely to the choice of MIT boundary conditions, the sum of the direct and reflected part at a boundary is finite. While this is encouraging, we do not regard this as a proof that there are no divergences on a *curved* surface.

In the next section we develop the 'z-form' of the cavity vacuum polarization, and see that it's numerical behaviour is free from any non-integrable term. Thus although not rigorously proven, we think that taken together, this evidence is convincing that the method is valid. Finally we present some results for the massless theory.

Figure 5.1: The Gauge Propagator

5.1 Free Space Vacuum Polarization

5.1.1 Renormalization

In this section we renormalize the photon propagator to one loop in free space [45], this sets the scene for a discussion of the cavity problem. If we sum over the irreducible vacuum polarization insertion, we get the two point function for the photon, as shown in figure 5.1

$$G^{\mu\nu}(s) = iD^{\mu\nu}(s) + iD^{\mu\alpha}(s)i\Pi_{\alpha\beta}(s)iD^{\beta\nu}(s) + \cdots \tag{5.1}$$

The photon propagator in an arbitrary gauge is discussed in section 2.3.2, and is given by equation (2.72),

$$D^{\mu\nu}(s) = -\left[\frac{g^{\mu\nu}}{s^2 + i0} + (\zeta - 1)\frac{s^\mu s^\nu}{(s^2 + i0)^2}\right]. \tag{5.2}$$

Here we use $\zeta = 1/\lambda$, where λ is the gauge fixing parameter. In free space in our gauge and most other gauges [39], the vacuum polarization tensor must be transverse,

$$s_\mu \Pi^{\mu\nu} = 0, \tag{5.3}$$

so we may write

$$\Pi^{\mu\nu}(s) = (s^2 g^{\mu\nu} - s^\mu s^\nu)\pi(s^2). \tag{5.4}$$

If we now perform the sum we get

$$-iG^{\mu\nu}(s) = -\left[\left(g^{\mu\nu} - \frac{s^\mu s^\nu}{s^2}\right)\frac{1}{s^2(1-\pi(s^2))} + \zeta\frac{s^\mu s^\nu}{s^4}\right]. \tag{5.5}$$

To renormalize we require

$$\frac{1}{1-\pi} = \frac{Z_3}{1-\pi_R}. \tag{5.6}$$

We must renormalize at some momentum squared, this means that we require

$$\pi_R(\nu^2) = 0, \tag{5.7}$$

which gives the wavefunction renormalization constant as

$$Z_3 = \frac{1}{1-\pi(\nu^2)} \tag{5.8}$$

and the renormalized vacuum polarization defined by

$$\pi(s^2) = (1 - Z_3^{-1}) + Z_3^{-1}\pi_R(s^2). \tag{5.9}$$

If we define the renormalized wave function, two point function, and gauge parameter as

$$A_R = Z_3^{-1/2}A, \tag{5.10}$$
$$G_R = Z_3^{-1}G, \tag{5.11}$$
$$\zeta_R = Z_3^{-1}\zeta, \tag{5.12}$$

then the renormalized propagator is given by

$$-iG_R^{\mu\nu}(s) = -\left[\left(g^{\mu\nu} - \frac{s^\mu s^\nu}{s^2}\right)\frac{1}{s^2(1-\pi_R(s^2))} + \zeta_R\frac{s^\mu s^\nu}{s^4}\right]. \tag{5.13}$$

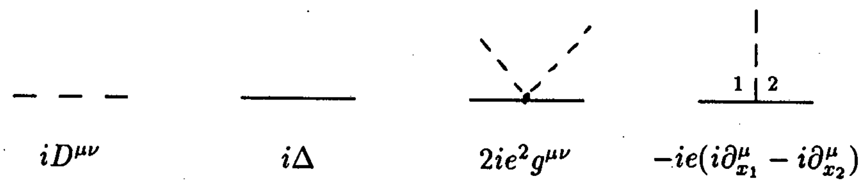

Figure 5.2: The Feynman Rules in Scalar QED

Figure 5.3: The Vacuum polarization $O(\alpha)$

5.1.2 Regularization : Scalar QED

The Lagrangian for this theory is given by

$$\begin{aligned}\mathcal{L} &= (\partial_\mu\phi - ieA_\mu\phi)^*(\partial^\mu\phi - ieA^\mu\phi) \\ &\quad -m^2\phi^*\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu),\end{aligned} \tag{5.14}$$

The Feynman rules [46], are given in figure 5.2. The order α vacuum polarization, $i\Pi = i\Pi_1 + i\Pi_2$, is given by the diagrams in figure 5.3, which evaluate to

$$\begin{aligned}i\Pi_1^{\mu\nu}(x,y) &= (-ie)^2(i\partial^\mu_{x_1} - i\partial^\mu_{x_2})(i\partial^\nu_{y_2} - i\partial^\nu_{y_1}) \\ &\qquad \times i\Delta(x_1,y_1)i\Delta(y_2,x_2),\end{aligned} \tag{5.15}$$

$$i\Pi_2^{\mu\nu}(x,y) = 2ie^2g^{\mu\nu}\delta(x,y)i\Delta(x,x). \tag{5.16}$$

It is understood that after the differentiation, we set $x = x_1 = x_2$ and $y = y_1 = y_2$. After a Fourier transform we get

$$\Pi_1^{\alpha\beta}(k) = -ie^2\mu^{2\epsilon}\int\frac{d^Dp}{(2\pi)^D}\frac{4p^\alpha p^\beta + 6p^\alpha k^\beta + 2k^\alpha k^\beta}{[p^2-m^2+i0][(p+k)^2-m^2+i0]}, \tag{5.17}$$

$$\Pi_2^{\alpha\beta}(k) = 2ie^2\mu^{2\epsilon}\int\frac{d^Dp}{(2\pi)^D}\frac{g^{\alpha\beta}}{p^2-m^2+i0}. \tag{5.18}$$

We now proceed to regularize in the framework of dimensional regularization. From the standard integrals in section C.4.1, we see that

$$\Pi_2^{\alpha\beta}(k) = \frac{e^2}{(4\pi)^{D/2}}2g^{\alpha\beta}m^2\left[\frac{1}{\varepsilon} - \gamma + 1 - \log\frac{m^2}{\mu^2}\right]. \tag{5.19}$$

We draw attention to the $m^2g^{\alpha\beta}$ behaviour, in other words by itself this term is not transverse. Only the sum of these two diagrams satisfies the transversality condition, equation (5.3), and gives the standard dimensional regularization result for the vacuum polarization in scalar QED [44,47],

$$\Pi^{\alpha\beta}(k) = \frac{e^2}{(4\pi)^{D/2}}\left(k^2 g^{\alpha\beta} - k^\alpha k^\beta\right) \times \left[\frac{-1}{3\varepsilon} + \frac{\gamma}{3} + \int_0^1 dt\,(1-2t)^2 \log\left(\frac{k^2 t(1-t) + m^2}{\mu^2}\right)\right]. \qquad (5.20)$$

In order to derive this result we need the identities associated with equation (C.64). If we now set the mass to zero, then $\Pi_2 = 0$, (see the discussion on massless tadpoles in Appendix C.4.1), and the above expression simplifies to

$$\Pi^{\alpha\beta}(k) = \frac{e^2}{(4\pi)^{D/2}}\left(k^2 g^{\alpha\beta} - k^\alpha k^\beta\right)\left[\frac{-1}{3\varepsilon} + \frac{\gamma}{3} - \frac{8}{9} + \frac{1}{3}\log\frac{k^2}{\mu^2}\right]. \qquad (5.21)$$

We note that the usual QED practice of choosing the renormalization point at $k^2 = 0$ is no longer appropriate when the theory is massless. In the massless case the tadpole term gives zero, and one might be tempted to ignore it. When we examine the case of a planar boundary we will see that this term is necessary even in the massless case, in order to get rid of an unwanted surface divergence.

5.1.3 Analytic Continuation

We intend to proceed to develop a method that may be implemented on a computer. We have to be extremely careful when dealing with undefined quantities. Both Π_1 and Π_2 contain poles in ε due to terms like $\Gamma(-1+\varepsilon)$. Why is this a problem? We must firstly consider the Gamma function (see Appendix C.1). The standard definition is

$$\Gamma(w) = a^w \int_0^\infty dz\, z^{w-1} e^{-az}, \qquad Re\, w > 0. \qquad (5.22)$$

The Gamma function terms $\Gamma(\varepsilon)$ or $\Gamma(-1+\varepsilon)$ always arise in the Feynman integrals in precisely the form of the right hand side of (5.22). But the latter term has $w < 0$ and the integral is divergent. The Gamma function itself is defined over the entire w plane, by analytic continuation of the above definition. Some alternative definitions of the Gamma function do not need to be analytically continued, as they are well defined for all w, except the poles.

In section 4.1.3 we fortunately had only $\Gamma(\varepsilon)$ which was defined for $\varepsilon > 0$, and we could safely take the limit $\varepsilon \to 0$. However we now see that our expression for $\Gamma(-1+\varepsilon)$ is only defined through an analytic continuation. It is of particular importance that we take care of this if we intend to do a computer calculation. A practical way of doing this analytic continuation is by subtracting out the $z^{-2+\varepsilon}$ divergence, using identity (C.8). In other words we replace our undefined expression

$$k^{1-\varepsilon}\Gamma(-1+\varepsilon) = \int_0^\infty dz\, z^{-2+\varepsilon} e^{-kz}, \qquad (5.23)$$

by an analytic continuation to an expression which is defined,

$$k^{1-\varepsilon}\Gamma(-1+\varepsilon) = \int_0^\infty dz\, [z^{-2+\varepsilon} e^{-kz} - D(z)], \qquad (5.24)$$

where the function $D(z)$ is given by

$$D(z) = z^{-2+\varepsilon}, \tag{5.25}$$

and $0 < \varepsilon < 1$. Clearly we now wish to extend this scheme to allow a subtraction in $D = 4$ dimensions, as we did previously in section 4.1.3. We consider the expression

$$X = \int_0^\infty dz\, z^{-2+\varepsilon} e^{-kz}. \tag{5.26}$$

Firstly we define it by analytic continuation by subtracting D(z).

$$X = \int_0^\infty dz\, [z^{-2+\varepsilon} e^{-kz} - D(z)] = \int dz\, X(z). \tag{5.27}$$

Next we separate it into a part containing the remaining singularity and a finite remainder,

$$X = S + F, \tag{5.28}$$
$$X(z) = S(z) + F(z) \tag{5.29}$$

where the singular part is designed to remove the leading singularity, and is given by

$$S(z) = -kz^{-1+\varepsilon} e^{-z}, \tag{5.30}$$

$$S = \int dz\, S(z) = -k\left[\frac{1}{\varepsilon} - \gamma\right], \tag{5.31}$$

where we ignore terms of order ε. The finite part may be taken to be defined by

$$F(z) = X(z) - D(z) - S(z) \tag{5.32}$$
$$= z^{-2+\varepsilon}(e^{-kz} - 1) + kz^{-1+\varepsilon} e^{-z}, \tag{5.33}$$

$$F = \int dz\, F(z) = -k + k \log k. \tag{5.34}$$

It is important to realize that because one could have chosen a different D, the finite part is in fact arbitrary. However the important thing is that we choose it to be finite, and therefore we are allowed to set $D = 4$. Finally we obtain the expected result, (which is not arbitrary),

$$X = F + S = -k\left[\frac{1}{\varepsilon} - \gamma + 1 - \log k\right]. \tag{5.35}$$

So once again we have developed a subtracted form for X that may safely be evaluated when $\varepsilon = 0$.

We may now proceed to apply these subtractions to the case of the vacuum polarization as given by equations (5.17,5.18). We use the standard integrals A to D from appendix C.4.1, in which the result is still an integral over some variable z, specifically equations (C.48,C.57,C.59,C.63). For $\Pi_2^{\alpha\beta}$ we get the factor $D_2^{\alpha\beta}(z)$ to be

$$D_2^{\alpha\beta}(z) = \frac{e^2\mu^2}{(4\pi)^{D/2}} 2g^{\alpha\beta} z^{-2+\varepsilon}. \tag{5.36}$$

We note that for massless particles this sets $\Pi_2^{\alpha\beta} = 0$ as expected. $D_1^{\alpha\beta}$ for $\Pi_1^{\alpha\beta}$ is opposite in sign,

$$D_1^{\alpha\beta}(z) = -\frac{e^2\mu^2}{(4\pi)^{D/2}} 2g^{\alpha\beta} z^{-2+\varepsilon}, \tag{5.37}$$

so that the total $D^{\alpha\beta}(z)$ turns out to be zero.

$$D^{\alpha\beta}(z) = 0. \tag{5.38}$$

Next we evaluate the singular part

$$S^{\alpha\beta}(k,z) = -\frac{e^2}{(4\pi)^{D/2}} e^{-z} z^{-1+\varepsilon} \frac{1}{3}\left(k^2 g^{\alpha\beta} - k^\alpha k^\beta\right), \tag{5.39}$$

and the total singular part gives

$$\begin{aligned} S^{\alpha\beta}(k) &= \int dz\, S^{\alpha\beta}(k;z) \\ &= \frac{e^2}{(4\pi)^{D/2}}\left(k^2 g^{\alpha\beta} - k^\alpha k^\beta\right)\left[\frac{-1}{3\varepsilon} + \frac{\gamma}{3}\right]. \end{aligned} \tag{5.40}$$

This completes the discussion of how to make a subtraction in free space. One important point needs to be made. Many algebraic manipulations, in particular cancelling out a k^2 from the denominator and numerator, whilst not affecting the final result, do not necessarily leave $D(z)$ or $S(z)$ unchanged. This is discussed in appendix C.4.2. Because we intend to subtract the corresponding cavity expression it is of vital importance to ensure that we do the cavity calculation in a *precisely analogous* fashion.

Later we will see that it is particularly convenient that $D^{\alpha\beta}(z) = 0$. (Naturally this could have been arranged, if had not turned out to be so.) We must forsee the possibility of surface divergences. The surface divergences are however intimately related to the quadratic divergence in the vacuum polarization, which causes the terms that we need to cancel out with the $D(z)$. It turns out that by arranging $D(z) = 0$, the surface divergences vanish and we may proceed to calculate the result. This we will see in the next section.

5.2 Half Space Vacuum Polarization

As we will see, there are difficulties with the vacuum polarization that were absent in the quark self-energy. In that problem no new singularities were encountered due to the presence of the surface. The vacuum polarization is quadratically divergent, and singularities do arise due to the presence of the surface. We believe that these singularities cancel out in the final result in such a way as to give a meaningful result in the cavity.

It would be a formidable problem to show that these singularities vanish on the curved surface that bounds the cavity, although the techniques used in [4] might be of use. For simplicity therefore we show how to solve the problem for an infinite flat surface, i.e. a half space. This treatment will draw on some of the ideas developed in sections 3.2.3 and 3.2.4.

Our 'cavity' is now a half space, an infinite D dimensional space, $x = \{x_0, x_1, ...x_D\}$, with a $D-1$ dimensional plane at $x_1 = 0$. The scalar field may once again obey either the Dirichlet or Neumann boundary condition,

$$\phi(x)|_{x_1=0} = 0, \tag{5.41}$$

$$\frac{\partial}{\partial x_1}\phi(x)|_{x_1=0} = 0. \tag{5.42}$$

The same boundary conditions would apply to the scalar propagators. The half space propagator will be the sum of a direct and reflected part,

$$\Delta^h(x,y) = \Delta^0(x,y) + \tilde{\Delta}(x,y). \tag{5.43}$$

For this simple system we may write down the reflection part immediately using the method of images [50],

$$\Delta^h(x,y) = \Delta^0(x,y) + \eta\Delta^0(x,y_\perp), \tag{5.44}$$

where $\eta = -1$ for Dirichlet, and $\eta = 1$ for Neumann boundary conditions.

The vector field we choose to obey the boundary conditions of the M.I.T. bag model, which are given by

$$\begin{array}{ll} \hat{r}\cdot\vec{\nabla}a^0(\vec{r}) = 0 & x_1 = 0, \\ \hat{r}\cdot\vec{a}(\vec{r}) = 0 & x_1 = 0, \\ \hat{r}\times(\vec{\nabla}\times\vec{a}(\vec{r})) = 0 & x_1 = 0. \end{array} \tag{5.45}$$

These boundary conditions reduce to

$$\begin{aligned} \partial_{x_1}A^0(x)|_{x_1=0} &= 0 \\ A^1(x)|_{x_1=0} &= 0 \\ \partial_{x_1}A^2(x)|_{x_1=0} &= 0 \\ \partial_{x_1}A^3(x)|_{x_1=0} &= 0 \end{aligned} \tag{5.46}$$

In other words the boundary conditions are Neumann for all polarizations of the photon, except for the perpindicular polarization, which obeys Dirichlet boundary conditions.

Can we calculate the full vacuum polarization near the boundary? The configuration space expression for the vacuum polarization remains valid, except that we now understand the propagators to include reflections,

$$\begin{aligned} i\Pi_1^{\mu\nu}(x,y) &= (-ie)^2(i\partial^\mu_{x_1} - i\partial^\mu_{x_2})(i\partial^\nu_{y_2} - i\partial^\nu_{y_1}) \\ &\qquad \times i\Delta(x_1,y_1)i\Delta(y_2,x_2), \qquad (5.47) \\ i\Pi_2^{\mu\nu}(x,y) &= 2ie^2g^{\mu\nu}\delta(x,y)i\Delta(x,x). \qquad (5.48) \end{aligned}$$

There are two serious problems with this expression

- From equation 5.40 we see that Π^0 contains a $\partial^2\delta$ term. As we have seen in section 3.2.4 when a $\partial^2\delta$ encounters wave functions obeying Neumann boundary conditions, the result is undefined.

- The reflection part of Π_2 may readily be seen to behave like $1/x_1^2$, and this is not integrable, when folded in with a wavefunction that goes like a constant at athe boundary.

These are the problems that we aim to address, by making a detailed study of the half space problem. We note that the first problem was anticipated in the conclusion to [4], and was implictly solved in the analytic calculation in a cubical cavity [10].

If we insert the half space propagator (5.44) into (5.48) and (5.47) we get (once again we consider massless scalar fields), the half space vacuum polarization for the two Feynamn diagrams to be the sum of a direct part and a part containing at least one reflection,

$$\Pi_2^{\mu\nu} = \Pi_2^{0,\mu\nu} + \tilde{\Pi}_2^{\mu\nu}, \tag{5.49}$$

$$\Pi_1^{\mu\nu} = \Pi_1^{0,\mu\nu} + \tilde{\Pi}_1^{\mu\nu}. \tag{5.50}$$

We firstly examine $\tilde{\Pi}_2^{\mu\nu}$, using (C.72). We proceed to set $\varepsilon = 0$, and get

$$\tilde{\Pi}_2^{\alpha\beta}(x,y) = \frac{e^2}{(4\pi)^2}\delta(x,y)g^{\alpha\beta}\frac{2}{x_1^2}. \tag{5.51}$$

This causes the second problem referred to above. $\Pi_1^{\alpha\beta}(k)$ is more difficult, and we evaluate it in steps. Since the half space propagator is symmetric with respect to interchanging x and y,

$$\Pi_1^{\alpha\beta}(x,y) = -2ie^2\left[i\partial_x^\alpha\Delta(x,y)i\partial_y^\beta\Delta(y,x) - \Delta(x,y)i\partial_x^\alpha i\partial_y^\beta\Delta(y,x)\right] \tag{5.52}$$

Noting that

$$\begin{aligned}\Delta(x,y)\Delta(y,x) &= \eta\left[\Delta^0(x-y_\perp)\Delta^0(y-x) + \Delta^0(x-y)\Delta^0(y_\perp - x)\right] + \\ &\quad \left[\Delta^0(x-y)\Delta^0(y-x) + \Delta^0(x-y_\perp)\Delta^0(y_\perp - x)\right]\end{aligned} \tag{5.53}$$

and that

$$\partial_{y_\perp}^\alpha = (-)^{n_\alpha}\partial_y^\alpha, \tag{5.54}$$

where $n^\alpha = \{0,1,0,0\}$, is a normal four-vector, and no summation over the index α is implied. We see that

$$\begin{aligned}\Pi_1^{\alpha\beta}(x,y) &= \left[\Pi_{1a}^{\alpha\beta}(x,y) + (-)^{n_\beta}\Pi_{1a}^{\alpha\beta}(x,y_\perp)\right] \\ &\quad +\eta\left[\Pi_{1b}^{\alpha\beta}(x,y) + (-)^{n_\beta}\Pi_{1b}^{\alpha\beta}(x,y_\perp)\right]\end{aligned} \tag{5.55}$$

where the term denoted by $\Pi_{1a}^{\alpha\beta}(x,y)$ is nothing other than the familiar free space vacuum polarization, $\Pi^{0,\alpha\beta}(x,y)$. The second term corresponds to a vacuum polarization from an image point, and the final two terms contain a mixture of reflection and direct propagators. If we insert the explicit forms for the propagators, and shift the momentum integrations we get

$$\begin{aligned}\Pi_{1b}^{\alpha\beta}(x,y) &= -ie^2\mu^{2\varepsilon}\int\frac{d^Dq}{(2\pi)^D}e^{-iq(y-x)}\int\frac{d^Dp}{(2\pi)^D}e^{-2ip_1y_1} \\ &\quad \times\frac{4p^\alpha p^\beta + 6p^\alpha q^\beta + 2q^\alpha q^\beta}{[p^2+i0][(p+q)^2+i0]}\end{aligned} \tag{5.56}$$

As a check we note that by removing the second exponential factor we recover the free space result. If we write

$$\Pi_{1b}^{\alpha\beta}(x,y) = -ie^2\int\frac{d^Dq}{(2\pi)^D}e^{-iq(y-x)}\Pi_{1b}^{\alpha\beta}(q,y), \tag{5.57}$$

then, with the help of the integrals (C.75,C.77,C.79) we may evaluate this expression,

$$\begin{aligned}\Pi_{1b}^{\alpha\beta}(q,y) &= \frac{i}{(4\pi)^{D/2}}\int_0^1 dt\, e^{-2itq_1y_1}\Big\{2q^\alpha q^\beta I(\varepsilon)\\ &+3ix^\alpha q^\beta I(-1+\varepsilon)-6iq^\alpha q^\beta tI(\varepsilon)\\ &-2g^{\alpha\beta}I(-1+\varepsilon)-x^\alpha x^\beta I(-2+\varepsilon)\\ &-2it(x^\alpha q^\beta+q^\alpha x^\beta)I(-1+\varepsilon)+4t^2q^\alpha q^\beta I(\varepsilon)\Big\}, \end{aligned}\tag{5.58}$$

where $I(\nu)=I(\beta,\rho;\nu)$, and $\beta=(2y_1)^2/4$, and $\rho=k^2t(1-t)$. By x^α we mean $\{0,2y_1,0,0\}$. We are not interested in the detailed result, but rather in any behaviour of $1/y$ or stronger, which will not be integrable. By use of Appendix C.3 we see that to leading order

$$I(0) \sim \log x \tag{5.59}$$

$$I(-1) \sim 1/x^2 \tag{5.60}$$

$$I(-2) \sim 1/x^4 \tag{5.61}$$

The question is : Do we have to worry about any of these terms ? All terms in x^α are suppressed by y_1^2, since they pick up only those polarizations which satisfy Dirichlet boundary conditions, and therefore do not cause non-integrable contributions. The only term which does, is the term in $g^{\alpha\beta}$, since $I(-1)\sim 4/x^2=1/y_1^2$ (note the q independence). Thus we get

$$\Pi_{1b}^{\alpha\beta}(q,y)\sim\frac{-2g^{\alpha\beta}}{y_1^2} \tag{5.62}$$

$$\Pi_{1b}^{\alpha\beta}(x,y)\sim-\frac{e^2}{(4\pi)^2}\int_0^1 dt\,\delta(y-x+2ty_1)\frac{2g^{\alpha\beta}}{y_1^2} \tag{5.63}$$

and since the first derivative of the wave functions that we will fold this function in with, $A^\mu(x)A^\nu(y)$, always vanishes, we may write for small y_1,

$$\Pi_{1b}^{\alpha\beta}(x,y)\sim-\frac{e^2}{(4\pi)^2}\delta(y-x)\frac{2g^{\alpha\beta}}{y_1^2}. \tag{5.64}$$

$\Pi_{1b}^{\alpha\beta}(x,y_\perp)$ clearly does not contribute, because the argument of the delta function cannot be zero. Therefore in total this non-integrable term cancels with that produced by $\tilde{\Pi}_2^{\alpha\beta}(x,y)$. This solves the second problem.

It remains to solve the problem caused by the $\partial^2\delta$. The solution to this comes from the term $\Pi_{1a}^{\alpha\beta}(x,y_\perp)$. This means that we will usually have a contribution to Π containing a term like

$$\partial^2(\delta(x-y)+\delta(x-y_\perp)) \tag{5.65}$$

This solves the problem as may be seen by considering the one dimensional problem, where we let $d(x-y)$ be some smooth distribution for the δ function. For an example we consider A where

$$\begin{aligned} A &= \lim_{d\to\delta}\int_0^\infty dx\int_0^\infty dy f(x)g(y)\partial_y^2(d(x-y)+d(x-y_\perp)), &(5.66)\\ &= \lim_{d\to\delta}\int_0^\infty dx\int_{-\infty}^\infty dy f(x)g(y)\partial_y^2 d(x-y), &(5.67)\\ &= \int_0^\infty dx f(x)\partial_x^2 g(x) &(5.68)\end{aligned}$$

We note that in equation (5.55) there is a $(-)^{n_\alpha}$, in other words we sometimes have $\partial_y^2(\delta(x,y) - \delta(x,y_\perp))$. However this is not a problem because it is sandwiched between the polarization that satisfies Dirchlet boundary conditions.

Thus there are no new singularities caused by the presence of the boundary. This proof is valid only for a flat surface, and we cannot yet rule out the possibility that subleading behaviour may be a problem near a curved surface, however we feel that this is unlikely.

A second point that may be noted is that if we rather do the space integration before the z integration, we can get the leading order surface z behaviour, as we did previously in section 3.2.3. It turns out to be of leading order $z^{-1/2}$, similar to the Dirichlet case in ϕ^4 theory. This is obviously related to the integrable nature of the reflections, that we have just shown.

5.3 Cavity Vacuum Polarization

5.3.1 Renormalization

The renormalized Green's function must conserve angular momentum and parity. This means that a photon in a Magnetic polarization, which has parity $(-1)^{J+1}$, cannot make a transition to any of the other polarizations, which have parity $(-1)^J$. There is no conservation law which prevents a Magnetic photon from changing to another radial quantum number, nor is a transition from Electric to Scalar polarization forbidden.

We will need the Green's function, the propagator, and the vacuum polarization, in cavity mode space, so we Fourier transform all of these quantities according to

$$G^C(\Sigma,q;\Sigma',q') = \int d^4x d^4y\, A^*_\mu(\Sigma,q;x) G^{C,\mu\nu}(x,y) A_\nu(\Sigma',q';y) \qquad (5.69)$$

Because the magnetic part does not mix, for simplicity we start by considering the Magnetic polarization on its own, and sum over the irreducible vacuum polarization insertions in the cavity, Π^C. The Green's function is given by

$$\begin{aligned} G^C(M,q;M,q') &= \frac{-ig^{MM}}{q^2}\delta(q,q') + \frac{-ig^{MM}}{q^2} i\Pi^C(M,q;M,q')\frac{-ig^{MM}}{q'^2} + \cdots \qquad (5.70) \\ &= \frac{-ig^{MM}\delta(\omega,\omega')}{(\omega^2 - \Omega^2_{Mp})\delta_{pp'} - g^{MM}\Pi^C(\omega,M,p;M,p')} \qquad (5.71) \end{aligned}$$

As usual we have diagonality in ω, so we drop the argument ω'. This quantity contains a divergence, so we must renormalize. We have already given the renormalization prescription for free space in equation (5.9). In free space, for transverse polarizations (denoted TT), this relation may be rewritten in tensor form as

$$\Pi^{TT}(s^2) = (1 - Z_3^{-1})s^2 g^{TT} + Z_3^{-1}\Pi_R^{TT}(s^2), \qquad (5.72)$$

where the definition of Z_3 is

$$(1 - Z_3^{-1}) = \frac{1}{s^2} g^{TT}\Pi^{TT}(s^2)|_{s^2=\nu^2} = \pi(\nu^2). \qquad (5.73)$$

The renormalization point is denoted by ν^2. Earlier on, in section 3.2.2, we took some care to point out that implicit in our approach is the choice of a specific mass scale $\mu^2 = 1/R^2$.

Thus in free space we see that in order to implement the renormalization prescription we see that we must have

$$(1 - Z_3^{-1}) = \frac{e^2}{(4\pi)^{D/2}} \left[-\frac{1}{3\varepsilon} + \frac{\gamma}{3} - \frac{8}{9} + \frac{1}{3} \log \frac{\nu^2}{\mu^2} \right]. \tag{5.74}$$

In the cavity we would define Z_{3C}, and a more natural definition that is not R dependent is simply,

$$(1 - Z_{3C}^{-1}) = \frac{e^2}{(4\pi)^{D/2}} \left[-\frac{1}{3\varepsilon} + \frac{\gamma}{3} \right] = (1 - Z_3^{-1}) - P(\nu, \mu). \tag{5.75}$$

This differs from the usual definition by a finite piece $P(\nu, \mu)$. We know now from the discussion of the Neumann piece that $< s^2 > \neq < q^2 >$. However from the discussion in the previous section it is clear that by adding some of the reflection the surface singularity disappears. So in the cavity theory we may set $< s^2 > = < q^2 >$ with the understanding that some of the reflection is included. Thus we may Fourier transform the free space singular part,

$$\begin{aligned} S^{\alpha\beta}(s; z) &= -\frac{e^2}{(4\pi)^{D/2}} e^{-z} z^{-1+\varepsilon} \frac{1}{3} \left(s^2 g^{\alpha\beta} - s^\alpha s^\beta \right) \\ \rightarrow S(\Sigma, q; \Sigma', q'; z) &= -\frac{e^2}{(4\pi)^{D/2}} e^{-z} z^{-1+\varepsilon} \frac{1}{3} \left(q^2 g^{\Sigma\Sigma'} - q^\Sigma q^{\Sigma'} \right) \delta(q, q') \end{aligned} \tag{5.76}$$

This suggests the following renormalization prescription in the cavity,

$$\Pi^C(\omega, M, p; M, p') = (1 - Z_{3C}^{-1}) q^2 g^{MM} + Z_{3C}^{-1} [P(\nu, \mu) q^2 g^{MM} + \Pi_R^C(\omega, M, p; M, p')] \tag{5.77}$$

In practice this means that we need to calculate

$$\Pi_R^C(\omega, M, p; M, p') = \int_0^\infty dz \, [\Pi^C(\omega, M, p; M, p'; z) - S(\omega, M, p; M, p'; z)] \tag{5.78}$$

This prescription will give us the Green's function

$$G_R^C(\omega, M, p; M, p') = \frac{-i g^{MM}}{(\omega^2 - \Omega_{Mp}^2)\delta_{pp'}[1 - P(\nu, \mu)] - g^{MM} \Pi_R^C(\omega, M, p; M, p')}. \tag{5.79}$$

Thus we see that the cavity effective coupling constant 'runs' as we change R, and that there is an additional part, independent of R which also renormalizes the effective coupling constant.

At this point we may summarize the consequences of this prescription

- The wavefunction renormalization constant, Z_3, is specified at some renormalization point in free space.
- Our definition of Z_{3C} is independent of ν^2.
- The cavity effective coupling constant 'runs' with R.
- The perturbative photon self-energy, deduced from the poles of G_R^C, is given by $g^{MM} \Pi^C(\omega, M, p; M, p)|_{\omega^2 = \Omega_{Mp}^2}$, and is independent of the renormalization point. (Except, of course, indirectly through α.)

This concludes the case of the magnetic mode. What are the self-energies of the other photon cavity modes? For the electric mode we simply use the same formalism as the above, which means that we have ignored mixing.

For the longitudinal and scalar modes we must recall the property of transversality. Since this is a local property we would expect that in the cavity

$$\partial_x^\mu \Pi^{C,\mu\nu}(x,y) = 0 \tag{5.80}$$

we must have

$$q_\Sigma \Pi^C(\Sigma,q;\Sigma',q') = 0 = \Pi^C(\Sigma,q;\Sigma',q')q'_{\Sigma'}. \tag{5.81}$$

For simplicity we now ignore the off diagonal contributions, except the degenerate energy Scalar to Longitudinal mixing. We can see that in this case for $\Sigma = S, L$, the tensor structure must be

$$\Pi^C(\Sigma,q;\Sigma',q) = (q^2 g^{\Sigma\Sigma'} - q^\Sigma q^{\Sigma'})\pi^C(q;q). \tag{5.82}$$

With this structure it is immediately obvious by analogy to the free space tensor structure that the Scalar and Longitudinal modes must have self-energy zero. We have not pursued the question of what effects the off-diagonal contributions will have.

5.3.2 Regularization and Results

Our next task is to regularize the cavity vacuum polarization, so that we may implement the above renormalization scheme. Thus we need the quantity, $\Pi^C(\Sigma,q;\Sigma',q';z)$. Naturally this calculation follows the same procedure that we have set out before, in the cases of the ϕ^4 self-energy, and the quark self-energy, so we defer a derivation of this quantity to Appendix B.4. However we do stress the point that there are some subtleties to the calculation.

The test of whether the method is working hinges on an examination of the small z behaviour of $\Pi^C(z)$. As usual it is convenient to convert to the variable $y^2 = z$. What do we see? The $\Pi_2^C(z)$ term goes like $1/z^2$, i.e. $y\Pi^C(y)$ goes like $1/y^3$. The leading part of the $\Pi_1^C(y)$ cancels this out as expected. In figure 5.4 we plot $y\Pi^C(y)$ vs y, and we see convincing evidence that it goes as $O(y^0)$, before the usual error term cuts in. This means that not only the leading $1/y^3$ term vanishes, but also remarkably the $1/y^2$ and $1/y$ terms as well.

We may compute this for off shell values of q^2 as well, and once again we discover this benign behaviour, although here we need to subtract $yS(y) \sim 1/y$, before we get $O(y^0)$ behaviour. We have also confirmed numerically in a couple of cases that the transversality equation (5.81) is indeed satisfied. We note that the electric mode does couple to the scalar and longitudinal modes, in a way that satisfies transversality.

Finally in table 5.1 we display a few results for the case of Dirichlet boundary conditions on the scalar cavity mode. The results are for Π^C i.e. *not* $g^{MM}\Pi^C$. The accuracy will clearly be dependent on the chosen E_{max}, and the energy of the cavity mode for which the self-energy is evaluated.

The Neumann case is avoided since we would presumably have to decide what treatment to give the zero cavity mode. The usual treatment may not be valid because it is now multiplied by another distribution. For the purposes of comparison we present a result obtained by dropping the zero energy cavity mode of the Neumann propagator, for the $1M1$ cavity mode, $\Pi^C = \alpha_s(3.9177 \pm 0.0005)/R$.

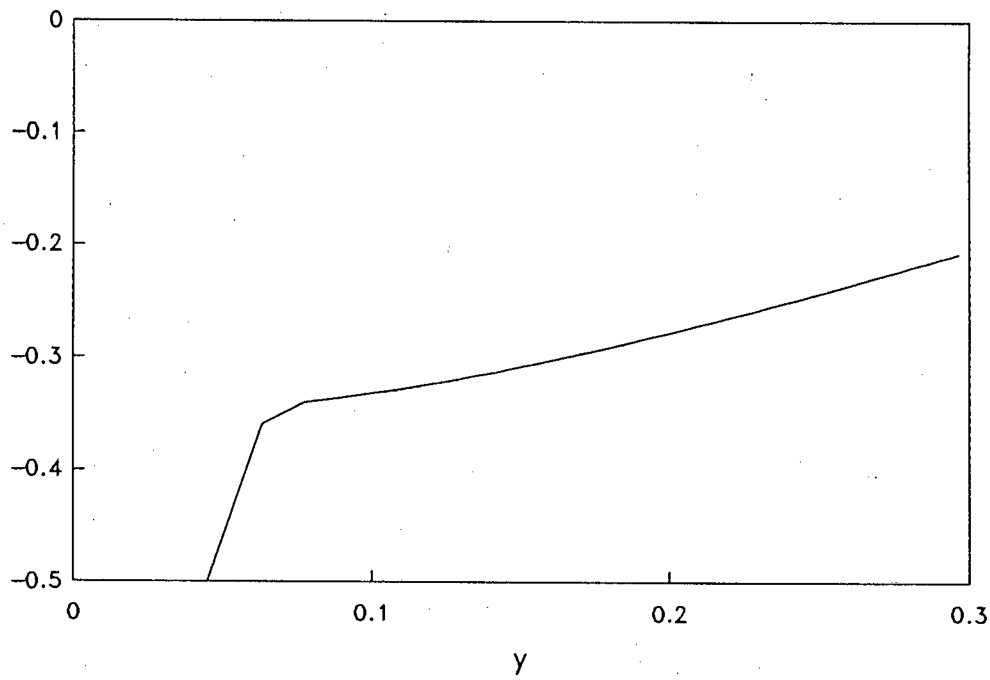

Figure 5.4: $y\Pi^C(y)$ vs y for $1M1$

Cavity Mode	Eigenenergy	Self-Energy
1M1	2.74371	−0.24698 ±0.00003
1M2	3.87024	−0.3919 ±0.0002
1M3	4.97342	−0.547 ±0.001
2M1	6.11676	−0.37 ±0.02
1E1	4.49341	−0.3217 ±0.0009
1E2	5.76346	−0.427 ±0.006
1E3	6.98793	−0.53 ±0.01
2E1	7.72525	−0.49 ±0.05

Table 5.1: Photon Self-Energies in Scalar QED

Chapter 6

Conclusion

6.1 Applications

In this thesis we have developed a method for subtracting out divergences from divergent loop diagrams. This may be applied to any problem that cannot be solved analytically, however it seems most practical for problems with discrete energy spectra, in which we know the wavefunction to a very high degree of accuracy, because it necessarily involves some numerical extrapolation. Thus, while we could tackle probelms with a background potential, only those potentials for which analytical solutions exist, like the Coulomb potential, would be practical.

At this stage, this is mainly an advance in our calculational capabilities, and it has taken a long time for such methods to arrive, since the need was first identified. However, while the method may be 'fun', the real physics lies in the applications.

A number of such applications await. Clearly the most immediate application is the gluon self-energy, and work is already in progress towards this goal. It could vary anywhere between $\pm 3\alpha/R$, and the consequences depend critically on where. A negative self-energy suggests the possibilty of a boson condensation, and a strong positive quantity would offer an explanation for the absence of exotic states.

Another interesting application would be to calculate the self-energy of quarks moving in the Coulomb field of a static heavy quark. Can this improve the fits achieved by Background field + Bag model studies of Hadrons containing a heavy quark [30]. Our running cavity coupling constant suggests that we may be able to fit Baryons and Mesons with the same coupling constant which would perhaps begin to address the wide range of α_s from 2.2 to 0.4 used in spectroscopy.

Our formalism contains information about off-shell self-energies, and although we do not present results, the calculation is straightforward. The 1M1 gluon that produces the hyperfine splitting in baryons is actually rather off-shell, it would of interest to see what effects this has on the size of α_s needed to fit the spectrum. More generally to explore the off-shell properties of the self-energy.

6.2 Further Work

Some work could usefully be done in providing a more rigorous justification of the method.

The understanding of the surface divergences in the case of Neumann boundary conditions could usefully be extended.

There is a certain amount that could be done to fine tune the computational methods used in calculating the presented numbers. The calculation of an accurate set of vertices up to some energy takes the bulk of the computational time and space, and is a quickly increasing function of the chosen maximum energy. The extrapolation technique is crucial for extracting a final result from this information, and it is clearly the first place to look for improvements. The method we have used so far is rather crude, and one might envisage more sophisticated numerical extrapolation techniques being applied to this problem.

6.3 Acknowledgements

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Appendix A

The Cavity Modes

In this appendix we present the cavity modes of scalar, Dirac and gauge fields confined to a static spherical cavity. We use a notation similar to [2]. We then briefly consider the planar momentum representation of cavity modes, and finally the $< \not{p} - m >$ expectation value of quark cavity modes. To begin with, we establish our conventions.

A.1 Conventions and Units

We use the Minkowski space metric, $g^{\mu\nu}$, with signature

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}\{1, -1, -1, -1\}, \tag{A.1}$$

$$g^{\mu}_{\nu} = \delta^{\mu}_{\nu}. \tag{A.2}$$

As a general rule, whenever we evaluate Feynman integrals, in particular those of section C.4.1, we convert to Euclidean space, and then evaluate the integral. In the main text we usually express the final result once again in Minkowski space. Any (hopefully local) sign errors, or apparent errors, may be due to this procedure, which is not always explicitly mentioned.

We use the 4×4 Dirac γ matrices satisfying

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}, \tag{A.3}$$

which may be represented as follows

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{A.4}$$

where the σ are the 2×2 Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.5}$$

We employ the usual $3j$ and $6j$ symbols according to the conventions of Edmonds [57]. We also use 'natural' units throughout with

$$\hbar = c = 1 \tag{A.6}$$

With this step quantities such as the energy, momentum, time all have units of length/mass to some power. We note that

$$\hbar c = 197.3285851 \text{ MeV fm}, \quad (A.7)$$

$$1\text{GeV} = 5.06768963\,\hbar c \ \ \text{fm}^{-1}. \quad (A.8)$$

Since we are working in the cavity, there is a natural unit of length, the cavity radius R. We can use this to relate dimensional and dimensionless quantities, where we distinguish the dimensionless variable by a bar, e.g.

$$\varepsilon = \bar{\varepsilon}/R, \quad (A.9)$$

The Normalization has dimensions such that

$$N = \bar{N}/R^{3/2}. \quad (A.10)$$

In exponentials and logs, where it is obvious that we mean a dimensionless quantity, we will often omit the bar, to impose a little less strain on the eye, e.g.

$$\exp\{\varepsilon^2 z\} \to \exp\{\bar{\varepsilon}^2 z\}. \quad (A.11)$$

Finally when we explicitly mention the cavity energy we may sometimes omit the $/R$, e.g.

$$E_{max} = 50 \to E_{max} = 50/R. \quad (A.12)$$

In dimensional regularization we will need to refer to the arbitrary mass scale μ which is must be introduced. In Appendix C.4.1 we simply assume that the integrals are dimensionless, whereas in the text we will usually retain the mass scale explicitly. When we start regularizing cavity results we set the dimensional regularization scale $\mu = 1/R$, for convenience (see discussion in section 3.2.2).

We note that in the literature the word Lagrang.an is spelt Lagrangean and Lagrangian with roughly equal frequency. We adopt this convention.

A.2 Scalar Fields

We firstly consider the time independent Klein Gordon equation, which is given by

$$(\Omega^2 + \nabla^2 - m^2)\phi(\vec{r}) = 0, \qquad \vec{r}\epsilon V. \quad (A.13)$$

The wave equation must be satisfied everyhere inside the cavity volume V, and we apply some boundary condition on the surface of the cavity. Below we give three possible boundary conditions for a scalar field, the Dirichlet, Neumann and mixed respectively,

$$\phi(\vec{r}) = 0, \qquad \vec{r}\epsilon S, \quad (A.14)$$

$$\hat{r}.\vec{\nabla}\phi(\vec{r}) = 0, \qquad \vec{r}\epsilon S, \quad (A.15)$$

$$(\hat{r}.\vec{\nabla} + a)\phi(\vec{r}) = 0, \qquad \vec{r}\epsilon S. \quad (A.16)$$

The solutions are given by

$$\phi(p;\vec{r}) = N_{nl}\, j_l(k_{nl}\, r)\, Y_{lm}(\hat{r}). \quad (A.17)$$

The normalization constants normalize the spatial integral to 1, and are given by

$$N_p^{-2} = R^3 \tfrac{1}{2} j_{l-1}^2(k_p R), \tag{A.18}$$

$$N_p^{-2} = R^3 \tfrac{1}{2} j_l^2(k_p R)[\bar{k}_p^2 - l(l+1)]/\bar{k}_p^2, \tag{A.19}$$

$$N_p^{-2} = R^3 \tfrac{1}{2} j_l^2(k_p R)[\bar{k}_p^2 - l(l+1) - a + a^2]/\bar{k}_p^2. \tag{A.20}$$

j_l stands for the spherical Bessel function, and the k_{nl} are those values of the (cavity) momentum that satisfy the boundary conditions (A.14, A.15, A.16). Note that we use the dimensionless quantity $\bar{k}_p$, where $k_p = \bar{k}_p/R$. For brevity we often summarize the radial, angular momentum and magnetic quantum labels by p,

$$p = \{n, l, m\}. \tag{A.21}$$

The eigenenergy of a massive scalar particle will be given by

$$\Omega_p^2 = k_p^2 + m^2. \tag{A.22}$$

This set of cavity modes forms a complete set in which can expand any function in the cavity. We can write the orthogonality and completeness relations as

$$\int d\vec{r}\, \phi^*(p;\vec{r})\phi(p';\vec{r}) = \delta_{p,p'}, \tag{A.23}$$

$$\sum_p \phi(p;\vec{r})\phi^*(p;\vec{r}\,') = \delta(\vec{r},\vec{r}\,'). \tag{A.24}$$

The completeness relation in the case of Neumann boundary conditions needs a zero energy mode as well, for some discussion of this point see Appendix D.

A.3 Dirac Fields

Next we consider the cavity modes for Dirac fields confined to a static spherical cavity. The time-independent wave equation for a massive spinor is

$$(\gamma^0 \varepsilon + i\vec{\gamma}.\vec{\nabla} - m)u(\vec{r}) = 0, \tag{A.25}$$

with the usual boundary conditions,

$$(i\vec{\gamma}.\hat{r} + 1)u(\vec{r}) = 0 \qquad \vec{r}\epsilon S, \tag{A.26}$$

$$\bar{u}(\vec{r})(i\vec{\gamma}.\hat{r} - 1) = 0 \qquad \vec{r}\epsilon S. \tag{A.27}$$

The eigenmodes are

$$u(p;\vec{r}) = \begin{bmatrix} g_p(r)\chi_{\kappa\mu}(\hat{r}) \\ i f_p(r)\chi_{-\kappa\mu}(\hat{r}) \end{bmatrix}, \tag{A.28}$$

where the radial wave functions are given by

$$g_p(r) = N_p\, j_l(k_p r), \tag{A.29}$$

$$f_p(r) = N_p\,\mathrm{sgn}(\kappa)k_p/(\varepsilon_p + m)\; j_{\bar{l}}(k_p r). \tag{A.30}$$

κ and μ are the Dirac and magnetic quantum numbers of the spinor spherical harmonics, $\chi_{\kappa\mu}$. k_p stands for the momentum satisfying the boundary condition (A.27) with radial quantum number ν. p summarizes these quantum labels

$$p = \{\nu, \kappa, \mu\}. \tag{A.31}$$

The other quantum labels are given in terms of κ by

$$j(\kappa) = \mathrm{mod}(\kappa) - 1/2, \tag{A.32}$$

$$l(\kappa) = j(\kappa) + \mathrm{sgn}(\kappa)/2, \tag{A.33}$$

$$\bar{l}(\kappa) = j(\kappa) - \mathrm{sgn}(\kappa)/2. \tag{A.34}$$

Using (A.28) the boundary condition (A.27) simplifies to

$$j_l(\bar{k}_p) + \mathrm{sgn}(\kappa)\; j_{\bar{l}}(\bar{k}_p)/(\bar{\varepsilon}_p + \bar{m}) = 0. \tag{A.35}$$

The normalization constant is given by

$$N_p^{-2} = R^3[2\bar{\varepsilon}_p(\bar{\varepsilon}_p + \kappa) + \bar{m}][j_l(\bar{k}_p)/\bar{k}_p]^2. \tag{A.36}$$

If we allow negative ν to label the negative energy states, the energy of the cavity mode is

$$\varepsilon_p = \mathrm{sgn}(\nu)\;[k_p^2 + m^2]^{1/2}. \tag{A.37}$$

The eigenmodes satisfy orthonormality and completeness relations in three dimensions, we show the Dirac index explicitly,

$$\sum_\alpha \int d\vec{r}\; u_\alpha^*(p, \vec{r}) u_\alpha(p'; \vec{r}) = \delta(p, p'), \tag{A.38}$$

$$\sum_p u_\alpha(p, \vec{r}) u_\beta^*(p, \vec{r}\,') = \delta(\vec{r}, \vec{r}\,')\delta_{\alpha\beta}. \tag{A.39}$$

A.4 Gauge Fields

Finally, we repeat the above treatment for the gauge field. We start with the time-independent wave equation for a massless vector field, which is

$$(\nabla^2 + \Omega^2)a^\mu = 0. \tag{A.40}$$

We make the choice that the gluons satisfy the boundary conditions of the M.I.T. bag model, i.e.

$$\begin{array}{ll} \hat{r}.\vec{\nabla}a^0(\vec{r}) = 0 & \vec{r}\epsilon S, \\ \hat{r}.\vec{a}(\vec{r}) = 0 & \vec{r}\epsilon S, \\ \hat{r} \times (\vec{\nabla} \times \vec{a}(\vec{r})) = 0 & \vec{r}\epsilon S. \end{array} \tag{A.41}$$

The eigenmodes come in four polarizations, the scalar, longitudinal, transverse magnetic, and transverse electric which we label by $\Sigma = S, L, M, E$ respectively. We introduce the label

$$p = \{N, J, M\}, \tag{A.42}$$

N, J, M being the radial, angular momentum and magnetic quantum numbers of the gluon, respectively. The Scalar mode has only the zeroth component of a^μ, and, including a phase, it is given by

$$a^0(\Sigma = S, p; \vec{r}) = N_{Sp} i\, j_J(\Omega_{Sp} r) Y_{JM}(\hat{r}). \tag{A.43}$$

The $\Sigma = L, M, E$ modes contain only a spatial part. These modes may be expanded in terms of the vector spherical harmonics,

$$\vec{a}(\Sigma, p; \vec{r}) = N_{\Sigma p} \sum_{L=|J-1|}^{L=J+1} \alpha^\Sigma_{JL}\, j_L(\Omega_{\Sigma p} r)\, \vec{Y}_{JLM}(\hat{r}). \tag{A.44}$$

The non-zero coefficients α^Σ_{JL} are given by

$$\begin{array}{ccc} \alpha^L_{J,J-1} = \sqrt{\frac{J}{2J+1}} & & \alpha^L_{J,J+1} = \sqrt{\frac{J+1}{2J+1}} \\ & \alpha^M_{J,J} = 1 & \\ \alpha^E_{J,J-1} = \sqrt{\frac{J+1}{2J+1}} & & \alpha^E_{J,J+1} = -\sqrt{\frac{J}{2J+1}} \end{array} \tag{A.45}$$

The cavity mode $\vec{a}(\Sigma, p; \vec{r})$ must satisfy whichever of the boundary conditions (A.41) is appropriate. These reduce to eigenvalue equations on the spherical Bessel functions as usual,

$$\frac{d}{dr} j_J(\Omega_{Sp} r)|_{r=R} = 0, \tag{A.46}$$

$$\frac{d}{dr}[r j_J(\Omega_{Mp} r)]|_{r=R} = 0, \tag{A.47}$$

$$j_J(\Omega_{Ep} r)|_{r=R} = 0, \tag{A.48}$$

and $\Omega_{Sp} = \Omega_{Lp}$. The normalization constants are given by

$$N^{-2}_{Sp} = N^{-2}_{Lp} = R^3 \tfrac{1}{2} j^2_J(\bar{\Omega}_{Sp})[1 - \frac{J(J+1)}{\bar{\Omega}^2_{Sp}}], \tag{A.49}$$

$$N^{-2}_{Mp} = R^3 \tfrac{1}{2} j^2_J(\bar{\Omega}_{Mp})[1 - \frac{J(J+1)}{\bar{\Omega}^2_{Mp}}], \tag{A.50}$$

$$N^{-2}_{Ep} = R^3 \tfrac{1}{2} j^2_{J+1}(\bar{\Omega}_{Ep}). \tag{A.51}$$

For compactness of notation we will usually refer to the four vector cavity mode $a^\mu(\Sigma, p; \vec{r})$. This includes only the time, or space components of the cavity mode, as appropriate for the particular polarization.

It is useful to introduce the metric tensor in polarization space,

$$g^{\Sigma\Sigma'} = \begin{cases} 1 & \Sigma = \Sigma' = S \\ -1 & \Sigma = \Sigma' = L, M, E \\ 0 & \Sigma \neq \Sigma' \end{cases}, \tag{A.52}$$

which enables us to write the orthonormality and completeness relations simply as (but see discussion in Appendix D)

$$\int d\vec{r}\, g^{\mu\nu} a^*_\mu(\Sigma', p; \vec{r}) a_\nu(\Sigma, p; \vec{r}) = g^{\Sigma\Sigma'} \delta(p, p'), \tag{A.53}$$

$$\sum_{\Sigma p} g^{\Sigma\Sigma} a^{\mu}(\Sigma, p; \vec{r}) a^{\nu *}(\Sigma, p; \vec{r}') = g^{\mu\nu} \delta(\vec{r}, \vec{r}'). \tag{A.54}$$

Finally we note a few miscellaneous identities. The scalar and longitudinal modes are related by

$$\vec{a}(L, p; \vec{r}) = \frac{-i}{\Omega_{Sp}} \vec{\nabla} a^0(S, p; \vec{r}), \tag{A.55}$$

$$a^0(S, p; \vec{r}) = \frac{-i}{\Omega_{Sp}} \vec{\nabla} \cdot \vec{a}(L, p; \vec{r}) \tag{A.56}$$

We may usefully define the phase η_Σ as

$$\eta_\Sigma = \begin{cases} +1 & \Sigma = L, E \\ -1 & \Sigma = S, M \end{cases} . \tag{A.57}$$

If we define $p^* = \{N, J, -M\}$ as changing the sign of the magnetic quantum number, then under complex conjugation the gluon cavity mode behaves as

$$[a^{\mu}(\Sigma, p; \vec{r})]^* = \eta_\Sigma (-1)^M a^{\mu}(\Sigma, p^*; \vec{r}). \tag{A.58}$$

A.5 The Cavity Modes in Momentum Space

It is sometimes useful to transform the usual cavity modes depending on the argument $\vec{r}$, to depend on the three-momentum $\vec{k}$. We have three different representations of any function defined in the cavity, namely configuration space $x = \{t, \vec{r}\}$, cavity mode space $q = \{\omega, p\}$, and plane wave momentum space $s = \{\omega, \vec{k}\}$. (Planar momentum space is, of course, overcomplete, for a description of functions defined only in the cavity.) We begin by defining the four vector transform

$$\phi(q; x) = \int \frac{d^4 s}{(2\pi)^{4/2}} \phi(q; s)\, e^{-isx}. \tag{A.59}$$

We choose a normalization that preserves $< \omega, p | \omega', p' >= \delta(\omega, \omega') \delta_{pp'}$. The time-frequency part is always trivial, so we concentrate on the space part,

$$\phi(p; \vec{r}) = \int \frac{d\vec{k}}{(2\pi)^{3/2}} \phi(p; \vec{k})\, e^{i\vec{k}.\vec{r}}. \tag{A.60}$$

$$\phi(p; \vec{k}) = \int \frac{d\vec{r}}{(2\pi)^{3/2}} \phi(p; \vec{r})\, e^{-i\vec{k}.\vec{r}} \beta(r). \tag{A.61}$$

The function $\beta(r) = 1 - \theta(r - R)$ is zero outside the cavity, and both integrations take place over all space. Once again we use the Raleigh relation (C.39)

$$e^{i\vec{k}.\vec{r}} = 4\pi \sum_{lm} i^l j_l(kr) Y^*_{lm}(\hat{k}) Y_{lm}(\hat{r}), \tag{A.62}$$

and can easily show that the Fourier transform of

$$f(\vec{r}) = j_l(k_p r) Y_{lm}(\hat{r}), \tag{A.63}$$

is simply

$$f(\vec{k}) = (-i)^l \sqrt{\frac{2}{\pi}} t_l(k, k_p) Y_{lm}(\hat{k}), \tag{A.64}$$

where we note the integral [4],

$$t_l(k, k_p) = \int_0^R r^2 dr\ j_l(kr) j_l(k_p r) \tag{A.65}$$

$$= \frac{R^2}{k^2 - k_p^2} \left[k_p j_l(kR) j_{l-1}(k_p R) - k j_{l-1}(kR) j_l(k_p R) \right]. \tag{A.66}$$

$t_l(k, k_p)$ has a maximum at k_p in the variable k, and has zeros at other eigenergies, corresponding to orthogonality. The Dirac case can be done similarly by noting the spinor generalization of the Raleigh relation.

$$e^{i\vec{k}.\vec{r}} \delta^{\alpha\beta} = 4\pi \sum_{\kappa\mu} i^l j_l(kr) \chi^{\dagger\alpha}_{\kappa\mu}(\hat{k}) \chi^{\beta}_{\kappa\mu}(\hat{r}) \tag{A.67}$$

A.6 $< \not{s} - m >$

In section 3.2.4 we considered the identity $< s^2 - m^2 >$ for scalar fields. We need the corresponding identity for Dirac fields, which we derive here, based on the derivation in [4]. If we let

$$A = \int d\vec{k}\ \bar{u}(p'; \vec{k}) [\ \not{s} - m] u(p; \vec{k}), \tag{A.68}$$

we would like to show

$$A = \delta(p', p)[\omega - \varepsilon_p], \tag{A.69}$$

where as usual $s^\mu = \{\omega, \vec{k}\}$. The usual wave function $u(p; \vec{r})$ may be considered as

$$u(p; \vec{r}) = U(p; \vec{r}) \beta(r) \tag{A.70}$$

where $U(p; \vec{r})$ is continuous at the boundary, and $\beta(r)$ is the usual step function. It may then be shown that

$$\left[\gamma^0 \varepsilon_p + i\vec{\nabla}.\vec{\gamma} - m \right] u(p; \vec{r}) = i\hat{r}.\vec{\gamma}\, U(p; \vec{r}) \delta(r - R) \tag{A.71}$$

After a Fourier transform equation (A.68) becomes

$$A = \int d\vec{r}\, \bar{u}(p'; \vec{r}) \left[\gamma^0 \omega + i\vec{\nabla}.\vec{\gamma} - m \right] u(p; \vec{r}), \tag{A.72}$$

which we now see gives

$$A = \delta(p', p)[\omega - \varepsilon_p] + \int d\vec{r}\, \bar{u}(p; \vec{r}) [i\hat{r}.\vec{\gamma}] u(p; \vec{r}) \delta(r - R). \tag{A.73}$$

From the boundary conditions the second term is zero, and the result follows.

Appendix B

Vertex Integrals

In this section we define various vertex integrals, and reduce them to a form appropriate for the computer. Before doing so, we list some useful angular integrals.

B.1 Angular Integrals

We need a number of spherical integrals in a cavity, for convenience we list them here. $\hat{J}$ is shorthand for $\sqrt{J(J+1)}$, and we use the usual 3j and 6j conventions of Edmonds, [57].

$$\int d\hat{r}\, Y_{lm} Y_{l'm'} Y_{l''m''} = \frac{1}{\sqrt{4\pi}} \hat{l}\hat{l}'\hat{l}'' \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}, \tag{B.1}$$

$$\begin{aligned} \int d\hat{r}\, \chi^{\dagger}_{\kappa\mu} Y_{JM} \chi_{\kappa'\mu'} &= (-1)^{\mu+1/2} \frac{1}{\sqrt{4\pi}} \hat{j}\hat{J}\hat{j}' \frac{(-1)^{l+J+l'}+1}{2} \\ &\times \begin{pmatrix} j & J & j' \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} j & J & j' \\ -\mu & M & \mu' \end{pmatrix} \end{aligned} \tag{B.2}$$

$$\begin{aligned} \int d\hat{r}\, \vec{Y}_{JLM} \cdot \vec{Y}_{J'L'M'} Y_{J''M''} &= (-1)^{J+L} \frac{1}{\sqrt{4\pi}} \hat{J}\hat{J}'\hat{J}''\hat{L}\hat{L}' \begin{pmatrix} J & J' & J'' \\ M & M' & M'' \end{pmatrix} \\ &\times \begin{pmatrix} L & J'' & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} J' & J'' & J \\ L & 1 & L' \end{Bmatrix}, \end{aligned} \tag{B.3}$$

$$\begin{aligned} \int d\hat{r}\, Y_{JM} Y_{J'M'} Y_{J''M''} Y_{J'''M'''} &= \sum_{k\kappa} \frac{1}{4\pi} (-1)^{\kappa} (2k+1) \hat{J}\hat{J}'\hat{J}''\hat{J}''' \\ \times \begin{pmatrix} J & J' & k \\ M & M' & \kappa \end{pmatrix} & \begin{pmatrix} J'' & J''' & k \\ M'' & M''' & -\kappa \end{pmatrix} \begin{pmatrix} J & J' & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} J'' & J''' & k \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{B.4}$$

$$\int d\hat{r}\, \vec{Y}_{JLM} \cdot \vec{Y}_{J'L'M'} Y_{J''M''} Y_{J'''M'''} = \sum_{k\kappa} \frac{1}{4\pi} (-1)^{\kappa+k+J+L'}$$

$$\times(2k+1)\hat{J}\hat{J}'\hat{J}''\hat{J}'''\hat{L}\hat{L}' \begin{pmatrix} J & J' & k \\ M & M' & \kappa \end{pmatrix} \begin{pmatrix} J'' & J''' & k \\ M'' & M''' & -\kappa \end{pmatrix}$$
$$\times \begin{pmatrix} J'' & J''' & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & L' & k \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} J & J' & k \\ L' & L & 1 \end{Bmatrix} \tag{B.5}$$

$$\int d\hat{r}\, \vec{Y}_{JLM} \cdot \vec{Y}_{J'L'M'}\, \vec{Y}_{J''L''M''} \cdot \vec{Y}_{J'''L'''M'''} = \sum_{k\kappa} \frac{1}{4\pi} (-1)^{\kappa+J+L'+J''+L'''}$$
$$\times(2k+1)\hat{J}\hat{J}'\hat{J}''\hat{J}'''\hat{L}\hat{L}'\hat{L}''\hat{L}''' \begin{pmatrix} J & J' & k \\ M & M' & \kappa \end{pmatrix} \begin{pmatrix} J'' & J''' & k \\ M'' & M''' & -\kappa \end{pmatrix}$$
$$\times \begin{pmatrix} L & L' & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L'' & L''' & k \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} J & J' & k \\ L' & L & 1 \end{Bmatrix} \begin{Bmatrix} J'' & J''' & k \\ L''' & L'' & 1 \end{Bmatrix} \tag{B.6}$$

B.2 The ϕ^4 Vertex

We define the vertex integral for the 4 scalar field vertex as

$$Q(p_1, p_2, p_3) = \sum_{m_3} \int d\vec{r}\, \phi^*(p_1; \vec{r}) \phi(p_2; \vec{r}) \phi^*(p_3; \vec{r}) \phi(p_3; \vec{r}). \tag{B.7}$$

If we note the identity

$$\frac{2l+1}{4\pi} = \sum_m Y_{lm}(\hat{r}) Y^*_{lm}(\hat{r}), \tag{B.8}$$

and exploit the orthogonality of the Y_{lm}, then we get

$$Q(p_1, p_2, p_3) = \delta_{l_1 m_1, l_2 m_2} \frac{2l_3+1}{4\pi} \int r^2 dr\, j_{l_1}(k_{p_1} r) j_{l_2}(k_{p_2} r) j^2_{l_3}(k_{p_3} r) \tag{B.9}$$

B.3 The qqg Vertex

The vertex integrals that describe the absorption or emission of a gluon by a quark in the cavity are defined as

$$Q^{\Sigma p_3}_{p_1 p_2} = i \int d\vec{r}\, \bar{u}(p_1; \vec{r}) \gamma_\mu u(p_2; \vec{r}) a^\mu(\Sigma, p_3; \vec{r}), \tag{B.10}$$

and a related factor for the complex conjugate gluon, (see (A.58),

$$\tilde{Q}^{\Sigma p_3}_{p_1 p_2} = i \int d\vec{r}\, \bar{u}(p_1; \vec{r}) \gamma_\mu u(p_2; \vec{r}) a^{\mu *}(\Sigma, p_3; \vec{r}) = (-1)^M \eta_\Sigma Q^{\Sigma p_3^*}_{p_1, p_2} = -Q^{\Sigma p_3}_{p_2 p_1}. \tag{B.11}$$

We use a notation consistent with [2], and below we present a short summary of how to calculate this quantity. The radial and angular dependence may be separated as follows

$$\begin{aligned} Q^{\Sigma p_3}_{p_1 p_2} &= R^{\Sigma p_3}_{p_1 p_2} \int d\hat{r}\, \chi^\dagger_{\kappa_1 \mu_1}(\hat{r}) Y_{J_3 M_3}(\hat{r}) \chi_{\kappa_2 \mu_2}(\hat{r}) \qquad \Sigma = S, L, E, \\ Q^{\Sigma p_3}_{p_1 p_2} &= R^{\Sigma p_3}_{p_1 p_2} \int d\hat{r}\, \chi^\dagger_{\kappa_1 \mu_1}(\hat{r}) Y_{J_3 M_3}(\hat{r}) \chi_{-\kappa_2 \mu_2}(\hat{r}) \qquad \Sigma = M. \end{aligned} \tag{B.12}$$

The angular integral may be found in section B.1. The radial matrix elements are given by

$$R^{Sp}_{p_1p_2} = -N_{Sp}\int_0^R dr\, r^2\, j_J(\Omega_{Sp}r)\, S_{p_1p_2}(r) \tag{B.13}$$

$$R^{Lp}_{p_1p_2} = \frac{\varepsilon_{p_2}-\varepsilon_{p_1}}{\Omega_{\Sigma p}} R^{Sp}_{p_1p_2} \tag{B.14}$$

$$R^{Mp}_{p_1p_2} = \frac{\kappa+\kappa'}{\sqrt{J(J+1)}} N_{Mp}\int_0^R dr\, r^2\, j_J(\Omega_{Mp}r)\, T_{p_1p_2}(r) \tag{B.15}$$

$$R^{Ep}_{p_1p_2} = \frac{N_{Ep}}{\Omega_{Ep}\sqrt{J(J+1)}}\int_0^R dr\, r^2\, \{J(J+1)\, j_J(\Omega_{Ep}r) U_{p_1p_2}(r) \tag{B.16}$$

$$+(\kappa-\kappa')[J j_J(\Omega_{Ep}r) - \Omega_{Ep} r\, j_{J-1}(\Omega_{Ep}r)] T_{p_1p_2}(r)\} \tag{B.17}$$

Here we have introduced the radial functions

$$S_{p_1p_2} = g_{p_1}g_{p_2} + f_{p_1}f_{p_2} \tag{B.18}$$

$$T_{p_1p_2} = g_{p_1}f_{p_2} + f_{p_1}g_{p_2} \tag{B.19}$$

$$U_{p_1p_2} = g_{p_1}f_{p_2} - f_{p_1}g_{p_2} \tag{B.20}$$

which are given in terms of the radial wave functions of the quarks in the initial and final state, as defined in equations (A.29) and (A.30). It is useful to attach the parity selection rule, which arises from the angular integral (B.2), to the radial integral, thereby defining

$$S^{\Sigma p}_{p_1p_2} = \frac{(-1)^{l+J+l'}+1}{2} R^{\Sigma p}_{p_1p_2} \tag{B.21}$$

B.3.1 The qqg Sum Rule

The factor $M(p,p',p_1,p_2,\Sigma_2)$, is the most vital ingredient in the numerical calculation of the quark self energy. To evaluate it is a fairly lengthy analytical and computational task, and it is therefore worthwhile to have a check on this quantity, if we wish to present our results with any degree of confidence. A simple sum rule provides a strong check. M is defined by

$$M(p,p',p_1,p_2,\Sigma_2) = \sum_{\mu_1 M_2} 4\pi g^{\Sigma_2\Sigma_2} Q^{\Sigma_2p_2}_{pp_1}\tilde{Q}^{\Sigma_2p_2}_{p_1p'}. \tag{B.22}$$

p and p' label the incoming and outgoing quark, p_1 labels the intermediate quark, and p_2, Σ_2 label the intermediate gluon. By using the results of the previous section, and simple angular momentum identities, we get for $j=j'$

$$M(p,p',p_1,p_2,\Sigma_2) = g^{\Sigma_2\Sigma_2} S^{\Sigma_2p_2}_{pp_1} S^{\Sigma_2p_2}_{p_1p'}(2j_1+1)(2J_2+1)\begin{pmatrix} j_1 & J_2 & j' \\ \frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix}^2 \delta_{jm,j'm'}. \tag{B.23}$$

This expression we may take directly to the computer. To provide a check on the results, we obtain a sum rule by summing over the intermediate quark quantum numbers p_1, and using the quark completeness relation, equation (A.39), followed by some trace algebra,

$$\sum_{p_1} M(p,p',p_1,p_2,\Sigma_2) = -4\pi\sum_{M_2}\int d\vec{r}\, \psi^\dagger(p;\vec{r})\psi(p';\vec{r})A^\mu(\Sigma_2,p_2;\vec{r})A^*_\mu(\Sigma_2,p_2;\vec{r}). \tag{B.24}$$

The angular momentum algebra is straight forward, and we get

$$\sum_{p_1} M(p,p',p_1,p_2,\Sigma_2) = -(2J_2+1)\int dr\, r^2\, \Phi(\Sigma_2,p_2;r) \times(g(p;r)g(p';r)+f(p;r)f(p';r)), \quad \text{(B.25)}$$

where the part due to the gluon, Φ, is given by

$$\begin{aligned}
\Phi(S,p) &= N_{Sp}^2\, j_J^2(\Omega_{Sp} r), \\
\Phi(L,p) &= -N_{Lp}^2\left[\frac{J+1}{2J+1} j_{J+1}^2(\Omega_{Lp} r) + \frac{J}{2J+1} j_{J-1}^2(\Omega_{Lp} r)\right], \\
\Phi(M,p) &= -N_{Mp}^2\, j_J^2(\Omega_{Mp} r), \\
\Phi(E,p) &= -N_{Ep}^2\left[\frac{J}{2J+1} j_{J+1}^2(\Omega_{Ep} r) + \frac{J+1}{2J+1} j_{J-1}^2(\Omega_{Ep} r)\right]. \qquad \text{(B.26)}
\end{aligned}$$

The right hand side of equation (B.24) is easy to calculate numerically, by an entirely independent computer program. The numerical agreement usually depends on how many quark modes are included in the sum on the left hand side. Summing up about 12 modes already gives agreement to 5 decimal places. In the calculation described in section 4.2.4, the sum is suppressed by an exponential factor, and the accuracy of Σ is considerably higher.

B.4 Vacuum Polarization

In this Appendix we calculate the quantity $\Pi^C(\Sigma,q;\Sigma',q';z)$. For convenience we define A, B, and C,

$$A^{\mu\nu}(x,y) = (-ie^2) i\partial_x^\mu \Delta(x,y) i\partial_y^\nu \Delta(y,x) \qquad \text{(B.27)}$$

$$B^{\mu\nu}(x,y) = (-ie^2)\Delta(x,y) i\partial_x^\mu i\partial_y^\nu \Delta(y,x) \qquad \text{(B.28)}$$

$$C^{\mu\nu}(x,y) = e^2 g^{\mu\nu} i\Delta(x,y)\delta(x,y) \qquad \text{(B.29)}$$

and the vacuum polarization is given by

$$\Pi^{\mu\nu}(x,y) = 2A^{\mu\nu}(x,y) - 2B^{\mu\nu}(x,y) + 2C^{\mu\nu}(x,y) \qquad \text{(B.30)}$$

What we need is the cavity mode version of each of the above, e. g. $A(\Sigma,q;\Sigma',q';z)$, and the derivation is lengthy but straightforward. We convert to cavity mode space, insert the scalar propagators, seperate into a (spatial) vertex part, and a part that will be z dependent. We give only the result, but introduce some notation to make it manageable. We start off by defining the vertex integral Q, in terms of the vertex integral that we will need,

$$\frac{1}{\sqrt{4\pi}} Q(\Sigma,p;p_1,p_2)\hat{J}\begin{pmatrix} J & J_1 & J_2 \\ M & M_1 & M_2 \end{pmatrix} = \int d\vec{r}\, a_\mu(\Sigma,p;\vec{r})[d^\mu\phi(p_1;\vec{r})]\phi(p_2;\vec{r}). \quad \text{(B.31)}$$

Here we use $d^\mu = \{i, \nabla\}$. Noting that $\vec{\nabla}\phi(p_1;\vec{r}) = \Omega_{p_1}\vec{a}(L,p_1;\vec{r})$ and using the angular integral (B.3), we get finally an expression for Q if $\Sigma = L, M, E$,

$$\begin{aligned} Q(\Sigma,p;p_1,p_2) &= \Omega_{p_1} N_{\Sigma p} N_{p_1} N_{p_2} \sum_{LL_1} \alpha^{\Sigma}_{JL}\alpha^{L}_{J_1 L_1} \int dr\, r^2\, j_L(\Omega_{\Sigma p} r) j_{L_1}(\Omega_{p_1} r) j_{J_2}(\Omega_{p_2} r) \\ &\quad \times(-)^{L+J} \hat{J}_1 \hat{J}_2 \hat{L}_1 \hat{L} \begin{pmatrix} L & J_2 & L_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} J_1 & J_2 & J \\ L & 1 & L_1 \end{Bmatrix} \end{aligned} \tag{B.32}$$

For the scalar mode we get instead

$$\begin{aligned} Q(S,p;p_1,p_2) &= -N_{Sp} N_{p_1} N_{p_2} \int dr\, r^2\, j_J(\Omega_{Sp} r) j_{J_1}(\Omega_{p_1} r) j_{J_2}(\Omega_{p_2} r) \\ &\quad \times \hat{J}_1 \hat{J}_2 \begin{pmatrix} J & J_1 & J_2 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{B.33}$$

Next we define the factors M_A, M_B,

$$\begin{aligned} M_A(\Sigma,p,\Sigma,p';p_1,p_2) &= -4\pi \sum_{M_1 M_2} \int d\vec{r}\, a^*_\mu(\Sigma,p;\vec{r})[d^\mu \phi(p_1;\vec{r})]\phi^*(p_2;\vec{r}) \\ &\quad \times \int d\vec{r}'\, a_\nu(\Sigma',p';\vec{r}')[d^\nu \phi^*(p_2;\vec{r}')]\phi(p_1;\vec{r}'), \end{aligned} \tag{B.34}$$

$$= -\eta_\Sigma\, \delta_{JM,J'M'}\, Q(\Sigma,p;p_1,p_2) Q(\Sigma',p';p_2,p_1). \tag{B.35}$$

$$\begin{aligned} M_B(\Sigma,p,\Sigma,p';p_1,p_2) &= -4\pi \sum_{M_1 M_2} \int d\vec{r}\, a^*_\mu(\Sigma,p;\vec{r})[d^\mu \phi^*(p_2;\vec{r})]\phi(p_1;\vec{r}) \\ &\quad \times \int d\vec{r}'\, a_\nu(\Sigma',p';\vec{r}')[d^\nu \phi(p_2;\vec{r}')]\phi^*(p_1;\vec{r}'), \end{aligned} \tag{B.36}$$

$$\begin{aligned} &= (-)^{1+J+J_1+J_2} \eta_\Sigma\, \delta_{JM,J'M'} \\ &\quad \times Q(\Sigma,p;p_2,p_1) Q(\Sigma',p';p_2,p_1). \end{aligned} \tag{B.37}$$

Now that we have set up some defintions for the vertex part, we turn to the z (or ω) part, we will need

$$\begin{aligned} K_i(\omega,\varepsilon_1,\varepsilon_2) &= \int \frac{d\omega_1}{2\pi} \frac{1}{[\omega_1^2 - \varepsilon_1^2]} \frac{1}{[(\omega-\omega_1)^2 - \varepsilon_2^2]} \\ &\quad \times \{1; (\omega-\omega_1); -\omega_1; (\omega_1 - \omega)\omega_1\} \end{aligned} \tag{B.38}$$

The index $i = 1..4$ labels the four options in the curly bracket. We follow the usual procedure as given in section 4.1.3, rotate to Euclidean, elevate the denominators, apply standard integrals, change the variables of integration, and rotate back to Minkowski. We get finally

$$K_i(\omega,\varepsilon_1,\varepsilon_2;z) = \left(\frac{z}{4\pi}\right)^{1/2} \left\{D; \omega E; -\omega[D+E]; -\frac{1}{2\gamma}D - \omega^2 E + \omega^2 F\right\}. \tag{B.39}$$

where the integrals D,E,F are defined as

$$D = D(a,b,c) = \int_0^1 dt\, e^{at^2+bt+c}, \quad (B.40)$$

$$E = E(a,b,c) = \int_0^1 dt\, te^{at^2+bt+c}, \quad (B.41)$$

$$F = F(a,b,c) = \int_0^1 dt\, t^2 e^{at^2+bt+c}, \quad (B.42)$$

and the variables a,b,c are always

$$a = -\omega^2 z, \quad (B.43)$$

$$b = (\omega^2 - \varepsilon_1^2 + \varepsilon_2^2)z, \quad (B.44)$$

$$c = -\varepsilon_2^2 z. \quad (B.45)$$

The calculation of the integrals D, E, F needs a little care because of under/overflow problems, and subtraction errors, depending on the values of a,b,c. We have a strong check since we may integrate K_i directly by contour integration, and compare this with a numerical integration of $K_i(z)$. We usually get 14 decimal precision for this test, and the explicit values for K_i are

$$K_1(\omega,\varepsilon_1,\varepsilon_2) = \frac{-1}{2\varepsilon_1[(\omega-\varepsilon_1)^2 - \varepsilon_2^2]} + \frac{-1}{2\varepsilon_2[(\omega-\varepsilon_2)^2 - \varepsilon_1^2]}, \quad (B.46)$$

$$K_2(\omega,\varepsilon_1,\varepsilon_2) = \frac{-1}{2\varepsilon_1[\omega - \varepsilon_1 - \varepsilon_2]} - \varepsilon_2 K_1(\omega,\varepsilon_1,\varepsilon_2), \quad (B.47)$$

$$K_3(\omega,\varepsilon_1,\varepsilon_2) = -K_2(\omega,\varepsilon_2,\varepsilon_1) \quad (B.48)$$

$$K_4(\omega,\varepsilon_1,\varepsilon_2) = \frac{-1}{2\varepsilon_2} + \varepsilon_1^2 K_1(\omega,\varepsilon_1,\varepsilon_2) + \omega K_3(\omega,\varepsilon_1,\varepsilon_2) \quad (B.49)$$

Now we may define the full K factors for the A and B terms. We let $V = L, M, E$ denote one of the vector photon polarizations, then, noting the delta function in ω,

$$K(\Sigma,q;\Sigma',q') = \delta(\omega,\omega')K(\Sigma,\omega,p;\Sigma',p') \quad (B.50)$$

we can define

$$\begin{aligned}
K_A(V,\omega,p;V',p';p_1,p_2;z) &= K_1(\omega,\Omega_1,\Omega_2;z) \\
K_A(V,\omega,p;S,p';p_1,p_2;z) &= K_2(\omega,\Omega_1,\Omega_2;z) \\
K_A(S,\omega,p;V',p';p_1,p_2;z) &= K_3(\omega,\Omega_1,\Omega_2;z) \\
K_A(S,\omega,p;S,p';p_1,p_2;z) &= K_4(\omega,\Omega_1,\Omega_2;z) \\
K_B(V,\omega,p;V',p';p_1,p_2;z) &= K_1(\omega,\Omega_1,\Omega_2;z) \\
K_B(V,\omega,p;S,p';p_1,p_2;z) &= K_2(\omega,\Omega_1,\Omega_2;z) \\
K_B(S,\omega,p;V',p';p_1,p_2;z) &= -K_2(\omega,\Omega_1,\Omega_2;z) \\
K_B(S,\omega,p;S,p';p_1,p_2;z) &= -K_4(\omega,\Omega_1,\Omega_2;z) - \omega K_2(\omega,\Omega_1,\Omega_2;z) \quad (B.51)
\end{aligned}$$

Using this notation we may finally write down A and B as a function of z,

$$\begin{aligned} A(\Sigma,q;\Sigma',q';z) &= \alpha\,\delta_{JM,J'M'}\sum_{p_1p_2} M_A(\Sigma,p;\Sigma,p';p_1,p_2) \\ &\quad \times K_A(\Sigma,q;\Sigma',q';p_1,p_2;z) \end{aligned} \tag{B.52}$$

$$\begin{aligned} B(\Sigma,q;\Sigma',q';z) &= \alpha\,\delta_{JM,J'M'}\sum_{p_1p_2} M_B(\Sigma,p;\Sigma,p';p_1,p_2) \\ &\quad \times K_B(\Sigma,q;\Sigma',q';p_1,p_2;z) \end{aligned} \tag{B.53}$$

The z dependent version of C has already been calculated in section 3.2.2, the result is

$$\begin{aligned} C(V,q;V',q';z) &= -\alpha\,\delta_{JM,J'M'}\delta(\omega,\omega')\sum_{p_1} N_{Vp}N_{V'p'}N^2_{p_1}(2J_1+1) \\ &\quad \times \sum_{LL'}\delta_{LL'}\alpha^V_{JL}\alpha^{V'}_{J'L'}\int dr\,r^2\,j_L(\Omega_{Vp}r)j_{L'}(\Omega_{V'p'}r)j_{J_1}(\Omega_{p_1}r) \\ &\quad \times \left(\frac{1}{4\pi z}\right)^{1/2} e^{-\Omega^2_{p_1}z} \end{aligned} \tag{B.54}$$

$$\begin{aligned} C(S,q;S,q';z) &= -\alpha\,\delta_{JM,J'M'}\delta(\omega,\omega')\sum_{p_1} N_{Sp}N_{Sp'}N^2_{p_1}(2J_1+1) \\ &\quad \times \int dr\,r^2\,j_J(\Omega_{Sp}r)j_{J'}(\Omega_{Sp'}r)j_{J_1}(\Omega_{p_1}r) \\ &\quad \times \left(\frac{1}{4\pi z}\right)^{1/2} e^{-\Omega^2_{p_1}z} \end{aligned} \tag{B.55}$$

This completes the set of formula necessary to evaluate $\Pi^C(\Sigma,q;\Sigma',q';z)$ on the computer.

B.4.1 The Vacuum Polarization Sum Rule

We already have shown that we may apply a strong test of the validity of the K functions. The larger part of the calculation is, however, the calculation of the factors M_A and M_B. It is of considerable importance to have a check on these quantities, and this is provided once again by a sum rule. It is obtained in the usual way and is given by

$$\begin{aligned} &\sum_{p_1} M_A(V,p;V',p';p_1,p_2) = \\ &\quad N_{Vp}N_{V'p'}N^2_{p_2}\Omega^2_{p_2}\sum_{LL_2L_2'L'}\alpha^V_{JL}\alpha^L_{J_2L_2}\alpha^L_{J_2L_2'}\alpha^{V'}_{J'L'}(-)^{L+L'+1+L_2+J_2} \\ &\quad \times \int dr\,r^2\,j_L(\Omega_{Vp}r) hello j_{L_2}(\Omega_{p_2}r)j_{L_2'}(\Omega_{p_2}r)j_{L'}(\Omega_{V'p'}r)\sum_k(2k+1)\hat{J}_2\hat{J}_2\hat{L}\hat{L}_2\hat{L}_2'\hat{L}' \\ &\quad \times \begin{pmatrix} L & L_2 & k \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L_2' & L & k \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} J & J_2 & k \\ L_2 & L & 1 \end{Bmatrix}\begin{Bmatrix} J_2 & J' & k \\ L' & L_2' & 1 \end{Bmatrix}. \end{aligned} \tag{B.56}$$

$$\begin{aligned} &\sum_{p_1} M_A(S,p;S,p';p_1,p_2) = \\ &\quad (2J_2+1)N_{Sp}N_{Sp'}N^2_{p_2}\int dr\,r^2\,j_J(\Omega_{Sp}r)j^2_{J_2}(\Omega_{p_2}r)j_{J'}(\Omega_{Sp'}r) \end{aligned} \tag{B.57}$$

We do not bother with a sum rule for S to V transitions, nor with a sum rule for M_B. All of these functions rely in a very simple way on the factors Q, which are strongly tested by the above.

Appendix C

Mathematical Appendices

In this Appendix we supply various useful mathematical identities. Some are rather common, and are included merely for convenience, or to establish notation. Others are a little more obscure.

C.1 The Gamma Function

The gamma function, $\Gamma(w)$, is usually defined as [55],

$$\Gamma(w) = a^w \int_0^\infty dz\, z^{w-1} e^{-az}, \quad Re\, w > 0, \tag{C.1}$$

and has the recurrence relation $\Gamma(w+1) = w\Gamma(w)$. The function is analytic in the $Re\, w > 0$ complex plane, and may be defined in the rest of the plane by analytic continuation. With this definition, the recurrence relation remains valid throughout the entire complex plane. This in turn may be used to define the value of the integral on the right hand side of equation (C.1). The value of the gamma function at some special points is given below

$$\Gamma(n+1) = n! \quad n = 0, 1, 2, \cdots, \tag{C.2}$$

$$\Gamma(1/2) = \sqrt{\pi}. \tag{C.3}$$

The gamma function has poles at $0, -1, -2, \ldots$ Near these poles it may be approximated by,

$$\Gamma(-1+\varepsilon) = -\frac{1}{\varepsilon} + \gamma - 1 + O(\varepsilon), \tag{C.4}$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon), \tag{C.5}$$

$$\Gamma(1+\varepsilon) = 1 - \gamma\varepsilon + O(\varepsilon^2). \tag{C.6}$$

γ is Eulers constant, and is about 0.57721 56649. It is defined as

$$\gamma = \lim_{n\to\infty} \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - ln\, n\right]. \tag{C.7}$$

It is important to note that the divergences encountered in the one loop graphs always crop up at the poles 0 and -1. The integral is however divergent for $w < 0$. This is something of a problem in a numerical calculation, and the following subtraction [54] is very useful for $w < 0$,

$$\Gamma(w) = \int_0^\infty dz\, z^{w-1} \left[e^{-z} - \sum_{k=0}^{n} (-)^k \frac{z^k}{k!} \right], \tag{C.8}$$

where n is the largest integer less than $-Re\, w$.

A function associated with the gamma function is the beta function $B(p,q)$,

$$B(p,q) = \int_0^1 dt\, t^{p-1}(1-t)^{q-1}, \tag{C.9}$$

$$= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{C.10}$$

C.2 Gaussian Integrals and Delta Distributions

The usual Gaussian integral is

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\{-p^2 z\} = \frac{1}{\sqrt{4\pi z}}. \tag{C.11}$$

We will often use the D dimensional version of the Gaussian integrals. Naturally in D-dimensions $p^2 = p_\mu p_\mu$, and we work in Euclidean space. (see discussion in section C.4)

$$\int_{-\infty}^{\infty} \frac{d^D p}{(2\pi)^D} \{1, p, p^2\} \exp\{-p^2 z\} = \left\{1, 0, \frac{1}{2z}\right\} \frac{1}{(4\pi z)^{D/2}}. \tag{C.12}$$

A useful generalization is

$$\int_0^\infty dp\, (p^2)^n \exp\{-p^2 z\} = \frac{1}{2} \frac{\Gamma(n+1/2)}{z^{n+1/2}} \tag{C.13}$$

Another useful generalization is

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\{-p^2 z + ipx\} = \frac{1}{\sqrt{4\pi z}} \exp\left\{-\frac{1}{4}\frac{x^2}{z}\right\} = f(z;x). \tag{C.14}$$

$f(z;x)$ is a useful δ distribution since

$$\lim_{z\to 0} \int_{-\infty}^{\infty} f(z;x) g(x) = g(0) \tag{C.15}$$

For a class of well-behaved functions, namely those that don't blow up for large x, and are smooth near the origin, along with their derivatives, using the above identities it may be shown that, for $z \to 0$

$$\int_{-\infty}^{\infty} f(z;x) g(x) = g(0) + z g''(0) + \frac{z^2}{2!} g^{(4)}(0) + \cdots. \tag{C.16}$$

In the presence of a boundary we would be interested to know

$$\int_0^\infty f(z;x) g(x) = \frac{1}{2} g(0) + \sqrt{\frac{z}{\pi}} g'(0) + \frac{z}{2} g''(0) + \cdots. \tag{C.17}$$

C.2.1 Delta Identities

In this section we record a few identities containing δ or θ funtions.

$$\theta'(x) = \frac{d}{dx}\theta(x) = \delta(x), \tag{C.18}$$

$$\delta'(x) = \frac{d}{dx}\delta(x) = \lim_{h\to 0}\frac{\delta(x+h)-\delta(x-h)}{2h}, \tag{C.19}$$

$$\int_{-\infty}^{\infty} dx\, \delta(x)\theta(x) = \frac{1}{2}, \tag{C.20}$$

$$\int_{-\infty}^{\infty} dx\, \delta'(x) f(x) = -f'(0), \tag{C.21}$$

$$\int_{-\infty}^{\infty} dx\, \delta'(x)\theta(x) = -\int_{-\infty}^{\infty} dx\, \delta^2(x) = \text{undefined}, \tag{C.22}$$

$$\int_{-\infty}^{\infty} dx\, \delta'(x)\theta(x)x = \frac{1}{2}, \tag{C.23}$$

$$\int_{-\infty}^{\infty} dx\, \delta'(x)\theta(x)x^2 = 0. \tag{C.24}$$

C.3 Bessel Functions

Problems with spherical symmetry in some arbitrary dimension usually demand solutions that are composed of Bessel functions [56]. Bessels differential equation is given by

$$\left[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + \left(1 - \frac{\nu^2}{z^2}\right)\right] F_\nu(z) = 0. \tag{C.25}$$

The solutions are given by Bessel functions of the first kind, $J_\nu(z)$, second kind (Neumann function) $N_\nu(z)$, or third kind (Hankel functions), $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, where

$$\begin{aligned} J_\nu(z) &= \left(\frac{z}{2}\right)^{\nu/2} \sum_{k=0}^{\infty} \left(-\frac{z^2}{4}\right)^k \frac{1}{k!\Gamma(\nu+k+1)}, && \text{(C.26)} \\ N_\nu(z) &= \frac{1}{\sin\nu\pi}\left[\cos(\nu\pi)J_\nu(z) - J_{-\nu}(z)\right], && \text{(C.27)} \\ H_\nu^{(1)}(z) &= J_\nu + iN_\nu(z), && \text{(C.28)} \\ H_\nu^{(2)}(z) &= J_\nu - iN_\nu(z). && \text{(C.29)} \end{aligned}$$

The modified Bessel equation is given by

$$\left[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} - \left(1 + \frac{\nu^2}{z^2}\right)\right] F_\nu(z) = 0. \tag{C.30}$$

The solutions are the modified Bessel functions of the first kind $K_\nu(z)$, and the second kind $I_\nu(z)$. We will in particular need $K_\nu(z)$, and these functions are given by

$$K_\nu(z) = \frac{i\pi}{2}\exp\left\{\frac{\pi\nu i}{2}\right\} H_\nu^{(1)}(iz). \tag{C.31}$$

$$I_\nu(z) = \exp\left\{-\frac{\pi\nu i}{2}\right\} J_\nu(iz). \tag{C.32}$$

We will need some properties of the modified Bessel functions of the first and second kind, which we list here for convenience.

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty}\left(\frac{z^2}{4}\right)^k \frac{1}{k!\Gamma(\nu+k+1)}, \tag{C.33}$$

$$K_\nu(z) = \frac{\pi}{2}\frac{I_\nu - I_{-\nu}}{\sin\nu\pi} \tag{C.34}$$

$$K_\nu(z) = K_{-\nu}(z) \tag{C.35}$$

$$\begin{aligned} K_\nu(z) = & \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^\nu\left(1+\frac{z^2}{4}\frac{1}{-\nu+1}+\cdots\right) \\ & -\frac{1}{2}(-\Gamma(-\nu))\left(\frac{z}{2}\right)^\nu\left(1+\frac{z^2}{4}\frac{1}{\nu+1}+\cdots\right) \end{aligned} \tag{C.36}$$

We will also need the integral formula [53],

$$I(\beta,\rho;\nu) = \int_0^\infty dz\, z^{\nu-1}\exp\{-\frac{\beta}{z}-\rho z\}, \qquad \mathrm{Re}\,\beta,\rho > 0, \tag{C.37}$$

$$= 2\left(\frac{\beta}{\rho}\right)^{\nu/2} K_\nu(2\sqrt{\beta\rho}). \tag{C.38}$$

C.4 Dimensional Regularization

We commence this section with some discussion of the D dimensional generalization of the Rayleigh identity. Although we do not actually use the identity, it is useful background to have in mind, when dealing with dimensional regularization. The usual Rayleigh identity, which we use in several places in this thesis is

$$e^{i\vec{k}.\vec{r}} = 4\pi\sum_{lm} i^l j_l(kr) Y_{lm}^*(\hat{k}) Y_{lm}(\hat{r}). \tag{C.39}$$

The identity is useful for Fourier transforming cavity modes into planar momentum space. A D dimensional generalization exists for Euclidean space D-vectors [37], (but note mistake in (A.10) of this reference),

$$e^{2ipy} = \Gamma(\lambda)\sum_{n=0}^{\infty} i^n(n+\lambda) C_n^\lambda(\hat{y}.\hat{p})\,(py)^n f_{n+\lambda}(p^2y^2), \tag{C.40}$$

where $\lambda = D/2-1$, and the function f is defined in terms of a Bessel function by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu f_\nu\left(\frac{z^2}{4}\right). \tag{C.41}$$

The C_n^λ are Gegenbauer polynomials with generating function,

$$(1 - 2rt + r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(t) r^n \tag{C.42}$$

A D dimensional integration may be transformed into spherical variables,

$$\int d^D y = \int dy\, y^{D-1} \int d\hat{y}\, \frac{2\pi^{d/2}}{\Gamma(D/2)}, \tag{C.43}$$

$$\int d\hat{y} = 1. \tag{C.44}$$

Further details may be found in [37]. We simply wish to remark that using these identities, we may provide a derivation of equation (C.12). In addition to this, the functions $(py)^n f_{n+\lambda}(p^2 y^2)$, are clearly the radial wavefunctions of scalar particles confined to a cavity in D dimensions, since e^{2ipy} is a solution of the D dimensional wave equation. The radial wave functions for spinor and vector particles are made up of identical functions. Thus, at least in principle, if not (yet) in practice, any cavity Feynman diagram can be calculated in D dimensions. This observation partially underpins our confidence in proceeding with this approach.

C.4.1 Feynman Integrals

In this section we evaluate some elementary Feynman integrals in the dimensional regularization scheme. Pascual and Tarrach, [34], present these and other integrals in detail, and also many useful identities. The final four integrals are more specialized, and needed only for reflections, similar integrals may be found in [22].

- *Special Note:* In this section the integrals are firstly written down in Minkowski space. Then we usually convert to Euclidean space, and remain there. In the text of the thesis final results are usually converted back to Minkowski space, but intermediate results remain in Euclidean space.

We work in $D = 4 - 2\varepsilon$ dimensions. In this section we assume that all quantities are dimensionless, to recover the dimensional variables one should make the substitution

$$k^2 \to \bar{k}^2 = \frac{k^2}{\mu^2}. \tag{C.45}$$

We start with an integral that we evaluated previously in section 3.1.2, and evaluate A where we let

$$A = \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 - m^2 + i0}. \tag{C.46}$$

We convert to Euclidean space, and elevate the denominator according to,

$$\frac{1}{l^2 + m^2} = \int_0^\infty dz\, e^{-zl^2 - zm^2}, \tag{C.47}$$

then using the Gaussian integral (C.12), we get

$$A = -i\int_0^\infty dz \left(\frac{1}{4\pi z}\right)^{D/2} e^{-m^2 z}, \tag{C.48}$$

$$= -i\left(\frac{1}{4\pi}\right)^{2-\varepsilon}(m^2)^{1-\varepsilon}\Gamma(-1+\varepsilon), \tag{C.49}$$

$$= \frac{i}{(4\pi)^2}m^2\left[\frac{1}{\varepsilon} - \gamma + 1 - \log\frac{m^2}{4\pi}\right]. \tag{C.50}$$

We should note that in the limit $m^2 \to 0$ this integral vanishes, which suggests that we take the massless tadpole integral equal to zero,

$$\int \frac{d^D p}{(2\pi)^D}\frac{1}{p^2} = 0, \tag{C.51}$$

However, strictly speaking, it is defined for no value of the dimensionality, and thus we cannot make an analytical continuation. We should note that the D-dimensional Gaussian integral really supplies a *definition* to the expression on the left hand side of

$$\int_{-\infty}^{\infty} \frac{d^D p}{(2\pi)^D}\exp\{-p^2 z\} = \frac{1}{(4\pi z)^{D/2}}. \tag{C.52}$$

This definition is part of a regularization procedure that *defines* (in a consistent way) rather than *evaluates* divergent integrals. The regularization scheme has been elaborated slightly to establish (C.51) more rigorously by Capper and Leibbrandt [38].

Next we evaluate the simplest two point loop diagram, given by B. A more detailed derivation is given in the text in section 4.1.3.

$$B = \int \frac{d^D l}{(2\pi)^D}\frac{1}{[l^2 - m^2 + i0][(l+k)^2 - m^2 + i0]}. \tag{C.53}$$

Once again we rotate to Euclidean space, and elevate the denominators,

$$B = i\int \frac{d^D l}{(2\pi)^D}\int dt_1 \int dt_2 \exp\{-l^2 t_1 - (l+k)^2 t_2 - m^2(t_1+t_2)\} \tag{C.54}$$

By a shift of the momentum variable

$$l' = l - k\frac{t_2}{t_1+t_2} = (l-k) + k\frac{t_1}{t_1+t_2}, \tag{C.55}$$

and change of variable $t_1 = zt$, $t_2 = z(1-t)$, we may apply the Gaussian integral again to get

$$B = i\int_0^\infty dz\, z\int_0^1 dt \left(\frac{1}{4\pi z}\right)^{D/2}\exp\{-k^2 z\alpha - m^2 z\}, \tag{C.56}$$

$$= \frac{i}{(4\pi)^{D/2}}\left[\frac{1}{\varepsilon} - \gamma - \int dt\, \log(m^2 + k^2\alpha)\right], \tag{C.57}$$

where the final expression is still in Euclidean space, and we use the shorthand $\alpha = t(1-t)$. We now proceed to consider the above integral with tensor additions, the method is identical,

$$\begin{aligned} C^\nu &= \int \frac{d^D l}{(2\pi)^D} \frac{l^\nu}{[l^2 - m^2 + i0][(l+k)^2 - m^2 + i0]}, & \text{(C.58)} \\ &= i\int_0^\infty dz\, z \int_0^1 dt \left(\frac{1}{4\pi z}\right)^{D/2} (-k^\nu t) \exp\{-k^2 z\alpha - m^2 z\}, & \text{(C.59)} \\ &= \frac{i}{(4\pi)^{D/2}} k^\nu \left[\frac{-1}{2\varepsilon} + \frac{\gamma}{2} + \int dt\, t \log(m^2 + k^2\alpha)\right], & \text{(C.60)} \end{aligned}$$

$$\begin{aligned} D^{\nu\sigma} &= \int \frac{d^D l}{(2\pi)^D} \frac{l^\nu l^\sigma}{[l^2 - m^2 + i0][(l+k)^2 - m^2 + i0]}, & \text{(C.61)} \\ &= i\int_0^\infty dz\, z \int_0^1 dt \left(\frac{1}{4\pi z}\right)^{D/2} \left[k^\nu k^\sigma t^2 + \delta^{\nu\sigma}/2z\right] \times & \text{(C.62)} \\ &\quad \exp\{-k^2 z\alpha - m^2 z\}, \\ &= \frac{i}{(4\pi)^{D/2}} \left\{ k^\nu k^\sigma \left[\frac{1}{3\varepsilon} - \frac{\gamma}{3} - \int dt\, t^2 \log(m^2 + k^2\alpha)\right] \right. & \text{(C.63)} \\ &\quad \left. - g^{\nu\sigma} \left[\left(m^2 + \frac{k^2}{6}\right)\left(\frac{1}{2\varepsilon} - \frac{\gamma - 1}{2}\right) - \frac{1}{2}\int dt\, (k^2\alpha + m^2)\log(m^2 + k^2\alpha)\right]\right\}. \end{aligned}$$

The tensor $D^{\nu\sigma}$ is used directly to calculate the vacuum polarization in scalar QED, but one needs in addition the identities (equation C.56 of [34]),

$$I_n = \int_0^1 dt\, t^n \log[u - t(1-t)]. \qquad \text{(C.64)}$$

Some results for this identity are given below, where $w = \sqrt{1 - 4u}$,

$$\begin{aligned} I_0 &= -2 + \log u + w \log \frac{w+1}{w-1}, & \text{(C.65)} \\ I_1 &= \frac{I_0}{2} & \text{(C.66)} \\ I_2 &= \frac{1}{3}\left[-\frac{13}{6} + 2u + \log u + (1-u) w \log \frac{w+1}{w-1}\right]. & \text{(C.67)} \end{aligned}$$

We need a slightly more complicated version of the above Feynman integrals to discuss the case of reflections. The method of evaluation is similar except that instead of the Gamma function the z integral gives a modified Bessel function (see section C.3),

$$\begin{aligned} I(\beta, \rho; \nu) &= \int_0^\infty dz\, z^{\nu-1} \exp\{-\frac{\beta}{z} - \rho z\}, & \text{(C.68)} \\ &= 2\left(\frac{\beta}{\rho}\right)^{\nu/2} K_\nu(2\sqrt{\beta\rho}). & \text{(C.69)} \end{aligned}$$

In the limit $\beta \to 0$ we naturally recover

$$\lim_{\beta \to 0} I(\beta, \rho; \nu) = \rho^{-\nu}\Gamma(\nu). \qquad \text{(C.70)}$$

The integrals we need are listed below. In all cases the first line is in Minkowski space, and subsequent lines in Euclidean space.

$$\begin{aligned} A(x) &= \int \frac{d^D l}{(2\pi)^D} e^{-ilx} \frac{1}{l^2 - m^2 + i0}, & \text{(C.71)} \\ &= -\frac{i}{(4\pi)^{D/2}} I(\frac{x^2}{4}, m^2; -1+\varepsilon), & \text{(C.72)} \end{aligned}$$

In the next three integrals, we shorten

$$I(\nu) = I(\frac{x^2}{4}, m^2 + k^2\alpha; \nu). \qquad \text{(C.73)}$$

$$\begin{aligned} B(x) &= \int \frac{d^D l}{(2\pi)^D} \frac{e^{-ilx}}{[l^2 - m^2 + i0][(l+k)^2 - m^2 + i0]}, & \text{(C.74)} \\ &= \frac{i}{(4\pi)^{D/2}} \int_0^1 dt\, e^{-iktx} I(\varepsilon). & \text{(C.75)} \end{aligned}$$

$$\begin{aligned} C^\sigma(x) &= \int \frac{d^D l}{(2\pi)^D} \frac{e^{-ilx} l^\sigma}{[l^2 - m^2 + i0][(l+k)^2 - m^2 + i0]}, & \text{(C.76)} \\ &= \frac{i}{(4\pi)^{D/2}} \int_0^1 dt\, e^{-iktx} \left[\frac{i}{2} x^\sigma I(-1+\varepsilon) - k^\sigma t I(\varepsilon)\right]. & \text{(C.77)} \end{aligned}$$

$$\begin{aligned} D^{\sigma\tau}(x) &= \int \frac{d^D l}{(2\pi)^D} \frac{e^{-ilx} l^\sigma l^\tau}{[l^2 - m^2 + i0][(l+k)^2 - m^2 + i0]}, & \text{(C.78)} \\ &= \frac{i}{(4\pi)^{D/2}} \int_0^1 dt\, e^{-iktx} \left[\frac{1}{2}\delta^{\sigma\tau} I(-1+\varepsilon) - \frac{1}{4} x^\sigma x^\tau I(-2+\varepsilon)\right. \\ &\quad \left. - \frac{it}{2}(k^\sigma x^\tau + x^\sigma k^\tau) I(-1+\varepsilon) + t^2 k^\sigma k^\tau I(\varepsilon)\right]. & \text{(C.79)} \end{aligned}$$

All these integrals may be checked by seeing that in the limit $x \to 0$ they give the correct result. We need the equation (C.70) to do this.

C.4.2 Ambiguities in Subtraction Factors

In section 5.1.3 we mentioned that the factor $D(z)$ that we use to generate the analytical continuation is in fact arbitrary. In this Appendix we show an example of this arbitraryness. We consider the tadpole diagram,

$$A = \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 - m^2 + i0}. \qquad \text{(C.80)}$$

As we have shown in the previous section the 'z-form' and the full result are given by

$$\begin{aligned} A &= -i \int_0^\infty dz \left(\frac{1}{4\pi z}\right)^{D/2} e^{-m^2 z}, & \text{(C.81)} \\ &= \frac{i}{(4\pi)^2} m^2 \left[\frac{1}{\varepsilon} - \gamma + 1 - \log\frac{m^2}{4\pi}\right]. & \text{(C.82)} \end{aligned}$$

Now, suppose that we rescale the variable of integration so that $z \rightarrow sz$. Now the new z form will look like

$$A = \int_0^\infty dz\, A(z) = -is^{1-D/2} \int_0^\infty dz \left(\frac{1}{4\pi z}\right)^{D/2} e^{-sm^2 z}, \tag{C.83}$$

If we proceed to choose $D(z)$ and $S(z)$ according to the usual procedure as laid out in section 5.1.3, then we will get

$$\begin{aligned} D(z) &= -is(4\pi s)^{-2+\varepsilon}(z^{-2+\varepsilon}), && \text{(C.84)} \\ S(z) &= -is(4\pi s)^{-2+\varepsilon}(-sm^2 z^{-1+\varepsilon} e^{-z}), && \text{(C.85)} \\ F(z) &= A(z) - D(z) - S(z). && \text{(C.86)} \end{aligned}$$

At this stage we see that $D(z)$ is multiplied by an arbitrary constant because we may choose the rescaling factor to be anything we like. In contrast $S(z)$ is multiplied by a factor s^ε, which will not affect the divergent piece, but will simply contribute a finite part. If we now evaluate the factors S and F, we get

$$\begin{aligned} S &= \frac{i}{(4\pi)^2} m^2 \left(\frac{1}{\varepsilon} - \gamma + \log 4\pi + \log s\right), && \text{(C.87)} \\ F &= \frac{i}{(4\pi)^2} m^2 \left(1 - \log m^2 - \log s\right), && \text{(C.88)} \\ A &= F + S, && \text{(C.89)} \\ &= \frac{i}{(4\pi)^2} m^2 \left(\frac{1}{\varepsilon} - \gamma + 1 - \log \frac{m^2}{4\pi}\right). && \text{(C.90)} \end{aligned}$$

Thus we finally see that even though the factors $D(z)$ and $S(z)$ have some arbitraryness, provided that one carries through the calculation consistently, the correct answer for A is obtained, and is independent of the rescaling factor s.

This is not the only way of changing the value one gets for $D(z)$ or $S(z)$. We note that the tadpole may also be obtained by using

$$A = \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 + i0} = g_{\mu\nu} \int \frac{d^D l}{(2\pi)^D} \frac{(l+k)^\mu (l+k)^\nu}{[l^2 + i0][(l+k)^2 + i0}], \tag{C.91}$$

and then using the standard integrals from the previous section, to get the 'z-form', followed by the appropriate choice of $D(z)$ and $S(z)$. By following this procedure one obtains a different $D(z)$ and $S(z)$, but the final result remains the same.

At this point the reader may object. The massless tadpole has a problem which we discussed in the previous section. Capper and Leibbrandt [38] point out that because of nonoverlapping or nonexistent regions of analyticity the above equation is fallacious. They offer a prescription to avoid this. In our case we note that for ease of computation, but perhaps not mathematical rigour, we retain the mass, and let this go to zero at the appropriate intermediate step. In this way we can get the desired answers out of the above expression.

But as we have seen the rescaling ambiguity exists for the massive (well defined) tadpole. Therefore we see that the ambiguity in D, and the problems with regions of analyticity are unrelated.

Appendix D

The Coulomb Interaction

There are some subtleties associated with the Coulomb interaction, usually relegated to footnote status in papers, in particular [4,9]. There are three things that we would like to note. The exact value of the quark self energy is in fact arbitrary, due the Coulomb part. There is an inconsistency in applying the boundary conditions to the Coulomb Green's function, and there is a zero energy scalar mode. A detailed understanding of the Coulomb interaction is necessary to resolve these problems.

We start by examining the free space case, and showing how the Coulomb interaction emerges from the covariant form [48]. The gauge propagator is given by

$$D^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2 + i0}. \tag{D.1}$$

We may formulate this propagator differently by introducing a complete set of polarization vectors ε_r:

$$\begin{aligned} \varepsilon_0^\mu &= n^\mu = \{1,0,0,0\} \\ \varepsilon_1^\mu &= \{0,1,0,0\} \\ \varepsilon_2^\mu &= \{0,0,1,0\} \\ \varepsilon_3^\mu &= \{0,0,0,1\} \end{aligned} \tag{D.2}$$

The z axis is always chosen as $\hat{k}$. These four modes correspond to a scalar, two transverse, and a longitudinal polarization respectively. The longitudinal mode may be written in covariant form as

$$\varepsilon_3^\mu = \frac{k^\mu - (kn)n^\mu}{\sqrt{(kn)^2 - k^2}}. \tag{D.3}$$

If we introduce a metric η^r, such that

$$\eta^0 = 1, \qquad \eta^1 = \eta^2 = \eta^3 = -1, \tag{D.4}$$

then the polarizations are complete

$$\sum_r \eta^r \varepsilon_r^\mu \varepsilon_r^\nu = g^{\mu\nu}. \tag{D.5}$$

We may now write the propagator as

$$D^{\mu\nu}(k) = \frac{1}{k^2 + i0}\left\{\sum_{r=1}^{2} \varepsilon_r^\mu \varepsilon_r^\nu + \frac{[k^\mu - (kn)n^\mu][k^\nu - (kn)n^\nu]}{(kn)^2 - k^2} - n^\mu n^\nu\right\}. \tag{D.6}$$

The propagator will be sandwiched between conserved currents, so that $k_\mu j^\mu = 0$, thus we may ignore the parts of the propagator that contain k_μ. We also ignore the part of the propagator corresponding to the exchange of physical transverse photons. The remainder becomes

$$D^{\mu\nu}_{Coul}(k) = \frac{n^\mu n^\nu}{(kn)^2 - k^2}. \tag{D.7}$$

After a Fourier transform, for $x = \{t, \vec{r}\}$ as usual, we get

$$\begin{aligned} D^{\mu\nu}_{Coul}(x) &= n^\mu n^\nu \int \frac{d\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}.\vec{r}}}{\vec{k}^2} \int \frac{dk_0}{2\pi} e^{-ik_0 t}, & \text{(D.8)} \\ &= n^\mu n^\nu \frac{1}{4\pi|\vec{r}|} \delta(t), & \text{(D.9)} \\ &= n^\mu n^\nu G(\vec{r}) \delta(t). & \text{(D.10)} \end{aligned}$$

We take this to define the instantaneous free space Coulomb interaction, denoted by G. We now repeat the treatment for the cavity, where we have a propagator given by (2.53),

$$D^{\mu\nu}(x, x') = \sum_{\Sigma q} -g^{\Sigma\Sigma} \frac{A^\mu(\Sigma, q; x) A^{\nu *}(\Sigma, q; x')}{(\omega^2 - \Omega^2_{\Sigma p} + i0)}. \tag{D.11}$$

This propagator will be sandwiched between conserved currents, $j_\mu(x)$,

$$\partial_\mu j^\mu(x) = 0 = \partial_0 \rho(x) + \vec{\nabla}.\vec{j}(x). \tag{D.12}$$

We consider the integral

$$A = \int d^4x \, j_\mu [A^\mu(S, q; x) + A^\mu(L, q; x)]. \tag{D.13}$$

Using the identity (A.55)

$$\vec{a}(L, p; \vec{r}) = \frac{-i}{\Omega_{Sp}} \vec{\nabla} a^0(S, p; \vec{r}). \tag{D.14}$$

and (D.12), and two partial integrations we may show that

$$A = (1 - \frac{\omega}{\Omega_{Sp}}) \int dx \, \rho(x) A^0(S, q; x). \tag{D.15}$$

Thus, if we once again ignore the transverse modes, and sum over the scalar and longitudinal modes of the cavity gauge propagator, we get the Coulomb part

$$\begin{aligned} D^{\mu\nu}_{Coul}(x, x') &= \delta^{\mu 0} \delta^{\nu 0} \delta(t, t') \sum_p \frac{a^0(S, p; \vec{r}) a^{0*}(S, p; \vec{r})}{\Omega^2_{Sp}}, & \text{(D.16)} \\ &= \delta^{\mu 0} \delta^{\nu 0} \delta(t, t') G_F(\vec{r}, \vec{r}'). & \text{(D.17)} \end{aligned}$$

We take this to define the Feynman gauge cavity Coulomb interaction. There is, however, a slight problem with this beast. By substitution, we may see that it obeys

$$\nabla^2 G_F(\vec{r}, \vec{r}') = -\delta(\vec{r}, \vec{r}'), \tag{D.18}$$

Yet, if we apply Green's theorem [51],

$$\int_V \nabla^2 G_F = \int_S \hat{r}.\vec{\nabla} G_F, \quad \text{(D.19)}$$
$$-1 = 0, \quad \text{(D.20)}$$

because the propagator, made up of scalar cavity modes, apparently satisfies Neumann boundary conditions. There is another version of a cavity Coulomb Green's function, presented by T. D. Lee in his Coulomb gauge presentation of cavity field theory [7]. It is well known [52], that we may expand the free Green's function in partial waves,

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi|\vec{r}-\vec{r}'|} = \sum_{lm} g_l(r,r')Y_{lm}(\hat{r})Y^*_{lm}(\hat{r}'), \quad \text{(D.21)}$$

where the partial wave l has radial dependence given by $g_l(r,r')$

$$g_l(r,r') = \frac{1}{2l+1}\frac{r^l_<}{r^{l+1}_>}. \quad \text{(D.22)}$$

The radial variables $r_>$ and $r_<$ are the greater and lesser of r and r' respectively. In the cavity we may simply add a homogeneous piece, in order to try and make a Coulomb Green's function that satisfies the Neumann boundary condition. If we do so the partial wave radial functions become

$$g^C_0(r,r') = \frac{1}{r_>}, \quad \text{(D.23)}$$

$$g^C_l(r,r') = r^l_< \left(\frac{r^l_>(l+1)}{l(2l+1)} + \frac{1}{(2l+1)r^{l+1}_>}\right), \quad \text{(D.24)}$$

and we get the Coulomb gauge cavity Green's function

$$G_C(\vec{r},\vec{r}') = \sum_{lm} g^C_l(r,r')Y_{lm}(\hat{r})Y^*_{lm}(\hat{r}'). \quad \text{(D.25)}$$

This function suffers from no inconsistency when we apply Green's theorem, and obeys the Neumann boundary condition for all partial waves, except the $l = 0$ one. In fact

$$\hat{r}\cdot\nabla G_C(\vec{r},\vec{r}')|_{\vec{r}\epsilon S} = \frac{-1}{4\pi} \quad \text{(D.26)}$$

So what went wrong with the Feynman gauge version? The answer lies in the completeness relationship (A.54). We lied! There is in fact a zero energy mode which we may label with $p = \{0,0,0\}$, and when properly normalized it is

$$a(S,p) = \sqrt{3}Y_{00}, \quad \text{(D.27)}$$

and we see that we cannot invert the operator ∇^2 anymore. The problems all crop up in the g_0 part, and we pose the question how are g^C_0 and g^F_0 related? In order to investigate we consider the orthogonal and complete set of scalar, angular momentum zero, radial wave functions,

$$R_n(r) = N_{Sn00}\, j_0(\Omega_{Sn00}r),$$
$$R_0(r) = \sqrt{3}. \quad \text{(D.28)}$$

By expanding the function

$$f(r,r') = \frac{1}{r_>} = \sum_n c_n(r)R_n(r'), \tag{D.29}$$

we discover that

$$g_0^C(r,r') = \frac{1}{r_>} = g_0^F(r,r') - g_0^F(1,r') + \frac{3-r^2}{2}. \tag{D.30}$$

If we are suitably inspired, and perform the exercise of expanding

$$r^2 = \sum_n c_n R_n(r), \tag{D.31}$$

then we get finally

$$g_0^C(r,r') = g_0^F(r,r') + \frac{18 - 5r^2 - 5r'^2}{10}. \tag{D.32}$$

This expression was first given by Chin et al [9], using a derivation based on taking the limit of $\Omega \to 0$ in the propagator mode expansion, where Ω is the energy of the 'zero energy' mode. We may finally discuss the gauge arbitrariness that this causes. The two body Coulomb interaction for two charge distributions in a cavity is given by

$$E_{int} = 2\int \rho_1(\vec{r})G(\vec{r},\vec{r}')\rho_2(\vec{r}') \tag{D.33}$$

The self energy contains a static part in which the intermediate quark is in the same cavity mode, and this part of the self energy may be written

$$E_{self,1} = \int \rho_1(\vec{r})G(\vec{r},\vec{r}')\rho_1(\vec{r}') \tag{D.34}$$

If the net charge in the cavity is zero, and the Neumann boundary condition may therefore be satisfied, then the total static Coulomb energy

$$E_{tot} = E_{int} + E_{self,1} + E_{self,2}, \tag{D.35}$$

does not depend on whether G^F or G^C is used. If $\rho_1 + \rho_2 = 0$ for all $\vec{r}$, $E_{tot} = 0$. This justifies our ignoring the zero energy cavity mode in all our calculations of the Coulomb energy, as is also done in [4].

In table D.1 we present the Coulomb gauge and Feynman gauge interaction energies between quarks in some low cavity modes. In the literature the numbers 0.0098 and 1.2784 are usually seen. The Coulomb energy of two quarks in the p state will be given by

$$E_{Coul} = \alpha C\mu(p,p)/R, \tag{D.36}$$

where the factors μ are given in the table, and C is the color factor $\lambda/2\cdot\lambda/2$. These numbers will allow for comparison between calculations done in the two different conventions.

Cavity mode	Feynman Gauge	Coulomb Gauge
$1s_{1/2}$	0.00979507	1.27840302
$1p_{3/2}$	1.34804619	2.52796050
$1p_{1/2}$	0.13210553	1.55526757
$2s_{1/2}$	0.22817215	1.67064058

Table D.1: Comparison of Coulomb Energy in Different Gauges

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