



*Efficient Monte Carlo Simulations
of Pricing Captions using
Libor Market Models*

By

M.A. Mkhwanazi

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Supervised By

Emeritus Prof. Ronald I. Becker

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Declaration

I know the meaning of plagiarism and declare that all of the work in the dissertation, save for what is properly acknowledged and cited in the references, is my own. This dissertation has not been accepted for any degree and is not concurrently submitted in candidature of any other degree.

Signed on the 06th December 2013,

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Mkhwanazi, Mpendulo Armstrong.

Abstract

The cap option (caption) is one of common European exotic options discussed in literature. This (interest rates) exotic option has no closed form solution and its accurate pricing and hedging in a volatile market is a challenge for traders. The reason for this is that, comparatively, the behaviour on an individual interest rate is more complex than that of a stock price. To price any interest rate product, it is essential to develop an interest rates model describing the behaviour of the entire zero coupon yield curve. The equity and yield curve, respectively, relate to the difference in the dynamics of a scalar variable and vector variable. Moreover, captions are second order with respect to the discount bonds in that they are options on caps (which are also options on bonds). These reasons make it of particular interest to study efficient numerical solutions to price captions. Monte Carlo simulation provides a simple method for pricing this option, and a suitable interest rate model to use is the Libor market model.

The approach of describing the behaviour of the entire zero coupon yield curve, in the era post the 2007 credit crunch crisis, is what is called a standard single-curve market practice, and Part I of this work is based on it. After introducing the framework for option pricing in the interest rate market, the theory and implementation procedure for Monte Carlo simulation using Libor market models is described. A detailed analysis of the results is presented together with a sensitivity analysis, and finally suggestions for efficient pricing of captions are given.

In Part II we review the recent financial market evolution, triggered by the credit crunch crisis towards double-curve approach. Unfortunately, such a methodology is not easy to build¹. In practice an empirical approach to price and hedge interest rate derivatives has prevailed in the market. Future cash flows are generated through multiple forwarding yield curves associated to the underlying rate tenors, and their net present value is calculated through discount factors from a single discounting yield curve.

¹Mercurio [67] and Morini [73] allude to this dilemma.

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Part I

Market Practices Before The Credit Crunch Crisis

The second half of the twentieth century has seen the financial markets, world-wide, go through major developments in various aspects. Today, trading is no longer restricted to a few hours during the day but goes on for 24-hours in foreign exchange markets. Also, trading has moved from being conducted face-to-face on a market floor to being done via telephone and computers from separate dealing rooms all over the world.

On the other hand, over the counter (OTC) market instruments that are traded, whilst becoming increasingly diverse and far more complex, have grown huge by market value. Examining activity in the derivatives market in June 2004, statistics published by the Bank of International Settlement (Galati et al. [36]) on their March 2005 “Triennial Central Bank Survey”, see Table 1.1 below, show that both gross market values and notional amounts outstanding stood at \$6.4 trillion and \$220 trillion, respectively, with interest rates contributing a bigger share.

Table 1.1: OTC Derivatives market activity by type of instrument.

% Contribution by instrument type	Gross market value		Notional amount	
	June 2001 : \$3.0 trillion	June 2004: \$6.4 trillion	June 2001: \$100 trillion	June 2004: \$220 trillion
Interest rates	57.4%	71.7%	76.1%	80.6%
Foreign exchange	31.8%	17.5%	20.5%	14.3%
Equity	7.2%	5.0%	2.0%	2.3%
Commodity	2.9%	2.8%	0.7%	0.7%
Credit linked & other	0.7%	3.0%	0.7%	2.1%

Such a growth is, to a greater extent, attributed to Black and Scholes [14]’s major breakthrough in the framework for pricing of stock (call or put) options (Hull [53]), also called derivatives, in 1973. A call (put) option gives the holder the right, but not the obligation, to buy (sell) an asset for a certain price. The option is classified as European (American) if the holder has the right to exercise it on (by) a certain time.

The Black and Scholes analytical solution is applicable for standard European instruments. For non-standard (exotic) options more computationally-intensive numerical techniques

are used to accurately price these options. For example, for European options with no analytical solution Monte Carlo simulation is used and has benefited from the advantage of speed and accuracy offered by computers to price such options. Options with early exercise opportunities (American) also have no closed form solution and are priced differently. The typical techniques used are binomial trees, trinomial trees, and finite difference methods.

Although the pricing framework was originally developed for equity and foreign exchange options, it soon had a contagion effect that extended to options on futures contracts, and interest rates. The interest rates derivatives market is divided into two sub-markets, caps and swaptions. Hence, the pricing of these options and the modelling of interest rates gained momentum through the years that followed. The models that surfaced started from instantaneous short rates and then evolved to instantaneous forward rates; departure from the latter models led to the market models. Throughout these developments, the mathematics involved in the pricing of interest rates derivatives has increased in complexity. Unlike with standard equity options where the Black and Scholes [14] model is universally accepted for pricing, no such unifying framework existed in interest rate models till the introduction of (Libor and Swap) market models in the 1990's. The Libor and Swap market models price standard caps and swaptions according to the Black's [13] model, and for this reason they allow automatic calibration to market data.

Exotic derivatives in both markets (equity and interest rates) are generally more profitable than the standard options, and are now becoming the fastest growing in the derivatives market (Hull [53]). The main challenge faced by derivatives traders is having an efficient model to price and hedge these exotic options. Having a model on itself is not enough. In order to derive maximal benefit from the model, a trader has to also understand its limitations in any market conditions. Sheldon Natenberg, a derivatives trader, (*op.cit.* Avraamides [7]) emphasised this point by saying:

“A trader who slavishly uses a model to make every trading decision is heading for disaster. Only a trader who fully understands what a model can and cannot do will be able to make the model his servant rather than his master.”

This challenge is even greater in a volatile market because it calls for the trader to react fast in pulling together ideas from the financial markets, mathematical tools, and technology (see Figure 1.1 on the facing page) to price these products within reasonable time bounds.

Mimicking this traders disposition and focusing on the caps sub-market, the aim of this thesis is to investigate an efficient way to price captions (a type of European exotic options). Hence, using the Libor market model and the Monte Carlo (MC) technique we are going to simulate interest rates and price captions. The resources are limited to material that has been published on the subject of derivatives markets, mathematical models, and on the aspect of technology the scope is on the implementation of the models using resources available in the financial mathematics (University of Cape Town) computer laboratory.

The content of the thesis is as depicted in Figure 1.1 and is organised as follows. In the next Chapter we review the theory of derivatives pricing and show how the pertinent

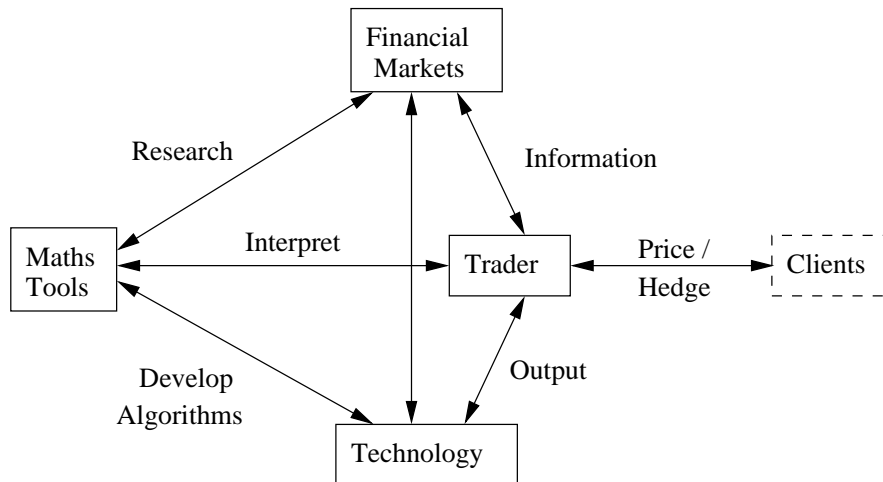


Figure 1.1: Developments in the financial markets.

assumptions combine to determine derivative prices. This is the link between financial markets and the maths-tools in our diagram. As mentioned earlier, the Black and Scholes model is based on equity products and is the building block for the overall derivatives market. In Chapter 3 we change focus to review interest rates products and their pertinent formulae. We also review approaches to interest rates modelling. In Chapter 4 we describe how computational efficiency can be brought about both in the numerical algorithms and Libor market models; this links maths-tools and technology in our diagram. In Chapter 5 we integrate ideas from the financial markets, maths-tools, and technology and describe the implementation procedure for pricing captions. In the same Chapter we also discuss the results. Finally, in Chapter 6 we conclude by suggesting an efficient way of pricing Captions and possible future work.

Review of the Theory of Derivatives Pricing

In the previous Chapter we alluded to the consistent theoretical framework for pricing equity options published by Black and Scholes [14] in the financial markets as a whole. Today, this framework is not confined to equity options, but is also applicable to other classes of derivatives (interest rates, foreign exchange etc.). Therefore, the aim of this Chapter is to lay a foundation for subsequent Chapters by introducing the essential terminology and mathematical tools for pricing derivatives. The main consulted sources for this pricing framework and mathematical foundation are Black and Scholes [14], Clewlow and Strickland [24], Hull [53], Harrison and Kreps [47], Harrison and Pliska [48], Shreve [95], and Brigo and Mercurio [20].

The principal assumption in any derivative pricing model is the mathematical description of how the asset price underlying the derivative evolves randomly through time. There are also certain conducive conditions (e.g. no-arbitrage, martingales) about the economy that are assumed in order to justify the use of certain mathematical techniques.

In this Chapter we start by reviewing these conditions as developed by Harrison and Kreps [47] and Harrison and Pliska [48]. Next we describe the change-of-numeraire technique as developed by Geman et al. [38]. Using a hedge approach, we then combine these principles to derive the famous Black and Scholes partial differential equation (PDE), with explicit solutions for European call and put options. Thereafter, since market exposure in options has risks associated with it, we discuss the “Greeks” as a risk management tool for options traders. We end the Chapter by summarising key concepts linking financial markets and mathematical-tools.

2.1 Model of the Financial Market

Consider a financial market with a finite trading interval $[0, T]$. Uncertainty in the financial market is modelled by a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, where Ω denotes a sample space with elements $\omega \in \Omega$, \mathbb{P} represents market probability, and the σ -field \mathcal{F}_t the information available at time t . In the given economy, we consider a model of a financial market

consisting of $K + 1$ non-dividend paying securities; one risk-less money market account S_t^0 , and K dimensional adapted semi-martingale S_t^j , $j = 1, 2, \dots, K$, $t \in [0, T]$, whose components are positive.

A predictable $K + 1$ dimensional stochastic process $\xi(t, \omega) = \{\xi_t^j : j = 0, 1, \dots, K\}$, $t \in [0, T]$, is called a dynamic portfolio process (trading strategy), where ξ^j represents the j^{th} asset holding at time t . Its corresponding realised market value and cumulative gains from trading are $\Xi(t, \xi) = \sum_{j=0}^K \xi_t^j S_t^j$ and $\sum_{j=0}^K \int_0^t \xi_u^j dS_u^j$, respectively, assuming that the gains are modelled as Itô integrals. A self-financing trading strategy is a strategy $\xi(t, \omega)$ such that $\Xi(t, \xi) = \Xi(0, \xi) + \sum_{j=0}^K \int_0^t \xi_u^j dS_u^j$ and requires no withdrawals or injection of cash throughout its life. For the portfolio Ξ to be only confined to non-negative values requires an admissible, self-financing strategy such that $\Xi(T, \xi) \geq 0$. An arbitrage opportunity is a self-financing trading strategy such that $\mathbb{P}(\Xi(T, \xi) > 0) = 1$ and $\Xi(0, \xi) \leq 0$.

An \mathcal{F}_T -measurable random variable $\Pi(T)$, which represents the uncertain payoff at time T , defines a corresponding derivative security. If there exists a self-financing trading strategy that has the property $\Xi(T, \xi) = \Pi(T)$, with probability one, then the derivative is said to be attainable, and such a strategy is called a replicating strategy. If in an economy all derivative securities are attainable, the economy is called complete. Therefore, in the absence of both transaction costs and arbitrage, the value of a replicating strategy at time t gives a unique value for $\Pi(T)$. This allows the valuation of the derivative security by valuing the replicating strategy, and this is a key concept in the Black and Scholes pricing framework called **arbitrage pricing**. Two questions are raised by this concept: under which conditions is a continuous trading economy (i) **arbitrage free** and (ii) **complete**?

2.2 Equivalent Martingale Measure

In what led to the Fundamental Theorem of Asset Pricing 1, FTAP-1, Harrison and Kreps [47] and Harrison and Pliska [48] were first to address these questions and show that both questions can be solved at once using the notion of martingale measure on *deflated* (*relative*) marketed asset prices. The one asset, say S_t^0 , that deflates the prices of other assets is called a numeraire. Its attribute is that it is strictly positive for all $t \in [0, T]$. The prices of other marketed assets denominated in S_t^0 are called deflated prices denoted by $\bar{S}_t^j = S_t^j / S_t^0$ where $j > 0$.

The main result in their work (*op.cit.* Brigo and Mercurio [20]) is the established connection between the economic concept of no-arbitrage and the mathematical property of existence of a probability measure, the equivalent martingale measure (or risk-neutral measure). In what led to FTAP-2, Delbaen and Schachermayer [28] refined this concept and, focusing on semi-martingales, proved the existence of an equivalent martingale measure in terms of the conditions of No Free Lunch with Vanishing Risk (NFLVR), and gave a new criterion (Theorem 2.2.1) for market completeness.

Theorem 2.2.1 (Unique Equivalent Martingale Measure) *In a continuous arbitrage-free economy where every derivative security is attainable, there exists a unique equivalent martingale measure for every choice of numeraire.*

The implication of this theorem is that for different choices of numeraire, there exists different unique *Equivalent Martingale Measures (EMM)* and is summarised in the following proposition, proved by Harrison and Pliska [48] (*op.cit.* Brigo and Mercurio [20]).

Proposition 2.2.1 *Assume there exists an equivalent martingale measure \mathbb{Q}^X , where X is the relevant numeraire, and let Ξ be an attainable contingent claim. Then there exists a unique price Π_t associated with Ξ given as*

$$\Pi(t) = X(t)\mathbb{E}^X \left[\frac{\Xi(T, \xi)}{X(T)} \mid \mathcal{F}_t \right], \quad t \in [0, T]. \quad (2.1)$$

2.3 Change of Numeraire Theorem

Equation (2.1) shows that the unique price $\Pi(t)$ of a derivative security is independent of the choice of numeraire. At the same time, Geman et al. [38] (*ibid.* [20]) noted that an equivalent martingale measure \mathbb{Q} , as in Proposition 2.2.1, is not necessarily the most natural and convenient measure for pricing the claim Ξ . Indeed, under stochastic interest rates, the presence of the stochastic discount factor $\frac{1}{B(t)} = \frac{1}{S^0(t)}$ complicates the calculation of the expectation considerably. In such cases, a change of measure can be quite helpful.

Considering two numeraires S^1 and S^2 with martingale measures \mathbb{Q}^{S^1} and \mathbb{Q}^{S^2} , the result of Proposition 2.2.1 applied to both numeraires yields

$$\Pi(t) = S_t^1 \mathbb{E}^{S^1} \left[\frac{\Xi(T, \xi)}{S_T^1} \mid \mathcal{F}_t \right] = S_t^2 \mathbb{E}^{S^2} \left[\frac{\Xi(T, \xi)}{S_T^2} \mid \mathcal{F}_t \right]. \quad (2.2)$$

Geman et al. [38] (*op.cit.* Brigo and Mercurio [20], Pelsler [80]) prove that the process defining \mathbb{Q}^{S^1} is given by $\frac{d\mathbb{Q}^{S^1}}{d\mathbb{Q}^{S^2}} = \frac{S_T^1/S_t^1}{S_T^2/S_t^2}$. Upon manipulation, the last two expressions of Equation (2.2) result in

$$\mathbb{E}^{S^1} [\gamma(T) \mid \mathcal{F}_t] = \mathbb{E}^{S^2} \left[\gamma(T) \frac{S_T^1/S_t^1}{S_T^2/S_t^2} \mid \mathcal{F}_t \right],$$

where $\gamma(T) = \Xi(T, \xi)$. Since Ξ , S^1 , and S^2 are general, this result holds for all random variables γ and all numeraires S^1 , and S^2 and is summarised in the following theorem (*ibid.* [20], Pelsler [80]).

Theorem 2.3.1 (Change of Numeraire) *The expectation of an arbitrary payoff $\gamma(T)$ under \mathbb{Q}^{S^1} is equal to the expectation of $\gamma(T)$ times the random variable $\frac{S_T^1/S_t^1}{S_T^2/S_t^2}$ (also referred to as the Radon Nikodym derivative $\Psi(T)$ under the measure \mathbb{Q}^{S^2}).*

2.4 Derivation of Black and Scholes PDE and Formula

Computational progress in any kind of engineering depends on the accurate modelling of the real life problem at hand. The Black and Scholes [14] option pricing model describes a two asset economy comprising a stock S and a risk-less money market account B , where $B(0) = 1$. S follows a geometric Brownian motion with drift μ and volatility σ , and B earns a risk less rate of return r , given as

$$dS = \mu(t, S)dt + \sigma(t, S)dW, \quad (2.3)$$

$$dB = rBdt. \quad (2.4)$$

dS and dB represents a change in the asset price and money market account, respectively, over a small time interval dt and $dW = \varepsilon\sqrt{dt}$ is a change in the 1-dimensional random variable $W(t)$, also called Wiener process. ε is a random number from the standard normal distribution $\phi(0, 1)$. Two key properties define the Wiener process: firstly the change dW is normally distributed with mean zero and standard deviation \sqrt{dt} , and secondly the values of dW for any two short intervals of time are independent. The value $\Xi(t, \xi)$ of a financial derivative depends on the price of S at every time instant t , and by Itô's Lemma the stochastic differential equation (SDE) followed by $\Xi(t, \xi)$ is

$$d\Xi = \left(\nabla_t(\Xi) + \mu\nabla_s(\Xi) + \frac{1}{2}\sigma^2\nabla_{ss}(\Xi) \right) dt + \sigma\nabla_s(\Xi)dW. \quad (2.5)$$

$\nabla(\Xi)$ is a vector differential operator on a function Ξ such that $\nabla_t(\Xi)$, $\nabla_s(\Xi)$ and $\nabla_{ss}(\Xi)$ refer to partial derivatives with respect to t , and S , i.e $\frac{\partial\Xi}{\partial t}$, $\frac{\partial\Xi}{\partial S}$ and $\frac{\partial^2\Xi}{\partial S^2}$. Replicating this financial derivative $\Xi(t, \xi)$ with a self-financing trading strategy results in

$$\Xi = \xi^B B + \xi^S S, \quad (2.6)$$

$$d\Xi = \xi^B dB + \xi^S dS. \quad (2.7)$$

Equation (2.7) is the differential form of Equation (2.6). Substituting Equation (2.3) and (2.4) into Equation (2.7) and using Equation (2.6) results in

$$d\Xi = [\xi^S \mu + r(\Xi - \xi^S S)] dt + \xi^S \sigma dW. \quad (2.8)$$

Equating (2.8) to (2.5) results in

$$\left[\nabla_t(\Xi) + \mu\nabla_s(\Xi) + \frac{1}{2}\sigma^2\nabla_{ss}(\Xi) - \xi^S \mu - r(\Xi - \xi^S S) \right] dt + \sigma [\nabla_s(\Xi) - \xi^S] dW = 0. \quad (2.9)$$

Choosing $\xi^S \equiv \nabla_s(\Xi)$ eliminates the uncertainty term in Equation (2.9) resulting in the famous Black and Scholes PDE:

$$\nabla_t(\Xi) + rS\nabla_s(\Xi) + \frac{1}{2}\sigma^2\nabla_{ss}(\Xi) - r\Xi = 0. \quad (2.10)$$

Since the expected return $\mu(t, S)$ on the asset does not appear on the PDE, this allowed Black and Scholes to use a strong argument that the value of the contingent claim is independent of investor risk preference. The return of the asset price S can be anything, and when assuming risk neutrality then all assets earn the risk-less rate of interest r . Replacing μ by r , the dynamics of asset price S in the risk-neutral world are given by

$$\frac{dS}{S} = r(t)dt + \sigma d\tilde{W}(t), \quad (2.11)$$

where \tilde{W} is the Brownian motion in the risk-neutral world. The implication of risk-neutrality is that the present value of any future random cash-flow of a derivative in the risk-neutral world is given by the expected value of the random future value discounted at the risk-less rate. This valuation is called risk-neutral pricing and is also known as the expectation approach to derivative pricing.

For example, suppose the contingent claim is a European call option to buy the asset S at price (strike) \mathcal{K} , at a certain time (maturity) T . The price for this option today, $t = 0$, is given by the expectation of the discounted payoff $S_T - \mathcal{K}$ with respect to a different probability measure \mathbb{Q} , called the risk-neutral measure $\Xi_0 = \mathbb{E}^B \left[e^{-\int_0^T r(s)ds} \max(S_T - \mathcal{K}, 0) \mid \mathcal{F}_0 \right]$, where \mathbb{E} is the expectation under the risk neutral probabilities, and \mathcal{F}_0 is the available market information today. The key concept with risk neutral measure is that it makes all discounted asset prices into martingales (Brigo and Mercurio [20]), a consequence of the martingale representation theorem.

Theorem 2.4.1 (Martingale representation) *Let $W(t)$, $t \in [0, T]$ be a Brownian motion on $(\Omega, \mathcal{F}_t, \mathbb{P})$, and let $Y(t)$ be a martingale under \mathbb{P} relative to \mathcal{F}_t . Then there exists an \mathcal{F}_t -measurable process $\delta(t)$ such that $dY(t) = \delta(t)dW(t)$.*

A martingale has the convenient property that its expected value at any future time T is equal to its value today, $\mathbb{E}[Y(T) \mid \mathcal{F}_0] = Y(0)$. Risk-neutral pricing is merely an artificial device for obtaining solutions to the Black and Scholes PDE (Hull [53]). The solutions obtained are valid not only in the risk-neutral world but also in the real world. Black and Scholes [14] solved the PDE for the call above and obtained the famous formula

$$\begin{aligned} \Xi_0 &= S_0 N(d_+) - e^{-rT} \mathcal{K} N(d_-), \quad \text{where} \\ d_{\pm} &= \frac{\ln(S_0/\mathcal{K}) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \end{aligned}$$

$N(d_{\pm})$ is the probability that a variable with a standard normal distribution, $\phi(0, 1)$, will be less than d_{\pm} . Equation (2.11) above shows that when moving from the real world to the risk neutral world the expected return on the stock changes, but its volatility remains the same. An important result that provides techniques necessary to move from the real world measure to the risk-neutral measure is given by Girsanov and is stated below.

Theorem 2.4.2 (Girsanov-Martin-Cameron - "Change of Numeraire Toolkit")

Let $\mathcal{W}^{S^2}(t)$ be an n -dimensional correlated standard Brownian motion (under numeraire S^2) on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$. Under a second numeraire (say S^1), the n -dimensional correlated standard Brownian motion $\mathcal{W}^{S^1}(t)$ is defined as

$$d\mathcal{W}^{S^1}(t) = \nu(t)dt + d\mathcal{W}^{S^2}(t), \quad (2.12)$$

where $\nu(t)$ is any n -dimensional column vector \mathcal{F}_t -adapted process that satisfies the Novikov regularity condition $\mathbb{E}^{S^2} \left(\exp \left[\int_0^t \|\nu(s)\|^2 ds \right] \right) < \infty$. Then a new probability measure \mathbb{Q}^{S^1} , equivalent to \mathbb{P}^{S^2} , can be defined via the relationship $\mathbb{Q}(A) = \int_A \Psi(T) d\mathbb{P}$, where $\Psi(t) = \exp \left[-\frac{1}{2} \int_0^t \|\nu(s)\|^2 ds - \int_0^t \nu'(s) \cdot d\mathcal{W}(s) \right]$ is a \mathbb{P} -martingale. Brigo and Mercurio [20, Chapter 2], rewrites Equation 2.12 as

$$d\mathcal{W}^{S^1}(t) = -\rho \left(\frac{\sigma^{S^1}(t)}{S^1} - \frac{\sigma^{S^2}(t)}{S^2} \right)' \cdot dt + d\mathcal{W}^{S^2}(t), \quad (2.13)$$

where ρ is the instantaneous correlation, σ^{S^1} and σ^{S^2} are instantaneous volatilities, respectively, of numeraire S^1 and S^2 .

Using this theorem, the transition of the asset dynamics from the real world to the risk-neutral world in Equation (2.11) is shown as follows:

$$\begin{aligned} \frac{dS}{S} &= \mu dt + \sigma dW = rdt + \sigma \left(\frac{\mu - r}{\sigma} dt + dW \right) \\ &= rdt + \sigma d\tilde{W}, \end{aligned} \quad (2.14)$$

where, by Girsanov's theorem, \tilde{W} is a standard Brownian motion with respect to a measure \mathbb{Q} . The implication is that the percentage change in the asset price in the risk neutral world, in Equation (2.14), is normally distributed $\phi \left(rdt, \sigma\sqrt{dt} \right)$ with mean rdt and standard deviation $\sigma\sqrt{dt}$.

2.5 Standard Hedge Parameters - "The Greeks"

The elimination of risk (often called hedging) in the portfolio Ξ over time dt , see Equation 2.9 becoming 2.10, in the development of the Black and Scholes PDE is a microcosm of the risk-management concerns traders are faced with when managing portfolios that contain options. After taking an option position, traders compute option-price sensitivities, within the life of the option, in order to know what corrective action to take in the midst of new market conditions. The associated risk measures are a set of factor sensitivities called the "Greeks".

Delta $\nabla_s(\Xi)$, Vega $\nabla_\sigma(\Xi)$, Theta $\nabla_t(\Xi)$ and Rho $\nabla_r(\Xi)$ measure the linear sensitivity of the option price to asset price, asset volatility, time, and risk-free interest rates, respectively.

Gamma $\nabla_{ss}(\Xi)$ measures the linear sensitivity of Delta (second order sensitivity). In cases where the option-price has no closed-form solution, these measures are estimated by finite difference methods.

2.6 Summary

In this Chapter we have introduced the mathematical tools for modelling the random movement of financial securities in the financial markets, and shown how equity options are priced in the Black and Scholes framework. The main concepts in pricing derivatives encompass: no-arbitrage, martingales, and numerical solution. These concepts are not mutually exclusive but are intrinsic in the pricing such that no price is derived at the exclusion of any-one (of them). The shaded region in the connected rings, in Figure 2.1, depicts the inextricable connectedness of the concepts (Pelsser [80]). The price of the derivative is unique, and hence independent of the choice of numeraire. Therefore the numeraire should be chosen such that pricing is made simpler.

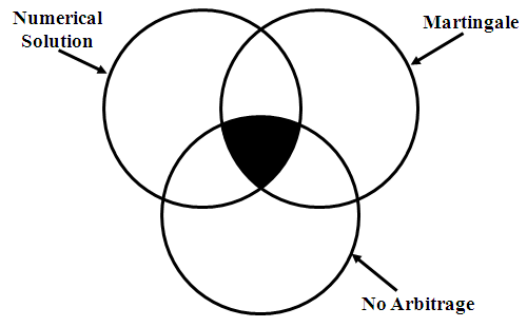


Figure 2.1: Tri-union concepts.

Extension from Equity Options to Interest Rates Options

Now that the three main concepts (martingale, no-arbitrage, and numerical solution) in valuing derivatives have been highlighted, how do they fit into the objective of pricing captions (interest rate derivatives)? In answering that question, this Chapter discusses the Libor Market Models (LMM) in order to show that they uphold the no-arbitrage principle among discount bonds, and that deflating all the (discount) bond prices by the terminal discount bond the martingale principle is also upheld among the deflated discount bonds. The concept of numerical solution is discussed separately in Chapter 4.

Interest rates have been in existence over the past centuries as the price for borrowing money. The party borrowing money can be viewed as someone who has sold a bond to the lender and is obliged to pay the bond holder the principal plus interest. Because money is demanded in different markets, there are also different types of interest rates corresponding to the various markets and different types of products. Interest rates derivatives are valued using a model, similar to that of Black and Scholes [14] in that it assumes log-normal behaviour of interest rates, referred to as Black's [13] model. Out of the various models used to simulate interest rates, the market models are ideal because of the inherent log-normal property of interest rates, compatible with Black's formula.

The rest of the Chapter is organised as follows. Before discussing the Libor market models, the building blocks of interest rate derivatives are discussed, followed by an examination of Black's [13] model highlighting its input that makes it compatible with interest rate models. Next is a discussion of some heavily traded interest rate derivatives, focusing on Captions, followed by a discussion on various approaches to interest rates modelling. Thereafter, we discuss in depth how the Libor market model upholds the martingale and no-arbitrage principle, and why they are compatible with Black formula. We end the Chapter by summarising key concepts.

3.1 Discount Bonds as Underlying Assets

In this thesis, we are considering an economy with a trading interval given by $[0, T]$. Also, other aspects of the economy that are considered are interest rates and their corresponding traded assets, pure discount bonds $P(t, T)$ with different maturities. These bonds are assumed to have a pre-specified collection of settlement dates $0 = T_0 < T_1 < \dots < T_N < T_{N+1} = T$. For simplicity, the interval between the reset dates, also called *tenor* or *day-count fraction*, is assumed constant and is denoted by $\tau_i = T_{i+1} - T_i = \tau$, where $i = 0, 1, \dots, N$. See Figure 3.1 below.

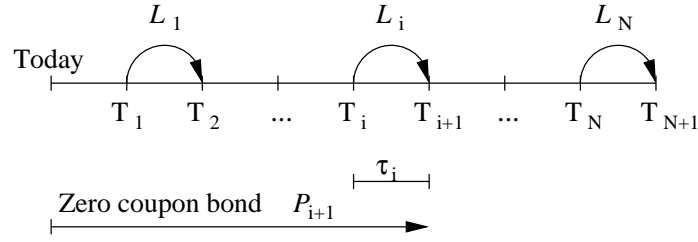


Figure 3.1: Tenor structure and typical market instruments.

Definition 3.1.1 *A zero coupon (discount) bond, is a contract that pays one unit at maturity T . The value of the bond at time t , where $t \leq T$, is denoted $P(t, T)$. Hence $P(T, T) = 1$. For $t < U < T$ the assumption is that $P(t, U) > P(t, T)$.*

In order to reduce notation when considering the discrete maturities of bonds in $[0, T]$ they will also be written as $P_i(t) = P_{T_i}(t) = P(t, T_i) \quad t \leq T_i, \quad i = 0, \dots, N + 1$ and thus $P_i(T_i) = 1$. This notation is adopted for all instruments in subsequent Chapters. With the bond market given as such, different types of interest rates (simple and continuously compounded) can now be defined for any time $t \leq T_i$ given two time points in the future, say $[T_i, T_{i+1}]$.

Definition 3.1.2 *Types of interest rates:*

1. *The simple forward Libor rate is defined as*

$$L(t, T_i, T_{i+1}) = L_i(t) = \frac{1}{\tau_i} \left(\frac{P_i(t)}{P_{i+1}(t)} - 1 \right),$$

and its corresponding spot rate is

$$L_i(T_i) = \frac{1}{\tau_i} \left(\frac{1}{P_{i+1}(T_i)} - 1 \right).$$

2. *The simple forward price is defined as*

$$F(t, T_i, T_{i+1}) = F_i(t) = 1 + \tau_i L_i(t) = \frac{P_i(t)}{P_{i+1}(t)}.$$

3. The instantaneous forward rate is defined as

$$f(t, T_i) = -\frac{\partial \ln P_i(t)}{\partial T_i},$$

implying that $P_i(t) = \exp \left[-\int_t^{T_i} f(t, s) ds \right]$.

4. The instantaneous short rate is defined as

$$r(t) = f(t, t).$$

Apart from the discount bonds, under consideration in this market, that pay only interest and principal at maturity, there are also bonds that pay interest (coupons) periodically between time of purchase and maturity (Hull [53]). The interest paid is either fixed or floating. Two parties having sold such different paying bonds, with the same principal, could be interested in swapping their interest payments. Such a deal defines an interest rate swap, with the exception that at maturity the principal is not exchanged.

Definition 3.1.3 *An interest swap is a contract where two parties agree to simultaneously pay and receive fixed and floating interest rates payments for an agreed period of time starting from a future time instant $T_i \geq T_1$. At every time instant $T_i \in [T_1, T_{N+1}]$ the floating interest rate side pays the amount $\mathcal{N}\tau_i L_i(T_i)$ corresponding to the Libor interest rate $L_i(T_i)$ reset at the time instant T_i and a principal amount \mathcal{N} , whereas the fixed interest rate side pays the amount $\mathcal{N}\tau_i \mathcal{R}$ corresponding to the fixed interest rate \mathcal{R} . The principal amount is never exchanged at expiry of the contract.*

An example of an interest rate swap is shown in Figure 3.2. Company A, agrees to pay cash-flow at a floating (Libor) rate and receives cash-flow at a fixed rate from company B.

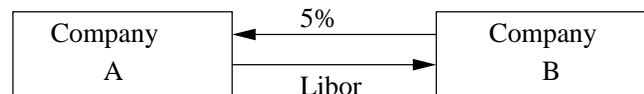


Figure 3.2: Interest rate swap between company A and B.

3.2 Extension from Black-Scholes to Black(1976)

In the introductory Chapter we alluded to the contagion effect of the Black and Scholes model. Fischer Black was first to extend this model to value options on futures contracts. Derivatives (including interest rates options) that surfaced in later years, similar to futures options, were also priced using the model suggested by Fischer Black.

In this Section we are going to discuss the assumptions inherent in Black's [13] model and the seemingly faulty logic leading to it.

3.2.1 Assumptions in Black(1976) Formula

Consider the call option discussed under “Derivation of Black and Scholes PDE and Formula”, in Section 2.4. Its price was derived as the expectation of the discounted payoff given in Equation (3.1) below,

$$\Xi_0 = \mathbb{E} \left[e^{-\int_0^T r(s) ds} \max(S_T - \mathcal{K}, 0) \mid \mathcal{F}_0 \right]. \quad (3.1)$$

Black [13], modifies the above equation as follows:

$$c_0 = e^{-rT} \mathbb{E} [\max(S_T - \mathcal{K}, 0)], \quad (3.2)$$

$$c_0 = P(0, T) \mathbb{E} [\max(S_T - \mathcal{K}, 0)], \quad (3.3)$$

assuming that the interest rates $r(s)$ are not stochastic. Equation (3.3) is derived using the arbitrage pricing theory which identifies the discount factor with the corresponding bond price $P(0, T) = e^{-rT}$.

At this stage the assumption made is that S_T has log-normal distribution with the standard deviation of $\log S_T$ equal to $\sigma\sqrt{T}$. Defining $F(t, T)$ as the forward price of S for a contract with maturity T , Black [13] gets the expected value of the forward price today be $\mathbb{E}[F(t, T) \mid \mathcal{F}_0] = F(0, T)$ by the martingale property and using Black and Scholes [14] formula the price of the option is

$$\begin{aligned} c_0 &= P(0, T) [F(0, T)N(d_+) - \mathcal{K}N(d_-)], \quad \text{where} \\ d_{\pm} &= \frac{\ln(F(0, T)/\mathcal{K}) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}. \end{aligned} \quad (3.4)$$

In contrast to Black and Scholes [14] pricing formula, where the underlying is assumed to follow a geometric Brownian motion, Black’s [13] model does not assume this feature for the evolution of either S or F . What is required is for S to be log-normal at time T .

3.2.2 Seemingly Faulty Logic Leading to Black’s Formula

The payoff function from this European futures call option is $(F - \mathcal{K})^+$. A similar call option in the interest rate market is a caplet with a payoff $(L_i(0) - \mathcal{R})^+$, where $L_i(\cdot)$ is the forward price and \mathcal{R} is the caplet strike rate. Black’s [13] assumption that interest rates are non-stochastic (deterministic) has the following implication for caplets

$$\mathbb{E} \left[e^{-\int_0^{T_{i+1}} r(s) ds} \tau_i (L_i(T_i) - \mathcal{R})^+ \right] = P_{i+1}(0) \tau_i \mathbb{E} [(L_i(T_i) - \mathcal{R})^+]. \quad (3.5)$$

Equation (3.5) is just analogous to the equality of Equations (3.1) and (3.3). In the interest rate market caplets are priced using stochastic Libor rates $L_i(\cdot)$ (see Subsection 3.5.1 for a detailed discussion).

A quick glance in Equation (3.5), seemingly, suggests an inconsistency in assumptions

between the implicit assumption in the last statement and Black's [13] assumption. A critical analysis of this equation within the change-of-numeraire setup proves otherwise. The left side of Equation (3.5) is based on the bank account numeraire $B(\cdot)$, associated with risk-neutral measure \mathbb{Q} , whereas the right side is based on the discount bond $P_{i+1}(\cdot)$ as a numeraire, associated with measure $\mathbb{Q}^{P_{i+1}}$ (hereafter, referred to as \mathbb{Q}^{i+1}). Thus, Black's [13] assumption of deterministic discount factor (i.e. interest rates are not stochastic) is essentially an indication that the expectation is under a measure associated with the bond price and not the bank account. By the change of numeraire theorem (2.3.1) we can switch from the bank account numeraire $B(\cdot)$ to the bond price numeraire $P_{i+1}(\cdot)$ and still have a unique price (Brigo and Mercurio [20]):

$$\mathbb{E}^B \left[e^{-\int_0^{T_{i+1}} r(s) ds} \tau_i (L_i(T_i) - \mathcal{R})^+ \right] = P_{i+1}(0) \tau_i \mathbb{E}^{P_{i+1}} [(L_i(T_i) - \mathcal{R})^+].$$

3.3 Heavily Traded Interest Rate Derivatives (IRD)

The underlying asset for all heavily traded IRDs is the discount bond. As such, the IRD market can be categorised into two types of options (Brigo et al. [23]) namely caps and swaptions. All these are priced using the Black's [13] formula.

3.3.1 Caps

An interest cap is an insurance against the floating rate of interest rising above a certain level called a cap rate (Hull [53]). A cap is made up of a sum of caplets with pay-off

$$\mathcal{N} \tau_i \max(L_i(T_i) - \mathcal{R}, 0), \quad (3.6)$$

where $\tau_i = T_{i+1} - T_i$, and $L_i(t)$ is the realised floating interest rate at time T_i . T_i is the time from the evaluation date t until the reset of the caplet, and T_{i+1} is the time when the caplet payoff is paid. \mathcal{R} is the cap strike rate and \mathcal{N} is the notional principal. The caplet pay-off is, in essence, a call option on $L_i(t)$.

Using the forward martingale measure \mathbb{Q}^{i+1} at time T_{i+1} the caplet price is given by

$$\begin{aligned} \mathbf{Caplet}_i(t) &= P_{i+1}(t) \mathbb{E}^{P_{i+1}} [\mathcal{N} \tau_i \max(L_i(T_i) - \mathcal{R}, 0)] \\ &= \mathcal{N} P_{i+1}(t) \tau_i \mathbb{E}^{P_{i+1}} [\max(L_i(T_i) - \mathcal{R}, 0)], \end{aligned} \quad (3.7)$$

since \mathbb{Q}^{i+1} is the measure associated with numeraire $P_{i+1}(t)$ and $P_{i+1}(T_{i+1}) = 1$. Assuming that $L_i(t)$ has log-normal distribution, with constant volatility, one may model the caplet as a European call option on the forward rate $L_i(t)$ using Black's [13] model. The value of

the call option is

$$\begin{aligned}\text{Caplet}_i(t) &= \mathcal{N}P_{i+1}(t)\tau_i [L_i(T_i)N(d_+) - \mathcal{R}N(d_-)], \quad \text{where} \\ d_{\pm} &= \frac{\ln(L_i(T_i)/\mathcal{R}) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.\end{aligned}$$

A floor is similar to a cap, except that it is bounded (capped) below. Floors are made up of floorlets with a payoff similar to a put option on $L_i(t)$.

Cap-Floor Parity

A portfolio of a long caplet and a short floorlet has the following cash-flow:

$$\max(L_i(T_i) - \mathcal{R}, 0) - \max(\mathcal{R} - L_i(T_i), 0) = L_i(T_i) - \mathcal{R}.$$

This is the same cash-flow as one payment of a swap. Therefore, there is a model-independent no-arbitrage relationship (Hull [53]) that exists, which is given as

$$\text{Cap price} = \text{Floor price} + \text{Swap value}.$$

A caplet as a portfolio of bond options

The pay-off for the caplet, Equation (3.6), is at time T_{i+1} . At time T_i , with a unit notional principal ($\mathcal{N} = 1$), the pay-off is equivalent to

$$\frac{\tau_i}{1 + \tau_i L_i(T_i)} \max[L_i(T_i) - \mathcal{R}, 0] = \max\left[1 - \frac{(1 + \tau_i \mathcal{R})}{1 + \tau_i L_i(T_i)}, 0\right]. \quad (3.8)$$

The term $\frac{1 + \tau_i \mathcal{R}}{1 + \tau_i L_i(T_i)}$ on the right hand side of Equation (3.8) is the value at T_i of a discount bond that pays $(1 + \tau_i \mathcal{R})$ at time T_{i+1} . So, the expression in Equation (3.8) is the pay-off from a put option with maturity T_i on a discount bond with maturity T_{i+1} when the face value of the bond is $(1 + \tau_i \mathcal{R})$ and the strike is 1 (Hull [53]).

3.3.2 Swap Options

Options on interest rate swaps give the holder the right to enter into a certain interest rate swap at a certain time in the future (Hull [53]). Based on the cap-floor parity, a swap option can be viewed as an option to exchange a fixed rate bond for the floating bond. If a swap option gives the holder the right to pay fixed and receive floating, it is a put option on the fixed rate bond with the strike price on the floating rate bond at the start of the swap, and vice-versa for a call option on the fixed-rate bond.

3.3.3 Options on Caps

In the market there are also options whose underlying asset is an option (Wilmott [98]). These options are second order with respect to the underlying assets. The typical instruments are captions (cap options) and floortions (floor options). Captions are the focus of this thesis. Brigo and Mercurio [20] defines the caption as follows.

Definition 3.3.1 *A caption is an option that gives its holder the right to enter at a future time T_{exp} a cap struck at \mathcal{R} and whose first caplet resets at date $T_\alpha \geq T_{\text{exp}}$, and whose subsequent caplets reset at times $T_{\alpha+1}, \dots, T_N$ with T_{N+1} the last payment date. The holder of the caption will pay a caption price struck at \mathcal{X} .*

The caption value at time $t = 0$ with maturity $t = T_{\text{exp}}$ is given by the discounted expectation of the caption's payoff, written as,

$$\mathbf{Caption\ Value}(t = 0) = P(0, T_{\text{exp}}) \mathbb{E}^{P_{\text{exp}}} \left[\mathbf{Cap}^{\mathbf{Black76}}(\mathbf{T}_{\text{exp}}) - \mathcal{X} \right]^+. \quad (3.9)$$

Assuming a unit notional principal, the T_{exp} value of the underlying cap is given by the usual Black formula (computed at time $t = T_{\text{exp}}$),

$$\mathbf{Cap}^{\mathbf{Black76}}(T_{\text{exp}}) = \sum_{i=\alpha}^N P_{i+1}(T_{\text{exp}}) \tau_i \mathbf{Black76}(L_i(T_{\text{exp}}), \mathcal{R}, v_i), \quad (3.10)$$

where $P_{i+1}(T_{\text{exp}})$ is the value at time T_{exp} of a discount bond maturing at the payment of the i^{th} Libor rate and $\mathbf{Black76}(\cdot)$ is defined as,

$$\begin{aligned} \mathbf{Black76}(L_i(T_{\text{exp}}), \mathcal{R}, v_i) &= \mathbb{E}^{P_{i+1}} \left[(L_i(T_{\text{exp}}) - \mathcal{R})^+ \right] \\ &= L_i(T_{\text{exp}}) N(d_{+,i}) - \mathcal{R} N(d_{-,i}), \quad \text{where} \end{aligned} \quad (3.11)$$

$d_{\pm,i} = \frac{\ln(L_i(T_{\text{exp}})/\mathcal{R}) \pm v_i^2/2}{v_i}$, $v_i^2 = \int_{T_{\text{exp}-1}}^{T_i} \sigma_i(s)^2 ds$ for $T_i > T_{\text{exp}-1}$ and $\sigma_i(s)$ is the instantaneous volatility of the i^{th} Libor rate at time s .

3.4 Various Approaches to Interest Rate Modelling

Since the advent of arbitrage-free pricing theory by Black and Scholes [14] in 1973, the modelling of the term structure of interest rates has occupied the efforts of both academics and practitioners and produced a variety of approaches (RWG [1]). One reason for this is that the phenomenon being modelled, the random fluctuation of the whole yield curve, is more complex than the movements of a single stock (Jackson and Staunton [57]). Intuitively, this can be related to the difference in dynamics between a scalar variable (in the case of stock) and a vector variable (representing the yield curve).

As a result, in the past three decades three separate ways for allowing for the term structure when pricing options on bonds have surfaced (Jackson and Staunton [57]). (i) Ignore the term structure. (ii) Model the term structure. (iii) Match the term structure. The first approach is the mere use of the Black's model. In the model Black ignores the evolution of the term structure and instead assumes the log-normal property for the forward price of the bond (see Section 3.2 for details). The second approach is the use of continuous interest rate models whereas the third approach is the use of market models.

In the following Subsections we discuss the various approaches to modelling interest rates. In Subsection 3.4.1, we describe the early interest rate models before market models. This is followed by an account for the birth of market models in Subsection 3.4.2. Finally, in Subsection 3.4.3, we describe several extensions of market models that are aimed at circumventing its well known pitfalls. Towards the end of Subsection 3.4.3, we select the particular market model we intend to use in this dissertation, and describe it in detail in Section 3.5.

3.4.1 Metamorphoses of Early Interest Rates Models

Up until the early 1990's, the classical approach adopted in interest rate models dealt with continuously compounded instantaneous rates. Rebonato [86] argues that this choice is due to a combination of mathematical convenience and numerical ease of implementation. As a result, the description of the evolution of interest rates through time could be explained by equilibrium and no-arbitrage models, see Figure 3.3 below.

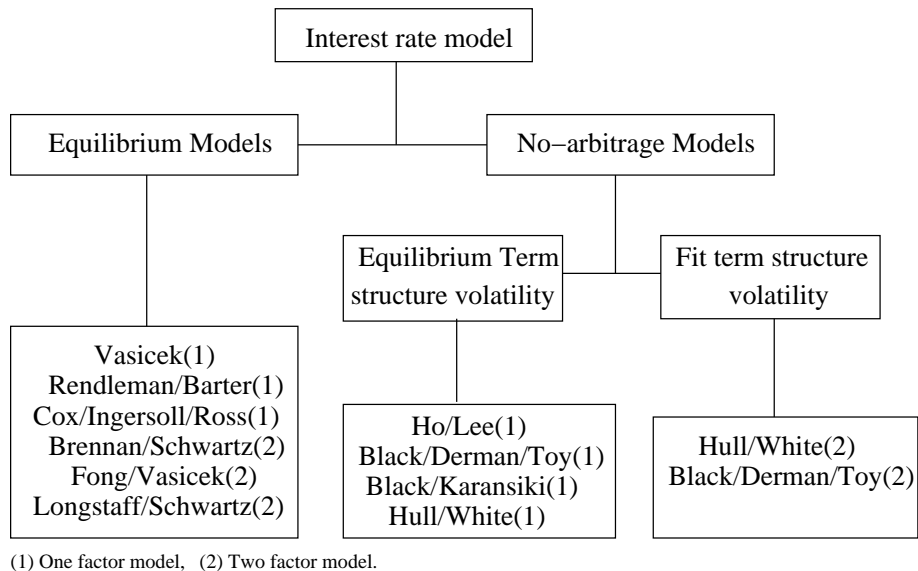


Figure 3.3: Different types of short rate models.

The numbers in parentheses give an indication as to whether the SDE describing the interest rate is a one factor or a two factor model. Equilibrium models are typified by the incapability of fitting today's term structure whereas the no-arbitrage models are designed

to be consistent with today's term structure of interest rates (Hull [53]). Within no-arbitrage models, some assume an equilibrium term structure of volatility and others fit the term structure of volatility.

In 1992, Heath et al. [49] (popularly known as Heath Jarrow and Morton, hence-after HJM) offered an alternative approach (*op.cit.* Jamshidian [58]). They regarded instantaneous forward rates as the dependent variable, providing conditions in a general framework that incorporates all the principles of arbitrage-free pricing and bond dynamics. Despite this theoretical success, Heath et al. [49] noted that the model exhibited unbounded behaviour with positive probability and the model was perceived to be difficult to calibrate.

Another approach that surfaced, during this period (1990's), is the potential approach by Rogers [88, 87]. This approach is based on the specification of the state price density. Its attractive features are that: (i) it is efficient and consistent in handling yield curves in multiple currencies, (ii) it is set-up under actual probabilities, and (iii) it ensures positive interest rates (Flesaker and Hughston [32, 33], *op.cit.* Nakamura and Yu [77]). Both approaches were later shown to be equivalent ways of generating positive interest rates in the arbitrage free HJM framework by Jin and Glasserman [60]. In spite of these attractive features there has been lack of effort in examining the pricing implications of the potential approach, possibly, due to its general framework (Nakamura and Yu [77]).

The fact that the HJM framework is based on continuously compounded rates whereas the rates used in interest rate derivatives are quoted for accrual periods of at least 3 months, made calibration difficult. This created a demand for simpler models that made reference to interest rates on accrual periods. The first step in addressing this demand was done by Sandmann and Sondermann [90, 91] (*op.cit.* Jamshidian [58]) in 1994 who proposed a log-normal model for the effective annual interest rate motivated by its stability (particularly over Eurodollar) in comparison to the earlier models.

3.4.2 Birth of Market Models

Goldys et al. [43] and Musiela [75] (*ibid.* [58]) later integrated the log-normal model for the effective rate within the HJM framework, and through the works of Miltersen and Sondermann [71], Miltersen et al. [72] and Brace et al. [16, popularly known as BGM] focus shifted from effective continuous interest rates to Libor rates. Jamshidian [58] developed a counterpart model to that of BGM by modelling swap rates instead of Libor rates (*op.cit.* Kagraoka [63]). The key feature of both models is that a term structure model of the log-normal Libor and Swap rates was constructed which priced caplets and swaptions by the industry standard Black's [13] formula. Because of this log-normal property of rates (and volatility), these models were termed Market Models.

A drawback of the BGM model was the reliance on the continuously compounded spot rate because the model dynamics were specified in the risk-neutral measure. Musiela and Rutkowski [76] (*op.cit.* Jamshidian [58]) later carried out a clear construction in the

terminal measure. Their model is a backward induction, an explicit recursive equation, for the term structure of forward Libor rates with a log-normal volatility (*ibid.* [58]). In this framework, Libor rates are modelled as the exponential of a Brownian motion under their corresponding forward measure, and this is what deems them log-normal Libor Market Models (*op.cit.* Papapantoleon [78]).

This model has become a standard model for the pricing of interest rate derivatives in recent years. The main advantage of the Libor model in comparison to other approaches, is that the evolution of discretely compounded, market-observable forward rates is modelled directly and not deduced from the evolution of unobservable factors. Moreover, the log-normal Libor model is consistent with the market practice of pricing caps according to a Black [13] type formula (*op.cit.* Papapantoleon et al. [79]).

Notwithstanding its apparent popularity, the Libor Market Model has certain well-known pitfalls.

3.4.3 Several Extensions of the Libor Market Models

On the one hand, the log-normal Libor model is driven by a Brownian motion; hence, it cannot be calibrated adequately to the observed market data (i.e. matching cap and swaption volatility smiles and skews observed in the markets). An interest rate model is typically calibrated to the implied volatility surface from the cap market and the correlation structure of at-the-money swaptions (*ibid.* [79]).

As a remedy, several variations to the standard Libor market model have been proposed. They can be roughly categorized into three streams: local volatility models, stochastic volatility models, and jump-diffusion models (*op.cit.* Belomestny et al. [10]). Especially jump-diffusion and stochastic volatility models are popular due to their economically meaningful behavior, and the greater flexibility they offer compared to local volatility models for instance. For local volatility Libor models we refer to Brigo and Mercurio [20], and for jump-diffusion models for assets allude to Merton [70] and Eberlein et al. [31] (*ibid.* [10]). Jamshidian [59] developed a general semi-martingale framework for the Libor process which covers the possibility of incorporating jumps as well as stochastic volatility. Specific jump-diffusion Libor models are proposed, among others, by Glasserman and Kou [39] and Belomestny and Schoenmakers [9]. Lévy Libor models are studied by Eberlein and Özkan [30]. Incorporation of stochastic volatility has been proposed by Andersen and Brotherton-Ratcliffe [5], Piterbarg [82], and Wu and Zhang [99], Zhu [100] (*ibid.* [10]). Recently, a Libor model with SABR stochastic volatility (Hagan et al. [46]) has attracted some attention (*op.cit.* Mercurio and Morini [69]).

On the other hand, the dynamics of Libor rates are not tractable under every forward measure due to the random terms that enter the dynamics of Libor rates during the construction of the model (*op.cit.* Papapantoleon et al. [79]). In particular, when the driving process has continuous paths the dynamics of Libor rates are tractable under

their corresponding forward measure, but they are not tractable under any other forward measure. When the driving process is a general semi-martingale, then the dynamics of Libor rates are not even tractable under their very own forward measure.

Consequently, if the driving process is a continuous semi-martingale caplets can be priced in closed form, but not swaptions or other multi-Libor derivatives. However, if the driving process is a general semi-martingale, then even caplets cannot be priced in closed form (*ibid.* [79]). In order to calibrate the model, closed-form solutions are necessary, and these typically involve approximations. The standard approximation is the so-called "frozen drift" approximation, where one replaces the random terms in the dynamics of Libor rates by their deterministic initial values; it was first proposed by Brace et al. [16] (*ibid.* [79]) for the pricing of swaptions and has been used by several authors ever since. The frozen drift approximation typically leads to closed-form solutions for caplet pricing in realistic Libor models (Eberlein and Özkan [30], Belomestny et al. [10], *ibid.* [79]). Although some authors (Brace et al. [17], Dun et al. [29], Schlögl [92]) argue that freezing the drift is justified in the log-normal Libor Market Models, Kurbanmuradov et al. [65] has shown that it does not yield acceptable results for exotic derivatives and longer time horizons (*ibid.* [79]).

Therefore, several alternative approximations have been developed in literature. In one line of research, Kurbanmuradov et al. [65] and Daniluk and Gatarek [25] derived log-normal approximations of the forward Libor dynamics (for deterministic volatility structures). Other authors (Hunter et al. [56] and Glasserman and Zhao [42]) have used linear interpolations and predictor-corrector Monte Carlo methods to obtain a more accurate discretisation of the drift term (*ibid.* [79]). For a detailed overview of the literature, and for some new approximation schemes and numerical experiments, we refer the interested reader to the work by Joshi and Stacey [61] and Gatarek et al. [37, Chapter 10]. Although most of this literature focuses on the log-normal Libor market model, Glasserman and Merener [40, 41] have developed approximation schemes for the pricing of caps and swaptions in jump-diffusion Libor market models, based on freezing the drift.

In this dissertation we are going to focus on the multi-factor Musiela and Rutkowski [76] Libor Market Model to model the Libor rates, and limit our discretisation to Glasserman and Zhao [42] and log-normal approximations. In the next Section we describe this model in detail, and in Subsection 4.3.5 we discuss these two discretisation types.

3.5 Musiela-Rutkowski Market Model

Before developing the dynamics in the terminal measure \mathbb{Q}^{N+1} , first, we are going to setup a model of the multi-factor Libor process and discuss why the dynamics of Libor rates are based in \mathbb{Q}^{i+1} measure. This will be followed by a discussion of the dynamics in the terminal measure \mathbb{Q}^{N+1} . We conclude this Section by explaining why market models are compatible Black formulas.

3.5.1 The Multi-Factor Libor Market Model (LMM)

A Libor rate $L_i(t)$ is the interest rate one can contract for at time t for the time period $[T_i, T_{i+1}]$. One unit of currency at time T_i has a future value of $1 + \tau_i L_i(t)$ at time T_{i+1} . By arbitrage argument, i.e. assuming a strictly decreasing and strictly positive initial term structure of bond prices $P_i(0)$, the present value of both amounts at time t is the same and is given by Equation (3.12).

$$1 \cdot P_i(t) = (1 + \tau_i L_i(t)) \cdot P_{i+1}(t). \quad (3.12)$$

Since our economy has, as tradable assets, $N + 1$ pure discount bonds $P_i(t)$ (see discussion in Section 3.1) N Libor rates are deduced via the relation

$$L_i(t) = \frac{1}{\tau_i} \cdot \left(\frac{P_i(t) - P_{i+1}(t)}{P_{i+1}(t)} \right). \quad (3.13)$$

Equation (3.13) is a ratio of tradable assets (strictly positive). Taking the discount bond $P_{i+1}(t)$ as a numeraire and given an arbitrage-free economy, by unique *EMM* (Theorem 2.2.1) there exists a unique martingale (forward) measure \mathbb{Q}^{i+1} associated with $P_{i+1}(t)$. Any tradable asset, $(P_i(t) - P_{i+1}(t))$, deflated by numeraire $P_{i+1}(\cdot)$ is a martingale under forward measure \mathbb{Q}^{i+1} . Since τ_i is constant, the Libor process $L_i(t)$ is a martingale under \mathbb{Q}^{i+1} .

The dynamics of the N Libor rates expressed in matrix notation and with respect to martingale measure \mathbb{Q}^{i+1} are given as

$$[\text{diag}\{L(t)\}]^{-1} \cdot dL(t) = \mu(t, L(t)) \cdot dt + \text{diag}\{\sigma(t)\} \cdot C \cdot dW^{i+1}, \quad (3.14)$$

where,

- $\text{diag}L(t) \sim (N + 1 \text{ by } N + 1)$ diagonal matrix for a $(N + 1 \text{ by } 1)$ column vector of $dL(t)$ Libor rates where the i^{th} element is the i^{th} Libor rate, and $[\dots]^{-1}$ is the inverse of a square matrix $[\dots]$,
- $\mu(t, L(t)) \sim (N + 1 \text{ by } 1)$ column vector of drifts, which may be functions of Libor rates themselves and time,
- $\text{diag}\{\sigma(t)\} \sim (N + 1 \text{ by } N + 1)$ diagonal matrix for a $(N + 1 \text{ by } 1)$ column vector of volatilities where the i^{th} element σ_i is equal to the instantaneous volatility of the i^{th} Libor rate.
- $dW^{i+1}(t) \sim (N + 1 \text{ by } 1)$ column vector of orthogonal (independent) standard Brownian motion under measure \mathbb{Q}^{i+1} .
- $C \sim (N + 1 \text{ by } N + 1)$ matrix. $C = \{c_{i,k} : i = 0, 1, \dots, N, k = 0, 1, \dots, N\}$ is a pseudo square root for the instantaneous correlation matrix ρ such that $C \cdot C' = \rho$ where $\rho_{i,j}$ is the i, j element of ρ , i.e. $\rho_{i,j} = \sum_{k=0}^N c_{i,k} c_{j,k}$. $C \cdot dW^{i+1}$ may also be interpreted as

an $N + 1$ by 1 column vector of correlated standard Brownian motions $d\mathcal{W}^{i+1}$ with correlation matrix $(d\mathcal{W}^{i+1}) \cdot (d\mathcal{W}^{i+1})' = \rho dt$, where $(\dots)'$ is the transpose for matrix (\dots) .

A simpler and useful specification of the Libor rate dynamics is one using each i^{th} element of the Libor rates $L_i(t)$ as follows

$$\frac{dL_i(t)}{L_i(t)} = \mu_i(t, L_i(t)) \cdot dt + \sigma_i d\mathcal{W}_i^{i+1}, \quad \text{re-written as} \quad (3.15)$$

$$= \mu_i(t, L_i(t)) \cdot dt + \sigma_i \sum_{k=0}^N c_{i,k} d\mathcal{W}_k^{i+1}, \quad (3.16)$$

where $c_{i,k} (= \frac{\sigma_{i,k}}{\sigma_i})$ is the proportion of total volatility of the Libor rate $L_i(t)$ attributable to independent standard Brownian shock k ($d\mathcal{W}_k^{i+1}$). Equation (3.16) is introduced here for completeness and will be discussed when dealing with numerical estimation issues of Libor Market Models in Section 4.3, on page 41.

The idea behind the Libor market model is the construction of an arbitrage-free interest rate model that implies a pricing formula for caplets that is compatible with Black's [13] market formula (De-Jong et al. [26]). Since the Black's pricing formula for a caplet is hinged on the assumption that the relevant Libor rate follows a log-normal process under *EMM*, the LMM has to imply (drift-less) log-normal dynamics (Brigo et al. [23]) for the Libor rates $L_i(t)$ under the forward measure \mathbb{Q}^{i+1}

$$\frac{dL_i(t)}{L_i(t)} = \sigma_i(t) d\mathcal{W}_i^{i+1} \quad \forall i \in [0, N], \quad (3.17)$$

where $\sigma_i(t)$ is deterministic volatility function, possibly different for different Libor rates, and \mathcal{W}_i^{i+1} is the i^{th} correlated standard Brownian motion under \mathbb{Q}^{i+1} measure. The solution to Equation (3.17) can be expressed as

$$L_i(t) = L_i(0) \exp \left[-\frac{1}{2} \int_0^t \sigma_i^2(s) ds + \int_0^t \sigma_i(s) d\mathcal{W}_i^{i+1} \right],$$

and the log-normal probability distribution for $L_i(t)$, under \mathbb{Q}^{i+1} measure, is

$$\frac{dL_i(t)}{L_i(t)} \sim \phi \left(-\frac{1}{2} \int_0^t \sigma_i^2(s) ds, \sqrt{\int_0^t \sigma_i^2(s) ds} \right),$$

where $-\frac{1}{2} \int_0^t \sigma_i^2(s) ds$, and $\sqrt{\int_0^t \sigma_i^2(s) ds}$ are the mean and standard deviation, respectively.

3.5.2 Dynamics in the Terminal Measure \mathbb{Q}^{N+1}

If we are to price a tradable asset depending on several Libor rates $L_i(t)$, Equation (3.17) is not ideal. The asset will have a particular measure associated with it whereas all the

$L_i(t)$ will have N different measures associated with each of the Libor rates. We need to fix a pricing measure, say the terminal measure \mathbb{Q}^{N+1} , and model all the rates $L_i(t)$ under this measure \mathbb{Q}^{N+1} . The implication of this terminal measure (\mathbb{Q}^{N+1}) set-up is that for each of the Libor rates $L_i(t)$ there exists one special Libor rates $L_N(t)$ for which the drift equals zero. For $i < N$ the drift is non-zero, and we are now going to derive it using the “Change of Numeraire Toolkit”, see Theorem 2.4.2

$$\begin{aligned} d\mathcal{W}^{S^1}(t) &= -\rho \left(\frac{\sigma^{S^1}(t)}{S^1} - \frac{\sigma^{S^2}(t)}{S^2} \right)' \cdot dt + d\mathcal{W}^{S^2}(t), \quad \text{re-written as} \\ d\mathcal{W}^{S^1}(t) &= -\rho \left(\frac{\text{DC}(S^1)}{S^1} - \frac{\text{DC}(S^2)}{S^2} \right)' \cdot dt + d\mathcal{W}^{S^2}(t), \end{aligned} \quad (3.18)$$

where “DC” refers to “Vector Diffusion Coefficient”. DC is a linear operator for diffusion processes that works as follows. $\text{DC}(S_t^1)$ is the row vector v in

$$dS^1 = (\dots)dt + v \cdot d\mathcal{W}^{S^1},$$

for diffusion processes S_t^1 with \mathcal{W} column vector Brownian motion. This is to say that given $dL_N(t) = \sigma_N(t)L_N(t)d\mathcal{W}^{N+1}$ then

$$\text{DC}(L_N(t)) = [0, \dots, 0, \sigma_N L_N(t)] = \sigma_N(t)L_N(t) \cdot \mathbf{e}_N,$$

where \mathbf{e}_N is a zero row vector except in the N^{th} position, where we have 1. The correlation matrix ρ is the instantaneous correlation among all shocks (under any measure) $d\mathcal{W}_i d\mathcal{W}_j = \rho_{i,j} dt$. Noticing that

$$\begin{aligned} \frac{\text{DC}(S^1)}{S^1} - \frac{\text{DC}(S^2)}{S^2} &= \text{DC}(\ln(S^1)) - \text{DC}(\ln(S^2)), \\ &= \text{DC}(\ln(S^1) - \ln(S^2)) = \text{DC}(\ln(S^1/S^2)), \end{aligned}$$

means that Equation (3.18) simplifies to

$$d\mathcal{W}^{S^1}(t) = -\rho (\text{DC}(\ln(S^1/S^2)))' \cdot dt + d\mathcal{W}^{S^2}(t). \quad (3.19)$$

Let us apply the toolkit: $S^1 = P_{i+1}(t)$ and $S^2 = P_{N+1}(t)$ results in

$$d\mathcal{W}^{i+1}(t) = -\rho (\text{DC}(\ln(P_{i+1}(t)/P_{N+1}(t))))' \cdot dt + d\mathcal{W}^{N+1}(t). \quad (3.20)$$

Now notice that

$$\begin{aligned} \ln \frac{P_{i+1}(t)}{P_{N+1}(t)} &= \ln \left[\frac{P_{i+1}}{P_{i+2}} \frac{P_{i+2}}{P_{i+3}} \dots \frac{P_N}{P_{N+1}} \right], \\ &= \ln [(1 + \tau_{i+1} L_{i+1}(t)) \cdot (1 + \tau_{i+2} L_{i+2}(t)) \cdots (1 + \tau_N L_N(t))], \\ &= \ln \left[\prod_{j=i+1}^N (1 + \tau_j L_j(t)) \right] = \sum_{j=i+1}^N \ln(1 + \tau_j L_j(t)). \end{aligned} \quad (3.21)$$

Thus, from linearity

$$\begin{aligned}
\text{DC} \left(\ln \left[\frac{P_{i+1}(t)}{P_{N+1}(t)} \right] \right) &= \sum_{j=i+1}^N \text{DC} (\ln(1 + \tau_j L_j(t))), \\
&= \sum_{j=i+1}^N \frac{\text{DC}(1 + \tau_j L_j(t))}{1 + \tau_j L_j(t)} = \sum_{j=i+1}^N \tau_j \frac{\text{DC}(L_j(t))}{1 + \tau_j L_j(t)}, \\
&= \sum_{j=i+1}^N \tau_j \frac{\sigma_j(t) L_j(t) \mathbf{e}_j}{1 + \tau_j L_j(t)}. \tag{3.22}
\end{aligned}$$

Substituting this term in Equation (3.22) back into Equation (3.20), results in

$$d\mathcal{W}^{i+1}(t) = -\rho \sum_{j=i+1}^N \tau_j \frac{\sigma_j(t) L_j(t) \mathbf{e}'_j}{1 + \tau_j L_j(t)} \cdot dt + d\mathcal{W}^{N+1}(t). \tag{3.23}$$

Pre-multiply both sides of Equation (3.23) by \mathbf{e}_i , we get

$$\begin{aligned}
d\mathcal{W}_i^{i+1}(t) &= -[\rho_{i,1}, \rho_{i,2}, \dots, \rho_{i,N}] \sum_{j=i+1}^N \tau_j \frac{\sigma_j(t) L_j(t) \mathbf{e}'_j}{1 + \tau_j L_j(t)} \cdot dt + d\mathcal{W}_i^{N+1}(t), \\
&= - \sum_{j=i+1}^N \tau_j \frac{\sigma_j(t) L_j(t) \rho_{i,j}}{1 + \tau_j L_j(t)} \cdot dt + d\mathcal{W}_i^{N+1}(t). \tag{3.24}
\end{aligned}$$

Substituting this in our usual equation $dL_i(t) = \sigma_i(t) L_i(t) d\mathcal{W}^{i+1}$, we get

$$\frac{dL_i(t)}{L_i(t)} = - \sum_{j=i+1}^N \tau_j \frac{\sigma_j(t) L_j(t) \rho_{i,j}}{1 + \tau_j L_j(t)} \sigma_i(t) \cdot dt + \sigma_i(t) d\mathcal{W}_i^{N+1}(t). \tag{3.25}$$

for $1 \leq i \leq N$ where the sum is taken as zero when $i = N$, and non-zero otherwise. This zero drift for $i = N$ derives from the fact that $L_N(t)$ is a martingale under measure \mathbb{Q}^{N+1} associated with numeraire $P_{N+1}(\cdot)$. This is similar to the result of the unique *EMM* theorem discussed in the previous Subsection 3.5.1 for $L_i(t)$, and $P_{i+1}(t)$ deflated tradable assets, being martingales under measure \mathbb{Q}^{i+1} .

In our economy we recall that our tradable assets are discount bonds $P_i(t)$ and deflating $P_i(t)$ by $P_{i+1}(t)$, $P_i(t)/P_{i+1}(t)$, is a martingale with respect to \mathbb{Q}^{i+1} . And deflating $P_i(t)$ by numeraire $P_{i+2}(t)$ instead, leaves $P_i(t)$ as a $P_{i+2}(t)$ based martingale with respect to \mathbb{Q}^{i+2} . Hence, all discount bonds $P_i(t)$ with exercise times $T_i, \forall i \in [0, N+1]$, are martingales in the terminal measure \mathbb{Q}^{N+1} .

Essentially the model that we want has all tradable assets being martingales, and the bonds are tradable assets. The above does not settle the question of what would be an appropriate model for $P(t, T)$ when T is not one of the tenor times. An extension to model these bonds can be specified, but this is not used in the numerical procedure.

As for existence and uniqueness of the solution for Equation (3.25), we use Itô's formula

to obtain

$$d\ln L_i(t) = \begin{cases} -\frac{\sigma_i^2(t)}{2}dt + \sigma_i(t)d\mathcal{W}_i^{N+1} & (i = N). \\ -\sum_{j=i+1}^N \tau_j \frac{\sigma_j(t)L_j(t)\rho_{i,j}}{1+\tau_j L_j(t)} \sigma_i(t) \cdot dt - \frac{\sigma_i^2(t)}{2}dt + \sigma_i(t)d\mathcal{W}_i^{N+1}(t) & (i < N). \end{cases} \quad (3.26)$$

The diffusion coefficient is deterministic and bounded. Moreover, since $0 < \frac{\tau_j L_j(t)\rho_{i,j}}{1+\tau_j L_j(t)} < 1$, the drift is also bounded. These equations are solved numerically as described under ‘‘Suitable Discretisation for Simulation of Libor Rates’’, in Subsection 4.3.5, in the next Chapter. The issue of how this market model is compatible with Black’s [13] model is discussed in the next Subsection.

3.5.3 Market Models Save Black Formulas

The main assumptions that Black [13] assumes are that

1. Interest rates are non-stochastic.
2. The forward price of $F(t, T)$ at time $t = 0$ is $F(0, T) = \mathbb{E}[F(t, T) | \mathcal{F}_0]$.
3. The volatility σ is defined by $\text{STD Dev}(\log F(t, T)) = \sigma\sqrt{T}$.
4. $F(T, T)$ is log-normally distributed at time T .

We are going to address these in reverse order bearing in mind that $F(t, T)$ in this context refers to the Libor rates $L_i(t)$, whose dynamics were discussed under ‘‘The Multi-Factor Libor Market Model’’, in Subsection 3.5.1.

In Black’s model $F(t, T)$ is only log-normal at maturity T and no emphasis is made on this property at time $t < T$. With the Libor market model this property is satisfied by all the N Libor rates $L_i(t)$. The respective Libor rates $L_i(t)$ have a log-normal distribution in the corresponding forward measure \mathbb{Q}^{i+1} .

Since the evolution of $L_i(t)$ follows a geometric Brownian motion, it is justified to define the volatility $\sigma_i(t)$ through the standard deviation (STD Dev) of the log-normal rates

$$\text{STD Dev}(\ln(L_i(t))) = \sqrt{\int_0^{T_i} \sigma_i^2(s) ds}, \quad s \leq T_i.$$

The Libor rates $L_i(t)$ are forward prices defined by discount bonds $P_i(t)$ and $P_{i+1}(t)$ and they are martingales under \mathbb{Q}^{i+1} measure. Hence, their price at time $t = 0$ of $L_i(0) = \mathbb{E}^{P_{i+1}}[L_i(T_i) | \mathcal{F}_0]$ is justified.

Interest rates are stochastic, but that does not affect the price because of the Change of Measure Theorem (2.3.1). The justification was discussed in Subsection 3.2.2.

Since these assumptions are justified, therefore, they have enabled the Libor market models to be compatible with Black’s model for caps. Although theoretically incompatible with

Black's swaption formula, the discrepancy between the swaption prices implied by the two models is often small, so that the Libor market model is practically compatible with swaption market formula, Brigo et al. [23].

3.6 Summary

In this Chapter we looked at discount bonds as the main underlying asset in the interest rates derivatives. Various types of interest rates are derived from the decreasing term structure of these bonds. Pricing of options on bonds has advanced from endeavours to modify the Black and Scholes [14] analytical solution through continuous models of interest rates, and is now focused on discrete log-normal models of interest rates that match the term structure. The latter model used in our case is the Libor market model proposed by Musiela and Rutkowski and has three attractive features. It prevents arbitrage among bonds, keeps interest rates positive, and intrinsically, prices caps according to Black's formula because each Libor rate $L_i(\cdot)$ is a martingale associated with numeraire $P_{i+1}(\cdot)$.

Tools for Numerical Implementation

We saw in Chapter 2 that the tri-union concepts are core in pricing derivatives. In the last Chapter we focused mainly on the martingale and no-arbitrage principles when building the Libor market models. In this Chapter an in depth analysis of the numerical implementation of the models is done starting from motivating why the Monte-Carlo (MC) simulation is ideal. The basic form of MC is computationally inefficient. After introducing it, we are going to show how quasi-random numbers and variance reduction techniques improve its efficiency. This is followed by looking at available estimation issues for the Libor market models that could enhance the numerical simulation whilst upholding the trinity of the derivative's pricing concepts. Finally, we end the Chapter by summarising key ideas.

4.1 Basic Monte Carlo Simulation

Monte Carlo (MC) methods are stochastic techniques based on the use of random numbers to investigate problems. The basic MC method is extremely easy to implement. In this (introductory) Section we describe the basic method of MC and some of its properties. Our objective is to use MC to value captions:

$$\mathbf{Caption\ Value}(t=0) = P(0, T_{\text{exp}}) \mathbb{E} \left[\left(\mathbf{Cap}^{\mathbf{Black76}}(T_{\text{exp}}) - \mathcal{X} \right)^+ \right].$$

In general the value of the derivative is given by an expression of the form

$$\Xi_0 = P(0, T_{\text{exp}}) \mathbb{E} [\Xi(T_{\text{exp}}, \omega)], \quad (4.1)$$

$$= P(0, T_{\text{exp}}) \int_{\Omega} \Xi(T_{\text{exp}}, \omega) d\mathbb{Q}, \quad (4.2)$$

where $\Xi(T_{\text{exp}}, \omega)$ is the payoff, and $\omega \in \Omega$ is a sample point in the path space. Equation (4.2) is another way of expressing the price where the expectation is written as an integral over Ω in the risk neutral measure \mathbb{Q} . Using the Riemann sum to approximate integration,

the approximation for Equation (4.2) is given by

$$\hat{\Xi}_0 \approx \frac{1}{M} \sum_{j=1}^M \Xi_{0,j}, \quad (4.3)$$

where $\Xi_{0,j}$ is the discounted payoff $P(0, T_{\text{exp}}) \Xi_{T_{\text{exp}},j}$, $\hat{\Xi}_0$ is an estimate of the true value of the option Ξ_0 , and M is the number of the sampled points. The MC lends itself naturally to the evaluation of randomly distributed prices represented as expectation if the payoff has no closed form solution in that its average value is what is sought (Wilmott [98]).

The average price $\hat{\Xi}_0$ in Equation (4.3) has an inherent standard error $\text{SE}(\cdot)$. It is estimated as the sample standard deviation $\text{SD}(\cdot)$ of $\Xi_{0,j}$ divided by the square root of the number of samples (Clewlow and Strickland [24]).

$$\text{SE}(\hat{\Xi}_0) = \frac{\text{SD}(\Xi_{0,j})}{\sqrt{M}}, \quad \text{where} \quad (4.4)$$

$$\text{SD}(\Xi_{0,j}) = \sqrt{\frac{1}{M-1} \sum_{j=1}^M (\Xi_{0,j} - \hat{\Xi}_0)^2}. \quad (4.5)$$

Valuing the Capion involves the following several steps (Wilmott [98]).

1. We simulate M random paths for the Libor rates $L_i(t)$ from the terminal measure world.
2. For each path j , we calculate the value of the Cap at time T_{exp} and the payoff of the Capion.

$$\begin{aligned} \mathbf{Cap}_j(T_{\text{exp}}) &= \sum_{i=\alpha}^N P_{i+1}(T_{\text{exp}}) \tau_i \mathbf{Black76}(L_i(T_{\text{exp}}), \mathcal{R}, v_i) \\ \Xi_{T_{\text{exp}},j} &= \max[\mathbf{Cap}_j(T_{\text{exp}}) - \mathcal{X}, 0] \end{aligned}$$

3. We compute the mean, at time $t = 0$, from the product of the M sampled payoffs and the discount bond $P(0, T_{\text{exp}})$ to obtain an estimate of the value of the Capion

$$\hat{\Xi}_0 \approx \frac{1}{M} \sum_{j=1}^M \Xi_{0,j},$$

4. Finally, we compute the standard error $\text{SE}(\cdot)$

$$\text{SE}(\hat{\Xi}_0) = \frac{\text{SD}(\Xi_{0,j})}{\sqrt{M}},$$

where $\text{SD}(\cdot)$ is as given in Equation (4.5).

The pros and cons for the basic MC simulation are given in Table 4.1 on the next page.

Table 4.1: Pros and Cons for (Basic) Monte Carlo.

Advantages	Disadvantages
- Error of convergence independent of the dimensions, $O(1/\sqrt{M})$.	- For very complex problems more replications are required to obtain precise results
- Flexible and easy to implement and modify.	- Slow compared with Finite difference solutions.

The critical part of the MC simulation (Clewlow and Strickland [24]) is the simulation of the Brownian motion $dW (= \varepsilon\sqrt{dt})$, for the Libor rates $L_i(t)$, via the the generation of the standard normal random numbers (obtained from converting pseudo-random numbers $[0, 1)$). Most computer programs provide built-in functions for generating pseudo-random numbers. The numbers generated by these functions are designed to appear random when subject to standard statistical tests for randomness. Some functions, however, are more efficient than others and so the pseudo-random number generator used should always be subjected to standard statistical tests (Clewlow and Strickland [24]).

4.2 Computational Efficiency

Using the basic form of MC simulations raises two issues. One is convergence time and the other is precision. In order to obtain precise results a large number of simulations are required because the error is of order $O(1/\sqrt{M})$. For example, to reduce the error by 10% for 100 simulations, 10 000 simulations have to be carried out (Galanti and Jung [35]). This large number of simulations come at the expense of computational time and, to solve this problem, different variance reduction techniques have been developed to increase precision (Boyle et al. [15]).

The pseudo random numbers generated by computer programs are not uniform as they have a high discrepancy that affects the standard error. We can improve convergence by using uniformly distributed, pre-selected deterministic numbers in the interval $[0, 1)$, hence-after, referred to as the low discrepancy sequences (LDS) or quasi-random numbers (QRN). These numbers are the basis used to define other variables such as Brownian motion $W(t)$, Libor rates $L_i(t)$, and Cap price $\mathbf{Cap}(T_{\text{exp}})$ by inverting them to standard normal random numbers ε . The LDS's improve convergence from the order $O(1/\sqrt{M})$ to nearly $O(1/M)$ (Jackson and Staunton [57], Joy et al. [62]).

In the next Subsections, we begin our discussion by first describing the LDS's followed by techniques for inverting them to standard normal random numbers. After that we describe the different variance reduction techniques pertinent to our theme. We end the Section by summarising important issues regarding convergence and precision as means of improving the inefficient basic MC simulation.

4.2.1 Low Discrepancy Sequences (Convergence time)

There are various procedures for generating multi-dimensional low discrepancy sequences and all of them are based on the variation of the concepts behind the one-dimensional low discrepancy sequence called Van der Corput sequences. The sequence in each dimension is generated by using prime numbers as base. In this context, a base is an integer $b \geq 2$, where b is the prime number.

4.2.1.1 Van der Corput (1935) Sequences

The idea behind the LDS's by Van der Corput (VDC) is that points in a line are repelled, maximally avoiding each other, and additional sequences fill larger gaps between existing points resulting in uniformly distributed points. The sequences are generated using the base number 2, hence called VDC base-2.

16 numbers starting from 0 to 15 are used to illustrate the dynamic optimal filling of these VDC sequences. The sequence point in the interval $[0, 1)$ for the decimal base (denary) whole number h is obtained by the following steps.

1. Expand the denary number h in base-2.

$$h = \sum_{j=0}^m a_j(h)2^j, \quad (4.6)$$

where $a_j(h)$ are intrinsic sequence integers $0 \leq a_j(h) < 2$ for base-2, and m is the lowest integer that makes $a_j(h) = 0$ for all $j > m$. For example, 8 in base-2 is $8 = 1000_2$.

2. Reverse the order of the digits and put a radix in front of the sequence to get the base-2 fraction $\psi_2(h)$:

$$\psi_2(h) = \sum_{j=0}^m \frac{a_j(h)}{2^{j+1}}. \quad (4.7)$$

Reversing $1000_2 \rightarrow 0.0001_2 = \frac{1}{16}$. These steps are done for all the remaining numbers, resulting in the distribution shown in Figure 4.1 below.

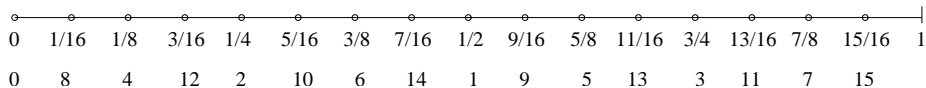


Figure 4.1: The sequence of the first 16 numbers ($h=0$ to 15).

The points on the number-line are the VDC sequences corresponding to the first 16 numbers shown below them.

The VDC sequence is the basic one-dimensional LDS. What sequences are used for multi-dimensional space? The Halton, Faure and Sobol sequences are the best known LDS and are described, respectively, in the next Subsections.

4.2.1.2 Halton Sequences

Halton sequence is the simplest LDS in multiple dimensions and it uses one different prime base for each dimension. The sequence for the first dimension is the VDC base-2 sequence, for the second dimension it is the VDC base-3 sequence, and for the third dimension it is the VDC base-5, and so on. In general, the sequence for the n^{th} -dimension is the VDC sequence using the n^{th} prime number as the base (Galanti and Jung [35]).

The algorithm to construct VDC sequence for multiple dimensions, $[0, 1]^n$ unit hypercube, is as follows.

1. For each dimension $k = 1, \dots, n$, find the representation of the denary number h in the associated prime number base b_k .

$$h = \sum_{j=0}^m a_{j,k}(h) b_k^j.$$

2. Reverse the order of the digits and put a radix in front of the sequence to get the base- b_k fraction $\psi_{b_k}(h)$:

$$\psi_{b_k}(h) = \sum_{j=0}^m \frac{a_{j,k}}{b_k^{j+1}}.$$

For example, the Halton sequence for the decimal number 50, $h = 50$, in the unit hypercube $[0, 1]^3$, for prime numbers 2, 3, and 5 is constructed as follows, see Table 4.2 below.

Table 4.2: Halton sequence for number 50 in 3-d.

Decimal number 50 ($h = 50$).			
k	b_k	h_{b_k}	$\psi_{b_k}(h)$
1	2	110010 ₂	0.010011 ₂ = 19/64
2	3	1212 ₃	0.2121 ₃ = 70/81
3	5	200 ₅	0.002 ₅ = 2/125

The sequences for higher dimensions are juxtaposed in Figure 4.2 on the facing page, contrasting the visible degradation and correlation.

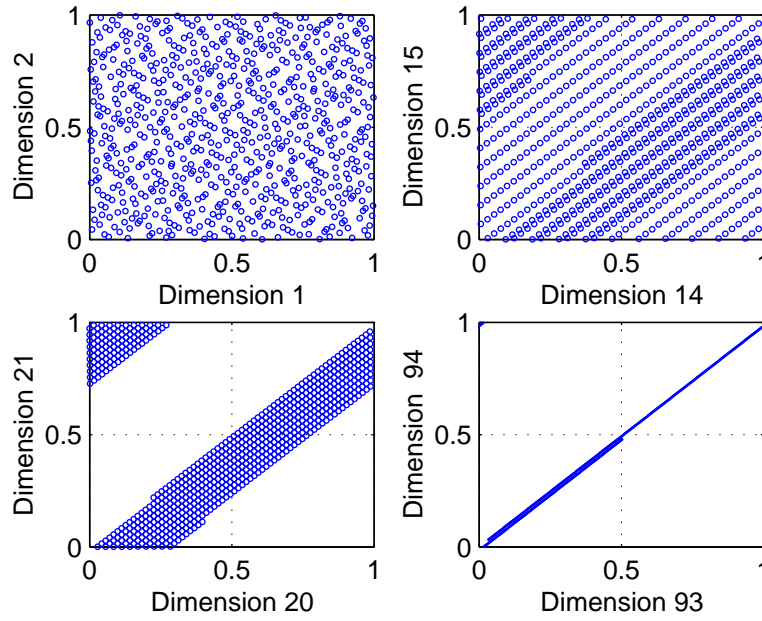


Figure 4.2: Filling in the gaps for 1000 simulations.

Higher base means higher cycle and higher computational time to fill the unit hypercube $[0, 1]^n$. This implies that the speed at which increasing finer grid points are generated decreases with increasing dimension. Improvements over these problems of low speed and correlation are offered by the Faure and Sobol sequences.

4.2.1.3 Faure Sequences

The Faure sequence is similar to the Halton sequence, however, it has two main differences (Galanti and Jung [35]). Firstly, it uses one base for all dimensions higher than the first dimension. Secondly, the sequence for each successive dimension, uses the re-ordering of the previous sequence, starting from the first dimension VDC sequence.

The base for the other dimensions is the smallest prime number that is larger than or equal to the number of dimensions in the problem. For example, if the dimensions of the problem are 40, Faure sequences use 41, the first prime number after 40, whereas Halton sequence use the 40th prime number 173. The algorithm for the Faure sequence uses the two equations, Equations (4.6) and (4.7), given in the topic of Van der Corput sequence, but before using the second equation, it re-orders the $a_j(h)$ digits. This is executed using the recursive formula from dimension $(k - 1)$ to the new dimension (k) .

$$a_{i,k}(h) = \left[\sum_{j \geq i}^m \binom{j}{i} a_{j,k-1} \right] \bmod b,$$

where $[\cdot] \bmod b$ is the remainder from dividing $[\cdot]$ by b .

For higher dimensions, Galanti and Jung [35] argues that the Faure sequence has problem

of start-up, because the points show crowding about zero. Faure (*op.cit.* Galanti and Jung [35]) suggests discarding the first $n = (b^4 - 1)$ points, where b is the base.

There are also some variations of Faure sequence available in literature. Tezuka and Tokuyama [97] formulated the generalised Faure sequence based on the Halton sequence but using polynomials for re-ordering.

4.2.1.4 Sobol Sequences

The generation of Sobol sequences use the same base for all dimensions like the Faure sequences (Galanti and Jung [35]) and proceed an intricate permutation of the vector elements within each dimension. However, instead of re-ordering sequences from previous dimensions as in the Faure sequences, the intricate re-ordering for each dimension (in the Sobol sequences) is carried out using different irreducible polynomials \mathcal{P}_d of modulo 2 (Bratley and Fox [18]):

$$\mathcal{P}_d = x^d + a_1x^{d-1} + \dots + a_{d-1}x + 1,$$

where d is the order of the polynomial, with coefficients $a_i \in \{0, 1\}$.

The sequence points in each dimension are based on a set of “direction numbers” v_i (Bratley and Fox [18]). Each v_i is a binary fraction given by equation $v_i = m_i/2^i$, where the m_i 's are odd integers, $0 < m_i < 2^i$, defined by a recurrence relation based on the coefficients of the irreducible polynomials

$$m_i = 2a_1m_{i-1} \oplus 2^2a_2m_{i-2} \oplus \dots \oplus 2^{d-1}a_{d-1}m_{i-d+1} \oplus 2^d m_{i-d} \oplus m_{i-d}.$$

The sign \oplus denotes a bit-wise exclusive-or (XOR).

Supposing that the polynomial $x^3 + x + 1$ is the generator for the first dimension, the algorithm for generating the Sobol sequences is as follows.

1. Generate a set of odd integers m_i for $i = 1, 2, 3, \dots$ using the irreducible polynomial, that is

$$m_i = 4m_{i-2} \oplus 8m_{i-3} \oplus m_{i-3}.$$

If $m_1 = 1, m_2 = 3, m_3 = 7$ then

$$\begin{aligned} m_4 &= 12 \oplus 8 \oplus 1 \\ &= 1100_2 \oplus 1000_2 \oplus 0001_2 \\ &= 0101_2 = 5. \end{aligned}$$

2. Calculate a set of direction numbers v_i using Equation (4.8), below.

$$v_i = m_i/2^i. \tag{4.8}$$

Hence, the first four direction numbers are

$$v_1 = 0.1_2, v_2 = 0.11_2, v_3 = 0.111_2, v_4 = 0.0101_2.$$

3. Expand the decimal base number h (i.e. $h = 0, 1, 2, \dots, 19 \times M$) into base-2 ($h = \sum_{j=0}^m a_j(h)2^j$) and represent it in binary form.

$$\mathbf{a}(h) = a_{m-1}(h)a_{m-2}(h) \dots a_1(h)a_0(h).$$

For $h = 13$, $\mathbf{a}(h) = 1101_2$.

4. Finally, generate the h^{th} Sobol sequence number:

$$\psi_{b=2}(h) = a_0(h)v_1 \oplus a_1(h)v_2 \oplus \dots \oplus a_{m-1}(h)v_m. \quad (4.9)$$

Hence,

$$\begin{aligned} \psi_{b=2}(h = 13) &= v_1 \oplus v_3 \oplus v_4 \\ &= 0.0001_2 = \frac{1}{16}. \end{aligned}$$

The Sobol sequence is faster than the Faure sequence in the sense that Sobol sequence uses base-2 for all dimensions, thus there is some computational time advantage due to the shorter cycle length.

Algorithms for generating uniform random numbers have been discussed. The question raised is: how good are these produced numbers, over pseudo-random numbers, in being uniformly distributed numbers $\mathbb{U}[0, 1]$? This question is addressed by doing statistical tests on the sequences generated by these algorithms and comparing their distribution moments.

4.2.1.5 Statistical Properties for Random Numbers.

The functions for the random number generators were used to generate 10^3 points in the interval $[0, 1)$ and several statistical properties were tested, see Table 4.3 on the next page. For all the four distribution moments the quasi-random numbers present a better performance than the sequence followed by pseudo-random numbers. Comparatively, this indicates that the sequence by the former closely matches the uniform distribution $\mathbb{U}[0, 1]$ than the latter.

This is key in estimating the average value of our payoff function $\Xi(T_{\text{exp}}, \omega)$ because if the sequence is not uniformly distributed the so calculated average value is distorted. But how is a point from the uniform sequence transformed into a standard random number ε ? This topic is discussed in the next Subsection.

Table 4.3: Statistical properties of random number generators.

Variable	Excel Rand () Function	VDC	Uniform
Minimum	0.0004	0.0060	0.0000
Maximum	0.9998	0.9990	1.0000
Mean	0.5026	0.5078	0.5000
Median	0.5003	0.5050	0.4995
Std Dev.	0.2927	0.2914	0.2888
Variance	0.0857	0.0849	0.0834
Skewness	-0.0093	-0.0499	0.0000
Kurtosis	-1.2254	-1.2340	-1.2002

4.2.2 Inversion of Uniform Distribution

The uniform distribution $\mathbb{U}[0,1]$ is ideal for simulation purposes due to the fact that other distributions are derived from it. Thus, it is viewed as the cumulative probability distribution function. Hence, pseudo random numbers and quasi-random numbers, thus generated, are proxies of the actual probability.

These probability proxies from $\mathbb{U}[0,1]$ are not our ultimate goal. What we are after are the corresponding samples from a distribution that gave rise to it. The important distribution for finance applications, in general, is the standard normal density $\phi(0,1)$ and its integral is the cumulative distribution $N(\varepsilon)$. Hence, $\mathbb{U}[0,1] \equiv N(\varepsilon)$ (Joy et al. [62]).

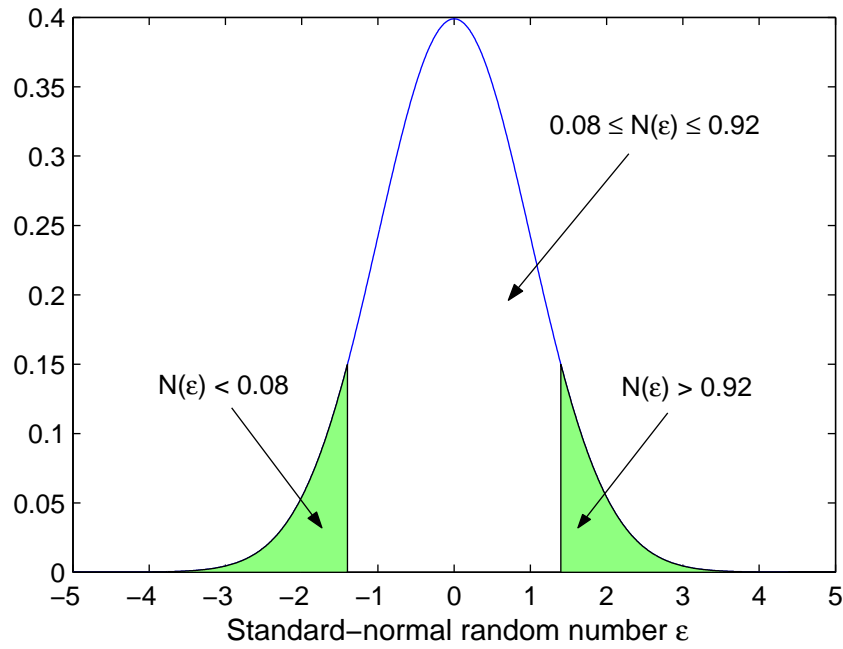
If $N(\varepsilon)$ is the standard normal distribution function, the independent variable ε is the desired sample number. This sample enables us to simulate the evolution of the Libor rates $L_i(t)$ through Brownian motion $W(t)$. From these rates the cap price $\mathbf{Cap}(T_{\text{exp}})$ and the caplet payoff $\Xi(T_{\text{exp}}, \varepsilon)$ are derived. The techniques that enables us to do the inversion are Moro's inversion and NormsInv() and are discussed next, respectively.

4.2.2.1 Moro's Inversion

The Box-Muller algorithm has been known as the general way to obtain the inverse of the normal distribution, and Galanti and Jung [35], however, suggests against the use of this algorithm since it alters the uniformity of the LDS's. Instead the proposition made is the use of the algorithm provided by Moro [74] (*op.cit.* Galanti and Jung [35]). It has been reported that Moro's inversion is faster than the usual Box-Muller algorithm and has a maximum error of 3×10^{-9} (see Press et al. [83], *ibid.* [35]).

Moro's algorithm is a hybrid of the Beasley and Springer [8] (*op.cit.* Joy et al. [62]) algorithm in that, except for the central part of the Normal distribution where it is used, the truncated Chebyshev series is used instead for the tails. Therefore, the domain for the algorithm is split into two regions, see Figure 4.3 on the facing page.

1. The central region of the distribution, $0.08 \leq N(\varepsilon) \leq 0.92$, is modelled with the

Figure 4.3: Standard normal distribution $\phi(0, 1)$.

traditional Beasley and Springer [8] algorithm

$$\varepsilon = [N(\varepsilon) - 0.5] \frac{\sum_{j=0}^3 a_j \cdot [N(\varepsilon) - 0.5]^{2j}}{\sum_{j=0}^4 b_j \cdot [N(\varepsilon) - 0.5]^{2j}}.$$

2. The tails of the distribution are modelled with the Chebyshev series, through $z = k_1 * [2 * \ln(-\ln(0.5 - |N(\varepsilon) - 0.5|)) - k_2]$:

$$\varepsilon = \begin{cases} \sum_{j=0}^8 c_j \cdot T_j(z) - \frac{c_0}{2} & \text{if } N(\varepsilon) > 0.92. \\ \frac{c_0}{2} - \sum_{j=0}^8 c_j \cdot T_j(z) & \text{if } N(\varepsilon) < 0.08. \end{cases}$$

The constants a_j , b_j , and c_j , are suitably chosen whereas k_1 , and k_2 are chosen such that $z = -1$ when $N(\varepsilon) = 0.92$, and $z = 1$ when $N(\varepsilon) = 1 - 10^{-12}$. Hence, the required constants for this exercise, as provided by Joy et al. [62], are given in Table 4.4, on the next page.

The function $T_j(z)$ is computed using the Chebyshev polynomials, Press et al. [83, Equation (5.8.2)]:

$$T_{j+1}(z) = 2 \cdot z \cdot T_j(z) - T_{j-1}(z) \quad j \geq 2,$$

where $T_0(z) = 1$, and $T_1(z) = z$. This hybrid algorithm has high accuracy for all values of $N(\varepsilon)$ in the interval $[10^{-10}, 1 - 10^{-10}]$.

Table 4.4: Constants for inverting the standard normal distribution.

j	a_j	b_j	j	c_j
0	2.50662823884	1.00	0	7.7108870705487895
1	-18.61500062529	-8.47351093090	1	2.7772013533685169
2	41.39119773534	23.08336743743	2	0.3614964129261002
3	-25.44106049637	-21.06224101826	3	0.0373418233434554
4	-	3.13082909833	4	0.0028297143036967
			5	0.0001625716917922
	k_1	k_2	6	0.0000080173304740
	0.4179886424926431	4.2454686881376569	7	0.0000003840919865
			8	0.0000000129707170

4.2.2.2 Excel Function: NormsInv()

Another popular algorithm may be used to calculate the inverse of the normal distribution. The program is called NormsInv and is a built-in function implemented in the widely available Excel package. Its accuracy is tested by comparing it with Moro's inversion against the exact numbers of the normal inversion, drawn from the paper of McCullough & Wilson (2001, p.6). The tests were two fold using the VDC sequence. In the first instance, statistical properties for all four probability moments were compared and the last test looked at the accuracy of the inversion at the tails.

Results from both tests, Table 4.5 and 4.6, respectively and below, show that both inversions produced acceptable accuracy.

Table 4.5: Statistical properties of VDC sequence to $\phi(0, 1)$.

Variable	NormsInv()	Moro's inversion	Normal (0,1)
Mean	-5.44075E-3	-5.44079E-3	0
Median	-4.89535E-3	-4.89577E-3	0
Std Dev	0.993072	0.993071	1
Variance	0.986191	0.986191	1
Skewness	-0.029555	-0.029557	0
Kurtosis	2.856627	2.856627	3

Table 4.6: Comparison of inversion of the tails.

$N(\varepsilon)$	Standard random number ε .		
	NormsInv	Moro	Exact
0.001	-3.09023	-3.09023	-3.09023
0.0001	-3.71902	-3.71902	-3.71902
0.00001	-4.26489	-4.26489	-4.26489
0.000001	-4.75342	-4.75342	-4.75342
0.0000001	-5.19934	-5.19934	-5.19934
0.000000001	-5.99781	-5.99781	-5.99781

Now that we have established the efficiency of our algorithms for inverting $N(\varepsilon)$ we are confident that the generated values of ε and values of other variables dependent on ε shall

be sound. In the next Subsection we look at techniques for increasing precision in our basic MC simulation.

4.2.3 Precision

Different variance reduction techniques have been developed by Boyle et al. [15] to increase precision when doing MC simulations. Two of the classical variance reduction techniques are the antithetic and control variate methods. More recently, moment matching and importance sampling have been introduced in finance applications (Boyle et al. [15]). These are discussed next.

4.2.3.1 Antithetic Variance Reduction - “Antivariate”

Unlike the Basic MC simulation where a single value is produced per simulation trial, in this technique two values are generated (Clewlow and Strickland [24], Hull [53]). The difference between the values lies in the sample ε used. If the first value used ε the second value uses $-\varepsilon$, that is $\Xi_{0,1} = f(\varepsilon)$, and $\Xi_{0,2} = f(-\varepsilon)$. The average $\bar{\Xi}_0$ of these values gives an estimate of the derivative per trial.

$$\bar{\Xi}_0 = \frac{\Xi_{0,1} + \Xi_{0,2}}{2}.$$

The final estimate of the value of the derivative is the average of the $\bar{\Xi}_0$'s. If $\bar{\varphi}$ is the standard deviation of the $\bar{\Xi}_0$'s, and M is the number of simulation trials (i.e., the number of pairs of the values calculated), the standard error of the estimates is $\bar{\varphi}/\sqrt{M}$ (Hull [53]). Often, this is much less than the standard error calculated using $2M$ random trials.

4.2.3.2 Control Variate Approach

The control variate is used when there are two similar derivatives, A and B, where A is the derivative whose price we are estimating using the analytical solution of derivative B (Hull [53]). Two simulations are run concurrently using the same random ε and time interval dt . Each simulation is used to estimate the value of A Ξ_A^* , and B Ξ_B^* , respectively and an unbiased estimator of the value of A Ξ_A is provided (Boyle et al. [15], Hull [53]) as

$$\Xi_A = \Xi_B + \Xi_A^* - \Xi_B^*,$$

where Ξ_B is the known true value of B. As with path-wise simulation for pricing, the method has the problem of depending on there being a good approximation to the product (security) having an analytic solution (Wilmott [98]).

4.2.3.3 Moment Matching

This technique is also referred to as *quadratic resampling* and it entails adjusting the samples taken from a standardised normal distribution so that the distribution moments (first, second, etc.) are matched (Boyle et al. [15], Hull [53]). Assuming that $\varpi_j (1 \leq j \leq n)$ are samples from the normal distribution $\phi(m, s)$ (with mean m and standard deviation s) used to calculate the change in the value of a particular variable over a particular time period, the standardised samples $Z_j (1 \leq j \leq n)$ are defined as (Hull [53])

$$Z_j = \frac{\varpi_j - m}{s}.$$

These adjusted samples Z_j 's match the standard normal distribution $\phi(0, 1)$ and we use them for all calculations. Moment matching saves on computation time, but has the disadvantage of memory problems because every number sampled must be stored until the end of the simulation (Hull [53]).

4.2.3.4 Importance Sampling

This technique entails using analytic probability distribution functions that sample non-zero payoff paths at maturity of an option [53]. Let us consider estimating the price of a call option on a stock. Assume that ϱ is the probability distribution of the stock price being greater than the strike price \mathcal{K} , and that \mathcal{G} is the probability that the stock price is greater than \mathcal{K} . The estimate of the value of the option is the average discounted payoff sampled from the distribution ϱ multiplied by \mathcal{G} .

4.2.4 Summary of Computational Efficiency

In this Section we discussed two issues regarding improving computational efficiency of the basic MC simulation and these were convergence and precision. Convergence is improved by using QRNs instead of pseudo random numbers and the three main QRN generators we discussed are: Halton, Faure, and Sobol sequences. They are preferred over pseudo-random numbers because in all four distribution moments they present better performance, see Table 4.3. These QRN are proxies to our ultimate objective of obtaining standard random numbers ε in $\phi(0, 1)$. In order to obtain ε we use either Moro's algorithm or the NormsInv function. On the other hand precision is improved by using variance reduction techniques and these are antithetic variate, control variate, moment matching and importance sampling. However, the last three techniques are not ideal for our task because of their inherent memory problems and analytical solution prerequisites.

4.3 Estimation Issues for Libor Market Models

Efficiency does not only focus on what can be improved on the basic MC simulation (variance reduction and quasi-random generators). The form the variables in the Libor market model take also affects efficiency. In order to implement Equation (3.25) for the i^{th} Libor rate by Monte Carlo, we need a decomposition of $\rho = C \cdot C'$ with C an $N+1$ by $N+1$ matrix. In other words, if we denote the i^{th} row of matrix C by C_i , then the decomposition reads $\langle C_i, C_j \rangle = \rho_{i,j}$ where $\langle \cdot, \cdot \rangle$ denotes the scalar product. Hence, the implementation of Equation (3.25) becomes:

$$\frac{dL_i(t)}{L_i(t)} = - \sum_{j=i+1}^N \frac{\tau_j \sigma_j(t) L_j(t) \rho_{i,j}}{1 + \tau_j L_j(t)} \sigma_i(t) dt + \sigma_i \sum_{k=0}^N c_{i,k} dW_k^{N+1}, \quad \text{where} \quad (4.10)$$

the W_k^{N+1} are now independent standard Brownian shocks. Indeed Equation (4.10) corresponds to Equation (3.25) since both volatility and correlation are implemented correctly. Inverting $N(\varepsilon)$ through use of QRNs allows us to compute $dW_k^{N+1} (= \varepsilon \sqrt{dt})$, see Subsection 4.2.2). τ_j and $L_i(0)$ can be deduced from market data and using Equation (4.10) $L_i(t)$ can be evaluated provided the volatility function $\sigma_i(t)$, percentage volatility from factor k ($c_{i,k}$), and correlation matrix ρ are known. Therefore in this Section we are going to look at estimation issues for these variables that would enhance simulation of the Libor rates $L_i(t)$.

We will start by discussing suitable functional forms for estimating $\sigma_i(t)$, followed by that of both $\rho_{i,j}$ and $c_{i,k}$ because of their inter-relationship. Next, we show how to jointly calibrate $\sigma_i(t)$ and $\rho_{i,j}$ to market data. Thereafter, we discuss suitable discretisation for simulating Libor rates $L_i(t)$. We end the Section by summarising key estimation issues pertaining to the Libor market model.

4.3.1 Choices for the Instantaneous Volatility Functions

Extensive material has covered this issue under the umbrella term of “regularisation”, and there already exist established ways to deal with this problem “of which Tikhonov’s approach is probably the most common” (Rebonato [86]). However, a more powerful approach is to require the solution belong to a class of functions identified on the basis of the modellers understanding of the financial characteristics of the problem (Rebonato [86]). A wide range of work has been done by Brigo and Mercurio [20], Brigo et al. [23], Rebonato [86] on the possible functions that the volatility could take and the nature of the term structure of volatilities. Their work form the basis of this thesis in this regard. Stressing the importance of the choice of the volatility function Rebonato [86, p.144] wrote:

“An important aspect in the construction of the Libor market model is the specification of what values volatilities of different forward rates should assume ... As much as I believe that the modern pricing framework is powerful and

practically useful, I must stress that it is only as good as the input volatilities (and, to a lesser extent, correlations) that are fed into it. A simplistic or ill-thought-out choice for the forward-rate volatility function can easily produce little more than a cumbersome and computationally expensive implementation of a poor one-factor short-rate-based model.”

In many instances the implied volatility of an option has an inverse relation to its strike price known as a volatility smile. In our case we are going to assume the absence of volatility smiles. However, an interested reader is referred to the works of Brigo et al. [22] for a detailed coverage of volatility smiles.

4.3.1.1 Time Homogeneity

Empirical observations over very extended periods of time show that the term structure of volatilities, despite change in level and occasional localised dramatic changes, tend to retain their qualitative shape and structural features (Rebonato [86]). Hence, in this thesis the volatility term structure is assumed to evolve in a time-homogeneous manner. This implies that the future term structure of volatilities at time λ (such that $T_i + \lambda < T_{N+1}$) will look exactly like today, see Figure 4.4,

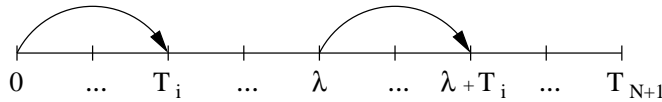


Figure 4.4: Time homogeneous term structure.

and is expressed as

$$\int_0^{T_i} \sigma_{\text{inst}}^2(s, T_i) ds = \int_{\lambda}^{T_i + \lambda} \sigma_{\text{inst}}^2(s, T_i + \lambda) ds, \quad (4.11)$$

where $\sigma_{\text{inst}}(s, \cdot)$ is the instantaneous volatility. The term structure of implied volatility $\sigma_{\text{Black76}}(T_i)$ for each caplet is linked to $\sigma_{\text{inst}}(s, T_i)$ by the market variance

$$\sigma_{\text{Black76}}^2(T_i) T_i = \int_0^{T_i} \sigma_{\text{inst}}^2(s, T_i) ds. \quad (4.12)$$

This is satisfied $\forall \lambda$ if the functional form $\sigma_{\text{inst}}(t, T_i) = \sigma_{\text{inst}}^*(T_i - t)$ holds, implying

$$\int_{\lambda}^{T_i + \lambda} \sigma_{\text{inst}}^2(s, T_i + \lambda) ds = \int_{\lambda}^{T_i + \lambda} \sigma_{\text{inst}}^{*2}(T_i + \lambda - s) ds, \quad (4.13)$$

$$= \int_0^{T_i} \sigma_{\text{inst}}^{*2}(T_i - s) ds. \quad (4.14)$$

From Equation (4.11), for a caplet of T_i maturity, its variance can also be expressed as a

sum of the caplet variance of the previous maturities.

$$\begin{aligned}
\int_0^{T_i} \sigma_{\text{inst}}^2(s, T_i) ds &= \int_0^{T_1} \sigma_{\text{inst}}^2(s, T_i) ds + \int_{T_1}^{T_2} \sigma_{\text{inst}}^2(s, T_i) ds + \dots \\
&\quad + \int_{T_{i-2}}^{T_{i-1}} \sigma_{\text{inst}}^2(s, T_i) ds + \int_{T_{i-1}}^{T_i} \sigma_{\text{inst}}^2(s, T_i) ds. \\
&= \sum_{k=1}^i \int_{T_{k-1}}^{T_k} \sigma_{\text{inst}}^2(s, T_i) ds.
\end{aligned} \tag{4.15}$$

This simplification is of great use when calibrating the instantaneous volatility to market data (see Subsection 4.3.4).

For the caplet maturing $T_i + \lambda$ years from today the market variance is

$$\int_0^{T_i+\lambda} \sigma_{\text{inst}}^2(s, T) ds = \sigma_{\text{Black76}}^2(T_i + \lambda)(T_i + \lambda). \tag{4.16}$$

Taking the ratio of Equation (4.16) to (4.12) and incorporating time homogeneous property one gets

$$\begin{aligned}
\frac{\sigma_{\text{Black76}}^2(T_i + \lambda)(T_i + \lambda)}{\sigma_{\text{Black76}}^2(T_i)(T_i)} &= \frac{\int_0^{T_i+\lambda} \sigma_{\text{inst}}^2(s, T_i) ds}{\int_{\lambda}^{T_i+\lambda} \sigma_{\text{inst}}^2(s, T_i + \lambda) ds}, \\
&= \frac{\int_0^{\lambda} \sigma_{\text{inst}}^2(s, T_i) ds + \int_{\lambda}^{T_i+\lambda} \sigma_{\text{inst}}^2(s, T_i) ds}{\int_{\lambda}^{T_i+\lambda} \sigma_{\text{inst}}^2(s, T_i + \lambda) ds}, \\
&= 1 + \frac{\int_0^{\lambda} \sigma_{\text{inst}}^2(s, T_i) ds}{\int_{\lambda}^{T_i+\lambda} \sigma_{\text{inst}}^2(s, T_i + \lambda) ds}.
\end{aligned} \tag{4.17}$$

$\int_0^{\cdot} \sigma_{\text{inst}}^2(s, \cdot) ds$ is strictly a positive quantity, hence, Equation (4.17) is always greater than unity. Therefore, to ensure that the term structure of volatilities evolves in a time homogeneous profile, $\sigma_{\text{Black76}}^2(T_i)T_i$ the market variance, as evaluated from today's term structure of volatilities should be a strictly increasing function of T_i . This implies that the summands in Equation 4.15 are all positive.

4.3.1.2 Plausible Choice for Volatility Function

Requiring a time homogeneous volatility term structure is not enough. The trader is also concerned about the financial characteristics of σ_{inst} . A detailed account on this is given by Rebonato [86, pp 158-166]. In particular Rebonato argues that the function of the general form

$$\sigma_{\text{inst}}(t, T_i) = f(T_i - t)\varphi(t)K(T_i) \quad t \leq T_i, \tag{4.18}$$

displays several positive features, from both the financial and the computational perspective.

1. $f(T_i - t)$: The desirable qualitative features for $f(T_i - t)$ are that it ensures correct pricing to the caplet market and best approximates time homogeneity. The proposed functional form is

$$f(T_i - t) = [a + b(T_i - t)] \exp[-c(T_i - t)] + d, \quad (4.19)$$

where a , b , c , and d are constants with the following financial and econometric constraints: $a + d > 0$, $c > 0$, and $d > 0$.

In addition, as $T_i - t$ tends to zero, instantaneous and average volatilities tend to coincide, and therefore the quantity $a + d$ should at least approximately assume the values given by the shortest-maturity implied volatility.

2. $\varphi(t)$: Suitable qualitative features for $\varphi(t)$ are less easy to identify in that they capture occurrence of time localised shocks, such as forthcoming elections. The proposed function is given by

$$\varphi(t) = \left[\sum_{j=1}^n \varphi_j \sin \left(\frac{t\pi j}{T_{Mat}} + \varphi_{j+1} \right) \right] \exp(-\varphi_{n+1}t) \quad (4.20)$$

with n as small as 4, and the quantity T_{Mat} is the time equal to the longest caplet maturity. This function is linear combination of sine waves (where the phases and amplitudes are optimised), multiplied by a decay factor (with optimised decay constant). The idea behind this functional form is to allow a sufficient number of frequencies to pick up inherent localised time shocks, but not too many so that excessive market noise obscures the specification.

3. $K(T_i)$: The requirement that the prices of the caplets should be exactly recovered uniquely determines the quantities

$$K(T_i) = (1 + \kappa_i), \quad \forall i \in [1, N]. \quad (4.21)$$

If the procedure has worked properly, the variation of the term κ as a function of the forward-rate index might contain high frequencies, but should be of very small amplitude.

This proposed volatility functional form, as is, is not yet suitable to modelling Libor rates. It has to be calibrated to market data first before it could be used to simulate Libor rates.

4.3.2 Choices for the Correlation Matrix

Apart from the volatility function, another equally important aspect for this LMM calibration problem is the (non-) parametric specification of the instantaneous correlation matrix. Considering that one must obviously preserve certain properties of a correlation matrix, renders the whole exercise of modelling of the correlation matrix a delicate

task (Brigo and Morini [21]). The main properties that one must preserve are namely: 1) symmetry: $\rho_{i,j} = \rho_{j,i}$, $\forall i, j$, 2) unitary diagonals: $\rho_{i,i} = 1$, 3) positive definiteness: $x' \rho x \geq 0$, $\forall x \in \mathbb{R}^M$, and 4) normalised entries: $\rho_{i,j} \leq 1$, $\forall i, j$.

The non-parametric form involves determining the $\frac{1}{2}N(N+1)$ unique elements of the $N+1$ by $N+1$ correlation matrix from historical forward Libor rates (Tanimura and Yamada [96], Alexander and Lvov [3]). When historical data on forward Libor rates are used, there are many sources of model risk. This is due to the fact that bootstrap method over fits the yield curve and magnifies the errors in the original data. Another decision that will affect historical forward rate correlations is the length of historical observation period. It should be short enough so that correlations represent current market conditions, but long enough so that correlations at the short end are not too unstable. Finally, which statistical model is to be used to forecast the correlations: an equally weighted moving average, an exponentially weighted moving average, or a GARCH model?

In view of this model risk, the alternative approach may be preferred, where having decided on a parametric form, a number of parameters that is smaller than $\frac{1}{2}N(N+1)$ is determined to best fit market prices. As a first possibility we may decide to maintain a *full-rank* correlation matrix and such an approach has been followed by Schoenmakers and Coffey [94] and Rebonato [85] (*op.cit.* Brigo [19]). Schoenmakers and Coffey recommend the two following parametric forms 1) $\rho = \exp \left[-\frac{|i-j|}{M-1} \left(-\ln \rho_\infty + \eta \frac{M-1-i-j}{M-2} \right) \right]$, and 2) $\rho = \exp \left[-\frac{|i-j|}{M-1} \left(-\ln \rho_\infty + \eta \frac{i^2+j^2+ij-3Mi+3i+3j+2M^2-M-4}{(M-2)(M-3)} \right) \right]$, whereas, Rebonato [85] recommends $\rho = \alpha + (1-\alpha) \exp[-|i-j|(\gamma - \zeta \max(i, j))]$.

Alternatively, until work produced by Tanimura and Yamada [96] (in 2006) *rank-reduced* correlations could be obtained from parametric forms posed by 1) De-Jong et al. [27] (*op.cit.* Pietersz and Groenen [81]):

$$\rho = \exp \left[-\gamma_1 |t_i - t_j| - \frac{\gamma_2 |t_i - t_j|}{\max(t_i, t_j)^{\gamma_3}} - \gamma_4 |\sqrt{t_i} - \sqrt{t_j}| \right],$$

where $\gamma_i > 0$, and 2) Rebonato [85], Rapisarda et al. [84]:

$$\rho_{i,j} = \sum_{k=0}^N c_{i,k} c_{j,k}, \quad \text{where}$$

$$c_{i,k} = \begin{cases} \cos(\theta_{i,k}) \prod_{j=0}^{k-1} \sin(\theta_{i,j}) & (0 \leq k < N) \\ \prod_{j=0}^{k-1} \sin(\theta_{i,j}) & (k = N) \end{cases}.$$

Tanimura and Yamada [96] propose a new parameterization for the correlation matrix using a decay function with respect to the starting times of the Libor rates. Using the Japanese market as a case study, Tanimura and Yamada, conclude that their parameterization explain the correlation structure reasonably well and that it is computationally tractable and provide a practical estimation for the implied correlation matrix. Their model form the

basis of this thesis in this regard. Let τ_{ij} be

$$\tau_{ij} = |T_j - T_i|, \quad 0 \leq i < j \leq N.$$

We set the (i, j) entries of the correlation matrix ρ as

$$\rho_{i,j}(\vartheta_i, \tau_{ij}) = \rho_{j,i}(\vartheta_i, \tau_{ij}) = \begin{cases} 1 & (i = j) \\ 1 - \exp\left(-\frac{\vartheta_i}{\tau_{ij}}\right) & (i \neq j), \end{cases} \quad (4.22)$$

where $\vartheta_i > 0$ is a decay factor such that the correlation range $0 < \rho_{i,j} \leq 1$ is satisfied, and $\rho_{i,j} \searrow 0$ as τ_{ij} increases. Hence, the matrix ρ is determined by $N + 1$ parameters, $\Theta = \{\vartheta_i : i = 0, 1, \dots, N\}$.

Now that we have a parametric model for the correlation matrix, we are in position to describe the techniques for reducing the rank of this matrix and a detailed account of this subject is given in the next Subsection.

4.3.3 Factor Reduction Techniques

For efficient pricing of derivatives the correlation matrix has to be reduced to a lower rank with a smaller number of random factors. A large number of factors increases the number of random numbers that have to be drawn when simulating, thereby costing expensive computational time. Another reason for keeping the number of factors rather small is trying to explain these factors with usual market movements. The first factor is interpreted as a shift of the yield curve (i.e. simultaneous up or down movement of the forward rates), the second as a tilt of the yield curve (i.e. the forward rates close to and far away from the reset date move in opposite directions), and the third factor as a butterfly movement (i.e. forward rates close to and far away from the reset date move stronger in the same direction than forward rates in between (Hull [53])). Thus increasing the number of factors far beyond these comprehensible market movements is usually avoided.

Instead of taking into account all Brownian motions in Equation (4.10), we want to do the simulation with a smaller number, d say, with $d \ll N + 1$. This process in essence is an optimisation problem of finding a rank- d correlation matrix $\rho^d = C_d \cdot C_d'$ (where C_d is a $N + 1$ by d matrix) closest to the original correlation matrix $\rho(\Theta, \tau)$. Unlike in scalar variables where the closest match between two points is obtained by finding the least distance between them, on the space of matrices to measure distance we require a matrix norm. The frequently used norm is the Frobenius matrix norm (Pietersz and Groenen [81]):

$$\begin{aligned} \|X\|_F^2 &= \text{trace}(XX') \\ &= \sum_{i=0}^N \sum_{j=0}^N |x_{i,j}|^2. \end{aligned}$$

$$\text{Subject to } \text{diag}(C_d \cdot C_d') = \text{diag}(I).$$

Thus, the closest matrix is interpreted as the one optimised to have minimum distance in the Frobenius norm. This optimization problem stated mathematically is:

$$\begin{aligned} & \text{Find } \Theta \in \mathbb{R}^{N+1}, \\ & \text{by minimizing } \mathcal{Z}_\rho(\Theta, \mathbf{d}) = \|\rho(\Theta, \tau) - \rho^{\mathbf{d}}\|_F^2. \\ & \text{Subject to } \text{diag}(C_{\mathbf{d}} \cdot C'_{\mathbf{d}}) = \text{diag}(I). \end{aligned} \quad (4.23)$$

In literature, several algorithms to minimize \mathcal{Z}_ρ have been proposed and a useful review of these is given in Pietersz and Groenen [81]. In addition to describing the various algorithms, they have also developed a novel algorithm based on iterative majorization that minimize \mathcal{Z}_ρ . After numerically comparing its performance with that of the other methods Pietersz and Groenen recommend the majorization algorithm over the others because it has the advantage of simultaneously being 1) simple to implement, 2) globally convergent, and 3) computational efficient. (An interested reader is referred to Pietersz and Groenen's work for a detailed comparison of the various algorithms.) For the VBA program code for this thesis, we are going to implement their novel algorithm.

The starting point for majorization is finding an initial estimate for the i^{th} row of matrix $C_{\mathbf{d}}$, i.e. $(C_{\mathbf{d}})_i$, through singular value decomposition (SVD) of $\rho(\Theta, \tau)$. The main reason for this is the non-optimality of the SVD method: generally one may find decompositions $C_{\mathbf{d}}$ (even locally) for which the associated correlation matrix $C_{\mathbf{d}} \cdot C'_{\mathbf{d}}$ is closer to the original matrix $\rho(\Theta, \tau)$ than the SVD-approximated correlation matrix $C_{\mathbf{d}}^{\text{svd}} \cdot (C_{\mathbf{d}}^{\text{svd}})'$ (Pietersz and Groenen [81]). Thus, in the next Sub-subsections before describing the majorization algorithm we are going to first discuss the SVD technique.

4.3.3.1 Singular Value Decomposition

SVD is popular among financial practitioners and implemented in numerous financial institutions. This technique is based on the common principal component analysis introduced by Flurry [34] (*op.cit.* Alexander and Lvov [3], Pietersz and Groenen [81]). It is easy to implement because almost all that is required is an eigenvalue decomposition. The calculation is almost instant, and the approximation is reasonably accurate (Pietersz and Groenen [81]).

A standard method that is advocated by Rebonato [85], Hull and White [54, 55] and many others (*op.cit.* Alexander and Lvov [3]) uses an orthogonal transformation of the correlated Brownian motions in Equation (3.25). We recall that the Libor rate dynamics are expressed in terms of independent Brownian motions as follows:

$$\frac{dL_i(t)}{L_i(t)} = - \sum_{j=i+1}^N \frac{\tau_j \sigma_j(t) L_j(t) \rho_{i,j}}{1 + \tau_j L_j(t)} \sigma_i(t) dt + \sum_{k=0}^N \sigma_{i,k} dW_k^{N+1}, \quad \text{where}$$

$\sigma_i d\mathcal{W}_i^{N+1} = \sum_{k=0}^N \sigma_{i,k} dW_k^{N+1}$. From this it follows that

$$\sigma_i = \sqrt{\sum_{k=0}^N \sigma_{i,k}^2}, \quad \text{and} \quad (4.24)$$

$$\rho_{i,j} = \frac{\sum_{k=0}^N \sigma_{i,k} \sigma_{j,k}}{\sigma_i \sigma_j}. \quad (4.25)$$

The $N + 1$ by $N + 1$ covariance matrix of Libor rates $[\text{diag}\{L(t)\}]^{-1} \cdot dL(t)$, \mathcal{V} contains instantaneous volatilities σ_i and correlation $\rho_{i,j}$ and is given by

$$\mathcal{V} = \mathcal{D}\rho\mathcal{D}', \quad \text{where} \quad (4.26)$$

\mathcal{D} is the diagonal matrix of the instantaneous volatilities, and ρ is the correlation matrix. Given these, the volatility components may then be determined from the spectral decomposition of the correlation matrix ρ :

$$\rho = \mathcal{B}\lambda\mathcal{B}', \quad \text{where} \quad (4.27)$$

λ is the $N + 1$ by $N + 1$ diagonal matrix of eigenvalues, and \mathcal{B} is the $N + 1$ by $N + 1$ matrix of eigenvectors of the Libor rate correlation matrix ρ . To derive the \mathbf{d} volatility components from Equation (4.27) denote by $\lambda_0, \dots, \lambda_{\mathbf{d}-1}$ the \mathbf{d} largest eigenvalues of ρ and by $\beta_0, \dots, \beta_{\mathbf{d}-1}$ their corresponding eigenvectors. Set

$$\mathcal{M}_i = \frac{\sigma_i}{\sqrt{\Upsilon_i}}, \quad \text{where} \quad (4.28)$$

$$\Upsilon_i = \sum_{k=0}^{\mathbf{d}-1} \beta_{i,k}^2 \lambda_k. \quad (4.29)$$

Then setting $\sigma_{i,k} = \mathcal{M}_i \beta_{i,k} \sqrt{\lambda_k}$ for $k \in [0, \mathbf{d} - 1]$ satisfies Equation (4.24) and (4.25), as required. Now that we have the expression for $\sigma_{i,k}$ (in terms of eigenvalues and eigenvectors) we can now express $c_{i,k}$ in terms of eigenvalues and eigenvectors. From Equation (3.16), we recall that $c_{i,k} = \frac{\sigma_{i,k}}{\sigma_i}$. Using Equation (4.28) and (4.29), and the expression for $\sigma_{i,k}$ results in

$$c_{i,k} = \frac{\beta_{i,k} \sqrt{\lambda_k}}{\sqrt{\sum_{j=0}^{\mathbf{d}-1} \beta_{i,j}^2 \lambda_j}}, \quad \forall k \in [1, \mathbf{d}].$$

Note that the columns of \mathcal{B} due to the orthogonality of eigenvectors $\beta_0, \dots, \beta_{\mathbf{d}-1}$ satisfies the following conditions:

$$\langle \beta_i, \beta_j \rangle = \begin{cases} 1, & i = j. \\ 0, & i \neq j. \end{cases}$$

Now that we have computed estimate entries for the matrix $C_{\mathbf{d}}$ we are in a position, using

the majorization algorithms, to find optimised entries in order to obtain C_d decompositions for which the correlation matrix $C_d \cdot C_d'$ is closer to the original matrix $\rho(\Theta, \tau)$.

4.3.3.2 Majorization

Suppose that we need to minimize the objective function $f(\cdot)$. The idea behind Pietersz and Groenen [81]'s method is to find a function $g(\cdot, \cdot)$, the majorization function, such that 1) it has the same function value at a point y and anywhere else in the domain is greater than or equal to the objective function i.e $g(x, y) \geq f(x)$, and 2) it is easier to minimize than the objective function. Figure 4.5 below, (as presented by Pietersz and Groenen [81, Figure 1]), gives a graphic illustration of the concept behind the majorization algorithm. Pietersz and Groenen argue that by minimizing the majorization function - which is an easier task since this function is 'simpler' - we obtain the next point of the algorithm. As a consequence from any point where the gradient of the objective function is non-negligible iterative majorization will be able to find a next point with a strictly smaller objective value. This generic fact for majorization algorithms had been pointed out in Hieser [52] (*ibid.* [81]).

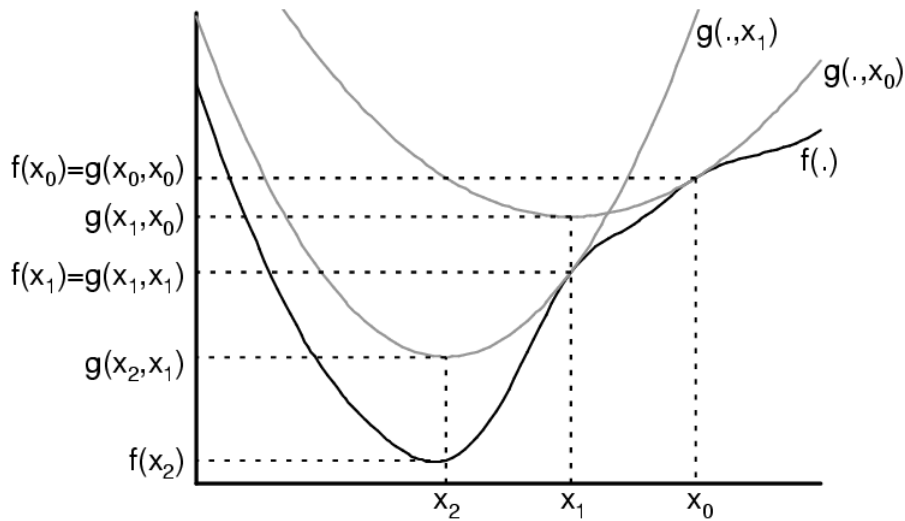


Figure 4.5: The idea of majorization algorithm sets out at x_0 . The majorization function $g(\cdot, x_0)$ is fitted by matching the value and first derivative of $f(\cdot)$ at x_0 . Subsequently the function $g(\cdot, x_0)$ is minimized to find the next point x_1 . This procedure is repeated to find the point x_2 etc.

In our case we recall that the objective function that has to be optimised, subject to the unit diagonal constraint, is $\mathcal{Z}(\Theta, \mathbf{d})$, see Equation (4.23). Pietersz and Groenen [81] derive the majorization function g for each row of the matrix C_d (for this and a more general objective function). The algorithm is, starting with the first row, to find the minimum of the majorization function, replace the row by the calculated minimum and loop over all the rows until convergence is achieved. The authors prove that this algorithm is globally convergent and has a sub-linear local rate of convergence. The authors also

provide MATLAB code containing the implementation of their algorithm.

4.3.4 Calibration to Market Data - Putting it all Together

Now that we have a sense of suitable functional forms for both volatility σ_{inst} and correlation $\rho_{i,j}$, we are in a position to calibrate them to market data. A soft copy of a detailed VBA and Matlab program code that does the calibration is found in the CD enclosed in the inside back cover of this thesis; however, in this Subsection we are going to summarise a sample calibration procedure that would enable us to get optimised parameters for these functional forms.

4.3.4.1 Volatility Parameters

Often the market quotes prices in terms of caps, hence, caplet spot volatilities are stripped from cap prices and the caplet variance $\sigma_{\text{Black76}}^2(T_i)T_i$ is computed using these stripped values. To guarantee time-homogeneity this quantity has to be a strictly increasing function of T_i . This market variance has to match the model variance through critical values of the parameters of $f(T_i - t)$, $\varphi(t)$, and $K(T_i)$. This implies that our objective function is to minimise the function

$$\mathcal{Z}_{\sigma_{\text{inst}}} = \sum_{i=1}^N \left[\sigma_{\text{Black76}}^2(T_i)T_i - \sum_{k=1}^i \int_{T_{k-1}}^{T_k} \sigma_{\text{inst}}^2(s, T_k) ds \right]^2, \quad (4.30)$$

where $\mathcal{Z}_{\sigma_{\text{inst}}}$ is defined as the objective function. Since the instantaneous volatility function σ_{inst} is defined by three functions, $f(T_i - t)$, $\varphi(t)$, and $K(T_i)$, the calibration is done in three stages for each function parameters.

The first minimisation is done on $f(T_i - t)$ over the parameters $\{a, b, c, d\}$ subject to the constraints that $a + d > 0$, $c > 0$, and $d > 0$.

$$\mathcal{Z}_{\sigma_{\text{inst}}}(a, b, c, d) = \sum_{i=1}^N \left[\sigma_{\text{Black76}}^2(T_i)T_i - \sum_{k=1}^i \int_{T_{k-1}}^{T_k} f^2(T_k - s) ds \right]^2, \quad \text{where} \quad (4.31)$$

$\varphi(t)$ and $K(T_i)$ are initialised to unity. At this time the time homogeneous functional form $f(T_i - t)$ best account for the observed term structure of volatilities by the optimal parameters. Next is the incorporation of the time dependent functional form $\varphi(t)$ to the model to improve the quality of the fit. This minimisation is done again but now over the parameters $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\}$ leaving the previously determined parameters $\{a, b, c, d\}$ unchanged;

$$\mathcal{Z}_{\sigma_{\text{inst}}}(\varphi_j : j \in [1, 5]) = \sum_{i=1}^N \left[\sigma_{\text{Black76}}^2(T_i)T_i - \sum_{k=1}^i \int_{T_{k-1}}^{T_k} f^2(T_k - s) \cdot \varphi^2(s) ds \right]^2. \quad (4.32)$$

With these two steps one attempts to account for the observed market prices by putting first

as much explanatory burden as possible on the time-homogeneous part of the evolution; and to explain then as much as possible of the residual discrepancies between market and model caplet prices in terms of purely time dependent (parametrized) separable component.

After carrying out this second step, the caplet prices will in general not be exactly recovered. To obtain the correct prices, one can at this point define a set of N -quantities $K(T_i)^2$ for $1 \leq i \leq N$:

$$K(T_i)^2 = \frac{\sigma_{\text{Black76}}^2(T_i)T_i}{\sum_{k=1}^i \int_{T_{k-1}}^{T_k} f^2(T_k - s) \cdot \varphi^2(s) ds}, \quad (4.33)$$

which by construction ensures that the correct variance to expiry is perfectly recovered. Hence, taking square-root $\forall i \in [1, N]$, results in optimal parameters for $K(T_i)$.

4.3.4.2 Correlation Parameters

Next we reduce the rank of the correlation matrix $\rho(\Theta, \tau)$ to a lower rank by finding a rank- d correlation matrix $\rho^d = C_d \cdot C_d'$, closest to the original matrix. We select the number of factors d and minimise the objective function $\mathcal{Z}_\rho(\Theta, d) = \|\rho(\Theta, \tau) - \rho^d\|_F^2$ over $\vartheta_i \forall i$ in Equation (4.22) subject to 1) correlation parameters $\vartheta_i > 0$, and 2) unit diagonal components of the resultant rank reduced correlation matrix. The estimate entries for C_d are further optimised by using the majorization algorithm to obtain the closest matrix ρ^d .

4.3.5 Suitable Discretisation for Simulation of Libor Rates

The volatility and correlation functions have been determined and initial values for the Libor rates $L_i(0)$ are directly from the market (through discount bonds $P_{i+1}(0)$). The question raised is: what are the rates at times other than $t = 0$? Since the market rates are not continuous but discrete, the rates are approximated by assuming a particular discretisation. The appealing approximations are the log-normal, and the Glasserman and Zhao because they ensure that the rates are positive. These are discussed next, respectively.

4.3.5.1 Log-normal Approximation

The traditional approach that has been commonly used is the Euler approximation of Equation (3.25). The inherent drawback when discretising this equation is that some of its paths allow $L_i(t) < 0$. Hence, it is preferable to use the SDE for $\log L_i(t)$, given in

Equation (4.34) below, since this compels the $L_i(t)$ to be positive.

$$\begin{aligned}
L_i(T_{n+1}) &= L_i(T_n) \exp \{A + B\}, \quad \forall n < i, \quad \text{where} & (4.34) \\
A &= \left[- \sum_{j=i+1}^N \frac{\tau_j \sigma_j(T_n) L_j(T_n) \rho_{i,j}}{1 + \tau_j L_j(T_n)} \sigma_i(T_n) - \frac{1}{2} \sigma_i^2(T_n) \right] \cdot (T_{n+1} - T_n), \\
B &= \sigma_i(T_n) \cdot \sum_{k=0}^{d-1} c_{i,k} d \left[W_k^{N+1}(T_{n+1}) - W_k^{N+1}(T_n) \right].
\end{aligned}$$

4.3.5.2 Glasserman and Zhao Approximation

Apart from modelling the log-normal Libor rates directly, Glasserman and Zhao [42] suggests simulating suitable martingales X directly and then recovering discretised Libor rates than vice-versa (simulating Libor rates first and enforcing the martingale property X indirectly). They begin with the perfect correlation specification of forward Libor in Equation (3.25), corresponding to the dynamics under the terminal measure and then set

$$X_i(t) = \frac{1}{\tau} \left(\frac{P_i(t)}{P_{N+1}(t)} - \frac{P_{i+1}(t)}{P_{N+1}(t)} \right) = L_i(t) \prod_{j=i+1}^N (1 + \tau L_j(t)), \quad (4.35)$$

for $1 \leq i \leq N$, where X_i is a martingale under the forward measure \mathbb{Q}^{N+1} associated with numeraire $P_{N+1}(\cdot)$. This follows from the fact that the product of both the constant $\frac{1}{\tau}$, and the difference of $P_{N+1}(\cdot)$ deflated discount bonds, in Equation 4.35, is a martingale. This fact is useful for numerical simulation (and also for option pricing).

X has following property (Glasserman and Zhao [42]):

Lemma 4.3.1 *Each X_i is a martingale and satisfies the SDE:*

$$\frac{dX_i}{X_i} = \left(\sigma_i + \sum_{j=i+1}^N \frac{\tau X_j \sigma_j}{1 + \tau X_j + \cdots + \tau X_N} \right) \cdot dW.$$

This immediately suggests an algorithm for arbitrage-free simulation. We then simulate a discretisation of the X_i , ensuring that the discretised process remains a martingale. Similarly, to get the evolution of $X_i(t)$ we use the relation for $\log X_i(t)$ given by Equation (4.36) below as

$$\begin{aligned}
X_i(T_{n+1}) &= X_i(T_n) \exp \{A\}, \quad \forall n < i, \quad \text{where} & (4.36) \\
A &= -\frac{1}{2} B^2 \cdot (T_{n+1} - T_n) + B \cdot (W(T_{n+1}) - W(T_n)), \\
B &= \left(\sigma_i + \sum_{j=i+1}^N \frac{\tau X_j \sigma_j}{1 + \tau X_j + \cdots + \tau X_N} \right).
\end{aligned}$$

The Libor rates are recovered using the following relation:

$$L_i = \frac{X_i}{1 + \tau X_{i+1} + \dots + \tau X_N}. \quad (4.37)$$

Because 1) the inherent assumption in the Glasserman and Zhao model is that the noise in the martingale property X is driven by a single factor ($d = 1$), and 2) that when deriving the Libor rates as per the relation in Equation (4.35) perfect correlation between these forward rates is inferred, directly modelling the martingale property X to recover Libor rates, for cases where $d > 1$ (as in our multi-factor Libor market model in Equation (3.25)), would not be plausible. Instead we will directly model these forward rates using the log-normal approximation in Equation (4.34) and recover X .

4.3.6 Summary of Estimation Issues for LMM

In this Section we looked at yet another aspect of improving efficiency. This time emphasis was on the estimation issues regarding the Libor model. The main issues are 1) the volatility functional form, 2) parametric model for the correlation matrix, 3) factor reduction techniques, 4) calibration to market data, and 5) suitable discretisation for simulation of Libor rates. The essential attribute for the volatility function is displaying positive features that encapsulates both the financial and computational perspectives, and for the Libor model the attribute entails discretisation that ensures positive interest rates. A poor choice of either renders the whole simulation futile.

4.4 Summary

In this Chapter we have shown that the basic MC simulation for computing the cap price is primarily hinged on the evolution of Libor rates $L_i(t) \forall t \in (0, T_N]$ generated using either $\log L_i(t)$ or $\log X_i(t)$ discretisation. The five main inputs to this Libor market model are the standard random number ε , initial term structure of Libor rates $L_i(0)$, correlation matrix ρ , number of random factors d , and the instantaneous volatility of the rates $\sigma_{\text{inst}}(s, T_i)$, see Figure 4.6 below. The standard random number ε is inferred from the inversion of QRN numbers generated by Halton, Faure, and Sobol algorithms. These numbers are preferred over the pseudo random numbers because they improve convergence of the estimated cap price from order $O(1/\sqrt{M})$ to $O(1/M)$. Also, the accuracy of cap price is improved by using variance reduction techniques. The market term structure of discount bonds and volatility, respectively, initialises the Libor rates at time $t = 0$, and the model's instantaneous volatility through time-homogeneous variance calibration.

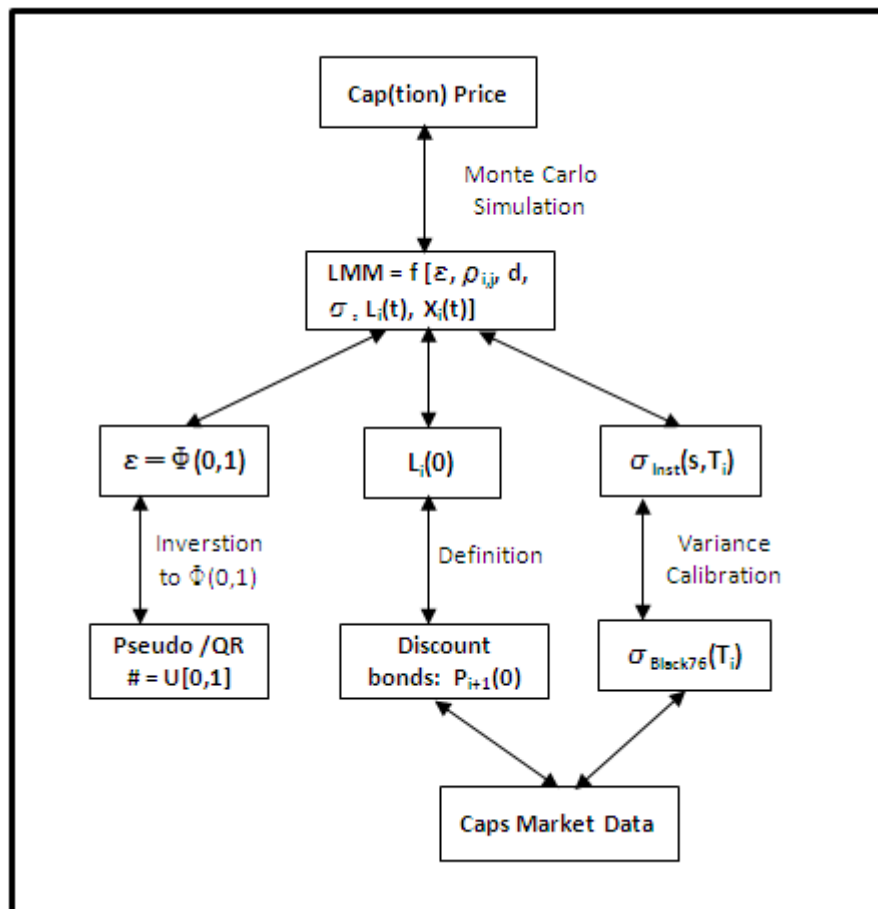


Figure 4.6: Numerical implementation procedure.

Implementation Procedure for Pricing Captions

In the summary of the previous Chapter we gave a schematic representation (see Figure 4.6) of the pertinent inputs in the whole exercise of pricing Captions using MC simulation. The implementation procedure we are going to follow stems from this scheme and can be categorised into three main themes namely; Initialisation of parameters, Calibration to market data, and Pricing of Captions. In the following Sections we describe the pertinent algorithms and present the relevant results under each theme. This is followed by a discussion of sensitivity analysis results and we end the Chapter by summarising key ideas.

This thesis is accompanied with the Excel program file “Pricing_Captions.xls”¹ and the results that we are going to give in this chapter were derived using the VBA program in the Microsoft Office Excel 2007 package. Before using the program to price Captions, one is advised to read instructions on the sheet “Read-Me”. Much of the actual program-code was written in VBA than in Matlab. The Matlab part of the program was limited only to function files that optimises the correlation objective function \mathcal{Z}_ρ . The way these function files are used in a VBA program is through an Excel Link software, which is an add-in that integrates Microsoft Excel and Matlab in a Microsoft Windows-based computing environment. By connecting Excel and Matlab, one is able to access the numerical, computational, and graphical power of Matlab from Excel worksheet and macro programming tools. This pricing program was executed on a Intel(R) Core(TM)2 Duo CPU E7400 computer processor @ 2.80GHz, 3.24 GB of RAM.

The assumed market data is that used by Glasserman and Zhao [42]: (i) The Libor rates $L_i(0)$ are given by $L_i(0) = \ln[\alpha + \beta i]$, with $\alpha = 1.05127$ and $\beta = 5.4455 \times 10^{-4}$, specifically chosen so that $L_0(0) = 5\%$ and $L_{39}(0) = 7\%$. (ii) The market cap flat volatilities are given as $\sigma(T_i) = 15\% + 0.25\%(i)$, for $i = 0, 1, \dots, 39$. These simplistic values for $L_i(0)$ and $\sigma(T_i)$ are broadly consistent with the US market term structure in the late 1997 [42].

¹The soft copy is found in the CD enclosed in the inside back cover of this thesis document.

5.1 Initialising Modelling Parameters

All parameters and information from the market are initialised. The caplet spot volatilities are stripped from cap prices and the caplet variance $\sigma_{\text{Black76}}^2(T_i)T_i$ is tested for time-homogeneity. These are effected by Algorithm 5.1, shown below.

Algorithm 5.1 Initialisation of parameters.

- 1: **Sub** INITIALISE()
 - 2: Get user defined parameters ($dt, T_{N+1}, \mathcal{X}, \mathcal{R}, T_{\text{exp}}, M, \mathbf{d}$, etc.).
 - 3: Get market information; $L_i(0), \sigma$, and compute $P_{i+1}(0)$ and Cap price.
 - 4: **End Sub**

 - 5: **Sub** STRIPPING()
 - 6: Use Bisection method to strip caplet volatility $\sigma_{\text{Black76}}(T_i)$ from cap prices.
 - 7: Check if caplet variance $\sigma_{\text{Black76}}^2(T_i)T_i$ is increasing. ▷ Testing homogeneity.
 - 8: **End Sub**
-

5.1.1 Initialisation Results

The market caplet variance function $\sigma_{\text{Black}}^2(T_i)T_i$ is increasing and it ensures the time homogeneity of the volatility functional form, see Figure 5.1 below.

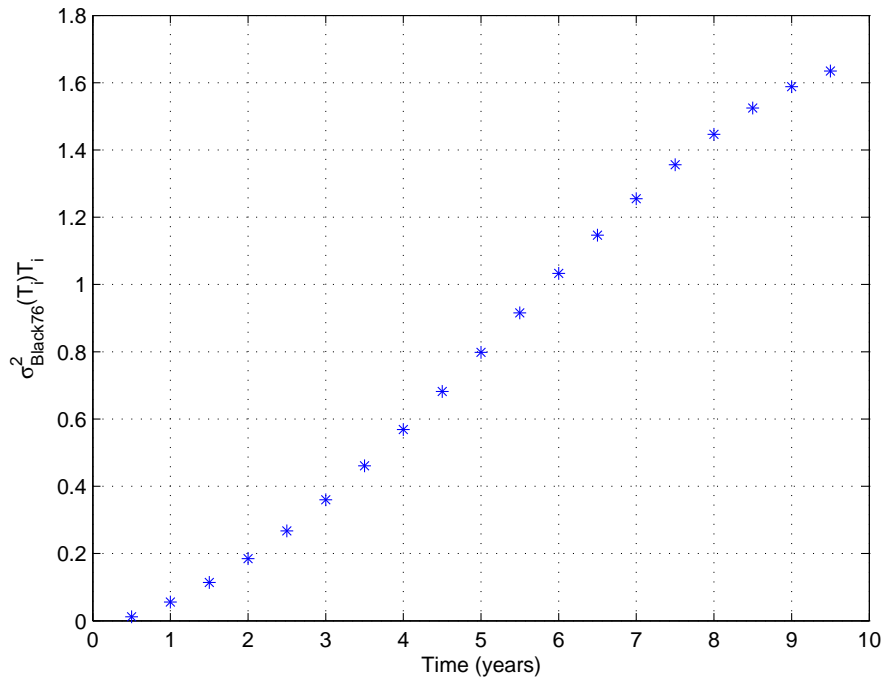


Figure 5.1: Homogeneity property.

5.2 Calibration to Market Data

Next, critical values for the parameters of the volatility $\sigma_{\text{inst}}(t, T_i)$ and correlation $\rho(\Theta, \tau)$ functional forms are computed. Algorithm 5.2, below gives a summary of the procedure.

Algorithm 5.2 Calibrating to market data.

- Volatility: Calibration
- 1: **Sub** SOLVER1()
 - 2: Initialise $K(T_i)$ and $\varphi(t)$ to 1 and compute $f(T_i - t)$ using Eq. (4.19).
 - 3: Minimise $\mathcal{Z}_{\sigma_{\text{inst}}}$ using Eq. (4.31). \triangleright Results: critical values for (a, b, c, d) .
 - 4: **End Sub**
 - 5: **Sub** SOLVER2()
 - 6: Compute $\varphi(t)$ using Eq. (4.20).
 - 7: Minimise $\mathcal{Z}_{\sigma_{\text{inst}}}$ again using Eq. (4.32). \triangleright Results: critical values for $(\varphi_1, \dots, \varphi_5)$.
 - 8: Finally compute $K(T_i)$ using Eq. (4.33).
 - 9: **End Sub**
- Correlation: Calibration
- 10: **Sub** SOLVER3()
 - 11: Initialise ϑ_i to 1.
 - 12: Minimise \mathcal{Z}_ρ using Eq. (4.23). \triangleright Results: critical ϑ_i and estimate C_d values.
 - 13: Minimise \mathcal{Z}_ρ again using majorization algorithm. \triangleright Results: critical values for C_d .
 - 14: **End Sub**
-

5.2.1 Calibration Results: Volatility.

The profile for the respective volatilities is also shown below, in Figure 5.2.

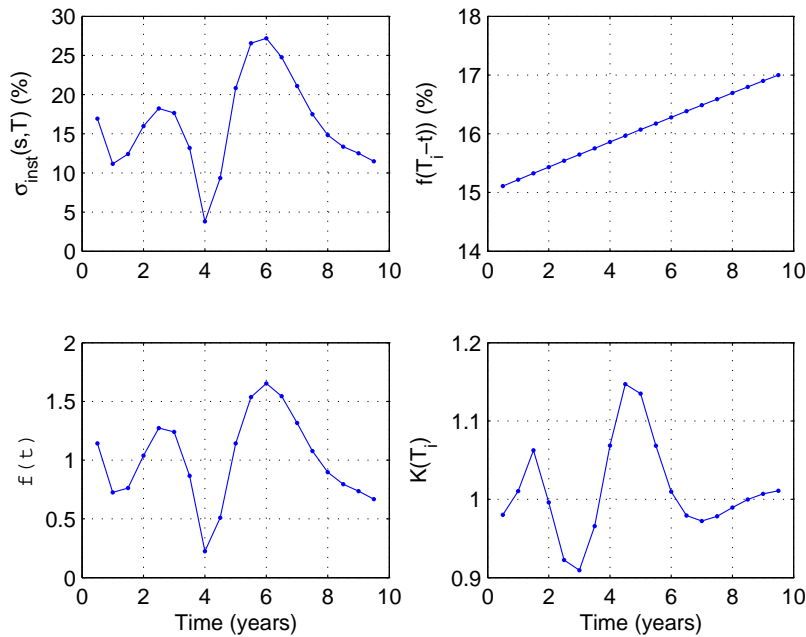


Figure 5.2: Calibration output.

The critical parameters of the instantaneous volatility functional form $\sigma_{\text{inst}}(t, T_i) = f(T_i - t)\varphi(t)K(T_i)$ are given below, in Table 5.1, together with the associated value of the objective function $Z_{\sigma_{\text{inst}}}$.

Table 5.1: Constants for volatility function

$Z_{\sigma_{\text{inst}}} = 2.4199\text{E-}01$. (Following from Equation (4.30).)

	$f(T_i - t)$	i	$K(T_i)$	i	$K(T_i)$
a	-2.2012E-01	0	1.0000	10	1.1350
b	1.1234E-03	1	0.9802	11	1.0685
c	4.7976E-03	2	1.0105	12	1.0098
d	3.7012E-01	3	1.0626	13	0.9792
$a + d$	1.5000E-01	4	0.9960	14	0.9722
	$\varphi(t)$	5	0.9226	15	0.9784
φ_1	4.3208E+00	6	0.9093	16	0.9895
φ_2	2.3673E+00	7	0.9658	17	0.9998
φ_3	-5.0223E-01	8	1.0687	18	1.0070
φ_4	-1.6348E+00	9	1.1470	19	1.0109
φ_5	2.1194E-01				

5.2.2 Calibration Results: Correlation

The first principal component explained 82.4% of the variation of the term structure while the second and the third components explained 13.0% and 3.0%, respectively. The correlation matrix ρ and its first three principal components are displayed below, in Figure 5.3.

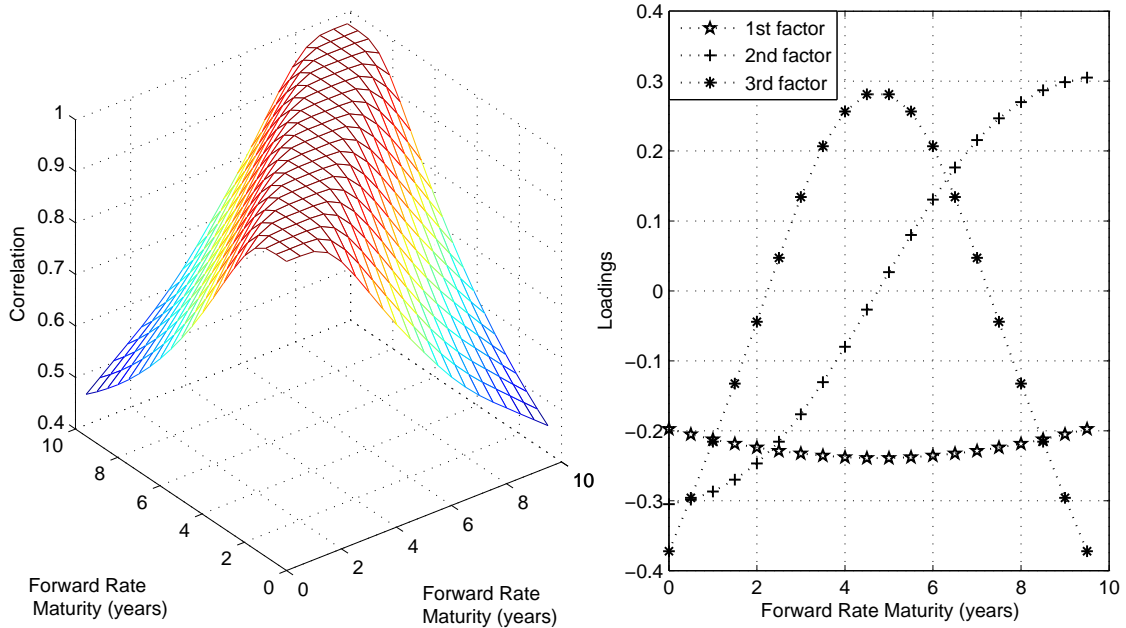


Figure 5.3: Forward rate correlation and its first three principal components.

The critical parameters of the correlation functional form $\rho(\Theta, \tau)$ are given in Table 5.2

below, together with the associated value of the objective function Z_ρ .

Table 5.2: Constants for correlation function
 $Z_\rho = 3.2272\text{E-}02$. (Following from Equation (4.23).)

i	ϑ_i	i	ϑ_i	i	ϑ_i
0	6.22893	7	6.28347	14	6.28505
1	6.28811	8	6.30628	15	6.30826
2	6.32557	9	6.32670	16	6.33158
3	6.33158	10	6.32670	17	6.32557
4	6.30826	11	6.30628	18	6.28811
5	6.28505	12	6.28347	19	6.22893
6	6.27621	13	6.27621		

5.3 Pricing Captions

By this time volatility σ_{inst} and correlation ρ are fully expressed by the functional forms. Hence, all the essential variables (ρ^d , C_d , etc.) to simulate multi-factor ($d > 1$) Libor rates are now determined. The Libor rates were simulated using $\log L_i$ discretisation, except for the case $d = 1$ where the Libor rates could either be simulated by using $\log X_i$ or $\log L_i$ discretisation for any Monte Carlo choice.

The normal random number is computed and the appropriate discretisation is carried out and at each time step the bond prices, volatility, and Brownian motion are computed and this completes a single run of the Monte-Carlo simulation for the Libor rates. With the Libor rates path established, payoffs for caplets, $\text{Cap}(T_{\text{exp}})$ prices, and Caption payoffs become feasible to compute in each simulation run. The Monte Carlo simulation then repeats this M times and finds the average price for the Caption and its associated standard error. Algorithm 5.3 below shows this procedure.

Algorithm 5.3 Pricing of Captions.

```

1: Sub PRICING CAPTIONS()
2:   Choose type of: discretisation, Monte Carlo, Rand # generator, and Inversion.
3:   Caption0 = 0.                                     ▷ Initialising Caption price.
4:   for sim = 1 : M(103) do                         ▷ M is number of simulations.
5:     Cap0 = 0.                                       ▷ Initialising cap price.
6:     for i = 1 : N(19) do
7:       Compute Pseudo/QR number and invert it to standard random number  $\varepsilon$ .
8:       Compute path ( $t \neq 0$ ) for;  $dW_i(t)$ ,  $L_i(t)$ ,  $P_i(t)$ ,  $\sigma_{\text{inst}}(t)$ , etc.
9:       Compute  $\text{Cap}(T_{\text{exp}})$  price using Eq. (3.10).   ▷  $\text{Cap}_i = \text{Cap}_{i-1} + \text{caplet}_i$ 
10:    End for
11:    Captionsim = Captionsim-1 +  $P(0, T_{\text{exp}})[\text{Cap}(T_{\text{exp}}) - \mathcal{X}]^+$   ▷ Using Eq. (3.9).
12:  End for
13:  Compute the average price and Standard error.    ▷ Using Eq. (4.3) and Eq. (4.4).
14: End Sub

```

5.3.1 Simulation Results: Typical path for Libor rates

A path-snippet of the Libor rates using $\log X_i$ and $\log L_i$ discretisation for both Basic and Antivariate Monte Carlo simulation is shown below, in Figure 5.4.

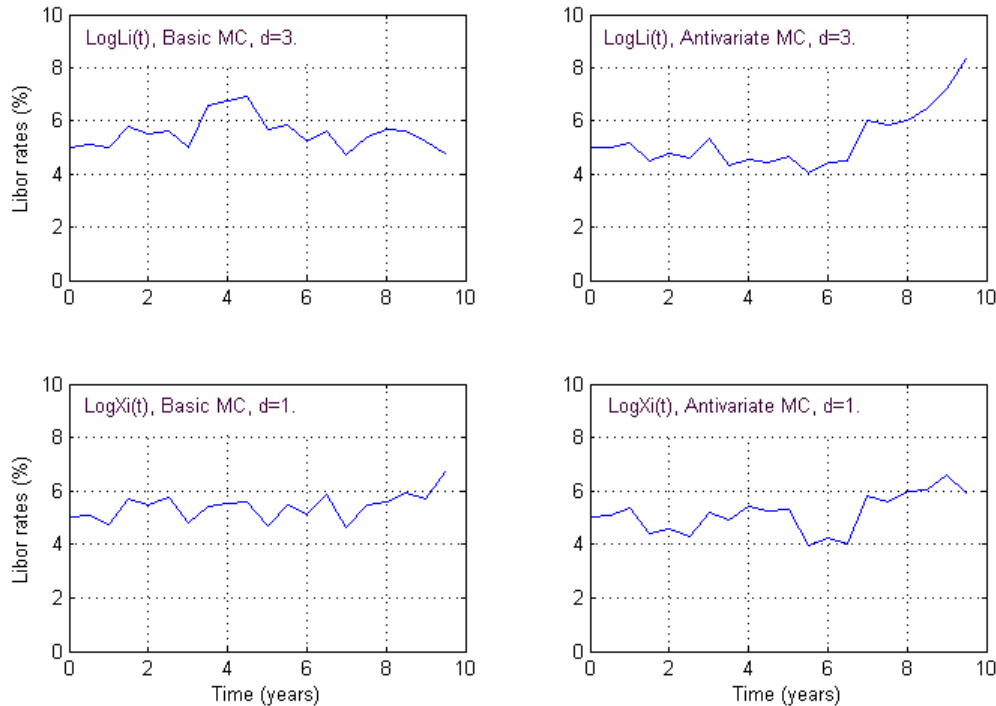


Figure 5.4: Simulated Libor rates

The appealing feature for both discretisation, as discussed in Subsection 4.3.5, is that they ensure positive interest rates and the above Libor rate paths, Figure 5.4, manifest this conjecture.

5.3.2 Simulation Results: Typical path for other variables

The different input variables resulted in various Caption prices. The properties for the Caption were the following: time for maturity T_{exp} is 5 years and assuming a unit notional principal ($\mathcal{N} = 1$), the strike for both the Caption and the Caplet is 5.1% and 3.1%, respectively. During each simulation run, using the $\log L_i$ discretisation, values of the different variables were updated and Table 5.3, on the facing page, shows a typical path evaluation of some variables leading to the computation of the cap price $\text{Cap}(T_{\text{exp}})$. A detailed table showing these values can be viewed in the accompanying Excel program file "Pricing_Captions.xls".

For a combination of Monte Carlo type (Basic or Antivariate) and random number generator (Excel default, QRNs) different Caption payoffs were computed. After M simulations

Table 5.3: Typical path evaluation (For: $M=8$, $\log L_i$, Antivariate, QRN=Faure, and using Moro's Inversion).

T_i	$T_0 = 0$	$T_1 = 0.5$	$T_2 = 1.0$	\cdots	$T_{\text{exp}=10} = 5.0$	$T_{11} = 5.5$	$T_{12} = 6.0$	$T_{13} = 6.5$	\cdots	$T_{17} = 8.5$	$T_{18} = 9.0$	$T_{19} = 9.5$
$\Delta W_0^{20}(t)$		-0.8101	0.3476	\cdots	0.3014	-0.0398	0.9038	-1.1259	\cdots	-0.6999	0.4198	-0.1874
$\Delta W_1^{20}(t)$		0.8168	-1.3075	\cdots	-0.8961	0.0433	-0.2976	0.6095	\cdots	0.9530	-1.4449	0.1482
$\Delta W_2^{20}(t)$		-0.3437	0.5514	\cdots	0.7406	-0.7346	0.3953	-0.1235	\cdots	-0.2751	0.5236	-0.5751
$C_{0,:}$	-0.8176	0.4990	0.2872									
$C_{1,:}$	-0.8451	0.4866	0.2215									
\vdots	\vdots	\vdots	\vdots									
$C_{18,:}$	-0.8444	-0.4875	0.2219									
$C_{19,:}$	-0.8169	-0.4999	0.2876									
$\rho_{0,:}^d$	1.0000	0.9977	0.9903	\cdots	0.7099	0.6693	0.6332	0.6028	\cdots	0.5243	0.5111	0.5000
$\rho_{1,:}^d$	0.9977	1.0000	0.9975	\cdots	0.7503	0.7096	0.6721	0.6389	\cdots	0.5440	0.5264	0.5111
\vdots	\vdots	\vdots	\vdots	\cdots	\cdots	\cdots	\cdots	\cdots	\cdots	\vdots	\vdots	\vdots
$\rho_{18,:}^d$	0.5111	0.5264	0.5440	\cdots	0.7927	0.8358	0.8785	0.9180	\cdots	0.9975	1.0000	0.9977
$\rho_{19,:}^d$	0.5000	0.5111	0.5243	\cdots	0.7535	0.7990	0.8453	0.8895	\cdots	0.9903	0.9977	1.0000
$L_0(t)$	5.000%											
$L_1(t)$	5.104%	5.866%										
$L_2(t)$	5.207%	6.449%	5.839%									
\vdots	\vdots	\vdots	\vdots	\cdots	\cdots	\cdots	\cdots	\cdots	\cdots	\cdots		
$L_{18}(t)$	6.848%	7.145%	7.544%	\cdots	6.968%	6.820%	6.385%	6.844%	\cdots	6.817%	7.161%	
$L_{19}(t)$	6.949%	7.098%	7.517%	\cdots	6.984%	6.813%	6.454%	6.848%	\cdots	6.846%	7.213%	7.130%
$P_0(t)$	1.0000											
$P_1(t)$	0.9756	1.0000										
$P_2(t)$	0.9513	0.9715	1.0000									
\vdots	\vdots	\vdots	\vdots	\cdots	\cdots	\cdots	\cdots	\cdots	\cdots	\cdots	\cdots	
$P_{19}(t)$	0.5742	0.5203	0.5527	\cdots	0.7493	0.7735	0.8196	0.8213	\cdots	0.9352	0.9654	1.0000
$P_{20}(t)$	0.5549	0.5025	0.5327	\cdots	0.7240	0.7480	0.7940	0.7941	\cdots	0.9043	0.9318	0.9656
Caplet(T_{exp})					0.0061	0.0054	-	0.0063	\cdots	0.0069	0.0079	0.0075
Cap(T_{exp})					0.0614							

the average price and associated error for each scenario were computed and Figure 5.5 below shows these various prices.

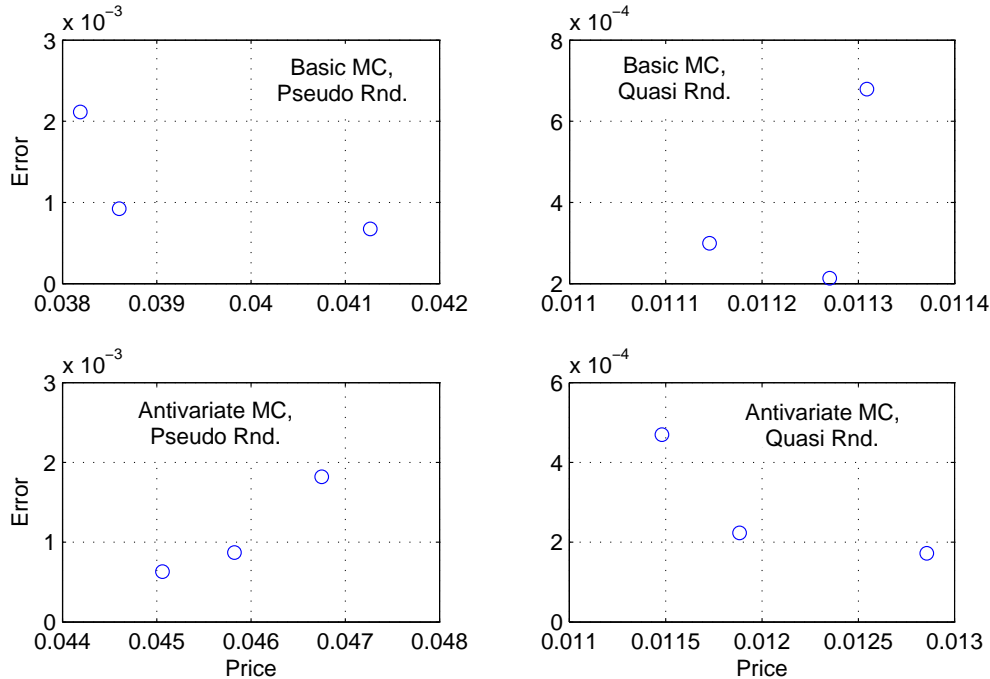


Figure 5.5: Caption prices as a result of different inputs.

What is apparent is that the price and error associated with the use of the QRNs is far smaller compared to that associated with pseudo-random numbers. Another observation is that, with the use of QRNs, the error associated with Antivariate MC is smaller than that associated with Basic MC. These observations complement our theory, discussed in Section 4.2, of improving computational efficiency by using; 1) QRNs instead of pseudo-random numbers, and 2) Antivariate MC simulations instead of Basic MC simulations. Hence, the preferred simulation for the multi-factor Libor Market Model is the Antivariate technique using QRNs.

Based on a unit notional principal ($\mathcal{N} = 1$), the average price derived from the use of pseudo-random numbers is 426 ± 12 basis points for both Monte Carlo methods (Basic and Antivariate), whereas for the quasi-random numbers it is 116 ± 3 basis points. The delta and gamma for these are $2.546\text{E} - 02$ and $2.736\text{E} - 02$. The absolute value of gamma is large and this implies that delta for the Caption is highly sensitive to the price of the cap. Therefore, it is very risky to leave a portfolio of these options delta-neutral for any length of time. Re-balancing of the portfolio should be done often.

5.4 Sensitivity Analysis

The main variables that were sensitised are shown in Table 5.4 on the next page. What is observed is that the Caption price is very sensitive to Caption strike \mathcal{X} , time to maturity

T_{exp} , and a flat term structure of interest rates. A change in any of these variables affects the Caption price by more than 50%. On the other hand, a change in the volatility functional form $\sigma_{\text{inst}}(t, T_i)$, and number of simulations M , comparatively, affect the price of the Caption by 12% at most. We also note that the change in delta corresponding to $\sigma_{\text{inst}}(t, T_i)$ is almost 100%, which implies that a portfolio of Captions should be hedged often during volatile market conditions.

Table 5.4: Sensitivity of model variables.

Variable	Change in Variable	Change in Price	Change in Delta
$\sigma_{\text{inst}}(t, T_i)$	-5%	-1.1%	-97.4%
	+5%	5.4%	2.5%
Strike \mathcal{X}	1.5%	70.4%	15.8%
	4.5%	-60.7%	-47.5%
M	5×10^4	3.5%	2.1%
	10^5	12.0%	10.3%
T_{exp}	4years	327.6%	162.4%
	6 years	-52.2%	-42.9%
$L_i(0)(=5\%)$	flat term structure 5%	-96.2%	-83.5%
$L_i(0)(=6\%)$	flat term structure 6%	-62.4%	-53.9%

5.5 Summary

In this Chapter we implemented the pricing procedure for Captions. This procedure has three main algorithms and these are initialisation of parameters, calibration to market data, and pricing Captions. We found that (i) the best MC simulation is using Antithetic variate method incorporating QRNs, (ii) holding a delta-neutral position in a portfolio of Captions for any length of time is not ideal due to the large absolute value of gamma which suggest re-balancing of the portfolio should be done often, and (iii) the Caption price is very sensitive to changes in the Caption strike price \mathcal{X} , time to maturity T_{exp} , and initial term structure of Libor rates $L_i(0)$ than to changes in the volatility functional form $\sigma_{\text{inst}}(t, T_i)$, and number of simulations M .

In the next Chapter we suggest the efficient way of pricing Captions and possible future work in this area.

The Efficient Way and Future Work

The Caption is one of common European style exotic options discussed in literature and is within the class of derivatives that has no closed form solution. Accurately pricing and hedging exotic derivatives is a challenge for traders and more so if the derivative is an interest rates product as opposed to equities product. This is because pricing any interest rate product requires the whole zero coupon yield curve (typifying a vector variable) whereas a stock option its just the stock typifying a scalar variable. Studying efficient numerical solutions to price Captions makes it of particular interest to study and was the reason behind the inception of this thesis. We studied Monte Carlo simulation in order to calculate the price and hedge parameters of this option.

6.1 Conclusion

The basic Monte Carlo simulation was computationally inefficient, hence, we improved its efficiency by using Antivariate MC simulation in conjunction with quasi random numbers coupled with either Moro's algorithm or Excel's NormsInv function. Efficiency was also improved by using;

- Time homogeneous volatility term structure whose functional form has positive financial an computational features.
- Rank reduced correlation matrix instead of full rank matrix, which costs expensive computational.
- Log-normal and Glasserman and Zhao Approximations because they ensure positive Libor rates (i.e. $L_i(t) > 0$).

A sensitivity analysis was carried and the following results were obtained

- A naive choice of the volatility functional form $\sigma_{\text{inst}}(t, T_i)$ can easily produce a inefficient computation that has no financial bearing on the position the trader takes, resulting in erroneous pricing and hedging strategies.

- The best deal for buying this option (Caption) can be struck by a good negotiation of the strike price \mathcal{X} and time to maturity T_{exp} .

6.2 Recommendations

When developing the simulation model, and when computing some parameters certain assumptions were made and to ascertain stability of our model we suggest certain considerations outlined below for future work around this topic.

- We limited our discretisation types to $\log L_i(t)$ and Glasserman and Zhao [42], and the latter model could only be used for the case $\mathbf{d} = 1$. Perhaps other drift approximation (predictor-corrector) methods as introduced by Joshi and Stacey [61] could be used.
- The “Greeks” or hedge parameters were computed using finite difference methods, perhaps computing these parameters using the “Malliavin Calculus” could be considered, Benhamou [11].
- Instead of using term structure of Libor rates recommended by Glasserman and Zhao [42] (i.e. $L_i(0) = \ln[\alpha + \beta i]$), use a Libor yield curve obtained by a bootstrap method on a combination of spot Libor rates, forward rate agreements (FRA), futures and swap rates (Alexander and Lvov [3]), using a stable interpolation method. Hagan and West [44] propose a monotone convex spline interpolation instead of the popular cubic spline because it overcomes the problem of negative forward rates.
- Rebonato [86] argued that, though the modern pricing framework is powerful and practically useful, it is only as good as the input volatilities and to a lesser extent on correlation. Going forward, we suggest the following:
 - Instead of assuming volatility skew is non-existent, incorporate constant elasticity of variance (CEV) model to handle this.
 - Instead of using Glasserman and Zhao [42] market data for calibration of forward rate volatilities, use current market data.
 - Instead of using Tanimura and Yamada [96] parsimonious parameterization (empirical market model) of the correlation matrix, use either historical or current market data.

Part II

Evolution Of Financial Markets After The Credit Crunch Crisis

Introduction

Before the credit crunch of 2007, the interest rates quoted in the market showed typical consistencies that we learned on books. These consistencies between rates allowed the construction of a well-defined zero-coupon curve, typically using bootstrapping techniques in conjunction with interpolation methods. Differences between similar rates were present in the market, but generally regarded as negligible. For instance, deposit (Xibor based) rates and OIS (EONIA based for EUR) rates for the same maturity would chase each other, but keeping a safety distance (the basis) of a few basis points. Similarly, swap rates with the same maturity, but based on different lengths for the underlying floating rates, would be quoted at a non-zero (but again negligible) spread.

Then, August 2007 arrived, and our convictions became questionable. The liquidity crisis widened the basis, so that market rates that were consistent with each other suddenly revealed a degree of incompatibility that worsened as time passed by. For instance, the forward rates implied by two consecutive deposits became different than the quoted FRA rates or the forward rates implied by OIS quotes. Remarkably, this divergence in values does not create arbitrage opportunities when credit or liquidity issues are taken into account. As an example, a swap rate based on semi-annual payments of the six-month Libor rate can be different (and higher) than the same-maturity swap rate based on quarterly payments of the three-month Libor rate.

These facts had a very strong impact on the financial community, since they questioned both our understanding of the working of the interest rate market during the credit crisis, and the techniques and relations used by all banks to construct the term structure of discount factors to be used for pricing all financial products. Various papers dealing with this new situation were recently put forward in the literature. Among the most relevant we recall Mercurio [67], Ametrano and Bianchetti [4], Bianchetti [12], Henrard [51]. Most of these works focus on finding methodologies for building consistent interest rate models or curves also in the context of anomalous interest rate quotes, and focus on abstract frameworks. Mercurio [67] takes FRA quotes as new separate inputs for a larger Libor market model, Bianchetti [12] recognizes an analogy between FX pricing and the pricing of interest rate derivatives when the discounting is decoupled from the indexation of the rates

in the payoff, Henrard [51] follows an axiomatic approach to make the standard framework to term structure bootstrapping consistent with the multicurve situation generated by the presence of large basis spreads.

These frictions have thus induced a sort of "segmentation" of the interest rate market into sub-areas, mainly corresponding to instruments with $1M$, $3M$, $6M$, $12M$ underlying rate tenors, characterized, in principle, by different internal dynamics, liquidity and credit risk premia, reflecting the different views and interests of the market players. This is a central problem in the interest rate market, with many consequences in trading, financial control, risk management and IT, which still lacks attention in the financial literature. The need to adopt corresponding fundamental and structural changes in the mathematical tools and technology (see Figure 1.1 on page 3), is therefore, not a one man's responsibility but all combined (i.e. market practitioners, quantitative researchers, and IT developers).

In order to find efficient Monte Carlo Simulations to price Captions, post the credit crunch crisis, we will require the flowing-together of financial markets, mathematical tools, and technology to be adopted. And whilst not yet adopted by all financial fraternity members, in this Part (II) of the thesis, we are going to show, theoretically, how to generalize the main (interest rate) market models so as to account for the new market practice of using multiple curves for each single currency, and the main consulted sources are: Morini [73], Ametrano and Bianchetti [4], Bianchetti [12], Mercurio [67], and Mercurio [68].

The rest of Part II is organized as follows. In Chapter 8 we fix the notation, and revisit some general concepts of standard no-arbitrage single-curve pricing and we formalize the double-curve pricing framework, showing how no-arbitrage is broken and formally recovered with the introduction of forward basis. In Chapter 9, we summarize the transition of market practices for pricing and hedging IRDs during the credit crunch crisis. In Chapter 10, we extend the market Black formulas for pricing cap(let)s to double-curve case. In Chapter 11, we introduce the extended log-normal Libor market models and derive the FRA and forward rate dynamics under terminal measure, and the pricing formula for Captions. Finally, in Chapter 12, we give concluding remarks.

Basic Definitions, Assumptions and Notation

Following the discussion above, we denote with \mathcal{M}_x , $x = \{d, f_1, \dots, f_n\}$ multiple distinct interest rate sub-markets, characterised by the same currency and by distinct bank account B_x , such that

$$B_x(t) = \exp \int_0^t r_x(u) du \quad (8.1)$$

where $r_x(t)$ are the associated short rates, and t is time within trading interval $[0, T]$. We also have multiple yield curves \mathfrak{C}_x in the form of a continuous term structure of discount factors.

$$\mathfrak{C}_x = \{T \rightarrow P_x(t, T), T \geq t\} \quad (8.2)$$

where t is the reference date of the curves (e.g. settlement date, or today) and $P_x(t, T)$ denotes the price at time $0 \leq t \leq T$ of the \mathcal{M}_x -zero coupon bond for maturity T , such that $P_x(T, T) = 1$. Similar to a discussion in Section 3.1, these bonds are assumed to have a pre-specified collection of settlement dates $0 = T_0 < T_1 < \dots < T_N < T_{N+1} = T$. And for simplicity, in each sub-market \mathcal{M}_x , the interval between the reset dates (also called *tenor*) is assumed constant and is denoted by $\tau_{x,i} = T_{i+1} - T_i = \tau_x$, where $i = 0, 1, \dots, N$.

In each sub-market \mathcal{M}_x we postulate the usual no arbitrage relation:

$$P_x(t, T_{i+1}) = P_{x,i+1}(t) = P_x(t, T_i)P_x(t, T_i, T_{i+1}), \quad t \leq T_i < T_{i+1}, \quad (8.3)$$

where $P_x(t, T_i, T_{i+1})$ denotes the \mathcal{M}_x forward discount factor from time T_{i+1} to time T_i , prevailing at time t . The financial meaning of Equation (8.3) is that, in each market \mathcal{M}_x , given a cash flow of one unit of currency at time T_{i+1} , its corresponding value at time $t < T_{i+1}$ must be unique, both if we discount in one single step from T_{i+1} to t , using the discount factor $P_x(t, T_{i+1})$, and if we discount in two steps, first from T_{i+1} to T_i , using the forward discount $P_x(t, T_i, T_{i+1})$ and then from T_i to t , using $P_x(t, T_i)$. Denoting with $F_x(t; T_i, T_{i+1})$ the simple compounded forward rate associated to $P_x(t, T_i, T_{i+1})$, resetting

at time T_i and covering the time interval $[T_i, T_{i+1}]$, we have

$$P_x(t, T_i, T_{i+1}) = \frac{P_{x,i+1}(t)}{P_{x,i}(t)} = \frac{1}{1 + \tau_{x,i} F_x(t, T_i, T_{i+1})}, \quad (8.4)$$

From Equation (8.3) we obtain the familiar no arbitrage expression

$$\begin{aligned} F_x(t; T_i, T_{i+1}) = F_{x,i}(t) &= \frac{1}{\tau_{x,i}} \left[\frac{1}{P_x(t, T_i, T_{i+1})} - 1 \right], \\ &= \frac{P_{x,i}(t) - P_{x,i+1}(t)}{\tau_{x,i} P_{x,i+1}(t)}. \end{aligned} \quad (8.5)$$

Equation (8.5) can be also derived (Brigo and Mercurio [20], Section 1.4) as the fair value condition at time t of the Forward Rate Agreement (**FRA**) contract with payoff at maturity T_{i+1} given by

$$\mathbf{FRA}_x(T_{i+1}; T_i, T_{i+1}, \mathcal{K}, \mathcal{N}) = \mathcal{N} \tau_{x,i} [L_x(T_i, T_{i+1}) - \mathcal{K}], \quad \text{where} \quad (8.6)$$

$$L_x(T_i, T_{i+1}) = L_{x,i}(T_i) = \frac{1 - P_x(T_i, T_{i+1})}{\tau_{x,i} P_x(T_i, T_{i+1})}. \quad (8.7)$$

\mathcal{N} is the nominal amount, $L_{x,i}(T_i)$ is the T_i -spot Xibor rate for maturity T_{i+1} and \mathcal{K} the (simply compounded) strike rate (sharing the same day-count convention for simplicity). Introducing expectations we have, $\forall t \leq T_i < T_{i+1}$,

$$\begin{aligned} \mathbf{FRA}_x(t; T_i, T_{i+1}, \mathcal{K}, \mathcal{N}) &= P_{x,i+1}(t) \mathbb{E}_t^{\mathbb{Q}_x^{i+1}} [\mathbf{FRA}(T_{i+1}; T_i, T_{i+1}, \mathcal{K}, \mathcal{N})] \\ &= \mathcal{N} P_{x,i+1}(t) \tau_{x,i} \left\{ \mathbb{E}_t^{\mathbb{Q}_x^{i+1}} [L_x(T_i, T_{i+1})] - \mathcal{K} \right\} \\ &= \mathcal{N} P_{x,i+1}(t) \tau_{x,i} [F_x(t; T_i, T_{i+1}) - \mathcal{K}]. \end{aligned}$$

\mathbb{Q}_x^{i+1} denotes the $\mathcal{M}_x - T_{i+1}$ -forward measure corresponding to the numeraire $P_x(t, T_{i+1})$, $\mathbb{E}_t^{\mathbb{Q}_x^{i+1}}[\cdot]$ denotes the expectation at time t with respect to measure \mathbb{Q}_x^{i+1} and filtration \mathcal{F}_t , encoding the market information available up to time t . We have assumed the standard martingale property of forward rates

$$F_x(t; T_i, T_{i+1}) = \mathbb{E}_t^{\mathbb{Q}_x^{i+1}} [F_x(T_i; T_i, T_{i+1})] = \mathbb{E}_t^{\mathbb{Q}_x^{i+1}} [L_x(T_i, T_{i+1})], \quad (8.8)$$

to hold in each interest rate market \mathcal{M}_x (Brigo and Mercurio [20]). We stress that the assumptions above imply that each sub-market \mathcal{M}_x is internally consistent and has the same properties of the "classic" interest rate market before the crisis. This is surely a strong hypothesis, that could be relaxed in more sophisticated frameworks (Bianchetti [12]).

Transition to Multiple-Curve Pricing Framework

Now that we have, in the previous chapter, fixed the notation, in this chapter we describe the evolution of the market practice for pricing and hedging interest rate derivatives through the credit crunch crisis. We start by reviewing market practices (single-curve framework) for pricing and hedging of IRDs before the credit crunch crisis. We end the chapter by doing same in the multiple-curve framework.

We use consistently the notation described in the previous chapter, considering a general single-currency interest rate derivative with $N + 1$ future coupons with payoffs $\pi = \{\pi_0, \dots, \pi_N\}$, with $\pi_i = \pi_i(F_x)$, generating $N + 1$ cash flows $c = \{c_1, \dots, c_{N+1}\}$, at future dates $\mathbf{T} = \{T_1, \dots, T_{N+1}\}$, with $t < T_1 < \dots < T_{N+1}$.

9.1 Single-Curve Framework

The pre-crisis standard market practice was based on a single-curve procedure, well known to the financial world, that can be summarised as follows (Ron [89], Hagan and West [44], Andersen [6], and Hagan and West [45], *op.cit.* Bianchetti [12]):

1. select a single finite set of the most convenient (i.e. liquid) interest rate vanilla instruments traded on the market with increasing maturities and build a single yield curve \mathfrak{C}_d using the preferred bootstrapping procedure (pillars, priorities, interpolation, etc.); for instance, a common choice in the EUR market is a combination of short term EUR deposits, medium-term Futures/FRA on Euribor3M and medium/long term swaps on Euribor6M;
2. for each interest rate coupon $i \in \{0, 1, \dots, N\}$ compute the relevant forward rates using the given yield curve \mathfrak{C}_d as in Equation (8.5),

$$F_{d,i}(t) = \frac{P_{d,i}(t) - P_{d,i+1}(t)}{\tau_{d,i}P_{d,i}(t)}, \quad t \leq T_i < T_{i+1},$$

compute cash flows c_{i+1} as expectations at time t of the corresponding coupon payoffs $\pi_i(F_d)$ with respect to the T_{i+1} -forward measure \mathbb{Q}_d^{i+1} , associated to the numeraire

$P_{d,i+1}(t)$ from the same yield curve \mathfrak{C}_d ,

$$c_{i+1}(t) = c(t, T_{i+1} \pi_i) = \mathbb{E}_t^{\mathbb{Q}_d^{i+1}} [\pi_i(F_d)]; \quad (9.1)$$

3. compute the relevant discount factors $P_{d,i+1}(t)$ from the same yield curve \mathfrak{C}_d ;
4. compute the derivative's price at time t as the sum of the discounted cash flows,

$$\pi(t; T) = \sum_{i=0}^N P_{d,i+1}(t) c_{i+1}(t) = \sum_{i=0}^N P_{d,i+1}(t, T_i) \mathbb{E}_t^{\mathbb{Q}_d^{i+1}} [\pi_i(F_d)];$$

5. compute the delta sensitivity with respect to the market pillars of yield curve \mathfrak{C}_d and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the same set of vanillas.

For instance, a 7.5Y maturity EUR floating swap leg on Euribor1M (not directly quoted on the market) is commonly priced using discount factors and forward rates calculated on the same depo-Futures-swap curve cited above. The corresponding delta risk is hedged using the suggested amounts (hedge ratios) of 7Y and 8Y Euribor6M swaps.

Notice that step 3 above has been formulated in terms of the pricing measure \mathbb{Q}_d^{i+1} associated to the numeraire $P_{d,i+1}(t)$. This is convenient in our context because it emphasizes that the numeraire is associated to the discounting curve. Obviously any other equivalent measure associated to different numeraires may be used as well.

We stress that this is a single-currency-single-curve approach, in that a unique yield curve is built and used to price and hedge any interest rate derivative on a given currency. Thinking in terms of more fundamental variables, e.g. the short rate, this is equivalent to assuming that there exist a unique fundamental underlying short rate process able to model and explain the whole term structure of interest rates of all tenors.

It is also a relative pricing approach, because both the price and the hedge of a derivative are calculated relatively to a set of vanillas quoted on the market. We notice also that it is not strictly guaranteed to be arbitrage-free, because discount factors and forward rates obtained from a given yield curve through interpolation are, in general, not necessarily consistent with those obtained by a no arbitrage model; in practice bid-ask spreads and transaction costs hide any arbitrage possibilities.

Finally, we stress that the first key point in the procedure is much more a matter of art than of science, because there is no unique financially sound recipe for selecting the bootstrapping instruments and, in principle, none is better than the others.

9.2 Multiple-Curve Framework

Unfortunately, the pre-crisis approach outlined in the previous section is no longer consistent, at least in its simple formulation, with the present market conditions. First, it does

not take into account the market information carried by basis swap spreads, now much larger than in the past and no longer negligible. Second, it does not take into account that the interest rate market is segmented into sub-areas corresponding to instruments with distinct underlying rate tenors, characterized, in principle, by different dynamics (e.g. short rate processes). Thus, pricing and hedging an interest rate derivative on a single yield curve mixing different underlying rate tenors can lead to “dirty” results, incorporating the different dynamics, and eventually the inconsistencies, of distinct market areas, making prices and hedge ratios less stable and more difficult to interpret. On the other side, the more the vanillas and the derivative share the same homogeneous underlying rate, the better ought to be the relative pricing and the hedging. Third, by no arbitrage, discounting must be unique: two identical future cash flows of whatever origin must display the same present value; hence we need a unique discounting curve.

In principle, a consistent credit and liquidity theory would be required to account for the interest rate market segmentation. This would also explain the reason why the asymmetries cited above do not necessarily lead to arbitrage opportunities, once counterparty and liquidity risks are taken into account. Unfortunately such a framework is not easy to construct (see Mercurio [67], and Morini [73]; *ibid.* [12]). In practice an empirical approach has prevailed on the market, based on the construction of multiple “forwarding” yield curves from plain vanilla market instruments homogeneous in the underlying rate tenor, used to calculate future cash flows based on forward interest rates with the corresponding tenor, and of a single “discounting” yield curve, used to calculate discount factors and cash flows’ present values. Consequently, interest rate derivatives with a given underlying rate tenor should be priced and hedged using vanilla interest rate market instruments with the same underlying rate tenor. The post-crisis market practice may thus be summarised in the following working procedure:

1. build one discounting curve \mathfrak{C}_d using the preferred selection of vanilla interest rate market instruments and bootstrapping procedure;
2. build multiple distinct forwarding curves $\mathfrak{C}_{f_1}, \dots, \mathfrak{C}_{f_n}$ using the preferred selections of distinct sets of vanilla interest rate market instruments, each homogeneous in the underlying Xibor rate tenor (typically with 1M, 3M, 6M, 12M tenors) and bootstrapping procedures;
3. for each interest rate coupon $i \in \{0, \dots, N\}$ compute the relevant forward rates with tenor $\tau_{x,i}$ using the corresponding yield curve \mathfrak{C}_x as in Equation (8.5),

$$F_{x,i}(t) = \frac{P_{x,i}(t) - P_{x,i+1}(t)}{\tau_{x,i} P_{x,i+1}(t)}, \quad t \leq T_i < T_{i+1}; \quad (9.2)$$

4. compute cash flows c_{i+1} as expectations at time t of the corresponding coupon payoffs $\pi_i(F_f)$ with respect to the discounting T_{i+1} -forward measure \mathbb{Q}_d^{i+1} , associated to the

numeraire $P_{d,i+1}(t)$, as

$$c_{i+1}(t) = c(t, T_{i+1}, \pi_i) = \mathbb{E}_t^{\mathbb{Q}_d^{i+1}} [\pi_i(F_f)];$$

5. compute the relevant discount factors $P_{d,i+1}(t)$ from the discounting yield curve \mathfrak{C}_d ;
6. compute the derivative's price at time t as the sum of the discounted cash flows,

$$\pi(t; T) = \sum_{i=0}^N P_{d,i+1}(t) c_{i+1}(t) = \sum_{i=0}^N P_{d,i+1}(t) \mathbb{E}_t^{\mathbb{Q}_d^{i+1}} [\pi_i(F_f)];$$

7. compute the delta sensitivity with respect to the market pillars of each yield curve $\mathfrak{C}_d, \mathfrak{C}_{f_1}, \dots, \mathfrak{C}_{f_n}$ and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the corresponding set of vanillas.

For instance, the 7.5Y floating swap leg cited in the previous Section, is currently priced using Euribor1M forward rates calculated on the \mathfrak{C}_{1M} forwarding curve, boot-strapped using Euribor1M vanillas only, plus discount factors calculated on the discounting curve \mathfrak{C}_d . The delta sensitivity is computed by shocking one by one the market pillars of both \mathfrak{C}_{1M} and \mathfrak{C}_d curves and the resulting delta risk is hedged using the suggested amounts (hedge ratios) of 7Y and 8Y Euribor1M swaps plus the suggested amounts of 7Y and 8Y instruments from the discounting curve \mathfrak{C}_d .

Such multiple-curve framework is consistent with the present market situation, but there is no free lunch, it is also more demanding. First, the discounting curve clearly plays a special and fundamental role, and must be built with particular care. This "pre-crisis" obvious step has become, in the present market situation, a very subtle and controversial point, that would require a whole paper in itself (Henrard [50], *ibid.* [12]). In fact, while the forwarding curves construction is driven by the underlying rate homogeneity principle, for which there is (now) a general market consensus, there is no longer, at the moment, general consensus for the discounting curve construction. At least two different practices can be encountered in the market: a) the old "pre-crisis" approach (e.g. the depo, Futures/FRA and swap curve cited before), that can be justified with the principle of maximum liquidity (plus a little of inertia), and b) the OIS curve, based on the overnight rate (Eonia for EUR), considered as the best proxy to a risk free rate available on the market because of its 1-day tenor, justified with collateralised (risk-less) counterparties (see Madigan [66], GSa [2] *ibid.* [12]). Second, building multiple curves requires multiple quotations: many more bootstrapping instruments must be considered (deposits, Futures, swaps, basis swaps, FRAs, etc., on different underlying rate tenors), which are available on the market with different degrees of liquidity and can display transitory inconsistencies. We refer the reader to the works of Ametrano and Bianchetti [4], for a detailed discussion on bootstrapping multiple yield curves. Third, non trivial interpolation algorithms are crucial to produce smooth forward curves (see Hagan and West [44, 45], Ametrano and Bianchetti [4], *ibid.* [12]). Fourth, multiple bootstrapping instruments implies multiple sensitivities,

so hedging becomes more complicated. Last but not least, pricing libraries, platforms, reports, etc. must be extended, configured, tested and released to manage multiple and separated yield curves for forwarding and discounting; and this is not a trivial task for quants, risk managers, developers and IT people.

The static multiple-curve pricing and hedging methodology described above can be extended, in principle, by adopting multiple distinct models for the evolution of the underlying interest rates with tenors $\tau_{f_1}, \dots, \tau_{f_n}$ to calculate the future dynamics of the yield curves $\mathfrak{C}_{f_1}, \dots, \mathfrak{C}_{f_n}$ and the expected cash flows. The volatility (and/or correlation) dependencies carried by such models imply, in principle, bootstrapping multiple distinct variance/covariance matrices and hedging the corresponding sensitivities using volatility (and correlation) dependent vanilla market instruments. Such more general approach has been carried by Mercurio [67] (*ibid.* [12]) in the context of generalised market models.

In the next chapter we show how to value cap(let)s under our assumption of distinct forward and discount curve by deriving pricing formulas, which will result in modifications of the corresponding Black-like formulas governed by our double-curve framework.

Market Formula for Pricing Cap(let)s

As is well known, the formal justifications for the use of Black-like formulas for cap(let)s come from the log-normal LMM of Brace et al. [16] and Miltersen et al. [72] (a detailed justification is given in Part I, Subsection 3.5.3). To be able to adapt such formulas to our double-curve case, we will have to reformulate accordingly the corresponding market models. Again, the choice of the discount curve P_d depends on the credit worthiness of the counterparty and on the possible presence of a collateral mitigating the credit risk exposure.

We first consider the case of a caplet paying out at time T_{i+1}

$$\tau_{x,i} [F_{x,i}(t) - \mathcal{K}]^+. \quad (10.1)$$

To price such payoff in the basic single-curve case, one notices that the forward rate $F_{x,i}$ is a martingale under the T_{i+1} -forward measure \mathbb{Q}_x^{i+1} for curve x , and then calculates the time- t caplet price

$$\mathbf{Caplet}(t, \mathcal{K}; T_i, T_{i+1}) = \tau_{x,i} P_{x,i+1}(t) \mathbb{E}_t^{\mathbb{Q}_x^{i+1}} \{ [F_{x,i}(t) - \mathcal{K}]^+ | \mathcal{F}_t \},$$

according to the chosen dynamics. For instance, the classic choice of a drift-less geometric Brownian motion

$$dF_{x,i}(t) = \sigma_i F_{x,i}(t) d\mathcal{W}_{x,i}^{i+1}, \quad t \leq T_i,$$

where σ_k is a constant and $\mathcal{W}_{x,i}^{i+1}$ is a \mathbb{Q}_x^{i+1} -Brownian motion, leads to Black's pricing formula:

$$\begin{aligned} \mathbf{Caplet}_i(t, \mathcal{K}; T_i, T_{i+1}) &= \tau_{x,i} P_{x,i+1}(t) \mathbf{Black76}(F_{x,i}(T_i), \mathcal{R}, \sigma_i \sqrt{T_i - t}) \\ &= \tau_{x,i} P_{x,i+1}(t) [F_{x,i}(T_i) N(\eta_+) - \mathcal{R} N(\eta_-)], \quad \text{where} \end{aligned} \quad (10.2)$$

$\eta_{\pm} = \frac{\ln(F_{x,i}(T_i)/\mathcal{R}) \pm \frac{1}{2}\sigma_i^2 \cdot (T_i - t)}{\sigma_i \sqrt{T_i - t}}$ and $N(\cdot)$ denotes the standard normal distribution function. In our double-curve setting, the caplet valuation requires more attention. In fact, since the pricing measure is now the forward measure \mathbb{Q}_d^{i+1} for curve \mathfrak{C}_d , the caplet price at time t

becomes

$$\mathbf{Caplet}(t, \mathcal{K}; T_i, T_{i+1}) = \tau_{x,i} P_{d,i+1}(t) \mathbb{E}_t^{\mathbb{Q}_d^{i+1}} \{ [F_{x,i}(T_i) - \mathcal{K}]^+ | \mathcal{F}_t \}.$$

The problem with this new expectation is that the forward rate $F_{x,i}$ is not, in general, a martingale under \mathbb{Q}_d^{i+1} . A possible way to value it is to model the dynamics of $F_{x,i}$ under its own measure \mathbb{Q}_x^{i+1} and then to model the Radon Nikodym derivative $d\mathbb{Q}_x^{i+1}/d\mathbb{Q}_d^{i+1}$ that defines the measure change from \mathbb{Q}_x^{i+1} to \mathbb{Q}_d^{i+1} . This is the approach proposed by Bianchetti [12], who uses a foreign-currency analogy and derives a quanto-like correction for the drift of $F_{x,i}$. Here, instead, we take a different route (Mercurio [67]).

Our idea is to follow a conceptually similar approach as in the classic LMM. There, the trick was to replace the Libor rate entering the caplet payoff with the equivalent forward rate, since the latter has “better” dynamics (a martingale) under the reference pricing measure. Here, we make a step forward, and replace the forward rate with its conditional expected value (the FRA rate). The purpose is the same as before, namely to introduce an underlying asset whose dynamics is easier to model. Since

$$L_{x,i}(t) = \mathbb{E}_t^{\mathbb{Q}_d^{i+1}} [F_{x,i}(T_i) | \mathcal{F}_t],$$

at the reset time T_i the two rates $F_{x,i}$ and $L_{x,i}$ coincides:

$$L_{x,i}(T_i) = F_{x,i}(T_i).$$

We can, therefore, replace the payoff (Equation (10.1)) with

$$\tau_{x,i} [L_{x,i}(T_i) - \mathcal{K}]^+,$$

and view the caplet as a call option no more on $F_{x,i}(T_i)$ but on $L_{x,i}(T_i)$. This leads to

$$\mathbf{Caplet}(t, \mathcal{K}; T_i, T_{i+1}) = \tau_{x,i} P_{d,i+1}(t) \mathbb{E}_t^{\mathbb{Q}_d^{i+1}} \{ [L_{x,i}(T_i) - \mathcal{K}]^+ | \mathcal{F}_t \}.$$

The FRA rate $L_{x,i}(t)$ is, by definition, a martingale under the measure \mathbb{Q}_d^{i+1} . If we smartly choose the dynamics of such a rate, we can value the last expectation analytically and obtain a closed-form formula for the caplet price. For instance, the obvious choice of a drift-less geometric Brownian motion

$$L_{x,i}(t) = v_i L_{x,i}(t) d\mathcal{W}^{i+1}(t), \quad t \leq T_i,$$

where v_i is a constant and \mathcal{W}^{i+1} is now a \mathbb{Q}_d^{i+1} Brownian motion, leads to the following pricing formula:

$$\mathbf{Caplet}_i(t, \mathcal{K}; T_i, T_{i+1}) = \tau_{x,i} P_{d,i+1}(t) \mathbf{Black76}(L_{x,i}(t), \mathcal{R}, v_i \sqrt{T_i - t}). \quad (10.3)$$

Therefore, under log-normal dynamics for the rate $L_{x,i}$, the caplet price is again given by Black’s formula with an implied volatility v_i . The differences with respect to the classic

formula (Equation (10.2)) are given by the underlying rate, which here is the FRA rate $L_{x,i}$, and by the discount factor, which here belongs to curve \mathfrak{C}_d .

Similar to Equation (3.10) and using the results of Equation (10.3) the cap price is now given as

$$\mathbf{Cap}^{\mathbf{Black76}}(T_{\text{exp}}) = \sum_{i=\alpha}^N \tau_{x,i} P_{d,i+1}(T_{\text{exp}}) \mathbf{Black76}(L_{x,i}(T_{\text{exp}}), \mathcal{R}, \nu_i \sqrt{T_i - t}). \quad (10.4)$$

Double Curve Log-Normal Libor Market Models

In the classic (single-curve) LMM, one models the joint evolution of a set of consecutive forward Libor rates under a common pricing measure, typically some “terminal” forward measure corresponding to the set of times defining the family of forward rates, see Section 3.5. Using measure change techniques, one finally derives pricing formulas for the main calibration instruments cap(let)s either in closed form or through efficient approximations.

To extend the LMM to the multi-curve case, we first need to identify the rates we need to model. The previous chapter suggests that the FRA rates $L_{x,i}$ are convenient rates to model as soon as we have to price a payoff, like that of a caplet, which depends on Libor rates belonging to the same curve \mathfrak{C}_x .

However, there is a major difference with respect to the single-curve case, namely that forward rates belonging to the discount curve need to be modelled too. We will show below that the dynamics of FRA rates under typical pricing measures depend on forward rates of curve \mathfrak{C}_d , so that also path-dependent payoffs on Libor rates will depend on the dynamics of the discount curve P_d .

11.1 The model dynamics

The LMM was introduced in the financial literature by Brace et al. [16] and Miltersen et al. [72] by assuming that forward Libor rates have a log-normal type diffusion coefficient. Here, we extend their approach to the case where the curve used for discounting is different than that used to generate the relevant future rates. For simplicity, we stick to the case where these rates belong to the same curve \mathfrak{C}_x .

We assume that each rate $L_{x,i}(t)$ evolves, under its canonical forward measure \mathbb{Q}_d^{i+1} , as a drift-less geometric Brownian motion:

$$dL_{x,i}(t) = \sigma_{x,i}(t)L_{x,i}(t)d\mathcal{W}_{d,i}^{i+1}(t), \quad t \leq T_i, \quad (11.1)$$

where the instantaneous volatility $\sigma_{x,i}(t)$ is deterministic and $\mathcal{W}_{d,i}$ is the i^{th} component

of an $N + 1$ dimensional \mathbb{Q}_d^{i+1} Brownian motion $\mathcal{W}_{d,i}^{j+1}$ with instantaneous correlation matrix $(\rho_{i,k}^{d,d})_{i,k=0,\dots,N}$, namely $d\mathcal{W}_{d,i}(t)d\mathcal{W}_{d,k}(t) = \rho_{i,k}^{d,d}dt$. This implies that each forward Libor rate evolves according to a geometric Brownian motion under its associated forward measure. In a double-curve setting, we also need to model the evolution of rates

$$\begin{aligned} F_{d,i}(t) &= F_d(t; T_i, T_{i+1}) = \frac{1}{\tau_{d,i}} \left[\frac{P_{d,i}(t)}{P_{d,i+1}(t)} - 1 \right], \text{ where} \\ \tau_{d,i} &= \tau_d(T_i, T_{i+1}). \end{aligned}$$

To this end, we assume that the dynamics of each rate $F_{d,j}$ under the associated forward measure \mathbb{Q}_d^{j+1} is given by:

$$dF_{d,j}(t) = \sigma_{d,j}(t)F_{d,j}(t)d\mathcal{W}_{d,j}^{j+1}(t), \quad t \leq T_i, \quad (11.2)$$

where the instantaneous volatility $\sigma_{d,j}(t)$ is deterministic and $\mathcal{W}_{d,j}^{j+1}$ is the j^{th} component of an $N + 1$ dimensional \mathbb{Q}_d^{j+1} Brownian motion whose correlations are

$$d\mathcal{W}_{d,i}(t)d\mathcal{W}_{d,j}(t) = \rho_{i,j}^{d,d}dt,$$

$$d\mathcal{W}_{x,i}(t)d\mathcal{W}_{d,j}(t) = \rho_{i,j}^{x,d}dt.$$

Clearly, correlations $\rho = (\rho_{i,k})_{i,k=0,\dots,N}$, $\rho^{d,d} = (\rho_{i,j}^{d,d})_{i,j=0,\dots,N}$ and $\rho^{x,d} = (\rho_{i,j}^{x,d})_{i,j=0,\dots,N}$ must be chosen so as to ensure that the global matrix

$$R \equiv \begin{bmatrix} \rho & \rho^{x,d} \\ (\rho^{x,d})' & \rho^{d,d} \end{bmatrix},$$

is positive (semi)definite.

Remark 11.1 *In some situations, it may be more realistic to resort to an alternative approach and model either curve \mathfrak{C}_x or \mathfrak{C}_d jointly with the spread between them, see e.g. Kijima et al. [64] or Schönbucher [93]. This happens, for instance, when one curve is above the other and there are sound financial reasons why the spread should be preserved positive in the future. In such a case, one can assume that each spread $\mathcal{S}_i(t) \equiv |L_{x,i}(t) - F_{d,i}(t)|$ evolves under the corresponding forward measure \mathbb{Q}_d^{i+1} , according to some*

$$d\mathcal{S}_i(t) = \sigma_i^{\mathcal{S}}(t, \mathcal{S}_i(t))d\mathcal{W}_i^{\mathcal{S}}(t),$$

whose solution is positively distributed. Sticking to Equation (11.1), the dynamics of forward rates $F_{d,i}$ ((11.2)) must then be replaced with

$$dF_{d,i}(t) = dL_{x,i}(t) \pm d\mathcal{S}_i(t),$$

where the sign \pm depends on the relative position of curves \mathfrak{C}_x and \mathfrak{C}_d . The analysis that follows can be equivalently applied to the new dynamics of rates $F_{d,i}$.

11.2 Dynamics under a general forward measure

To derive the dynamics of the FRA rate $L_{x,i}(t)$ in the terminal measure \mathbb{Q}_d^{N+1} we start from Equation (11.1) and perform a change of measure from \mathbb{Q}_d^{i+1} to \mathbb{Q}_d^{N+1} , whose associated numeraires are $P_{d,i+1}(t)$ and $P_{d,N+1}(t)$, respectively. We recall that in Subsection 3.5.2 the dynamics of $L_i(t)$ in the terminal measure \mathbb{Q}^{N+1} were derived in detail using the ‘‘Change of Numeraire Toolkit’’ (see Theorem 2.4.2) and were given as

$$\frac{dL_i(t)}{L_i(t)} = - \sum_{k=i+1}^N \tau_k \frac{\sigma_k(t)L_k(t)\rho_{i,k}}{1 + \tau_k L_k(t)} \sigma_i(t) \cdot dt + \sigma_i(t) d\mathcal{W}_i^{N+1}(t).$$

Similarly, the dynamics of $L_{x,i}(t)$ and $F_{d,i}$ in the terminal measure \mathbb{Q}_d^{N+1} are derived perfectly analogous as

$$\frac{dL_{x,i}(t)}{L_{x,i}(t)} = - \sum_{k=i+1}^N \tau_k \frac{\sigma_{x,k}(t)L_{x,k}(t)\rho_{i,k}^{x,d}}{1 + \tau_{x,k}L_{x,k}(t)} \sigma_{x,i}(t) \cdot dt + \sigma_{x,i}(t) d\mathcal{W}_{d,i}^{N+1}(t), \quad (11.3)$$

$$\frac{dF_{d,i}(t)}{F_{d,i}(t)} = - \sum_{k=i+1}^N \tau_k \frac{\sigma_{d,k}(t)L_{d,k}(t)\rho_{i,k}^{d,d}}{1 + \tau_{d,k}L_{d,k}(t)} \sigma_{d,i}(t) \cdot dt + \sigma_{d,i}(t) d\mathcal{W}_{d,i}^{N+1}(t). \quad (11.4)$$

Remark 11.2 *Following the same arguments used in the standard single-curve case, see Equation (3.26) in Subsection 3.5.2, we can easily prove that the above SDEs (Equation (11.3) and (11.4)) for the rates $L_{x,i}(t)$ and $F_{d,i}(t)$ all admit a unique strong solution if the coefficients $\sigma_{\cdot,k}$ are bounded.*

Now that we have a bounded profile of $L_{x,i}$ rates, with the use of Equation (10.4), the Monte Carlo Caption estimate price can be computed as

$$\mathbf{Caption\ Value}(t=0) = \frac{1}{M} \sum_{j=0}^M P_x^j(0, T_{\text{exp}}) \mathbb{E}^{\mathbb{Q}_x^{F_x(0, T_{\text{exp}})}} \left[\mathbf{Cap}_j^{\mathbf{Black76}}(T_{\text{exp}}) - \mathcal{X} \right]^+.$$

When curves \mathfrak{C}_x and \mathfrak{C}_d coincide, we have already noticed that the FRA rates $L_{x,i}$ coincide with the corresponding $F_{x,i}$. As a further sanity check, we can also see that the FRA rate dynamics reduce to those of the corresponding forward rates since

$$\mathfrak{C}_x = \mathfrak{C}_d \Rightarrow \begin{cases} \rho_{i,k}^{x,d} \rightarrow \rho_{i,k} \\ \tau_{d,i} \rightarrow \tau_{x,i} \\ \sigma_{d,i} \rightarrow \sigma_i(t) \\ F_{d,i} \rightarrow F_{x,i} \end{cases},$$

for each i, k .

The extended dynamics of the log-normal LMM in Equation (11.3) and (11.4) may raise some concern on numerical issues. In fact, having doubled the number of rates to simulate, the computational burden of the log-normal LMM is doubled with respect to that of the

single-curve case, since the SDEs for the homologues $L_{x,i}$ and $F_{d,i}$ share the same structure. However, some smart selection of the correlations between rates can reduce the simulation time. For instance, assuming that $\rho_{i,k}^{x,d} = \rho_{i,k}^{d,d}$ for each i, k , leads to the same drift rates for $L_{x,i}$ and the corresponding $F_{d,i}$, thus halving the number of drifts to be calculated at each simulation time. This gives a valuable advantage since it is well known that the drift calculations in a LMM are extremely time consuming.

We have started by describing the change in value of the market interest rate quotes, which occurred since August 2007. We noticed that once-compatible rates began to diverge sensibly, producing a clear segmentation of market rates. Practitioners tackled the issue by building different yield curves for different rate tenors.

The new double-curve framework involves the bootstrapping of multiple yield curves using separated sets of vanilla interest rate instruments homogeneous in the underlying rate (typically with 1M, 3M, 6M, 12M tenors). Prices, sensitivities and hedge ratios of interest rate derivatives on a given underlying rate tenor are calculated using the corresponding forward curve with the same tenor, plus a second distinct curve for discount factors.

We have also shown how to price the main (linear and plain vanilla) interest rate derivatives under the assumption of two distinct curves for generating future Libor rates and for discounting. The pricing formulas for caps result in a simple modification of the corresponding Black formulas used by the market in the single-curve setting. We have then extended the basic log-normal LMM and derived its dynamics under the terminal measure, from which we were able to derive the cap price.

Besides the lack of information about volatility and correlation, the present framework has the advantage of introducing a minimal set of parameters with a transparent financial interpretation and leading to familiar pricing formulas, thus constituting a simple and easy-to-use tool for practitioners and traders to promptly intercept possible market evolutions.

Glossary of Notation

$B(t)$

Amount in bank account.

$F_x(\cdot)$

Curve x simple compounded forward rate.

$L_i(t)$

Simply compounded Libor rate at time t for the expiry-maturity pair T_i and T_{i+1} .

$N(\varepsilon)$

Cumulative probability distribution for a standard normal distribution function.

$P(t, T)$

Price at time t of a zero coupon bond that provides a unit payoff at time T .

$P_x(\cdot)$

Curve x discount factor.

$S(t)$

Price for a non-dividend paying stock at time $t \in [0, T]$.

U by V

Size of a matrix with U rows and V columns.

$W^y(t)$

The value of a Brownian motion at time t under the EMM of numeraire y .

Ω

Sample space.

$\Pi(t)$

Value of a derivative security at time t .

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- $\Psi(t)$
The Radon-Nikodym derivative.
- Ξ
Value of a financial instrument contingent on a particular asset.
- \approx
Approximately equal.
- λ
This symbol is used interchangeably. In one context, it is used as a scalar variable of Time. In another context, it refers to a vector variable of $N + 1$ by $N + 1$ diagonal matrix of eigenvalues.
- $\langle a, b \rangle$
Given two vectors of equal size, say N , a scalar product is a sum of the products of their corresponding components i.e $\sum_{i=1}^N a_i b_i$.
- \mathbb{E}^y
Expectation taken under the EMM of numeraire y . If y is a bank account \mathbb{E}^B is denoted \mathbb{E} .
- \mathbb{P}
Market probability measure.
- \mathbb{Q}^y
Martingale measure under numeraire y . If y is a bank account, the martingale measure is called a risk neutral measure \mathbb{Q} .
- \mathbb{Q}_x^{i+1}
Martingale measure associated with numeraire $P_x(t, T_{i+1})$ in sub-market \mathcal{M}_x .
- $\mathbb{U}[0, 1]$
Uniform distribution in the interval $[0, 1]$.
- \mathcal{F}_t
Information about the path of the process up to time t .
- \mathcal{K}
Strike price for asset S_t .
- \mathcal{M}_x
Multiple distinct interest rate sub-markets.
- \mathcal{N}
Notional principal amount.
- \mathcal{P}_d
Irreducible polynomials of modulo 2, and order d .
-

-
- \mathcal{R}
Cap(let) strike rate.
- $\mathcal{W}^y(t)$
The value of a correlated Brownian motion at time t under the EMM of numeraire y .
- $\mathcal{W}_{x,i}^{i+1}$
 i^{th} Correlated Brownian motion associated with numeraire $P_x(t, T_{i+1})$ in sub-market \mathcal{M}_x .
- \mathcal{X}
Caption strike price.
- \mathcal{Z}_x
Represents an objective function to be minimised with respect to variable x .
- \mathfrak{C}_x
Multiple yield curves.
- d
Number of random, Brownian Motion, factors.
- μ
The coefficient of the time (dt) term of a stochastic process.
- $\nabla_x(f)$
Partial derivative of function f with respect to x i.e. $\frac{\partial f}{\partial x}$.
- ν
Girsanov's kernel adapted to \mathcal{F}_t .
- \oplus
Bit-by-bit exclusive-or XOR operator.
- $\phi(a, b)$
Normal distribution with mean a and standard deviation b .
- $\prod_{j=1}^N a_j$
A product of N terms of variable a_j .
- $\psi_{b_k}(h)$
Fraction corresponding to the prime number base b_k for denary number h , in dimension $k \in [1, n]$.
- ρ
A positive definite symmetric correlation matrix with entries $0 < \rho_{i,j} \leq 1$ and a unit diagonal.
-

σ

The coefficient of the Brownian motion (dW) term of a stochastic process.

 $\sum_{j=1}^N a_j$

Summation of N terms of variable a_j .

 τ_i

The year fraction between T_{i+1} and T_i .

 $\tau_{x,i}$

Time interval between reset dates $[T_i, T_{i+1}]$ in sub-market \mathcal{M}_x .

 ε

Standard random number from $\phi(0, 1)$.

 ξ_t^y

Asset holding with respect to asset y at time t .

 $a(h)$

Sequence of digits in base (modulo) b_k , $0 \leq a_j(h) < b_k$, for denary number h .

 b

Prime number ≥ 2 .

 dt

An infinitesimal (very small) time interval.

 h

Decimal-base number $h(= h_{b_k} = \sum_{j=0}^m a_{j,k}(h)b_k^j)$ re-written in increasing prime number base $b_k \geq 2$, for each increase in dimension $k \in [1, n]$.

 $r(t)$

risk-less rate of interest.

Glossary of Terms

Itô's formula

A stochastic version of the "chain rule" which expresses the volatility and drift of the function of a stochastic process in terms of the volatility and drift of the process itself and the derivatives of the function.

DC

Vector Diffusion Coefficient.

EMM

Equivalent Martingale Measure.

FTAP

Fundamental Theorem of Asset Pricing.

GARCH

Generalized Autoregressive Conditional Heteroscedasticity.

HJM

Heath Jarrow and Morton.

LDS

Low Discrepancy Sequences.

LMM

Libor Market Model.

MC

Monte Carlo (simulation).

NFLVR

No Free Lunch with Vanishing Risk.

PDE

Partial Differential Equation.

QRN

Quasi-Random Numbers.

SDE

Stochastic Differential Equation.

SD

Standard Deviation.

SE

Standard Error.

URL

Uniform Resource Locator, internet address.

VBA

Visual Basic Application.

VDC

Van der Corput (sequence in base 2).

Arbitrage

An opportunity when an instrument has a guaranteed non-negative payoff and a negative price i.e. investors generate money without any initial investment.

Cap

An option that provides a payoff when a specified interest rate is above a certain level. The interest rate is a floating rate that is reset periodically.

Caption

Call option on a Cap.

Contingent claim

A claim whose amount is determined by the behaviour of market securities up until the time it is paid.

Derivative

An instrument which derive its price from another.

EONIA

Euro OverNight Index Average.

EUR

Pertaining to the European Markets.

Expectation

The mean of a random variable, which will be the limiting value of the average of an infinite number of identical trials.

FRA

Floating Rate Agreement.

FX

Foreign Exchange.

Hedge

A trade position intended to reduce risk.

Martingale

Mathematical formalisation of the concept of a fair game.

Maturity

The time at which any claim pays off.

Mean

Synonym for expectation.

Multi-factor

A market model which is driven by more than one Brownian motion.

Numeraire

Any asset that has strictly positive prices for all $t \in [0, T]$.

OIS

Overnight Index Swaps.

Over-the-counter

An agreement between two parties concluded without mediation of an exchange.

Payoff

A payment.

Volatility Skew

The variation of implied volatility with strike price.

Xibor

Generic Interbank Offered Rate.

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