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BISIGMA FRAMES

by

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Thesis prepared in partial fulfilment of the requirements for the Degree of Doctor of Philosophy under the joint supervision of

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and

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FEBRUARY 1999

ACKNOWLEDGEMENTS

I wish to thank Professor C. Brink, the head of the Maths and Applied Maths Department at the University of Cape Town for the role he played in enabling and encouraging me to register for a Ph.D.

I am grateful to my supervisor Professor C.R.A. Gilmour for his continuous assistance and guidance towards this thesis. My appreciation is also directed to my co-supervisor and the leader of our research group, Professor G.C.L. Brümmer, who has set a good example of a researcher. I am also thankful to Dr A. Schauerte who was of great help when my supervisor was away on Sabbatical leave from March to June in 1996. I am especially grateful to my husband, Sizwe for the proof reading of this thesis.

The financial support of the University of Cape Town, the Foundation for Research and Development, and Mrs P. Groves through the Chisnall endowed tutorship, is gratefully acknowledged.

The pleasant environment and the friendly attitude of the staff members at the University of Stellenbosch where I work contributed positively to the completion of this thesis.

I must sincerely thank Ms H. Oberholzer and Mrs L. Adams for their excellent work and patience during the typing of this thesis. I am grateful to Dr J. Vermuelen for the time consuming production of the diagrams which appear in the text.

This effort would not have been possible without the dedicated support of my mother, husband, family and friends.

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ABSTRACT

We introduce and investigate the concept of a bi σ-frame. The cozero part of a biframe, itself a bi σ-frame, is defined and used to construct the compact regular, and regular Lindelöf coreflections for biframes. Pseudocompactness for biframes is defined in a natural way and characterised in terms of the cozero part.

Finally we obtain the σ-frame analogue of the result that characterises the stably continuous frames in terms of the compact regular biframes.

INTRODUCTION

As a lattice, the open sets of a topological space form a frame and thus frames provide an algebraic tool for the study of topology. Some authors (see [17]) use frames as a convenient setting for choice free construction in topology. Our main concern in this thesis is to exploit frames as a tool in the settings of bitopological spaces and biframes. Frames have a countable-join generalisation called σ-frames. These were considered by Reynolds [24], Charalambous [8] and were explored by, amongst others, Banaschewski [2], Gilmour [13] and Walters [29]. They naturally occur in various contexts, the cozero set lattices of topological spaces and the Boolean σ-algebras being typical examples. Cozero sets of a completely regular space form a basis for the topology. In the setting of σ-frames we have assumed throughout the axiom of countable dependent choice, as this appears to be inherent in proving frames (respectively biframes) with cozero basis to be completely regular.

Kelly [18] followed by Lane [19] initiated the study of bitopological spaces. An important motivation for studying bitopological spaces is given by the quasi-uniform spaces in topology. The quasi-uniform spaces are the asymmetric version of the uniform spaces. A quasi-uniform space naturally gives rise to a bitopological space as well as to an ordered space (see [23]). Biframes are the frame counterpart of bitopological spaces. They were first defined by Banaschewski, Brümmer and Hardie (see [3]), and have been extensively studied by several authors including Frith [11] and Schauerte [26]. Of particular note is the remarkable result that characterises the stably continuous frames as first parts of compact regular biframes [4]. The bi σ-frames simultaneously generalise σ-frames and biframes. Bi σ-frames are extensively studied in this thesis, and are used to study biframes through the latter's cozero part.

A particularly appropriate class of spaces for the investigation of σ-frames and the cozero part of frames is that of the Alexandroff spaces. They were introduced by Alexandroff [1], who called them completely normal spaces. Gordon [14] reinvented them under the name of zero-set spaces and later Gilmour placed them in the setting of frames (see [13]).

Hager [15] shows that Alexandroff spaces arise naturally when one studies the algebras (in the sense of Henrikson and Johnson [16], and Mrówka [22]) of real-valued functions on topological spaces. Alexandroff bispaces extend both Alexandroff spaces and bitopological spaces, and are defined here for the first time.

We give an outline of contents of each chapter.

CHAPTER 0

The first section is of introductory nature, giving the basic definitions and results of what is to follow. In the second section, using the characterisation of compactifiable biframes as those with strong inclusions (see [27]), we show that completely regular biframes are exactly those biframes which are compactifiable. The compact completely regular coreflection of a biframe is constructed using the completely regular ideals.

CHAPTER 1

Alexandroff bispaces are defined in terms of regular bi σ-frames. The adjunction between Alexandroff bispaces and regular bi σ-frames is defined and also that between Alexandroff bispaces and bitopological spaces. The latter adjunction extends that between Alexandroff spaces and completely regular topological spaces [13]. To do this we need and prove a Urysohn's lemma for Alexandroff bispaces.

CHAPTER 2

The interaction between biframes and bi σ-frames is investigated by considering the cozero part of a biframe. As in frames (see [1]), the cozero part of a biframe is the largest regular sub bi σ-frame of a biframe. For a completely regular biframe its cozero part generates it as a bi σ-frame. An alternative construction of the compact completely regular coreflection of a biframe is given, using its cozero part.

CHAPTER 3

The biframe of reals is presented in a similar way to the frame of reals (see [7]). Pseudocompactness of a biframe is defined in terms of bounded maps and we show that a biframe is pseudocompact if and only if the first and second parts of its cozero part are compact.

CHAPTER 4

We show that the categories of stably continuous σ-frames and compact regular bi σ-frames are equivalent. This is the analogue of the remarkable result of Banaschewski and Brümmer [4] linking the stably continuous frames and the compact regular biframes.

CHAPTER 0

PRELIMINARIES, COMPLETELY REGULAR BIFRAMES AND COMPACTIFICATION

In the first section we give the basic definitions and concepts associated with frames, σ-frames, biframes and ideals.

In the second section of this chapter we prove that completely regular biframes are precisely those which have a compactification. We will do this by making use of the characterisation of the compactifiable biframes as those which admit strong inclusions. This latter result is due to Schauerte [27]. Finally, by considering the completely regular ideals, we obtain the compact completely regular coreflection of a biframe. A normal regular biframe is completely regular, and in this case, the compact regular and compact completely regular coreflections, coincide.

0.1. Background

FRAMES

A *frame* is a complete lattice L satisfying the distributive law, $x \wedge \bigvee S = \bigvee \{x \wedge s | s \in S\}$ for binary meet $\wedge$, and arbitrary join $\bigvee$, $x \in L, S \subseteq L$. A *frame homomorphism* $h : L \to M$ is a map between frames preserving finite meets (including the unit, or top element e) and arbitrary joins (including zero, or bottom element 0). The resulting category is denoted by **FRM**.

A frame morphism $h : L \to M$ is called *dense* if $h(x) = 0$ implies $x = 0$. It is called *codense* iff $h(x) = e$ implies $x = e$.

Let L be a frame, and a, b, c elements of L.

- *Regular frames.* We say that a is rather below b, written $a \prec b$ if there exists $c \in L$

such that $a \wedge c = 0$ and $b \vee c = e$. Such an element c is called a separating element. L is *regular* if $a = \bigvee\{b|b \prec a\}$ for all $a \in L$. Frame homomorphisms preserve $\prec$ and the full subcategory of **FRM** consisting of the regular frames will be denoted by **Reg FRM**.

- *Completely regular frames.* We say that a is completely below b, written $a \prec\!\prec b$, if there is a family $\{x_i|\ i \in \mathbb{Q} \cap [0,1]\}$ of elements in L satisfying $x_0 = a$, $x_1 = b$, $i \leq j$ implying that $x_i \prec x_j$. L is completely regular if the relation $\prec\!\prec$ is approximating, i.e. for each $a \in L$, $a = \bigvee\{x|x \prec\!\prec a\}$. Frame homomorphisms preserve $\prec\!\prec$. The category of completely regular frames will be denoted by **CReg FRM**.

Lemma 0.1.1: *If $h : L \to M$ is a codense frame homomorphism with L regular, then it is one-one.*

Proof: Assume that h is codense. Consider $h(a) = h(b)$ for $a, b \in L$. Since L is regular, $a = \bigvee\{x|\ x \prec a\}$ and $b = \bigvee\{y|\ y \prec b\}$. This implies that

$$\begin{aligned} h(a) &= \bigvee\{h(x)|\ h(x) \prec h(a)\} \\ &= \bigvee\{h(y)|\ h(y) \prec h(b)\} = h(b). \end{aligned}$$

Take $y \prec b$ with separating element c, then $h(c)$ separates $h(y) \prec h(b)$. Since $h(a) = h(b)$, $h(c)$ separates $h(y) \prec h(a)$. Thus $e = h(c) \vee h(a) = h(c \vee a)$. Since h is codense, $c \vee a = e$ and therefore $y \prec a$. This holds for all y rather below b, thus $b \leq a$. The result follows by symmetry. ■

- *Compact frames.* An element $c \in L$ is called compact if, whenever $c \leq \bigvee X$ for some $X \subseteq L$, it follows that $c \leq \bigvee F$ for some finite subset $F \subseteq X$. L is compact iff $e \in L$ is compact.

- *Normal frames.* L is normal if whenever $a \vee b = e$ in L, there exist $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = e = b \vee v$.

Lemma 0.1.2: *For any dense frame homomorphism $h : N \to L$, if N is regular and L is compact, then h is codense.*

Proof: Suppose that $a \in N$ such that $h(a) = e$. By regularity $a = \bigvee\{x | \ x \prec a\}$, so that $h(\bigvee\{x | \ x \prec a\}) = e$. Thus $\bigvee\{h(x) | \ h(x) \prec h(a)\} = e$, hence $h(y) = e$ for some $y = \bigvee\limits_{i \in F} x_i \prec a$, F finite, by compactness of L.

Thus there exists a separating element $b \in N$ for $y \prec a$ such that $b \vee a = e$, moreover $h(b) = h(b) \wedge h(y)$ since $h(y) = e$. Hence $h(b) = h(b \wedge y)$. But $b \wedge y = 0 = h(b \wedge y)$ so that $h(b) = 0$. Density of h gives us that $b = 0$, consequently $a = e$. ∎

σ – FRAMES

A *σ-frame* is a lattice L which has countable joins, finite meets, a top element e, a bottom element 0 and satisfies the (countable) distributive law : $x \wedge \bigvee\limits_n x_n = \bigvee\limits_n (x \wedge x_n)$ ($n \in I$, countable) for binary meet $\wedge$, countable join $\bigvee$, $x \in L$ and any sequence $\{x_n\}$ in L. A *σ-frame homomorphism* $h : L \to M$ is a map between σ-frames preserving countable joins and finite meets, in particular preserving 0 and e. The resulting category is denoted by **σ-FRM**.

- *Ideal.* In any lattice, an ideal is a subset I such that $\bigvee E \in I$ for any finite subset $E \subseteq I$ and $x \in I$ whenever $x \leq z$ for some $z \in I$.

- *σ-ideal.* A σ-ideal is an ideal that is closed under countable joins.

- *completely prime filter, prime filter, σ-prime filter.* A filter is a dual-ideal. A filter F is called completely prime if $\bigvee X \in P$ implies $X \cap F \neq \emptyset$ for all subsets X of the lattice, prime if this condition holds for all finite subsets X, σ-prime if it holds for all countable subsets X.

The definitions of regularity, complete regularity, compactness and normality are similar to those for frames except that arbitrary joins are replaced by countable joins in σ-frames.

BIFRAMES

A *biframe* $L = (L_0, L_1, L_2)$ is a triple in which L_0 is a frame and, L_1 and L_2 are subframes of L_0 which together generate L_0. A *biframe homomorphism* $h : L \to M$ between biframes is a frame homomorphism $h = h_0 : L_0 \to M_0$ for which the restrictions $h|_{L_i} = h_i : L_i \to M_i$ $(i \in \{1, 2\})$ are also frame homomorphisms. The category of biframes and their homomorphisms is denoted by **BIFRM**. We refer to L_0 as the total part of L, L_1 and L_2 as its first and second parts, respectively.

A biframe homomorphism $h : L \to M$ is said to be

- *dense* if h_0 is dense, that is, $a = 0$ whenever $h(a) = 0$, for any $a \in L_0$,
- *codense* if h_0 is codense, that is, $a = e$ whenever $h(a) = e$, for any $a \in L_0$,
- *onto* if h_1 and h_2 are both onto,
- *one-one* if h_0 is one-one,
- *an isomorphism* if h_0 is both one-one and onto.

Observe that a dense, one-one (respectively, codense) biframe homomorphism has first and second parts dense, one-one (respectively, codense). An onto biframe homomorphism has a total part which is onto.

Let $L = (L_0, L_1, L_2)$ be a biframe.

For $\{x, y\} \subseteq L_i$, $i \in \{1, 2\}$, we write $x \prec_i y$ (and say that x is i-rather below y) if there exists $c \in L_k$, $i \neq k$ such that $x \wedge c = 0$ and $y \vee c = e$, where $k \in \{1, 2\}$. We note the following basic properties of $\prec_i$:

For $a, c, x, z \in L_i$,

- $x \leq a \prec_i c \leq z$ implies $x \prec_i z$,
- if $a \prec_i c$ and $x \prec_i z$, then $a \wedge x \prec_i c \wedge z$ and $a \vee x \prec_i c \vee z$,
- any biframe homomorphism preserves $\prec_i$.

- *Regularity.* L is called regular if $x = \bigvee\{z \in L_i | \ z \prec_i x\}$ for all $x \in L_i$ where $i \in \{1, 2\}$, (we say that the relation $\prec_i$ is approximating). Observe that the total part of a regular biframe is regular. We will denote the rather below relation on the total part L_0 of L, by $\prec_0$ where both the elements and separating elements come from L_0.

The homomorphic image of a regular biframe is regular.

Any biframe L has a largest regular sub biframe $\mathcal{R}L$, generated by a family of regular sub biframes of L. This defines a coreflection functor $\mathcal{R}$ from **BIFRM** to its full subcategory of regular biframes, see [3].

Complete regularity. For $x, y \in L_i$ we say x is i-completely below y, written $x \prec\prec_i y$ if there exists an interpolating sequence $\{c_{nk}\}_{n=0,1,2,\ldots;\ k=0,1,\ldots 2^n}$ in L_i between x and y, where $x \leq c_{00}$, $c_{01} \leq y$, $c_{nk} = c_{n+1\ 2k}$, $c_{nk} \prec c_{n\, k+1}$.

Observe that any interpolating sequence $\{c_{nk}\}$ between x any y determines a scale between x and y, that is, a family $\{c_q : q \in \mathbb{Q} \cap [0, 1]\}$ such that $x \leq c_0$, $c_1 \leq y$ and $c_r \prec_i c_s$ whenever $r < s$: put $c_q = \bigvee\{c_{nk} | \ \frac{k}{2^n} \leq q\}$. L is completely regular if the relation $\prec\prec_i$ is approximating for each i. Clearly $\prec\prec_i$ inherits the properties of $\prec_i$ listed above.

Since biframe homomorphisms preserve these relations, this gives full subcategories **Reg BIFRM** and **CReg BIFRM**.

- *Compactness.* A biframe L is called compact if L_0 is a compact frame. The full subcategory of **BIFRM** consisting of the compact completely regular biframes will be denoted by **KCReg BIFRM**.

- *Lindelöfness.* A biframe is called Lindelöf if L_0 is Lindelöf, that is whenever $\bigvee N = e$, $N \subseteq L_0$ there exists a countable subset M of N with $\bigvee M = e$.

- *Normality.* A biframe is called normal if whenever $x \vee y = e$ for some $x \in L_i$, $y \in L_k$ $i \neq k$, there exists $a \in L_k$, $b \in L_i$ with $a \wedge b = 0$ and $b \vee y = e = a \vee x$.

- *Compactification.* A compactification of a biframe L is a dense, onto biframe homomorphism $h : M \to L$ from a compact, regular biframe M to L.

- *Strong inclusion.* A strong inclusion on a biframe L is a pair of $\lhd = (\lhd_1, \lhd_2)$ of relations on L_1 and L_2 respectively such that for $i, k \in \{1, 2\}, i \neq k$

(SI 1) $x, y \in L_i$ and $y \leq x \lhd_i a \leq b$ imply $y \lhd_i b$

(SI 2) $\lhd_i$ is a sublattice of $L_i \times L_i$

(SI 3) $x \lhd_i a$ implies that $x \prec_i a$

(SI 4) $x \lhd_i$ a implies that there exists $y \in L_i$ with $x \lhd_i y \lhd_i a$

(SI 5) If $x \lhd_i a$, then there exist $u, v \in L_k$ such that $u \lhd_k v$, $x \wedge v = 0$ and $a \vee u = e$

(SI 6) $a = \bigvee\{x : x \lhd_i a\}, a \in L_i$.

The functor $\Im$: **BIFRM** → **BIFRM** is given by the following:

If $\Im L_0$ denotes the frame of ideals of L_0, then

$(\Im L)_i = \{J \in \Im L_0 | J \text{ is generated by } J \cap L_i\}$ for $i \in \{1, 2\}$.

$(\Im L)_0 =$ the subframe of $\Im L_0$ generated by $(\Im L)_1 \cup (\Im L)_2$. For a biframe homomorphism $h : L \to M$ the associated homomorphism $\Im h : \Im L \to \Im M$ is given by the rule that $\Im h(J)$ is the ideal generated in $(\Im M)_0$ by the image $h(J)$.

- *Regular ideal.* An ideal $J \in (\Im L)_i$ is called regular if $a \in J \cap L_i$ implies that there exists $b \in J \cap L_i$ such that $a \prec_i b$.

- *Completely regular ideal.* An ideal $J \in (\Im L)_i$ is called completely regular if $a \in J \cap L_i$ implies that there exists $b \in J \cap L_i$ such that $a \prec\prec_i b$.

Bitopological Spaces

A bitopological space [18] is a triple (X, P, Q) in which X is a set, P and Q are topologies on X. A bicontinuous map $f : (X, P_1, Q_1) \to (Y, P_2, Q_2)$ is a map $f : X \to Y$ such that $f : (X, P_1) \to (X, P_2)$ and $f : (X, Q_1) \to (Y, Q_2)$ are continuous. The bitopological spaces together with bicontinuous maps form the category **BITOP**.

The following are the open set and the spectrum functors, respectively, see [3]. The contravariant functor $\mathcal{O}$: **BITOP** $\to$ **BIFRM** is given by: $\mathcal{O}(X, P, Q) = (P \vee Q, P, Q)$ and $\mathcal{O}f = f^{-1}$ which delivers the f-preimages of open sets.

The contravariant spectrum functor $\sum$: **BIFRM** $\to$ **BITOP** is described in a number of ways, of which we use the following two:
$\sum(L_0, L_1, L_2) = (\sum L_0, \{\sum_a : a \in L_1\}, \{\sum_b : b \in L_2\})$ where

1. $\sum L_0$ is the set of all completely prime filters in the frame L_0, $\sum_a = \{P \in \sum L_0 : a \in P\}$. For a biframe map $h : L \to M$, the continuous map $\sum h : \sum M \to \sum L$ is obtained by taking h-preimages of completely prime filters.

2. $\sum L_0$ is the set of all frame homomorphisms $\xi : L_0 \to \mathbf{2}$ where $\sum_a = \{\xi : \xi(a) = 1\}$, $\sum h(\xi) = \xi \circ h$ and $\mathbf{2}$ is the two element frame $\{0, 1\}$.

0.2. Completely regular biframes and compactifications

Definition: Let $L = (L_0, L_1, L_2)$ be a frame, for $x \in L_i$, $i \in \{1,2\}$ we denote by x^* the largest $v \in L_j$ for which $v \wedge x = 0$, that is, $x^* = \bigvee\{v \in L_j | v \wedge x = 0\}$

We show that $x \prec\prec_i y$ implies that $y^* \prec\prec_j x^*$. It suffices to check that $x \prec_i y$ implies $y^* \prec_j x^*$. Let z be a separating element for $x \prec_i y$. Since $z \wedge x$ then $z \leq x^*$, but $z \vee y = e$ thus $x^* \vee y = e$. $y \wedge y^* = 0$ holds, so that $y^* \prec_j x^*$ with separating element y.

Lemma 0.2.1: *For any completely regular biframe* (L_0, L_1, L_2), $\prec\prec = (\prec\prec_1, \prec\prec_2)$ *is a strong inclusion.*

Proof:

(SI 1) Suppose $x \leq a \prec\prec_i b \leq y$ holds in L_i. Since any interpolating sequence for a and b will be an interpolating sequence for x and y, we have $x \prec\prec_i y$.

(SI 2) (a) $0 \prec\prec_i 0$ because $0 \prec_i 0$ with separating element $e \in L_j$, $i \neq j$.

(b) $e \prec\prec_i e$ since $e \prec_i e$ with separating element $0 \in L_j$, $i \neq j$.

(c) Let $a \prec\prec_i b$ and $x \prec\prec_i y$, with interpolating sequences $\{c_{nk}\}$ and $\{z_{nk}\}$ respectively. As $\prec_i$ satisfies $(d \prec_i f, \quad g \prec_i h \Longrightarrow d \wedge g \prec_i f \wedge h)$, it is easy to see that $\{c_{nk} \wedge z_{nk}\}$ is an interpolating sequence for $a \wedge x \prec\prec_i b \wedge y$.

Similarly $a \vee x \prec\prec_i b \vee y$.

(SI 3) The fact that $x \prec\prec_i a$ implies $x \prec_i a$, follows immediately from the definition of $x \prec\prec_i a$.

(SI 4) $\prec\prec_i$ interpolates since if $a \prec\prec_i b$ with interpolating sequence $\{c_{nk}\}$, then $a \prec\prec_i c_{11} \prec\prec_i b$.

(SI 5) If $x \prec\prec_i a$, then there exists $y \in L_i$ with $x \prec\prec_i y \prec\prec_i a$. It follows that $a \vee y^* = e$ and, as always $x \wedge x^* = 0$. Since $y^* \prec\prec_k x^*$ we are done.

(SI 6) Statement SI 6 is the complete regularity of L. ■

Our main result will follow from the above observation and the next proposition from [27].

Proposition 0.2.2: *A biframe has a compactification if and only if it has a strong inclusion.*

Proof: We present a summary of the proof given in [27].
$\Longrightarrow$: Given a compactification $h : M \to L$ for a biframe L, $\prec = (\prec_1, \prec_2)$ is a strong inclusion on M because M is compact and regular. Since h is onto,
$\prec = ((h \times h)[\prec_1], (h \times h)[\prec_2])$ is a strong inclusion on L.

$\Longleftarrow$: Let $(\lhd_1, \lhd_2)$ be a strong inclusion on L. Recall that an ideal $J \in (\Im L)_i$ is strongly regular if and only if $x \in J \cap L_i$ implies that there exists a $y \in J \cap L_i$ with $x \lhd_i y$. Let $\mathcal{I}_i$ consist of the strongly regular ideals in $(\Im L)_i$ and $\mathcal{I}_0 \subseteq \Im(L_0)$ be the subframe generated by $\mathcal{I}_1 \cup \mathcal{I}_2$. The join map $J_L : \mathcal{I} \longrightarrow L$ provides the required compactification for L, where $\mathcal{I} = (\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2)$. ■

Proposition 0.2.3: *A biframe L has a compactification if and only if L is completely regular.*

Proof: $\Longleftarrow$: This implication follows from the previous two results.

$\Longrightarrow$: Since L has a compactification, it admits a strong inclusion $(\lhd_i, \lhd_j)$ say, and $a = \bigvee\{x : x \lhd_i a\}$ for each $a \in L_i$. Given that $x \lhd_i a$ then we can find a scale $\{c_q\}$ between x and a such that $x = c_0, c_1 = a$ and $c_r \lhd_i c_s$ whenever $r < s$, but $x \lhd_i a$ and $c_r \lhd_i c_s$ imply that $x \prec_i a$ and $c_r \prec_i c_s$, hence $x \lhd_i a$ implies that $x \prec\prec_i a$. ■

We will construct the compact completely regular coreflection of a biframe L by considering completely regular ideals in $(\Im L)_1$ and $(\Im L)_2$.

Definition: *The biframe* $\mathbf{CR}\Im L$ *of completely regular ideals of a biframe* L *is*

$$\mathbf{CR}\Im L = ((\mathbf{CR}\Im L)_0, (\mathbf{CR}\Im L)_1, (\mathbf{CR}\Im L)_2)$$

where $(\mathbf{CR}\Im L)_i = \{J \in (\Im L)_i : J \quad \textit{is completely regular}\}, i \in \{1,2\}$ *and* $(\mathbf{CR}\Im L)_0$ *is the subframe of* $(\Im L)_0$ *generated by* $(\mathbf{CR}\Im L)_1 \cup (\mathbf{CR}\Im L)_2$.

Proposition 0.2.4: *For a biframe* L, $\mathbf{CR}\Im L$ *is a compact completely regular biframe.*

Proof: We first show that $\mathbf{CR}\Im L$ is a biframe:

(a) Each $(\mathbf{CR}\Im L)_i$ is closed under binary meets and joins by (SI 2)(c) in the proof of Lemma 0.2.1.

(b) Both $\{0\}$ and $\downarrow e = L_0$ are in $(\mathbf{CR}\Im L)_i$ by (SI 2)(a) and (b) in the proof of Lemma 0.2.1, respectively.

(c) We check that $(\mathbf{CR}\Im L)_i$ is closed under updirected joins, which is equivalent to checking that it is closed under arbitrary joins because it is closed under finite joins : Suppose that $J = \bigcup J_\ell$, where $\{J_\ell | \ell \in I\}$ is an updirected family in $(\mathbf{CR}\Im L)_i$. Let $a \in J \cap L_i$. Then $a \in J_\ell \cap L_i$ for some ℓ. But $a \in J_\ell \cap L_i$ implies $a \prec\prec_i b_\ell$ for some $b_\ell \in J_\ell \cap L_i \subseteq J \cap L_i$ and so $J \in (\mathbf{CR}\Im L)_i$.

$\mathbf{CR}\Im L$ is compact as $(\mathbf{CR}\Im L)_0$ is a subframe of $(\Im L)_0$ which is compact.

Finally we check complete regularity: Consider $r_i : L_i \longrightarrow (\mathbf{CR}\Im L)_i$ where $r_i(a) = \langle\{x \in L_i : x \prec\prec_i a\}\rangle$, where $\langle \ldots \rangle$ denotes the ideal generated in L_0. Then $r_i(a) \in (\mathbf{CR}\Im L)_i$ since $\prec\prec_i$ interpolates.

Claim: If $a \prec\prec_i b$, then $r_i(a) \prec\prec_i r_i(b)$.

Proof of Claim: Suppose that $a \prec\prec_i b$. Then $b^* \prec\prec_j a^*$. From the interpolating sequence $\{c_{nk}\}$ for $a \prec\prec_i b$ we have $a \prec\prec_i c_{21} \prec\prec_i c_{22} \prec\prec_i b$ with $c_{22} \vee c_{21}^* = e$ resulting in $r_i(b) \vee r_j(a^*) = L_0$ and $r_j(a) \wedge r_j(a^*) = \{0\}$. Thus $r_i(a) \prec_i r_i(b)$. Since $\prec\prec_i$ interpolates this argument actually shows that $r_i(a) \prec\!\prec_i r_i(b)$. Thus $a \prec\!\prec_i b \in J \cap L_i$ implies that $r_i(a) \prec\!\prec_i r_i(b) \subseteq J$.

For any $J \in (\mathbf{CR}\Im L)_i, J = \bigvee\{r_i(a) | a \in J \cap L_i, r_i(a) \prec\!\prec_i J\}$ and hence $\mathbf{CR}\Im\mathbf{L}$ is completely regular. ■

Proposition 0.2.5: $\mathbf{CR}\Im$ *is functorial.*

Proof: For a biframe homomorphism $h : L \longrightarrow M$ and $J \in (\mathbf{CR}\Im L)_i$, $i \in \{1, 2\}$, we define $(\mathbf{CR}\Im h)(J) = \langle h(J) \rangle$, the ideal generated by $h(J)$ in M_0.

Suppose $J \in (\mathbf{CR}\Im L)_i$ and $x \in \langle h(J) \rangle$. Then $x \leq h(a)$ for some $a \in J$. Since J is completely regular, there exists $y \in J \cap L_i$ such that $a \prec\!\prec_i y$. Hence $x \leq h(a) \prec\!\prec_i h(y) \in \langle h(J) \rangle$. Thus $(\mathbf{CR}\Im h)(J) = \langle h(J) \rangle$ is completely regular in M_i. ■

Lemma 0.2.6: *If M is a compact completely regular biframe, then the join map $J_M : \mathbf{CR}\Im M \longrightarrow M$ is an isomorphism.*

Proof: Compactness of $(\mathbf{CR}\Im M)_0$ implies codensity of J_M. Hence $J_M : (\mathbf{CR}\Im M)_0 \longrightarrow M_0$ is one-one. If $a \in M_i$, then $\bigvee r_i(a) = \bigvee(\{x \in M_i | x \prec\!\prec_i a\}) = a$, and, consequently, J_M is an isomorphism. ■

Remark: In Lemma 0.2.6 we only need regularity, not complete regularity. The reason for this being that compact regular biframes are normal [28] and normality implies that the i-rather below relation interpolates in biframes (as seen in the following Lemma), resulting in complete regularity.

Proposition 0.2.7: **KCReg BIFRM** *is coreflective in* **BIFRM** *with coreflection maps* $J_L : \mathbf{CR}\Im L \longrightarrow L$.

Proof: Consider $h : M \longrightarrow L$ in **BIFRM**, where M is compact completely regular. Then applying $\mathbf{CR}\Im$ we obtain:

$$\begin{tikzcd}\mathrm{CR}\Im M \arrow[r, "\cong", "J_M"'] \arrow[d, "\mathrm{CR}\Im h"'] & M \arrow[d, "h"] \\ \mathrm{CR}\Im L \arrow[r, "J_L"] & L\end{tikzcd}$$

Thus h factors through $\mathbf{CR}\Im L$ and density of J_L implies that this factorisation is unique. ∎

Our aim is to prove that for normal regular biframes the compact completely regular and the compact regular coreflections are isomorphic, assuming the Countable Dependent Choice, of course.

Lemma 0.2.8: *For a normal biframe L, $\prec_i$ interpolates.*

Proof: Suppose $a \prec_i b$, with $a, b \in L_i$. Then there exists $c \in L_j$ satisfying $a \wedge c = 0$ and $b \vee c = e$. Normality gives us $u \in L_j, v \in L_i$ with $u \wedge v = 0$ and $b \vee u = e = c \vee v$. Thus c and u are separating elements for $a \prec_i v$ and $v \prec_i b$ so that $\prec_i$ interpolates. ∎

The following two results appear in [28].

Lemma 0.2.9: *If a biframe L is normal and $x \vee y = e$ for some $x \in L_i, y \in L_j$ $i \neq j$, then $r_i(x) \vee r_j(y) = L_0$.*

Proof: Let $x \vee y = e$, for $x \in L_i, y \in L_j$. By normality there exists $u \in L_j$, $v \in L_i$ with $u \wedge v = 0, x \vee u = e = y \vee v$. In particular, $u \prec_j y$. Using normality and $x \vee u = e$, we can find $s \in L_j$, $t \in L_i$ such that $s \wedge t = 0$, $x \vee s = e = u \vee t$. Thus $t \prec_i x$. Once more, using normality of L and the axiom of countable dependent choice, we get $u \prec\!\prec_j y$ and $t \prec\!\prec_i x$. Thus, we have $u \vee t = e$ for $u \in r_j(y)$ and $t \in r_i(x)$ and thus $r_i(x) \vee r_j(y) = L_0$. ∎

Recall that for a biframe L, an ideal J of L_0 generated by $J \cap L_i$ is called regular if $x \in J \cap L_i$ implies that there exists an element $y \in J \cap L_i$ for which $x \prec_i y$. Now $(\mathbf{R}\Im L)_i$ consists of these regular ideals and $(\mathbf{R}\Im L)_0$ is the subframe of $\Im L_0$ generated by $(\mathbf{R}\Im L)_1 \cup (\mathbf{R}\Im L)_2$. The compactification is given by the join map $\bigvee : \mathbf{R}\Im L \longrightarrow L$.

Proposition 0.2.10. *Let $h : M \longrightarrow L$ be a biframe compactification and q_i the right adjoint of $h|_{M_i}$. Suppose that $x \vee y = e$ for $x \in L_i$, $y \in L_j$ implies that $q_i(x) \vee q_j(y) = e$. Then L is a normal regular biframe and the compactification $h : M \longrightarrow L$ is isomorphic to $\bigvee : \mathbf{R}\Im L \longrightarrow L$.*

Proof: We give a brief summary of the proof.

Normality of L is a consequence of normality of M and the identity satisfied by the right adjoint in the hypothesis. The universal property of this compactification gives us a biframe homomorphism $g : M \longrightarrow \mathbf{R}\Im L$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{R}\Im L \arrow[r, "\vee"] & L \\
 & M \arrow[u, "h"'] \arrow[ul, "g"]
\end{tikzcd}$$

It is shown that g is an isomorphism. ■

As a consequence of the above two results the compactifications $J_L : \mathbf{CR}\Im L \longrightarrow L$ and $\bigvee : \mathbf{R}\Im L \longrightarrow L$ are isomorphic whenever L is normal and regular.

Remark: The definition of $\Im$ used here is different to the one used in [3]. There

$$(\tilde{\Im} L)_i = \{J \in \Im L_0 : \bigvee J \in L_i\}$$

$(\tilde{\Im} L)_0 =$ the subframe of $\Im L_0$ generated by $(\tilde{\Im} L)_1 \cup (\tilde{\Im} L)_2$. Then $(\Im L)_i \subseteq (\tilde{\Im} L)_i$ but $\Im$ and $\tilde{\Im}$ need not coincide even for regular biframes.

CHAPTER 1

ALEXANDROFF BISPACES

We study closely the category **BIσFRM** of bi σ-frames and bi σ-frame homomorphisms. Alexandroff bispaces are defined using bi σ-frames and then characterized in terms of cozero sets (cf. [14]). The dual adjunction between regular σ-frames and Alexandroff spaces [13] is extended to one between regular bi σ-frames and Alexandroff bispaces. The largest duality contained in this dual adjunction defines the realcompact Alexandroff bispaces and we prove that an Alexandroff bispace X is realcompact iff the Alexandroff space generated by the union of its first and second parts is realcompact.
The adjunction between Alexandroff bispaces and completely regular bispaces [13] is given. This is achieved by invoking a Urysohn's lemma for Alexandroff bispaces.

1.1 Regular bi σ-frames and Alexandroff bispaces

We start by defining the category **BIσFRM**.

Definition 1.1.1: *A bi σ-frame is a triple $M = (M_0, M_1, M_2)$ in which M_0 is a σ-frame and M_i, $i \in \{1,2\}$ are sub σ-frames of M_0 such that $M_1 \cup M_2$ generates M_0 (i.e each $m \in M_0$ is a countable join of finite meets from $M_1 \cup M_2$.) We write $M_0 = M_1 \vee M_2$. A bi σ-frame homomorphism $h : M \longrightarrow N$ is a σ-frame homomorphism $h : M_0 \longrightarrow N_0$ such that $h(M_i) \subseteq N_i$ $i \in \{1,2\}$.* **BIσFRM** *will denote the category of bi σ-frames and bi σ-frame homomorphisms.*

The definition of regularity is similar to that for biframes except that arbitrary joins are replaced by countable joins. For the sake of clarity we write the following:

Definition 1.1.2: *A bi σ-frame M is regular if there is a sequence $\{z_n\}$ such that $z_n \prec_i x$ $(n \in \mathbb{N})$ and $x = \bigvee\{z_n | n \in \mathbb{N}\}$ for all $x \in L_i, i \in \{1,2\}$.*

Lemma 1.1.3: *Any homomorphic image of a regular bi σ-frame is regular.*

Proof. Suppose $h : M \longrightarrow L$ is a bi σ-frame homomorphism and onto. Then h preserves $\prec_i$ for $i \in \{1,2\}$. Indeed, if $x \prec_i a$ with separating element $y \in M_j$, then $h(x) \prec_i h(a)$ with separating element $h(y) \in L_j$. Since L is an image of M, $\ell \in L_i$ implies that $\ell = h(m)$ for some $m \in M$. Furthermore, since M is regular, $h(m) = h\left(\bigvee\{x_n | x_n \prec_i m\}\right)$. Consequently $h(m) = \ell = \bigvee\{h(x_n) | h(x_n) \prec_i h(m) = \ell\}$. Thus L is regular. ■

Let **Reg BIσFRM** denote the full subcategory of **BIσFRM** whose objects are the regular bi σ-frames.

The next result extends the one (in [5]) for σ-frames and the proof is similar.

Proposition 1.1.4: *Any regular bi σ-frame L is normal.*

Proof. Suppose $a \vee b = e$ for some $a \in L_j$ and $b \in L_i$. Since L is regular we may write $a = \bigvee\{a_n | a_n \prec_j a\}$ and $b = \bigvee\{b_n | b_n \prec_i b\}$ in such a way that $a_n \leq a_{n+1}$ and $b_n \leq b_{n+1}$. For each n let $x_n \in L_i$ and $y_n \in L_j$ be separating elements for $a_n \prec_j a$ and $b_n \prec_i b$, respectively. Put $x = \bigvee(x_n \wedge b_n), y = \bigvee(y_n \wedge a_n)$. Observe that $x \in L_i$ and $y \in L_j$. Then $x \vee a = e$ and $b \vee y = e$. We have that

$$\begin{aligned} x \wedge y &= \bigvee_{n \in \mathbb{N}} (x_n \wedge b_n) \wedge \bigvee_{m \in \mathbb{N}} (y_m \wedge a_m) \\ &= \bigvee_{n,m \in \mathbb{N}} [(x_n \wedge a_m) \wedge (b_n \wedge y_m)] \\ &= 0, \end{aligned}$$

for if $n \leq m$, then $b_n \leq b_m$ and $a_n \leq a_m$, thus $b_n \wedge y_m \leq b_m \wedge y_m = 0$, similarly $m \leq n$ implies that $x_n \wedge a_m = 0$, which completes the proof.

Corollary 1.1.5: *For a regular bi σ-frame L, $\prec_i$ interpolates.*

Proof. Suppose that $a, b \in L_i$ with $a \prec_i b$. Take $c \in L_j$, $i \neq j$ such that $a \wedge c = 0$ and $b \vee c = e$. Normality of L gives us $u \in L_j, v \in L_i$ with $u \wedge v = 0$ and $b \vee u = e = c \vee v$. Thus c and u are separating elements for $a \prec_i v$ and $v \prec_i b$ so that $\prec_i$ interpolates. ■

As a consequence of the previous result along with the axiom of countable dependent choice, $\prec_i = \prec\!\prec_i$ and we conclude that regularity coincides with what would be called complete regularity in bi σ-frames as in σ-frames.

We now define the category **BIAlex** of Alexandroff bispaces and cozero maps. These bispaces will be shown to be the appropriate spatial objects in this setting.

Definition 1.1.6: *An* **Alexandroff bispace** *is defined to be a triple* $X = (|X|, \mathcal{A}_1, \mathcal{A}_2)$ *with* $\mathcal{A}_i$ *subframes of* $\mathcal{P}X$ *such that* $(\mathcal{A}_0X, \mathcal{A}_1, \mathcal{A}_2)$ *is a regular bi* σ*-frame, where* $\mathcal{A}_0X = \mathcal{A}_1 \vee \mathcal{A}_2$ *is taken in* $\mathcal{P}X$ *and* $|X|$ *is the underlying set of* $\mathcal{A}_0X$*. A cozero map* $f : (|X|, \mathcal{A}_1, \mathcal{A}_2) \longrightarrow (|Y|, \mathcal{B}_1, \mathcal{B}_2)$ *is a map* $f : |X| \longrightarrow |Y|$ *such that* $f^{-1}(B) \in \mathcal{A}_i \, (B \in \mathcal{B}_i)$*. We call members of* $\mathcal{A}_0X$ *cozero sets.*

In the sequel we will write X for $|X|$ as no confusion should arise.

The latter terminology will be justified in Theorem 1.3.8 below.

Since the total part of a regular bi σ-frame is regular, it is immediate from the above definition that if X is an Alexandroff bispace, then $\mathcal{A}_0X$ is a regular σ-frame.

The following characterisation of Alexandroff bispaces is the bispace analogue of the characteristic properties of the zero set spaces of Gordon [14]. Here it makes sense to look at the properties expressed in terms of the complements of what Gordon calls zero sets with the separation of points axiom omitted.

Proposition 1.1.7: $(|X|, \mathcal{A}_1, \mathcal{A}_2)$ *is an Alexandroff bispace if and only if the following properties are satisfied for* $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{P}X$*:*

(1) $\mathcal{A}_1, \mathcal{A}_2$ *contain* $\emptyset$ *and* X*, and are closed under finite intersections and countable unions.*

(2) If $A \in \mathcal{A}_i$*,* $B \in \mathcal{A}_j, j \neq i$ *such that* $X = A \cup B$*, then there exist* $A' \in \mathcal{A}_i$ *and* $B' \in \mathcal{A}_j$*, such that* $A' \cap B' = \emptyset$ *and* $X \setminus A \subseteq B'$*,* $X \setminus B \subseteq A'$ *(i.e.,* $A \cup B' = X = B \cup A'$*).*

(3) If $A \in \mathcal{A}_i$*, then there exists a sequence* $\{A_n\}$ *in* $\mathcal{A}_j$*,* $i \neq j$ *such that* $A = \bigcup\limits_n X \setminus A_n$*.*

Proof: $\Longrightarrow$: (1) and (2) follow immediately from the fact that $(\mathcal{A}_0X, \mathcal{A}_1, \mathcal{A}_2)$ is a regular bi σ-frame and consequently normal.

For (3), let $A \in \mathcal{A}_i$. Then there exists a sequence $\{A_n\} \subset \mathcal{A}_i$ such that $A = \bigcup\limits_n A_n$ and $A_n \prec_i A$. Thus, there exists $C_n \in \mathcal{A}_j$ with $A_n \cap C_n = \emptyset$ and $C_n \cup A = X$. Thus, $A_n \subseteq X \setminus C_n \subseteq A$, consequently $A = \bigcup X \setminus C_n$.

$\Longleftarrow$: By (1) it only remains to prove that $(\mathcal{A}_0X, \mathcal{A}_1, \mathcal{A}_2)$ is regular. Suppose $A \in \mathcal{A}_i$. Then there exists a sequence $\{A_n\}$ in $\mathcal{A}_j$, $i \neq j$ such that $A = \bigcup\limits_n X \setminus A_n$ by (3). This implies that $A_n \cup A = X$, for all $n \in \mathbb{N}$. By (2) there exist $\{C_n\} \subset \mathcal{A}_i$ and $\{D_n\} \subset \mathcal{A}_j$ such that $C_n \cap D_n = \emptyset$ and $A \cup D_n = X = A_n \cup C_n$ for each n. Now $A = \bigcup\limits_n C_n$ since we have

$X \setminus A_n \subseteq C_n \subseteq X \setminus D_n \subseteq A$ for all n. ∎

1.2 Adjoint functors

We define functors between **BIAlex** and **Reg BIσFRM** and prove that they are adjoint on the right. These are analogous to the open set and spectrum functors between **BIFRM** and **BITOP** introduced in [3].

The contravariant functors $\mathcal{A}$ and $\overline{\underline{\Psi}}$ are defined by:

$$\begin{array}{rrcl} \mathcal{A}: & \mathbf{BIAlex} & \longrightarrow & \mathbf{Reg\,BI\sigma FRM} \\ & (X, \mathcal{A}_1, \mathcal{A}_2) & \longmapsto & (\mathcal{A}_0X, \mathcal{A}_1, \mathcal{A}_2) \\ & f & \longmapsto & f^{-1} \end{array}$$

$$\begin{array}{rrcl} \overline{\underline{\Psi}}: & \mathbf{Reg\,BI\sigma FRM} & \longrightarrow & \mathbf{BIAlex} \\ & (M_0, M_1, M_2) & \longmapsto & (|\overline{\underline{\Psi}}M_0|, \overline{\underline{\Psi}}_{M_1}, \overline{\underline{\Psi}}_{M_2}) \\ & f & \longmapsto & f^{-1} \end{array}$$

Where $\overline{\underline{\Psi}}M_0$ consists of all σ-prime filters on M_0, $\overline{\underline{\Psi}}_{M_i} = \{\overline{\underline{\Psi}}_a\}_{a \in M_i}$ and $\overline{\underline{\Psi}}_a = \{F : a \in F \in \overline{\underline{\Psi}}M_0\}$. $\mathcal{A}$ is easily seen to be functorial, and, with the following result, so is $\overline{\underline{\Psi}}$.

Proposition 1.2.1: *$(|\overline{\underline{\Psi}}M_0|, \overline{\underline{\Psi}}_{M_1}, \overline{\underline{\Psi}}_{M_2})$ is an Alexandroff bispace.*

Proof: We first show that $\overline{\underline{\Psi}}_a \cap \overline{\underline{\Psi}}_b = \overline{\underline{\Psi}}_{a\wedge b}$ for $a, b \in M_0$. Let $J \in \overline{\underline{\Psi}}_a \cap \overline{\underline{\Psi}}_b$. Then $a, b \in J$ implies that $a \wedge b \in J$ therefore $J \in \overline{\underline{\Psi}}_{a\wedge b}$.

Conversely, let $G \in \overline{\underline{\Psi}}_{a\wedge b}$, then $a, b \in G$ since G is an upset. Thus, $G \in \overline{\underline{\Psi}}_a \cap \overline{\underline{\Psi}}_b$.

Next, we show that $\bigcup\limits_n \overline{\underline{\Psi}}_{a_n} = \overline{\underline{\Psi}}_{\bigvee a_n}$. Let $J \in \bigcup\limits_n \overline{\underline{\Psi}}_{a_n}$ for $\{a_n\} \subseteq M_0$. Then $J \in \overline{\underline{\Psi}}_{a_k}$ for some k. Then $\bigvee\limits_n a_n \in J$ because J is an upset. Thus, $J \in \overline{\underline{\Psi}}_{\bigvee a_n}$ and, consequently, $\bigcup\limits_n \overline{\underline{\Psi}}_{a_n} \subseteq \overline{\underline{\Psi}}_{\bigvee a_n}$.

On the other hand, if $G \in \overline{\underline{\Psi}}_{\bigvee a_n}$ for $\{a_n\} \subseteq M_0$, then $\bigvee\limits_n a_n \in G$, which implies that $a_k \in G$ for some k since G is σ-prime. Thus, $G \in \bigcup\limits_n \overline{\underline{\Psi}}_{a_n}$.

We prove that $\overline{\underline{\Psi}}_{M_1} \vee \overline{\underline{\Psi}}_{M_2} = \overline{\underline{\Psi}}_{M_0}$. Let $\overline{\underline{\Psi}}_a \in \overline{\underline{\Psi}}_{M_0}$, then $\overline{\underline{\Psi}}_a = \overline{\underline{\Psi}}_{\bigvee(c_n \wedge d_n)}$ for $\{c_n\} \subseteq M_i$, $\{d_n\} \subseteq M_j, i \neq j$ and $i, j \in \{1, 2\}$

$$\iff \overline{\underline{\Psi}}_a = \bigcup_n \overline{\underline{\Psi}}_{(c_n \wedge d_n)}$$

$$\iff \overline{\underline{\Psi}}_a = \bigcup_n (\overline{\underline{\Psi}}_{c_n} \cap \overline{\underline{\Psi}}_{d_n})$$

$$\iff \overline{\underline{\Psi}}_a \in \overline{\underline{\Psi}}_{M_1} \vee \overline{\underline{\Psi}}_{M_2}.$$

The fact that $\overline{\underline{\Psi}}_{M_i}$ $(i \in \{1, 2\})$ are sub σ-frames of $\overline{\underline{\Psi}}_{M_0}$ is a consequence of the above identities and that (M_0, M_1, M_2) is a bi σ-frame.

Lastly, regularity of $(|\overline{\underline{\Psi}}_{M_0}|, \overline{\underline{\Psi}}_{M_1}, \overline{\underline{\Psi}}_{M_2})$ follows immediately from the observation that the map $\varepsilon_M : M \longrightarrow \mathcal{A}\overline{\underline{\Psi}}M$ given by $a \mapsto \overline{\underline{\Psi}}_a$ is a homomorphism as the above identities show and the fact that M is regular. ∎

Let $\eta_X : X \longrightarrow \overline{\underline{\Psi}}\mathcal{A}X$

$\qquad x \mapsto \mathcal{A}(x)$

where X is an Alexandroff bispace and $\mathcal{A}(x) = \{U : x \in U \in \mathcal{A}_0X\}$ is a σ-prime filter of $\mathcal{A}_0X$. Let ε_M be the homomorphism defined in the above proposition, then:

Proposition 1.2.2: *$\mathcal{A}$ and $\overline{\underline{\Psi}}$ are adjoint on the right with adjunctions η_X and ε_M.*

Proof: η_X is a cozero map:

If $C \in \mathcal{A}_i \quad i \in \{1,2\}$, then

$$\begin{aligned} \eta_X^{-1}(\overline{\underline{\Psi}}_C) &= \{x : \eta_X(x) \in \overline{\underline{\Psi}}_C\} \\ &= \{x : \mathcal{A}(x) \in \overline{\underline{\Psi}}_C\} \\ &= \{x : C \in \mathcal{A}(x)\} \\ &= C. \end{aligned}$$

Naturality of η:

Suppose $g : X \longrightarrow Y$ is any cozero map and $x \in X$. We show that the following diagram commutes,

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ \overline{\underline{\Psi}}\mathcal{A}X \arrow[r, "\overline{\underline{\Psi}}\mathcal{A}g"] & \overline{\underline{\Psi}}\mathcal{A}Y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & g(x) \arrow[d, mapsto] \\ \mathcal{A}(x) \arrow[r, mapsto] & \mathcal{A}(g(x)) \end{tikzcd} .$$

We have that

$$\begin{aligned} \overline{\underline{\Psi}}\mathcal{A}g(\mathcal{A}(x)) &= (\mathcal{A}g)^{-1}(\mathcal{A}(x)) \\ &= \{C \in \mathcal{A}Y : (\mathcal{A}g)(C) \in \mathcal{A}(x)\} \\ &= \{C \in \mathcal{A}Y : g^{-1}(C) \in \mathcal{A}(x)\} \\ &= \{C \in \mathcal{A}Y : x \in g^{-1}(C)\} \\ &= \{C \in \mathcal{A}Y : g(x) \in C\} \\ &= \mathcal{A}(g(x)). \end{aligned}$$

Naturality of ε:

If $f : M \longrightarrow L$ is any homomorphism and $a \in M$ i.e., $a \in |M_0|$, we must check that the square

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\varepsilon_M"'] & L \arrow[d, "\varepsilon_L"] \\ \mathcal{A}\overline{\underline{\Psi}}M \arrow[r, "\mathcal{A}\overline{\underline{\Psi}}f"] & \mathcal{A}\overline{\underline{\Psi}}L \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & f(a) \arrow[d, mapsto] \\ \overline{\underline{\Psi}}_a \arrow[r, mapsto] & \overline{\underline{\Psi}}_{f(a)} \end{tikzcd} .$$

commutes. Thus,

$$\begin{aligned}
(\mathcal{A}\overline{\underline{\Psi}}f)(\overline{\underline{\Psi}}_a) &= (\overline{\underline{\Psi}}f)^{-1}(\overline{\underline{\Psi}}_a) \\
&= \{P \in \overline{\underline{\Psi}}L : (\overline{\underline{\Psi}}f)(P) \in \overline{\underline{\Psi}}_a\} \\
&= \{P \in \overline{\underline{\Psi}}L : f^{-1}(P) \in \overline{\underline{\Psi}}_a\} \\
&= \{P \in \overline{\underline{\Psi}}L : a \in f^{-1}(P)\} \\
&= \{P \in \overline{\underline{\Psi}}L : f(a) \in P\} \\
&= \overline{\underline{\Psi}}_{f(a)}.
\end{aligned}$$

We verify the following identities:

(1) $\overline{\underline{\Psi}}\varepsilon_M \circ \eta_{\overline{\underline{\Psi}}M} = id_{\overline{\underline{\Psi}}M}$, for each $M \in$ **Reg BIσFRM**, and

(2) $\mathcal{A}\eta_X \circ \varepsilon_{\mathcal{A}X} = id_{\mathcal{A}X}$, for each $X \in$ **BIAlex**.

For (1) let $P \in \overline{\underline{\Psi}}M_0$, then $\eta_{\overline{\underline{\Psi}}M}(P) = \mathcal{A}(P)$

$$\begin{aligned}
&= \{\overline{\underline{\Psi}}_a : a \in P\}. \\
\text{Thus } \overline{\underline{\Psi}}\varepsilon_M \circ \eta_{\overline{\underline{\Psi}}M}(P) &= (\varepsilon_M)^{-1}\{\overline{\underline{\Psi}}_a : a \in P\} \\
&= \{b : \varepsilon_M(b) = \overline{\underline{\Psi}}_a, \quad \text{for some} \quad a \in P\} \\
&= \{b : \overline{\underline{\Psi}}_b = \overline{\underline{\Psi}}_a, \quad \text{for some} \quad a \in P\} \\
&= P
\end{aligned}$$

(since if $\overline{\underline{\Psi}}_a = \overline{\underline{\Psi}}_b$ for $a \in P$, then $P \in \overline{\underline{\Psi}}_b$ and hence $b \in P$. The reverse inclusion is easy).

For (2) take $C \in \mathcal{A}_0X$ where $X = (|X|, \mathcal{A}_1, \mathcal{A}_2)$. Then

$$\begin{aligned}
\mathcal{A}\eta_X \circ \varepsilon_{\mathcal{A}X}(C) &= \mathcal{A}\eta_X(\overline{\underline{\Psi}}_C) \\
&= (\eta_X)^{-1}(\overline{\underline{\Psi}}_C) \\
&= \{x \in X : \eta_X(x) \in \overline{\underline{\Psi}}_C\} \\
&= \{x \in X : \mathcal{A}(x) \in \overline{\underline{\Psi}}_C\} \\
&= \{x \in X : C \in \mathcal{A}(x)\} \\
&= \{x \in X : x \in C\} \qquad \blacksquare
\end{aligned}$$

1.3 Completely regular bitopological spaces and Alexandroff bispaces.

Pairwise completely regular bitopological spaces were first introduced by Lane [19] and Fletcher [10] independently and later studied by, amongst others, Nachbin and Salbany [25]. We will denote the category of pairwise completely regular bitopological spaces and continuous maps by **BICRG**. We now show that the adjunction between **ALEX** (Alexandroff spaces) and **CRG** (completely regular topological spaces) in [13] lifts to the one between **BIAlex** and **BICRG**.

Notation: We denote by $(\mathbb{R}, \ell, u)$ the real line with the lower topology $\ell = \{(a, \infty) : a \in \mathbb{R}\}$ and the upper topology $u = \{(-\infty, b) : b \in \mathbb{R}\}$. (I, ℓ, u) is the unit interval $I = [0, 1]$ together with the lower topology $\ell = \{(a, 1] : a \in I\}$ and the upper topology $u = \{[0, b) : b \in I\}$. Basically (I, ℓ, u) is the subspace of $(\mathbb{R}, \ell, u)$. In the sequel we will use ℓ and u for both I and $\mathbb{R}$ where no confusion will arise.

Definition 1.3.1: *We say that a bitopological space (X, P, Q) is pairwise completely regular if*

(a) *for each P-closed set W and x_0 with $x_0 \notin W$, there is a continuous map $f : (X, P, Q) \longrightarrow (I, \ell, u)$ such that $f(x_0) = 1$ and $f(x) = 0$ on W and,*

(b) *for each Q-closed set V not containing x_0, there is a continuous map $g : (X, P, Q) \longrightarrow (I, \ell, u)$ such that $g(x_0) = 0$ and $g(x) = 1$ on V.*

We will show that $a_t X$ is an Alexandroff bispace whenever X is a pairwise completely regular bitopological space, and conversely, that tX is a pairwise completely regular bitopological space whenever X is an Alexandroff bispace.

Definition 1.3.2: *We define $a_t(X, P, Q) = (X, \mathcal{C}_P, \mathcal{C}_Q)$ where*
$\mathcal{C}_P = \{h^{-1}(0, \infty) | \, h : (X, P, Q) \longrightarrow (\mathbb{R}, \ell, u)$ *is a continuous map*$\}$ *and*
$\mathcal{C}_Q = \{h^{-1}(-\infty, 0) | \, h : (X, P, Q) \longrightarrow (\mathbb{R}, \ell, u)$ *is a continuous map*$\}$, *also*
$t(X, \mathcal{A}_1, \mathcal{A}_2) = (X, \mathcal{A}_1^*, \mathcal{A}_2^*)$ *where* $\mathcal{A}_i^* (i \in \{1, 2\})$ *are topologies on X generated by $\mathcal{A}_i, i \in$*

$\{1, 2\}$, *respectively.*

(We will show that for all $X \in$ **BICRG** $a_t X \in$ **BIAlex** and for all $X \in$ **BIAlex** $tX \in$ **BICRG**.

The justifications for the above definition are given below.

We require the following definitions for a bitopological space (X, P, Q) for the next result.

Definition 1.3.3: [25]. *Let $\mathcal{B}_P$ and $\mathcal{B}_Q$ be bases for P and Q, respectively.*

1. *If $\mathcal{B}_P$ and $\mathcal{B}_Q$ are closed under finite intersections and finite unions and whenever $N_1 \in \mathcal{B}_P$ and $N_2 \in \mathcal{B}_Q$ are such that $N_1 \cup N_2 = X$, then there are $N_3 \in \mathcal{B}_P$ and $N_4 \in \mathcal{B}_Q$ such that $X \setminus N_1 \subseteq N_4 \subseteq X \setminus N_3 \subseteq N_2$, then $(\mathcal{B}_P, \mathcal{B}_Q)$ is called a pairwise normal base for (X, P, Q).*

2. *$(\mathcal{B}_P, \mathcal{B}_Q)$ is called pairwise $\Re_0$ if for each $x \in N \in \mathcal{B}_P$, repectively $\mathcal{B}_Q$, there is $N' \in \mathcal{B}_Q$, respectively $\mathcal{B}_P$, such that $x \notin N'$ and $N' \cup N = X$.*

For the properties of lower and upper semicontinuous maps which ensure that the maps constructed in the next proof are again continuous from (X, P, Q) to $(\mathbb{R}, \ell, u)$, see [9], 1.7.14.

Proposition 1.3.4: *$(X, \mathcal{C}_P, \mathcal{C}_Q)$ is an Alexandroff bispace.*

Proof: We check that $(\mathcal{C}_P \vee \mathcal{C}_Q, \mathcal{C}_P, \mathcal{C}_Q)$ is a regular bi σ-frame. We show that $\mathcal{C}_P$ (and, similarly, $\mathcal{C}_Q$) is a sub σ-frame of $\mathcal{C}_P \vee \mathcal{C}_Q$.

(a) If $g^{-1}(0, \infty), f^{-1}(0, \infty) \in \mathcal{C}_P$ then $g^{-1}(0, \infty) \cap f^{-1}(0, \infty) = ((g \vee \mathbf{0})(f \vee \mathbf{0}))^{-1}(0, \infty) \in \mathcal{C}_P$.

(b) Suppose $f_n^{-1}(0, \infty) \in \mathcal{C}_P, n \in J$ and J countable. We may assume that $f_n \geq 0$ for each n. By [9], 1.7.14 we have that $g = \sum f_n \wedge 2^{-n}$ is continuous from (X, P, Q) to

$(\mathbb{R}, \ell, u)$. Thus $\bigcup_n f_n^{-1}(0,\infty) = g^{-1}(0,\infty)$, where g is continuous. Hence $\mathcal{C}_P$ is closed under countable unions and is thus a σ-frame.

For regularity, let $U \in \mathcal{C}_P$. Then $U = f^{-1}(0,\infty)$ for some continuous $f : (X, P, Q) \to (\mathbb{R}, \ell, u)$. Observe that $U = \bigcup_n f^{-1}(\frac{1}{n},\infty)$, $f^{-1}(\frac{1}{n},\infty) \cap f^{-1}(-\infty,\frac{1}{2n}) = \emptyset$ and $f^{-1}(-\infty,\frac{1}{2n}) \cup f^{-1}(0,\infty) = X$. ■

The following theorem is vital for our proof that $(X, \mathcal{A}_1^*, \mathcal{A}_2^*) \in$ **BICRG**.

Theorem 1.3.5: [25]. *A bitopological space X is pairwise completely regular iff it has a pairwise normal and pairwise $\Re_0$ base.*

Proof: We outline the proof given in [25]. The proof uses a base of closed sets.

$\Longrightarrow$: Let (X, P, Q) be pairwise completely regular. Suppose, $\mathcal{B}_P = \{Z(f) | f : (X, P, Q) \longrightarrow (I, \ell, u)\}$ and $\mathcal{B}_Q = \{Z(g) | g : (X, P, Q) \longrightarrow (I, u, \ell)\}$ where $Z(f) = f^{-1}(0)$ and $Z(g) = g^{-1}(0)$. It is shown that $(\mathcal{B}_P, \mathcal{B}_Q)$ is a pairwise normal and $\Re_0$ base for (X, P, Q).

$\Longleftarrow$: We start with a pairwise normal and $\Re_0$ base $(\mathcal{B}_P, \mathcal{B}_Q)$ for (X, P, Q). It is shown that a disjoint point and a P-closed or Q-closed set can be separated by a continuous map $g : (X, P, Q) \longrightarrow (I, u, \ell)$. The construction of the map g is modelled on that of the proof of Urysohn's Lemma. ■

Let $(X, \mathcal{A}_1, \mathcal{A}_2)$ be an Alexandroff bispace. $(\mathcal{A}_1, \mathcal{A}_2)$ is pairwise normal, for if $N_1 \in \mathcal{A}_1$ and $N_2 \in \mathcal{A}_2$ such that $N_1 \cup N_2 = X$, then there exist $N_3 \in \mathcal{A}_1$ and $N_4 \in \mathcal{A}_2$, such that $N_3 \cap N_4 = \emptyset$ and $X \setminus N_1 \subseteq N_4, X \setminus N_2 \subseteq N_3$. $N_3 \cap N_4 = \emptyset$ implies $N_4 \subseteq X \setminus N_3$ and $X \setminus N_2 \subseteq N_3$ implies $X \setminus N_3 \subseteq N_2$, we thus have $X \setminus N_1 \subseteq N_4 \subseteq X \setminus N_3 \subseteq N_2$. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ are closed under finite intersections and unions. $(\mathcal{A}_1, \mathcal{A}_2)$ is an $\Re_0$ base, since if $x \in N \in \mathcal{A}_1$, then there is a sequence $\{A_n\}$ in $\mathcal{A}_2$ such that $N = \bigcup_n (X \setminus A_n)$. $x \in N$ implies that there exists n_0 such that $x \in X \setminus A_{n_0}$, thus $x \notin A_{n_0} \in \mathcal{A}_2$, but $X \setminus A_{n_0} \subseteq N$ implies $N \cup A_{n_0} = X$.

Since $(\mathcal{A}_1, \mathcal{A}_2)$ is both a pairwise normal and $\Re_0$ base whenever $(X, \mathcal{A}_1, \mathcal{A}_2)$ is an Alexandroff bispace, $(X, \mathcal{A}_1^*, \mathcal{A}_2^*) \in$ **BICRG** by the above theorem.

It is now clear that a_t and t defined above are functors whose action on maps is to leave them fixed as set maps. We need a number of results before we are able to prove that a_t is left adjoint to t.

Proposition 1.3.6: *$(\mathbb{R}, \ell, u)$ is an Alexandroff bispace.*

Proof: It suffices to prove that $(\mathcal{O}\mathbb{R}, \ell, u)$ is a regular bi σ-frame where $\mathcal{O}\mathbb{R}$ denotes all open sets in $\mathbb{R}$. We already know that $\mathcal{O}\mathbb{R}$ is a σ-frame and u, ℓ are sub σ-frames of $\mathcal{O}\mathbb{R}$. We show regularity: Suppose that $A \in u$, then $A = (-\infty, b)$ for some $b \in \mathbb{R}$, thus $A = \bigcup\limits_n (-\infty, b - \frac{1}{n})$. Note that $(-\infty, b - \frac{1}{n}) \in u$, also for each n there exists $(b - \frac{1}{2n}, \infty) \in \ell$ such that $(-\infty, b - \frac{1}{n}) \cap (b - \frac{1}{2n}, \infty) = \emptyset$ and $(-\infty, b) \cup (b - \frac{1}{2n}, \infty) = \mathbb{R}$. This yields $(-\infty, b - \frac{1}{n}) \prec_2 A \ \ \forall \, n \in \mathbb{N}$. ■

Notation. (1) We denote the set of all cozero maps $f : (X, \mathcal{A}_1, \mathcal{A}_2) \longrightarrow (\mathbb{R}, \ell, u)$ by $S(X)$ and the bounded (i.e., bounded as maps) cozero maps from X to $\mathbb{R}$ by $S^*(X)$.

(2) Let $(X, \mathcal{B}_1, \mathcal{B}_2)$ be an Alexandroff bispace. We denote the collection of all sets of the form $X \setminus A$ where $A \in \mathcal{B}_i$ by $\mathcal{B}_i^c$ with $i \in \{1, 2\}$.

We extend Urysohn's Lemma for Alexandroff spaces to Alexandroff bispaces. The proof is modelled on that for pairwise normal bitopological spaces [25].

Urysohn's Lemma 1.3.7: *Suppose $(X, \mathcal{B}_1, \mathcal{B}_2)$ is an Alexandroff bispace, let $A \in \mathcal{B}_1, B \in \mathcal{B}_2$ be such that $A \cup B = X$. Then there exists a cozero map $f : (X, \mathcal{B}_1, \mathcal{B}_2) \longrightarrow (I, \ell, u)$ such that $f^{-1}(0, 1] \subseteq A$ and $f^{-1}[0, 1) \subseteq B$.*

Proof: Assume the hypotheses of the lemma and that $A' = X \setminus A, B' = X \setminus B$, then $A' \cap B' = \emptyset$. By property (2) in 1.1.7, there exists $C_{\frac{1}{2}} \in \mathcal{B}_1, D_{\frac{1}{2}} \in \mathcal{B}_2$ such that $C_{\frac{1}{2}} \cap D_{\frac{1}{2}} = \emptyset$ and $A' \subset D_{\frac{1}{2}}, B' \subset C_{\frac{1}{2}}$. In short $A' \subset D_{\frac{1}{2}} \subset X \setminus C_{\frac{1}{2}} \subset B$. Note that $C_{\frac{1}{2}} \in \mathcal{B}_1$ and $B \in \mathcal{B}_2$ also $D_{\frac{1}{2}} \cup A = X$. Another application of (2) yields $D_{\frac{1}{4}} \in \mathcal{B}_2$ and $C_{\frac{1}{4}} \in \mathcal{B}_1$ such that

$C_{\frac{1}{4}} \cap D_{\frac{1}{4}} = \emptyset, C'_{\frac{1}{4}} \subset D_{\frac{1}{2}}$ and $A' \subset D_{\frac{1}{4}}$. Thus $A' \subset D_{\frac{1}{4}} \subset X \setminus C_{\frac{1}{4}} \subset D_{\frac{1}{2}} \subset X \setminus C_{\frac{1}{2}} \subset B$. Continuing inductively we can find $C_{\frac{k}{2^n}} \in \mathcal{B}_1$ and $D_{\frac{k}{2^n}} \in \mathcal{B}_2 (k = 1, 2, \ldots, 2^n - 1)$ such that

$$A' \subset D_{\frac{1}{2^n}} \subset X \setminus C_{\frac{1}{2^n}} \subset D_{\frac{2}{2^n}} \subset \ldots \subset X \setminus C_{\frac{(k-1)}{2^n}} \subset D_{\frac{k}{2^n}} \subset \ldots \subset B.$$

Set $D_{\frac{2m}{2^{n+1}}} = D_{\frac{m}{2^n}}$ and $C_{\frac{2m}{2^{n+1}}} = C_{\frac{m}{2^n}}$ and introduce $D_{\frac{(2m+1)}{2^{n+1}}} \in \mathcal{B}_2$ and $C_{\frac{(2m+1)}{2^{n+1}}} \in \mathcal{B}_1$ by requiring that

$$A' \subset D_{\frac{2m+1}{2^{n+1}}} \subset X \setminus C_{\frac{2m+1}{2^{n+1}}} \subset D_{\frac{2m+2}{2^{n+1}}}$$

and that

$$X \setminus C_{\frac{2m}{2^{n+1}}} \subset D_{\frac{(2m+1)}{2^{n+1}}} \subset X \setminus C_{\frac{(2m+1)}{2^{n+1}}} \subset D_{\frac{2m+2}{2^{n+1}}}.$$

The subscripts of D_r and C_r form a set:

$$\underline{R} = \{\frac{m}{2^n} : 1 \leq m \leq 2^n, m, n \in \mathbb{N}\}.$$

Let $D_r \in \mathcal{B}_2$ and $C_r \in \mathcal{B}_1$ with $r, s \in \underline{R}$ such that $r < s$, then

$$A' \subset D_r \subset X \setminus C_r \subset D_s \subset X \setminus C_s \subset B.$$

Define $f(x) = \begin{cases} \inf\{r : r \in \underline{R}, x \in D_r\} & x \in B \\ 1 & \text{otherwise.} \end{cases}$

Then $f(x) = 0$ for all $x \in A'$, $f(x) = 1$ for all $x \in B'$, and $0 \leq f(x) \leq 1$ for all $x \in X$. Thus $f(x) > 0$ implies that $x \in A$ since $X = A \cup B$, hence $f^{-1}(0, 1] \subseteq A$. We also have $f(x) < 1$ implies $x \in B$, consequently $f^{-1}[0, 1) \subseteq B$. We show that f is a cozero map. We must check that $f^{-1}[0, a) \in \mathcal{B}_2$ and $f^{-1}(b, 1] \in \mathcal{B}_1$ for $a, b \in (0, 1)$. This follows from the observation that

(1) $f^{-1}[0, a) = \bigcup\{D_r : r \in \underline{R}, r < a\}$ and

(2) $f^{-1}(b, 1] = \bigcup\{C_r : r \in \underline{R}, r > b\}$.

We prove (1): If $x \in D_r, r < a$, then $f(x) \leq r < a$. Conversely, if $f(x) < a$, then by density of $\underline{R}$ in $[0, 1]$, there exists $r \in \underline{R}$ such that $f(x) < r < a$, thus $x \in D_r$.

For (2): If $b < f(x) \leq 1$, there exists $r_1, r_2 \in \underline{R}$ such that $b < r_1 < r_2 < f(x)$. In view of the fact that $r_2 < f(x)$ and the definition of f we conclude that $x \notin D_{r_2}$. Since $r_1 < r_2$ implies $X \setminus C_{r_1} \subseteq D_{r_2}$ we get that $x \notin X \setminus C_{r_1}$ (i.e. $x \in C_{r_1}$). Conversely let $x \in C_{r_x}$ with $r_x > b$. Now $r < r_x$ implies $D_r \subset D_{r_x} \subset X \setminus C_{r_x}$, also $x \notin X \setminus C_{r_x}$ implies $x \notin D_r$, for $r < r_x$ which yields $b < r_x \leq f(x)$ (i.e. $x \in f^{-1}(b, 1]$). ■

The collection of all lower (respectively, upper) cozero sets of an Alexandroff bispace $(X, \mathcal{B}_1, \mathcal{B}_2)$ (i.e., sets of form $f^{-1}(-\infty, 0)(respectively, f^{-1}(0, \infty))$ for some $f \in S(X)$, is denoted by $coz_\ell X(respectively, coz_u X)$.

Theorem 1.3.8: *Whenever* $(X, \mathcal{B}_1, \mathcal{B}_2) \in$ **BIAlex**, $\mathcal{B}_1 = coz_\ell X$ *and* $\mathcal{B}_2 = coz_u X$.

Proof: We will only show that $\mathcal{B}_1 = coz_\ell X$; the remaining proof is similar and therefore omitted.

$\supseteq$: Let $A \in coz_\ell X$, then there exists an $f \in S(X)$ such that $A = f^{-1}(-\infty, 0)$. Hence $A \in \mathcal{B}_1$.

$\subseteq$: Suppose $A \in \mathcal{B}_1$, then by property (3) of (1.1.7) $A = \bigcup_n X \setminus A_n$ for a sequence $\{A_n\}$ in $\mathcal{B}_2$. Thus $X \setminus A = \bigcap_n A_n$ and hence $(X \setminus A) \cap (X \setminus A_n) = \emptyset$ for each n. By Urysohn's Lemma, there exists $f_n \in S^*(X)$ such that

$$f_n(x) = \begin{cases} 0 & x \in X \setminus A \quad \text{and} \\ 1 & x \in X \setminus A_n \quad \text{for each} \quad n. \end{cases}$$

Observe that $X \setminus A_n \subseteq f_n^{-1}(0, \infty) \subseteq A$ and $A = \bigcup_n X \setminus A_n$, hence $A = \bigcup_n f_n^{-1}(0, \infty)$. Note that $f_n^{-1}(0, \infty) \in coz_\ell X$, for all $n \in \mathbb{N}$ and $coz_\ell X$ is closed under countable unions, (the proof is similar to that of 1.3.4(b) and therefore omitted), consequently $A \in coz_\ell X$.■

We now come to one of the crucial results in this section.

Proposition 1.3.9: $ta_t = id_{\mathbf{BICRG}}$ *and* a_t *is left adjoint to* t.

Proof: If $(X, P, Q) \in$ **BICRG**, then $ta_t(X, P, Q) = t(X, \mathcal{C}_P, \mathcal{C}_Q) = (X, P, Q)$. The last equality follows from the fact that $(\mathcal{C}_P, \mathcal{C}_Q)$ is a base for (X, P, Q). Next we prove that a_t

is left adjoint to t. Define the natural transformation

$$\eta : \mathbf{1}_{\mathbf{BICRG}} \longrightarrow ta_t$$

to be the identity. Let $X = (|X|, P, Q) \in \mathbf{BICRG}$, $Y = (|Y|, \mathcal{B}_1, \mathcal{B}_2) \in \mathbf{BIAlex}$ and suppose that

$$\begin{array}{rcl} f : X & \longrightarrow & tY \\ (|X|, P, Q) & \mapsto & (Y, \mathcal{B}_1^*, \mathcal{B}_2^*) \end{array}$$

is a continuous map.

It remains to show that there exists a cozero map $\overline{f} : a_tX \longrightarrow Y$ making the following diagram to commute,

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & ta_tX \arrow[dl, dashed, "t\overline{f}"] \\ tY & \end{tikzcd}$$

Applying the functor a_t on $f : X \longrightarrow tY$ gives us $a_tf : a_tX \longrightarrow Y$. We verify that this is indeed a cozero map.

Let $U \in \mathcal{B}_i$, then there exists a cozero map $g : (Y, \mathcal{B}_1, \mathcal{B}_2) \longrightarrow (\mathbb{R}, \ell, u)$ such that

$$U = \begin{cases} g^{-1}(0, \infty) & \text{if } \ U \in \mathcal{B}_1 \\ \\ g^{-1}(-\infty, 0) & \text{if } \ U \in \mathcal{B}_2, \end{cases}$$

by the foregoing theorem.

We have $f^{-1}(U) = (gf)^{-1}(0, \infty)$ or $(gf)^{-1}(-\infty, 0)$. In proving that $gf : (X, P, Q) \longrightarrow (\mathbb{R}, \ell, u)$ is continuous it will be sufficient to show that $f^{-1}(U) \in \mathcal{C}_P$ or $\mathcal{C}_Q$ respectively. $V \in \ell$ implies that $g^{-1}(V) \in \mathcal{B}_1 \subseteq \mathcal{B}_1^*$ since $g \in S(Y)$. Thus $(gf)^{-1}(V) \in P$ by continuity of $f : X \longrightarrow tY$. It can be similarly proved that $gf : (X, Q) \to (R, u)$ is continuous. Clearly $\overline{f} = a_tf$ is the unique map which makes the triangle commute. ■

From the above proposition we conclude that **BICRG** is embedded as a coreflective subcategory of **BIAlex** and we shall regard constructions in **BIAlex** as extensions of constructions in **BICRG**.

The next proposition follows immediately from Propositions 1.2.2 and 1.3.9.

Proposition 1.3.10: *The functors* $\sum = t\underline{\Psi} : \mathbf{Reg\ BI\sigma FRM} \longrightarrow \mathbf{BICRG}$ *and* $P = \mathcal{A}a_t : \mathbf{BICRG} \longrightarrow \mathbf{Reg\ BI\sigma FRM}$ *are adjoint on the right.*

1.4 Realcompact Alexandroff bispaces

We introduce a functor between **BIAlex** and **ALEX**.

Definition 1.4.1: *The* **wedding functor** *is defined by*

$$\begin{array}{rcl} W : \mathbf{BIAlex} & \longrightarrow & \mathbf{ALEX} \\ (X, \mathcal{A}_1, \mathcal{A}_2) & \mapsto & (X, \mathcal{A}_0 X) \\ f & \mapsto & f. \end{array}$$

An Alexandroff space X is *separated* if for any two points in X we can find a cozero set containing just one of them, i.e., the cozero sets separate points.

Gilmour showed that a separated Alexandroff space X is realcompact iff $\eta_X : X \longrightarrow \Psi\mathcal{U}X$ is an isomorphism where η_X, Ψ and $\mathcal{U}$ are analogues of our $\eta_X, \underline{\Psi}$ and $\mathcal{A}$ for Alexandroff spaces and σ-frames respectively. We will say that an Alexandroff bispace X is separated if its total part is separated. We bypass other well known characterisations of realcompactness and define realcompactness as follows:

Definition 1.4.2: *A separated Alexandroff bispace X is* **realcompact** *if $\eta_X : X \longrightarrow \underline{\Psi}\mathcal{A}X$ is an isomorphism.*

Denote by T the functor which takes the total part of a regular bi σ-frame and leaves morphisms unchanged.
The fact that the following squares commute will be useful in proving the next proposition.

$$\begin{tikzcd} \mathbf{BIAlex} \arrow[r, "\mathcal{A}"] \arrow[d, "W"'] \arrow[dr, phantom, "(1)"] & \mathbf{Reg\ BI\sigma FRM} \arrow[d, "T"] \\ \mathbf{ALEX} \arrow[r, "\mathcal{U}"'] & \mathbf{Reg\,\sigma FRM} \end{tikzcd} \qquad \begin{tikzcd} (X, \mathcal{A}_1, \mathcal{A}_2) \arrow[r, mapsto] \arrow[d, mapsto] & (\mathcal{A}_0 X, \mathcal{A}_1, \mathcal{A}_2) \arrow[d, mapsto] \\ (X, \mathcal{A}_0 X) \arrow[r, mapsto] & \mathcal{A}_0 X \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{BIAlex} \arrow[d, "W"'] \arrow[dr, phantom, "(2)"] & \mathbf{Reg\ BI\sigma FRM} \arrow[l, "\underline{\Psi}"'] \arrow[d, "T"] \\ \mathbf{ALEX} & \mathbf{Reg\,\sigma FRM} \arrow[l, "\Psi"] \end{tikzcd} \qquad \begin{tikzcd} (\underline{\Psi} M_0, \underline{\Psi}_{M_1}, \underline{\Psi}_{M_2}) \arrow[d, mapsto] & (M_0, M_1, M_2) \arrow[l, mapsto] \arrow[d, mapsto] \\ (\Psi M_0, \Psi_{M_0}) & M_0 \arrow[l, mapsto] \end{tikzcd} .$$

This is the main result in this section.

Proposition 1.4.3: *A separated Alexandroff bispace X is realcompact if and only if WX is realcompact.*

Proof:

$\Longrightarrow$: Suppose $X = (X, \mathcal{A}_1, \mathcal{A}_2)$ is realcompact, then

$$\begin{aligned} X &\cong \overline{\underline{\Psi}}\mathcal{A}X \\ WX &\cong W\overline{\underline{\Psi}}\mathcal{A}X \quad \text{(by applying the functor } W) \\ WX &\cong \Psi T\mathcal{A}X \quad (W\overline{\underline{\Psi}} = \Psi T \text{ in (2) above}) \end{aligned}$$

and as is shown in [13], ΨL is realcompact for any $L \in \mathbf{Reg}\,\sigma\mathbf{FRM}$.

$\Longleftarrow$: Let $X = (X, \mathcal{A}_1, \mathcal{A}_2)$ and WX be realcompact. Then $\eta_{WX} : WX \longrightarrow \Psi\mathcal{U}(WX)$ is an isomorphism and hence onto. We prove that $\eta_X : X \longrightarrow \overline{\underline{\Psi}}\mathcal{A}X$ is onto. Let $P \in \overline{\underline{\Psi}}\mathcal{A}X$, then $P \in \overline{\underline{\Psi}}\mathcal{A}_0X$. But the underlying sets of $\overline{\underline{\Psi}}\mathcal{A}X$ and $\Psi\mathcal{A}(WX)$ are the same hence $P \in \Psi\mathcal{U}(WX)$. But η_{WX} is onto so that there exists $x \in X$ such that $\eta_{WX}(x) = P = \eta_X(x)$. Since η_{WX} is always one-one (WX is separated) it quickly follows that $\eta_X : X \to \overline{\underline{\Psi}}\mathcal{A}X$ is one-one. ∎

CHAPTER 2

THE COZERO PART OF A BIFRAME

We will define the cozero part of a biframe analogously to that for cozero sets of a bitopological space. The following interesting properties extend those of the cozero part of a frame. As in frames (see [24]), the cozero part of a completely regular biframe generates it as a biframe. The cozero part of a biframe is the largest regular sub bi σ-frame of that biframe. This cozero part determines the compact completely regular coreflection of a biframe. We supply a useful characterisation of the elements of a cozero part of a biframe.

2.1 The definition of the cozero part of a biframe

For a biframe $L = (L_0, L_1, L_2)$ we say $cozL = (coz_\ell L_1 \vee coz_u L_2, coz_\ell L_1, coz_u L_2)$ where

$$\begin{aligned} \mathrm{coz}_\ell L_1 &= \{a \in L_1 | a = h(0,\infty),\ h : \mathcal{O}(\mathbb{R},\ell,u) \longrightarrow L \text{ in } \mathbf{BIFRM}\}, \text{ and} \\ \mathrm{coz}_u L_2 &= \{b \in L_2 | b = h(-\infty,0),\ h : \mathcal{O}(\mathbb{R},\ell,u) \longrightarrow L \text{ in } \mathbf{BIFRM}\} \end{aligned}$$

and $coz_\ell L_1 \vee coz_u L_2$ is the σ-frame generated in L_0. It is easy to see that $coz_\ell L_1$ and $coz_u L_2$ are sub σ-frames of L_1 and L_2,respectively. Given a biframe homomorphism $h: L \to M$, $cozh : cozL \to cozM$ is the restriction of h to $cozL$, which is clearly a bi σ-frame homomorphism.

The next proposition shows that this is indeed a good extension of the classical definition of cozero sets.

From [3], we have the natural transformations σ and O. For a bispace X, $\sigma_X : X \longrightarrow \sum \mathcal{O}X$ is defined by $\sigma_X(x) = \mathcal{O}(x)$ which is the filter of open neigbourhoods of x in the topology $(\mathcal{O}X)_0$. For a biframe L, $O_L : L \longrightarrow \mathcal{O}\sum L$ is a biframe map with

$$O_L(a) = \textstyle\sum_a = \{P \in \sum L_0 : a \in P\}.$$

Proposition 2.1.1: *For (X,P,Q) a bitopological space, $coz(\mathcal{O}(X,P,Q)) = \mathcal{A}a_t(X,P,Q) = (\mathcal{C}_P \vee \mathcal{C}_Q, \mathcal{C}_P, \mathcal{C}_Q)$.*

Proof: We only check equality of the first (and effectively the second) part of $coz(\mathcal{O}(X,P,Q))$ and $\mathcal{A}a_t(X,P,Q)$. Since both joins are σ-frame generated in $P \vee Q$, it follows that the total parts are the same.

$\subseteq$: Observe that $\mathcal{A}a_t(X,P,Q) = (\mathcal{C}_{P\vee Q}, \mathcal{C}_P, \mathcal{C}_Q)$. Suppose $A_1 = h(0,\infty)$ for some $h : \mathcal{O}(\mathbb{R},\ell,u) \longrightarrow \mathcal{O}(X,P,Q)$. By naturality of σ and spatiality of $(\mathbb{R},\ell,u)$ (see [3]) the diagram

$$\begin{tikzcd}
\sum \mathcal{O}X \arrow[r, "\Sigma h"] & \sum \mathcal{O}\mathbb{R} \\
(X,P,Q) \arrow[u, "\sigma_X"] \arrow[r, dashed, "f"] & (\mathbb{R},\ell,u) \arrow[u, "\sigma_{\mathbb{R}}"']
\end{tikzcd}$$

commutes for $f = (\sigma_{\mathbb{R}})^{-1} \circ \sum h \circ \sigma_X$. Hence $f : (X,P,Q) \longrightarrow (\mathbb{R},\ell,u)$ is continuous. Now $f^{-1}(0,\infty) = \mathcal{O}f(0,\infty)$

$$= \mathcal{O}(\sigma_{\mathbb{R}}^{-1} \circ \textstyle\sum h \circ \sigma_X)(0,\infty)$$

$$= (\mathcal{O}\sigma_X \circ \mathcal{O}\textstyle\sum h \circ \mathcal{O}\sigma_{\mathbb{R}}^{-1})(0,\infty)$$

$$\overset{(1)}{=} ((O_{\mathcal{O}X})^{-1} \circ \mathcal{O}\textstyle\sum h \circ O_{\mathcal{O}\mathbb{R}})(0,\infty)$$

$$\overset{(2)}{=} h(0,\infty)$$

$$= A_1.$$

We check that $O_{\mathcal{O}X} \circ \mathcal{O}_{\sigma_X} = Id_{\mathcal{O}\sum \mathcal{O}X}$,

$$\begin{aligned} O_{\mathcal{O}X} \circ \mathcal{O}_{\sigma_X}(\textstyle\sum_U) &= O_{\mathcal{O}X}\sigma_X^{-1}(\textstyle\sum_U) \text{ for U} \in (\mathcal{O}\text{X})_0 \\ &= O_{\mathcal{O}X}\{x \in X : \sigma_X(x) \in \textstyle\sum_U\} \\ &= O_{\mathcal{O}X}\{x \in X : \mathcal{O}(x) \in \textstyle\sum_U\} \\ &= O_{\mathcal{O}X}\{x \in X : U \in \mathcal{O}(x)\} \\ &= O_{\mathcal{O}X}(U) \\ &= \textstyle\sum_U . \end{aligned}$$

Equation (1) follows from the fact that $\mathcal{O}\sigma_X \circ O_{\mathcal{O}X} = Id_{\mathcal{O}X}$ (see [3]) and $O_{\mathcal{O}X} \circ \mathcal{O}_{\sigma_X} = Id_{\mathcal{O}\sum \mathcal{O}X}$. Equation (2) is a consequence of naturality of O and spatiality of $\mathcal{O}(X, P, Q)$:

$$\begin{tikzcd} \mathcal{O}X \arrow[r, "O_{\mathcal{O}X}"] & \mathcal{O}\sum \mathcal{O}(X,P,Q) \\ \mathcal{O}\mathbb{R} \arrow[u, "h"] \arrow[r, "O_{\mathcal{O}\mathbb{R}}"] & \mathcal{O}\sum \mathcal{O}\mathbb{R} \arrow[u, "\mathcal{O}\sum h"'] \end{tikzcd}$$

This yields $coz_\ell P \subseteq \mathcal{C}_P$. Similarly for the upper cozero part.

$\supseteq$: Next we let $B_1 = g^{-1}(0, \infty)$ for some continuous $g : (X, P, Q) \longrightarrow (\mathbb{R}, \ell, u)$. Applying the functor $\mathcal{O}$ immediately leads to $B_1 \in coz_\ell P$ and similarly for the second parts. Our result follows. ∎

Consider a spatial biframe M. Then $M \cong \mathcal{O}\sum M$ [3]. Applying the functor coz one gets, $cozM \cong coz(\mathcal{O}\sum M) \cong \mathcal{A}a_t \sum M$ by the last result. We conclude that the cozero part of a spatial biframe is isomorphic to the cozero part of its spectrum.

2.2 Properties of the cozero part of a biframe

The next proposition proves that elements of the cozero part of a biframe are images of subsets of the unit interval under a particular biframe homomorphism.

Proposition 2.2.1: *The following hold in a biframe* $L = (L_0, L_1, L_2)$,

(i) $a \in coz_\ell L_1 \Longleftrightarrow a = h(0,1]$ *for some* $h : \mathcal{O}(I,\ell,u) \longrightarrow (L_0, L_1, L_2)$ *in* **BIFRM**,

(ii) $b \in coz_u L_2 \Longleftrightarrow b = h[0,1)$ *for some* $h : \mathcal{O}(I,\ell,u) \longrightarrow (L_0, L_1, L_2)$ *in* **BIFRM**.

Proof: We prove (i). The proof for (ii) is similar.

$\Longrightarrow$: Define $g : (\mathbb{R},\ell,u) \longrightarrow (I,\ell,u)$ by $g(\alpha) = (\alpha \wedge \mathbf{1}) \vee \mathbf{0}$, where $\mathbf{1}$ is the map which sends every element to 1 and $\mathbf{0}$ sends every element to 0. Now $g(\alpha) > 0 \Longleftrightarrow \alpha > 0$, consequently $\mathcal{O}g : \mathcal{O}(I,\ell,u) \longrightarrow \mathcal{O}(\mathbb{R},\ell,u)$ with $(\mathcal{O}g)(0,1] = (0,\infty)$. Since $a \in coz_\ell L_1, a = k(0,\infty)$ for some biframe homomorphism $k : \mathcal{O}(\mathbb{R},\ell,u) \longrightarrow L$ yielding $a = h(0,1]$ with $h = (k \circ \mathcal{O}g) : \mathcal{O}(I,\ell,u) \longrightarrow L$.

$\Longleftarrow$: Let $a = h(0,1]$ for some $h : \mathcal{O}(I,\ell,u) \longrightarrow (L_0, L_1, L_2)$ in **BIFRM**. Applying the functor $\mathcal{O}$ to the inclusion map $c : (I,\ell,u) \longrightarrow (\mathbb{R},\ell,u)$ gives the biframe homomorphism $\mathcal{O}c : \mathcal{O}(\mathbb{R},\ell,u) \longrightarrow \mathcal{O}(I,\ell,u)$ with $(\mathcal{O}c)(0,\infty) = (0,1]$. Hence $a = (h \circ \mathcal{O}c)(0,\infty)$ and $(h \circ \mathcal{O}c) : \mathcal{O}(\mathbb{R},\ell,u) \longrightarrow L$. ∎

Remark: Proofs in the setting of biframes often involve what amounts to a repetition of some computation of a proof for the first part to obtain an analogous result for the second part. In these cases we will only prove the result for the one part and will omit any mention of the second part.

Lemma 2.2.2: *If* L *and* M *are biframes and* $h : L \longrightarrow M$ *is a biframe homomorphism, then* $h(cozL) \subseteq cozM$.

Proof: Since the total part of a biframe is generated by its first and second parts we need only show that the image of the first (second) part of $cozL$ under h is contained in the first (second) part of $cozM$ respectively.

Suppose $a \in coz_\ell L_1$, then there exists $g : \mathcal{O}(\mathbb{R},\ell,u) \longrightarrow L$ in **BIFRM** such that $a = g(0,\infty)$. Thus $h \circ g : \mathcal{O}(\mathbb{R},\ell,u) \longrightarrow M$ in **BIFRM** with $h(a) = (h \circ g)(0,\infty)$, consequently $h(a) \in coz_\ell M_1$. ∎

Since I with the usual topology is compact, $\mathcal{O}(I,\ell,u)$ is also compact. Recall that the functor $\mathcal{R}$ takes a biframe L to its largest regular sub biframe $\mathcal{R}L$. The latter is generated by all regular sub biframes of L. Schauerte [26] shows that the compact regular biframes are coreflective in **BIFRM** with coreflection maps $\tau_L : \mathcal{R}\Im L \longrightarrow L$ given by join.

Lemma 2.2.3: *If* $L \in$ **BIFRM**, *and* $\mathcal{K}L$ *is the compact regular coreflection of* L *with coreflection map* τ_L *given by join, then* $cozL = \tau_L(coz\mathcal{K}L)$.

Proof: The inclusion $\tau_L(coz\mathcal{K}L) \subseteq cozL$ follows from Lemma 2.2.2.
For the reverse inclusion, suppose that $a \in coz_\ell L_1$. Then $a = h(0,1]$ for some $h : \mathcal{O}(I,\ell,u) \longrightarrow L$ in **BIFRM** by Proposition 2.2.1. But $\mathcal{O}(I,\ell,u)$ is compact and regular (see Proposition 1.3.6), so there exists a unique $h' : \mathcal{O}(I,\ell,u) \longrightarrow \mathcal{K}L$ with $\tau_L \circ h' = h$:

$$\begin{tikzcd}
 & \mathcal{K}L \arrow[d, "\tau_L"] \\
\mathcal{O}(I,\ell,u) \arrow[ur, dashed, "h'"] \arrow[r, "h"'] & L .
\end{tikzcd}$$

We obtain $a = \tau_L \circ h'(0,1]$. Also $h'(0,1] \in coz_\ell(\mathcal{K}L)_1$ by Proposition 2.2.1 and, consequently, $a \in \tau_L(coz_\ell(\mathcal{K}L)_1)$ as required. Inclusion of the total part follows from the fact that τ_L is a biframe homomorphism and the total part is generated by the first and second parts. ∎

Proposition 2.2.4: *If* L *is a biframe, then* $cozL$ *is a regular sub bi* σ*-frame of* L *as a bi* σ*-frame.*

Proof: Consider any biframe homomorphism $h : \mathcal{O}(\mathbb{R},\ell,u) \to L$. Regularity of $\mathcal{O}\mathbb{R}$ implies that of $h(\mathcal{O}\mathbb{R})$. But $\{h(\mathcal{O}\mathbb{R}) \,|\, h : \mathcal{O}(\mathbb{R},\ell,u) \to L\}$ generates $cozL$, hence $cozL$ is regular. ∎

We have shown that the cozero part of a biframe is a regular sub bi σ-frame and we will see in 2.2.10 below that it is the largest such.
We say $a \in L$ is a Lindelöf element if $a \leq \bigvee S (S \subseteq L)$ implies $a \leq \bigvee T$ for some countable

subset T of S. Whenever X is a topological space, then a Lindelöf element of the frame $\mathcal{O}X$ is precisely a Lindelöf subspace of X.

Lemma 2.2.5: *Every upper or lower cozero set of a compact bitopological space* (X, P, Q) *has the Lindelöf property.*

Proof: Observe that $h^{-1}(0,\infty) = \bigcup_{n=1}^{\infty} h^{-1}[\frac{1}{n},\infty)$ and $g^{-1}(-\infty,0) = \bigcup_{n=1}^{\infty} g^{-1}(-\infty,-\frac{1}{n}]$ for some continuous maps $g : (X, P, Q) \longrightarrow (\mathbb{R}, \ell, u)$ and $h : (X, P, Q) \longrightarrow (\mathbb{R}, \ell, u)$. Thus every upper or lower cozero set is a countable join of lower or upper zero sets. Since upper or lower zero sets are closed (see [19]) and hence compact in $P \vee Q$, then cozero sets have the Lindelöf property. ■

Definition 2.2.6: *An ideal* $J \in (\Im L)_i \quad (i \in \{1,2\})$ *is countably generated if there exists a sequence* $\{x_n\} \subseteq J \cap L_i \quad (i \in \{1,2\})$ *such that for each* $a \in J$, $a \leq x_n$ *for some* n.

Lemma 2.2.7: $J \in coz_\ell(\mathbf{CR}\Im L)_1 \quad$ *(resp.* $coz_u(\mathbf{CR}\Im L)_2$*) if and only if* J *is a countably generated completely regular ideal of* L_1 *(resp.* L_2*).*

Proof:

$\Longrightarrow$: The total part of $(\mathbf{CR}\Im L)$ is compact regular and hence spatial so that $(\mathbf{CR}\Im L)$ is spatial [1]. Let $J \in coz_\ell(\mathbf{CR}\Im L)_1$. Then for each $b \in J \cap L_1$, choose a sequence $b = b_1 \prec\!\prec_1 b_2 \prec\!\prec_1 b_3 \ldots$ in $J \cap L_1$. Let J_b be the completely regular ideal generated by this sequence. Then we obtain $J = \bigvee\{J_b : b \in J \cap L_1\}$. But J has the Lindelöf property by Lemma 2.2.5 and hence is a countable join of such elements and, consequently, countably generated.

$\Longleftarrow$: Let $J \in (\mathbf{CR}\Im L)_1$ be countably generated. Since $(\mathbf{CR}\Im L)$ is spatial and completely regular ([19]), its cozero elements form a base for the open elements. Hence J is a union of cozero sets, precisely a countable union since it is countably generated. Since cozero elements are closed under countable unions, J is a cozero element. ■

The following Corollary gives the characterisation of elements of the cozero part.

Corollary 2.2.8: *For any biframe L, $a \in coz_\ell L_1$ iff $a = \bigvee\limits_{n\in\mathbb{N}} a_n$ for some sequence $\{a_n\}$ in L with $a_k \prec\prec_1 a_{k+1}$ for all $k \in \mathbb{N}$.*

Proof: $\Longrightarrow$: Suppose that $a \in coz_\ell L_1$. Then $a = \tau_L(J)$ for some $J \in coz_\ell(\mathbf{CR}\Im L)_1$ by Lemma 2.2.3. Therefore, by Lemma 2.2.7, J is countably generated by $\{c_n\} \subseteq L_1$, say. Define a sequence $\{a_n\}$ in $L_1 \cap J$ by letting $a_1 = c_1$, and choosing $a_{n+1} \in L_1 \cap J$ such that $a_n \vee c_{n+1} \prec\prec_1 a_{n+1}$. Complete regularity of J enables us to make this construction. Consequently $a = \tau_L(J) = \bigvee a_n$ and $a_k \prec\prec_1 a_{k+1}$ for all $k \in \mathbb{N}$.
$\Longleftarrow$: Suppose $a = \bigvee a_n$ with a and a sequence $\{a_n\}$ in L_1 and $a_k \prec\prec_1 a_{k+1}$ for all k. Then $J = \{x \in L_0 : x \leq a_n, \text{ for some } n\}$ is countably generated in $(\mathbf{CR}\Im L)_1$, so that $J \in coz_\ell(\mathbf{CR}\Im L)_1$ by Lemma 2.2.7. Thus $a = \tau_L(J)$, which implies that $a \in \tau_L(coz_\ell(\mathbf{CR}\Im L)_1) = coz_\ell L_1$. ∎

Lemma 2.2.9: *Let L be a regular bi σ-frame. If $a \in L_i$, then $a = \bigvee b_n$ where $b_n \prec_i b_{n+1}$ for all $n \in \mathbb{N}$.*

Proof: Given $a \in L_i$, one gets $a = \bigvee\limits_{a_n \prec_i a} a_n$ by regularity. But the rather below relation interpolates, so we construct a sequence $\{b_n\}$ such that $b_0 = a_0$. Choose $b_1 \in L_i$ such that $a_0 \vee a_1 \prec_i b_1 \prec_i a$; pick $b_2 \in L_i$ such that $b_1 \vee a_2 \prec_i b_2 \prec_i a$; continuing the process pick $b_n \in L_i$ such that $b_{n-1} \vee a_n \prec_i b_n \prec_i a$. It is not difficult to see that for each n, $b_n \prec_i b_{n+1}$ and $a = \bigvee b_n$. ∎

We define the forgetful functor

$$\begin{array}{rcl} U : \mathbf{BIFRM} & \longrightarrow & \mathbf{BI\sigma FRM} \\ (L_0, L_1, L_2) & \mapsto & ((UL)_0, (UL)_1, (UL)_2) \\ f & \mapsto & f|_{(UL)_0} \end{array}$$

where $(UL)_i \quad (i \in \{1,2\})$ is the σ-frame obtained by forgetting arbitrary joins in L_i $(i \in \{1,2\})$ and $(UL)_0$ is the σ-frame generated by $(UL)_1$ and $(UL)_2$ in L_0 as a σ-frame.

Notation: In what follows, $(cozL)_i$, $i = 1,2$ will denote $coz_\ell L_1$ or $coz_u L_2$, *respectively.*

The next result tells us that the cozero part of a completely regular biframe is its largest regular sub bi σ-frame as a bi σ-frame.

Corollary 2.2.10: *For* $L \in$ **BIFRM**, $cozL \cong \mathcal{R}UL$.

Proof: Regularity of $cozL$ implies that $cozL \subseteq \mathcal{R}UL$. For the reverse inclusion, $a \in (\mathcal{R}UL)_i$ implies that $a = \bigvee a_n$ with $a_n \prec_i a_{n+1}$, so that $a_n \prec\!\!\prec_i a_{n+1}$ by the interpolation property. Consequently $a \in (cozL)_i$. ■

Proposition 2.2.11: For $L \in$ **CRegBIFRM**, $cozL$ generates L as a biframe.

Proof: Let $a \in L_1$, then $a = \tau_L(J)$ for some $J \in (\mathcal{K}L)_1$ since τ_L is onto. But $J = \bigvee\{J_b : b \in J \cap L_1\}$ with $J_b \in coz_\ell(\mathcal{K}L)_1$ as in the proof of Lemma 2.2.7. Hence $a = \tau_L(\bigvee\{J_b : b \in J \cap L_1\}) = \bigvee\{\tau_L(J_b) : b \in J \cap L_1\}$ since τ_L is a biframe homomorphism. But $\tau_L(J_b) \in coz_\ell L_1$ for each b, by Lemma 2.2.3.

2.3 The cozero part and compactification of a biframe

We construct the compact completely regular coreflection of a biframe by considering its cozero part.

Definition 2.3.1: *The biframe* **RegIdl** A *of regular ideals for a bi σ-frame A is given by:*

$$\mathbf{RegIdl}\ A = ((\mathbf{RegIdl}\ A)_0, (\mathbf{RegIdl}\ A)_1, (\mathbf{RegIdl}\ A)_2)$$

where $(\mathbf{RegIdl}\ A)_i = \{J \in (\Im A)_i : J \textit{ is regular }\}$, $(i \in \{1,2\})$ *and* $(\mathbf{RegIdl}\ A)_0$ *is the subframe of* $(\Im A)_0$ *generated by* $(\mathbf{RegIdl}\ A)_1 \cup (\mathbf{RegIdl}\ A)_2$.

Let us check that **RegIdl** A is indeed a biframe. By the above definition it suffices to check that $(\mathbf{RegIdl}\ A)_1$ is a frame. We know that $(\Im A)_1$ is a frame so we will check the following points which prove that $(\mathbf{RegIdl}\ A)_1$ is its subframe:

(a) $A_0 \in (\mathbf{RegIdl}\ A)_1$ because $e \prec_1 e$ with a separating element 0, so that $a \in A_0$

implies that $a \leq e \prec_1 e$ consequently $a \prec_1 e$. Since $0 \prec_1 0$ with separating element e we conclude that $\{0\} \in (\mathbf{RegIdl}\ A)_1$.

(b) We check that $J_1 \cap J_2 \in (\mathbf{RegIdl}\ A)_1$ whenever both J_1 and J_2 are there. For $a \in J_1 \cap J_2$, we can find $b_1 \in J_1$ and $b_2 \in J_2$ such that $a \prec_1 b_1$ and $a \prec_1 b_2$, hence $a \prec_1 b_1 \wedge b_2 \in J_1 \cap J_2$.

(c) Suppose $J = \bigvee_{j \in \Lambda} J_j$ where each $J_j \in (\mathbf{RegIdl}\ A)_1$, then we must show that $J \in (\mathbf{RegIdl}\ A)_1$. If $a \in J$, then $a \leq b_1 \vee b_2 \vee \ldots \vee b_k$ where each $b_i \in J_j$ for some $j \in \Lambda$. By regularity of each J_j, there exists $c_i \in J_j$ with $b_i \prec_1 c_i$, hence $a \prec_1 c_1 \vee c_2 \vee \ldots \vee c_k \in J$. Consequently J is regular.

Another thing which is of interest to us is whether $(\mathbf{RegIdl}\ A)_0$ is contained in the set of regular ideals of A_0. Suppose $J \in (\mathbf{RegIdl}\ A)_0$, then $J = \bigvee_j (J_{1j} \wedge J_{2j})$ where $J_{ij} \in (\mathbf{RegIdl}\ A)_i$ with $(i \in \{1, 2\})$. Consider $a \in J$, then $a \leq b_1 \vee b_2 \vee \ldots \vee b_k$ where each $b_i \in J_{1j} \cap J_{2j}$ for some j. By regularity of each J_{ij} there exists $c_i \in J$ with $b_i \prec_0 c_i$, hence $a \leq b_1 \vee b_2 \vee \ldots \vee b_k \prec_0 c_1 \vee c_2 \vee \ldots \vee c_k \in J$. Consequently J is a regular ideal of A_0.

We now consider **RegIdl** in combination with *coz*:

$$\mathbf{BIFRM} \xrightarrow{coz} \mathbf{Reg\,BI\sigma FRM} \xrightarrow{\mathbf{RegIdl}} \mathbf{KCReg\ BIFRM}.$$

Proposition 2.3.2: *For any regular bi σ-frame L,* **RegIdl** *L is a compact completely regular biframe.*

The proof of this proposition is similar to that of Proposition 0.2.4 and therefore omitted. Since $coz : \mathbf{BIFRM} \longrightarrow \mathbf{Reg\,BI\sigma FRM}$ is functorial, to prove that $\mathbf{RegIdl} \circ coz : \mathbf{BIFRM} \longrightarrow \mathbf{KCReg\ BIFRM}$ is functorial it suffices to show that **RegIdl**: $\mathbf{Reg\,BI\sigma FRM} \longrightarrow \mathbf{KCReg\ BIFRM}$ is functorial. ■

Proposition 2.3.3: **RegIdl**: $\mathbf{Reg\,BI\sigma FRM} \longrightarrow \mathbf{KCReg\ BIFRM}$ *is functorial.*

Proof: Given a regular bi σ-frame homomorphism $h : L \longrightarrow M$ and $J \in (\mathbf{RegIdl}\ L)_i$ where $i \in \{1, 2\}$, we define $(\mathbf{RegIdl}\ h)(J) = \langle h(J) \rangle$ to be the ideal generated by $h(J)$ in M_0. We check that $\langle h(J) \rangle$ is regular in M_i. Suppose $x \in \langle h(J) \rangle$, then $x \leq h(a)$ for some $a \in J$. But J is regular, so there exists $y \in J \cap L_i$ such that $a \prec_i y$, hence $x \leq h(a) \prec_i h(y) \in \langle h(J) \rangle \cap M_i$. ∎

Proposition 2.3.4: *If M is a compact regular biframe, then the join map $J_M : \mathbf{RegIdl}(cozM) \longrightarrow M$ is an isomorphism.*

Proof: Since $(\mathbf{RegIdl}(cozM))_0$ is regular, $(J_M)_0$ dense and M_0 compact, $(J_M)_0$ is codense by the Lemma 0.1.2 and hence one-one. For onto, suppose $a \in M_i$, then

$$\bigvee r_i(a) = \bigvee \langle x \in M_i : x \prec_i a \rangle = a,$$

hence J_M is an isomorphism. ∎

Proposition 2.3.5: KCReg BIFRM *is coreflective in* **BIFRM** *with coreflection maps $J_L : \mathbf{RegIdl}(cozL) \longrightarrow L$.*

Proof: Consider $h : M \longrightarrow L$ in **BIFRM** where M is compact and regular. Note that a compact biframe is regular iff it is completely regular. (assuming the axiom of countable dependent choice.) Then applying the functor $\mathbf{RegIdl}(coz)$ we obtain:

$$\begin{tikzcd}
\mathrm{RegIdl}(cozM) \arrow[r, "\cong", "J_M"'] \arrow[d, "\mathrm{RegIdl}(cozh)"'] & M \arrow[d, "h"] \\
\mathrm{RegIdl}(cozL) \arrow[r, "J_L"] & L
\end{tikzcd}$$

So h factors through $\mathbf{RegIdl}(cozh)$ and density of J_L gives us that this factorisation is unique. ∎

Remark: $\mathbf{RegIdl}(cozL)$ is the compact completely regular coreflection of a completely regular biframe L.

2.4 The regular Lindelöf coreflection of a biframe and pseudocompactness

Our objective in this section is to define the regular Lindelöf coreflection of a biframe. We later give a characterisation of pseudocompactness of a biframe in terms of compactness of its regular Lindelöf coreflection.

We start by defining a functor $\mathcal{H} : \mathbf{BI\sigma FRM} \longrightarrow \mathbf{BIFRM}$.

Definition 2.4.1: *Let L be a* bi σ-frame, *$\mathcal{H}L_i$ the collection of* σ-ideals in L_i *and* $J^* = \{a \in L_0 : a \leq b \text{ for some } b \in J\}$, $J \in \mathcal{H}L_i$. *We define $\mathcal{H}L$ by* $\mathcal{H}L = (\mathcal{H}L_0, (\mathcal{H}L_1)^*, (\mathcal{H}L_2)^*)$, *where* $(\mathcal{H}L_i)^* = \{J^* : J \in \mathcal{H}L_i\}$.

J^* is indeed a σ-ideal on the σ-frame L_0 for each $J \in \mathcal{H}L_i$.

Proof: (a) Suppose $a \leq b \in J^*$, then we can find $x \in J$ satisfying $b \leq x$ so that $a \leq x \in J$ yielding $a \in J^*$.

(b) For $a_n \in J^*$, we can find $x_n \in J$ such that $a_n \leq x_n$ for each n, consequently $\bigvee\limits_n a_n \leq \bigvee\limits_n x_n$. But $\bigvee\limits_n x_n \in J$ since J is a σ-ideal. Hence $\bigvee\limits_n a_n \in J^*$.

Proposition 2.4.2: *If L is a bi σ-frame, then $\mathcal{H}L = (\mathcal{H}L_0, (\mathcal{H}L_1)^*, (\mathcal{H}L_2)^*)$ is a biframe.*

Proof: Firstly we check that for $i \in \{1,2\}$ $(\mathcal{H}L_i)^*$ is a frame.

(a) Take $H^*, J^* \in (\mathcal{H}L_i)^*$, then we show that $H^* \cap J^* = (H \cap J)^*$.

$\supseteq$: is straightforward.

$\subseteq$: Given $a \in H^* \cap J^*$, then $a \leq h$ and ℓ for $h \in H$, $\ell \in J$, hence $a \leq h \wedge \ell$ because $a, h, \ell \in L_0$, now $a \in (H \cap J)^*$ since $h \wedge \ell \in H \cap J$.

In view of the fact that $H \cap J \in \mathcal{H}L_i$ and $H^* \cap J^* = (H \cap J)^*$, we have $H^* \cap J^* \in (\mathcal{H}L_i)^*$.

(b) Consider an indexing set Λ, and $J^*_\alpha \in (\mathcal{H}L_i)^*$ for each $\alpha \in \Lambda$. Then we prove that $\bigvee\limits_\alpha J^*_\alpha = (\bigvee\limits_\alpha J_\alpha)^*$

$\subseteq$: Pick $a \in \bigvee\limits_\alpha J^*_\alpha$, then $a = \bigvee\limits_n a_n$ for $a_n \in J^*_{\alpha_n}$ where $\{\alpha_n | n \in \mathbb{N}\}$ is a countable subset of Λ. For each n there exists $x_n \in J_{\alpha_n}$ such that $a_n \leq x_n$, hence $a = \bigvee\limits_n a_n \leq \bigvee\limits_n x_n \in \bigvee\limits_\alpha J_\alpha$, yielding $a \in (\bigvee\limits_\alpha J_\alpha)^*$.

$\supseteq$: Let $a \in (\bigvee\limits_\alpha J_\alpha)^*$, then there exists $x \in \bigvee\limits_\alpha J_\alpha$ such that $a \leq x = \bigvee\limits_n x_n$, where $x_n \in J_{\alpha_n}$. As a consequence of the fact that $J_{\alpha_n} \subseteq J^*_{\alpha_n}$ for each n, we have $\bigvee\limits_n x_n \in \bigvee\limits_\alpha J^*_\alpha$. From $\bigvee\limits_\alpha J^*_\alpha \in \mathcal{H}L_0$ we deduce that $a \in \bigvee\limits_\alpha J^*_\alpha$.

Hence $\bigvee\limits_\alpha J^*_\alpha \in (\mathcal{H}L_i)^*$.

(c) Since $(\mathcal{H}L_i)^* \subseteq \mathcal{H}L_0$ and $\mathcal{H}L_0$ is a frame, we have distributivity in $(\mathcal{H}L_i)^*$.

Next we prove that $\mathcal{H}L_0$ is generated by $(\mathcal{H}L_1)^*$ and $(\mathcal{H}L_2)^*$. We have $J = \bigvee\limits_{x \in J} \downarrow x$ for each $J \in \mathcal{H}L_0$. (In the remainder of this section $\downarrow$ will be taken in L_0, unless otherwise stated.) For each $x \in J, x \in L_0$ so that $x = \bigvee\limits_n b_n \wedge c_n$ where $\{b_n\} \subseteq L_1$, $\{c_n\} \subseteq L_2$, then

$$\begin{aligned} \downarrow x &= \downarrow (\bigvee_n (b_n \wedge c_n)) \\ &= \bigvee_n (\downarrow b_n \cap \downarrow c_n). \end{aligned}$$

Certainly $\downarrow b_n \in (\mathcal{H}L_1)^*$ and $\downarrow c_n \in (\mathcal{H}L_2)^*$ as confirmed by the fact that $\downarrow b_n = (\downarrow_{L_1} b_n)^*$ and $\downarrow c_n = (\downarrow_{L_2} c_n)^*$ where $(\downarrow_{L_1} b_n) \in \mathcal{H}L_1$ and $(\downarrow_{L_2} c_n) \in \mathcal{H}L_2$. Consequently $(\mathcal{H}L_0, (\mathcal{H}L_1)^*, (\mathcal{H}L_2)^*)$ is a biframe. ∎

Proposition 2.4.3: *For a bi σ-frame homomorphism $h : L \longrightarrow M$ define $\mathcal{H}h : \mathcal{H}L \longrightarrow \mathcal{H}M$ by $\mathcal{H}h(J) = \langle h(J) \rangle$, the σ-ideal generated by $h(J)$ in M_0. Then $\mathcal{H}h$ is a biframe homomorphism whenever h is a bi σ-frame homomorphism.*

Proof: We know that $\mathcal{H}h : \mathcal{H}L_0 \longrightarrow \mathcal{H}M_0$ is a frame homomorphism. If J is a σ-ideal on M_0 and $h : M \to L$ is a bi σ-frame homomorphism, then the σ-ideal generated by $h(J)$ is just the ideal generated by $h(J)$, namely $\{b \in M|$ there exists $a \in J$ such that $b \leq h(a)\}$. Since $\mathcal{H}L_i \subseteq \mathcal{H}M_i$, then $(\mathcal{H}L_i)^* \subseteq (\mathcal{H}M_i)^*$ for $i \in \{1, 2\}$. ■

Our aim is to prove that $\mathcal{H}coz$ is the regular Lindelöf coreflection functor for biframes. A step in this direction is given by

Lemma 2.4.4: *Whenever L is a bi σ-frame, $\mathcal{H}L$ is a Lindelöf biframe.*

Proof: For a bi σ-frame(L_0, L_1, L_2), $\mathcal{H}L$ is Lindelöf since the total part $\mathcal{H}L_0$ is Lindelöf [21]. ■

Proposition 2.4.5: *If L is a regular bi σ-frame, then $(\mathcal{H}L_0, (\mathcal{H}L_1)^*, (\mathcal{H}L_2)^*)$ is a regular biframe.*

Proof: Consider $J^* \in (\mathcal{H}L_i)^*$, then

$$J^* = \bigvee_{a \in J^*} \downarrow a = \bigvee_{b \in J} \downarrow b.$$

We check that the last equality holds.

$\geq$: This inequality follows from $J \subseteq J^*$.

$\leq$: Suppose $a \in J^*$, then there exists $b \in J$ such that $a \leq b$, which gives $\downarrow a \subseteq \downarrow b$. Hence the above equation holds.

Now $J^* = \bigvee_{a \in J} \downarrow a$. Furthermore if $a \in J$, then $\downarrow a = \bigvee_{n} \downarrow a_n$, where $\downarrow a_n \prec_i \downarrow a$ for each n by regularity of L. Thus J^* can be written as a join of σ-ideals in $(\mathcal{H}L_i)^*$ which are i-rather below $\downarrow a$ and hence i-rather below J^*. From this we deduce that $\mathcal{H}L$ is regular.

■

Lemma 2.4.6: *Whenever L is a biframe, $\bigvee_L : \mathcal{H}cozL \longrightarrow L$ is natural in L.*

Proof: For each biframe L, $\bigvee_L : (\mathcal{H}cozL)_0 \longrightarrow L_0$ is a biframe homomorphism since $J^* \in (\mathcal{H}(cozL)_i)^*$ implies that $\bigvee J^* \in L_i$. Take a biframe M, then for each frame morphism $h : M_0 \longrightarrow L_0$ we check that the following diagram commutes:

$$\begin{tikzcd}
M_0 \arrow[r, "h"] & L_0 \\
\mathcal{H}(cozM)_0 \arrow[u, "\bigvee_M"] \arrow[r, "\mathcal{H}cozh"] & \mathcal{H}(cozL)_0 \arrow[u, "\bigvee_L"'] .
\end{tikzcd}$$

Consider $J \in \mathcal{H}(cozM)_0$, then $(\mathcal{H}cozh)(J) = \mathcal{H}h(J)$

$$= \bigvee_{j \in J} \downarrow h(j) \; (\bigvee \text{ and } \downarrow \text{ in } (cozL)_0).$$

Now taking a join in L_0 we obtain

$$\begin{aligned}
\bigvee_L (\mathcal{H}(cozh))(J) &= \bigvee_L \bigvee_{j \in J} \downarrow h(j) \\
&= \bigvee_{j \in J} \bigvee_L \downarrow h(j) \\
&= \bigvee_{j \in J} h(j) \\
&= h(\bigvee J). \qquad \blacksquare
\end{aligned}$$

Lemma 2.4.7: *For any regular Lindelöf biframe $L, \bigvee : (\mathcal{H}(cozL)_i)^* \longrightarrow L_i$ is onto.*

Proof: Schauerte [28] proves that any regular Lindelöf biframe is normal. Moreover it is not difficult to show that for a normal regular biframe L, $\prec_i$ interpolates and consequently L is completely regular. By Corollary 2.2.11 *cozL* generates L whenever L is completely regular.

Suppose $a \in L_i$, then $a = \bigvee J_a$ where $J_a = \downarrow a \cap (cozL)_i \in \mathcal{H}(cozL)_i$, hence $a = \bigvee J_a^*$

Proposition 2.4.8: *If L is a regular Lindelöf biframe, then $\bigvee_L : \mathcal{H}cozL \longrightarrow L$ is an isomorphism.*

Proof: It suffices to prove that $\bigvee : \mathcal{H}(cozL)_0 \longrightarrow L_0$ is one-one and $\bigvee : (\mathcal{H}(cozL)_i)^* \longrightarrow L_i$ is onto for $i \in \{1,2\}$.

Since codense implies one-one in **Reg FRM** we check that $\bigvee : \mathcal{H}(cozL)_0 \longrightarrow L_0$ is codense. If $\bigvee J = e$, then since L is Lindelöf, $\bigvee_{a_n \in J} a_n = e$, which yields $e \in J$ because J is a σ-ideal, hence $J = (cozL)_0$. The fact that $\bigvee : (\mathcal{H}(cozL)_i)^* \longrightarrow L_i$ is onto has been proved in Lemma 2.4.7. ■

Proposition 2.4.9: *The regular Lindelöf biframes are coreflective in* **BIFRM**, *with coreflection maps $\bigvee_L : \mathcal{H}cozL \longrightarrow L$.*

Proof: Consider a biframe homomorphism $h : M \longrightarrow L$ where M is a regular Lindelöf biframe. Since $\bigvee_L : \mathcal{H}cozL \longrightarrow L$ is a natural transformation, by Lemma 2.4.6, the following square commutes:

$$\begin{tikzcd}
M_0 \arrow[r, "h"] & L_0 \\
\mathcal{H}(cozM)_0 \arrow[u, "\bigvee_M"] \arrow[r, "(\mathcal{H}cozh)"'] & (\mathcal{H}(cozL)_0. \arrow[u, "\bigvee_L"']
\end{tikzcd}$$

We can find a unique $h' : M_0 \longrightarrow \mathcal{H}(cozL)_0$ such that $h = \bigvee_L \circ h'$ see [29].

Lastly we prove that $\mathcal{H}$ gives an equivalence of the category of regular bi σ-frames and the category of regular Lindelöf biframes.

Proposition 2.4.10: *The category of regular Lindelöf biframes and biframe homomorphisms* **Reg Lind BIFRM**, *and* **Reg BIσFRM** *are equivalent as categories.*

Proof: We check that the functor $\mathcal{H}$: **Reg BIσFRM** $\longrightarrow$ **Reg Lind BIFRM** is full, faithful and isomorphism dense.

Fullness: If $f : M \longrightarrow L$ is a morphism in **Reg Lind BIFRM**, then $cozf : cozM \longrightarrow cozL$ is a morphism in **Reg BIσFRM** and $\mathcal{H}cozf : \mathcal{H}cozM \longrightarrow \mathcal{H}cozL$ is precisely $f : M \longrightarrow L$.

Faithfulness: Suppose that $\mathcal{H}f = \mathcal{H}g$, where $f : M \longrightarrow L$ and $g : M \longrightarrow L$ are in **Reg BIσFRM**. Let $x \in M_0$, then since

$$\mathcal{H}f = \mathcal{H}g, \mathcal{H}f(\downarrow x) = \mathcal{H}g(\downarrow x), \quad \text{thus}$$

$$\langle f(\downarrow x)\rangle = \langle g(\downarrow x)\rangle, \quad \text{which leads to}$$

$$\langle \downarrow f(x)\rangle = \langle \downarrow g(x)\rangle, \quad \text{consequently}$$

$\bigvee\langle \downarrow f(x)\rangle = \bigvee\langle \downarrow g(x)\rangle$, where the join is taken in L_0, this yields $f(x) = g(x)$. We conclude that $\mathcal{H}f = \mathcal{H}g$ implies that $f = g$.

Isomophism-dense: Let L be a regular Lindelöf biframe. Then we have $L \cong \mathcal{H}cozL$ by Proposition 2.4.8.

This proves the result. ■

CHAPTER 3

PSEUDOCOMPACTNESS AND THE COZERO PART OF A BIFRAME

Here we extend some of the results of [6] for frames to biframes. We start by defining the biframe $\mathcal{L}(\mathbb{R}_b)$ of reals. Pseudocompactness of a biframe L is defined in terms of the bounded maps $\mathcal{L}(\mathbb{R}_b) \to L$. We then give a characterisation of pseudocompactness which does not mention the reals. The latter is used to give a characterisation of pseudocompactness of a biframe in terms of its cozero part.

3.1 The biframe of reals

Banaschewski and Mulvey [7] describe the frame of reals by means of a set of relations in $\mathbb{Q} \times \mathbb{Q}$. We follow their presentation in the next definition.

Definition 3.1.1: *The biframe of reals is the biframe* $\mathcal{L}(\mathbb{R}_b) = (\mathcal{L}_0(\mathbb{R}_b), \mathcal{L}_1(\mathbb{R}_b), \mathcal{L}_2(\mathbb{R}_b))$ *where* $\mathcal{L}_0(\mathbb{R}_b)$ *is the frame generated by ordered pairs* $(p,q) \in \mathbb{Q}\times\mathbb{Q}$ *satisfying the following relations:*

R1) $(p,q) \wedge (s,t) = (p \vee s, q \wedge t)$

R2) $(p,q) \vee (s,t) = (p,t)$ whenever $p \leq s < q \leq t$

R3) $(p,q) = \bigvee\{(s,t) : p < s < t < q\}$

R4) $e = \bigvee\{(p,q) \in \mathbb{Q} \times \mathbb{Q}\}$

and $\mathcal{L}_1(\mathbb{R}_b)$ *is the frame generated by elements* $(p,-) = \bigvee\{(p,q) : q \in \mathbb{Q}\}$ *inside* $\mathcal{L}_0(\mathbb{R}_b)$, $\mathcal{L}_2(\mathbb{R}_b)$ *is the frame generated by elements* $(-,q) = \bigvee\{(p,q) : p \in \mathbb{Q}\}$ *inside* $\mathcal{L}_0(\mathbb{R}_b)$.

$\mathcal{L}(\mathbb{R}_b)$ is indeed a biframe, because $(p,q) = (p,-) \wedge (-,q)$ for any $p,q \in \mathbb{Q}$.

Let $[\mathcal{L}_0(\mathbb{R}_b)]$ denote the set of generators of $\mathcal{L}_0(\mathbb{R}_b)$. A map $h : [\mathcal{L}_0(\mathbb{R}_b)] \to L$ can be uniquely extended to a biframe homomorphism from $\mathcal{L}(\mathbb{R}_b)$ into L if and only if h transforms the above relations into identities which hold in L and takes the generators of $\mathcal{L}_i(\mathbb{R}_b)$ to L_i, $i \in \{1,2\}$.

We now show that the spectrum of the biframe of reals is homeomorphic to $(\mathbb{R}, \ell, u)$, moreover this homeomorphism satisfies a special property.

Notation: Let $\mathbf{2}$ be the two element frame with top 1 and bottom $0, L_1 = L_2 = \{0, 1\}$. We will denote $(\mathbf{2}, L_1, L_2)$ by $\mathbf{2}_b$.

The real line together with the lower and upper topologies $(\mathbb{R}, \ell, u)$ will be denoted by $\mathbb{R}_b$.

Let $]\!]p, q[\![= \{x \in \mathbb{R} : p < x < q\}$, which is an open interval in $\mathbb{R}$ with usual topology.

Proposition 3.1.2: *There exists a homeomorphism* $\rho : \sum(\mathcal{L}(\mathbb{R}_b)) \longrightarrow \mathbb{R}_b$ *such that* $\rho(\zeta) \in]\!]p, q[\![$ *iff* $\zeta(p, q) = 1$.

Proof: Given $\zeta \in \sum(\mathcal{L}(\mathbb{R}_b))$, our first task is to define an open Dedekind cut (V, W) which determines a $\lambda \in \mathbb{R}$.

Suppose $\zeta \in \sum(\mathcal{L}(\mathbb{R}_b))$, thus $\zeta : \mathcal{L}_0(\mathbb{R}_b) \longrightarrow \mathbf{2}$, define $V = \{r \in \mathbb{Q} : \zeta(r, -) = 1\}$ and $W = \{s \in \mathbb{Q} : \zeta(-, s) = 1\}$, then we check that (V, W) is an open Dedekind cut.

(1) $e = \bigvee\{(p, q) : p, q \in \mathbb{Q}\}$

$\Longrightarrow 1 = \bigvee\{\zeta(p, q) : p, q \in \mathbb{Q}\}$

$\Longrightarrow$ there are $p, q \in \mathbb{Q}$ such that $\zeta(p, q) = 1$

$\Longrightarrow \zeta(p, -) = 1$ and $\zeta(-, q) = 1$

$\Longrightarrow p \in V$ and $q \in W$.

(2) We show that V is a downset. If $p \in V$ and $p' < p$, then $(p, -) < (p', -)$, thus $\zeta(p, -) = 1$ implies that $\zeta(p', -) = 1$. Similarly it can be shown that W is an upset.

(3) Let $p \in V$, then $(p,-) = \bigvee_{n\in\mathbb{N}} (p+\frac{1}{n}, -)$. Since $\zeta(p,-) = 1$, then $\bigvee_{n} \zeta(p+\frac{1}{n}, -) = 1$ which implies that there exists $n_0 \in \mathbb{N}$ such that $\zeta(p+\frac{1}{n_0}, -) = 1$. In a similar manner it can be shown that for any $q \in W$ there is $q' < q$ with $q' \in W$.

(4)
$$
\begin{aligned}
p < q &\Longrightarrow (-,q) \vee (p,-) = e \\
&\Longrightarrow \zeta(-,q) \vee \zeta(p,-) = 1 \\
&\Longrightarrow \zeta(-,q) = 1 \quad \text{or} \quad \zeta(p,-) = 1 \\
&\Longrightarrow p \in V \quad \text{or} \quad q \in W.
\end{aligned}
$$

(5)
$$
\begin{aligned}
r \in V \cap W &\Longrightarrow \zeta(r,-) = 1 = \zeta(-,r) \\
&\Longrightarrow \zeta[(r,-) \wedge (-,r)] = 1.
\end{aligned}
$$

But $(r,-) \wedge (-,r) = 0$ which is a contradiction.

Hence (V,W) is a Dedekind cut. Now (V,W) represents $\lambda \in \mathbb{R}$ which means that $p, q \in \mathbb{Q}$ such that $p < \lambda$ iff $p \in V$, $\lambda < q$ iff $q \in W$. This implies that $p \in V$ and $q \in W$ iff $\lambda \in \,]p,q[$.

We check that $\zeta(p,q) = 1$ iff $p \in V$ and $q \in W$.

$\Longrightarrow$: $\zeta(p,q) = 1$ implies that $\zeta(p,-) = 1 = \zeta(-,q)$

$\Longleftarrow$: $p \in V$ and $q \in W$ mean that $\zeta(p,-) = 1 = \zeta(-,q)$, thus $\zeta(p,r) = \zeta(s,q) = 1$ for some $r, s \in \mathbb{Q}$, hence $1 = \zeta(p,r) \wedge \zeta(s,q)$

$$
\begin{aligned}
&= \zeta((p,r) \wedge (s,q)) \\
&= \zeta(p \vee s, r \wedge q) \\
&\leq \zeta(p,q).
\end{aligned}
$$

We conclude that (V,W) determines λ such that $\lambda \in \,]p,q[$ iff $\zeta(p,q) = 1$.

Next we define a map which will be uniquely extended to the inverse of ρ. Given $\lambda \in \mathbb{R}$, define $\zeta_\lambda : [\mathcal{L}_0(\mathbb{R}_b)] \longrightarrow \mathbf{2}$ by

$$\zeta_\lambda(p,q) = \begin{cases} 1 & \text{if} \quad \lambda \in \,]\!]p,q[\![\\ 0 & \text{otherwise.} \end{cases}$$

We check that the definition turns (R1) - (R4) into true identities in $\mathbf{2}_b$, preservation of the generators of $\mathcal{L}_i(\mathbb{R}_b)$ are automatic since the first and second parts of $\mathbf{2}_b$ are the same as its total part and ζ_λ is thus a biframe homomorphism. We will only show these identities on generators of $\mathcal{L}_0(\mathbb{R}_b)$ as $\mathcal{L}_1(\mathbb{R}_b)$ and $\mathcal{L}_2(\mathbb{R}_b)$ are frames generated in $\mathcal{L}_0(\mathbb{R}_b)$. For generators of $\mathcal{L}_0(\mathbb{R}_b)$:

(i) $\zeta_\lambda(p,q) \wedge \zeta_\lambda(r,s) = 1$

$\Longleftrightarrow \zeta_\lambda(p,q) = 1 = \zeta_\lambda(r,s)$

$\Longleftrightarrow \lambda \in \,]\!]p,q[\![$ and $\lambda \in \,]\!]r,s[\![$

$\Longleftrightarrow \lambda \in \,]\!]p \vee r, q \wedge s[\![$

$\Longleftrightarrow \zeta_\lambda(p \vee r, q \wedge s) = 1.$

(ii) If $p \leq r < q \leq s$, then $\zeta_\lambda(p,s) = 1$

$\Longleftrightarrow \lambda \in \,]\!]p,s[\![$

$\Longleftrightarrow \lambda \in \,]\!]p,q[\![$ or $\lambda \in \,]\!]r,s[\![$

$\Longleftrightarrow 1 = \zeta_\lambda(p,q) \vee \zeta_\lambda(r,s).$

(iii) If $p < p' < q' < q$, then $\zeta_\lambda(p,q) = 1$

$$\iff \lambda \in \,]\!]p, q[\![$$

$$\iff \exists p', q' \text{ such that } p < p' < q' < q \text{ with } \lambda \in \,]\!]p', q'[\![$$

$$\iff \exists p', q' \text{ with } p < p' < q' < q \text{ such that } \zeta_\lambda(p', q') = 1$$

$$\iff \bigvee\{\zeta_\lambda(p', q') : p < p' < q' < q\} = 1.$$

(iv) $\bigvee\{\zeta_\lambda(p, q) : p, q \in \mathbb{Q}\} = 1$ because if p and q are rational numbers such that $p < r < q$, then $\zeta_\lambda(p, q) = 1$, hence $1 = \zeta_\lambda(p, q) \leq \bigvee\{\zeta_\lambda(r, s) : r, s \in \mathbb{Q}\} \leq \zeta_\lambda(e) = 1$, so equality must hold.

We have thus defined a map which can be extended to a biframe homomorphism $\zeta_\lambda : \mathcal{L}(\mathbb{R}_b) \longrightarrow \mathbf{2}_b$ such that $\lambda \in]\!]p, q[\![$ iff $\zeta_\lambda(p, q) = 1$. Clearly $\rho(\zeta_\lambda) = \lambda$. We check that ρ is indeed an isomorphism:

(i) Now $\zeta = \zeta_{\rho(\zeta)}$ since $\zeta_{\rho(\zeta)}(p, q) = 1$ iff $p < \rho(\zeta) < q$ iff $\zeta(p, q) = 1$.

Lastly we show that $\rho : \sum(\mathcal{L}(\mathbb{R}_b)) \longrightarrow \mathbb{R}_b$ is a homeomorphism.

On $\mathcal{L}_0(\mathbb{R}_b)$: $\rho^{-1}(]\!]p, q[\![) = \{\zeta : \zeta(p, q) = 1\} = \sum_{(p,q)}$, also

$\rho(\sum_{(p,q)}) = \{\lambda : \lambda \in]\!]p, q[\![\} =]\!]p, q[\![$.

On $\mathcal{L}_1(\mathbb{R}_b)$:

$$\begin{aligned}
\rho^{-1}(\rrbracket p, \infty\llbracket) &= \rho^{-1}(\bigcup_{q\in\mathbb{Q}} \rrbracket p, q\llbracket) \\
&= \bigcup_{q\in\mathbb{Q}} \rho^{-1}(\rrbracket p, q\llbracket) \\
&= \{\zeta : \zeta(p,q) = 1 \quad \text{for some} \quad q \in \mathbb{Q}\} \\
&= \bigcup_{q\in\mathbb{Q}} \Sigma_{(p,q)} \\
&= \Sigma_{\bigvee_{q\in\mathbb{Q}} (p,q)} \\
&= \Sigma_{(p,-)} \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\rho(\Sigma_{(p,-)}) &= \rho(\Sigma_{\bigvee_{q\in\mathbb{Q}} (p,q)}) \\
&= \rho(\bigcup_{q\in\mathbb{Q}} \Sigma_{(p,q)}) \\
&= \bigcup_{q\in\mathbb{Q}} \rho(\Sigma_{(p,q)}) \\
&= \bigcup_{q\in\mathbb{Q}} (\rrbracket p, q\llbracket) \\
&= \rrbracket p, \infty\llbracket.
\end{aligned}$$

Similarly it can be shown that

$$\begin{aligned}
\rho^{-1}(\rrbracket -\infty, q\llbracket) &= \Sigma_{(-,q)} \quad \text{and} \\
\rho(\Sigma_{(-,q)}) &= \rrbracket -\infty, q\llbracket.
\end{aligned}$$

■

We now show that we have a one-one onto map from **BIFRM** $(\mathcal{L}(\mathbb{R}_b), \mathcal{O}X)$ to **BITOP** $(X, \mathbb{R}_b)$.

Proposition 3.1.3: *For a bitopological space* X, **BIFRM** $(\mathcal{L}(\mathbb{R}_b), \mathcal{O}X) \xrightarrow{\simeq}$ **BITOP** $(X, \mathbb{R}_b)$.

Proof: We define a map

$$\begin{aligned} \mathbf{BIFRM}(\mathcal{L}(\mathbb{R}_b), \mathcal{O}X) &\longrightarrow \mathbf{BITOP}(X, \mathbb{R}_b) \\ \varphi &\mapsto \tilde{\varphi} \end{aligned}$$

by $\tilde{\varphi}(x) = \rho(\sum \varphi)\varepsilon_X(x)$ where

$$\begin{aligned} \varepsilon_X : X &\longrightarrow \textstyle\sum \mathcal{O}X \\ x &\mapsto \hat{x} \end{aligned}$$

is the adjunction of $\sum$ and $\mathcal{O}$, and for any $U \in (\mathcal{O}X)_0$, $\hat{x}(U) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$. We check that $\tilde{\varphi} : X \longrightarrow \mathbb{R}_b$ is continuous:

We will show that $\tilde{\varphi}(x) \in \rrbracket p, q \llbracket \Longleftrightarrow x \in \varphi(p, q)$,

$$\tilde{\varphi}(x) \in \rrbracket p, \infty \llbracket \Longleftrightarrow x \in \varphi(p, -),$$

and finally $\tilde{\varphi}(x) \in \rrbracket -\infty, q \llbracket \Longleftrightarrow x \in \varphi(-, q)$.

But $\tilde{\varphi}(x) = \rho(\sum \varphi)\varepsilon_X(x)$

$= \rho(\sum \varphi)\hat{x}$

$= \rho(\hat{x}\varphi)$, thus

$\tilde{\varphi}(x) \in \rrbracket p, q \llbracket$

$\Longleftrightarrow \hat{x}\varphi(p, q) = 1$ (property of ρ)

$\Longleftrightarrow x \in \varphi(p, q)$ (property of $\hat{x}$).

Next $\tilde{\varphi}(x) \in \rrbracket p, \infty \llbracket$

$$\Longleftrightarrow \hat{x}\varphi(p,-) = 1$$

$$\Longleftrightarrow x \in \varphi(p,-).$$

We can show the last one in a similar way. Thus $\tilde{\varphi}$ is continuous.

We define a map which will be the inverse of $\sim$:

Given $f : X \longrightarrow \mathbb{R}_b$ define

$$\bar{f} : [\mathcal{L}_0(\mathbb{R}_b)] \longrightarrow (\mathcal{O}X)_0 \text{ by } \bar{f}(p,q) = f^{-1}(\rrbracket p,q \llbracket).$$

We show that $\bar{f}$ determines a frame homomorphism on $\mathcal{L}_0(\mathbb{R}_b)$:

(i) $\bar{f}(p,q) \wedge \bar{f}(s,t)$

$$= f^{-1}\rrbracket p,q \llbracket \cap f^{-1}\rrbracket s,t \llbracket$$

$$= f^{-1}(\rrbracket p,q \llbracket \cap \rrbracket s,t \llbracket)$$

$$= f^{-1}\rrbracket p \vee s, q \wedge t \llbracket$$

$$= \bar{f}(p \vee s, q \wedge t),$$

(ii) $\bar{f}(p,q) \vee \bar{f}(s,t)$ (where $p \leq s < q \leq t$)

$$= f^{-1}\rrbracket p,q \llbracket \cup f^{-1}\rrbracket s,t \llbracket$$

$$= f^{-1}(\rrbracket p,q \llbracket \cup \rrbracket s,t \llbracket)$$

$$= f^{-1}\rrbracket p,t \llbracket$$

$= \bar{f}(p,t),$

(iii) $\bigvee\{\bar{f}(s,t) : p < s < t < q\}$

$= \bigcup\{f^{-1}\,]\!]s,t[\![: p < s < t < q\}$

$= f^{-1}(\bigcup\{\,]\!]s,t[\![: p < s < t < q\})$

$= f^{-1}\,]\!]p,q[\![$

$= \bar{f}(p,q)$ and

(iv) $\bigvee\{\bar{f}(p,q) : (p,q) \in \mathbb{Q} \times \mathbb{Q}\}$

$= \bigcup\{f^{-1}\,]\!]p,q[\![: (p,q) \in \mathbb{Q} \times \mathbb{Q}\}$

$= f^{-1}(\bigcup\{\,]\!]p,q[\![: (p,q) \in \mathbb{Q} \times \mathbb{Q}\})$

$= f^{-1}(\mathbb{R})$

$= e.$

Furthermore
$$\begin{aligned}\bar{f}(p,-) = \bar{f}(\bigvee\{(p,q) : q \in \mathbb{Q}\}) &= \bigvee\{\bar{f}(p,q) : q \in \mathbb{Q}\} \\ &= \bigcup\{f^{-1}\,]\!]p,q[\![: q \in \mathbb{Q}\} \\ &= f^{-1}\,]\!]p,\infty[\![\\ &\in (\mathcal{O}X)_1 \quad (\text{since } f : X \longrightarrow \mathbb{R}_b \quad \text{is continuous}).\end{aligned}$$

Similary $\bar{f}(-,q) \in (\mathcal{O}X)_2$. The above properties show that $\bar{f}$ can be uniquely extended into a biframe homomorphism from $\mathcal{L}(\mathbb{R}_b)$ to $\mathcal{O}X$.

Lastly we check that $\sim$ and $-$ are inverses. $\tilde{\bar{f}}(x) = \rho(\hat{x}\bar{f})$ by definition, and then

$$\begin{aligned}
\tilde{\bar{f}} \in \rrbracket p,q\llbracket \quad &\text{iff} \quad \hat{x}\bar{f}(p,q) = 1 \\
&\text{iff} \quad x \in \bar{f}(p,q) \\
&\text{iff} \quad x \in f^{-1}\rrbracket p,q\llbracket \\
&\text{iff} \quad f(x) \in \rrbracket p,q\llbracket
\end{aligned}$$

hence $\tilde{\bar{f}}(x) = f(x)$. We show that $\overline{\tilde{\varphi}} = \varphi$:

$$\begin{aligned}
\overline{\tilde{\varphi}}(p,q) &= \tilde{\varphi}^{-1}\rrbracket p,q\llbracket \\
&= \varphi(p,q) \text{ for all } (p,q) \in \mathcal{L}_0(\mathbb{R}_b),
\end{aligned}$$

hence $\overline{\tilde{\varphi}} = \varphi$. ■

3.2 Pseudocompactness in biframes

Definition 3.2.1: *For any biframe L,*

(a) *A biframe homomorphism $\varphi : \mathcal{L}(\mathbb{R}_b) \longrightarrow L$ is called bounded if $\varphi(p,q) = e$ for some $p, q \in \mathbb{Q}$, and*

(b) *L is called pseudocompact if all biframe homomorphisms $\varphi : \mathcal{L}(\mathbb{R}_b) \longrightarrow L$ are bounded.*

Recall that for a biframe L and $x \in L_i$, $x^* = \bigvee\{z \in L_j : x \wedge z = 0\}$ $(i \neq j)$.

We show that our definition of pseudocompactness is compatible with that for bitopological spaces.

Proposition 3.2.2: *Let X be a bitopological space, then X is pseudocompact iff $\mathcal{O}X$ is pseudocompact.*

Proof: $\Longrightarrow$: Let $\varphi : \mathcal{L}(\mathbb{R}_b) \longrightarrow \mathcal{O}X$ be a biframe homomorphism, then there is a corresponding continuous function $\tilde{\varphi} : X \longrightarrow \mathbb{R}_b$ such that φ is bounded iff $\tilde{\varphi}$ is bounded (see proof of 3.1.3.) By pseudocompactness of X, φ is bounded.

$\Longleftarrow$: Let $f : X \longrightarrow \mathbb{R}_b$ be continuous, then with notation as in 3.1.3 $\bar{f} : [\mathcal{L}_0(\mathbb{R}_b)] \longrightarrow \mathcal{O}X$ determines a biframe homomorphism such that f is bounded iff the unique extension of $\bar{f}$ is bounded. Now by pseudocompactness of $\mathcal{O}X$, f is bounded. ■

We will need the following lemmas, which will assume the following hypotheses. Given that $a_0 \prec\!\prec_2 a_1 \prec\!\prec_2 a_2 \prec\!\prec_2 \ldots$ is a sequence in a biframe L with $\{a_n\} \subseteq L_2$ and $\bigvee a_n = e$. Let $(c_{nq} : q \in \mathbb{Q} \cap [0,1])$ be an interpolating sequence between a_n and a_{n+1} for each n, put

$$c_r = \begin{cases} 0 & (r < 0) \\ c_{nr-n} & (n \leq r \leq n+1), \end{cases}$$

then define

$$\varphi(p,q) = \bigvee \{c^*_{p'} \wedge c_{q'} : p < p' < q' < q\}.$$

Lemma 3.2.3(a): For any $q \in \mathbb{Q}$, $\displaystyle\bigvee_{p,p',q' \in \mathbb{Q}} \{c^*_{p'} \wedge c_{q'} : p < p' < q' < q\} = \bigvee_{q' \in \mathbb{Q}} \{c_{q'} : q' < q\}$.

Proof: $\leq$: is clear.

$\geq$: Consider $c_{q'}$ with $q' < q$, choose p and p' such that $p' < 0$ and $p < p' < q' < q, q$ in $\mathbb{Q}$. Then $c_{p'} = 0$, hence $c^*_{p'} = e$. Thus $c^*_{p'} \wedge c_{q'} = c_{q'}$, resulting in

$$\bigvee_{p \in \mathbb{Q}} \bigvee \{c^*_{p'} \wedge c_{q'} : p < p' < q' < q\} \geq \bigvee \{c_{q'} : q' < q\}$$

which completes the proof. ■

Lemma 3.2.3(b): For any $p \in \mathbb{Q}$, $\displaystyle\bigvee_{p',q'q \in \mathbb{Q}} \{c^*_{p'} \wedge c_{q'} : p < p' < q' < q\} = \bigvee_{p' \in \mathbb{Q}} \{c^*_{p'} : p < p'\}$.

Proof: $\leq$: is straightforward.

$\geq$: For any $p \in \mathbb{Q}$,

$$\bigvee_{p',q',q\in\mathbb{Q}} \{c^*_{p'} \wedge c_{q'} : p < p' < q' < q\}$$

$\geq c^*_{p'} \wedge \bigvee_{q',q\in Q} \{c_{q'} : p' < q' < q\}$ for each fixed p' such, that $p < p'$

$= c^*_{p'} \wedge e$ for each p', since $\bigvee_n a_n = e$

$= c^*_{p'}$.

Consequently, for any $p \in \mathbb{Q}$

$$\bigvee_{p',q',q\in\mathbb{Q}} \{c^*_{p'} \wedge c_{q'} : p < p' < q' < q\} \geq \bigvee_{p'\in\mathbb{Q}} \{c^*_{p'} : p < p'\}$$

Lemma 3.2.3(c): For any $p, q \in \mathbb{Q}$, $\varphi(p,q) = \bigvee_{p'q'\in\mathbb{Q}} \{c^*_{p'} \wedge c_{q'} : p < p' < q' < q\}$ *transforms the relations* (R1)-(R4) *in the definition of the biframe of reals into identities in* L_0.

Proof: The proofs are essentially those for the case of frames (see [6]) and are given here for completeness.

(i) We show that $\varphi(p,q) \wedge \varphi(s,t) = \varphi(p \vee s, q \wedge t)$:

$\varphi(p,q) \wedge \varphi(s,t)$

$= \bigvee\{c^*_{p'} \wedge c_{q'} : p < p' < q' < q\} \wedge \bigvee\{c^*_{s'} \wedge c_{t'} : s < s' < t' < t\}$

$= \bigvee\{(c^*_{p'} \wedge c^*_{s'}) \wedge (c_{q'} \wedge c_{t'}) : p < p' < q' < q, s < s' < t' < t\}$

$\overset{(a)}{=} \bigvee\{c^*_{p'\vee s'} \wedge c_{q'\wedge t'} : p < p' < q' < q, s < s' < t' < t\}$

$= \varphi(p \vee s, q \wedge t)$.

We show that (a) holds by checking that $c^*_p \wedge c^*_q = c^*_{p\vee q}$ and $c_p \wedge c_q = c_{p\wedge q}$:

Indeed $c_p^* \wedge c_q^* \geq c_{p\vee q}^*$, since

$$p \leq p \vee q \implies c_p^* \geq c_{p\vee q}^*, \text{ and}$$

$$q \leq p \vee q \implies c_q^* \geq c_{p\vee q}^*.$$

Next we show that $c_p^* \wedge c_q^* \leq c_{p\vee q}^*$: If $b \geq c_{p\vee q}^* = \bigvee\{a \in L_1 : a \wedge (p \vee q) = 0\}$

$$= \bigvee\{a \in L_1 : (a \wedge p) \vee (a \wedge q) = 0\},$$

then $b \geq \bigvee\{a \in L_1 : a \wedge p = 0\} \wedge \bigvee\{a \in L_1 : a \wedge q = 0\} = c_p^* \wedge c_q^*$. Consequently $c_p^* \wedge c_q^* \leq c_{p\vee q}^*$.

Lastly we check that $c_p \wedge c_q = c_{p\wedge q}$:

Clearly $c_{p\wedge q} \leq c_p \wedge c_q$.

Next observe that $p \wedge q = p$ or q, hence $c_{p\wedge q} = c_p$ or c_q, thus $c_p \wedge c_q \leq c_{p\wedge q}$. This completes the proof of relation (R1).

(ii) Next we check that $\varphi(p,q) \vee \varphi(r,s) = \varphi(p,s)$ whenever $p \leq r < q \leq s$. Since, $[\varphi(p,q) \vee \varphi(r,s)] \wedge \varphi(p,s)$

$= [\varphi(p,q) \wedge \varphi(p,s)] \vee [\varphi(r,s) \wedge \varphi(p,s)]$

$= \varphi(p,q) \vee \varphi(r,s)$ (by (i)),

we have that $\varphi(p,q) \vee \varphi(r,s) \leq \varphi(p,s)$.

For the reverse inequality: Consider $p < p' < s' < s$ such that $r < p'$ or $s' < q$, then $c_{p'}^* \wedge c_{s'} \leq \varphi(r,s) \vee \varphi(p,q)$, since $r < p' < s' < s$ or $p < p' < s' < q$.

It remains to show the inequality when $p' \leq r$ and $q \leq s'$. For this, choose t and t' such that $r < t < t' < q$

we have,

$$\begin{aligned}\varphi(p,q) \vee \varphi(r,s) &\geq (c_{p'}^* \wedge c_{t'}) \vee (c_t^* \wedge c_{s'})\\ &= (c_{p'}^* \vee c_t^*) \wedge (c_{t'} \vee c_t^*) \wedge (c_{p'}^* \vee c_{s'}) \vee (c_{t'} \vee c_{s'})\\ &\geq c_{p'}^* \wedge c_{s'}.\end{aligned}$$

Hence $\varphi(p,q) \vee \varphi(r,s) > \varphi(p,s)$.

(iii) $\bigvee\{\varphi(s,t) : p < s < t < q\}$

$= \bigvee\{c_{s'}^* \wedge c_{t'} : s < s' < t' < t, p < s < t < q\}$

$= \bigvee\{c_s^* \wedge c_t : p < s < t < q\}$

$= \varphi(p,q)$.

(iv) $\bigvee\{\varphi(p,q) : (p,q) \in \mathbb{Q} \times \mathbb{Q}\}$

$= \bigvee\{c_{p'}^* \wedge c_{q'} : p < p' < q' < q, \text{ all } p, q\}$

$= \bigvee\{c_p^* \wedge c_q : \text{ all } p, q\}$

$= (\bigvee_{p\in\mathbb{Q}} c_p^*) \wedge (\bigvee_{q\in\mathbb{Q}} c_q)$

$= e \wedge e$

$= e.$ ∎

The above 3 lemmas will be used in the proof of the following characterisation of pseudocompactness. This proposition will be used in proving that pseudocompactness of a biframe is equivalent to compactness of the first and second parts of its cozero part.

The arguments in the above three lemmas apply in the case where the given sequence $\{a_n\}$ is in L_2 now consider the case where $a_0 \prec\prec_1 a_1 \prec\prec_1 a_2 \prec\prec_1 \cdots$ is a sequence in L with $\{a_n\}$ in L_1 and $\bigvee a_n = e$. Let $(c_{nq} : q \in \mathbb{Q} \cap [0,1])$ be an interpolating sequence between a_n and a_{n+1}, set

$$c_r = \begin{cases} 0 & (r > 0) \\ c_{n(-r-n)} & n \leq -r \leq n+1 \end{cases}$$

and define

$$\tau(p,q) = \bigvee \{c^*_{p'} \wedge c_{q'} : p < p' < q' < q\}.$$

The arguments given above can be modified for this situation.

Proposition 3.2.4: *A biframe L is pseudocompact iff any sequence $a_0 \prec\prec_i a_1 \prec\prec_i a_2 \prec\prec_i \ldots$ such that $\bigvee_n a_n = e$ in L_i terminates, that is $a_m = e$ for some m.*

Proof: $\Longrightarrow$: Given $a_0 \prec\prec_i a_1 \prec\prec_i a_2 \prec\prec_i \ldots$ such that $\bigvee a_n = e$, let c_{nq} and $\varphi(p,q)$ be as in the above 3 lemmas. By Lemma 3.2.3(c), φ transforms the relations (R1) - (R4) into identities in L_0.

We now check that φ preserves the generators of the first and second parts of $\mathcal{L}(\mathbb{R}_b)$ and thus determines a biframe homomorphism.

$$\begin{aligned} \varphi(-,q) &= \bigvee \{\varphi(p,q) : p \in \mathbb{Q}\} \\ &= \bigvee_{p \in \mathbb{Q}} \bigvee \{c^*_{p'} \wedge c_{q'} : p < p' < q' < q\} \\ &= \bigvee \{c_{q'} : q' < q\} \in L_i \quad \text{by Lemma 3.2.3(a), and} \\ \varphi(p,-) &= \bigvee \{\varphi(p,q) : q \in \mathbb{Q}\} \\ &= \bigvee_{q \in \mathbb{Q}} \bigvee \{c^*_{p'} \wedge c_{q'} : p < p' < q' < q\} \\ &= \bigvee \{c^*_{p'} : p < p'\} \in L_k \quad i \neq k \quad \text{by Lemma 3.2.3.(b).} \end{aligned}$$

Since L is pseudocompact, $\varphi : \mathcal{L}(\mathbb{R}_b) \longrightarrow L$ is bounded hence $\varphi(p,q) = e$ for some $p, q \in \mathbb{Q}$.

Now, for $m \geq q, m \in \mathbb{N}$

$$\begin{aligned} a_m &\geq \bigvee\{c_{q'} : p < q' < q\} \\ &\geq \bigvee\{c^*_{p'} \wedge c_{q'} : p < p' < q' < q\} \\ &= \varphi(p,q). \quad \text{Hence } a_m = e. \end{aligned}$$

$\Longleftarrow$: Consider a biframe homomorphism $h : \mathcal{L}(\mathbb{R}_b) \longrightarrow L$. We must show that h is bounded: Let $a_n = h(-,n)$, observe that $a_n \prec\prec_2 a_{n+1} \prec\prec_2 \ldots$ and $\bigvee\limits_n a_n = e$. By hypotheses $a_m = e$ for some m. Next consider $b_n = h(-n,-)$ which gives us that $b_n \prec\prec_1 b_{n+1} \prec\prec_1 \ldots$ and $\bigvee\limits_n b_n = e$ hence $b_j = e$ for some j.

$$\begin{aligned} \text{Thus } e &= a_m \wedge b_j \\ &= h(-,m) \wedge h(-j,-) \\ &= h(-j,m) \end{aligned}$$

■

We recall that $(cozL)_i$ denotes $coz_\ell L_1$ or $coz_u L_2$.

We introduce a characterisation of pseudocompactness of a biframe which involves compactness of both the first and second parts of its cozero part.

Proposition 3.2.5: *L is a pseudocompact biframe iff both $coz_\ell L_1$ and $coz_u L_2$ are compact.*

Proof: $\Rightarrow$: Suppose that $e = \bigvee\limits_n\{a_n : a_n \in (cozL)_i\}$, then $a_n = \bigvee a_{nk}$ for some sequence $\{a_{nk}\} \subseteq L_i$ such that $a_{nk} \prec\prec_i a_{nk+1}$ for all k (see 2.2.8). Put $c_n = a_{1n} \vee a_{2n} \vee \ldots \vee a_{nn}$. Then $c_n \prec\prec_i c_{n+1}$ and $\bigvee c_n = e$, hence $c_n = e$ for some n by pseudocompactness, consequently $a_1 \vee a_2 \vee \ldots \vee a_n = e$ as required. ■

$\Longleftarrow$: Suppose that $a_0 \prec\prec_i a_1 \prec\prec_i a_2 \prec\prec_i \ldots$ such that $\bigvee a_n = e$ in L_i. Now $a_0 \prec\prec_i a_1$ implies that there exists $b_1 \in (cozL)_i$ such that $a_0 \prec\prec_i b_1 \prec\prec_i a_1$. This gives us a sequence $b_1 \prec\prec_i b_2 \prec\prec_i \ldots$ in $(cozL)_i$ such that $\bigvee b_n = e$. Compactness of $(cozL)_i$ implies that $b_1 \vee b_2 \vee \ldots \vee b_n = e$ for some n. But there is an a_n in the above sequence such that $b_m \prec\prec_i a_n$ for all m where $1 \leq m \leq n$, yielding $a_n = e$. ■

CHAPTER 4

COMPACT REGULAR BI σ-FRAMES AND STABLY CONTINUOUS σ-FRAMES

Stably continuous σ-frames have been studied by among others Madden and Walters-Wayland. Our intention here is to prove that **St Cont σFRM** is equivalent to **KReg BIσFRM**. This equivalence is accomplished by extending the results of [4].

4.1 The Lawson dual and the congruence lattice of a stably continuous σ-frame

We give the necessary tools to construct functors between the above categories.

For the next definitions let $L \in \sigma$**FRM**.

Definition 4.1.1: (a) *An ideal $J \subseteq L \in \sigma$**FRM** is called countably generated if there is a sequence $\{x_n\}$ such that $a \in J$ implies that $a \leq x_n$ for some n.*

(b) *For $x, y \in L \in \sigma$**FRM**, x is said to be σ-way below y, written $x \ll_\sigma y$, if for any countable $X \subseteq L$ with $y \leq \bigvee X$, there exists finite $E \subseteq X$ with $x \leq \bigvee E$. In particular this means if $y \in \bigvee J$, for some countably generated ideal J, then $x \in J$.*

(c) *L is called continuous if $\ll_\sigma$ is a σ-approximating relation, equivalently for each $a \in L$ there exists a sequence $\{a_n\}$ such that $a_n \ll_\sigma a$ for each n and $a = \bigvee_n a_n$.*

(d) *L is stably continuous if L is continuous and the relation $\ll_\sigma$ is closed under finite meets in $L \times L$, i.e., $a \ll_\sigma b, c$ implies $a \ll_\sigma b \wedge c$ and $e \ll_\sigma e$.*

(e) *The σ-proper σ-frame homomorphisms $h : L \longrightarrow M$, are the σ-frame homomorphisms which preserve the $\ll_\sigma$ relation.*

The corresponding categories will be denoted by **Cont σFRM** *and* **St Cont σFRM** respectively.

Remark: Let a, b and c be elements of a continuous lattice L.

(a) If $a \ll_\sigma c$ and $b \ll_\sigma c$, then $a \vee b \ll_\sigma c$.

(b) There is a sequence $\{a_n\}$ such that $a = \bigvee_n a_n$ and $a_n \ll_\sigma a_{n+1} \ll_\sigma a$ for all n.

(c) The $\ll_\sigma$ relation interpolates. That is, if $a \ll_\sigma b$, then there is some c for which $a \ll_\sigma c \ll_\sigma b$.

We now aim to introduce and study the Lawson dual of a σ-frame.

Definition 4.1.2: *A filter $F \subseteq L \in \sigma$***FRM** *is called countably generated if there is a sequence $\{x_n\}$ in F such that $a \in F$ implies $x_n \leq a$ for some n.*

Lemma 4.1.3: *The collection $\Phi_\sigma L$ of all countably generated filters in $L \in \sigma$***FRM** *is a σ-frame.*

Proof: (a) If $F, G \in \Phi_\sigma L$, it is straightforward to show that $F \cap G \in \Phi_\sigma L$.

(b) Suppose $\{F_n\} \subseteq \Phi_\sigma L$, we first show that

$$\bigvee_n F_n = \{a \in L : a \geq f_{n_1} \wedge f_{n_2} \wedge \ldots \wedge f_{n_k}, \quad f_{n_j} \in F_{n_j},\ 1 \leq j \leq k\}.$$

Let F designate the right hand side displayed above. The fact that $F_n \subseteq F$ for each n follows immediately. Next we check that F is the smallest filter that contains $\bigcup_n F_n$. Suppose G is a filter containing $\bigcup_n F_n$, then $a \in F$ implies that $a \geq f_{n_1} \wedge f_{n_2} \wedge \ldots \wedge f_{n_k}$, $f_{n_j} \in F_{n_j}$, $1 \leq j \leq k$. But $a \in G$ since $f_{n_j} \in G$, for all j and G is a filter. Thus $F \subseteq G$. Consequently $\bigvee_n F_n = F$.

Next we check that $\bigvee_n F_n$ is indeed a countably generated filter. $\bigvee_n F_n$ is countably

generated for if we take finite meets of generators of the F_n's then we obtain a countable set which generates $\bigvee_n F_n$. If $a, b \in \bigvee_n F_n$, then there exist f_{n_α}'s and f_{m_β}'s such that $a \geq f_{n_1} \wedge f_{n_2} \wedge \ldots \wedge f_{n_\ell}$ and $b \geq f_{m_1} \wedge f_{m_2} \wedge \ldots \wedge f_{m_u}$. Thus
$a \wedge b \geq \bigwedge_{1 \leq \alpha \leq \ell} \{f_{n_\alpha} \wedge f_{m_\beta} : 1 \leq \beta \leq u\}$, hence $a \wedge b \in \bigvee_n F_n$. It is easily seen that $\bigvee_n F_n$ is an upset.

(c) Lastly we check that the distribution law holds. We show that $G \cap \bigvee_n F_n = \bigvee_n (G \cap F_n)$ whenever $G, \{F_n\} \subseteq \Phi_\sigma L$.

$\subseteq$: $a \in G \cap \bigvee_n F_n$ implies that $a \in G$ and $a \geq f_{n_1} \wedge \ldots \wedge f_{n_\ell}$, $f_{n_j} \in F_{n_j}$, $1 \leq j \leq \ell$.

Thus $a \leq a \vee (f_{n_1} \wedge f_{n_2} \wedge \ldots \wedge f_{n_\ell}) = \bigwedge_{j=1}^{\ell} (a \vee f_{n_j})$.

But $(a \vee f_{n_j}) \in G \cap F_{n_j}$ for each j. Therefore $a \in \bigvee_n (G \cap F_n)$.

$\supseteq$: $b \in \bigvee_n (G \cap F_n)$ implies that $b \geq h_{n_1} \wedge h_{n_2} \wedge \ldots \wedge h_{n_\ell}$, $h_{n_j} \in G \cap F_{n_j}$, $1 \leq j \leq \ell$. Thus $b \in G$ and $b \in \bigvee_n F_n$ resulting in $b \in G \cap \bigvee_n F_n$. ■

We have the following characterisation of the σ-way below relation in the σ-frame $\Phi_\sigma L$.

Lemma 4.1.4: Let $L \in \sigma\mathbf{FRM}$, then *$F \ll_\sigma G$ in $\Phi_\sigma L$ iff $F \subseteq \uparrow a \subseteq G$ for some $a \in L$.*

Proof: Clearly $G = \bigvee \{\uparrow x_n | n \in \mathbb{N}\}$ for a generating sequence $\{x_n\}$.

$\Longrightarrow$: Since $G = \bigvee_{x_n \in G} \uparrow x_n$ and $F \ll_\sigma G$ we have
$F \subseteq \uparrow x_{n_1} \vee \uparrow x_{n_2} \vee \ldots \vee \uparrow x_{n_m} = \uparrow (x_{n_1} \wedge x_{n_2} \wedge \ldots \wedge x_{n_m}) \subseteq G$ and $x_{n_1} \wedge x_{n_2} \wedge \ldots \wedge x_{n_m} \in L$.

$\Longleftarrow$: Suppose $F \subseteq \uparrow a \subseteq G$ for some $a \in L$ and $G \subseteq \bigvee_n G_n$. Since $a \in \uparrow a \subseteq \bigvee_n G_n$, we have that $a \geq g_{n_1} \wedge g_{n_2} \wedge \ldots \wedge g_{n_m}$, $g_{n_j} \in G_{n_j}$, $1 \leq j \leq m$. Now $b \in \uparrow a$ implies $b \geq g_{n_1} \wedge \ldots \wedge g_{n_m}$. Hence $b \in \bigvee_{j=1}^{m} G_{n_j}$ which results in $\uparrow a \subseteq \bigvee_{j=1}^{m} G_{n_j}$. Hence $F \subseteq \bigvee_{j=1}^{m} G_{n_j}$ which completes the proof. ■

Definition 4.1.5: *A subset $U \subseteq L \in \sigma$**FRM** is called σ-Scott open if U is an upset and for any countable $X \subseteq L$, $\bigvee X \in U$ implies that there exists a finite $E \subseteq X$ such that $\bigvee E \in U$.*

In **Cont** σ**FRM** we have the following characterisation of σ-Scott open subsets:

Proposition 4.1.6: [29] *In a continuous σ-frame L, an upset U is σ-Scott open iff for each x in U, there exists $y \in U$ such that $y \ll_\sigma x$.*

Proof: $\Longrightarrow$: Let U be σ-Scott open in a continuous σ-frame L. Because L is continuous $x \in U$ implies $x = \bigvee\limits_n x_n$ where $x_n \ll_\sigma x$. Since U is σ-Scott open, there exists a $y = \bigvee\limits_{i=1}^{m} x_{n_i}$ such that $y \in U$ and $y \ll_\sigma x$.

$\Longleftarrow$: Assume that the above hypotheses holds for all x in some upset U of L. If $\bigvee X \in U$ for some countable $X \subseteq L$, then we can find $y \in U$ such that $y \ll_\sigma \bigvee X$ in U. Hence there exists a finite $E \subseteq X$ with $y \leq \bigvee E$. But $\bigvee E \in U$ because U is an upset. Thus U is σ-Scott open. ■

Notation: The Lawson dual $\Lambda_\sigma L$ designates the set of countably generated σ-Scott open filters of the σ-frame L.

Lemma 4.1.7: *The correspondence $L \rightsquigarrow \Lambda_\sigma L$ determines a covariant functor*
Λ_σ : **St Cont** σ**FRM** $\longrightarrow$ **St Cont** σ**FRM**.

Proof: We show that for any stably continuous σ-frame L, $\Lambda_\sigma L$ is a sub σ-frame of $\Phi_\sigma L$, and hence a σ-frame. If F and G are σ-Scott open in $\Phi_\sigma L$ it is easy to see that $F \cap G$ and $F \vee G$ are also σ-Scott open filters using the above characterisation and stable continuity of L. This implies, that the join $\bigvee\limits_n F_n$ of any σ-Scott open filters F_n in L is σ-Scott open as it is the union of the finite subjoins of the F_n's. Now $\bigvee \emptyset = \{e\}$ which is σ-Scott open because $e \ll_\sigma e$, by stable continuity of L.

Secondly we check that $\Lambda_\sigma L$ is stably continuous. If $G \in \Lambda_\sigma L$ and $\{x_n\}$ is a generating set, let $x_1 = a_1$, then there exists $b_1 \in G$ such that $b_1 \ll_\sigma a_1$ since G is σ-Scott open. Let $a_2 = x_2 \wedge b_1 \leq b_1 \ll_\sigma a_1$. Continuing the process, choose $b_n \in G$ such that $b_n \ll_\sigma a_n$ and set $a_{n+1} = x_{n+1} \wedge b_n$. Note that the a_n's generate G, because the x_n's do. We have constructed a generating set $\{a_n\}$ for G such that $a_{n+1} \ll_\sigma a_n$. In between a_{n+1} and a_n we can interpolate a countable set $a_{n+1} \ll_\sigma \dots \ll_\sigma b_{n(m+1)} \ll_\sigma b_{nm} \ll_\sigma \dots \ll_\sigma b_{n1} \ll_\sigma a_n$. Let $\{b_{n_m}\}$ generate the filter F_n in G then by Proposition 4.1.6, $F_n \in \Lambda_\sigma L$ and $G = \bigvee\{F_n : n \in \mathbb{N}\}$, for $\uparrow a_n \subseteq F_n \subseteq \uparrow a_{n+1}$ for all $n \in \mathbb{N}$. By exactly the same argument as that in Lemma 4.1.4, we have that $F_n \ll_\sigma G$ in $\Lambda_\sigma L$ iff $F_n \subseteq \uparrow a_{n+1} \subseteq G$.

Suppose $F \ll_\sigma G, H$ in $\Lambda_\sigma L$, then $F \ll_\sigma F \cap H$ since if $F \subseteq \uparrow a \subseteq G$ and $F \subseteq \uparrow b \subseteq H$ for some $a, b \in L$, then $F \subseteq \uparrow (a \vee b) \subseteq G \cap H$. Furthermore the unit $\uparrow 0$ of $\Lambda_\sigma L$ is compact so $\Lambda_\sigma L$ is stably continuous.

Lastly we check morphisms. If $h : L \longrightarrow M$ is a morphism in **St Cont σFRM**, then the filter $\langle h(F) \rangle$ generated by the image of any $F \in \Lambda_\sigma L$ is σ-Scott open and countably generated. If $a \in \langle h(F) \rangle$, then $h(f_1) \leq a$ for some $f_1 \in F$. But F is σ-Scott open so there exists $f_2 \in F$ such that $f_2 \ll_\sigma f_1$, then $h(f_2) \ll_\sigma h(f_1) \leq a$, thus $h(f_2) \ll_\sigma a$ and $h(f_2) \in \langle h(F) \rangle$, consequently $\langle h(F) \rangle$ is σ-Scott open. $\langle h(F) \rangle$ is countably generated for if $\{x_n\}$ generates F, $a \in \langle h(F) \rangle$ implies $h(f) \leq a$ for some $f \in F$, thus $x_n \leq f$ for some n, so that $h(x_n) \leq h(f) \leq a$. Consequently $\langle h(F) \rangle$ is countably generated by $\{h(x_n)\}$. $\Lambda_\sigma h : \Lambda_\sigma L \longrightarrow \Lambda_\sigma M$ is σ-proper for if $F \ll_\sigma G$ in $\Lambda_\sigma L$, then $F \subseteq \uparrow a \subseteq G$ for some $a \in L$, which implies that $\langle h(F) \rangle \subseteq \uparrow h(a) \subseteq \langle h(G) \rangle$, this yields $\langle h(F) \rangle \ll_\sigma \langle h(G) \rangle$.

We conclude that Λ_σ : **St Cont σFRM**$\longrightarrow$ **St Cont σFRM** is indeed functorial. ■

We now concentrate on congruences on a σ-frame.

Definition 4.1.8: *A congruence on a σ-frame L is an equivalence relation on L which is a sub σ-frame of the product $L \times L$.*

Notation: The lattice denoted by CL of all congruences on a σ-frame L, partially ordered by inclusion, is a frame generated by the congruences of the form $\triangle_a = \{(x,y) : x \wedge a = y \wedge a\}$ and $\nabla_b = \{(x,y) : x \vee b = y \vee b\}$ for $a, b \in L$ [6].

$\triangle = \{(x,x) : x \in L\}$, $\nabla = \{(x,y) : x, y \in L\}$.

Definition 4.1.9: *For a σ-frame L and* $F \in \Phi_\sigma L$ *we define* $\triangle_F = \bigvee \triangle_a (a \in F)$.

Proposition 4.1.10: *For any σ-frame L, the map* $F \longmapsto \triangle_F = \bigvee\limits_{a \in F} \triangle_a$ *is a σ-frame homomorphism* $\Phi_\sigma L \longrightarrow CL$.

Proof: The zero and the unit of $\Phi_\sigma L$ are $\uparrow e$ and $\uparrow 0$, respectively, and $\triangle_{\uparrow e} = \triangle_e = \triangle$ also $\triangle_{\uparrow 0} = \triangle_0 = \nabla$, hence this map preserves zero and unit.

Given $F, G \in \Phi_\sigma L$, then

$$\begin{aligned} \triangle_F \cap \triangle_G &= \bigvee_{a \in F} \triangle_a \cap \bigvee_{b \in G} \triangle_b \\ &= \bigvee_{\substack{a \in F \\ b \in G}} (\triangle_a \cap \triangle_b) \quad \text{(since } CL \text{ is a frame)} \\ &= \bigvee_{\substack{a \in F \\ b \in G}} (\triangle_{a \vee b}) \\ &= \triangle_{F \cap G} \quad \text{(because } F \cap G = \{a \vee b : a \in F, b \in G\}). \end{aligned}$$

For any $\{F_n : n \in \mathbb{N}\} \subseteq \Phi_\sigma L$ we have

$$\begin{aligned}
\triangle_{\bigvee_n F_n} &= \bigvee \triangle_a (a \in \bigvee_n F_n) \\
&= \bigvee \{\triangle_a | \exists a_1, \ldots, a_m, a_i \in F_{n_i}, a \geq a_1 \wedge \ldots \wedge a_m\} \\
&= \bigvee \{\triangle_{a_1 \wedge a_2 \wedge \ldots \wedge a_m} | a_i \in F_{n_i}, i \in \mathbb{N}\} \\
&= \bigvee \{\triangle_{a_1} \vee \ldots \vee \triangle_{a_m} | a_i \in F_{n_i}, i \in \mathbb{N}\} \\
&= \bigvee_n \{\triangle_a | a \in F_n\} \\
&= \bigvee_n \bigvee_{a \in F_n} \triangle_a \\
&= \bigvee_n \triangle_{F_n}
\end{aligned}$$

■

Notation: Θ_{ab} is the smallest congruence in a σ-frame L which contains (a, b).

Proposition 4.1.11: *Let L be a σ-frame $a, b \in L$, then $a \geq b$ iff $\Theta_{ab} = \nabla_a \cap \triangle_b$.*

Proof:

$\Longrightarrow$: $\subseteq$: For $a \geq b$ we have $a \wedge b = b = b \wedge b$ hence $(a, b) \in \triangle_b$. Also $a \vee a = a \vee b = a$ thus $(a, b) \in \nabla_a$. Hence $(a, b) \in \nabla_a \cap \triangle_b$, yielding $\Theta_{ab} \subseteq \nabla_a \cap \triangle_b$.

$\supseteq$: Conversely if $(x,y) \in \nabla_a \cap \triangle_b$, then

$$\begin{aligned} x &= x \wedge (x \vee a) \\ &= x \wedge (y \vee a) \quad (x,y) \in \nabla_a \\ &\equiv x \wedge (y \vee b) \quad (\text{since} \quad (a,b) \in \Theta_{ab}) \\ &= (x \wedge y) \vee (x \wedge b) \\ &= (x \wedge y) \vee (y \wedge b) \quad (\text{because} \quad (x,y) \in \triangle_b) \\ &= y \wedge (x \vee b) \\ &\equiv y \wedge (x \vee a) \\ &= y \wedge (y \vee a) \quad (\text{since} \quad (x,y) \in \nabla_a) \\ &= y. \end{aligned}$$

Observe that $\equiv$ is with respect to Θ_{ab}. Thus $(x,y) \in \Theta_{ab}$.

$\Longleftarrow$: $\Theta_{ab} = \nabla_a \cap \triangle_b$ implies that $(a,b) \in \nabla_a$, hence $a \vee a = a = a \vee b$ resulting in $a \geq b$. ■

Remark: In general for a, b in a σ-frame L $\nabla_a \cap \triangle_b = \nabla_{a\vee b} \cap \triangle_b = \nabla_a \cap \triangle_{a\wedge b}$.

Corollary 4.1.12: $\Theta_{e0} = \nabla$

Proof: $\Theta_{e0} = \nabla_e \cap \triangle_0$

$= \nabla \cap \nabla$

$= \nabla$

Definition 4.1.13: 1) *For $L \in \sigma$**FRM**, an element $a \in L$ is called compact if for every countable $X \subseteq L$ such that $a \leq \bigvee X$, there exists a finite $E \subseteq X$ with $a \leq \bigvee E$.* Thus $a \in L \in \sigma$**FRM** is compact iff $a \ll_\sigma a$.

2) *Let KL be the set of all compact elements of L. L is coherent if KL is a sublattice*

generating L, that is, KL is closed under finite meets and finite joins and each element of L is a countable join of compact elements.

3) *$h : L \longrightarrow M$ is coherent if h is a σ-frame homomorphism which preserves compact elements, that is, if $c \in KL$ then $h(c) \in KM$.*

The category of coherent σ-frames and coherent σ-frame homomorphisms is denoted by **Coh σFRM**.

4) *A σ-frame L is called noetherian when each of its elements is compact.*

The following proposition appears in [6]. We will omit the proof.

Proposition 4.1.14: *Let L be a σ-frame, then CL is compact iff L is noetherian.*

The following appears in [29]. The functor $\Im_\sigma : \mathcal{D} \longrightarrow \sigma\mathbf{FRM}$ assigns to any distributive lattice A the σ-frame consisting of all countably generated ideals of A, denoted $\Im_\sigma(A)$, and any lattice morphism $h : A \longrightarrow B$ to $\Im_\sigma h : \Im_\sigma A \longrightarrow \Im_\sigma B$ which takes any ideal in $\Im_\sigma A$ to the ideal generated by its image under h in B.

Lemma 4.1.15: [29] *For $A \in \mathcal{D}$, the principal ideals in $\Im_\sigma(A)$ are precisely the compact elements of $\Im_\sigma(A)$.*

Proof: Consider a principal ideal $\downarrow a \subseteq \bigvee J_n$ for some sequence $\{J_n\}$ of countably generated ideals of A, then $a \in \bigvee J_n$ so that $a \leq x_{i_1} \vee \ldots \vee x_{i_n}$ for $x_{i_j} \in J_j$. Hence $\downarrow a \subseteq J_{i_1} \vee \ldots \vee J_{i_n}$, and thus $\downarrow a$ is compact. Conversely if J is compact it is generated by principal ideals and is thus a countable join of principal elements, so that J is a finite join of principal elements and hence is itself principal. ■

Proposition 4.1.16: [29] $\Im_\sigma : \mathcal{D} \longrightarrow$ **Coh σFRM** *is functorial.*

Proof: We first show that $\Im_\sigma(A)$ is coherent for each $A \in \mathcal{D}$. Each $J \in \Im_\sigma A$ is countably generated, say by $\{a_n\}$. J can be expressed as a countable join of the principal ideals $\downarrow a_n$

which are compact by the above lemma. These are closed under finite intersections and unions and the top $\downarrow e$ is compact, hence $K(\Im_\sigma(A))$ is a sublattice generating $\Im_\sigma(A)$.

If $h : A \longrightarrow B$ in $\mathcal{D}$ then $\Im_\sigma h : \Im_\sigma A \longrightarrow \Im_\sigma B$ is coherent: $\Im_\sigma h$ is a σ-frame homomorphism since $\Im_\sigma : \mathcal{D} \longrightarrow \sigma\mathbf{FRM}$ is functorial, and $h(\downarrow a) = \downarrow h(a)$ so $\Im_\sigma h$ is coherent. ■

Proposition 4.1.17: *If $L \in$ **Coh** σ**FRM**, then $\Im_\sigma KL \cong L$.*

Proof: For each $L \in$ **Coh** σ**FRM** define $\sigma_L : \Im_\sigma KL \longrightarrow L$ by join, which is a σ-frame homomorphism.

σ_L is onto: For any $a \in L, a = \bigvee\limits_{a_n \in KL} a_n$ since L is coherent.

Let $J = \langle a_n \rangle$, i.e., the ideal generated by $\{a_n\}$ in KL. Then $\bigvee J = a$. We check that σ_L is one-one. First observe that for any $J \in \Im_\sigma KL$ and $c \in KL$, $c \ll_\sigma c$ means that $c \leq \bigvee J$ if and only if $c \in J$.

Suppose that $\bigvee J = \bigvee G$, for $J, G \in \Im_\sigma KL$. We show that $J = G$. $c \in J$ implies that $c \leq \bigvee J = \bigvee G$. This implies that $c \in G$ since $c \in KL$, and hence $c \ll_\sigma c$. The reverse inclusion is similar. We conclude that σ_L is one-one, and hence an isomorphism. ■

Lemma 4.1.18: *For any coherent σ-frame L, the subframe $K'L$ of CL generated by all $[a, b] = \nabla_a \cap \Delta_b$ for $a, b \in KL$, is compact.*

Proof: To show that $K'L$ is compact it suffices to check that $\nabla = \bigvee\limits_i [a_i, b_i]$ implies that $\nabla = [a_{i_1}, b_{i_1}] \vee \ldots \vee [a_{i_n}, b_{i_n}]$ for some $i_1, \ldots, i_n$, since for any $a, b, c, d \in KL$,

$$\begin{aligned}
[a, b] \cap [c, d] &= (\nabla_a \cap \Delta_b) \cap (\nabla_c \cap \Delta_d) \\
&= \nabla_{a \wedge c} \cap \Delta_{b \vee d} \\
&= [a \wedge c, b \vee d]
\end{aligned}$$

and $K'L$ consists of joins of such elements.

Given such $[a_i, b_i](i \in I)$ (i.e $\{a_i, b_i\} \subseteq KL$) let their restrictions to $KL \times KL$ be $[a_i, b_i]|_{KL} = [a_i, b_i] \cap (KL \times KL)$. Our aim is to show that $\bigvee\limits_{i \in I}[a_i, b_i]|_{KL} = \nabla$. But since ∇ is compact in $C(KL)$ the result follows. Suppose all $[a_i, b_i]|_{KL} \subseteq \Theta \neq \nabla$ in $C(KL)$.

We must show that the latter assumption creates a contradiction hence $\bigvee\limits_{i \in I}[a_i, b_i]|_{KL} = \nabla$. Now one has the following commuting square

$$\begin{tikzcd}
KL \arrow[r, "i"] \arrow[d, "\nu"'] & L \cong \Im_\sigma(KL) \arrow[d, "\overline{\nu}"] \\
KL/_\Theta \arrow[r, "j"] & \Im_\sigma(KL/_\Theta),
\end{tikzcd}$$

where ν is the quotient homomorphism (see [20]), i the natural embedding, j the standard embedding of KL/Θ into its σ-ideal lattice taking elements to principal σ-ideals $(x \longmapsto \downarrow x \cap (KL/\Theta))$, and $\overline{\nu}$ is a σ-frame homomorphism since $\Im_\sigma$ is functorial.

Next we check that $\nabla = Ker(\overline{\nu})$, by showing that $\Theta \subseteq Ker(\overline{\nu})$ and $[a_i, b_i]$ is generated by $(b_i, a_i \vee b_i) \in KL \times KL$, hence $[a_i, b_i] \subseteq Ker(\overline{\nu})$.

$$\begin{aligned}
(x, y) \in \Theta &\Longrightarrow \nu(x) = \nu(y) \\
&\Longrightarrow j \circ \nu(x) = j \circ \nu(y) \\
&\Longrightarrow \overline{\nu} \circ i(x) = \overline{\nu} \circ i(y) \\
&\Longrightarrow \overline{\nu}(x) = \overline{\nu}(y), \quad \text{thus} \quad (x, y) \in Ker(\overline{\nu}).
\end{aligned}$$

$$\begin{aligned}
\text{Now } [a_i, b_i] &= \nabla_{a_i} \cap \Delta_{b_i} \\
&= \Theta_{a_i b_i} \quad \text{iff} \quad a_i \geq b_i \\
&= \Theta_{b_i\, b_i \vee a_i}
\end{aligned}$$

hence $(b_i, a_i \vee b_i) \in KL \times KL$ generates $[a_i, b_i]$.

We conclude that $[a_i, b_i] \subseteq Ker(\overline{\nu})$, $(i \in I)$, hence $\bigvee_i [a_i, b_i] = \bigtriangledown = Ker(\overline{\nu})$, thus

$$\begin{aligned} (0,e) \in Ker(\overline{\nu}), \quad \text{so that} \qquad \overline{\nu}(0) &= \overline{\nu}(e) \\ \Longrightarrow \qquad i \circ \overline{\nu}(0) &= i \circ \overline{\nu}(e) \\ \Longrightarrow \qquad j \circ \overline{\nu}(0) &= j \circ \overline{\nu}(e) \\ \Longrightarrow \qquad \nu(0) &= \nu(e) \end{aligned}$$

consequently $(0,e) \in Ker(\nu) = \Theta$ a contradiction to the fact that $(0,e) \notin \Theta$, so $\Theta = \bigtriangledown = \Theta_{0e}$ by Corollary 4.1.12. We have $\bigvee_i [a_i, b_i]|_{KL} = \bigtriangledown$ in $C(KL)$, by compactness of $\bigtriangledown$ in $C(KL)$ one gets

$$\bigtriangledown = [a_{i_1}, b_{i_1}]|_{KL} \vee \ldots \vee [a_{i_n}, b_{i_n}]|_{KL}$$

containing $(0,e)$. Consequently

$$\bigtriangledown = [a_{i_1}, b_{i_1}] \vee \ldots \vee [a_{i_n}, b_{i_n}] \in CL$$

since it contains $(0,e)$. Our result follows. ∎

Lemma 4.1.19: *For any regular σ-frame L, if two congruences Θ and Ψ on L have the same e-blocks, that is $\Theta[e] = \{x | (x,e) \in \Theta\} = \Psi[e]$, then $\Theta = \Psi$.*

Proof: Let $(a,b) \in \Theta$ and $a \leq b$. For any $x_n \prec b$ we can find $u \in L$ such that $u \wedge x_n = 0$ and $b \vee u = e$. Observe that $(a \vee u, e) = (a,b) \vee (u,u)$ belongs to Θ and hence to Ψ since $\Theta[e] = \Psi[e]$, consequently $(a \wedge x_n, x_n) = (a \vee u, e) \wedge (x_n, x_n)$ is in Ψ. Taking joins over x_n's we obtain $(\bigvee_n (a \wedge x_n), \bigvee_n x_n)$ is in Ψ, thus $(\bigvee_n x_n \wedge a, b) = (b \wedge a, b) = (a,b)$ is in Ψ by regularity of L. ∎

4.2 The equivalence between KReg BIσFRM and St Cont σFRM

We will denote the functors from **BIσFRM** to **σFRM** taking first and second parts by H and S respectively, and acting by restrictions on bi σ-frame homomorphisms.

KReg BIσFRM is the category of compact regular bi σ-frames with morphisms which are just the bi σ-frame homomorphisms.

Lemma 4.2.1: *H takes* **KReg BIσFRM** *into* **St Cont σFRM**.

Proof: We first show that for any compact regular bi σ-frame L, $a \ll_\sigma b$ holds in L_1 iff $a \prec_1 b$.

($\Longrightarrow$): Let $a \ll_\sigma b$, then $b = \bigvee\limits_{s_n \prec_1 b} s_n$ by the regularity of L. So $a \leq \bigvee\limits_{i=1}^{m} s_{n_i} = x$. But $\prec_i$ is closed under finite joins, hence $a \leq x \prec_1 b$, yielding $a \prec_1 b$.

($\Longleftarrow$): Conversely given $a \prec_i b$ and $b \leq \bigvee\limits_{n} s_n$ for some $\{s_n\} \subseteq L_1$. Take $c \in L_2$ such that $c \wedge a = 0$ and $c \vee b = e$. Then $\bigvee\limits_{n} s_n \vee c = e$ and hence $\bigvee\limits_{i=1}^{m} s_{n_i} \vee c = e$ by compactness of L. Therefore $a \prec_1 \bigvee\limits_{i=1}^{m} s_{n_i}$, which implies that $a \leq \bigvee\limits_{i=1}^{m} s_{n_i}$, and so $a \ll_\sigma b$.

Secondly regularity of L gives us continuity of L_1. Closure of $\prec_1$ under binary meets implies that of $\ll_\sigma$ in L_1. L_1 is compact since L is compact. Consequently L_1 is a stably continuous σ-frame.

Lastly, for any homomorphism $h : L \longrightarrow K$ between compact regular bi σ-frames, the map $h : L_1 \longrightarrow K_1$ induced by h preserves $\prec_1$, hence $\ll_\sigma$ and is thus proper. ∎

The following result connects the functors S and H on **KReg BiσFRM**.

Lemma 4.2.2: *On* **KReg BiσFRM**, *S and $\Lambda_\sigma H$ are naturally equivalent.*

Proof: Suppose that $L \in$ **KReg BiσFRM**, $F_b = \{a \in L_1 : a \vee b = e\}$ for each $b \in L_2$, we check that $F_b \in \Lambda_\sigma L_1$.

We show that F_b is a filter for each $b \in L_2$: Let $a, c \in F_b$, then $a \vee b = e = c \vee b$. Now $(a \wedge c) \vee b = (a \vee b) \wedge (c \vee b) = e$, which yields $a \wedge c \in F_b$. Given that $a \in F_b$ and $a \leq c$ in

L_1, then $c \vee b = e$ which yields $c \in F_b$.

We check that F_b is σ-Scott open for each $b \in L_2$: If $\bigvee D \in F_b$ for any countable subset D of L_1, then $(\bigvee D) \vee b = e$ implies $(\bigvee E) \vee b = e$ for a finite subset $E \subseteq D$ by compactness of L. Hence $\bigvee E \in F_b$ by the definition of F_b. We thus conclude that F_b is σ-Scott open.

Lastly we check that F_b is countably generated: Regularity of L and the fact that $\prec_2$ interpolates give us $b = \bigvee\limits_{b_n \prec_2 b_{n+1}} b_n$ for each b. Suppose c_n is a separating element for $b_n \prec_2 b_{n+1}$ for each n. Then we have that $c_{n+1} \prec_1 c_n$ with a separating element b_{n+1}.

The set $\{c_n : n \in \mathbb{N}\}$ generates F_b: Firstly $\{c_n\} \subseteq F_b$ since $c_n \vee b_{n+1} = e$ implies that $c_n \vee b = e$, for each n. If $a \in F_b$, then $a \vee b = e$ which implies that $a \vee \bigvee\limits_{b_n \prec_2 b_{n+1}} b_n = e$, hence by compactness $a \vee \bigvee\limits_{i=1}^{m} \{b_i : b_i \prec_2 b_{i+1}\} = e$, i.e., $a \vee b_m = e$, for some m. Now c_m is the separating element of $b_m \prec_2 b_{m+1}$, thus

$$\begin{aligned}
c_m &= c_m \wedge e \\
&= c_m \wedge (a \vee b_m) \\
&= (c_m \wedge a) \vee (c_m \wedge b_m) \\
&= (c_m \wedge a) \vee 0 \\
&= (c_m \wedge a),
\end{aligned}$$

therefore $c_m \leq a$, hence F_b is countably generated by $\{c_m\}$ such that $c_{m+1} \prec_1 c_m$, consequently $F_b \in \Lambda_\sigma L_1$.

Our next task is to show that

$$\begin{aligned}
\phi_L :\ & L_2 \longrightarrow \Lambda_\sigma L_1 \\
& b \longmapsto F_b
\end{aligned}$$

is an isomorphism.

To prove that ϕ_L is one-one it suffices to check that $F_b \subseteq F_c$ with $b, c \in L_2$ implies that $b \leq c$. If $x \prec_2 b$ then there exists $a \in L_1$ such that $a \wedge x = 0$ and $a \vee b = e$. Observe that

$a \in F_b$ and hence $a \in F_c$ thus $a \vee c = e$. This results in $x \prec_2 c$.

Since $b = \bigvee_{x_n \prec_2 b} x_n$, we conclude that $b \leq c$. We show that ϕ_L is indeed onto.

Let $F \in \Lambda_\sigma L_1$ and $\{y_n\}$ a set which generates F. We show that we can find a generating set $\{x_n\}$ for F such that $\ldots \prec_1 x_3 \prec_1 x_2 \prec_1 x_1$.

Put $y_1 = x_1$, then there exists $b_1 \in F$ such that $b_1 \prec_1 x_1$. Also suppose $x_2 = (y_2 \wedge b_1)$, then $y_2 \wedge b_1 \leq b_1 \prec_1 x_1$. Continuing the process, we let $x_m = (y_m \wedge b_{m-1})$ for some $b_{m-1} \in F$ such that $b_{m-1} \prec_1 x_{m-1}$, hence $x_m \prec_1 x_{m-1}$. Note that $\{x_n\}$ generates F for if $a \in F$ then there exists $y_m \leq a$, but $x_m = (y_m \wedge b_{m-1}) \leq y_m \leq a$.

Given a sequence $\{x_n\}$ in L_1 such that $x_{n+1} \prec_1 x_n$, for each n choose a single c_n which is a separating element for $x_{n+1} \prec_1 x_n$. Then $c_{n-1} \prec_2 c_n$ with a separating element x_n, hence $c_{n-1} \leq c_n$, thus the separating elements are increasing.

Define $S = \{c_n : c_n \text{ is a separating element of } x_{n+1} \prec_1 x_n\}$. Put $b = \bigvee S$, then $b \in L_2$ and we show that in fact $F_b = F$:

$\supseteq$: Pick $a \in F$, then there exists $x_n \in F$ such that $x_n \leq a$, so that $x_{n+1} \prec_1 x_n \leq a$, thus we can find $c_n \in S$ such that $c_n \wedge x_{n+1} = 0$ and $c_n \vee a = e$, which gives that $b \vee a = e$, hence $a \in F_b$, consequently $F \subseteq F_b$.

$\subseteq$: Let $a \in F_b$, then $a \vee b = e$, thus $a \vee (\bigvee S) = e$, which implies that $a \vee (\bigvee_{i=1}^{m} c_i) = e$ by compactness of L. But then S is an increasing sequence so $a \vee c_m = e$.

Now $x_{m+1} = x_{m+1} \wedge (a \vee c_m)$

$$= (x_{m+1} \wedge a) \vee (x_{m+1} \wedge c_m)$$

$$= (x_{m+1} \wedge a) \vee 0$$

$$= (x_{m+1} \wedge a),$$

which yields $x_{m+1} \leq a$, thus $a \in F$. We have shown that ϕ_L is an isomorphism for each $L \in$**KReg BIσFRM**, and it remains to show that this determines a natural equivalence.

We check that given any $h : L \longrightarrow M$ in **KReg BIσFRM** the following diagram commutes.

$$\begin{tikzcd}
L_2 \arrow[r, "\phi_L"] \arrow[d, "h_2"'] & \Lambda_\sigma(L_1) \arrow[d, "\Lambda_\sigma(h_1)"] \\
M_2 \arrow[r, "\phi_M"] & \Lambda_\sigma(M_1)
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, mapsto] \arrow[d, mapsto] & F_b \arrow[d, mapsto] \\
h(b) \arrow[r, mapsto] & F_{h(b)} = \langle h(F_b) \rangle
\end{tikzcd}$$

All we need to do is check that $F_{h(b)} = \langle h(F_b) \rangle$ for each $b \in L_2$.

$\supseteq$: If $a \in F_b$, then $a \vee b = e$, which implies that $h(a) \vee h(b) = e$, hence $h(a) \in F_{h(b)}$. Since the generating set $h(F_b) \subseteq F_{h(b)}$, then the whole filter $\langle h(F_b) \rangle \subseteq F_{h(b)}$.

$\subseteq$: If $c \in F_{h(b)}$, then $c \vee h(b) = e$, thus $c \vee h(\bigvee_{x_n \prec_2 x_{n+1}} x_n) = e$ (by regularity of L) $c \vee \bigvee_{x_n \prec_2 x_{n+1}} h(x_n) = e$ (since h is a homomorphism), there exists m such that $c \vee h(x_m) = e$ by compactness and the fact that $\{x_n\}$ is increasing. Take a separating element $d \in L_1$ with $x_m \wedge d = 0$ and $b \vee d = e$. Then $h(x_m) \wedge h(d) = 0$, hence $h(d) \prec_1 c$ which implies $h(d) \leq c$. But $d \in F_b$ so that $h(d) \in \langle h(F_b) \rangle$, consequently $c \in \langle h(F_b) \rangle$. ∎

Lemma 4.2.3: *For any $h : L \longrightarrow M$ in* **KReg BIσFRM** *if Hh is an isomorphism, then so is h.*

Proof: Whenever Hh is an isomorphism $\Lambda_\sigma Hh$ will be also an isomorphism since functors preserve isomorphisms. We have $Sh \cong \Lambda_\sigma Hh$ by the previous lemma. h is onto since both Sh and Hh are.

We show that h is one-one: We need only prove that $h : L_0 \longrightarrow M_0$ is one-one. Both L_0 and M_0 are regular σ-frames so we only check that h is codense. Suppose $h(x) = e$ for $x = \bigvee_n (a_n \wedge b_n)$ where $\{a_n\} \subseteq L_1$ and $\{b_n\} \subseteq L_2$. Then $\bigvee_n (h(a_n) \wedge h(b_n)) = e$, and by compactness we can find a finite number of terms with indices $1, 2, \ldots, m$ such that

$$(h(a_1) \wedge h(b_1)) \vee \ldots \vee (h(a_m) \wedge h(b_m)) = e. \tag{1}$$

We want to conclude from this equation that

$$(a_1 \wedge b_1) \vee \ldots \vee (a_m \wedge b_m) = e. \tag{2}$$

We prove this fact by mathematical induction.

Suppose $m = 1 : h(a_1) \wedge h(b_1) = e$ implies

$$h(a_1) = h(b_1) = e = h(e), \text{ hence}$$

$$a_1 = b_1 = e \text{ because}$$

Hh is one-one.

If (1) $\Longrightarrow$ (2) holds for any natural number less than m we have:

(1) implies that $h(a_1) \vee (h(a_2) \wedge h(b_2)) \vee \ldots \vee (h(a_m) \wedge h(b_m)) = e$, therefore

$$[(h(a_1) \vee h(a_2)) \wedge (h(a_1) \vee h(b_2))] \vee \ldots \vee [(h(a_1) \vee h(a_m)) \wedge (h(a_1) \vee h(b_m))] = e,$$

then

$$(h(a_1 \vee a_2) \wedge h(a_1 \vee b_2)) \vee \ldots \vee (h(a_1 \vee a_m) \wedge h(a_1 \vee b_m)) = e. \tag{3}$$

Note that for any $c \in L_1$, the compact bi σ-frame $L' = (\uparrow c, L_1 \cap \uparrow c, \{x \vee c : x \in L_2\})$ is regular.

On the first part consider $a \in L_1 \cap \uparrow c$, then $a \in L_1$ and $a \geq c$. Since L is regular $a = \bigvee_{x_n \prec_1 a} x_n$ with separating element $s_n \in L_2$ for each $x_n \prec_1 a$. Now $a = \bigvee_{x_n \vee c \prec_1 a} (x_n \vee c)$.

For each n, $(s_n \vee c) \vee a = e$ and

$(x_n \vee c) \wedge (s_n \vee c) = (x_n \wedge s_n) \vee c = 0 \vee c = c$

which is the bottom element of $\uparrow c$. We conclude that $(s_n \vee c) \in (L')_2$ is the separating element of $(x_n \vee c) \prec_1 a$ for each n. On the second part

$x \vee c = (\bigvee_{y_n \prec_2 x} y_n) \vee c$ for $x \in L_2$. Thus $x \vee c = \bigvee_{y_n \vee c \prec_2 x \vee c} (y_n \vee c)$. Suppose that z_n is a separating element of $y_n \prec_2 x$ for each n. Then we show that $z_n \vee c \in (L')_1$ is the separating element of $y_n \vee c \prec_2 x \vee c$ for each $n \in \mathbb{N}$.

$$(x \vee c) \vee (z_n \vee c) = c \vee (x \vee z_n) = c \vee e = e.$$

Also $(y_n \vee c) \wedge (z_n \vee c) = (y_n \wedge z_n) \vee c = 0 \vee c = c$

Consequently L' is regular. For any $c \in L_2$ we can prove in a similar manner that

$$N' = (\uparrow c, \uparrow c \cap L_2, \{c \vee x : x \in L_1\})$$

is a regular bi σ-frame. We can view (3) as an analogue of (1) with $m-1$ terms, for the homomorphism induced by h between compact regular bi σ-frames

$$L' = (\uparrow a_1, L_1 \cap \uparrow a_1, \{x \vee a_1 : x \in L_2\}) \quad \text{and}$$

$$M' = (\uparrow h(a_1), M_1 \cap \uparrow h(a_1), \{y \vee h(a_1) : y \in M_2\}).$$

But this homomorphism induces an isomorphism on first parts, it follows that

$$((a_1 \vee a_2) \wedge (a_1 \vee b_2)) \vee \ldots \vee ((a_1 \vee a_m) \wedge (a_1 \vee b_m)) = e$$

$$\text{i.e. } a_1 \vee (a_2 \wedge b_2) \vee \ldots \vee (a_m \wedge b_m) = e. \tag{4}$$

Using b_1 instead of a_1 and the homomorphism induced by h between compact regular bi σ-frames $N' = (\uparrow b_1, L_2 \cap \uparrow b_1, \{x \vee b_1 : x \in L_1\})$ and

$W' = (\uparrow h(b_1), M_2 \cap \Uparrow h(b_1), \{y \vee h(b_1) : y \in M_1\})$, we obtain the corresponding analogue (4′) of (4). Taking meets of the left hand sides of (4) and (4′), produces (2) by distributivity. ■

Notation: If L is a σ-frame, then let $\mathcal{L}_1 = \{\nabla_a : a \in L\}$, $\mathcal{L}_2 = \{\Delta_F : F \in \Lambda_\sigma L\}$ and $\mathcal{L}_0$ be the σ-frame generated in CL as a σ-frame by $\mathcal{L}_1$ and $\mathcal{L}_2$.

Proposition 4.2.4: *For any σ-frame L, $\mathcal{B}L = (\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ is bi σ-frame.*

Proof: This follows immediately from the fact that $\mathcal{L}_1$ and $\mathcal{L}_2$ are both σ-frames. ■

The map $\mathcal{B}h : \mathcal{B}L \longrightarrow \mathcal{B}M$ for any $h : L \longrightarrow M$ in **St Cont σFRM** is induced by the σ-frame homomorphism $Ch : CL \longrightarrow CM$, where $Ch(\Theta)$ is the congruence generated by $(h \times h)(\Theta)$ in $M \times M$. Then $(\mathcal{B}h)_1(\nabla_a) = \nabla_{h(a)}$, $a \in L$ and $(\mathcal{B}h)_2(\Delta_F) = \Delta_{\langle h(F)\rangle}$, $F \in \Lambda_\sigma L$ where $\langle h(F)\rangle$ is the filter generated by $h(F)$ in M.

Lemma 4.2.5: *For each $L \in$ **St Cont σFRM**, $\mathcal{B}L$ is a compact regular bi σ-frame.*

Claim: For each $L \in$ **St Cont σFRM**, $x \ll_\sigma a$ in L implies that $\nabla_x \prec_1 \nabla_a$ in $\mathcal{B}L$.

Proof of claim: If $x \ll_\sigma a$, then there exists $c_1 \in L$ such that $x \ll_\sigma c_1 \ll_\sigma a$, continuing the process there exists $c_{n+1} \in L$ such that
$x \ll_\sigma c_{n+1} \ll_\sigma c_n \ll_\sigma c_{n-1} \ll_\sigma a$. Let F be the filter generated by $\{c_n : n \in \mathbb{N}\}$ in L. Then $F \in \Lambda_\sigma L$. Note that $a \in F \subseteq \uparrow x$, thus $\Delta_a \subseteq \Delta_F \subseteq \Delta_{\uparrow x} = \Delta_x$. Consequently $\nabla_x \cap \Delta_F = \Delta$ and $\nabla_a \vee \Delta_F = \nabla$ which means $\nabla_x \prec_1 \nabla_a$.

Proof of Lemma: Now $x \ll_\sigma a$ implies $\nabla_x \prec_1 \nabla_a$ and by continuity $a = \bigvee_{x_n \ll_\sigma a} x_n$, this tells us that $\nabla_a = \bigvee_{\nabla_{x_n} \prec_1 \nabla_a} \nabla_{x_n}$.

For $\mathcal{L}_2$ we check that $\Delta_F \prec_2 \Delta_G$ whenever $F \ll_\sigma G$ in $\Lambda_\sigma L$.

$F \ll_\sigma G$ implies that $F \subseteq \uparrow a \subseteq G$ for some $a \in L$ by Lemma 4.1.4, hence $\Delta_F \subseteq \Delta_a \subseteq \Delta_G$

which gives us that $\Delta_F \cap \nabla_a = \Delta$ and $\Delta_G \vee \nabla_a = \nabla$, this shows that $\Delta_F \prec_2 \Delta_G$. Continuity of $\Lambda_\sigma L$ gives us that $\mathcal{B}L$ is regular.

For compactness of $\mathcal{L}_0$, consider the σ-frame homomorphism

$$\begin{array}{rccc} k: & L & \longrightarrow & \Im_\sigma L \\ & a & \longmapsto & \downarrow a \end{array}$$

Applying the congruence functor we obtain

$$Ck : CL \quad \longrightarrow \quad C(\Im_\sigma L).$$

We check that $Ck(\nabla_a \cap \Delta_F) = \bigvee\limits_{b \in F} [\downarrow a, \downarrow b]$,

$$\begin{aligned} Ck(\nabla_a \cap \Delta_F) &= \bigvee_{b \in F} (\nabla_{k(a)} \cap \Delta_{k(b)}) \\ &= \bigvee_{b \in F} (\nabla_{\downarrow(a)} \cap \Delta_{\downarrow(b)}) \\ &= \bigvee_{b \in F} [\downarrow a, \downarrow b]. \end{aligned}$$

The principal ideals of L are compact elements of $\Im_\sigma L$ so Ck embeds $\mathcal{L}_0$ into the sub σ-frame of $C(\Im_\sigma L)$ which is compact by Lemma 4.1.18. Hence $\mathcal{L}_0$ is compact. ■

We now state a number of preliminary results which will be used in the proof of the following lemma. Consider $L \in$ **KReg BIσFRM** and a frame homomorphism

$$\begin{array}{ccc} CL_1 & \longrightarrow & CL_0 \\ \Theta & \longmapsto & \overline{\Theta} \end{array}$$

induced by the natural embedding $L_1 \longrightarrow L_0$, where $\overline{\Theta}$ is the congruence generated by Θ in $L_0 \times L_0$. If ∇_a and Δ_a are in CL_1, then ∇_a^0 and Δ_a^0 are the analogous congruences of L_0.

Result 1: $\overline{\Delta_a} = \Delta_a^0$ for $a \in L_1$

Proof:

$\subseteq$: This follows immediately from the fact that $\Delta_a \subseteq \Delta_a^0$

$\supseteq$: Δ_a^0 is generated by (a, e). $\overline{\Delta_a}$ is generated by $\{(x, y) : x \wedge a = y \wedge a\}$ in $L_0 \times L_0$. Since (a, e) belongs to this set we have $\Delta_a^0 \subseteq \overline{\Delta_a}$.

Result 2: $\overline{\Delta_F} = \bigvee\limits_{a \in F} \Delta_a^0$

Proof: All we need to show is that $\overline{\Delta_F} = \bigvee\limits_{a \in F} \overline{\Delta_a}$, then the above equality follows from result (1). We have $\Delta_a \subseteq \overline{\Delta_a}$, for all $a \in F$, then $\bigvee\limits_{a \in F} \Delta_a \subseteq \bigvee\limits_{a \in F} \overline{\Delta_a}$, hence $\overline{\bigvee\limits_{a \in F} \Delta_a} \subseteq \bigvee\limits_{a \in F} \overline{\Delta_a}$ because $\bigvee\limits_{a \in F} \overline{\Delta_a}$ is a congruence in L_0. For each $a \in F, \Delta_a \subseteq \overline{\bigvee\limits_{a \in F} \Delta_a}$, so that $\overline{\Delta_a} \subseteq \overline{\bigvee\limits_{a \in F} \Delta_a}$, hence $\bigvee\limits_{a \in F} \overline{\Delta_a} \subseteq \overline{\bigvee\limits_{a \in F} \Delta_a}$.

Result 3: If $a \vee b = e$, then $\nabla_a^0 \vee \nabla_b^0 = \nabla^0$.

Proof: This result follows from the fact that $a \mapsto \nabla_a$ is a σ-frame homomorphism for a σ-frame.

Lemma 4.2.6: *For each $L \in$ **KReg BIσFRM**, there is an isomorphism*

$$\begin{array}{rccc} \tau_L : & \mathcal{B}HL & \longrightarrow & L \\ & \nabla_a & \longmapsto & a \end{array}$$

for each $a \in HL$.

Proof: We claim that $\overline{\Delta_F} = \nabla_b^0$ where $b = \bigvee\{c_n : c_n$ is one of the separating elements of $x_{n+1} \prec_1 x_n$ for a generating set $\{x_n\} \subseteq F\}$ and $F = F_b = \{a \in L_1 : a \vee b = e\}$ by proof of Lemma 4.2.2.

$\subseteq$: If $a \in F$, then $a \vee b = e$ implies $\nabla_a^0 \vee \nabla_b^0 = \nabla^0$ by result 3. Taking intersections with Δ_a^0 on both sides we obtain $\Delta_a^0 = (\Delta_a^0 \cap \nabla_b^0) \subseteq \nabla_b^0$ for each $a \in F$. This yields $\bigvee\limits_{a \in F} \Delta_a^0 = \overline{\Delta_F} \subseteq \nabla_b^0$ by result 2, which implies that $\overline{\Delta_F}[e] \subseteq \nabla_b^0[e]$.

$\supseteq$: Next we check that $\nabla_b^0[e] \subseteq \overline{\Delta_F}[e]$. Suppose $(x, e) \in \nabla_b^0$, then $x \vee b = e$.

By regularity $x \vee (\bigvee\limits_{c_n \prec_2 b} c_n) = e$. This yields $x \vee c = e$ for $c \prec_2 b$ by compactness. If

$a \in L_1$ satisfies $a \wedge c = 0$ and $a \vee b = e$, then $a \in F_b$ and $a \wedge (x \vee c) = a \wedge e$ results in $a \wedge x = a \wedge e$ which shows that $(x, e) \in \triangle^0_a$. Thus $(x, e) \in \overline{\triangle_F}$ confirming that $\nabla^0_b[e]$ and $\overline{\triangle_F}[e]$ are the same and hence $\nabla^0_b = \overline{\triangle_F}$ by Lemma 4.1.19. This shows that the σ-frame homomorphism

$$CL_1 \longrightarrow CL_0$$

maps the sub σ-frame of CL_1 as a σ-frame consisting of $\Theta = \bigvee \nabla_{a_i} \cap \triangle_{F_i}$ where $a_i \in L_1$ and $F \in \Lambda_\sigma L_1$, that is the total frame of $\mathcal{B}HL$ to

$$\begin{aligned}
\overline{\Theta} &= \overline{\bigvee_{\substack{a_i \in L_1 \\ F_i \in \Lambda_\sigma L_1}} \nabla_{a_i} \cap \triangle_{F_i}} \\
&= \bigvee_{\substack{a_i \in L_1 \\ F_i \in \Lambda_\sigma L_1}} \overline{\nabla_{a_i}} \cap \overline{\triangle_{F_i}} \\
&= \bigvee_{a_i \in L_1} \nabla^0_{a_i} \cap \nabla^0_{b_i} \quad \text{(for } b_i \in L_2 \text{ which corresponds to } F_i \in \Lambda_\sigma L_1 \text{ as at beginning of proof)}, \\
&= \nabla^0_c \quad \text{where} \quad c = \bigvee \{(a_i \wedge b_i) : a_i \in L_1, b_i \in L_2\}
\end{aligned}$$

Since $L_0 \longrightarrow CL_0$ is an embedding taking $c \longmapsto \nabla^0_c$ we have a bi σ-frame homomorphism

$$\begin{array}{rccc}
\tau_L : & \mathcal{B}HL & \longrightarrow & L \\
& \Theta & \longmapsto & c
\end{array}$$

whenever $\overline{\Theta} = \nabla^0_c$. Now $\tau_L(\nabla_a) = a$ when $a \in L_1$ since $\overline{\nabla_a} = \nabla^0_a$. Similarly $\tau_L(\triangle_F) = b$ if $F = F_b$ for $F \in \Lambda_\sigma L_1$. The first part of τ_L is obviously an isomorphism. By Lemma 4.2.3 we conclude that τ_L is an isomorphism. ■

Proposition 4.2.7: *The functor H induces a right adjoint equivalence between* **KReg BIσFRM** *and* **St Cont σFRM**.

Proof: We firstly show that

$$\begin{array}{rccc}
\tau_L : & \mathcal{B}HL & \longrightarrow & L \\
& \nabla_a & \longmapsto & a
\end{array}$$

for each $a \in L_1$ is natural in L.

Consider the diagram

$$\begin{tikzcd}
L \arrow[d, "h"'] & \mathcal{B}HL \arrow[l, "\tau_L"'] \arrow[d, "\mathcal{B}Hh"] \\
M & \mathcal{B}HM. \arrow[l, "\tau_M"']
\end{tikzcd}$$

On the first parts we have that $h \circ \tau_L(\nabla_a) = h(a)$ and

$$\tau_M \circ \mathcal{B}Hh(\nabla_a) = \tau_M(\nabla_{h(a)}) = h(a).$$

On second parts $h \circ \tau_L(\triangle_F) = h(b)$ whenever $F = F_b$.

The proof of the following relies on this diagram which has been previously shown to commute

$$\begin{tikzcd}
L_2 \arrow[r, "\phi_L"] \arrow[d, "h_2"'] & \Lambda_\sigma(L_1) \arrow[d, "\Lambda_\sigma(h_1)"] \\
M_2 \arrow[r, "\phi_M"] & \Lambda_\sigma(M_1)
\end{tikzcd}$$

So $\tau_M \circ \mathcal{B}Hh(\triangle_F) = \tau_M \circ \mathcal{B}Hh(\triangle_{F_b})$

$= \tau_M(\triangle_{\Lambda_\sigma h_1(F_b)})$

$= \tau_M(\triangle_{F_{h_2(b)}})$ (by naturality of ϕ_L as shown in the square)

$= h_2(b) = h(b).$

Hence $h \circ \tau_L$ and $\tau_M \circ \mathcal{B}Hh$ coincide on the first and second parts of $\mathcal{B}HL$, and are thus equal.

If $\nu_{HL}(a) = \nabla_a$, for all $a \in F$, then the next diagram commutes.

$$\begin{tikzcd}
& H\mathcal{B}HL \arrow[dl, "H\tau_L"'] & \\
HL \arrow[rr, equal] & & HL \arrow[ul, "\nu_{HL}"']
\end{tikzcd}$$

Lastly we show that the following triangle commutes.

$$\begin{tikzcd}
& \mathcal{B}H\mathcal{B} \arrow[dl, "\tau\mathcal{B}"'] & \\
\mathcal{B} \arrow[rr, equal] & & \mathcal{B} \arrow[ul, "\mathcal{B}\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& \nabla_a \arrow[dl, mapsto] & \\
a \arrow[rr, equal] & & a \arrow[ul, mapsto]
\end{tikzcd}$$

We must check that $\tau\mathcal{B} \circ \mathcal{B}\nu = (\mathbf{1})_{\mathcal{B}}$, thus show that for each $K \in$ **St Cont** σ**FRM**

$$\tau_{\mathcal{B}K} \circ \mathcal{B}\nu_K = \mathbf{1}_{\mathcal{B}K}.$$

It suffices to show that the latter equation holds on the first and second parts of $\mathcal{B}K$.

On the first part of $\mathcal{B}K$:

$$\begin{aligned}
\tau_{\mathcal{B}K}(\mathcal{B}\nu_K)(\nabla_b) &= \tau_{\mathcal{B}K}(\nabla_{\nu_K(b)}) \quad \text{for each} \quad b \in K \\
&= \nu_K(b) \\
&= \nabla_b
\end{aligned}$$

On the second part of $\mathcal{B}K$: Note that $H\mathcal{B}K = \{\nabla_a : a \in K\}$ and $\mathcal{B}H\mathcal{B}K = (\mathcal{M}_0, \{\nabla_{\nabla_a} : a \in K\}, \{\triangle_F : F \in \Lambda_\sigma(H\mathcal{B}K)\})$, where $\mathcal{M}_0$ is the sub σ-frame of $C(H\mathcal{B}K)$ generated by the first and second parts of $\mathcal{B}H\mathcal{B}K$. Whenever $F \in \Lambda_\sigma K$,

$$\begin{aligned}
\tau_{\mathcal{B}K}(\mathcal{B}\nu_K)(\triangle_F) &= \tau_{\mathcal{B}K}(\mathcal{B}\nu_K(\bigvee_{b\in F} \triangle_b)) \\
&= \tau_{\mathcal{B}K}(\bigvee_{b\in F} \triangle_{\nabla_b}) \\
&= \tau_{\mathcal{B}K}(\triangle_{\mathcal{F}}) \qquad \text{(where} \quad \mathcal{F} = \{\nabla_a : a \in F\}) \\
&= \tau_{\mathcal{B}K}(\triangle_{\mathcal{F}_{\triangle_G}}) \\
&= \triangle_G,
\end{aligned}$$

where $\mathcal{F}_{\triangle_G} = \{\bigtriangledown_b : b \in K \text{ and } \bigtriangledown_b \vee \triangle_G = \bigtriangledown\}$ and $\triangle_G = \bigvee\{\triangle_{G_n} : G_n \in \Lambda_\sigma K$ where $\triangle_{G_n}$ is a separating element of $\bigtriangledown_{a_{n+1}} \prec_1 \bigtriangledown_{a_n}$ and $\{a_n\}$ a generating set for $F\}$, check this construction in the proof of Lemma 4.2.2.

We only have to show that $\triangle_F = \triangle_G$.

$\subseteq$: $\triangle_F = \triangle_{\bigvee\{F_n : F_n \ll_\sigma F\}}$ (constructing F_n's as in Lemma 4.1.4.)

$$= \bigvee\{\triangle_{F_n} : F_n \ll_\sigma F\}.$$

To prove this inclusion it suffices to show that $\triangle_{F_n} \subseteq \triangle_G$ whenever $F_n \ll_\sigma F$. We check that $\triangle_{F_n}$ is a separating element of $\bigtriangledown_{a_{n+1}} \prec_1 \bigtriangledown_{a_n}$ for $\{a_n, a_{n+1}\}$ contained in the generating set of F. Observe that $\uparrow a_n \subseteq F_n \subseteq \uparrow a_{n+1}$, thus $\triangle_{\uparrow a_n} \subseteq \triangle_{F_n} \subseteq \triangle_{\uparrow a_{n+1}}$, but $\triangle_{\uparrow a_n} = \triangle_{a_n}$ and $\triangle_{\uparrow a_{n+1}} = \triangle_{a_{n+1}}$. Consequently $\triangle_{a_n} \subseteq \triangle_{F_n} \subseteq \triangle_{a_{n+1}}$.

Therefore $\bigtriangledown_{a_{n+1}} \cap \triangle_{a_{n+1}} = \triangle$ implies that $\bigtriangledown_{a_{n+1}} \cap \triangle_{F_n} = \triangle$, and $\bigtriangledown_{a_n} \vee \triangle_{a_n} = \bigtriangledown$ implies that $\bigtriangledown_{a_n} \vee \triangle_{F_n} = \bigtriangledown$.

For the reverse inclusion.

$\supseteq$: We know that $\triangle_{a_n}$ is the largest congruence in K such that $\triangle_{a_n} \cap \bigtriangledown_{a_n} = \triangle$ for some $a_n \in F$. Now $\triangle_{G_i} \cap \bigtriangledown_{a_n} = \triangle$ implies that $\triangle_{G_i} \subseteq \triangle_{a_n}$ for some $a_n \in F$.
Thus $\triangle_{G_i} \subseteq \bigvee_{c \in F} \triangle_c = \triangle_F$, hence $\triangle_G \subseteq \triangle_F$. ■

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