A study of the representational use of aggregates in the pedagogic elaboration of addition and subtraction in the Department of Basic Education Grades 1 to 3 Numeracy Workbooks, prescribed for use in state-funded South African schools

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ABSTRACT

Cognitive science demonstrates that a sensitivity to aggregates (groups, collections, classes, categories) forms part of the biologically endowed human (core domain) capacity for dealing with quantity, along with an ability to compute using aggregates, both approximately and exactly. Core domain computations using aggregates serve as a basis for the growth of noncore mathematical computations and principles, following exposure to number enculturation and the counting algorithm, both of which are enhanced by the growth of linguistic competence.

The study focuses on the pedagogic use of the class of small, discrete aggregates in the teaching and learning of natural number addition and subtraction across the Foundation Phase of schooling. The central concern is the computational processes that use discrete aggregates, and operations over such aggregates. The six 2021 Department of Basic Education numeracy workbooks (*Mathematics in English*) for Grades One to Three, prescribed for use in state-funded SA schools, constitute the archive of information from which the data is produced for the study.

The study adopts a computational analytic approach conditioned by the proposition that all thought is computational, entailing the use of operations over domains of objects that serve as arguments (inputs) and values (outputs). A mathematised notion of *representation*—as a structure-preserving mapping—comprised the chief analytical resource for describing computations related across *representing* and *represented* computational structures.

The analysis, firstly, proceeds descriptively. The *unit of analysis* is a *Task*, made up of *Subtasks* containing *Exercises*, so that the analysis of a *Task* proceeds by way of an analysis of its *Exercises*. Only *Tasks* employing discrete aggregates for the purposes of teaching addition and subtraction are analysed to reveal the representations used by identifying the computational structures and the relations between such structures. Typically, the representations used in *Tasks* entail mappings from operations over discrete aggregates to operations over the natural numbers. As a further means of gauging the extent of the range of mappings/operations and structures identified across the workbooks, the descriptive data is extended by the use of quantitative databases, summarising and totalling all identified mappings/operations and structures.

The study found that: (1) operations over discrete aggregates are used extensively as a ground for addition, subtraction, natural number order relations, and number partitions, including the use of partitions for teaching place value in the base-ten natural number system; (2) counting is the primary computational resource for relating operations over discrete aggregates to operations over the natural numbers; (3) addition and subtraction are often derived from operations over discrete aggregates in a manner that privileges a unary rather than binary form; (4) the treatment of discrete aggregates, together with the use of partitioning, suggests that aggregates are conceived of in a manner that has more affinity with fusions than with sets; (5) the general semantic basis for addition, subtraction and partitioning appears to be the universal cognitive operation referred to as *merge* (and its derivatives, *unmerge* and *purge*) as used by the human conceptual-intentional system.

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LIST OF ABBREVIATIONS

ANS	Approximate Number System
AR	Additive Reasoning
AS	Addition and subtraction
CAG	Continuous Aggregate
CAPS	Curriculum and Assessment Policy Statement (Mathematics)
CPr	Comprehension Principle
CPR	Class of Comprehension Principles
DAG	Discrete aggregate
DBE	Department of Basic Education
FINSET	Finite Set
FP	Foundation Phase
ICM	Integrated Causal Model
MD	Multiplication and division
MR	Multiplicative Reasoning
$\mathbb{N}$	Natural Numbers
NUM	Numbers
OBE	Outcomes Based Education
OTS	Object Tracking System
RATP	Revised Annual Teaching Plan
SA	South African
SACMEQ	Southern African Consortium for Monitoring Educational Quality
SSSM	Standard Social Science Model
SYM	Number symbols
TIMSS	Trends in International Mathematics and Science Study

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CHAPTER ONE

Introduction

1.1 Background and rationale

Ongoing reports of the early mathematics competence of South African learners indicate a range of difficulties in the teaching and learning of the contents of the curriculum strand, *Number, Operations, and Relationships*, a topic where the use of sets is particularly concentrated. What is curious is the recurrence of poor performance outcomes despite the amount of time available for teaching and learning the topic in the Department of Basic Education (DBE) curriculum (CAPS), namely, 58% of time in Grade 1, rising to 65% by Grade 3. We can be reasonably certain that early numerical competence, which includes counting, is a good predictor of later success in school mathematics (Carpenter & Moser 1984; Ginsburg 1977; Graven 2014; Jordan, Glutting & Ramineni 2010; Sarama & Clements 2009; Spaull 2013; Starkey & Gelman 1982; Steffe & Cobb 1988; Venkat & Spaull 2015). Therefore, it behoves us to understand reasons for our collective failure in the mathematics education of our youth.

Interest has been recently growing on learning in the Foundation Phase (FP). A number of local studies signal a strong trend of increasingly poor performance of learners in mathematics as they move through higher phases of schooling from the Foundation Phase (FP) (Davis, Jaffer & Sapire 2022; Ensor *et al.* 2008; Fleisch 2008; Schollar 2008, 2015; Spaull 2013; Spaull & Kotze 2015;). This is an ongoing research interest, with many researchers arguing that the situation results from the use of so-called 'concrete counting' as the major computational tool, which registers a retardation in learners' transition to arithmetic proper (Ensor, Hoadley, Jacklin, Kühne, Schmitt & Lombard. 2008; Fleisch 2008; Hoadley 2007; Schollar 2008, 2015). The poor socioeconomic situation of a great many of learners has also been highlighted as a primary reason for ongoing poor performance (Hoadley 2007; Mathews, Mdluli & Ramsingh 2014; Spaull 2013). Other research points to the racially stratified education policies implemented during the apartheid era to be contributary factors (Adler 2005; Adler & Davis 2006). More recent research, exploring the fast pacing of the curriculum in relation to the slow pace of learning, calls for curriculum redesign (Davis, Jaffer & Sapire 2022; Tyatya 2021).

Curriculum design focusses on the political and socio-economic needs to be met by schooling, in relation to how children learn, what content to include, and how it should be relayed. Yet, what is generally not considered in a sufficiently serious manner are the implications that follow from the collection of innate cognitive systems that constitute the biological ground for the growth of mathematics in humans (Davis, Jaffer & Sapire 2022). This study responds to the call to condition research in mathematics education by considering cognitive affordances and constraints that issue from biological endowment.

1.2 Formulating the research question

During the 1960s and 1970s, sets were explicitly inserted into Western school curricula as part of restructuring of mathematics curricula, intended to serve as a ground for learning school mathematics (Geddes & Lipsey 1968). This was sparked by the launch of the Sputnik satellite by the USSR (1958), which served as a catalyst for an education crisis in the West and lead to collaboration between the education and mathematics fraternities for the delivery of broader, more advanced mathematics curricula (Baker, Knipe, Collins, Leon, Cummings, Blair and Gamson 2010; Van Engen 1993). The reformation of mathematics curricula was also strongly influenced by the emergence of the Bourbaki group and its 1968 manifesto, produced by a cohort of French mathematicians. They campaigned for a mathematics grounded in Georg Cantor's theory of sets, since set theory was believed to provide a natural ground and unifying basis for much of mathematics.

1.2.1 The problem in its immediacy

Given the brief, but seriously flawed and heavily criticised inclusion of sets as a topic in school mathematics curricula, one might be tempted to conclude they are no longer engaged within school mathematics. While set theory is no longer a named strand in elementary and secondary school mathematics curricula, an examination of the curriculum documents and school textbooks, especially those written for the early grades, suggest that sets are still used. Contemporary elementary texts are peppered with objects that might be viewed as sets. For example, the DBE FP numeracy workbooks referred to in this study are saturated with objects that look like sets (collections of objects familiar to young children), which are accompanied by operations implicitly defined over them. Such objects are intended to function pedagogically as the precursors to numbers and the basic arithmetic operations.

We thus arrive at the research problem in its immediacy:

> *How are sets used for the teaching and learning of addition and subtraction in the DBE FP workbooks?*

Currently, how sets are used in elementary mathematics may appear to be rather obvious. Learners are provided with collections of objects that can be manipulated, or with illustrations of familiar collections of objects that can be related in a brief story (the so-called ‘word sums’), from which engagement with, they initially learn how to count, add and subtract, and later, to multiply and divide. The process is often referred to as a developmental progression from the concrete (operations using manipulatives/illustrations of objects) to the abstract (operations using numbers and mathematics notations/symbolism). Such a response to the initial formulation of the research problem does, however, beg the general question of how it is possible for humans to exploit cognitive operations apposite to one computational system as a ground for operations in a different computational system. Various direct and indirect answers have been offered, as encountered in the prolific work of pioneers like Piaget and Bruner. However, before accepting what is familiar to students of mathematics education and currently available, a moment of reflection is needed to consider the objects and conditions of the research problem more carefully.

1.2.2 The problem upon reflection

Going forward, I shall mostly use the term *aggregate* rather than the more commonly used term, *set*, because: (i) a set, taken as the mathematical object referred to in set theory, has properties that are not always shared by other conceptions of aggregates[1], and (ii) some of the ways in which aggregates are worked with in schooling imply that the aggregates are not, strictly speaking, conceived of in a manner consistent with the mathematical notion of *set*. I will only use the mathematical term *set* when it is helpful to do so for descriptive or explanatory purposes.

In its presentation of prescriptions for the teaching of school mathematics, the DBE CAPS for FP mathematics implicitly asserts the importance of engagement with finite discrete aggregates for the development of learners’ understanding of numbers, as well as the importance of counting small discrete aggregates as central resources for the development of early arithmetic competence (cf. DBE 2011:13, 18).

Gelman & Gallistel (1986: vi-vii) argue that the growth of number-related concepts in children is guided by two principles: one is concerned with counting the members of finite discrete aggregates, and another is concerned with the relationships between adding, subtracting, ordering and number

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[1] For example: there exists an empty set; the empty set is part of every set; altering elements of a set is impossible; the formation of a set does not always require the existence of a rule for belonging to a set; elements of a set are not the same thing as subsets of a set. The term *set* is often used in the psychology and cognitive science literature without any discussion of whether or not the entities referred to in studies are sets in the mathematical sense, which is a potential source of confusion and unintended misdirection.

equivalence. In other words, discrete aggregates, along with the operations implicitly defined over them, are somehow translated into number and the arithmetic operations defined over numbers.

Cognitive scientific studies of infants and of pre-school children provide strong evidence in support of the proposition that biologically endowed specialised knowledge systems (core domain systems) exist for quantification that serve as a basis for mathematics. The latter comprises a constellation of culturally acquired noncore domain systems of knowledge (Gallistel & Gelman 2005; Gelman 2009a, 2009b; Spelke & Kinzler 2007; Spelke 2022; Xu & Spelke 2000). Gelman (2009a: 247) describes a knowledge domain as ‘a set of coherent principles that form a structure and contains domain -specific entities…that can combine and form other entities within the domain.’ Core domain systems are the result of biological adaptation through natural selection and are biologically ancient, whilst noncore domain systems constitute ‘organized knowledge […] acquired later’ (Gelman 2009a: 248).

The research literature shows clear evidence of the core domain ability of preverbal infants to subitise[2] small collections of objects based upon their reaction to aggregates of different cardinal values (Halberda & Feigenson 2008) and has been demonstrated to be associated with core domain systems known as the *Object Tracking System* (OTS) and the *Approximate Number System* (ANS) (Dehaene 2011). The OTS and ANS contribute centrally to quantitative sense-making. Moreover, the OTS and ANS are mapped to systems of number words and number symbols in literate cultures which use them (Mazzocco, Feigenson & Halberda 2011). However, a literate culture is not a prerequisite for the OTS and ANS to function (Gordon 2004; Everett, Berlin, Gonçalves, Kay, Levinson & Pawley 2005; Pica *et al.* 2004).

Davis (2016, 2018) argues that a sensitivity to aggregates is part of human core domain knowledge and furthermore, that one is obliged to consider the structuring effects that core domain conceptions of aggregates have on the teaching and learning of school mathematics. It would be difficult to deny the existence of core domain conceptions of aggregates since even societies whose languages carry very rudimentary number systems and who offer no formal schooling, like the Amazonian Mundurucu (Gordon 2004; Pica, Lemer, Izard & Dehaene 2004) and Pirahã tribes (Everett, Berlin, Gonçalves *et al.* 2005), recognise, compare and order aggregates in the absence of stable and extended number systems. Such is also the case with infants and slightly older preverbal children (Halberda & Feigenson 2008; Jordan & Brannon 2006). We now arrive at the research problem upon reflection:

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[2] ‘The direct apprehension of the numerosity of a group’ (Kaufman *et al.*1949:521); that is, without counting.

> *How are discrete aggregates and operations over collections of discrete aggregates used in the teaching and learning of addition and subtraction in the DBE FP numeracy workbooks?*

Once again, an answer to the research problem as currently conceived can be constructed in a variety of ways. What needs to be done next is to map out an approach to the problem that converges on a necessary methodological path.

1.2.3 A methodological path converging on the research problem

A host of scholars who study cognition use the proposition that all thought is constituted by operations over domains of objects that serve as arguments and values for computational activity. For example: Boyer (2020) on the formation of human societies; Chomsky (2006, 2007) on the human capacity for language; Dehaene (2020) on mathematics; Gallistel & King (2010) on the general mechanisms of cognition; Gelman & Gallistel (1986, 1978) on number knowledge in young children; Pinker (2007) on language as a window to cognition; and Spelke (2022) on all of the core domain systems of which we currently have knowledge.

In mathematics education, work by Davis (2010, 2011, 2012, 2013, 2016, 2018; Davis, Jaffer & Sapire 2022) demonstrates the productivity of mathematical descriptions of the computational activity of learners, teachers, and those embedded in tasks, as resources for detailing the computational structures used explicitly and implicitly during teaching and learning. Central to the approach, therefore, is the necessary identification and description of computational structures, as well as the relations between such structures. A computational structure is to be thought of as simply an aggregate or set along with one or more operations defined over the aggregate/set. The research problem can now be stated in its necessity:

> *How are computational structures that are constructed from discrete aggregates and operations over collections of discrete aggregates used as a ground for the teaching and learning of the computational structures that constitute addition and subtraction in the DBE FP numeracy workbooks?*

1.3 Overview of the research framework

The study employs a computational-analytic approach based on the proposition that all thought is computational, entailing the uses of operations over domains of objects that serve as arguments (inputs) and values (outputs) for operations.

A mathematised notion of *representation,* as put forward by Gallistel & King (2010), serves as the analytical tool to produce data for the study. *Representations* foreground the idea of structure-preservation (homomorphisms) to render explicit computational details by which to address the research question. Thus, the study examines the relations between computational structures to analyse the use of aggregates in the six workbooks prescribed by the DBE for Grades 1 to 3, relative to how operations over finite sets or discrete aggregates are mapped to operations over numbers. In order to achieve this, I have drawn from related fields, like mathematics, mathematics education, cognitive science and biolinguistics in an attempt to construct a descriptively adequate analytic resource.

1.4 A summary of the dissertation

Chapter Two reviews the research of relevance to the study, much of which is situated in cognitive science rather than mathematics education.

Chapter Three provides a detailed account of the mathematical resources used in the study.

Chapter Four details the analytic protocol used in the production and analysis of data.

Chapters Five, Six and Seven present the results of the analysis, with Chapter Seven providing a quantitative overview of the computational features of the *Tasks* analysed.

Chapter Eight presents the concluding remarks on the study.

The Appendix contains additional information of pertinence to the study.

CHAPTER TWO

Literature Review

2.1 Introduction

This chapter includes a review of the pertinent literature on how young children use aggregates. One goal is to highlight views of scholars, across mathematics education and cognitive science, on core domain cognition and to ascertain the likely influences of core domain conceptions on learning outcomes in school mathematics. Another goal is to highlight useful research approaches which assist with constructing the analytical framework for the study.

The review takes its direction from the proposition that the human mind comes pre-fitted by natural selection with specialised domains of knowledge, referred to as *core domains* in the literature (Gelman 2009; Spelke 2022; Spelke & Kinzler 2007). In general, a domain constitutes 'a set of coherent principles that form a structure … of domain-specific entities … that can combine to form other entities within the domain' (Gelman 2009: 247). Among such knowledge systems, initially skeletal, are those that carry structures for quantification and which, with the appropriate formal instruction, become extensively organised to constitute sophisticated *noncore domains* of knowledge, of which mathematics is one such domain (Chomsky 1983; Gelman 1972; Gelman & Gallistel 1978; Gelman & Meck 1983; Hauser, Chomsky & Fitch 2002; Spelke 2022). Both general categories—*core* and *noncore domains*—are now well-established as descriptive and explanatory resources in cognitive science (Gelman 2009).

The proposition stated above is foundational and should be understood as part of what Tooby and Cosmides (1992) refer to as an *integrated causal model* (ICM) approach to research.[3] An ICM approach inevitably compels one to consider what we know of innate cognitive functioning and its structuring effects on the computational activity of humans when we study what comes to be constituted as mathematical knowledge in pedagogic situations. In the research context of this study, an ICM approach that focuses on computational activity informs both the 'generation of descriptions of mental activity as well as the production and analysis of data' (Davis 2018:3). Given the general approach taken in this study, a review of the literature requires one to pay attention to the fundamental, biologically endowed basis for numerical cognition; otherwise, one implicitly adopts a blank slate

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[3] By which they mean that researchers explain phenomena by drawing on scientific work from diverse fields of enquiry that are able to shed light on a research problem.

conception of mind (Pinker 2002), the result of which is unproductive empiricism that lacks both descriptive and explanatory adequacy.

The arithmetic operations of addition and subtraction (AS) have been extensively researched in mathematics education and related fields, these traditionally being the first arithmetic operations introduced to learners in elementary school curricula and which are also used as bases for multiplication and division in the intermediate phase and beyond (Romberg 1982; Sarama & Clements 2009; Verschaffel, Greer & De Corte 2007). As should be expected, a related research focus has been on counting since the growth of number concepts depends on the growth of knowledge of counting (Batchelor, Inglis & Gilmore 2015; Davis 2018; Gelman 1972; Gelman & Gallistel 1986; Gilmore, McCarthy & Spelke 2007). Numerous studies of quantification and number knowledge in neophytes have been, and continue to be, carried out in cognitive science and neuroscience (Dehaene 2015; Fuson 1982; Gelman 1972; Gelman & Gallistel 1986; McCarthy & Spelke 2007; Spelke 2022).

Such research is important for this study because it shows us something of the emergent mathematical conceptualisations employed when computing with aggregates and operations on aggregates in the FP. Furthermore, cognitive science research conditioned by the proposition that thought is computational in nature (Chomsky 2006, 2007; Gallistel & King 2010; Pinker 1997), often referred to as a *computational* or *representational theory of mind*, is pertinent to this study. Consequently, this study draws on what is known about the biologically endowed systems for number and the advances made in that area of research. Studies that assume a computational approach to mind are, however, rather limited in the mathematics education literature, but there is some work upon which one can draw (Carpenter, Moser & Romberg 1982; Davis 2010, 2011, 2013, 2016, 2018; Davis, Jaffer & Sapire 2022; Gilmore, Göbel & Inglis 2018).

The ICM approach to research adopted here requires an exploration of the computational foundations for AS, which entails engaging with a great deal of material that does not have as its immediate goal a discussion of AS. Much of what is presented in mathematics education research on AS does not move beyond the scope of the edited collection by Carpenter, Moser & Romberg (1982), and much does not even draw on the advances presented in that collection of work.

Before engaging with the research, two necessary caveats should be noted:

(i) The study of sets as aggregates with (often counterintuitive) mathematical characteristics is clarified in the field of mathematics, pointedly in the work on set theory by Cantor (1952) and as subsequently refined by others. It would be expected that an analysis of aggregates as a ground for elementary arithmetic might refer to set theoretic ideas about what constitutes a set. However, there is very little explicit reference to the mathematical notion of sets in mathematics education research, or in psychology and cognitive science research. Rather, the term *set* is used loosely to refer to collections of things irrespective of the operations used to manipulate such collections, and irrespective of the related conceptual characteristics of such collections as conceived of by young children and their teachers. Consequently, one has to be particularly vigilant when encountering the term *set* in mathematics education and psychology research, and one should always attend to the syntactic uses of collections referred to as *sets* by considering the computational processes they are subjected to if one is to make appropriate sense of the arguments presented and of what kind of aggregates are being used.

(ii) The terms *number* and *set* are very often used interchangeably in mathematics education and psychology research. For example, researchers often refer to a particular set of discrete objects as a number, conflating a set and its cardinality and thereby failing to distinguish two levels of analysis; namely, the level of operations over sets and the level of operations over (natural) numbers. In some instances, such conflation of sets and numbers is not problematic and so does not matter to the results of analysis; in other instances it is detrimental to the ultimate goal of producing accounts of phenomena that meet the criteria for explanatory adequacy in scientific research.

2.2 Early research, subitising, and Piaget

The education literature documents the early studies of number and quantity by theorists like Dewey (1898), Thorndike (1922) and Brownell (1947), whose work on subtraction and addition refer to aggregates/sets as a natural ground for counting and basic arithmetic. Collectively, their research was seminal because they influenced how numbers came to be conceived of with respect to early mathematics teaching and learning. They distilled theoretical premises for formal instruction from the then under-theorised morass of available research of their day (Sarama & Clements 2009). They also continue to have far-reaching effects on present day curricula, which generally still prescribe AS as the first arithmetic operations that children encounter, and as bases upon which multiplication and division are introduced (Fuson 1982; Gelman 1972; Gelman & Gallistel 1986; Romberg 1982; Sarama & Clements 2009; Verschaffel, Greer & De Corte 2007), CAPS being no exception.

Other seminal work of the period that focussed on number and counting is that of Kaufman, Lord, Rees & Volkman (1949), who coined the term *subitising*. Subitising, described as 'the direct apprehension of the numerosity of a group' (Douglas 1999: 400), without counting, is now accepted by many researchers as a feature of core domain computational systems and linked to the ability to track small collections (up to four objects) without resorting to explicit counting (Gallistel & Gelman 1991; Spelke 2022; Spelke & Kinzler 2007). The idea was defended as part of the nativist position at the time. However, a change of focus driven by the Piagetian movement inclined ideas of number concept development heavily towards a Piagetian constructivist position. Only much later, after Piagetian ideas started unravelling, did research on subitising resume and become extended (Douglas 1999; Mandler & Shebo 1982; Meck & Church 1983; von Glasersfeld 1982).

Piagetian theory advocated that number concepts are constructed through experiences with number, rather than being a derivative from innate systems. Opposing the work of Kaufman, Lord, Rees and Volkmann (1949) on subitising, Piaget (1964, 1965) and Piaget & Szeminska (1952) argued that counting is initially rather rote-like, becoming elaborated for computation only at the stage of *concrete operations* (around seven years old), following cognitive development and practice with numbers. Moreover, the development of natural number counting was based upon logico-mathematical operations, resting on the notions of classification, seriation and conservation.[4]

Piaget's (1964, 1965) work on number conservation, using discrete aggregates and equivalence/non-equivalence as a basis for comparing aggregates by one-to-one correspondences between object pairings, is particularly noteworthy for my study. Firstly, it aligns with foundational aspects of Cantorian set theory, one-to-one correspondence being the simplest arithmetical process constituting the basis for the construction of number (Pica, Lemer, Izard & Dehaene 2004). Secondly, it foregrounded the recognition of structure-preservation between operations defined over domains of objects as used by mathematicians and later extended into mathematics education research by Krause (1969), Baker, Bruckheimer & Flegg (1971) and Davis (2012, 2013, 2016, 2018). Thirdly, in mathematics education, the constructivist movement opened up consideration to the possibility that core domain number knowledge is culturally exploited to contribute to the growth of formal, noncore conceptions of numerical computation (cf. Dehaene 2020; Spelke 2022), potentially serving as a catalyst for a shift toward an amalgamation of nativist and constructivist positions, bereft of the empiricist retreat implicit to Piagetian constructivism.

Piaget's findings regarding children's numerical cognition were upheld until new perspectives later in the 20th century and early in the 21st century overturned them (Baker *et al.* 2010; Sarama &

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[4] That is, categorise objects according to differences/similarities; arrange objects by size/use; understand how objects remain unchanged across transformations, respectively.

Clements 2009). The growing corpus of cognitive science research unravelled aspects of his theory and his influence in psychology (but perhaps not in education), by demonstrating the existence of extended cognitive capabilities of pre-school children beyond limitations specified by Piaget in respect of comparing and conserving cardinality across transformations[5] on aggregate pairs when one-to-one correspondence was offset by contour length or when varying container height or shape; and, in respect of approximating judgements only as more/less (Becker 1989; Brainerd 1978; Gelman and Gallistel 1978).[6]

2.3 The human sensitivity to aggregation

Underlying an innate sensitivity to number is an innate sensitivity to aggregates, clearly demonstrated in experiments on the activity of infants (Antell & Keating 1983; Chomsky & McGilvray 2012; Hauser 2003; Izard, Sann, Spelke & Streri 2009; Markman 1973; Markman & Siebert 1976; Mix, Huttenlocher & Levine 2002; Piazza & Eger 2016; Tooby & Cosmides 1992; Uller, Carey, Huntley-Fenner & Klatt. 1999; Verschaffel, Greer & De Corte 2007; Wynn 1998). Humans recognise, compare and order aggregates even in cultures whose languages carry crude number schemes, like the Amazonian tribes that do not have a stable counting algorithm—the Mundurucu (Gordon 2004; Pica *et al.* 2004) and Pirahã (Everett 2005). The same situation is also found to be the case in preverbal children in industrialised societies (Halberda, Mazzocco & Feigenson 2008; Jordan & Brannon 2006).

More profound evidence that incipient number-related core domain knowledge exists is provided by studies which show infants intuitively operating with small collections in ways that have the same abstract structure as adding and subtracting (Feigenson, Carey & Hauser 2002; Feigenson, Carey & Spelke 2002; Mix, Huttenlocher & Levine 2002; Starkey & Cooper 1980; Uller *et al.*1999; Wynn 1992, 1995; Xu & Spelke 2000). Wynn's (1992) landmark study, which uses aggregates that might be reasonably taken to represent '1 + 1 = 2 or 3', or '2 – 1 = 1 or 2', documents infants surprise with violations that correspond to '1 + 1 = 3', or '2 - 1 = 2', respectively. Furthermore, infants have been shown to 'add' and 'subtract' across multi-modal contexts (Brannon & Jordan 2006; Clearfield & Mix 1999; Gilmore 2015; Halberda & Feigenson 2008; Lipton & Spelke 2003, 2004; McCrink & Wynn 2007; Wynn 1996; Xu & Spelke 2000), matching the mode representations on a one-to-one basis (Sarama & Clements 2009), which corresponds to the recognition of instances of different phenomena being the same in number.

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[5] Using 'adding' or 'subtracting', object displacement and replacement, or removal.

[6] Piaget (1964, 1965) believed children had no ordinal perspective or true enumeration facility until age seven.

Given the implications that flow from the body of work just referenced, it is unsurprising that curricula like the CAPS (DBE 2011) use small aggregates in recognition, ordering, and comparison tasks given that these computations exist in core domain systems, thus providing a basis for their use in noncore domain systems. The research also shows that the growth of noncore domain number knowledge relies on language to carry the count lexicon. Moreover, the extended research shows that growth relies upon computing with aggregates of ever-increasing size, beyond the core competence that is restricted to aggregates of about four objects (Gilmore 2015; Gilmore, McCarthy & Spelke 2007; Spelke 2022; Spelke & Kinzler 2007).

Extended work in cognitive science further demonstrates that spontaneous conceptions of aggregates by humans follows an intuitive conceptual developmental sequence (Markman 1973, 1978, 1979, 1983; Markman, Horton & McClenaghan 1980; Markman & Siebert 1976). The sequence ranges from initial attention to aggregates as objects, to aggregates as collections of similar discrete objects, and finally, to aggregates as classes of objects. Each of the intuitive conceptions of aggregate is identified with one or other membership criterion (also referred to as a *comprehension principle* (CPr) in mathematics, meta-mathematics and philosophy). CPrs are used to code aggregates on the basis of common properties (a membership criterion); for example, having beaks, wings, feathers and two legs for the class of birds (Markman 1983).

CPrs are perceptually complex when they are considered in relation to aggregates since they place demands on counting and operating—What to count? What to operate with? CPrs also entail an implicit conception of part-whole relationships, drawing attention towards relations of entailment between objects (Davis 2016; Markman & Siebert 1976). Part-whole relations imply mereological *fusions* (Davis 2016; Potter 2004), which are aggregations of a type quite different from the mathematical notion *set*. A *set* is a special kind of aggregate that keeps 'things distinct' to comprise a 'further entity over and above' the things (Potter 2004: 21-22) and is thus uniquely constituted by its elements. A fusion has the interesting property that a whole need not be constituted from unique parts, but rather can be constituted in multiple part combinations, including the single part that is the whole. Whenever part-whole relations are employed in computations we are very often in the presence of fusions, thus suggesting an additional conception of an aggregate to be added to the categories of aggregate discussed earlier, namely, aggregate-as-fusions (Davis 2016).

The general mathematical conception of *set* eschews the use of a CPr as a necessary component because it leads to serious problems, such as Russell's and Cantor's paradoxes (Potter 2004). In other words, it is possible to construct well-formed CPrs that lead to incoherence in mathematics.[7] In the context of learning and teaching, learners and educators mostly fail to recognise the distinction between sets and fusions. Principally, because they do not understand sets in the mathematical sense, they generally appear to treat aggregates in a manner that suggests the use of fusions. At the heart of the problem is a failure to distinguish elements of a set from subsets of a set, with elements generally treated as 'parts'. In set theory, it is the subsets of a set that are the parts of the set, and there are more parts/subsets of a given set than there are elements of the set.[8]

The extensive use of part-whole relations in schooling and in much of mathematics education research concerned with the teaching and learning of arithmetic, connects quite naturally with the fusion conception of aggregates. The work of Carpenter & Moser (1982), in which part-whole relations are used to research and discuss AS, is exemplary in this regard. More contemporary work in mathematics education, especially work directed at teacher education, continues to use frameworks similar to that proposed by Carpenter and Moser when discussing AS (see Askew 2016; Morrison & Askew 2022; Naudé, Meier & Bosman 2014; van de Walle, Karp & Bay-Williams 2016).

2.4 Core Number: Object Tracking System (OTS) and Approximate Number System (ANS)

The finding that dual core systems constitute central conceptual primitives of human number cognition is now well-documented by cognitive science. One system, the OTS, enables us to keep track of small collections of discrete objects and contributes to the conception of exact number (natural numbers) and counting. The other, the ANS, supports approximate computations with large collections of discrete objects (like piles of food) and with continuous quantity (like distance/length; two-dimensional space/area; three-dimensional space/volume; duration/time) (Carpenter, Moser & Romberg 1982; Davis 2011; Fuson 1982; Gelman & Gallistel 1978). More specifically, whereas the OTS enables precise tracking by subitising up to four discrete objects (Clements 1999; Klein & Starkey 1988; Le Corre*,* van de Walle, Brannon & Carey 2006; Starkey 1992; Starkey & Gelman 1987; Trick & Pylyshyn 1993), and slightly more when the objects are arranged in recognisable patterns (Gelman & Gallistel 1986; Gelman & Meck 1983); the imprecise ANS computes with large sets beyond OTS subitising for approximating quantity (magnitude/numerosity) (Barth, La Mont, Lipton, Dehaene, Kanwisher & Spelke 2006; Carey 2009; Dehaene 1997; Feigenson, Dehaene & Spelke 2004; Gelman 1972; Gelman 2015; Gilmore 2015; Izard, Pica, Spelke & Dehaene 2008; Spelke & Kinzler 2007). For example, comparative set-pair studies have shown pre-counting children

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[7] For example, consider the CPr "the set of all sets that do not contain themselves"—is such a set possible?

[8] The collection of all the subsets of a finite set constituting the *power set* of the set. If a set has n elements, then its power set has 2^n elements.

verbally demonstrating approximate use of ‘more’ or ‘less’, and adults revealing computations like ‘about six’ when disallowed counting (Feigenson, Dehaene & Spelke 2004; Izard, Streri & Spelke 2014; Lipton & Spelke 2003; Xu & Spelke 2000).

Even more is now known about ANS functioning. One such discovery is the ANS capability to support computational efficiency with the use of repetitions of learnt number patterns by exploiting the OTS to quickly compute object numbers in a collection, by partitioning the collection according to the constraints of the OTS. In this vein, subitising as Kaufman *et al.* (1949) described it and with the constraints of the ANS/OTS, is extended by enculturation. It is now referred to in research literature as *conceptual subitising* —with Kaufman’s original understanding referred to as *perceptual subitising* (Gelman & Gallistel 1978; Steffe & Cobb 1988; von Glasersfeld 1982). Significantly then, this body of work on subitising suggests the ANS not only supports core domain quantitative computation, but also noncore domain number knowledge acquisition.

The literature consistently reports the universal nature of ANS acuity (Gordon 2004; Halberda, Mazzocco & Feigenson 2009; Jordon & Brannon 2006; Pica, Lemer, Izard & Dehaene 2004; Xu & Spelke 2000). There is reasonable consensus that by the age of 6 months, all infants can discriminate between aggregates differing in cardinality in a 1:2 ratio (e.g., 16 vs 32, 8 vs 16, 4 vs 8) when presented with dot arrays, sound or action-sequences (Izard, Sann, Spelke & Streri 2009). The ratio, indexing the ability to distinguish aggregates, approaches 9:10 by adulthood (Halberda & Feigenson 2008) and reaches its apogee by age 30 (Halberda, Ly, Wilmer, Naiman & Germine 2012; Libertus & Brannon 2010; Lipton & Spelke 2003).

The literature has recently begun documenting the inter-variable nature of the ANS, correlating ANS acuity with number enculturation experiences (Gilmore 2010; Halberda & Feigenson 2008; Halberda *et al* 2012; Libertus & Brannon 2010; Lipton & Spelke 2003; Lourenco, Bonnie, Fernandez & Rao 2010). For instance, ANS acuity ratios have been measured as closer to 1 with higher achievers as opposed to children of similar age who struggle to calculate, and also when preschool attendees are compared with non-attendees (Halberda & Feigenson 2008; Hyde, Khanum & Spelke 2014). Thus, whilst the ANS is a universal feature of cognitive functioning, it can be impacted by experiences with number.

2.5 Counting: from core domain beginnings to true enumeration

The emergence of counting requires environmental conditions that call for precise enumeration, like record keeping and the growth of trading systems. The latter imply the emergence of the formal enculturation of number knowledge alongside language (Gordon 2004; Pica *et al.* 2004). Additionally, there must be an awareness of the concept *one* and of the relation of exact numerical equality with one-to-one correspondence (Leslie, Gelman & Gallistel 2008; Mix, Huttenlocher & Levine 2002; Pica *et al.* 2004).

We know that children recognise small object collections quantitatively ahead of understanding how to count (Gelman *et al.*, 1978; Gelman & Gallistel 1980; Hauser, Chomsky & Fitch 2002; Starkey & Cooper 1980), and when they acquire language and learn number words and number symbols alongside the count sequence, counting ideas grow. Therefore, as previously mentioned, the core systems alone cannot generate counting. This is even true of pre-counters in industrialised societies, who have been shown to fail at 'how many' and 'Give-A-Number' tasks (Schaeffer, Eggleston & Scott 1974; Wynn 1990, 1992). When asked for a number, pre-counters provide out-of-sequence number answers or merely show a handful of objects (Ansari 2008; Clements & Sarama 2004, 2009; Sarnecka & Gelman 2004).

The idea that counting requires five implicit counting principles, partially evident in core functioning, is widely accepted by cognitive science. These are: (i) the cardinal principle, where the last number of a counted sequence indicates the number of things in an aggregate; (ii) the one-one principle, where only one tag is associated with an object and all the objects are tagged; (iii) the stable order principle, where order in which objects are tagged is fixed; (iv) the abstraction principle, where any number can be associated with aggregates of any object kind; and (v) the order irrelevance principle, enabling tagging in any way provided that the other principles are observed (Gallistel & Gelman 1986; Gelman & Gallistel 1978; Gelman & Meck 1983). Characteristically, the ordinal meanings remain implicit when learning the count sequence; yet, grasping the cardinality entails that learners realise that a cardinal value represents an abbreviation of the quantitative and ordinal aspect of the prior numbers in the sequence (Fuson 1982).

Mapping a number word to its related concept of quantity and symbol as a sequence is a mentally arduous task, even though that does not seem to be the case once one has the competence (Baroody, Li & Lai 2008; Batchelor, Inglis & Gilmore 2015; Gilmore, McCarthy & Spelke 2007; Schaeffer, Eggleston & Scott 1974; Trick & Pylyshyn 1993). Learning to count successfully must engage a *successor function*, where the sequenced order of counting always generates one more number (Izard *et al.* 2008; Siegler & Booth 2004).

It is competence in counting which provides the impetus for knowledge of number to be extended from small discrete aggregates to noncore number knowledge (Dehaene, Izard, Spelke & Pica 2008). Noncore number acquisition is important for this study since incompetent counting negatively impacts success in later schooling (Carpenter & Moser 1984; Ginsburg 1977; Sarama & Clements 2009; Saxe 1982; Starkey & Gelman 1982; Steffe & Cobb 1988).

Chomsky (1995) draws our attention to a binary operation, *merge*, as the simplest core operation necessary for the production of language, and as a core and foundational operation for processing quantity. *Merge* takes two already formed things and combines them to form a new thing, without changing the original things. Thus, two distinct objects, X and Y, subjected to *merge*, generate some object Z i.e., *merge*(X,Y) = Z.

Chomsky (1983, 2016) proposes the hypothesis that exact number is abstracted from the language faculty, conceived of as derived from the necessary recursive linguistic use of *merge*. When *merge* operates over a domain consisting of a single object, a sequence that has the same formal properties as the natural numbers is generated, specifically, that of the use of a successor function, as elaborated in Peano's (1889) axioms for the natural numbers.

While Chomsky's primary interest is in the production of language, mine is concerned with aspects of the learning of elementary arithmetic. However, *merge* is defined in sufficient generality to be of use across computational systems, including those that differ from language. Chapter Three will elaborate further on my use of *merge*, but for now it is sufficient to note that *merge* can potentially be used to describe the computations performed by infants in the studies referred to in this chapter.

2.6 Concluding remarks

This chapter provided an overview of research on the innate competence of humans to deal with quantity. It included a discussion of the features of core and noncore domain number-related knowledge and the significance of counting for the study. The literature strongly advocates that noncore domain number acquisition is rooted in core domain knowledge systems and driven by the intervariable nature of the ANS affected by enculturation, often in formal education settings. Moreover, in schooling, the use of discrete aggregates is a well-established basis for extending natural number counting as instantiated by the OTS.

The cognitive resources described in this chapter are necessary conceptual prerequisites for learning elementary school arithmetic and are made available by biological endowment.

CHAPTER THREE
Theoretical Framework

3.1 Introduction

This chapter sets out the various theoretical concepts from which an analytic methodology is generated in Chapter Four. Some research reviewed in Chapter Two offers up theoretical propositions that are relevant to this study and are used as part of the general approach described in what follows.

The general orientation to research by this study is that of an *Integrated Causal Model* (ICM) approach. The ICM is a rationalist correction to what is referred to as the *Standard Social Science Model* (SSSM), in which the mind is implicitly (and often explicitly) conceived of as a blank slate (Pinker 2002), formed almost exclusively by socio-cultural practices to which the individual is exposed (Tooby & Cosmides 1992). A rationalist approach to human thought takes it as foundational that the mind/brain is pre-fitted with specialised cognitive systems by natural selection that have to be considered when studying any aspect of human activity (Chomsky 2016), like the learning of mathematics.

The texts that are the objects of analysis in this study— *Tasks* in the DBE FP numeracy workbooks— are saturated with linguistic and pictorial references to objects and processes that are very familiar to young children. What is often distracting for many researchers with an SSSM orientation when confronted with such tasks are references to objects that are, in general, considered to be non-mathematical (for example, chickens or dots). In fact, there is a body of research literature in mathematics education and curriculum studies predicated on drawing a hard boundary between the mathematical and the non-mathematical (or, ‘everyday’) in mathematics education, where references to what are seen as non-mathematical objects and processes in school mathematics texts and teaching are severely criticised as unproductive.

Arguably, the most voluble and insistent of such criticisms is to be found in the sociological work of Dowling (2008). Many of the criticisms of the use of ‘references to the everyday’ by Dowling and others (Cooper 1998; Cooper & Dunne 1999; Davis 1995; Ensor & Galant 2005; Hoadley 2005; Ensor, Hoadley, Jacklin, Kühne, Schmitt, Lombard & van der Heuvel-Panhuizen 2008; Venkat 2010) have some merit, specifically with regard to questioning the ethics of distributing mathematically impoverished content to children from working class families. However, what is largely excluded from analytic consideration in those studies is the function of apparently non-mathematical objects and processes as *computational objects and processes* in the various pedagogic situations under analysis. References to objects and processes familiar to young children might be more productively interpreted as one way in which formal schooling attempts to exploit core domain systems. This does not mean that teachers, curriculum designers or textbook authors necessarily have knowledge of the relevant cognitive science and neuroscience research, but rather that there is a *recognition of the necessity to engage the intuitive computational competence of young children* if we are to teach them school mathematics. Such recognition is part of our intuitive psychology—another core domain system produced by evolutionary processes—which enables us to interact meaningfully with other people, as well as to grasp and form social relations (Boyer 2020; Spelke 2022; Tooby & Cosmides 1992).

There are key methodological problems with the general approach that generates the various criticisms of references to the 'everyday' mounted by Dowling and others. First, language has no reference function—by which is meant that there is no necessary relation between any lexical item and things in the world, including any objects of the imagination (Chomsky 2016; Strawson 1950). An implication that flows from the absence of a reference function in language is that the associationist analytic resources of referentialism succumb to the problems of weak observational adequacy (failure to recognise research objects appropriate to the problem—the *what*) and weak descriptive adequacy (failure to recognise the appropriate internal relations of a phenomenon—the *how*), resulting in weak explanatory adequacy (failure to appropriately explain the form taken by a phenomenon—the *why*). This is not a declaration of naïve realism because the ICM approach adopted requires that one considers the ways in which the human cognition constructs observation, description and explanation.

Second, and closely related to the first methodological problem, is the problem of the strong externalism that is central to the SSSM. Holding to an externalist position, even if only implicitly, makes it impossible to generate a productive theory of learning because the human apparatuses that constitute learnability are not treated seriously (cf. Pinker 2013). Consequently, approaches falling within the frame of the SSSM cannot explain pedagogic technologies other than in a misleading manner. Unfortunately, externalist descriptions of pedagogic technologies are found in abundance in mathematics education.

One way of ameliorating the methodological difficulties discussed above is to construct a theoretical orientation that draws appropriate propositions from whichever fields of enquiry are helpful. In this study, evolutionary psychology (e.g., Tooby & Cosmides 1992), cognitive science (e.g., Spelke

2022), cognitive anthropology (e.g., Boyer 2020), biolinguistics (e.g., Chomsky 2016), mathematics education (e.g., Davis 2016), meta-mathematics/philosophy (e.g., Potter 2004) and mathematics (set theory and abstract algebra) have been drawn on to construct a theoretical orientation.

3.2 A computational approach: Constituents of a Representation

A computational approach to the study of the growth of elementary arithmetic in the mind requires one to exhibit the relations between core domain computational resources (i.e., biologically endowed computational systems) and noncore domain computational systems (in this instance, aspects of elementary arithmetic). However, the archive from which the data are generated in this study is a collection of elementary arithmetic workbooks rather than the analytic engagement with young children, so the first step is to show how the primary computational resources used in the workbooks (namely, operations on aggregates) link with core domain computational resources. To that end, a computational account of *representation* is required.

To specify in computational terms what a *representation* is, the study minimally requires a series of inter-related concepts, namely, that of a *function*, an *operation*, a *structure*, and *structure-preserving mappings*.

3.2.1 Functions

A function is a particular type of relation (process, mapping) between a set of inputs (a *domain*) and a set of possible outputs (a *codomain*), such that every element of the domain is associated with one and only one element of the codomain. The specific set of outputs of a particular function is its *range*, which is a subset of its codomain. The *process* that associates the input and output entities of a function is generally referred to in mathematical terms as a *rule*. One of the interesting things about such rules is that they are not unique, by which is meant that for a given set of inputs, an expected set of outputs can be obtained by different rules (processes). A common-sense way of thinking about the previous statement can be expressed thus: a desired outcome can be arrived at in different ways.

More abstractly, one can think of a function as a set of ordered pairs of the form $\{(d_1, c_1), (d_2, c_2), (d_3, c_3), \ldots, (d_n, c_n), \ldots\}$, where the d_i and c_i are, respectively, the elements of the domain and codomain related by some or other rule (process). Note that while for each (d_i, c_i) the d_i are unique, the c_i need not be, which merely means that more than one d_i can be associated with the same c_i, but not more than one c_i can be associated with a given d_i. Any rule (process) that realises the set of associations $\{(d_1, c_1), (d_2, c_2), (d_3, c_3), \ldots, (d_n, c_n), \ldots\}$ might be considered suitable if one sets to one side criteria like computational efficiency, concept fidelity and any others that may have the effect of restricting the range of potential rules for a given function.

3.2.2 Operations

An *operation* can be thought of as a basic, universal function. One such operation is the process of combining entities of the same kind to produce a new entity of the same kind. For example, consider having two heaps of flour and combining them to form a new, larger heap of flour. Such an operation, freed of its connection to a specific application (like combining heaps of flour), might generally be referred to as *merge* and recognised as being rather common and universal across diverse human activities and thought (see Chomsky 2016 for an account of the centrality of *merge* to the human capacity for language).

Every operation is defined over a specific domain of entities and has one or more properties. Typically, an operation exhibits closure; that is, the entities of its domain and codomain are of the same kind and there is always a result for the operation, as in the example of heaps of flour subjected to the operation *merge*. Often it is the case that the *order* in which the domain entities are used does not matter—it does not matter which heap of flour is used first, the resultant heap (codomain entity) of *merge* remains the same. There are other operations where order does matter, but the simplest situation is one in which order does not matter. Where the order of the domain entities does not matter, the operation is referred to as *commutative*. Where we have three domain entities to *merge*—like three heaps of flour—say, h_1, h_2 and h_3, it does not matter if we *merge* h_1 and h_2 first and then *merge* the result with h_3, or if we *merge* h_1 with the result of merging h_2 and h_3. The resultant heap of flour is the same in each case. In such cases we refer to the operation as *associative*. We might even imagine the existence of an 'empty heap' of flour, h_0, (acknowledging that such a construction is rather contrived in the situation). If we *merge* the 'empty heap' with any actual heap of flour, what results is the existing heap of flour. If the domain of *merge* in a particular situation does contain an entity like h_0, it is referred to as the *identity element* of the domain.

From the description of *merge*, it should be recognised that it is a *binary* operation, meaning that it takes as its input *two* objects. The number of entities an operation (and more generally, a function) takes as its input material is referred as its *arity*, so *merge* has an arity of 2. I return to a discussion of various features of operations across the subsequent sections of this chapter.

3.2.3 Structures

The collection of properties of *merge* as it acts on heaps of flour as just described constitutes a *structure*. Accepting that heaps of all sorts of things can be treated in the manner described, we might postulate the existence of a general class, HEAP, where a particular member of the class is indicated by a suitable existential marker—like *flour*, for example, so that $\text{HEAP}_{\text{flour}}$ might be a suitable name for the example of heaps discussed here. The ordered pair ($\text{HEAP}_{\text{flour}}$, *merge*) is the symbol for the *structure* consisting of in: (i) heaps of flour (domain), with (ii) the operation *merge* defined over the set of all possible heaps of flour. As elaborated above, the structure ($\text{HEAP}_{\text{flour}}$, *merge*) has the following properties, which are now presented in more formal terms:

1. Combining any two heaps results in a heap (*closure*: for all $h_i, h_j \in \text{HEAP}_{\text{flour}}$, there always exists a $h_n \in \text{HEAP}_{\text{flour}}$ such that $[h_i \textit{ merge } h_j] = h_n$).
2. The order in which two heaps are merged does not matter (*commutativity*: for all $h_i, h_j \in \text{HEAP}_{\text{flour}}$, $[h_i \textit{ merge } h_j]= [h_j \textit{ merge } h_i]$).
3. Given any three heaps, combining the first heap with the heap that results from combining the second and third heaps produces the same result as combining the result of combining the first two heaps with the third heap (*associativity*: for all $h_i, h_j, h_k \in \text{HEAP}_{\text{flour}}$, $[h_i \textit{ merge } [h_j \textit{ merge } h_k]] = [[h_i \textit{ merge } h_j] \textit{ merge } h_k]$).
4. There exists a null heap which is such that combing the null heap with a non-null heap results in the non-null heap. Similarly, combining a non-null heap with the null heap results in the non-hull heap once again (*identity element*: there exists a $h_0 \in \text{HEAP}_{\text{flour}}$ such that for all $h_i \in \text{HEAP}_{\text{flour}}$, $[h_0 \textit{ merge } h_i] = h_i = [h_i \textit{ merge } h_0]$). Of course, this is just another way of saying that we have not *merged* a particular heap with any other heap.

The kind of structure described is very simple. More complex structures can be described in which more than one operation are constituents of a structure (e.g., rings and fields). However, the example discussed here is sufficient to communicate the idea of a structure for now.

3.2.4 Structure-preserving mappings

Note that in Section 3.2.1.3 the *merging* of heaps of flour can be thought about without any reference to number, which simply means that numerical quantification is not a necessary feature of *merge*. In other words, recognising distinct heaps of flour does not require that any numerical quantification accompany such recognition, neither does *merging* heaps to form new heaps require numerical quantification.

Imagine now that our heaps of flour must be processed in accordance with some or other system of quantification, like when following a recipe. Once a system of quantification is introduced into the situation, a *quantification procedure* is necessarily introduced—like *counting/measuring*. A baking recipe typically treats heaps of flour in terms of units like *cups*, or *grams*, or *millilitres* (or any other culturally established units).

Now, processes of numerical quantification of entities in the world/imagination imply the existence of systems in which numbers can serve as inputs to arithmetic operations, where the assigning of numerical values to the entities that are referenced is a mediating device that has the potential to enable a stable connection between operations on things in the world/imagination and arithmetic operations. The system of processes involved are more easily recognised in diagrammatic form. A simple but illuminating example can be constructed by way of reference to the discussion of *merge*. Suppose that we have heaps of flour, h_i and h_j, that are to be *merged* to produce a new heap, h_k. Suppose further that we have some procedure for quantifying the heaps—let's refer to it as *measure*—that associates a heap with a number. Intuitively, it appears that the operation *merge* has the same general features as those possessed by the *addition* of numbers, so we might reasonably imagine a mapping from *merge* to *addition*.

Given that I have been discussing *merge* in relation to heaps of flour (including the idea of a null heap), I will restrict *addition* to operating over the set of non-negative real numbers ($\mathbb{R}_0^+$) for this discussion. Figure 3.1 describes the relations between the structures ($\text{HEAP}_{\text{flour}}$, *merge*) and ($\mathbb{R}_0^+$,+) in diagrammatic form in terms of the mapping *measure*:($\text{HEAP}_{\text{flour}}$, *merge*)→($\mathbb{R}_0^+$,+), which nicely expresses the 'translation' of operations on heaps to operations on numbers.

$$\begin{tikzcd}
h_i, h_j \arrow[r, "merge"] \arrow[d, "measure"'] & h_k \arrow[d, "measure"] \\
m,n \arrow[r, "+"'] & p
\end{tikzcd}$$

Figure 3.1: Details of the mapping *measure*:($\text{HEAP}_{\text{flour}}$, *merge*) → ($\mathbb{R}_0^+$,+). Here, h_i, h_j are heaps, while m, n and p are non-negative real numbers. The operation (process) *measure* assigns numbers to heaps, thereby quantifying the latter.

The details of Figure 3.1 can be made more intelligible by imagining a commonly occurring domestic situation, like a step in baking a cake.[9] Suppose we have a recipe calling for 250g of flour and that we have a scale atop of which sits a measuring bowl into which ingredients are to be inserted to be weighed. Imagine that after filling the bowl with flour (the first heap, h_1) we find that the scale registers the weight of the flour as 208g (*measure*). We then empty the contents of the measuring bowl into a mixing bowl. Since we are short of 250g by 42g, we pour more flour into the now empty measuring bowl until the scale registers 42g on its display (the second heap, h_2). We then pour the second heap of flour into the mixing bowl (thereby producing a third heap, h_3).

The operation *merge* is instantiated in the act of combing the two heaps of flour in the mixing bowl. Figure 3.2 describes the system of computations in the manner of Figure 3.1.

$$\begin{tikzcd}
h_1,h_2 \arrow[r, "merge"] \arrow[d, "measure"'] & h_3 \arrow[d, "measure"] \\
208,42 \arrow[r, "+"'] & 250
\end{tikzcd}$$

Figure 3.2: The mapping *measure*:($\text{HEAP}_{\text{flour}}$, *merge*) → ($\mathbb{R}_0^+$,+) with reference to the specificities of the example discussed above.

Now consider the properties of *addition* defined over the non-negative real numbers ($\mathbb{R}_0^+$)[10]:

1. For every two non-negative real numbers, m and n, there exists a non-negative real number p such that $m + n = p$ (*closure*).
2. For every two non-negative real numbers, m and n, it is the case that $m + n = n + m$ (*commutativity*).
3. Given any three non-negative real numbers, m, n and p, it is the case that $m + (n + p) = (m + n) + p$ (*associativity*).
4. There exists a non-negative real number 0 such that for all non-negative real numbers, n, $0 + n = n = n + 0$ (*identity element*).

Referring to the properties of the structure ($\text{HEAP}_{\text{flour}}$, *merge*) detailed earlier, it should be apparent that it has the same general properties as the structure ($\mathbb{R}_0^+$,+). In other words, *measure* preserves

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[9] The example is somewhat contrived for the purpose of clarifying the system described in Figure 3.1. When baking one would probably arrive at the desired result in a more efficient manner, like placing the mixing bowl directly on the scale and pouring flour into the bowl from a packet until the scale registers 250g.

[10] For this study, we speak of the properties of addition defined over the natural numbers, where zero is included. Thus, $\mathbb{N} = \{0,1,2,3, \dots\}$

structure when mapping ($\text{HEAP}_{\text{flour}}$, *merge*) to ($\mathbb{R}_0^+$,+) and we can, therefore, refer to *measure* as a structure-preserving mapping in this instance. Structure-preserving mappings are also referred to as *homomorphisms* in mathematics.

3.3 Representations

Gallistel & King (2010) argue that homomorphisms are essential to the construction of *representations* and are thus central to cognitive processing enabling of the mental computations we employ to manipulate data derived from the environments we inhabit. The processes we recognise as existing in the environment can be thought of as compositions of operations defined over apposite varieties of entities perceived in the world. Such processes along with the classes of entities they act on are referred to as *represented systems* by Gallistel and King. The conceptual-intentional systems internal to the mind/brain that process the data derived from represented systems are referred to as *representing systems*. Such *represented* and *representing* systems correspond to what I have referred to as *structures*.

More generally, a *representation* is defined as a relation between a *representing system* and a *represented system* meeting three conditions:

1. Causality: The mapping from entities in the represented system to their symbols in the representing system is *causal* (as, for example, ‘when light reflected off an object in the world acts on sensory receptors in an eye causing neural signals that eventuate in a percept of the object)’ (Gallistel & King 2010:55; italics in the original).
2. Structure-preservation: ‘The mapping from entities in the represented system to their symbols is such that functions defined on the represented entities are mirrored by functions of the same mathematical form between their corresponding symbols. Structure-preserving mappings are called *homomorphisms*’ (*ibid.* 2010; italics in the original).
3. Efficaciousness: ‘Symbolic operations (procedures) in the representing systems are (at least sometimes) behaviourally *efficacious*: they control and direct appropriate behaviour within, or with respect to, the represented system’ *(ibid*. 2010; italics in the original).

Returning to the mapping *measure*:($\text{HEAP}_{\text{flour}}$, *merge*) → ($\mathbb{R}_0^+$,+) and the specific example outlined, we can check it against the conditions for the realisation of a representation. First, causality: *measure* associates numbers (indicating grams of flour) with heaps in a manner that preserves order—heaps arranged pairwise as smaller vs. larger are associated with numbers arranged pairwise as lesser vs. greater. Second, structure-preservation: the *merging* of heaps corresponds in a consistent manner to the *addition* of the numbers associated with the heaps. Third, efficaciousness: computing the shortfall in grams of the first heap from the target (in grams) of the required amount of flour directs the baker in producing an appropriate second heap of flour to make up the shortfall. We can thus conclude that the mapping $\textit{measure}:(\mathrm{HEAP}_{\mathrm{flour}}, \textit{merge})\rightarrow(\mathbb{R}_0^+,+)$ is an appropriate representation for the situation under consideration.

Note, once again, that humans (and many other species) can compare and arrange groups of entities pairwise in increasing or decreasing orders of magnitude without recourse to number. The core domain resources that figure in such computations are, at minimum, the ANS and the OTS, as described earlier in Chapter Two. The existence of the ANS and the OTS presupposes a sensitivity to aggregates since both systems take as their computational material (inputs and outputs) groups of entities perceived as such (Davis 2016; 2018). In the context of the pedagogic situations of schooling concerned with the introduction of elementary arithmetic to young children, curriculum designers, textbook authors and teachers routinely exploit the incipient potential for the growth of mathematics inherent to the ANS and OTS as grounds for teaching and learning arithmetic. In other words, the intuitive computations that young children can perform on aggregates are exploited for the teaching of elementary arithmetic. This is achieved through the construction of representations that relate core domain computations on aggregates to presentations of computations on aggregates, together with counting, that constitute an additional layer of representations linking to the basic operations of elementary arithmetic, a typical example of which is shown in Figure 3.3.

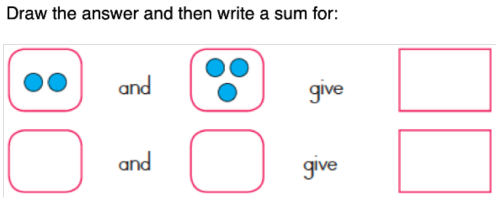

Draw the answer and then write a sum for:

Figure 3.3: Task 19, Subtask 4, Grade 1, Workbook 1 (DBE 2021a: 41).

The computation required in the first row of the problem shown in Figure 3.3 is an application of *merge*, the result of which is a third aggregate consisting of all the blue disks contained in the first two aggregates, reading left to right.

Figure 3.4: Solution to Task 19, Subtask 4, Grade 1, Workbook 1 (DBE 2021a:41).

The computations in the second row require the application of *counting*, giving the cardinality of each of the aggregates in the first row. Figure 3.4 shows the intended series of results.

The term *give* used in the task refers to an operation akin to *merge* when dealing with the intended operation implicitly defined over the aggregates of blue discs, and it refers to *addition* defined over natural numbers (ℕ) when dealing with the cardinal values of the aggregates. Figure 3.5 shows the system of intended computations, illuminating the details that constitute the representation.

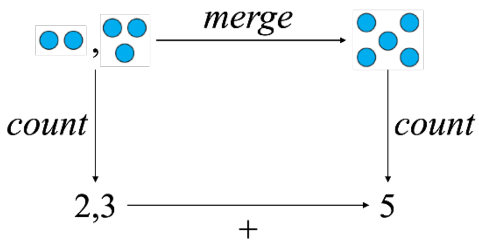


Figure 3.5: The representation for Figure 3.4.

We can consider the aggregates used in *Subtask* 4 as members of the class of finite sets or discrete aggregates (FINSET or DAG, respectively) and the instance of *merge* used as the set-theoretic operation *disjoint union* (⊔). The process *count* is the mapping that relates the structures (DAG,⊔) and (ℕ,+) in a structure-preserving manner, and we can write *count*:(DAG,⊔)→(ℕ,+) as a general description of the type of representation used in *Subtask* 4. Using the symbol # for *count*, capitalised letters for finite sets, and lowercase letters for natural numbers, we can diagram the general representation #:(DAG,⊔)→(ℕ,+) used in problems like *Subtask* 4, illustrated in Figure 3.6.

$$\begin{tikzcd}
A,B \arrow[r, "\sqcup"] \arrow[d, "\#"'] & C \arrow[d, "\#"] \\
m,n \arrow[r, "+"'] & p
\end{tikzcd}$$

Figure 3.6: The general representation used in the design of problems like Subtask 4, Task 19.

That the mapping #:(DAG,⊔)→(ℕ,+) is structure-preserving can be checked by comparing the properties of (DAG,⊔) and (ℕ,+), as in Table 3.1. It is a relatively easy matter to check that the three conditions for a representation—causality, structure-preservation, and efficaciousness—are also satisfied by #:(DAG,⊔)→(ℕ,+).

Table 3.1: A comparison of the properties of the structures (DAG,⊔) and (ℕ,+).

	Properties of (DAG,⊔)	Properties of (ℕ,+)
Closure:	Given any two finite sets, A and B, there exists a finite set C such that $A \sqcup B = C$.	Given any two natural numbers, m and n, there exists a natural number p such that $m + n = p$.
Commutativity:	Given any two finite sets, A and B, $A \sqcup B = B \sqcup A$.	Given any two natural numbers, m and n, $m + n = n + m$.
Associativity:	Given any three finite sets, A, B and C, we have $A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C$.	Given any three natural numbers, m, n and p, $m + (n + p) = (n + m) + p$.
Identity element:	There exists a set $\emptyset$, called the *empty set*, which is such that, for any finite set A, $A \sqcup \emptyset = A = \emptyset \sqcup A$.	If we include 0 in the natural numbers, then for any natural number n, it is the case that $n + 0 = n = 0 + n$.

Similar descriptions of the representations linking operations defined over DAG to the various basic arithmetic operations will be discussed in the chapters that follow (i.e., those used for progressing AS). For instance, Chapter Five explains the intuitive notion *unmerge* (or *purge*), a derivative of *merge*. *Unmerge* refers to a kind of an 'undoing of *merge*', pertaining to the idea of intuitive partitioning. It will also later become clear how attempts at representations that map relative complement over DAG to subtraction over natural number via counting is more complex than representations mapping disjoint union over DAG to addition over natural number, since the former requires more mathematical restrictions (refer to Chapter Five) on the relations between the arguments or operations than the latter. Thus, for the sake of retaining simplicity and clarity for the reader at this point in the thesis, only *merge* which translates intuitive addition is articulated.

3.4 Concluding remarks

Mathematical tasks presented to learners in the FP of schooling range from learning how to count; associating finite sets with numbers, with number words and with numerals (number symbols); learning the prescribed notational features for referring to number words, numerals and operations; solving problems focused on a single application of a particular basic operation, as well as multiple applications of a particular operation; and solving problems that require the composition of different basic operations.

The theoretical framework articulated in this chapter sets out the ideas necessary for a computational-analytical approach that meets the disciplinary requirements of mathematics. Chapter Four establishes the analytic protocols for describing the computational characteristics of the mathematical tasks that use operations on discrete aggregates in the workbooks as a ground for elaborating elementary arithmetic. To that end, additional mathematical resources will be introduced as they are required. The analytic protocol will be developed by way of reference to the different types of tasks that use discrete aggregates.

CHAPTER FOUR

Analytic Framework

4.1 Introduction

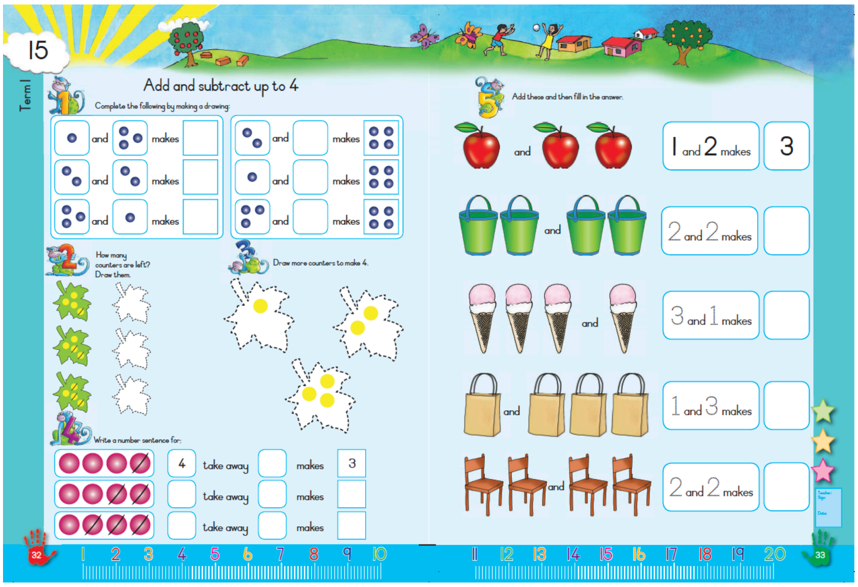
The study is primarily *descriptive*, using *document analysis* to generate an account of some of the central features of the computational universe of FP mathematics intended to be realised in pedagogical treatments of elementary arithmetic in state-funded SA schools. The descriptive data will be extended by the use of tabulations recording totals of all identified aggregate mappings, arithmetic mappings, and structures across the tasks analysed.

4.2 Research design

The central concern of this study is the computational processes that use discrete aggregates, and operations over such aggregates, in the constitution of *addition and subtraction* (AS) in elementary arithmetic.

4.2.1 Information archive

The archive of information from which data will be produced is the complete set of six 2021 DBE numeracy workbooks (*Mathematics in English*) allocated for Grades One to Three of FP, as prescribed for use in state-funded SA schools. Two workbooks are made available per grade, the first of which covers content to be covered in terms one and two, and the second, terms three and four. Thus, six workbooks constitute the information archive.



15

Term 1

Add and subtract up to 4

1 Complete the following by making a drawing:

and makes

and makes

and makes

and makes

and makes

and makes

2 How many counters are left? Draw them.

3 Draw more counters to make 4.

4 Write a number sentence for:

4 take away makes 3

take away makes

take away makes

5 Add these and then fill in the answer.

and 1 and 2 makes 3

and 2 and 2 makes

and 3 and 1 makes

and 1 and 3 makes

and 2 and 2 makes

Teacher: Sign: Date:

Figure 4.1: Grade 1, Workbook 1, Task 15: ‘Add and subtract up to 4’ (*source:* DBE 2021a:32).

Workbooks typically consist of a series of numbered *Tasks*, each of which contain *Subtasks* and within the *Subtasks*, *Exercises*. *Tasks* tend to be spread over two facing pages and are headed by a descriptive name announcing the global topic of the task. For example, in Figure 4.1, *Task* 15 of Grade 1, Workbook 1 is titled, ‘Add and subtract up to 4’, which is, presumably, meant to function as an indicator of the semantic focus of the *Task*, its five *Subtasks*, and related *Exercises*.

4.2.2 The unit of analysis; production of descriptive data

Since this study is concerned with the way operations over discrete aggregates are used to ground the growth of knowledge of AS, there are a few observations required to produce the descriptive data.

The primary *unit of analysis* in this study is referred to as a *Task* made up of *Subtasks* which contain *Exercises*. For example, *Subtask* 5 of Figure 4.1 consists of five *Exercises*. Each *Exercise*, together with the *Subtask* and *Task* of which it is a part, constitutes a *workspace* populated by *computational objects*. Computational objects are things that function as inputs and outputs of processes, together with the processes indicated in an exercise and in the broader contexts of *Subtasks* and *Tasks*.

The processes, inputs and outputs indicated in a workspace consist of those indicated explicitly (in text, diagrams, and pictures, as well as other typographical resources used to present an *Exercise*) and those that are incipient, emerging from the application of processes. Note that processes can themselves be treated as inputs and outputs of processes in the workspace of an *Exercise*.

In the construction of solutions to *Exercises*, *any* of the computational objects can be selected for use as needed:

(1) When considering an *Exercise*, the first question asked is whether or not discrete aggregates are used. If no discrete aggregates appear, it is not considered for producing data. Discrete aggregates are always taken as members of the class of finite sets (DAG). For example, in *Task* 15 shown in Figure 4.1 it is evident that various collections of discrete aggregates (like discs/balls, apples, buckets, bags, chairs) are used in the design of the *Subtasks*.

(2) If discrete aggregates are used in an exercise, the operation/s or mappings (implicitly) defined over DAG are noted. For example, in *Subtask* 1 of *Task* 15, shown in Figure 4.1, the term *and* is intended to behave like the binary set-theoretic operation *disjoint union*, $\sqcup$. In other words, the depicted aggregates are meant to be merged to produce a third aggregate consisting of all the members

of the given disjoint aggregates. An implicitly defined computational structure, $(\mathrm{DAG},\sqcup)$, thus emerges in *Subtask* 1 of *Task* 15, and is noted.

(3) In many *Exercises* one expects to find operations over the natural numbers, $\mathbb{N}$, as well as operations over DAG. Where that is the case, the computational structures implicitly defined over $\mathbb{N}$ are also noted. For example, in *Subtask* 4 of *Task* 15, the term ‘take away’ refers to subtraction defined over $\mathbb{N}$, so that we have the computational structure $(\mathbb{N},-)$ emerging implicitly in the *Subtask*. This structure and its correlate DAG-related structure have a few complications that will be discussed as they emerge in the analysis of subtasks. The terms FINSET (finite sets) and DAG (discrete aggregates) are synonymous where they are referenced across the study.

(4) Where *Exercises* consist of operations over discrete aggregates and over $\mathbb{N}$, there must be some mapping connecting the distinct computational structures, and so should be identified. In most such instances the connecting mapping is *counting* or is a composition of counting with some or other operation/mapping. Again, *Subtasks* 4 and 5 of *Task* 15 show examples of such instances.

(5) Once the computational structures used in an exercise have been identified and described in suitably precise mathematical terms, and where one or more mappings connecting the structures are present, the connection/s between the mappings is/are assessed for structure-preservation. For example, in *Subtask* 5 of *Task* 15 (Figure 4.1), the structures $(\mathrm{DAG},\sqcup)$ and $(\mathbb{N},+)$ are connected by *counting*, which we can write in mathematical terms as $\#:(\mathrm{DAG},\sqcup)\rightarrow(\mathbb{N},+)$, where the symbol # refers to *counting*. The mapping $\#:(\mathrm{DAG},\sqcup)\rightarrow(\mathbb{N},+)$ can then be assessed for structure-preservation. Table 3.1 of Chapter Three shows the properties of the structures, a comparison of which demonstrates that structure is preserved as $(\mathrm{DAG},\sqcup)$ is mapped to $(\mathbb{N},+)$ by *counting*.

(6) Given that the use of operations over aggregates mapped to operations over $\mathbb{N}$ is central to the constructions of representations used for the purposes of teaching and learning elementary arithmetic, each of the exercises analysed are measures against the criteria for a representation: (i) *causality*, (ii) *structure-preservation*, and (iii) *efficaciousness*. Primarily, (ii) is the focus since (i) and (iii) rely upon (ii).

(7) As learners progress beyond the most elementary *Tasks* in the workbooks they encounter *Exercises* that require the composition of two or more operations/mappings. Compositionality of operations/mappings is another way of saying that the learner is required to deal with *recursion* in the sense that the value of an operation/mapping is used as the argument (or one element of an argument)

of a subsequent operation/mapping. In such instances it would sometimes be the case that an exercise requires the learner to select appropriate arguments from the values (outputs) of previous operations/mappings. Such selections will be indicated by an appropriate *inclusion mapping* as necessitated by the computational context. Inclusion mappings enable selections of whichever computational objects in the workspace are needed, as they are needed.

(8) In the process of data production object-and-arrow diagrams (examples of which are shown in the figures of Chapter Three) will often be used to diagrammatically display the computational structures, the connections between structures, and extended chains[11] of compositions (where required). It is not possible to include all detailed object-and-arrow diagrams of all *Exercises* analysed in the body of the text. Only illustrative diagrams of specific instances that capture a class of instances of a particular computational type will be highlighted, accompanied by global diagrams where necessary.

4.2.3 Quantitative trend data

As a means of gauging the extent of the use of the range of mappings/operations and structures identified for *representations* across the *Tasks* analysed, the descriptive (qualitative) data will be extended by summary tables totalling all mappings/operations and structures used across the workbooks.

Simple grid systems of the form *Task* × *Mappings/Operations* will record the presence or absence of computational mappings noted across the workbook *Tasks*, per grade and per workbook. The symbols ‘1’ and ‘0’ record the presence or absence, respectively, of the mapping corresponding to a grid cell. The cumulative data will be used to generate and interpret global quantitative and trend data across the workbooks with respect to AS.

4.2.4 Notable exclusions from analysis

Tasks where multiplication is developed by the use of ‘repeated addition’ are excluded from analysis, since the development of multiplication is beyond the scope of this study. However, where repeated addition is inserted into *Tasks* aimed at developing addition and subtraction, they are considered in the context of iterative disjoint union and discussed under Section 6.4, Chapter Six, ‘Chains of operations.’

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[11] For example, as in the case of ‘3 + 3 + 3’ which substantiates iterations/recursions/chains of the operation addition over N. A recursive process is one in which the output of one operation/mapping serves as the input of a subsequent operation/mapping.

4.3 Analytic reliability

The object-and-arrow diagrams that emerge from the analytic protocol outlined in Section 4.2.2 exhibit the pertinent computational data in the form of a visual summary that is very revealing since it shows the domain, codomains, operations, computational structures, computational paths (including compositions of operations/mappings), thereby providing a succinct description of the realised curriculum *content* and (aspects of) the mathematical universe generated by the workbooks. Such data can then be used as a productive basis for further considerations of curriculum design, teaching, and learning.

Reliability references the extent of an analytical tool or method to consistently produce a similar result when conducted by researchers across different contexts at different times (Lincoln & Guba 1985; Opie 2004). The reliability of the method used by this study rests upon the mathematical descriptions of computational activity, in line with meeting the conditions for descriptive and observational adequacy. A foundational premise of this study rests upon the notion that all thought is computational, as is now well-documented by contemporary cognitive science research. Thus, choosing to construct computational descriptions of mathematical activity defined upon explicit, unambiguous principles which are easily comprehensible and replicable by other researchers becomes an essential prerequisite for this study.

One major challenge to reliability is entailed in the infinite ways in which rules connecting inputs with outputs of any operation, function or mapping may vary. For example, substituting an operation by a chain of operations is one such variation. Consequently, this ineradicable capacity for rules to vary demand that a researcher be sensitive to potential variations in the process of generating computational descriptions.

Analytic reliability in this study was strengthened by having two additional mathematically adept researchers use the protocol to generate object-and-arrow diagrams of examples of each of the computational types that emerge from my analysis. Where discrepancies emerged, they were deliberated, and the data were revised—if necessary—to the satisfaction of myself and the additional researchers.

4.4 Validity

Firstly, qualitative research requires consideration of descriptive and interpretive validity, where accurate reporting of facts and accurate interpretation is essential. This is achieved by the study’s rigorous approach to analysis. Suitable mathematical descriptions were generated for computational objects used by the discrete aggregates across the six workbooks. These were then sorted on the basis of common descriptions to capture a class of instances of computational processes used in problem-solving and organised to reflect emergent patterns, revealing the big ideas for reporting upon.

Secondly, qualitative research also requires attention to theoretical and explanatory validity which is achieved when theoretical and analytical frameworks of a study become systematically integrated/coupled (Maxwell 1992). Adopting an ICM approach for the study enabled me to freely adopt relevant resources and phenomena across multiple fields of enquiry as was necessary. Notwithstanding, over time the descriptions and research are likely to become outmoded and need reconsideration in the context of emergent new knowledge, but this is an ineradicable component of all scientific endeavour.

4.5 Ethical considerations

Being a descriptive textual analysis of workbooks in the public domain, with no involvement of human subjects, ethical permissions from any institution outside the University are not required.

4.6 Concluding remarks

This chapter set out the methodological protocol for generating the data pursuant to the main research question framed in Chapter One. Chapters Five, Six and Seven next present the data analysis.

CHAPTER 5

Analysis: Part One

5.1 Introduction

Chapters Five and Six presents the descriptive analysis pursuant to the analytic protocol outlined in Chapter Four. An analysis of *Tasks* across the six DBE FP numeracy workbooks enabled descriptions of computational features of *Exercises* that employ discrete aggregates for the development of natural number AS. One hundred and three (24.9%) of 414 *Tasks* constituted across the six workbooks met the criteria for inclusion for analysis. See Table 5.1a.

Table 5.1a: Summary of the number of *Tasks* analysed, per workbook, per grade.

Grade	Workbook	Number of *Tasks* analysed	Percentage of total
1	*Mathematics in English, Book 1*	29/69	42,0%
1	*Mathematics in English, Book 2*	18/64	28,1%
Total		47/133	35,3%
2	*Mathematics in English, Book 1*	21/68	30,9%
2	*Mathematics in English, Book 2*	13/75	17,3%
Total		34/143	23,8%
3	*Mathematics in English, Book 1*	7/71	9,6%
3	*Mathematics in English, Book 2*	15/67	22,4%
Total		22/138	15,9%
Cumulative Total		103/414	24,9%

The intended goal is to discuss the constitution of natural number addition and subtraction (AS). However, learners are required to master preliminary[12] computations prior to being confronted with AS more explicitly and implicitly achieved through the use of discrete aggregates (DAG). Furthermore, the association of discrete aggregates with natural numbers (ℕ) entails the recognition

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[12] These are peppered across Grades One to Three, but saturated in Grade Reception workbooks and the main reason Grade Reception is excluded from the study. It would have also required including an additional analytical framework for the study expanded upon in Davis (2018). Some of the mappings are covered in Appendix A.1.1. and A.1.2.

of ℕ as markers of equivalence classes of discrete aggregates. That is, any natural number, *n*, refers to an entire class of discrete aggregates of cardinality *n*. The construction of the equivalence classes of discrete aggregates associated with the natural numbers implies the recognition of bijective mappings (1-1 correspondences) between the discrete aggregates that populate an equivalence class. Classes, and so equivalence classes, are collections that are governed by CPRs, like the class of all sets of cardinality '3'. A learner could select and display any discrete aggregate of cardinality 'three' to demonstrate the concept of 'threeness' is understood. The ability to perform such a task successfully implies that a learner can connect number words, numbers and discrete aggregates, and that they can count. When such tasks are presented in written text that uses numerals, then connections with the latter have to be included. Thus, it should be clear, these minimum computational resources foreground the learning of AS. Appendix A.1.1 and A.1.2 lists the mappings which occur between discrete aggregates (DAG) and number concepts (NUM), number symbols (SYM), and number word (WRD), not forgetting continuous aggregates (CAG), relating domains with codomains when learners begin to count (Davis 2018: 17-18).

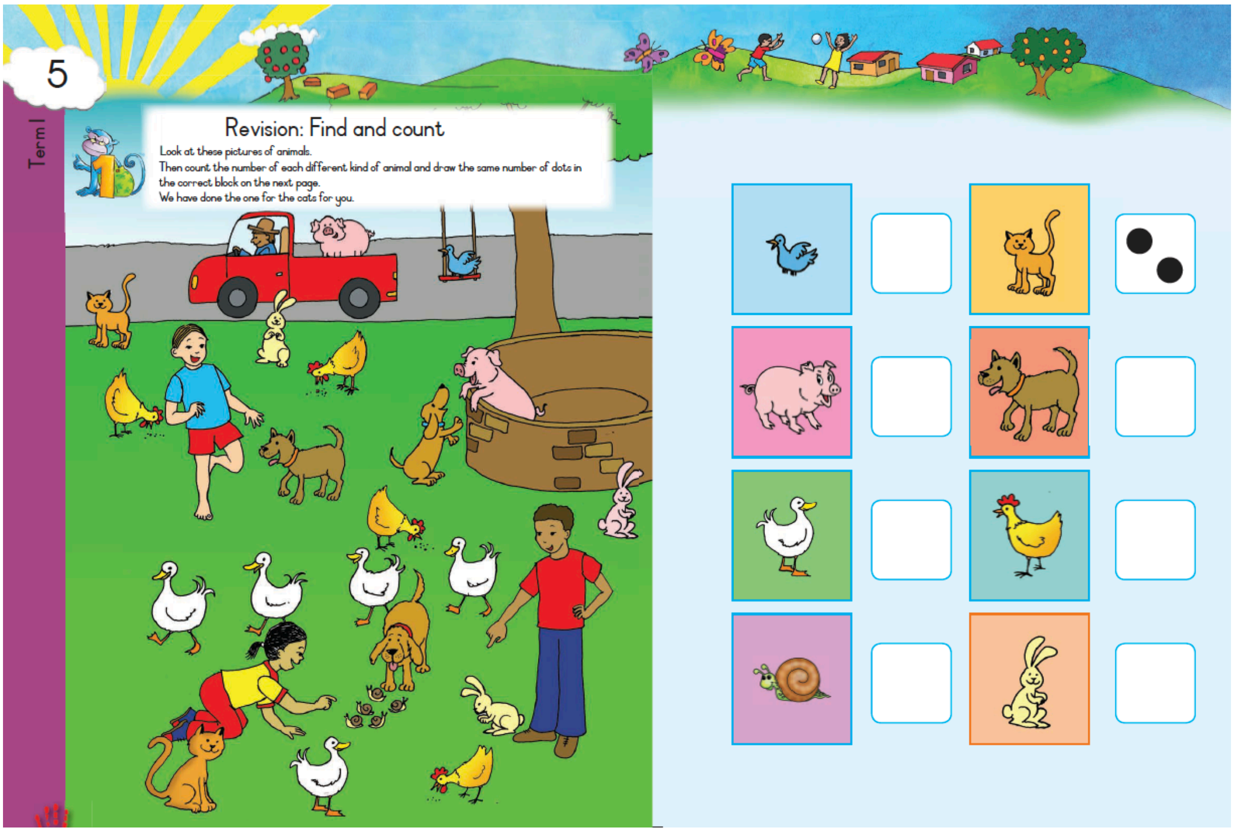
5.2 The use of Comprehension Principles

When discrete aggregates are used in the workbooks, it is invariably the case that the use of a CPr can be discerned. Often more than one CPr is used. For example, both object type and colour often appear simultaneously as CPrs. The use of CPrs is apparent throughout the six workbooks. This is unsurprising given that core domain conceptions of aggregates appear to be bound by membership rules (see Chapter Two).

5.2.1 The formation of discrete aggregates by means of CPrs

The intuitive idea that one can form a discrete aggregate (DAG) by use of a CPr is affirmed and exploited from the outset, and constitutes a fundamental computational mapping used in the FP mathematics curriculum. The mapping, let us name it GRP, can be indicated generally in symbolic terms as GRP: CPR → DAG, where GRP is intended to suggest the formation of a group of discrete things. CPR refers to the class of CPrs indicated by its rules of membership, (domain) while DAG refers to the class of discrete finite sets, as codomain. The '→' translates the mapping between domain and codomain.

Figure 5.1 illustrating *Task* 5 of Grade 1, Workbook 1, involves a set of instructions which require learners construct groups of discrete things, employing a CPr that refers to a specific type of creature. The completed *Exercise* in *Subtask* 1 implicitly entails the computation GRP:(*cat*) → {∀creatures in Task 5 s.t. the creature is a *cat*}, with the CPr, *cat*, indicated pictorially (rather than in words as indicated here). Learners are not required to list the groups of creatures to which the CPrs refer by drawing them but constitute the necessary discrete aggregates by, at least, recognising and selecting images of creatures of the correct kind.



Revision: Find and count

Look at these pictures of animals.
Then count the number of each different kind of animal and draw the same number of dots in the correct block on the next page.
We have done the one for the cats for you.

Figure 5.1: Task 5, Grade 1, Workbook 1 (*Source*: DBE 2021a:10-11)

5.2.2 The use of bijections (1-1 correspondences) between discrete aggregates

Referring once more to *Task* 5 in Figure 5.1, note the set of instructions: ‘Look at these pictures of animals. Then count the number of each different kind of animal and draw the same number of dots in the correct block on the next page. We have done the one for the cats for you’.

Each *Exercise* requires the production of discrete aggregates—groups of dots—standing for the aggregate indicated by the pictorial CPrs, by explicitly counting dots as part of the process. Yet, simply using 1-1 correspondence/bijections without counting is equally possible.

Consider GRP:(*chicken*) → {∀ creatures in Task 5 s.t. the creature is a *chicken*}. For each chicken in the picture, the learner can draw a dot in the space provided and can do so without counting either chickens or dots. When that happens, the mapping employed is one that uses the construction of a bijection between two discrete aggregates, the mapping which one might describe in symbolic terms as

> BIJ: {$\forall$ creatures in Task 5 s.t. the creature is a *chicken*} $\rightarrow$ {*dots* s.t. for every *dot* $\exists$ a unique *chicken* in Task 5},

where BIJ indicates a 1-1 correspondence between one group of things and another. Since the aggregate {*dots* s.t. for every *dot* $\exists$ a unique *chicken* in Task 5} is the same thing as {●,●,●,●}, we can write,

> BIJ:{$\forall$creatures in Task 5 s.t. the creature is a *chicken*} $\rightarrow$ {●,●,●,●}.

More abstractly, we might write the mapping, BIJ, as BIJ: DAG$\rightarrow$ DAG, indicating that both the domain and codomain of BIJ, as defined here, is the class DAG. Since there are usually very specific discrete aggregates that are to be related by a bijection in the exercises, as in this example, it might be helpful to specify them in the notation, indicating the originating aggregate first (A) and the target aggregate second (B), as $\mathrm{BIJ}_{A,B}$: DAG$\rightarrow$ DAG.

5.2.3 The use of counting to construct quantifications of discrete aggregates (DAG)

Still referring to *Task* 5 in Figure 5.2, we note that the instructions call for the use of counting (let us refer to using symbol #). Considering once more the exercise that corresponds to the mapping GRP:(chicken) $\rightarrow$ {$\forall$creatures in Task 5 s.t. the creature is a chicken}, we can see that the learner is required to construct the mapping #:{$\forall$creatures in Task 5 s.t. the creature is a chicken} $\rightarrow$ 4, since # maps discrete aggregates to their cardinal values (natural numbers, $\mathbb{N}$). We can describe the mapping in general symbolic terms as #:DAG $\rightarrow$ $\mathbb{N}$, indicating that DAG is the domain of # with $\mathbb{N}$ as its codomain.

5.2.4 The use of $\mathbb{N}$ and counting to construct discrete aggregates aligned with a CPr

Once again, refer to the chicken *Exercise* in *Task* 5, Figure 5.1. for this discussion. To complete each of the *Exercises* in strict accordance with the instructions provided, the learner would be required to use a slightly more complex *binary* version of GRP, which has as its domain not just a specific CPr (*dot*, as the membership rule of the specific discrete aggregate required in this instance), but also the cardinality ($\mathbb{N}$) of the discrete aggregate to be produced. Defining CPR as distinct from CPr, as the class of all CPrs, here the domain of GRP is the set of ordered pairs, (CPR$\times\mathbb{N}$), which has as its first argument a particular CPr and, as its second, a particular natural number, as in (*dot*,4).

For the exercise referencing chickens we can write, GRP:(*dot*,4) → {●,●,●,●}, indicating that the domain of GRP is the cross product of the class of CPrs and the natural numbers, CPR×ℕ, and that its codomain is DAG. More generally, GRP:CPR×ℕ → DAG. One can quibble about whether the pairs need to be ordered. For instance, suppose that a learner thinks of the solution as requiring *4 dots* to be produced, then it may seem reasonable that we can describe the domain as the set of pairs of the form (ℕ×CPR). However, since the specific natural number in use is always to be thought of as an abstract quality of a particular discrete aggregate, a count of the aggregate is implied, that is, #: DAG → ℕ. It therefore appears to be more appropriate to use the cross-product (CPR×ℕ), where CPR indicates elements of DAG of a certain type (that is, use of a particular CPr), and ℕ indicates the number of objects of the type of object indexed by the CPr.

5.2.5 The composition of mappings and the use of the inclusion mapping

The previous sections have revealed a series of related mappings that constitute solutions to the subtasks of *Task* 5, Figure 5.1. Given a CPr as an input (initial bit of information), subjected to the computational demands of the subtasks, a discrete aggregate is to be generated as a result (terminal bit of information; output). The analysis of *Task* 5 suggests two paths to the required result, described here in the most general terms as follows:

i. First path: the learner uses the computation GRP$_1$:CPr → DAG followed by the computation BIJ$_{A,B}$:DAG → DAG.

   Referring to the chicken exercise again, as an example, we have:

   GRP$_1$:(*chicken*) → {∀creatures in *Task* 5 s.t. the creature is a *chicken*}, followed by
   BIJ$_{chicken,\bullet}$:{∀creatures in Task 5 s.t. the creature is a *chicken*} → {●,●,●,●}.

ii. Second path: the learner uses GRP$_1$:CPR → DAG, followed by #:DAG → ℕ, followed by GRP$_2$:CPR×ℕ → DAG.

   Again, referring to the chicken exercise:

   GRP$_1$:(*chicken*) → {∀creatures in *Task* 5 s.t. the creature is a *chicken*}, followed by
   #:{∀creatures in Task 5 s.t. the creature is a *chicken*} → 4, followed by
   GRP$_2$:(●,4) → {●,●,●,●}.

It should be apparent that the series of mappings are chained by using the codomain elements (outputs) of prior mappings as domain elements (inputs) of subsequent mappings, and that the process is halted once the required result (final output) is obtained.

In the case of the first path to the result we can easily construct an object-and-arrow diagram showing the chain of mappings, as in Figure 5.2. The dotted arrow, which is the composite produced by taking $\mathrm{BIJ}_{chicken,\bullet}$ following (or, after) GRP_1 (usually written symbolically as $\mathrm{BIJ}_{chicken,\bullet} \circ \mathrm{GRP}_1$).

$$\begin{tikzcd}
chicken \arrow[rr, "\mathrm{GRP}_1"] \arrow[dr, dashed, "\mathrm{BIJ}_{chicken,\bullet} \circ \mathrm{GRP}_1"'] & & \{\forall \text{creatures in Task 5 s.t. the creature is a } chicken\} \arrow[dl, "\mathrm{BIJ}_{chicken,\bullet}"] \\
& \{\bullet,\bullet,\bullet,\bullet\} &
\end{tikzcd}$$

Figure 5.2: Object-and-arrow diagram for the first path for the chicken exercise of *Task* 5.

For the second path, which includes counting, the construction of an object-and-arrow diagram is slightly more complicated because of the presence of $\mathrm{GRP}_2 : (\bullet, 4) \rightarrow \{\bullet,\bullet,\bullet,\bullet\}$. Since the latter mapping is binary, we need some mechanism for constructing a two-argument domain element for it. To that end, the *inclusion mapping*, $i_x : x \rightarrow x$, is introduced. The inclusion mapping takes as its domain and codomain any given object, x, available in the workspace and includes it where it is deemed to be required in a computation. See Figure 5.3.

$$\begin{tikzcd}
chicken \arrow[rr, "\mathrm{GRP}_1"] \arrow[dd, dashed, "\mathrm{GRP}_2 \circ i_x \circ \# \circ \mathrm{GRP}_1"'] & & \{\forall \text{creatures in Task 5 s.t. the creature is a } chicken\} \arrow[d, "\#"] \\
& \bullet \arrow[dr, "i_x"'] & 4 \arrow[d, "i_x"] \\
\{\bullet,\bullet,\bullet,\bullet\} & & (\bullet,4) \arrow[ll, "\mathrm{GRP}_2"]
\end{tikzcd}$$

Figure 5.3: Object-and-arrow diagram for the second path for the chicken subtask of *Task* 5.

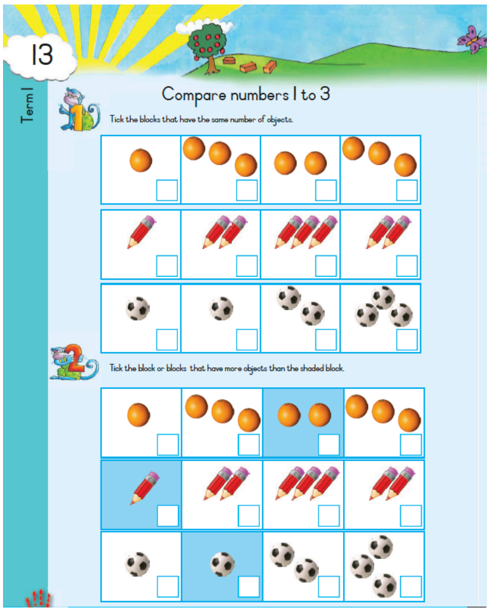
The use of the inclusion mapping may appear to be unsatisfactory because of its apparent arbitrariness. However, every task/subtask must supply details, implicitly or explicitly, of the computational objects to be used in its workspace. The images, symbols and words that may be reasonably taken to connote objects that are familiar to everyday life but used in mathematical tasks, are computational objects in the sense that the familiar arithmetic objects are. In the workspace of a task, all computational objects—directly displayed or implied—are available for use in the construction of a solution. The inclusion mapping enables the problem solver to select whichever of the computational objects in the workspace are perceived of as helpful in the construction of a solution.

5.2.6 Concluding comments of the use of CPrs

The discussion in Section 5.2 has demonstrated how CPrs are used to present, construct and compute aggregates as a global feature employed in the use of discrete aggregates as a ground for the growth of elementary arithmetic in the workbooks. To simplify the analysis going forward, references to the use of CPrs will be taken as given and generally excluded from future analytic descriptions.

5.3 The ordering of discrete aggregates and numerical order

Whilst this study does not consider the use of continuous aggregates indexed by the presence of terms like ‘length’ or ‘wide’, evident in comparison Tasks in the workbooks, where counting rather indexes counting-as-measuring, discrete aggregates appear. For example, consider Figure 5.4. showing *Subtasks* 1 and 2, *Task* 13 of Grade 1, Workbook 1.



13

Term 1

Compare numbers 1 to 3

1 Tick the blocks that have the same number of objects.

2 Tick the block or blocks that have more objects than the shaded block.

Figure 5.4: *Task* 13 titled ‘Compare numbers 1 to 3’. Grade 1, Workbook 1 (*Source*: DBE 2021a:28).

5.3.1 Discrete finite aggregates can be ordered in terms of size without counting

Task 13, Figure 5.4, is titled 'Compare numbers 1 to 3'. The intention is that learners compare discrete aggregates by counting objects in each aggregate, yet the option is also available to construct 1-1 correspondences (bijections) between pairs of aggregates, which entail the construction of 1-1 mappings (injections).[13] In this instance, finite sets can be ordered in terms of size without recourse to number or counting. This is described as a mathematical theorem by Davis (2015), for which the justification is inserted here:

Suppose that A and B are two disjoint elements of DAG (the class of finite sets), and that f is a function mapping A to B. Further, suppose that f is restricted to the construction of an injection, in so far as that is possible, for the purposes of effecting a comparison of A and B with respect to set size. The size of a set, $|S|$, is indicated using the notation $|S|$. Here, f is always generated by an attempt to construct injections between sets. Under such a condition, f is either (1) injective, (2) surjective, or (3) bijective. Each case is discussed in turn:

(1) $f{:}A \rightarrow B$, where f is injective (i.e., 1-to-1, or, into).

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. Since f is injective, for all pairs $a_i, a_j \in A$, $i \neq j$, $f(a_i) \neq f(a_j)$. That is, there is no $b_k \in B$ such that $b_k = f(a_i)$ AND $b_k = f(a_j)$. If it is the case that for all $b_k \in B$ there exists an $a_i \in A$ such that $f(a_i) = b_k$, then f is also bijective, and A and B are equinumerous. [1]

However, if there exist one or more $b_k \in B$ for which there does not exist an $a_i \in A$ such that $f(a_i) = b_k$, then A and B are not equinumerous. A partition of B, namely, $(B_1|B_2) \vdash B$, can be constructed such that B_1 contains all the b_k for which there exists an $a_i \in A$ such that $f(a_i) = b_k$ and B_2 contains all the b_k for which no such a_i exists. It follows that, with respect to B, B_2 is in excess of B_1. [2]

Now consider the set $C = A \cup B_2$. C is then an element of DAG. By transitivity, B and C are equinumerous since A and B_1 are equinumerous and $B = B_1 \cup B_2$. $(A|B_2) \vdash C$ is a partition of C. This means that, with respect to C, B_2 is in excess of A. We can thus conclude that $|B|$ is greater than $|A|$, by which is meant that there exist elements of B—namely, the set B_2— for which there are no associated elements in A when we construct an injective mapping from A to B. [3]

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[13] Given the computational resources supplied by the OTS as part of biological endowment, it is very probable that learners use subitising (the direct apprehension of number in the absence of #), since the aggregates are small—all in the range of core domain functioning (one to three objects). This does not mean that bijections and counting are not used, only that they are used implicitly, which is the case with subitising (Gallistel & Gelman 1991).

It follows from [1], [2] and [3] that, if f is injective, then $|A| \leq |B|$, and that if f is not also bijective, then $|A| < |B|$.

(2) $f{:}A \rightarrow B$, where f is surjective (i.e., onto).

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. Since f is surjective, for all $b_k \in B$ there exists an a_k such that $b_k = f(a_k)$. If it is the case that for all $b_k \in B$, then there exists only one $a_i \in A$ such that $f(a_i) = b_k$, then f is also bijective, and A and B are equinumerous. [4]

However, if for any pair b_i, $b_j \in B$, there exists an $a_k \in A$ such that $b_i = f(a_k)$ AND $b_j = f(a_k)$, $i \neq j$, then A and B are not equinumerous. A partition of A, $(A_1|A_2) \vdash A$, can be constructed such that, for some bijective function g,$g{:}A_1 \rightarrow B$. Now consider the set $C = B \cup A_2$ where C is disjoint with respect to A and B. C is then an element of DAG. By transitivity, A and C are equinumerous since B and A_1 are equinumerous and $A = A_1 \cup A_2$. Now, $(B|A_2) \vdash C$ is a partition of C. This means that, with respect to C, A_2 is in excess of B. We can thus conclude that $|A|$ is greater than $|B|$, by which is meant that there exist elements of A—viz., the set A_2—for which there are no associated elements in B when we construct an injective mapping from B to A. [5]

It follows from [4] and [5] that, if f is surjective, then $|A| \geq |B|$, and if f is not bijective, that $|A| > |B|$.

(3) $f{:}A \rightarrow B$, where f is bijective (i.e., into and onto).

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. Since f is bijective, for all pairs a_i, $a_j \in A$, $i \neq j$, $f(a_i) \neq f(a_j)$, and for all $b_k \in B$ there exists an a_k such that $b_k = f(a_k)$. Therefore, there exists an inverse function, $f^{-1}{:}B \rightarrow A$ which is necessarily bijective. Since f and $f^{-1}$are bijective, A and B are equinumerous. That is, $|A| = |B|$.

In summary, given two finite sets A and B, and f is a function from A to B subject to the listed retractions, then we can say:

(1) if $f{:}A \rightarrow B$ is injective but not bijective, then $|A| < |B|$

(2) if $f{:}A \rightarrow B$ is surjective but not bijective, then $|A| > |B|$

(3) if $f{:}A \rightarrow B$ is bijective, then $|A| = |B|$.

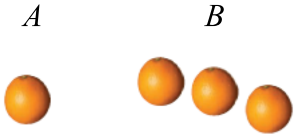
Taking the elements of any collection of finite sets pairwise, we can use (1), (2) and (3) to generate a monotonically increasing ordering of the sets with respect to size.

5.3.2 Structure-preservation for order

The discussion continues in relation to structure-preservation for order. When using bijections, the *Exercises* of *Subtask* 1 of *Task* 13 (Figure 5.4.) are solved by selecting only those aggregates for which a bijection can be constructed; whilst in *Subtask* 2, the learner is to recognise where an attempt to construct a bijection will fail. An alternative is to recognise aggregates of equal cardinality by referring to the results of counting.

For *Subtask* 2, using counting, the learner can use their knowledge of numerical order to solve the problems. Whether the learner uses comparison by constructing 1-1 mappings or numerical order, or both, *Task* 13 implicitly suggests the existence of a structure-preserving mapping between the ordering of pairs of discrete aggregates and numerical order, using counting as the mapping that mediates between order defined over finite discrete aggregates and order defined over the natural numbers.

Consider the problem: Suppose that we have two discrete aggregates, *A* and *B*, and a binary order relation, *more*, defined over discrete aggregates (as is implied in *Task* 13). Suppose further that when *A* and *B* are taken as arguments of the order relation, that *B* is the result. For example, let *A* and *B* be the sets,



and let the order relation be one that selects the larger of the two aggregates. Since it is impossible to construct a bijection between *A* and *B* because there are insufficient oranges in *A* to be associated in a one-to-one manner with all the oranges in *B*, we say that *B* contains *more* oranges than *A*. It is important to note that, by this method, we conclude that *B* contains *more* than *A* without recourse to counting. Now, suppose that we count the oranges in each of *A* and *B* to obtain 1 and 3, respectively. Since '3 – 1 = 2' and 2 > 0, we can conclude that 3 is *greater_than* 1 and, consequently, that there are *more* oranges in *B* than in *A*.[14] The implied mappings can be put together to produce the object-and-arrow diagram shown in Figure 5.5. Note that the inputs to the ordered relations should not be treated as ordered pairs. That is, it does not matter whether the inputs are the pairs (*A*,*B*) or (*B*,*A*) for *more*; (1,3) or (3,1) for *greater_than*.

---

[14] Note that if a and b are two real numbers, then $a > b \Leftrightarrow (a - b) \in \mathbb{R}^{+}$.

$$\begin{tikzcd} A,B \arrow[r, "\mathit{more}"] \arrow[d, "\#"'] & B \arrow[d, "\#"] \\ 1,3 \arrow[r, "\mathit{greater\_than}"'] & 3 \end{tikzcd}$$

Figure 5.5: A structure-preserving mapping, using counting, from the order relation *more* implicitly defined over discrete aggregates in *Task* 13, to the order relation *greater_than* defined over ℕ.

We can now check on the requirements for a good representation for the mapping shown in Figure 5.5. First, causality: counting (#) maps finite discrete aggregates to natural numbers in a consistent and stable manner and one can always find a natural number that is the count of such an aggregate. Second, structure preservation: Figure 5.5 clearly shows that the objects (aggregates and natural numbers) and operations (*more* and *greater_than*) are related by # in a manner that echoes the structure of (FINSET, *more*) in (ℕ, *greater_than*). Third, efficaciousness: the ordering of discrete aggregates grounds the ordering of the natural numbers when related by #.

To deal with the full suite of order relations that can potentially be used in tasks like *Task* 13, we might consider taking as input of an order relation an ordered pair, (*x*,*y*), and as output a truth value, namely, an element of the set {TRUE, FALSE}. For example, if the order relation is more and its input is the (*A*,*B*) where *A* and *B* are the sets of oranges referred to above, then *more*: (*A*,*B*) → FALSE and *more*:(*B*,*A*) → TRUE. With reference to the correlate order relation defined over ℕ, *greater_than*:(1,3) → FALSE and *greater_than*:(3,1) → TRUE. In this use of the order relations, we can refer to the general codomain of *more*, and to that of *greater_than*, as BOOL (which is a reference to the term Boolean, implying logical relations) since it is the set {TRUE, FALSE} in both cases. Therefore, *more*:DAG×DAG → BOOL and *greater_than*:ℕ×ℕ → BOOL. Similar arguments can easily be made for the other correlate pairs of order relations defined over DAG and ℕ used in the FP (e.g., *less* and *less_than*; *same_number_of_things* and *equal*). In each instance the order relation defined over DAG and its correlate defined over ℕ are mapped to the same truth value, thereby demonstrating consistency across the structures (DAG, $\ast$), and (ℕ, $\circ$), where $\ast$ and $\circ$ are correlate order relations.

Similar arguments apply for the other correlate pairs of order relations defined over DAG and ℕ used in the FP (for example, *same_number_of_objects* and *equal*). In each instance the order relation defined over DAG and its correlate defined over ℕ are mapped to the same truth value, demonstrating that structure is preserved by counting.

5.3.3 Concluding comments on order of discrete aggregates and numerical order

While the focus of this study is centred on the use of operations defined over discrete aggregates as a ground for AS, a discussion of the use of order relations defined over discrete aggregates becomes necessary because it serves as a ground for establishing numerical order. In turn, AS requires recognition of the order of the natural numbers. Natural number order is implicitly introduced during rote learning of the number sequence (order of number words). However, it is the innate human ability to distinguish between the relative sizes of collections of discrete things that serve as a semantic bases for understanding both counting, ordering of ℕ, and AS.

In Chapter Seven, when discussing representations for order used in various *Tasks*, the symbols $\mathrm{ORD}_{\mathrm{set}}$ and $\mathrm{ORD}_{\mathrm{num}}$ are used to indicate the use of order relations over discrete aggregates ($\mathrm{ORD}_{\mathrm{set}}$) and over natural numbers ($\mathrm{ORD}_{\mathrm{num}}$), respectively. In other words, for the purpose of recording the use of representations concerned with order, $\mathrm{ORD}_{\mathrm{set}}$ and $\mathrm{ORD}_{\mathrm{num}}$ do not discriminate between the different possible order relations but merely record the explicit use of order relations in *Tasks*.

5.4 Union, disjoint union and addition defined over $(\mathbb{N}, +)$

In Chapter Three it was demonstrated that there exists a structure-preserving mapping between disjoint union defined over DAG and addition defined over ℕ, as **(ℕ,+).** What remains is to point out that not all unions of aggregates are appropriate as a ground for **(ℕ,+).** Let us consider the cases.

5.4.1 Why counting cannot always map union to (ℕ,+)

Consider, for example, two discrete aggregates, *A* and *B*. Then the following cases arise:

(i) $A \cap B = B$ (where $A,B \neq \emptyset$ and all the elements of B are also elements of A)
(ii) $A \cap B = C$ (where $A,B \neq \emptyset$ and some elements of B are also elements of A)
(iii) $A \cap B = \emptyset$ (where $A,B \neq \emptyset$ and A and B have no elements in common)
(iv) $A \cap B = \emptyset$ (where $A \neq \emptyset$ and $B = \emptyset$).

If (i), then $\#A + \#B > \#(A \cup B)$ and we do not have a structure-preserving mapping.
If (ii), then $\#A + \#B > \#(A \cup B)$ and, once again, we do not have a structure-preserving mapping.
If (iii), then $\#A + \#B = \#(A \cup B)$. In this case, we do have a structure-preserving mapping, $\#:(\mathrm{FINSET},\cup) \rightarrow (\mathbb{N},+)$, as was demonstrated in Chapter Three.

If (iv), then $\#A + \#B = \#A = \#B$, and we do have a structure-preserving mapping. Mappings of this type are rarely used in the workbooks.

5.4.2 Concluding comments on union, disjoint union and $(\mathbb{N},+)$

The representations used in the workbooks implicitly use disjoint union because the aggregates are presented pictorially, so the potential problems indicated in (i) and (ii) of Section 5.4.1 are generally averted.

5.5 Aggregates and subtraction

Recall the discussion of *merge* in Chapter Three. There it was demonstrated that *merge* can be thought of as a core domain computation that grounds noncore domain operations like *union*, *disjoint union* and *addition*. An intuitive conception of subtraction is as the undoing of *merge*, implying the recognition of entities that have been merged and can, so to speak, be *unmerged*, along with an operation that removes one of the *unmerged* entities. The latter operation might be referred to as *purge*. The cognitive operation entails the recognition of at least three entities, viz., a primary entity and at least two entities that are conceived as contained in the primary entity and which can be extracted from it, like in the so-called Part-Part-Whole relations that are routinely used in FP mathematics teaching.

5.5.1 Relative complement ($\backslash$) and subtraction ($-$)

The discussion refers to Figures 5.6 through 5.8. Again, consider problem-solving *Exercises* indicated in *Subtask* 4, *Task* 15, Grade 1, Workbook 1, titled ‘Add and subtract up to 4’ (Figure 5.6). One, two and three discs, respectively, of a series of three discrete aggregates are struck through marking them for removal.

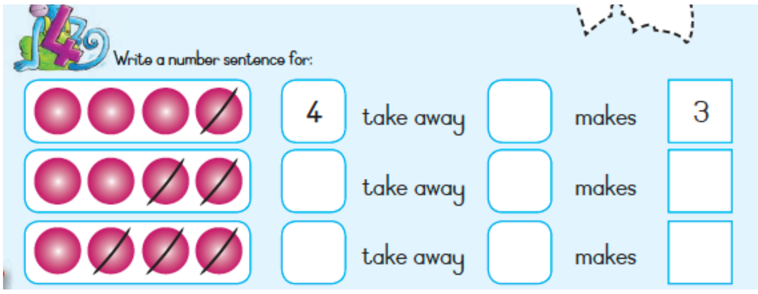


Figure 5.6: Subtask 4 of *Task* 15 of the Grade 1, Workbook 1 (*Source:* DBE 2021a:32).

The struck through discs in each set constitute aggregates that are contained entirely within each whole aggregate, as rows of four discs. For each *Exercise*, the learner must count the elements of three aggregates—first, all the discs in a row; second, all the marked/struck-through discs in a row indicated by '/'; and third, all the unmarked discs in a row. Next, they must indicate the results of counting in the appropriate blocks to the right of the discs, such that arithmetically '4 take away 1 makes 3', and so on. Thus, mappings exist from discrete aggregates to ℕ via counting such that the relations between the numbers correspond to subtraction over the natural numbers, $(\mathbb{N}, -)$. The exercises implicitly employ the set theoretic operation *relative complement* as defined over DAG, indicated by the usual symbol \ (not to be confused with the forward slash '/' used in the *Exercises*). The terms *take away* and *make* are correlates for *subtraction* and *equal to*, respectively. Now consider the following cases with respect to structure-preservation:

> For two aggregates A and B, the relative complement of B with respect to A is the subset of A containing all the elements of A that are not in B. If this subset is referenced as C, we can write $A \setminus B = C$ to express the fact that C is the relative of B with respect to A. Once again, the following cases arise for two discrete aggregates, A and B:
>
> (i) $A \cap B = B$ (where $A,B \neq \emptyset$ and *all* the elements of B are also elements of A);
> (ii) $A \cap B = D$ (where $A,B \neq \emptyset$ and *some* elements of B are also elements of A);
> (iii) $A \cap B = \emptyset$ (where $A,B \neq \emptyset$ and *none* of the elements of B are also elements of A);
> (iv) $A \cap B = \emptyset$ (where $A \neq \emptyset$ and $B = \emptyset$).

If (i), then there exists an aggregate C such that $A \cap C = C$, $B \cup C = A$ and $\#A - \#B = \#C$. In this case, we have a structure-preserving mapping of the type $\#:(\text{FINSET},\setminus) \rightarrow (\mathbb{N},-)$.
If (ii), then there exists an aggregate C such that $A \cap C = C$, but $B \cup C \neq A$, and $\#A - \#B \neq \#C$. In this case, we *do not* have a structure-preserving mapping.
If (iii), then the relative complement of B with respect to A is A. In this case, we *do not* have a structure-preserving mapping.
If (iv), then the relative complement of $\emptyset$ with respect to A is A. In this case, we *do* have a structure-preserving mapping. However, such representations are not explicitly used in the DBE workbooks.

Juxtaposed to Figure 5.6, consider Figure 5.7 illustrating a structure-preserving mapping from relative complement defined over finite discrete aggregates $(\text{DAG},\setminus)$ to subtraction defined over the natural numbers $(\mathbb{N},-)$, via #. The binary equation translates '4 – 1 = 3'. Note that the second argument of \ (i.e., the aggregate consisting of a single disc) *must be a subset of the first argument* for structure-preservation to be ensured (corresponding to case (i)).

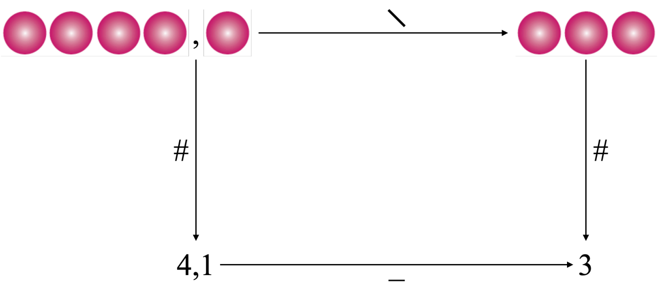


Figure 5.7: An example of #:(DAG,\) → (ℕ,−), translating '4 – 1 = 3', where the arguments of \ are restricted to ensure that the second argument is a subset of the first.

We now consider the conditions for a representation. First, causality: counting maps finite discrete aggregates to natural numbers in a consistent and stable manner and one can always find a ℕ that is the count of such an aggregate. Second, structure preservation: Figure 5.7 clearly shows that the objects (aggregates and natural numbers) and operations (\ and −) are related by # in a manner that echoes the structure of (DAG,\) in (ℕ,−). Third, efficaciousness: the relative complement of discrete aggregates grounds the difference of natural numbers when related by #.

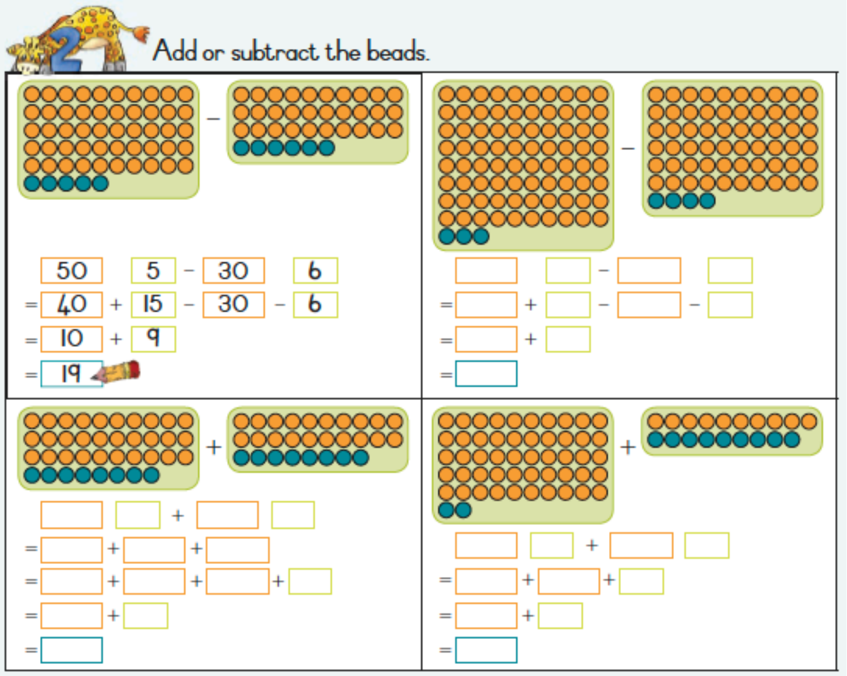
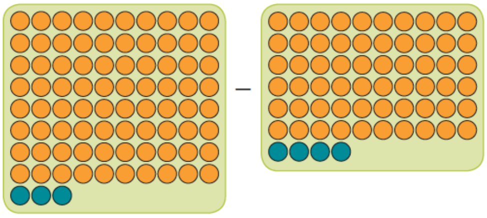


Figure 5.8: *Subtask* 2 of *Task* 101 of the Grade 2, Workbook 2 (*Source*: DBE 2021d:80).

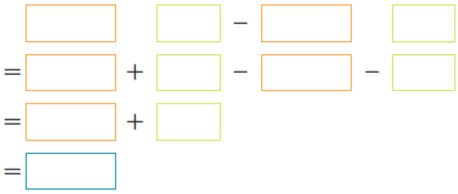
The analysis now considers the completed exercise in *Subtask* 2 of *Task* 101, Grade 2, Workbook 2, illustrated in Figure 5.8. The title, 'Addition and Subtraction', alerts us to use of the combination of these operations. Secondly, from the worked *Exercise* included in *Subtask* 2 (serving as a guide for learners), it can be deduced learners are to use these operations to fill blocks to produce a sequence of expressions terminating in cardinality, 19.

Numerous problematic features appear in the attempted representations, but only one is focussed on. Consider the image,



,

which is intended to depict the aggregate correlate of the computation ‘83 – 64’. By referring to the image, mappings of the type #:DAG → ℕ, are to be employed by learners to generate the contents of the first row of spaces in the solution template,



,

and then to complete the computation to the model of the example. Read in the context of other exercises of *Subtask* 2 featuring addition problems, the image referring to the aggregates in the subtraction exercise is ambiguous. We know that the second aggregate (cardinality of orange and blue balls as 10s or 1s) is to be thought of as a subset of the first (orange and blue balls as 10s and 1s) for the image to function as part of a reasonable representation that maps relative complement to subtraction. However, the remaining *Exercises*, which refer to addition, require the learner to treat the aggregates in the associated images as disjoint. If the aggregates depicted in the image associated with the subtraction exercise are treated as disjoint, then we have an instance of case (iii; under section 5.5.1) and, consequently, there does not exist a structure-preserving mapping to support a transition from relative complement over aggregates to subtraction over natural numbers. In general, images of the type used in the subtraction exercise discussed here make for problematic attempts at representing subtraction.

When compared with the addition over ℕ, subtraction has a relatively impoverished structure. To ensure subtraction can function as an operation over the ℕ, commutativity and associativity must be relinquished, and special conditions must be put in place to ensure closure and use of an identity element. Table 5.2 details the features of the structures (DAG,\) and (ℕ,−), which the reader should compare with the features of (DAG,⊔) and (ℕ,+) illustrated in Table 3.1 of Chapter Three.

Table 5.2: A comparison of the properties of the structures (DAG,\) and (ℕ,−).

	Properties of (DAG,\)	Properties of (ℕ,−)
Forced closure:	Given any two finite sets, A and B, where $A \cap B = B$, there exists a finite set C such that $A \setminus B = C$. Closure is forced by excluding of pairs (A,B) where B is not a subset of A.	Given any two natural numbers, m and n, where $m \geq n$, there exists a natural number p such that $m - n = p$. Closure is forced by excluding all pairs (m,n) where $n \geq m$.
Right identity element:	There exists a set ∅, called the *empty set*, which is such that, for any finite set A, $A \setminus \emptyset = A$. We have a restriction on the identity element because $A \setminus \emptyset \neq \emptyset \setminus A$.	Taking 0 as an element of natural numbers, for any natural number n, it is the case that $n - 0 = n$. We have a restriction on the identity element because $n - 0 \neq 0 - n$.
Inverse elements :	For all finite sets A, it is the case that $A \setminus A = \emptyset$, meaning that every discrete aggregate is its own inverse with respect to relative complement.	For any natural number n, it is the case that $n - n = 0$, which means that every natural number is its own inverse with respect to subtraction.

5.5.2 Concluding remarks on relative complement defined over DAG in relation to (ℕ,−)

We have seen that attempts at *representations* that map relative complement over DAG to subtraction over ℕ via counting require more restrictions on the relations between the arguments of operations than is the case for addition over ℕ. The common, automatic association of discrete aggregates with natural numbers appears to encourage a conflation of operations over aggregates and operations over natural numbers, one effect of which may well be the kinds of ambiguities in representations that we observed in the exercise in Figure 5.8.

5.6 Concluding remarks: Part One

The significance of discrete aggregates as a ground for AS was established and counting was demonstrated as a central resource in the construction of structure-preserving mappings from operations/mappings over discrete aggregates to operations over natural numbers. Table 5.3 shows a summary of the mappings encountered across the *exercises* of *Tasks* analysed. Chapter Six continues the analysis presenting more complex numerical computations.

Table 5.3: A summary of the mappings/operations discussed in Chapter 5.

MAPPING	DOMAIN	CODOMAIN	DESCRIPTION
GRP (Grouping)	CPR; or CPR×$\mathbb{N}$	DAG	Forming a discrete aggregate (group of discrete objects), A, by means of a comprehension principle; GRP:CPR $\rightarrow A$; forming a discrete aggregate, A, of a specified cardinality by means of a discrete aggregate. GRP:CPR×$\mathbb{N} \rightarrow A$.
BIJ; $\text{BIJ}_{A,B}$ (Bijection/1-1 correspondence)	DAG	DAG	A 1-1 correspondence (bijection) between a pair of aggregates, A and B. BIJ:$A \rightarrow B$; Sometimes the discrete aggregates, A and B, are explicitly indicated. $\text{BIJ}_{A,B}$:$A \rightarrow B$;
# (Count)	DAG	$\mathbb{N}$	Cardinality of a discrete aggregate, A. #: $A \rightarrow \mathbb{N}$.
i_x (Inclusion function)	x	x	Inclusion of some object, x, in a computation—usually as an argument of a mapping/operation. i_x: $x \rightarrow x$.
MORE	DAG×DAG	DAG	An order relation on discrete aggregates, A and B. *more*:$(A,B) \rightarrow A$ if $\#A > \#B$; *more*:$(A,B) \rightarrow A$ if there exists a non-bijective surjection, f, such that f:$A \rightarrow B$; *more*:$(A,B) \rightarrow B$ if $\#A < \#B$; *more*:$(A,B) \rightarrow B$ if there exists a non-bijective surjection, g, such that g:$B \rightarrow A$.
MORE	DAG×DAG	BOOL	An order relation on discrete aggregates, A and B, treated as a logical relation. *more*:$(A,B) \rightarrow$ TRUE if $\#A > \#B$; *more*:$(A,B) \rightarrow$ TRUE if there exists a non-bijective surjection, f, such that f:$A \rightarrow B$; *more*:$(A,B) \rightarrow$ FALSE if $\#A < \#B$;

Table 5.3: A summary of the mappings/operations discussed in Chapter 5.

MAPPING	DOMAIN	CODOMAIN	DESCRIPTION
			$\mathit{more}:(A,B) \rightarrow$ FALSE if there exists a non-bijective surjection, g, such that $g:B \rightarrow A$.
LESS	DAG×DAG	DAG	An order relation on discrete aggregates, A and B. $\mathit{less}:(A,B) \rightarrow A$ if $\#A < \#B$; $\mathit{less}:(A,B) \rightarrow A$ if there exists a non-bijective injection, f, such that $f:A \rightarrow B$; $\mathit{less}:(A,B) \rightarrow B$ if $\#A > \#B$; $\mathit{less}:(A,B) \rightarrow B$ if there exists a non-bijective injection, g, such that $g:B \rightarrow A$.
LESS	DAG×DAG	BOOL	An order relation on discrete aggregates, A and B, treated as a logical relation. $\mathit{less}:(A,B) \rightarrow$ TRUE if $\#A < \#B$; $\mathit{less}:(A,B) \rightarrow$ TRUE if there exists a non-bijective injection, f, such that $f:A \rightarrow B$; $\mathit{less}:(A,B) \rightarrow$ FALSE if $\#A > \#B$; $\mathit{less}:(A,B) \rightarrow$ TRUE if there exists a non-bijective injection, g, such that $g:B \rightarrow A$.
SAME_NUMBER_OF_THINGS	DAG×DAG	DAG	An order relation on discrete aggregates, A and B. $\mathit{same\_number\_of\_things}:(A,B) \rightarrow A$ or B if $\#A = \#B$; $\mathit{same\_number\_of\_things}:(A,B) \rightarrow A$ or B if there exists a bijection, f, such that $f:A \rightarrow B$; $\mathit{same\_number\_of\_things}:(A,B)$ is undefined if $\#A \neq \#B$; $\mathit{same\_number\_of\_things}:(A,B)$ is undefined if there does not exist a bijection, f, such that $f:A \rightarrow B$.
SAME_NUMBER_OF_THINGS	DAG×DAG	BOOL	An order relation on discrete aggregates, A and B, treated as a logical relation. $\mathit{same\_number\_of\_things}:(A,B) \rightarrow$ TRUE if $\#A = \#B$; $\mathit{same\_number\_of\_things}:(A,B) \rightarrow$ TRUE if there exists a bijection, f, such that $f:A \rightarrow B$; $\mathit{same\_number\_of\_things}:(A,B) \rightarrow$ FALSE if $\#A \neq \#B$; $\mathit{same\_number\_of\_things}:(A,B) \rightarrow$ FALSE if there does not exist a bijection, f, such that $f:A \rightarrow B$.

Table 5.3: A summary of the mappings/operations discussed in Chapter 5.

MAPPING	DOMAIN	CODOMAIN	DESCRIPTION
GREATER_THAN	$\mathbb{N}\times\mathbb{N}$	$\mathbb{N}$	An order relation on natural numbers, m and n. $greater\_than:(m,n) \rightarrow m$ if $m - n > 0$; $greater\_than:(m,n) \rightarrow n$ if $n - m > 0$;
GREATER_THAN	$\mathbb{N}\times\mathbb{N}$	BOOL	An order relation on natural numbers, m and n, treated as a logical relation. $greater\_than:(m,n) \rightarrow$ TRUE if $m - n > 0$; $greater\_than:(m,n) \rightarrow$ FALSE if $n - m > 0$.
LESS_THAN	$\mathbb{N}\times\mathbb{N}$	$\mathbb{N}$	An order relation on natural numbers, m and n. $less\_than:(m,n) \rightarrow m$ if $n - m > 0$; $less\_than:(m,n) \rightarrow n$ if $m - n > 0$.
LESS_THAN	$\mathbb{N}\times\mathbb{N}$	BOOL	An order relation on natural numbers, m and n, treated as a logical relation. $less\_than:(m,n) \rightarrow$ TRUE if $n - m > 0$; $less\_than:(m,n) \rightarrow$ FALSE if $m - n > 0$.
EQUAL	$\mathbb{N}\times\mathbb{N}$	$\mathbb{N}$	An order relation on natural numbers, m and n. $equal:(m,n) \rightarrow m$ or n if $n = m$; $equal:(m,n)$ is undefined if $m \neq n$.
EQUAL	$\mathbb{N}\times\mathbb{N}$	BOOL	An order relation on natural numbers, m and n, treated as a logical relation. $equal:(m,n) \rightarrow$ TRUE or n if $m = n$; $equal:(m,n) \rightarrow$ FALSE if $m \neq n$.
$\cup$ (Union)	DAG×DAG	DAG	The merging of two (discrete) aggregates, A and B, in a manner that excludes duplicate elements. $\cup:(A,B) \rightarrow C$; $C \in$ DAG.
$\sqcup$ (Disjoint union)	DAG×DAG	DAG	The merging of two (discrete) aggregates, A and B, in a manner that includes duplicate elements, if they exist, with markers distinguishing the duplicated elements. The problem of duplicate elements does not arise when the arguments, A and B, are disjoint. $\sqcup:(A,B) \rightarrow C$; $C \in$ DAG.
+ (Addition)	$\mathbb{N}\times\mathbb{N}$	$\mathbb{N}$	The addition of natural numbers, m and n. $+:(m,n) \rightarrow p$; $p \in \mathbb{N}$.
\	DAG×DAG	DAG	What remains of a discrete aggregate, A, after all elements common to A and another

Table 5.3: A summary of the mappings/operations discussed in Chapter 5.

MAPPING	DOMAIN	CODOMAIN	DESCRIPTION
(Relative complement)			aggregate, B, are removed from A. If A and B are disjoint or partially disjoint, then relative complement is not analogous to the subtraction of natural numbers. That is, B must be a subset of A. $\backslash:(A,B) \rightarrow C$; $C \in$ DAG.
$-$ (Subtraction)	$\mathbb{N}\times\mathbb{N}$	$\mathbb{N}$	The subtraction of natural numbers, m and n. $-:(m,n) \rightarrow p$; $p \in \mathbb{N}$; subtraction is undefined for all ordered $(m,n) \in \mathbb{N}\times\mathbb{N}$ when $m < n$.

CHAPTER 6

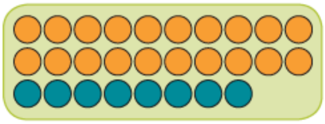
Analysis: Part Two

6.1 Introduction

This chapter continues with Part Two of the analysis.

6.2 The use of aggregate partitions and number partitions

The workbooks are saturated with aggregate and number partitions distinguishing blocks of 100s, 10s, 1s and even 500s. Those partitions depicted in *Subtask* 2 of *Task* 101 (Section 5.5, Chapter Five) show set partitions, apparently conditioned by the idea of distinguishing 'tens' and 'units.' For example, the image



shows two rows of ten orange discs and one row of eight blue discs, implicitly suggesting a set partition of the form shown in Figure 6.1.

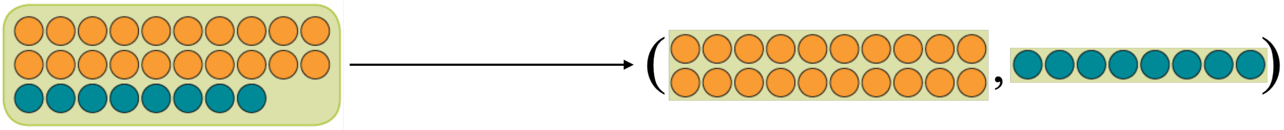


Figure 6.1: A mapping showing a partition of an aggregate of 28 objects.

6.2.1 Relating aggregate and number partitions in a structure-preserving manner

Let the first aggregate be A and the aggregates that make up the partition of A as A_1 and A_2 in Figure 6.1, then we can use the expression $P_{\text{set}}: A \rightarrow (A_1, A_2)$ to describe the mapping that produces the set partition. P_{set} is defined over DAG. Note that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$; that is, A_1 and A_2 are disjoint, and A is made up of A_1 and A_2.

If we use $\#: \text{DAG} \rightarrow \mathbb{N}$ to associate natural numbers with A, A_1 and A_2, we can arrive at the number partition $P_{\text{num}}: 28 \rightarrow (20,8)$. P_{num} is defined over $\mathbb{N}$. Figure 6.2 shows the representation that relates P_{set} and P_{num} via counting.

$$\begin{tikzcd}
A \arrow[r, "P_{\text{set}}"] \arrow[d, "\#"'] & (A_1, A_2) \arrow[d, "\#"] \\
28 \arrow[r, "P_{\text{num}}"'] & (20,8)
\end{tikzcd}$$

Figure 6.2: A representation of the relation between P_{set} and P_{num} shown in Figure 6.1.

The general case is one in which a partition of some discrete aggregate, A, is related to the partition of some natural number, k, by observing the following criteria:

(i) No subset that is a part of a partition is the empty set, $\emptyset$. The union of all the subsets of A results in A and the subsets are pairwise disjoint. Stated more formally, $(A_1, A_2, \ldots, A_n)$ is a partition of A if $A_1 \cup A_2 \cup \ldots \cup A_n = A$, and for all possible pairs of subsets of A, (A_i, A_j), we have $A_i \cap A_j = \emptyset$. We indicate that $(A_1, A_2, \ldots, A_n)$ is a partition of A by writing $(A_1, A_2, \ldots, A_n) \vdash A$. The members of the list constituting a set partition are referred to as the *blocks* of the partition.

(ii) Each of A and $A_1, A_2, \ldots, A_n$ are taken as arguments of $\#:\text{DAG} \rightarrow \mathbb{N}$ to produce natural numbers associated with each aggregate. If we have $\#:A \rightarrow k$, and $\#:A_i \rightarrow k_i$, where $i = 1, 2, \ldots, n$, then we arrive at a partition of k, with the form $(k_1, k_2, \ldots, k_n) \vdash k$. Of course, we must have $\sum_{i=1}^{n} k_i = k$. For example, $(3, 2, 5, 2) \vdash 12$ since $3 + 2 + 5 + 2 = 12$. As was the case with set partitions, the members of the list constituting a number partition are referred to as the *blocks* of the partition. Also, since no $A_i = \emptyset$, zero is not a block of any number partition.

From (i) and (ii) we derive the general mapping, $\#:(\text{DAG},P_{\text{set}}) \rightarrow (\mathbb{N},P_{\text{num}})$. Refer to Figure 6.3.

$$\begin{tikzcd}
A \arrow[r, "P_{\text{set}}"] \arrow[d, "\#"'] & (A_1, A_2, \ldots, A_n) \arrow[d, "\#"] \\
k \arrow[r, "P_{\text{num}}"'] & (k_1, k_2, \ldots, k_n)
\end{tikzcd}$$

Figure 6.3: The general mapping, $\#:(\text{DAG},P_{\text{set}}) \rightarrow (\mathbb{N},P_{\text{num}})$.

Table 6.1 details the properties of (FINSET, P_{set}) and (ℕ, P_{num}), showing that they have the same structural features. Table 6.1 should be read in conjunction with the properties of (FINSET,⊔) and (ℕ,+) detailed in Table 3.1 of Chapter 3.

Table 6.1: A comparison of the properties of the structures (FINSET, P_{set}) and (ℕ, P_{num}).

	Properties of (FINSET, P_{set})	Properties of (ℕ, P_{num})
Partitions always exist:	Given any discrete finite set, *A*, by definition of a subset of a set, we can always select elements of *A* to form a new set consisting of only those elements selected. It follows that we can construct a series of subsets of *A* such that the subsets are pairwise disjoint. The simplest partitions of *A* are the lists (*A*) and (A_1, A_2, …, A_n) where each A_i consists of a unique element of *A*.	Given any natural number, *n*, we can always find one or more natural numbers such that the sum of those numbers is equal to *n*. The simplest partitions of *n* are the lists (*n*) and (1, 1, 1, …, 1) consisting of *n* 1s. By definition, any sum of natural numbers equal to *n* constitutes a partition of *n*.
The ordering of the blocks of a partition does not matter:	Since disjoint union over FINSET is commutative, any ordering of a given the list of subsets of *A* is taken to be the same partition of *A*.	Since addition over ℕ is commutative, any ordering of natural numbers that sum to *n* form the same partition of *n*.

We now consider the conditions for a representation. Causality: counting maps finite discrete aggregates to natural numbers in a consistent and stable manner and one can always find a natural number that is the count of such an aggregate. Structure-preservation: Figure 6.3 clearly shows that the objects (aggregates and natural numbers) and mappings (P_{set} and P_{num}) are related by # in a manner that echoes the structure of (DAG, P_{set}) in (ℕ, P_{num}). Efficaciousness: the partitioning of discrete aggregates grounds the partitioning of natural numbers when related by #.

6.2.2 Concluding remarks on the use of aggregate and number partitions

The comparison of the properties of the structures (DAG, P_{set}) and (ℕ, P_{num}), read in conjunction with the properties of (DAG,⊔) and (ℕ,+), demonstrate that $\#:(\text{DAG}, P_{\text{set}}) \rightarrow (\mathbb{N}, P_{\text{num}})$ is structure-preserving. Both P_{set} and P_{num} are used subject to the demands of the particular computational contexts in which they are required. It is, therefore, good practice to specify the contextual demands on P_{set} and P_{num} whenever required.

6.3 The substitution of binary operations/mappings with unary operations/mappings

The *Tasks* used for the purpose of elaborating subtraction exploit the core domain properties of *merge* in respect to the related notions of *unmerge* and *purge*. The learner is confronted with an aggregate from which some of the elements need to be taken away. Operationally, the process often entails the act of removing marked elements from an original aggregate, and thus has formal features of an operation that behaves like a unary operation: a single argument is subjected to a process and produces a single value as a result.

6.3.1 The use of aggregates in binary to unary substitutions: curried subtraction

Consider a subtraction task, *Subtask* 1 of *Task* 23b of Grade 2, Workbook 1, shown in Figure 6.4. A series of five distinct aggregates, consisting of sweets, are shown. Some sweets in each aggregate are marked with backslashes. The exercises of Subtask 1 entail the use of #:(DAG,\) → (ℕ,−), subject to the restrictions discussed in Section 5.5.1.

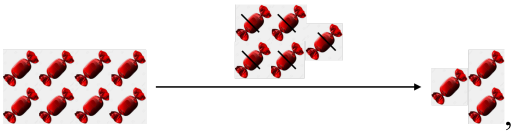
Subtraction

Look at the picture and write minus sums.

red sweets = 8 − 5 = 3
green sweets = ☐ − ☐ = ☐
yellow sweets = ☐ − ☐ = ☐
orange sweets = ☐ − ☐ = ☐
pink sweets = ☐ − ☐ = ☐

Figure 6.4: *Subtask* 1 of *Task* 23b, ‘Subtraction’, Grade 2, Workbook 1 (*Source*: DBE 2021c:32).

Recall that for such problems the learner must recognise three aggregates: the entire collection of sweets, the collection of sweets to be removed, and the collection of sweets that remain. The computations on aggregates in the example provided—the ‘red sweets’ exercise—are to be thought of in manner that corresponds to the mapping,



where

indicates the operation, \, to be used. Rewriting the aforementioned 'operation' as  is neater. Such an 'operation' differs across the exercises in the sense that different amounts of elements are conjoined with the symbol indicating the removal of elements. The situation is, therefore, a bit different from subtraction conceived of as a binary operation, as described in Section 5.5.1. Figure 6.5 shows an example of the effect of the use of 'operations' like on arithmetic operations if they are expected to map in a structure-preserving manner to basic arithmetic.

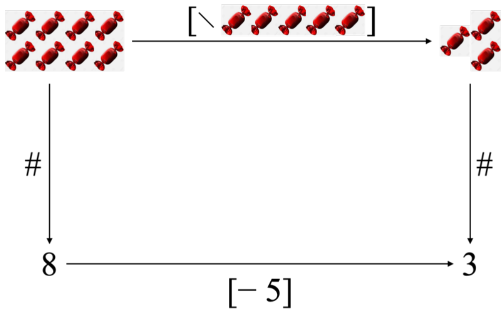


Figure 6.5: A representation of the relation between currying-like transformations on relative complement and subtraction.

The effect is something like the process of *currying* or *schönfinkelisation*[15] —familiar to readers who have knowledge of formal programming languages like Haskell—where mappings of arity two or more are routinely replaced by chains of unary mappings.

With respect to relative complement mapped to subtraction, currying-like transformations of the kind shown in Figure 6.5 replace the mapping $\backslash:(A,B)\rightarrow C$ with the mapping $[\backslash B]:(A) \rightarrow C$ for discrete aggregates, where $[\backslash B]$ is the 'operation', and the mapping $-:(m,n)\rightarrow p$ for natural numbers with the mapping $[-n]:(m)\rightarrow p$, where $[-n]$ is the 'operation'. We can describe the transformed computational

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[15] The terms *currying* and *schönfinkelisation* are named after Haskell Curry (1900 − 1982) and Moses Schönfinkel (1888 − 1942), respectively. Curry popularised the process of replacing functions of more than one argument by chains of functions of only one argument, although the process is found in even earlier work by Frege (1848 − 1925) published in 1893.

structures in general terms as $(\mathrm{DAG},[\backslash S])$ and $(\mathbb{N},[-n])$, respectively, where S is an element of the class DAG, and n is an element of $\mathbb{N}$ such that $\#{:}S \rightarrow n$.

Table 6.2, which should be read with reference to Table 5.2, details the properties of the structures $(\mathrm{DAG},[\backslash S])$ and $(\mathbb{N},[-n])$, showing structure-preservation. Prefix notation is used in the table rather than the more familiar infix notation for operations to deal with notational problems that arise when the second argument of an operation/mapping is treated as part of the operation/mapping. Unfortunately, the use of prefix notation often results in statements that are opaquer than those expressed using infix notation.[16]

The conditions of causality, structure-preservation and efficaciousness for a representation hold for $\#(\mathrm{DAG}, [\backslash S]) \rightarrow (\mathbb{N}, [-n])$, just as in the case $\#(\mathrm{DAG},\backslash) \rightarrow (\mathbb{N},-)$.

Table 6.2: A comparison of the properties of the structures $(\mathrm{DAG}, [\backslash S])$ and $(\mathbb{N}, [-n])$.

	Properties of $(\mathrm{DAG}, [\backslash S])$	Properties of $(\mathbb{N}, [-n])$
Forced closure:	Given any finite sets, A and B, such that $A \cap B = B$, there exists a finite set C such that $[\backslash B](A) = C$.	Given any natural number, m, such that $m \geq n$, where $n \in \mathbb{N}$, there exists a natural number p such that $[-n](m) = p$.
Right identity element:	There exists a set $\emptyset$, called the *empty set*, which is such that, for any finite set A, $[\backslash\emptyset](A) = A$, which corresponds to the right identity element for $(\mathrm{DAG},\backslash)$.	For any natural number n, it is the case that $[-0]n = n$, which corresponds to the right identity element for $(\mathbb{N},-)$.
"Inverses":	For any finite set A, $[\backslash A](A) = \emptyset$, which corresponds to "inverses" for $(\mathrm{DAG},\backslash)$.	For any natural number m, it is the case that $[-m](m) = 0$, which corresponds to "inverses" for $(\mathbb{N},-)$.

6.3.2 Binary to unary operations/mappings: currying

Since this study is concerned with the use of aggregates in the teaching and learning of AS, it is necessary to detail the features of $\#(\mathrm{DAG},\sqcup) \rightarrow (\mathbb{N},+)$ when subjected to a currying-like transformation.

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[16] '2 + 3' is an example of infix notation with '+' positioned between its arguments, whereas '+(2,3)' uses prefix notation, with '+' positioned before its arguments.

It is easy to construct *Exercises* that suggest addition subjected to a currying-like transformation. For example, a typical problem like: ‘Mary has a basket containing 5 apples. Thandi places 3 more apples in Mary’s basket. How many apples does Mary have in her basket now?’ Here the description of Thandi’s actions suggests an ‘operation’ [+ 3], which takes 5 as its argument and produces 8 as its value. Using prefix notation, we can write [+ 3](5) = 8 to describe the solution. A FP teacher might discuss the computation ‘5 + 3 = 8’ with learners by pointing at the elements of the expression and saying something like, ‘there’s a five and we have to add three to it, and so we get eight’.

Table 6.3 shows a comparison of the currying-like transformations of the structures related to addition, which should be read in conjunction with Table 3.1, Chapter 3.

Table 6.3: A comparison of the properties of the structures (DAG, $[\sqcup S]$) and ($\mathbb{N}$, $[+\, n]$).

	Properties of (DAG, $[\sqcup S]$)	Properties of ($\mathbb{N}$, $[+\, n]$)
“Closure”:	Given any finite set, *A* and *B*, there exists a finite set *C* such that $[\sqcup B](A) = C.$	Given any two natural numbers, *m* and *n*, there exists a natural number *p* such that $[+\, n](m) = p.$
“Commutativity”:	Given any two finite sets, *A* and *B*, $[\sqcup B](A) = [\sqcup B](A).$	Given any two natural numbers, *m* and *n*, $[+\, n](m) = [+\, m](n).$
“Associativity”:	Given any three finite sets, *A*, *B* and *C*, we have $[\sqcup[[\sqcup C](B)]](A) = [\sqcup C]([\sqcup B](A)).$	Given any three natural numbers, *m*, *n* and *p*, $[+\, [[+\, p](n)]](m) = [+\, p]([+\, n](m)).$
“Identity element”:	There exists a set ∅, called the *empty set*, which is such that, for any finite set *A*, $[\sqcup \emptyset](A) = A = [\sqcup A](\emptyset).$	If we include 0 in the natural numbers, then for any natural number *n*, it is the case that $[+\, 0](n) = n = [+\, n](0).$

The use of infix notation probably contributes to fixing the perception of the operation indicated in the expression ‘5 + 3 = 8’ as [+ 3] since it emphasises the linear order of the sequence of symbols, echoing the linear order of speech in natural language expression.[17] Prefix notation, which more clearly marks out an operation from its arguments, is more difficult to decode when computations described using it entail the use of more than a single operation.

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[17] See Chomsky (2016) for a non-technical discussion of the problems that arise from the necessary linear ordering of externalised natural language.

6.3.3 Concluding remarks on the substitutions of binary with unary operations

In general, where currying-like transformations are used for operations defined over DAG, similar transformations are effected for operations defined over the natural numbers. This occurs to preserve the general features of the related natural number computational structures, thereby implicitly encouraging the continued perception of the basic operations of arithmetic as unary rather than binary. Further, it is rather commonplace for mathematics expressions to be rendered in speech in a manner implying that one of the arguments of a binary operation are part of the operation.

6.4 Chains of operations/mappings and recursion

The use of discrete aggregates in chains of operations make an appearance in *Tasks* entailing 'decomposition,' where aggregates are partitioned and mapped to correlate number partitions. Other uses of aggregates in chains of operations appear in *Exercises* concerned with teaching place value or when mapping non-equinumerous or equinumerous partitions of aggregates to correlate number partitions. See Figure 6.6.

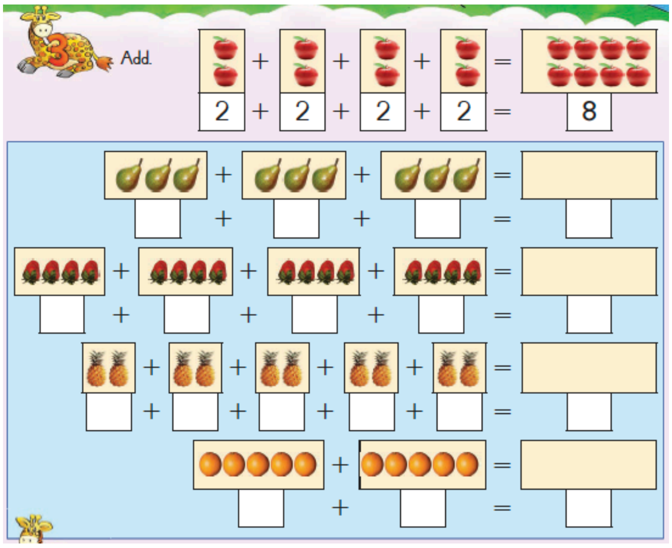


Figure 6.6: *Subtask* 3 of *Task* 5, 'Addition and subtraction,' of Grade 2, Workbook 1

(*Source*: DBE 2021c:10)

When arithmetic problems require the use of chains of operations, we have instances of recursion. Understood in its most general sense, a recursive process is one in which the output of one operation/mapping serves as the input of a subsequent operation/mapping.

6.4.1 Recursion and the use of the inclusion mapping

When the operations/mappings are not unary—as is the case with the basic arithmetic operations—after the application of the first operation/mapping, we are confronted with the problem of constituting the appropriate arguments for the next operation/mapping in the chain of computations because we have only a single value available as a result. For example, when calculating ‘2 + 2 + 2 + 2’, we take the result of ‘2 + 2’, and then compute ‘4 + 2’, followed by ‘6 + 2’. Using mapping notation, to more clearly show the arguments of addition used in the chain of computations, we compute,

$$\begin{array}{ll} & +:(2,2) \rightarrow 4, \\ \text{then} & +:(4,2) \rightarrow 6, \\ \text{followed by} & +:(6,2) \rightarrow 8. \end{array}$$

Note that the output of an addition computation is not sufficient to constitute the argument for the subsequent addition computation. Implicitly, we use the inclusion mapping to constitute the required pairs of arguments by selecting the appropriate computational objects from the workspace of the problem, as shown in Figure 6.7.

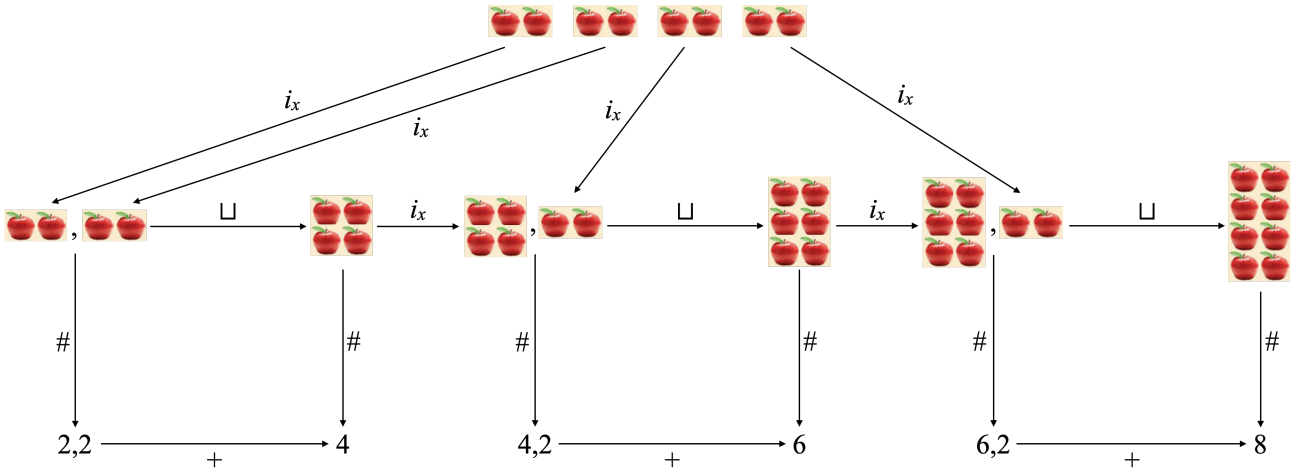


Figure 6.7: The non-curried form of the representation for the completed exercise, *Subtask* 3,*Task* 5 shown in Figure 6.6.

The representation in Figure 6.7 shows the chain of structure-preserving mappings but is rather convoluted. However, if we think of disjoint union and addition in curried forms, the representation is much simpler. For example, as a curried chain of addition computations,

$$2 \xrightarrow{[+\,2]} 4 \xrightarrow{[+\,2]} 6 \xrightarrow{[+\,2]} 8.$$

A similar description of the computations using the aggregates can be constructed and put in relation with the chain of addition computations, using $\#:\mathrm{DAG}\rightarrow\mathbb{N}$, to show structure-preservation in a more compact diagram, as in Figure 6.8.

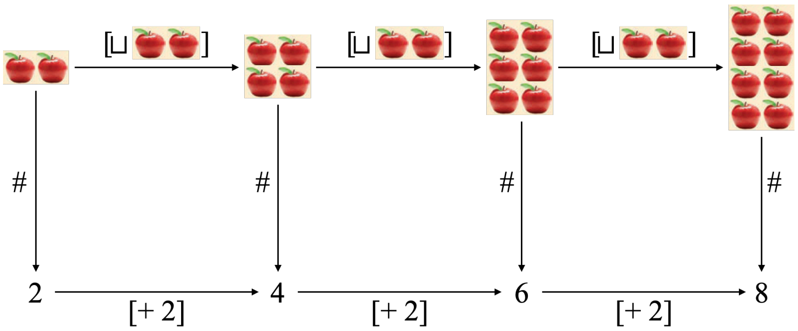


Figure 6.8: A curried form of the representation for the completed *Exercise* of *Subtask* 3, *Task* 5.

6.4.2 Concluding remarks on recursion

Any use of recursion can be described computationally in the manner described in Section 6.4.1. Comparing Figures 6.7 with 6.8, applying currying-like transformations to the operations that are part of the computational structures used can often make for simpler representations of computational chains.

6.5 The use of aggregates to teach place value

An understanding of place value is necessary for dealing with computations involving numbers expressed using more than single digits. For the most part, in schooling a base-ten system is used in computations. It is unsurprising that the earliest engagements with place-value systems for referring to numbers in the FP use the base-ten system.

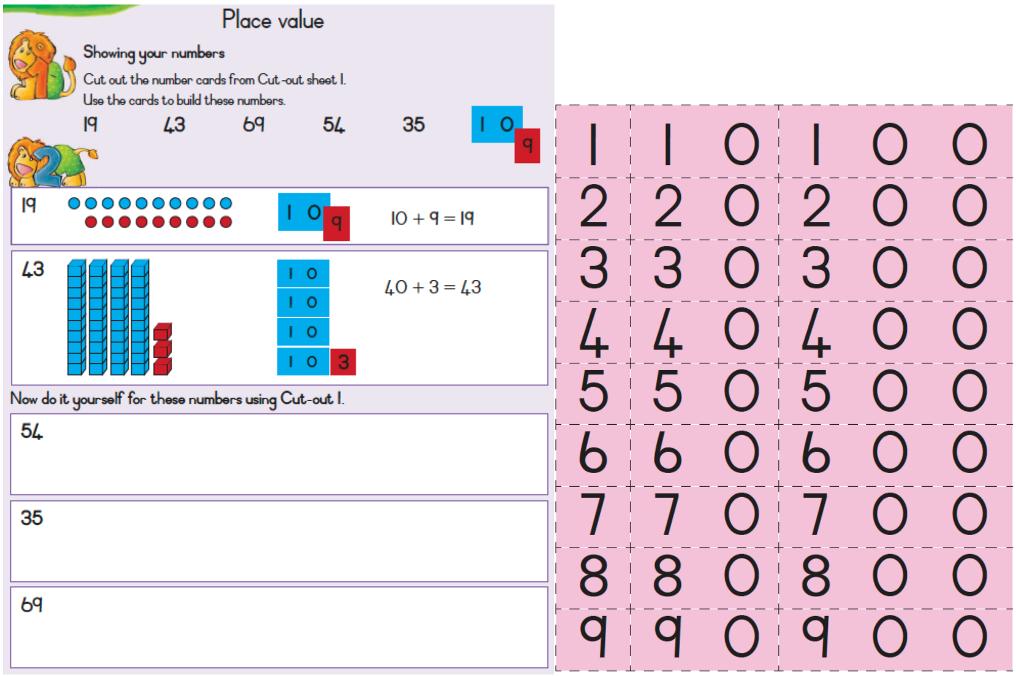
6.5.1 The use of aggregate and number partitions in the teaching of base-ten place value

Consider *Task* 4, ‘Place value’, of the Grade 3, Workbook 1, shown in Figure 6.9. The *Exercises* require learners to use ‘tens’ and ‘units’ cards supplied with the workbook to select the appropriate cards corresponding to a two-digit natural number. The cards are to be right aligned, with the smaller card placed on the larger, so that when the learner reads the digits from left to right, they see the digits of the intended number. Implicitly, the manipulatives restrict the learner’s activity to the production of number partitions that correspond to the place values of the base-ten system.

The *Exercise* provided in *Subtask* 1 is equivalent to the mapping $P_{\text{num}}:19 \rightarrow (10,9)$. However, since the number partitions used in *Subtasks* 1 and 2 are restricted to the subclass of partitions that correspond to the base-ten system, let the symbol P_{base} indicate such partitions. Thus, the exercise in

Subtask 1 is described as $P_{\text{base}}{:}19 \rightarrow (10,9)$. With respect to aggregate partitions, the same symbol, P_{base}, will be used since it is usually clear from the context of its usage what kinds of objects it has as arguments and values. In other words, for $P_{\text{base}}{:}x \rightarrow (x_1, x_2)$, if x refers to a number, we are dealing with number partitions in a particular base; and when x refers to an aggregate, we are dealing with set/aggregate partitions, subject to the base of the numbers used to indicate the cardinality of the sets.

Consider the use of aggregates in the first *Exercise* of *Subtask* 2, in Figure 6.9:



Place value

Showing your numbers

Cut out the number cards from Cut-out sheet 1.
Use the cards to build these numbers.

19 43 69 54 35

19 10 + 9 = 19

43 40 + 3 = 43

Now do it yourself for these numbers using Cut-out 1.

54

35

69

1	1	0	1	0	0
2	2	0	2	0	0
3	3	0	3	0	0
4	4	0	4	0	0
5	5	0	5	0	0
6	6	0	6	0	0
7	7	0	7	0	0
8	8	0	8	0	0
9	9	0	9	0	0

Figure 6.9: *Subtasks* 1 and 2 of *Task* 4, 'Place value', of Grade 3, Workbook 1. Cut-out sheet 1 is shown alongside Task 4 (*Source*: DBE 2021e:10).

Note that the example used at the head of in *Task* 1 is repeated in first example of *Subtask* 2, namely, $P_{\text{base}}{:}19 \rightarrow (10,9)$. Here the implicit mappings[18]

$$\mathrm{NUMDAG}{:}19 \rightarrow \begin{array}{c}\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\\ \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\end{array}$$

and

$$P_{\text{base}}{:}\begin{array}{c}\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\\ \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\end{array} \rightarrow (\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet, \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet)$$

---

[18] NUMDAG is a symbolisation of the relation, (concept: symbol) abbreviated as NUM for NUMBER concept as domain and DAG for DISCRETE AGGREGATE as codomain. An arrow maps the relation.

are used, with the effects of P_{base} indicated by colour referencing CPrs (blue and red). In order to present the resulting partition a consisting of blue (10s) and red (1s) balls, an implicit bijection (BIJ) is used, namely,

BIJ: 🔵🔵🔵🔵🔵🔵🔵🔵🔵 → 🔴🔴🔴🔴🔴🔴🔴🔴🔴.

The resulting partition is indicated as

(🔵🔵🔵🔵🔵🔵🔵🔵🔵🔵, 🔴🔴🔴🔴🔴🔴🔴🔴🔴).

The number cards are marked by colour, indicating associations with the appropriate aggregate, and thus suggesting the base ten structured number partition (10, 9) ⊢ 19, which is also shown as the sum $10 + 9 = 19$. Putting the various computations involving the aggregates together, including the implicit computations, the representation shown in Figure 6.10 can be constructed. Strictly speaking, one can associate *any* suitable discrete aggregate with a natural number, so the very specific aggregates shown in Figure 6.10 enjoy no necessity as values for NUMDAG having 19 as its argument. *Subtask* 1 deals with this difficulty by indicating exactly which discrete aggregates are to be used.

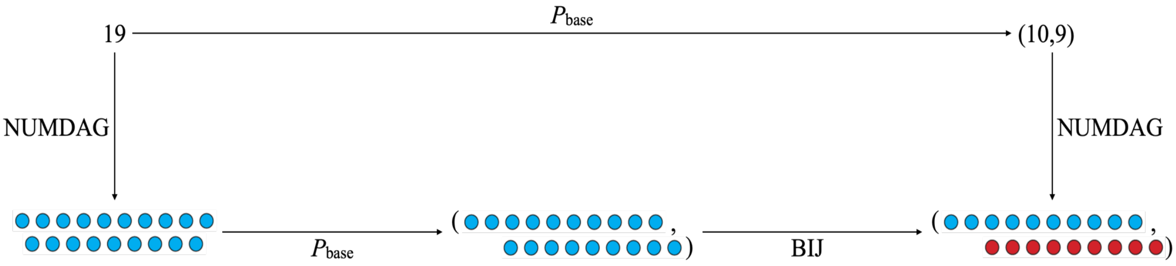
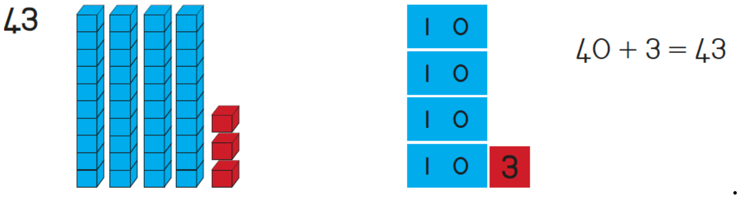
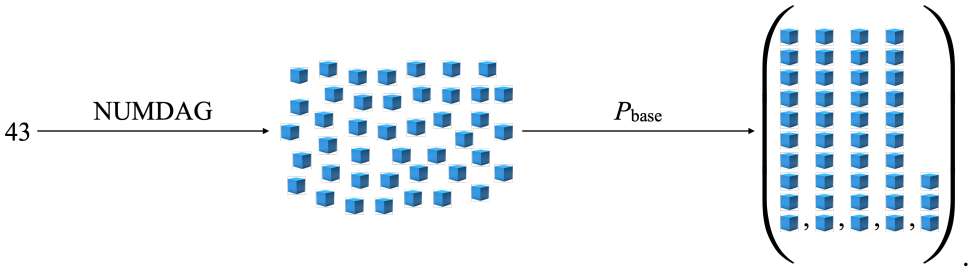


Figure 6.10: A representation modelling the use of aggregates in the first example of *Subtask* 2 shown in Figure 6.9.

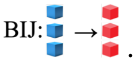
Consider the second example of *Subtask* 2 of *Task* 4, which has 43 as its focus natural number:



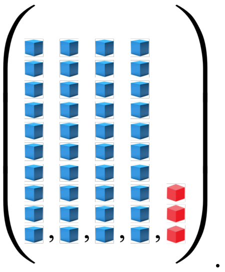
There are number of implicit mappings at work in the example. First, given 43 as its argument, NUMDAG generates a discrete aggregate consisting of forty-three blue cubes. The resultant collection of cubes then serves as the argument to P_{base},



The block of the resulting partition corresponding to the '1s' is replaced by an aggregate consisting of red cubes by means of a bijection conditioned by a CPr requiring the selection of a set of red cubes:



We are then left with the result,



Note that the latter partition really corresponds to the number partition (10, 10, 10, 10, 3) ⊢ 43 rather than (40, 3) ⊢ 43. The recursive use of disjoint union, ⊔, applied to the aggregates consisting of blue cubes in the workspace generated by the exercise example, produces an aggregate partition that can be mapped to the number partition (40, 3) ⊢ 43, shown in Figure 6.11.

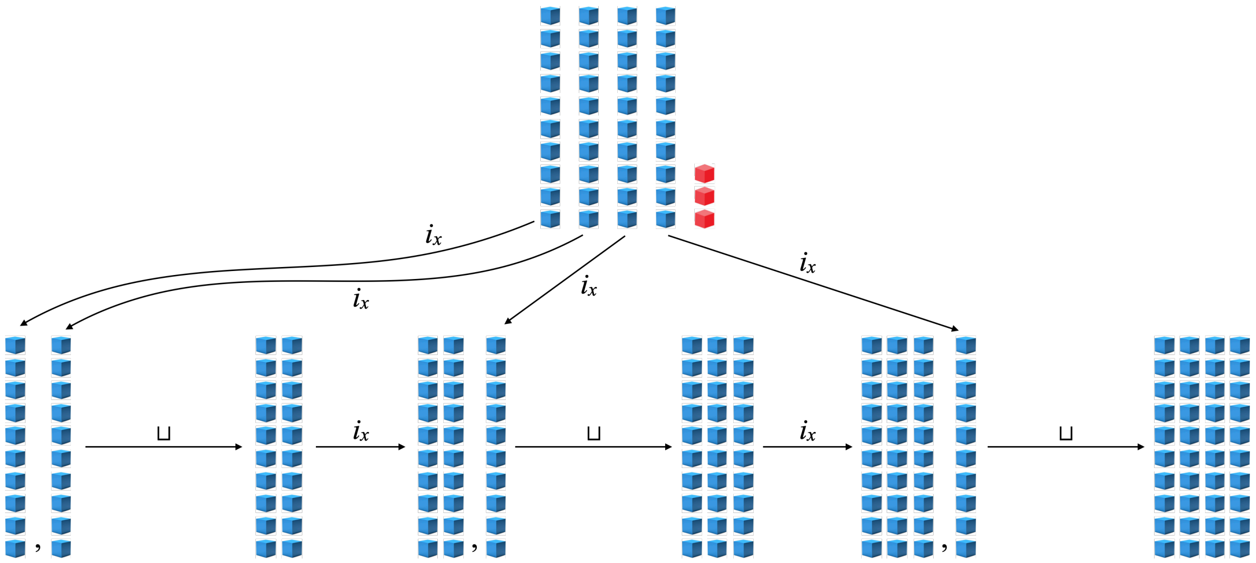


Figure 6.11: The recursive use of disjoint union applied to the individual 'tens' to arrive at an aggregate of cardinality 40.

The process described in Figure 6.11 alters the workspace to include the aggregate of cardinality 40, thereby allowing for the formation of the desired partition, shown in Figure 6.12.

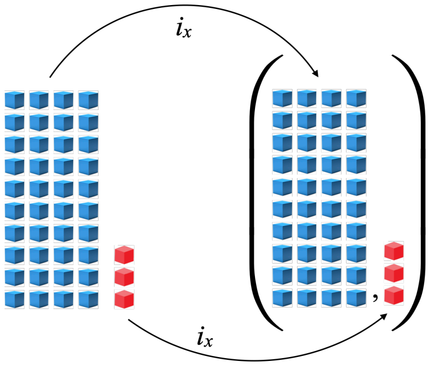


Figure 6.12: The use of the inclusion mapping to construct the aggregate partition corresponding to the number partition (40, 3) ⊢ 43.

The system of computations discussed collectively generate the *representation* in Figure 6.13. The symbol $\sqcup^*$ is employed to indicate $\sqcup$ is used along with other implicit mappings shown in Figure 6.11 and Figure 6.12.

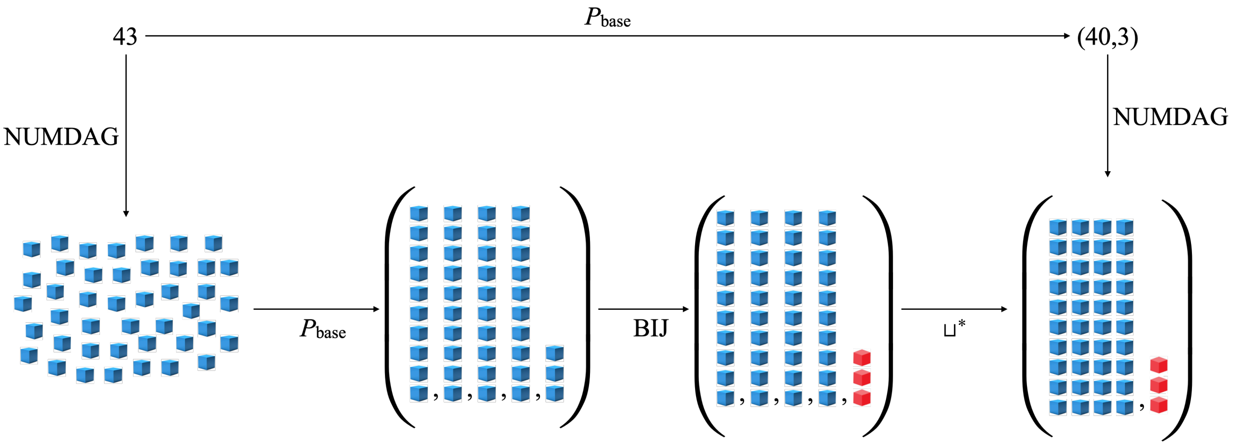


Figure 6.13: A Representation showing the system of mappings relating aggregate partitions to the base-ten number partition (40, 3) ⊢ 3. Recall that $\sqcup^*$ is a composite of implicit mappings.

Recursion works in an analogous way in the workbooks when CPrs code blocks of numbers, as 500s and 100s (Grade 3), aligned with the base-ten number system. The base-ten number system is extended into grades Two and Three by the use of partitioning featuring 500s, 100s in object groupings.

6.5.2 Concluding remarks on aggregates and place value

The use of aggregates to introduce base-ten place value to FP learners exploits the core domain ability to perform intuitive operations using aggregates, as should be apparent from the analysis conducted in Section 6.5.1. The growing ability of learners to count small collections of objects is enabling of the construction of mappings from aggregates to numbers, and opens access to the concepts of elementary arithmetic, including understanding number symbols as a place value notation.

6.6 Concluding remarks

A summary of the mappings discussed in Chapter Six are included in Table 6.4. A discussion of the analysis is next presented in Chapter Seven.

Table 6.4: A summary of the mappings discussed in Chapter 6.

MAPPING	DOMAIN	CODOMAIN	DESCRIPTION
P_{set} (Set partition)	DAG	DAG	Partitioning a discrete aggregate, A, to produce a collection of disjoint subsets of A, namely., $A_1, A_2, \ldots A_n$, such that $A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n = A$ and for all $i \neq j$, $A_i \cap A_j = \emptyset$; we write $(A_1, A_2, \ldots, A_n) \vdash A$; $P_{\text{set}}: A \rightarrow (A_1, A_2, \ldots, A_n)$.
P_{num} (Number partition)	$\mathbb{N}$	$\mathbb{N}$	Partitioning a natural number, k, to produce a collection of natural numbers, namely, k_1, k_2, …, k_n, such that $k_1 + k_2 + \ldots + k_n = k$; we write $(k_1, k_2, \ldots, k_n) \vdash k$; $P_{\text{num}}: k \rightarrow (k_1, k_2, \ldots, k_n)$.
P_{base} (Base partition)	DAG; $\mathbb{N}$	DAG; $\mathbb{N}$	Partitioning a discrete aggregate, A, to produce a collection of disjoint subsets of A, each of which has a cardinality corresponding to a power of a given number base, starting with subsets that have cardinalities corresponding to the largest natural number power of the base, then the next largest and so forth, until all the elements of A have been assigned to a subset; $P_{\text{base}}: A \rightarrow (A_1, A_2, \ldots, A_n)$. Partitioning a natural number, k, to produce a collection of natural numbers, $k_1, k_2, \ldots, k_n$, such that each k_i is a multiple of a power of a given base; $P_{\text{base}}: k \rightarrow (k_1, k_2, \ldots, k_n)$.
$[\backslash S]$ ('curried' relative complement)	DAG	DAG	A version of relative complement where the second argument, S, is treated as though it is part of the operation; $[\backslash S]: A \rightarrow B$, where S, A and B are discrete aggregates.

Table 6.4: A summary of the mappings discussed in Chapter 6.

MAPPING	DOMAIN	CODOMAIN	DESCRIPTION
$[-n]$ (‘curried’ subtraction)	$\mathbb{N}$	$\mathbb{N}$	A version of subtraction over the natural numbers where the second argument, n, is treated as though it is part of the operation; $[-n]:m \rightarrow p$, where m, n and p are natural numbers.
$[\sqcup S]$ (‘curried’ disjoint union)	DAG	DAG	A version of disjoint union where the second argument, S, is treated as though it is part of the operation; $[\sqcup S]:A \rightarrow B$, where S, A and B are discrete aggregates.
$[+n]$ (‘curried’ addition)	$\mathbb{N}$	$\mathbb{N}$	A version of addition over the natural numbers where the second argument, n, is treated as though it is part of the operation; $[+n]:m \rightarrow p$, where m, n and p are natural numbers.

CHAPTER 7
Discussion of the analysis

7.1 Introduction

Chapters Five and Six presented analytic descriptions of the range of operations, mappings, computational structures, and mappings between computational structures used by the workbooks in the elaboration of *addition and subtraction* (AS). This chapter subjects the results obtained in the previous two chapters to further analysis and discussion.

Table 5.1 of Chapter Five reported that 103 *Tasks* used discrete aggregates (DAGS) of relevance to the research focus of this study: 47(35,5%), 34(23,8%) and 22(15,9%) of *Tasks*, respectively, across Grades One, Two and Three. The general reduction in the use of DAG as a computational resource for AS as learners progress through the grades is consistent with the expectation that they move towards using mainly numerical computational resources by the latter stages of Grade Three.

7.2 Grade by grade commentary on the data generated from the analyses of *Tasks*

The extent of the use of the range of mappings/operations across *Tasks* in each of workbooks is now discussed. Recall that the unit of analysis of the computational demands made in the workbooks is the *Task* made up of *Subtasks* and *Exercises*. However, not all *Exercises* are relevant to the research focus. The data have been generated and organised as follows:

(1) for each workbook, where an *Exercise* of a *Task/Subtask* shows the use of a relevant operation/mapping, the *Task* is marked as using the operation/mapping by inserting ‘1’ in a database/grid cell that corresponds to the particular operation/mapping. Any further uses of the particular operation/mapping across the *Exercises* are noted in the analysis and captured in object-and-arrow diagrams, but not recorded in the generated databases.

(2) where an operation/mapping listed in the database is not used in any of the *Exercises* of a *Task*, ‘0’ is inserted in the grid cell corresponding the operation/mapping.

(3) *Tasks* not concerned with AS have been excluded from analyses and therefore from workbook databases—for example, *Tasks* featuring repeated addition to introduce multiplication are excluded.

(4) the *Task*-level data contained in each of the workbook databases are summarised in charts showing the information visually and numerically (as percentages).

Table 7.1 shows a summary of the numbers of mappings/operations used in *Tasks* of relevance to the teaching and learning of AS across the workbooks. Appendices A.3 to A.7 contain the databases of *Tasks* analysed, workbook-by-workbook and grade-by-grade.

Table 7.1: Numbers of *Tasks* of relevance to AS in terms of use of operations/mappings for Grades 1, 2 and 3 (excl. repeated addition/subtraction).

GRADES 1 TO 3 WORKBOOKS	OPERATIONS/MAPPINGS OVER DISCRETE AGGREGATES										OPERATIONS/MAPPINGS OVER NATURAL NUMBERS						
	GRP	$\mathrm{BIJ}_{A,B}$	$\mathrm{ORD}_{\mathrm{set}}$	$\sqcup$	$[\sqcup S]$	$\setminus$	$[\setminus S]$	$\mathrm{P}_{\mathrm{set}}$	$\mathrm{P}_{\mathrm{base}}$	#	$\mathrm{ORD}_{\mathrm{num}}$	(+)	$[+ n]$	$-$	$[- n]$	P_{num}	P_{base}
Grade 1, Workbook 1 TOTALS	29	22	7	20	3	0	4	4	0	28	7	20	3	0	4	2	0
Grade 1, Workbook 2 TOTALS	18	11	1	11	2	1	3	17	13	18	13	7	1	1	3	5	13
Grade 1 Workbooks 1 and 2 TOTALS	47	33	8	31	5	1	7	21	13	46	20	27	4	1	7	7	13
Grade 2, Workbook 1 TOTALS	18	9	3	11	4	2	6	12	11	21	4	11	3	5	6	1	10
Grade 2, Workbook 2 TOTALS	12	8	0	9	2	3	3	1	11	13	4	9	2	4	3	1	12
Grade 2, Workbooks 1 and 2 TOTALS	30	17	3	20	6	5	9	13	22	34	8	20	5	9	9	2	22
Grade 3, Workbook 1 TOTALS	7	7	0	7	0	0	1	1	6	7	0	7	0	1	0	0	6
Grade 3, Workbook 2 TOTALS	14	14	0	14	0	1	0	0	14	14	11	15	0	2	0	1	14
Grade 3 Workbook 1 and 2 TOTALS	21	21	0	21	0	1	1	1	20	21	11	22	0	3	0	1	20
TOTALS	98	71	11	72	11	7	17	35	55	101	39	69	9	13	16	10	55
Percentages (wrt *Tasks* analysed, $N = 103$)	95,1%	68,9%	10,7%	69,9%	10,7%	6,8%	16,5%	34,0%	53,4%	98,1%	37,9%	67,0%	8,7%	12,6%	15,5%	9,7%	53,4%
Percentages (wrt all *Tasks*, $N = 414$)	23,7%	17,1%	2,7%	17,4%	2,7%	1,7%	4,1%	8,5%	13,3%	24,4%	9,4%	16,7%	2,2%	3,1%	3,9%	2,4%	13,3%

7.2.1 Grade 1, Workbook 1

Chart 7.1 details the operation/mapping level data across the relevant *Tasks* use in Grade 1, Workbook 1. At this stage the attention of learners is chiefly focussed on:

(1) the recognition and constitution of discrete aggregates (GRP and BIJA,B)
(2) the counting of discrete aggregates (#)
(3) the disjoint union of discrete aggregates (⊔ and [⊔S])
(4) the addition of natural numbers (+ and [+n]).

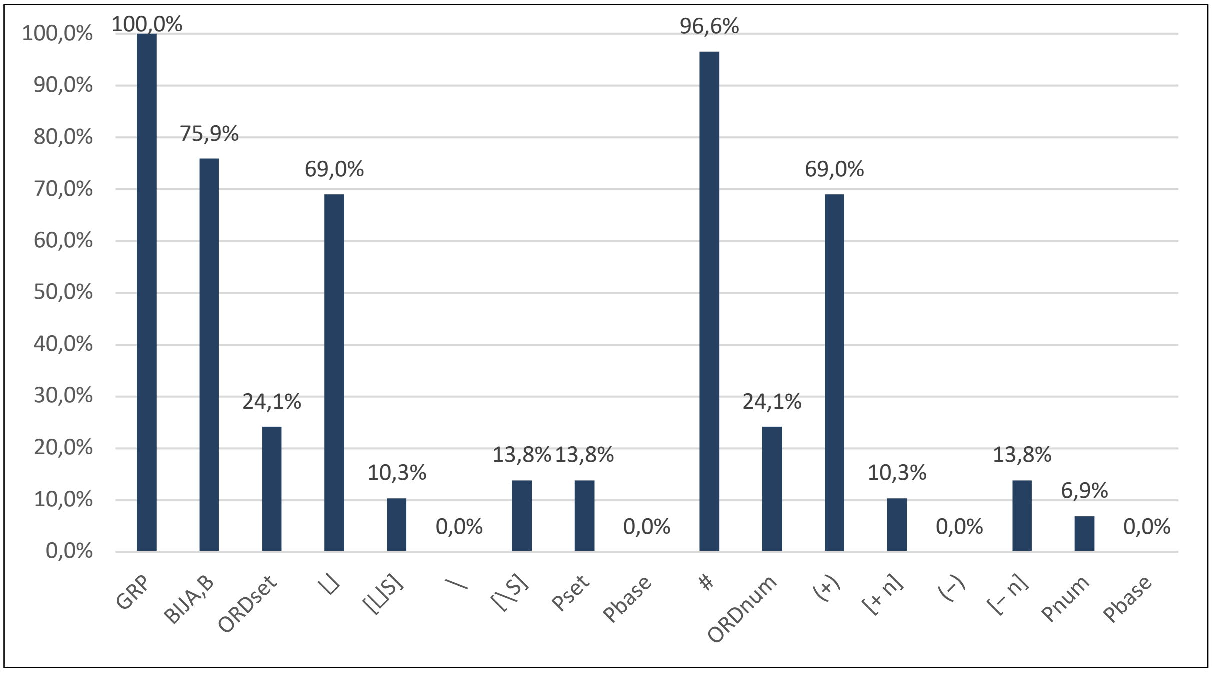


Chart 7.1: Percentages of Tasks ($N = 29$) across Grade 1, Workbook 1 (DBE 2021a) of relevance to addition and subtraction in terms of the operations/mappings used.

Significant attention is given to the order of the natural numbers (ORD_{num}), grounded in an ordering of aggregates (ORD_{set}).

Learners are introduced to subtraction, but in a manner that implies the curried form that derives directly from the *representation* relating relative complements of aggregates to subtraction over the natural numbers. However, the dominant operation in the representation of additive relations is of addition rather than subtraction.

The attention given to subtraction gives rise to a more explicit use of partitions of discrete aggregates, with the latter used as a ground for partitions of natural numbers.

7.2.2 Grade 1, Workbook 2

Chart 7.2 details the operation/mapping level data across the relevant Tasks use in Grade 1, Workbook 2. The attention of learners is chiefly focussed on:

(1) the recognition and constitution of discrete aggregates (GRP and BIJA,B)
(2) the counting of discrete aggregates (#)
(3) the disjoint union of discrete aggregates (⊔ and [⊔S]
(4) the addition of natural numbers (+ and [+n])
(5) the partitioning of discrete aggregates (P_{set} and P_{base})
(6) the partitioning of natural numbers (P_{num} and P_{base})
(7) the order of the natural numbers (ORD_{num}).

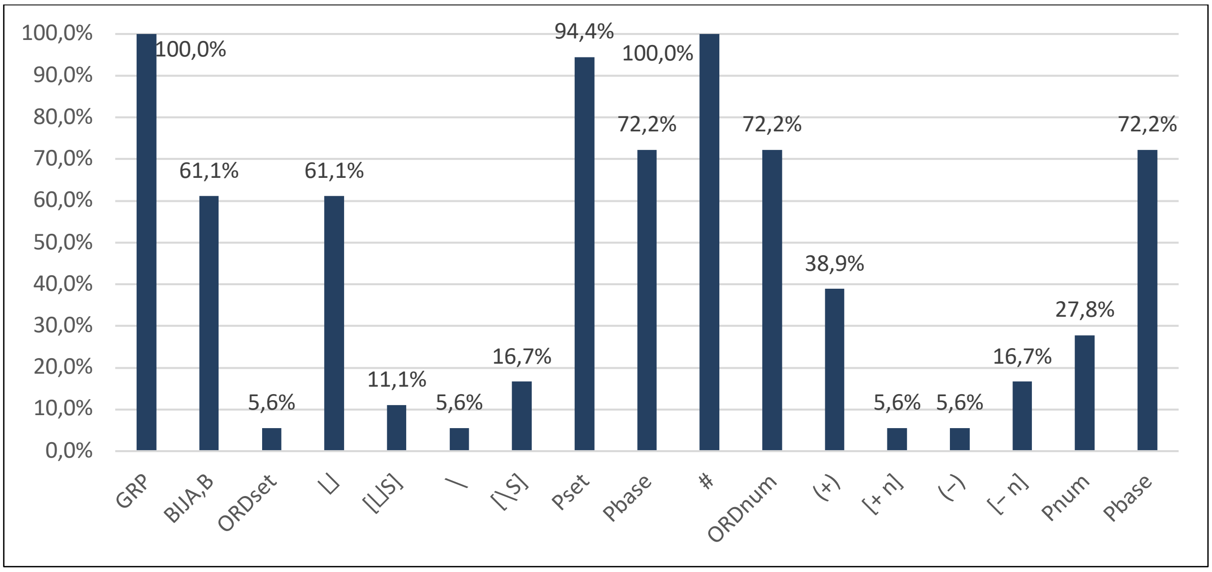


Chart 7.2: Percentages of Tasks (*N* = 18) across Grade 1, Workbook 2 (DBE 2021b) of relevance to addition and subtraction in terms of the operations/mappings used.

Significantly, we see the introduction of the use of base partitioning due to the engagement with quantities greater than 9, which is when place value becomes important in a base 10 number notation system. Once again, subtraction is mostly treated in a manner that implies a curried form of the operation, whilst addition is predominantly introduced using the binary form of the operation.

7.2.3 Grade 2, Workbook 1

Chart 7.3 details the operation/mapping level data across the relevant Tasks use in Grade 2, Workbook 1. The attention of learners is chiefly focussed on:

(1) the recognition and constitution of discrete aggregates (GRP and BIJA,B)
(2) the counting of discrete aggregates (#)
(3) the disjoint union of discrete aggregates (⊔ and [⊔S])
(4) the addition of natural numbers (+ and [+n])
(5) the partitioning of discrete aggregates (Pset and Pbase)
(6) the partitioning of natural numbers (Pnum and Pbase)
(7) the subtraction natural numbers (- and [-n]).

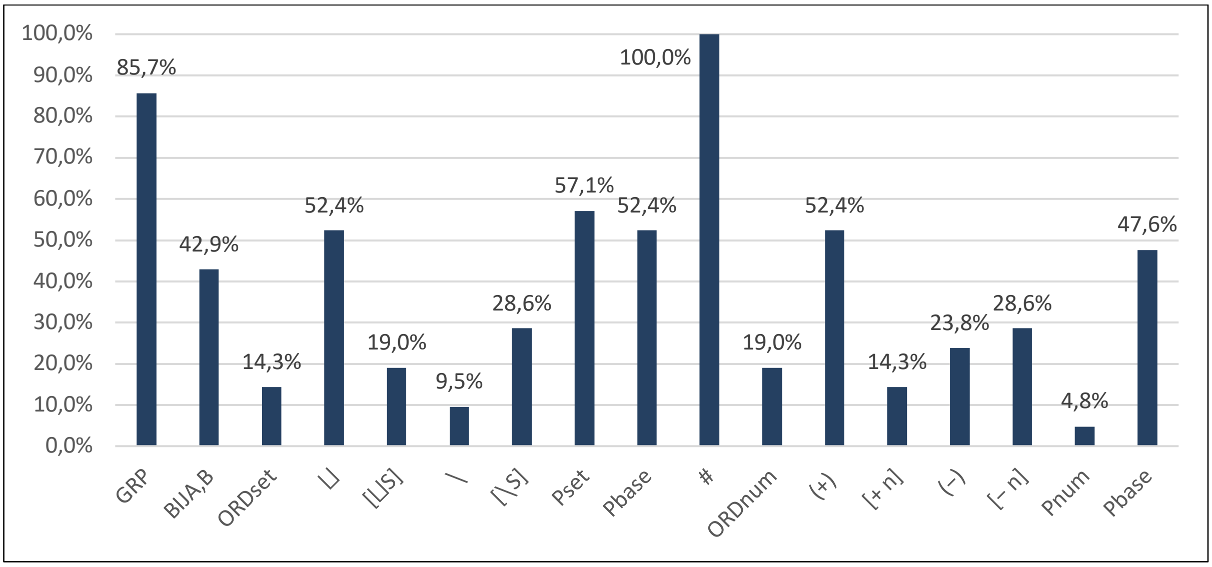


Chart 7.3: Percentages of Tasks (*N* = 21) across Grade 2, Workbook 1 (DBE 2021c) relevance to addition and subtraction in terms of the operations/mappings used.

The treatment of subtraction in a curried form continues, but there is a significant increase in the use of an unambiguously binary form of subtraction over the natural numbers. However, the use of a curried form of relative complement over discrete aggregates continues the practice established in Grade One. The use of base-ten partitioning is becoming more extensive through this workbook.

7.2.4 Grade 2, Workbook 2

Chart 7.4 details the operation/mapping level data across the relevant Tasks use in Grade 2, Workbook 2. The attention of learners is chiefly focussed on:

(1) the recognition and constitution of discrete aggregates (GRP and BIJA,B)
(2) the counting of discrete aggregates (#)
(3) the disjoint union of discrete aggregates (⊔ and [⊔S])
(4) the addition of natural numbers (+ and [+n])

(5) the partitioning of discrete aggregates (P_{set} and P_{base})

(6) the partitioning of natural numbers (P_{num} and P_{base})

(7) the order of the natural numbers (ORD_{num})

(8) the subtraction natural numbers (- and [-n]).

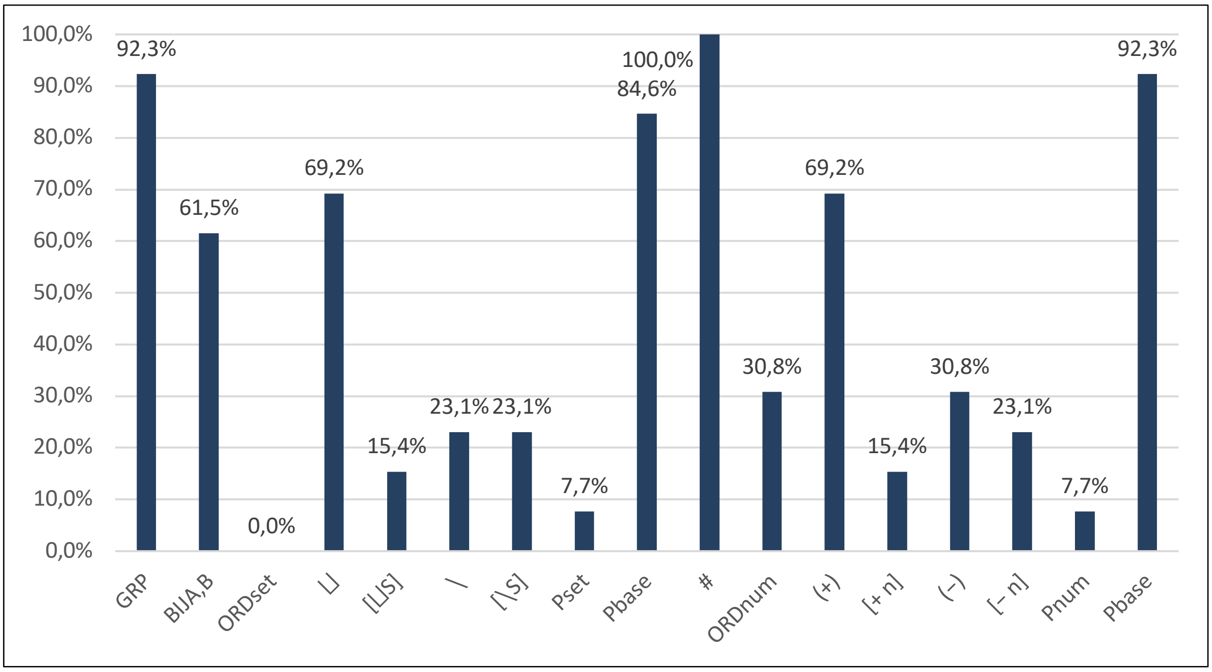


Chart 7.4: Percentages of Tasks ($N = 13$) across Grade 2, Workbook 2 (DBE 2021d) of relevance to addition and subtraction in terms of the operations/mappings used.

The use of base-ten partitions is, once again, fairly extensive because of the need to have learners master the base-ten place value notation for the natural numbers.

Subtraction is treated as both binary and in a curried manner, whilst addition appears in both binary and unary forms, although it is mostly binary.

7.2.5 Grade 3, Workbook 1

Chart 7.5 details the operation/mapping level data across the relevant Tasks use in Grade 3, Workbook 1. The attention of learners is chiefly focussed on:

(1) the recognition and constitution of discrete aggregates (GRP and BIJA,B)

(2) the counting of discrete aggregates (#)

(3) the disjoint union of discrete aggregates (⊔)

(4) the addition of natural numbers (+)

(5) the partitioning of discrete aggregates (P_{set} and P_{base})

(6) the partitioning of natural numbers (P_{num} and P_{base}).

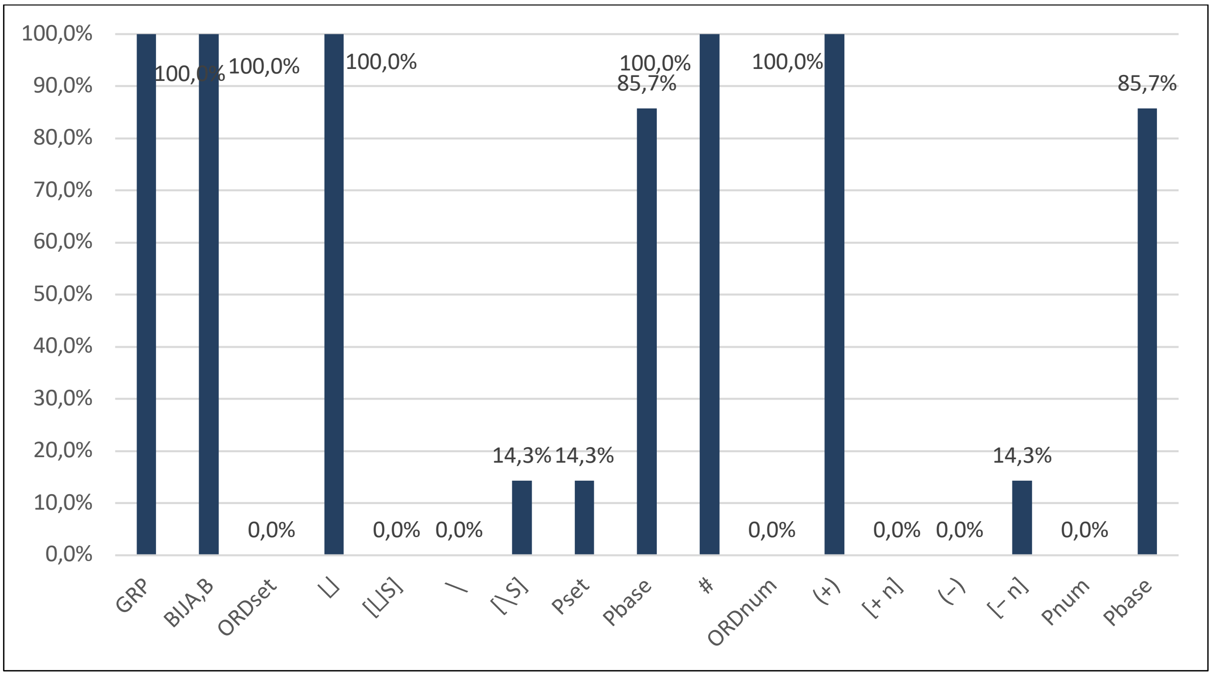


Chart 7.5: Percentages of Tasks ($N = 7$) across Grade 3, Workbook 1 (DBE 2021e) of relevance to addition and subtraction in terms of the operations/mappings used.

The trends observed in the previous workbooks for Grade Two continue. Base-ten partitioning appears again, extensively. Subtraction continues to be treated in a curried manner while addition is treated as binary.

7.2.6 Grade 3, Workbook 2

Chart 7.6 details the operation/mapping level data across the relevant *Tasks* use in Grade 3, Workbook 2. The attention of learners is chiefly focussed on:

(1) the recognition and constitution of discrete aggregates (GRP and $\text{BIJ}_{A,B}$)

(2) the counting of discrete aggregates (#)

(3) the disjoint union of discrete aggregates (⊔)

(4) the addition of natural numbers (+)

(5) the partitioning of discrete aggregates (P_{set} and P_{base})

(6) the partitioning of natural numbers (P_{num} and P_{base}).

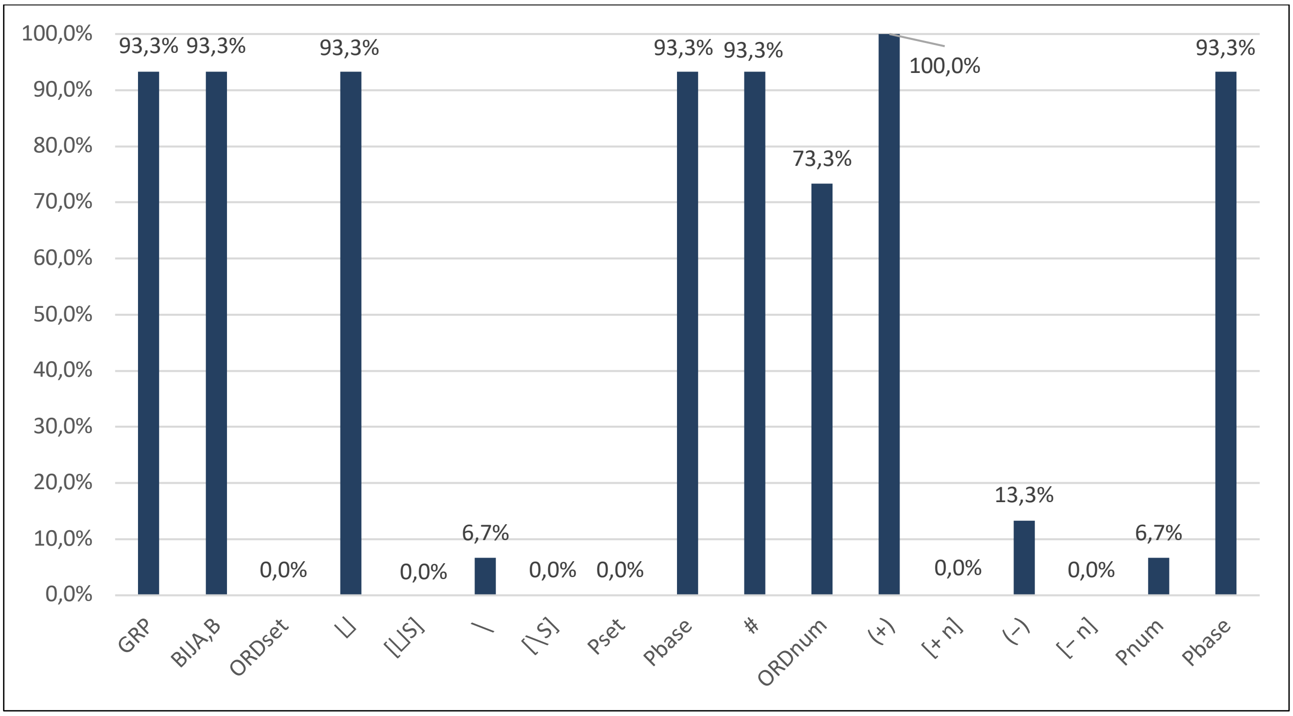


Chart 7.6: Percentages of *Tasks* (*N* = 15) across Grade 3, Workbook 2 (DBE 2021f) of relevance to addition and subtraction in terms of the operations/mappings used.

Again, base-ten partitioning is extensive. Both addition and subtraction are unambiguously extensive in the binary form.

7.2.7 Representations used across the FP workbooks for the teaching of AS

Recall that *representations* are structure-preserving mappings between computational structures, one of which is a representing system and the other a represented system. The discussion is restricted to representations for addition, subtraction, partitioning and order because of the centrality of those representations to the teaching and learning of AS, as elaborated in the FP workbooks.

The use of disjoint union and relative complementation over DAG leads quite naturally to an awareness of the computational utility of the partitioning of discrete aggregates. From the use of disjoint union, aggregates can be recognised in terms of collections of constitutive aggregates. Relative complementation very directly marks out a partition of an aggregate in the sorts of *Exercises* used in the workbooks.

The use of order relations over aggregates as a ground for ordinal relations between natural numbers is important for thinking about subtraction over the natural numbers in a manner that keeps the computational problem of the lack of closure at bay.

7.2.7.1 Representations for addition: $\#:(\mathrm{DAG},\sqcup)\rightarrow(\mathbb{N},+)$ & $\#:(\mathrm{DAG},[\sqcup S])\rightarrow(\mathbb{N},[+n])$

The data shown in Table 7.2 indicates that a relatively large number of *Tasks* (70) use representations mapping disjoint union over DAGS to addition over $\mathbb{N}$ via counting.

Table 7.2: Numbers of *Tasks* using representations for addition across the analysed *Tasks*

Workbook	$(\mathrm{DAG},\sqcup) \rightarrow (\mathbb{N},+)$	$(\mathrm{DAG},[\sqcup S]) \rightarrow (\mathbb{N},[+n])$	$(\mathrm{DAG},\sqcup) \rightarrow (\mathbb{N},+)$ AND $(\mathrm{DAG},[\sqcup S]) \rightarrow (\mathbb{N},[+n])$
Grade 1, Workbook 1	19	3	3
Grade 1, Workbook 2	6	1	1
Grade 1 Totals	25	4	4
Grade 2, Workbook 1	11	3	0
Grade 2, Workbook 2	9	2	1
Grade 2 Totals	20	5	1
Grade 3, Workbook 1	7	0	0
Grade 3, Workbook 2	14	0	0
Grade 3 Totals	21	0	0
FP Workbook Totals	66	9	5

Addition tends to be set up as binary in most *Tasks*, but there are small number of cases (9) where a curried form of disjoint union is used in Grades 1 and 2, leading to a curried form of addition.

In Grade 1 we see that there are *Tasks* that employ both forms of addition. In Grade 3, addition is unambiguously elaborated as binary.

7.2.7.2 Representations for subtraction: $\#:(\mathrm{DAG},\setminus)\rightarrow(\mathbb{N},-)$ & $\#:(\mathrm{DAG},[\setminus S])\rightarrow(\mathbb{N},[-n])$

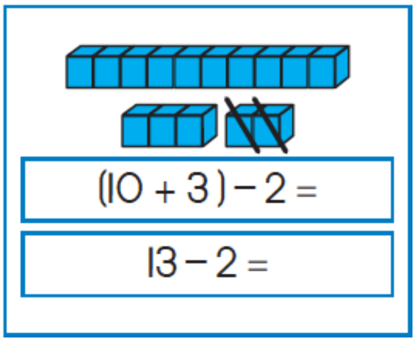
As was the case with *Representations* for addition, counting is the mapping that mediates between relative complement over discrete aggregates and subtraction over $\mathbb{N}$. Table 7.3 shows that, in comparison to the extent of the use of representations for addition, the use of representations for subtraction relatively low in number (20). It is curious that there are fewer *Tasks* that use *Representations* for subtraction than there are *Tasks* using *Representations* for addition.

Interestingly, the use of a curried form of subtraction occurs in around twice as many *Tasks* (15) as those that use the binary form (7). Over Grades 1 and 2, the curried form of subtraction is consistently used more frequently than is the binary form.

Table 7.3: Numbers of *Tasks* using representations for subtraction across the analysed *Tasks*

Workbook	$(\mathrm{DAG},\setminus) \to (\mathbb{N},-)$	$(\mathrm{DAG},[\setminus S]) \to (\mathbb{N},[-n])$	$(\mathrm{DAG},\setminus) \to (\mathbb{N},-)$ AND $(\mathrm{DAG},[\setminus S]) \to (\mathbb{N},[-n])$
Grade 1, Workbook 1	0	4	0
Grade 1, Workbook 2	1	3	0
Grade 1 Totals	1	7	0
Grade 2, Workbook 1	2	5	0
Grade 2, Workbook 2	3	3	2
Grade 2 Totals	5	8	2
Grade 3, Workbook 1	0	0	0
Grade 3, Workbook 2	1	0	0
Grade 3 Totals	1	0	0
FP Workbook Totals	7	15	2

There are a number of instances of *Exercises* where *Representations* for subtraction are confusing because they violate the graphical notation initially used to indicate relative complements. This appears to be problem that derives from an over-determining effect of the arithmetic notation for subtraction, namely, $a - b = c$ (where $a,b,c \in \mathbb{N}$ and $a \geq b$). For example, consider an *exercise* in Subtask 6 of Task 103 in the Grade 1, Workbook 2:



.

While the attempted representation implicitly claims that the presented use of relative complement ultimately maps to the numerical computation 13 – 2, it should be mapped to 15 – 2 to be consistent with the initial uses of the graphical notation for relative complementation. A consistent mapping would be (10 + 3 + 2) – 2, rather than (10 + 3) – 2. The *Exercise* nicely demonstrates the over-determining effect of the arithmetic notation on the graphical notation used with discrete aggregates. The confusion might also be an effect of the very extensive use of curried forms of relative complementation: the aggregate made of two unit cubes along with the diagonals drawn through them is read as mapped to a unary operation, [–2], rather than as merely marking out a part of an aggregate that needs to be excluded. For the learner to construct a structure-preserving mapping in relation to the data provided in the *Exercise* they would need to change their understanding of the graphical notation.

7.2.7.3 Representations-partitioning: $\#:(\mathrm{DAG},P_{\mathrm{set}})\rightarrow(\mathbb{N},P_{\mathrm{num}})$; $\#:(\mathrm{DAG},P_{\mathrm{base}})\rightarrow(\mathbb{N},P_{\mathrm{base}})$

Table 7.4 shows the extensive use of base partitioning that increases from Grades One to Three. As was indicated earlier, base partitioning is used in order to establish a ground for the numerical place value notation and as a resource for computing sums and differences of two- and three-digit natural numbers.

Once again, counting is the mediating mapping taking DAG-partitions to ℕ-partitions. Partitions are used as computational resources throughout the primary school curriculum, far beyond their uses in place value, and AS.

Table 7.4: Numbers of *Tasks* using representations for partitioning across the analysed *Tasks*

Workbook	$(\mathrm{DAG},P_{\mathrm{set}}) \rightarrow (\mathbb{N},P_{\mathrm{num}})$	$(\mathrm{DAG},P_{\mathrm{base}}) \rightarrow (\mathbb{N},P_{\mathrm{base}})$	$(\mathrm{DAG},P_{\mathrm{set}}) \rightarrow (\mathbb{N},P_{\mathrm{num}})$ AND $(\mathrm{DAG},P_{\mathrm{base}}) \rightarrow (\mathbb{N},P_{\mathrm{base}})$
Grade 1, Workbook 1	2	0	0
Grade 1, Workbook 2	5	13	3
Grade 1 Totals	7	13	3
Grade 2, Workbook 1	0	3	0
Grade 2, Workbook 2	1	11	1
Grade 2 Totals	1	14	1
Grade 3, Workbook 1	0	6	0
Grade 3, Workbook 2	0	14	0
Grade 3 Totals	0	20	0
FP Workbook Totals	8	47	4

7.2.7.4 Representations for order: $\#:(\mathrm{DAG},\mathrm{ORD}_{\mathrm{set}})\rightarrow(\mathbb{N},\mathrm{ORD}_{\mathrm{num}})$

Representations for order are sparsely used explicitly across the FP workbooks and are concentrated in Grade One, where learners are still new to school mathematics (see Table 7.5).

Table 7.5: Numbers of *Tasks* using representations for order across the analysed *Tasks*

Workbook	$(DAG, ORD_{set}) \rightarrow (\mathbb{N}, ORD_{num})$
Grade 1, Workbook 1	2
Grade 1, Workbook 2	5
Grade 1 Totals	7
Grade 2, Workbook 1	0
Grade 2, Workbook 2	1
Grade 2 Totals	1
Grade 3, Workbook 1	0
Grade 3, Workbook 2	0
Grade 3 Totals	0
FP Workbook Totals	8

Ordinality, while important in mathematics, is treated rather simplistically in all school mathematics. One effect of the simplistic treatment of order in schooling is a poor understanding of the use of inequalities, which ultimately affects secondary and tertiary education.

As indicated earlier, ordinality appears to be used to deal with the lack of closure with respect to natural number subtraction.

7.3 *Merge* as a deep semantic basis for addition, subtraction and partitioning

The analyses of the workbooks reveal the use of discrete aggregates as foundational in representations for addition, subtraction and partitioning, all of which are implicated in what has become known as *additive reasoning* in the mathematics education literature. It can reasonably be argued that each of the classes of representations are specific realisations of the use of a universal cognitive operation which is referred to as *merge* in the work of Chomsky and in the field of biolinguistics (see Chapter Three).

Figure 7.1 presents a system of representations showing *merge* as the deep semantic basis for natural number addition, which becomes the model for additive processes in general. α and β are already formed objects that are merged to form a new object, γ. The context-specific mediator from *merge* is *DAG*, which specialises the abstract arguments of *merge*, the objects α, β and γ, as discrete aggregates (sets), *A*, *B* and *C*, subject to the set-theoretic operation of disjoint union. Disjoint union is in turn mapped to addition over the natural numbers ($m + n = p$) by means of counting (#).

$$\begin{tikzcd}
\mathrm{Obj}(\alpha),\mathrm{Obj}(\beta) \arrow[r, "merge"] \arrow[dd, "DAG"'] & \mathrm{Obj}(\gamma) \arrow[dd, "DAG"] & \left.\right\}\text{Core semantic basis} & \\
 & & \left.\right\}\text{Mediating mapping} & \left.\right\}\text{Conceptual-intentional system} \\
A,B \arrow[r, "\sqcup"] \arrow[dd, "\#"'] & C \arrow[dd, "\#"] & \left.\right\}\text{Intermediate semantic basis for addition} & \\
 & & \left.\right\}\text{Mediating mapping} & \\
m,n \arrow[r, "+"'] & p & \left.\right\}\text{Arithmetic} &
\end{tikzcd}$$

Figure 7.1: *Merge* as a semantic basis for disjoint union and natural number addition.

To deal with subtraction, we need to ask how it might result from the existence of *merge* as a universal cognitive operation. First, we note that for any given object, γ, which is conceived of as formed from the merging of two prior objects, α and β, one of the initial objects can be purged from γ to leave one of the initial objects as a result. We can, therefore, define an operation, *purge*, auxiliary to *merge*, and which enables us to recover an initial object of interest.

$$\begin{tikzcd}
(\mathrm{Obj}(\gamma),\mathrm{Obj}(\beta)) \arrow[r, "purge"] \arrow[dd, "DAG"'] & \mathrm{Obj}(\alpha) \arrow[dd, "DAG"] & \left.\right\}\text{Core semantic basis} & \\
 & & \left.\right\}\text{Mediating mapping} & \left.\right\}\text{Conceptual-intentional system} \\
(A,B) \arrow[r, "\setminus"] \arrow[dd, "\#"'] & C \arrow[dd, "\#"] & \left.\right\}\text{Intermediate semantic basis for subtraction} & \\
 & & \left.\right\}\text{Mediating mapping} & \\
m,n \arrow[r, "-"'] & p & \left.\right\}\text{Arithmetic} &
\end{tikzcd}$$

Figure 7.2: *Purge* as a semantic basis for relative complement and natural number subtraction.

Figure 7.2 presents the system of representations that shows how *purge* functions as a semantic basis for subtraction over the natural numbers and which, like *merge*, serves as a general cognitive basis for subtraction beyond the natural numbers. Once again, *DAG* and # are the mediating mappings enabling us to arrive at natural number arithmetic.

In the case of partitioning, we start by asking how partitions might be related to *merge*? Starting with some already formed object, which is conceived of as constituted by a series of existing objects, we can imagine the undoing of *merge* to reveal the constituents of the object formed by merge. An obvious name for such a process is *unmerge*, which takes a given object as its argument and produces the list of computational objects that were used to constitute the argument (input) as its value (output). Thus, *erge* and *purge* are interchangeable ideas. *DAG* and # are mediating mappings enabling us to arrive at natural number partitions and, once again, at natural number arithmetic. Figure 7.3 presents the system of representations that enables the use of partitions as a computational resource.

$$\begin{tikzcd}
\mathrm{Obj}(\lambda) \arrow[r, "\mathit{unmerge}"] \arrow[dd, "\mathit{DAG}"'] & (\mathrm{Obj}(\lambda_1), \mathrm{Obj}(\lambda_2), \ldots, \mathrm{Obj}(\lambda_n)) \arrow[dd, "\mathit{DAG}"] & \left.\right\}\ \textit{Core semantic basis} & \\
 & & \left.\right\}\ \textit{Mediating mapping} & \left.\right\}\ \textit{Conceptual-intentional system} \\
A \arrow[r, "P_{\mathrm{set}}"] \arrow[dd, "\#"'] & (A_1, A_2, \ldots, A_n) \arrow[dd, "\#"] & \left.\right\}\ \textit{Intermediate semantic basis for partitioning} & \\
 & & \left.\right\}\ \textit{Mediating mapping} & \\
k \arrow[r, "P_{\mathrm{num}}"'] & (k_1, k_2, \ldots, k_n) & \left.\right\}\ \textit{Arithmetic} &
\end{tikzcd}$$

Figure 7.3: *Unmerge* as a semantic basis for set and natural number partitioning.

P_{base} is a special case of P_{set} and of P_{num}, which can be thought of as a realisation of *unmerge* subject to a constraint that picks out a specific conditioning rule, namely, powers of some natural number (the base).

7.4 Concluding remarks

Object-and-arrow diagrams similar to those shown in Figures 7.1 to 7.3 can be constructed for the curried forms of set and arithmetic operations.

The analysis points to *merge*, and a couple of obvious variations on *merge*, serving as a deep semantic basis for the key computational processes exploited in FP arithmetic, establishing additive reasoning as a foundational resource in school mathematics. The system of representations in each instance proceeds from a universal property of cognition, grounded in *merge*, translated into operations over discrete aggregates and then to operations over number. In cultures that do not use a counting algorithm, like those of some Amazonian tribes, basic arithmetic as we know it is not present. Precise counting appears to be foundational to the emergence of elementary arithmetic and the much more complex mathematical systems that are generated from arithmetic.

It is important to appreciate that *merge*, as an intuitive core domain resource, proves useful for negotiating the world around us. However, for considering the production of noncore domain mathematics, it is inadequate without the insertion of an algorithm for precise counting and the use of structure-preserving mappings that enable us to realise universal additive properties in specific, diverse computational contexts.

CHAPTER EIGHT

Conclusion

8.1 Introduction

The general problematic in which this study is situated is that of the production and growth of specialised, noncore domain knowledge by genetically endowed computational resources (core domain knowledge/systems) on the basis of contextual data. More specifically, the study is concerned with exploring the production and growth of school mathematics by genetic endowment (species-specific computational resources) in the context of formal schooling (contextual data). The research problem specialises the focus even further:

> *How are computational structures that are constructed from discrete aggregates and operations over collections of discrete aggregates, used as a ground for the teaching and learning of the computational structures that constitute addition and subtraction in the DBE FP numeracy workbooks?*

The study used the six 2021 DBE FP workbooks as a source of data on the pedagogical recruitment of discrete aggregates for the purposes of the teaching and learning of addition and subtraction.

In Chapter Two a sketch of the genetically endowed, species-specific computational resources typically available to humans and of relevance to the study was constructed, mainly from a review of contemporary cognitive science research.

Chapter Three presented the general mathematical resources that would be used in the study, and Chapter Four detailed the procedures for the production of data and the mode of analysis.

The primary data for the study was generated in Chapters Five and Six, revealing the range of computational resources used in the workbooks in some detail.

Finally, the discussion in Chapter Seven focussed the data to get some measure of the extent of the use of computational structures constructed from operations over collections of discrete aggregates as a ground for the teaching and learning of the computational structures that constitute addition and subtraction.

8.2 A review of the main findings

An overview of the main findings are presented next.

8.2.1 The operational range of *merge*

The major focus of CAPS (2011) in the FP is on the content strand *numbers, operations and relationships*, where the use of discrete aggregates are concentrated. This strand focuses on addition and subtraction being the first arithmetic operations introduced (CAPS 2011:21-23), alongside counting as the major resource associating discrete aggregates with natural numbers (CAPS2-11:18, 73-74). The analysis also revealed that set and natural number partitioning are used extensively, with the former serving as a ground for the latter. Addition, subtraction and partitioning all use *merge* or a derivative of it (*purge* or *unmerge*), which was to be expected from the review of the cognitive science research. In other words, much of FP mathematics derives from the exploitation of a computationally simple cognitive operation, *merge*, used creatively to fashion a variety of computational resources.

8.2.2 Representations that suggest arity changes in operations

A notable finding across the workbooks is the use of representations that imply an arity of one for addition and subtraction, thereby obscuring the binary nature (arity of two or more) of the operations. This feature of the texts is almost certainly unintended by the workbook authors and the curriculum advisors. It can be easily corrected by ensuring that operations over aggregates are always presented as unambiguously binary. The mappings from operations over aggregates to arithmetic operations would then not unintentionally obscure the binary nature of addition and subtraction.

8.2.3 Comprehension principles, part-whole relations, fusions and sets

The analysis revealed that the formation of discrete aggregates and collections of such aggregates is extensive (95,1% of *Tasks* analysed). The *Tasks* invariably required the use of a CPr to generate the required aggregates. Alongside the constant use of CPrs we find the extensive use of partitioning for teaching place value. Partitions are also used for addition and subtraction when dealing with computations that involve two-digit and three-digit numbers, and very often for subtraction presented in a curried form.

Both features of the workbooks—the extensive use of CPrs and partitioning—give coherence to discrete aggregates because they align with intuitions that derive from core domain computational resources. However, Davis (2016, 2018) demonstrated that the use such resources has the potential to align teaching and learning with aggregates conceived of in terms of *fusions* rather than *sets*.

This result, perhaps, may be of little consequence in primary school mathematics. However, it can function as a source of conceptual difficulties when learners have to understand sets when studying computational structures explicitly, especially in tertiary mathematics courses.

8.2.4 Speculative remarks on additive and multiplicative reasoning

Given the limitations on space, this study excluded reporting on the analysis of *Tasks* for the teaching of multiplication, division and fractions. What can be said is that the workbooks consistently treat multiplication as repeated addition, or recursive *merge*. CAPS (2011:21-22) prescriptions align. The implications point to a FP mathematics with a semantic basis that is essentially additive (Davis, Jaffer & Sapire 2022). An additive approach encourages the settling of a problematic semantic basis for multiplication, division and fractions—one which is difficult to shift in later years and which results in generally poor performances on tasks that require proportional reasoning, or an understanding of multiplicative processes. Data from assessment studies like ANAs (2012), SACMEQ (2015), and TIMSS (2012) clearly show that SA school children have significant shortcomings when required to deal with multiplicative processes. What this suggests, in the context of this study, is that the use of *merge* as *the* foundational computational resource for generating FP mathematics needs to be carefully reconsidered.

8.3 Study implications and recommendations

The outcomes of the study point to the need for workbooks that engage knowledge of the principles of mathematics in the treatment of sets and of operations used to constitute arithmetic computational structures. In so doing, the hope is that the emergence of understandings consistent with mathematics are prioritised over the unintended growth of misconceptions of mathematical objects and processes in initial mathematics education.

The uses of aggregates in the workbooks are strongly aligned with the prescriptions of the curriculum strand, *Number, Operations and Relationships*. However, due to the inherent structural problems highlighted by the analysis, the implications suggest a review of the FP CAPS (Grades One to Three) that accounts for the distinction between core and noncore domain systems in relation to the construction of representations used in the teaching of FP mathematics. An intention of this small-scale study is to encourage conversations amongst the education and mathematics community. Firstly, conversations should foreground the pedagogical use of mathematical relations between computational structures. Secondly, conversations should consider how core domain resources for dealing with quantity can more effectively be used to support learning in a noncore domain like mathematics.

Another implication from the study is that the use of aggregates, along with all the complications that emerge once they are treated as mathematical objects, become an explicit component of the training of both pre- and in-service teachers. Such training would need to be accompanied by exposure to some of the relevant research in contemporary cognitive science, rather than relying solely on the typical psychology courses offered on teacher training programmes (often restricted to vague treatments of the work of Piaget and Vygotsky).

8.4 Limitations and potential of the study

The central achievement of this study has been the application of a computational-analytic approach which revealed the range of the creative use of the core domain operation, *merge*, with precision that derives from the use of mathematically conditioned analytic resources.

Small-scale studies suffer inevitable limitations. Extending this study to include an examination of the learning of all four basic arithmetic operations, as well as including the uses of continuous aggregates evident across the workbooks, would yield a more comprehensive data set and possibly reveal trends that are significant for understanding what happens in schooling beyond the FP. Furthermore, learner and teacher interview data would have been of great assistance in revealing the computational structures produced by learners in response to the *Tasks* analysed. Such data would elucidate the conceptual-intentional of learners which might well change the conclusions arrived at by the study.

The insights offered by the study suggests potential factors that contribute to the ongoing decline in the mathematical performances of South African learners on high stakes assessments, particularly the decline evident amongst high performing Grade Three learners when transitioning to Grade Four (Spaull, Courtney & Qvist 2022).

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APPENDIX A

A.1.1 Relations between aggregates, natural/rational numbers, number words and number symbols (Source: Davis 2018:17)

A.1.2 An abridged version of the description of the mappings between discrete aggregates (DAG), number words (NUM), words (WRD), number symbol (SYM), continuous aggregates (CAG) with respect to the relational mappings indicated in appendix A.1.1. (Source: Davis 2018:18)

A.2 Operations/mappings of relevance to addition and subtraction in Grade 1, Workbook 1 (DBE 2021a). Workbook *Tasks* (N=69)

A.3 Operations/mappings of relevance to addition and subtraction in Grade 1, Workbook 2 (DBE 2021b). Workbook *Tasks* (N=64).

A.4 Operations of relevance to addition and subtraction in Grade 2, Workbook 1 (DBE 2021c). Workbook *Tasks* (N= 68).

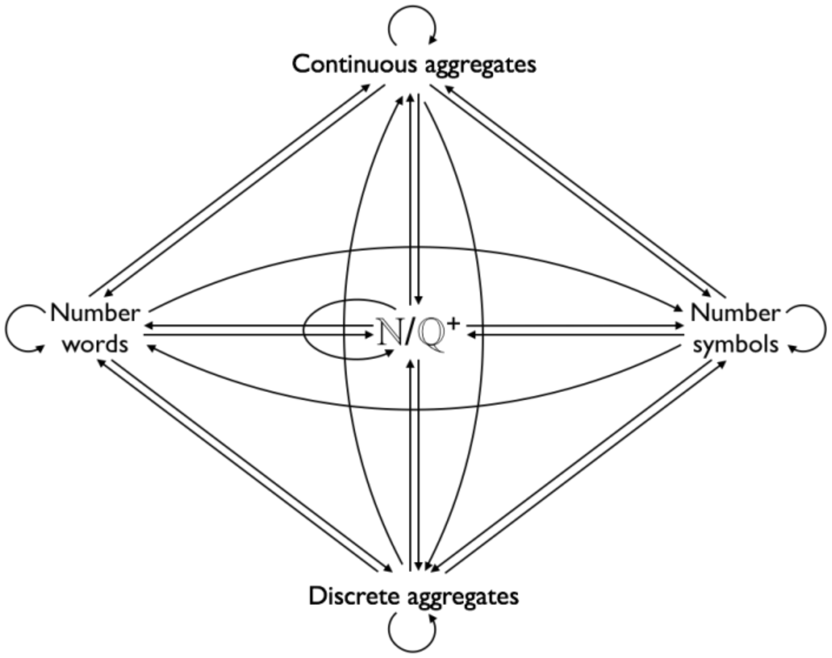
A.5 Operations/mappings of relevance to addition and subtraction in Grade 2 Workbook 2 (DBE 2021d). Workbook *Tasks (*N=75).

A.6 Operations/mappings of relevance to addition and subtraction in Grade 3 Workbook 1 (DBE 2021e). Workbook *Tasks (*N=71).

A.7 Operations/mappings of relevance to addition and subtraction in Grade 3 Workbook 2 (DBE 2021e). Workbook *Tasks (*N=67).

APPENDIX A.1.1

Relations between aggregates, natural/rational numbers, number words and number symbols
(Source: Davis 2018:17)



APPENDIX A.1.2

An abridged version of the description of the mappings between discrete aggregates (DAG), number words (NUM), words (WRD), number symbol (SYM), continuous aggregates (CAG) with respect to the relational mappings indicated in appendix A.1.1. (Source: Davis 2018:18)

KEY to naming the relations is as follows (Davis 2018:17):

1. Identify the domain (set of inputs) and codomain (set of potential outputs) of the relation.
2. Use three letters, suitably descriptive, to refer to the domain.
3. Use three letters, suitably descriptive, to refer to the codomain.
4. Concatenate the letters selected in (1) and (2), always listing those for (1) first, and then those for (2). Create a mapping using '→.'

RELATION	DOMAIN	CODOMAIN	SYMBOLISATION
CAGDAG	continuous aggregates	discrete aggregates	CAGDAG: CAG → DAG
DAGWRD	discrete aggregates	number words	DAGWRD: DAG → WRD
DAGSYM	discrete aggregates	number symbols	DAGSYM: DAG → SYM
DAGNUM	discrete aggregates	numbers	DAGNUM: DAG → NUM
DAGCAG	discrete aggregates	continuous aggregates	DAGCAG: DAG → CAG
DAGDAG	discrete aggregates	discrete aggregates	DAG*: DAG → DAG
WRDNUM	number words	numbers	WRDNUM: WRD → NUM
WRDDAG	number words	discrete aggregates	WRDDAG: WRD → DAG
SYMDAG	number symbols	discrete aggregates	SYMDAG: SYM → DAG
NUMDAG	numbers	discrete aggregates	NUMDAG: NUM → DAG
NUMWRD	numbers	number words	NUMWRD: NUM → WRD

APPENDIX A.2: Operations/mappings of relevance to addition and subtraction in Grade 1, Workbook 1 (DBE 2021a). Workbook *Tasks* (N=69).

Operations/mappings of relevance to addition and subtraction in the 2021a DBE Grade 1, Workbook 1 *Tasks* ($N = 69$)																			
TASKS			OPERATIONS/ MAPPINGS OVER DISCRETE AGGREGATES										OPERATIONS/MAPPINGS OVER NATURAL NUMBERS						
No.	Title	Page	GRP	BIJ$_{A,B}$	ORD$_{set}$		[S]		[S]	P_{set}	P_{base}	#	ORD$_{num}$	(+)	[+ n]	(−)	[− n]	P_{num}	P_{base}
5	Revision: Find and count	10	1	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0
8	Revision: Shapes, sizes and colours	16	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	One	18	1	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
10	Two	20	1	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
11	Three	22	1	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
12a	Length and position	24	1	1	1	1	0	0	0	0	0	1	1	0	0	0	0	0	0
12b	Length	26	1	1	1	0	0	0	0	0	0	1	1	1	0	0	0	0	0
13	Compare numbers 1 to 3	28	1	1	1	1	0	0	0	0	0	1	1	1	0	0	0	0	0
14	Four	30	1	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
15	Add and subtract up to 4	32	1	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0
17	Five	36	1	1	0	1	0	0	0	0	0	1	1	1	0	0	0	0	0
18	Revise numbers 1 to 5	38	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
19	Add up to 5	40	1	1	1	1	0	0	0	0	0	1	1	1	0	0	0	0	0
20	Subtract from 5 and add up to 5	42	1	1	1	0	0	0	1	0	0	1	1	0	0	0	1	0	0
21	Add and subtract up to 5	44	1	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0
22	Addition and subtraction 1 to 5	46	1	0	0	1	0	0	1	1	0	1	0	1	0	0	1	1	0
25	Building up and breaking down numbers	54	1	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
33	Six	70	1	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
34	Seven	72	1	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
35	Eight	74	1	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
36	Nine	76	1	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
38	Ten	80	1	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
39	Numbers 1 to 10	82	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
41	Number 1 to 10	86	1	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
42	More, equal and less	88	1	1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
43	Adding	90	1	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
44	Collecting and organising	92	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0
45	Addition to 10: counting on	94	1	0	0	1	1	0	0	0	0	1	0	1	1	0	0	0	0
46	Addition: building and breaking to 10	96	1	0	0	1	0	0	0	1	0	1	0	1	0	0	0	1	0
29	TOTALS		29	22	7	20	3	0	4	4	0	28	7	20	3	0	4	2	0
42,0%			100,0%	75,9%	24,1%	69,0%	10,3%	0,0%	13,8%	13,8%	0,0%	96,6%	24,1%	69,0%	10,3%	0,0%	13,8%	6,9%	0,0%
	Percentages wrt the total number of *Tasks* (69)		42,0%	31,9%	10,1%	29,0%	4,3%	0,0%	5,8%	5,8%	0,0%	40,6%	10,1%	29,0%	4,3%	0,0%	5,8%	2,9%	0,0%

APPENDIX A.3 : Operations/mappings of relevance to addition and subtraction in Grade 1, Workbook 2 (DBE 2021b). Workbook *Tasks* (N=64).

Operations/mappings of relevance to addition and subtraction in the 2021b Grade 1, Workbook 2 *Tasks* (N = 64)																			
TASKS			OPERATIONS/MAPPINGS OVER DISCRETE AGGREGATES										OPERATIONS/MAPPINGS OVER NATURAL NUMBERS						
No.	Title	Page	GRP	BIJ$_{A,B}$	ORD$_{\text{set}}$		[S]		[S]	P_{set}	P_{base}	#	ORD$_{\text{num}}$	(+)	[+ n]	(−)	[− n]	P_{num}	P_{base}
65	Understand number 11	2	1	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	1
66	Understand number 12	4	1	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	1
67	Understand number 13	6	1	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	1
68	Understand number 14	8	1	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	1
69	Understand number 15	10	1	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	1
70	Addition up to 20 – counting on	12	1	0	0	1	0	0	1	0	0	1	0	1	0	0	1	0	0
71	Addition – building up and breaking down numbers up to 10	14	1	0	0	1	0	0	0	1	0	1	0	1	0	0	0	1	0
72	Addition – building up and breaking down to 20	16	1	0	0	1	0	0	0	1	0	1	0	1	0	0	0	1	0
73	Addition and subtraction – building up and breaking down	18	1	0	0	1	0	0	1	1	1	1	0	1	0	0	1	1	1
97	Number 16	66	1	1	0	1	0	0	0	1	1	1	1	0	0	0	0	0	1
98	Number 17	68	1	1	0	1	0	0	0	1	1	1	1	0	0	0	0	0	1
99	Number 18	70	1	1	0	1	0	0	0	1	1	1	1	0	0	0	0	0	1
100	Number 19	72	1	1	0	1	0	0	0	1	1	1	1	0	0	0	0	0	1
101	Number 20	74	1	1	0	1	0	0	0	1	1	1	1	0	0	0	0	0	1
102	Addition	76	1	0	0	1	0	0	0	1	1	1	1	1	0	0	0	1	1
103	Subtraction	78	1	0	0	0	1	0	1	1	1	1	0	1	0	0	1	1	1
104	Addition and subtraction	80	1	1	0	1	1	1	0	1	0	1	1	1	1	1	0	0	0
105	Ordinal numbers	82	1	0	1	0	0	0	0	1	0	1	1	0	0	0	0	0	0
18 28,1%	TOTALS		18 100,0%	11 61,1%	1 5,6%	11 61,1%	2 11,1%	1 5,6%	3 16,7%	17 94,4%	13 72,2%	18 100,0%	13 72,2%	7 38,9%	1 5,6%	1 5,6%	3 16,7%	5 27,8%	13 72,2%
	Percentages wrt the total number of *Tasks* (64)		28,1%	17,2%	1,6%	17,2%	3,1%	1,6%	4,7%	26,6%	20,3%	28,1%	20,3%	10,9%	1,6%	1,6%	4,7%	7,8%	20,3%

APPENDIX A.4: Operations of relevance to addition and subtraction in Grade 2, Workbook 1 (DBE 2021c). Workbook *Tasks* (N= 68).

68	Operations/mappings of relevance to addition and subtraction in the 2021c DBE Grade 2, Workbook 1 *Tasks* (N = 68)																			
	TASKS			OPERATIONS/MAPPINGS OVER DISCRETE AGGREGATES										OPERATIONS/MAPPINGS OVER NATURAL NUMBERS						
	No.	Title	Page	GRP	$BIJ_{A,B}$	ORD_{set}		[S]		[S]	P_{set}	P_{base}	#	ORD_{num}	(+)	[+ n]	(−)	[− n]	P_{num}	P_{base}
	2	Counting	4	1	1	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0
	3	Numbers	6	1	1	0	1	0	0	0	1	1	1	0	1	0	0	0	0	0
	4	More Numbers	8	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
	5	Addition and subtraction	10	0	0	0	1	0	0	1	0	0	1	0	1	0	0	1	0	0
	16	Read and interpret	32	1	0	1	0	0	0	0	1	0	1	1	0	0	0	0	0	0
*	17	Before, after and between	34	1	1	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0
	18	Numbers 1-30	36	1	1	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
	21	Addition and subtraction	42	1	0	0	1	0	0	1	1	0	1	0	1	0	0	1	0	0
	23a	Addition	46	1	0	0	1	1	0	0	1	0	1	0	1	0	0	0	1	0
*	23b	Subtraction	48	1	0	0	0	0	0	1	1	0	1	0	0	0	0	1	0	0
	24	Some more addition	50	0	0	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
	33	Order and compare numbers: 1 - 40	68	1	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
	34	Order and compare numbers: 40 - 50	70	1	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
	35	Numbers: 40 - 50	72	1	1	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
*	37	Addition and subtraction up to 20	76	1	1	0	1	0	1	0	0	1	1	0	1	0	1	0	0	1
*	38	Addition and subtraction up to 50	78	1	1	0	1	0	1	0	0	1	1	0	1	0	1	0	0	1
	39a	More addition	80	1	0	0	0	1	0	0	1	1	1	0	0	1	0	0	0	1
	39b	More addition (continued)	82	1	0	0	0	1	0	0	1	1	1	0	0	1	0	0	0	1
	41	Subtraction	86	1	0	0	0	0	0	1	1	1	1	0	0	0	1	1	0	1
	42a	More subtraction	88	1	0	0	0	1	0	1	1	1	1	0	0	1	1	1	0	1
	42b	Even more subtraction	90	1	0	0	0	0	0	1	1	1	1	0	0	0	1	1	0	1
	21	TOTALS		18	9	3	11	4	2	6	12	11	21	4	11	3	5	6	1	10
	30,9%			85,7%	42,9%	14,3%	52,4%	19,0%	9,5%	28,6%	57,1%	52,4%	100,0%	19,0%	52,4%	14,3%	23,8%	28,6%	4,8%	47,6%
		Percentages wrt the total number of *Tasks* (68)		26,5%	13,2%	4,4%	16,2%	5,9%	2,9%	8,8%	17,6%	16,2%	30,9%	5,9%	16,2%	4,4%	7,4%	8,8%	1,5%	14,7%

APPENDIX A.5 Operations/mappings of relevance to addition and subtraction in Grade 2 Workbook 2 (DBE 2021d). Workbook *Tasks (*N=75).

Operations/mappings of relevance to addition and subtraction in the 2021d DBE Grade 2, Workbook 2 *Tasks* (N = 75)																			
TASKS			OPERATIONS/MAPPINGS OVER DISCRETE AGGREGATES										OPERATIONS/MAPPINGS OVER NATURAL NUMBERS						
No.	Title	Page	GRP	BIJ$_{A,B}$	ORD$_{\text{set}}$		[S]		[S]	P_{set}	P_{base}	#	ORD$_{\text{num}}$	(+)	[$+n$]	(−)	[$-n$]	P_{num}	P_{base}
65	Numbers 50 to 99	2	1	1	0	0	0	0	0	0	1	1	1	0	0	0	0	0	1
66	Numbers 100 to 150	4	1	1	0	0	0	0	0	0	1	1	1	0	0	0	0	0	1
69	Numbers 150 to 170	10	1	1	0	0	0	0	0	0	1	1	1	0	0	0	0	0	1
70	Counting and estimating (0 - 100)	12	1	0	0	1	0	1	1	0	1	1	0	1	0	1	1	0	1
72	Addition 0 to 50	16	1	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
73	Addition 0 to 75	18	1	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	1
74	More addition: 0 to 75	20	1	1	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
77	More addition and subtraction 0 to 75	26	0	1	0	1	0	0	0	0	1	1	0	1	0	1	0	0	1
97	Numbers 150 to 180	72	1	1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
98	Numbers 170 to 200	74	1	1	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
101	Addition and subtraction	80	1	1	d	1	0	1	0	0	1	1	0	1	0	1	0	0	1
102	Addition and subtraction again	82	1	0	0	0	1	0	1	0	1	1	0	0	1	0	1	0	1
104	More addition and subtraction	86	1	0	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1
13 17,3%	TOTALS		12 92,3%	8 61,5%	0 0,0%	9 69,2%	2 15,4%	3 23,1%	3 23,1%	1 7,7%	11 84,6%	13 100,0%	4 30,8%	9 69,2%	2 15,4%	4 30,8%	3 23,1%	1 7,7%	12 92,3%
	Percentages wrt the total number of *Tasks* (68)		16,0%	10,7%	0,0%	12,0%	2,7%	4,0%	4,0%	1,3%	14,7%	17,3%	5,3%	12,0%	2,7%	5,3%	4,0%	1,3%	16,0%

APPENDIX A.6: Operations/mappings of relevance to addition and subtraction in Grade 3 Workbook 1 (DBE 2021e). Workbook *Tasks (*N=71).

Operations/mappings of relevance to addition and subtraction in the 2021e DBE Grade 3, Workbook 1 *Tasks* ($N = 71$)																			
TASKS			OPERATIONS/MAPPINGS OVER DISCRETE AGGREGATES										OPERATIONS/MAPPINGS OVER NATURAL NUMBERS						
No.	Title	Page	GRP	$\mathrm{BIJ}_{A,B}$	$\mathrm{ORD}_{\mathrm{set}}$		$[\ S]$		$[\ S]$	P_{set}	P_{base}	#	$\mathrm{ORD}_{\mathrm{num}}$	$(+)$	$[+n]$	$(-)$	$[-n]$	P_{num}	P_{base}
1	Count, sort and show	2	1	1	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0
4	Place value	10	1	1	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
18	Place value to 99	38	1	1	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
19	Putting tens together when we add to 99	40	1	1	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
35a	Putting tens together and taking them apart	80	1	1	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
35b	Putting tens together and taking them apart	82	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1
42	Adding and subtracting with 100s	98	1	1	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
7	TOTALS		7	7	0	7	0	0	1	1	6	7	0	7	0	0	1	0	6
9,9%			100,0%	100,0%	0,0%	100,0%	0,0%	0,0%	14,3%	14,3%	85,7%	100,0%	0,0%	100,0%	0,0%	0,0%	14,3%	0,0%	85,7%
	Percentages wrt the total number of *Tasks* (71)		9,9%	9,9%	0,0%	9,9%	0,0%	0,0%	1,4%	1,4%	8,5%	9,9%	0,0%	9,9%	0,0%	0,0%	1,4%	0,0%	8,5%

APPENDIX A.7: Operations/mappings of relevance to addition and subtraction in Grade 3 Workbook 2 (DBE 2021f). Workbook Tasks (N=67).

Operations/mappings of relevance to addition and subtraction in the 2021f DBE Grade 3, Workbook 2 *Tasks* ($N = 67$)																			
TASKS			OPERATIONS/MAPPINGS OVER DISCRETE AGGREGATES										OPERATIONS/MAPPINGS OVER NATURAL NUMBERS						
No.	Title	Page	GRP	$BIJ_{A,B}$	ORD_{set}		$[\ S]$		$[\ S]$	P_{set}	P_{base}	#	ORD_{num}	(+)	$[+n]$	(−)	$[-n]$	P_{num}	P_{base}
65	Numbers 500 to 600	2	1	1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
66	More numbers 500 to 600	4	1	1	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
67	Numbers 600 to 700	6	1	1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
69	More numbers 600 to 700	10	1	1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
70	Numbers 650 to 750	12	1	1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
71	Numbers 700 to 750	14	1	1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
98	Numbers 700 to 800	70	1	1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
99	More numbers 800 to 800	72	1	1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
100	Numbers 800 to 900	74	1	1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
101	More numbers 800 to 900	76	1	1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
103	Numbers 900 to 1000	82	1	1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
104	More numbers 900 to 1000	84	1	1	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1
105	Addition and subtraction to 999	86	1	1	0	1	0	1	0	0	1	1	0	1	0	1	0	0	1
108	More addition and subtraction to 999	92	1	1	0	1	0	0	0	0	1	1	0	1	0	0	0	0	1
109	Addition and subtraction to 999 again	94	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0
15	TOTALS		14	14	0	14	0	1	0	0	14	14	11	15	0	2	0	1	14
22,4%			93,3%	93,3%	0,0%	93,3%	0,0%	6,7%	0,0%	0,0%	93,3%	93,3%	73,3%	100,0%	0,0%	13,3%	0,0%	6,7%	93,3%
	Percentages wrt the total number of *Tasks* (67)		20,9%	20,9%	0,0%	20,9%	0,0%	1,5%	0,0%	0,0%	20,9%	20,9%	16,4%	22,4%	0,0%	3,0%	0,0%	1,5%	20,9%