A Space-Time Approach to Quantum Mechanics

A Thesis
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by

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Abstract

We present a systematic development and application of Geometric Algebra, an extended vector calculus.

The entire algebraic structure, which is a graded Clifford algebra, is developed.

To illustrate the derived results, examples are given for two and three dimensions. Here it becomes clear, how rotations and Lorentz boosts can be fomulated in the Geometric Algebra. Further we realize that the Geometric Algebra contains elements, which can be used as representations of the complex unit j.

Having derived the necessary tools, we turn our attention to physics. We give applications to classical mechanics, quantum mechanics, field theory, curved manifolds, electromagnetism, and gravity as a gauge theory.

Contents

Introduction and Motivation

Motivation

Geometric Algebra is an extended vector calculus with the structure of a graded Clifford algebra. A few axioms for an associative product lead to an algebra, which includes the conventional vector calculus as a substructure, but goes far beyond this point. The major property, and *advantage*, of this algebra is the existence of an associative product — *i.e.*, suddenly the power and the inverse[1] of a vector are well-defined. This alone makes this algebra interesting enough from the mathematical point of view, however, it falls to physics to fully reveal the power of Geometric Algebra.

The incorporation of the Dirac and Pauli algebras in a natural way has to be mentioned here as one of the most impressive features. This leads to a geometric interpretation of the Dirac and Pauli matrices — they are nothing else than orthonormal unit vectors in four and three dimensions, respectively. This does not change the mathematical structure of quantum mechanics, but gives a much clearer geometrical interpretation — to my mind, exactly what is missing in conventional approaches.

In conventional treatments the *complex unit* is taken as a pure mathematical object, with no geometrical or physical interpretation. Clearly this is an unsatisfactory situation. Geometric Algebra, however, implies geometrically meaningful quantities with exactly the same behaviour[2]. Indeed, the complex unit appearing in the Dirac Theory can now be identified as a *bivector*, representing the plane of the spin — so the complex structure of Dirac Theory is directly related to the spin of the treated particles.

A Geometric Algebra representation of the Dirac spinor employes only real coefficients and yields a physically meaningful decomposition — in each space-time point the spinor defines a Lorentz-rotation and dilation. It is this new interpretation, which leads to a different understanding of quantum mechanics. Operators and state vectors change their role, *e.g.*, this explains the single sided transformation law for spinors. The ease with which one derives this results is striking.

These are just a few examples of the ability of the Geometric Algebra to clarify geometrical relations. Through the algebra the mathematical or physical content becomes clearer and this makes it more accessible for interpretation.

Outline of the Thesis

In Section 1 we show how the Geometric Algebra arises from simple axioms for an associative Geometric Product for vectors. Vector identities are derived and objects of a higher grade are associated with a geometrical interpretation. A number of useful mathematical applications are considered, *e.g.*, linear functions, determinants, rotations, etc. A similar treatment can be found in [1]. In Section 2 examples are given for vector spaces of two and three dimensions. In Section 3 we construct the Geometric Algebra of our four-dimensional space-time, *i.e.*, the Minkowski space.

In the following Sections (4 to 9) we examine possible applications to physics. Starting with classical mechanics we find a representation of Hamilton's equations, which unifies the p and q-equations.

In Section 5 we take a short look at electromagnetism. We show how Maxwell's equations are reduced to one equation and how this can be extended to magnetic monopoles. Further, the constraint of invariance under local phase transformations of the Dirac equation leads us to a curvature closely related to the Faraday tensor.

Turning our attention to quantum mechanics, we find representations and interpretations of spinors and operators. We translate the Dirac equation into a Geometric Algebra form and show how the Schrödinger equation arises as the non-relativistic limit.

Going a step further to quantum field theory, we show how the variational principle and Noether's theorem can be applied to derive the energy-momentum and angular-momentum tensor for quantum fields. For the Dirac case, conjugate currents are calculated and the Dirac current is shown to be conserved. Most of the results presented here were derived before in [2].

[1] With the exception of null vectors, which do not have an inverse.

[2] An excellent, simple presentation of the relation between the complex unit and Geometric Algebra is given in [26]. But you might also be referred to Section 2.1.1 in the work.

For the Dirac theory we discuss in detail the decomposition of the spinor into spin-energy states, find a Fourier representation of the wave-function and subsequently give the momentum in terms of the Fourier amplitudes.

An application to curved manifolds shows how General Relativity can be formulated in a coordinate free manner. Here the quantities which occur depend on vectors, not on their representations, and are thus automatically covariant. We derive a number of useful translations; *e.g.*, the Bianchi identities are given in a compact form, demonstrating the power of Geometric Algebra.

In Section 9 we give a review of gravity as a gauge theory. This theory was developed by the group around A. Lasenby, S. Gull, and C. Doran in Cambridge [3], [4] and [5]. We construct new derivative operators, which are invariant under arbitrary local position and rotation transformations. An associated curvature tensor is defined. We assume a certain Lagrangian and derive the resulting field equations, one of which becomes the analogue of the Einstein field equation. Further more, the Weyl tensor is identified.

Large parts of this thesis review known results, with appropriate citations given in the text. To the best of my knowledge new results are the definition of determinants of non-commuting quantities given in Section 1.3.1 with the associated results, Hamilton's equations in the form (483), the (anti-) particle spin up and down sections of the Dirac spinor, expressed with the projection operator (599), the explicit formula for the mixing angle of the whole wave (837), and the generalized duality relations of inner and outer product with the vector derivative (883).

Words of Thanks

Most of all I have to thank my loved parents, who supported me in every decision I made. It was their trust in my abilities that took me to where I am.

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1 Mathematical Foundation

1.1 Geometric Algebra

1.1.1 The Geometric Product

The Idea The common vector calculus is not always very satisfactory. As the most important example we might look at the three-dimensional case, where we have two vector products.

- The scalar, inner (or dot) product $\vec{a} \cdot \vec{b}$, which defines orthogonality via

$$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b} \quad . \tag{1}$$

- The outer or cross product $\vec{a} \times \vec{b}$.

Both are not totally satisfactory, for the following reasons:

- Neither product is associative. $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ does not make sense at all, since $\vec{a} \cdot \vec{b}$ is a scalar, and for the cross product we have in general

$$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c}) \quad . \tag{2}$$

- The cross product can only be defined in three dimensions. Otherwise two linearly independent vectors do not define a unique direction orthogonal to them.

In more than three dimensions we have problems to define an analogue to the cross product. Hence it appears to be a natural step to look for a more fundamental product, *a product which is associative.*

To do this, we employ a *graded algebra.* Scalars and vectors are elements of certain grade in this algebra, but a general element, which is called a *multivector*, can be a mixture of different grades.

The Geometric Product Our vector calculus should at least be able to handle scalars and vectors. To avoid confusion we will denote scalars by small Greek letters (*e.g.* α, β, γ) and vectors by small Latin letters[3] (*e.g.* a, b, c).

After demanding, that *scalars and vectors* should *commute*, *i.e.*,

$$\lambda a = a\lambda \quad , \tag{3}$$

we define the **geometric product of vectors** by postulating

- **associativity** *i.e.* $(ab)c = a(bc)$ for arbitrary vectors a, b, c,
- **distributivity** $a(b + c) = ab + ac$,
- that the **square of a vector is a scalar**[4], that is

$$aa = a^2 = \text{scalar} \quad , \tag{4}$$

- given n linearly independent vectors, we should not be able to construct more linearly independent vectors by multiplication[5].

The Geometric Algebra Thus the linear space of all geometric products of vectors is a *ring.* We call the associated algebra over the real numbers (scalars), obeying for any geometric products of vectors X, Y

$$\left.\begin{array}{rclcrcl} (\alpha+\beta)X &=& \alpha X + \beta X &;& \alpha(X+Y) &=& \alpha X + \alpha Y \\ (\alpha\beta)X &=& \alpha(\beta X) &;& 1X &=& X \\ \alpha(XY) &=& (\alpha X)Y &=& X(\alpha Y) && \end{array}\right\} , \tag{5}$$

the **Geometric Algebra.**

[3]We will later use an arrow to indicate a vector of a three-dimensional space (*e.g.* $\vec{a}, \vec{b}, \vec{c}$).

[4]Note that this does not fix the signature. Vectors are still allowed to have positive, negative or zero square.

[5]The usual cross product would not satisfy this axiom, since $a \times b$ would be linearly independent to a and b. Even when we are in three dimensions, $a \times b$ should be part of the algebra of two vectors and should thus not contain any information about a third linearly independent vector. Demanding this will lead to a graded structure of the algebra, in which directed plane segments have their own representation.

1.1.2 Inner and Outer Product of Vectors

The Inner Product of Vectors The most important thing we need is a Inner Product, which maps two vectors to a scalar. Only with this product are we able to define orthogonality and a metric. According to (4) we know for two *vectors* $\mathfrak{a}$ and $\mathfrak{b}$

$$\text{scalar} = (\mathfrak{a}+\mathfrak{b})^2 = \underbrace{\mathfrak{a}^2+\mathfrak{b}^2}_{\text{scalar}} + \underbrace{\mathfrak{a}\mathfrak{b}+\mathfrak{b}\mathfrak{a}}_{\Rightarrow\text{scalar}} \tag{6}$$

and define thus the **inner product of vectors** as

$$\mathfrak{a}\cdot\mathfrak{b} \overset{\text{def}}{=} \frac{\mathfrak{a}\mathfrak{b}+\mathfrak{b}\mathfrak{a}}{2} = \text{scalar} \quad . \tag{7}$$

Commutation and Anticommutation Relations The inner product of vectors (7) defines now orthogonality via

$$\mathfrak{a} \text{ orthogonal to } \mathfrak{b} \quad \overset{\text{def}}{\Leftrightarrow} \quad \mathfrak{a}\cdot\mathfrak{b} = 0 \quad \overset{(7)}{\Leftrightarrow} \quad \mathfrak{a}\mathfrak{b} = -\mathfrak{b}\mathfrak{a} \quad . \tag{8}$$

If $\mathfrak{a}$ and $\mathfrak{b}$ are *parallel*, *i.e.*, $\mathfrak{b} = \lambda\mathfrak{a}$ with a scalar λ, we get on the other side

$$\mathfrak{a}\mathfrak{b} = \mathfrak{a}\lambda\mathfrak{a} \overset{(3)}{=} \lambda\mathfrak{a}\mathfrak{a} = \mathfrak{b}\mathfrak{a} \overset{(4)}{=} \text{scalar} \tag{9}$$

and define thus

$$\mathfrak{a} \text{ parallel } \mathfrak{b} \quad \Leftrightarrow \quad \mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a} \quad \overset{(7)}{\Leftrightarrow} \quad \mathfrak{a}\cdot\mathfrak{b} = \mathfrak{a}\mathfrak{b} \quad . \tag{10}$$

For parallel vectors we have

$$(\mathfrak{a}\cdot\mathfrak{b})^2 \overset{(10)}{=} \mathfrak{a}\mathfrak{b}\mathfrak{a}\mathfrak{b} \overset{(10)}{=} \mathfrak{a}\mathfrak{b}\mathfrak{b}\mathfrak{a} = \mathfrak{a}^2\mathfrak{b}^2 \quad . \tag{11}$$

The Outer Product The scalar product of vectors (7) is the symmetric part of the geometric product. Thus we define the **outer product** of vectors as the remaining antisymmetric part

$$\mathfrak{a}\wedge\mathfrak{b} \overset{\text{def}}{=} \frac{\mathfrak{a}\mathfrak{b}-\mathfrak{b}\mathfrak{a}}{2} \tag{12}$$

and write

$$\mathfrak{a}\mathfrak{b} = \mathfrak{a}\cdot\mathfrak{b} + \mathfrak{a}\wedge\mathfrak{b} \quad . \tag{13}$$

Thus

$$\mathfrak{a}\wedge\mathfrak{b} \overset{(12)}{=} \tfrac{\mathfrak{a}\mathfrak{b}-\mathfrak{b}\mathfrak{a}}{2} = 0 \quad \overset{(10)}{\Leftrightarrow} \quad \mathfrak{a} \parallel \mathfrak{b} \quad . \tag{14}$$

The outer product is totally defined by the orthogonal parts of the vectors, as can be seen by substituting $\mathfrak{b} = \mathfrak{b}_{\parallel} + \mathfrak{b}_{\perp}, \mathfrak{b}_{\parallel} \parallel \mathfrak{a}, \mathfrak{b}_{\perp} \perp \mathfrak{a}$ in (12) to get

$$\mathfrak{a}\wedge\mathfrak{b} = \mathfrak{a}\wedge(\mathfrak{b}_{\parallel}+\mathfrak{b}_{\perp}) \overset{(14)}{=} \mathfrak{a}\wedge\mathfrak{b}_{\perp} \overset{(13)}{=} \mathfrak{a}\mathfrak{b}_{\perp} \quad . \tag{15}$$

Thus $\mathfrak{a}\wedge\mathfrak{b}$ *is always equivalent to the product of two orthogonal vectors.* On the other side (8) and (13) tell us that the product of two *orthogonal* vectors $\mathfrak{a},\mathfrak{b}$ is always

$$\mathfrak{a}\mathfrak{b} \overset{(13)}{=} \mathfrak{a}\wedge\mathfrak{b} \quad . \tag{16}$$

1.1.3 The graded Structure

More than Vectors and Scalars $\mathfrak{a}\wedge\mathfrak{b}$ cannot be a scalar, since there are vectors with which it does not commute:

$$\left.\begin{aligned} (\mathfrak{a}\wedge\mathfrak{b})\mathfrak{a} &= \tfrac{1}{2}\Big[\mathfrak{a}\mathfrak{b}\mathfrak{a} - \mathfrak{b}\underbrace{\mathfrak{a}\mathfrak{a}}_{\text{scalar}}\Big] = \tfrac{\mathfrak{a}\mathfrak{b}\mathfrak{a}-\mathfrak{a}\mathfrak{a}\mathfrak{b}}{2} = -\mathfrak{a}(\mathfrak{a}\wedge\mathfrak{b}) \\ (\mathfrak{a}\wedge\mathfrak{b})\mathfrak{b} &= -\mathfrak{b}(\mathfrak{a}\wedge\mathfrak{b}) \end{aligned}\right\} \quad . \tag{17}$$

If $a \wedge b$ would be a vector, (17) would tell us that this vector is (according to (8)) orthogonal to a and b. But according to our last axiom, $a \wedge b$ cannot be a vector linearly independent[6] of a and b. We thus have to conclude, that $a \wedge b$ is a new object, which we will call **bivector**. We say it has **grade** 2 in compare to **grade** 0 **for scalars** and **grade** 1 **for vectors**.

But even without assuming the last axiom, we can show, that the assumption of $a \wedge b$ being a vector leads to a contradiction, if we assume, that a Geometric Algebra exists for each choice of signature. Let's for the minute assume a space with

$$a^2 \geq 0 \qquad \forall \text{ vectors } a \quad . \tag{18}$$

Let a and b be *orthogonal* vectors[7]. We assume now, that

$$c \stackrel{\text{def}}{=} ab \tag{19}$$

is a vector. (19) yields then

$$c^2 = cc \stackrel{(19)}{=} abab = -abba = -b^2a^2 \quad . \tag{20}$$

But we assumed with (18) that $c^2 > 0$. Thus ab can not be a vector.

The product of two *orthogonal* vectors a and b is then always a bivector, since here $a \cdot b = 0$ and so $ab \stackrel{(13)}{=} a \wedge b$. Bivectors can be seen as representing the plane spanned by two vectors. A bivector anticommutes with vectors in this plane, since (17) yields

$$(a \wedge b) \overbrace{(\alpha a + \beta b)}^{\text{arbitrary vector in plane}} = -(\alpha a + \beta b)(a \wedge b) \quad . \tag{21}$$

On the other side it commutes with vectors orthogonal to the plane. If c is orthogonal to a and b

$$(a \wedge b)c = \frac{abc - bac}{2} \stackrel{(8)}{=} \frac{-acb + bca}{2} \stackrel{(8)}{=} \frac{cab - cba}{2} = c(a \wedge b) \quad . \tag{22}$$

Similarly we can conclude that the geometric product of three *orthogonal* vectors is no scalar, vector or bivector. Let a, b, c be orthogonal to each other. If $abc = \lambda$ would be a scalar, then λc must be a vector, but

$$\lambda c = (abc)c = abc^2 = \text{bivector} \;\Rightarrow abc \text{ no scalar.} \tag{23}$$

Further we can conclude from

$$(abc)a \stackrel{(8)}{=} a(abc) \;\; ; \;\; (abc)b \stackrel{(8)}{=} b(abc) \;\; ; \;\; (abc)c \stackrel{(8)}{=} c(abc) \quad , \tag{24}$$

that abc is no vector, since a vector can not be parallel to two orthogonal vectors, and no bivector, since bivectors anticommute with vectors in their plane[8].

This process can be continued to any number n of orthogonal vectors. Each time we have to identify the product as a new object and we say it is of **grade** n.

Simple n-Vectors We call the geometric product of n *orthogonal* vectors a **simple n-vector** A_n and say it is of grade n. Objects of *different grade are always linearly independent.* Obviously A_0 is a scalar and A_1 a vector. Further we identify according to above discussion A_2 as a bivector, A_3 as a trivector, etc.

The Pseudoscalar In a n-dimensional linear vector space we cannot find more than n orthogonal vectors. Thus we cannot construct a simple m-vector of grade m higher than n. A simple n-vector is of highest possible grade. Given an orthonormal basis a_j, we call

$$I_n \stackrel{\text{def}}{=} a_1 \ldots a_n \tag{25}$$

the **pseudoscalar** of the n-dimensional vector space. Since every vector is a linear combination of the basis vectors, *each object of grade* n *must be a multiple of the pseudoscalar* I_n.

[6] If $a \wedge b$ would be a vector, then there would not be a algebra of a two-dimensional space. $a \wedge b$ would always give a third linearly independent vector.

[7] Remember that $a \wedge b$ is always the product of two orthogonal vectors.

[8] We assumed three orthogonal vectors. The product abc commutes with all of them. For a bivector there would have to be at least two anticommuting vectors, which span the plane of the bivector.

Interpretation of higher grade Objects Every simple r-vector $A_r = a_1 \dots a_r$ can be seen as the pseudoscalar of a r-dimensional subspace spanned by $\{a_1, \dots, a_r\}$. Later we will see, how the pseudoscalar can be used to define the (sub-)space in an easy way.

Dimensions Given a n-dimensional linear vector space, we can find n orthonormal vectors. Out of them we can construct $\binom{n}{2} = \frac{n(n-1)}{2}$ bivectors, $\binom{n}{3}$ trivectors, etc. Thus the objects of grade r span a linear vector space with $\binom{n}{r}$ dimensions. Altogether we have then

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n \tag{26}$$

linearly independent objects. This can be seen as a 2^n-dimensional linear space.

Multivectors Going back to the decomposition (13) we recognize that the product ab contains a scalar and a bivector, objects of different grade. We want to introduce a generalized quantity of this kind.

A **multivector** is an arbitrary sum of simple vectors of possibly different grade. So

$$A = \text{scalar} + \text{vector} + \text{bivector} + \text{trivector} + \dots \tag{27}$$

is the most general form of a multivector. We will denote multivectors by capital Latin letters (*e.g.* M), except where a physical interpretation suggests another name.

We say a multivector is of grade r, if it is the sum of simple r-vectors (for just one r). So it can not contain any other grade than r.

The Grade Operator According to its definition, a general multivector appears as a linear combination of simple vectors of different grade. Often it is useful to project out the part of a multivector, which is of a certain grade. This motivates us to define the **grade operator**

$$\langle M \rangle_n \quad ; \quad \langle M \rangle \stackrel{\text{def}}{=} \langle M \rangle_0 \quad , \tag{28}$$

which gives just the part of a multivector M with grade n. We can then write any multivector M as

$$M = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \dots = \sum_{n=0}^{\infty} \langle M \rangle_n \quad . \tag{29}$$

With this definition a multivector of grade r is defined by

$$M = \langle M \rangle_r \Leftrightarrow \langle M \rangle_i = 0 \quad \forall i \neq r \quad . \tag{30}$$

1.1.4 Basis

The different Spaces We will denote by $\mathcal{A}_n$ a n-dimensional linear vector space. The linear space of all multivectors constructed out of all vectors in $\mathcal{A}_n$ by multiplication and addition is then called $\mathcal{G}(\mathcal{A}_n)$. The linear vector space of grade r multivectors is further denoted by $\mathcal{G}^r(\mathcal{A}_n)$.

Basis of the Vector Space We call a set $\{e_\mu\}$ of n linearly independent vectors in $\mathcal{A}_n$ a basis of $\mathcal{A}_n$. If this is further an orthonormal set of vectors[9], *i.e.*,

$$|e_\mu \cdot e_\nu| = \delta_{\mu\nu} \quad , \tag{31}$$

we call it an **orthonormal basis** or, in four dimensions, a **tetrad**.

[9]Note that this does not exclude the case $e_\mu e_\mu = -1$, as it employs the norm.

The Reciprocal Basis Let $\{e_\nu : 1 \leq \nu \leq n\}$ be a set of n linearly independent vectors, *i.e.*, a basis of $\mathcal{A}_n$. We call then the set of n vectors defined by

$$\{e^\nu : 1 \leq \nu \leq n, e^\mu \cdot e_\nu = \delta^\mu_\nu\} \tag{32}$$

the **reciprocal basis**. It obeys by construction the orthogonality relation

$$\sum_{\nu,\mu=1}^{n} e^\nu \cdot e_\mu = \sum_{\nu,\mu=1}^{n} \delta^\nu_\mu = n \quad . \tag{33}$$

For an *orthonormal* system $e_\mu \cdot e_\nu = \pm\delta^\mu_\nu$ and consequently we can construct the reciprocal vectors by

$$e^\mu = e_\mu^2 e_\mu = \pm e_\mu \quad . \tag{34}$$

We define thus the quantity

$$\eta^{\mu\nu} \stackrel{\text{def}}{=} \eta(e^\mu, e^\nu) = e^\mu \cdot e^\nu \tag{35}$$

and allow the raising and lowering of indices, *i.e.*, the transition from the basis to the reciprocal basis,

$$e^\nu = \textstyle\sum_{\mu=1}^{n} e_\mu \eta^{\mu\nu} \quad ; \quad e_\nu = \sum_{\mu=1}^{n} e^\mu \eta_{\mu\nu} \quad . \tag{36}$$

Each vector a can then be written as a linear combination of the basis or the reciprocal basis, *i.e.*,

$$a = \sum_{\nu=1}^{n} a^\nu e_\nu = \sum_{\nu=1}^{n} a_\nu e^\nu \quad , \tag{37}$$

where we defined the scalar vector components with respect to the basis or reciprocal basis, respectively, by[10]

$$a^\nu \stackrel{\text{def}}{=} a \cdot e^\nu \quad ; \quad a_\nu \stackrel{\text{def}}{=} a \cdot e_\nu \quad . \tag{38}$$

Basis for higher Grade Multivectors and Multi Indices Each combination of r orthogonal vectors gives a linearly independent simple r-vector. Thus the basis of grade r-multivectors has $\binom{n}{r}$ elements. For example the $\binom{n}{2}$ bivectors $\{e_\nu \wedge e_\mu : 1 \leq \nu < \mu \leq n\}$ give a basis of the linear space of grade two multivectors.
To abbreviate, we define the multi index notation

$$e_0 \stackrel{\text{def}}{=} 1 \quad ; \quad e_{\mu\cdots\nu} \stackrel{\text{def}}{=} e_\mu \wedge \cdots \wedge e_\nu \tag{39}$$

and define the **multi index** J_r, which labels the basis of grade r elements. Thus

$$J_0 = \{0\} \quad ; \quad J_1 = \{1, \ldots, n\} \quad ; \quad J_2 = \{\mu\nu : 1 \leq \mu < \nu \leq n\} \quad ; \quad \ldots \tag{40}$$

and J_r runs over $\binom{n}{r}$ elements. A grade r-multivector $\langle M \rangle_r$ becomes now

$$\langle M \rangle_r = \sum_{J_r} e_{J_r} M^{J_r} \tag{41}$$

with scalar components M^{J_r}. A general multivector M is then given by

$$M = \sum_{r=0}^{n} \sum_{J_r} e_{J_r} M^{J_r} = \sum_{J} e_J M^J \quad , \tag{42}$$

where we defined the multi index J, which runs over the set $\{J_0, \ldots, J_n\}$.

[10]Here we do not follow the rule to indicate scalars by Greek letters. This is just to indicate, that the a^μ are the components of the vector a.

1.1.5 Generalization of Inner and Outer Product

The Product with a Vector Taking the geometric product of a simple r-vector $A_r = a_1 \dots a_r$, where the a_i are orthogonal to each other, with an arbitrary vector b, which can be decomposed

$$b = b_\perp + \textstyle\sum_{n=1}^{r} b_n \quad ; \quad b_n \stackrel{\text{def}}{=} \frac{(b \cdot a_n)}{a_n^2} a_n \quad \Rightarrow \quad b_n a_n = \text{scalar} \quad , \tag{43}$$

so that b_r is parallel to a_r, yields

$$\begin{aligned} a_1 \cdots a_r b &= a_1 \cdots a_r b_1 + \cdots + a_1 \cdots a_r b_r + a_1 \cdots a_r b_\perp \\ &\stackrel{(8)}{=} \sum_{s=1}^{r} (-1)^{r-s} \underbrace{a_1 \cdots \breve{a}_s \cdots a_r}_{\text{grade } r-1} \overbrace{(a_s b_s)}^{\stackrel{(43)}{=} \text{scalar}} + \underbrace{a_1 \cdots a_r b_\perp}_{\text{grade } r+1} \quad , \end{aligned} \tag{44}$$

where the $\breve{a}_s$ has to be skipped. So the product of a simple r-vector with a vector contains the grades $r-1$ and $r+1$. This extends via linearity immediately to every multivector of grade r. Remember, that the inner product of two vectors (7) is a scalar, which is grade 0, and the outer product (12) is a bivector, this is grade 2. It is thus natural to extend the definition of the outer and inner product to

$$a \cdot A_r \stackrel{\text{def}}{=} \langle a A_r \rangle_{r-1} \quad ; \quad a \wedge A_r \stackrel{\text{def}}{=} \langle a A_r \rangle_{r+1} \quad , \tag{45}$$

so that for every multivector M and vector a

$$aM = a \cdot M + a \wedge M \quad . \tag{46}$$

The interpretation of the outer product as raising the grade and of the inner product as lowering the grade allows us to generalize this further for arbitrary *simple* multivectors to

$$A_r \cdot B_s \stackrel{\text{def}}{=} \langle A_r B_s \rangle_{|r-s|} \quad ; \quad A_r \wedge B_s \stackrel{\text{def}}{=} \langle A_r B_s \rangle_{r+s} \quad . \tag{47}$$

Via linearity these products are now also *defined for arbitrary multivectors.*

Associativity of the Outer Product It is important to note, that the associativity of the geometric product yields for simple n, m, l-vectors A_n, B_m, C_l

$$\begin{aligned} \langle A_n B_m C_l \rangle_{n+m+l} &\stackrel{(47)}{=} \langle A_n B_m \rangle_{n+m} \wedge C_l \stackrel{(47)}{=} (A_n \wedge B_m) \wedge C_l \\ &\stackrel{(47)}{=} A_n \wedge \langle B_m C_l \rangle_{m+l} \stackrel{(47)}{=} A_n \wedge (B_m \wedge C_l) \quad . \end{aligned} \tag{48}$$

This extends via linearity to outer products of multivectors. So *the outer product is associative,* that means for any multivectors M, N, K

$$(M \wedge N) \wedge K = M \wedge (N \wedge K) \quad . \tag{49}$$

Together with the anticommutation relation $a \wedge b = -b \wedge a$ this yields

$$a_1 \wedge \ldots \wedge a_r = 0 \Leftrightarrow \exists\, i, j \quad \text{with} \quad a_i \wedge a_j = 0 \quad . \tag{50}$$

Thus *the outer product of* n *vectors is zero, if one of them is not linearly independent of the others.*

Decomposition of the Geometric Product We will show now, that the geometric product can be decomposed as

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \ldots + \langle A_r B_s \rangle_{r+s-2} + \langle A_r B_s \rangle_{r+s} \quad . \tag{51}$$

The proof can be done by induction. According to (46) the decomposition (51) is true for $r = 1, s \geq r$. For the case $r+1$, $s \geq r+1$ it follows then with $A_{r+1} \stackrel{\text{def}}{=} a_{r+1} A_r$

$$\begin{aligned} a_{r+1} A_r B_s &\stackrel{(51)}{=} a_{r+1} \{ \langle A_r B_s \rangle_{s-r} + \langle A_r B_s \rangle_{s-r+2} + \cdots + \langle A_r B_s \rangle_{r+s} \} \\ &\stackrel{(46)}{=} \underbrace{a_{r+1} \cdot \langle A_r B_s \rangle_{s-r}}_{\stackrel{(47)}{=} \langle a_{r+1} A_r B_s \rangle_{s-r-1}} + \underbrace{a_{r+1} \wedge \langle A_r B_s \rangle_{s-r} + a_{r+1} \cdot \langle A_r B_s \rangle_{s-r+2}}_{\stackrel{(47)}{=} \langle a_{a+1} A_r B_s \rangle_{s-r+1}} + \cdots \\ &\stackrel{(45)}{=} \langle a_{r+1} A_r B_s \rangle_{s-r-1} + \langle a_{r+1} A_r B_s \rangle_{s-r+1} + \cdots + \langle a_{r+1} A_r B_s \rangle_{s+r+1} \quad . \end{aligned} \tag{52}$$

But this is (51) for $r+1$. Thus (51) is proved.

Let us assume $r < s$. The geometric product between two arbitrary simple r, s-vectors can then be written as

$$A_r B_s = a_1 \dots a_r b_1 \dots b_s = a_1 \dots a_{r-1} \overbrace{a_r b_1}^{=\text{grade } 0 \,+\, \text{grade } 2} b_2 \dots b_s \quad . \tag{53}$$

Hence the part of lowest grade, *i.e.*, the inner product between A_r and B_s, becomes

$$A_r \cdot B_s \overset{(47)}{=} \langle A_r B_s \rangle_{s-r} = (a_r \cdot b_1) \dots (a_1 \cdot b_{r-s}) b_{r-s+1} \dots b_s \quad , \tag{54}$$

while the part of highest grade of the product is simply

$$A_r \wedge B_s \overset{(47)}{=} \langle A_r B_s \rangle_{s+r} = a_1 \wedge \dots \wedge a_r \wedge b_1 \wedge \dots \wedge b_s \quad . \tag{55}$$

Thus for an algebra of a n-dimensional vector space

$$A_r \wedge B_s = 0 \qquad \text{if } r+s > n \quad , \tag{56}$$

since in this case at least $r+s-n$ vectors must be linearly dependent on the others.

Reversion We saw that geometric relations expressed themself in commutation and anticommutation relations, *e.g.*, parallel vectors commute, while perpendicular vectors anticommute. The reversion of the order of a geometric product seems thus a useful operation. Hence we define the **operation of reversion** by

$$\left.\begin{array}{lclclcl} \widetilde{(AB)} & = & \widetilde{B}\widetilde{A} & ; & \widetilde{(A+B)} & = & \widetilde{A}+\widetilde{B} \\ \widetilde{\lambda} & = & \lambda & ; & \widetilde{a} & = & a \end{array}\right\} \quad . \tag{57}$$

For the geometric product of n arbitrary vectors this leads to

$$(a_1 \widetilde{a_2 a_3 \dots a_n}) \overset{(57)}{=} a_n \dots a_3 a_2 a_1 \quad . \tag{58}$$

Since a simple r-vector can be represented as the product of r *orthogonal* vectors a_i, we have

$$\begin{aligned} \widetilde{A}_r & \overset{(58)}{=} a_r \dots a_2 a_1 \overset{(8)}{=} (-1)^{r-1} a_1 a_r \dots a_2 \\ & \overset{(8)}{=} (-1)^{(r-1)+(r-2)} a_1 a_2 a_r \dots a_3 \overset{(8)}{=} (-1)^{\sum_{n=1}^{r}(r-n)} \underbrace{a_1 \dots a_r}_{=A_r} \\ & = (-1)^{rr-\frac{1}{2}r(r+1)} A_r = (-1)^{\frac{1}{2}r(r-1)} A_r \quad . \end{aligned} \tag{59}$$

Since any multivector M can be expressed as a sum of simple r-vectors $\langle M \rangle_r$ it follows

$$\left.\begin{array}{lcl} \langle \widetilde{M} \rangle_r & = & (-1)^{\frac{1}{2}r(r-1)} \langle M \rangle_r \\ \widetilde{M} & \overset{(29)}{=} & \sum_{r=0} (-1)^{\frac{1}{2}r(r-1)} \langle M \rangle_r \end{array}\right\} . \tag{60}$$

If we can write a multivector as the product of vectors[11], *i.e.*, $M = a_1 \dots a_n$, we have

$$M\widetilde{M} \overset{(58)}{=} a_1 \cdots a_n a_n \cdots a_1 = \Pi_{r=1}^{n} a_r^2 = \text{scalar} \quad . \tag{61}$$

and define thus for products of vectors, and especially so for simple r-vectors

$$|A_k|^2 \overset{\text{def}}{=} A_k \widetilde{A}_k \overset{(60)}{=} \text{scalar} \quad , \tag{62}$$

or more generally,

$$|M|^2 \overset{\text{def}}{=} \langle M\widetilde{M} \rangle = \sum_r \langle M_r \widetilde{M}_r \rangle = \text{scalar} \tag{63}$$

for an arbitrary multivector $M = \sum_r M_r$. Note, that (62) does not exclude the case $|A_r|^2 = 0$. But in this case we can conclude that A_r is zero or contains at least one null-vector as a factor.

[11] Which can be nonorthogonal to each other.

The Inverse of Simple Multivectors Now (62) allows us to define the inverse of a simple r-vector by

$$A_r^{-1} \stackrel{\text{def}}{=} \frac{\widetilde{A}_r}{A_r\widetilde{A}_r} = (-1)^{\frac{r(r-1)}{2}} \frac{A_r}{|A_r|^2} \text{ if } A_r\widetilde{A}_r \neq 0 \quad . \tag{64}$$

Since $A_r\widetilde{A}_r$ is by (62) a scalar, the division is well defined if $|A_r|^2 \neq 0$ and the condition

$$A_r A_r^{-1} \stackrel{(64)}{=} \frac{A_r\widetilde{A}_r}{A_r\widetilde{A}_r} = 1 \tag{65}$$

is satisfied. We can extend this definition for all multivectors with

$$M\widetilde{M} = \text{scalar} \neq 0 \quad . \tag{66}$$

Thus the inverse of a vector a in a positive definite vector space[12] is a vector in the same direction, but with magnitude $\frac{1}{|a|}$.

Symmetries of Outer and Inner Product From (59) it follows for $r \geq s$

$$\begin{aligned} A_r \cdot B_s &= \langle A_r B_s\rangle_{|r-s|} = \langle \widetilde{\widetilde{B}_s\widetilde{A}_r}\rangle_{|r-s|} \\ &\stackrel{(59)}{=} (-1)^{\frac{1}{2}(r-s)(r-s-1)}(-1)^{-\frac{1}{2}r(r-1)}(-1)^{-\frac{1}{2}s(s-1)} B_s \cdot A_r \\ &= (-1)^{(r+1)s} B_s \cdot A_r \end{aligned} \tag{67}$$

and

$$\begin{aligned} A_r \wedge B_s &= \langle A_r B_s\rangle_{r+s} = \langle \widetilde{\widetilde{B}_s\widetilde{A}_r}\rangle_{r+s} \\ &\stackrel{(59)}{=} (-1)^{\frac{1}{2}(r+s)(r+s-1)}(-1)^{-\frac{1}{2}r(r-1)}(-1)^{-\frac{1}{2}s(s-1)} B_s \wedge A_r \\ &= (-1)^{rs} B_s \wedge A_r \quad . \end{aligned} \tag{68}$$

For a general product *of a vector* a and a simple r-vector B_r one gets from (67) and (68) for $r \geq 1$

$$a \cdot B_r \stackrel{(67)}{=} \frac{1}{2}(aB_r - (-1)^r B_r a) = (-1)^{r+1} B_r \cdot a \tag{69}$$

$$a \wedge B_r \stackrel{(68)}{=} \frac{1}{2}(aB_r + (-1)^r B_r a) = (-1)^r B_r \wedge a, \tag{70}$$

since the symmetry is different and the split in symmetric and antisymmetric part is unique. So that we get by adding both equations

$$a \cdot B_r + a \wedge B_r = \frac{1}{2}(aB_r - (-1)^r B_r a) + \frac{1}{2}(aB_r + (-1)^r B_r a) = aB_r \tag{71}$$

as demanded.

Products with Scalars Please note, that for a scalar λ we get with $s = 0$ in (67) and (68)

$$\lambda \cdot A_r = A_r \cdot \lambda \quad ; \quad \lambda \wedge A_r = A_r \wedge \lambda \quad . \tag{72}$$

To extend the relations (69) and (70) to the scalar case, we define

$$\lambda \cdot A_r \stackrel{\text{def}}{=} 0 \quad ; \quad \lambda \wedge A_r \stackrel{\text{def}}{=} A_r \quad . \tag{73}$$

Thus *the inner product with a scalar is by definition always zero.*

[12]This means $a^2 \geq 0 \forall a$. For $a^2 \leq 0$ we get as the inverse $a^{-1} = \frac{a}{a^2} = -\frac{1}{|a|}$.

Precedence of Outer and Inner Product We will use the convention that *inner and outer products have precedence* over the geometrical product. So

$$AB \cdot C \overset{\text{def}}{=} A(B \cdot C) \quad ; \quad AB \wedge CD \overset{\text{def}}{=} A(B \wedge C)D \quad . \tag{74}$$

1.1.6 Generalized Orthogonality

With (8) we defined orthogonality between vectors. It is possible, and useful, to *extend* this definition to arbitrary multivectors. We will call two *multivectors* A and B orthogonal, if they obey

$$A\widetilde{B} = -B\widetilde{A} \quad . \tag{75}$$

This is an obvious generalization, since now for $M \overset{\text{def}}{=} \alpha A + \beta B$

$$M\widetilde{M}' = (\alpha A + \beta B)(\alpha'\widetilde{A} + \beta'\widetilde{B}) = \alpha\alpha' A\widetilde{A} + \beta\beta' B\widetilde{B} + (\alpha\beta' - \alpha'\beta)A\widetilde{B} \quad , \tag{76}$$

which can be seen as a generalized Pythagoras Rule. For vectors this definition yields

$$a \perp b \overset{(75)}{\Leftrightarrow} ab = -ba \overset{(7)}{\Leftrightarrow} a \cdot b = 0 \tag{77}$$

and is thus equivalent to the condition (8).

Orthogonality with non-scalar coefficients To be able to construct an analogue to complex coefficients, we say two multivectors A and B are orthogonal with coefficients α and β, which are *not scalars*, if

$$A\alpha\widetilde{B\beta} = A\alpha\widetilde{\beta}\widetilde{B} = -B\beta\widetilde{A\alpha} \quad . \tag{78}$$

This can be written as

$$Ac\widetilde{B} = -B\widetilde{c}\widetilde{A} \quad . \tag{79}$$

1.1.7 Scalar-, Cross- and other Products

The Scalar Product Since the inner product $A_r \cdot B_s = \langle A_r B_s \rangle_{|r-s|}$ has not necessarily to be a scalar[13], we define in addition the **scalar product of multivectors** A, B

$$A * B \overset{\text{def}}{=} \langle AB \rangle \quad . \tag{80}$$

We see that for $r \neq s \Rightarrow |r - s| > 0$

$$A_r * B_s \overset{(80)}{=} \langle A_r B_s \rangle \overset{(51)}{=} \langle \langle A_r B_s \rangle_{|r-s|} + \ldots + \langle A_r B_s \rangle_{r+s} \rangle = 0 \quad . \tag{81}$$

So for general multivectors A,B we have

$$\begin{aligned} A * B &\overset{(29)}{=} \sum_r \langle A \rangle_r * B \overset{(81)}{=} \sum_r \langle A \rangle_r * \langle B \rangle_r \\ &= \langle A \rangle \langle B \rangle + \langle A \rangle_1 \cdot \langle B \rangle_1 + \ldots \quad . \end{aligned} \tag{82}$$

Since it only involves inner products of multivectors of the same grade, it has to be symmetric and linear, according to (67)

$$\left.\begin{aligned} A * B &\overset{(67)}{=} B * A \\ A * (\alpha B + \beta C) &= \alpha A * B + \beta A * C \end{aligned}\right\} . \tag{83}$$

(83) gives us the very important *cyclic identity*

$$\langle AB \rangle = A * B = B * A = \langle BA \rangle \quad . \tag{84}$$

Since A and B can be any multivector this yields

$$\langle AB \ldots YZ \rangle = \langle ZA \ldots Y \rangle = \langle B \ldots YZA \rangle \quad . \tag{85}$$

[13] This is the case for $r \neq s$.

The Commutator Product We define the commutator product *for arbitrary multivectors* as

$$A \times B \stackrel{\text{def}}{=} \frac{1}{2}(AB - BA) = -B \times A \quad . \tag{86}$$

Obviously it obeys the identity

$$\begin{aligned} A \times (BC) &\stackrel{(86)}{=} \frac{1}{2}(ABC - BCA) = \frac{1}{2}(ABC - BAC + BAC - BCA) \\ &= \frac{1}{2}\left[(AB - BA)C + B(AC - CA)\right] \\ &= (A \times B)C + B(A \times C) \quad , \end{aligned} \tag{87}$$

which is well known for the three-dimensional cross-product[14] . Further its antisymmetric structure leads to the **Jacobi Identity** [15]

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) \stackrel{(86)}{=} 0 \quad . \tag{88}$$

Symmetric and antisymmetric Product We will further introduce the symmetric and antisymmetric product. We define

$$A \odot B \stackrel{\text{def}}{=} \tfrac{A\widetilde{B}+B\widetilde{A}}{2} \quad ; \quad A \oslash B \stackrel{\text{def}}{=} \tfrac{A\widetilde{B}-B\widetilde{A}}{2} \quad . \tag{89}$$

Both are obviously linear. But it is important to note here

$$\widetilde{A \odot B} = A \odot B \quad ; \quad \widetilde{A \oslash B} = -A \oslash B \quad , \tag{90}$$

but in general

$$\widetilde{A \odot B} \neq \widetilde{B} \odot \widetilde{A} \quad ; \quad \widetilde{A \oslash B} \neq \widetilde{B} \oslash \widetilde{A} \quad ! \tag{91}$$

Since $A \odot B$ reverses to give itself, we can conclude from (60) that it only contains grades r for which $\frac{1}{2}r(r-1) = 2n$, *i.e.*,

$$A \odot B = \sum_{\frac{r(r-1)}{2}=2n} \langle A \odot B \rangle_r = \sum_r \frac{\langle A\widetilde{B}\rangle_r + \langle B\widetilde{A}\rangle_r}{2} = \sum_r \langle A\widetilde{B}\rangle_r \tag{92}$$

and analogously

$$A \oslash B = \sum_{\frac{r(r-1)}{2}=2n+1} \langle A \oslash B \rangle_r = \sum_r \frac{\langle A\widetilde{B}\rangle_r - \langle B\widetilde{A}\rangle_r}{2} = \sum_r \langle A\widetilde{B}\rangle_r \quad . \tag{93}$$

For instance for the algebra of a three-dimensional linear vector space[16], *i.e.*, $n = 1$ in (92) and (93),

$$\left.\begin{aligned} A \odot B &= \langle A\widetilde{B}\rangle_0 + \langle A\widetilde{B}\rangle_1 \\ A \oslash B &= \langle A\widetilde{B}\rangle_2 + \langle A\widetilde{B}\rangle_3 \end{aligned}\right\} . \tag{94}$$

For the square of a sum $M \stackrel{\text{def}}{=} \sum_i M_i$ so we get

$$M\widetilde{M} = \sum_{i,j} M_i\widetilde{M}_j = \sum_{i,j} \frac{M_i\widetilde{M}_j + M_j\widetilde{M}_i}{2} = \sum_{i,j} M_i \odot M_j = \sum_i M_i\widetilde{M}_i + \sum_i \sum_{j \neq i} M_i \odot M_j \quad . \tag{95}$$

Orthogonality of two multivectors A, B can now be expressed by

$$A \odot B = 0 \quad \Leftrightarrow A\widetilde{B} = -B\widetilde{A} \Leftrightarrow A \text{ orthogonal to } B \quad . \tag{96}$$

Thus if the M_i are orthogonal multivectors, (95) simplifies to

$$M\widetilde{M} = \sum_i M_i\widetilde{M}_i \quad . \tag{97}$$

[14]Please note, that the commutator product is related, but not identical, to the three-dimensional crossproduct. The relation will become clear with (402).

[15]To prove the identity, we just use the definition (86) to eliminate all commutator products and see that the terms cancel each other.

[16]So the pseudoscalar is a trivector. No higher grades than trivectors exist.

	$A_r \cdot B_s$	$A_r \wedge B_s$	$A_r * B_s$	$A \times B$
Definition	$\langle A_r B_s \rangle_{\lvert r-s \rvert}$	$\langle A_r B_s \rangle_{r+s}$	$\langle A_r B_s \rangle$ $(= 0$ if $r \neq s)$	$\frac{AB-BA}{2}$
Description	Lowest grade contained in geometric product.	Highest grade contained in geometric product. It is the *only product besides the geometric product that is always associative.*	Scalar part of geometric product. Always zero, if A and B do not contain the same grades.	Commutator, *can* be equivalent to an other product. This depends on the involved grades. Especially $\lambda \times A_n = \lambda \cdot A_N = 0$, $a \times b = a \wedge b$ and $a \times B_2 = a \cdot B_2$.
Symmetries for $r \geq s, s \neq 0$	$= (-1)^{(r+1)s} B_r \cdot A_s$	$= (-1)^{rs} B_s \wedge A_r$	$= B_s * A_r$	$= -B \times A$
Product with scalars	$\lambda \cdot M = 0$	$\lambda \wedge M = \lambda M$	$\lambda * M = \lambda \langle M \rangle$	$\lambda \times M = 0$

Table 1: Table of the different, frequently occurring products for simple r, s-vectors A_r, B_s. All products extend via linearity to arbitrary multivectors.

1.1.8 Subalgebras

Definition A subalgebra $\mathcal{S}$ of an algebra $\mathcal{G}(\mathcal{A}_n)$ is a set of multivectors obeying

$$1 \in \mathcal{S} \quad ; \quad X, Y \in \mathcal{S} \Rightarrow XY \in \mathcal{S} \quad . \tag{98}$$

We call the linear space of scalars (spanned by $\{1\}$) and $\mathcal{G}(\mathcal{A}_n)$ itself trivial subalgebras.

Subalgebra of a Subspace Let $\mathcal{A}_m$ be a subspace of $\mathcal{A}_n$ with $m < n$. Each multivector $M \in \mathcal{G}(\mathcal{A}_m)$ constructed out of vectors in A_m is thus also a multivector in $\mathcal{G}(\mathcal{A}_n)$. Accordingly $\mathcal{G}(\mathcal{A}_m)$ is a subalgebra. We conclude, that for each subspace of $\mathcal{A}_n$ there is an associated subalgebra $\mathcal{G}(\mathcal{A}_m)$.

Even Subalgebra For a n-dimensional linear vector space $\mathcal{A}_n$ we have according to (26) a 2^n-dimensional linear vector space of multivectors, giving a closed algebra with the geometric product $\mathcal{G}(\mathcal{A}_n)$. We call a multivector $M \in \mathcal{G}(\mathcal{A}_n)$ even, if

$$M = \sum_{r=0}^{\infty} \langle M \rangle_{2r} \quad . \tag{99}$$

Since the product of even elements is according to (51) even and $1 \in \mathcal{G}^0(\mathcal{A}_n)$ is even we have a subalgebra, which we will denote by $\mathcal{E} \stackrel{\text{def}}{=} \mathcal{E}_\mathcal{G} \stackrel{\text{def}}{=} \mathcal{E}(\mathcal{G}(\mathcal{A}_n))$. The linear space of multivectors in $\mathcal{E}_\mathcal{G}$ has[17]

$$\sum_{r=0}^{\infty} \binom{n}{2r} = \frac{1}{2} \sum_{r=0}^{\infty} \binom{n}{r} \stackrel{(102)}{=} 2^{n-1} \tag{103}$$

dimensions. The number of dimensions shows already, that this space is spanned by $n-1$ basis vectors.

We can construct a basis of $\mathcal{E}_\mathcal{G}$ in the following way. Given an orthonormal basis e_μ of $\mathcal{A}_n$ we choose one of the n basis vectors, lets say e_n. Multiplying e_n with all others gives all $n-1$ possible

[17] Here assume $\binom{n}{r} = 0$ for $r > n$. Further, note that

$$0 = (1-1)^n = \sum_{r=0}^{\infty} (-1)^r \binom{n}{r} = \sum_{r=0}^{\infty} \left\{ \binom{n}{2r} - \binom{n}{2r+1} \right\} \tag{100}$$

and thus

$$\sum_{r=0}^{\infty} \binom{n}{2r} = \sum_{r=0}^{\infty} \binom{n}{2r+1} \quad . \tag{101}$$

This yields

$$\sum_{r=0}^{\infty} \binom{n}{2r} \stackrel{(101)}{=} \frac{1}{2} \sum_{r=0}^{\infty} \left\{ \binom{n}{2r} + \binom{n}{2r+1} \right\} = \frac{1}{2} \sum_{r=0}^{\infty} \binom{n}{r} = \frac{2^n}{2} = 2^{n-1} \quad . \tag{102}$$

bivectors in $\mathcal{G}(\mathcal{A}_n)$ containing e_n

$$B_\mu \stackrel{\text{def}}{=} e_\mu e_n \quad ; \mu \in \{1, \ldots, n-1\} \quad . \tag{104}$$

Since for $1 \leq \mu, \nu \leq n-1$

$$B_\mu B_\nu \stackrel{(104)}{=} e_\mu e_n e_\nu e_n = -e_\mu (e_1)^2 e_\nu = \pm e_\mu e_\nu \tag{105}$$

all other bivectors, which do not contain e_n, can be constructed out of the others.

Thus we choose the $n-1$ bivectors B_μ as our basis of vectors in $\mathcal{E}_\mathcal{G}$. The other bivectors in $\mathcal{G}(\mathcal{A}_n)$ appear then also as bivectors in $\mathcal{E}_\mathcal{G}$.

1.1.9 Vector Identities

After having derived the various products, we take now a closer look on some possible products of more than two vectors. This vector identities will be very useful for calculations in the following Sections.

From the scalar product of vectors $a \cdot b = \frac{1}{2}(ab + ba)$ we can deduce that

$$ab = 2a \cdot b - ba \quad . \tag{106}$$

This allows us to reverse the order of the geometric product of two vectors by adding an extra term. Applying this procedure r times we derive

$$\begin{aligned} ab_1 b_2 b_3 \ldots b_r &\stackrel{(106)}{=} 2a \cdot b_1 (b_2 b_3 \ldots b_r) - b_1 a b_2 b_3 \ldots b_r \\ &\stackrel{(106)}{=} 2a \cdot b_1 (b_2 b_3 \ldots b_r) - 2b_1 (a \cdot b_2) b_3 \ldots b_r + \\ &\quad + b_1 b_2 a b_3 \ldots b_r \\ &\;\vdots \\ &= (-1)^r b_1 \ldots b_r a + 2 \sum_{k=1}^{r} (-1)^{k+1} (a \cdot b_k) b_1 \ldots \check{b}_k \ldots b_r \quad , \end{aligned} \tag{107}$$

where $\check{b}_k$ is skipped. Since, according to (51), the geometric product of r vectors *contains either odd or even grades, but never both*, we can substitute (107) in the expression for the inner product of simple r-vectors with a vector to derive

$$\begin{aligned} a \cdot (b_1 b_2 \ldots b_r) &\stackrel{(69)}{=} \frac{1}{2} [ab_1 \ldots b_r - (-1)^r b_1 \ldots b_r a] \\ &\stackrel{(107)}{=} \sum_{k=1}^{r} (-1)^{k+1} (a \cdot b_k) b_1 \ldots \check{b}_k \ldots b_r \quad . \end{aligned} \tag{108}$$

By looking at the highest grade contained in (108) we establish the fundamental relation

$$\begin{aligned} a \cdot (b_1 \wedge b_2 \wedge \ldots \wedge b_r) &\stackrel{(45)}{=} a \cdot \langle b_1 \ldots b_r \rangle_r \stackrel{(45)}{=} \langle a \cdot (b_1 \ldots b_r) \rangle_{r-1} \\ &\stackrel{(108)}{=} \langle \sum_{k=1}^{r} (-1)^{k+1} (a \cdot b_k) b_1 \ldots \check{b}_k \ldots b_r \rangle_{r-1} \\ &= \sum_{k=1}^{r} (-1)^{k+1} (a \cdot b_k) b_1 \wedge \ldots \wedge \check{b}_k \wedge \ldots \wedge b_r \quad . \end{aligned} \tag{109}$$

For $r = 2$ we derive so the result

$$a \cdot (b \wedge c) \stackrel{(108)}{=} (a \cdot b) c - (a \cdot c) b \quad . \tag{110}$$

We are now looking at the inner product of a vector with the geometric product of two r, s-vectors A_r, B_s. Since the geometric product of A_r and B_s contains either odd or even grades, but not both, we can use (69) to derive identities

$$\begin{aligned} a\cdot(A_rB_s) &\overset{(69)}{=} \frac{1}{2}\left[aA_rB_s-(-1)^{r+s}A_rB_sa\right] \\ &= \frac{1}{2}\left[aA_rB_s-(-1)^rA_raB_s\right]+\frac{1}{2}\left[(-1)^rA_raB_s-(-1)^{r+s}A_rB_sa\right] \\ &\overset{(69)}{=} (a\cdot A_r)B_s+(-1)^r\frac{1}{2}\left[A_raB_s-(-1)^sA_rB_sa\right] \\ &\overset{(69)}{=} (a\cdot A_r)B_s+(-1)^rA_r(a\cdot B_s) \end{aligned} \tag{111}$$

$$\begin{aligned} a\cdot(A_rB_s) &\overset{(69)}{=} \frac{1}{2}\left(aA_rB_s-(-1)^{r+s}A_rB_sa\right) \\ &= \frac{1}{2}\left(aA_rB_s+(-1)^rA_raB_s-(-1)^rA_raB_s-(-1)^{r+s}A_rB_sa\right) \\ &\overset{(70)}{=} (a\wedge A_r)B_s-(-1)^rA_r(a\wedge B_s) \quad . \end{aligned} \tag{112}$$

In the same way one derives for the outer product the two relations

$$\begin{aligned} a\wedge(A_rB_s) &\overset{(70)}{=} \frac{1}{2}\left[aA_rB_s+(-1)^{r+s}A_rB_sa\right] \\ &= \frac{1}{2}\left[aA_rB_s+(-1)^rA_raB_s-(-1)^rA_raB_s+(-1)^{r+s}A_rB_sa\right] \\ &\overset{(70)}{=} a\wedge A_rB_s-(-1)^r\frac{1}{2}\left[A_raB_s-(-1)^sA_rB_sa\right] \\ &\overset{(70)}{=} a\wedge A_rB_s-(-1)^rA_ra\cdot B_s \end{aligned} \tag{113}$$

$$\begin{aligned} a\wedge(A_rB_s) &\overset{(70)}{=} \frac{1}{2}(aA_rB_s+(-1)^{r+s}A_rB_sa) \\ &= \frac{1}{2}\left[aA_rB_s-(-1)^rA_raB_s+(-1)^rA_raB_s+(-1)^{r+s}A_rB_sa\right] \\ &\overset{(69)}{=} (a\cdot A_r)B_s+(-1)^rA_ra\wedge B_s \quad . \end{aligned} \tag{114}$$

With the help of the grade operator one derives then for the highest grade contained in (111)

$$\begin{aligned} a\cdot(A_r\wedge B_s) &= a\cdot\langle A_rB_s\rangle_{r+s}=\langle a\cdot(A_rB_s)\rangle_{r+s-1} \\ &\overset{(111)}{=} \langle a\cdot A_rB_s+(-1)^rA_r(a\cdot B_s)\rangle_{r+s-1} \\ &= (a\cdot A_r)\wedge B_s+(-1)^rA_r\wedge(a\cdot B_s) \end{aligned} \tag{115}$$

and for (114) with $s\geq r>1$

$$\begin{aligned} a\wedge(A_r\cdot B_s) &= a\wedge\langle A_rB_s\rangle_{s-r}=\langle a\wedge(A_r\cdot B_s)\rangle_{s-r+1} \\ &\overset{(114)}{=} \langle a\cdot A_rB_s+(-1)^rA_ra\wedge B_s\rangle_{s-r+1} \\ &= (a\cdot A_r)\cdot B_s+(-1)^rA_r\cdot(a\wedge B_s) \quad . \end{aligned} \tag{116}$$

If $B_s=b$ is a vector we derive instead

$$a\wedge(A_R\cdot b)=\langle aA_rb\rangle_r-a\cdot(A_r\wedge b)=(a\cdot A_r)\wedge b+(a\wedge A_r)\cdot b-a\cdot(A_r\wedge b) \quad . \tag{117}$$

Further, if $A_r=c$ is also a vector this identity becomes identical to (110), *i.e.*,

$$a(b\cdot c)=(a\cdot c)b+\underbrace{(a\wedge c)\cdot b+(c\wedge b)\cdot a}_{=-(b\wedge a)\cdot c}=(a\cdot c)b-(b\wedge a)\cdot c \quad . \tag{118}$$

Further, for $n>s$ we have

$$\begin{aligned} a\cdot(A_n\cdot B_s) &= a\cdot\langle A_nB_s\rangle_{n-s}=\langle aA_nB_s\rangle_{n-s-1} \\ &= \langle aA_n\rangle_{n-1}\cdot B_s=(a\cdot A_n)\cdot B_s \end{aligned} \tag{119}$$

$a\cdot(b_1\ldots b_r)$	$\stackrel{(108)}{=}$	$\sum_{k=1}^{r}(-1)^{k+1}(a\cdot b_k)b_1\ldots\check{b}_k\ldots b_r$
$a\cdot(b_1\wedge\ldots\wedge b_r)$	$\stackrel{(109)}{=}$	$\sum_{k=1}^{r}(-1)^{k+1}(a\cdot b_k)b_1\wedge\ldots\wedge\check{b}_k\wedge\ldots\wedge b_r$
$a\cdot(A_rB)$	$\stackrel{(111)}{=}$	$a\cdot A_rB+(-1)^rA_r(a\cdot B)$
	$\stackrel{(112)}{=}$	$a\wedge A_rB-(-1)^rA_r(a\wedge B)$
$a\wedge(A_rB)$	$\stackrel{(113)}{=}$	$a\wedge A_rB-(-1)^rA_ra\cdot B$
	$\stackrel{(114)}{=}$	$a\cdot A_rB+(-1)^rA_ra\wedge B$
$a\cdot(A_r\wedge B)$	$\stackrel{(115)}{=}$	$(a\cdot A_r)\wedge B+(-1)^rA_r\wedge(a\cdot B)$
$a\wedge(A_r\cdot B)$	$\stackrel{(116)}{=}$	$(a\cdot A_r)\cdot B+(-1)^rA_r\cdot(a\wedge B)$with $r>1$

(126)

Table 2: Overview of the most important identities.

and thus *the inner product is in this case associative*! Further, the following identity will be essential for the treatment of bivectors, which are especially important:

$$\langle abA_r\rangle_{r-2} \stackrel{(13)}{=} \langle a\wedge bA_r+a\cdot bA_r\rangle_{r-2} \stackrel{(47)}{=} (a\wedge b)\cdot A_r \tag{120}$$

$$\begin{aligned}\langle abA_r\rangle_{r-2} &\stackrel{(71)}{=} \langle a(b\cdot A_r)+a(b\wedge A_r)\rangle_{r-2} \stackrel{(45)}{=} a\cdot(b\cdot A_r) && (121)\\ &\stackrel{(67)}{=} (-1)^{r+1}a\cdot(A_r\cdot b) && (122)\\ &\stackrel{(119)}{=} (-1)^{r+1}(a\cdot A_r)\cdot b \quad . && (123)\end{aligned}$$

So we derived the equivalence relation

$$(a\wedge b)\cdot A_r=a\cdot(b\cdot A_r)=(-1)^{r+1}a\cdot(A_r\cdot b)=(-1)^{r+1}(a\cdot A_r)\cdot b \quad . \tag{124}$$

This yields the useful expression

$$\begin{aligned}(a_1\wedge a_2)\cdot(b_1\wedge b_2) &\stackrel{(124)}{=} a_1\cdot[a_2\cdot(b_1\wedge b_2)]\\ &= a_1\cdot[(a_2\cdot b_1)b_2-(a_2\cdot b_2)b_1]\\ &= (a_2\cdot b_1)(a_1\cdot b_2)-(a_2\cdot b_2)(a_1\cdot b_1)\\ &= -\begin{vmatrix} a_1\cdot b_1 & a_1\cdot b_2\\ a_2\cdot b_1 & a_2\cdot b_2\end{vmatrix} \quad .\end{aligned} \tag{125}$$

The results are listed in Table 2.

Outer Product of Vectors The outer product of n vectors is associative and thus it changes sign under exchange of two neighbouring ones. We would thus expect the outer product of n vectors to be a totally skew combination of the vectors with the geometric product, but this is not obvious when looking at (70). We will limit the discussion to the case of the outer product of three vectors. Here we derive by successive application of (70)

$$\begin{aligned}(a\wedge b)\wedge c &\stackrel{(70)}{=} \frac{1}{2}\{(a\wedge b)c+c(a\wedge b)\} \stackrel{(70)}{=} \frac{1}{4}\{abc-bac+cab-cba\}\\ &= \frac{1}{3!}\{abc+bca+cab-cba-bac-acb\}\\ &\quad +\frac{1}{12}\{abc+cab-cba-bac-2bca+2acb\} \quad .\end{aligned} \tag{127}$$

But now we find for the second term under use of (7)

$$abc+cab-cba-bac-2bca+2acb=-b(a\cdot c)+a(b\cdot c)+\underbrace{cab+acb}_{=(a\cdot c)b}\underbrace{-cba-bca}_{=-(c\cdot b)a}=0 \quad . \tag{128}$$

Substituting this in (127) yields

$$a\wedge b\wedge c=\frac{1}{3!}\{abc+bca+cab-cba-bac-acb\} \quad . \tag{129}$$

Thus the outer product of three vectors is indeed their totally skew combination.

Duality Relations Setting in (112) and (113) $A_r = I_r$, with $a \wedge I_r = 0 \forall a \in \mathcal{A}_n$ we find the important *duality relations* for inner and outer product[18]

$$a \cdot (IB) \overset{(112)}{=} (-1)^{r+1} I_r (a \wedge B) \quad ; \quad a \wedge (I_r B) \overset{(113)}{=} (-1)^{r+1} I_r a \cdot B \quad . \tag{130}$$

We can extend this relation to simple multivectors $A_s = a_1 \wedge \cdots \wedge a_s$ $(s < r)$ by repeated application of (130), *i.e.*,

$$\begin{aligned} A_s \cdot (I_r B) &= (a_1 \wedge \cdots \wedge a_{s-1}) \cdot [a_s \cdot (I_r B)] \\ &\overset{(130)}{=} (a_1 \wedge \cdots \wedge a_{s-1}) \cdot \left[(-1)^{r+1} I_r (a_s \wedge B)\right] \\ &\;\vdots \\ &= (-1)^{s(r+1)} I_r (A_s \wedge B) \end{aligned} \tag{131}$$

and similarly

$$A_s \wedge (I_r B) = (-1)^{s(r+1)} I_r (A_s \cdot B) \quad . \tag{132}$$

1.1.10 The Commutator Product with Bivectors

The commutator product mostly appears in connection with bivectors. Thus it is worth taking a closer look at the special results in this case.

Since for a bivector $A_2 = a_1 a_2$, with $a_1 a_2 = a_2 a_1$, and a simple r-vector B_r with $r \neq 1$

$$\begin{aligned} A_2 B_r &= a_1 a_2 B_r \overset{(46)}{=} a_1 (a_2 \cdot B_r + a_2 \wedge B_r) \\ &\overset{(46)}{=} a_1 \cdot (a_2 \cdot B_r + a_2 \wedge B_r) + a_1 \wedge (a_2 \cdot B_r + a_2 \wedge B_r) \\ &= \underbrace{a_1 \cdot (a_2 \cdot B_r)}_{=\langle A_2 B_r \rangle_{r-2}} + \underbrace{a_1 \cdot (a_2 \wedge B_r) + a_1 \wedge (a_2 \cdot B_r)}_{\langle A_2 B_r \rangle_r} + \\ &\quad + \underbrace{a_1 \wedge a_2 \wedge B_r}_{\langle A_2 B_r \rangle_{r+2}} \quad , \end{aligned} \tag{133}$$

we gain the identification

$$\left.\begin{aligned} A_2 \cdot B_r &= \langle A_2 B_r \rangle_{r-2} = a_1 \cdot (a_2 \cdot B_r) \\ A_2 \wedge B_r &= \langle A_2 B_r \rangle_{r+2} = a_1 \wedge a_2 \wedge B_r \end{aligned}\right\} . \tag{134}$$

Since for $r \geq 2$

$$B_r \cdot A_2 \overset{(67)}{=} (-1)^{2(r+1)} A_2 \cdot B_r \tag{135}$$

$$A_2 \wedge B_r \overset{(68)}{=} (-1)^{2s} B_r \wedge A_2 = B_r \wedge A_2 \tag{136}$$

$$\langle A_2 B_r \rangle_r = \langle \widetilde{\tilde{B}_r \tilde{A}_2} \rangle_r \overset{(59)}{=} (-1)^{\frac{2(2-1)}{2}} \langle B_r A_2 \rangle_r = -\langle B_r A_2 \rangle \tag{137}$$

we have

$$\begin{aligned} A_2 \times B_r &= \frac{1}{2}(A_2 B_r - B_r A_2) \\ &\overset{(133)}{=} \frac{1}{2}\left[A_2 \cdot B_r + \langle A_2 B_r \rangle_r + A_2 \wedge B_r - \overbrace{B_r \cdot A_2}^{\overset{(135)}{=} A_2 \cdot B_r} - \underbrace{\langle B_r A_2 \rangle_r}_{\overset{(137)}{=} \langle B_r A_2 \rangle_r} - \overbrace{B_r \wedge A_2}^{\overset{(136)}{=} A_2 \wedge B_r}\right] \\ &= \langle A_2 B_r \rangle_r \quad . \end{aligned} \tag{138}$$

Further, for $r = 1$ we have $A_2 \times a = A_2 \cdot a = \langle A_2 a \rangle_1$ and finally for a scalar $A_2 \times \lambda = 0$. Thus we see that *the commutator product with a bivector conserves the grade* and we can write

$$A_2 \times B_r = \langle A_2 B_r \rangle_r \quad . \tag{139}$$

[18] Remember, that a has to be a vector, but B can be any multivector, even with $I_r \wedge B \neq 0$!

This gives in (51)

$$A_2 B_r \overset{(51)}{=} \langle A_2 B_r\rangle_{r-2} + \langle A_2 B_r\rangle_r + \langle A_2 B_r\rangle_{r+2} \overset{(139)}{=} A_2 \cdot B + A_2 \times B + A_2 \wedge B \quad . \tag{140}$$

The grade preserving property yields further for a bivector A

$$\begin{aligned} A \times (B_s \cdot C_r) &\overset{(47)}{=} A \times \langle B_s C_r\rangle_{|s-r|} \overset{(139)}{=} \langle A \times (B_s C_r)\rangle_{|s-r|} \\ &\overset{(87)}{=} \langle (A \times B_s) C_r + B_s (A \times C_r)\rangle_{|s-r|} \\ &\overset{(47)}{=} (A \times B_s) \cdot C_r + B_s \cdot (A \times C_r) \end{aligned} \tag{141}$$

and

$$\begin{aligned} A \times (B_s \wedge C_r) &\overset{(47)}{=} A \times \langle B_s C_r\rangle_{s+r} \overset{(139)}{=} \langle A \times (B_s C_r)\rangle_{s+r} \\ &\overset{(87)}{=} \langle (A \times B_s) C_r + B_s (A \times C_r)\rangle_{s+r} \\ &\overset{(47)}{=} (A \times B_s) \wedge C_r + B_s \wedge (A \times C_r) \quad . \end{aligned} \tag{142}$$

Iteration of (142) gives immediately

$$\begin{aligned} B \times (a_1 \wedge \ldots \wedge a_r) &= \overbrace{(B \times a_1)}^{\overset{(69)}{=} B \cdot a_1} \wedge a_2 \wedge \ldots a_r + + a_1 \wedge (B \times a_2) \wedge a_3 \wedge \ldots a_r + \ldots \\ &\overset{(69)}{=} \sum_{k=1}^{r} (-1)^{k+1} (B \cdot a_k) \wedge a_1 \wedge \ldots \wedge \check{a}_k \wedge \ldots \wedge a_r \quad . \end{aligned} \tag{143}$$

1.1.11 The Projection Operator

Subspaces and Pseudoscalar Let $\{e_\nu : 1 \le \nu \le n\}$ be an orthonormal basis of a subspace $\mathcal{A}_n$ of $\mathcal{A}_m$ $(m > n)$, which can be extended to a basis $\{e_\mu : 1 \le \mu \le m\}$ of $\mathcal{A}_m$. Since the pseudoscalar is unique, it can be represented as the product of *any* n orthonormal vectors in $\mathcal{A}_n$. Thus

$$I_n \overset{\text{def}}{=} e_1 e_2 \cdots e_n \overset{(16)}{=} e_1 \wedge \cdots \wedge e_n \tag{144}$$

is the unique pseudoscalar of $\mathcal{G}(\mathcal{A}_n)$. If $a \in \mathcal{A}_n$ we can express it as a linear combination of basis vectors, *i.e.*,

$$a = \sum_{\nu=1}^{n} a^\nu e_\nu \quad . \tag{145}$$

In this case the outer product with the pseudoscalar of $\mathcal{A}_n$ becomes

$$a \wedge I_n \overset{(145)}{=} \sum_{\nu=1}^{n} a^\nu e_\nu \wedge e_1 \wedge \cdots \wedge e_n \overset{(50)}{=} 0 \quad . \tag{146}$$

If on the other side $a \notin \mathcal{A}_n$, we can conclude that a has a component $a_\perp$ orthogonal to all vectors in $\mathcal{A}_n$, *i.e.*,

$$a = \underbrace{\sum_{\nu=1}^{n} a^\nu e_\nu}_{\overset{\text{def}}{=} a_\parallel} + \underbrace{\sum_{\mu=n+1}^{m} a^\mu e_\mu}_{\overset{\text{def}}{=} a_\perp} = a_\parallel + a_\perp \quad , \tag{147}$$

and now

$$a \wedge I_n \overset{(147)}{=} (a_\parallel + a_\perp) \wedge I_n \overset{(146)}{=} \underbrace{a_\perp \wedge I_n}_{\text{grade } n+1} \neq 0 \quad . \tag{148}$$

Thus we can summarize

$$a \wedge I_n \begin{cases} = 0 & \forall a \in \mathcal{A}_n \\ \neq 0 & \forall a \notin \mathcal{A}_n \end{cases} \quad , \tag{149}$$

i.e., the pseudoscalar defines uniquely $\mathcal{A}_n$.

The Product with the Pseudoscalar Let $A_r = a_r \dots a_1 \in \mathcal{G}(\mathcal{A}_n)$ be a *simple r-vector*. Since the pseudoscalar is unique, we can expand it as

$$I_n = \lambda a_1 \dots a_r a_{r+1} \dots a_n \quad , \tag{150}$$

where λ is chosen, so that $I^2 = \pm 1$. Thus the geometric product

$$A_r I_n = a_r \dots a_1 a_1 \dots a_r a_{r+1} \dots a_n = \left[\Pi_{\nu=1}^{r} a_\nu^2\right] a_{r+1} \dots a_n \tag{151}$$

is a simple $(n-r)$-vector and hence (according to (47))

$$A_r I_n \overset{(151)}{=} A_r \cdot I_n \qquad \forall A_r \in \mathcal{G}(\mathcal{A}_n) \quad . \tag{152}$$

If $A_r \notin \mathcal{G}(\mathcal{A}_n)$, it must contain $l \geq 1$ factors $b_\nu \notin \mathcal{A}_n$. Then with $r = l + k$

$$A_r \cdot I_n = \langle A_r I_n \rangle_{|n-r|} = \langle a_1 \dots a_k \underbrace{b_1 \dots b_l I_n}_{\text{grade } n+l} \rangle_{|n-r|} = 0 \quad . \tag{153}$$

Under use of $I_n I_n^{-1} = 1$ we derive the identity

$$a = a I_n I_n^{-1} = (a \cdot I_n + a \wedge I_n) I_n^{-1} = a \cdot I_n I_n^{-1} + a \wedge I_n I_n^{-1} \quad . \tag{154}$$

The first term on the right can be seen as the projection into the space represented by I_n, while the second term (according to (149)) projects out the part of a orthogonal to this space. This suggests to define as a generalization to arbitrary multivectors

$$\left.\begin{array}{lllll} A_\parallel & = & P_{I_n}(A) & \overset{\text{def}}{=} & (A \cdot I_n) I_n^{-1} \quad \text{projection onto } \mathcal{A}_n(I) \\ A_\perp & = & P_{I_n}^{\perp}(A) & \overset{\text{def}}{=} & (A \wedge I_n) I_n^{-1} \quad \text{rejection} \end{array}\right\} . \tag{155}$$

Example It is instructive to consider here the simplest possible case. Let a, b be vectors in $\mathcal{A}_n$. We can treat b as the pseudoscalar of a one-dimensional subspace. Now

$$P_b(a) \overset{(155)}{=} (a \cdot b) b^{-1} \tag{156}$$

is a vector parallel to b, while

$$P_b^{\perp}(a) \overset{(155)}{=} a \wedge b b^{-1} = a \wedge \underbrace{b \wedge b^{-1}}_{=0} + (a \wedge b) \cdot b^{-1} \tag{157}$$

is perpendicular to b, since

$$P_b^{\perp}(a) \cdot b = \left[(a \wedge b) \cdot b^{-1}\right] \cdot b = (a \wedge b) \cdot \underbrace{(b^{-1} \wedge b)}_{=0} = 0 \quad . \tag{158}$$

Properties The with (155) defined **projection operator** is linear and grade preserving

$$\left.\begin{array}{lll} P(\mu A + \nu B) & = & \mu P(A) + \nu P(B) \\ P(\langle A \rangle_r) & = & \langle P(A) \rangle_r \end{array}\right\} , \tag{159}$$

as can be seen from the definition. Further it has the property of a projection

$$P(P(A)) = P(A) \quad . \tag{160}$$

Another obvious consequence of the definition is

$$P_{I_n}(B_r) = 0 \quad \text{if } r > n \quad , \tag{161}$$

since the simple r-vector B_r must contain vectors linearly independent of any choice of the n basis vectors giving $I_n = e_1 \wedge \dots \wedge e_n$.

The projection operator is an *outermorphism*, since for $n \geq s+t$ we have

$$\begin{aligned} P(A_s \wedge B_t) &\overset{(155)}{=} (A_s \wedge B_t) \cdot I_n I_n^{-1} = A_s \cdot \overbrace{(B_t \cdot I_n)}^{=P(B_t)\cdot I_n} I_n^{-1} = A_s \cdot (P(B_t) \cdot I_n) I_n^{-1} \\ &= [A_s \wedge P(B_t)] \cdot I_n I_n^{-1} \overset{(68)}{=} (-1)^{ts} [P(B_t) \wedge A_s] \cdot I_n I_n^{-1} \\ &= (-1)^{ts} P(B_t) \cdot \overbrace{(A_s \cdot I_n)}^{=P(A_s)\cdot I_n} I_n^{-1} = (-1)^{ts} P(B_t) \cdot (P(A_s) \cdot I_n) I_n^{-1} \\ &= (P(A_s) \wedge P(B_t)) \cdot I_n I_n^{-1} = P(A_s) \wedge P(B_t) \end{aligned} \tag{162}$$

and for $n < s+t$ it follows immediately from (161).

For $P(a) = a$ we have

$$\begin{aligned} P(a \wedge B_r) &\overset{(162)}{=} P(a) \wedge P(B_r) = a \wedge P(B_r) \\ P(a \cdot B_r) &= (a \cdot B_r) \cdot I_n I_n^{-1} = a \wedge \overbrace{(B_r \cdot I_n)}^{=P(B_r)I_n} I_n^{-1} = a \wedge (P(B_r) I_n) I_n^{-1} = a \cdot P(B_r) \\ \Rightarrow P(aB_r) &= P(a \cdot B_r) + P(a \wedge B_r) = a \cdot P(B_r) + a \wedge P(B_r) = aP(B_r) \quad . \end{aligned} \tag{163}$$

This allows us to conclude under use of the linearity of P and by successive application for multivectors A, B

$$P(AB) = AP(B) \quad \text{if } P(A) = A \quad . \tag{164}$$

1.2 Differentiation

1.2.1 Definitions

Directional and Vector Derivative The usual derivative extends in a natural way to the geometric algebra, if one defines, in analogy to the vector calculus, the directional derivative by

$$a \cdot \partial_x F(x) \overset{\text{def}}{=} \lim_{\epsilon \to 0} \frac{F(x + \epsilon a) - F(x)}{\epsilon} \quad . \tag{165}$$

Let $\{e_\nu\}$ be an *orthonormal* basis of $\mathcal{A}_n$. The directional derivative of the μ-component in the direction of e^ν becomes

$$e^\nu \cdot \partial_x (x \cdot e^\mu) \overset{(165)}{=} \frac{x^\mu + \epsilon e^\nu \cdot e^\mu - x^\mu}{\epsilon} = e^\nu \cdot e^\mu \quad , \tag{166}$$

while in direction of e_ν

$$e_\nu \cdot \partial_x (x \cdot e^\mu) \overset{(165)}{=} \frac{x^\mu + \epsilon e_\nu \cdot e^\mu - x^\mu}{\epsilon} = \delta^\mu_\nu \quad . \tag{167}$$

Thus we relate the directional derivative and the component derivative by

$$e_\nu \cdot \partial_x = \frac{\partial}{\partial x^\nu} \overset{\text{def}}{=} \partial_{x^\nu} \quad \Rightarrow \quad e^\nu \cdot \partial_x = \partial_{x_\nu} \quad . \tag{168}$$

(166) and (167) read now

$$\partial_{x^\nu} x^\mu = \delta^\mu_\nu \quad ; \quad \partial_{x_\nu} x^\mu = e^\nu \cdot e^\mu \tag{169}$$

and we see that for an orthonormal basis

$$\partial_{x_\nu} \overset{(168)}{=} e^\nu \cdot \partial_x \overset{(36)}{=} \sum_{\mu=1}^{n} \eta^{\nu\mu} e_\mu \cdot \partial_x \overset{(168)}{=} \sum_{\mu=1}^{n} \eta^{\nu\mu} \partial_{x^\mu} \quad . \tag{170}$$

We define ∂_x as an operator, which has the properties of a vector, so that the operator $a \cdot \partial_x$ in (165) can be treated as a scalar. Thus we can represent ∂_x with the help of an orthonormal basis e_ν (of the linear vector space[19] $\mathcal{A}_n$) and the directional derivative (165) as

$$\partial_x \overset{\text{def}}{=} \sum_\nu e^\nu \, (e_\nu \cdot \partial_x) = \sum_\nu e_\nu \partial_{x_\nu} \quad . \tag{171}$$

[19] That is a flat space.

We define the derivative operators to be *acting on the function to the right.* If is acting on something else we will use a dot, *e.g.*, $\dot{\partial}_x f(x)\dot{g}(x)$, to indicate the scope of the operator. In such cases it is useful to substitute the representation (171)

$$\dot{\partial}_x A(x)\dot{B}(x) \overset{(171)}{=} \sum_\nu e^\nu (e_\nu \cdot \dot{\partial}_x) A(x)\dot{B}(x) = \sum_\nu e^\nu A(x)\left[(e_\nu \cdot \partial_x) B(x)\right] \quad . \tag{172}$$

The Differential We will call the linear function

$$\underline{F}(x, a) \overset{\text{def}}{=} a \cdot \partial_x F(x) = \frac{1}{2}\left(a\dot{\partial}_x + \dot{\partial}_x a\right)\dot{F} \tag{173}$$

the **differential** of $F(x)$[20]. Obviously one has

$$\underline{F}(x, a) = a \cdot \partial_x F(x) = F(a) \Leftrightarrow F(x) \text{ is a linear function.} \tag{174}$$

Product and Chain Rule for Directional Derivative and Differential The differential obeys the product rule

$$\underline{FG} = \underline{F}G + G\underline{F} \tag{175}$$

as can be seen from the definition of the directional derivative (165) in the usual way

$$\begin{aligned} a \cdot \partial_x (FG) &\overset{(165)}{=} \lim_{\epsilon \to 0} \frac{F(x+\epsilon a)G(x+\epsilon a) - F(x)G(x)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{F(x+\epsilon a)G(x+\epsilon a) - F(x)G(x+\epsilon a) + F(x)G(x+\epsilon a) - F(x)G(x)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \left[\frac{F(x+\epsilon a) - F(x)}{\epsilon} G(x+\epsilon a) + F(x)\frac{G(x+\epsilon a) - G(x)}{\epsilon}\right] \\ &\overset{(165)}{=} \underline{F}G + G\underline{F} \quad . \end{aligned} \tag{176}$$

We derive further the chain rule

$$\begin{aligned} a \cdot \partial_x G(f(x)) &\overset{(165)}{=} \lim_{\epsilon \to 0} \frac{G(f(x+\epsilon a)) - G(f(x))}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{G(f(x) + \epsilon a \cdot \partial_x f(x)) - G(f(x))}{\epsilon} \\ &= (a \cdot \partial_x f(x)) \cdot \partial_f G(f(x)) && (177) \\ &\overset{(173)}{=} \underline{G}(f(x), \underline{f}(x)) \quad . && (178) \end{aligned}$$

First we consider the derivative of the inner product to find the most important identity

$$\partial_x a \cdot x \overset{(171)}{=} e_\nu\, e^\nu \cdot \partial_x\, a \cdot x \overset{(165)}{=} e_\nu \lim_{\epsilon \to 0} \frac{a \cdot (x + \epsilon e^\nu) - a \cdot x}{\epsilon} = e_\nu a \cdot e^\nu = a \quad . \tag{179}$$

If a is in the vector space $\mathcal{A}_n$ spanned by $\{e_\nu\}$, we can now use (179) to express the vector derivative operator as

$$\partial_x F \overset{(179)}{=} \partial_a a \cdot \partial_x F \overset{(173)}{=} \partial_a \underline{F}(x, a) = \partial_a \underline{F}(a) \quad . \tag{180}$$

One can extend these results to the vector derivative, *i.e.*, for the product rule

$$\partial_x (FG) = \dot{\partial}_x \dot{F} G + \dot{\partial}_x F \dot{G} \quad , \tag{181}$$

and the chain rule for a vector-valued function $g(x)$

$$\begin{aligned} \partial_x F(g(x)) \overset{(180)}{=} \partial_a a \cdot \partial_x F(g(x)) &\overset{(178)}{=} \partial_a \underline{F}(\underline{g}(a)) = \partial_a \left\{\underline{g}(a) \cdot \partial_g\right\} F(g) \\ &= \bar{g}(\partial_g) F(g) \end{aligned} \tag{182}$$

with the adjoint[21]

$$\bar{g}(x, a) \overset{\text{def}}{=} \partial_x \left[g(x) \cdot a\right] \Rightarrow \bar{g}(x, a) \cdot b = a \cdot \underline{g}(x, b) \quad . \tag{183}$$

[20] We can even have $a \notin \mathcal{G}^1$, but then the dot product projects out the part in $\mathcal{G}^1$, so that

$$a \cdot \partial_x = P(a) \cdot \partial_x \quad ,$$

where $P(a)$ is the projection operator.

[21] We will have a closer look at differential and adjoint in a Section 1.3.2.

Second Differential and Integrability Condition The directional derivative, defined by (165), behaves like a scalar and thus commutes with every multivector[22], *i.e.*,

$$(a \cdot \dot{\partial}_x) A\dot{F}(x)B = A(a \cdot \dot{\partial}_x)\dot{F}(x)B = A\dot{F}(x)(a \cdot \dot{\partial}_x)B \quad . \tag{184}$$

Since directional derivatives commute with themselves, we derive for the *second* differential

$$F_{ab} = F_{ba} \quad , \tag{185}$$

where we defined F_a as an abbreviation for the differential, *i.e.*,

$$F_a \stackrel{\text{def}}{=} \underline{F}(x, a) \quad . \tag{186}$$

Using (180) we find for the outer product of two vector derivative operators

$$\partial_x \wedge \partial_x F \stackrel{(180)}{=} \underbrace{\overbrace{\partial_b \wedge \partial_a}^{\text{antisymm.}} \overbrace{F_{ab}}^{\text{symm.}}}_{\text{independent of } a,b} = -\partial_a \wedge \partial_b F_{ba} = -\partial_b \wedge \partial_a F_{ab} = 0 \quad , \tag{187}$$

where we renamed a, b in the second last step and compared with the second step from the left. Thus

$$\partial_b \wedge \partial_a F_{ab} \stackrel{(180)}{=} \partial_x \wedge \partial_x F = 0 \tag{188}$$

and we derived the so-called *integrability condition*

$$\partial_x \wedge \partial_x = 0 \quad . \tag{189}$$

This allows us finally to conclude that the **Laplacian**

$$\partial_x \partial_x = \partial_x \cdot \partial_x + \overbrace{\partial_x \wedge \partial_x}^{\stackrel{(189)}{=} 0} = \partial_x \cdot \partial_x \tag{190}$$

is a *scalar operator*.

1.2.2 Derivatives of some simple Functions

We will derive the derivatives for some functions, which allow us to construct more complicated derivatives with the help of the chain and product rule. We start with the differential of the identity, which is easily calculated by

$$a \cdot \partial_x x = \underline{x}(a) \stackrel{(165)}{=} \lim_{\epsilon \to 0} \frac{x + \epsilon a - x}{\epsilon} = a \quad . \tag{191}$$

Thus

$$\partial_x x = e^\nu \overbrace{e_\nu \cdot \partial_x x}^{\stackrel{(191)}{=} e_\nu} \stackrel{(191)}{=} e^\nu e_\nu = n \quad , \tag{192}$$

where $\{e_\nu\}$ is a basis of the n-dimensional space. By using the product rule this yields

$$\partial_x x^2 = \partial_a a \cdot \partial_x (xx) = \partial_a (\underline{x}x + x\underline{x}) \stackrel{(191)}{=} \partial_a (ax + xa) \stackrel{(69)}{=} 2\partial_a a \cdot x \stackrel{(179)}{=} 2x \quad . \tag{193}$$

This result allows us to see that

$$\partial_x \wedge x \stackrel{(193)}{=} \frac{1}{2} \partial_x \wedge \partial_x x^2 \stackrel{(189)}{=} 0 \quad . \tag{194}$$

The chain rule with a scalar function helps to find

$$\partial_x |x| \;=\; \partial_x \sqrt{x^2} = \frac{1}{2\sqrt{x^2}} \partial_x x^2 \stackrel{(193)}{=} \frac{x}{|x|} \quad . \tag{195}$$

[22] Note that the scope of the operator stays the same.

We can apply this result, to find with the help of the chain rule

$$\partial_x|x|^k \overset{(181)}{=} k|x|^{k-1}\partial_x|x| \overset{(195)}{=} k|x|^{k-1}\frac{x}{|x|} = k|x|^{k-2}x \quad . \tag{196}$$

A further application of the product rule yields

$$\partial_x\left[\frac{x}{|x|^k}\right] \overset{(181)}{=} |x|^{-k}\overbrace{\partial_x x}^{\overset{(192)}{=}n} + x\partial_x|x|^{-k} \overset{(196)}{=} n|x|^{-k} + (-k)x|x|^{-k-2}x = \frac{n-k}{|x|^k} \quad . \tag{197}$$

This allows us finally to calculate

$$\partial_x \log|x| = \frac{1}{|x|}\partial_x|x| \overset{(195)}{=} \frac{1}{|x|}\frac{x}{|x|} = x^{-1} \quad . \tag{198}$$

Given a simple r-vector A_r we have[23]

$$\left.\begin{array}{lclcl} \partial_x \wedge (x \cdot A_r) & = & \partial_x \wedge \left[\sum_{k=1}^{r}(-1)^{k+1} x \cdot a_k (a_1 \wedge \cdots \wedge \check{a}_k \wedge \cdots \wedge a_r)\right] & = & rA_r \\ \partial_x \cdot (x \wedge A_r) & = & \partial_x x A_r - \partial_x \wedge (x \cdot A_r) & = & (n-r)A_r \end{array}\right\} . \tag{199}$$

Writing ∂_x and x in components yields

$$\textstyle\sum_\nu e_\nu \wedge (e^\nu \cdot A_r) \overset{(199)}{=} rA_r \quad ; \quad \sum_\nu e_\nu \cdot (e^\nu \cdot A_r) \overset{(199)}{=} 0 \quad . \tag{200}$$

1.2.3 Derivative with respect to Simple Multivectors

Given a simple multivector $A_r \overset{\text{def}}{=} a_1 \wedge \ldots \wedge a_r$ we derive by using (109) and (180)

$$\partial_{a_r} \wedge \ldots \wedge \partial_{a_1} \underbrace{(a_1 \wedge \ldots \wedge a_r) * (b_r \wedge \ldots \wedge b_1)}_{r!\ \text{combinations}} = r!\, b_r \wedge \ldots \wedge b_1 \quad . \tag{201}$$

We identify thus the derivative with respect to A_r as

$$\partial_{A_r} \overset{\text{def}}{=} \frac{1}{r!}\partial_{a_r} \wedge \ldots \wedge \partial_{a_1} \quad . \tag{202}$$

This definition leads to the generalization of (179) to simple r-vectors, *i.e.*,

$$\partial_{A_r} A_r * B = \langle B \rangle_r \quad . \tag{203}$$

1.2.4 The Multivector Derivative

We can extend the definition of the directional derivative to arbitrary multivectors by defining

$$A * \partial_X F(X) \overset{\text{def}}{=} \frac{F(X + \epsilon A) - F(X)}{\epsilon} \quad , \tag{204}$$

where $F(X)$ is dependent on a multivector X containing all grades. That means, $F(\langle X \rangle_r)$ must be defined for all r. By definition $A * \partial_X$ is a scalar-valued operator and commutes with everything. If $F(X)$ does not depend on grade r of its argument, it follows from (204)

$$\langle A \rangle_r * \partial_X F(X) = 0 \quad . \tag{205}$$

Thus if $F(X) = F(\langle X \rangle_r)$, *i.e.*, $F(X)$ depends only on grade r,

$$A * \partial_X F(\langle X \rangle_r) = \langle A \rangle_r * \partial_X F(\langle X \rangle_r) \quad . \tag{206}$$

The derivative of the identity becomes

$$A * \partial_X X \overset{(204)}{=} \lim_{\epsilon \to 0} \frac{X + \epsilon A - X}{\epsilon} = A \quad . \tag{207}$$

[23] For the case $r = 0$ remember the definition $x \cdot \lambda = 0$.

Expressing A in a basis $\{e_J\}$ of the algebra [24] $\mathcal{G}(\mathcal{A}_n)$ we have

$$A * \partial_X = \left[e_J e^J * A\right] * \partial_X = A * \left[e^J e_J * \partial_X\right] \quad . \tag{208}$$

Thus we derive the identification

$$\partial_X \stackrel{\text{def}}{=} e^J e_J * \partial_X \stackrel{\text{def}}{=} e^J \partial_J \quad , \tag{209}$$

with

$$\partial_K \stackrel{\text{def}}{=} e_K * \partial_X \quad , \tag{210}$$

which is the analogue to (171).

From the definition (204) we derive easily the fundamental relation

$$e_J * \partial_X \langle XA \rangle = \frac{\langle [X + \epsilon e_J] A \rangle - \langle XA \rangle}{\epsilon} = \langle e_J A \rangle = e_J * A \qquad \forall J \tag{211}$$

and thus for the multivector derivative of the scalar product (as a further generalization of (203))

$$\partial_X X * A = \partial_X \langle XA \rangle = A \quad . \tag{212}$$

This yields immediately

$$\partial_X \langle X \rangle_r * A = \partial_X X * \langle A \rangle_r \stackrel{(212)}{=} \langle A \rangle_r \tag{213}$$

and as an obvious generalization for $X \in \mathcal{G}(\mathcal{A}_n)$,

$$\partial_X X * A = P(A) \quad . \tag{214}$$

As an application we will consider the derivative of the commutator product. Let A, C be arbitrary multivectors. Then

$$\begin{aligned}
\partial_B \langle (A \times \langle B \rangle_r) C \rangle &\stackrel{(86)}{=} \frac{1}{2} \partial_B \langle (A \langle B \rangle_r - \langle B \rangle_r A) C \rangle = \frac{1}{2} \langle A \langle B \rangle_r C - \langle B \rangle_r A C \rangle \\
&\stackrel{(85)}{=} \frac{1}{2} \partial_B \langle A \langle B \rangle_r C - A C \langle B \rangle_r \rangle \stackrel{(213)}{=} \frac{1}{2} \langle CA - AC \rangle_r \\
&\stackrel{(86)}{=} \langle C \times A \rangle_r \quad .
\end{aligned} \tag{215}$$

Let C be a simple multivector of grade $r \leq n$ in an algebra $\mathcal{G}(\mathcal{A}_n)$, spanned by n basis vectors. For the grade r we need then $\binom{n}{r}$ basis vectors e^{K_r}. Then with

$$C^{K_r} \stackrel{\text{def}}{=} C \cdot e^{K_r} \quad ; \quad C = C^{J_r} e_{J_r} \tag{216}$$

the derivative of the identity becomes

$$\partial_C C \stackrel{(209)}{=} e^{K_r} \overbrace{\partial_{C_{K_r}} C^{J_r}}^{=\delta^{J_r}_{K_r}} e_{J_r} = e^{K_r} e_{K_r} = \binom{n}{r} \quad . \tag{217}$$

This is an obvious generalization of (192) and we can conclude

$$\partial_A A = \langle \partial_A A \rangle = \partial_A * A \quad . \tag{218}$$

If X contains all grades and we label the basis of *the whole algebra* $\mathcal{G}(\mathcal{A}_n)$ as e^{J_r}, we derive[25]

$$\partial_x X = \sum_{r,s=0}^{n} e^{J_r} \overbrace{\partial_{X^{J_r}} X^{K_s}}^{=\delta^{K_s}_{J_r}} e_{K_s} = \sum_{r=0}^{n} e^{J_r} e_{J_r} = \sum_{r=0}^{n} \binom{n}{r} = 2^n \quad . \tag{221}$$

[24] Note that J is a multi-index! It labels basis vectors for scalar, vectors, bivectors etc. at once.

[25] Here we use the binomial coefficients, defined by

$$\binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k!(n-k)!} \quad , \tag{219}$$

and the relation

$$2^n = (1+1)^n = \sum_{r=0}^{n} \binom{n}{r} \quad . \tag{220}$$

1.2.5 Operator Identities

Let us define the Lie bracket

$$[a(x), b(x)]_{\mathrm{Lie}} \stackrel{\mathrm{def}}{=} a(x) \cdot \dot{\partial}_x \dot{b}(x) - b(x) \cdot \dot{\partial}_x \dot{a}(x) = \underline{b}(a) - \underline{a}(b) \quad . \tag{222}$$

With this definition we gain the identity

$$\begin{aligned}
\{b(x) \wedge a(x)\} \cdot \left\{\dot{\partial}_x \wedge \ddot{\partial}_x\right\} \ddot{f}(x) &= \left[a(x) \cdot \dot{\partial}_x b(x) \cdot \ddot{\partial}_x - b(x) \cdot \dot{\partial}_x a(x) \cdot \ddot{\partial}_x\right] \ddot{f}(x) \\
&= a(x) \cdot \partial_x \left[b(x) \cdot \partial_x f(x)\right] - b(x) \cdot \partial_x \left[a(x) \cdot \partial_x f(x)\right] \\
&\quad - \left[a(x) \cdot \dot{\partial}_x \dot{b}(x) - b(x) \cdot \dot{\partial}_x \dot{a}(x)\right] \cdot \dot{\partial}_x \dot{f}(x) \\
&= \left[a(x) \cdot \partial_x, b(x) \cdot \partial_x\right] f(x) - [a(x), b(x)]_{\mathrm{Lie}} \cdot \dot{\partial}_x \dot{f}(x) \quad .
\end{aligned} \tag{223}$$

Please note, that this expression vanishes according to (189), if we work in a flat vector space. But we will see later, that the extension to a curved manifold changes the integrability condition and (223) can be nonzero.

1.3 Linear Functions

Definitions We call any function $F(X)$ obeying

$$F(\lambda X + \kappa Y) = \lambda F(X) + \kappa F(Y) \tag{224}$$

a linear function.

Outermorphism If $f(x)$ is a linear function of a *vector* valued argument x, we can always extend $f(x)$ to the whole algebra by defining

$$f(a_1 \wedge \cdots \wedge a_n) \stackrel{\mathrm{def}}{=} f(a_1) \wedge \cdots \wedge f(a_n) \quad . \tag{225}$$

We call $f(A_r)$ the **outermorphism** of $f(a)$.

1.3.1 Determinants

For a linear vector valued transformation $h(a)$ in a n-dimensional vector space with the matrix elements

$$h_i \stackrel{\mathrm{def}}{=} h(e_i) \quad ; \quad h_{ij} \stackrel{\mathrm{def}}{=} h_i \cdot e_j \tag{226}$$

we define

$$\begin{aligned}
|h| \stackrel{\mathrm{def}}{=} \det(h) &\stackrel{\mathrm{def}}{=} (h_n \wedge \ldots \wedge h_1) \cdot (e_1 \wedge \ldots \wedge e_n) \\
&= h_n \cdot (\ldots (h_1 \cdot (e_1 \wedge \ldots \wedge e_n)) \ldots) \\
&= h(I_n) \cdot I_n^{-1} \quad .
\end{aligned} \tag{227}$$

This is consistent, since

1. it is *linear* in rows (e_i) and columns (h_j)
2. it *changes sign under exchange* of rows or columns
3. if rows or columns are *linearly dependent* it is zero, since the outer product is then zero.
4. it *does not change under exchange of rows with columns.*

As explicit examples we give the case of $n = 1, 2$:

- **Example $n = 1$:**

$$\det h \stackrel{(227)}{=} h_1 \cdot e_1 = h_{11} \tag{228}$$

- **Example** $n = 2$:

$$\det h \overset{(227)}{=} (h_2 \wedge h_1) \cdot (e_1 \wedge e_2) \overset{(125)}{=} (h_1 \cdot e_1)(h_2 \cdot e_2) - (h_1 \cdot e_2)(h_2 \cdot e_1) \overset{(226)}{=} h_{11}h_{22} - h_{12}h_{21} = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \tag{229}$$

General Proof The inner product of a vector with a simple n-vector is given by (109), *i.e.*, [26]

$$h_i \cdot (e_1 \wedge \ldots \wedge e_n) = \sum_{j=1}^{n} (-1)^{j+1} h_i \cdot e_j (e_1 \wedge \ldots \wedge \check{e}_j \wedge \ldots \wedge e_n) \quad . \tag{230}$$

Together with the skew symmetry of the outer product

$$h_n \wedge \ldots \wedge h_1 = (-1)^{i+1} h_n \wedge \ldots \wedge \check{h}_i \wedge \ldots \wedge h_1 \wedge h_i \tag{231}$$

this yields

$$(h_n \wedge \ldots \wedge h_1) \cdot (e_1 \wedge \ldots \wedge e_n) = \sum_{j=1}^{n} (-1)^{i+j} h_i \cdot e_j (h_n \wedge \ldots \wedge \check{h}_i \wedge \ldots \wedge h_1) \cdot (e_1 \wedge \ldots \wedge \check{e}_j \wedge \ldots \wedge e_n) \quad . \tag{232}$$

By interpreting the last term as a determinant of grade $n-1$, which is constructed out of the grade n determinant by taking out row i and column j according to (227), this proves recursively the equivalence of the definition (227) to the ordinary one, since we can reduce higher-order determinants with the help of (232) to the case $n = 2$, which is shown with (229) to be the usual determinant.

Determinant of non-commuting Quantities We define here the determinant of non-commuting quantities σ_{nm} by demanding that the determinant can be developed in the usual way, if we start with the top row, and that for any σ_{11}

$$|\sigma_{11}| = \sigma_{11} \quad . \tag{233}$$

The determinant of four vectors is as an example

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad . \tag{234}$$

Note that the determinant *changes now under exchange of rows and columns.* Also the exchange of two rows or columns has a more complicated effect than just a sign change.

A determinant of commuting quantities, like scalars, is zero, if two rows or columns are linearly dependent. But for a determinant of non-commuting quantities, like vectors, this is not the case. Instead we have for instance for two vectors

$$\begin{vmatrix} b_1 & b_2 \\ b_1 & b_2 \end{vmatrix} = b_1 b_2 - b_2 b_1 = 2 b_1 \wedge b_2 \quad . \tag{235}$$

This generalizes to

$$\begin{vmatrix} b_1 & \cdots & b_n \\ \vdots & & \vdots \\ b_1 & \cdots & b_n \end{vmatrix} = n!\, b_1 \wedge \cdots \wedge b_n \quad . \tag{236}$$

A proof via induction is given in Appendix A. We use the opportunity here to show how this formalism can be usefully applied to express the inner product $A_r \cdot B_s$ for two *simple* r, s-*vectors*

[26] Quantities with a check are skipped.

$A_r \stackrel{\text{def}}{=} a_r \wedge \cdots \wedge a_1$ and $B_s \stackrel{\text{def}}{=} b_1 \wedge \cdots \wedge b_s$. Under use of the derivative with respect to a simple r-vector (202) we derive for $s > r$

$$\begin{aligned} A_r \cdot B_s &= \partial_{C_{s-r}} C_{s-r} \cdot (A_r \cdot B_s) = \partial_{C_{s-r}} (C_{s-r} \wedge A_r) * B_s \\ &= \partial_{C_{s-r}} (c_s \wedge \cdots \wedge c_{r+1} \wedge a_r \wedge \cdots \wedge a_1) * (b_1 \wedge \cdots \wedge b_s) \\ &= \partial_{C_{s-r}} \begin{vmatrix} a_1 \cdot b_1 & \cdots & a_1 \cdot b_s \\ \vdots & & \vdots \\ a_r \cdot b_1 & \cdots & a_r \cdot b_s \\ c_{r+1} \cdot b_1 & \cdots & c_{r+1} \cdot b_s \\ \vdots & & \vdots \\ c_s \cdot b_1 & \cdots & c_s \cdot b_s \end{vmatrix} \stackrel{(202)}{=} \frac{1}{(s-r)!} \begin{vmatrix} a_1 \cdot b_1 & \cdots & a_1 \cdot b_s \\ \vdots & & \vdots \\ a_r \cdot b_1 & \cdots & a_r \cdot b_s \\ b_1 & \cdots & b_s \\ \vdots & & \vdots \\ b_1 & \cdots & b_s \end{vmatrix} \quad . \end{aligned} \tag{237}$$

1.3.2 Differential and Adjoint

Definitions As defined with (173) we call

$$\underline{F}(A) \stackrel{\text{def}}{=} A * \partial_X F(X) \tag{238}$$

the **differential**. It is still dependent on X, even if we only indicate the argument in which it is linear. We define now the **adjoint** as

$$\bar{F}(A) \stackrel{\text{def}}{=} \partial_X (F(X) * A) \quad , \tag{239}$$

so that the relation

$$\underline{F}(A) * B \stackrel{(238)}{=} (A * \partial_X)(F(X) * B) = A * \bar{F}(B) \tag{240}$$

holds.

Fundamental Relation between Differential and Adjoint For $s \geq r$ adjoint and differential obey

$$\begin{aligned} A_r \cdot \bar{f}(B_s) &= \frac{1}{\binom{n}{s-r}} [\partial_C \cdot (C_{s-r} \wedge A_r)] \cdot \bar{f}(B_s) = \frac{1}{(s-r)!} \langle \partial_C C \wedge A_r \bar{f}(B_s) \rangle_{s-r} \\ &= \frac{1}{(s-r)!} \langle \partial_C \underline{f}(C \wedge A_r) B_s \rangle_{s-r} = \frac{1}{(s-r)!} \langle \partial_C \underline{f}(C) \wedge \underline{f}(A_r) B_s \rangle_{s-r} \\ &= \frac{1}{(s-r)!} \partial_C \underline{f}(C) \cdot [\underline{f}(A_r) \cdot B_s] = \frac{1}{(s-r)!} \partial_C C \bar{f}(\underline{f}(A_r) \cdot B_s) \\ &= \bar{f}(\underline{f}(A_r) \cdot B_s) \end{aligned} \tag{241}$$

and analogously for $s \leq r$

$$\underline{f}(A_r) \cdot B_s = \underline{f}(A_r \cdot \bar{f}(B_s)) \quad . \tag{242}$$

Transformed Derivative Often we will encounter equations involving a linear function of the derivative operator. Let $\underline{F}(A)$ and $\underline{h}(A)$ be linear functions. Since

$$\bar{h}(\partial_A) = \partial_C C * \bar{h}(\partial_A) = \partial_C \underline{h}(C) * \partial_A \tag{243}$$

we can conclude

$$\bar{h}(\partial_A) \underline{F}(A) \stackrel{(243)}{=} \partial_C \underline{h}(C) * \partial_A \underline{F}(A) = \partial_C \underline{F}(\underline{h}(C) * \partial_A A) = \partial_C \underline{F}(\underline{h}(C)) \quad . \tag{244}$$

In a relation of this kind we can thus exchange

$$\underline{h}(A), \partial_A \leftrightarrow A, \bar{h}(\partial_A) \quad . \tag{245}$$

This relation is of high value for further calculations, especially gravity gauge theory.

Determinants The fundamental property (240) yields immediately

$$\det \underline{f} \overset{(227)}{=} I^{-1}\underline{f}(I) = \bar{f}(I^{-1})I = I^{-1}\bar{f}(I) = \det \bar{f} \quad . \tag{246}$$

Thus *differential and adjoint have the same determinant.*

1.3.3 Symmetric and antisymmetric Linear Functions

Definition We will call a linear function $\underline{f}$ **symmetric,** if

$$\underline{f} = \bar{f} \tag{247}$$

and **antisymmetric or skew symmetric** if

$$\underline{f} = -\bar{f} \quad . \tag{248}$$

Unique Decomposition We can decompose every linear function in a symmetric and antisymmetric part by defining

$$\underline{f}_+ \overset{\text{def}}{=} \tfrac{\underline{f}+\bar{f}}{2} \quad ; \quad \underline{f}_- \overset{\text{def}}{=} \tfrac{\underline{f}-\bar{f}}{2} \quad , \tag{249}$$

so that

$$\underline{f} = \underline{f}_+ + \underline{f}_- \quad ; \quad \bar{f}_\pm = \pm\underline{f}_\pm \quad . \tag{250}$$

Under use of the definition of the adjoint (239)

$$\begin{aligned} \underline{f}_+(a) &= \frac{1}{2}\left\{\underline{f}(a) + \bar{f}(a)\right\} = \frac{1}{2}\left\{\partial_b b \cdot \underline{f}(a) + \partial_b b \cdot \bar{f}(a)\right\} = \frac{1}{2}\{\partial_b b \cdot \underline{f}(a) + \partial_b a \cdot \underline{f}(b)\} \\ &= \frac{1}{2}\partial_a\,[a \cdot \underline{f}(a)] \end{aligned} \tag{251}$$

$$\begin{aligned} \underline{f}_-(a) &= \frac{1}{2}\left[\underline{f}(a) - \bar{f}(a)\right] = \frac{1}{2}\left[a \cdot \partial_b \underline{f}(b) - \partial_b b \cdot \bar{f}(a)\right] = \frac{1}{2}\,[a \cdot \partial_b \underline{f}(b) - \partial_b a \cdot \underline{f}(b)] \\ &= \frac{1}{2} a \cdot [\partial_b \wedge \underline{f}(b)] \quad . \end{aligned} \tag{252}$$

Thus

$$\dot{\partial}_x \wedge \underline{f}(\dot{x}) \overset{(245)}{=} \dot{x} \wedge \underline{f}(\dot{\partial}_x) = 0 \Leftrightarrow \underline{f}_-(x) = 0 \Leftrightarrow b \cdot \underline{f}(a) = a \cdot \underline{f}(b) \quad . \tag{253}$$

For a symmetric function and a bivector B_2 we derive the useful identity

$$\partial_a \cdot f_+(a \cdot B_2) = (a \cdot B_2) \cdot f_+(\partial_a) = -B_2 \cdot \underbrace{[a \wedge f_+(\partial_a)]}_{\overset{(253)}{=} 0} \quad . \tag{254}$$

Further the divergence of a linear function, that is it's trace,

$$\begin{aligned} \dot{\partial}_x \cdot \underline{f}(\dot{x}) &= \frac{1}{2}\left\{\partial_x \cdot \underline{f}(x) + \bar{f}(\partial_x) \cdot x\right\} = \frac{1}{2}\left\{\partial_x \cdot \underline{f}(x) + \dot{\bar{f}}(x) \cdot \dot{\partial}_x\right\} \\ &= \partial_x \cdot \underline{f}_+(x) \end{aligned} \tag{255}$$

is completely determined by the symmetric part of $\underline{f}$. This is just the statement, that the trace of a matrix is determined by its symmetric part, since the antisymmetric part cannot have any diagonal elements. Thus an antisymmetric function will always have $\partial_x \cdot \underline{f} = 0$. But *a symmetric function can also be trace-free!* This relation translates trivially to linear functions of a multivector argument.

Let now $F(a_1 \wedge \cdots \wedge a_n)$ be a linear function of grade n with

$$\partial_{a_1} \wedge F(a_1 \wedge \cdots \wedge a_n) = 0 \quad . \tag{256}$$

Then obviously

$$(\partial_{a_n} \wedge \cdots \wedge \partial_{a_1}) \wedge F(a_1 \wedge \cdots \wedge a_n) = 0 \quad . \tag{257}$$

The antisymmetry in the the a_r ensures[27]

$$\partial_{A_n} F(A_n) = \langle \partial_{A_n} F(A_n) \rangle_{0,2n} \quad , \tag{259}$$

so that $F(a_1 \wedge \cdots \wedge a_n)$ is a symmetric function. It follows

$$A_n * F(B_n) = B_n * F(A_n) \quad . \tag{260}$$

Note that in general there is no function f^* with

$$a \wedge f(b) = f^*(a) \wedge b \qquad \forall a, b \in \mathcal{A}_n \quad . \tag{261}$$

To see this, let a and b be orthogonal unit vectors. Taking then the outer product of (261) with b yields

$$a \wedge f(b) \wedge b = f^*(a) \wedge b \wedge b = 0 \qquad \forall\, a \in \mathcal{A}_n \quad . \tag{262}$$

Since in $n \leq 2$ dimensions there can not be more than two linearly independent vectors, (262) is fulfilled for all functions f. For $n > 2$ we choose a orthogonal to $f(b), b$ and conclude that

$$f(b) \wedge b = 0 \Leftrightarrow f(b) = \lambda b \quad , \tag{263}$$

with an arbitrary scalar λ.

Example Projection Tensor With the tools developed we are now in the position to calculate the adjoint of the projection tensor. Since $P(X) \in \mathcal{G}(\mathcal{A}_n)$ it follows

$$\bar{P}(A) = \partial_X P(X) * A \overset{(214)}{=} P(A) \quad . \tag{264}$$

Thus the projection tensor $P(A)$ is a symmetric linear function.

1.3.4 The Inverse of a Linear Function - Cramer's Rule

Often it arises the need to invert a linear function $\underline{f}$ with non-zero determinant. Geometric Algebra supplies us with the necessary tools to give $\underline{f}^{-1}$ in a neat algebraic form. We start with substituting in (242) $A_r \overset{\text{def}}{=} I_n, B \overset{\text{def}}{=} B_s$ to derive

$$\underline{f}(I_n) \cdot B = \underline{f}(I_n \cdot \bar{f}(B)) \quad . \tag{265}$$

Using the definition of the determinant and substituting $B \overset{\text{def}}{=} \bar{f}^{-1}(C)$ yields

$$(\det f)\bar{f}^{-1}(C) I_n = \underline{f}(I_n C) \quad . \tag{266}$$

Thus the desired formula for the inverse of the adjoint becomes

$$\bar{f}^{-1}(C) = \frac{\underline{f}(I_n C) I_n^{-1}}{\det f} \quad . \tag{267}$$

In the same manner we derive for the inverse of the differential

$$\underline{f}^{-1}(C) = \frac{\bar{f}(I_n C) I_n^{-1}}{\det f} \quad . \tag{268}$$

This is the analgue of Cramer's Rule, well known from linear algebra. But it is Geometric Algebra, which allows us to formulate it without any need to introduce a matrix representation.

[27] For all grades except 0 and $2n$ we have a term like

$$\partial_{a_2} \wedge [\partial_{a_1} \cdot F(a_1 \wedge \cdots \wedge a_n)] = -\partial_{a_1} \cdot \underbrace{(\partial_{a_2} \wedge F(\cdots))}_{\overset{(256)}{=}0} + \underbrace{\partial_{a_1} \cdot \partial_{a_2} F(\cdots)}_{=0} = 0 \quad . \tag{258}$$

1.3.5 Tensors

Lets assume a linear vector space $\mathcal{A}_n$. We will call a multivector valued function

$$T(a_1,\dots,a_r) \quad ; \quad a_n = \langle a_n \rangle_1 \quad , \tag{269}$$

a tensor of degree r, if it is linear in the vector arguments a_n[28]. Since T and a_n are not necessarily in $\mathcal{G}(\mathcal{A}_n)$, we define in addition a *tensor field* T_f, for which

$$P\left(T_f(P(a_1),\dots,P(a_r))\right) = T_f(a_1,\dots,a_r) \quad . \tag{270}$$

So for $a_j \in \mathcal{A}_n \forall j$ we have $T_f \in \mathcal{G}(\mathcal{A}_n)$.

If $T(a_1,\dots,a_r)$ is of grade s we say T has rank $s+r$. If T is scalar valued, one gets the relation to usual tensor index notation

$$T_{i\dots j} = T(a_i,\dots,a_j) \quad ; \quad T^{i\dots j} = T(a^i,\dots,a^j) \quad . \tag{271}$$

We can define a **contraction** via

$$\begin{aligned}
\partial_{a_k}\cdot\partial_{a_i}T(a_1,\dots,a_r) &= T(a_1,\dots,a_{i-1},\partial_{a_k},\dots,a_r) \\
&= \sum_\mu T(a_1,\dots,a_{i-1},e^\mu\partial_{a_k^\mu},\dots,a_r) \\
&= \sum_\mu T(a_1,\dots,a_{i-1},e^\mu,\dots,\partial_{a_k^\mu}a_k,\dots,a_r) \\
&= \sum_\mu T(a_1,\dots,a_{i-1},e^\mu,\dots,e_\mu,\dots,a_r) \quad .
\end{aligned} \tag{272}$$

Contraction reduces the degree (and so the rank) by two. A contraction of two tensors over a_i and b_k is thus given by

$$S(a_1,\dots,a_{i-1},\partial_{b_k},\dots,a_n)T(b_1,\dots,b_m) \quad . \tag{273}$$

Let us in addition define the **traction** by

$$\begin{aligned}
\partial_{a_k}T(a_1,\dots,a_k,\dots,a_n) &= \sum_\mu e^\mu T(a_1,\dots,\partial_{a_k^\mu}a_k,\dots,a_n) \\
&= \sum_\mu e^\mu T(a_1,\dots,e_\mu,\dots,a_n) \quad ,
\end{aligned} \tag{274}$$

which is equivalent to the contraction of

$$S \stackrel{\text{def}}{=} a_{n+1}T(a_1,\dots,a_n) \quad , \tag{275}$$

over a_{n+1} and a_k as can be seen with (272).

Split in symmetric and antisymmetric Parts The skew symmetric part of a tensor is given by

$$T_-(a_1,\dots,a_n) \stackrel{\text{def}}{=} \frac{1}{n!}(a_1\wedge\dots\wedge a_n)\cdot(\partial_{b_n}\wedge\dots\wedge\partial_{b_1})T(b_1,\dots,b_n) \quad , \tag{276}$$

while the symmetric part is

$$T_+ = \frac{1}{n!}(a_1\cdot\partial_a)\dots(a_n\cdot\partial_a)T(a,a,\dots,a) \quad . \tag{277}$$

[28] One can extend the definition and allow multivector valued arguments instead of vectors. We then call $T(A_1,\dots,A_r)$ an **extensor**.

The effect of both operations becomes quite clear, if one calculates it explicitly. Under use of the definition of the determinant, we can rewrite[29]

$$\begin{aligned}
T_- &= \frac{1}{n!}\begin{vmatrix} a_1\cdot\partial_{b_1} & a_1\cdot\partial_{b_2} & \dots & a_1\cdot\partial_{b_n} \\ a_2\cdot\partial_{b_1} & a_2\cdot\partial_{b_2} & \dots & \dots \\ \vdots & & & \vdots \\ a_n\partial_{b_1} & \dots & \dots & a_n\cdot\partial_{b_n} \end{vmatrix} T(b_1,\dots,b_n) \\
&= \frac{1}{n!}\sum_{j=1}^{n!}\mathrm{sign}(\pi_j)(a_{\pi_j^1}\cdot\partial_{b_1})(a_{\pi_j^2}\cdot\partial_{b_2})\dots(a_{\pi_j^n}\cdot\partial_{b_n})T(b_1,\dots,b_n) \\
&= \frac{1}{n!}\sum_{j=1}^{n!}\mathrm{sign}(\pi_j)T(a_{\pi_j^1},a_{\pi_j^2},\dots,a_{\pi_j^n}) \quad ,
\end{aligned} \tag{278}$$

what shows nicely how the antisymmetric part is derived. The symmetrization operator works very similar

$$\begin{aligned}
T_+ &= \frac{1}{n!}(a_1\cdot\partial_a)\dots(a_n\cdot\partial_a)T(a,\dots,a) \\
&= \frac{1}{n!}(a_1\cdot\partial_a)\dots(a_{n-1}\cdot\partial_a)[T(a_n,a,\dots,a)+T(a,a_n,a,\dots,a)+\dots+T(a,\dots,a,a_n)] \\
&= \frac{1}{n!}\sum_{j=1}^{n!}T(a_{\pi_j^1},a_{\pi_j^2},\dots,a_{\pi_j^n}) \quad .
\end{aligned} \tag{279}$$

As an example, the usual results are easily obtained for $n = 2$. With

$$\begin{matrix} \pi_1 = \{1,2\} & , & \pi_2 = \{2,1\} \\ \mathrm{sign}(\pi_1) = 1 & , & \mathrm{sign}(\pi_2) = -1 \end{matrix} \quad , \tag{280}$$

it follows immediately

$$\begin{aligned}
T_- &= \frac{1}{2!}\sum_{j=1}^{2!}\mathrm{sign}(\pi_j)T(a_{\pi_j^1},a_{\pi_j^2}) = \frac{1}{2}[T(a_1,a_2)-T(a_2,a_1)] \\
T_+ &= \frac{1}{2!}[T(a_1,a_2)+T(a_2,a_1)] \quad .
\end{aligned} \tag{281}$$

1.3.6 Decomposition of second order Tensors

Given an arbitrary linear function $T(a,b)$, depending on two vectors, we can split it into symmetric and antisymmetric parts by defining

$$T_\pm = \frac{1}{2}[T(a,b)\pm T(b,a)] \Rightarrow T(a,b) = T_+(a,b)+T_-(a,b) \quad . \tag{282}$$

The skew symmetric part depends only on $a\wedge b$, thus

$$T_-(a\wedge b) \stackrel{\mathrm{def}}{=} T_-(a,b) \quad . \tag{283}$$

Since T_- is trace free we derive for the trace

$$T \stackrel{\mathrm{def}}{=} T(\partial_a, a) = T_+(\partial_a, a) \quad . \tag{284}$$

This allows us to define the *trace free* symmetric part by

$$\sigma(a,b) = T_+(a,b) - T(a\cdot b) \quad ; \quad T(a\cdot b) \stackrel{\mathrm{def}}{=} \tfrac{a\cdot b}{\partial_d\cdot d}T \quad , \tag{285}$$

so that

$$T(a,b) = T_-(a\wedge b)+\sigma(a,b)+T(a\cdot b) \quad . \tag{286}$$

When looking at (286) we realize now, that $T_-(a\wedge b)$ and $T(a\cdot b)$ depend only on parts of the geometric products between a and b. We thus define the **extensor**

$$T(ab) \stackrel{\mathrm{def}}{=} T(a\wedge b + a\cdot b) = T_-(a\wedge b)+T(a\cdot b) \quad . \tag{287}$$

[29] Here π_j denotes a set of natural numbers, which is derived from $\{1,\dots,n\}$ by j exchanges of neighbouring elements. There are just $n!$ possible combinations, so that j runs from 1 to $n!$. $\mathrm{sign}(\pi_j)$ is then defined as $(-1)^j$. π_j^i denotes then the i-th element of the set π_j.

1.3.7 Derivative of Tensors

Given a tensor $T = T(x, a_1, \ldots, a_n)$ we define the differential by

$$T_a \stackrel{\text{def}}{=} a \cdot \dot{\partial}_x \dot{T}(x, a_1, \ldots, a_n) \quad . \tag{288}$$

Since the a_j are normally dependent on x, the only way to write it as a total differential is

$$\begin{aligned} T_a &= a \cdot \dot{\partial}_x \dot{T}(x, a_1(x), \ldots, a_n(x)) \\ &= a \cdot \partial_x T(x, a_1(x), \ldots, a_n(x)) - T(x, a \cdot \partial_x a_1(x), \ldots, a_n(x)) - \ldots \\ &\quad \ldots - T(x, a_1(x), \ldots, a \cdot \partial_x a_n(x)) \quad , \end{aligned} \tag{289}$$

so that T_a does not depend on the $a \cdot \partial_x a_j$. In matrix representation this means, that we are only looking at the differentials of the matrix components, but not at the argument, which gets multiplied by the matrix.

1.4 Integration

1.4.1 The n-dimensional Volume Element

We call

$$d^n x \stackrel{\text{def}}{=} d\vec{x}_1 \wedge \cdots \wedge d\vec{x}_n \tag{290}$$

the n-dimensional volume element. Since it is of grade n, it is proportional to the unit pseudoscalar of the n-dimensional space. Thus we define

$$d^n x = |d^n x| I_n \quad . \tag{291}$$

If $f(\vec{x})$ is a vector valued function, we define the vector $\vec{z}$ by

$$\vec{z} \stackrel{\text{def}}{=} f(\vec{x}) \quad . \tag{292}$$

The differentials of $\vec{z}$ become

$$d\vec{z} = \underline{f}(d\vec{x}) \quad . \tag{293}$$

Extending $\underline{f}$ via outermorphism to the whole algebra gives for the new volume element

$$d^n z = \underline{f}(d^n x) = |d^n x|\, \underline{f}(I_n) \quad . \tag{294}$$

Hence we gain the identification

$$d^n x = d^n x (d^n z)^{-1} d^n z = \frac{d^n z}{d^n z (d^n x)^{-1}} = \frac{d^n z}{\underline{f}(I_n) I_n^{-1}} = \frac{d^n z}{\det \underline{f}} \quad . \tag{295}$$

It is remarkable, how the substitution rule arises nearly trivially if the volume element is not a scalar, but a pseudoscalar valued quantity. It is only one of many points, where Geometric Algebra leads to known results, but with a much clearer geometric understanding.

1.5 Functional Differentiation

1.5.1 Derivative with respect to a vector valued linear Function

Derivative with respect to a vector valued Function Given an orthonormal basis $\{e_j\}$ of $\mathcal{A}_n$, the matrix elements of a vector valued linear function $\underline{f}(a)$ are defined by

$$f_{ij} \stackrel{\text{def}}{=} e_i \cdot \underline{f}(e_j) \quad ; \quad f(a) = e^i a^j f_{ij} \quad . \tag{296}$$

We define now the derivative with respect to $\underline{f}$ as[30]

$$\partial_{\underline{f}(a)} \stackrel{\text{def}}{=} (a \cdot e_j) e_i \partial_{f_{ij}} = a_j e_i \partial_{f_{ij}} = a^j e^i \partial_{f^{ij}} \quad . \tag{297}$$

This leads to

$$\partial_{\underline{f}(a)} \underline{f}(b) \cdot c \stackrel{(297)}{=} a_j e_i \underbrace{\partial_{f_{ij}} f_{kl}}_{=\delta^i_k \delta^j_l} b^l c^k = a_j e_i c^i b^j = (a \cdot b) c \quad . \tag{298}$$

[30]It might have been more logical to define $\partial_{\underline{f}(a)} = \frac{1}{a} \cdot e_j e_i \partial_{f_{ij}}$, since then in analogy to $\frac{\partial}{\partial(\lambda x)} = \frac{1}{\lambda} \frac{\partial}{\partial x}$ we would have $\partial_{\underline{f}(\lambda a)} = \partial_{\lambda \underline{f}(a)} = \frac{1}{\lambda} \partial_{\underline{f}(a)}$. But to maintain compatibility to other publications we decide here for the definition (297).

Geometric Interpretation of the Functional Derivative For constant a, b the functional derivative $\partial_{f(a)}$ can be seen as the multivector derivative with respect to $\underline{f}(a^{-1})$. So, *e.g.*, consider (298)

$$\partial_{\underline{f}(e_\nu)}\underline{f}(b)\cdot c = b\cdot e_\nu \underbrace{\partial_{\underline{f}(e_\nu)}\underline{f}(e^\nu)\cdot c}_{\overset{(179)}{=}c} + \underbrace{\partial_{\underline{f}(e_\nu)}f(b_\perp)\cdot c}_{=0} = b\cdot e_\nu c \quad . \tag{299}$$

Multiplying with a^ν and summing over ν gives (298).

Some useful Results Extending $\underline{f}(a)$ via outermorphism over the whole algebra, we get for a *bivector* B

$$\begin{aligned}\partial_{\underline{f}(a)}\langle\underline{f}(b\wedge c)B\rangle &= \partial_{\underline{f}(a)}\langle\underline{f}(b)\wedge\underline{f}(c)B\rangle = \dot{\partial}_{\underline{f}(a)}\langle\dot{\underline{f}}(b)\cdot(\underline{f}(c)\cdot B) - \dot{\partial}_{\underline{f}(a)}\langle\dot{\underline{f}}(c)\cdot(\underline{f}(b)\cdot B)\rangle\rangle \\ &= a\cdot b\underline{f}(c)\cdot B - a\cdot c\underline{f}(b)\cdot B = \underline{f}(a\cdot bc - a\cdot cb)\cdot B \\ &= \underline{f}(a\cdot(b\wedge c))\cdot B \quad .\end{aligned} \tag{300}$$

This allows us to prove the general relation for $A_n \overset{\text{def}}{=} a_n\wedge\ldots\wedge a_1 = a_n\wedge A_{n-1}$ by induction. For $n-1=2$ the relation

$$\partial_{\underline{f}(a)}\langle\underline{f}(A_{n-1})B_{n-1}\rangle \overset{(300)}{=} \underline{f}(a\cdot A_{n-1})\cdot B_{n-1} \tag{301}$$

holds, as is seen with (300). But if (301) holds for $n-1$, it follows for n

$$\begin{aligned}\partial_{\underline{f}(a)}\langle\underline{f}(A_n)B_n\rangle &= \partial_{\underline{f}(a)}\langle[\underline{f}(a_n)\wedge f(A_{n-1})]\,B_n\rangle \\ &= a\cdot a_n\underline{f}(A_{n-1})\cdot B_n + (-1)^{n-1}\underbrace{\dot{\partial}_{\underline{f}(a)}\langle\dot{\underline{f}}(A_{n-1})\cdot[\underline{f}(a_n)\cdot B_n]\rangle}_{\overset{(300)}{=}\underline{f}(a\cdot A_{n-1})\cdot[\underline{f}(a_n)\cdot B_n]=\underline{f}([a\cdot A_{n-1}]\wedge a_n)\cdot B_n} \\ &= \underline{f}(a\cdot a_nA_{n-1} + (-1)^{n-1}[a\cdot A_{n-1}]\wedge a_n)\cdot B_n \\ &= \underline{f}(a\cdot A_n)\cdot B_n \quad .\end{aligned} \tag{302}$$

Thus we have for all $n\geq 2$

$$\partial_{\underline{f}(a)}\langle\underline{f}(A_n)B_n\rangle = \underline{f}(a\cdot A_n)\cdot B_n \quad . \tag{303}$$

Hence the derivative of the determinant becomes

$$\partial_{\underline{f}(a)}\det\underline{f} \overset{(227)}{=} \partial_{\underline{f}(a)}\underline{f}(I_n)I_n^{-1} \overset{(303)}{=} \underline{f}(a\cdot I_n)I_n^{-1} \overset{(266)}{=} \bar{f}^{-1}(a)\det\underline{f} \quad . \tag{304}$$

Further the derivative of the adjoint is

$$\partial_{\underline{f}(a)}\bar{f}(b) \overset{(239)}{=} \partial_{\underline{f}(a)}\langle\underline{f}(c)b\rangle\partial_c \overset{(298)}{=} (a\cdot\dot{c})b\dot{\partial}_c \overset{(179)}{=} ba \quad . \tag{305}$$

The Chain Rule The chain rule for the directional derivative (178) can be extended to act on linear functions with (nonlinear) position dependence.

$$\begin{aligned}b\cdot\partial_x &\overset{(178)}{=} \sum_{i,j}(b\cdot\dot{\partial}_x)\dot{f}_{ij}\partial_{f_{ij}} = \sum_{i,j,k}(b\cdot\dot{\partial}_x)\underbrace{\partial_{a_j}a_k}_{=\delta^{jk}}\dot{f}_{ij}\partial_{f_{ik}} \\ &= \sum_{i,j,k,l}(b\cdot\dot{\partial}_x)\left[\partial_{a_j}e^i\dot{f}_{ij}\right]\cdot[a_ke_l\partial_{f_{lk}}] \\ &\overset{(297)}{=} (b\cdot\dot{\partial}_x)\left[\dot{f}(\partial_a)\cdot\partial_{f(a)}\right] \quad .\end{aligned} \tag{306}$$

Derivative with respect to the Differential of a Linear Function We now ask, how we can formulate a functional derivative with respect to a function

$$b\cdot\partial_x\underline{f}(a) \quad . \tag{307}$$

Using our definition (297) for the dependence on a gives

$$\partial_{b\cdot\partial_x\underline{f}(a)} = a_je_i\partial_{b\cdot\partial_xf_{ij}} \quad . \tag{308}$$

Applying now the same translation to the b-dependence yields

$$\partial_{b\cdot\partial_x \underline{f}(a)} = a_j b_k e_i \partial_{\partial_k f_{ij}} \quad . \tag{309}$$

This leads to the fundamental relation

$$\begin{aligned} \partial_{b\cdot\partial_x \underline{f}(a)} \langle d \cdot \partial_x \underline{f}(c) M \rangle &\overset{(309)}{=} e_i a_j b_k \partial_{\partial_k \underline{f}_{ij}} \langle d^l \dot{\partial}_l c^n e^m \dot{\underline{f}}_{mn} M \rangle \\ &= (a \cdot c)(b \cdot d) \langle M \rangle_1 \quad . \end{aligned} \tag{310}$$

1.5.2 Derivative with respect to Linear Functions of arbitrary Grade

The definitions and calculations of the previous section are easily extended to linear functions of arbitrary grade. Let F be a linear function of grade r. Given an orthonormal basis e_{J_r} of this grade, we can decompose F as

$$F^{J_r k} \overset{\text{def}}{=} F(e^k) * e^{J_r} \quad ; \quad F(a) = F^{J_r k} e_{J_r} a_k \quad . \tag{311}$$

Thus all the relations derived for vector valued linear functions are also valid for linear functions of arbitrary grade. Especially if $F_r(a)$ is a linear function of grade r

$$\left.\begin{aligned} \partial_{F_r(a)} \langle F_r(b) M \rangle &= a * b \langle M \rangle_r \\ \partial_{b*\partial_x F_r(a)} \langle d * \partial_x F_r(c) M \rangle &= (a*c)(b*d)\langle M \rangle_r \end{aligned}\right\} . \tag{312}$$

1.6 Extension of Scalar Functions

Assume $f(x)$ to be a scalar function expandable by a Taylor series as

$$f(x) = e^{x\partial_h} f(h)\Big|_{h=0} = f(0) + x\partial_h f(h)\Big|_{h=0} + \frac{x^2}{2}\partial_h^2 f(h)\Big|_{h=0} + \dots \quad . \tag{313}$$

For the original function x was assumed to be a real or complex scalar. We now allow x to be any multivector, which obeys a cyclic identity of the kind

$$x^n = \lambda^n = \text{scalar} \quad . \tag{314}$$

This splits our series (313) in n linearly independent sums.

Let us make this explicit with the very useful example of a vector in a positive definite space. A split in a unit vector $\hat{\phi}$, giving the direction, and the magnitude ϕ is given by

$$x = \phi\hat{\phi} \quad ; \quad \hat{\phi}^2 = 1 \quad ; \quad x^2 = \phi^2\hat{\phi}^2 = \phi^2 \quad . \tag{315}$$

In this case we have $n = 2$ and our function splits into two parts. From (315) it follows, that the even powers are scalars, while the odd powers become vectors[31]. So we divide f into even and odd parts

$$f_+(x) = \tfrac{1}{2}[f(x) + f(-x)] \quad ; \quad f_-(x) = \tfrac{1}{2}[f(x) - f(-x)] \quad , \tag{316}$$

so that $f(x) = f_+(x) + f_-(x)$ and we write

$$f(\phi\hat{\phi}) = f_+(\phi\hat{\phi}) + f_-(\phi\hat{\phi}) = f_+(\phi) + \hat{\phi} f_-(\phi) \quad , \tag{317}$$

since f_+ can be expressed by the even powers and f_- by the odd powers.

Let B be any multivector with $B^2 = -1$. Then

$$B^n = \begin{cases} (-1)^{\frac{n}{2}} & \text{for } n \text{ even} \\ (-1)^{\frac{n-1}{2}} B & \text{for } n \text{ odd} \end{cases} \tag{318}$$

behaves like the complex unit and so

$$\begin{aligned} e^{\varphi B} &= \sum_{n=0}^{\infty} \frac{1}{n!}\varphi^n B^n = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}\varphi^{2n} + \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)!}\varphi^{2m+1} B \\ &= \cos(\varphi) + B\sin(\varphi) \quad . \end{aligned} \tag{319}$$

If instead $B^2 = +1$, we derive in the same way $e^{\varphi B} = \cosh(\varphi) + B\sinh(\varphi)$.

[31] Note that the associativity of the geometrical product is here of fundamental importance.

Rotations and Duality Rotations If B is a bivector, (319) leads to rotations and Lorentz boosts, as will be further discussed in Section 1.7.5 and Section 2.1. But if $B = I_n$ is the pseudoscalar of $\mathcal{A}_n$ we derive a **continuous duality transformation**. For example, if $I_n^2 = -1$ and $M \in \mathcal{G}(\mathcal{A}_n)$ is any multivector,

$$M' \overset{\text{def}}{=} e^{\varphi I_n} M = \cos(\varphi) M + I_n M \sin(\varphi) \tag{320}$$

gives a transformed multivector $M' \in \mathcal{G}(\mathcal{A}_n)$. We recognize that

1. orthogonal multivectors stay orthogonal, since from $A\widetilde{B} = -B\widetilde{A}$ it follows

$$A'\widetilde{B'} \overset{(320)}{=} e^{\alpha I_n} A\widetilde{B} e^{\alpha \widetilde{I}_n} = -e^{\alpha I_n} B\widetilde{A} e^{\alpha \widetilde{I}_n} = -B'\widetilde{A'} \quad . \tag{321}$$

2. if $\widetilde{I}_n = -I$, that is $\frac{n(n-1)}{2}$ is odd, the norm of a multivector is invariant under (320), since

$$|A'|^2 = \langle A'\widetilde{A'} \rangle \overset{(320)}{=} \langle e^{\alpha I_n} A\widetilde{A} e^{\alpha \widetilde{I}_n} \rangle = \langle A\widetilde{A} \rangle = |A|^2 \quad . \tag{322}$$

3. if $I_n^2 = -1$, then $e^{2\pi I_n} = 1$.

1.6.1 Square Roots of Multivectors

We can define the square root in the same way as is usually done, by

$$A_n^2 = B \Leftrightarrow A_n = \sqrt{B} \quad . \tag{323}$$

But we have to note that there will be mostly more than one A_n. It can even be a continuous set. So for example

$$A^2 = 1 \tag{324}$$

allows the two scalar solutions $A = \pm 1$, but also $A = \hat{n}$, with $\hat{n}^2 = 1$ a unit vector, and other multivectors! We can limit the set of solutions, if we demand A to be of a certain type.

As an important example we consider the square root of a multivector of the kind $(\alpha + \vec{a})^2$. We demand that the solution is of the type

$$\sqrt{(\alpha + \vec{a})^2} = \beta + \vec{b} \quad . \tag{325}$$

In order to find the possible relations between $(\alpha, \vec{a})$ and $(\beta, \vec{b})$, we square both sides and use the definition (323) to get

$$(\alpha + \vec{a})^2 = (\beta + \vec{b})^2 \quad . \tag{326}$$

Separating into scalar and vector parts we get the two relations

$$\alpha^2 + \vec{a}^2 = \beta^2 + \vec{b}^2 \tag{327}$$

$$\alpha \vec{a} = \beta \vec{b} \quad . \tag{328}$$

(328) tells us that $\vec{a} \parallel \vec{b}$. We can thus write

$$\hat{e} \overset{\text{def}}{=} \frac{\vec{a}}{|\vec{a}|} \quad ; \quad \vec{a} \overset{\text{def}}{=} a\hat{e} \quad ; \quad \vec{b} \overset{\text{def}}{=} b\hat{e} \tag{329}$$

and get so from (327) and (328)

$$\alpha^2 + a^2 = \beta^2 + b^2 \tag{330}$$

$$\alpha a = \beta b \Rightarrow b = \frac{\alpha a}{\beta} \quad . \tag{331}$$

Using (331) to eliminate b in (330) and multiplying by β^2 gives

$$0 = \beta^4 - \beta^2(\alpha^2 + a^2) + \alpha^2 a^2 \quad . \tag{332}$$

Solving the quadratic equation for β^2 in the usual way yields

$$\begin{aligned}\beta_{1,2}^2 &= \frac{\alpha^2+a^2}{2} \pm \sqrt{\frac{(\alpha^2+a^2)^2}{4} - \alpha^2 a^2} \\ &= \frac{\alpha^2+a^2 \pm (\alpha^2-a^2)}{2} = \begin{cases}\alpha^2 \\ a^2\end{cases} \quad .\end{aligned} \tag{333}$$

Thus for β and b there are the four solutions

$$\beta_n = \begin{cases}\pm\alpha \\ \pm a\end{cases} \Rightarrow \vec{b}_n = \begin{cases}\pm a\hat{e} \\ \pm\alpha\hat{e}\end{cases} = \begin{cases}\pm\vec{a} \\ \pm\frac{\alpha}{|\vec{a}|}\vec{a}\end{cases} \quad . \tag{334}$$

This example shows, how multivalued square roots occur. From case to case one has to consider carefully, for which type of solutions one is looking. But it is this extension of the definition to the whole graded algebra, which allows us, *e.g.*, to define a sphere of radius r by the set of solutions $\left\{\langle\sqrt{r^2}\rangle_1\right\}$.

1.7 Geometric Interpretations

1.7.1 The Interpretation of simple r-Vectors

The interpretation of vectors and scalars should be given in the usual way: vectors are quantities with direction and magnitude, scalars are real numbers. Simple r-vectors define a r dimensional subspace $\mathcal{A}_r$ of $\mathcal{A}_n$ by the relation

$$A_r \wedge a = 0 \Leftrightarrow a \in \mathcal{A}_r \quad . \tag{335}$$

A_r is the pseudoscalar for this subspace and can be seen as representing $\mathcal{A}_r$. A bivector $a \wedge b$ represents the plane in which a and b lie, a trivector a volume and so on.

1.7.2 Bivectors as Operators on Vectors

A bivector B defines with (335) a plane $\mathcal{A}_2$, for which it is the pseudoscalar. Since the pseudoscalar is unique any product of orthogonal vectors in $\mathcal{A}_2$ is proportional to B.

We consider first a positive definite space. Let B be the product of two *orthonormal* vectors $n, m \in \mathcal{A}_2$, that is

$$B = n \wedge m = nm \quad ; \quad nm = -mn \quad ; \quad n^2 = m^2 = 1 \quad . \tag{336}$$

Every $a \in \mathcal{A}_2$ can now be written as

$$a = \alpha n + \beta m \quad , \tag{337}$$

and it is thus sufficient to consider the action of B on n and m, for which we get

$$\left.\begin{array}{llllll} Bn & \overset{(336)}{=} & n\,\overbrace{mn}^{\overset{(336)}{=}-B} & = & -nB & = -m \\ Bm & \overset{(336)}{=} & nmm = -mnm & \overset{(336)}{=} & -mB & = +n \end{array}\right\} . \tag{338}$$

Thus B *represents a* $\pi/2$ *rotation of vectors in* $\mathcal{A}_2$, which rotates m to n and n to $-m$. For vectors k orthogonal to $\mathcal{A}_2$ we get on the other hand

$$k \perp n, m \Rightarrow Bk \overset{(336)}{=} nmk = -nkm = knm \overset{(336)}{=} kB \quad . \tag{339}$$

Combining both results yields that for *any* vector c

$$\frac{Bc - cB}{2} = B \times c \tag{340}$$

is a rotation[32] of $\frac{\pi}{2}$ in the plane of B.

[32] Note that this is in agreement with the result that the commutator prodct with a bivector conserves the grade.

For mixed Signature For the case $n^2 = -1, m^2 = 1$ we get instead of (338)

$$Bn = m \quad ; \quad Bm = n \quad . \tag{341}$$

Thus B exchanges the components with positive and negative signature. Since the commutation relations of B with vectors are independent of the signature, and stay thus unchanged, we can formulate the exchange of components in n and m direction of an arbitrary vector c by $B \times c$.

1.7.3 Interpretation of Products of Vectors

Let $\mathcal{A}_n$ be a positive definite linear vector space. Given a unit vector n we can split any vector a in parallel and orthogonal parts

$$a_{\|} = (a \cdot n)n \quad ; \quad a_{\perp} = a - a_{\|} \quad , \tag{342}$$

for which we find after multiplying with n from the right

$$\left.\begin{array}{lllllllll} a_{\|} \cdot n & = & a \cdot n & ; & a_{\|} \wedge n & = & 0 & \Leftrightarrow & a_{\|}n = na_{\|} \\ a_{\perp} \cdot n & = & 0 & ; & a_{\perp} \wedge n & = & a \wedge n & \Leftrightarrow & a_{\perp}n = -na_{\perp} \end{array}\right\} . \tag{343}$$

With the help of these commutation relations we derive

$$-nan = -n(a_{\|} + a_{\perp})n = -(a_{\|} - a_{\perp})nn = a_{\perp} - a_{\|} \quad , \tag{344}$$

which is just the inversion of the parallel component of a. In three dimensions this is a *reflection on the plane with normal* n.

1.7.4 The Product of Unit Vectors

In a positive definite space a *unit* vector n is defined by $n^2 = 1$. We can split a second unit vector m in parts parallel and perpendicular to n, *i.e.*,

$$m = m_{\|} + m_{\perp} \quad ; \quad n \wedge m_{\|} = 0 \quad ; \quad n \cdot m_{\perp} = 0 \quad . \tag{345}$$

Squaring the decomposition yields

$$1 = m^2 = (m_{\|} + m_{\perp})^2 = m_{\|}^2 + m_{\perp}^2 \Rightarrow |m_{\|}|, |m_{\perp}| \leq 1 \quad . \tag{346}$$

Thus there is a unique $\varphi \in (-\pi, \pi]$, for which

$$|m_{\|}| = \cos(\varphi) \quad ; \quad |m_{\perp}| = \sin(\varphi) \quad . \tag{347}$$

This gives immediately

$$m \cdot n = |m_{\|}| \stackrel{(347)}{=} \cos\varphi \quad ; \quad |m \wedge n| = |m_{\perp} \wedge n| = \sin\varphi \quad . \tag{348}$$

Let $n_{\perp} \stackrel{\text{def}}{=} m_{\perp}/|m_{\perp}|$ be an unit vector orthogonal to n, so that $m_{\perp} = n_{\perp}|m_{\perp}|$. Since $(n \wedge n_{\perp})^2 = (nn_{\perp})^2 = -1$ and $m \wedge n = m_{\perp} \wedge n = |m_{\perp}|n_{\perp} \wedge n$, we can combine and extend (348) to

$$mn = e^{\varphi n_{\perp} \wedge n} \quad . \tag{349}$$

Thus for arbitrary vectors enclosing an angle $\varphi \neq 0$

$$ab = |a||b| \exp\left(\varphi \frac{a \wedge b}{|a \wedge b|}\right) \quad . \tag{350}$$

Here the bivector $B \stackrel{\text{def}}{=} \frac{a \wedge b}{|a \wedge b|}$ is a unit bivector, *i.e.*, $B\widetilde{B} = 1$, defining the plane $\mathcal{A}_2$, in which a and b lie.

(350) is not defined for $\varphi = 0$, since we divide by the magnitude of a bivector, which is proportional to $\sin\varphi$.Nevertheless we can extend (350) to this case, since the magnitude of the numerator also goes with $\sin\varphi$. For two parallel vectors a, b we have indeed $ab = a \cdot b = |a||b|e^0$.

2 Explicit Examples

2.1 The two-dimensional Case

In two dimensions we can always choose an orthonormal basis $\{e_1, e_2\}$. The orthonormality is expressed by

$$e_1 \cdot e_2 = 0 \quad ; \quad |e_1^2| = |e_2^2| = 1 \quad . \tag{359}$$

The orthogonality yields

$$e_1 e_2 = -e_2 e_1 \quad . \tag{360}$$

There are only two combinations of the basis vectors for the unit pseudoscalar possible

$$I_2 = e_1 e_2 \quad ; \quad I_2' = e_2 e_1 = -e_1 e_2 = -I_2 \quad . \tag{361}$$

They represent the same object, just differ in sign. If we call I_2 the pseudoscalar of a right handed system, I_2' will then be the pseudoscalar of the corresponding left-handed system. Since for both basis vectors

$$e_1 I_2 = e_1 e_1 e_2 = -e_1 e_2 e_1 = -I_2 e_1 \quad ; \quad e_2 I_2 = e_2 e_1 e_2 = -I_2 e_2 \quad , \tag{362}$$

the pseudoscalar anticommutes with all linear combinations of e_1 and e_2, but this are all vectors in $\mathcal{A}_2$. We know that I_2 is a bivector, thus

$$\widetilde{I}_2 = -I_2 \quad ; \quad I_2^2 = I_2 I_2 = -I_2 \widetilde{I}_2 = -(e_1)^2 (e_2)^2 \tag{363}$$

and so I_2^2 depends on the signature of our space. Since vectors anticommute with I_2, we have for the geometric product of $a'(\alpha) \stackrel{\text{def}}{=} e^{-I_2 \alpha} a, b'(\alpha) \stackrel{\text{def}}{=} e^{-I_2 \alpha}$

$$a'b' = \left(e^{-I_2 \alpha} a\right)\left(e^{-I_2 \alpha} b\right) = a e^{I_2 \alpha} e^{-I_2 \alpha} b = ab \quad . \tag{364}$$

Thus the full geometrical product is preserved. We split the product in its two parts, *i.e.*,

$$a'b' = a' \cdot b' + a' \wedge b' = ab = a \cdot b + a \wedge b \quad , \tag{365}$$

and compare grades to obtain

$$a' \cdot b' = a \cdot b \quad ; \quad a' \wedge b' = a \wedge b \quad . \tag{366}$$

This has the important consequence that *bivectors are invariant under rotations in their plane* in all spaces, since they are spanned in a two-dimensional subspace. For the case of more than two dimensions this yields, that *the normal with the magnitude contains all the information.*

Since the inner product is conserved, the points of constant distance to the origin passing through r_0 (spheres) are given by

$$r(\alpha) = e^{-I_2 \alpha} r_0 \Rightarrow (r(\alpha))^2 = \left(e^{-I_2 \alpha} r_0\right)^2 = r_0^2 \quad \forall \alpha \quad . \tag{367}$$

2.1.1 With positive Signature

An orthonormal basis $\{\sigma_1, \sigma_2\}$ with positive signature obeys

$$\sigma_1 \cdot \sigma_2 \stackrel{\text{def}}{=} 0 \quad ; \quad \sigma_1^2 = \sigma_1 \cdot \sigma_1 = \sigma_2^2 \stackrel{\text{def}}{=} +1 \quad ; \quad I_2^2 = -1 \quad . \tag{368}$$

We *define*, that

$$\sigma_2 = \sigma_1 \text{ rotated by } +\frac{\pi}{2}. \tag{369}$$

The pseudoscalar represents with[33]

$$I_2 \sigma_1 = -\sigma_2 \quad ; \quad I_2 \sigma_2 = \sigma_1 \tag{370}$$

according to the definition (369) a $-\frac{\pi}{2}$-rotation. Since $I_2^2 \stackrel{(368)}{=} -1$ we can write $e^{I_2 \alpha} = \cos\alpha + I_2 \sin\alpha$. Applying it on a basis vector yields

$$\left.\begin{array}{rcl} e^{I_2 \alpha}\sigma_1 & = & \sigma_1 \cos\alpha + I_2 \sigma_1 \sin\alpha \stackrel{(370)}{=} \sigma_1 \cos\alpha - \sigma_2 \sin\alpha \\ e^{I_2 \alpha}\sigma_2 & = & \sigma_2 \cos\alpha + I_2 \sigma_2 \sin\alpha \stackrel{(370)}{=} \sigma_2 \cos\alpha + \sigma_1 \sin\alpha \end{array}\right\} \tag{371}$$

and shows thus, that it *gives a rotation* by[34] $-\alpha$.

[33] This is for the pseudoscalar of the right handed system. I_2' leads to a rotation in opposite direction.
[34] Remember definition (369).

A Representation of Complex Numbers Since I_2 squares like the complex unit j to -1, the mapping

$$\alpha + \beta I_2 \mapsto f(\alpha + \beta I_2) \stackrel{\text{def}}{=} \alpha + j\beta \tag{372}$$

is an *isomorphism to the complex numbers.* This isomorphism allows us to represent a complex number as the multivector

$$z \stackrel{\text{def}}{=} \alpha + I_2\beta = \alpha + \sigma_1\sigma_2\beta = \sigma_1(\alpha\sigma_1 + \beta\sigma_2) \quad , \tag{373}$$

but this is according to our previous considerations just a rotation in the σ_1, σ_2-plane. One can use the property of the pseudoscalar as a bivector $\widetilde{I} = -I$ to define the complex conjugation as

$$z^* \stackrel{\text{def}}{=} \widetilde{z} = \alpha + \widetilde{I}_2\beta = \alpha - I_2\beta \quad . \tag{374}$$

Real and imaginary Parts We introduce thus the two operators[35]

$$\Re[z] \stackrel{\text{def}}{=} \langle z \rangle = \frac{z+z^*}{2} \quad ; \quad \Im[z] \stackrel{\text{def}}{=} \frac{z-z^*}{2} I_2^{-1} \quad , \tag{375}$$

which give the equivalent of real and imaginary parts so that we can write z as

$$z = \Re[z] + \Im[z]\ I_2 \quad . \tag{376}$$

Cauchy-Riemann Conditions With the vector derivative operator in $\mathcal{A}_2$

$$\partial_x \stackrel{\text{def}}{=} \sigma_1 \frac{\partial}{\partial\alpha} + \sigma_2 \frac{\partial}{\partial\beta} \tag{377}$$

one finds by applying it to the equivalent of a complex number $\alpha + j\beta$, *i.e.*, $f^{-1}(\alpha + j\beta) = \alpha + I_2\beta$,

$$\partial_x z = (\sigma_1\partial_\alpha + \sigma_2\partial_\beta)(\alpha + I_2\beta) = \sigma_1 + \underbrace{\sigma_2 I_2}_{=-\sigma_1} = 0 \tag{378}$$

and so[36] $\partial_x z^n = 0$. For

$$f(z) = f(\alpha, \beta) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \alpha_n z^n \stackrel{\text{def}}{=} f_r + I_2 f_i \quad ; \quad f_r \stackrel{\text{def}}{=} \Re[f] \quad ; \quad f_i \stackrel{\text{def}}{=} \Im[f] \tag{380}$$

it follows thus

$$\begin{aligned} 0 &= \partial_x f = (\sigma_1\partial_\alpha + \sigma_2\partial_\beta)(f_r + I_2 f_i) = \sigma_1\partial_\alpha f_r + \sigma_2\partial_\beta f_r + \overbrace{\sigma_1 I_2}^{=\sigma_2}\partial_\alpha f_i + \overbrace{\sigma_2 I_2}^{=-\sigma_1}\partial_\beta f_i \\ &= \sigma_1[\partial_\alpha f_r - \partial_\beta f_i] + \sigma_2[\partial_\beta f_r + \partial_\alpha f_i] \quad , \end{aligned} \tag{381}$$

which are just the Cauchy-Riemann conditions for analytic functions!

2.1.2 Two Dimensions with mixed Signature

We will examine the structure of a two-dimensional geometric algebra generated by the orthonormal basis-vectors γ_0, γ_1 with the signature

$$\gamma_0^2 = +1 \quad ; \quad \gamma_1^2 = -1 \quad . \tag{382}$$

According to the different signature we say γ_0 is the time and γ_1 the space direction. Now the signature induces

$$I_2^2 = \gamma_0\gamma_1\gamma_0\gamma_1 = -\gamma_0^2\gamma_1^2 \stackrel{(382)}{=} +1 \quad , \tag{383}$$

and so I_2 has no similarity with the complex unit j but can be seen as a *hyper-complex unit.* By calculating the products with the basis vectors, *i.e.*,

$$\left.\begin{array}{lllllll} I_2\gamma_0 &=& \gamma_0\gamma_1\gamma_0 &=& -\gamma_0^2\gamma_1 &=& -\gamma_1 \\ I_2\gamma_1 &=& \gamma_0\gamma_1\gamma_1 &=& \gamma_0\gamma_1^2 &=& -\gamma_0 \end{array}\right\} , \tag{384}$$

we recognize, that I_2 maps the time part to the space part and vice versa.

[35]Please note, that the second definition is not quite standard. The usual definition would be given without the factor of I_2^{-1}.

[36]z commutes with I_2, thus

$$\partial_x(z\cdots z\dot{z}\cdots z) = \sigma_1(z\cdots z\underbrace{\dot{\partial}_\alpha\dot{z}}_{=1}z\cdots z) + \sigma_2(z\cdots z\underbrace{\dot{\partial}_\beta\dot{z}}_{=I_2}z\cdots z) = \underbrace{(\sigma_1 + \sigma_2 I_2)}_{=0}(z\cdots z) = 0 \quad . \tag{379}$$

Lorentz Boosts But what is the significance of $e^{I_2\alpha}$? Since $I_2^2 = 1$, we can write

$$e^{I_2\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} I_2^n = \cosh\alpha + I_2 \sinh\alpha \tag{385}$$

and the action on a basis vector becomes

$$\left.\begin{array}{lllll} \gamma_0'(\alpha) \stackrel{\text{def}}{=} e^{I_2\alpha}\gamma_0 & \stackrel{(385)}{=} & \gamma_0\cosh\alpha + I_2\gamma_0\sinh\alpha & \stackrel{(384)}{=} & \gamma_0\cosh\alpha - \gamma_1\sinh\alpha \\ \gamma_1'(\alpha) \stackrel{\text{def}}{=} e^{I_2\alpha}\gamma_1 & \stackrel{(385)}{=} & \gamma_1\cosh\alpha + I_2\gamma_1\sinh\alpha & \stackrel{(384)}{=} & \gamma_1\cosh\alpha - \gamma_0\sinh\alpha \end{array}\right\} . \tag{386}$$

Here we recognize the well known *Lorentz Transformation*. We can thus interpret α by

$$\alpha = \tanh^{-1}\frac{v}{c} = \tanh^{-1}\beta \quad , \tag{387}$$

where v is the velocity of the inertial frame we are transforming to, c the speed of light, and $\beta \stackrel{\text{def}}{=} \frac{v}{c}$. From $\cosh^2\alpha - \sinh^2\alpha = 1$ we derive

$$\left.\begin{array}{lllll} \cosh\alpha & = & \frac{1}{\sqrt{1-\tanh^2\alpha}} & = & \frac{1}{\sqrt{1-\beta^2}} \stackrel{\text{def}}{=} \gamma \\ \sinh\alpha & = & \frac{\tanh\alpha}{\sqrt{1-\tanh^2\alpha}} & = & \frac{\beta}{\sqrt{1-\beta^2}} = \beta\gamma \end{array}\right\} . \tag{388}$$

We can now write (386) in matrix notation, which becomes under use of (388) for a vector $a = t\gamma_0 + x\gamma_1$

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \quad . \tag{389}$$

Since $\gamma_0'^2 \stackrel{(365)}{=} \gamma_0^2 = 1$ the lines of constant distance to the origin are given by $\pm\gamma_j'(\alpha)$ with the curve parameter α.

For compatibility we will define the operator of a proper Lorentz transformation by

$$L \stackrel{\text{def}}{=} e^{-I_2\alpha/2} \quad ; \quad \tilde{L} = e^{I_2\alpha/2} \quad , \tag{390}$$

so that

$$x' = Lx\tilde{L} \stackrel{(390)}{=} e^{-I_2\alpha/2} x e^{I_2\alpha/2} = e^{-I_2\alpha} x \tag{391}$$

gives a Lorentz transformation. When generalizing this to more dimensions, I_2 gets replaced by a bivector, constructed out of a spatial vector a (*i.e.*, $\gamma_0 \cdot a = 0$) and a time like vector γ_0. The direction of the boost is then defined by a. The use of definition (390) becomes now clear, when we realize that space like vectors perpendicular to a commute with $B = \gamma_0 \wedge a$, and so

$$Lb\tilde{L} = bL\tilde{L} = b \quad \forall b \perp a, \gamma_0 \quad . \tag{392}$$

The outer product between the transformed and original vector is

$$\gamma_1' \wedge \gamma_1 = \left(e^{-I_2\alpha}\gamma_1\right) \wedge \gamma_1 = \gamma_0 \wedge \gamma_1 \sinh\alpha = \gamma_0 \wedge \gamma_1 \beta\gamma \quad . \tag{393}$$

2.2 The three-dimensional Case

2.2.1 Without mixed Signature

Not too much new happens here, but this case is of special importance, since our rest-space has three *spatial* dimensions.

Let $\{\sigma_1, \sigma_2, \sigma_3\}$ be a right handed orthonormal basis with positive signature, *i.e.*,

$$\sigma_i \cdot \sigma_j = \delta_{ij} \quad . \tag{394}$$

The unit pseudoscalar for the right handed system is then given by

$$I_3 = \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = \sigma_1\sigma_2\sigma_3 \tag{395}$$

and has the properties

$$\widetilde{I}_3 = (-1)^{3\cdot 2/2} I_3 = -I_3 \quad ; \quad I_3^2 = -I_3\widetilde{I}_3 = -1 \quad ; \quad \sigma_k I_3 = I_3 \sigma_k \quad . \tag{396}$$

There are $\binom{3}{2} = 3$ combinations of the basis vectors to bivectors, given by

$$\left.\begin{array}{rcccl} \sigma_1 \wedge \sigma_2 & = & \sigma_1\sigma_2 & = & I_3\sigma_3 \\ \sigma_2 \wedge \sigma_3 & = & \sigma_2\sigma_3 & = & I_3\sigma_1 \\ \sigma_3 \wedge \sigma_1 & = & \sigma_3\sigma_1 & = & I_3\sigma_2 \end{array}\right\} . \tag{397}$$

Here we realize, that the pseudoscalar acting on a vector $\hat{a}$ gives the bivector of the plane orthogonal to $\hat{a}$. Together with the orthonormality relation this can be written as

$$\sigma_i\sigma_j = \sigma_i \cdot \sigma_j + \sigma_i \wedge \sigma_j = \delta_{ij} + I_3\epsilon_{ijk}\sigma_k \quad . \tag{398}$$

But this is exactly the *relation for the Pauli spin matrices*! Thus these matrices are no more than just a matrix-representation of an orthonormal basis of the three-dimensional space. The commutator and anticommutator relations follow immediately from (398), namely

$$[\sigma_i, \sigma_j] = 2\sigma_i \wedge \sigma_j = 2I_3\epsilon_{ijk}\sigma_k \quad ; \quad \{\sigma_i, \sigma_j\} = 2\sigma_i \cdot \sigma_j = 2\delta_{ij} \quad , \tag{399}$$

and express thus just the orthonormality. We take this similarity as the reason to call this algebra the **Pauli-algebra** $\mathcal{P}$.

The Cross Product Given (398) we find for two arbitrary vectors in $\mathcal{P}$

$$\vec{a} \stackrel{\text{def}}{=} \sum_{i=1}^{3} a^i\sigma_i \quad \text{and} \quad \vec{b} \stackrel{\text{def}}{=} \sum_{j=1}^{3} b^j\sigma_j \tag{400}$$

as the outer product

$$\begin{aligned} \vec{a} \wedge \vec{b} &\stackrel{(400)}{=} \sum_{i,j=1}^{3} a^i b^j \sigma_i \wedge \sigma_j \stackrel{(398)}{=} \sum_{i,j,k=1}^{3} a^i b^j \epsilon_{ijk} I_3 \sigma^k = I_3 \begin{vmatrix} a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \\ \sigma^1 & \sigma^2 & \sigma^3 \end{vmatrix} \\ &= I_3 \vec{a} \otimes \vec{b} \quad , \end{aligned} \tag{401}$$

where we denote the ordinary cross product with $\vec{a} \otimes \vec{b}$. We thus found a representation of the three-dimensional cross product by

$$\vec{a} \otimes \vec{b} = -I_3 \vec{a} \wedge \vec{b} \quad . \tag{402}$$

Hence one can see the outer product as a generalization of the cross product. The advantages of the outer product have to be pointed out clearly. The cross product between two vectors can only be defined in three dimensions, while the outer product appears naturally in the Geometric Algebra of a linear space of *any* dimension. It is also the outer product, which simplifies calculations by its associativity.

Subspaces in $\mathcal{P}$ and Normals In $\mathcal{P}$ each bivector A_2 defines a two-dimensional subspace $\mathcal{S}$ of $\mathcal{A}_3$ by

$$A_2 \wedge \vec{a} = 0 \qquad \forall \vec{a} \in \mathcal{S} \quad . \tag{403}$$

Expressing A_2 with it's dual vector $\vec{c}$ gives

$$\vec{c} \stackrel{\text{def}}{=} I_3^{-1} A_2 \quad ; \quad A_2 = I\vec{c} \quad . \tag{404}$$

Substituting this in (403) yields

$$0 = (I_3\vec{c}) \wedge \vec{a} = I_3(\vec{c} \cdot \vec{a}) \quad . \tag{405}$$

Thus the plane is given by all vectors perpendicular to $\vec{c}$, and we call $\vec{c}$ the **normal** of A_2. Thus *in three dimensions* the dual vector of a bivector gives the normal.

The even subalgebra is spanned by scalars and bivectors and has thus $1+3$ dimensions. A basis would be given by $\{1, i\sigma_j\}$.

Rotations Since a bivector A_2 in $\mathcal{S}$ is a multiple of the pseudoscalar of $\mathcal{S}$, we can conclude from the discussion of the two-dimensional case, that $e^{A_2\alpha}$ is a rotation in the plane $\mathcal{S}$ for vectors in $\mathcal{S}$. Here we can express A_2 by its dual vector, defined by

$$\vec{a} \stackrel{\text{def}}{=} I_3^{-1}A_2 \Rightarrow A_2 = I_3\vec{a} \quad , \tag{406}$$

and a rotation in the plane becomes

$$e^{-I_3\vec{a}\alpha} \quad . \tag{407}$$

Vectors perpendicular to the plane are parallel, vectors in the plane are orthogonal to $\vec{a}$. So we can split each vector $\vec{x}$ in a part parallel and a part orthogonal to $\vec{a}$. Thus $\vec{x}$ can be written as

$$\vec{x} = \vec{x}_{\|} + \vec{x}_{\perp} \quad ; \quad \vec{x}_{\|} = P_{\vec{a}}(\vec{x}) \quad ; \quad \vec{x}_{\perp} = P_{\vec{a}}^{\perp}(\vec{x}) \quad , \tag{408}$$

where $\vec{x}_{\|}$ and $\vec{x}_{\perp}$ satisfy the commutation relations

$$\vec{x}_{\|}\vec{a} = \vec{a}\vec{x}_{\|} \qquad ; \qquad \vec{x}_{\perp}\vec{a} = -\vec{a}\vec{x}_{\perp} \quad . \tag{409}$$

Now note that

$$\begin{aligned} e^{-I_3\vec{a}\alpha/2}\vec{x}e^{I_3\vec{a}\alpha/2} &= e^{-I_3\vec{a}\alpha/2}(\vec{x}_{\|} + \vec{x}_{\perp})e^{I_3\vec{a}\alpha/2} \\ &= e^{-I_3\vec{a}\alpha/2}e^{I_3\vec{a}\alpha/2}\vec{x}_{\|} + e^{-I_3\vec{a}\alpha/2}e^{-I_3\vec{a}\alpha/2}\vec{x}_{\perp} \\ &= \vec{x}_{\|} + e^{-I_3\vec{a}\alpha}\vec{x}_{\perp} \end{aligned} \tag{410}$$

rotates just the part in the plane and leaves the rest as it is. Thus we identify a rotation in the plane orthogonal to a vector $\vec{\theta}$ with

$$R = e^{-I_3\vec{\theta}/2} \quad . \tag{411}$$

Addition of Bivectors in three Dimensions But how can we add bivectors, directed plane segments? What does it mean to write $e^{A_2+B_2}$? The answer is easy to find in three dimensions, since two planes are always intersecting or parallel[37].

1. If A_2 and B_2 represent parallel plane-segments, we can just add them, since $B_2 = \lambda A_2$ and so $A_2 + B_2 = (1+\lambda)A_2$ lies in the same plane. Just the magnitude of the segment changed.

2. If A_2 and B_2 are non-parallel planes, they are intersecting, since we are in three dimensions. For $A_2 \stackrel{\text{def}}{=} \vec{a}_1 \wedge \vec{a}_2$ and $B_2 \stackrel{\text{def}}{=} \vec{b}_1 \wedge \vec{b}_2$ we have

$$a_1 \wedge a_2 \wedge b_1 \wedge b_2 = 0 \quad , \tag{412}$$

and thus the set $\{a_1, a_2, b_1, b_2\}$ must be linearly dependent in three dimensions and we can thus construct a vector lying in both planes. To represent A_2 and B_2 we have the freedom to rotate the vectors $\{a_1, a_2\}$ and $\{b_1, b_2\}$, respectively [38]. Thus we can choose [39] $\hat{z} = \vec{a}_2 = \vec{b}_2$ with $\hat{z}^2 = 1$ to be a vector parallel to the intersection of the planes. Now

$$A_2 = \vec{a} \wedge \hat{z} \quad ; \quad B_2 = \vec{b} \wedge \hat{z} \quad \Rightarrow \quad |A_2| = |\vec{a}| \quad ; \quad |B_2| = |\vec{b}| \quad . \tag{413}$$

The sum

$$A_2 + B_2 = (\vec{a} + \vec{b}) \wedge \hat{z} \tag{414}$$

is now easily to interpret as a new plane segment (which is rotated around the intersection) with the magnitude

$$|A_2 + B_2|^2 = (\vec{a} + \vec{b})\hat{z}\hat{z}(\vec{a} + \vec{b}) = \vec{a}^2 + \vec{b}^2 + 2\vec{a}\cdot\vec{b} = |A_2|^2 + |B_2|^2 + 2|A_2||B_2|\cos\theta \quad . \tag{415}$$

One can also see it as an addition of the normal vectors. A representation of A_2 and B_2 by their normals is given by $A_2 = I_3 a, B_2 = I_3 b$. After multiplying by I_3^{-1}

$$\vec{a} \stackrel{\text{def}}{=} I_3^{-1}A_2 \quad ; \quad \vec{b} \stackrel{\text{def}}{=} I_3^{-1}B_2 \quad . \tag{416}$$

Thus the addition of two bivectors becomes

$$A_2 + B_2 = I_3(\vec{a} + \vec{b}) \quad . \tag{417}$$

[37] We raise this topic here, since there is a remarkable difference to the four-dimensional case, where planes do not have to intersect.

[38] Remember that the full geometric product is invariant under a rotation of both vectors.

[39] We can choose $\hat{z}$ to be normalized, since a scalar factor can always be put into the other vector $\vec{a}_1$ or $\vec{b}_1$, respectively.

2.2.2 With mixed Signature

Let $\{\gamma_0, \gamma_1, \gamma_2\}$ be an orthonormal basis with

$$\gamma_0^2 = +1 \quad ; \quad \gamma_1^2 = \gamma_2^2 = -1 \quad . \tag{418}$$

We call γ_0 a time-like vector and $\{\gamma_1, \gamma_2\}$ space-like vectors. The pseudoscalar

$$I_3 \stackrel{\text{def}}{=} \gamma_0\gamma_1\gamma_2 \tag{419}$$

commutes with vectors and has $\widetilde{I}_3 = -I_3$, as always in three dimensions. The choice of signature (418) leads to

$$I_3^2 = -I_3\widetilde{I}_3 = -\gamma_0\gamma_1\gamma_2\gamma_2\gamma_1\gamma_0 \stackrel{(418)}{=} -1(-1)^2 = -1 \quad . \tag{420}$$

Note that the choice $\gamma_0^2 = -1, \gamma_1^2 = \gamma_2^2 = +1$ would lead to a hyper-complex unit $I_3^2 = +1$.

Major difference to the two-dimensional case with mixed signature is the fact that the space of bivectors has three dimensions. A basis of this space is given by $\{I_3\gamma_0, I_3\gamma_1, I_3\gamma_2\}$. With (149) each of this basis bivectors defines a two-dimensional subspace — $I_3\gamma_1$ and $I_3\gamma_2$ define two-dimensional subspaces with mixed signature, *i.e.*, space-times, while $I_3\gamma_0$ defines a space with not mixed signature. With (371) and (385) we saw, how bivectors generate rotations and Lorentz boosts in this subspaces. But now the question arises, how we can interpret the exponential of a linear combination of bivectors. As an example consider $\alpha I_3\gamma_0 + \beta I_3\gamma_1 = I_3(\alpha\gamma_0 + \beta\gamma_1)$. Clearly, the behaviour will be determined by the signature of the square, but can we find a deeper insight?

Since bivectors are invariant under rotations in their plane, each unit *bivector* σ,*i.e.*, any linear combination σ of the basis bivectors with $|\sigma\widetilde{\sigma}| = 1$, can be represented by

$$\sigma = nm \quad \text{with} \quad n \cdot m = 0 \quad ; \quad |n|^2 = |m|^2 = 1 \quad ; \quad m \cdot \gamma_0 = 0 \quad . \tag{421}$$

Thus the square of σ depends with

$$\sigma^2 = nmnm = -n^2m^2 = n^2 = \pm 1 \tag{422}$$

on the square of the unit vector n. Hence we derive for n time-like and space-like, respectively,

$$e^{\sigma\phi} = \begin{cases} \cosh\phi + \sigma\sinh\phi & \text{for } n^2 = +1 \\ \cos\phi + \sigma\sin\phi & \text{for } n^2 = -1 \end{cases} \quad . \tag{423}$$

We can write n as

$$n = n_\| + n_\perp \quad \text{with} \quad n_\| \wedge \gamma_0 = 0 \quad ; \quad n_\perp \cdot \gamma_0 = 0; n \cdot \gamma_0 > 0 \quad , \tag{424}$$

so that $n_\perp$ is a pure space-vector and $n_\|$ a pure time-vector. If $n^2 = +1$ we can express n as a Lorentz transformation applied to the timelike basis vector γ_0, *i.e.*,

$$n = L_{n_\perp}(\alpha)\gamma_0\widetilde{L}_{n_\perp}(\alpha) \quad \text{with } L_{n_\perp} \stackrel{\text{def}}{=} \exp\left(-\frac{n_\| n_\perp \alpha}{2\sqrt{(n_\| n_\perp)^2}}\right) \quad ; \quad \tanh\alpha = \frac{|n_\perp|}{|n_\||} \tag{425}$$

and[40] for $n^2 = -1$

$$n = L_{n_\perp}(\alpha)\frac{n_\perp}{|n_\perp|}\widetilde{L}_{n_\perp}(\alpha) \quad \text{with } \tanh\alpha = \frac{|n_\||}{|n_\perp|} \quad . \tag{426}$$

Since $n \perp m$ we have $n_\perp \perp m$ and thus[41]

$$L_{n_\perp} m = e^{\gamma_0 n_\perp \cdots} m = m L_{n_\perp} \quad . \tag{427}$$

Substituting this results into (423) yields

$$e^{\sigma\phi} = e^{nm\phi} = \exp\left(L_{n_\perp}\frac{n_{\|/\perp}}{|n_{\|/\perp}|}\widetilde{L}_{n_\perp} m\phi\right) = L_{n_\perp} e^{\gamma m\phi}\widetilde{L}_{n_\perp} \quad , \tag{428}$$

where we defined $\gamma \stackrel{\text{def}}{=} \frac{n_{\|/\perp}}{|n_{\|/\perp}|}$. Thus $e^{\sigma\phi}$ represents always a pure rotation or Lorentz transformation, but not necessarily in the rest-frame!

[40] While in the last equation $n_\|/|n_\|| = \gamma_0$ we can now not say what the direction of $n_\perp$ is.

[41] Remember that by construction $m \perp \gamma_0$.

3 Application to the flat Minkowski Space

3.1 The four-dimensional Space-Time

Given an orthonormal basis $\{e_\mu\}$ of a linear vector space $\mathcal{A}_n$, we call $\mathcal{A}_n$ a **space-time**, if *one* of the basis vectors is distinguished from all the others by a different signature, that is with $k \stackrel{\text{def}}{=} \pm 1$ there is a $\nu \in \{1, \ldots, n\}$, so that

$$e_\mu^2 = \begin{cases} k & \text{for} \quad \mu \neq \nu \\ -k & \text{for} \quad \mu = \nu \end{cases} \quad . \tag{429}$$

Normally ν is chosen to be 0 or n. We will adopt the convention $k = -1$ and $\nu = 0$.

3.1.1 Four-dimensional Space-Time Algebra

Turning our attention to physical problems, we are mostly concerned with the four-dimensional flat Minkowski space $\mathcal{M}_4$. We choose the three *spatial coordinates* to have *negative signature*, while *time-like vectors have a positive square.* We call the resulting Geometric Algebra $\mathcal{G}(\mathcal{M}_4)$ the **space-time algebra** or Dirac algebra $\mathcal{D}$.

The Basis We introduce a set of four orthonormal basis vectors $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, obeying

$$\gamma_0^2 = +1 \quad ; \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1 \quad . \tag{430}$$

The orthogonality of the γ_μ is according to (8) for $\mu \neq \nu$ expressed by

$$\gamma_\mu \cdot \gamma_\nu = 0 \quad \Leftrightarrow \quad \gamma_\mu \gamma_\nu \quad = \quad -\gamma_\nu \gamma_\mu \quad ; \quad \text{for } \mu \neq \nu \quad . \tag{431}$$

With

$$\eta_{\mu\nu} \stackrel{(35)}{=} \gamma_\mu \cdot \gamma_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \delta_{\mu\nu} \gamma_\mu^2 \tag{432}$$

we see, that the reciprocal basis vectors are given by

$$\gamma^\mu \stackrel{(36)}{=} \sum_{\nu=0}^{3} \eta^{\mu\nu} \gamma_\nu \quad . \tag{433}$$

Summation Convention We will often be concerned with sums over indices like in (433). We thus employ the *Einstein Summation Convention.* That means, if not stated differently, an index appearing upstairs *and* downstairs represents a sum over this index. For example (433) would read now $\gamma^\mu = \eta^{\mu\nu} \gamma_\nu$.

Time-like and Space-like Vectors We call a vector a time-like, if $a^2 > 0$, space-like if $a^2 < 0$ and a null-vector, if $a^2 = 0$. Thus γ_0 represents a time-like direction.

Relation to the Dirac Matrices We see with (431) and (430), that the γ_μ obey the same anticommutation relations, that are imposed on the Dirac Matrices $\hat{\gamma}_\mu$, *i.e.*,

$$\{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} \stackrel{\text{def}}{=} \hat{\gamma}_\mu \hat{\gamma}_\nu + \hat{\gamma}_\nu \hat{\gamma}_\mu = 2\eta_{\mu\nu} \quad . \tag{434}$$

Here we realize that this relation expresses nothing else than orthonormality, which is imposed with (430) and (431). This analogy is the reason for us to call the space-time algebra the **Dirac algebra** $\mathcal{D}$.

The Basis of the Dirac Algebra With the Geometric Product of the basis vectors we can construct a basis of bivectors by

$$\{\gamma_1\gamma_0, \gamma_2\gamma_0, \gamma_3\gamma_0, \gamma_2\gamma_1, \gamma_3\gamma_1, \gamma_3\gamma_2\} \quad . \tag{435}$$

A further multiplication yields a basis of trivectors

$$\{\gamma_2\gamma_1\gamma_0, \gamma_3\gamma_1\gamma_0, \gamma_3\gamma_2\gamma_0, \gamma_3\gamma_2\gamma_1\} \quad . \tag{436}$$

Now there is only one (up to a sign) possible combination for an element of grade four, that we will call **pseudoscalar** i [42]

$$i \stackrel{\text{def}}{=} \gamma_0\gamma_1\gamma_2\gamma_3 \quad . \tag{437}$$

From this definition we derive immediately the fundamental relations

$$\tilde{i} = i \quad ; \quad i^2 = -1 \quad ; \quad i\gamma_j = -\gamma_j i \quad . \tag{438}$$

With the pseudoscalar i we can write the basis of trivectors (436) as $\{i\gamma_\mu\}$. If we define

$$\sigma_k \stackrel{\text{def}}{=} \gamma_k\gamma_0 \tag{439}$$

the basis of bivectors (435) is further given by

$$\{\sigma_k, i\sigma_k\} \quad . \tag{440}$$

So the full basis of the Dirac algebra $\mathcal{D}$ becomes

Grade in $\mathcal{D}$	**grade 0**	**grade 1**	**grade 2**	**grade 3**	**grade 4**
Interpretation	*scalars*	*vectors*	*bivectors*	*trivectors*	*pseudoscalar*
Basis	$\{1\}$	$\{\gamma_\mu\}$	$\{\sigma_k, i\sigma_k\}$	$\{i\gamma_\mu\}$	$\{i\}$

3.2 The Pauli Algebra

3.2.1 Physical Interpretations

The basis of the Minkowski space contains one distinguished basis vector, *i.e.*, the time-like vector γ_0. But in four dimensions a vector can be seen as a normal of a three-dimensional vector space $\mathcal{A}_3$, which is defined by

$$0 = i(a \cdot b) = (ia) \wedge b = 0 \quad ; \quad \forall b \in \mathcal{A}_3 \quad . \tag{441}$$

Since a is time-like it follows, that every vector $b \in \mathcal{A}_3$ is space-like. We thus employ the following interpretation:

The vector γ_0 defines the four-velocity of a frame. So the inner product with γ_0 gives just the time-like part, while the outer product gives the projection in the three-dimensional rest-space. Thus we identify

$$\sigma_k \stackrel{\text{def}}{=} \gamma_k\gamma_0 = \gamma_k \wedge \gamma_0 \tag{442}$$

as **basis vectors of the three-dimensional rest-space** of the frame corresponding to γ_0.

[42] To avoid misunderstandings we will use j for the complex unit. The representation of j is not unique, since in general there are many elements in $\mathcal{G}(\mathcal{A}_n)$ which square to -1. So j has to be identified in each single case seperately. In certain cases j can be represented by i. But especially in the action of j on a non-scalar quantum mechanical wave-function $|\psi\rangle$ one has to differentiate between them as will be seen later.

3.2.2 The Pauli Algebra as the even Subalgebra of $\mathcal{D}$

The σ_k, defined with (439), satisfy

$$\sigma_i * \sigma_j \overset{(442)}{=} \langle \gamma_i\gamma_0\gamma_j\gamma_0 \rangle = \delta_{ij} \quad \Rightarrow \quad \sigma^i = \sigma_i \quad , \tag{443}$$

and are thus orthonormal. *They can be seen as basis vectors of a three-dimensional space.* Accordingly they *satisfy* with (399) *the Pauli commutation relations.* This gives the motivation for calling the associated algebra the **Pauli algebra** $\mathcal{P}$.

A basis of $\mathcal{P}$ can be constructed by multiplying the orthogonal basis vectors σ_k with each other. The results are collected in the following table.

Grade in $\mathcal{P}$	**grade 0**	**grade 1**	**grade 2**	**grade 3**
Interpretation	*scalars*	*vectors*	*bivectors*	*pseudoscalar*
Basis	$\{1\}$	$\{\sigma_i\}$	$\{i\sigma_k\}$	$\{i\}$
Grade in $\mathcal{D}$	0	2	2	4

We realize, that $\mathcal{P}$ contains all even elements of $\mathcal{D}$, so that it is the even subalgebra of $\mathcal{D}$. It is also worth noting, that the space of bivectors in $\mathcal{D}$ splits into two disjoint linear spaces, the spaces of vectors and bivectors in $\mathcal{P}$! On the other side we note, that the pseudoscalars $i_\mathcal{P}$ and $i_\mathcal{D}$ of $\mathcal{P}$ and $\mathcal{D}$, respectively, are with

$$i_\mathcal{P} = \sigma_1\sigma_2\sigma_3 \overset{(439)}{=} \gamma_1\gamma_0\gamma_2\gamma_0\gamma_3\gamma_0 = \gamma_0\gamma_1\gamma_2\gamma_3 = i_\mathcal{D} \tag{444}$$

identical to each other. Thus $\mathcal{D}$ and $\mathcal{P}$ share the space of scalars and pseudoscalars.

Since vectors in $\mathcal{P}$ occur frequently, we will adopt here the convention, to denote them with an arrow, to distinguish them from vectors in $\mathcal{D}$. So, *e.g.*,

$$\vec{a} = \sum_{i=1}^{3} \sigma_i a^i \quad . \tag{445}$$

Products in $\mathcal{P}$ The vectors of $\mathcal{P}$ satisfy itself

$$\vec{a}^2 = \sum_{i,j=1}^{3} a^i a^j \sigma_i\sigma_j = \sum_{i=1}^{3}(a^i)^2\sigma_i\sigma_i + \underbrace{\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{3} \underbrace{a^i a^j}_{\text{symm.}} \overbrace{\sigma_i\sigma_j}^{\text{skew symm.}}}_{=0} = \sum_{i=1}^{3}(a_i)^2 = \text{scalar} \quad . \tag{446}$$

Thus we can construct a inner and outer product of $\mathcal{P}$ as usual[43]

$$\left.\begin{array}{lclcl} \vec{a}\cdot_\mathcal{P}\vec{b} & \overset{\text{def}}{=} & \frac{\vec{a}\vec{b}+\vec{b}\vec{a}}{2} & = & \langle\vec{a}\vec{b}\rangle \\ \vec{a}\wedge_\mathcal{P}\vec{b} & \overset{\text{def}}{=} & \frac{\vec{a}\vec{b}-\vec{b}\vec{a}}{2} & = & \langle\vec{a}\vec{b}\rangle_2^\mathcal{P} \end{array}\right\} . \tag{447}$$

Since scalars of $\mathcal{P}$ and $\mathcal{D}$ are identical we get

$$\vec{a}\cdot_\mathcal{P}\vec{b} = \vec{a}\cdot_\mathcal{D}\vec{b} = \langle\vec{a}\vec{b}\rangle \quad . \tag{448}$$

But for the outer product

$$\vec{a}\wedge_\mathcal{P}\vec{b} = \vec{a}\times\vec{b} = \langle\vec{a}\vec{b}\rangle_2^\mathcal{P} = \text{bivector in } \mathcal{D} \quad , \tag{449}$$

while

$$\vec{a}\wedge_\mathcal{D}\vec{b} = \sum_{i,j=1}^{3} a^i b^j (\gamma_i\wedge_\mathcal{D}\gamma_0)\wedge_\mathcal{D}(\gamma_j\wedge_\mathcal{D}\gamma_0) = 0 \quad . \tag{450}$$

We thus have to indicate if we use the product of the subalgebra or the ordinary one.

Let us further generalize the definitions (447) in the usual way for simple multivectors $A_r, B_s \in \mathcal{P}$ (where r, s are the grades in $\mathcal{P}$) to

$$A_r \cdot_\mathcal{P} B_s \overset{\text{def}}{=} \langle A_r B_s\rangle_{|r-s|}^\mathcal{P} \quad ; \quad A_r \wedge_\mathcal{P} B_s \overset{\text{def}}{=} \langle A_r B_s\rangle_{r+s}^\mathcal{P} \quad , \tag{451}$$

where we defined with $\langle M\rangle_s^\mathcal{P}$ the grade operator with respect to the grade in $\mathcal{P}$.

[43] With (446) the σ_i fulfill the axioms of the geometric product. Thus we can follow the same steps given in the introduction.

Grade in $\mathcal{P}$	grade 0	grade 1	grade 2	grade 3
	scalar λ	*vector* $\vec{a} = a^k\sigma_k$	*bivector* $B = b^k i\sigma_k = i\vec{b}$	*pseudoscalar* i
Herm. Adj.	$\lambda^\dagger = \lambda$	$\vec{a}^\dagger = \vec{a}$	$B^\dagger = (i\vec{b})^\dagger = -i\vec{b}$	$i^\dagger = -i$
Reversion in $\mathcal{D}$	$\widetilde{\lambda} = \lambda$	$\widetilde{\vec{a}} = -\vec{a}$	$\widetilde{B} = -B$	$\widetilde{i} = i$

Table 3: Effects of the operations of hermitian adjoint and reversion in $\mathcal{P}$.

Reversion in $\mathcal{P}$ Let $A \in \mathcal{P}$ be a multivector. We denote the reversed quantity by $A^\dagger$. That means by definition (57)

$$[AB]^\dagger = B^\dagger A^\dagger \quad ; \quad [\sigma_i\sigma_j]^\dagger = \sigma_j\sigma_i \quad . \tag{452}$$

But defining the reversion in $\mathcal{P}$ in this way is not satisfactory, since the action on odd elements in $\mathcal{D}$ stays undefined. To remove this complication we prove by induction

$$A \in \mathcal{P} \Rightarrow A^\dagger = \gamma_0\widetilde{A}\gamma_0 \quad . \tag{453}$$

Clearly (453) is proved, when it is proved for each grade in $\mathcal{P}$. Thus we define $A_r \in \mathcal{P}$ to be a simple r-vector[44]. First we note, that for $r = 0$, that is for a scalar A_0, (453) is true. But if it is true for grade $n-1$, then it follows for a simple multivector of grade n that

$$\gamma_0\widetilde{A}_n\gamma_0 = \gamma_0\widetilde{A_{n-1}\sigma_m}\gamma_0 = \gamma_0\widetilde{\sigma}_m\gamma_0\underbrace{\gamma_0\widetilde{A}_{n-1}\gamma_0}_{=A^\dagger_{n-1}} = \sigma_m A^\dagger_{n-1} = A^\dagger_n \quad . \tag{454}$$

Thus we proved (453) by induction[45].

Given the reversion in $\mathcal{P}$ by (453), we encounter no further problems in extending the definition to all multivectors in $\mathcal{D}$. Thus we define the **hermitian adjoint** $A^\dagger$ of a multivector $A \in \mathcal{D}$ as

$$A^\dagger \stackrel{\text{def}}{=} \gamma_0\widetilde{A}\gamma_0 \quad . \tag{455}$$

For vectors in $\mathcal{D}$ we derive so

$$\gamma_\mu\gamma_\mu^\dagger = \gamma_\mu\gamma_0\gamma_\mu\gamma_0 = \begin{cases} 1 & \text{for} \quad \mu = 0 \\ -\gamma_0^2\gamma_\mu^2 & \text{for} \quad \mu \neq 0 \end{cases} \tag{456}$$

and thus if γ_0 and $\gamma_\mu; \mu \neq 0$ have different signatures

$$\gamma_\mu^\dagger = \gamma^\mu \quad . \tag{457}$$

In Table 3 we compare the hermitian adjoint with the reversion in $\mathcal{D}$, when applied on elements of $\mathcal{P}$. It shows, that there is a one-to-one relation between sign-changes and the grade in $\mathcal{P}$. The grade operator in $\mathcal{P}$ can thus always be expressed as a certain combination of reversions and hermitian adjungations. As an example, let $A \in \mathcal{P}$ be arbitrary. $B \stackrel{\text{def}}{=} (A + A^\dagger)/2$ contains then only the scalar and vector parts of A. Using the reversion in $\mathcal{D}$ allows us now to get with $C \stackrel{\text{def}}{=} (B + \widetilde{B})/2$ just the scalar part of A.

The Subalgebra of Pauli-even Elements The same arguments yield that the even elements in $\mathcal{P}$ define a subalgebra, which we will call $\mathcal{P}_2$. It is spanned by the basis vectors

$$\{1, \sigma_i\sigma_j\} \quad \text{or} \quad \{1, i\sigma_i\} \quad . \tag{458}$$

That this is closed under the geometric product is easily seen by

$$i\sigma_j i\sigma_k = -\sigma_j\sigma_k = -(\delta_{jk} + i\epsilon_{jkl}\sigma_l) \quad .$$

[44] That means with grade r in $\mathcal{P}$.

[45] Please note, that the proof given here is valid for arbitrary dimensional spaces, as long as an even subalgebra is constructed in the same way as here. The only assumption made is that $\gamma_0^2 = +1$. If $\gamma_0^2 = -1$ the right side of (453) gets an additional minus sign.

We can construct an orthonormal basis of $\mathcal{P}_2$ in the same way, as we did for $\mathcal{P}$. First we have to choose one of the basis vectors of $\mathcal{P}$, for instance σ_3. Now we define the vectors of $\mathcal{P}_2$ by

$$\mathbb{I} \stackrel{\text{def}}{=} \sigma_2\sigma_3 = i\sigma_1 \quad ; \quad \mathbb{J} \stackrel{\text{def}}{=} \sigma_1\sigma_3 = -i\sigma_2 \quad . \tag{459}$$

They are orthonormal, since

$$\mathbb{I} * \mathbb{J} = 0 \quad ; \quad \mathbb{I}^2 = \mathbb{J}^2 = -1 \quad . \tag{460}$$

The pseudoscalar $\mathbb{K}$ of $\mathcal{P}_2$ becomes then

$$\mathbb{K} \stackrel{\text{def}}{=} \mathbb{I}\mathbb{J} = i\sigma_1(-i\sigma_2) = i\sigma_3 \tag{461}$$

and is thus not equivalent with the pseudoscalar of $\mathcal{P}$ and $\mathcal{D}$. Indeed, since the pseudoscalar of $\mathcal{P}$ is of odd grade, and so not part of the even subalgebra, both pseudoscalars *cannot* be equivalent. It obeys

$$\mathbb{K}^2 = -1 \quad ; \quad \mathbb{K}\mathbb{I} = -\mathbb{I}\mathbb{K} \quad ; \quad \mathbb{K}\mathbb{J} = -\mathbb{J}\mathbb{K} \quad . \tag{462}$$

It should be pointed out, that there is nothing that distinguishes the pseudoscalar $\mathbb{K}$ from the vectors $\mathbb{I}$ and $\mathbb{J}$. It is pure convention to take one of them as the pseudoscalar. Writing down all the derived relations

$$\mathbb{I}^2 = \mathbb{J}^2 = \mathbb{K}^2 = -1 \quad ; \quad \mathbb{I}\mathbb{J}\mathbb{K} = -1 \tag{463}$$

shows, that $\mathbb{I}, \mathbb{J}$ and $\mathbb{K}$ give a **representation of quarternions.**

Subalgebras of $\mathcal{P}_2$ If $\mathbb{K}$ is the pseudoscalar of $\mathcal{P}_2$, it must have grade two in $\mathcal{P}_2$. Therefore the even subalgebra is spanned by the basis $\{1, \mathbb{K}\}$. As discussed above, also $\mathbb{I}$ and $\mathbb{J}$ can be taken as the pseudoscalar. Thus we have three subalgebras, spanned by

$$\{1, \mathbb{I}\} \quad ; \quad \{1, \mathbb{J}\} \quad ; \quad \{1, \mathbb{K}\} \quad . \tag{464}$$

But given two of them, *e.g.*, the first two, we find the multiplication table

$$\left.\begin{array}{lllllll} 11 & = & 1 & ; & 1\mathbb{J} & = & \mathbb{J} \\ \mathbb{I}1 & = & \mathbb{I} & ; & \mathbb{I}\mathbb{J} & = & \mathbb{K} \end{array}\right\} . \tag{465}$$

Thus only two of the algebras are independent.

4 Application to Classical Mechanics

This short section contains just two examples of applications of the geometric algebra in physics. All results given here are my own derivations. Nevertheless, I cannot exclude the possibility, that these results have been given before.

4.1 The Principle of Virtual Work

Let us consider first a particle, which can only move on a straight line. For a constant force $\vec{F}$ the quantity

$$\vec{F}\vec{x} = \vec{F}\cdot\vec{x} + \vec{F}\wedge\vec{x} \tag{466}$$

can be interpreted as the work done against the force (scalar part) and the torque currently exerted by the force (bivector part). For an arbitrary displacement $\delta\vec{x}$, which does not violate the constraints of the system, we have then

$$\delta(\vec{F}\vec{x}) = \vec{F}\delta\vec{x} = \delta W + \delta M \tag{467}$$

or for a whole system of n particles in equilibrium

$$\sum_{i=1}^{n} \vec{F}_i\delta\vec{x}_i = \sum_i \vec{F}\cdot\delta\vec{x}_i + \sum_i \vec{F}_i\wedge\delta\vec{x}_i = \delta W + \delta M \quad . \tag{468}$$

The *Principle of Virtual Work* is now

$$\sum_i \vec{F}_i\cdot\delta\vec{x}_i = \sum_i \langle\vec{F}_i\delta\vec{x}_i\rangle = \delta W = 0 \quad . \tag{469}$$

But what can we say about the term

$$\sum_i \vec{F}\wedge\delta\vec{x}_i = \sum_i \langle\vec{F}\delta\vec{x}_i\rangle_2 = \delta M \quad ? \tag{470}$$

If there is *rotational freedom around the origin* and the system is in equilibrium, the Principle of Virtual Work tells us that there is no resulting torque, *i.e.*, $M = 0$. Then δM is the change in torque exerted by the external forces. In order to have a *stable solution*, the change in torque should give a restoring force.

4.2 Reformulation of Hamiltons Equations

Let us assume, we have a system with n degrees of freedom and a Hamiltonian H. For generalized coordinates q^μ and their momenta p^μ, *Hamiltons equations of motion* are given by[46]

$$\partial_{q^k} H = -\partial_t p^k \quad ; \quad \partial_{p^k} H = \partial_t q^k \quad . \tag{471}$$

We *define* n *orthogonal* basis vectors e_k, which we associate to the generalized coordinates and for which we define $\partial_{q^k} e_k = 0$. Thus this basis vectors are in general not physically meaningful vectors. Only for the special case that the q^k are components with respect to a fixed, orthogonal basis, we can identify the e_k with this basis vectors. We introduce an additional basis vector e_0 with

$$e_0^2 \stackrel{\text{def}}{=} +1 \quad ; \quad e_0\cdot e_k = 0 \quad \text{for} \quad k \neq 0 \tag{472}$$

and define the multivector

$$z \stackrel{\text{def}}{=} \sum_{k=1}^{n} e_k(q^k + p^k e_0) \quad . \tag{473}$$

So the vector part contains the coordinates and the bivector part the momenta, *i.e.*,

$$Q \stackrel{(473)}{=} \langle z\rangle_1 \quad ; \quad P \stackrel{(473)}{=} \langle z\rangle_2 \quad . \tag{474}$$

[46]Here $\partial_{q^k} \stackrel{\text{def}}{=} \frac{\partial}{\partial q^k}$.

The Hamiltonian H now becomes a scalar-valued function of z

$$H(Q,P) \to H(z) = \langle H(z) \rangle \quad . \tag{475}$$

As an example, by assuming $e_k^2 = +1$, the Hamiltonian of a harmonic osciallator can be written as

$$H = \frac{1}{2}\langle z\tilde{z} \rangle = \frac{1}{2}(q^2 + p^2) \quad . \tag{476}$$

Further we define coordinate- and momenta-derivative operators by

$$\partial'_q \stackrel{\text{def}}{=} \sum_{k=1}^{n} e^k \partial_{q^k} \quad ; \quad \partial'_p \stackrel{\text{def}}{=} \sum_{k=1}^{n} e^k e_0 \partial_{p^k} \quad , \tag{477}$$

so that

$$\partial'_q Q = n \quad ; \quad \partial'_p P \stackrel{(477)}{=} e^k e_0 \partial_{p^k} p^i e_i e_0 \stackrel{(472)}{=} -n \quad . \tag{478}$$

We can now express Hamilton's equations in vector notation by

$$\left.\begin{array}{llll} \partial'_q H & \stackrel{(477)}{=} & \sum_{k=1}^{n} e^k \partial_{q^k} H & \stackrel{(471)}{=} & -\dot{P} e_0 \\ \partial'_p H & \stackrel{(477)}{=} & \sum_{k=1}^{n} e^k e_0 \partial_{p^k} H & \stackrel{(471)}{=} & \dot{Q} e_0 \end{array}\right\} . \tag{479}$$

We can combine both equations by writing

$$(\partial'_q + \partial'_p) H = (\dot{Q} - \dot{P}) e_0 = -e_0 \partial_t (Q - P) = -e_0 \partial_t \tilde{z} \quad . \tag{480}$$

Since the basis vector e_0 can be associated to the time derivative ∂_t, we interpret e_0 as the basis vector representing the *time-direction*. We can rewrite our result by taking the second term from the right to the left

$$\begin{aligned} 0 & \stackrel{(480)}{=} (e_0\partial_t + \partial'_q + \partial'_p)(H + Q - P) - e_0\partial_t H - \underbrace{(\partial'_q + \partial'_p)(Q - P)}_{=2n} \\ & = (e_0\partial_t + \partial'_q + \partial'_p + \partial_H)(\underbrace{e_0 t + Q}_{\stackrel{\text{def}}{=} q} \underbrace{-P + H}_{\stackrel{\text{def}}{=} e_0 p}) - 2n - \underbrace{e_0\partial_t e_0 t}_{=1} - \underbrace{\partial_H H}_{=1} - e_0\partial_t H \\ & = (\partial_q + \partial_p e_0)(\widetilde{q + p e_0}) - 2(n+1) - e_0\partial_t H \quad , \end{aligned} \tag{481}$$

where we defined

$$\left.\begin{array}{llllll} p & \stackrel{\text{def}}{=} & \sum_{\mu=0}^{n} p^\mu(t) e_\mu & ; & p^0 & \stackrel{\text{def}}{=} & H(t,Q,P) \\ q & \stackrel{\text{def}}{=} & \sum_{\mu=0}^{n} q^\mu(t) e_\mu & ; & q^0 & \stackrel{\text{def}}{=} & t \end{array}\right\} . \tag{482}$$

If the Hamiltonian is time-independent, that is $\partial_t H = 0$, Hamilton's equation takes the simple form

$$2(n+1) = (\partial_q + \partial_p e_0)(\widetilde{q + p e_0}) \quad . \tag{483}$$

That (483) contains still the same information as (471) is seen, when we split it in even and odd parts, which lead to (479). Taking the scalar product with the unit vectors then leads to the coordinate representation (471).

5 Electromagnetism

The electromagnetic interaction, described by the Maxwell equations, is one of the most important physical concepts. Geometric Algebra helps to simplify the picture, as was shown, *e.g.*, in [11] or [25]. Here I will collect and present some of the known results, which again show the power of Geometric Algebra in simplifying matter. As will be seen, Maxwells equations reduce to one multivector equation!

5.1 Maxwells Equations

5.1.1 Formulation in flat Space

First we will define the differential operator for our flat space by

$$\partial_x = \gamma^\mu \partial_\mu \quad ; \quad \partial_\mu \stackrel{\text{def}}{=} \frac{\partial}{\partial x^\mu} \quad , \tag{484}$$

where the γ^μ are the reciprocal vectors to γ_μ, which obey the relation

$$\gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu \quad . \tag{485}$$

We work in Minkowski space and choose the signature (+ - - -). Thus we can construct with

$$\gamma^0 = \gamma_0 \quad , \quad \gamma^j = -\gamma_j \tag{486}$$

the reciprocal basis vectors, which fulfill (485). Lowering the indices in (484) gives accordingly to (486)

$$\partial_x = \gamma_0 \partial_0 - \sum_{j=1}^{3} \gamma_j \partial_j \quad \Rightarrow \quad \partial_x \gamma_0 = \partial_0 - \vec{\nabla} \stackrel{\text{def}}{=} \partial_t - \vec{\nabla} \quad , \tag{487}$$

where we defined the basis of the three-dimensional even subalgebra, *i.e.*, the Pauli algebra $\mathcal{P}$, by

$$\sigma_j \stackrel{\text{def}}{=} \gamma_j \gamma_0 \quad ; \quad \sigma_j = \sigma^j \quad ; \quad \vec{\nabla} \stackrel{\text{def}}{=} \sum_{j=1}^{3} \sigma_j \partial_j = \sigma^j \partial_j \quad . \tag{488}$$

If we define the electromagnetic fields and the charge current as three-dimensional vector fields, *i.e.*, fields in the Pauli algebra $\mathcal{P}$ expressed with the help of the consructed basis by

$$\vec{E} \stackrel{\text{def}}{=} E^i \sigma_i \quad ; \quad \vec{B} \stackrel{\text{def}}{=} B^i \sigma_i \quad ; \quad \vec{J} = J^i \sigma_i \quad , \tag{489}$$

Maxwells equations can be written in the form

$$-(\partial_t - \vec{\nabla})(\vec{E} - i\vec{B}) = q + \vec{J} \quad . \tag{490}$$

Introducing the **Faraday multivector**

$$F \stackrel{\text{def}}{=} \vec{E} + i\vec{B} \quad \Rightarrow \quad F^\dagger = \vec{E} - i\vec{B} \quad , \tag{491}$$

which is an analogue of the conventional electromagnetic field tensor F_{ab}, allows us to rewrite Maxwells equations (490) as

$$-(\partial_t - \vec{\nabla})F^\dagger = q + \vec{J} \quad . \tag{492}$$

The usual form of the equations is easily derived by equating grades in (490), which is written out (all outer products in $\mathcal{P}$)

$$\begin{aligned} q + \vec{J} &\stackrel{(490)}{=} \vec{\nabla} \cdot \vec{E} + \vec{\nabla} \wedge \vec{E} + i\partial_t \vec{B} - \partial_t \vec{E} - \vec{\nabla} \cdot (i\vec{B}) - \vec{\nabla} \wedge (i\vec{B}) \\ &= \overbrace{\vec{\nabla} \cdot \vec{E}}^{\text{scalar}} \overbrace{- i\vec{\nabla} \wedge \vec{B} - \dot{\vec{E}}}^{\text{vector}} + \overbrace{\vec{\nabla} \wedge \vec{E} + i\partial_t \vec{B}}^{\text{pseudovector}} - \overbrace{i\vec{\nabla} \cdot \vec{B}}^{\text{pseudoscalar}} \quad . \end{aligned} \tag{493}$$

This is equivalent to the four equations

$$\left. \begin{array}{rclcrclcl} \vec{\nabla} \cdot \vec{E} &=& q &;& -\partial_t \vec{E} - i\vec{\nabla} \wedge \vec{B} &=& \vec{\nabla} \otimes \vec{B} - \partial_t \vec{E} &=& \vec{J} \\ \vec{\nabla} \cdot \vec{B} &=& 0 &;& \partial_t \vec{B} - i\vec{\nabla} \wedge \vec{E} &=& \vec{\nabla} \otimes \vec{E} + \partial_t \vec{B} &=& 0 \end{array} \right\} , \tag{494}$$

which are the well known Maxwell Equations. With (490) they are obviously formulated in the three-dimensional Pauli algebra $\mathcal{P}$. To deduce the form in the four-dimensional Dirac algebra $\mathcal{D}$, we remember, that $\gamma_0\gamma_0 = 1, i_3 = i_4 \stackrel{\text{def}}{=} i, \sigma_j = \gamma_j\gamma_0$ and write

$$-(\partial_t - \vec{\nabla})F^\dagger = -\overbrace{(\partial_t - \vec{\nabla})}^{\stackrel{(487)}{=}\partial_x\gamma_0}(\vec{E} - i\vec{B}) = -\partial_x \overbrace{(-\vec{E} - i\vec{B})}^{=-F}\gamma_0 = \partial_x F\gamma_0 \stackrel{(490)}{=} q + \vec{J} = J\gamma_0 \quad , \tag{495}$$

where

$$J \stackrel{\text{def}}{=} q\gamma_0 + J^k\gamma_k = (q + \vec{J})\gamma_0 \quad . \tag{496}$$

Multiplying with γ_0 on the right yields thus

$$\partial_x F = J \quad ; \quad J \stackrel{\text{def}}{=} q\gamma_0 + \sum_{i=1}^{3} J^i\gamma_i \quad . \tag{497}$$

This surprisingly simple equation contains all four Maxwells equations! The geometric product incorporates the whole algebraic structure of the equations. For me one of the strongest supports for Geometric Algebra — to bring one of the most fundamental equations of physics in such compact form.

5.1.2 The Vector Potential

Since $\vec{\nabla}\cdot\vec{B} = 0$, there exists a vector field $\vec{A}$, also called **vector potential**, so that we can write $\vec{B}$ as the curl of this field, *i.e.*,

$$\vec{B} = \vec{\nabla}\otimes\vec{A} = -i\vec{\nabla}\wedge_{\mathcal{P}}\vec{A} \quad . \tag{498}$$

Substituting into the Maxwell equations we find

$$\begin{aligned} 0 &\stackrel{(494)}{=} \partial_t\vec{B} - i\vec{\nabla}\wedge\vec{E} = -i\vec{\nabla}\wedge\partial_t\vec{A} - i\vec{\nabla}\wedge\vec{E} \\ &= -i\vec{\nabla}\wedge(\partial_t\vec{A} + \vec{E}) \quad . \end{aligned} \tag{499}$$

Thus there exists a *scalar* potential Φ, so that

$$\partial_t\vec{A} + \vec{E} = -\vec{\nabla}\Phi = -\vec{\nabla}\wedge_{\mathcal{P}}\Phi \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}\Phi - \partial_t\vec{A} \tag{500}$$

We combine the vector potential $\vec{A}$ and the scalar potential Φ to form the so called **four-vector potential**

$$A \stackrel{\text{def}}{=} \Phi\gamma_0 + \sum_{j=1}^{3} A^j\gamma_j \quad . \tag{501}$$

Taking the outer derivative of it yields[47]

$$\begin{aligned} \partial_x\wedge A &\stackrel{(487)}{=} (\partial_t\gamma_0 - \sum_{i=1}^{3}\partial_i\gamma_i)\wedge(A^0\gamma_0 + A^j\gamma_j) = \partial_t A^j\overbrace{\gamma_0\wedge\gamma_j}^{=-\sigma_j} - \partial_i A^0\overbrace{\gamma_i\wedge\gamma_0}^{=\sigma_i} - \partial_i A^j\overbrace{\gamma_i\wedge\gamma_j}^{=-\sigma_i\wedge_{\mathcal{P}}\sigma_j} \\ &= -\partial_t A^j\sigma_j - \vec{\nabla}\wedge_{\mathcal{P}}A^0 + \vec{\nabla}\wedge_{\mathcal{P}}\vec{A} = \overbrace{-\dot{\vec{A}} - \vec{\nabla}A^0}^{\stackrel{(500)}{=}\vec{E}} + i\overbrace{\vec{\nabla}\otimes\vec{A}}^{\stackrel{(498)}{=}\vec{B}} \\ &= \vec{E} + i\vec{B} = F \quad . \end{aligned} \tag{503}$$

We see immediately, that there is a gauge invariance, since for an arbitrary differentiable *scalar* function χ

$$A' \stackrel{\text{def}}{=} A - \partial_x\wedge\chi \stackrel{(487)}{=} A - \partial_t\gamma_0\chi + \sum_{j=1}^{3}\gamma_j\partial_j\chi \tag{504}$$

[47] Here we use

$$\gamma_i\wedge\gamma_j = \frac{\gamma_i\gamma_j - \gamma_j\gamma_i}{2} = \frac{\gamma_i\gamma_0\gamma_0\gamma_j - \gamma_j\gamma_0\gamma_0\gamma_i}{2} = \frac{\sigma_i\tilde{\sigma}_j - \sigma_j\tilde{\sigma}_i}{2} = -\sigma_i\wedge_{\mathcal{P}}\sigma_j \quad . \tag{502}$$

gives with

$$\partial_x \wedge A' \overset{(504)}{=} \partial_x \wedge A - \overbrace{\partial_x \wedge \partial_x \wedge}^{=0} \chi = \partial_x \wedge A \overset{(503)}{=} F \qquad (505)$$

the same physical fields. Thus the gauge transformation for the scalar and vector potential is

$$A'^0 \overset{(504)}{=} A^0 - \partial_t \chi = A'^0 - \partial^0 \chi \quad ; \quad A'^j \overset{(504)}{=} A^j + \partial_j \chi = A^j - \partial^j \chi \quad . \qquad (506)$$

We can always choose the so called **Lorentz gauge**, *i.e.*,

$$0 = \partial_x \cdot A = \sum_{\mu=0}^{3} \partial_\mu A^\nu \overbrace{\gamma^\mu \cdot \gamma_\nu}^{\overset{(485)}{=} \delta^\mu_\nu} = \partial_t A^0 + \vec{\nabla} \cdot \vec{A} \quad , \qquad (507)$$

which yields in (503)

$$\partial_x A = \partial_x \wedge A \overset{(503)}{=} F \quad . \qquad (508)$$

This allows us to write the Maxwell Equations (503) in the compact form

$$\partial_x \partial_x A \overset{(508)}{=} \partial_x F \overset{(503)}{=} J \quad . \qquad (509)$$

5.1.3 Maxwell Equation for Magnetic Mono-Poles

Given above results it is not very difficult to include magnetic charge and current. Magnetic mono-poles are not observed. Nevertheless it seems useful to explore the possibility of extending Maxwell equations.

Assuming magnetic mono-poles is equivalent to the assumption of magnetic charges and currents, *i.e.*,

$$\vec{\nabla} \cdot \vec{B} = q_m \quad ; \quad \partial_t \vec{B} - i \vec{\nabla} \wedge \vec{E} = -\vec{J}_m \quad . \qquad (510)$$

Adding the extra terms to Maxwells Equations (492) gives

$$\vec{J}_g = q + \vec{J} - i\vec{J}_m - iq_m = -(\partial_t - \vec{\nabla}) F^\dagger \quad , \qquad (511)$$

where we defined the generalized current

$$\vec{J}_g \overset{\text{def}}{=} q - iq_m + (\vec{J} - i\vec{J}_m) = (J - iJ_m)\gamma_0 \overset{\text{def}}{=} J_g \gamma_0 \quad . \qquad (512)$$

Multiplying (511) with γ_0 on the right yields

$$\partial_x F = J_g \quad ; \quad J_g \overset{\text{def}}{=} J - iJ_m \quad . \qquad (513)$$

5.2 Electromagnetism as a Theory of Curvature

5.2.1 Definition of a Curvature Tensor

Demanding *local* gauge invariance of the Dirac action under arbitrary local phase transformations leads[48] to the new (directional-)derivative operators

$$\mathcal{D} \overset{\text{def}}{=} \partial_x - \frac{je}{\hbar c} A \quad ; \quad \mathcal{D}_a \overset{\text{def}}{=} a \cdot \partial_x - \frac{je}{\hbar c} a \cdot A \quad , \qquad (514)$$

so that

$$\begin{aligned} \mathcal{D}_a \mathcal{D}_b f &= \left(a \cdot \partial_x - \frac{je}{\hbar c} a \cdot A\right) \left(b \cdot \partial_x - \frac{je}{\hbar c} b \cdot A\right) f \\ &= (a \cdot \dot{\partial}_x)(b \cdot \dot{\partial}_x)\ddot{f} - \frac{e^2}{\hbar^2 c^2}(a \cdot A)(b \cdot A) f - \frac{je}{\hbar c}\left[(a \cdot \dot{\partial}_x)(b \cdot \dot{A})\dot{f} + (a \cdot A)(b \cdot \dot{\partial}_x)\dot{f}\right] \end{aligned}$$

[48] See for example the gauge gravity section.

and thus

$$\begin{aligned}
[\mathcal{D}_a, \mathcal{D}_b]f &= (\mathcal{D}_a\mathcal{D}_b - \mathcal{D}_b\mathcal{D}_a)f \\
&= \frac{je}{\hbar c}\left[-(a\cdot\dot{\partial}_x)(b\cdot\dot{A})\dot{f} - (a\cdot A)(b\cdot\dot{\partial}_x)\dot{f} + (b\cdot\dot{\partial}_x)(a\cdot\dot{A})\dot{f} + (b\cdot A)(a\cdot\dot{\partial}_x)\dot{f}\right] \\
&= \frac{je}{\hbar c}\left\{a\cdot\left[\left(b\cdot\dot{\partial}_x\right)\dot{A}\right] - b\cdot\left[\left(a\cdot\dot{\partial}_x\right)\dot{A}\right]\right\}f \quad . \qquad (515)
\end{aligned}$$

We thus define the curvature as

$$R(a,b) = \frac{je}{\hbar c}\left\{a\cdot\left[\left(b\cdot\dot{\partial}_x\right)\dot{A}\right] - b\cdot\left[\left(a\cdot\dot{\partial}_x\right)\dot{A}\right]\right\} \quad . \qquad (516)$$

Since R is skew symmetric and linear in a and b we can say $R(a,b) = R(a\wedge b)$. It should be noted that R is rather an operator than just a linear function! The j that is still present is acting as $i\sigma_3$ from the right!

5.2.2 The Matrix Elements

Calculating the matrix elements gives

$$\begin{aligned}
R(\gamma_j\wedge\gamma_0) &= \frac{je}{\hbar c}\left\{\overbrace{(\gamma_0\cdot\dot{\partial}_x)}^{=\partial_0}\overbrace{(\gamma_j\cdot\dot{A})}^{=A_j} - \overbrace{(\gamma_j\cdot\dot{\partial}_x)}^{=\partial_j}\overbrace{(\gamma_0\cdot\dot{A})}^{=A_0}\right\} \\
&= \frac{je}{\hbar c}\left\{\dot{\partial}_0\dot{A}_j - \dot{\partial}_j\dot{A}_0\right\} = \frac{je}{\hbar c}\vec{E}\cdot\sigma_j \qquad (517)
\end{aligned}$$

and

$$\begin{aligned}
R(\gamma_i\wedge\gamma_j) &= \frac{je}{\hbar c}\left\{\overbrace{(\gamma_j\cdot\dot{\partial}_x)}^{=\partial_j}\overbrace{(\gamma_i\cdot\dot{A})}^{=A_i} - \overbrace{(\gamma_i\cdot\dot{\partial}_x)}^{=\partial_i}\overbrace{(\gamma_j\cdot\dot{A})}^{=A_j}\right\} \\
&= \frac{je}{\hbar c}\{\partial_j A_i - \partial_i A_j\} = \frac{je}{\hbar c}(i\vec{B})\cdot(\gamma_i\wedge\gamma_j) \quad . \qquad (518)
\end{aligned}$$

Thus we derived the compact form

$$R(\gamma_\nu\wedge\gamma_\mu) = \frac{je}{\hbar c}F\cdot(\gamma_\nu\wedge\gamma_\mu) \quad . \qquad (519)$$

Since j is a right-sided multiplication with $i\sigma_3$ the curvature operates as

$$[\mathcal{D}_a, \mathcal{D}_b]\,\psi = R\psi = \lambda(a\wedge b)\psi i\sigma_3 \qquad (520)$$

with the scalar

$$\lambda(a\wedge b) \overset{(519)}{=} \frac{e}{\hbar c}F * (a\wedge b) \quad . \qquad (521)$$

6 Formulation of Quantum Mechanics

This section must be seen as a review. Most results have been derived, *e.g.* [8]. Nevertheless, it also contains some original ideas: (559) and (593), which finally lead to (599), seem to be new.

6.1 Non-relativistic Spin 1/2 Particles - The Pauli Spinor

6.1.1 Representation of Pauli Spinors and Pauli Matrices

We saw that the σ-basis vectors of our three-dimensional rest-frame obey the anticommutation relation of the σ-matrices. So the σ-matrices can be seen as a matrix representation of our three-dimensional geometric algebra. We will now try to find an equivalent representation of the spinors and σ-operators in our Dirac algebra $\mathcal{D}$, connected by an isomorphism to the standard representation of quantum mechanics by matrices.

The Spinor The Pauli spinor for fermions (spin 1/2) is written as

$$|\phi\rangle = \begin{pmatrix} \phi_\uparrow \\ \phi_\downarrow \end{pmatrix} \quad , \tag{522}$$

where $\phi_\uparrow$ describes the state with spin up and $\phi_\downarrow$ the state with spin down. Since both wave-functions are complex, one can split them in real and imaginary parts. With j as the complex unit we define

$$\phi_\uparrow \stackrel{\text{def}}{=} a^0 + ja^3 \quad ; \quad \phi_\downarrow \stackrel{\text{def}}{=} -a^2 + ja^1 \tag{523}$$

and find after substituting in (522) the representation for the Pauli spinor

$$|\phi\rangle \stackrel{(522)}{=} \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \end{pmatrix} \quad . \tag{524}$$

These are four real, independent components. Since the even subalgebra $\mathcal{P}_2$ of the Pauli algebra $\mathcal{P}$ is the only subalgebra with four dimensions, a mapping onto this subalgebra would seem to be natural. This would guarantee that the product of two spinors is again a spinor, *i.e.*,

$$\phi, \chi \in \mathcal{P}_2 \Rightarrow \phi\chi \in \mathcal{P}_2 \quad . \tag{525}$$

We will see that the mapping

$$|\phi\rangle = \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \end{pmatrix} \leftrightarrow \phi = a^0 + a^k i\sigma_k \tag{526}$$

gives a convenient representation.

The Operators Now we have to find a representation of the σ-operators[49] $\hat{\sigma}$. In standard quantum mechanics they are given by the σ-matrices

$$\hat{\sigma}_1 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_2 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix} \qquad \hat{\sigma}_3 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{527}$$

Let us first consider $\hat{\sigma}_1$. Starting with the action on the Pauli spinor in the usual representation we derive

$$\begin{aligned}
\hat{\sigma}_1|\phi\rangle &\stackrel{(527)}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b^0 + jb^3 \\ -b^2 + jb^1 \end{pmatrix} = \begin{pmatrix} -b^2 + jb^1 \\ b^0 + jb^3 \end{pmatrix} \\
&\stackrel{(526)}{\leftrightarrow} -b^2 + b^3 i\sigma_1 - b^0 i\sigma_2 + b^1 i\sigma_3 \\
&= \sigma_1 \left[-\sigma_1\sigma_3 b^2 + b^3 i\sigma_3 - b^0 i \overbrace{\sigma_1\sigma_2\sigma_3}^{=i} + b^1 i\sigma_1 \right] \sigma_3 \\
&= \sigma_1 \left[b^0 + b^1 i\sigma_1 + b^2 i\sigma_2 + b^3 i\sigma_3 \right] \sigma_3 \\
&= \sigma_1 \phi \sigma_3 \quad .
\end{aligned} \tag{528}$$

[49] They are different to the basis vectors σ_μ, so we will mark them with a hat as $\hat{\sigma}_\mu$.

Analogously one gets for $k \in \{1,2,3\}$

$$\hat{\sigma}_k|\phi\rangle \leftrightarrow \sigma_k\phi\sigma_3 \quad . \tag{529}$$

Multiplication by j yields for the Pauli spinor

$$j|\phi\rangle \overset{(524)}{=} j\begin{pmatrix} b^0+jb^3 \\ -b^2+jb^1 \end{pmatrix} = \begin{pmatrix} -b^3+jb^0 \\ -b^1-jb^2 \end{pmatrix} \quad , \tag{530}$$

which transforms with (526) to

$$\begin{aligned} j\phi &\overset{(526)}{=} -b^3 - b^2 i\sigma_1 + b^1 i\sigma_2 + b^0 i\sigma_3 \\ &= \left[-b^3(-i\sigma_3) - b^2\sigma_1\sigma_3 + b^1\sigma_2\sigma_3 + b^0\right] i\sigma_3 \\ &= \left[b^0 + b^1 i\sigma_1 + b^2 i\sigma_2 + b^3 i\sigma_3\right] i\sigma_3 \\ &= \phi i\sigma_3 \quad . \end{aligned} \tag{531}$$

Thus the complex unit is replaced by a the bivector $i\sigma_3 = -\gamma_1\gamma_2$ — *representing the spin plane* in the rest-frame of the fermion. This surprising result shows how the spin is related to the complex nature of the Dirac theory.

We continue by considering the operation of complex conjugation. Following the same translation procedure yields

$$\begin{aligned} |\phi\rangle^* &\overset{(526)}{=} \begin{pmatrix} a^0 - ja^3 \\ -a^2 - ja^1 \end{pmatrix} \\ &\updownarrow \\ \phi^* &\overset{(526)}{=} a^0 - a^1 i\sigma_1 + a^2 i\sigma_2 - a^3 i\sigma_3 \\ &= \sigma_2\left[a^0\sigma_2 - a^1 i\sigma_2\sigma_1 + a^2 i\sigma_2\sigma_2 - a^3 i\sigma_2\sigma_3\right] = \sigma_2\phi\sigma_2 \quad . \end{aligned} \tag{532}$$

6.1.2 The Scalar Product for Pauli Spinors

The Construction of a Scalar Product The standard inner product between two spinors, which is in bra-ket notation $\langle\phi|\chi\rangle = \lambda$, gives a complex number λ and it obeys

$$\langle\phi|\chi\rangle = \langle\chi|\phi\rangle^* \quad ; \quad \langle\phi|j\chi\rangle = j\langle\phi|\chi\rangle \quad . \tag{533}$$

This allows us to separate it in real and imaginary part through

$$\langle\phi|\chi\rangle = \Re\langle\phi|\chi\rangle - j\Re\langle\phi|j\chi\rangle \quad . \tag{534}$$

It is thus sufficient to find a translation for the real part. Inspired by (63) we define

$$\Re\langle\phi|\chi\rangle \leftrightarrow \langle\phi^\dagger\chi\rangle \quad , \tag{535}$$

with the hermitian adjoint defined by (455)[50]. This yields

$$\begin{aligned} \langle\phi^\dagger\chi\rangle &\overset{(455)}{=} \langle(\gamma_0\tilde{\phi})(\gamma_0\chi)\rangle = \langle(\gamma_0\cdot\tilde{\phi} + \gamma_0\wedge\tilde{\phi})(\gamma_0\cdot\chi + \gamma_0\wedge\chi)\rangle \\ &= (\gamma_0\cdot\tilde{\phi})(\gamma_0\cdot\chi) + (\gamma_0\wedge\tilde{\phi})\cdot(\gamma_0\wedge\chi) \quad . \end{aligned} \tag{536}$$

Using (535) we can construct the full inner product with the help of (534) and (531) as

$$\langle\phi|\chi\rangle \leftrightarrow \langle\phi^\dagger\chi\rangle - \langle\phi^\dagger\chi i\sigma_3\rangle i\sigma_3 \quad . \tag{537}$$

It is easy to check that the requirements (533) are fulfilled[51]:

$$\begin{aligned} \langle\phi|j\chi\rangle &\leftrightarrow \langle\phi^\dagger\chi i\sigma_3\rangle - \langle\phi^\dagger\chi \overbrace{i\sigma_3 i\sigma_3}^{=-1}\rangle i\sigma_3 \\ &= \left[\langle\phi^\dagger\chi\rangle - \langle\phi^\dagger\chi i\sigma_3\rangle i\sigma_3\right] i\sigma_3 \leftrightarrow j\langle\phi|\chi\rangle \end{aligned} \tag{538}$$

$$\begin{aligned} \langle\phi|\chi\rangle &\leftrightarrow \langle\phi^\dagger\chi\rangle - \langle\phi^\dagger\chi i\sigma_3\rangle i\sigma_3 = \langle\phi^\dagger\chi\rangle^\dagger - \langle\phi^\dagger\chi i\sigma_3\rangle^\dagger i\sigma_3 \\ &= \langle\chi^\dagger\phi\rangle + \langle\chi^\dagger\phi i\sigma_3\rangle i\sigma_3 \leftrightarrow \langle\phi|\chi\rangle^* \quad . \end{aligned} \tag{539}$$

[50] We will later see that Pauli spinors commute with γ_0 and so $\phi^\dagger \overset{(455)}{=} \tilde{\phi}$. Therefore, there is no ambiguity in the definition (535).

[51] In fact, they have to be satisfied by construction.

Position Dependence The standard inner product for wave-functions would be

$$\int |d^3x| \; \{\langle \phi^\dagger \chi \rangle - \langle \phi^\dagger \chi i\sigma_3 \rangle i\sigma_3\} \quad . \tag{540}$$

However, this integral is in comparison with (537) dependent on the choice of the rest-space of the observer. One could express this best by replacing the scalar volume element with a directed volume element

$$d^3x \overset{\text{def}}{=} |d^3x| \gamma_3 \wedge \gamma_2 \wedge \gamma_1 = |d^3x| i\gamma_0 \quad . \tag{541}$$

It is important to note that in (537) we are not involving any integral and so our inner product is *still position dependent.* Thus this inner product (537) is defined for each point in the Minkowski space.

6.1.3 Interpretation of the Pauli Spinor

Rotations and Lorentz transformations are given by

$$R = nm \qquad \text{with} \qquad n^2 = m^2 = 1 \quad . \tag{542}$$

For a *pure rotation* the rotation-plane is given by n and m must *not* contain any time component. Thus

$$\begin{aligned} n \cdot \gamma_0 &= m \cdot \gamma_0 = 0 & (543)\\ &\Rightarrow \; n\gamma_0 = -\gamma_0 n \quad , \quad m\gamma_0 = -\gamma_0 m \\ &\Rightarrow \; R^\dagger \overset{(455)}{=} \gamma_0 \widetilde{R} \gamma_0 \overset{(542)}{=} \gamma_0 m n \gamma_0 = -m\gamma_0 n\gamma_0 = mn = \widetilde{R} \quad . & (544) \end{aligned}$$

On the other side *pure Lorentz transformations* should be "rotations" in a plane with the "time" axis γ_0. Thus[52]

$$L = \gamma_0 m \Rightarrow L^\dagger \overset{(455)}{=} \gamma_0 \widetilde{L} \gamma_0 \overset{(542)}{=} \gamma_0 m \gamma_0 \gamma_0 = L \quad . \tag{545}$$

Rotations and Lorentz transformations obey

$$R\widetilde{R} = L\widetilde{L} = 1 \quad . \tag{546}$$

The Square of the Wave Function Since

$$\rho \overset{\text{def}}{=} \phi\phi^\dagger = (\phi\phi^\dagger)^\dagger = \rho^\dagger \quad , \tag{547}$$

we can conclude, from the above Table 3, that ρ has only scalar and vector parts[53]. But since the Pauli spinor is an element of the *even subalgebra* $\mathcal{P}_2$ of $\mathcal{P}$, spanned by $\{1, i\sigma_k\}$, the product cannot contain a vector part. Thus ρ is a *real number*, which suggests defining

$$R = \frac{\phi}{\sqrt{\phi\phi^\dagger}} = \frac{\phi}{\sqrt{\rho}} \quad , \tag{548}$$

so that

$$RR^\dagger = \frac{\phi\phi^\dagger}{\rho} = 1 \quad . \tag{549}$$

Thus R is a rotation in the three-dimensional space[54]. The wave function is now given by

$$\phi = \sqrt{\rho} R \quad , \tag{550}$$

which tells us that ϕ contains the information to rotate and dilate[55]. Now we can see that expressions like

$$s = \phi \sigma_3 \phi^\dagger \quad , \tag{551}$$

which first seem to be strangely dependent on the fixed $\{\sigma_k\}$-frame, become physically meaningful. ϕ contains the information to rotate the σ-frame in the frame of observables. In the example of the spin vector, σ_3 gets rotated in the spin direction.

[52] To define the rotation only the plane and angle between n and m is significant. So we can rotate n and m in the plane. This allows us to set $n \overset{\text{def}}{=} \gamma_0$, since γ_0 is in the plane. Note that we cannot make any predictions about m.

[53] A similar argument will be true for the Dirac spinor, where ρ can contain a pseudoscalar part.

[54] Since the hermitian adjoint is the reversion in the three-dimensional rest-space.

[55] Note that rotation and dilation are depending on x.

Spin Eigenstates It is important to note from (526) that a Pauli spinor ϕ commutes with γ_0

$$\gamma_0\phi = \phi\gamma_0 \tag{552}$$

and thus

$$\phi^\dagger \overset{(455)}{=} \gamma_0\tilde{\phi}\gamma_0 = \tilde{\phi} \quad . \tag{553}$$

Another important point in the interpretation of the derived spinor representation is revealed by writing

$$|\phi\rangle = \begin{pmatrix} |\phi_\uparrow\rangle \\ |\phi_\downarrow\rangle \end{pmatrix} \Rightarrow \left.\begin{array}{lcl} \phi_\uparrow & \overset{(526)}{=} & a^0 + i\sigma_3 a^3 \\ \phi_\downarrow & \overset{(526)}{=} & a^1 i\sigma_1 + a^2 i\sigma_2 \quad = \quad i(a^1\sigma_1 + a^2\sigma_2) \end{array}\right\} , \tag{554}$$

and noting

$$\left.\begin{array}{lcl} \sigma_3\phi_\uparrow & = & \phi_\uparrow\sigma_3 \\ \sigma_3\phi_\downarrow & = & -\phi_\downarrow\sigma_3 \end{array}\right\} . \tag{555}$$

Together with (529) we find thus the *eigenvalue equation*

$$\hat{\sigma}_3\phi_{\uparrow/\downarrow} = \sigma_3\phi_{\uparrow/\downarrow}\sigma_3 = \pm\phi_{\uparrow/\downarrow} \quad , \tag{556}$$

which is the expected eigenvalue of the spin operator! $\phi_{\uparrow/\downarrow}$ are thus both associated eigenstates. Note that since $\phi_{\uparrow/\downarrow}$ commutes with γ_0 and i, we can also see $i\gamma_3$ or γ_3 as the spin operator

$$i\gamma_3\phi_{\uparrow/\downarrow}\gamma_3 i = \pm\phi_{\uparrow/\downarrow} \quad ; \quad \gamma_3\phi_{\uparrow/\downarrow}\gamma_3 = \mp\phi_{\uparrow/\downarrow} \quad , \tag{557}$$

but γ_3 is not an element of the Pauli algebra $\mathcal{P}$. Since ϕ is even in $\mathcal{P}$, and $\phi = \phi_\uparrow + \phi_\downarrow$, finally the relation follows:

$$\left.\begin{array}{lclcl} \sigma_3 \cdot_{\mathcal{P}} \phi & \overset{(69)}{=} & \frac{\sigma_3(\phi_\uparrow+\phi_\downarrow)-(\phi_\uparrow+\phi_\downarrow)\sigma_3}{2} & \overset{(555)}{=} & \sigma_3\phi_\downarrow \\ \sigma_3 \wedge_{\mathcal{P}} \phi & \overset{(70)}{=} & \frac{\sigma_3(\phi_\uparrow+\phi_\downarrow)+(\phi_\uparrow+\phi_\downarrow)\sigma_3}{2} & \overset{(555)}{=} & \sigma_3\phi_\uparrow \end{array}\right\} . \tag{558}$$

After multiplying with σ_3, (558) yields the equivalent relations[56]

$$\phi_\downarrow \overset{(558)}{=} P_{\sigma_3}(\phi) \quad ; \quad \phi_\uparrow \overset{(558)}{=} P^\perp_{\sigma_3}(\phi) \quad . \tag{559}$$

Orthogonality of the Spin Parts We will establish now that $\phi_\uparrow$ and $\phi_\downarrow$ are "orthogonal" *multivectors* in the sense of (75), *i.e.*, they obey

$$\phi_\uparrow\tilde{\phi}_\downarrow = -\phi_\downarrow\tilde{\phi}_\uparrow \quad . \tag{560}$$

Using the representation (526), *i.e.*,

$$\phi_\uparrow = a^0 + i\sigma_3 a^3 \quad , \quad \phi_\downarrow = a^1 i\sigma_1 + a^2 i\sigma_2 \quad , \tag{561}$$

we find for the reversion in the Dirac algebra[57] $\mathcal{D}$

$$\tilde{\phi}_\uparrow = a^0 - i\sigma_3 a^3 \quad , \quad \tilde{\phi}_\downarrow = -\phi_\downarrow \quad . \tag{562}$$

With the help of the commutation relations

$$a^0\tilde{\phi}_\downarrow = -\phi_\downarrow a^0 \quad , \quad i\sigma_3\tilde{\phi}_\downarrow = -\tilde{\phi}_\downarrow i\sigma_3 = \phi_\downarrow i\sigma_3 \tag{563}$$

we derive the desired result:

$$\phi_\uparrow\tilde{\phi}_\downarrow = (a^0 + i\sigma_3 a^3)\tilde{\phi}_\downarrow = \tilde{\phi}_\downarrow(a^0 - i\sigma_3 a^3) = -\phi_\downarrow\tilde{\phi}_\uparrow \quad . \tag{564}$$

Since the Pauli spinors commute with γ_0, (564) yields immediately with (553) the orthogonality relation in $\mathcal{P}$, *i.e.*, $\phi_\uparrow\phi_\downarrow^\dagger = -\phi_\downarrow\phi_\uparrow^\dagger$.

[56] Here we understand the projection operator as acting in $\mathcal{P}$.

[57] But since $i\sigma_j$ is a bivector in $\mathcal{D}$ *and* $\mathcal{P}$, it changes sign under reversion and hermitian adjungation.

6.2 Relativistic Spin 1/2 Particles - The Dirac Spinor

6.2.1 A Representation of the Dirac Spinor and the related Operators

The Spinor Representation The Dirac spinor

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\chi\rangle \end{pmatrix} = \begin{pmatrix} |\phi_\uparrow\rangle \\ |\phi_\downarrow\rangle \\ |\chi_\uparrow\rangle \\ |\chi_\downarrow\rangle \end{pmatrix} \tag{565}$$

consists of two Pauli spinors and thus has eight real components. The mapping of them onto the Pauli algebra $\mathcal{P}$, which is the even subalgebra of the space-time algebra $\mathcal{D}$, would seem to be natural, since it has eight dimensions. Using this subalgebra ensure that

$$\psi, \psi' \in \mathcal{P} \Rightarrow \psi\psi' \in \mathcal{P} \quad . \tag{566}$$

It turns out to be convenient to map one Pauli spinor onto the even part of the Pauli algebra, and the other one onto the odd. Thus we can write the transformation by using the mapping of Pauli spinors, given by (526), as

$$|\psi\rangle \overset{(565)}{=} \begin{pmatrix} |\phi\rangle \\ |\chi\rangle \end{pmatrix} \leftrightarrow \psi = \phi + \chi\sigma_3 \quad . \tag{567}$$

Using (526), we have, in full

$$|\psi\rangle \overset{(565)}{=} \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \\ b^0 + jb^3 \\ -b^2 + jb^1 \end{pmatrix} \leftrightarrow \psi \overset{(526)}{=} a^0 + a^k i\sigma_k + \left[b^0 + b^k i\sigma_k\right]\sigma_3 \quad . \tag{568}$$

Translation of the Quantum Operators Again we have to ask how the action of the quantum operators translate into our space-time algebra description. We start with the Dirac matrices $\hat{\gamma}_\mu$, given by

$$\hat{\gamma}_0 \overset{\text{def}}{=} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \hat{\gamma}_k \overset{\text{def}}{=} \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix} \quad \hat{\gamma}_5 \overset{\text{def}}{=} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad . \tag{569}$$

Applying $\hat{\gamma}_0$ on a Dirac spinor in the matrix representation (565) and then translating the resulting spinor with (567) gives

$$\begin{aligned} \hat{\gamma}_0|\psi\rangle &\overset{(569)}{=} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \phi \\ -\chi \end{pmatrix} \\ &\updownarrow \\ \hat{\gamma}_0\psi &= \phi - \chi\sigma_3 = \gamma_0(\phi + \chi\sigma_3)\gamma_0 = \gamma_0\psi\gamma_0 \quad . \end{aligned} \tag{570}$$

For $\hat{\gamma}_j$ it follows that

$$\begin{aligned} \hat{\gamma}_j|\psi\rangle &\overset{(569)}{=} \begin{pmatrix} 0 & -\hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} -\hat{\sigma}_j\chi \\ \hat{\sigma}_j\phi \end{pmatrix} = \begin{pmatrix} -\sigma_j\chi\sigma_3 \\ \sigma_j\phi\sigma_3 \end{pmatrix} \\ &\updownarrow \\ \hat{\gamma}_j\psi &\overset{(567)}{=} -\sigma_j\chi\sigma_3 + \sigma_j\phi\sigma_3\sigma_3 = -\gamma_j\gamma_0\chi\gamma_3\gamma_0 + \gamma_j\gamma_0\phi \\ &= \gamma_j\phi\gamma_0 - \gamma_j\chi\gamma_0\gamma_3\gamma_0 = \gamma_j\phi\gamma_0 + \gamma_j\chi\sigma_3\gamma_0 \\ &= \gamma_j(\phi + \chi\sigma_3)\gamma_0 = \gamma_j\psi\gamma_0 \quad . \end{aligned} \tag{571}$$

$\hat{\gamma}_5$ has to be treated separately. Applying the same procedure we find

$$\begin{aligned} \hat{\gamma}_5|\psi\rangle &\overset{(569)}{=} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \chi \\ \phi \end{pmatrix} \\ &\updownarrow \\ \hat{\gamma}_5\psi &= \chi + \phi\sigma_3 = (\chi\sigma_3 + \phi)\sigma_3 = \psi\sigma_3 \quad . \end{aligned} \tag{572}$$

But also the action of j has to be translated. Here it follows directly from the discussion of the Pauli spinor case that

$$j\psi \overset{(567)}{=} j(\phi + \chi\sigma_3) \overset{(531)}{=} (\phi + \chi\sigma_3)i\sigma_3 = \psi i\sigma_3 \quad . \tag{573}$$

We also have to translate the operation of *complex conjugation* into a space-time algebra operation. Here we use the result for the Pauli spinor (523) and find

$$\begin{aligned}\psi^* &\overset{(568)}{=} \phi^* + \chi^*\sigma_3 \overset{(523)}{=} \sigma_2\phi\sigma_2 + \sigma_2\chi\sigma_2\sigma_3 = \sigma_2(\phi - \chi\sigma_3)\sigma_2 \\ &\overset{(570)}{=} \sigma_2\gamma_0\psi\gamma_0\sigma_2 = -\gamma_2\psi\gamma_2 \quad .\end{aligned} \tag{574}$$

6.2.2 The Dirac Equation in Space-Time Algebra Form

With the derived translation rules for the quantum operators we encounter no further problems to translate the Dirac equation

$$\hat{\gamma}^\mu(j\partial_\mu - eA_\mu)|\psi\rangle = m|\psi\rangle \tag{575}$$

to

$$\gamma^\mu\partial_\mu\psi i\sigma_3\gamma_0 - e\gamma^\mu A_\mu\psi\gamma_0 = m\psi \quad . \tag{576}$$

By multiplication with γ_0 from the right we transfer the result in the neat form

$$\partial_x\psi i\sigma_3 - eA\psi = m\psi\gamma_0 \quad . \tag{577}$$

6.2.3 Relation to Klein-Gordon Theory

If there is *no vector potential, i.e.*, $A = 0$, (577) becomes

$$\psi = \frac{1}{m}\partial_x\psi i\gamma_3 \quad . \tag{578}$$

Substituting for itself on the right-hand side yields

$$\psi = \frac{1}{m}\partial_x\left\{\frac{1}{m}\partial_x\psi i\gamma_3\right\}i\gamma_3 = -\frac{1}{m^2}\partial_x^2\psi \quad . \tag{579}$$

Since this equation involves only scalar-like operators each component of the Dirac spinor $\psi = e_J\psi^J$ satisfies

$$0 = \underbrace{(\partial_x^2 + m^2)}_{\text{scalar operator}}\psi \quad , \tag{580}$$

which is the Klein-Gordon equation.

6.2.4 The Schrödinger Equation as the Non-Relativistic Limit

Substituting the plane-wave solution

$$\psi = (\phi_p + \chi_p\sigma_3)e^{-i\sigma_3 p\cdot x} \tag{581}$$

into the Dirac equation (577) yields

$$m(\phi_p + \chi_p\sigma_3)\gamma_0 = p(\phi_p + \chi_p\sigma_3) - eA(\phi_p + \chi_p\sigma_3) = \underbrace{(p - eA)}_{\overset{\text{def}}{=} P}(\phi_p + \chi_p\sigma_3) \quad . \tag{582}$$

We define now

$$P \overset{\text{def}}{=} p - eA \quad ; \quad P\gamma_0 = P_0 + \vec{P} \quad , \tag{583}$$

so that (582), after multiplying with γ_0 from the right, gives

$$m(\phi_p + \chi_p\sigma_3) = (P^0 + \vec{P})(\phi_p - \chi_p\sigma_3) \quad . \tag{584}$$

Equating Pauli odd and even terms yields the two equations

$$\left.\begin{array}{lcl} m\phi_p & = & P^0\phi_p - \vec{P}\chi_p\sigma_3 \\ m\chi_p\sigma_3 & = & \vec{P}\phi_p - P^0\chi_p\sigma_3 \end{array}\right\} . \tag{585}$$

With the help of the second one we can express $\chi_p\sigma_3$ as

$$\chi_p\sigma_3 = \frac{\vec{P}\phi_p}{m + P^0}. \tag{586}$$

So for $\vec{P} = 0$ we find $\chi_p = 0$! We thus interpret χ_p as the antiparticle solution. In the non-relativistic limit the mass contribution towards the energy is dominating, *i.e.*, $P^0 \approx m$. Hence we derive as approximation for (586)

$$\chi_p\sigma_3 \approx \frac{\vec{P}\phi_p}{2m} \quad . \tag{587}$$

This gives finally in (585)

$$(m - P^0)\phi_p \overset{(585)}{=} -\vec{P}\chi_p\sigma_3 \overset{(587)}{\approx} -\vec{P}\frac{\vec{P}\phi_p}{2m} \quad , \tag{588}$$

which is with[58] $E \overset{\text{def}}{=} P^0 - m$ the desired **Schrödinger Equation**

$$\frac{\vec{P}^2}{2m}\phi_p = E\phi_p \quad . \tag{589}$$

6.2.5 The Interpretation of the Dirac Spinor

The Dirac Spinor as a Lorentz-Rotation In analogy to the interpretation of the Pauli spinor, we see that[59] $\psi\tilde{\psi} = \widetilde{\psi\tilde{\psi}} = \psi\tilde{\psi}$ can only contain *scalar and pseudoscalar* parts. Thus we define

$$\rho e^{i\beta} \overset{\text{def}}{=} \psi\tilde{\psi} \tag{590}$$

and define a Lorentz transformation by[60]

$$L \overset{\text{def}}{=} \frac{\psi}{\pm\sqrt{\rho}e^{i\beta/2}} \quad . \tag{591}$$

Particle and Antiparticle Parts Since in (567) ϕ and χ commute with γ_0, but σ_3 anticommutes with γ_0, we have

$$\psi\gamma_0 \overset{(567)}{=} (\phi + \chi\sigma_3)\gamma_0 = \gamma_0(\phi - \chi\sigma_3) \quad . \tag{592}$$

Therefore, we can express the Pauli even and odd parts of the Dirac spinor by

$$\left.\begin{array}{lcl} \phi & = & \gamma_0\frac{(\gamma_0\psi+\psi\gamma_0)}{2} = \frac{\psi+\tilde{\psi}^\dagger}{2} \overset{(68)}{=} \gamma_0(\gamma_0 \wedge_{\mathcal{D}} \psi) = P^{\perp}_{\gamma_0}(\psi) \\ \chi\sigma_3 & = & \gamma_0\frac{(\gamma_0\psi-\psi\gamma_0)}{2} = \frac{\psi-\tilde{\psi}^\dagger}{2} \overset{(67)}{=} \gamma_0(\gamma_0 \cdot_{\mathcal{D}} \psi) = P_{\gamma_0}(\psi) \end{array}\right\} , \tag{593}$$

where ϕ and χ are usually interpreted as *particle and antiparticle solutions.*

[58] Thus the energy E contains also the contribution of the electromagnetic field.

[59] $\psi\psi^\dagger$ could be a vector here, since ψ is not out of the even subalgebra, but contains elements of all grades in $\mathcal{P}$.

[60] Note that the root gives two solutions, which differ in sign, but since the Lorentz transformation acts double sided

$$LA\tilde{L} = (-L)A(-\tilde{L}) \quad ,$$

both represent the same physical transformation.

Spin up and down Parts Since ϕ and χ are both Pauli spinors, we can split them into spin up and down parts by using (559). But this relation can now be expressed in the Dirac algebra $\mathcal{D}$. Since ϕ and χ are both even in $\mathcal{P}$ and $\mathcal{D}$ and since both commute with γ_0 we have

$$\left.\begin{array}{rclcl} \sigma_3 \cdot_{\mathcal{P}} \phi & = & \gamma_3 \cdot_{\mathcal{D}} \phi\gamma_0 & = & \gamma_3\phi_\downarrow\gamma_0 \\ \sigma_3 \wedge_{\mathcal{P}} \phi & = & \gamma_3 \wedge_{\mathcal{D}} \phi\gamma_0 & = & \gamma_3\phi_\uparrow\gamma_0 \end{array}\right\} . \tag{594}$$

Multiplying with γ_0 from the right and γ^3 from the left yields

$$\left.\begin{array}{rclcl} \phi_\downarrow & = & \gamma^3\gamma_3 \cdot \phi & = & P_{\gamma_3}(\phi) \\ \phi_\uparrow & = & \gamma^3\gamma_3 \wedge \phi & = & P^\perp_{\gamma_3}(\phi) \end{array}\right\} . \tag{595}$$

For $\chi\sigma_3$ we find instead

$$\begin{aligned} P_{\gamma_3}(\chi\sigma_3) &= \gamma^3\gamma_3 \cdot \overbrace{(\chi\sigma_3)}^{\text{even in } \mathcal{D}} = \gamma^3\left[\gamma_3\chi\gamma_3\gamma_0 - \chi\gamma_3\gamma_0\gamma_3\right] \\ &= \gamma^3\left[\gamma_3\chi + \chi\gamma_3\right]\gamma_3\gamma_0 = P^\perp_{\gamma_3}(\chi)\sigma_3 \\ &= \chi_\uparrow\sigma_3 \end{aligned} \tag{596}$$

and accordingly

$$P^\perp_{\gamma_3}(\chi\sigma_3) = \chi\sigma_3 - P_{\gamma_3}(\chi\sigma_3) = P_{\gamma_3}(\chi)\sigma_3 = \chi_\downarrow\sigma_3 \quad . \tag{597}$$

Thus we derive for the Dirac spinor $\psi = \phi + \chi\sigma_3$ the relations

$$\left.\begin{array}{rcl} P_{\gamma_3}(\psi) & = & \phi_\downarrow + \chi_\uparrow\sigma_3 \\ P^\perp_{\gamma_3}(\psi) & = & \phi_\uparrow + \chi_\downarrow\sigma_3 \end{array}\right\} . \tag{598}$$

Combining the projections (593) and (598) leads to

$$\left.\begin{array}{rclcrcl} \phi_\uparrow & = & P^\perp_{\gamma_3}(P^\perp_{\gamma_0}(\psi)) & ; & \phi_\downarrow & = & P_{\gamma_3}(P^\perp_{\gamma_0}(\psi)) \\ \chi_\uparrow\sigma_3 & = & P_{\gamma_3}(P_{\gamma_0}(\psi)) & ; & \chi_\uparrow\sigma_3 & = & P^\perp_{\gamma_3}(P_{\gamma_0}(\psi)) \end{array}\right\} . \tag{599}$$

This allows us to project out any one of the four Pauli spinors contained in the Dirac spinor. Projecting with respect to the time direction distinguishes particle and antiparticle parts, while the projection with respect to the rest-frame spin direction γ_3 distinguishes spin up and down parts.

The Inner Product for Dirac Spinors In order to find an inner product, we follow the same argument as then given for Pauli spinors, which leads to (537), but with the hermitian adjoint replaced by the reversion in $\mathcal{D}$, *i.e.*,

$$\langle\bar{\psi}|\phi\rangle = \langle\tilde{\psi}\phi\rangle - \langle\tilde{\psi}\phi i\sigma_3\rangle i\sigma_3 \quad , \tag{600}$$

where $\bar{\psi}$ is the Dirac adjoint. Note that this product defines a scalar field – to obtain the analogue of the quantum mechanical inner product we would have to integrate (600) over the whole rest-space. Since the operations of reversion in $\mathcal{D}$ and hermitian adjungation are identical in $\mathcal{P}_2$, the *definitions match for Pauli spinors.*

The Dirac Current Given this inner product, we are now able to look at the (local) expectation value of operators. Considering the $\hat{\gamma}_\mu$-operators we find

$$\langle\bar{\psi}|\hat{\gamma}_\mu|\psi\rangle = \langle\tilde{\psi}\gamma_\mu\psi\gamma_0\rangle - \langle\tilde{\psi}\gamma_\mu\psi\underbrace{\gamma_0 i\sigma_3}_{=i\gamma_3}\rangle i\sigma_3 \quad . \tag{601}$$

Scalars do not change sign under reversion, but $\widetilde{\langle\tilde{\psi}\gamma_\mu\psi i\gamma_3\rangle} = \langle\gamma_3 i\tilde{\psi}\gamma_\mu\psi\rangle = -\langle\tilde{\psi}\gamma_\mu\psi i\gamma_3\rangle$. Therefore, we conclude $\langle\tilde{\psi}\gamma_\mu\psi i\gamma_3\rangle = 0$, and obtain after substituting in (601)

$$\langle\bar{\psi}|\hat{\gamma}_\mu|\psi\rangle = \langle\tilde{\psi}\gamma_\mu\psi\gamma_0\rangle = \gamma_\mu \cdot \langle\psi\gamma_0\tilde{\psi}\rangle_1 \quad . \tag{602}$$

This is just the μ-component of a frame free current — the **Dirac Current**

$$J \stackrel{\text{def}}{=} \langle\psi\gamma_0\tilde{\psi}\rangle_1 \quad . \tag{603}$$

This current also appears as a conjugate current of the Dirac Lagrangian, as will be shown in the following section. From there the conservation equation, *i.e.*, $\partial_x \cdot J = 0$, follows.

6.3 Concluding Remarks

We gave a representation for the Pauli and Dirac spinors in the space-time algebra — they become multivector fields in $\mathcal{P}_2$ and $\mathcal{P}$, respectively. Therefore they define a Lorentz-rotation and dilation at each point. By translating the quantum mechanics matrix operators into space-time algebra form we found that they are closely related to the basis vectors of $\mathcal{P}$ and $\mathcal{D}$ — they differ just by a right-sided multiplication with σ_3 (for the Pauli spinor) and γ_0 (for the Dirac spinor).

But these results do not fit the standard interpretation of quantum mechanics. The spinor does not appear as a vector, but rather as an *operator* on a vector (or multivector). But it is this change in perspective that allows us to understand the *transformation rule for spinors*. Since a spinor is itself identified as a Lorentz-rotation, it also transforms like a rotation — *single sided.*

The representation of the Dirac spinor in the space-time algebra does not employ the complex unit. Indeed, we identified the action of the complex unit j as[61] a *right-sided* multiplication with $i\sigma_3 = \sigma_1\sigma_2$. But this is just the unit bivector of the spin-plane. Hence, the complex structure of the Dirac equation expresses nothing else than the spin of the particle.

We used the commutation relations of γ_0 and γ_3 with the Dirac spinor to represent (anti-)particle spin up and down parts in the neat form (599).

We can conclude that Geometric Algebra does not lead to new solutions, but to deeper insight and new interpretations.

[61]This result clearly depends on the chosen representation, *e.g.*, with (567) we assign a special role to σ_3 — it becomes the spin vector in the rest-frame.

7 Field Theory

I will now go a step further and take a closer look at quantum field theory. Here I follow a similar approach as given in [2]. After considering general relations for fields I discuss the scalar, the electromagnetic and the Dirac field.

7.1 Basics

7.1.1 Euler-Lagrange Equations for Fields

For a scalar-valued Lagrangian, depending on fields ψ_n and their *first derivative* $\partial_x\psi_n$, *i.e.*,

$$\mathcal{L} = \mathcal{L}(\psi_n, \partial_x\psi_n) = \langle \mathcal{L}(\psi_n, \partial_x\psi_n) \rangle \tag{604}$$

we define the *action* by the space-time integral

$$\mathcal{S} \overset{\text{def}}{=} \int |d^4x| \mathcal{L} \quad . \tag{605}$$

In the following discussion we just consider the case for one field ψ, but the generaliztion to the dependence of $\mathcal{L}$ on n independent fields is straight forward by replacing ψ by ψ_n below. If we assume that the action is extremal for physical states, we have

$$\delta_\psi \mathcal{S}(\psi) = 0 \quad . \tag{606}$$

Let ψ make $\mathcal{S}$ extremal and let ϕ be an arbitrary differentiable field, containing the same grades as ψ. We define

$$\psi' \overset{\text{def}}{=} \psi + \epsilon\phi \quad , \tag{607}$$

which makes the action extremal for $\epsilon = 0$, and thus

$$\begin{aligned} 0 \quad &= \quad \frac{\partial}{\partial\epsilon} S(\psi')\Big|_{\epsilon=0} \overset{(605)}{=} \partial_\epsilon \int |d^4x| \mathcal{L}(\psi', \partial_x\psi')\Big|_{\epsilon=0} \\ &\overset{(607)}{=} \quad \partial_\epsilon \int |d^4x| \mathcal{L}(\psi + \epsilon\phi, \partial_x\psi + \epsilon\partial_x\phi)\Big|_{\epsilon=0} \\ &= \quad \int |d^4x| \left[\phi * (\partial_\psi \mathcal{L}) + (\partial_x\phi) * (\partial_{\partial_x\psi}\mathcal{L})\right] \quad . \end{aligned} \tag{608}$$

Now we make use of

$$\begin{aligned} (\partial_x\phi) * (\partial_{\partial_x\psi}\mathcal{L}) \quad &= \quad \langle \dot{\partial}_x \dot{\phi} \partial_{\partial_x\psi}\mathcal{L} \rangle = \partial_x \cdot \langle \phi \partial_{\partial_x\psi}\mathcal{L} \rangle_1 - \langle \dot{\partial}_x \phi \partial_{\partial_x\psi} \dot{\mathcal{L}} \rangle \\ &= \quad \partial_x \cdot \langle \phi \partial_{\partial_x\psi}\mathcal{L} \rangle_1 - \phi * \left([\partial_{\partial_x\psi}\mathcal{L}]^\bullet \, \dot{\partial}_x \right) \end{aligned} \tag{609}$$

and derive

$$0 \overset{(608)}{=} \int |d^4x| \left[\phi * (\partial_\psi \mathcal{L}) + \partial_x \cdot \langle \phi \partial_{\partial_x\psi}\mathcal{L} \rangle_1 - \phi * \left(\partial_{\partial_x\psi} \dot{\mathcal{L}} \dot{\partial}_x \right) \right] \quad . \tag{610}$$

Since ϕ can be chosen *arbitrarily*, the assumption that the boundary term vanishes, yields

$$0 = \partial_\psi \mathcal{L} - [\partial_{\partial_x\psi}\mathcal{L}]^\bullet \, \dot{\partial}_x \quad \text{or} \quad 0 = \partial_{\tilde{\psi}} \mathcal{L} - \partial_x(\partial_{\widetilde{\partial_x\psi}} \mathcal{L}) \quad , \tag{611}$$

where we assumed ψ to be a general element of the algebra. But these are just the Euler-Lagrange equations in space-time algebra form.

7.1.2 Noether's Theorem

Let $\psi'_n \overset{\text{def}}{=} s(\psi_n, M) = s$ be a symmetry, parametrized by the *multivector* M, so that

$$\mathcal{L}' \overset{\text{def}}{=} \mathcal{L}(\psi'_n, \partial_x\psi'_n) \tag{612}$$

obeys again the Euler-Lagrange equations, *i.e.*,

$$0 = \partial_{\psi'_n} \mathcal{L}' - (\partial_{\partial_x\psi'_n} \dot{\mathcal{L}}') \dot{\partial}_x \quad . \tag{613}$$

With the multivector differential

$$\left.\begin{array}{rcl} s_A \stackrel{\text{def}}{=} A * \partial_M s(\psi_n, M) & = & A * \partial_M \psi_n'(M) \\ \partial_A \ldots s_A \ldots & = & \partial_M \ldots \psi_n'(M) \ldots \end{array}\right\} \tag{614}$$

we find for the derivative of the Lagrangian with respect to the symmetry parameter M

$$\begin{aligned} A * \partial_M \mathcal{L}' &= \sum_n \left[\underbrace{(A * \partial_M \psi_n')}_{\stackrel{(614)}{=} s_A(\psi_n, M)} * \partial_{\psi_n'} \mathcal{L}' + \left[A * \dot{\partial}_M (\partial_x \psi_n')^{\bullet} \right] * \partial_{\partial_x \psi_n'} \mathcal{L}' \right] \\ &= \sum_n \left[s_A * \partial_{\psi_n'} \mathcal{L}' + \langle \left[\dot{\partial}_x \dot{s}_A \right] \partial_{\partial_x \psi_n'} \mathcal{L}' \rangle \right] \\ &= \sum_n \left[s_A * \partial_{\psi_n'} \mathcal{L}' + \partial_x * \left[s_A \partial_{\partial_x \psi_n'} \mathcal{L}' \right] - \langle s_A \left[\dot{\partial}_{\partial_x \psi_n'} \ddot{\mathcal{L}}' \ddot{\partial}_x \right] \rangle \right] \\ &= \sum_n s_A * \underbrace{\left[\partial_{\psi_n'} \mathcal{L}' - (\partial_{\partial_x \psi_n'} \dot{\mathcal{L}}') \dot{\partial}_x \right]}_{\stackrel{(613)}{=} 0} + \sum_n \partial_x * \left[s_A \partial_{\partial_x \psi_n'} \mathcal{L}' \right] \end{aligned} \tag{615}$$

and after multiplying with ∂_A the relation

$$\partial_M \mathcal{L}' = \partial_A \partial_x \cdot \langle s_A(\psi_n, M) \partial_{\partial_x \psi_n'} \mathcal{L}' \rangle_1 \stackrel{(614)}{=} \dot{\partial}_M \partial_x \cdot \langle \dot{\psi}_n' \partial_{\partial_x \psi_n'} \mathcal{L}' \rangle_1 \quad , \tag{616}$$

which is Noether's theorem in a generalized version for multivectors. One should note the ease with which we derive this important result.

Symmetries with a scalar Parameter If the parameter $M \stackrel{\text{def}}{=} \alpha$ is a scalar, (616) yields

$$\partial_\alpha \mathcal{L}' = \partial_x \cdot \langle \dot{\partial}_\alpha \dot{\psi}_n' \partial_{\partial_x \psi_n'} \mathcal{L}' \rangle_1 \quad . \tag{617}$$

Therefore, we define the current associated to α by

$$j \stackrel{\text{def}}{=} \langle \partial_\alpha \psi' \partial_{\partial_x \psi_n'} \mathcal{L}' \rangle_1 \quad , \tag{618}$$

and derive, after substituting back into (617)

$$\partial_x \cdot j \stackrel{(617)}{=} \partial_\alpha \mathcal{L}' \quad . \tag{619}$$

Thus the current obeys a *conservation equation* if $\mathcal{L}'$ is independent of α. We can define an associated charge by

$$Q \stackrel{\text{def}}{=} \int |d^3x| \, j \cdot \gamma_0 \quad . \tag{620}$$

If $\mathcal{L}'$ is not independent of α it is often useful to consider (617) for $\alpha = 0$, *i.e.*,

$$\partial_\alpha \mathcal{L}' \Big|_{\alpha=0} \stackrel{(617)}{=} \partial_x \cdot \langle \dot{\partial}_\alpha \dot{\psi}' \partial_{\partial_x \psi'} \mathcal{L}' \rangle_1 \Big|_{\alpha=0} \quad . \tag{621}$$

7.1.3 The Energy-Momentum Tensor

As the first important example we will derive the general expression for the energy-momentum tensor of an arbitrary Poincare-invariant Lagrangian. We start by considering the transformation

$$x' \stackrel{\text{def}}{=} x + \alpha n \quad , \quad \psi' \stackrel{\text{def}}{=} \psi(x') \tag{622}$$

and a *Poincare-invariant Lagrangian* $\mathcal{L}(\psi, \partial_x \psi)$, depending only on the first derivative of the field. We denote the transformed Lagrangian by

$$\mathcal{L}' \stackrel{\text{def}}{=} \mathcal{L}(\psi_i', \partial_x \psi_i') \quad . \tag{623}$$

With the help of the chain rule we are able to express ∂_α by

$$\partial_\alpha = (\partial_\alpha x') * \partial_{x'} \stackrel{(622)}{=} n \cdot \partial_{x'} \quad . \tag{624}$$

substituting this in (621) yields

$$\begin{aligned} \partial_\alpha \mathcal{L}'\Big|_{\alpha=0} &\stackrel{(617)}{=} \partial_{x'} \cdot \langle \partial_\alpha \psi_i(x') \partial_{\partial_{x'}\psi_i'} \mathcal{L}' \rangle_1 \Big|_{\alpha=0} \\ &\stackrel{(624)}{=} \partial_x \cdot \langle (n \cdot \dot{\partial}_x) \dot{\psi}_i \partial_{\partial_x \psi_i} \mathcal{L} \rangle_1 \quad . \end{aligned} \tag{625}$$

But looking at $\mathcal{L}' = \mathcal{L}(x')$ as a function of x', rather than ψ', gives

$$\partial_\alpha \mathcal{L}'\Big|_{\alpha=0} \stackrel{(624)}{=} n \cdot \partial_{x'} \mathcal{L}(x')\Big|_{\alpha=0} = n \cdot \partial_x \mathcal{L} = \dot{\partial}_x \cdot \langle \dot{\mathcal{L}} n \rangle_1 \quad . \tag{626}$$

Thus we obtain by subtracting (626) from (625)

$$0 = \partial_\alpha \mathcal{L}'\Big|_{\alpha=0} - n \cdot \partial_x \mathcal{L} = \partial_x \cdot \langle (n \cdot \dot{\partial}_x) \dot{\psi}_i \partial_{\partial_x \psi_i} \mathcal{L} - n\mathcal{L} \rangle_1 \tag{627}$$

and we define the *adjoint* of the energy-momentum tensor

$$\bar{T}(n) \stackrel{\text{def}}{=} \langle n \cdot \dot{\partial}_x \dot{\psi}_i \partial_{\partial_x \psi_i} \mathcal{L} - n\mathcal{L} \rangle_1 \quad . \tag{628}$$

This is a conserved current, since (627) becomes now

$$\dot{\partial}_x \cdot \dot{\bar{T}}(n) \stackrel{(627)}{=} 0 \quad . \tag{629}$$

We define the **energy-momentum tensor** T as the adjoint of $\bar{T}$. We use $\langle r\bar{T}(n) \rangle = \langle T(r) n \rangle$ to derive the general expression, which becomes

$$\begin{aligned} T(r) &= \partial_n \langle r \bar{T}(n) \rangle \stackrel{(628)}{=} \underbrace{\partial_n n \cdot \dot{\partial}_x}_{=\partial_x} \langle \dot{\psi}_i \partial_{\partial_x \psi_i} \mathcal{L} r \rangle - \underbrace{\partial_n (r \cdot n)}_{=r} \mathcal{L} \\ &= \dot{\partial}_x \langle \dot{\psi}_i \partial_{\partial_x \psi_i} \mathcal{L} r \rangle - \mathcal{L} r. \end{aligned} \tag{630}$$

The conservation equation for the adjoint of the energy-momentum tensor (629) becomes now the important relation

$$\dot{T}(\dot{\partial}_x) = 0 \quad . \tag{631}$$

Here it should be noted that the energy-momentum tensor is with (630) a vector-valued linear function of a vector argument, which is *not necessarily symmetric.*

Momentum and Energy Having defined the energy-momentum tensor, we can now proceed by defining the *four-momentum and energy density* as

$$\mathcal{P}_x \stackrel{\text{def}}{=} T(\gamma_0) \quad ; \quad \mathcal{E}_x \stackrel{\text{def}}{=} \gamma_0 \cdot T(\gamma_0) = \gamma_0 \cdot \mathcal{P}_x \tag{632}$$

The *four-momentum* of the field in the γ_0-frame is given by the spatial integral over the momentum density (632), *i.e.*,

$$p = \int |d^3x| T(\gamma_0) \quad , \tag{633}$$

and thus the *energy* as the γ_0-component

$$E = \gamma_0 \cdot p \stackrel{(633)}{=} \int |d^3x| \, \gamma_0 \cdot T(\gamma_0) \quad . \tag{634}$$

7.1.4 The Symmetry of the Energy-Momentum Tensor

The energy-momentum tensor, defined by (630), can contain an antisymmetric part. Indeed, as will be seen later, the antisymmetric part of the energy-momentum tensor for the Dirac field contributes directly to the angular-momentum and can thus be seen as a spin contribution..

General Condition for Symmetry If $\mathsf{T}(n)$ is a symmetric linear function, *i.e.*, $\mathsf{T}(n) = \bar{\mathsf{T}}(n)$, (628) and (630) yield the condition

$$n * \dot\partial_x \langle \dot\psi_i \partial_{\partial_x \psi_i} \mathcal{L} \rangle_1 = \dot\partial_x \langle \dot\psi_i \partial_{\partial_x \psi_i} \mathcal{L} \rangle_1 * n \quad , \tag{635}$$

and after multiplying with ∂_n from the left

$$\dot\partial_x \langle \dot\psi_i \partial_{\partial_x \psi_i} \mathcal{L} \rangle_1 = \langle \dot\psi_i \partial_{\partial_x \psi_i} \mathcal{L} \rangle_1 \dot\partial_x \quad . \tag{636}$$

Using (12) we bring the condition in the neat form

$$\dot\partial_x \wedge \langle \dot\psi_i (\partial_{\partial_x \psi_i} \mathcal{L}) \rangle_1 = 0 \quad . \tag{637}$$

Taking the inner product with n leads back to (635), which is in turn equivalent to the symmetry of $\mathsf{T}(n)$. Therefore, $\mathsf{T}(n)$ is symmetric if, *and only if*, (637) is satisfied.

7.1.5 A possible Relation to the Einstein-Field Equation

The following considerations reflect some ideas that I had. They might turn out to be *physically* wrong, but nevertheless it is interesting to see how the definition of the quantum-field energy-momentum tensor can be used to construct a general-relativity-like structure.

Split of T We define the vector u by

$$u \stackrel{\text{def}}{=} \langle \psi_i (\partial_{\partial_x \psi_i} \mathcal{L}) \rangle_1 \tag{638}$$

and rewrite (628) and (630) as

$$\left.\begin{array}{llll} \mathsf{T}(r) & \stackrel{(630)}{=} & \partial_x u \cdot r - \mathcal{L} r - \dot\partial_x \langle \psi_i (\partial_{\partial_x \psi_i} \mathcal{L})^{\bullet} r \rangle & \stackrel{\text{def}}{=} \mathsf{T}_g(r) - \mathsf{T}_q(r) \\ \bar{\mathsf{T}}(r) & \stackrel{(628)}{=} & r \cdot \partial_x u - \mathcal{L} r - r \cdot \dot\partial_x \langle \psi_i (\partial_{\partial_x \psi_i} \mathcal{L})^{\bullet} \rangle_1 & = \bar{\mathsf{T}}_g(r) - \bar{\mathsf{T}}_q(r) \end{array}\right\} , \tag{639}$$

where we defined

$$\left.\begin{array}{lllll} \mathsf{T}_g(r) & \stackrel{\text{def}}{=} \partial_x u \cdot r - \mathcal{L} r & \Rightarrow \bar{\mathsf{T}}_g(r) & = & r \cdot \partial_x u - \mathcal{L} r \\ \mathsf{T}_q(r) & \stackrel{\text{def}}{=} \dot\partial_x \langle \psi_i (\partial_{\partial_x \psi_i} \mathcal{L})^{\bullet} r \rangle & \Rightarrow \bar{\mathsf{T}}_q(r) & = & r \cdot \dot\partial_x \langle \psi_i (\partial_{\partial_x \psi_i} \mathcal{L})^{\bullet} \rangle_1 \end{array}\right\} . \tag{640}$$

The same argument that led to (637) yields the following equivalent statements

$$\mathsf{T}_g = \bar{\mathsf{T}}_g \Leftrightarrow \partial_x \wedge u = 0 \Leftrightarrow \partial_x u = \partial_x \cdot u \Leftrightarrow (n \cdot \partial_x) u = \partial_x (u \cdot n) \quad . \tag{641}$$

If T_g *is symmetric*, any skew contribution can only come from T_q. Since symmetric functions have vanishing protraction, it follows[62]

$$\mathsf{T}_q(\dot\partial_x) \wedge \dot x = -\bar{\mathsf{T}}_q(\dot\partial_x) \wedge \dot x = \partial_x^* \cdot \dot\partial_x \langle \psi_i (\partial_{\partial_x \psi_i} \mathcal{L})^{\bullet} \rangle_1 \wedge x^* = -\dot\partial_x \wedge \langle \psi_i (\partial_{\partial_x \psi_i} \mathcal{L})^{\bullet} \rangle_1 \quad . \tag{642}$$

As we will see later, this quantity will contribute to the angular-momentum tensor.

Examples We will check the symmetry of T_g for the two most important Lagrangians:

- For the **Klein-Gordon field** with $\mathcal{L} = \frac{1}{2}\left\{(\partial_x \phi)^2 - m^2 \phi^2\right\}$ it follows

$$u = \phi (\partial_x \phi) = \frac{1}{2} \partial_x \phi^2 \Rightarrow \partial_x \wedge u = \frac{1}{2} \partial_x \wedge \partial_x \phi^2 = 0 \quad , \tag{643}$$

  and thus, according to (641), T_g *is symmetric*. The protraction of T_q becomes with $\partial_{\partial_x \phi} \mathcal{L} = \partial_x \phi$

$$\mathsf{T}_q(\dot\partial_x) \wedge \dot x \stackrel{(642)}{=} -\dot\partial_x \wedge \langle \phi (\partial_x \phi)^{\bullet} \rangle_1 = -\phi (\partial_x \wedge \partial_x) \phi = 0 \quad . \tag{644}$$

  Therefore also T_q is symmetric.

[62] Here we also use a $*$ to indicate the scope of the differental operator.

- For the **Dirac field** with $\mathcal{L} = (\dot{\partial}_x\dot{\psi})i\gamma_3\widetilde{\psi} - m\psi\widetilde{\psi} - eA\psi\gamma_0\widetilde{\psi}$ we get

$$u \overset{(638)}{=} \langle \psi i\gamma_3\widetilde{\psi}\rangle_1 = 0 \Rightarrow \partial_x \wedge u = 0 \quad , \tag{645}$$

and thus T_g *is symmetric.* Here T_q has an antisymmetric part, since with $\partial_{\partial_x\psi}\mathcal{L} = i\gamma_3\widetilde{\psi}$ the protraction

$$T_q(\dot{\partial}_x) \wedge \dot{x} = -\dot{\partial}_x \wedge \langle \psi i\gamma_3\dot{\widetilde{\psi}}\rangle_1 = \dot{\partial}_x \wedge \langle \dot{\psi} i\gamma_3\widetilde{\psi}\rangle_1 \tag{646}$$

is non-zero.

Ricci Tensor and Scalar Comparing the definition of T_g in (640) with the Einstein-Field equation

$$T(a) = R(a) - \frac{1}{2}aR \tag{647}$$

suggests to identify the Ricci scalar by

$$R \overset{\text{def}}{=} 2\mathcal{L}(\psi_i, \partial_x\psi_i) \tag{648}$$

and the Ricci tensor by

$$R(a) = \partial_x\langle \psi_i \dot{\partial}_{\partial_x\psi_i}\dot{\mathcal{L}}a\rangle \overset{(638)}{=} \partial_x u \cdot a \quad . \tag{649}$$

We have to note here the important fact that the occuring vectors in (648) and (649) are vectors defined in a *flat* background space. Ricci tensor and scalar are still position dependent with x, but x is a vector in a linear vector space without intrinsic curvature. This is a fundamental difference to general relativity, where the tensors are functions on a curved manifold. It should be mentioned that in gravity as a gauge theory (see Section 9), a flat background space is also used.

If T_g is symmetric, it follows from (641)

$$R(a) = a \cdot \partial_x u \overset{(638)}{=} a \cdot \partial_x \langle \psi_i \dot{\partial}_{\partial_x\psi_i}\dot{\mathcal{L}}\rangle_1 \quad . \tag{650}$$

Now we have to check, if the Ricci scalar is really the contraction of the so defined Ricci tensor. From (648) and (649) follows the condition

$$2\mathcal{L} \overset{(648)}{=} R = \partial_a \cdot R(a) = \partial_a \cdot (\partial_x u \cdot a) = \partial_x \cdot u \quad . \tag{651}$$

In full length this condition becomes

$$\begin{aligned} 2\mathcal{L} &\overset{(649)}{=} \langle \partial_x (\psi_i \dot{\partial}_{\partial_x\psi_i}\dot{\mathcal{L}})\rangle \\ &= \langle \dot{\partial}_x \dot{\psi}_i (\partial_{\partial_x\psi_i}\mathcal{L})\rangle + \langle \psi_i \{\partial_{\partial_x\psi_i}\mathcal{L}\}^{\bullet}\, \dot{\partial}_x\rangle \\ &= \langle \dot{\partial}_x \dot{\psi}_i \partial_{\partial_x\psi_i}\mathcal{L}\rangle + \langle \psi_i \dot{\partial}_{\psi_i}\dot{\mathcal{L}}\rangle \quad . \end{aligned} \tag{652}$$

Therefore, in order that the Ricci-scalar (648) is the contraction of the Ricci-tensor (649), the Lagrangian has to satisfy the condition (652). It is interesting enough, that the Klein-Gordon and the Dirac Lagrangian satisfy this condition!

Example Scalar Field We will first look at the Klein-Gordon-Lagrangian

$$\mathcal{L} \overset{\text{def}}{=} \underbrace{\frac{1}{2}(\partial_x\phi)^2}_{\overset{\text{def}}{=}\mathcal{L}_p} - \underbrace{\frac{1}{2}m^2\phi^2}_{\overset{\text{def}}{=}\mathcal{L}_m} = \mathcal{L}_p - \mathcal{L}_m \quad . \tag{653}$$

(652) is for each term itself satisfied, since

$$\left.\begin{aligned} \langle \dot{\partial}_x \dot{\phi} \partial_{\partial_x\phi}\mathcal{L}_p\rangle + \underbrace{\langle \phi \dot{\partial}_\phi \dot{\mathcal{L}}_p\rangle}_{=0} &\overset{(653)}{=} (\partial_x\phi)^2 = 2\mathcal{L}_p \\ \underbrace{\langle \dot{\partial}_x \dot{\phi} \partial_{\partial_x\phi}\mathcal{L}_m\rangle}_{=0} + \langle \phi \dot{\partial}_\phi \dot{\mathcal{L}}_m\rangle &\overset{(653)}{=} m^2\phi^2 = 2\mathcal{L}_m \end{aligned}\right\} . \tag{654}$$

Example Dirac Field For the terms of the full Dirac Lagrangian, *i.e.*, including electromagnetic interaction,

$$\mathcal{L} = \underbrace{\langle(\dot{\partial}_x\dot{\psi})i\gamma_3\tilde{\psi}\rangle}_{\stackrel{\text{def}}{=}\mathcal{L}_s} - \underbrace{\langle m\psi\tilde{\psi}\rangle}_{\stackrel{\text{def}}{=}\mathcal{L}_m} - \underbrace{\langle eA\psi\gamma_0\tilde{\psi}\rangle}_{\stackrel{\text{def}}{=}\mathcal{L}_e} = \mathcal{L}_s - \mathcal{L}_m - \mathcal{L}_e \quad , \tag{655}$$

we get for each term

$$\left.\begin{array}{lcl} \langle\dot{\partial}_x\dot{\psi}\partial_{\partial_x\psi}\mathcal{L}_s\rangle + \langle\psi(\partial_\psi\mathcal{L}_s)\rangle & \stackrel{(655)}{=} & \langle(\partial_x\psi)i\gamma_3\tilde{\psi}\rangle - \langle\psi i\gamma_3\dot{\tilde{\psi}}\dot{\partial}_x\rangle \\ & = & 2\langle\partial_x\psi i\gamma_3\tilde{\psi}\rangle = 2\mathcal{L}_s \\ \underbrace{\langle\dot{\partial}_x\dot{\psi}\partial_{\partial_x\psi}\mathcal{L}_e\rangle}_{=0} + \langle\psi(\partial_\psi\mathcal{L}_e)\rangle & \stackrel{(655)}{=} & \langle\psi\left(\gamma_0\tilde{\psi}eA + \widetilde{eA\psi\gamma_0}\right)\rangle = 2\mathcal{L}_e \\ \underbrace{\langle\dot{\partial}_x\dot{\psi}\partial_{\partial_x\psi}\mathcal{L}_m\rangle}_{=0} + \langle\psi(\partial_\psi\mathcal{L}_m)\rangle & \stackrel{(655)}{=} & 2m\psi\tilde{\psi} = 2\mathcal{L}_m \end{array}\right\} . \tag{656}$$

Accordingly the Dirac Lagrangian satisfies the condition (652).

The Equation of State Contracting $T_g(a)$ we derive the Equation of State, which becomes[63]

$$\partial_a \cdot T_g(a) \stackrel{(647)}{=} R - \underbrace{\frac{\partial_a \cdot a}{2}}_{=2} R = -R \stackrel{(648)}{=} -2\mathcal{L}(\psi_i, \partial_x\psi_i) \quad . \tag{657}$$

The contracted Bianchi Identity Ricci tensor and scalar obey in General-Relativity the Bianchi identity $0 = \dot{R}(\dot{\delta}_x) - \frac{1}{2}\dot{\delta}_x\dot{R}$. But substituting (648) and (649) we derive

$$\dot{R}(\dot{\partial}_x) - \frac{1}{2}\dot{\partial}_x\dot{R} = \partial_x(\partial_x \cdot u) - \frac{1}{2}\partial_x 2\mathcal{L} \stackrel{(651)}{=} \partial_x\mathcal{L} \stackrel{(648)}{=} \frac{1}{2}\partial_x R \quad . \tag{658}$$

Therefore, the analogue of the contracted Bianchi identity is only satisfied if the Lagrangian (or Ricci scalar) is a constant in space-time. Clearly a very strong condition, which will in general not be satisfied. However, if one considers a solution of the quantum field for a localized matter distribution, one would have to check if $\partial_x\mathcal{L} \to 0$ for increasing distance from the matter. If this is true, the Bianchi identity would appear as a natural limit of (658).

Note that the Dirac Lagrangian fulfills (658), since it vanishes under the equations of motion.

7.1.6 The Angular-Momentum Tensor

The Angular Momentum of Fields, which transform double sided The transformation[64]

$$\left.\begin{array}{lcl} x' & \stackrel{\text{def}}{=} & e^{-\frac{\alpha B}{2}} x e^{\frac{\alpha B}{2}} \\ \psi' & \stackrel{\text{def}}{=} & e^{\frac{\alpha B}{2}} \psi(x') e^{-\frac{\alpha B}{2}} \end{array}\right\} \quad , \tag{659}$$

where B is an arbitrary bivector, yields

$$\partial_\alpha x'\Big|_{\alpha=0} \stackrel{(659)}{=} \frac{x'B - Bx'}{2}\Big|_{\alpha=0} = -B \times x = -B \cdot x \tag{660}$$

and so

$$\partial_\alpha\psi'\Big|_{\alpha=0} \stackrel{(659)}{=} B \times \psi'\Big|_{\alpha=0} + (\partial_\alpha x')\Big|_{\alpha=0} * \partial_x\psi(x) \stackrel{(660)}{=} B \times \psi - \underbrace{(B \cdot x) * \partial_x}_{B\cdot(x\wedge\partial_x)}\psi(x) \quad . \tag{661}$$

From Noether's theorem we derive

$$\partial_\alpha\mathcal{L}'\Big|_{\alpha=0} \stackrel{(621)}{=} \partial_x \cdot \langle\partial_\alpha\psi'\partial_{\partial_x\psi'}\mathcal{L}'\rangle_1\Big|_{\alpha=0} \stackrel{(661)}{=} \partial_x \cdot \langle\{B \times \psi - [B \cdot (x \wedge \partial_x)]\psi\}\partial_{\partial_x\psi}\mathcal{L}\rangle_1 \tag{662}$$

[63] Here we consider a four dimensional space. Thus $\partial_a \cdot a = 4$.

[64] Note that here ψ transforms double sided and is thus *no* spinor.

But $\mathcal{L}$ can also be seen as a function of x, *i.e.*,

$$\mathcal{L}(\psi(x), \partial_x\psi(x)) = \mathcal{L}(x) \quad , \tag{663}$$

and the derivative with respect to the scalar symmetry parameter α becomes

$$\partial_\alpha \mathcal{L}'\Big|_{\alpha=0} = \partial_\alpha x'\Big|_{\alpha=0} * \partial_x \mathcal{L} \overset{(660)}{=} (-B\cdot x)\cdot\partial_x\mathcal{L}(x) = \dot{\partial}_x \cdot \left\{x\cdot B\dot{\mathcal{L}}(x)\right\} = \partial_x\cdot\{x\cdot B\mathcal{L}(x)\} \tag{664}$$

Equating (662) and (664) yields

$$0 = \partial_x \cdot \left\{\langle\left[B\times\psi - \left[B\cdot(x\wedge\dot{\partial}_x)\right]\dot{\psi}\right]\partial_{\partial_x\psi}\mathcal{L}\rangle_1 - x\cdot B\mathcal{L}\right\} \tag{665}$$

and thus we define the conserved current as

$$\bar{J}(B) \overset{\text{def}}{=} \langle\left\{B\times\psi - \left[B\cdot(x\wedge\dot{\partial}_x)\right]\dot{\psi}\right\}\partial_{\partial_x\psi}\mathcal{L}\rangle_1 - x\cdot B\,\mathcal{L} \quad . \tag{666}$$

As for the energy-momentum tensor follows the conservation equation

$$0 = \partial_x\cdot\bar{J}(B) \quad \forall B \quad \Leftrightarrow \quad 0 = \underline{J}(\partial_x)\cdot B \quad \forall B \quad \Leftrightarrow \quad \underline{J}(\partial_x) = 0 \quad . \tag{667}$$

To derive an expression for $J \overset{\text{def}}{=} \underline{J}$ we use

$$\left.\begin{array}{rcl} \partial_B\langle(B\times\psi)\,\partial_{\partial_x\psi}\mathcal{L}r\rangle & \overset{(215)}{=} & \langle\psi\times(\partial_{\partial_x\psi}\mathcal{L}r)\rangle_2 \\ \partial_B\langle[B\cdot(x\wedge\partial_x)]\,\psi\partial_{\partial_x\psi}\mathcal{L}r\rangle & = & \partial_B\langle B\cdot(x\wedge\partial_x)\rangle\langle\psi\partial_{\partial_x\psi}\mathcal{L}r\rangle = (x\wedge\partial_x)\langle\psi\partial_{\partial_x\psi}\mathcal{L}r\rangle \\ \partial_B(x\cdot B)\cdot r\mathcal{L} & = & -\partial_B B\cdot(x\wedge r)\mathcal{L} = r\wedge x\mathcal{L} = -x\wedge(r\mathcal{L}) \end{array}\right\} \tag{668}$$

to obtain

$$\begin{aligned} \underline{J} &= \partial_B\langle\bar{J}(B)r\rangle \overset{(666)}{=} \partial_B\{\langle[B\times\psi - [B\cdot(x\wedge\partial_x)]\,\psi]\,\partial_{\partial_x\psi}\mathcal{L}r\rangle - (x\cdot B)\cdot r\mathcal{L}\} \\ &\overset{(668)}{=} \langle\psi\times(\partial_{\partial_x\psi}\mathcal{L}r)\rangle_2 \underbrace{-(x\wedge\partial_x)\langle\psi\partial_{\partial_x\psi}\mathcal{L}r\rangle + x\wedge(r\mathcal{L})}_{\overset{(630)}{=} -x\wedge T(r)} \\ &\overset{(630)}{=} \langle\psi\times(\partial_{\partial_x\psi}\mathcal{L}r)\rangle_2 + T(r)\wedge x \quad . \end{aligned} \tag{669}$$

The second term is the expected momentum contribution but the first term seems to be related to the spin. Thus we define

$$S(r) \overset{\text{def}}{=} \underline{J}(r) - T(r)\wedge x = \langle\psi_i\times(\partial_{\partial_x\psi_i}\mathcal{L}r)\rangle_2 \quad . \tag{670}$$

Angular-Momentum of Spinor Fields If $\psi \overset{\text{def}}{=} \phi$ is a spinor, *i.e.*, it transforms single sided, the transfromation (659) has to be replaced by

$$x' \overset{\text{def}}{=} e^{-\frac{\alpha B}{2}} x e^{\frac{\alpha B}{2}} \quad ; \quad \phi' \overset{\text{def}}{=} e^{\frac{\alpha B}{2}}\phi(x') \quad . \tag{671}$$

A very similar calculation as given above gives now the **angular momentum for the spinor field**

$$\underline{J} = \tfrac{1}{2}\langle\psi_i\partial_{\partial_x\psi_i}\mathcal{L}r\rangle_2 + T(r)\wedge x \quad ; \quad S(r) \overset{(670)}{=} \tfrac{1}{2}\langle\psi_i\partial_{\partial_x\psi_i}\mathcal{L}r\rangle_2 \quad . \tag{672}$$

The skew Part of T as total Divergence (667) yields under use of (631)

$$0 \overset{(667)}{=} \dot{J}(\dot{\partial}_x) = \dot{S}(\dot{\partial}_x) + T(\dot{\partial}_x)\wedge\dot{x} \quad . \tag{673}$$

The protraction is completely determined by the skew part, so that

$$a\cdot\left[T(\dot{\partial}_x)\wedge\dot{x}\right] = a\cdot T_-(\dot{\partial}_x)\dot{x} - T_-(\dot{\partial}_x)a\cdot\dot{x} = -2T_-(a) \quad . \tag{674}$$

Thus (673) gives

$$0 = a\cdot\dot{S}(\dot{\partial}_x) + a\cdot\left[T(\dot{\partial}_x)\wedge\dot{x}\right] \tag{675}$$

and we see that the skew part of T is always a total divergence.

7.2 The Scalar Field

The simplest case is given by a scalar[65] field obeying the **Klein-Gordon equation**

$$(\partial_x^2 + m^2)\phi = 0 \quad . \tag{676}$$

The corresponding Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left[(\partial_x\phi)^2 - m^2\phi^2\right] \quad . \tag{677}$$

Differentiating with respect to the scalar field and its derivative gives

$$\partial_\phi \mathcal{L} \;=\; -m^2\phi \quad ; \quad \partial_{\partial_x\phi}\mathcal{L} \;=\; \partial_x\phi \quad , \tag{678}$$

so that in this particular case the Euler-Lagrange equation (611) becomes with

$$0 \stackrel{(611)}{=} \partial_\phi\mathcal{L} - (\partial_{\partial_x\phi}\dot{\mathcal{L}})\dot{\partial}_x \stackrel{(678)}{=} -m^2\phi - (\partial_x\dot{\phi})\dot{\partial}_x = -m^2\phi - \partial_x^2\phi \tag{679}$$

the Klein-Gordon equation (676).

7.2.1 The Momentum Representation

Complex Numbers in the Klein-Gordon Theory Before we can apply a Fourier transformation, we have to find a representation of the complex unit j. In principle, this can be any multivector, which squares to -1 and commutes with the wavefunction. But since a scalar field represents particles without spin, j should not prefer any direction! The only multivector, which fulfills this conditions is the pseudoscalar of our four-dimensional space-time, *i.e.*,

$$i = \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3 \quad . \tag{680}$$

We identify thus

$$j \leftrightarrow i \quad . \tag{681}$$

Since $i^\dagger = -i$, the operation of complex conjugation can then be represented as the hermitian adjungation, *i.e.*, reversion in $\mathcal{P}$. Thus we define

$$[\alpha + i\beta]^* \stackrel{\text{def}}{=} [\alpha + i\beta]^\dagger \quad . \tag{682}$$

The Fourier Transformation and the δ-Function Since $i = \gamma_0\gamma_1\gamma_2\gamma_3$ commutes with scalars and pseudoscalars, we have no problem in defining the Fourier transformation of a scalar and pseudoscalar valued function f as

$$f(k) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}^n}\int d^n x\, f(x)e^{ik\cdot x} \quad . \tag{683}$$

The orthonormality relation is expressed by

$$\delta^n(k-k') = \frac{1}{(2\pi)^n}\int d^n x\, e^{i(k-k')\cdot x} \quad , \tag{684}$$

and gives us a representation of the Dirac δ-function, which will be frequently used. As usual one has

$$\int d^n k'\, f(k')\,\delta^n(k'-k) = \int d^n k'\, f(k')\,\delta^n(k-k') = f(k) \quad . \tag{685}$$

Substituting $x' \stackrel{\text{def}}{=} -x$ we find

$$\int_{-\infty}^{\infty} dx\, e^{-ik\cdot x} = \int_{\infty}^{-\infty}(-dx')e^{ik\cdot x'} = \int_{-\infty}^{\infty} dx' e^{ik\cdot x'} \tag{686}$$

and therefore we can extend (684) to[66]

$$\int d^n x\, e^{\pm i(k-k')\cdot x} = (2\pi)^n\,\delta^n(k-k') \quad . \tag{687}$$

[65]Please note that this excludes pseudoscalar valued functions and corresponds so to a real field.
[66]One can see that the δ-function is real, since complex conjugation does not change anything.

The Fourier Representation of the Field We can write the field as a *Fourier Integral*

$$\phi(x) = \frac{1}{\sqrt{2\pi}^4}\int |d^4k| e^{ik\cdot x}\phi'(k) \quad . \tag{688}$$

Now $\phi'(k)$ has scalar and pseudoscalar parts[67] and, since i anticommutes with vectors, we have for an arbitrary vector x

$$\phi'(k)x = x\phi'^*(k) \quad . \tag{689}$$

Since $\phi(x)$ is a pure scalar, *i.e.*,

$$\phi(x) = \phi^*(x) \quad . \tag{690}$$

Taking the complex conjugation[68] of (688) we derive

$$\phi^*(x) = \frac{1}{\sqrt{2\pi}^4}\int |d^4k| e^{-ik\cdot x}\,[\phi'(k)]^* = \frac{1}{\sqrt{2\pi}^4}\int |d^4k| e^{ik\cdot x}\,[\phi'(-k)]^* \quad . \tag{691}$$

Equating (688) and (691) according to (690) yields the reality condition

$$\phi'^*(k) = \phi'(-k) \quad . \tag{692}$$

Substituting (688) in the Klein-Gordon equation (676) yields the condition

$$k^2 - m^2 = k_0^2 - \vec{k}^2 - m^2 = 0 \quad . \tag{693}$$

$\phi'(k)$ must thus be of the form[69]

$$\phi'(k) = \sqrt{2\pi}\,\delta(k^2 - m^2)\phi(k) \quad ; \quad \phi^*(k) \overset{(692)}{=} \phi(-k) \quad , \tag{694}$$

where the δ-function can be seen as fixing the value of k_0. In (688) we get

$$\phi(x) \overset{(688)}{=} \frac{1}{\sqrt{2\pi}^3}\int |d^4k|\, \delta(k^2 - m^2) e^{ik\cdot x}\phi(k) \quad . \tag{695}$$

Split into positive and negative Frequency Parts [70] The integration over k_0 in (695) can be carried out by the replacement

$$k_0 = \pm\sqrt{\vec{k}^2 + m^2} \quad . \tag{696}$$

So the integration is over two disjoint three-dimensional hypersurfaces, one in the future light cone and one in the past light cone in the k-space. We define

$$\phi^{\pm}(k) \overset{\text{def}}{=} \theta(k_0)\phi(\pm k) = \begin{cases} 0 & \text{for} \quad k_0 < 0 \\ \phi(\pm k) & \text{for} \quad k_0 \geq 0 \end{cases} \tag{697}$$

$$\phi^{\pm}(x) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi}^3}\int d^4k\, e^{\pm ik\cdot x}\delta(k^2 - m^2)\phi^{\pm}(k) \quad . \tag{698}$$

Adding the positive and negative frequency parts gives

$$\begin{aligned}
\phi^+(x) + \phi^-(x) &\overset{(698)}{=} \frac{1}{\sqrt{2\pi}^3}\int d^4k\, \delta(k^2 - m^2)\left[e^{ik\cdot x}\phi^+(k) + e^{-ik\cdot x}\phi^-(k)\right] \\
&\overset{(697)}{=} \frac{1}{\sqrt{2\pi}^3}\int d^4k\, \delta(k^2 - m^2)\left[e^{ik\cdot x}\theta(k_0)\phi(k) + e^{-ik\cdot x}\theta(k_0)\phi(-k)\right] \\
&= \frac{1}{\sqrt{2\pi}^3}\int d^4k\, \delta(k^2 - m^2)\left[e^{ik\cdot x}\theta(k_0)\phi(k) + e^{+ik\cdot x}\theta(-k_0)\phi(k)\right] \\
&= \frac{1}{\sqrt{2\pi}^3}\int d^4k\, \delta(k^2 - m^2)e^{ik\cdot x}\phi(k) \\
&\overset{(695)}{=} \phi(x) \quad ,
\end{aligned} \tag{699}$$

[67] According to (681) it is thus a "complex number".
[68] In the sense of (682).
[69] The factor of $\sqrt{2\pi}$ appears only for compatibility reasons.
[70] A similar treatment can be found in [6].

where we substituted

$$k'_n \stackrel{\text{def}}{=} -k_n \quad \Rightarrow \quad dk_n = -dk'_n \quad \Rightarrow d^4k = d^4k' \tag{700}$$

and then renamed $k' \to k$. Now (697) yields

$$[\phi^{\pm}(k)]^* \stackrel{(697)}{=} \theta(k_0)\,[\phi(\pm k)]^* \stackrel{(694)}{=} \theta(k_0)\phi(\mp k) \stackrel{(697)}{=} \phi^{\mp}(k) \quad . \tag{701}$$

Since the value of k_0 is fixed by (696) we define[71]

$$\phi^{\pm}_{\vec{k}} \stackrel{\text{def}}{=} \phi^{\pm}(\vec{k}) \stackrel{\text{def}}{=} \sqrt{2k_0}\phi^{\pm}(\left[k_0+\vec{k}\right]\gamma_0)\Big|_{k_0=\sqrt{\vec{k}^2+m^2}} = \sqrt{2k_0}\phi^{\pm}(k)\Big|_{k_0=\sqrt{\vec{k}^2+m^2}} \quad , \tag{702}$$

for which again

$$\left[\phi^{\pm}(\vec{k})\right]^* \stackrel{(701)}{=} \phi^{\mp}(\vec{k}) \quad . \tag{703}$$

Replacing $\phi^{\pm}(k)$ in (698) and carrying out the integration over k_0 yields

$$\phi^{\pm}(x) \stackrel{(698)}{=} \frac{1}{\sqrt{2\pi}^3}\int \frac{d^3k}{\sqrt{2k_0}^5} e^{\pm ik\cdot x}\phi^{\pm}(\vec{k}) = \frac{1}{\sqrt{2\pi}^3}\int \frac{d^3k}{\sqrt{2k_0}^5} e^{\pm ik_0x_0}e^{\mp i\vec{k}\cdot\vec{x}}\phi^{\pm}(\vec{k}) \tag{704}$$

with[72]

$$k_0 = +\sqrt{\vec{k}^2+m^2} \quad . \tag{705}$$

7.2.2 The Energy-Momentum Tensor of the Klein-Gordon Field

The Lagrangian for the Klein-Gordon field (677) does only depend on the field and its derivative and is therefore Poincare-invariant. Hence the energy-momentum tensor is given by (630) and we derive by substituting the Lagrangian (677)

$$\begin{aligned} T(n) &\stackrel{(630)}{=} \dot{\partial}_x\langle\dot{\phi}\,\underbrace{\partial_{\partial_x\phi}\mathcal{L}}_{\stackrel{(678)}{=}\partial_x\phi}\,n\rangle - n\mathcal{L} \stackrel{(678)}{=} \dot{\partial}_x\dot{\phi}\langle(\partial_x\phi)n\rangle - n\mathcal{L} \\ &\stackrel{(677)}{=} \dot{\partial}_x\dot{\phi}\,[(\partial_x\phi)*n] - \frac{n}{2}\left[(\partial_x\phi)^2 - m^2\phi^2\right] \quad . \end{aligned} \tag{706}$$

Obviously $h\cdot T(n) = n\cdot T(h)$ and hence T is symmetric, *i.e.*,

$$\underline{T} = \bar{T} \quad . \tag{707}$$

The Energy-Momentum Tensor for a plane Wave A plane wave solution of the Klein-Gordon equation is given by

$$\phi = \phi_0\cos(k\cdot x+\alpha) \quad ; \quad k^2 = m^2 \quad . \tag{708}$$

For this case the energy-momentum tensor becomes

$$\begin{aligned} T(n) &\stackrel{(706)}{=} -\phi_0 k\sin(k\cdot x+\alpha)\,[-\phi_0\sin(k\cdot x+\alpha)k\cdot n] \\ &\quad -\frac{n}{2}\left[\phi_0^2k^2\sin^2(k\cdot x+\alpha) - m^2\phi_0^2\cos^2(k\cdot x+\alpha)\right] \\ &= \phi_0^2 k\sin^2(k\cdot x+\alpha)k\cdot n - \frac{n}{2}\phi_0^2\left[m^2\sin^2(k\cdot x+\alpha) - m^2\left(1-\sin^2(k\cdot x+\alpha)\right)\right] \\ &= \phi_0^2 k\sin^2(k\cdot x+\alpha)k\cdot n - \frac{n}{2}m^2\phi_0^2\left[-1+2\sin^2(k\cdot x+\alpha)\right] \quad . \end{aligned} \tag{709}$$

This leads to

$$k\cdot T(k) = \phi_0^2k^4 - \frac{k^2}{2}\phi_0^2\left[-m^2+2m^2\sin^2(k\cdot x+\alpha)\right] = \frac{\phi_0^2m^4}{2}\left[1-2\sin^2(k\cdot x+\alpha)\right] \quad , \tag{710}$$

[71] Note that the difference to (697) is that it depends only on the three-dimensional $\vec{k}$, not on k. Here is $k_0 = k_0(\vec{k}) > 0$

[72] Note that the signature of the energy is already encoded in the definition of $\phi^{\pm}(x)$.

thus for the time-like unit vector $\hat{k} \stackrel{\text{def}}{=} k/m$

$$\hat{k} \cdot T(\hat{k}) = \frac{\phi_0^2 m^2}{2} \left[1 - 2\sin^2(k \cdot x + \alpha)\right] \quad . \tag{711}$$

Since $\hat{k}^2 = +1$, we can express $\hat{k}$ in terms of the time-like basis vector γ_0 by $\hat{k} = R\gamma_0\tilde{R}$, where R is a Lorentz boost.

Split in T_g and T_q Given the Lagrangian (677) for the Klein-Gordon field we find

$$u \stackrel{(638)}{=} \langle \phi \partial_{\partial_x \phi} \mathcal{L} \rangle_1 \stackrel{(677)}{=} \phi \partial_x \phi = \frac{1}{2} \partial_x \phi^2 \quad . \tag{712}$$

The symmetry of T_g follows then immidiately from

$$\partial_x \wedge u \stackrel{(712)}{=} \frac{1}{2} \partial_x \wedge \partial_x \phi^2 \stackrel{(189)}{=} 0 \quad . \tag{713}$$

The analogue of Ricci tensor and scalar become according to (650) and (651)

$$\left.\begin{array}{llll} R(a) & \stackrel{(650)}{=} & a \cdot \partial_x u & \stackrel{(712)}{=} & \frac{1}{2} a \cdot \partial_x \partial_x \phi^2 \\ R & \stackrel{(651)}{=} & \partial_a \cdot R(a) & = & \frac{1}{2} \partial_x^2 \phi^2 \end{array}\right\} . \tag{714}$$

Since $R = 2\mathcal{L}$ for solutions of the Euler-Lagrange equations the Lagrangian has the value

$$\mathcal{L} = \frac{1}{4} \partial_x^2 \phi^2 \quad . \tag{715}$$

We derive so for T_g

$$T_g(a) = R(a) - \frac{a}{2} R = \frac{1}{2} \left[a \cdot \partial_x \partial_x \phi^2 - \frac{a}{2} \partial_x^2 \phi^2\right] \quad . \tag{716}$$

The four-momentum density $\mathcal{P}_g$ represented by T_g becomes then under use of $\partial_x^2 = \partial_0^2 - \vec{\partial}_x^2$

$$\begin{aligned} \mathcal{P}_g &\stackrel{\text{def}}{=} T_g(\gamma_0) \stackrel{(716)}{=} \frac{1}{2} \left[\partial_0 \partial_x \phi^2 - \frac{\gamma_0}{2} \partial_x^2 \phi^2\right] \\ &= \frac{1}{2} \left[\partial_0 \partial_x \gamma_0 \phi^2 - \frac{1}{2} (\partial_0^2 - \vec{\partial}_x^2) \phi^2\right] \gamma_0 \\ &= \frac{1}{2} \left[\frac{1}{2} \partial_0^2 \phi^2 - \partial_0 \vec{\partial}_x \phi^2 + \frac{1}{2} \vec{\partial}_x^2 \phi^2\right] \gamma_0 \quad . \end{aligned} \tag{717}$$

Taking the inner product with γ_0 gives us finally the Energy density $\mathcal{E}_g$ represented by T_g

$$\mathcal{E}_g \stackrel{\text{def}}{=} \gamma_0 \cdot T_g(\gamma_0) \stackrel{(717)}{=} \frac{1}{4} \left(\partial_0^2 \phi^2 + \vec{\partial}_x^2 \phi^2\right) \quad . \tag{718}$$

7.2.3 The Momentum of the Klein-Gordon Field

Given the energy-momentum tensor of the scalar field (706), we find for the *four-momentum density*[73]

$$\mathcal{P}_{\phi_x} \stackrel{\text{def}}{=} T(\gamma_0) \stackrel{(706)}{=} (\partial_x \phi_x)(\partial_0 \phi_x) - \frac{1}{2} \gamma_0 \left[(\partial_x \phi_x)^2 - m^2 \phi_x^2\right] \quad . \tag{719}$$

The γ_0-part is the *energy density*, which becomes

$$\begin{aligned} \mathcal{E}_{\phi_x} &\stackrel{\text{def}}{=} \gamma_0 \cdot \mathcal{P}_{\phi_x} = \gamma_0 \cdot T(\gamma_0) \\ &\stackrel{(719)}{=} (\partial_0 \phi_x)^2 - \frac{1}{2} \left[(\partial_0 \phi_x)^2 - (\vec{\partial}_x \phi_x)^2 - m^2 \phi_x^2\right] \\ &= \frac{1}{2} \left[(\partial_0 \phi_x)^2 + (\vec{\partial}_x \phi_x)^2 + m^2 \phi_x^2\right] \quad . \end{aligned} \tag{720}$$

[73]Here $\partial_\nu \stackrel{\text{def}}{=} \partial_{x^\nu}$.

The *total four-momentum* is given by the spatial integral in the rest-frame over the momentum density (719), *i.e.*,

$$p_{\phi_x} = \int |d^3x|\, \mathcal{P}_{\phi_x} \quad . \tag{721}$$

As shown in Appendix B, the momentum of ϕ^+ and ϕ^- vanishes[74], *i.e.*,

$$p_{\phi^\pm} = 0 \quad . \tag{722}$$

Thus, when substituting $\phi(x) = \phi^+(x) + \phi^-(x)$ into (721), only mixed terms survive, and we obtain, as is shown in Appendix B,

$$p_{\phi_x} = \int |d^3k|\, \frac{k}{2} \left[\phi^+_{\vec{k}} \phi^-_{\vec{k}} + \phi^-_{\vec{k}} \phi^+_{\vec{k}} \right] \quad , \tag{723}$$

which is in accordance with standard results, like in [6].

7.3 The electromagnetic Field

Here we consider the four-vector potential A, obeying Maxwells equations, as discussed in Section 5. In the Lorentz gauge, *i.e.*,

$$\partial_x \cdot A = 0 \quad , \tag{724}$$

the Faraday multivector $F = \vec{E} + i\vec{B}$ can be written in terms of the four-vector potential as

$$F \overset{\text{def}}{=} \partial_x \wedge A \overset{(724)}{=} \partial_x A = \vec{E} + i\vec{B} \quad . \tag{725}$$

7.3.1 The Lagrangian

Maxwells equations, *i.e.*, $\partial_x F = J$, can now be derived from the Lagrangian

$$\mathcal{L} \overset{\text{def}}{=} \langle -A \cdot J + \frac{1}{2} F^2 \rangle = \langle -\widetilde{A} \cdot J + \frac{1}{2} \widetilde{F}^2 \rangle \overset{(725)}{=} \langle -\widetilde{A} \cdot J + \frac{1}{2} (\widetilde{\partial_x A})^2 \rangle \quad , \tag{726}$$

where the four-vector potential A is the independent field. The variational principle yields under use of $\widetilde{\partial_x A} = \widetilde{F}$ directly

$$0 = \partial_{\widetilde{A}} \mathcal{L} - \partial_x \partial_{\widetilde{F}} \mathcal{L} = -J - \partial_x \underbrace{\widetilde{F}}_{=-F} = -J + \partial_x F \quad \Rightarrow \quad \partial_x F = J \quad . \tag{727}$$

This is the form of Maxwells equations derived in Section 5. It is impressive, how Maxwells equations reduce to this one simple relation. The geometric product contains the whole algebraic structure of the equations — surely a stong support for the Geometric Algebra approach.

7.3.2 The Energy-Momentum Tensor

The Lagrangian (726) is clearly Poincare-invariant. Hence we can employ the definition (630) and derive for the energy-momentum tensor

$$T(n) = \dot{\partial}_x \langle \dot{A} \partial_{\partial_x A} \mathcal{L} n \rangle - n\mathcal{L} = \dot{\partial}_x \langle \dot{A} F n \rangle - n \langle -A \cdot J + \frac{1}{2} F^2 \rangle \quad . \tag{728}$$

But here the first term contains still a total divergence. To see this we rewrite

$$\begin{aligned}
\dot{\partial}_x \langle \dot{A}(F \cdot n) \rangle &= \langle \overbrace{\dot{\partial}_x \dot{A}}^{\overset{(725)}{=} F} (F \cdot n) \rangle_1 - \dot{\partial}_x \cdot \langle \dot{A}(F \cdot n) \rangle_2 \overset{(725)}{=} F \cdot (F \cdot n) - \dot{\partial}_x \cdot (\dot{A} \wedge (F \cdot n)) \\
&= F \cdot (F \cdot n) - \underbrace{(\dot{\partial}_x \cdot \dot{A})}_{\overset{(724)}{=} 0} \wedge (F \cdot n) - \dot{A} \wedge (\dot{\partial}_x \cdot (F \cdot n)) \\
&\overset{(724)}{=} F \cdot (F \cdot n) - \left[(\dot{\partial}_x \cdot \dot{F}) \cdot n \right] \dot{A} + \left[(\dot{\partial}_x \cdot \dot{F}) \cdot n \right] A \\
&\overset{(727)}{=} F \cdot (F \cdot n) - \underbrace{\left[(\dot{\partial}_x \cdot \dot{F}) \cdot n \right] \dot{A}}_{\text{total divergence!}} + (J \cdot n) A \quad .
\end{aligned} \tag{729}$$

[74]This means substituting $\phi^\pm$ into (719) for ϕ and integrating over the rest-space yields zero.

The second term is a total divergence. Thus we can drop this term under the assumption that its spatial integral vanishes, *i.e.*, has a vanashing boundary term. So the energy-momentum tensor (728) becomes

$$T(n) = F\cdot(F\cdot n) + (J\cdot n)A + n(A\cdot J) - \frac{1}{2}n\langle F^2\rangle \quad . \tag{730}$$

We can still simplify this expression by noting that F is a pure bivector, *i.e.*, $\widetilde{F} = -F$, and thus

$$\begin{aligned} F\cdot(F\cdot n) - \frac{n}{2}\langle F^2\rangle &\overset{(69)}{=} \frac{1}{2}F\cdot\langle Fn - nF\rangle_1 - \frac{n}{2}\langle F^2\rangle \overset{(69)}{=} \frac{1}{4}\langle F^2 n - FnF - FnF + nF^2\rangle_1 - \frac{n}{2}\langle F^2\rangle \\ &= -\frac{1}{2}FnF = \frac{1}{2}Fn\widetilde{F} \quad . \end{aligned} \tag{731}$$

Substituting in (730) yields the final form of the energy-momentum tensor

$$T(n) = (J\cdot n)A + n(A\cdot J) + \frac{1}{2}Fn\widetilde{F} \quad . \tag{732}$$

For the vacuum case, *i.e.*, $J = 0$, this takes the neat form $T(n) = \frac{1}{2}Fn\widetilde{F}$.

7.4 Dirac Theory

7.4.1 The Dirac Lagrangian and the Dirac Equation

The Lagrangian

$$\mathcal{L} = \langle \dot{\partial}_x \dot{\psi} i\gamma_3\widetilde{\psi} - eA\psi\gamma_0\widetilde{\psi} - m\psi\widetilde{\psi}\rangle \tag{733}$$

yields with[75]

$$\partial_{\widetilde{\psi}}\mathcal{L} \overset{(733)}{=} \partial_x\psi i\gamma_3 - 2eA\psi\gamma_0 - 2m\psi \quad ; \quad \partial_{\widetilde{\partial_x\psi}}\mathcal{L} \overset{(733)}{=} \widetilde{i\gamma_3\widetilde{\psi}} = \psi\gamma_3 i \tag{734}$$

for the Euler-Lagrange equation (611)

$$\begin{aligned} 0 &\overset{(611)}{=} \partial_{\widetilde{\psi}}\mathcal{L} - \partial_x(\partial_{\widetilde{\partial_x\psi}}\mathcal{L}) \overset{(734)}{=} \partial_x\psi i\gamma_3 - 2eA\psi\gamma_0 - 2m\psi - \partial_x(\psi \underbrace{\gamma_3 i}_{=-i\gamma_3}) \\ &= 2\partial_x\psi i\gamma_3 - 2eA\psi\gamma_0 - 2m\psi \end{aligned} \tag{735}$$

and thus the Dirac equation

$$0 = \partial_x\psi i\sigma_3 - eA\psi - m\psi\gamma_0 \quad . \tag{736}$$

Here it should be noted that in difference to standard approaches, ψ and $\widetilde{\psi}$ are *not* taken as independent when applying the variational principle. Why should two different representations of the same spinor ψ be independent from each other? Geometric Algebra removes the need for this assumption — to my mind, this reveals a much more satisfying picture.

7.4.2 Application of Noether's Theorem and conjugate Currents

We can extract further interesting information with the help of Noether's theorem. The most of the results here were derived in [2].

A general *position independent* transformation is given by

$$\psi' \overset{\text{def}}{=} \psi e^{\alpha M} \tag{737}$$

for any *multivector* M. Noether's theorem in the form (621) yields

$$\partial_\alpha \mathcal{L}(\psi')\Big|_{\alpha=0} \overset{(621)}{=} \partial_x \cdot \langle \psi M i\gamma_3\widetilde{\psi}\rangle_1 \quad . \tag{738}$$

[75] Here ψ and $\widetilde{\psi}$ are not independent of each other:

$$\begin{aligned} \partial_{\widetilde{\psi}}\langle A\psi\gamma_0\widetilde{\psi}\rangle &= \dot{\partial}_{\widetilde{\psi}}\langle A\psi\gamma_0\dot{\widetilde{\psi}}\rangle + \dot{\partial}_{\widetilde{\psi}}\underbrace{\langle A\dot{\psi}\gamma_0\widetilde{\psi}\rangle}_{=\widetilde{\langle\ldots\rangle}} = A\psi\gamma_0 + \dot{\partial}_{\widetilde{\psi}}\langle\psi\gamma_0\dot{\widetilde{\psi}}A\rangle \\ &= A\psi\gamma_0 + \dot{\partial}_{\widetilde{\psi}}\langle A\psi\gamma_0\dot{\widetilde{\psi}}\rangle = 2A\psi\gamma_0 \quad . \end{aligned}$$

The scalar Case If $M \overset{\text{def}}{=} \lambda$ is a scalar, we have

$$\widetilde{\psi\lambda i\gamma_3\tilde{\psi}} = \lambda\psi\gamma_3 i\tilde{\psi} = -\psi\lambda i\gamma_3\tilde{\psi} \quad , \tag{739}$$

so that $\psi\lambda i\gamma_3\tilde{\psi}$ cannot contain a scalar and vector part. (738) yields thus

$$0 = \partial_\alpha \mathcal{L}(\psi') = \partial_\alpha \mathcal{L}(e^{\alpha\lambda}\psi)\Big|_{\alpha=0} \quad . \tag{740}$$

Since $\mathcal{L}$ is quadratic[76] in ψ and $\partial_x\psi$ and $e^{\alpha\lambda}$ commutes with everything we have $\mathcal{L}(e^{\alpha\lambda}\psi) = e^{2\alpha\lambda}\mathcal{L}(\psi)$ and so

$$0 = \partial_\alpha e^{2\alpha\lambda}\mathcal{L}(\psi)\Big|_{\alpha=0} = 2\lambda\mathcal{L}(\psi) \quad . \tag{741}$$

We conclude that *the Lagrangian (733) vanishes for solutions* of the Dirac equation, *i.e.*,

$$\mathcal{L}(\psi) = 0 \quad . \tag{742}$$

The pseudoscalar Case Since $\langle\psi M i\gamma_3\tilde{\psi}\rangle_1 = 0$ if M is odd, we consider as the next case $M = i$, so that (738) gives

$$\begin{aligned}
-\partial_x \cdot \langle\psi\gamma_3\tilde{\psi}\rangle_1 &\overset{(738)}{=} \partial_\alpha \mathcal{L}(\psi e^{i\alpha}, \partial_x\psi e^{i\alpha})\Big|_{\alpha=0} \\
&\overset{(733)}{=} \partial_\alpha \langle \partial_x(\psi e^{i\alpha}) i\gamma_3 e^{i\alpha}\tilde{\psi} - eA\psi e^{i\alpha}\gamma_0 e^{i\alpha}\tilde{\psi} - m\psi e^{i\alpha}e^{i\alpha}\tilde{\psi}\rangle\Big|_{\alpha=0} \\
&= \partial_\alpha \langle \overbrace{\partial_x\psi i\gamma_3\tilde{\psi}e^{i\alpha}e^{-i\alpha} - eA\psi\gamma_0\tilde{\psi}e^{i\alpha}e^{-i\alpha}}^{\text{independent of } \alpha} - m\psi\tilde{\psi}e^{2i\alpha}\rangle\Big|_{\alpha=0} \\
&= \langle -2ime^{2i\alpha}\psi\tilde{\psi}\rangle\Big|_{\alpha=0} \\
&\overset{(590)}{=} -\langle 2im\rho e^{i\beta}\rangle = 2m\rho\sin\beta \quad .
\end{aligned} \tag{743}$$

Thus we define the **spin current**

$$\rho s \overset{\text{def}}{=} \psi\gamma_3\tilde{\psi} = \langle\psi\gamma_3\tilde{\psi}\rangle_1 \tag{744}$$

and write the result (743) as

$$\partial_x \cdot (\rho s) = -2m\rho\sin\beta \quad . \tag{745}$$

The $\mathcal{P}$-Vector Case If M is a bivector B, we have with $\tilde{B} = -B$

$$\begin{aligned}
\partial_\alpha \mathcal{L}(\psi')\Big|_{\alpha=0} &\overset{(737)}{=} \partial_\alpha \mathcal{L}(\psi e^{\alpha B})\Big|_{\alpha=0} \\
&\overset{(733)}{=} \partial_\alpha \langle \partial_x\psi e^{\alpha B} i\gamma_3 e^{-\alpha B}\tilde{\psi} - eA\psi e^{\alpha B}\gamma_0 e^{-\alpha B}\tilde{\psi} - m\psi\tilde{\psi}\rangle\Big|_{\alpha=0} \\
&= \langle \partial_x\psi\,[Bi\gamma_3 - i\gamma_3 B]\,\tilde{\psi} - eA\psi\,[B\gamma_0 - \gamma_0 B]\,\tilde{\psi}\rangle \\
&= 2\langle \partial_x\psi i(B\cdot\gamma_3)\tilde{\psi} - eA\psi(B\cdot\gamma_0)\tilde{\psi}\rangle \quad .
\end{aligned} \tag{746}$$

From the Dirac equation (736) it follows

$$\partial_x\psi i \overset{(736)}{=} m\psi\gamma_0\sigma_3 + eA\psi\sigma_3 = eA\psi\sigma_3 - m\psi\gamma_3 \quad , \tag{747}$$

which gives in (746)

$$\partial_\alpha \mathcal{L}(\psi')\Big|_{\alpha=0} \overset{(746)}{=} 2\langle eA\psi\sigma_3(B\cdot\gamma_3)\tilde{\psi} - m\psi \underbrace{\gamma_3(B\cdot\gamma_3)}_{=1/2[\gamma_3 B\gamma_3 + B]} \tilde{\psi} - eA\psi(B\cdot\gamma_0)\tilde{\psi}\rangle \quad . \tag{748}$$

---

[76]Note that this condition is crucial for the result.

Using

$$2\left[\sigma_3(B\cdot\gamma_3) - B\cdot\gamma_0\right] = \sigma_3(B\gamma_3 - \gamma_3 B) - B\gamma_0 + \gamma_0 B = \sigma_3 B\sigma_3\gamma_0 - \underbrace{\sigma_3\gamma_3}_{=\gamma_0} B - B\gamma_0 + \gamma_0 B$$
$$= (\sigma_3 B\sigma_3 - B)\gamma_0 \quad , \tag{749}$$

we thus get[77]

$$\partial_\alpha \mathcal{L}(\psi')\Big|_{\alpha=0} \overset{(748)}{=} \langle eA\psi(\sigma_3 B\sigma_3 - B)\gamma_0\widetilde{\psi} - m\underbrace{\psi(\gamma_3 B\gamma_3 + B)\widetilde{\psi}}_{\text{no scalar-part}}\rangle$$
$$= \langle eA\psi(\sigma_3 B\sigma_3 - B)\gamma_0\widetilde{\psi}\rangle \quad . \tag{751}$$

Thus (738) gives

$$\partial_x \cdot \langle \psi B i\gamma_3\widetilde{\psi}\rangle_1 \overset{(738)}{=} \langle eA\psi(\sigma_3 B\sigma_3 - B)\gamma_0\widetilde{\psi}\rangle \quad . \tag{752}$$

We will first consider the case $B = \sigma_n$. Here we have

$$\partial_x \cdot \langle \psi\sigma_n i\gamma_3\widetilde{\psi}\rangle_1 = \partial_x \cdot \langle \psi\sigma_n\sigma_1\sigma_2\gamma_0\widetilde{\psi}\rangle_1 \overset{(752)}{=} \langle eA\psi(\sigma_3\sigma_n\sigma_3 - \sigma_n)\gamma_0\widetilde{\psi}\rangle \tag{753}$$

and so for $B = \sigma_{1/2}$

$$\pm\partial_x \cdot \langle \psi\gamma_{2/1}\widetilde{\psi}\rangle_1 = \langle eA\psi(-\sigma_{1/2} - \sigma_{1/2})\gamma_0\widetilde{\psi}\rangle = -2e\langle A\psi\gamma_{1/2}\widetilde{\psi}\rangle \quad . \tag{754}$$

For $B = \sigma_3$ follows with

$$\partial_x \cdot \underbrace{\langle \psi i\gamma_0\widetilde{\psi}\rangle_1}_{=0} = 0 \overset{(753)}{=} \langle eA\psi(\sigma_3 - \sigma_3)\gamma_0\widetilde{\psi}\rangle = 0 \quad . \tag{755}$$

only a trivial identity.

The $\mathcal{P}$-Bivector Case We come now to a discussion of the case $B = i\sigma_n$. Here

$$\langle eA\psi B\gamma_0\widetilde{\psi}\rangle = \langle eA\psi i\sigma_n\gamma_0\widetilde{\psi}\rangle = e\langle A\psi i\gamma_n\widetilde{\psi}\rangle = 0 \quad , \tag{756}$$

so that (752) becomes

$$\partial_x \cdot \langle \psi B i\gamma_3\widetilde{\psi}\rangle_1 \overset{(752)}{=} e\langle A\psi\sigma_3 B\sigma_3\gamma_0\widetilde{\psi}\rangle \quad . \tag{757}$$

For $B = i\sigma_{1/2}$ we have further[78]

$$\left.\begin{aligned} \langle \psi i\sigma_{1/2} i\gamma_3\widetilde{\psi}\rangle_1 &= -\langle \psi\sigma_{1/2}\gamma_3\widetilde{\psi}\rangle_1 &= 0 \\ \langle eA\psi\sigma_3 i\sigma_{1/2}\sigma_3\gamma_0\widetilde{\psi}\rangle &= -\langle A\psi i\gamma_{1/2}\widetilde{\psi}\rangle &= 0 \end{aligned}\right\} , \tag{758}$$

and so the identity (757) becomes trivial. For $B = i\sigma_3$ we get instead the important result

$$\partial_x \cdot \langle \psi i\sigma_3 i\gamma_3\widetilde{\psi}\rangle_1 = -\partial_x \cdot \langle \psi\sigma_3\gamma_3\widetilde{\psi}\rangle_1 = -\partial_x \cdot \langle \psi\gamma_0\widetilde{\psi}\rangle_1$$
$$= \langle eA\psi\sigma_3 i\sigma_3\sigma_3\gamma_0\widetilde{\psi}\rangle = \underbrace{\langle eA\psi i\gamma_3\widetilde{\psi}\rangle}_{=0} = 0 \quad . \tag{759}$$

This is the missing *conservation equation for the Dirac current*

$$J \overset{\text{def}}{=} \langle \psi\gamma_0\widetilde{\psi}\rangle_1 \quad ; \quad \partial_x \cdot J = \partial_x \cdot \langle \psi\gamma_0\widetilde{\psi}\rangle = 0 \quad . \tag{760}$$

The following table collects all our results

$$\left.\begin{aligned} \rho s \overset{\text{def}}{=} \psi\gamma_3\widetilde{\psi} \quad &; \quad J \overset{\text{def}}{=} \psi\gamma_0\widetilde{\psi} \\ \partial_x \cdot (\rho s) = -2m\rho\sin\beta \quad &; \quad \partial_x \cdot J = 0 \\ \partial_x \cdot \langle \psi\gamma_2\widetilde{\psi}\rangle = -2eA\cdot\langle \psi\gamma_1\widetilde{\psi}\rangle \quad &; \quad \partial_x \cdot \langle \psi\gamma_1\widetilde{\psi}\rangle = +2eA\cdot\langle \psi\gamma_2\widetilde{\psi}\rangle \end{aligned}\right\} . \tag{761}$$

[77] Here we use

$$\left.\begin{aligned} \psi\gamma_3 B\gamma_3\widetilde{\psi} &= -\widetilde{\psi\gamma_3 B\gamma_3\widetilde{\psi}} \\ \psi B\widetilde{\psi} &= -\psi B\widetilde{\psi} \end{aligned}\right\} \Rightarrow \text{No scalar part.} \tag{750}$$

[78] Both expressions reverse to give minus itself. So they cannot contain a scalar part.

7.4.3 The Momentum Representation of the Dirac Spinor

The Fourier Decomposition A plane-wave solution of the Dirac equation is given by

$$\psi_p(x) = \psi'_p e^{-i\sigma_3 p\cdot x} \quad . \tag{762}$$

As recognized before, we have to identify the complex unit j with a multiplication with $i\sigma_3$ from the right. So (762) is just a term of a Fourier expansion, which now becomes

$$\psi(x) = \int \frac{d^4p}{\sqrt{2\pi}^4}\psi_p(x) = \int \frac{d^4p}{\sqrt{2\pi}^4}\psi'(p)\, e^{-i\sigma_3 p\cdot x} \quad . \tag{763}$$

Let us assume a constant vector field A. (763) gives after substituting in the Dirac equation (736) the momentum Dirac equation

$$\begin{aligned} 0 &\overset{(736)}{=} \int \frac{d^4p}{\sqrt{2\pi}^4}\left[\partial_x\psi_p(x)i\sigma_3 - eA\psi_p(x) - m\psi_p(x)\gamma_0\right] \\ &= \int \frac{d^4p}{\sqrt{2\pi}^4}\left[-p\psi'(p)\overbrace{i\sigma_3 i\sigma_3}^{=-1} - eA\psi'(p) - m\psi'(p)\gamma_0\right] e^{-i\sigma_3 p\cdot x} \\ \Rightarrow 0 &= p\psi'(p) - eA\psi'(p) - m\psi'(p)\gamma_0 \quad . \end{aligned} \tag{764}$$

Positive and negative Energy Solutions Let us now concentrate on the *free Dirac field*, without electromagnetic interaction, *i.e.*, $A = 0$. As discussed in Section 6.2.3, each component of the spinor has to satisfy the Klein-Gordon equation. Thus we have, as for the scalar field, the condition

$$p^2 = p_0^2 - \vec{p}^2 = m^2 \Rightarrow p_0 = \pm\sqrt{\vec{p}^2 + m^2} \tag{765}$$

and define in analogy to (694)[79]

$$\psi'(p) \overset{\text{def}}{=} 2p_0\sqrt{2\pi}\, \delta^4(p^2 - m^2)\, \psi(p) \quad , \tag{766}$$

so that (763) becomes

$$\psi(x) \overset{(763)}{=} \int \frac{d^4p}{\sqrt{2\pi}^3} 2p_0\, \delta^4(p^2 - m^2)\, \psi(p) e^{-i\sigma_3 p\cdot x} \quad . \tag{767}$$

Decomposition into positive and negative Frequency Parts In analogy to (697) and (698) we write the $\pm p_0$-parts of the integral (767) as

$$\left.\begin{aligned} \psi^\pm(p) &\overset{\text{def}}{=} \theta(p_0)\psi(\pm p) \\ \psi^\pm(x) &\overset{\text{def}}{=} \tfrac{1}{\sqrt{2\pi}^3}\int |d^4p| 2p_0\, \delta^4(p^2 - m^2)\, \psi^\pm(p) e^{\mp i\sigma_3 p\cdot x} \end{aligned}\right\} . \tag{768}$$

In the same way as (699) we get

$$\psi(x) = \psi_x^+ + \psi_x^- \quad . \tag{769}$$

Carrying out the integration over p_0 in (768) yields with the substitution

$$\xi \overset{\text{def}}{=} p_0^2 \quad \Rightarrow \quad d\xi = 2p_0 dp_0 \quad \Rightarrow \quad dp_0 = \tfrac{d\xi}{2p_0} \tag{770}$$

the result

$$\begin{aligned} \psi^\pm(x) &\overset{(768)}{=} \frac{1}{\sqrt{2\pi}^3}\int |d^4p| 2p_0 \delta^4(p_0^2 - \vec{p}^2 - m^2)\, \psi_p^\pm e^{\mp i\sigma_3 p\cdot x} \\ &\overset{(770)}{=} \frac{1}{\sqrt{2\pi}^3}\int d\xi \int |d^3p|\, \psi_p^\pm e^{\mp i\sigma_3 p\cdot x} \delta^4(\xi - \vec{p}^2 - m^2) \\ &= \frac{1}{\sqrt{2\pi}^3}\int |d^3p|\, \psi_p^\pm e^{\mp i\sigma_3 p\cdot x}\Big|_{p_0 = \sqrt{\vec{p}^2 + m^2}} \quad . \end{aligned} \tag{771}$$

[79]The factor of $2p_0$ occurs here, to cancel terms, resulting from the nonlinear argument of the δ-function.

Thus we define

$$\psi^{\pm}(\vec{p}) \stackrel{\text{def}}{=} \psi^{\pm}_{\vec{p}} \stackrel{\text{def}}{=} \psi^{\pm}_{p}\Big|_{p_0=\sqrt{\vec{p}^2+m^2}} \tag{772}$$

and write

$$\psi^{\pm}(x) = \frac{1}{\sqrt{2\pi}^3}\int d^3p\,\psi^{\pm}(\vec{p})e^{\mp i\sigma_3 p\cdot x} \quad . \tag{773}$$

7.4.4 Decomposition into Spin-Energy States

Decomposition of the Spinor Substituting (773) in the Dirac equation (736) gives

$$\begin{aligned} 0 &= \partial_x\psi^{\pm}_x i\sigma_3 - eA\psi^{\pm}_x - m\psi^{\pm}_x\gamma_0 \\ &\stackrel{(773)}{=} \frac{1}{\sqrt{2\pi}^3}\int|d^3p|\left\{\mp p\psi^{\pm}_{\vec{p}} i\sigma_3 e^{\mp i\sigma_3 p\cdot x} i\sigma_3 - eA\psi^{\pm}_{\vec{p}}e^{\mp i\sigma_3 p\cdot x} - m\psi^{\pm}_{\vec{p}}e^{\mp i\sigma_3 p\cdot x}\gamma_0\right\} \end{aligned} \tag{774}$$

and we get so the *momentum Dirac equation* for $\psi^{\pm}_{\vec{p}} \stackrel{\text{def}}{=} \psi^{\pm}(\vec{p})$

$$0 \stackrel{(774)}{=} \pm\psi^{\pm}_{\vec{p}} - eA\psi^{\pm}_{\vec{p}} - m\psi^{\pm}_{\vec{p}}\gamma_0 \quad . \tag{775}$$

Multiplying this with $\tilde{\psi}^{\pm}_{\vec{p}}$ on the right yields

$$\pm(p - eA)\psi^{\pm}_{\vec{p}}\tilde{\psi}^{\pm}_{\vec{p}} - mJ^{\pm}_{\vec{p}} = 0 \quad , \tag{776}$$

with

$$\psi^{\pm}_{\vec{p}} \stackrel{\text{def}}{=} \psi^{\pm}(\vec{p}) \quad ; \quad J^{\pm}_{\vec{p}} \stackrel{\text{def}}{=} \psi^{\pm}_{\vec{p}}\gamma_0\tilde{\psi}^{\pm}_{\vec{p}} \quad . \tag{777}$$

The derived condition $\mathcal{L} \stackrel{(742)}{=} 0$ would give us the same equation as can be seen by substituting (773) in (733). Since J_p is a pure vector[80], we can conclude from (776), that $\psi^{\pm}_{\vec{p}}\tilde{\psi}^{\pm}_{\vec{p}}$ *cannot contain a pseudoscalar part*[81]. This enables us to write $\psi^{\pm}_{\vec{p}}$ as

$$\psi^{\pm}_{\vec{p}}\tilde{\psi}^{\pm}_{\vec{p}} = \pm\rho^{\pm}_{\vec{p}} \Rightarrow \psi^{\pm}_{\vec{p}} = \sqrt{\rho^{\pm}_{\vec{p}}}L_{\vec{p}}\Phi_{\vec{p}}N \tag{778}$$

with

$$\left.\begin{array}{lll} \textbf{a rotation } \Phi_{\vec{p}}: & \Phi^{\dagger}_{\vec{p}} = \tilde{\Phi}_{\vec{p}} \quad ; & \Phi_{\vec{p}}\tilde{\Phi}_{\vec{p}} = 1 \\ \textbf{a proper Lorentz-boost } L_{\vec{p}}: & L^{\dagger}_{\vec{p}} = L_{\vec{p}} \quad ; & L_{\vec{p}}\tilde{L}_{\vec{p}} = 1 \\ \textbf{a dilation}: & \rho_{\vec{p}} > 0 & \\ \text{and any other multivector } N: & N\tilde{N} = \pm 1 & \end{array}\right\} \quad . \tag{779}$$

Positive Energy Solutions Since ψ^{+}_{0} *must be even in* $\mathcal{P}$ and $L_0 = 1$ is even, we can conclude, that N must be even in $\mathcal{P}$. Thus we *choose* $N = 1$, *i.e.*, $\psi^{+}_{\vec{p}} = \sqrt{\rho^{+}_{\vec{p}}}L_{\vec{p}}\Phi_{\vec{p}}$. This definition implies

$$\psi^{+}_{\vec{p}}\tilde{\psi}^{+}_{\vec{p}} \stackrel{(778)}{=} \rho_{\vec{p}}L_{\vec{p}}\Phi_{\vec{p}}\tilde{\Phi}_{\vec{p}}\tilde{L}_{\vec{p}} = \rho_{\vec{p}} > 0 \quad . \tag{780}$$

For $J^{+}_{\vec{p}}$ we obtain now

$$J^{+}_{\vec{p}} = \psi^{+}_{\vec{p}}\gamma_0\tilde{\psi}^{+}_{\vec{p}} \stackrel{(455)}{=} \psi^{+}_{\vec{p}}\left[\psi^{+}_{\vec{p}}\right]^{\dagger}\gamma_0 \stackrel{(778)}{=} \rho^{+}_{\vec{p}}L_{\vec{p}}\Phi_{\vec{p}}NN^{\dagger}\Phi^{\dagger}_{\vec{p}}L^{\dagger}\gamma_0 = \rho^{+}_{\vec{p}}L^2_{\vec{p}}\gamma_0 \quad . \tag{781}$$

This yields in (776)

$$p\rho^{+}_{\vec{p}} \stackrel{(781)}{=} m\rho^{+}_{\vec{p}}L^2_{\vec{p}}\gamma_0 \Rightarrow L^2_{\vec{p}} = \frac{p\gamma_0}{m} = \frac{\vec{p}+E}{m} \quad , \tag{782}$$

where we used

$$p\gamma_0 = \vec{p} + E \quad ; \quad E > 0 \quad ; \quad E^2 = \vec{p}^2 + m^2 \quad . \tag{783}$$

[80] Since ψ'_p is even in $\mathcal{D}$, $J_p = \psi^{\pm}_{\vec{p}}\gamma_0\tilde{\psi}^{\pm}_{\vec{p}}$ must be odd. But it reverses to give itself and is thus a pure vector. Therefore it cannot be of grade 3.

[81] It cannot contain a bivector part, since it does reverse to give itself.

This can be solved for L as follows[82]

$$L_{\vec{p}} \overset{(782)}{=} \sqrt{\frac{p\gamma_0}{m}} = \sqrt{\frac{2(E+m)(E+\vec{p})}{2m(E+m)}}$$

$$= \sqrt{\frac{2E^2+2E\vec{p}+2mE+2m\vec{p}}{2m(E+m)}} = \sqrt{\frac{(E+m+\vec{p})^2}{2m(E+m)}}$$

$$\overset{(334)}{=} \begin{cases} \pm\frac{1}{\sqrt{2m(E+m)}}[(E+m)+|\vec{p}|\hat{p}] \\ \pm\frac{1}{\sqrt{2m(E+m)}}[|\vec{p}|+(E+m)\hat{p}] \end{cases} , \tag{785}$$

where we get a *four valued square-root.* The second solution obeys $L_{\vec{p}}\widetilde{L}_{\vec{p}} = -1$ and is thus (according to (779)) excluded. To unite this result in inverse hyperbolic form first note

$$|\vec{p}| = \sqrt{E^2-m^2} = \sqrt{(E+m)(E-m)} \quad . \tag{786}$$

Therefore, after substituting this in (785),

$$L_{\vec{p}} = \pm\frac{\sqrt{E+m}+\sqrt{E-m}\hat{p}}{\sqrt{2m}} = \frac{E+m+\vec{p}}{\sqrt{2m(E+m)}} = \pm e^{\alpha\hat{p}} , \tag{787}$$

where

$$\alpha \overset{\text{def}}{=} \tanh^{-1}\left(\sqrt{\frac{E-m}{E+m}}\right) \quad . \tag{788}$$

This results are the same as given in [8].

Negative Energy Solutions For negative energies $-E < 0$ we demand ψ_0^- to be odd and set thus $N = \sigma_3$. Now we have

$$\psi_{\vec{p}}^-\widetilde{\psi}_{\vec{p}}^- = -\rho_{\vec{p}}^- \quad ; \quad \psi_{\vec{p}}^-\gamma_0\widetilde{\psi}_{\vec{p}}^- = \rho_{\vec{p}}^- L_{\vec{p}}^2\gamma_0 \quad , \tag{789}$$

so that (776) becomes

$$0 = -P(-\rho_{\vec{p}}^-) - m\psi_{\vec{p}}^-\gamma_0\widetilde{\psi_{\vec{p}}^-} \quad . \tag{790}$$

We obtain thus the same solution as for positive energies

$$L_{\vec{p}}^2 = \frac{P\gamma_0}{m} = \frac{E+\vec{p}}{m} \quad . \tag{791}$$

Thus

$$L_{\vec{p}}^- = L_{\vec{p}}^+ = \frac{\sqrt{E+m}+\sqrt{E-m}\hat{p}}{\sqrt{2m}} \quad . \tag{792}$$

The Energy States as Lorentz Boosts Every Lorentz boost L obeys

$$L\widetilde{L} = 1 \quad ; \quad L^\dagger(\vec{p}) = L(\vec{p}) \quad ; \quad \widetilde{L}(\vec{p}) = L(-\vec{p}) \quad . \tag{793}$$

Further we can represent every Lorentz boost in exponential form and find in our case

$$L_{\vec{p}}^\pm = \frac{\sqrt{E+m}+\sqrt{E-m}\hat{p}}{\sqrt{2m}} = e^{\alpha\hat{p}} \quad ; \tag{794}$$

with

$$\alpha \overset{\text{def}}{=} \tanh^{-1}\left(\sqrt{\frac{E-m}{E+m}}\right) \quad . \tag{795}$$

[82] Here we use $A = 0$ and write

$$\hat{p} \overset{\text{def}}{=} \frac{\vec{p}}{|\vec{p}|} \quad ; \quad p\gamma_0 = p_0 + \vec{p} = E + \vec{p} \quad ; \quad E^2 = \vec{p}^2 + m^2 \quad . \tag{784}$$

For the case $A \neq 0$ we just have to replace p by $p - eA$.

Decomposition into Spin States Thus our spinor is determined up to a scalar factor $\rho^\pm_{\vec p} > 0$ and a rotation $\Phi_{\vec p}$. But since $\Phi_{\vec p} \in \mathcal{P}_2$ we can decompose it with scalars c_1 and c_2 as

$$\rho^\pm_{\vec p}\Phi_{\vec p} = 1c_1 e^{i\sigma_3\alpha} + (-i\sigma_2)c_2 e^{i\sigma_3\beta} \quad . \tag{796}$$

We call $c_1 e^{i\sigma_3\alpha}, c_2 e^{i\sigma_3\beta}$ as *complex coefficients* and define the **spin eigenstates**

$$\varphi_0 \stackrel{\text{def}}{=} 1 = (-i\sigma_2)^0 \quad ; \quad \varphi_1 \stackrel{\text{def}}{=} -i\sigma_2 = (-i\sigma_2)^1 \quad . \tag{797}$$

They have the obvious properties

$$\left.\begin{array}{rclcrcl} \varphi_0 i\sigma_3 &=& i\sigma_3\varphi_0 &;& \varphi_1 i\sigma_3 &=& -i\sigma_3\varphi_1 \\ \widetilde{\varphi}_0 &=& \varphi_0^\dagger = \varphi_0 &;& \widetilde{\varphi}_1 &=& \varphi_1^\dagger = -\varphi_1 \\ \varphi_0\varphi_1 &=& \varphi_1\varphi_0 = \varphi_1 &;& \varphi_0\varphi_0 &=& -\varphi_1\varphi_1 = \varphi_0 \end{array}\right\} . \tag{798}$$

φ_0 and φ_1 can be seen to be orthonormal with complex coefficients

$$\varphi_0\widetilde{\varphi}_0 = \varphi_1\widetilde{\varphi}_1 = 1 \quad ; \quad \varphi_0 c\widetilde{\varphi}_1 \stackrel{(798)}{=} -\varphi_0 c\varphi_1 \stackrel{(798)}{=} -\varphi_1 c^*\widetilde{\varphi}_0 \quad . \tag{799}$$

We call them spin eigenstates, since for the spin vector γ_3

$$\varphi_0\gamma_3\widetilde{\varphi}_0 = \gamma_3 \quad ; \quad \varphi_1\gamma_3\widetilde{\varphi}_1 = -i\sigma_2\gamma_3 i\sigma_2 = -\gamma_3 \quad . \tag{800}$$

We now define the **orthonormal spin energy states** by

$$v_s^+(p) = \sqrt{\tfrac{m}{E}}L_p\varphi_s \quad ; \quad v_s^- = v_s^+\sigma_3 = \sqrt{\tfrac{m}{E}}L_p\varphi_s\sigma_3 \quad . \tag{801}$$

They are normalized in $\mathcal{P}$ to give

$$\langle v_s^\pm(\vec p)|v_s^\pm(\vec p)\rangle \mapsto \langle v_s^\pm(\vec p)\left[v_s^\pm(\vec p)\right]^\dagger\rangle = \frac{m}{E}\langle L_{\vec p}^2\rangle \stackrel{(782)}{=} \frac{m}{E}\frac{E}{m} = 1 \quad . \tag{802}$$

The momentum representation of the wave-function can be written as

$$\psi^\pm(\vec p) = \sum_s v_s^\pm(p)a_s^\pm(\vec p) \quad , \tag{803}$$

where the $a_s^\pm$ are *complex* coefficients, *which have to be on the right*[83]! This is in agreement with our translation of j as a right-sided multiplication with $i\sigma_3$ and shows us that the complex coefficients represent a rotation around the spin-axis σ_3 in the rest-frame.

So we derive finally for the wave function

$$\psi^\pm(x) = \frac{1}{\sqrt{2\pi}^3}\sum_s \int |d^3p|\, v_s^\pm(\vec p)a_s^\pm(\vec p)e^{\mp i\sigma_3 p\cdot x} \quad . \tag{804}$$

The explicit representation shows

$$v_s^-(\vec p) \stackrel{(801)}{=} v_s^+(\vec p)\sigma_3 \quad ; \quad v_s^+(\vec p) \stackrel{(801)}{=} v_s^-(\vec p)\sigma_3 \tag{805}$$

and as relation between the spin-states

$$\left.\begin{array}{rcl} v_1^+(\vec p) &=& -v_0^+(\vec p)i\sigma_2 \\ v_1^-(\vec p) &=& v_0^-(\vec p)i\sigma_2 \end{array}\right\} \Leftrightarrow v_1^\pm(\vec p) = \mp v_0^\pm(\vec p)i\sigma_2 \quad . \tag{806}$$

The explicit representation (801) shows further

$$\left.\begin{array}{rclclcl} [v_s^+(\vec p)]^\dagger &=& \sqrt{\tfrac{m}{E}}\left[L_{\vec p}^+\varphi_s\right]^\dagger &=& \sqrt{\tfrac{m}{E}}\widetilde{L_{-\vec p}^+\varphi_s} &=& \widetilde{v_s^+(-\vec p)} \\ [v_s^-(\vec p)]^\dagger &=& \sqrt{\tfrac{m}{E}}\left[L_{\vec p}^+\varphi_s\sigma_3\right]^\dagger &=& -\sqrt{\tfrac{m}{E}}\widetilde{L_{-\vec p}^+\varphi_s\sigma_3} &=& -\widetilde{v_s^-(-\vec p)} \end{array}\right\} , \tag{807}$$

so that

$$\left[v_r^\pm(\vec p)\right]^\dagger = \pm\widetilde{v}_r^\pm(-\vec p) \quad . \tag{808}$$

[83] Remember that $i\sigma_3$ does in general not commute with a Lorentz boost.

Products of Coefficients To simplify the calculations involving the expansion into spin-states, we will introduce some definitions for the products of the coefficients. We define

$$\left.\begin{array}{lcl} c_{rs}^{++}(\vec{p},\vec{p}\,') & \stackrel{\text{def}}{=} & a_r^+(\vec{p})e^{-i\sigma_3(p-p')\cdot x}\,[a_s^+(\vec{p}\,')]^* \\ c_{rs}^{--}(\vec{p},\vec{p}\,') & \stackrel{\text{def}}{=} & a_r^-(\vec{p})e^{+i\sigma_3(p-p')\cdot x}\,[a_s^-(\vec{p}\,')]^* \\ c_{rs}^{\pm\mp}(\vec{p},\vec{p}\,') & \stackrel{\text{def}}{=} & a_r^\pm(\vec{p})e^{\mp i\sigma_3(p+p')\cdot x}\,[a_s^\mp(\vec{p}\,')]^* \end{array}\right\} . \tag{809}$$

Integrating over x gives under use of (687)

$$\left.\begin{array}{lcl} \int|d^3x|\,c_{rs}^{\pm\pm}(\vec{p},\vec{p}\,') & \stackrel{(809)}{=} & (2\pi)^3\delta^3(\vec{p}-\vec{p}\,')\overbrace{a_r^\pm(\vec{p})\,[a_s^\pm(\vec{p})]^*}^{\stackrel{\text{def}}{=}\eta_{rs}^{\pm\pm}(\vec{p})} \\ & \stackrel{\text{def}}{=} & (2\pi)^3\delta^3(\vec{p}-\vec{p}\,')\eta_{rs}^{\pm\pm}(\vec{p}) \\ \int|d^3x|\,c_{rs}^{\pm\mp}(\vec{p},\vec{p}\,') & \stackrel{(809)}{=} & (2\pi)^3\delta^3(\vec{p}+\vec{p}\,')\overbrace{a_r^\pm(\vec{p})e^{\mp 2i\sigma_3 p_0x_0}\,[a_s^\mp(-\vec{p})]^*}^{\stackrel{\text{def}}{=}\eta_{rs}^{\pm\mp}(\vec{p})} \\ & \stackrel{\text{def}}{=} & (2\pi)^3\delta^3(\vec{p}+\vec{p}\,')\eta_{rs}^{\pm\mp}(\vec{p}) \end{array}\right\} . \tag{810}$$

Here we defined

$$\eta_{rs}^{\pm\pm} \stackrel{\text{def}}{=} a_r^\pm(\vec{p})\,[a_s^\pm(\vec{p})]^* \quad ; \quad \eta_{rs}^{\pm\mp} \stackrel{\text{def}}{=} a_r^\pm(\vec{p})e^{\mp 2i\sigma_3 p_0x_0}\,[a_s^\mp(-\vec{p})]^* \quad . \tag{811}$$

We can write the result in the neat form

$$\int|d^3x|\,c_{rs}^{AB}(\vec{p},\vec{p}\,') = (2\pi)^3\delta^3(\vec{p}-AB\vec{p}\,')\eta_{rs}^{AB}(\vec{p}) \quad . \tag{812}$$

If we assume, that the operation of complex conjugation is equivalent to the operation of reversion[84] in $\mathcal{P}$, *i.e.*, hermitian adjungation, we have further the relations

$$\left.\begin{array}{lcl} [c_{rs}^{\pm\pm}(\vec{p},\vec{p}\,')]^* & = & c_{sr}^{\pm\pm}(\vec{p}\,',\vec{p}) \\ [c_{rs}^{\pm\mp}(\vec{p},\vec{p}\,')]^* & = & c_{sr}^{\mp\pm}(\vec{p}\,',\vec{p}) \\ [\eta^{+-}(\vec{p})]^* & = & \eta^{-+}(-\vec{p}) \end{array}\right\} . \tag{814}$$

With the help of this definitions the square of the wave function can be written as

$$\begin{array}{lcl} \psi_x^\pm\,[\psi_x^\pm]^\dagger & = & \frac{1}{(2\pi)^3}\int\int d^3p\,d^3p'\,v_r^\pm(\vec{p})a_r^\pm(\vec{p})e^{\mp i\sigma_3 p\cdot x}e^{\pm i\sigma_3 p'\cdot x}\,[a_s^\pm(\vec{p}\,')]^*\,[v_s^\pm(\vec{p}\,')]^\dagger \\ & = & \frac{1}{(2\pi)^3}\int\int d^3p\,d^3p'\,v_r^\pm(\vec{p})c_{rs}^{\pm\pm}(\vec{p},\vec{p}\,')\,[v_s^\pm(\vec{p}\,')]^\dagger \quad . \end{array} \tag{815}$$

And an integration over x gives for $A,B\in\{+,-\}$

$$\begin{array}{lcl} \int|d^3x|\,\psi_x^A\,[\psi_x^B]^\dagger & \stackrel{(815)}{=} & \frac{1}{(2\pi)^3}\int\int d^3p\,d^3p'\,v_r^A\left(\int|d^3x|\,c_{rs}^{AB}\right)\,[v_s^B(\vec{p}\,')]^\dagger \\ & \stackrel{(812)}{=} & \int\int d^3p\,d^3p'\,v_r^A(\vec{p})\eta_{rs}^{AB}(\vec{p})\delta^3(\vec{p}-AB\vec{p}\,')\,[v_s^B(\vec{p}\,')]^\dagger \\ & = & \int|d^3p|\,v_r^A(\vec{p})\eta_{rs}^{AB}(\vec{p})\,[v_s^B(AB\vec{p})]^\dagger \\ & \stackrel{(808)}{=} & B\int|d^3p|\,v_r^A(\vec{p})\eta_{rs}^{AB}(\vec{p})\tilde{v}_s^B(-AB\vec{p}) \quad . \end{array} \tag{816}$$

7.4.5 Products of Spin-States

Relation between Spin-States Relations between spin-states of positive energy follow from the explicit representation (801) and the spin relations (798). They are given by

$$\left.\begin{array}{lcl} v_1^+(\vec{p})c\tilde{v}_1^+(\vec{p}\,') & \stackrel{(801)}{=} & v_0^+(\vec{p})c^*\tilde{v}_0^+(\vec{p}\,') \\ v_1^+(\vec{p})c\tilde{v}_0^+(\vec{p}\,') & \stackrel{(801)}{=} & -v_0^+(\vec{p})c^*\tilde{v}_1^+(\vec{p}\,') \end{array}\right\} . \tag{817}$$

[84]That means independent of the interpretation of $a_r^\pm$ we understand

$$[a_r^\pm(\vec{p})a_s^\pm(\vec{p}\,')]^* = [a_s^\pm(\vec{p}\,')]^*\,[a_r^\pm(\vec{p})]^* \quad . \tag{813}$$

This is always true, when the $a_r^\pm$ commute with each other, i.e. when they are just complex numbers of the form $\alpha + i\sigma_3\beta$, but if the $a_r^\pm$ are non commuting objects this is an additional assumption.

$\langle AcB\rangle^{\mathcal{P}}_{0,1,3}$	$\widetilde{\nu^+_0(\vec{p})}$	$\widetilde{\nu^+_1(\vec{p})}$	$\widetilde{\nu^-_0(\vec{p})}$	$\widetilde{\nu^-_1(\vec{p})}$
$\nu^+_0(\vec{p})$	$\frac{m}{E}\Re[c]$ $\frac{i\vec{p}\wedge\sigma_3}{2E}\Im[c]$ 0	0 $\frac{i\vec{p}\wedge(\sigma_2 c)}{2E}$ 0	0 $-C_{\vec{p},\vec{p}}(\sigma_3)\Re[c]$ $-i\frac{m}{E}\Im[c]$	0 $-C_{\vec{p},\vec{p}}(c\sigma_1)$ 0
$\nu^+_1(\vec{p})$	0 $-\frac{i\vec{p}\wedge(\sigma_2 c)}{2E}$ 0	$\frac{m}{E}\Re[c]$ $-\frac{i\vec{p}\wedge\sigma_3}{2E}\Im[c]$ 0	0 $-C_{\vec{p},\vec{p}}(\sigma_1 c)$ 0	0 $C_{\vec{p},\vec{p}}(\sigma_3)\Re[c]$ $-i\frac{m}{E}\Im[c]$
$\nu^-_0(\vec{p})$	0 $C_{\vec{p},\vec{p}}(\sigma_3)\Re[c]$ $i\frac{m}{E}\Im[c]$	0 $C_{\vec{p},\vec{p}}(c\sigma_1)$ 0	$-\frac{m}{E}\Re[c]$ $-\frac{i\vec{p}\wedge\sigma_3}{2E}\Im[c]$ 0	0 $-\frac{i\vec{p}\wedge(\sigma_2 c)}{2E}$ 0
$\nu^-_0(\vec{p})$	0 $C_{\vec{p},\vec{p}}(\sigma_1 c)$ 0	0 $C_{\vec{p},\vec{p}}(\sigma_3)\Re[c]$ $i\frac{m}{E}\Im[c]$	0 $\frac{i\vec{p}\wedge(\sigma_2 c)}{2E}$ 0	$-\frac{m}{E}\Re[c]$ $\frac{i\vec{p}\wedge\sigma_3}{2E}\Im[c]$ 0

Table 4: Table of the products $\langle \nu^\pm_s(\vec{p})c\widetilde{\nu^\pm_r(\vec{p})}\rangle^{\mathcal{P}}_{0,1,3}$, where c is any complex number $c = \Re[c] + \Im[c]\ i\sigma_3$ and $C_{\vec{p},\vec{p}'}(\vec{a}) = \frac{1}{2\sqrt{EE'}}\left\{[(E+m)(E'+m)]^{\frac{1}{2}}\,\vec{a} - [(E+m)(E'+m)]^{-\frac{1}{2}}\,\langle\vec{p}\vec{a}\vec{p}'\rangle^{\mathcal{P}}_1\right\}$.

$\langle AcB\rangle^{\mathcal{P}}_{0,1,3}$	$\widetilde{\nu^+_0(-\vec{p})}$	$\widetilde{\nu^+_1(-\vec{p})}$	$\widetilde{\nu^-_0(-\vec{p})}$	$\widetilde{\nu^-_1(-\vec{p})}$
$\nu^+_0(\vec{p})$	$\Re[c]$ $\frac{\vec{p}}{2E}\Re[c]$ $i\frac{\vec{p}\cdot\sigma_3}{2E}\Im[c]$	0 0 $i\frac{\vec{p}\cdot(c\sigma_2)}{2E}$	0 $-C_{\vec{p},-\vec{p}}(\sigma_3)\Re[c]$ $-i\Im[c]$	0 $-C_{\vec{p},-\vec{p}}(c\sigma_1)$ 0
$\nu^+_1(\vec{p})$	0 0 $-i\frac{\vec{p}\cdot(\sigma_2 c)}{2E}$	$\Re[c]$ $\frac{\vec{p}}{2E}\Re[c]$ $-i\frac{\vec{p}\cdot\sigma_3}{2E}\Im[c]$	0 $-C_{\vec{p},vp}(\sigma_1 c)$ 0	0 $C_{\vec{p},-\vec{p}}(\sigma_3)\Re[c]$ $-i\Im[c]$
$\nu^-_0(\vec{p})$	0 $C_{\vec{p},-\vec{p}}(\sigma_3)\Re[c]$ $i\Im[c]$	0 $C_{\vec{p},-\vec{p}}(c\sigma_1)$ 0	$-\Re[c]$ $-\frac{\vec{p}}{2E}\Re[c]$ $-i\frac{\vec{p}\cdot\sigma_3}{2E}\Im[c]$	0 0 $i\frac{\vec{p}\cdot(c\sigma_2)}{2E}$
$\nu^-_0(\vec{p})$	0 $C_{\vec{p},-\vec{p}}(\sigma_1 c)$ 0	0 $-C_{\vec{p},-\vec{p}}(\sigma_3)\Re[c]$ $i\Im[c]$	0 0 $i\frac{\vec{p}\cdot(\sigma_2 c)}{2E}$	$-\Re[c]$ $-\frac{\vec{p}}{2E}\Re[c]$ $i\frac{\vec{p}\cdot\sigma_3}{2E}\Im[c]$

Table 5: Table of the products $\langle \nu^\pm_s(\vec{p})c\widetilde{\nu^\pm_r(-\vec{p})}\rangle^{\mathcal{P}}_{0,1,3}$, where c is any complex number $c = \Re[c] + \Im[c]\ i\sigma_3$ and $C_{\vec{p},\vec{p}'}(\vec{a}) = \frac{1}{2\sqrt{EE'}}\left\{[(E+m)(E'+m)]^{\frac{1}{2}}\,\vec{a} - [(E+m)(E'+m)]^{-\frac{1}{2}}\,\langle\vec{p}\vec{a}\vec{p}'\rangle^{\mathcal{P}}_1\right\}$.

For states of mixed energy we get in the same manner

$$\left.\begin{aligned} \nu^-_1(\vec{p})c\tilde{\nu}^+_1(\vec{p}') &\overset{(801)}{=} -\nu^-_0(\vec{p})c^*\tilde{\nu}^+_0(\vec{p}') \\ \nu^-_1(\vec{p})c\tilde{\nu}^+_0(\vec{p}') &\overset{(801)}{=} \nu^-_0(\vec{p})c^*\tilde{\nu}^+_1(\vec{p}') \end{aligned}\right\} \tag{818}$$

Relations between Energy States Pure negative energy products are related to pure positive energy products by

$$\nu^-_r(\vec{p})c\tilde{\nu}^-_s(\vec{p}') \overset{(801)}{=} -\nu^+_r(\vec{p})c\tilde{\nu}^+_s(\vec{p}') \quad , \tag{819}$$

while mixed products obey

$$\nu^-_r(\vec{p})c\tilde{\nu}^+_s(\vec{p}') \overset{(801)}{=} -\nu^+_r(\vec{p})c\tilde{\nu}^-_s(\vec{p}') \quad . \tag{820}$$

These products are explicitly calculated in Appendix C and collected in Tables 4 and 5.

7.4.6 The Energy-Momentum Tensor for the Dirac Lagrangian

The Dirac-Lagrangian $\mathcal{L} \overset{(733)}{=} \langle \partial_x\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle$ is Poincare-invariant and yields with

$$\partial_{\partial_x\psi}\mathcal{L} \overset{(733)}{=} i\gamma_3\tilde{\psi} \quad , \quad \mathcal{L}(\psi) = 0 \tag{821}$$

in (630) the energy-momentum tensor

$$\begin{aligned} T(n) &\overset{(630)}{=} \dot{\partial}_x \langle \dot{\psi} \partial_{\partial_x \psi} \mathcal{L} n \rangle - n\mathcal{L} \\ &\overset{(821)}{=} \dot{\partial}_x \langle \dot{\psi} i\gamma_3 \widetilde{\psi} n \rangle = \dot{\partial}_x \langle \dot{\psi} i\sigma_3 \psi^\dagger \gamma_0 n \rangle \quad . \end{aligned} \tag{822}$$

This will be the most useful form, but we can also write it as[85]

$$T(n) = \dot{\partial}_x \wedge \left(n \cdot \langle \dot{\psi} i\gamma_3 \widetilde{\psi} \rangle_1 \right) = i\dot{\partial}_x \cdot \left(n \wedge \langle \dot{\psi} \gamma_3 \widetilde{\psi} \rangle_3 \right) \quad . \tag{823}$$

In Terms of the Fourier Coefficients Substituting our Fourier expansion in positive and negative frequency terms yields

$$\begin{aligned} T &= \dot{\partial}_x \langle (\dot{\psi}_x^+ + \dot{\psi}_x^-) i\sigma_3 (\psi_x^+ + \psi_x^-)^\dagger \gamma_0 n \rangle \\ &= T_x^{++} + T_x^{--} + T_x^{+-} + T_x^{-+} = \sum_{A=\pm} \sum_{B=\pm} T_x^{AB} \quad , \end{aligned} \tag{824}$$

where with $A, B \in \{+,-\}$

$$\begin{aligned} T_x^{AB} &\overset{\text{def}}{=} \dot{\partial}_x \langle \dot{\psi}_x^A i\sigma_3 (\psi_x^B)^\dagger \gamma_0 n \rangle \\ &\overset{(773)}{=} \iint \frac{d^3p\, d^3p'}{(2\pi)^3} \dot{\partial}_x \langle \psi_{\vec{p}}^A (e^{-Ai\sigma_3 p\cdot x})^* i\sigma_3 e^{Bi\sigma_3 p'\cdot x} (\psi_{\vec{p}'}^B)^\dagger \gamma_0 n \rangle \\ &= \iint \frac{d^3p\, d^3p'}{(2\pi)^3} p \langle \psi_{\vec{p}}^A e^{-Ai\sigma_3 p\cdot x} (-A) \underbrace{i\sigma_3 i\sigma_3}_{=-1} e^{Bi\sigma_3 p'\cdot x} (\psi_{\vec{p}'}^B)^\dagger \gamma_0 n \rangle \\ &\overset{(815)}{=} \sum_{r,s} A \iint \frac{d^3p\, d^3p'}{(2\pi)^3} p \langle \nu_r^A(\vec{p})\, c_{rs}^{AB}(\vec{p},\vec{p}') \left[\nu_s^B(\vec{p}')\right]^\dagger \gamma_0 n \rangle \quad . \end{aligned} \tag{825}$$

The total Momentum Under use of the calculated spin eigenstate products in table 4 and 5 we find

$$\begin{aligned} p^{AA} &= \int |d^3x|\, T^{AA}(\gamma_0) \\ &\overset{(825)}{=} A \iint d^3p\, d^3p'\, p \langle \nu_r^A(\vec{p}) \eta_{rs}^{AA}(\vec{p}) \delta^3(\vec{p}-\vec{p}') \left[\nu_s^A(\vec{p}')\right]^\dagger \rangle \\ &= \int |d^3p|\, p \langle \nu_r^A(\vec{p}) \eta_{rs}^{AA}(\vec{p}) \widetilde{\nu}_s^A(-\vec{p}) \rangle \\ &= A \sum_r \int |d^3p|\, p \Re \left[\eta_{rr}^{AA}(\vec{p})\right] \end{aligned} \tag{826}$$

$$\begin{aligned} p^{+-} &= -\sum_{r,s} \int |d^3p|\, p \langle \nu_r^+(\vec{p}) \eta_{rs}^{+-} \widetilde{\nu_s^-(\vec{p})} \rangle = 0 \\ p^{-+} &= 0 \quad . \end{aligned} \tag{827}$$

Thus the total momentum is given by

$$\begin{aligned} P &= P^{++} + P^{--} = \int |d^3p|\, p \Re \left[\eta_{rr}^{++}(\vec{p}) - \eta_{rr}^{--}(\vec{p})\right] \\ &= \sum_r \int |d^3p|\, p \left\{ a_r^+(\vec{p}) \left[a_r^+(\vec{p})\right]^* - a_r^-(\vec{p}) \left[a_r^-(\vec{p})\right]^* \right\} \quad . \end{aligned} \tag{828}$$

7.4.7 The Square of the Wave Function

As discussed before, the quantum mechanical inner product has to be translated as

$$\langle \phi | \chi \rangle \mapsto \langle \phi^\dagger \chi \rangle - \langle \phi^\dagger \chi i\sigma_3 \rangle i\sigma_3 \quad . \tag{829}$$

[85] Note that $\langle \dot{\psi} i\gamma_3 \widetilde{\psi} \rangle_1$ can be non-zero, since it does *not* reverse to give minus itself.

So the square of the wave-function becomes

$$\langle\psi_x|\psi_x\rangle \leftrightarrow \langle\psi_x^\dagger\psi_x\rangle - \underbrace{\langle\psi_x^\dagger\psi_x i\sigma_3\rangle}_{=0} i\sigma_3 = \langle\psi_x\psi_x^\dagger\rangle \tag{830}$$

We get so for the total inner product

$$\int |d^3x|\ \langle\psi_x\psi_x^\dagger\rangle \overset{(769)}{=} \int |d^3x|\ \left\{\langle\psi_x^+\,[\psi_x^+]^\dagger\rangle + \langle\psi_x^-\,[\psi_x^-]^\dagger\rangle + \langle\psi_x^+\,[\psi_x^-]^\dagger\rangle + \langle\psi_x^-\,[\psi_x^+]^\dagger\rangle\right\} \quad . \tag{831}$$

From (816) we find under use of Table 4 and 5 for $A \neq B$

$$\left.\begin{array}{rcl} \int |d^3x|\ \langle\psi_x^A\,[\psi_x^A]^\dagger\rangle & = & A\int|d^3p|\ \langle\nu_r^A(\vec{p})\eta_{rs}^{AA}(\vec{p})\tilde{\nu}_s^A(-\vec{p})\rangle = \sum_r \int|d^3p|\ \Re\left[\eta_{rr}^{AA}(\vec{p})\right] \\ & = & \sum_r\int|d^3p|\ a_r^A(\vec{p})\left[a_r^A(\vec{p})\right]^* = \sum_r\int|d^3p|\ |a_r^A|^2 \\ \int |d^3x|\ \langle\psi_x^A\,[\psi_x^B]^\dagger\rangle & = & B\int|d^3p|\ \langle\nu_r^A(\vec{p})\eta_{rs}^{AB}(\vec{p})\tilde{\nu}_s^B(\vec{p})\rangle = 0 \quad . \end{array}\right\} . \tag{832}$$

The final result is thus

$$\int |d^3x|\ \langle\psi_x\,[\psi_x]^\dagger\rangle = \sum_r \int |d^3p|\ \left\{|a_r^+(\vec{p})|^2 + |a_r^-(\vec{p})|^2\right\} \quad . \tag{833}$$

The mixing Angle of the whole Wave We know, that $\psi_x\tilde{\psi}_x = \rho(x)e^{i\beta(x)}$, thus

$$\int |d^3x|\ \psi_x\tilde{\psi}_x = \rho_t e^{i\beta_t} \quad , \tag{834}$$

where the index indicates the time dependence. With the help of the Tables 4 and 5 we derive the integrals of the products of ψ_x^+ and ψ_x^-

$$\begin{array}{rcl} \int |d^3x|\ \langle\psi_x^+\tilde{\psi}_x^+\rangle_{0,4}^{\mathcal{D}} & = & \sum_{r,s}\int|d^3p|\ \langle\nu_r^+(\vec{p})\eta_{rs}^{++}\tilde{\nu}_s^+(\vec{p})\rangle_{0,4}^{\mathcal{D}} \\ & = & \int|d^3p|\ \frac{m}{p_0}\Re\left[\eta_{00}^{++} + \eta_{11}^{++}\right] \\ \int |d^3x|\ \langle\psi_x^-\tilde{\psi}_x^-\rangle_{0,4}^{\mathcal{D}} & = & \sum_{r,s}\int|d^3p|\ \langle\nu_r^-(\vec{p})\eta_{rs}^{--}\tilde{\nu}_s^-(\vec{p})\rangle \\ & = & -\int|d^3p|\ \frac{m}{p_0}\Re\left[\eta_{00}^{--} + \eta_{11}^{--}\right] \\ \int |d^3x|\ \langle\psi_x^+\tilde{\psi}_x^-\rangle_{0,4}^{\mathcal{D}} & = & \sum_{r,s}\int|d^3p|\ \langle\nu_r^+(\vec{p})\eta_{rs}^{+-}(\vec{p})\tilde{\nu}_s^-(-\vec{p})\rangle_{0,4}^{\mathcal{D}} \\ & = & -\int|d^3p|\ i\Im\left[\eta_{00}^{+-} + \eta_{11}^{+-}\right] \\ \int |d^3x|\ \langle\psi_x^-\tilde{\psi}_x^+\rangle_{0,4}^{\mathcal{D}} & = & \int|d^3p|\ i\Im\left[\eta_{00}^{-+} + \eta_{11}^{-+}\right] \end{array} \tag{835}$$

and conclude thus

$$\int |d^3x|\ \langle\psi_x\tilde{\psi}_x\rangle_{0,4}^{\mathcal{D}} \overset{(835)}{=} \int|d^3p|\ \left\{\frac{m}{p_0}\Re\left[\eta_{rr}^{++}(\vec{p}) - \eta_{rr}^{--}(\vec{p})\right] + i\Im\left[\eta_{rr}^{-+}(\vec{p}) - \eta_{rr}^{+-}(\vec{p})\right]\right\} \quad . \tag{836}$$

We derive so the explicit representation of the mixing angle

$$\tan\beta_t \overset{(834)}{=} \frac{i^{-1}\int|d^3x|\ \langle\psi_x\tilde{\psi}_x\rangle_4^{\mathcal{D}}}{\int|d^3x|\ \langle\psi_x\tilde{\psi}_x\rangle_0^{\mathcal{D}}} \overset{(836)}{=} \frac{\sum_r\int|d^3p|\ \Im\left[\eta_{rr}^{-+}(\vec{p}) - \eta_{rr}^{+-}(\vec{p})\right]}{\sum_s\int|d^3p|\ \frac{m}{p_0}\left[\eta_{ss}^{++}(\vec{p}) - \eta_{ss}^{--}(\vec{p})\right]} \quad . \tag{837}$$

This shows us two important points:

- As long as all $a^-(\vec{p})$ or $a^+(\vec{p})$ vanish , we have $\eta_{rs}^{+-} = 0$, $\eta_{rs}^{-+} = 0$ and thus $\beta_t = 0$ or π.
- β_t will in general be time dependent, since the η_{rs}^{+-} contain the factor $e^{\mp 2i\sigma_3 p_0 x_0}$. Since $p_0 = \sqrt{\vec{p}^2 + m^2}$ the oscillation of β_t depends on three-momentum and rest-mass.

7.4.8 The Angular-Momentum Tensor of the Dirac Field

Since ψ is a spinor, the angular-momentum tensor is given by (672). Substituting $\partial_{\partial_x\psi}\mathcal{L} \overset{(733)}{=} \partial_{\partial_x\psi}\langle\partial_x\psi i\gamma_3\tilde{\psi}\rangle = i\gamma_3\tilde{\psi}$ yields

$$\underline{J} \overset{(672)}{=} \frac{1}{2}\langle\psi\partial_{\partial_x}\mathcal{L}r\rangle_2^{\mathcal{P}} + T(r)\wedge x = \frac{1}{2}\langle\psi i\gamma_3\tilde{\psi}r\rangle_2 + T(r)\wedge x = i\rho(s\wedge r) + T(r)\wedge x \quad . \tag{838}$$

We interpret the first term as a *spin contribution*, while the second term is the angular momentum of the field itself. Remember that the total angular momentum is derived by integrating (838) over the rest-space. The first term is only depending on the fields, but not on x itself. Thus the spin contribution to the total angular momentum is independent of the choice of origin in the rest-space. The second term is a function of x and thus dependent on the choice of the origin.

7.5 Quantization of the Fields

7.5.1 The Klein-Gordon Field

Given the momentum in terms of the Fourier amplitudes

$$p \overset{(723)}{=} \int |d^3k| \frac{k}{2}\left[\phi_{\vec{k}}^+\phi_{\vec{k}}^- + \phi_{\vec{k}}^-\phi_{\vec{k}}^+\right] \quad , \tag{839}$$

the quantization procedure can be applied in the standard way. We will not discuss this topic here in depth, since the Geometric Algebra has no further influence. It should just be mentioned, that $\phi_{\vec{k}}^+$ gets replaced by a particle creation operator, while $\phi_{\vec{k}}^-$ becomes an annihilation operator. They create or annihilate a particle of momentum $\vec{k}$. Central point of the quantization is the introduction of the equal time commutation relations

$$\left[\phi_{\vec{k}}^{\pm}, \phi_{\vec{k}'}^{\pm}\right] = 0 \quad ; \quad \left[\phi_{\vec{k}}^{\mp}, \phi_{\vec{k}'}^{\pm}\right] = \delta^3(\vec{k}-\vec{k}') \quad . \tag{840}$$

7.5.2 The Dirac Field

We showed in the previous section that[86]

$$\psi(x) = \frac{1}{\sqrt{2\pi}^3}\int |d^3p| \sum \left\{ v_r^+ a_r^+(\vec{p}) e^{-i\sigma_3 p\cdot x} + v_r^-\left[a_r^-(\vec{p})\right]^\dagger e^{i\sigma_3 p\cdot x}\right\} \tag{841}$$

and

$$P = \int |d^3p|\, p\left\{\left[a_s^+\right]^\dagger a_s^+ - a_s^-\left[a_s^-\right]^\dagger\right\} \quad . \tag{842}$$

Thus we can apply the second quantization procedure and interpret $[a_r^{\pm}]^\dagger$ as creation and $a_r^{\pm}$ as anhiliation operators of fermions. They obey the anticommutation relations

$$\{a_r^{\pm}(\vec{p}), a_s^{\pm}(\vec{p}')\} = 0 \quad ; \quad \left\{a_r^{\pm}(\vec{p}), [a_s^{\pm}(\vec{p}')]^\dagger\right\} = \delta^3(\vec{p}-\vec{p}')\delta_{rs} \quad . \tag{843}$$

Using the anticommutation relation and neglecting an infinite constant gives the familiar result

$$P = \int |d^3p|\, p\left\{\left[a_s^+\right]^\dagger a_s^+ + \left[a_s^-\right]^\dagger a_s^-\right\} \quad . \tag{844}$$

[86]Note that we redefine in (828) $a_r^- \to [a_r^-]^\dagger$

8 Space-Time Algebra on a curved Manifold

I will give here a similar approach as given by Hestenes and Sobczyk in [1]. Most of the derived equations are straight forward translations of the standard tensor relations, but often with a much clearer geometric interpretation.

8.1 Calculus on Manifolds

8.1.1 Points on a Manifold

We treat the points of a manifold as vectors in a higher-dimensional embedding space, which can even have infinite dimensions. The geometry of this embedding space, *e.g.*, its dimensionality, is of no further importance for the following definitions and considerations. We just need this embedding space to define the subtraction of points on the manifold, which are infinitesimal close to each other with respect to chosen coordinates. This is in turn necessary to enable us to define the tangent space.

8.1.2 The covariant Derivative

The Tangent Space Let $\mathcal{M}$ be a differentiable manifold. The tangent space in a point $x_0 \stackrel{\text{def}}{=} x(\tau_0) \in \mathcal{M}$ is given by all vectors

$$a = \left.\frac{dy(\tau)}{d\tau}\right|_{\tau_0} \quad ;\forall y(\tau) \in \mathcal{M} \text{ with } y(\tau_0) = x_0 \quad . \tag{845}$$

Projection Tensor and Fields In this n-dimensional tangent space we can choose a basis and construct a pseudoscalar $I_n(x)$. As discussed before, $I_n(x)$ defines the whole tangent space $\mathcal{T}_n(x)$ via

$$a \wedge I_n(x) = 0 \quad ;\forall a \in \mathcal{T}_n(x) \quad . \tag{846}$$

We call $A(x)$ a **field** if it is tangent to $\mathcal{M}$ in each point, that is

$$A(x) \text{ is a field} \Leftrightarrow A(x) = P_{I_n(x)}(A(x)) \quad \forall x \in \mathcal{M} \quad . \tag{847}$$

All fields on $\mathcal{M}$ give an algebra, since for two fields $A(x)$ and $B(x)$

$$A(x) = P_{I_n(x)}(A(x)), B(x) = P_{I_n(Cx)}(B(x)) \Rightarrow \begin{cases} A(x) + B(x) = P_{I_n(x)}(A(x) + B(x)) \\ A(x)B(x) = P_{I_n(x)}(A(x)B(x)) \end{cases} \quad . \tag{848}$$

We will call this the **tangent algebra** $\mathcal{G}(\mathcal{M})$.

The projection of a multivector A into the tangent algebra $\mathcal{G}(\mathcal{T}_n)$ is given by

$$P_{I_n(x)}(A) = I_n^{-1}(x)\,[I_n(x) \cdot A] \tag{849}$$

and becomes so dependent on x.

Keeping in mind that the pseudoscalar is position dependent and defines a n-dimensional linear space, we abbreviate

$$I \stackrel{\text{def}}{=} I_n(x) \quad ; \quad P(M) \stackrel{\text{def}}{=} P_I(M) \quad . \tag{850}$$

The covariant Derivative We have to extend the definition of the directional derivative, since we do not have a linear space. This is easily done for a tangent vector $a = \frac{dx(\tau)}{d\tau}$ by

$$a \cdot \partial_x F \stackrel{\text{def}}{=} \lim_{\tau \to 0} \frac{F(x(\tau)) - F(x(0))}{\tau} \quad . \tag{851}$$

We can use this to define the differential operator ∂_x by

$$\partial_x \stackrel{\text{def}}{=} \sum_\mu e^\mu e_\mu \cdot \partial_x \quad , \tag{852}$$

where $e_\mu = e_\mu(x)$ gives a basis in $\mathcal{T}_n(x)$ with reciprocal frame e^μ. Since ∂_x is constructed as a linear combination of the basis vectors of $\mathcal{T}_n(x)$ we have

$$\partial_x = P(\partial_x) \quad . \tag{853}$$

Integrability Condition The integrability condition becomes thus[87]

$$P(\partial_x) \wedge P(\partial_x) = P(\partial_x \wedge \partial_x) = 0 \quad , \tag{854}$$

which is with $a, b \in \mathcal{T}_n(x) \Leftrightarrow P(a) = a, P(b) = b$ equivalent to

$$\begin{aligned} (b \wedge a) \cdot P(\partial_x \wedge \partial_x) &= P(b \wedge a) \cdot (\partial_x \wedge \partial_x) = (b \wedge a) \cdot (\partial_x \wedge \partial_x) \\ &= a \cdot \partial_x b \cdot \partial_x - b \cdot \partial_x a \cdot \partial_x + a \cdot \dot{\partial}_x \dot{b} \cdot \partial_x - b \cdot \dot{\partial}_x \dot{a} \cdot \partial_x \\ &= [a \cdot \partial_x, b \cdot \partial_x] - [a, b]_{\text{Lie}} \cdot \partial_x \quad . \end{aligned} \tag{855}$$

Differential of the Projection Operator The identity function on a manifold

$$x = (x) \tag{856}$$

has as its first differential the projection operator

$$(x)_a = a \cdot \dot{\partial}_x \dot{x} \overset{(853)}{=} a \cdot P(\dot{\partial}_x)\dot{x} = P(a) \cdot \dot{\partial}_x \dot{x} = P(a) = (a \cdot I)I^{-1} \quad . \tag{857}$$

The projection operator can be seen as a tensor field. Therefore it has as the first differential[88]

$$\underline{P}_b(a) \overset{\text{def}}{=} b \cdot \dot{\partial}_x \dot{P}(a) \overset{(289)}{=} b \cdot \partial_x P(a) - P(b \cdot \dot{\partial}_x \dot{a}) \quad , \tag{858}$$

This becomes clearer by the following consideration

$$\begin{aligned} a \cdot \partial_x P(A(x)) &= a \cdot \partial_x \left[(A(x) \cdot I)I^{-1}\right] \\ &= a \cdot \dot{\partial}_x (\dot{A}(x) \cdot I)I^{-1} + a \cdot \dot{\partial}_x (A(x) \cdot \dot{I})I^{-1} + a \cdot \dot{\partial}_x (A(x) \cdot I)\dot{I}^{-1} \\ &= P(a \cdot \dot{\partial}_x \dot{A}(x)) + a \cdot \dot{\partial}_x \dot{P}(A(x)) \quad . \end{aligned} \tag{859}$$

Taking the first term to the left leads to (858). Thus the vector derivative of the projection tensor becomes

$$\partial_x P(A(x)) = \partial_b b \cdot \partial_x P(A(X)) \overset{(859)}{=} \partial_b \underline{P}_b(A(x)) + \partial_b P(b \cdot \partial_x A(x)) \quad . \tag{860}$$

The *linearity and grade preserving property, i.e.*,

$$\underline{P}_b(\alpha A + \beta B) = \alpha \underline{P}_b(A) + \beta \underline{P}_b(B) \quad ; \quad \underline{P}_b(\langle M \rangle_r) = \langle \underline{P}_b(\langle M \rangle_r) \rangle_r \quad , \tag{861}$$

follows directly from the properties of $P(a)$, since the differential preserves this. That scalars cannot have a part orthogonal to $\mathcal{T}_n(x)$ is expressed by $P(\langle B \rangle) = \langle B \rangle$ and leads for $B = \langle B \rangle$ constant on $\mathcal{M}$ as expected to

$$\underline{P}_b(\langle B \rangle) \overset{(858)}{=} b \cdot \partial_x P(\langle B \rangle) - P(\underbrace{b \cdot \dot{\partial}_x \langle \dot{B} \rangle}_{=0}) = b \cdot \partial_x \langle B \rangle - b \cdot \partial_x \langle B \rangle = 0 \quad . \tag{862}$$

From (264) we know that $P(A)$ is a symmetric function. Hence by differentiating $A * P(B) = P(A) * B$ for constant multivectors A, B we derive

$$A * \underline{P}_b(B) = \underline{P}_b(A) * B \tag{863}$$

and (162) yields together with the product rule

$$\underline{P}_b(A \wedge B) = [P(A) \wedge P(B)]_b = \underline{P}_b(A) \wedge P(B) + P(A) \wedge \underline{P}_b(B) \quad . \tag{864}$$

Further gives $P(P(A)) = P(A)$ the relation[89]

$$\underline{P}_b(P(A)) + P(\underline{P}_b(A)) = \underline{P}_b(A) \quad . \tag{865}$$

[87] We abbreviate here $P(M) \overset{\text{def}}{=} P_{I(x)}(M)$.

[88] Here the index does *not* indicate the pseudoscalar of the subspace, onto which $P(A)$ projects. It only *indicates here the differential of the projection operator*. This slightly misleading notation was introduced in [1].

[89] Remember that $P(A)$ is constructed out of multiplications with the pseudoscalar. Thus the product rule applies to $P(P(A))$.

Since ∂_x behaves like a vector in $\mathcal{T}_n$, *i.e.*, , $P(\partial_x) = \partial_x$, we find

$$\underline{P}_b(A) = b\cdot\partial_x P(A) \overset{(853)}{=} b\cdot P(\partial_x)P(A) = P(b)\cdot\partial_x P(A) = \underline{P}_{P(b)}(A) \quad . \tag{866}$$

Defining for an arbitrary multivector A

$$A_\| \overset{\text{def}}{=} P(A) \quad ; \quad A_\perp \overset{\text{def}}{=} P^\perp(A) = A - A_\| \Rightarrow P(A_\|) = A_\|, P(A_\perp) = 0 \tag{867}$$

shows

$$\underline{P}_b(A_\|) \overset{(865)}{=} \underline{P}_b(\overbrace{P(A_\|)}^{=A_\|}) + P(\underline{P}_b(A_\|)) = \underline{P}_b(A_\|) + P(\underline{P}_b(A_\|)) \quad , \tag{868}$$

so that after subtracting $\underline{P}_b(A)$ on both sides

$$P(\underline{P}_b(A_\|)) = 0 \quad . \tag{869}$$

On the other side we obtain

$$\underline{P}_b(A_\perp) \overset{(865)}{=} \underline{P}_b(\underbrace{P(A_\perp)}_{=0}) + P(\underline{P}_b(A_\perp)) = P(\underline{P}_b(A_\perp)) \quad . \tag{870}$$

In words: $\underline{P}_b(A_\|)$ is always orthogonal and $\underline{P}_b(A_\perp)$ parallel to the manifold $\mathcal{M}$. Using the property (865) this yields

$$\left.\begin{aligned}
\underline{P}_b(A_\| \wedge B_\|) &= \underline{P}_b(A_\|)\wedge P(B_\|) + P(A_\|)\wedge\underline{P}_b(B_\|) \\
&= \underline{P}_b(A_\|)\wedge B_\| + A_\| \wedge \underline{P}_b(B_\|) \\
\underline{P}_b(A_\perp \wedge B_\perp) &= \underline{P}_b(A_\perp)\wedge \overbrace{P(B_\perp)}^{=0} + \overbrace{P(A_\perp)}^{=0}\wedge\underline{P}_b(B_\perp) = 0 \\
\underline{P}_b(A_\perp \wedge B_\|) &= \underline{P}_b(A_\perp)\wedge P(B_\|) + P(A_\perp)\wedge\underline{P}_b(B_\|) \\
&= \underline{P}_b(A_\perp)\wedge B_\|
\end{aligned}\right\} . \tag{871}$$

8.1.3 The Shape Operator

We define now the **shape operator** by

$$S(A) \overset{\text{def}}{=} \dot\partial_x \dot P(A) = \partial_b \underline{P}_b(A) \quad . \tag{872}$$

For any multivector A of the tangent algebra $\mathcal{G}(\mathcal{T}_n)$ $\underline{P}_b(A)$ is orthogonal to the tangent space $\mathcal{T}_n$, as was seen by (869). Therefore we obtain

$$\partial_b \cdot \underline{P}_b(A) = P(\partial_b)\cdot\underline{P}_b(A) = \partial_b \cdot P(\underline{P}_b(A)) \overset{(869)}{=} 0 \quad . \tag{873}$$

We conclude that for any multivector *field* M_r of grade r the shape operator is of grade $r+1$, *i.e.*,

$$S(M_r) = \langle S(M_r)\rangle_{r+1} \quad . \tag{874}$$

For any *field* $A(x) = P(A(x))$ we have

$$\partial_x A(x) \overset{(847)}{=} \partial_x P(A(x)) = \dot\partial_x \dot P(A(x)) + \dot\partial_x P(\dot A(x)) \overset{(872)}{=} S(A(x)) + P(\dot\partial_x \dot A(x)) \tag{875}$$

and thus

$$S(A(x)) = \partial_x A(x) - P(\dot\partial_x \dot A(x)) = P^\perp(\partial_x A(x)) \quad . \tag{876}$$

This means $S(A(x))$ is *always orthogonal to the tangent space* $\mathcal{T}_n$, *i.e.*,

$$P(S(A(x))) = 0, \tag{877}$$

and hence contains information about the geometry of the *embedding space*. The shape operator gives so all information of the derivative which are orthogonal to the manifold and which can thus not be handled in other approaches. For example, in tensor calculus the derivative of the totally antisymmetric tensor η^{abcd}, which generates duality transformations, is zero. In the space-time algebra approach the pseudoscalar I overtakes the role of η^{abcd}, but $a\cdot\partial_x I \neq 0$! This is no contradiction, since the projection onto the tangent space still vanishes, *i.e.*,

$$P(a\cdot\dot\partial_x \dot I) = 0 \quad . \tag{878}$$

The Derivative of the Pseudoscalar Taking the directional derivative of $I^2 = I \cdot I = I * I = \pm 1$ gives the identity

$$0 = (a \cdot \partial_x I) * I + I * (a \cdot \partial_x I) = 2(a \cdot \partial_x I) \cdot I \quad . \tag{879}$$

Therefore, after multiplying with I^{-1}

$$0 = P(a \cdot \partial_x I) = a \cdot \dot{\partial}_x P(\dot{I}) \tag{880}$$

and consequently

$$\partial_x I = \partial_x P(I) = \dot{\partial}_x \dot{P}(I) + \underbrace{\dot{\partial}_x P(\dot{I})}_{=0} \overset{(872)}{=} S(I) \quad . \tag{881}$$

Hence the vector derivative of the pseudoscalar is just the shape operator.

8.1.4 Generalized Duality of Inner and Outer Derivative

Let us now consider a multivector field B_s of grade[90] $s \geq 1$. The vector derivative of the inner product with the pseudoscalar becomes for a n-dimensional tangent space

$$\begin{aligned} \partial_x (I_n \cdot B_s) &= \partial_x (I_n B_s) = \dot{\partial}_x \dot{I}_n B_s + \dot{\partial}_x I_n \dot{B}_s \\ &\overset{(881)}{=} S(I_n) B_s + (-1)^{n-1} I_n \dot{\partial}_x \dot{B}_s \quad . \end{aligned} \tag{882}$$

This is equivalent to the two equations

$$\left. \begin{aligned} \partial_x \cdot (I_n \cdot B_s) &= \langle \partial_x I_n B_s \rangle_{n-s-1} = \overbrace{\langle S(I_n) \cdot B_s \rangle_{n-s-1}}^{\overset{(874)}{=} 0} - (-1)^n I_n \cdot (\dot{\partial}_x \wedge \dot{B}_s) \\ \partial_x \wedge (I_n \cdot B_s) &= \langle \partial_x (I_n B_s) \rangle_{n-s+1} = \underbrace{\langle S(I_n) B_s \rangle_{n-s+1}}_{= S(I_n) \cdot B_s} - (-1)^n I_n \cdot (\dot{\partial}_x \cdot \dot{B}_s) \end{aligned} \right\} . \tag{883}$$

For a scalar field φ we find further

$$\partial_x (I_n \varphi) = S(I_n) \varphi - (-1)^n I_n (\partial_x \varphi) \quad . \tag{884}$$

The projections of (883) and (884) onto the algebra of the tangent space $\mathcal{G}(\mathcal{T}_n)$ removes all terms containing the shape operator.

8.2 Tetrad Basis, Fiducial Tensor and Metric

8.2.1 Coordinate and Tetrad Basis

Coordinate Basis We call the complete set of vectors e_μ a **coordinate basis** of $\mathcal{T}_n$, if for an associated set of coordinates x^μ the basis vectors can be written as

$$e_\mu = \frac{\partial x}{\partial x^\mu} \quad . \tag{885}$$

That means[91]

$$e_\mu \cdot \partial_x = \frac{\partial x}{\partial x^\mu} \cdot \frac{\partial}{\partial x} = \frac{\partial}{\partial x^\mu} \tag{887}$$

and leads in (852) to

$$\partial_x \overset{(852)}{=} e^\mu e_\mu \cdot \partial_x \overset{(887)}{=} e^\mu \partial_\mu \quad ; \quad \partial_\mu \overset{\text{def}}{=} \frac{\partial}{\partial x^\mu} \quad . \tag{888}$$

[90] The derivation holds for $s = 1$, *i.e.*, for a vector field B_1, since $I_n \wedge B_1 = 0$.

[91] Since vectors have an inverse in our algebra we can understand (887) as follows. ∂_x is an infinitesimal vector. $\frac{1}{\partial_x} = (\partial_x)^{-1}$ is the associated inverse, which is parallel to ∂_x. ∂ and ∂x^μ behave like scalars. Then

$$\left(\frac{\partial x}{\partial x^\mu} \right) \cdot \left(\frac{\partial}{\partial x} \right) = \frac{1}{\partial x^\mu} \underbrace{(\partial x) \cdot (\partial x)^{-1}}_{=1} \partial = \frac{\partial}{\partial x^\mu} \quad . \tag{886}$$

Tetrad Basis A tetrad $\{\gamma_\mu\}$ is a *complete set of orthonormal vectors*, defined in the tangent space $\mathcal{T}_n$ of each point of the manifold $\mathcal{M}$. We demand that the tetrad is differentiable on $\mathcal{M}$, *i.e.*, the directional derivative of each tetrad vector is well defined in each point. In general there are *no* coordinates, so that the γ_μ could be expressed by (885) and thus the tetrad is not a coordinate basis.

The orthonormality condition is expressed by

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \pm\delta_{\mu\nu} \quad , \tag{889}$$

where the $\eta_{\mu\nu}$ define the signature of the coordinates by $\gamma_\mu^2 = \eta_{\mu\mu} = \pm 1$. That

$$\gamma^\mu \stackrel{\text{def}}{=} \eta^{\mu\nu}\gamma_\nu \quad ; \quad \eta^{\mu\nu} \stackrel{\text{def}}{=} \gamma^\mu \cdot \gamma^\nu \tag{890}$$

defines the reciprocal tetrad is seen by

$$\gamma_\mu \cdot (\eta^{\nu\kappa}\gamma_\kappa) \stackrel{(890)}{=} (\gamma^\nu \cdot \gamma^\kappa)(\gamma_\kappa \cdot \gamma_\mu) = \gamma^\nu \cdot [\gamma^\kappa(\gamma_\kappa \cdot \gamma_\mu)] = \gamma^\nu \cdot \gamma_\mu \quad . \tag{891}$$

Relation between Tetrad and Coordinate Basis A tetrad $\{\gamma_\nu\}$ can be written as a linear combination of the $\{e_\mu\}$ or the other way around, *i.e.*,

$$e_\mu = h_\mu{}^\nu\gamma_\nu \quad ; \quad \gamma_\nu = h^\mu{}_\nu e_\mu \quad ; \quad h_\mu{}^\nu h^\mu{}_\nu = n \quad . \tag{892}$$

Not referring to coordinates this becomes

$$e_\mu = h(\gamma_\mu) \quad ; \quad \gamma_\mu = h^{-1}(e_\mu) \tag{893}$$

with a linear function h, which we call the **Fiducial Tensor**. Here the $h_\mu{}^\nu$, and so h, are still dependent on $x \in \mathcal{M}$, so it is a *tensor field* over $\mathcal{M}$. In each point it defines on the manifold $\mathcal{M}$ a linear mapping of the tetrad to the coordinate basis. The adjoint function is then given by[92]

$$\gamma^\lambda = \bar{h}(e^\lambda) = \bar{h}^\lambda{}_\nu e^\nu \quad ; \quad e^\nu = \bar{h}^{-1}(\gamma^\nu) = \bar{h}_\lambda{}^\nu\gamma^\lambda \quad . \tag{895}$$

Multiplying (893) with γ^λ and (895) with e_μ gives the matrix elements

$$h(\gamma_\mu) \cdot \gamma^\lambda = h_\mu{}^\lambda = e_\mu \cdot \gamma^\lambda = e_\mu \cdot \bar{h}(e^\lambda) = \bar{h}^\lambda{}_\mu \quad . \tag{896}$$

Thus h is symmetric in the sense[93]

$$h_\mu{}^\lambda = \bar{h}^\lambda{}_\mu \quad ; \quad \bar{h}_\lambda{}^\nu = h^\nu{}_\lambda \quad , \tag{898}$$

but in general we have still

$$\bar{h}(a) \neq h(a) \quad , \tag{899}$$

since the components of h are with respect to the tetrad $\{\gamma_\mu\}$, while the components of $\bar{h}$ are with respect to the coordinate basis $\{e_\mu\}$. We can find a relation between the $\bar{h}$ and h^{-1}, since

$$\gamma^\nu = \bar{h}(e^\nu) = \bar{h}(g(e_\nu)) = \bar{h}(g^{\nu\lambda}e_\lambda) = g^{\nu\lambda}\bar{h}(e_\lambda) \tag{900}$$

and thus after multiplying by $\eta_{\mu\nu}$

$$\gamma_\mu = \eta_{\mu\nu}g^{\nu\lambda}\bar{h}(e_\lambda) \quad . \tag{901}$$

Therefore, we thus identify

$$h^{-1}(e_\mu) = \eta_{\mu\nu}g^{\nu\lambda}\bar{h}(e_\lambda) \quad . \tag{902}$$

[92] Multiplying (893) with e^λ yields

$$\delta_\mu^\lambda = e_\mu \cdot e^\lambda \stackrel{(893)}{=} h(\gamma_\mu) \cdot e^\lambda = \gamma_\mu \cdot \bar{h}(e^\lambda) \Leftrightarrow \bar{h}(e^\lambda) = \gamma^\lambda \quad . \tag{894}$$

[93] The second relation is derived in the same manner with

$$\bar{h}_\lambda{}^\nu = e^\nu \cdot \gamma_\lambda = e^\nu \cdot \bar{h}^{-1}(e_\lambda) = e^\nu \cdot e_\mu h^\mu{}_\lambda = h^\nu{}_\lambda \quad . \tag{897}$$

8.2.2 The Determinant of the Fiducial Tensor

Given two tetrads by the mappings

$$\gamma_\mu = h^{-1}(e_\mu) \quad ; \quad \gamma'_\mu = h'^{-1}(e_\mu) \quad , \tag{903}$$

we know that the mapping f between them

$$\gamma'_\mu \overset{\text{def}}{=} f(\gamma_\mu) \tag{904}$$

must *preserve orthonormality*. Thus $\det f = \pm 1$. If both tetrads are of the same handedness finally[94]

$$\det f = +1 \quad . \tag{906}$$

Decomposing h' as

$$h'^{-1}(e_\mu) = \gamma'_\mu = f(\gamma_\mu) = f(h^{-1}(e_\mu)) \tag{907}$$

shows with

$$\det h' = \underbrace{\det f^{-1}}_{=1} \det h = \det h \quad , \tag{908}$$

that the determinant does not depend in any way on the chosen tetrad, but only on the chosen coordinate basis!

8.2.3 The Metric of the Coordinate Basis

We can now write the metric in terms of h as follows

$$g_{\mu\nu} \overset{\text{def}}{=} e_\mu \cdot e_\nu = h(\gamma_\mu) \cdot h(\gamma_\nu) = \gamma_\mu \cdot \bar{h}(h(\gamma_\nu)) \quad . \tag{909}$$

By writing

$$e_\mu g^{\mu\nu} a_\nu \overset{(909)}{=} e_\mu (e^\mu \cdot e^\nu) a_\nu = e_\mu (e^\mu \cdot a) = e_\mu a^\mu = a \tag{910}$$

we see that the metric lowers and raises indices of vector components. Applied to the coordinate basis vectors we find

$$e_\nu g^{\nu\mu} = e_\nu (e^\nu \cdot e^\mu) = e^\mu \quad . \tag{911}$$

Thus the metric transforms the basis vectors to their reciprocal ones. We can thus see the metric as a linear function defined by

$$g(e_\mu) = e^\mu \quad . \tag{912}$$

The Determinant of the metric is given by

$$|g| = (e_n \wedge \cdots \wedge e_1) \cdot (e_1 \wedge \cdots \wedge e_n) = \widetilde{h(I)} \cdot h(I) = \left[|h|\tilde{I}\right] \cdot [|h| I] = |h|^2 I\tilde{I} \quad . \tag{913}$$

A product of coordinate basis vectors becomes

$$e_\mu e^\nu = \delta_\mu^\nu + e_\mu \wedge e^\nu \quad , \tag{914}$$

where the second term is always non-zero for $\mu \neq \nu$. The magnitude of the bivector part is

$$|e_\mu \wedge e^\nu| = |e_\mu||e^\nu| = |h(\gamma_\mu)||\bar{h}^{-1}(\gamma^\nu)| \quad . \tag{915}$$

[94] This relations are easy to prove with the help of our algebraic tools. Since the pseudoscalar I is unique, it can be expressed in both tetrads in the same way. If the tetrads would differ in handedness we would just get an additional minus sign. If both tetrads have the same handedness

$$\det f = I^{-1} f(I) = (\gamma'_1 \wedge \cdots \wedge \gamma'_n)^{-1} * f(\gamma_1 \wedge \cdots \wedge \gamma_n) = (\gamma'_1 \wedge \cdots \wedge \gamma'_n)^{-1} * (\gamma'_1 \wedge \cdots \wedge \gamma'_n) = 1 \quad . \tag{905}$$

8.2.4 The covariant Derivative in Tetrad Components

We are now in the position to express the covariant vector derivative in tetrad components. Using (895) gives

$$\partial_x \overset{(852)}{=} e^\mu \partial_\mu \overset{(895)}{=} \bar{h}^{-1}(\gamma^\nu)\partial_\mu \overset{(898)}{=} \gamma^\nu \left[\bar{h}_\nu{}^\mu \partial_\mu\right] = \gamma^\nu h^\mu{}_\nu \partial_\mu = \gamma^\nu \gamma_\nu \cdot \partial_x = \gamma^\nu \partial_{\gamma_\nu} \quad , \tag{916}$$

where

$$\partial_{\gamma_\nu} \overset{\text{def}}{=} \gamma_\nu \cdot \partial_x = h^\mu{}_\nu \partial_\mu = h^\mu{}_\nu \frac{\partial}{\partial x^\mu} \quad . \tag{917}$$

It is now important to note that the ∂_{γ_ν} do not necessarily commute with each other, since the $h^\mu{}_\nu$ depend on the coordinates.

We can also see this result as follows. Working in an orthonormal basis we have to modify the derivative in order to take care of the curvature and thus non-integrability of our coordinates. This is done by replacing the ordinary derivative ∂_x by a new one

$$h^{-1}(\partial_x) \quad . \tag{918}$$

The so introduced linear function h determines the curvature of our space-time. This should be compared to what happens in the Gravity Gauge Theory, which is discussed in Section 9.

8.3 The Curvature Tensor

8.3.1 Tetrad Frames in Flat Space

Our tetrad frame should be orthonormal everywhere. Thus with the directional derivative operator[95]

$$\bar{\delta}_a M(x) \overset{\text{def}}{=} P(a \cdot \partial_x M(x)) \tag{919}$$

we have for any vector $a \in \mathcal{T}_n$

$$\gamma_\nu \cdot \gamma_\mu = \pm\delta_{\nu\mu} \quad \Rightarrow \quad 0 = (\bar{\delta}_a \gamma_\nu) \cdot \gamma_\mu + \gamma_\nu \cdot (\bar{\delta}_a \gamma_\mu) \quad . \tag{920}$$

We can express the vector $\bar{\delta}_a \gamma_\nu$ as a linear combination of our tetrad vectors by

$$\bar{\delta}_a \gamma_\nu = \sum_\mu f_{\nu\mu} \gamma^\mu \quad . \tag{921}$$

Substituting this in (920) yields

$$f_{\nu\mu} = -f_{\mu\nu} \quad . \tag{922}$$

Thus[96]

$$\bar{\delta}_a \gamma_\nu \overset{(921)}{=} \sum_\mu f_{\nu\mu}\gamma^\mu = \sum_\mu f_{\nu\mu}\gamma^\mu \underbrace{\gamma^\nu \gamma_\nu}_{=1} = \sum_\mu f_{\nu\mu}(\gamma^\mu \wedge \gamma^\nu)\cdot\gamma_\nu = \left[\sum_{\lambda,\mu} \frac{1}{2} f_{\nu\mu} \gamma^\mu \wedge \gamma^\lambda\right] \cdot \gamma_\nu \tag{923}$$

and we write

$$\bar{\delta}_a \gamma_\nu = \omega_a \cdot \gamma_\nu = \omega_a \times \gamma_\nu \tag{924}$$

with a bivector $\omega_a \overset{\text{def}}{=} \omega(a)$. Hence we describe the movement of our tetrad by going in direction a by

$$\bar{\delta}_a \gamma_\nu = P(a \cdot \partial_x \gamma_\nu) = \omega_a \cdot \gamma_\nu = \omega_a \times \gamma_\nu \quad . \tag{925}$$

The projection of the covariant vector derivative into the tangent space is thus

$$\delta_x \overset{\text{def}}{=} \gamma^\nu \frac{\partial}{\partial a^\nu} \bar{\delta}_{a^\mu \gamma_\mu} = \gamma_\nu \bar{\delta}_{\gamma^\nu} \quad . \tag{926}$$

[95] To distinguish the directional derivative from the later introduced covariant vector derivative δ_x, we employ the convention that directional derivatives have a bar like $\bar{\delta}_a$ or $\bar{d}_a$.

[96] No summation convention in the following calculation. The last step holds, since $(\gamma^\mu \wedge \gamma^\lambda)\cdot\gamma_\nu = 0$ for $\mu, \lambda \neq \nu$.

This yields for the basis of bivectors

$$\begin{aligned}\bar{\delta}_a[\gamma_\nu\wedge\gamma_\mu] &\overset{(919)}{=} P(a\cdot\partial_x[\gamma_\nu\wedge\gamma_\mu]) = P((a\cdot\partial_x\gamma_\nu)\wedge\gamma_\mu+\gamma_\nu\wedge(a\cdot\partial\gamma_\mu))\\ &= (\omega_a\times\gamma_\nu)\wedge\gamma_\mu+\gamma_\nu\wedge(\omega_a\times\gamma_\nu)\\ &= \omega_a\times(\gamma_\nu\wedge\gamma_\mu)\end{aligned} \tag{927}$$

and extends in the same way to the basis of any grade. Therefore we can conclude that for *any* multivector B

$$\bar{\delta}_a B = \bar{d}_a B + \omega_a \times B \quad , \tag{928}$$

where $\bar{d}_a$ is the **differential of our tetrad components**, for which we define

$$\bar{d}_a\gamma_\nu \overset{\text{def}}{=} 0 \quad . \tag{929}$$

The associated vector derivative is

$$d_x \overset{\text{def}}{=} \gamma_\nu\bar{d}_{\gamma^\nu} \quad ; \quad d_x\gamma_\mu \overset{\text{def}}{=} 0 \quad . \tag{930}$$

Since scalars commute with all elements of the algebra, we obtain for a scalar field φ directly $\omega_a\times\varphi = 0$ and so

$$\bar{\delta}_a\varphi \overset{(928)}{=} \bar{d}_a\varphi \quad . \tag{931}$$

8.3.2 Extension to Curved Space-Time

Now we just drop the condition of a flat space-time and allow a curved manifold to be the physical arena. The tetrad frame is now an orthonormal frame defined in the tangent space $\mathcal{T}(x)$ of each point of the manifold. Above considerations are then still valid and the covariant derivative of a multivector, given in terms of the tetrad, is

$$\bar{\delta}_a A(x) = \bar{d}_a A(x) + \omega_a \times A(x) \quad . \tag{932}$$

Functions of the covariant derivative Up to this point an expression like $\dot{f}(\dot{\delta}_x)$ is undefined, since it is not clear how the $\omega\times$-term acts on f. But with

$$\dot{F}(\dot{\delta}_x) \overset{\text{def}}{=} a\cdot\dot{\delta}_x\dot{F}(\partial_a) = \dot{\bar{\delta}}_a\dot{F}(\partial_a) = P(a\cdot\dot{\partial}_x\dot{F}(P(\partial_a))) \tag{933}$$

we can employ a useful definition, which will allow us to keep results in a compact form.

It should be pointed out here that covariant derivatives do in general not commute, *i.e.*,

$$\dot{\bar{\delta}}_{\gamma_\mu}\dot{\bar{\delta}}_{\gamma_\nu} = [\omega_{\gamma_\mu}\cdot\gamma_\nu]\cdot\delta_x = \omega_{\gamma_\mu}\cdot(\gamma_\nu\wedge\delta_x) \neq \dot{\bar{\delta}}_{\gamma_\nu}\dot{\bar{\delta}}_{\gamma_\mu} \quad . \tag{934}$$

The commutator of two covariant directional derivatives becomes

$$[\bar{\delta}_{\gamma_\mu},\bar{\delta}_{\gamma_\nu}] = \bar{\delta}_{\gamma_\mu}\bar{\delta}_{\gamma_\nu} - \bar{\delta}_{\gamma_\nu}\bar{\delta}_{\gamma_\mu} \overset{(934)}{=} [\omega_{\gamma_\mu}\cdot\gamma_\nu - \omega_{\gamma_\nu}\cdot\gamma_\mu]\cdot\delta_x \quad . \tag{935}$$

If the $\{\gamma_\mu\}$ give a *coordinate* basis, *e.g.*, in a flat space,

$$\dot{\bar{\delta}}_{\gamma_\mu}\dot{\bar{\delta}}_{\gamma_\nu} - \dot{\bar{\delta}}_{\gamma_\nu}\dot{\bar{\delta}}_{\gamma_\mu} = 0 \tag{936}$$

and hence $\omega_{\gamma_\mu}\cdot\gamma_\nu = \omega_{\gamma_\nu}\cdot\gamma_\mu$.

Contractions Since the covariant derivative includes curvature effects, we cannot use it to formulate contractions. Instead we have to use the tetrad derivative operator as defined with (929). As an example the contraction of $M(a\wedge b)$ can now be written as

$$d_b\cdot M(a\wedge b) \quad . \tag{937}$$

Relation to the Ricci Rotation Coefficients In standard general relativity the Ricci-Rotation-Coefficient $\Gamma^a{}_{bc}$ are defined as the a-component of the derivative of the basis vector γ_c in direction of γ_b for an orthonormal frame $\{\gamma_\nu\}$. Thus

$$\Gamma_{\alpha\beta\delta} = \gamma_\alpha \cdot \left[\bar{\delta}_{\gamma_\beta}\gamma_\delta\right] = \gamma_\alpha \cdot \left[\omega_{\gamma_\beta} \cdot \gamma_\delta\right] = \omega_{\gamma_\beta} \cdot (\gamma_\delta \wedge \gamma_\alpha) \tag{938}$$

and we define the compact notation

$$\Gamma(\gamma_\mu, \gamma_\nu) \overset{\text{def}}{=} \bar{\delta}_{\gamma_\mu}\gamma_\nu = \omega_{\gamma_\mu} \cdot \gamma_\nu \quad . \tag{939}$$

Here the antisymmetry in the first and last index becomes explicit. Developing ω_a in the basis of bivectors gives

$$\omega_a = \frac{1}{2}\omega_{a^\mu\gamma_\mu} \cdot (\gamma_\nu \wedge \gamma_\lambda)(\gamma^\nu \wedge \gamma^\lambda) \overset{(938)}{=} \underbrace{\frac{1}{2}a^\mu \Gamma_{\lambda\mu\nu}}_{(\omega_a)_{\nu\lambda}}(\gamma^\nu \wedge \gamma^\lambda) \quad . \tag{940}$$

Given a coordinate basis $\{e_\nu\}$ we find with

$$d_a \Gamma(a, e_\nu) \overset{(939)}{=} d_a\left(\bar{\delta}_a e_\nu\right) \overset{(926)}{=} \delta_x e_\nu \overset{(893)}{=} \delta_x h(\gamma_\nu) \tag{941}$$

a relation to the fiducial tensor.

The Commutation Coefficients The Commutation Coefficients for any basis $\{e_\mu\}$ are defined by[97]

$$\gamma^\mu{}_{\nu\lambda} \overset{\text{def}}{=} e^\mu \cdot [e_\nu, e_\lambda]_{\text{Lie}} = e^\mu \cdot (\bar{\delta}_{e_\nu} e_\lambda - \bar{\delta}_{e_\lambda} e_\nu) = \Gamma^\mu{}_{\nu\lambda} - \Gamma^\mu{}_{\lambda\nu} \quad . \tag{942}$$

We write this in the form

$$\gamma(e_\nu, e_\lambda) = \gamma(e_\nu \wedge e_\lambda) \overset{\text{def}}{=} \gamma^\mu{}_{\nu\lambda} e_\mu = \bar{\delta}_{e_\nu} e_\lambda - \bar{\delta}_{e_\lambda} e_\nu \quad , \tag{943}$$

where we used the antisymmetry in the arguments. Note that $\gamma(e_\nu \wedge e_\lambda)$ is *not* a linear function of $e_\nu \wedge e_\lambda$! If the $\{e_\mu\}$ are a *coordinate basis, i.e.*, $e_\mu = \partial_\mu x$, we find

$$\gamma(e_\mu \wedge e_\nu) = \partial_\mu \partial_\nu x - \partial_\nu \partial_\mu x = 0 \quad . \tag{944}$$

8.3.3 Killing Vectors

We call k a **Killing Vector**, if[98]

$$\gamma_\nu \cdot \left[\bar{\delta}_{\gamma_\mu} k\right] = -\gamma_\mu \cdot \left[\bar{\delta}_{\gamma_\nu} k\right] \Leftrightarrow (a \cdot \dot{\delta}_x)(b \cdot k) = -(b \cdot \dot{\delta}_x)(a \cdot k) \quad . \tag{946}$$

Setting $b \overset{\text{def}}{=} d_a$ yields[99]

$$0 = (d_a \cdot \dot{\delta}_x)(a \cdot \dot{k}) = \delta_x \cdot k \quad . \tag{947}$$

By multiplying (946) with γ^ν we obtain

$$\bar{\delta}_{\gamma_\mu} k = -\gamma^\nu \gamma_\mu \cdot \left[\dot{\bar{\delta}}_{\gamma_\nu} \dot{k}\right] = -\dot{\delta}_x \gamma_\mu \cdot \dot{k} \quad . \tag{948}$$

Multiplying with γ^μ from the right gives[100]

$$\dot{k}\dot{\delta}_x = -\dot{\delta}_x \dot{k} \quad \Leftrightarrow \quad \delta_x \cdot k = 0 \quad \Leftrightarrow \quad \delta_x k(x) = \delta_x \wedge k(x). \tag{950}$$

[97] For compatibility we will call the commutation coefficients γ and show the distinction to the tetrad basis vectors γ_ν by the bivector argument.

[98] This is equivalent to the conventional

$$k_{\mu;\nu} = -k_{\nu;\mu} \quad . \tag{945}$$

[99] Remember that we can exchange d_a and a.

[100] This result can also be derived in the following way:

$$\delta_x \cdot k = \left[\gamma_\mu \bar{\delta}_{\gamma^\mu}\right] \cdot k \overset{(946)}{=} -\left[\gamma_\mu \bar{\delta}_{\gamma^\mu}\right] \cdot k \Rightarrow \delta_x \cdot k = 0 \quad . \tag{949}$$

From (948) we derive further

$$\bar{\delta}_{\gamma_\mu} k = -\dot{\delta}_x \gamma_\mu \cdot \dot{k} = \gamma_\mu \cdot [\delta_x \wedge k] - \bar{\delta}_{\gamma_\mu} k \tag{951}$$

and thus

$$\bar{\delta}_{\gamma_\mu} k \overset{(951)}{=} \frac{1}{2} \gamma_\mu \cdot [\delta_x \wedge k] \overset{(950)}{=} \frac{1}{2} \gamma_\mu \cdot [\delta_x k] \quad . \tag{952}$$

With (950) we found that the vector derivative of a Killing vector is always a bivector.

8.3.4 Curvature

Curvature The Jacobi identity

$$(\omega_b \times \omega_a) \times B + (B \times \omega_b) \times \omega_a + (\omega_a \times B) \times \omega_b = 0 \tag{953}$$

gives

$$\omega_b \times (\omega_a \times B) - \omega_a \times (\omega_b \times B) = (\omega_b \times \omega_a) \times B \quad . \tag{954}$$

We use this result to derive the commutator of two directional derivatives and find for two vectors a, b obeying[101] $\bar{\delta}_a b = \bar{\delta}_b a$ the result[102]

$$\begin{aligned}
[\bar{\delta}_a, \bar{\delta}_b] B &= \bar{\delta}_a \bar{\delta}_b B - \bar{\delta}_b \bar{\delta}_a B = \bar{\delta}_a \left(\bar{d}_b B + \omega_b \times B \right) - \bar{\delta}_b \left(\bar{d}_a B + \omega_a \times B \right) \\
&= \bar{d}_a \bar{d}_b B + \bar{d}_a (\omega_b \times B) + \omega_a \times (\omega_b \times B) + \omega_a \times (\bar{d}_b B) \\
&\quad - \bar{d}_b \bar{d}_a B - \bar{d}_b (\omega_a \times B) - \omega_b \times (d_a B) - \omega_b \times (\omega_a \times B) \\
&= \bar{d}_a (\omega_b \times B) - \bar{d}_b (\omega_a \times B) + \omega_a \times (\omega_b \times B) - \omega_b \times (\omega_a \times B) \\
&\quad + \omega_a \times (\bar{d}_b B) - \omega_b \times (\bar{d}_a B) \\
&= \left[\bar{d}_a \omega_b - \bar{d}_b \omega_a \right] \times B + \omega_b \times (\bar{d}_a B) - \omega_a \times (\bar{d}_b B) \\
&\quad - (\omega_b \times \omega_a) \times B + \omega_a \times (d_b B) - \omega_b \times (\bar{d}_a B) \\
&= \left[\bar{d}_a \omega_b - \bar{d}_b \omega_a + \omega_a \times \omega_b \right] \times B = R(a \wedge b) \times B
\end{aligned} \tag{956}$$

with

$$R(a \wedge b) \overset{\text{def}}{=} \bar{d}_a \omega_b - \bar{d}_b \omega_a + \omega_a \times \omega_b \quad . \tag{957}$$

This is the **curvature tensor** in our space-time algebra representation. To give it in components and to allow a, b to be functions of x, one has to remember

$$\dot{\bar{d}}_a \dot{\omega}(b(x)) \overset{(289)}{=} \bar{d}_a \omega(b) - \omega(\bar{d}_a b) \quad , \tag{958}$$

since ω_b is a tensor. Thus by expressing ω_a in the basis of bivectors

$$\begin{aligned}
\dot{\bar{d}}_a \dot{\omega}_b - \dot{\bar{d}}_b \dot{\omega}_a &= \frac{1}{2} \left[a \cdot d_x \omega_b^{ij} - b \cdot d_x \omega_a^{ij} \right] \gamma_i \wedge \gamma_j \underbrace{- \omega(\bar{d}_a b) + \omega(\bar{d}_b a)}_{= -\omega_{[a,b]_{\text{Lie}}}} \\
&= \frac{1}{2} \left[a \cdot \partial_x \omega_b^{ij} - b \cdot \partial_x \omega_a^{ij} \right] \gamma_i \wedge \gamma_j - \omega_{[a,b]_{\text{Lie}}} \quad ,
\end{aligned} \tag{959}$$

where

$$[a, b]_{\text{Lie}} \overset{\text{def}}{=} a \cdot \partial_x b - b \cdot \partial_x a \quad . \tag{960}$$

So we derive the final expression for the curvature tensor

$$\begin{aligned}
R(a \wedge b) &= \bar{d}_a \omega_b - \bar{d}_b \omega_a + \omega_a \times \omega_b \\
&= \frac{1}{2} \left[a \cdot d_x \omega_b^{ij} - b \cdot d_x \omega_a^{ij} \right] \gamma_i \wedge \gamma_j - \omega_{[a,b]} + \omega_a \times \omega_b \quad .
\end{aligned} \tag{961}$$

We conclude immediately that for a scalar function ϕ

$$\left[\dot{\bar{d}}_a, \dot{\bar{d}}_b \right] \ddot{\phi} = R(a \wedge b) \times \phi = 0 \quad . \tag{962}$$

[101] This saves us the overdot notation.

[102] In operator notation

$$\begin{aligned}
[\bar{\delta}_a, \bar{\delta}_b] &= [\bar{d}_a + \omega_a \times, \bar{d}_b + \omega_b \times] \\
&= [\omega_a \times, \omega_b \times] + [\bar{d}_a, \omega_b \times] - [\bar{d}_b, \omega_a \times] = (\omega_a \times \omega_b) \times + (\bar{d}_a \omega_b) \times - (\bar{d}_b \omega_a) \times \quad .
\end{aligned} \tag{955}$$

Matrix Elements As is seen with (961), the Curvature tensor is a bivector valued linear function of a bivector argument. Thus we define the matrix elements as

$$R_{\alpha\beta\gamma\delta} \stackrel{\text{def}}{=} R(\gamma_\alpha, \gamma_\beta, \gamma_\gamma, \gamma_\delta) = R(\gamma_\alpha \wedge \gamma_\beta) * (\gamma_\delta \wedge \gamma_\gamma) \quad . \tag{963}$$

We see immediately that $R_{\alpha\beta\gamma\delta}$ must be *anisymmetric in the first and last both indices.*

With the help of this matrix elements we can develop $R_{\alpha\beta\gamma\delta}$ in the basis of bivectors[103]

$$R(\gamma_\alpha \wedge \gamma_\beta) = \frac{1}{2} R(\gamma_\alpha \wedge \gamma_\beta) * (\gamma_\delta \wedge \gamma_\gamma)(\gamma^\gamma \wedge \gamma^\delta) \stackrel{(963)}{=} \frac{1}{2} R_{\alpha\beta\gamma\delta}(\gamma^\gamma \wedge \gamma^\delta) \quad . \tag{965}$$

This can be written as

$$R(a \wedge b) = \frac{1}{2}(d_c \wedge d_d)\{R(a \wedge b) * (d \wedge c)\} \quad . \tag{966}$$

8.3.5 Assumption of vanishing Torsion

The assumption of vanishing torsion is expressed by

$$\delta_x \wedge \delta_x \varphi(x) = 0 \quad , \tag{967}$$

for any *scalar valued* function $\varphi(x)$. Using the definition of the covariant derivative we get for any multivector M

$$\begin{aligned} \delta_x \wedge \delta_x M &= (\gamma^\mu \bar{\delta}_{\gamma_\mu}) \wedge (\gamma^\nu \bar{\delta}_{\gamma_\nu}) M = \gamma^\mu \wedge \gamma^\nu \bar{\delta}_{\gamma_\mu} \bar{\delta}_{\gamma_\nu} M + \gamma^\mu \wedge (\dot{\bar{\delta}}_{\gamma_\mu} \dot{\gamma}^\nu) \bar{\delta}_{\gamma_\nu} M \\ &= \frac{1}{2} \gamma^\mu \wedge \gamma^\nu [R(\gamma_\mu \wedge \gamma_\nu) \times M] + \gamma^\mu \wedge \underbrace{(\dot{\bar{\delta}}_{\gamma_\mu} \dot{\gamma}^\nu)}_{\omega_{\gamma_\mu} \cdot \gamma^\nu} \bar{\delta}_{\gamma_\nu} M \\ &\quad + \frac{1}{2} \gamma^\mu \wedge \gamma^\nu \bar{\delta}_{[\gamma_\mu, \gamma_\nu]_{\text{Lie}}} M \quad . \end{aligned} \tag{968}$$

Using $\omega_{\gamma_\mu} \cdot \delta_x = \gamma^\lambda (\omega_{\gamma_\mu} \cdot \delta_x) \cdot \gamma_\lambda$ and $\gamma^\mu \wedge \gamma^\nu \bar{\delta}_{[\gamma_\mu, \gamma_\nu]_{\text{Lie}}} = 2\gamma^\mu \wedge \gamma^\nu (\omega_{\gamma_\mu} \cdot \gamma_\nu) \cdot \delta_x$ we see that the last both terms cancel each other and derive

$$\delta_x \wedge \delta_x M = \frac{1}{2} \gamma^\mu \wedge \gamma^\nu [R(\gamma_\mu \wedge \gamma_\nu) \times M] \quad . \tag{969}$$

Taking the inner product with a constant bivector $a \wedge b$ verifies (956). For a general multivector field M we have

$$\delta_x \wedge \delta_x \wedge M = \delta_x \wedge P(\partial_x \wedge M) = P(S(\partial_x \wedge M)) + P(\partial_x \wedge \partial_x M) = 0 \quad . \tag{970}$$

For M a vector we derive so

$$\delta_x \wedge \delta_x \wedge b(x) = 0 \qquad \forall b(x) \in \mathcal{G}_1(\mathcal{M}) \tag{971}$$

and hence

$$d_a \wedge d_b \wedge \{R(a \wedge b) \cdot c\} = \gamma^\mu \wedge \gamma^\nu \wedge \{R(\gamma_\mu \wedge \gamma_\nu) \cdot c\} = 0 \quad . \tag{972}$$

8.3.6 Symmetries of the Curvature Tensor

The antisymmetry of $R_{\alpha\beta\gamma\delta}$ in the first and last both indices

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} \Leftrightarrow R(a, b) = R(a \wedge b) = c \wedge d \tag{973}$$

[103]This is consistent with (963), since

$$(\gamma_\mu \wedge \gamma_\nu) * (\gamma^\kappa \wedge \gamma^\lambda) = \begin{vmatrix} \gamma_\nu \cdot \gamma^\kappa & \gamma_\nu \cdot \gamma^\lambda \\ \gamma_\mu \cdot \gamma^\kappa & \gamma_\mu \cdot \gamma^\lambda \end{vmatrix} = \delta^\kappa_\nu \delta^\lambda_\mu - \delta^\lambda_\nu \delta^\kappa_\mu \quad . \tag{964}$$

became already obvious when we identified $R(a\wedge b)$ with (961) as a bivector valued linear function of a bivector argument. To find further symmetries we look at the assumption of no torsion. From (971) it follows

$$
\begin{aligned}
0 &\overset{(971)}{=} \delta_x\wedge\delta_x\wedge\gamma_\kappa \overset{(969)}{=} \frac{1}{2}\gamma^\mu\wedge\gamma^\nu\wedge[R(\gamma_\mu\wedge\gamma_\nu)\cdot\gamma_\kappa] \\
&= \gamma^\mu\wedge\gamma^\nu\wedge\Bigg[\underbrace{R_{\mu\nu\alpha\beta}(\gamma^\alpha\wedge\gamma^\beta)\cdot\gamma_\kappa}_{=2R_{\mu\nu\alpha\beta}\gamma^\alpha(\gamma^\beta\cdot\gamma_\kappa)}\Bigg]
\end{aligned}
\tag{974}
$$

and thus

$$
0=\gamma^\mu\wedge\gamma^\nu\wedge\gamma^\alpha R_{\mu\nu\alpha\kappa} \quad . \tag{975}
$$

Taking the scalar product with $\gamma^\beta\wedge\gamma^\gamma\wedge\gamma^\delta$ gives

$$
0=R_{\beta\gamma\delta\kappa}-R_{\gamma\beta\delta\kappa}+R_{\gamma\delta\beta\kappa}-R_{\delta\gamma\beta\kappa}+R_{\delta\beta\gamma\kappa}-R_{\beta\delta\gamma\kappa} \quad . \tag{976}
$$

Using the antisymmetry in the first index (973) gives finally the cyclic identity

$$
0=R_{\beta\gamma\delta\kappa}+R_{\gamma\delta\beta\kappa}+R_{\delta\beta\gamma\kappa}=R(\gamma_\beta\wedge\gamma_\gamma)*(\gamma_\kappa\wedge\gamma_\delta)+R(\gamma_\gamma\wedge\gamma_\delta)*(\gamma_\kappa\wedge\gamma_\beta)+R(\gamma_\delta\wedge\gamma_\beta)*(\gamma_\kappa\wedge\gamma_\gamma) \quad , \tag{977}
$$

which can be written as

$$
0=R(a\wedge b)*(c\wedge d)+R(b\wedge c)*(a\wedge d)+R(c\wedge a)*(b\wedge d) \quad . \tag{978}
$$

But now

$$
\begin{aligned}
R_{\alpha\beta\gamma\delta} &\overset{(977)}{=} -R_{\beta\gamma\alpha\delta}-R_{\gamma\alpha\beta\delta} \overset{(973)}{=} R_{\beta\gamma\delta\alpha}+R_{\gamma\alpha\delta\beta} \overset{(977)}{=} -(R_{\gamma\delta\beta\alpha}+R_{\delta\beta\gamma\alpha})-(R_{\alpha\delta\gamma\beta}+R_{\delta\gamma\alpha\beta}) \\
&\overset{(973)}{=} 2R_{\gamma\delta\alpha\beta}+\underbrace{R_{\delta\beta\alpha\gamma}+R_{\alpha\delta\beta\gamma}}_{\overset{(977)}{=}-R_{\beta\alpha\delta\gamma}\overset{(973)}{=}-R_{\alpha\beta\gamma\delta}} = 2R_{\gamma\delta\alpha\beta}-R_{\alpha\beta\gamma\delta}
\end{aligned}
\tag{979}
$$

and thus

$$
R_{\alpha\beta\gamma\delta}=R_{\gamma\delta\alpha\beta} \quad . \tag{980}
$$

The curvature tensor is thus a symmetric bivector valued linear function, *i.e.*,

$$
R(a\wedge b)*(c\wedge d)=R(c\wedge d)*(a\wedge b) \quad . \tag{981}
$$

(975) shows directly the relation to the well known tensor relations (976). Nevertheless, given the symmetry (980), we can rewrite (975) as

$$
0\overset{(975)}{=}\gamma^\mu\wedge\gamma^\nu\wedge\gamma^\alpha R_{\mu\nu\alpha\kappa}=(R_{\alpha\kappa\mu\nu}\gamma^\mu\wedge\gamma^\nu)\wedge\gamma^\alpha=R(\gamma_\alpha\wedge\gamma_\kappa)\wedge\gamma^\alpha \quad . \tag{982}
$$

8.3.7 Identities of the Curvature Tensor

Starting up with (971) we can derive a number of useful identities. First we find

$$
\begin{aligned}
0 &\overset{(971)}{=} \delta_x\wedge\delta_x\wedge\gamma_\kappa \overset{(969)}{=} \frac{1}{2}\gamma^\mu\wedge\gamma^\nu\wedge\{R(\gamma_\mu\wedge\gamma_\nu)\cdot\gamma^\kappa\} \\
&\overset{(965)}{=} \frac{1}{2}\gamma^\mu\wedge\gamma^\nu\wedge\{R_{\mu\nu\alpha\beta}(\gamma^\alpha\wedge\gamma^\beta)\cdot\gamma_\kappa\}=R(\gamma_\alpha\wedge\gamma_\beta)\wedge\gamma^\alpha\delta^\beta_\kappa \\
&= R(\gamma_\alpha\wedge\gamma_\kappa)\wedge\gamma^\alpha \quad ,
\end{aligned}
\tag{983}
$$

which agrees with (982). Further, taking the inner product with γ_β yields the identity

$$
0\overset{(983)}{=}\{R(\gamma_\alpha\wedge\gamma_\kappa)\wedge\gamma^\alpha\}\cdot\gamma_\beta=R(\gamma_\beta\wedge\gamma_\kappa)-[R(\gamma_\alpha\wedge\gamma_\kappa)\cdot\gamma_\beta]\wedge\gamma^\alpha \tag{984}
$$

and thus

$$
[R(\gamma_\alpha\wedge\gamma_\kappa)\cdot\gamma_\beta]\wedge\gamma^\alpha=R(\gamma_\beta\wedge\gamma_\kappa) \quad . \tag{985}
$$

8.3.8 Contractions of the Curvature Tensor

Given the bivector valued curvature tensor we can define the so-called **Ricci Tensor** as its first contraction[104], *i.e.*,

$$R(b) \stackrel{\text{def}}{=} d_a \cdot R(a \wedge b) = \gamma^\mu \cdot R(\gamma_\mu \wedge a) \quad . \tag{986}$$

This is a vector valued linear function of a vector argument. R is *symmetric*, as can be seen by

$$c \cdot R(b) \stackrel{(986)}{=} c \cdot [d_a \cdot R(a \wedge b)] = (c \wedge d_a) * R(a \wedge b) = (a \wedge b) * R(c \wedge d_a) \stackrel{(981)}{=} (b \wedge d_a) * R(a \wedge c) = b \cdot R(c) \quad . \tag{987}$$

To see the relation to the standard index notation we calculate the matrix elements

$$R^\lambda{}_\mu = \gamma^\lambda * R(\gamma_\mu) \stackrel{(986)}{=} (\gamma^\lambda \wedge \gamma^\nu) * R(\gamma_\nu \wedge \gamma_\mu) = R^{\nu\lambda}{}_{\nu\mu} \quad . \tag{988}$$

Since $R(a)$ is symmetric, *i.e.*, $R(a) = \bar{R}(a)$, we find

$$\gamma^\mu a \cdot R(\gamma_\mu) \stackrel{(987)}{=} \gamma^\mu \gamma_\mu \cdot R(a) = R(a) \quad , \tag{989}$$

but on the other hand

$$\gamma^\mu a \cdot R(\gamma_\mu) = -a \cdot \{\gamma^\mu \wedge R(\gamma_\mu)\} + \underbrace{(a \cdot \gamma^\mu) R(\gamma_\mu)}_{=R(a)} = -a \cdot \{\gamma^\mu \wedge R(\gamma_\mu)\} + R(a) \quad . \tag{990}$$

Since a can be chosen arbitrarily, equating (989) and (990) yields

$$d_a \wedge R(a) = \gamma^\mu \wedge R(\gamma_\mu) = 0 \quad . \tag{991}$$

Note that *this is true for any symmetric function*! Thus a symmetric function has zero protraction.

Contracting the Ricci tensor yields the **Ricci Scalar**, *i.e.*,

$$R \stackrel{\text{def}}{=} d_b * R(b) = (d_b \wedge d_a) * R(a \wedge b) = (\gamma^\nu \wedge \gamma^\mu) * R(\gamma_\mu \wedge \gamma_\nu) = R^{\mu\nu}{}_{\mu\nu} \quad . \tag{992}$$

We see how the tetrad component derivative allows a neat index free notation. However, we can always express results in the standard index notation, *i.e.*, giving components with respect to the tetrad.

Identities for the Ricci Tensor Since $R(a)$ is symmetric its protraction vanishes and thus

$$\begin{aligned} 0 &\stackrel{(991)}{=} d_a \wedge R(a) = d_a \wedge [d_b \cdot R(b \wedge a)] \\ &= (d_a \wedge d_b) \times R(b \wedge a) - d_a \cdot \underbrace{[d_b \wedge R(b \wedge a)]}_{=0} \\ &= (d_a \wedge d_b) \times R(b \wedge a) \quad . \end{aligned} \tag{993}$$

Therefore, we derive the identity

$$(d_a \wedge d_b) R(b \wedge a) = \underbrace{(d_a \wedge d_b) \wedge R(b \wedge a)}_{=0} + \underbrace{(d_a \wedge d_b) \times R(b \wedge a)}_{=0} + \underbrace{(d_a \wedge d_b) \cdot R(b \wedge a)}_{=R} = R \quad . \tag{994}$$

Dimensionality of the Curvature Tensor All constraints are contained in $\partial_a \wedge R(a \wedge b)$, which gives

$$\frac{n^2(n^2-1)}{12} \tag{995}$$

degrees of freedom.

[104] We use the tetrad component derivative to formulate contractions. The distinction from the vector derivative is necessary, since no $(\omega\times)$-terms should be included.

Spaces of constant Curvature Here the curvature tensor takes the form

$$R(a \wedge b) = K a \wedge b \quad . \tag{996}$$

The usually given matrix components are derived by writting

$$R(e_\mu \wedge e_\nu)\cdot(e_\lambda \wedge e_\kappa) \overset{(996)}{=} K(e_\mu \wedge e_\nu)\cdot(e_\lambda \wedge e_\kappa) = K \begin{vmatrix} e_\mu \cdot e_\kappa & e_\mu \cdot e_\lambda \\ e_\nu \cdot e_\kappa & e_\nu \cdot e_\lambda \end{vmatrix} = K(g_{\mu\kappa} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\kappa}) \quad . \tag{997}$$

The Ricci tensor associated with (996) becomes so

$$R(b) \overset{(986)}{=} d_a \cdot R(a \wedge b) \overset{(996)}{=} K d_a \cdot (a \wedge b) = K \left[n b - \underbrace{b \cdot d_a a}_{=b} \right] = (n-1) K b \quad . \tag{998}$$

For the Ricci scalar follows with (192)

$$R \overset{(992)}{=} d_b \cdot R(b) \overset{(998)}{=} K n(n-1) \quad . \tag{999}$$

8.3.9 Parallel Transfer and Geodesics

Parallel Transfer We say a vector a is parallel transfered in direction b, if

$$\bar{\delta}_b a(x) = P(b \cdot \partial_x a(x)) = 0 \quad . \tag{1000}$$

That means for an "observer" going in direction b, the vector a does not change its direction relative to him.

Parallel Transfer around a closed Loop If we transfer a vector v along the infinitesimal vector da it changes to

$$v'(x) = v(x + da) = [da \cdot \delta_x]\, v(x) + v(x) \quad . \tag{1001}$$

Transferring now v' along $db' = [da \cdot \delta_x]\, db$ gives

$$v'' = [db' \cdot \delta_x]\, v'(x) + v'(x) \overset{(1001)}{=} [db' \cdot \delta_x]\, [da \cdot \delta_x v] + [db' \cdot \delta_x]\, v + [da \cdot \delta_x]\, v + v \quad . \tag{1002}$$

Reversing the order, *i.e.*, first going along db and then along $da' = [db \cdot \delta_x]\, da$, yields in the same way

$$v''' = [da' \cdot \delta_x]\, [db \cdot \delta_x v] + da' \cdot \delta_x v + db \cdot \delta_x v + v \quad . \tag{1003}$$

If v is now *parallel transfered* along da and db this equations become under use of (1000)

$$\left. \begin{array}{lllll} v'' & = & [db' \cdot \delta_x]\, v + v & = & (da \cdot \delta_x db) \cdot \delta_x v + v \\ v''' & = & [da' \cdot \delta_x]\, v + v & = & (db \cdot \delta_x da) \cdot \delta_x v + v \end{array} \right\} . \tag{1004}$$

The difference of v'' and v''' is thus

$$v'' - v''' = [da \cdot \delta_x db - db \cdot \delta_x da] \cdot \delta_x v \overset{(956)}{=} R(da \wedge db) \cdot v \quad . \tag{1005}$$

Geodesics We call $a(x)$ a **geodesic vector field**, if

$$\bar{\delta}_{a(x)} a(x) = P(a(x) \cdot \partial_x a(x)) = \lambda(x) a(x) \quad . \tag{1006}$$

That means by going in direction a, the vector $a(x)$ itself does not change its direction, but might change its magnitude. We call $a(x)$ a **normalized geodesic vector field**, if it obeys (1006) with $\lambda = 0$, *i.e.*,

$$\bar{\delta}_{a(x)} a(x) = 0 \quad . \tag{1007}$$

Each geodesic vector field obeying (1006) can be normalized to obey (1007). Suppose $a(x)$ is a geodesic vector field obeying (1006). Then

$$b(x) \overset{\text{def}}{=} \frac{a(x)}{|a(x)|} \tag{1008}$$

obeys

$$[b(x)\cdot\delta_x]\,b(x) \overset{(1008)}{=} \frac{1}{|a(x)|}[a(x)\cdot\delta_x]\frac{a(x)}{|a(x)|} \overset{(1006)}{=} \frac{1}{|a(x)|}\left\{\frac{\lambda a(x)}{|a(x)|} - \frac{a(x)\lambda|a(x)|}{|a(x)|^2}\right\} = 0 \qquad (1009)$$

and thus (1007), so that $b(x)$ is a normalized geodesic vector field.

For each point in $\mathcal{M}$ a geodesic vector field defines with (920) a unique one-dimensional curve, which has in x the tangent vector proportional to $a(x)$. This curve we call **geodesic.** Thus we can write each geodesic as

$$x(\tau), \qquad (1010)$$

so that the geodesic vector field is given by

$$a(x(\tau)) = \frac{\partial}{\partial\tau}x(\tau) \quad . \qquad (1011)$$

A *field* obeying (1006) defines a geodesic going through each point of the manifold $\mathcal{M}$. Given an one-dimensional curve $\mathcal{G} = \{x_0(\alpha)\}$ on the manifold, non-parallel to the geodesics passing through the curve[105], we can parameterize a family of geodesics by

$$x(\tau,\alpha) \quad . \qquad (1012)$$

Geodesic Deviation Let us consider a family of geodesics $x(\tau,\alpha)$, where α is a parameter labelling the geodesics of the family in a smooth manner and τ is an affine parameter along them. The tangent vector

$$v \overset{\text{def}}{=} \frac{\partial}{\partial\tau}x(\tau,\alpha) \qquad (1013)$$

is parallel transfered along the geodesic [106], *i.e.*,

$$\bar\delta_v v = 0 \quad . \qquad (1014)$$

We define further the connecting vector between nearby geodesics by

$$w \overset{\text{def}}{=} \frac{\partial}{\partial\alpha}x(\tau,\alpha) \quad . \qquad (1015)$$

Two geodesics labeled by α and $\alpha + d\alpha$ can be seen as separated by $w d\alpha$. The chain rule yields

$$\left.\begin{array}{lllllll} P(\frac{\partial}{\partial\tau}M) & = & P(\frac{\partial x}{\partial\tau}\cdot\partial_x M) & = & P(v\cdot\partial_x M) & = & \bar\delta_v M \\ P(\frac{\partial}{\partial\alpha}M) & = & P(\frac{\partial x}{\partial\alpha}\cdot\partial_x M) & = & P(w\cdot\partial_x M) & = & \bar\delta_w M \end{array}\right\} . \qquad (1016)$$

Since the scalar derivatives commute

$$\bar\delta_v w = P(\frac{\partial}{\partial\tau}\frac{\partial}{\partial\alpha}x) = P(\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\tau}x) = \bar\delta_w v \quad . \qquad (1017)$$

Applying this to $\frac{\partial^2}{\partial\tau^2}w$ yields

$$P(\frac{\partial}{\partial\tau}P(\frac{\partial}{\partial\tau}w)) \overset{(1016)}{=} \bar\delta_v\bar\delta_v w \overset{(1017)}{=} \bar\delta_v\bar\delta_w v \overset{(1014)}{=} \bar\delta_v\bar\delta_w v - \bar\delta_w \underbrace{\bar\delta_v v}_{\overset{(1014)}{=}0} \overset{(956)}{=} R(v\wedge w)\cdot v \quad . \qquad (1018)$$

This reveals the relation of the curvature to the second derivative with respect to the curve parameter.

[105]If the curve would be parallel to the geodesics, the labelling of the geodesics could be not a one-to-one mapping.
[106]This is equivalent to

1. v is tangent to the geodesic and
2. $|v|$ is constant.

Example of a Sphere Consider the geodesics on a sphere. According to the symmetry of the sphere we have constant curvature, *i.e.*,

$$R(a \wedge b) = K a \wedge b \quad . \tag{1019}$$

We *choose* the geodesic connection vector in such a way, that with the definitions (1013) and (1015)

$$w \cdot v = 0 \quad . \tag{1020}$$

We parameterize the great circles by the path length, *i.e.*,

$$l = \varphi r \quad ; \quad |\tfrac{\partial x}{\partial l}| = |v| = 1 \quad \Rightarrow \quad v^2 = 1 \quad ; \quad \partial l = r\,\partial\varphi \quad , \tag{1021}$$

where r is the radius of the sphere and φ the angle relative to an arbitrary chosen starting point. Thus the geodesic deviation equation (1018) yields

$$\frac{\partial^2}{\partial l^2} w = \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} w = -K w \quad . \tag{1022}$$

This is easily integrated by imposing the boundary condition $w(0) = w_0, \partial_\varphi w\Big|_{\varphi=0} = 0$ to obtain

$$w = \cos(\sqrt{K} r \varphi) w_0 \quad . \tag{1023}$$

In order to have a dimensionless argument of the cosine we need

$$K \propto \frac{1}{r^2} \quad . \tag{1024}$$

At $\varphi = 0$ the both geodesics are at their greatest distance, since the maximum of the cosine is reached. They are also parallel since $\partial_\varphi w\Big|_{\varphi=0} = 0$. Both geodesics meet when

$$0 = \cos(\sqrt{K} r \varphi) \Rightarrow \sqrt{K} r \varphi = \pm\frac{\pi}{2} \quad . \tag{1025}$$

If we are really on a sphere, then both crossing points must differ by an angle of π. We have thus for a sphere

$$K = \frac{1}{r^2} \quad . \tag{1026}$$

Here we can read of one further result. If we choose $K < 0$ the solutions to (1022) are $e^{\pm\sqrt{|K|} r \varphi}$. Looking at initially (for $\varphi = 0$) parallel geodesics, the solution becomes $w = \cosh(\sqrt{-K} r \varphi) w_0$. Thus this geodesics never meet.

8.3.10 Integrability Conditions

The Bianchi Identity From (971) it follows

$$\begin{aligned} 0 &= \dot{\delta}_x \wedge (\delta_x \wedge \delta_x)^{\bullet} M \\ &\overset{(969)}{=} [\delta_x \wedge R(\gamma^\mu \wedge \gamma^\nu)]\,(\gamma_\mu \wedge \gamma_\nu) \times M \end{aligned} \tag{1027}$$

and thus, since M can be chosen arbitrary,

$$\dot{\delta}_x \wedge \dot{R}(a \wedge b) = 0 \quad . \tag{1028}$$

This is the so-called Bianchi identity in its remarkable simple space-time algebra form.

The contracted Bianchi Identity Contracting the Bianchi Identity (1028) gives

$$0 \overset{(1028)}{=} d_a \cdot \{\delta_x \wedge R(a \wedge b)\} = d_a \cdot \delta_x R(a \wedge b) - \delta_x \wedge \underbrace{[d_a \cdot R(a \wedge b)]}_{=R(b)} = R(\delta_x \wedge b) - \delta_x \wedge R(b) \tag{1029}$$

Contracting again yields with

$$0 \overset{(1029)}{=} d_b \cdot (R(\delta_x \wedge b) - \delta_x \wedge R(b)) = -R(\delta_x) - d_b \cdot \delta_x R(b) + \delta_x d_b \cdot R(b) = -2R(\delta_x) + \delta_x R \tag{1030}$$

the contracted Bianchi Identity

$$0 = \dot{R}(\dot{\delta}_x) - \frac{1}{2}\dot{\delta}_x \dot{R} \quad . \tag{1031}$$

8.4 Physics in Curved Space-Time

Above derived results are valid for any number of dimensions and any signature. When looking at the real world, we can limit ourself to a four-dimensional manifold with signature (+ - - -), where the time is distinguished by a positive square.

Units For simplicity we use natural units, *i.e.*, we define

$$c = \hbar = 1 \quad , \tag{1032}$$

e.g., space and time have the same units.

8.4.1 Proper Time and Four-Velocity

Let $x(t)$ be a curve so that

$$\tfrac{\partial x}{\partial t} = \lambda(t)u(x) \quad ; \quad \lambda > 0 \quad ; \quad [u(x)]^2 = +1 \quad , \tag{1033}$$

i.e., the tangent to the curve is time-like in each point. Then we call

$$\tau \overset{\text{def}}{=} \int_{t_0}^{t} \lambda(t)\, dt \quad \Rightarrow \quad \partial\tau = \lambda(t)\, \partial t \tag{1034}$$

the **proper time**, for which (1033) becomes now[107]

$$\tfrac{\partial x}{\partial \tau} = u \quad ; \quad u^2 = +1 \tag{1035}$$

and we call u the **four velocity** of the particle.

8.4.2 The Energy Momentum Tensor

The energy-momentum tensor T is a symmetric[108], vector valued linear function of a vector argument, *i.e.*,

$$T(a) = \bar{T}(a) \quad . \tag{1036}$$

The *energy-momentum conservation* is given by

$$\dot{T}(\dot{\delta}_x) = 0 \quad . \tag{1037}$$

Given a Killing vector k we have

$$\delta_x \cdot T(k) = k \cdot \overbrace{\dot{T}(\dot{\delta}_x)}^{\overset{(1037)}{=}0} + \dot{\delta}_x \cdot T(\dot{k}) = d_a \cdot T(a \cdot \dot{\delta}_x \dot{k}) \overset{(952)}{=} \frac{1}{2} d_a \cdot T(a \cdot (\delta_x \wedge k)) = 0 \quad . \tag{1038}$$

Thus the quantity $T(k)$ is a conserved current

$$J \overset{\text{def}}{=} T(k) \Rightarrow \delta_x \cdot J \overset{(1038)}{=} 0 \quad . \tag{1039}$$

The Perfect Fluid Case With

$$\left.\begin{array}{lcc} \mu & \overset{\text{def}}{=} & \text{relativistic energy density} \\ p & \overset{\text{def}}{=} & \text{pressure} \end{array}\right\} \tag{1040}$$

the energy-momentum tensor is in standard index notation given[109] by

$$T^{\mu\nu} = (\mu + p)u^{\mu}u^{\nu} - pg^{\mu\nu} \quad , \tag{1041}$$

[107] Please note that vectors are frame independent. The following relation is thus true for every coordinate system.

[108] Remember the definition (631), where $T(a)$ could have a antisymmetric part. But as we have seen with (675) this is always a total divergence.

[109] This is the form for the signature convention (+ - - -).

where u^μ is the matter four-velocity with $u^\mu u_\mu = 1$. Applying the space-time algebra we can rewrite (1041) as

$$T(a) = (\mu + p)u(u \cdot a) - pa \quad ; \quad u^2 = +1 \quad . \tag{1042}$$

Since $u^2 = 1$ we conclude $u^{-1} = u$ and write (1042) as

$$\begin{aligned} T(a) &\overset{(1042)}{=} (\mu + p)\underbrace{u^{-1}(u \cdot a)}_{=P_u(a)} - pa = \mu P_u(a) - p\underbrace{[a - P_u(a)]}_{=P_u^\perp(a)} \\ &= \mu P_u(a) - pP_u^\perp(a) \quad . \end{aligned} \tag{1043}$$

This is a remarkable simplification if one compares this with (1041). Furthermore, the physical interpretation of μ and p becomes more explicit — the energy density μ is related to four velocity of the frame, while the pressure is associated with the space orthogonal to u, *i.e.*, the rest-space defined by u.

8.4.3 The Einstein Field Equation

The Einstein Tensor We define the Einstein tensor by

$$G(a) \overset{\text{def}}{=} R(a) - \frac{1}{2}aR \quad . \tag{1044}$$

According to the contracted Ricci identity (1030) it obeys

$$\dot{G}(\dot{\delta}_x) \overset{(1030)}{=} 0 \quad , \tag{1045}$$

while the symmetry, *i.e.*,

$$G(a) \cdot b = a \cdot G(b) \quad , \tag{1046}$$

follows from the symmetry of the Ricci tensor $R(a)$. The contraction of the Einstein tensor follows directly from (1044)

$$d_a \cdot G(a) = R - \frac{1}{2}4R = -R \quad . \tag{1047}$$

The Field Equation Having defined the Einstein tensor (1044) and the energy-momentum tensor we can formulate the Einstein Field equation[110]

$$G(a) = \kappa T(a) \quad . \tag{1048}$$

Contracting and using (1047) gives $-R = \kappa d_a \cdot T(a)$. This yields when substituting (1044) into the Einstein Field equation the equivalent relation

$$\frac{1}{\kappa}R(a) = T(a) - \frac{a}{2}\, d_b \cdot T(b) \quad . \tag{1049}$$

8.5 Conluding Remarks

In this section we showed, how a Geometric Algebra approach can be adapted in the treatment of smooth, curved manifolds. We were able to express the connection with (932) in a surprisingy neat form — as a commutator product with a position dependent bivector. The covariant directional derivative lead to the curvature tensor, for which symmetries and identities were discussed.

Having developed this tools, it is not much more than a formality, to apply it to physics. With (1043) we derived the perfect fluid energy-momentum tensor and note the clear structure in this representation. The Einstein tensor and the Einstein field equation translate in a direct way.

The main advantage of the given approach is its independence of coordinates. Occuring tensors, like the curvature, depend on *vectors* and not their components.

[110] We do not add a cosmological constant here. But this can of course be done.

9 Gravity as a Gauge Theory

This section is a review of an approach to gravity as a gauge theory as given in [3], but a slightly different notation is employed. Contractions are expressed with the same differential operator ∂_x as is used for the vector derivative and *not* as in [3] with ∇.

We show that the demand of gauge invariance of physical relations under arbitrary position and rotation transformations leads to modified derivatives, which are covariant. Assuming a certain Lagrangian and the gauge fields to be independent variables, the variational principle leads to two field equations. One of them can be identified as an analogue of the Einstein field equation. Thus the few assumptions about gauge invariance and a Lagrangian lead to a GR-like structure. But one should note that there is still a remarkable difference. The theory is developed in a *flat background space*.

It should be mentioned that only the geometric algebra makes it possible to keep the following calculations so easy and clear.

9.1 The Gauge Fields

9.1.1 The Position Gauge

If all *physical* relations have the form

$$\mathcal{A}(x) = \mathcal{B}(x) \quad ; \quad \forall x \tag{1050}$$

for fields $\mathcal{A}$ and $\mathcal{B}$, they should also hold for

$$x' \stackrel{\text{def}}{=} f(x) \quad , \tag{1051}$$

if f covers the whole space and is nonsingular. Accordingly we say that $\mathcal{A}$ and $\mathcal{B}$ are **covariant under position transformations**, if

$$\mathcal{A}(x) = \mathcal{B}(x) \Leftrightarrow \mathcal{A}(x') = \mathcal{B}(x') \quad . \tag{1052}$$

If the fields do not involve derivatives, the relation (1052) is automatically fulfilled. But the chain rule for the directional derivative tells us

$$a \cdot \partial_x \phi(x') = [a \cdot \partial_x x'] \cdot \partial_{x'} \phi(x') \stackrel{(1051)}{=} \underline{f}(a) \cdot \partial_{x'} \phi(x') = a \cdot \bar{f}(\partial_{x'}) \phi(x') \quad . \tag{1053}$$

Since this is valid for all a, we get after multiplying with ∂_a

$$\partial_x = \bar{f}(\partial_{x'}) \tag{1054}$$

and so

$$\partial_x \phi(x') \stackrel{(1054)}{=} \bar{f}(\partial_{x'}) \phi(x') \quad . \tag{1055}$$

Thus we can define a covariant[111] vector derivative of a function $\phi(x)$ by

$$\psi = \bar{h}(\partial_x) \phi(x) \tag{1056}$$

with a *position dependent* linear function $\bar{h}(a) = \bar{h}(a, x)$, which transforms under (1051) as

$$h(a,x) \stackrel{x \mapsto f(x)}{\longmapsto} h'(a,x) = \bar{h}(\bar{f}^{-1}(a), f(x)) = \bar{h}_{x'}\left(\bar{f}^{-1}(a)\right) \quad . \tag{1057}$$

Fields, which contain the vector derivative only in the form $\bar{h}(\partial_x)$, will now transform covariant under position transformations (1051), *i.e.*,

$$M(x) \mapsto M'(x) = M(x') \quad , \tag{1058}$$

and we define thus the new derivative

$$\bar{h}_x(\partial_x) \quad . \tag{1059}$$

[111] Here we mean only "covariant under arbitrary position transformation".

9.1.2 The Rotation Gauge

We call a multivector **covariant under rotations**, if it transforms under a local rotation transformation $x \mapsto Rx\tilde{R}$ like

$$A \mapsto RA\tilde{R} \quad . \tag{1060}$$

Since also gradients are physical meaningful *orientated* quantities, we demand the covariance of (1059), *i.e.*,

$$\bar{h}(a) \mapsto R\bar{h}(a)\tilde{R} \quad . \tag{1061}$$

We write

$$\bar{h}(\partial_x) = \bar{h}(\partial_a a \cdot \partial_x) = \bar{h}(\partial_a) a \cdot \partial_x \tag{1062}$$

and consider the action of the directional derivative $a \cdot \partial_x$ on a position dependent rotation

$$a \cdot \partial_x (RA\tilde{R}) = \underline{R}(a) A\tilde{R} + RA\underline{\tilde{R}}(a) + R\underline{A}(a)\tilde{R} \quad . \tag{1063}$$

But since R is a unitary transformation[112]

$$R\tilde{R} = 1 \Rightarrow a \cdot \partial_x (R\tilde{R}) = \underline{R}\tilde{R} + R\underline{\tilde{R}} = 0 \tag{1064}$$

and thus after multiplying with $\tilde{R}$ from the left

$$\underline{\tilde{R}} = -\tilde{R}\underline{R}\tilde{R} \quad . \tag{1065}$$

This yields in (1063)

$$\begin{aligned} a \cdot \partial_x RA\tilde{R} &= \underline{R}A\tilde{R} - RA\tilde{R}\underline{R}\tilde{R} + R\underline{A}\tilde{R} \\ &= R\underline{A}\tilde{R} + \underline{R}\underbrace{\tilde{R}R}_{=1}A\tilde{R} - RA\tilde{R}\underline{R}\tilde{R} \\ &= R\underline{A}\tilde{R} + 2(\underline{R}\tilde{R}) \times (RA\tilde{R}) \quad . \end{aligned} \tag{1066}$$

Therefore, to make the derivative covariant under rotations, we have to add an extra term. We do this by the replacement

$$a \cdot \partial_x \mapsto a \cdot \partial_x + \Omega(a)\times \quad , \tag{1067}$$

were Ω is a pure bivector valued[113] function, which transforms as

$$\Omega(a) \mapsto \Omega'(a) = R\Omega\tilde{R} - 2\underline{R}\tilde{R} \quad . \tag{1068}$$

Putting everything together we derive our directional derivative for covariant multivectors, which is covariant under rotations,

$$\mathcal{D}_a^O \stackrel{\text{def}}{=} (a \cdot \partial_x + \Omega(a)\times) \quad . \tag{1069}$$

Rotation Gauge for Spinors In contrast to covariant multivectors, which transform under rotations according to (1060), *spinors transform single sided* as

$$\psi \mapsto R\psi \quad . \tag{1070}$$

So the directional derivative becomes

$$a \cdot \partial_x (R\psi) = \underline{R}(a)\psi + R\underline{\psi}(a) = \underline{R}(a)\tilde{R}(R\psi) + R\underline{\psi} \tag{1071}$$

and hence we define the *directional derivative for spinors, which is covariant under rotations,* as

$$\mathcal{D}_a^S \stackrel{\text{def}}{=} a \cdot \partial_x + \frac{1}{2}\Omega(a) \quad . \tag{1072}$$

Now the transformation law (1067) ensures the covariance under arbitrary local rotation transformations.

---

[112] We abbreviate $\underline{R} \stackrel{\text{def}}{=} \underline{R}(a)$.

[113] (1064) tells us that $\underline{R}\tilde{R}$ reverses to minus itself and is thus a bivector (R is even).

9.1.3 Totally covariant Derivatives

The Operators Since

$$\underline{R}(a) = a \cdot \partial_x R(x) \mapsto a \cdot \bar{f}(\partial_x) R(x') = \underline{f}(a) \cdot \partial_{x'} R(x') \tag{1073}$$

Ω must transform under an arbitrary position transformation (1051) according to

$$\Omega(a,x) \mapsto \Omega'(a,x) \overset{(1068)}{=} \Omega(\underline{f}(a), x') \quad , \tag{1074}$$

which is clearly *not covariant.* The observable derivative (1069) transforms now under position transformations (1051)

$$\mathcal{D}^O_a \mapsto a \cdot \bar{f}(\partial_{x'}) + \Omega'(a)\times = \underline{f}(a) \cdot \partial_{x'} + \Omega(\underline{f}(a))\times = \mathcal{D}^O_{\underline{f}(a)} \tag{1075}$$

and in the same way does the spinor derivative (1072). Thus (1069) and (1072) transform covariant under rotations, but *not* under position transformations! But given the transformation property (1075) we can easily construct totally covariant directional derivatives by

$$\left.\begin{array}{lllll} a \cdot \mathcal{D}^O & \overset{\text{def}}{=} \mathcal{D}^O_{\underline{h}(a)} & \overset{(1069)}{=} & a \cdot \bar{h}(\partial_x) + \Omega(\underline{h}(a))\times & = & a \cdot \bar{h}(\partial_x) + \omega(a)\times \\ a \cdot \mathcal{D}^S & \overset{\text{def}}{=} \mathcal{D}^S_{\underline{h}(a)} & \overset{(1072)}{=} & a \cdot \bar{h}(\partial_x) + \frac{1}{2}\Omega(\underline{h}(a)) & = & a \cdot \bar{h}(\partial_x) + \frac{1}{2}\omega(a) \end{array}\right\} , \tag{1076}$$

with the **covariant rotation gauge field** ω, defined by

$$\omega \overset{\text{def}}{=} \Omega(\underline{h}(a), x) \quad . \tag{1077}$$

We are now in the position to define associated totally covariant vector derivatives by

$$\left.\begin{array}{lllllll} \mathcal{D}^O & \overset{\text{def}}{=} & \partial_a a \cdot \mathcal{D}^O \overset{(1076)}{=} \bar{h}(\partial_x) + \partial_a \omega(a)\times & = & \bar{h}(\partial_a)\left[a \cdot \partial_x + \Omega(a)\times\right] & = & \bar{h}(\partial_a)\mathcal{D}^O_a \\ \mathcal{D}^S & \overset{\text{def}}{=} & \partial_a a \cdot \mathcal{D}^S \overset{(1076)}{=} \bar{h}(\partial_x) + \frac{1}{2}\partial_a \omega(a) & = & \bar{h}(\partial_a)\left[a \cdot \partial_x + \frac{1}{2}\Omega(a)\right] & = & \bar{h}(\partial_a)\mathcal{D}^S_a \end{array}\right\} . \tag{1078}$$

With this definitions

$$\mathcal{D}^O_a = \underline{h}^{-1}(a) \cdot \mathcal{D}^O \quad ; \quad \mathcal{D}^S_a = \underline{h}^{-1}(a) \cdot \mathcal{D}^S \quad . \tag{1079}$$

Applications Since scalars commute with all multivectors, we have for a scalar valued function φ

$$a \cdot \mathcal{D}^O \varphi \overset{(1076)}{=} \left[a \cdot \bar{h}(\partial_x)\right] \varphi \quad ; \quad \mathcal{D}^O_a \varphi \overset{(1076)}{=} \left[a \cdot \partial_x\right] \varphi \tag{1080}$$

and thus for the covariant vector derivative

$$\mathcal{D}^O \varphi = \bar{h}(\partial_x)\varphi \quad . \tag{1081}$$

For later use, we will derive here one more relation. For a vector valued function $f(x)$ we have

$$\begin{array}{rcl} \mathcal{D}^O \bar{h}(f(x)) & \overset{(1078)}{=} & \bar{h}(\dot{\partial}_x)\left[\bar{h}(f(x))\right]^\bullet + \partial_a \omega_a \times \bar{h}(f(x)) \\ & = & \bar{h}(\partial_a) a \cdot \partial_x \left[\bar{h}(f(x))\right]^\bullet + \partial_a \omega_a \times \bar{h}(f(x)) \\ & = & \bar{h}(\partial_a)\left[a \cdot \dot{\partial}_x \dot{\bar{h}}(f(x)) + a \cdot \dot{\partial}_x \bar{h}(\dot{f}(x))\right] + \partial_a \omega_a \times \bar{h}(f(x)) \\ & \overset{(1078)}{=} & \dot{\mathcal{D}}^O \dot{\bar{h}}(f(x)) + \bar{h}(\dot{\partial}_x)\bar{h}(\dot{f}(x)) \quad . \end{array} \tag{1082}$$

The bivector part yields as a special case

$$\mathcal{D}^O \wedge \bar{h}(f(x)) \overset{(1082)}{=} \dot{\mathcal{D}}^O \wedge \dot{\bar{h}}(f(x)) + \bar{h}(\dot{\partial}_x \wedge \dot{f}(x)) \quad . \tag{1083}$$

9.2 A Gauge invariant Dirac Action

After having derived covariant derivatives, we turn our attention to physical applications. It is easily seen that the most important action integral, the Dirac action, is not gauge invariant — it changes its value under position and rotation transformations. Thus we modify each part of the Dirac action in order to find a totally covariant action integral.

9.2.1 The Construction

In order to derive a gauge invariant analogue to the Dirac equation we start with the Dirac action in space-time algebra form

$$S = \int |d^4x| \, \langle (\partial_x \psi) i\gamma_3 \tilde{\psi} - m\psi\tilde{\psi} \rangle \tag{1084}$$

and consider the *modifications on each part* to make it gauge invariant.

The Derivative As discussed before, the derivative ∂_x is not gauge invariant. Since it is acting on a spinor we replace it by the covariant spinor derivative $\mathcal{D}^S$ defined by (1078).

The Volume Element If we define a coordinate frame by

$$e_\mu \stackrel{\text{def}}{=} \partial_{x^\mu} x \tag{1085}$$

it transforms under displacements by

$$e'_\mu = \partial_{x^\mu} f(x) = e_\mu \cdot \partial_x f(x) = \underline{f}(e_\mu) \quad . \tag{1086}$$

The volume element

$$|dx^4| = -i e_0 \wedge e_1 \wedge e_2 \wedge e_3 dx^0 dx^1 dx^2 dx^3 \tag{1087}$$

transforms to

$$|dx'^4| = -i\underline{f}(e_0 \wedge e_1 \wedge e_2 \wedge e_3) dx'^0 dx'^1 dx'^2 dx'^3 = \det \underline{f}\, dx'^0 dx'^1 dx'^2 dx'^3 \tag{1088}$$

and is thus not invariant under arbitrary position transformations. To make it invariant, we replace each e_μ by $\underline{h}^{-1}(e_\mu)$, so that

$$\underline{h}_{x'}^{-1}(e'_\mu) = \underline{h}_{x'}^{-1}(\underline{f}^{-1}\underline{f}(e_\mu)) = \underline{h}_{x'}^{-1}(e_\mu) \quad . \tag{1089}$$

Thus our invariant volume element is given by

$$|d^4x| \det\left(\underline{h}^{-1}\right) = \frac{|d^4x|}{\det \underline{h}} \quad . \tag{1090}$$

The new Action After replacing volume element and derivative by their covariant equivalents we derive the *fully covariant action integral*

$$\begin{aligned} \mathcal{S} &= \int \frac{|d^4x|}{\det \underline{h}} \langle (\mathcal{D}^S\psi) i\gamma_3\tilde{\psi} - m\psi\tilde{\psi} \rangle \\ &\stackrel{(1078)}{=} \int \frac{|d^4x|}{\det \underline{h}} \langle \bar{h}(\dot{\partial}_x)\dot{\psi} i\gamma_3\tilde{\psi} + \frac{1}{2}\partial_a \omega(a)\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi} \rangle \quad . \end{aligned} \tag{1091}$$

9.2.2 The Phase Gauge Field

The Dirac action is invariant under a *global* phase rotation

$$\psi \mapsto \psi' = \psi e^{i\sigma_3 \phi} \quad . \tag{1092}$$

We assume now invariance of the action under local transformations of the kind

$$\psi \mapsto \psi' = \psi e^{i\sigma_3 \varphi(x)} \quad . \tag{1093}$$

Since

$$\begin{aligned} \bar{h}(\dot{\partial}_x)\dot{\psi}' &= \bar{h}(\dot{\partial}_x)\dot{\psi} e^{i\sigma_3\varphi(x)} + \bar{h}(\dot{\partial}_x \dot{\varphi}(x))\psi e^{i\sigma_3\varphi(x)} i\sigma_3 \\ &= \left[\bar{h}(\dot{\partial}_x)\dot{\psi} + e\bar{h}(A)\psi i\sigma_3\right] e^{i\sigma_3\varphi(x)} \quad , \end{aligned} \tag{1094}$$

where

$$eA \stackrel{\text{def}}{=} \partial_x \varphi(x) \quad , \tag{1095}$$

we introduce a correction term

$$-e\bar{h}(A)\psi\gamma_0\tilde{\psi} \tag{1096}$$

to our action and demand of A to transform under phase transformations (1093) as

$$eA \mapsto eA' = eA - \partial_x\varphi(x) \quad . \tag{1097}$$

Under this condition the action becomes also covariant under phase transformations (1093) and gets the final form

$$\begin{aligned} \mathcal{S} &= \int \frac{|d^4x|}{\det \underline{h}} \langle (\mathcal{D}^S\psi)i\gamma_3\tilde{\psi} - e\mathcal{A}\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi}\rangle \\ &\overset{(1078)}{=} \int \frac{|d^4x|}{\det \underline{h}} \langle \bar{h}(\partial_a)\left[\underline{\psi}i\gamma_3\tilde{\psi} + \frac{1}{2}\Omega(a)\psi i\gamma_3\tilde{\psi}\right] - e\mathcal{A}\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi}\rangle \quad , \end{aligned} \tag{1098}$$

where

$$\mathcal{A} \overset{\text{def}}{=} \bar{h}(A) \tag{1099}$$

is the **covariant vector potential** and represents the electromagnetic interaction with an external field.

9.2.3 Lagrangian and Euler-Lagrange Equations

From the Lagrangian (1098) we find the Lagrangian

$$\begin{aligned} \mathcal{L} &\overset{(1098)}{=} \tfrac{1}{\det \underline{h}}\langle \bar{h}(\partial_a)\left[a\cdot\dot{\partial}_x + \tfrac{1}{2}\Omega(a)\right]\dot{\psi}i\gamma_3\tilde{\psi} - e\mathcal{A}\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi}\rangle \\ &= \tfrac{1}{\det \underline{h}}\langle \bar{h}(\partial_a)\left[\underline{\psi}i\gamma_3\tilde{\psi} + \tfrac{1}{2}\Omega(a)\psi i\gamma_3\tilde{\psi}\right] - e\mathcal{A}\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi}\rangle \quad . \end{aligned} \tag{1100}$$

Hence we obtain for the derivatives with respect to the field ψ and its differential

$$\begin{aligned} \det(\underline{h})\partial_\psi\mathcal{L} &= \widetilde{\bar{h}(\partial_a)\underline{\psi}}i\gamma_3 + \frac{1}{2}i\gamma_3\tilde{\psi}\bar{h}(\partial_a)\Omega(a) + \frac{1}{2}\bar{h}(\partial_a)\widetilde{\Omega(a)}\psi i\gamma_3 \\ &\quad -e\widetilde{\bar{h}(A)}\psi\gamma_0 - e\gamma_0\tilde{\psi}\bar{h}(A) - 2m\tilde{\psi} \\ &= -i\gamma_3\underline{\tilde{\psi}}\bar{h}(\partial_a) + i\gamma_3\tilde{\psi}\frac{\bar{h}(\partial_a)\Omega(a) + \dot{\Omega}(a)\bar{h}(\dot{\partial}_a)}{2} - 2e\gamma_0\tilde{\psi}\mathcal{A} - 2m\tilde{\psi} \\ &= -i\gamma_3\underline{\tilde{\psi}}\bar{h}(\partial_a) + i\gamma_3\tilde{\psi}\left[\bar{h}(\partial_a)\wedge\Omega(a)\right] - 2e\gamma_0\tilde{\psi}\mathcal{A} - 2m\tilde{\psi} \end{aligned} \tag{1101}$$

and

$$\partial_{\underline{\psi}}\mathcal{L} = \frac{1}{\det \underline{h}}\left\{i\gamma_3\tilde{\psi}\bar{h}(\partial_a)\right\} \quad , \tag{1102}$$

respectively. The Euler-Lagrange equation is[114]

$$\partial_\psi\mathcal{L} = \partial_a\cdot\partial_x(\partial_{\underline{\psi}(a)}\mathcal{L}) \quad \Leftrightarrow \quad \widetilde{\partial_\psi\mathcal{L}} = a\cdot\partial_x\widetilde{\partial_{\underline{\psi}}\mathcal{L}} \tag{1103}$$

and becomes with (1101) and (1102) after reversing

$$a\cdot\partial_x\frac{\bar{h}(\partial_a)\psi\overbrace{\gamma_3 i}^{=-i\gamma_3}}{\det \underline{h}} = \frac{1}{\det \underline{h}}\left\{\frac{1}{2}\underbrace{\tilde{\Omega}(a)}_{=-\Omega}\bar{h}(\partial_a)\psi\gamma_3 i + \bar{h}(\partial_a)\underline{\psi}i\gamma_3 + \frac{1}{2}\bar{h}(\partial_a)\Omega(a)\psi i\gamma_3 - 2e\mathcal{A}\psi\gamma_0 - 2m\psi\right\}. \tag{1104}$$

Taking the first term on the right side to the left and adding $\frac{1}{2}\bar{h}(\partial_a)\Omega(a)\psi i\gamma_3$ on both sides gives

$$\begin{aligned} &-\det \underline{h}\;\underbrace{\left\{\left[a\cdot\dot{\partial}_x + \Omega\times\right]\right\}}_{=\mathcal{D}^O_a}\left[\frac{\bar{h}(\partial_a)}{\det \underline{h}}\right]^{\bullet}\psi i\gamma_3 - \bar{h}(\partial_a)a\cdot\partial_x(\psi i\gamma_3) = \\ &\qquad = \bar{h}(\partial_a)\left[a\cdot\partial_x + \Omega(a)\right]\psi i\gamma_3 - 2e\mathcal{A}\psi\gamma_0 - 2m\psi \quad . \end{aligned} \tag{1105}$$

[114]This form can be derived by applying the variational principle in one direction. Then the directional derivative replaces the full derivative.

Taking the last term on the left side to the right yields

$$-\det \underline{h}\dot{\mathcal{D}}^{O}_{a}\left[\frac{\bar{h}(\partial_a)}{\det \underline{h}}\right]^{\bullet}\psi i\gamma_3 = \underbrace{2\,\bar{h}(\partial_a)\left[a\cdot\partial_x+\frac{1}{2}\Omega(a)\right]}_{\overset{(1078)}{=}\mathcal{D}^S}\psi i\gamma_3 - 2e\mathcal{A}\psi\gamma_0 - 2m\psi \quad , \tag{1106}$$

so that we derive as the final result

$$-\frac{\det \underline{h}}{2}\left(\dot{\mathcal{D}}^{O}_{a}\frac{\bar{h}(\dot{\partial}_a)}{\det \underline{h}}\right)\psi i\gamma_3 = \mathcal{D}^S\psi i\gamma_3 - e\mathcal{A}\psi\gamma_0 - m\psi \quad . \tag{1107}$$

As pointed out in [3], this result *is different* to what we would have got by making the Dirac equation gauge invariant! Instead this would give us (1107) but with the left side equal to zero! As argued in [3] the left side should be zero in order to derive the minimal coupled Dirac equation from the minimal coupled action. We demand thus

$$\det \underline{h}\left\{\mathcal{D}^{O}_{a}\frac{\bar{h}(\partial_a)}{\det \underline{h}}\right\} = 0 \quad . \tag{1108}$$

We will see later that the same condition arises in order to obtain a trace free spin!

9.2.4 Observables

Observables are of the form

$$\mathcal{O} = \psi A\tilde{\psi} \quad , \tag{1109}$$

where A is a *constant* multivector. Demanding invariance under phase rotations is equivalent to the condition

$$A i\sigma_3 = i\sigma_3 A \quad . \tag{1110}$$

Thus A can only be constructed out of $\{\gamma_0, \gamma_3, i\}$ or with other words A cannot contain γ_1 or γ_2 alone, but $\gamma_1\gamma_2 = \gamma_0\gamma_3 i$. This shows that γ_1 and γ_2 are indistinguishable in the particle rest-frame.

Since the spinor transforms single sided, all observables are automatically covariant under rotations and translations, *i.e.*,

$$\psi(x)A\tilde{\psi}(x) \begin{array}{c}\mapsto\\ \mapsto\end{array} \left.\begin{array}{c} R\psi A\tilde{\psi}\tilde{R} \\ \psi(x')A\tilde{\psi}(x')\end{array}\right\}. \tag{1111}$$

9.3 Curvature and Field Equations

9.3.1 The Curvature Tensor

The commutator of directional derivatves gives the curvature. But by defining

$$\begin{aligned} R(a\wedge b) &= 2\left[\mathcal{D}^S_a, \mathcal{D}^S_b\right] \\ &= 2\left[a\cdot\partial_x+\frac{1}{2}\Omega(a), b\cdot\partial_x+\frac{1}{2}\Omega(b)\right] \\ &= a\cdot\partial_x\Omega(b) - b\cdot\partial_x\Omega(a) + \Omega(a)\times\Omega(b) \end{aligned} \tag{1112}$$

we derive a *non-covariant* quantity. This follows directly from the transformation property (1075), which gives[115]

$$R'(a\wedge b) \overset{(1075)}{=} 2\left[\mathcal{D}^S_{\underline{f}(a)}, \mathcal{D}^S_{\underline{f}(b)}\right] = R(\underline{f}(a\wedge b)) \quad . \tag{1113}$$

But this transformation property already shows, how we have to define a covariant curvature[116]:

$$\mathcal{R}(a\wedge b) \overset{\text{def}}{=} R(\underline{h}(a\wedge b)) \quad . \tag{1114}$$

Substituting the explicit representation (1112) yields

$$\begin{aligned} \mathcal{R}(a\wedge b) &= a\cdot\bar{h}(\partial_x)\omega(b) + b\cdot\bar{h}(\partial_x)\omega(a) + \omega(a)\times\omega(b) \\ &\quad -\omega(\underline{h}^{-1}[\underline{h}(a)\cdot\partial_x\underline{h}(b) - \underline{h}(b)\cdot\partial_x\underline{h}(a)]) \quad . \end{aligned} \tag{1115}$$

[115]Here we cannot use (1112) directly, since $\underline{f}(a)$ is not a *constant* vector. Indeed we would need an extra term just like in (961). Nevertheless, the result still holds, since $a\cdot\partial_x\underline{f}(b) - b\cdot\partial_x\underline{f}(a) = 0$.

[116]Here we extend $\underline{h}(a)$ via outermorphism to the space of bivectors.

Ricci Tensor and Scalar The bivector valued field strength $\mathcal{R}$ is the analogue of the Riemann tensor. We thus define the Ricci tensor

$$\mathcal{R}(b) \stackrel{\text{def}}{=} \partial_a \cdot \mathcal{R}(a \wedge b) \tag{1116}$$

and the Ricci scalar

$$\begin{aligned}\mathcal{R} &\stackrel{\text{def}}{=} \partial_b \mathcal{R}(b) = (\partial_b \wedge \partial_a) \cdot \mathcal{R}(a \wedge b) = \bar{h}(\partial_b \wedge \partial_a) \cdot R(a \wedge b) \\ &\stackrel{(1112)}{=} \bar{h}(\partial_b \wedge \partial_a)\left[a \cdot \dot{\partial}_x \dot{\Omega}(b) - b \cdot \dot{\partial}_x \dot{\Omega}(a) + \Omega(a) \times \Omega(b)\right] \quad .\end{aligned} \tag{1117}$$

With this definitions we can construct the Einstein tensor in the usual way

$$\mathcal{G}(a) = \mathcal{R}(a) - \frac{1}{2} a \mathcal{R} \quad . \tag{1118}$$

Of this tensors only the Ricci scalar is invariant under rotations. Therefore, we assume a Lagrangian of the form

$$\mathcal{L} = \frac{\frac{1}{2}\mathcal{R} - \kappa \mathcal{L}_m}{\det \underline{h}} \quad . \tag{1119}$$

Here $\bar{h}(a)$ and $\Omega(a)$ are the independent dynamical variables. $\mathcal{L}_m$ describes the matter and here we will *assume* that it contains *no second order derivatives* with respect to x, so that it contains no derivatives of $\bar{h}$ and Ω.

9.3.2 The $\bar{h}$-Field Equation

Assuming $\bar{h}$ and Ω to be independent dynamical variables, we can apply the variational principle to the Lagrangian (1119) to derive associated field equations.

In order to calculate the Euler-Lagrange equation for the $\bar{h}$-field, we derive

$$\partial_{\bar{h}(a)} \frac{1}{\det \bar{h}} = -\frac{1}{(\det \bar{h})^2} (\partial_{\bar{h}(a)} \det \bar{h}) = -\frac{1}{(\det \bar{h})^2} \det(\bar{h}) \underline{h}^{-1}(a) \stackrel{(304)}{=} -\frac{\underline{h}^{-1}(a)}{\det \bar{h}} \tag{1120}$$

$$\begin{aligned}\partial_{\bar{h}(a)} \mathcal{R} &\stackrel{(1117)}{=} \partial_{\bar{h}(a)} \langle \bar{h}(\partial_c \wedge \partial_b) R(b \wedge c) \rangle \\ &= \bar{h}(a \cdot [\partial_c \wedge \partial_b]) \cdot R(b \wedge c) = \bar{h}(\partial_b) \cdot R(b \wedge a) - \bar{h}(\partial_c) \cdot R(a \wedge c) \\ &= 2\bar{h}(\partial_b) \cdot R(b \wedge a) = 2\partial_b \cdot R\left(\underline{h}(b) \wedge \underline{h}\left(\underline{h}^{-1}(a)\right)\right) \\ &\stackrel{(1116)}{=} 2\mathcal{R}(\underline{h}^{-1}(a))\end{aligned} \tag{1121}$$

and use the assumption

$$\partial_{\partial_x \bar{h}(a)} \mathcal{L}_m = 0 \tag{1122}$$

to obtain for the Euler-Lagrange equation

$$\begin{aligned}0 &= \partial_{\bar{h}(a)} \mathcal{L} \stackrel{(1121)}{=} -\frac{\mathcal{R}\underline{h}^{-1}(a)}{2\det(\bar{h})} + \frac{\mathcal{R}(\underline{h}^{-1}(a))}{\det(\bar{h})} - \kappa \partial_{\bar{h}(a)} \frac{\mathcal{L}_m}{\det(\bar{h})} \\ &= \frac{\mathcal{G}(\underline{h}^{-1}(a))}{\det(\bar{h})} - \kappa \partial_{\bar{h}(a)} \frac{\mathcal{L}_m}{\det(\bar{h})} \quad .\end{aligned} \tag{1123}$$

If we define the **matter energy-momentum tensor** by

$$\mathcal{T}(\underline{h}^{-1}(a)) \stackrel{\text{def}}{=} \det(\bar{h}) \partial_{\bar{h}(a)} \frac{\mathcal{L}_m}{\det(\bar{h})} \tag{1124}$$

the $\bar{h}$-field equation becomes the analogue of the **Einstein Field Equation**

$$\mathcal{G}(a) = \kappa \mathcal{T}(a) \quad . \tag{1125}$$

While in ordinary GR the Einstein field equation has to be postulated, we derive it here as a direct result of the variational principle applied to a gauge invariant Lagrangian of the form (1119). Further this result is independent of the choice of the matter Lagrangian $\mathcal{L}_m$. But it is just this matter Lagrangian, which defines with (1124) the energy-momentum tensor.

9.3.3 The Ω-Field Equation

We first calculate the functional derivatives of the Ricci scalar with respect to $\Omega(a)$ and $b \cdot \partial_x \Omega(a)$:

$$\begin{aligned}
\partial_{\Omega(a)}\mathcal{R} &\overset{(1117)}{=} \partial_{\Omega(a)}\langle \bar{h}(\partial_c \wedge \partial_b)\Omega(b) \times \Omega(c)\rangle = 2\dot{\partial}_{\Omega(a)}\langle \bar{h}(\partial_c \wedge \partial_b)\dot{\Omega}(b) \times \Omega(c)\rangle \\
&= \dot{\partial}_{\Omega(a)}\langle \left[\dot{\Omega}(b)\Omega(c) - \Omega(c)\dot{\Omega}(b)\right] \bar{h}(\partial_c \wedge \partial_b)\rangle \\
&= 2a \cdot b\langle \Omega(c) \times \bar{h}(\partial_c \wedge \partial_b)\rangle_2 = 2\langle \omega(c) \times (\partial_c \wedge \bar{h}(a))\rangle_2 \\
&\overset{(142)}{=} 2\left\{[\omega(c) \times \partial_c] \wedge \bar{h}(a) + \partial_c \wedge [\omega(c) \times \bar{h}(a)]\right\} \\
&\overset{(1078)}{=} 2\left\{[\Omega(c) \times \bar{h}(\partial_c)] \wedge \bar{h}(a) + \mathcal{D}^O \wedge \bar{h}(a) - \bar{h}(\dot{\partial}_x) \wedge [\bar{h}(a)]^\bullet\right\} \qquad (1126)
\end{aligned}$$

$$\begin{aligned}
\partial_{b\cdot\partial_x\Omega(a)}\mathcal{R} &\overset{(1117)}{=} \partial_{b\cdot\partial_x\Omega(a)}\langle \bar{h}(\partial_c \wedge \partial_d)\,[d \cdot \partial_x\Omega(c) - c \cdot \partial_x\Omega(d)]\rangle \\
&= (b \cdot d)(a \cdot c)\langle \bar{h}(\partial_c \wedge \partial_d)\rangle_2 - (b \cdot c)(a \cdot d)\langle \bar{h}(\partial_c \wedge \partial_d)\rangle_2 \\
&= 2\bar{h}(a \wedge b) \quad . \qquad (1127)
\end{aligned}$$

The Euler-Lagrange equation is *for constant vectors* a *and* b

$$0 = \partial_{\Omega(a)}\mathcal{L} - \partial_b \cdot \partial_x \partial_{b\cdot\partial_x\Omega(a)}\mathcal{L} \quad . \qquad (1128)$$

Now we define the **matter-spin bivector**

$$\mathcal{S}(a) = S(\bar{h}^{-1}(a)) \quad ; \quad S(a) \overset{\text{def}}{=} \partial_{\Omega(a)}\mathcal{L}_m \qquad (1129)$$

and calculate the derivatives of the Lagrangian (1119) with respect to $\Omega(a)$ and $b \cdot \partial_x \Omega(a)$, respectively,

$$\left.\begin{aligned}
\partial_{\Omega(a)}\mathcal{L} &= \frac{1}{\det \underline{h}}\left\{\frac{\partial_{\Omega(a)}\mathcal{R}}{2} - \kappa \underbrace{\partial_{\Omega(a)}\mathcal{L}_m}_{\overset{(1129)}{=} S(a)}\right\} \\
&= \frac{1}{\det \underline{h}}\left\{\frac{\partial_{\Omega(a)}\mathcal{R}}{2} - \kappa S(a)\right\} \\
\partial_{b\cdot\partial_x\Omega(a)}\mathcal{L} &= \frac{1}{\det \underline{h}}\frac{\partial_{b\cdot\partial_x\Omega(a)}\mathcal{R}}{2} \overset{(1127)}{=} \frac{\bar{h}(a \wedge b)}{\det \underline{h}}
\end{aligned}\right\} . \qquad (1130)$$

So the Euler-Lagrange equation (1128), *i.e.*, the Ω-field equation, becomes

$$\begin{aligned}
\kappa S(a) &\overset{(1128)}{=} \frac{\partial_{\Omega(a)}\mathcal{R}}{2} - (\det \underline{h})\partial_b \cdot \partial_x \partial_{b\cdot\partial_x\Omega(a)}\mathcal{L} \\
&\overset{(1126)}{=} [\Omega(b) \times \bar{h}(\partial_b)] \wedge \bar{h}(a) + \mathcal{D}^O \wedge \bar{h}(a) - \bar{h}(\dot{\partial}_x) \wedge [\bar{h}(a)]^\bullet \\
&\quad -(\det \underline{h}) b \cdot \dot{\partial}_x \bar{h}(a) \wedge \left[\frac{\bar{h}(\partial_b)}{(\det \underline{h})}\right]^\bullet - \partial_b \cdot \dot{\partial}_x\, [\bar{h}(a)]^\bullet \wedge \bar{h}(b) \\
&= \mathcal{D}^O \wedge \bar{h}(a) + (\det \underline{h})\left\{b \cdot \partial_x \left[\frac{\bar{h}(\partial_b)}{\det \underline{h}}\right] + \Omega(b) \times \frac{\bar{h}(\partial_b)}{\det \underline{h}}\right\} \wedge \bar{h}(a) \\
&= \dot{\mathcal{D}}^O \wedge [\bar{h}(a)]^\bullet + (\det \underline{h})\left[\mathcal{D}^O_b \frac{\bar{h}(\partial_b)}{\det \underline{h}}\right] \wedge \bar{h}(a) \quad . \qquad (1131)
\end{aligned}$$

Trace of the Spin In order to contract (1131) with $\underline{h}^{-1}(\partial_a)$ we start with

$$\begin{aligned}
\underline{h}^{-1}(\partial_a) \cdot \left[\dot{\mathcal{D}}^O \wedge \dot{\bar{h}}(a)\right] &= \underline{h}^{-1}(a) \cdot \left[\dot{\mathcal{D}}^O \wedge \dot{\bar{h}}(\partial_a)\right] \\
&\overset{(1079)}{=} \dot{\mathcal{D}}^O_a \dot{\bar{h}}(\partial_a) - \underbrace{\dot{\mathcal{D}}^O}_{\overset{(1078)}{=} \bar{h}(\partial_b)\mathcal{D}^O_b} \underline{h}^{-1}(a) \cdot \dot{\bar{h}}(\partial_a) \\
&\overset{(1078)}{=} \dot{\mathcal{D}}^O_a \dot{\bar{h}}(\partial_a) - \bar{h}(\partial_b)\left\{\underline{h}^{-1}(\partial_a) \cdot \left[\dot{\mathcal{D}}^O_b \dot{\bar{h}}(a)\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&\stackrel{(1069)}{=} \dot{\mathcal{D}}^{\circ}_{a}\dot{\bar{h}}(\partial_a) - \bar{h}(\partial_b)\underbrace{\underline{h}^{-1}(\partial_a)\cdot(b\cdot\dot{\partial}_x)\dot{\bar{h}}(a)}_{\stackrel{(1134)}{=}-(\det\underline{h})\mathcal{D}^{\circ}_{b}\frac{1}{\det\underline{h}}} - \\
&\quad - \bar{h}(\partial_b)\underbrace{\underline{h}^{-1}(\partial_a)\cdot\left[\Omega(b)\times\bar{h}(a)\right]}_{\stackrel{(1133)}{=}0} \\
&= \dot{\mathcal{D}}^{\circ}_{b}\dot{\bar{h}}(\partial_b) + (\det\underline{h})\mathcal{D}^{\circ}_{b}\left[\frac{\bar{h}(\partial_b)}{\det\underline{h}}\right] = (\det\underline{h})\mathcal{D}^{\circ}_{b}\left[\frac{\bar{h}(\partial_b)}{\det\underline{h}}\right] \qquad (1132)
\end{aligned}
$$

where we used

$$
\underline{h}^{-1}(\partial_a)\cdot\left[\Omega(b)\times\bar{h}(a)\right] = \partial_a\cdot[\Omega(b)\times a] = -\partial_a\cdot[a\cdot\Omega(b)] = -\underbrace{(\partial_a\wedge a)}_{\stackrel{(194)}{=}0}\cdot\Omega(b) = 0 \quad . \qquad (1133)
$$

Further

$$
\begin{aligned}
\langle\underline{h}^{-1}(\partial_a)b\cdot\dot{\partial}_x\dot{\bar{h}}(a)\rangle &= \frac{1}{\det\underline{h}}\langle(\dot{\partial}_{\bar{h}(\partial_a)}\det\underline{h})b\cdot\dot{\partial}_x\dot{\bar{h}}(a)\rangle \stackrel{(306)}{=} \frac{b\cdot\dot{\partial}_x}{\det\underline{h}}(\det\underline{h})^{\bullet} \\
&= -(\det\underline{h})b\cdot\partial_x\frac{1}{\det\underline{h}} \stackrel{(1080)}{=} -(\det\underline{h})\mathcal{D}^{\circ}_{b}\frac{1}{\det\underline{h}} \quad . \qquad (1134)
\end{aligned}
$$

Thus

$$
\underline{h}^{-1}(\partial_a)\cdot\left\{\left[\mathcal{D}^{\circ}_{b}\frac{\bar{h}(\partial_b)}{\det\underline{h}}\right]\wedge\bar{h}(a)\right\} = \partial_a\cdot\left\{\left[\mathcal{D}^{\circ}_{b}\frac{\bar{h}(\partial_b)}{\det\underline{h}}\right]\wedge a\right\} = -(n-1)\left[\mathcal{D}^{\circ}_{b}\frac{\bar{h}(\partial_b)}{\det\underline{h}}\right] \qquad (1135)
$$

and the contraction of the Ω-field equation (1131) becomes then in a n-dimensional space ($\partial_a a = n$)

$$
\begin{aligned}
\kappa\underline{h}^{-1}(\partial_a)\cdot S(a) &\stackrel{(1131)}{=} \underline{h}^{-1}(\partial_a)\cdot\left[\dot{\mathcal{D}}^{\circ}\wedge\dot{\bar{h}}(a)\right] + (\det\underline{h})\underbrace{\underline{h}^{-1}(\partial_a)\cdot\left\{\left[\mathcal{D}^{\circ}_{b}\frac{\bar{h}(\partial_b)}{\det\underline{h}}\right]\wedge\bar{h}(a)\right\}}_{\stackrel{(1135)}{=}-(n-1)\mathcal{D}^{\circ}_{b}\left[\frac{\bar{h}(\partial_b)}{\det\underline{h}}\right]} \\
&\stackrel{(1132)}{=} -(n-2)(\det\underline{h})\mathcal{D}^{\circ}_{b}\left[\frac{\bar{h}(\partial_b)}{\det\underline{h}}\right] \quad . \qquad (1136)
\end{aligned}
$$

Thus if we are not in two dimensions ($n = 2$) *we need*

$$
\mathcal{D}^{\circ}_{b}\left[\frac{\bar{h}(\partial_b)}{\det\underline{h}}\right] = 0 \quad , \qquad (1137)
$$

in order to have a spin tensor with zero contraction. But (1137) is exactly the condition (1108), which leads to the minimal coupled Dirac equation! Substituting with

$$
\mathcal{L}_m \stackrel{\text{def}}{=} \bar{h}(\partial_a)\left\{\underline{\psi}i\gamma_3\tilde{\psi} + \frac{1}{2}\Omega(a)\psi i\gamma_3\tilde{\psi}\right\} + \underline{h}(A)\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi} \qquad (1138)
$$

the covariant Dirac Lagrangian as the matter Lagrangian yields for the spin bivector

$$
\begin{aligned}
S(a) &\stackrel{(1129)}{=} \partial_{\Omega(a)}\mathcal{L}_m = \frac{a\cdot b}{2}\langle\psi i\gamma_3\tilde{\psi}\bar{h}(\partial_b)\rangle_2 \\
&= \frac{1}{2}(\psi i\gamma_3\tilde{\psi})\cdot\bar{h}(a) \qquad (1139)
\end{aligned}
$$

and hence

$$
\mathcal{S}(a) \stackrel{(1129)}{=} S(\bar{h}^{-1}(a)) = \frac{1}{2}(\psi i\gamma_3\tilde{\psi})\cdot a \quad . \qquad (1140)
$$

It is remarkable that this spin tensor has indeed zero contraction

$$
\partial_a\cdot\mathcal{S}(a) \stackrel{(1140)}{=} (\partial_a\wedge a)\frac{1}{2}\psi i\gamma_3\tilde{\psi} = 0 \quad . \qquad (1141)
$$

The Ω-Field Equation for a tracefree Spin With a spin of this type, the Ω-field equation (1131) becomes for a *constant* vector a

$$\kappa\mathcal{S}(\bar{h}(a)) \overset{(1137)}{=} \mathcal{D}^{O} \wedge \bar{h}(a) \quad . \tag{1142}$$

If we have with a_x a not constant vector as the argument in (1142), we can use (1083) to find

$$\dot{\mathcal{D}}^{O} \wedge \dot{\bar{h}}(a_x) = \dot{\mathcal{D}}^{O} \wedge [\bar{h}(a_x)]^{\bullet} - \bar{h}(\dot{\partial}_x \wedge \dot{a}_x) \quad . \tag{1143}$$

Substituting this in (1142) yields

$$\kappa\mathcal{S}(\bar{h}(a_x)) \overset{(1143)}{=} \dot{\mathcal{D}}^{O} \wedge [\bar{h}(a_x)]^{\bullet} - \bar{h}(\dot{\partial}_x \wedge \dot{a}_x) \quad . \tag{1144}$$

Therefore, if we have *vanishing spin, i.e.*, $\mathcal{S}(a) = 0$, it follows

$$\mathcal{D}^{O} \wedge \bar{h}(a_x) = \bar{h}(\dot{\partial}_x \wedge \dot{a}_x) \tag{1145}$$

and so

$$\mathcal{D}^{O} \wedge \mathcal{D}^{O} \wedge a(x) \overset{(1145)}{=} \mathcal{D}^{O} \wedge \bar{h}(\partial_x \wedge \bar{h}^{-1}(a_x)) \overset{(1145)}{=} \bar{h}(\partial_x \wedge \partial_x \wedge \bar{h}^{-1}(a_x)) = 0 \quad . \tag{1146}$$

9.3.4 The Curvature in Terms of ω

Assuming *no spin* the field equation (1142) becomes for a *constant* vector a

$$0 \overset{(1142)}{=} \mathcal{D}^{O} \wedge \bar{h}(a) \overset{(1078)}{=} \bar{h}(\dot{\partial}_x) \wedge \dot{\bar{h}}(a) + \partial_b \wedge [\omega(b) \times \bar{h}(a)] \tag{1147}$$

and we conclude

$$\bar{h}(\dot{\partial}_x) \wedge \dot{\bar{h}}(a) \overset{(1147)}{=} -\partial_b \wedge [\omega(b) \times \bar{h}(a)] \quad . \tag{1148}$$

Now

$$\langle (b \wedge a) \left\{ \bar{h}(\dot{\partial}_x) \wedge \dot{\bar{h}}(d) \right\} \rangle = a \cdot \bar{h}(\dot{\partial}_x) b \cdot \dot{\bar{h}}(d) - b \cdot \bar{h}(\dot{\partial}_x) a \cdot \dot{\bar{h}}(d) \quad , \tag{1149}$$

but also

$$\begin{aligned} \langle (b \wedge a) \left\{ \bar{h}(\dot{\partial}_x) \wedge \dot{\bar{h}}(d) \right\} \rangle &\overset{(1148)}{=} -\langle (b \wedge a) \partial_b \wedge [\omega(b) \times \bar{h}(d)] \rangle \\ &= -\langle a \cdot \partial_b b \cdot [\omega(b) \cdot \bar{h}(d)] \rangle + \langle b \cdot \partial_b a \cdot [\omega(b) \cdot \bar{h}(d)] \rangle \\ &= -[b \cdot \omega(a)] \cdot \bar{h}(d) + [a \cdot \omega(b)] \cdot \bar{h}(d) \\ &= [a \cdot \omega(b) - b \cdot \omega(a)] \cdot \bar{h}(d) \quad . \end{aligned} \tag{1150}$$

Thus for two vector fields a_x, b_x

$$\begin{aligned} [a_x \cdot \bar{h}(\partial_x), b_x \cdot \bar{h}(\partial_x)] f &= a_x \cdot \bar{h}(\partial_x) b_x \cdot \bar{h}(\partial_x) f - b_x \cdot \bar{h}(\partial_x) a_x \cdot \bar{h}(\partial_x) f \\ &= \overbrace{a_x \cdot \bar{h}(\dot{\partial}_x) b_x \cdot \dot{\bar{h}}(\partial_x) f - b_x \cdot \bar{h}(\dot{\partial}_x) a_x \cdot \dot{\bar{h}}(\partial_x) f}^{\overset{(1149)}{=} \langle (b_x \wedge a_x) \{ \bar{h}(\dot{\partial}_x) \wedge \dot{\bar{h}}(\partial_x) \} \rangle f} + \\ &\quad + a_x \cdot \bar{h}(\dot{\partial}_x) \dot{b}_x \cdot \bar{h}(\partial_x) f - b_x \cdot \bar{h}(\dot{\partial}_x) \dot{a}_x \cdot \bar{h}(\partial_x) f \\ &\overset{(1149)}{=} \langle (b_x \wedge a_x) \left\{ \bar{h}(\dot{\partial}_x) \wedge \dot{\bar{h}}(\partial_x) \right\} \rangle f + \left[a_x \cdot \bar{h}(\dot{\partial}_x) \dot{b}_x - b_x \cdot \bar{h}(\dot{\partial}_x) \dot{a}_x \right] \cdot \bar{h}(\partial_x) f \\ &\overset{(1150)}{=} [a_x \cdot \omega(b_x) - b_x \cdot \omega(a_x)] \cdot \bar{h}(\partial_x) f + \left[a_x \cdot \bar{h}(\dot{\partial}_x) \dot{b}_x - b_x \cdot \bar{h}(\dot{\partial}_x) \dot{a}_x \right] \cdot \bar{h}(\partial_x) f \\ &= \left[a_x \cdot \omega(b_x) - b_x \cdot \omega(a_x) + a_x \cdot \bar{h}(\dot{\partial}_x) \dot{b}_x - b_x \cdot \bar{h}(\dot{\partial}_x) \dot{a}_x \right] \cdot \bar{h}(\partial_x) f \\ &= c_x \cdot \bar{h}(\partial_x) f \quad , \end{aligned} \tag{1151}$$

where

$$c_x \overset{\text{def}}{=} a_x \cdot \bar{h}(\dot{\partial}_x) \dot{b}_x - b_x \cdot \bar{h}(\dot{\partial}_x) \dot{a}_x + a_x \cdot \omega(b_x) - b_x \cdot \omega(a_x) \overset{(1078)}{=} a_x \cdot \dot{\mathcal{D}}^{O} \dot{b}_x - b_x \cdot \dot{\mathcal{D}}^{O} \dot{a}_x \quad . \tag{1152}$$

This enables us to express the curvature fully in terms of ω as

$$\begin{aligned}
\mathcal{R}(a_x \wedge b_x) &= R(\underline{h}(a_x) \wedge \underline{h}(b_x)) \\
&\overset{(1112)}{=} \underline{h}(a_x)\cdot\dot{\partial}_x\dot{\omega}(b_x) - \underline{h}(b_x)\cdot\dot{\partial}_x\dot{\omega}(a_x) + \omega(a_x)\times\omega(b_x) \\
&\quad -\omega(\underline{h}^{-1}\left[a_x\cdot\bar{h}(\dot{\partial}_x)\dot{\underline{h}}(b_x) - b_x\cdot\bar{h}(\dot{\partial}_x)\dot{\underline{h}}(a_x)\right] \\
&= \underline{h}(a_x)\cdot\partial_x\omega(b_x) - \underline{h}(b_x)\cdot\partial_x\omega(a_x) + \omega(a_x)\times\omega(b_x) \\
&\quad -\omega\left(\underline{h}^{-1}\left(\underline{h}(a_x)\cdot\partial_x\underline{h}(b_x) - \underline{h}(b_x)\cdot\partial_x\underline{h}(a_x)\right)\right) \\
&\overset{(1151)}{=} \underline{h}(a_x)\cdot\partial_x\omega(b_x) - \underline{h}(b_x)\cdot\partial_x\omega(a_x) + \omega(a_x)\times\omega(b_x) - \omega(c_x)\ .
\end{aligned} \tag{1153}$$

From (1146) it follows now

$$0 \overset{(1146)}{=} \mathcal{D}^O \wedge \mathcal{D}^O \wedge A \overset{(1078)}{=} \bar{h}(\partial_a)\wedge\bar{h}(\partial_b)\wedge\left[\mathcal{D}^O_a\mathcal{D}^O_b A\right] \tag{1154}$$

and we derive by exchanging a and b and subtracting

$$0 = \bar{h}(\partial_a\wedge\partial_b)\wedge\{R(a\wedge b)\times A\} = \partial_a\wedge\partial_b\wedge[\mathcal{R}(a\wedge b)\times A]\quad . \tag{1155}$$

Setting $A \overset{\text{def}}{=} c$ a vector and protracting over c yields

$$\begin{aligned}
0 &= \partial_c\wedge\partial_a\wedge\partial_b\wedge[\mathcal{R}(a\wedge b)\cdot c] \\
&= -\partial_a\wedge\partial_b\wedge\{\partial_c\wedge[c\cdot\mathcal{R}(a\wedge b)]\} \\
&= -2\partial_a\wedge\partial_b\wedge\mathcal{R}(a\wedge b)\quad .
\end{aligned} \tag{1156}$$

Taking the inner product with a vector c and using (1155) gives finally

$$\begin{aligned}
0 &= c\cdot[\partial_a\wedge\partial_b\wedge\mathcal{R}(a\wedge b)] \\
&= (c\cdot\partial_a)\partial_b\wedge\mathcal{R}(a\wedge b) - \partial_a\wedge[(c\cdot\partial_b)\mathcal{R}(a\wedge b)] + \underbrace{\partial_a\wedge\partial_b\wedge[c\cdot\mathcal{R}(a\wedge b)]}_{\overset{(1155)}{=}0} \\
&= \partial_b\wedge\mathcal{R}(c\wedge b) - \partial_a\wedge\mathcal{R}(a\wedge c) \\
&= 2\partial_b\wedge\mathcal{R}(c\wedge b)\quad .
\end{aligned} \tag{1157}$$

This condition also expresses the symmetry of $\mathcal{R}(a\wedge b)$. This is seen by

$$\begin{aligned}
\bar{\mathcal{R}}(a\wedge b) &= \frac{1}{2}\partial_d\wedge\partial_c(a\wedge b) * \mathcal{R}(c\wedge d) \\
&= \frac{1}{2}(a\wedge b)\cdot\overbrace{[\partial_d\wedge\partial_c\wedge\mathcal{R}(c\wedge d)]}^{\overset{(1157)}{=}0} - \frac{1}{2}\overbrace{(a\wedge b)\cdot(\partial_d\wedge\partial_c)\mathcal{R}(c\wedge d)}^{=2\mathcal{R}(a\wedge b)} \\
&\quad -\frac{1}{2}[a\cdot(\partial_d\wedge\partial_c)]\wedge[b\cdot\mathcal{R}(c\wedge d)] + \frac{1}{2}[b\cdot(\partial_d\wedge\partial_c)]\wedge[a\cdot\mathcal{R}(c\wedge d)] \\
&= -\mathcal{R}(a\wedge b) - \underbrace{\partial_c\wedge[b\cdot\mathcal{R}(c\wedge a)]}_{=-b\cdot[\partial_c\wedge\mathcal{R}(c\wedge a)]+b\cdot\partial_c\mathcal{R}(c\wedge a)} + \partial_c\wedge[a\cdot\mathcal{R}(c\wedge b)] \\
&= -\mathcal{R}(a\wedge b) - \mathcal{R}(b\wedge a) + \mathcal{R}(a\wedge b) = \mathcal{R}(a\wedge b)
\end{aligned} \tag{1158}$$

and thus

$$\mathcal{R}(a\wedge b) = \bar{\mathcal{R}}(a\wedge b)\quad . \tag{1159}$$

9.3.5 The Weyl Tensor

Lets consider a n-dimensional space, *i.e.*, $\partial_a\cdot a = n$. Since

$$\begin{aligned}
\partial_a\cdot[a\wedge\mathcal{R}(b) - b\wedge\mathcal{R}(a)] &= n\mathcal{R}(b) - \mathcal{R}(b) - \mathcal{R}(b) + b\wedge\overbrace{[\partial_a\cdot\mathcal{R}(a)]}^{=\mathcal{R}} \\
&= (n-2)\mathcal{R}(b) + b\mathcal{R}
\end{aligned} \tag{1160}$$

and

$$\partial_a \cdot (a \wedge b)\mathcal{R} = nb\mathcal{R} - b\mathcal{R} = (n-1)b\mathcal{R} \quad , \tag{1161}$$

we have

$$\partial_a \cdot \left\{ \frac{1}{n-2} [a \wedge \mathcal{R}(b) - b \wedge \mathcal{R}(a)] - \frac{1}{(n-1)(n-2)} (a \wedge b)\mathcal{R} \right\} = \mathcal{R}(b) \quad . \tag{1162}$$

Thus we define the two bivector fields

$$\left.\begin{array}{lcc} \mathcal{U}(a \wedge b) & \stackrel{\text{def}}{=} & \frac{1}{(n-2)} [a \wedge \mathcal{R}(b) - b \wedge \mathcal{R}(a)] - \frac{1}{(n-1)(n-2)} (a \wedge b)\mathcal{R} \\ \mathcal{W}(a \wedge b) & \stackrel{\text{def}}{=} & \mathcal{R}(a \wedge b) - \mathcal{U}(a \wedge b) \end{array}\right\} \tag{1163}$$

and write

$$\mathcal{R}(a \wedge b) = \mathcal{W}(a \wedge b) + \mathcal{U}(a \wedge b) \tag{1164}$$

with the **Weyl Tensor** $\mathcal{W}(a \wedge b)$. This definition ensures with

$$\mathcal{R}(b) = \partial_a \cdot \mathcal{R}(a \wedge b) \stackrel{(1164)}{=} \partial_a \cdot \mathcal{W}(a \wedge b) + \partial_a \cdot \mathcal{U}(a \wedge b) \stackrel{(1162)}{=} \partial_a \cdot \mathcal{W}(a \wedge b) + \mathcal{R}(b) \tag{1165}$$

that the Weyl tensor has vanishing contraction, *i.e.*,

$$\partial_a \cdot \mathcal{W}(a \wedge b) = 0 \quad . \tag{1166}$$

Since $\partial_a \wedge \mathcal{R}(a \wedge b) = \partial_a \wedge \mathcal{U}(a \wedge b) = 0$, it follows further that $\partial_a \wedge \mathcal{W}(a \wedge b) = 0$. Combined with (1166) this gives the result

$$\partial_a \mathcal{W}(a \wedge b) = 0 \quad . \tag{1167}$$

Thus, the Weyl tensor is tractionless. The vanashing contraction implies that, in the Einstein field equation, the Weyl tensor does thus not contribute to the *trace* of the energy-momentum tensor.

9.4 Concluding Remarks

This section gave a short and compact overview of Gravity-Gauge theory. By demanding position and rotation gauge invariance of the physical fields we constructed a General-Relativity-like theory, but in a flat background space.

It incorporates the matter Lagrangian, which defines the energy-momentum and spin tensor — this goes clearly beyond the scope of General-Relativity.

Centre of the theory are the field equations for the gauge fields. The position gauge field $\bar{h}$ can be seen as the analogue of the fiducial tensor, while the rotation gauge field ω appears like the connection.

10 Conclusion

The aim of this thesis is a systematic exploration of space-time algebra. In Section 1 we developed the algebraic structure and established the mathematical tools to be used. Most of the results here have been derived before in [1]. In Section 2 we applied the developed tools to the easiest possible cases. It was shown, how the complex unit can be replaced by bivectors and how rotations and Lorentz transformations occur in two and three dimensions.

In Section 3 we discussed the application to the four-dimensional Minkowski space. Here we saw, how the Dirac and Pauli algebras arise naturally as the orthonormality condition.

Section 4 gives a short discussion of the Principle of Virtual Work. When we replace the inner product with the geometrical product we obtain an extra term corresponding to the change in torque. With (483) we found an unified notation for Hamilton's equations. It is only the graded structure of the Geometric Algebra, that allows this surprisingly compact form. Maybe the given ideas can be extended to a more general application to symplectic forms.

In Section 5 we showed, how Maxwell equations become one single equation in this formulation. The vector potential was introduced. These were standard results. Further we extended Maxwells equations to incorporate magnetic monopoles. This discussion could be continued by finding a transformation between magnetic and electric charges and current.

In the following Sections 6 and 7 it was shown how Geometric Algebra enters quantum mechanics. The discussion of the Dirac spinor revealed the bivector nature of the employed complex unit, which now becomes the unit bivector of the spin plane. Further we saw, how particle and antiparticle solutions are projection and rejection onto γ_0, respectively. In the same way, the spin up and down solutions are related to the spin direction in the rest-frame. All of this did not lead to new results, but to a much clearer geometric understanding.

In the discussion of field theory it was shown, how the Euler-Lagrange equations and Noether's theorem generalize to multivector valued fields and parameters. This allowed us to derive general expressions for the energy-momentum and angular-momentum tensor in a remarkably easy way. These results were given before in [2].

By looking at the derived expression for the energy-momentum tensor we realized the similarity to the algebraic structure of the Einstein field equation. We were able to construct the analogue of Ricci tensor and scalar for the Dirac and scalar field. But as was seen with (658) the analogue of the contracted Bianchi identity was not satisfied. It would be interesting to examine possible relations to the Einstein-Cartan-Kibble theory.

We discussed the scalar and Dirac field explicitly and derived the momentum in terms of the Fourier coefficients. This lays the basis for second quantization. It remains open to examine a possible relation of Geometric Algebra to particle creation and annihilation operators. This could indeed lead to a much deeper geometrical insight.

In Section 8 we looked at curved manifolds. We were able to find a generalization of the duality relations for the inner and outer derivative. By introducing a tetrad basis we translated the conventional tensor treatment into geometric algebra form. Here we used some ideas developed in [1], especially the derivation of the curvature tensor. We derived a number of identities for the curvature tensor, *i.e.*, the Bianchi and the contracted Bianchi identity. This are merely translations of well known relations. Some of the results were derived in [3] in a gauge-gravity approach. We gave a derivation of the Geodesic Deviation equation in Geometric Algebra form.

In Section 9 we finally gave a review of gravity as a gauge theory, as was developed in [3]. We introduced position and rotation gauge fields and constructed covariant derivatives. This leads to a curvature *on a flat background space* and to a general-relativity-like theory. In [3] this theory has already been applied to cosmology.

In all the discussed topics it could be seen, how Geometric Algebra not only simplifies calculations, but also reveals clearly the underlying geometric interpretations. In my opinion Geometric Algebra is one of the most exciting mathematical tools for physics, with an unlimited range of possible applications.

A More Vector Identities

A.1 Helpful Vector Identity

In order to be able to develop a determinant of vectors, we need the identity

$$0=\sum_{k=1}^{n}(-1)^{k+1}b_k\cdot\left(b_1\wedge\cdots\wedge\check{b}_k\wedge\cdots\wedge b_n\right)\quad, \tag{1168}$$

which we will prove by induction. For $n=1$ (1168) is true, since $b_1\cdot 1=0$. But if (1168) is true for n, it follows for $n+1$

$$\begin{aligned}
&\sum_{k=1}^{n+1}(-1)^{k+1}b_k\cdot(b_1\wedge\cdots\wedge\check{b}_k\wedge\cdots\wedge b_{n+1})\\
&= (-1)^{n+2}b_{n+1}\cdot(b_1\wedge\cdots\wedge b_n)+\sum_{k=1}^{n}(-1)^{k+1}b_k\cdot(b_1\wedge\cdots\wedge\check{b}_k\wedge\cdots\wedge b_{n+1})\\
&= (-1)^{n}b_{n+1}\cdot(b_1\wedge\cdots\wedge b_n)+\overbrace{\sum_{k=1}^{n}(-1)^{k+1}\left[b_k\cdot(b_1\wedge\cdots\wedge\check{b}_k\wedge\cdots\wedge b_n)\right]}^{\overset{(1168)}{=}0}\wedge b_{n+1}\\
&\quad\underbrace{+\sum_{k=1}^{n}(-1)^{k+1}(-1)^{n+1}b_k\cdot b_{n+1}(b_1\wedge\cdots\wedge\check{b}_k\wedge\cdots\wedge b_{n+1})}_{=-(-1)^{n}b_{n+1}\cdot(b_1\wedge\cdots\wedge b_{n+1})}\\
&= 0\quad.
\end{aligned} \tag{1169}$$

Hence (1168) is also true for $n+1$. Thus (1168) is proved by induction for all integers n.

A.2 Determinant of Vectors

Here I want to give the prove for (236), *i.e.*, that for a set of *vectors* $\{a_n\}$

$$\begin{vmatrix} a_1 & \cdots & a_n\\ \vdots & & \vdots\\ a_1 & \cdots & a_n\end{vmatrix}=n!\;a_1\wedge\cdots\wedge a_n\quad. \tag{1170}$$

Now (1170) is true for $n=1$, since by definition $|\,a_1\,|\overset{(233)}{=}a_1$. But if (1170) is true for any n it follows for $n+1$

$$\begin{aligned}
\begin{vmatrix} a_1 & \cdots & a_n\\ \vdots & & \vdots\\ a_1 & \cdots & a_n\end{vmatrix}
&= \sum_{k=1}^{n+1}(-1)^{k+1}a_k\begin{vmatrix} \breve{a}_1 & \cdots & \breve{a}_k & \cdots & \breve{a}_{n+1}\\ a_1 & \cdots & \breve{a}_k & \cdots & a_{n+1}\\ \vdots & & \vdots & & \vdots\\ a_1 & \cdots & \breve{a}_k & \cdots & a_{n+1}\end{vmatrix}\\
&= n!\sum_{k=1}^{n+1}(-1)^{k+1}a_k(a_1\wedge\cdots\wedge\breve{a}_k\wedge\cdots\wedge a_{n+1})\\
&= (n+1)n!\;a_1\wedge\cdots\wedge a_{n+1}+n!\underbrace{\sum_{k=1}^{n+1}(-1)^{k+1}a_k\cdot(a_1\wedge\cdots\wedge\breve{a}_k\wedge\cdots\wedge a_{n+1})}_{\overset{(1168)}{=}0}\\
&= (n+1)!\;a_1\wedge\cdots\wedge a_{n+1}\quad.
\end{aligned} \tag{1171}$$

Therefore, (1170) is then also true for $n+1$. Hence we proved (1170) by induction for all integers n.

B The Momentum of the Scalar Field

I want to give here the explicit calculation of the four-momentum of the scalar field in terms of the Fourier coefficients. We can simplify further discussions of the $(\partial_x\phi_x^\pm)^2$-term by noting[117]

$$\begin{aligned}(\partial_x\phi_x)(\partial_x\phi_x) &= \langle\partial_x\phi_x\partial_x\phi_x\rangle = \langle\partial_x\gamma_0\gamma_0\phi_x\partial_x\phi_x\rangle = \langle\partial_x\gamma_0\phi_x\gamma_0\partial_x\phi_x\rangle \\ &= \langle(\partial'_x\phi_x)(\widetilde{\partial'_x}\phi_x)\rangle \end{aligned} \qquad (1173)$$

with

$$\partial'_x \stackrel{\text{def}}{=} \partial_x\gamma_0 = \gamma^\nu\partial_\nu\gamma_0 = \partial_0 - \sigma^n\partial_n \Rightarrow \widetilde{\partial}'_x = \partial_0 + \sigma^n\partial_n \quad . \qquad (1174)$$

The fields total four-momentum is obtained by integrating the momentum density over the three-dimensional rest-space. Using (1173) we find so for the γ_0 frame ($\phi_x^\pm \stackrel{\text{def}}{=} \phi^\pm(x)$)

$$\begin{aligned} p_\phi &\stackrel{(633)}{=} \int |d^3x|\,\mathcal{P} \\ &\stackrel{(719)}{=} \int |d^3x| \left\{(\partial_x\phi_x)(\partial_0\phi_x) - \frac{1}{2}\gamma_0\left[(\partial'_x\phi_x)(\widetilde{\partial}'_x\phi_x) - m^2\phi_x^2\right]\right\} \\ &\stackrel{(699)}{=} \int |d^3x|\,\partial_x\left[\phi_x^+ + \phi_x^-\right]\partial_0\left[\phi_x^+ + \phi_x^-\right] \\ &\quad -\frac{\gamma_0}{2}\int |d^3x| \left\{[\partial'_x\phi_x^+ + \partial'_x\phi_x^-][\widetilde{\partial}'_x\phi_x^+ + \widetilde{\partial}'_x\phi_x^-] - m^2[\phi_x^+ + \phi_x^-]^2\right\} \\ &= p_{\phi_x^+} + p_{\phi_x^-} + \int |d^3x|\,\left[\partial_x\phi_x^+\partial_0\phi_x^- + \partial_x\phi_x^-\partial_0\phi_x^+\right] \\ &\quad -\frac{\gamma_0}{2}\int |d^3x|\,\{(\partial'_x\phi_x^+)(\widetilde{\partial}'_x\phi_x^-) + (\partial'_x\phi_x^-)(\widetilde{\partial}'_x\phi_x^+) \\ &\quad -m^2(\phi_x^+\phi_x^- + \phi_x^-\phi_x^+)\} \quad . \end{aligned} \qquad (1175)$$

Now let us first consider the integrals over the pure $\phi_x^\pm$-terms. Here we have

$$p_{\phi^\pm} = \int |d^3x| \left\{(\partial_x\phi_x^\pm)(\partial_0\phi_x^\pm) - \frac{\gamma_0}{2}\left[(\partial'_x\phi_x^\pm)(\widetilde{\partial}'_x\phi_x^\pm) - m^2(\phi_x^\pm)^2\right]\right\} \quad . \qquad (1176)$$

We have for any operator D_n obeying

$$D_n e^{ik\cdot x} = d_n(ik)e^{ik\cdot x} \qquad (1177)$$

the relation

$$\begin{aligned}\int |d^3x|\,\left[D_1\phi^\pm(x)\right]\left[D_2\phi^\pm(x)\right] &\stackrel{(704)}{=} \frac{1}{(2\pi)^3}\iint \frac{d^3k\,d^3k'}{2\sqrt{k_0k'_0}} \\ &\quad \int |d^3x|\,\left[D_1 e^{\pm ik\cdot x}\phi^\pm(\vec{k}))(D_2 e^{\pm ik'\cdot x}\phi^\pm(\vec{k}')\right] \\ &= \iint \frac{d^3k\,d^3k'}{2\sqrt{k_0k'_0}} F^\pm_{D_1,D_2}(k,k') \quad , \end{aligned} \qquad (1178)$$

where $F^\pm_{D_1,D_2}$ is defined by

$$\begin{aligned} F^\pm_{D_1,D_2}(k,k') &\stackrel{\text{def}}{=} \frac{1}{(2\pi)^3}\int |d^3x|\, d_1(\pm ik)e^{\pm ik\cdot x}\phi^\pm_{\vec{k}}\, d_2(\pm ik')e^{\pm ik'\cdot x}\phi^\pm_{\vec{k}'} \\ &= \frac{1}{(2\pi)^3}\int |d^3x|\,\left(D_1 e^{\pm ik\cdot x}\right)\phi^\pm_{\vec{k}}\left(D_2 e^{\pm ik'\cdot x}\right)\phi^\pm_{\vec{k}'} \quad . \end{aligned} \qquad (1179)$$

[117]This only works, since ϕ_x is a pure scalar. Note that $\phi_x^\pm$ can have a pseudoscalar part. But from (699) we know

$$\langle\phi_x^+\rangle_4 = -\langle\phi_x^-\rangle_4 . \qquad (1172)$$

For $D_1 = D_2 = 1$ we obtain

$$F^{\pm}_{1,1} = \frac{1}{(2\pi)^3}\int |d^3x|\; e^{\pm i(k+k')\cdot x}\phi^{\pm}(\vec{k})\phi^{\pm}(\vec{k}')$$
$$= e^{\pm i(k_0+k_0')x_0}\delta^3(\vec{k}+\vec{k}')\phi^{\pm}(\vec{k})\phi^{\pm}(\vec{k}')$$
$$= e^{\pm 2ik_0x_0}\delta^3(\vec{k}+\vec{k}')\phi^{\pm}(\vec{k})\phi^{\pm}(-\vec{k}) \quad . \tag{1180}$$

Further, for $D_1 = \partial_x; D_2 = \partial_0$ we find[118]

$$F^{\pm}_{\partial_x,\partial_0} = \frac{1}{(2\pi)^3}\int |d^3x|\; (\mp ik)e^{\pm ik\cdot x}\phi^{\pm}(\vec{k})(\pm ik_0')e^{\pm ik'\cdot x}\phi^{\pm}(\vec{k}')$$
$$= -k_0' k e^{\pm i(k_0+k_0')x_0}\delta^3(\vec{k}+\vec{k}')\phi^{\pm}(\vec{k})\phi^{\pm}(\vec{k}')$$
$$= -k_0 k e^{\pm 2ik_0x_0}\delta^3(\vec{k}+\vec{k}')\phi^{\pm}(\vec{k})\phi^{\pm}(-\vec{k}) \quad . \tag{1182}$$

We have now the advantage that ∂_x' commutes with i and thus

$$F^{\pm}_{\partial_x',\widetilde{\partial_x'}} = \frac{1}{(2\pi)^3}\int |d^3x|\; (\partial_0 - \sigma^n\partial_n)e^{\pm ik\cdot x}\phi^{\pm}(\vec{k})(\partial_0 - \widetilde{\sigma^n}\partial_n)e^{\pm ik'\cdot x}\phi^{\pm}(\vec{k}')$$
$$= \frac{1}{(2\pi)^3}(\pm ik_0 \pm i\vec{k})(\pm ik_0' \mp i\vec{k}')e^{\pm i(k_0+k_0')x_0}\phi^{\pm}(\vec{k})\phi^{\pm}(\vec{k}')\int |d^3x|\; e^{\mp i(\vec{k}+\vec{k}')\cdot\vec{x}}$$
$$= -(k_0+\vec{k})(k_0'-\vec{k}')e^{\pm i(k_0+k_0')x_0}\phi^{\pm}(\vec{k})\phi^{\pm}(\vec{k}')\delta^3(\vec{k}+\vec{k}')$$
$$= -(k_0+\vec{k})(k_0'+\vec{k})e^{\pm 2ik_0x_0}\phi^{\pm}(\vec{k})\phi^{\pm}(-\vec{k})\delta^3(\vec{k}+\vec{k}')$$
$$= -\left(k_0^2 + \sum_{n=1}^{3} k_n^2 - 2k_0\vec{k}\right) e^{\pm 2ik_0x_0}\phi^{\pm}(\vec{k})\phi^{\pm}(-\vec{k})\delta^3(\vec{k}+\vec{k}') \quad . \tag{1183}$$

Thus we derive the total momentum of the $\phi^{\pm}_x$-field ($\alpha \stackrel{\text{def}}{=} 2k_0x_0$)

$$p_{\phi^{\pm}_x} \stackrel{(1176)}{=} \iint \frac{d^3k\, d^3k'}{2\sqrt{k_0k_0'}}\left[F^{\pm}_{\partial_x,\partial_0} - \frac{\gamma_0}{2}\left(F^{\pm}_{\partial_x',\widetilde{\partial_x'}} - m^2F^{\pm}_{1,1}\right)\right]$$
$$= \iint \frac{d^3k\, d^3k'}{2\sqrt{k_0k_0'}}\left[-k_0k - \frac{\gamma_0}{2}(-k_0^2 - \vec{k}^2 + \overbrace{2k_0\vec{k}}^{\text{antisym.}}) + \frac{\gamma_0}{2}m^2\right]$$
$$\underbrace{e^{\pm i\alpha}\phi^{\pm}(\vec{k})\phi^{\pm}(-\vec{k})}_{\text{symmetric}}\delta^3(\vec{k}+\vec{k}')$$
$$= \int |d^3k|\; \frac{1}{2k_0}\left[-k_0k + \frac{\gamma_0}{2}(k_0^2 + \underbrace{\vec{k}^2}_{=k_0^2-m^2} + m^2)\right] e^{\pm i\alpha}\phi^{\pm}(\vec{k})\phi^{\pm}(-\vec{k})$$
$$= \int |d^3k|\; \frac{1}{2k_0}\left[-k_0k + \gamma_0k_0^2\right] e^{\pm i\alpha}\phi^{\pm}(\vec{k})\phi^{\pm}(-\vec{k})$$
$$= \int |d^3k|\; \left[-k_0^2 + \underbrace{k_0\vec{k}}_{\text{antisym.}} + k_0^2\right]\gamma_0\underbrace{\frac{e^{\pm i\alpha}\phi^{\pm}(\vec{k})\phi^{\pm}(-\vec{k})}{2k_0}}_{\text{sym.}}$$
$$= 0 \quad . \tag{1184}$$

We come now to a discussion of the remaining terms for the momentum, which now contain ϕ^+_x *and* ϕ^-_x. In each case we substitute the Fourier representation and obtain a δ-function.

$$\int |d^3x|\; \phi^{\pm}_x\phi^{\mp}_x = \frac{1}{(2\pi)^3}\iint \frac{d^3k\, d^3k'}{2\sqrt{k_0k_0'}}\int |d^3x|\; e^{\pm ik\cdot x}\phi^{\pm}_{\vec{k}}e^{\mp ik'\cdot x}\phi^{\mp}_{\vec{k}'}$$

[118]Note that

$$\partial_x e^{ik\cdot x} = \gamma^\nu\partial_\nu e^{ik\cdot x} = \gamma^\nu ik_\nu e^{ik\cdot x} = kie^{ik\cdot x} = -ike^{ik\cdot x}. \tag{1181}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^3}\iint \frac{d^3k\, d^3k'}{2\sqrt{k_0k_0'}}\, e^{\pm i(k_0-k_0')x_0}\phi^\pm_{\vec k}\phi^\mp_{\vec k'} \underbrace{\int |d^3x|\, e^{\mp i(\vec k-\vec k')\cdot\vec x}}_{=(2\pi)^3\delta^3(\vec k-\vec k')} \\
&= \int \frac{d^3k}{2k_0}\phi^\pm_{\vec k}\phi^\mp_{\vec k}
\end{aligned} \tag{1185}
$$

$$
\begin{aligned}
\int |d^3x|\, \partial_x\phi^\pm_x\partial_0\phi^\mp_x &= \frac{1}{(2\pi)^3}\iint \frac{d^3k\, d^3k'}{2\sqrt{k_0k_0'}}\int |d^3x|\, \partial_x e^{\pm ik\cdot x}\phi^\pm_{\vec k}\partial_0 e^{\mp ik'\cdot x}\phi^\mp_{\vec k'} \\
&= \frac{1}{(2\pi)^3}\iint \frac{d^3k\, d^3k'}{2\sqrt{k_0k_0'}}\int |d^3x|\, (\pm ki)e^{\pm ik\cdot x}\phi^\pm_{\vec k}(\mp k_0 i)e^{\mp ik'\cdot x}\phi^\mp_{\vec k'} \\
&= \frac{1}{(2\pi)^3}\iint \frac{d^3k\, d^3k'}{2\sqrt{k_0k_0'}}\int |d^3x|\, k_0 k e^{\pm i(k_0-k_0')x_0} \underbrace{e^{\mp i(\vec k-\vec k')\cdot\vec x}}_{\overset{(684)}{=}(2\pi)^3\delta^3(\vec k-\vec k')} \phi^\pm_{\vec k}\phi^\mp_{\vec k'} \\
&= \int |d^3k|\, \frac{k_0 k}{2k_0}\phi^\pm_{\vec k}\phi^\mp_{\vec k} = \int |d^3k|\, \frac{k}{2}\phi^\pm_{\vec k}\phi^\mp_{\vec k}
\end{aligned} \tag{1186}
$$

$$
\begin{aligned}
\int |d^3x|\, (\partial'_x\phi^\pm_x)(\tilde\partial'_x\phi^\mp_x) &= \frac{1}{(2\pi)^3}\iint \frac{d^3k\, d^3k'}{2\sqrt{k_0k_0'}}\int |d^3x|\, \partial'_x e^{\pm ik\cdot x}\phi^\pm_{\vec k}\tilde\partial'_x e^{\mp ik\cdot x}\phi^\mp_{\vec k'} \\
&= \frac{1}{(2\pi)^3}\iint \frac{d^3k\, d^3k'}{2\sqrt{k_0k_0'}}\int |d^3x|\, (\pm i)(k_0+\vec k)e^{\pm ik\cdot x}\phi^\pm_{\vec k}(\mp i)(k_0'-\vec k')e^{\mp ik\cdot x}\phi^\mp_{\vec k'} \\
&= \frac{1}{(2\pi)^3}\iint \frac{d^3k\, d^3k'}{2\sqrt{k_0k_0'}}(k_0+\vec k)(k_0'-\vec k')\phi^\pm_{\vec k}\phi^\mp_{\vec k'}e^{\pm i(k_0-k_0')x_0} \underbrace{\int |d^3x|\, e^{\mp i(\vec k-\vec k')\cdot\vec x}}_{\overset{(684)}{=}(2\pi)^3\delta^3(\vec k-\vec k')} \\
&= \int |d^3k|\, \frac{1}{2k_0}\underbrace{(k_0^2-\vec k^2)}_{=k^2\overset{(693)}{=}m^2}\phi^\pm_{\vec k}\phi^\mp_{\vec k}\ .
\end{aligned} \tag{1187}
$$

So we finally obtain for the total momentum under use of (1184)

$$
\begin{aligned}
p_{\phi_x} &\overset{(1184)}{=} \int |d^3x|\, \Big\{\partial_x\phi^+_x\partial_0\phi^-_x + \partial_x\phi^-_x\partial_0\phi^+_x - \frac{\gamma_0}{2}\left(\partial'_x\phi^+_x\tilde\partial'_x\phi^-_x + \partial'_x\phi^-_x\tilde\partial'_x\phi^+_x\right) \\
&\qquad +\gamma_0\frac{m^2}{2}\left(\phi^+_x\phi^-_x + \phi^-_x\phi^+_x\right)\Big\} \\
&= \frac{1}{2}\int |d^3k|\, \Big\{k\left[\phi^+_{\vec k}\phi^-_{\vec k} + \phi^-_{\vec k}\phi^+_{\vec k}\right] - \frac{\gamma_0 k^2}{2k_0}\left[\phi^+_{\vec k}\phi^-_{\vec k} + \phi^-_{\vec k}\phi^+_{\vec k}\right] \\
&\qquad + \frac{\gamma_0 m^2}{2k_0}\left[\phi^+_{\vec k}\phi^-_{\vec k} + \phi^-_{\vec k}\phi^+_{\vec k}\right]\Big\} \\
&= \int |d^3k|\, \frac{k}{2}\left[\phi^+_{\vec k}\phi^-_{\vec k} + \phi^-_{\vec k}\phi^+_{\vec k}\right] \quad ,
\end{aligned} \tag{1188}
$$

which is in accordance with standard results, like in [6].

C Explicit Calculation of Spin State Products

C.1 Products of Lorentz Boosts

We first look at general products $l_{\vec{p}} M l_{-\vec{p}'}$, involing Lorentz boosts of the type

$$l_{\vec{p}} = \frac{\sqrt{E+m} + \sqrt{E-m}\hat{p}}{\sqrt{2m}} = \frac{E+m+\vec{p}}{\sqrt{2m(E+m)}} \quad . \tag{1189}$$

1. If M is a scalar, we have

   - scalar part

   $$\langle l_{\vec{p}} l_{-\vec{p}'} \rangle \overset{(1189)}{=} \frac{(E+m)(E'+m) - \vec{p}\cdot\vec{p}'}{2m\sqrt{(E+m)(E'+m)}} \overset{\text{def}}{=} \frac{\sqrt{EE'}}{m} B(\vec{p},\vec{p}') \quad . \tag{1190}$$

   In the special case of $\vec{p}' = \pm\vec{p}$ this becomes

   $$B(\vec{p},\pm\vec{p}) = \begin{cases} \frac{m}{E} & \text{for } + \\ 1 & \text{for } - \end{cases} \quad . \tag{1191}$$

   - $\mathcal{P}$-vector part

   $$\langle l_{\vec{p}} l_{-\vec{p}'} \rangle_1^{\mathcal{P}} \overset{(1189)}{=} \frac{\langle (E+m+\vec{p})(E'+m-\vec{p}') \rangle_1^{\mathcal{P}}}{2m\sqrt{(E+m)(E'+m)}} = \frac{(E'+m)\vec{p} - (E+m)\vec{p}'}{2m\sqrt{(E+m)(E'+m)}} = \frac{\sqrt{EE'}}{m}\vec{k}^- \tag{1192}$$

   with

   $$\vec{k}^-_{\vec{p},\vec{p}'} \overset{\text{def}}{=} \vec{k}^-(\vec{p},\vec{p}') \overset{\text{def}}{=} \frac{1}{2\sqrt{EE'}} \left\{ \sqrt{\frac{E'+m}{E+m}}\vec{p} - \sqrt{\frac{E+m}{E'+m}}\vec{p}' \right\} \quad . \tag{1193}$$

   It is antisymmetric in $\vec{p}$ and $\vec{p}'$ and obeys

   $$\vec{k}_{\vec{p},\pm\vec{p}} = \begin{cases} 0 & \text{for } + \\ \frac{\vec{p}}{2E} & \text{for } - \end{cases} \quad . \tag{1194}$$

   - $\mathcal{P}$-bivector part

   $$\begin{aligned} \frac{m}{\sqrt{EE'}} \langle l_{\vec{p}} l_{-\vec{p}'} \rangle_2^{\mathcal{P}} &= \frac{\langle (E+m+\vec{p})(E'+m-\vec{p}') \rangle_2^{\mathcal{P}}}{2\sqrt{EE'(E+m)(E'+m)}} \\ &= -\frac{\vec{p}\wedge\vec{p}'}{2\sqrt{EE'(E+m)(E'+m)}} \overset{\text{def}}{=} A(\vec{p},\vec{p}') \end{aligned} \tag{1195}$$

   - pseudoscalar part Since $l_{\vec{p}}$ contains only grade 0 and 1, we can immediately conclude

   $$\langle l_{\vec{p}} l_{-\vec{p}'} \rangle_3^{\mathcal{P}} = 0 \quad . \tag{1196}$$

2. If M is a $\mathcal{P}$-vector

   - scalar part

   $$\langle l_{\vec{p}} \vec{a} l_{-\vec{p}'} \rangle = \frac{1}{2m} \left\{ \sqrt{\frac{E'+m}{E+m}}\vec{p} - \sqrt{\frac{E+m}{E'+m}}\vec{p}' \right\} \cdot \vec{a} = \frac{\sqrt{EE'}}{m} \vec{k}^-_{\vec{p},\vec{p}'} \cdot \vec{a} \quad . \tag{1197}$$

   - $\mathcal{P}$-vector part

   $$\begin{aligned} \langle l_{\vec{p}} \vec{a} l_{-\vec{p}'} \rangle_1^{\mathcal{P}} &= \frac{\langle (E+m+\vec{p})\vec{a}(E'+m-\vec{p}') \rangle_1^{\mathcal{P}}}{2m\sqrt{(E+m)(E'+m)}} = \frac{(E+m)(E'+m)\vec{a} - \langle \vec{p}\vec{a}\vec{p}' \rangle_1^{\mathcal{P}}}{2m\sqrt{(E+m)(E'+m)}} \\ &= \frac{\sqrt{(E+m)(E'+m)}\vec{a} - \sqrt{(E-m)(E'-m)}\langle \hat{p}\vec{a}\hat{p}' \rangle_1^{\mathcal{P}}}{2m} \overset{\text{def}}{=} \frac{\sqrt{EE'}}{m} C(\vec{p},\vec{p}',\vec{a}) \quad , \end{aligned} \tag{1198}$$

	$\frac{m}{\sqrt{EE'}}\langle l_{\vec{p}} l_{-\vec{p}'}\rangle^{\mathcal{P}}_n$	$\frac{m}{\sqrt{EE'}}\langle l_{\vec{p}} \vec{a} l_{-\vec{p}'}\rangle^{\mathcal{P}}_n$	$\frac{m}{\sqrt{EE'}}\langle l_{\vec{p}} i\vec{a} l_{-\vec{p}'}\rangle^{\mathcal{P}}_n$	$\frac{m}{\sqrt{EE'}}\langle l_{\vec{p}} \vec{i} l_{-\vec{p}'}\rangle^{\mathcal{P}}_n$
$n=0$	$B_{\vec{p},\vec{p}'}$	$\vec{k}^-_{\vec{p},\vec{p}'}\cdot\vec{a}$	$-V_{\vec{p},\vec{p}'}(\vec{a})$	0
$n=1$	$\vec{k}^-_{\vec{p},\vec{p}'}$	$C_{\vec{p},\vec{p}'}(\vec{a})$	$i\vec{k}^-_{\vec{p},-\vec{p}'}\wedge\vec{a}$	$iA_{\vec{p},\vec{p}'}$
$n=2$	$A_{\vec{p},\vec{p}'}$	$\vec{k}^-_{\vec{p},-\vec{p}'}\wedge\vec{a}$	$iC_{\vec{p},\vec{p}'}(\vec{a})$	$i\vec{k}^-_{\vec{p},\vec{p}'}$
$n=3$	0	$iV_{\vec{p},\vec{p}'}(\vec{a})$	$i\vec{k}^-_{\vec{p},\vec{p}'}\cdot\vec{a}$	$iB_{\vec{p},\vec{p}'}$

Table 6: Table of the different grades in $\mathcal{P}$ of $l_{\vec{p}} M l_{-\vec{p}'}$.

with

$$C(\vec{p},\vec{p}',\vec{a}) \stackrel{\text{def}}{=} \frac{\sqrt{(E+m)(E'+m)}\vec{a} - \sqrt{(E-m)(E'-m)}\langle\hat{p}\vec{a}\hat{p}\rangle^{\mathcal{P}}_1}{2\sqrt{EE'}} \quad . \tag{1199}$$

For $\vec{p}' = \pm\vec{p}$ it takes the form

$$\begin{aligned} C(\vec{p},\pm\vec{p},\vec{a}) &\stackrel{(1199)}{=} \frac{1}{2E}\{(E+m)\vec{a} \mp (E-m)\hat{p}\vec{a}\hat{p}\} \\ &= \begin{cases} \frac{\vec{a}_\parallel + \frac{E}{m}\vec{a}_\perp}{E} & \text{for} \quad - \\ \frac{\vec{a}_\perp + \frac{E}{m}\vec{a}_\parallel}{E} & \text{for} \quad + \end{cases} = \begin{cases} \frac{1}{E}\vec{a}_\parallel + \frac{1}{m}\vec{a}_\perp & \text{for} \quad - \\ \frac{1}{E}\vec{a}_\perp + \frac{1}{m}\vec{a}_\parallel & \text{for} \quad + \end{cases} \end{aligned} \tag{1200}$$

- $\mathcal{P}$-bivector part

$$\begin{aligned} \frac{m}{\sqrt{EE'}}\langle l_{\vec{p}}\vec{a}l_{-\vec{p}'}\rangle^{\mathcal{P}}_2 &\stackrel{(1189)}{=} \frac{m}{\sqrt{EE'}}\frac{\langle(E+m+\vec{p})\vec{a}(E'+m-\vec{p}')\rangle^{\mathcal{P}}_2}{2m\sqrt{(E+m)(E'+m)}} = \frac{(E'+m)\vec{p}\wedge\vec{a}-(E+m)\vec{a}\wedge\vec{p}'}{2\sqrt{EE'(E+m)(E'+m)}} \\ &\stackrel{(1193)}{=} \vec{k}^-_{\vec{p},-\vec{p}'}\wedge\vec{a} \end{aligned} \tag{1201}$$

- pseudoscalar part

$$\langle l_{\vec{p}}\vec{a}l_{-\vec{p}'}\rangle^{\mathcal{P}}_3 = \frac{-\vec{p}\wedge\vec{a}\wedge\vec{p}'}{2m\sqrt{(E+m)(E'+m)}} \stackrel{\text{def}}{=} \frac{\sqrt{EE'}}{m} iV(\vec{p},\vec{p}',\vec{a}) \quad . \tag{1202}$$

Here we have now

$$V(\vec{p},\pm\vec{p},\vec{a}) = 0 \quad . \tag{1203}$$

C.2 Products of Spin-States

We can now turn our attention to the calculation of the products of spin energy states. It is sufficient to calculate $v^+_0 c\tilde{v}^+_0, v^+_1 c\tilde{v}^+_0, v^-_0 c\tilde{v}^+_0$ and $v^-_0 c\tilde{v}^+_1$, since all other products can be derived with above relations.

1. $v^+_0(\vec{p})c\tilde{v}^+_0(\vec{p}')$

   - scalar part

$$\begin{aligned} \langle v^+_0(\vec{p})c\tilde{v}^+_0(\vec{p}')\rangle &\stackrel{(801)}{=} \frac{m}{\sqrt{EE'}}\langle l_{\vec{p}}cl_{-\vec{p}'}\rangle = \frac{m}{\sqrt{EE'}}\langle l_{\vec{p}}l_{-\vec{p}'}\rangle\mathfrak{R}[c] + \frac{m}{\sqrt{EE'}}\langle l_{\vec{p}}i\sigma_3 l_{-\vec{p}'}\rangle\mathfrak{I}[c] \\ &= B(\vec{p},\vec{p}')\mathfrak{R}[c] + V(\vec{p},\vec{p}',\sigma_3)\mathfrak{I}[c] \quad . \end{aligned} \tag{1204}$$

   - $\mathcal{P}$-vector part

$$\begin{aligned} \langle v^+_0(\vec{p})c\tilde{v}^+_0(\vec{p}')\rangle^{\mathcal{P}}_1 &\stackrel{(801)}{=} \frac{m}{\sqrt{EE'}}\{\langle l_{\vec{p}}l_{-\vec{p}'}\rangle^{\mathcal{P}}_1\mathfrak{R}[c] + \langle l_{\vec{p}}i\sigma_3 l_{-\vec{p}'}\rangle^{\mathcal{P}}_1\mathfrak{I}[c]\} \\ &= \vec{k}^-_{\vec{p},\vec{p}'}\mathfrak{R}[c] + i\vec{k}^+_{\vec{p},\vec{p}'}\wedge\sigma_3\mathfrak{I}[c] \quad . \end{aligned} \tag{1205}$$

   - pseudoscalar part

$$\begin{aligned} \langle v^+_0(\vec{p})cv^+_0(\vec{p}')\rangle^{\mathcal{P}}_3 &\stackrel{(801)}{=} \frac{m}{\sqrt{EE'}}\langle l_{\vec{p}}cl_{-\vec{p}'}\rangle^{\mathcal{P}}_3 = \frac{m}{\sqrt{EE'}}\langle l_{\vec{p}}i\sigma_3 l_{-\vec{p}'}\rangle^{\mathcal{P}}_3\mathfrak{I}[c] \\ &= i\vec{k}^-_{\vec{p},\vec{p}'}\cdot\sigma_3\mathfrak{I}[c] \end{aligned} \tag{1206}$$

2. $\nu_1^+(\vec{p})c\tilde{\nu}_0^+(\vec{p}\,')$

- scalar part

$$\langle\nu_1^+(\vec{p})c\tilde{\nu}_0^+(\vec{p}\,')\rangle \overset{(801)}{=} -\frac{m}{\sqrt{EE'}}\langle l_{\vec{p}}\ \overbrace{i\sigma_2 c}^{\text{bivector}}\ l_{-\vec{p}\,'}\rangle = -V_{\vec{p},\vec{p}\,'}(\sigma_2 c) \tag{1207}$$

- $\mathcal{P}$-vector part

$$\begin{aligned}\langle\nu_1^+(\vec{p})c\tilde{\nu}_0^+(\vec{p}\,')\rangle_1^{\mathcal{P}} &\overset{(801)}{=} -\tfrac{m}{\sqrt{EE'}}\langle l_{\vec{p}} i\sigma_2 c l_{-\vec{p}\,'}\rangle_1^{\mathcal{P}} \\ &= -i\tfrac{m}{\sqrt{EE'}}\langle l_{\vec{p}}\sigma_2 c l_{-\vec{p}\,'}\rangle_2^{\mathcal{P}} = -i\vec{k}^+_{\vec{p},\vec{p}\,'}\wedge(\sigma_2 c)\end{aligned} \tag{1208}$$

- pseudoscalar part

$$\langle\nu_1^+(\vec{p})c\tilde{\nu}_0^+(\vec{p}\,')\rangle_3^{\mathcal{P}} = -\frac{m}{\sqrt{EE'}}i\langle l_{\vec{p}}\sigma_2 c l_{-\vec{p}\,'}\rangle = -i\vec{k}^-_{\vec{p},\vec{p}\,'}\cdot(\sigma_2 c) \tag{1209}$$

3. $\nu_0^-(\vec{p})c\tilde{\nu}_0^+(\vec{p}\,') = \frac{m}{\sqrt{EE'}} l_{\vec{p}}\sigma_3 c l_{-\vec{p}\,'} = \frac{m}{\sqrt{EE'}}(l_{\vec{p}}\sigma_3 l_{-\vec{p}\,'}\Re[c] + l_{\vec{p}} i l_{-\vec{p}\,'}\Im[c])$

- scalar part

$$\langle\nu_0^-(\vec{p})c\tilde{\nu}_0^+(\vec{p}\,')\rangle = \vec{k}^-\cdot\sigma_3\Re[c] \tag{1210}$$

- $\mathcal{P}$-vector part

$$\langle\nu_0^-(\vec{p})c\tilde{\nu}_0^+(\vec{p}\,')\rangle_1^{\mathcal{P}} = C(\vec{p},\vec{p}\,',\sigma_3)\Re[c] + iA(\vec{p},\vec{p}\,')\Im[c] \tag{1211}$$

- pseudoscalar part

$$\langle\nu_0^-(\vec{p})c\tilde{\nu}_0^+(\vec{p}\,')\rangle_2^{\mathcal{P}} = iV_{\vec{p},\vec{p}\,'}(\sigma_3)\Re[c] + iB_{\vec{p},\vec{p}\,'}\Im[c] \tag{1212}$$

4. $\nu_0^-(\vec{p})c\tilde{\nu}_1^+(\vec{p}\,') = \frac{m}{\sqrt{EE'}} l_{\vec{p}}\sigma_3 c i\sigma_2 l_{-\vec{p}\,'} = \frac{m}{\sqrt{EE'}} l_{\vec{p}}\ \overbrace{c\sigma_1}^{\text{vector}}\ l_{-\vec{p}\,'}$

- scalar part

$$\langle\nu_0^-(\vec{p})c\tilde{\nu}_1^+(\vec{p}\,')\rangle = \vec{k}^-\cdot(c\sigma_1) \tag{1213}$$

- $\mathcal{P}$-vector part

$$\langle\nu_0^-(\vec{p})c\tilde{\nu}_1^+(\vec{p}\,')\rangle_1^{\mathcal{P}} = \frac{m}{\sqrt{EE'}}\langle l_{\vec{p}} c\sigma_1 l_{-\vec{p}\,'}\rangle_1^{\mathcal{P}} = C_{\vec{p},\vec{p}\,'}(c\sigma_1) \tag{1214}$$

- pseudoscalar part

$$\langle\nu_0^-(\vec{p})c\tilde{\nu}_1^+(\vec{p}\,')\rangle_3^{\mathcal{P}} = iV_{\vec{p},\vec{p}\,'}(c\sigma_1) \tag{1215}$$

The results are collected in table 7.

$\langle AcB\rangle^{\mathcal{P}}_{0,1,3}$	$\widetilde{\nu_0^+(\vec{p}')}$	$\widetilde{\nu_1^+(\vec{p}')}$	$\widetilde{\nu_0^-(\vec{p}')}$	$\widetilde{\nu_1^-(\vec{p}')}$
$\nu_0^+(\vec{p})$	$B\Re[c] + V(\sigma_3)\Im[c]$ $\vec{k}^-\Re[c] + i\vec{k}^+\wedge\sigma_3\Im[c]$ $i\vec{k}^-\cdot\sigma_3\Im[c]$	$V(c\sigma_2)$ $i\vec{k}^+\wedge(c\sigma_2)$ $i\vec{k}^-\cdot(c\sigma_2)$	$-\vec{k}^-\cdot\sigma_3\Re[c]$ $-C(\sigma_3)\Re[c] - iA\Im[c]$ $-iV(\sigma_3)\Re[c] - iB\Im[c]$	$-\vec{k}^-\cdot(c\sigma_1)$ $-C(c\sigma_1)$ $-iV(c\sigma_1)$
$\nu_1^+(\vec{p})$	$-V(\sigma_2 c)$ $-i\vec{k}^+\wedge(\sigma_2 c)$ $-i\vec{k}^-\cdot(\sigma_2 c)$	$B\Re[c] - V(\sigma_3)\Im[c]$ $\vec{k}^-\Re[c] - i\vec{k}^+\wedge\sigma_3\Im[c]$ $-i\vec{k}^-\cdot\sigma_3\Im[c]$	$-\vec{k}^-\cdot(\sigma_1 c)$ $-C(\sigma_1 c)$ $-iV(\sigma_1 c)$	$\vec{k}^-\cdot\sigma_3\Re[c]$ $+C(\sigma_3)\Re[c] - iA\Im[c]$ $+iV(\sigma_3)\Re[c] - iB\Im[c]$
$\nu_0^-(\vec{p})$	$\vec{k}^-\cdot\sigma_3\Re[c]$ $C(\sigma_3)\Re[c] + iA\Im[c]$ $iV(\sigma_3)\Re[c] + iB\Im[c]$	$\vec{k}^-\cdot(c\sigma_1)$ $C(c\sigma_1)$ $iV(c\sigma_1)$	$-B\Re[c] - V(\sigma_3)\Im[c]$ $-\vec{k}^-\Re[c] - i\vec{k}^+\wedge\sigma_3\Im[c]$ $-i\vec{k}^-\cdot\sigma_3\Im[c]$	$-V(c\sigma_2)$ $-i\vec{k}^+\wedge(c\sigma_2)$ $-i\vec{k}^-\cdot(c\sigma_2)$
$\nu_0^-(\vec{p})$	$\vec{k}^-\cdot(\sigma_1 c)$ $C(\sigma_1 c)$ $iV(\sigma_1 c)$	$-\vec{k}^-\cdot\sigma_3\Re[c]$ $-C(\sigma_3)\Re[c] + iA\Im[c]$ $-iV(\sigma_3)\Re[c] + iB\Im[c]$	$V(\sigma_2 c)$ $i\vec{k}^+\wedge(\sigma_2 c)$ $i\vec{k}^-\cdot(\sigma_2 c)$	$-B\Re[c] + V(\sigma_3)\Im[c]$ $-\vec{k}^-\Re[c] + i\vec{k}^+\wedge\sigma_3\Im[c]$ $+i\vec{k}^-\cdot\sigma_3\Im[c]$

Table 7: Table of the products $\langle \nu_s^\pm(\vec{p})c\widetilde{\nu_r^\pm(\vec{p}')}\rangle^{\mathcal{P}}_{0,1,3}$, where c is any complex number $c = \Re[c] + \Im[c]\; i\sigma_3$, $\vec{k}^\pm \stackrel{\text{def}}{=} \frac{\sqrt{(E+m)(E'+m)}}{2\sqrt{EE'}}\left\{\frac{\vec{p}}{E+m} \pm \frac{\vec{p}'}{E'+m}\right\}$, $A \stackrel{\text{def}}{=} \frac{\vec{p}\wedge\vec{p}'}{2\sqrt{EE'(E+m)(E'+m)}}$, $B \stackrel{\text{def}}{=} \frac{[(E+m)(E'+m)-\vec{p}\cdot\vec{p}']}{2\sqrt{EE'(E+m)(E'+m)}}$, $C(\vec{a}) = \frac{\sqrt{(E+m)(E'+m)}\vec{a}-\sqrt{(E-m)(E'-m)}\langle\hat{p}\vec{a}\hat{p}'\rangle^{\mathcal{P}}_1}{2\sqrt{EE'}}$ and $V(\vec{a}) = \frac{i\vec{p}\wedge\vec{p}'\wedge\vec{a}}{2\sqrt{EE'(E+m)(E'+m)}}$.

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