ASPECTS OF VIBRATION IN BRIDGE-GIRDERS

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RESUME

A theoretical comparative study of methods for determining natural frequencies of vibration and their corresponding mode shapes is presented for the case of cantilevers, simply supported and continuous beam-structures. Different methods are used, such as classical, energy and matrix methods. Effects of mass lumpings are investigated. A literature review is given for the general problem of dynamic behaviour of highway bridges, including the passage of heavy vehicles.

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LIST OF SYMBOLS

A	area of cross-section
A	system matrix
B	connection matrix
c	dimensionless parameter = p/p_w
G	total energy of a system
d	deflection
d_{st}	static deflection
EI	flexural rigidity
f	frequency of vibration (section 2.3.1 and Chapter 6)
f	flexibility coefficient
F	flexibility matrix
F_m	flexibility matrix for member ends
g	gravitational constant
G	impulse
I	impact factor (Chapter 6)
I	unity matrix
k	stiffness coefficient or spring constant
k	a constant (section 3.3.4)
K	stiffness matrix
KE	kinetic energy
l	length of span of a continuous beam
L	length of beam
L_i	length of beam segment
m	mass per unit length
m_i	lumped mass
M	bending moment
M	mass matrix
M	mass of beam (Chapter 4)
M_i	i's mass on beam
M_B	mass of beam
M_L	mass of a central load
M_t	total mass
n	number of beam sections
n	number of natural modes
N	number of spans
N	axial force
p	natural circular frequency

List of Symbols (continued)

p_w	$(EI/M_t L^3)^{\frac{1}{2}}$
P	point matrix
P(t)	disturbing force
Po	maximum value of P(t)
PE	potential energy
Q	shear force
r	$(\rho A p^2/EI)^{\frac{1}{4}}$
R	reaction
R_A RA	reaction at support A — See pages 4.8 & 4.9
R_B RB	reaction at support B
s	arbitrary parameter in Rayleigh-Ritz Method
t	time
t_1	pulse length
T	period of vibration
T	field matrix
U	overall transfer matrix
v	velocity
W	static force or load matrix
x	displacement of vibrating mass (Chapter 2)
x	coordinate following the beam axis
x_{st}	static deflection
X	maximum value of half amplitude
y	transverse displacement
y	displacement vector
Y	maximum value of y
z	state vector
α	end rotation (section 3.3.7)
β	frequency parameter $= L^3 M p^2/EI$
ϵ	error
ρ	density
λ	eigenvalue
θ	end rotation

List of Symbols (continued)

Appendix B

A	general square matrix
α	arbitrary coefficient
H	Hotelling's matrix
T	transformation matrix
U	eigenvector of transpose of A
V	arbitrary vector
W	Wielandt's matrix
X	eigenvector of A
λ	eigenvalue

CHAPTER 1

INTRODUCTION

At the outset the writer intended to investigate the vibration of structures in general. It was soon realized that this is a very wide field. After studying some of the fundamentals of vibration, given in Chapter 2, the writer decided to confine the studies to the dynamic behaviour of beam-structures which vibrate freely, i.e. the periodic motion after the removal of a disturbance force.

The first aim in bridge design calculations with regard to dynamic behaviour will be to estimate the two or three lowest natural frequencies of the structure. The fundamental or lowest frequency is of the greatest importance because beam-structures vibrate mostly in the fundamental mode after excitation (depending on type and instance of the excitation). For highway bridges the knowledge of the natural frequencies is important because it is desirable that these do not fall within the range of natural frequencies of heavy vehicles, caused by suspension characteristics and/or axle spacing. This can lead to large dynamic deflections and stresses.

Damping of the structure reduces the amplitude of vibration to some extent and causes the vibration to die out eventually. However, damping is small in beam-structures and is therefore neglected in most practical cases. Furthermore, if the disturbance is caused by an impact, the largest and most dangerous amplitudes occur immediately afterwards, before damping could develop.

There exists a considerable number of methods for determining the natural frequencies of a structural system and some of them are investigated in greater detail in Chapter 3. These methods could be classified as follows:-

(i) classical method, which involves the solution of partial differential equations;

(ii) energy methods, which require for the solution a guess of the mode shape;

(iii) matrix methods, which

(a) require the solution of the eigenvalue problem

or (b) require the solution of a polynomial, called the frequency equation.

Numerical integration methods are often used in practical problems where a disturbing force acts repetitively, since this complicates the solution of the differential equations of motion.

Since 1945 matrix methods became increasingly popular because they provided not only an easy way to calculate the behaviour of complicated systems, but also an organized method of computation which lends itself to computerization. Whilst the classical methods can be applied only to systems having uniformly distributed mass, the energy methods can cope with both continuous and point masses. For the matrix methods, however, certain assumptions with respect to the mass distribution have to be made because these methods cannot handle uniformly distributed mass. The actual, continuous structure is approximated by a "lumped system". This is a system consisting of rigid bodies elastically connected. The rigid bodies represent the concentrations of the continuous mass and the elastic connections are weightless but possess the flexural properties of the system. This leads to the question of how many lumped masses the structure should be approximated by, to give accurate results for the natural frequencies. The effect of lumping the distributed mass of a beam is investigated in Chapter 4.

Certain matrix operations and matrix methods are necessary in order to solve the eigenvalue problem which arises from the use of matrix methods. The writer felt that a brief section on the relevant matrix calculus should be included in this thesis and it can be found in Appendix B.

The methods for determining the natural frequencies are first discussed in connection with statically determinate structures.

However, since many structures are not statically determinate, the assessment of the natural frequencies for two-span continuous beams is also discussed. Some of the methods considered can be extended to multi-span continuous structures or other more complicated beam-systems.

Finally a review of papers concerning the general problem of dynamic behaviour of highway bridges is given, because this is the obvious next step, i.e. to consider the vehicle-bridge system. This is quite a complex problem and was subject to many investigations during the last twenty years. Many factors influence the dynamic response, such as:

(i) bridge characteristics

(ii) road characteristics

(iii) vehicle characteristics.

Obviously, simplifying assumptions have to be made in order to solve the problem. A large number of field and model tests have been carried out in order to find the most important causes of undesirable vibrations due to heavy traffic and to develop design procedures.

CHAPTER 2

FORMULATION OF THE VIBRATION PROBLEM

Section 2.1 - Introduction

The response of a structure to dynamic loading depends on:

(i) the mass of the structure,
(ii) the resistance of the structure to deflections,
(iii) the definition of the loading.

Of course, if the supports of the structure are not immovable, their motion must be defined as well.

In order to determine the vibration behaviour of a structural system, we divide the analytical approach into a number of stages:

(i) the choice of the mathematical model which is to be used to represent the structure,

(ii) the derivation of the equations of motion of the mathematical model,

(iii) the determination of the physical constants such as lengths, masses, stiffnesses, etc. occurring in the mathematical equations,

(iv) the solution of the mathematical equations.

The following section will outline these stages in some detail. With regard to the equations of motion, damping will be neglected, since in many practical systems the amount of inherent damping is small or not relevant.

Section 2.2 - Vibrating Systems

2.2.1 - Degrees of freedom

In order to study the vibration behaviour of a system, one must be able to define its position at any instant. If this can be done using one coordinate only, the system is said to have one degree of freedom. An example is the mass-spring-system in fig. (2.1).

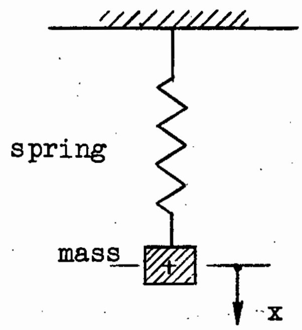


Figure 2.1

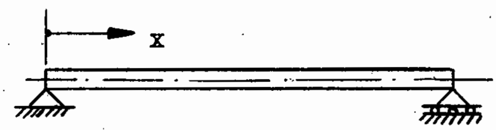


Figure 2.2

In general, a vibrating system has as many degrees of freedom as there are coordinates required to describe its motion. A rigid body in space has six degrees of freedom, three in rotation and three in translation. A body constrained to move in a plane has three degrees of freedom, two in translation and one in rotation about an axis perpendicular to the plane.

A system of two rigid bodies, elastically connected to each other can have as many as 12 degrees of freedom. Such systems of elastically connected rigid bodies are called "lumped systems". They are composed of concentrations of mass connected by massless springs.

Distributed systems, on the other hand, are composed of an infinite number of infinitely small masses, i.e. they have distributed mass and distributed elasticity. Such systems have an infinite number of degrees of freedom. Beams (see fig. 2.2) and plates are distributed systems and in fact most actual vibrating systems are of this type.

To simplify the vibration analysis, an actual, continuous structure may be approximated by a lumped system as shown in fig. 2.3. The mass of the members is concentrated at certain points and the beam or the columns are considered to be weightless but to have structural strength and stiffness.

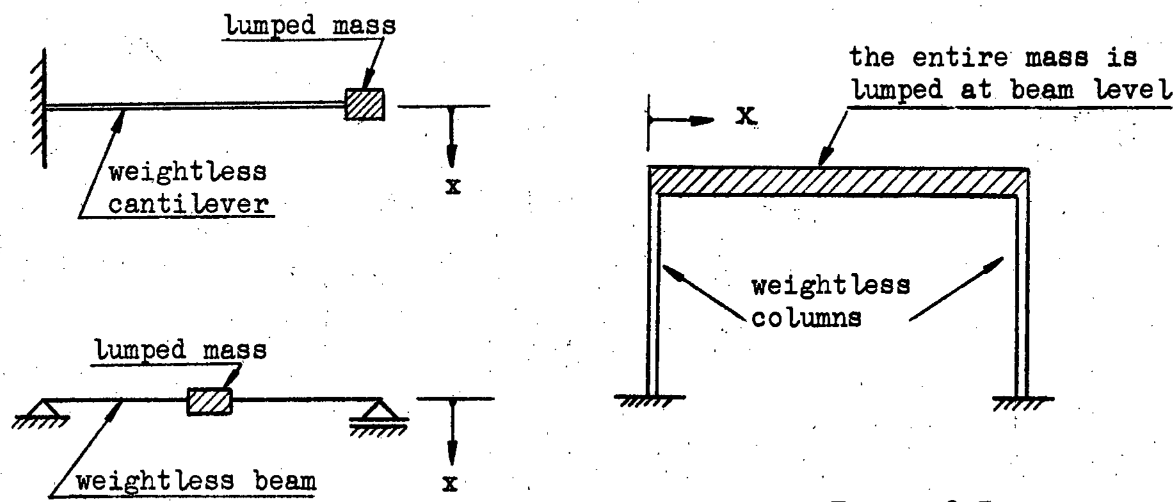


Figure 2.3

In most cases, the structural members are assumed to undergo no rotation and no axial deformation since they have negligible effect on the deflection and only bending deflections are considered. On this assumption the frame shown in fig. 2.4 has five degrees of freedom as indicated.

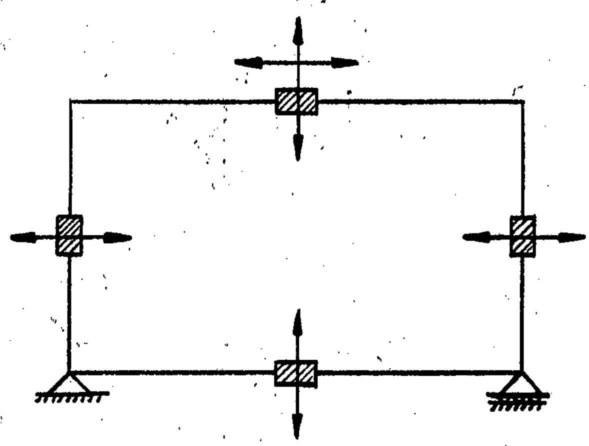


Figure 2.4

To summarise, the number of degrees of freedom possessed by a lumped mass system depends on:

(i) the number of lumped masses, and
(ii) the number of degrees of freedom attributed to each lumped mass.

Vibrating systems may conveniently be divided into two types:

(i) lumped systems

(ii) distributed systems.

and obviously this distinction is mathematical and not physical.

2.2.2 - Lumped systems

A lumped system is one whose exact equations of motion may be expressed as a set of ordinary differential equations in a finite number of unknowns. These unknowns are functions of time only. For small vibrations, in which we are interested, the equations of motion will be linear and have constant coefficients. Lumped masses will then be characterized by having a finite number of natural frequencies. Although lumped systems never occur exactly in practice, many structural systems may be assumed to be lumped for most practical purposes.

2.2.3 - Distributed systems

A distributed system is one whose exact equation of motion may be expressed as one or more partial differential equations governing certain displacements. These displacements are functions of the space variables x, y, z and the time t. For small vibrations, the equations will be linear, but the coefficients will not necessarily be constant. Distributed systems have an infinite number of natural frequencies and corresponding modes of vibration.

2.2.4 - Criterion of choice for the type of mathematical model

It is clear that although it is possible to make an absolute distinction between lumped and distributed systems, no such clear-cut distinction is possible in the real world. In practice, the decision whether to treat a system as lumped or distributed depends on arguments involving convenience and accuracy. An exact treatment of the system increases the accuracy but also the complication of the equation. However, in many cases the increase in accuracy does not justify the increase in mathematical complexity.

Sometimes it is easier to think of a system as being continuous rather than lumped. This is the case when the partial differential equation can be solved exactly. The class of systems for which this is possible is comparatively small, but includes important systems such as uniform beams and uniform plates. The partial differential equation and its solution for simple beam structures will be found in section 3.2.

When it is impossible, or difficult, to obtain an exact solution of the partial differential equations governing a distributed system, the system is reduced to a discrete form. So far only one method of approximating a real structure was mentioned, which was the lumped-mass approach (see section 2.2.1). There is however, another method by which an actual system can be approximated which is known as "assumed mode method". In this method the system is assumed to vibrate in a certain deflected shape. Both methods will be discussed in more detail in the sections 2.3 and 3.3.

Section 2.3 - The Equations of Motion and their Solution

2.3.1 - Single-degree-of-freedom problem

The equation of motion for a system as shown in fig. 2.5 may be obtained by considering the forces acting on the mass m at any time t. The coördinate x represents the displacement of the mass m from a

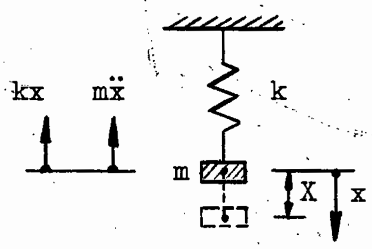


Figure 2.5

selected reference position (generally the position of static equilibrium). The force exerted by the spring on the mass will be $-kx$ (acting in the negative x-direction), where k is the elastic stiffness of the spring (defined as a force per unit displacement). The accelerated mass exerts an inertia force on the spring equal to $-m\ddot{x}$.

For dynamic equilibrium:-

$$F = 0 = -m\ddot{x} - kx$$

therefore,
$$m\ddot{x} + kx = 0 \qquad (2.1)$$

This is called the "<u>equation of motion</u>" for a single-degree-of-freedom undamped, freely vibrating system. The general solution is:-

$$x = A \sin pt + B \cos pt \qquad (2.2)$$

where
$$p^2 = k/m \qquad (2.3)$$

The period, or time taken for one complete cycle of vibration, is:

$$T = 2\pi/p = 2\pi\sqrt{m/k}$$

where p is called the natural circular frequency; it has the units "radians per unit of time". In the subsequent discussions, we shall speak of the <u>natural circular frequency</u> p simply as the natural frequency.

The number of cycles per unit time, say 1 second, is called the frequency of vibration, and will be denoted by f, where

$$f = 1/T$$

The constants A and B depend on the initial conditions of the problem. Let us assume the following initial conditions:-

$$\text{at } t = 0 \quad x = X \quad \text{and} \quad \dot{x} = 0$$

then
$$x = X \cos pt \qquad (2.4)$$

and
$$\ddot{x} = -X p^2 \cos pt = -p^2 x$$

Substituting $\ddot{x}$ into the equation of motion gives:-

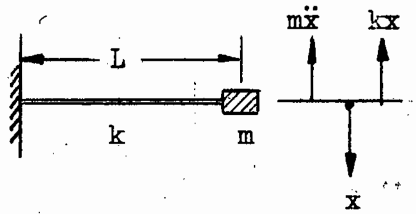
$$p^2 mx - kx = 0 \qquad (2.5)$$

which is called the "frequency equation". It will be noted that the natural frequency p depends only on the parameter k and m, and not on the amount of the initial displacement X.

Let us determine the natural frequency for the following systems, in which the actual mass of the beam is approximated by one lumped mass only:-

(a) Cantilever

The forces acting on the mass m are:-



inertia force $-m\ddot{x}$

restoring force $-kx$

The stiffness of a cantilever is known to be:-

$$k = 3\,EI/L^3$$

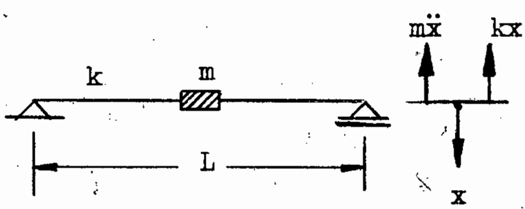
hence from equation (2.5)

$$p = \sqrt{k/m} = \sqrt{3\,EI/L^3m}$$

$$= 1{,}73\sqrt{EI/L^3m}$$

(b) Simply supported beam

Using the relationship given in Appendix A, the stiffness is found to be:-



$$k = \frac{48\,EI}{L^3}$$

$$p = \sqrt{\frac{48\,EI}{L^3m}} = 6{,}92\sqrt{\frac{EI}{L^3m}}$$

It will be seen in a later section, that the two natural frequencies obtained in the above examples are not good approximations to the exact values. This is due to the fact, that the one-degree-of-freedom model is not a good representation of the actual structure, which has a distributed mass.

2.3.2 - Multi-degree-of-freedom problem

When there is more than one degree of freedom, the complexity of the problem increases, since there are as many natural or normal nodes of vibration as there are degrees of freedom and the motion is not necessarily harmonic in form. However, any possible motion can be broken down into its harmonic components, each component being related to a normal mode with its associated natural frequency.

As an example, consider the two-degree-of-freedom system shown in fig. 2.7. The cantilever is approximated by a mass-spring-system consisting of two masses m_1 and m_2 and two weightless beam sections of stiffnesses k_1 and k_2. If this system were disturbed in some arbitrary manner and then allowed to vibrate freely, the movement of the two masses might have the form suggested in fig. 2.7(a) and obviously such a form is not the simple harmonic pattern of a single-degree-of-freedom problem. Since the system has two degrees of freedom, we know that it has two modes of vibrations, and each mode has its own natural frequency.

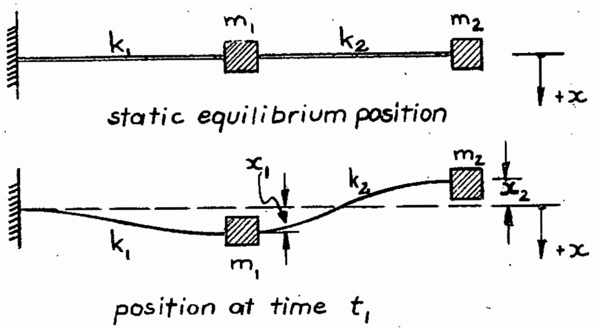
The two equations of motion for the free undamped vibration of the system can be written down by considering the equilibrium of each mass under inertia and restoring forces:

$$\begin{aligned} \text{for } m_1: \quad & m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0 \\ \text{for } m_2: \quad & m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0 \end{aligned} \qquad (2.6)$$

These lead to the frequency equation from which we can solve for p^2 and find the two natural frequencies and corresponding modes of vibration.

For simplicity let us assume that $k_1 = k_2 = k$ and $m_1 = m_2 = m$ and let the solution be of the form $x = X \cos pt$; then the two equations become:

$$\begin{aligned} x_1 \left(p^2 - \frac{2k}{m}\right) + x_2 \left(\frac{k}{m}\right) &= 0 \\ x_1 \left(\frac{k}{m}\right) + x_2 \left(p^2 - \frac{k}{m}\right) &= 0 \end{aligned} \qquad (2.7)$$



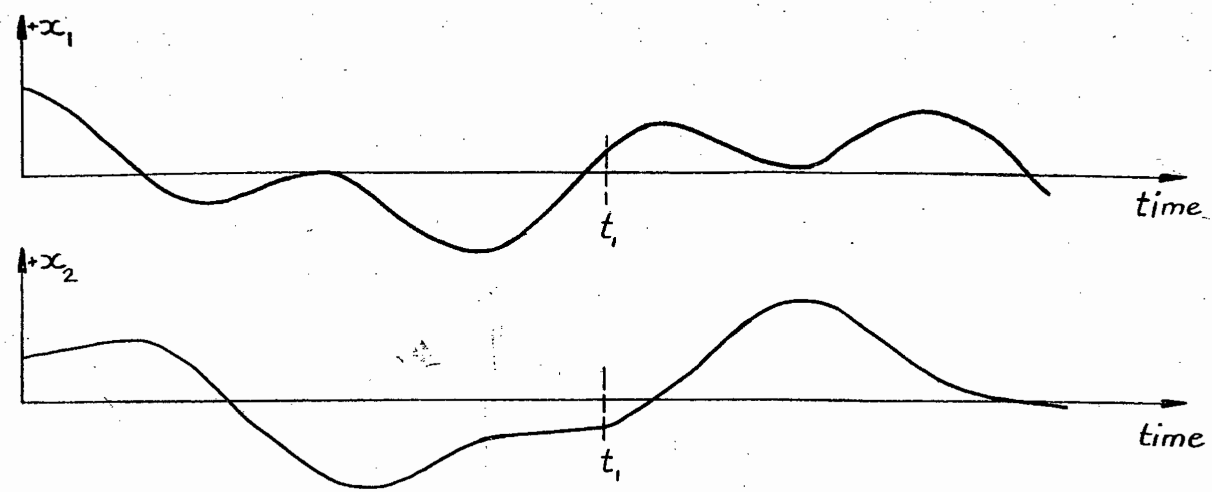


Fig. 2.7(a) Free vibration due to an initial random displacement of m_1 & m_2

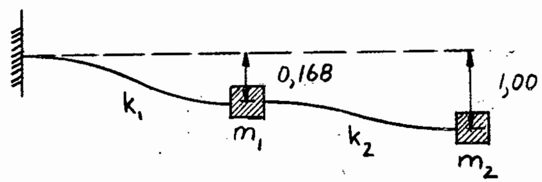
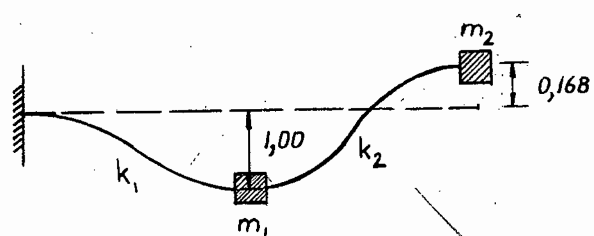


Fig. 2.7(b) Free vibration in the first mode



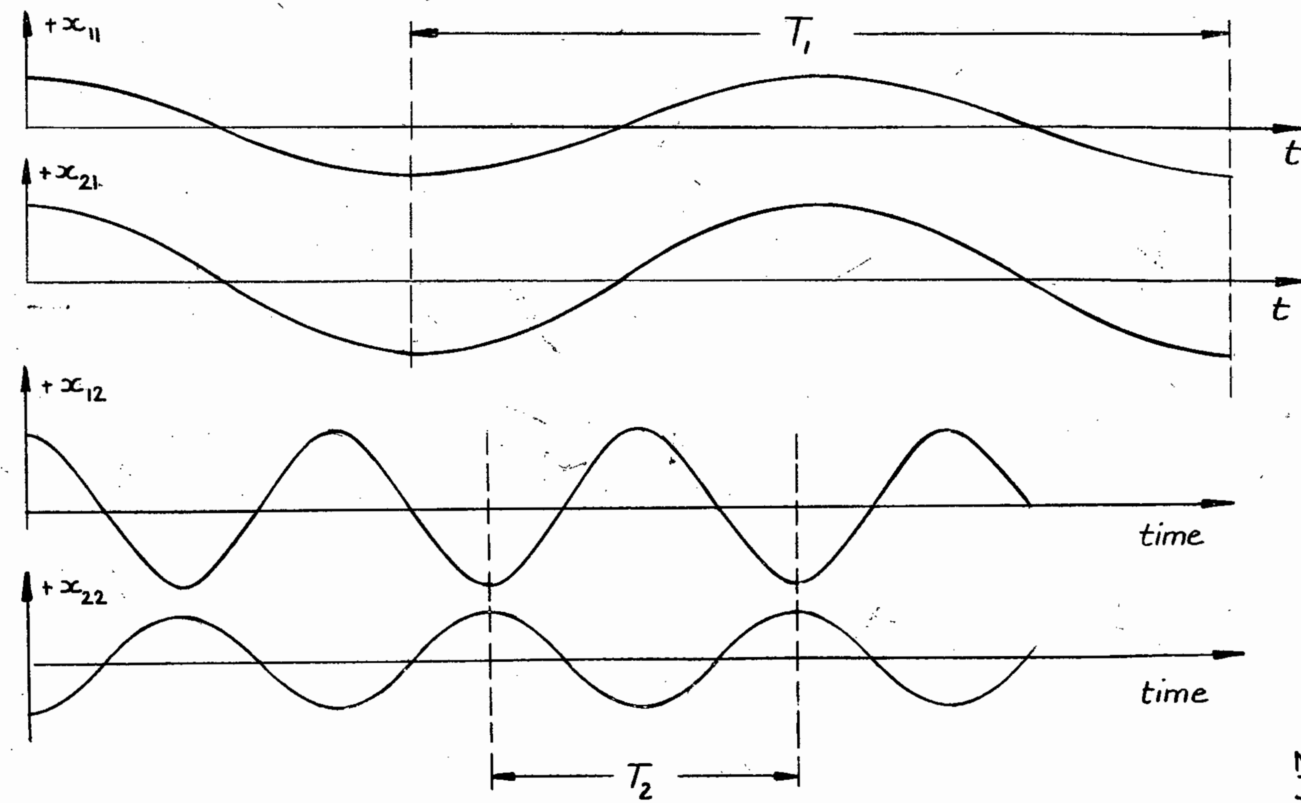


Fig. 2.7(c) Free vibration in the second mode

Elimination of the amplitude ratio x_1/x_2 leads to the frequency equation:

$$p^4 - p^2\left(\frac{3k}{m}\right) + \left(\frac{k}{m}\right)^2 = 0 \qquad (2.8)$$

hence,

$$p^2 = 0{,}382\ (k/m) \quad \text{or} \quad 2{,}618\ (k/m) \qquad (2.9)$$

and

$$p_1 = 0{,}618\sqrt{k/m} \qquad p_2 = 1{,}618\sqrt{k/m}$$

These two values of the circular frequency p are the first and second natural frequencies, p_1 being taken as the lower of the two, which is also called "fundamental frequency".

Substituting equation (2.9) in equation (2.7) we get two relationships between x_1 and x_2, one for each value of p. These relationships give the normal modes of vibration as follows:-

$$\text{mode 1:} \qquad x_{11} = +\,0{,}618\,x_{21} \qquad (2.10)$$

$$\text{mode 2:} \qquad x_{22} = -\,1{,}618\,x_{12} \qquad (2.11)$$

where the second subscript designates the mode.

The precise meaning of harmonic motion as applied to a multi-mass-system is as follows: every point in the structure has as its time-displacement relationship an expression of the form;

$$x = X\cos pt$$

where X is the maximum displacement at that point, and (cos pt) is common to the entire structure. This implies that all points reach their maximum displacement at the same time, and also pass through zero at the same time. These results are shown in figs. 2.7(b) and 2.7(c).

Depending on the form of the initial disturbance given to the system, the free vibration pattern can be related to mode 1 and mode 2, and the corresponding frequencies. If the initial displacements correspond to equation (2.10) on releasing, the system will vibrate in the first mode; if the initial displacements correspond to equation (2.11) the system will vibrate in the second mode. Both modes are equally possible. In the more general case of a random initial displacement of the two masses, this displacement can be considered as being composed of appropriate amounts of mode 1 and 2. The free vibration pattern will then adopt the forms shown in fig. 2.7(a).

There are several important properties of normal modes of vibration:-

(i) The absolute values of displacements are unimportant; what is important is the shape of the mode or the relative displacement.

(ii) Since the absolute values of the displacements are arbitrary, they can have any convenient value. It may be convenient for each mode to have the sum for all mass points of the product of each mass and the square of the corresponding displacement add up to unity:-

$$\text{for } p_1: \quad m_1 x_{11}^2 + m_2 x_{21}^2 = 1$$
$$\text{for } p_2: \quad m_1 x_{12}^2 + m_2 x_{22}^2 = 1$$

When the displacements have been adjusted in such a manner they are said to have been normalized.

(iii) The third property is called orthogonality property, which is given by the equation:-

$$x_{11} x_{12} m_1 + x_{21} x_{22} m_2 = 0 \tag{2.12}$$

This property is very important for determining higher natural frequencies and their associated modes; this will be shown in the next chapter.

It should be mentioned at this point, that in the above discussion only one particular way of obtaining the equations of motion was used, i.e. the equations were obtained by considering the equilibrium of the forces acting on the mass. There are, however, other ways to set up the equations of motion, which will be given in Chapter 3.

2.3.3 - Practical importance of the natural frequency

The values of the natural frequencies are of great importance because they represent dangerous frequencies in most structural systems. If a harmonic disturbing force is applied to the system at one of these frequencies, the deflections could become large as the system develops what is known as resonance. Without sufficient damping the amplitude of vibration at resonance would theoretically approach infinity. In practice, with beam structures it could become dangerously large (since damping may be small), causing discomfort, damage to the structure or fatigue failure in certain materials.

In practice, it is difficult to predict the amplitude of vibration to be expected at resonance because the actual amount of damping present in any given system can often only be guessed. This presents no real difficulty; it is sufficient to know the frequency at which resonance is expected to occur and to avoid exciting the system at any of the natural frequencies. The design calculations therefore first involve estimating natural frequencies and damping is always neglected.

In a multi-degree-of-freedom structure the frequency equation can become quite complicated. However, in most structural problems only the lower natural frequencies, and more specifically the fundamental frequency, are of practical importance, the latter only is relatively easy to obtain.

2.3.4 - Limiting criteria of the mathematical idealization

The resistance of a structure to deflections can be defined more or less satisfactorily. In general, it depends on:

(i) the simplicity of the structure,

(ii) the properties of the material.

A structure composed of a few simple elements, made of homogeneous isotropic and linearly elastic material like steel (with strains not exceeding the proportional limit) and subjected to small displacements which involve no significant changes in the geometry of the structure, behaves very much like its mathematical idealization, and its elastic properties can be estimated quite accurately. On the other hand, a complicated assemblage of structural elements made of non-homogeneous, non-isotropic and non-linearly elastic material like concrete exhibits a behaviour which is difficult to estimate.

The mass distribution of a structure is easily defined. However, in many cases it would be very tedious to include in the computation the exact distribution of the mass, as it was pointed out in sub-section 2.2.4.

We are now left to consider the various dynamic forces which can act on a structure and how they are treated mathematically. However, the description of types of forces will be very brief, since this thesis is concerned with free vibrations of beam-structures.

Section 2.4 - Types of Dynamic Loading

In the foregoing we considered undamped, free vibrations of an elastic system which are defined by:

(i) the elastic resistance and inertia forces of the structure,

(ii) the initial conditions of the state of motion,

(iii) the boundary conditions of the structure,

(iv) the absence of any external and internal friction forces,

(v) the absence of any external impressed force.

If a disturbing force is applied to a structure, the resulting motion is called an undamped, forced vibration. The disturbing force $P(t)$ is a function of the independent variable time and there are two main types of forces:-

(i) periodic forces,

(ii) transient forces.

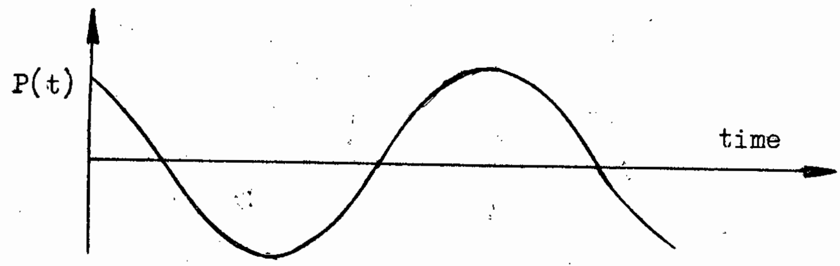
Such dynamic loads, applied to a structure, produce a certain motion which is called the response of the system. Depending on the applied force, we get the following types of responses:-

(i) steady-rate response,

(ii) transient response.

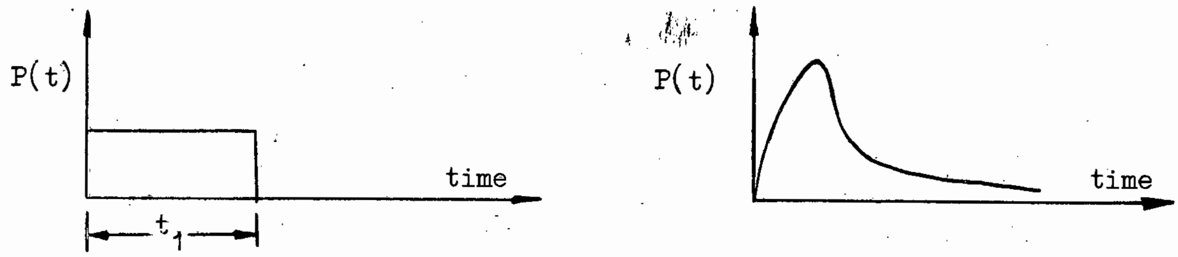
The first type of force repeats itself many times with constant maximum force at regular time intervals. If a sufficiently great number of cycles has occurred, the responding system will be found to vibrate in a steady state. Periodic forces may be caused by an unbalanced machine running at a constant speed.

The second type are forces which are applied suddenly or forces applied for a short interval of time, such as caused by an earthquake, blasting or an impact. The following figures represent the forcing functions graphically:-



Example for a periodic forcing function

Figure 2.8(a)



pulse load of duration t_1 — blast pressure pulse

Example for transient forcing function

Figure 2.8(b)

2.4.1 - Undamped forced vibrations

For simplicity, let us again consider the one-degree-of-freedom system as shown in fig. 2.9. The system has the following equation of motion:-

$$m\ddot{x} + kx = P(t) \qquad (2.13)$$

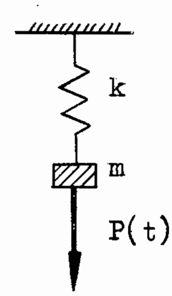


Figure 2.9

The solution of equation (2.13) is made up of two parts;

(i) the complementary function which is the solution of the homogeneous equation, i.e. the equation for free vibrations,

(ii) the particular integral, which must satisfy the complete equation (2.13).

Therefore

$$x = A \sin pt + B \cos pt + \text{particular integral.}$$

Obviously, if there is no disturbing force the particular integral is zero and the complete solution is given by equation (2.2) in sub-section 2.3.1.

2.4.2 - Damped, forced vibrations

It has already been pointed out that in many practical problems damping has a relatively small effect on the response. The usual procedure, therefore, is to solve for the natural frequencies and corresponding normal modes of vibration neglecting damping. However, when such compromises by means of idealizing the actual structure are made, this has to be in accordance with the aim of the particular study. Thus, where design loadings involve transient forces, it is justifiable to neglect frictional forces. Here the maximum response during or immediately after the application of the force is of greatest interest, i.e. before damping could have much effect. On the other hand, if the design is controlled by fatigue conditions, occurring under steady-state loadings, the maximum response during steady state conditions may be significantly affected by damping.

Not only should damping be considered in such steady-state cases produced by periodic loading, but it may also be significant in response produced by relatively long non-periodic loadings, as for example, those involved by earthquake response.

The damping present in real structures may take various forms, but mostly viscous damping is assumed, that is, the damping force is proportional to the velocity of the mass. The general equation of motion for a damped, forced, vibration becomes:

$$m\ddot{x} + c\dot{x} + kx = P(t)$$

The complete solution consists of the complementary function and the particular integral of the form:

$$x = \exp(-ct/2m)(A \sin pt + B \cos pt) + \text{particular integral} \qquad (2.14)$$

Physically, the complete response is the sum of the starting transient (the complementary function), which dies away exponentially, and the steady-state response (the particular integral). If the vibration during the first few cycles is of interest, then equation (2.14) must be investigated, but in problems, where the response is required after the starting transient has died out, further consideration only has to be given to the particular integral.

We will now return to undamped, forced, vibration and investigate the effect of various types of time-varying forces.

2.4.3 - Harmonic forces

Let us consider the system shown in fig. 2.9, which is subjected to a disturbing force of the form;

$$P = P_o \sin \alpha t$$

where α is the circular frequency of the periodic load. In order to evaluate the particular integral, $x = X \sin \alpha t$ is used as a solution and we get for the complete solution;

$$x = A \sin pt + B \cos pt + \frac{P_o/k}{1 - (\alpha/p)^2} \cdot \sin \alpha t$$

The constants A and B are determined from initial conditions. The first two terms of the equation above describe the <u>free</u> vibration, and the third term is known as the <u>steady-state forced</u> vibration. The amplitude of the forced vibration is:-

$$X = \frac{P_o/k}{1 - (\alpha/p)^2}$$

However, $X_{st} = P_o/k$, where X_{st} is the displacement for a static load P_o, then

$$\frac{X}{X_{st}} = \frac{1}{1 - (\alpha/p)^2} \qquad (2.15)$$

The ratio X/X_{st} is called <u>magnification factor</u>. In fig. 2.10 the absolute value of the magnification factor is plotted against the frequency ratio and the figure illustrates the phenomenon of resonance - that when the frequency of the disturbing force equals the natural frequency of the system, the amplitude of the forced vibration tends to infinity, if there is no damping. When the frequency of the disturbing force increases beyond the condition of resonance, the magnification factor again becomes finite, and its absolute value diminishes as α/p increases. This means that a dynamic load having a high frequency produces forced vibrations of very small amplitudes.

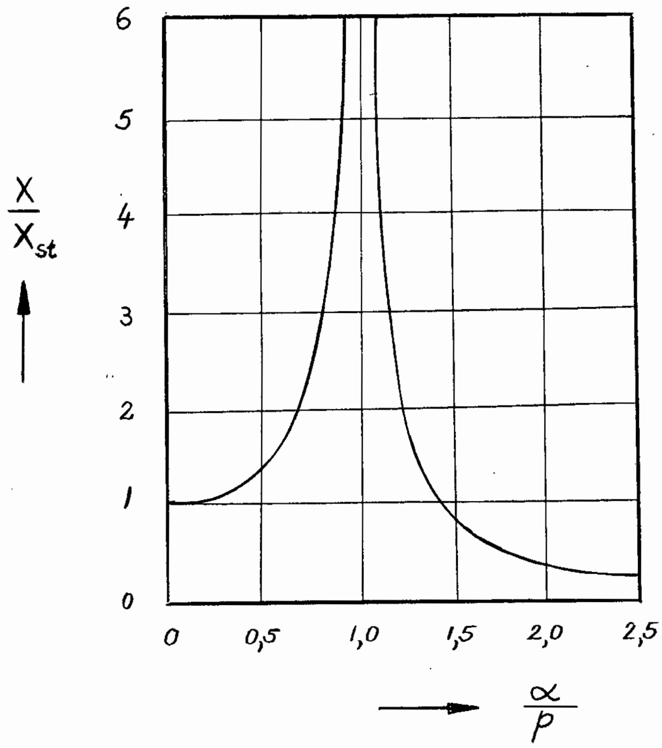


Figure 2.10

2.4.4 - Transient forces

If a force is applied to a one-degree-of-freedom system for a short interval of time and then removed, the subsequent motion will be free vibrations, which will decrease in amplitude, if damping is present in the system. Usually the maximum displacement of the mass is of interest; this will occur during the time of application of the force or in the first cycle of free vibration after the removal of the force. Thus considering transient disturbances of short duration, only a few cycles of vibration of the system are of interest. However, damping is small in many structural problems, so that its effect on the vibration during the first few cycles can be neglected.

In practice, the variation of the transient force with time is often not known exactly, and hence only approximations to the maximum displacement can be determined.

First we shall consider the effect of an impulsive load, from which the response due to other time-varying forces can be derived.

A) <u>Effect of an impulse</u>

If at a time $t = 0$ an impulse G is applied to the mass in the $+x$ direction, the body instantaneously acquires a velocity G/m. If no forces are applied thereafter, i.e. $P(t) = 0$ for $t > 0$, there is no particular integral and the complete solution of the differential equation is:-

$$x = A \sin pt + B \cos pt$$

applying the initial conditions:-

at $t = 0$ $x = 0$ and $\dot{x} = G/m$

then $A = G/m$

hence $x = G/(mp) \,.\, \sin pt$ for $t > 0$

By similar reasoning, if the mass had been at rest until time $t = t'$ and an impulse had been applied at $t = t'$, then the resulting motion would be:-

$$x = \frac{G}{mp} \sin p(t - t') \quad \text{for} \quad t > t'$$

B) <u>General case of a time-varying force</u>

Let us consider a variation of force $P(t)$ with time as shown in fig. 2.11.

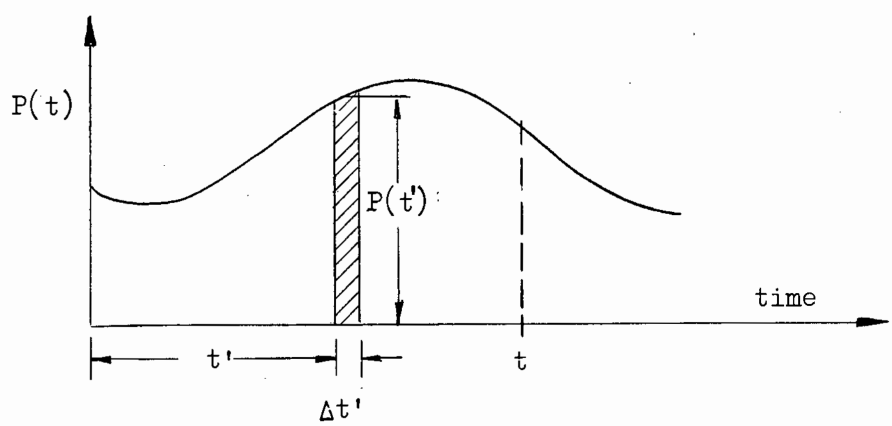


Figure 2.11

The general continuous force $P(t)$ can be considered as a series of impulses. The shaded area $\Delta G = P(t')\,\Delta t'$ is a typical impulse occurring at time $t = t'$. The increment in displacement Δx at time $t (t > t')$ due to ΔG is:-

$$\Delta x = \frac{P(t')\,\Delta t'}{m\,.\,p} \;.\; \sin p\,(t - t')$$

Summing for all the impulses into which the curve $P(t)$ is divided between $t = 0$ and $t = t$, the response at time t is:-

$$x = \frac{1}{m.p} \int_0^t P(t') \sin p\,(t - t')dt' \qquad (2.16)$$

This integral is also known as Duhamel integral. If the initial conditions at time $t = 0$ are $x = X$ and $\dot{x} = \dot{X}$, the solution of the homogeneous equation must be added so that the complete motion is described by:-

$$x = X \cos pt + \frac{\dot{X}}{p} \sin pt + \frac{1}{m\,.\,p} \int_0^t P(t') \sin p(t - t')\,dt'$$

The total displacement is simply the superposition of three separate contributions:

(1) The free vibration $X \cos pt$, which would result if at $t = 0$ the system were displaced by X but had no initial velocity and subsequently were not acted upon by an external force $P(t)$. This contribution is identical with the free vibration given by equation (2.2) in section 2.3.1.

(2) The free vibration $\frac{\dot{X}}{p} \sin pt$, which would result if, at $t = 0$ the system were subjected to an initial velocity $\dot{X}$ but had no initial displacement and subsequently were not acted upon by an external force $P(t)$.

(3) The dynamic response which would be produced by the time-varying force $P(t)$ to a system which had neither a displacement nor a velocity at $t = 0$.

C) Stepwise disturbance

Let us now consider the response of the system shown in fig. 2.9 produced by a rectangular pulse load as given by fig. 2.12, the system being initially at rest. This case is of interest because the maximum displacement x_{max} is dependent on the ratio of pulse length t_1 to the period of the system T as will be shown.

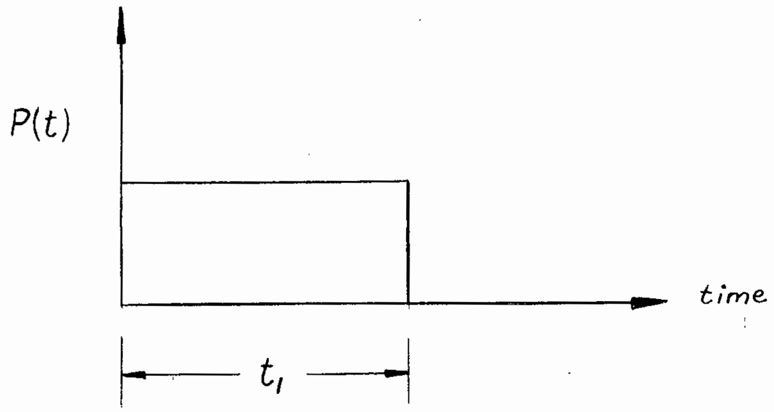


Fig. 2.12

For $0 < t < t_1$, the motion is governed by

$$x = \frac{1}{m \cdot p} \int_0^t P \sin p\,(t - t')\,dt'$$

$$x = \frac{P}{k}(1 - \cos pt) \qquad \text{since } p^2 = \frac{k}{m}$$

and with $P = k\,x_{st}$

$$\frac{x}{x_{st}} = 1 - \cos pt \tag{2.17}$$

For $t > t_1$, the motion will be a free vibration and is governed by:-

$$x = \frac{P}{m.p} \int_0^{t_1} \sin p(t - t')dt' + \frac{0}{m.p} \int_{t_1}^{t} \sin p(t - t')dt'$$

The integral must be split into two portions,

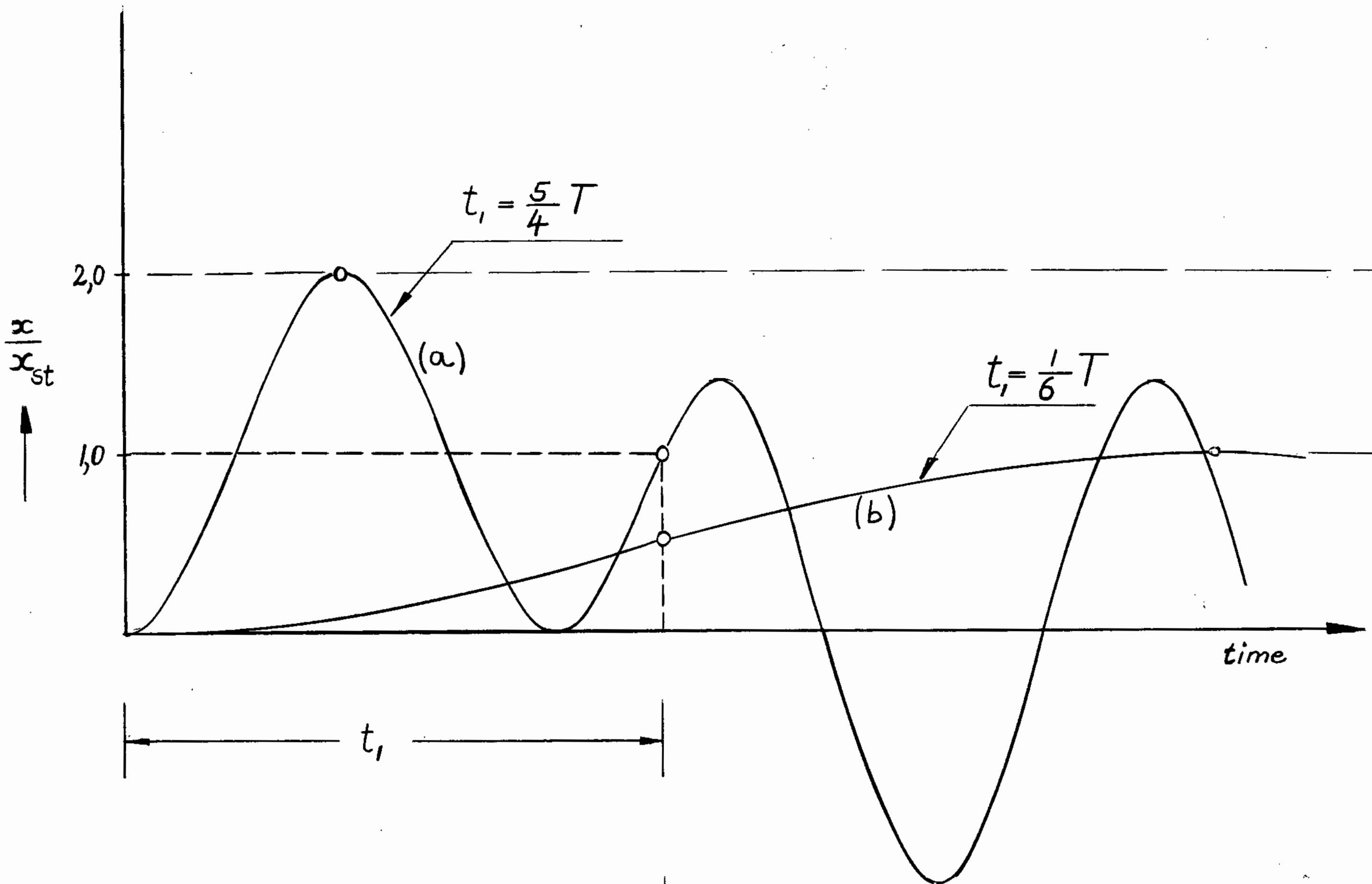
since $P(t') = P$ for $0 < t < t_1$

and $P(t') = 0$ for $t > t_1$

Evaluating the above equation leads to

$$\frac{x}{x_{st}} = 2 \sin \tfrac{1}{2}\,pt_1 \sin p(t - \tfrac{1}{2}t_1) \tag{2.18}$$

From equation (2.17), the maximum displacement x_{max} is given by



motion is governed by equation:

$$\frac{x}{x_{st}} = (1 - \cos pt)$$

motion is governed by equation:

$$\frac{x}{x_{st}} = 2 \sin \tfrac{1}{2} pt_1 \sin p (t - \tfrac{1}{2} t_1)$$

Figure 2.13 (a)

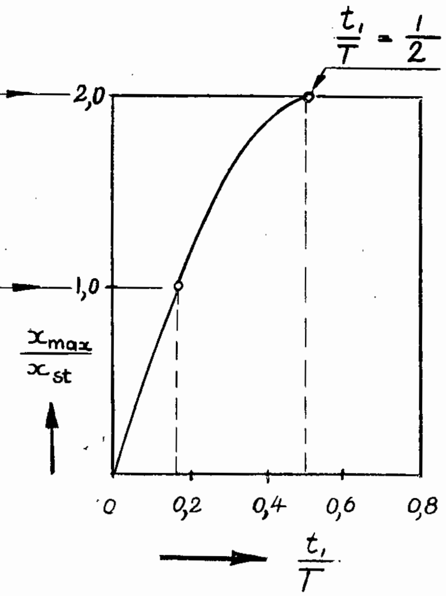


Figure 2.13 (b)

$$\frac{x_{max}}{x_{st}} = 2 \tag{2.19}$$

provided $pt_1 \geqslant \pi$

that is $t_1/T \geqslant \frac{1}{2}$

where T is the period of the system.

However, if $t_1/T < \frac{1}{2}$, then the maximum displacement is governed by equation (2.18) and the ratio x_{max}/x_{st} is given by:-

$$\frac{x_{max}}{x_{st}} = 2 \sin\left(\tfrac{1}{2}\, pt_1\right) \tag{2.20}$$

These two relationships are shown in fig. 2.13. The curves (a) and (b) represent the motion of two one-degree-of-freedom systems with different periods of vibration T, i.e.

for curve (a) $\quad T = \frac{4}{5}\, t_1$

for curve (b) $\quad T = 6\, t_1$

The curves were obtained by a method known as "Phase-plane Method", which will not be explained here and may be found in ref. (8).

Comparing the two values for x_{max}/x_{st} from equations (2.19) and (2.20) it can be inferred that whether the maximum transient response occurs during or after the period of application of the disturbing force depends on the ratio:-

$$\frac{t_1}{T} = \frac{\text{pulse length}}{\text{period of system}}$$

and

if $t_1/T \geqslant 1/2$ then the maximum displacement x_{max} occurs during the pulse and is given by equation (2.19), i.e. twice the static displacement;

if $t_1/T < 1/2$ then the maximum displacement x_{max} occurs after the end of the pulse and is given by equation (2.20).

The variation of the dynamic magnification factor x_{max}/x_{st} with the ratio t_1/T is shown in fig. 2.13(b)

CHAPTER 3

VIBRATIONS OF STATICALLY DETERMINATE BEAM STRUCTURES

Section 3.1 - Introduction

In nearly all structural systems such as beams, bridges and plates the mass and elastic properties are distributed rather than lumped. However, these parameters are assumed to be lumped in many vibration analyses since distributed systems are governed by partial differential equations and relatively few can be solved. But for some important cases, as for instance, beams, it is easy to write and solve the partial differential equation, governing the system.

In the following section only simple cases such as cantilevers and simply supported beams will be investigated. The exact solution will be found so as to form a basis of comparison with other mathematical models presented in section 3.3 of this chapter.

Section 3.2 - Transverse Bending Vibration of Distributed Systems

3.2.1 - General assumptions

The following analysis is applicable only to slender beams, for which the effects of rotary inertia and shear deformation are negligible. The material is assumed to be homogeneous and linearly elastic. Damping is neglected, and the beams are not subjected to axial forces. It is also assumed that the displacements are small and that they are referred to the static equilibrium position, i.e. they do not include the displacements due to gravity. Furthermore, the beams have uniform cross section.

3.2.2 - The equation of motion and its solution

Consider the forces and moments acting on a beam-element of length dx as shown in fig. 3.1.

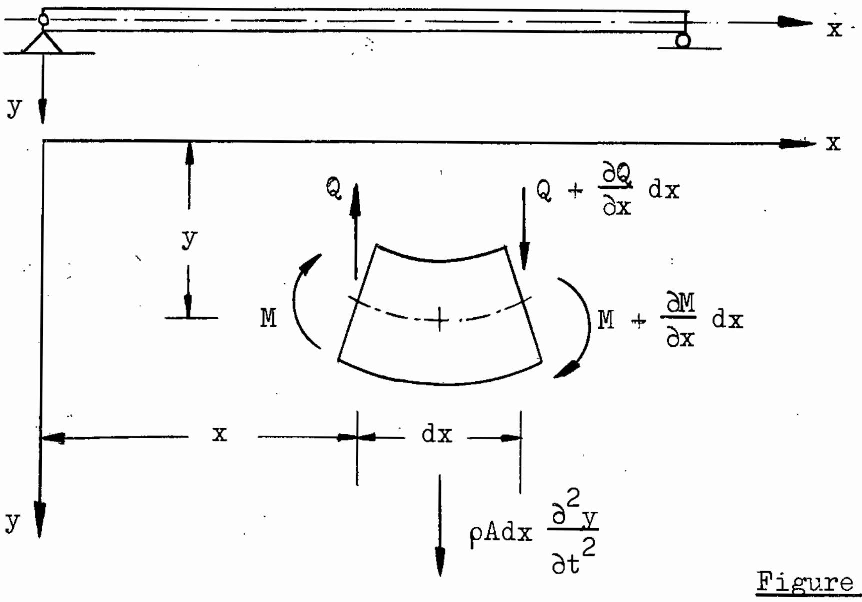


Figure 3.1

Q and M are the shear force and bending moment at section x. The inertia force on the element is $\rho A\, dx\, (\partial^2 y/\partial t^2)$, where ρ is the density of the beam and A is the cross-sectional area.

Taking moments about the centre line of the element (neglecting products of small quantities), and resolving for forces in the y-direction, we find:

$$Q = \frac{\partial M}{\partial x}$$

and

$$\frac{\partial Q}{\partial x} = \rho A \frac{\partial^2 y}{\partial t^2}$$

From the relationship between bending moment and curvature and the approximate curvature-displacement relationship:

$$M = -EI \frac{\partial^2 y}{\partial x^2}$$

where EI is the flexural rigidity. Combining the above three equations leads to the partial differential equation of motion:

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2} = 0 \qquad (3.1)$$

When a beam performs a normal mode of vibration the deflection varies harmonically with the time and can be represented as follows:

$$y = Y\,(A \sin pt + B \cos pt) \qquad (3.2)$$

where Y is a function of the coordinate x only. Y determines the shape of the normal mode under consideration and is called the "normal function". Substituting equation (3.2) into equation (3.1) we obtain:

$$\frac{d^4Y}{dx^4} - \frac{\rho A p^2}{EI} Y = 0 \qquad (3.3)$$

Equation (3.3) is of fourth order and the sine, cosine, sinh and cosh will each satisfy the equation. Consequently, the general solution will contain four constants and may be written in the following form:

$$Y = C \sin rx + D \cos rx + E \sinh rx + F \cosh rx \qquad (3.4)$$

where $r^4 = \rho A p^2/EI$

and the constants C, D, E and F will be determined from the end conditions of the beam. For the two ends of a vibrating beam we always have four end-conditions from which the ratios between the arbitrary constants of the general solution (3.4) can be obtained. In this manner the normal modes of free vibration and their natural frequencies will be established.

3.2.3 - End conditions

(a) Simply supported beam

In the case of a simply supported beam (fig. 3.2), the displacement and bending moment are both zero at each end of the beam. The end conditions are therefore:

$$\left.\begin{array}{llll} \text{at } x = 0 & Y = 0 & \text{i.e. } 0 & = D + F \\ \text{at } x = 0 & Y'' = 0 & \text{i.e. } 0 & = -r^2D + r^2F \end{array}\right\} \text{ thus } D = F = 0$$

(Primes will be used to designate differentiation w.r.t. x)

$$\begin{array}{llll} \text{at } x = L & Y = 0 & \text{i.e. } 0 & = C \sin rL + E \sinh rL \\ \text{at } x = L & Y'' = 0 & \text{i.e. } 0 & = -Cr^2 \sin rL + Er^2 \sinh rL \end{array}$$

The only non-trivial solution is E = 0 and sin rL = 0 for $C \neq 0$ leading to:

$$rL = n\pi \qquad n = 1,2,3,\ldots$$

and

$$p_n = (n\pi)^2\sqrt{\frac{EI}{\rho A L^4}} \qquad n = 1,2,3,\ldots \qquad (3.5)$$

For convenience let us introduce the symbol

$$p_w = \sqrt{\frac{EI}{\rho A L^4}} = \sqrt{\frac{EI}{M_B L^3}}$$

p_w incorporates all the properties for a particular beam, i.e. its total mass M_B, its length L and flexural rigidity EI.

For the first three natural frequencies, we have:

$$p_1 = 9{,}8696\ p_w$$

$$p_2 = 39{,}4784\ p_w$$

$$p_3 = 88{,}8264\ p_w$$

In the following sections, these values will be referred to as "exact" values of the natural frequencies for a beam having a continuous, uniform mass distribution.

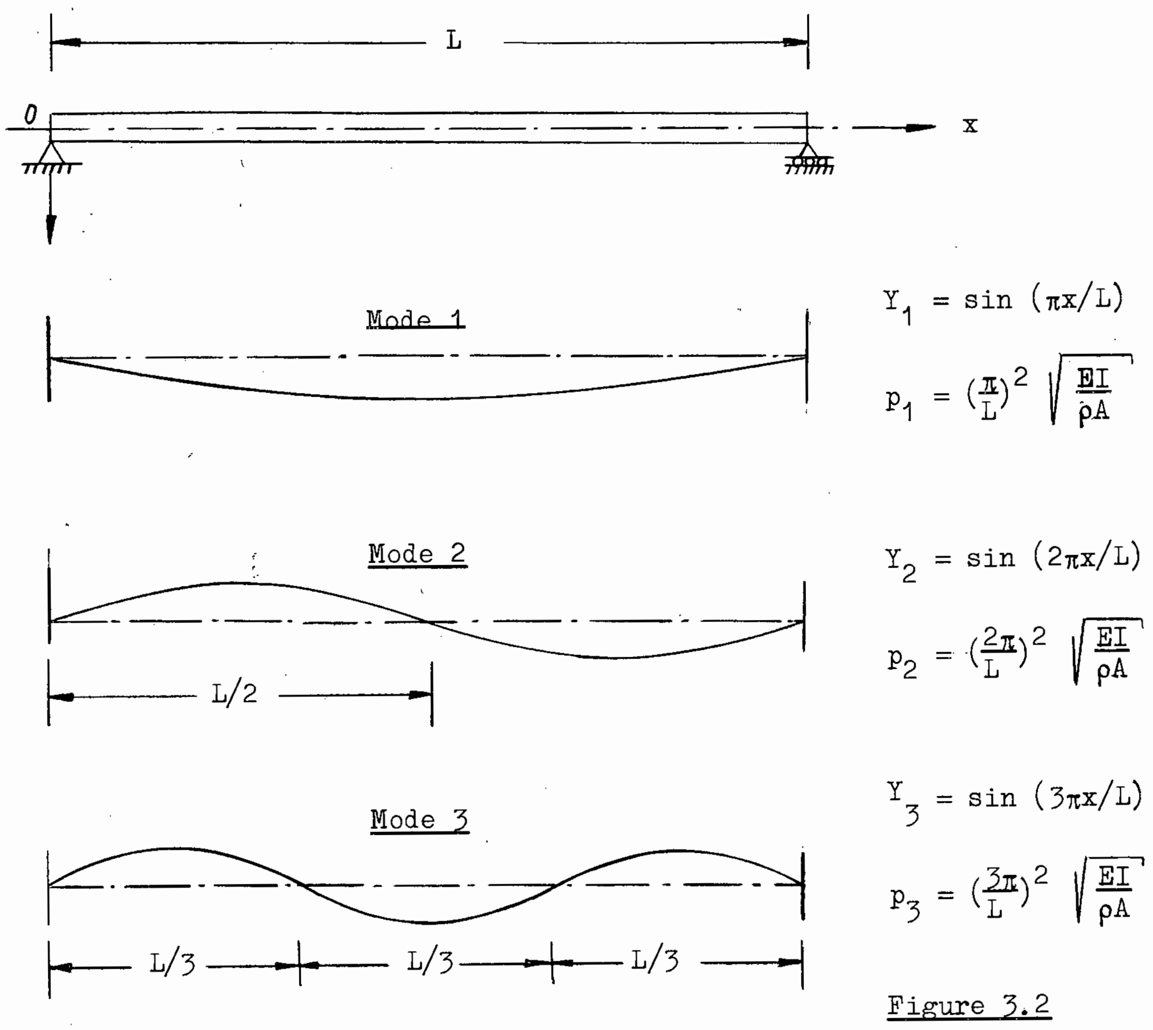


Figure 3.2

In section 2.3.1, example (b), the fundamental frequency of a simply supported beam having one degree of freedom was calculated to be:

$$p = 6{,}92\ p_w$$

This value is very much lower than that of p_1 given above and illustrates the magnitude of error which occurs when the total mass of the beam is concentrated at only one point.

From equation (3.5) we see that there is an infinite number of values of p that satisfy this equation. As mentioned previously this is true of all distributed systems and is in agreement with the fact that the systems are composed of an infinite number of mass particles. It should be noted that the natural frequencies increase with the square of the natural mode number.

The shape of the normal mode of vibration corresponding to the natural frequency is given by the normal function:

$$Y_n = \sin \frac{n\pi x}{L} \qquad 0 \leqslant x \leqslant L$$

The shapes of the normal modes therefore consist of integer numbers of "loops" of a sine curve, as shown in fig. 3.2. Hence for a uniform beam with simply supported ends the expression for the modes of vibration is given by:

$$y(x,t) = \sin \frac{n\pi x}{L} (A_n \sin p_n t + B_n \cos p_n t) \tag{3.6}$$

However, there is an infinite number of normal modes, and a general free vibration may involve all of them. Since the system is linear, we may, by superposition, write the following summation of an infinite number of expressions of the type of equation (3.6):

$$y = \sum_{n=1,2,..}^{\infty} \sin \frac{n\pi x}{L} (A_n \sin p_n t + B_n \cos p_n t) \tag{3.7}$$

In order to evaluate A_n and B_n, expressions are needed for the displacement y and the velocity $\dot{y}$ at zero time; thus:

$$y_o = \sum_n^{\infty} A_n \sin \frac{n\pi x}{L}$$

$$\dot{y}_o = \sum_n^{\infty} B_n \sin \frac{n\pi x}{L}$$

If we multiply both sides of each equation by $\sin(m\pi x/L)$, where m is an integer, and integrate between the limits $x = 0$ and $x = L$, then the following formulas are obtained:-

$$A_n = \frac{2}{L} \int_0^L y_o \sin \frac{n\pi x}{L} \, dx$$

$$B_n = \frac{2}{p_n L} \int_0^L \dot{y}_o \sin \frac{n\pi x}{L} \, dx \qquad (3.8)$$

<u>Example</u>:

Let us assume that the beam is subjected to an impulsive starting condition, $y_o = 0$ and $\dot{y}_o = v$ = constant. Hence from equation (3.8):

$$A_n = 0$$

$$B_n = \frac{2v}{p_1 n^2 L} \int_0^L \sin \frac{n\pi x}{L} \, dx = \begin{cases} \dfrac{4v}{\pi p_1 n^3} & \text{for } n = 1,3,5,\ldots \\ 0 & \text{for } n = 2,4,6,\ldots \end{cases}$$

noting, that we substituted $p_n = p_1 n^2$, for a simply supported beam.

Substituting A_n and B_n in equation (3.7), we find that the displacement series resulting from the impulsive starting condition is given by:

$$y = \frac{4v}{\pi p_1} \sum_{n=1,3,5}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L} \sin p_n t \qquad (3.9)$$

Only the odd-numbered or symmetrical normal modes are excited by the symmetrical starting condition. Furthermore, we see that their amplitudes are proportional to $1/n^3$, that is, to 1, 1/27, 1/125, etc. This indicates that the displacement response is largely in the fundamental mode.

(b) <u>Cantilever</u>

Consider the cantilever shown in fig. 3.3; with the origin of the axes at the fixed end, the end-conditions are:-

$$\begin{array}{llll} \text{at } x = 0 & Y = 0 & \text{i.e.} & 0 = D + F \\ \text{at } x = 0 & Y' = 0 & \text{i.e.} & 0 = rC + rE \\ \text{at } x = L & Y'' = 0 & \text{i.e.} & 0 = -r^2C \sin rL - r^2D \cos rL \\ & & & \quad + r^2E \sinh rL + r^2F \cosh rL \\ \text{at } x = L & Y''' = 0 & \text{i.e.} & 0 = -r^3C \cos rL + r^3D \sin rL \\ & & & \quad + r^3E \cosh rL + r^3F \sinh rL \end{array}$$

Hence we obtain the set of equations:

$$C(-\sin rL - \sinh rL) + D(-\cos rL - \cosh rL) = 0$$

$$C(-\cos rL - \cosh rL) + D(\sin rL - \sinh rL) = 0$$

These equations will have a non-trivial solution for C and D only if the determinant of the coefficients is equal to zero. Hence we get the frequency equation:

$$(\cosh rL + \cos rL)^2 + (\sin rL + \sinh rL)\cdot(\sin rL - \sinh rL) = 0$$

After expanding and simplifying this leads to:

$$\cos rL \,.\, \cosh rL = -1 \tag{3.10}$$

There exists an infinite number of values of rL which satisfy equation (3.10), as can be seen from fig. 3.4 where the graphs $\eta_1 = -1/\cosh rL$ and $\eta_2 = \cos rL$ are plotted. The first three roots of equation (3.10) are:

$$r_1L = 0{,}597\,\pi = 1{,}875$$

$$r_2L = 1{,}494\,\pi = 4{,}694$$

$$r_3L = 2{,}500\,\pi = 7{,}855$$

Since $1/\cosh rL$ becomes infinitely small for large values of rL, the higher roots are given with satisfactory accuracy by the equation:

$$r_nL = (n - 1/2) \qquad \text{for } n \geqslant 3$$

The natural frequency is given by:

$$p_n = r_n^2 \sqrt{\frac{EI}{\rho A}}$$

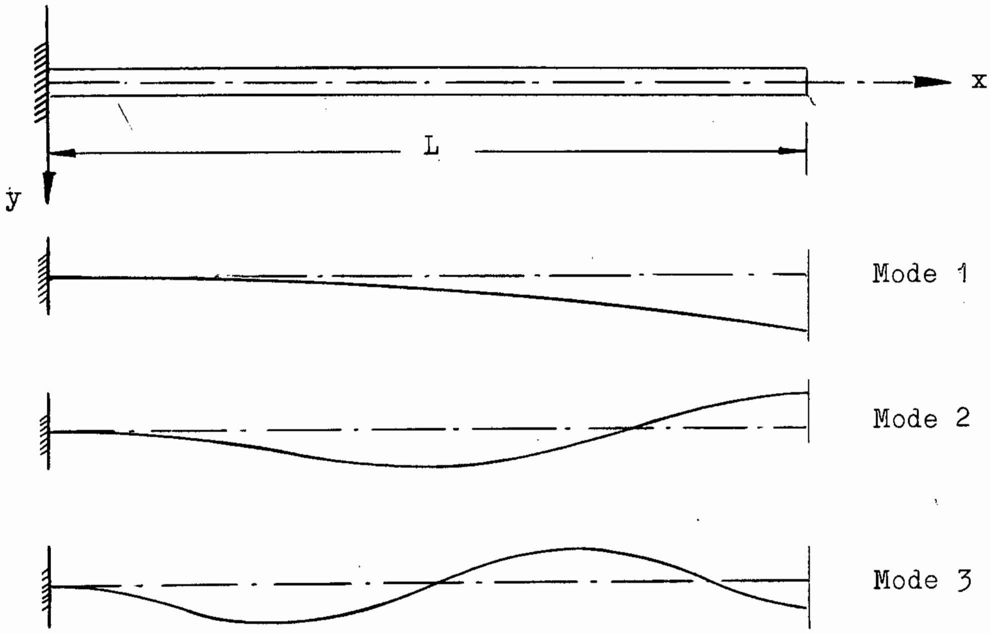


Figure 3.3

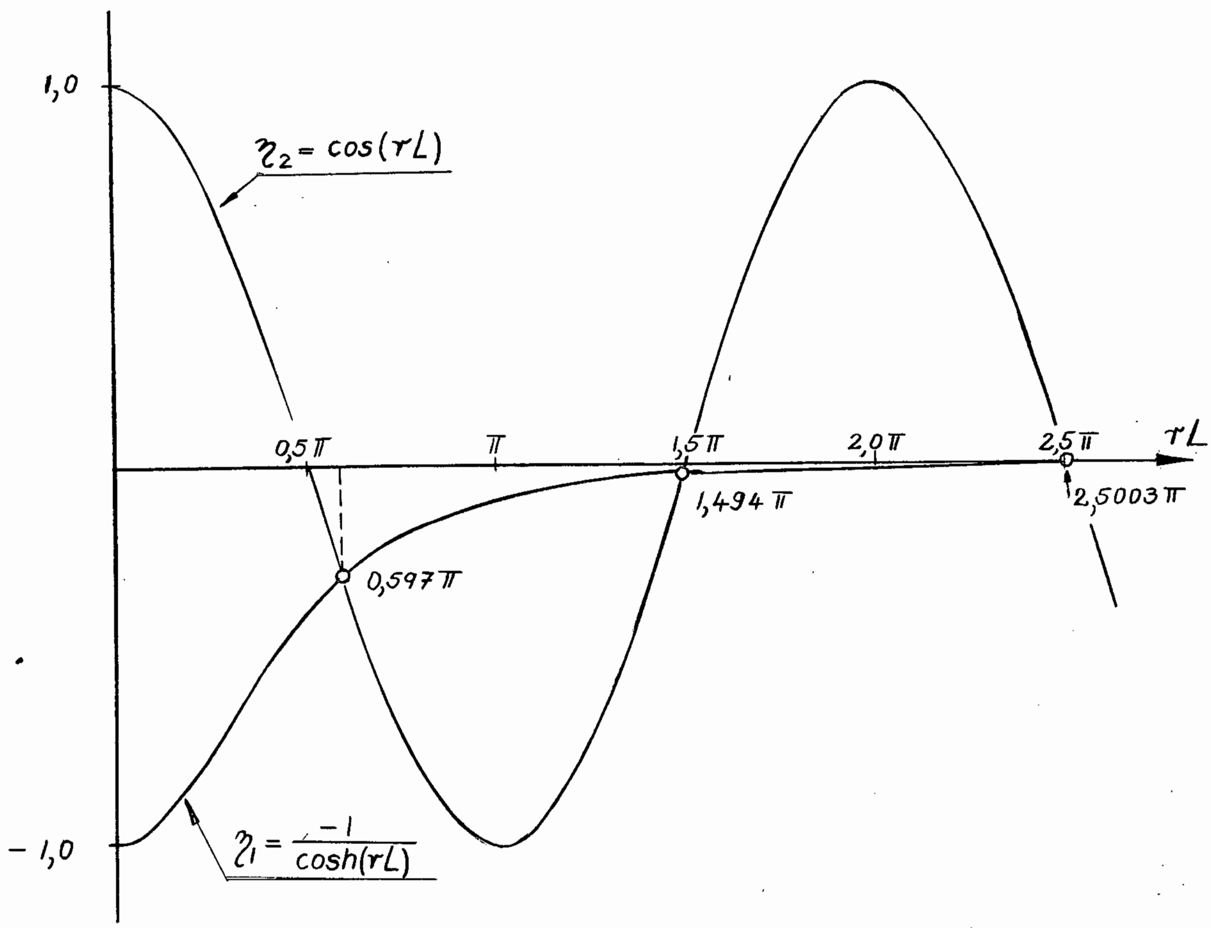


Figure 3.4

and for higher frequencies, they can be approximated by:

$$p_n = \frac{(n - 1/2)^2}{L^2} \pi^2 \sqrt{\frac{EI}{\rho A}} \qquad (3.11)$$

The values for the first three natural frequencies are as follows:-

$$p_1 = 3{,}516 \; p_w$$

$$p_2 = 22{,}03 \; p_w$$

$$p_3 = 61{,}70 \; p_w$$

Comparing the value of the fundamental frequency with the one obtained in the example (a) in section 2.3.1, it can be seen that the exact value is roughly twice as much as the value obtained when using the lumped-mass approximation with one degree of freedom.

Equation (3.11) shows that the natural frequencies of the higher harmonics increase with the squares of the successive integers n.

Section 3.3 - Approximate Methods of Determing the Natural Frequencies

3.3.1 - Introduction

We saw in the previous section that exact solutions can be obtained for simple beam structures having a constant cross-section, i.e. uniform mass distribution by solving the partial differential equation of motion and by substituting the appropriate end-conditions. However, for beams with non-uniform mass distribution or non-uniform cross-sectional area, the complexity of the calculation becomes excessive and with the exception of a few special cases such solutions do not exist and approximate methods are required to determine the natural frequencies and their associated mode shapes. There are many methods of determining approximately one or more natural frequencies and in the subsequent section follows a survey of some of these methods. In section 2.2.4 it was mentioned that there are two methods of approximating the actual structure:

(i) the lumped mass method; in this method the equations of motion are set up. There will be as many equations as there are degrees of freedom in the system. In order to obtain the frequencies and mode shapes, the set of n simultaneous equations has to be solved;

(ii) the assumed mode method; this approach is developed from considerations involving the total energy of the vibrating system. It is necessary to guess a normal mode shape and the success of this method depends on the choice of these assumed shapes. The Rayleigh and Rayleigh-Ritz methods fall into this group.

Another way, which will not be discussed in this thesis, is to obtain an approximate solution of the partial differential equations using finite-difference methods.

By lumping the distributed mass at certain points a possible non-uniformity of the beam mass can easily be taken into consideration. At the end of this chapter the question will arise, into how many concentrated masses a system should be divided in order to obtain a good approximation to the natural frequencies of the original system.

On the other hand, the self-weight of a beam might be negligible compared with the applied (point) loads and the approximate methods are very useful in assessing the natural frequencies of this type of system.

3.3.2 - Rayleigh's approximation

A) The Energy Equation

Rayleigh's method is based on the principle that in a freely vibrating system a periodic exchange of kinetic and potential energy takes place.

The energy equation for a vibrating system is easily obtained from the equation of motion:

$$m\ddot{y} + ky = 0$$

multiplying by $\dot{y}$

$$m\ddot{y}\dot{y} + ky\dot{y} = 0$$

and integrating w.r.t. time gives

$$\tfrac{1}{2}m\dot{y}^2 + \tfrac{1}{2}ky^2 = C \qquad (3.12)$$

The first term of this expression gives the instantaneous kinetic energy of the motion of the mass, and the second term represents the instantaneous potential energy of the restoring element relative to the potential energy at the static equilibrium position of the system. C is the total energy of the system and it remains constant throughout the motion provided there is no damping. The magnitude of C is determined by the initial conditions of the motion.

When the displacement y is a maximum Y, the velocity will be zero and

$$\tfrac{1}{2}kY^2 = C$$

Similarly, when the displacement y is zero, the velocity $\dot{y}$ will be a maximum $\dot{Y}$ and

$$\tfrac{1}{2}m\dot{Y}^2 = C$$

The maximum kinetic energy is thus equal to the maximum potential energy. The maximum values of the two forms of energy occur a quarter of a cycle apart, which leads to the assumption that the motion of the free vibration can be described as harmonic so that the velocity of the motion is related to the displacement.

Consider for example again the single-mass-spring system (fig. 3.5)

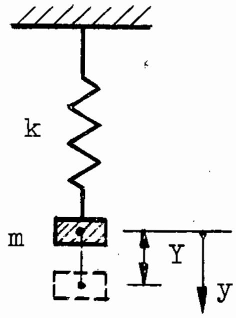


Figure 3.5

The displacement of the mass m is chosen to be:

$$y = Y \sin pt$$

The kinetic and potential energies are given by

$$KE = \tfrac{1}{2} m\dot{y}^2 = \tfrac{1}{2} mY^2 p^2 \cos^2 pt$$

$$PE = \int_0^y ky\, dy = \tfrac{1}{2} ky^2 = \tfrac{1}{2} kY^2 \sin^2 pt$$

At the position of maximum displacement we have:

$$PE_{max} = \tfrac{1}{2} kY^2 \quad ; \; KE = 0$$

and at the position of zero displacement we have:

$$KE_{max} = \tfrac{1}{2} mY^2 p^2 ; \; PE = 0$$

Knowing that the maximum kinetic energy is equal to the maximum potential energy, we get

$$\tfrac{1}{2} kY^2 = \tfrac{1}{2} mY^2 p^2$$

thus $\quad p^2 = k/m \qquad$ as before in section 2.3.1

B) Bending vibration of a simple beam

In the case of heavy beams, the expressions for the kinetic and potential energies are more complicated. Consider an element of a beam (fig. 3.6) deflected by the amount η from its equilibrium position. If a moment bends this element of length dx through an angle $d\theta$, the elastic potential energy stored in the elementary length of the beam will be $Md\theta/2$. Since $d\theta = dx/r$, and since the relationship between moment M and radius of curvature r of a beam is given by $M = EI/r$, the above expression for the potential energy of the element becomes

$$PE = \tfrac{1}{2} \frac{EI}{r^2} dx$$

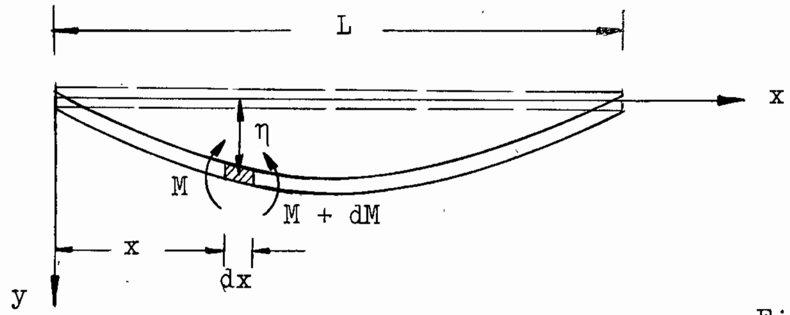


Figure 3.6

Since we consider only small displacements, i.e. small slopes, we may write:

$$\frac{1}{r} = \frac{d^2\eta}{dx^2}$$

so that the maximum potential energy of the entire beam will be given by an integration over its length, L, hence

$$PE = \tfrac{1}{2}\,EI \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx \qquad (3.13)$$

The kinetic energy of the element is $\Delta KE = \frac{1}{2}\, m\dot{y}^2 dx$. Assuming that the element has a simple harmonic motion about its static equilibrium position, then at x

$$\eta = y \sin pt$$

hence $\quad \max \dot{\eta} = -\,py$

and $\quad \Delta KE_x = \frac{1}{2}\, mp^2\, y^2\, dx$

The maximum kinetic energy of the whole beam will be obtained by integrating over its length L,

hence $\quad KE = \frac{1}{2} \times \int_0^L mp^2\, y^2\, dx$

Since p is constant at all points along the beam, and m is constant for a uniform beam, we rewrite

$$KE = \tfrac{1}{2}\, mp^2 \int_0^L y^2\, dx \qquad (3.14)$$

and thus equating the maximum kinetic and maximum potential energy given by equations (3.14) and (3.13) respectively, we get the following expression for the lowest natural frequency

$$p^2 = \frac{EI \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx}{m \int_0^L y^2\, dx} \qquad (3.15)$$

In the case of a non-uniform beam we get.

$$p^2 = \frac{\int_0^L EI \left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^L my^2\, dx} \qquad (3.16)$$

This is a general expression that cannot be integrated unless we can make a proper guess about y as a function of x. The procedure is to guess a deflection curve which satisfies the end-conditons and corresponds to the mode of vibration concerned.

It can be shown mathematically that for a uniform, simply supported beam the sine curve is the curve of all possible ones, satisfying the end-conditions of zero displacement, that will make the numerical value of equation (3.15) a minimum.

C) Examples: (1) Simply supported beam

Consider the system shown in fig. 3.7, consisting of a heavy uniform beam simply supported at each end and carrying a concentrated load at the centre of the span

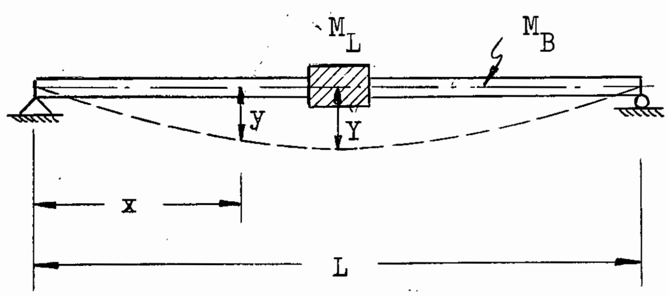


Figure 3.7

let m = mass per unit length of the beam

M_B = total mass of beam = $m.L$

M_L = mass of central load

Y = maximum deflection at mid-span.

The end-conditions to be satisfied are that the displacements and bending moments at each end are zero, i.e.

$$\text{at } x = 0 \text{ and } x = L: \quad y = 0; \quad \frac{d^2y}{dx^2} = 0$$

For the fundamental natural frequency the curve $y = Y \sin \pi x/L$ satisfies the end-conditions. From equation (3.14) the maximum kinetic energy of the beam is:

$$KE = \tfrac{1}{2} mp^2 \int_0^L Y^2 \sin^2 \frac{\pi x}{L}\, dx$$

$$= \tfrac{1}{4} M_B\, p^2\, Y^2$$

and for central load:

$$KE = \tfrac{1}{2} M_L p^2 Y^2$$

Hence for the whole system,

$$KE = \tfrac{1}{2} p^2 Y^2 (\tfrac{1}{2} M_B + M_L)$$

From equation (3.13) the maximum potential energy is:

$$PE = \frac{EI\, Y^2 \pi^4}{4 L^3}$$

Equating the above two expressions and solving for p^2, gives:

$$p^2 = \frac{48{,}7\ EI}{L^3 (\tfrac{1}{2} M_B + M_L)} \tag{3.17}$$

Two special cases may be noted.

(i) <u>If the central load is zero</u>, the system reduces to a heavy, simply supported beam, with total mass M_B:

$$p^2 = \frac{97{,}4\ EI}{M_B L^3}$$

hence $$\underline{p = 9{,}87 \sqrt{\frac{EI}{M_B L^3}}}$$

This agrees extremely well with the exact expression of the fundamental frequency derived in section 3.2.3, where

$$\underline{p_1 = 9{,}8686\ p_w}$$

(ii) If the beam is light compared with the central load, i.e. $M_B = 0$, then

$$p^2 = \frac{48{,}7\ EI}{M_L L^3}$$

or $$p = 6{,}98 \sqrt{\frac{EI}{M_L\, L^3}}$$

Comparing this with $p^2 = k/m$ for the system of fig. 3.7 and example (b) in section 2.3.1, the effective spring constant of the light beam is given as

$$K = \frac{48{,}7\ EI}{L^3}$$

as compared with the well known flexibility coefficient

$$K = \frac{48\ EI}{L^3}$$

Thus Rayleigh's method gives a value for p^2 which is 1,4% in excess of the true value for a light beam with a central load.

Rayleigh's method is very useful when it is desired to know the effect of neglecting the mass of a beam carrying concentrated loads. It can be seen from equation (3.17) that half the mass of the beam should be added to the central load if it is required to find the lowest natural frequency of a one-degree-of-freedom system having one lumped mass M_L and distributed mass $m = M_B/L$

(2) Cantilever

Consider the cantilever shown in fig. 3.8

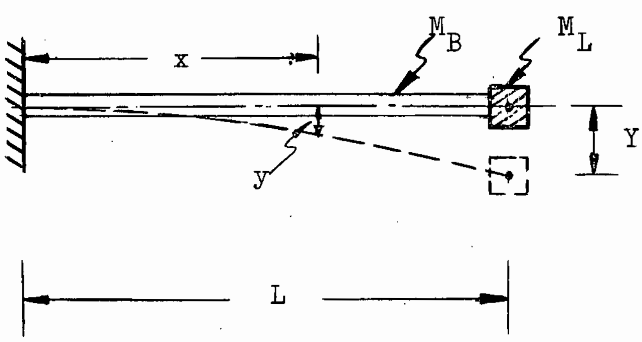


Figure 3.8

The end-conditions to be satisfied are the following:-

at $x = 0$ $\quad y = 0 \quad y' = 0$

at $x = L$ $\quad y'' = 0$

The following assumed function will satisfy the end-conditions:

$$y = Y(1 - \cos \frac{\pi x}{2L})$$

Similarly as above, we obtain for the kinetic and potential energies:

$$KE = p^2 Y^2 (\tfrac{1}{2} M_L + 0,113 M_B)$$

$$PE = \frac{\pi^4 EI}{64 L^3} Y^2$$

therefore $$p^2 = \frac{3,04 EI}{L^3 (M_L + 0,226 M_B)}$$

Again two special cases may be considered:

(i) $M_L = 0$, the system is reduced to a heavy beam with

$$p^2 = \frac{13,46 EI}{M_B L^3}$$

$$\therefore p = 3,66 \sqrt{EI/M_B L^3}$$

compared with the exact result in section 3.2.3

$$p = 3,516 \sqrt{EI/M_B L^3}$$

which is a percentage error of 4,1

(ii) $M_B = 0$, the system is reduced to a mass-spring system with

$$p^2 = \frac{3,04 EI}{M_L L^3}$$

comparing this with $p^2 = k/m$ for the system shown in example (a) section 2.3.1, the effective spring constant is given as

$$k = \frac{3,04 EI}{L^3}$$

as compared with the well known flexibility coefficient

$$k = \frac{3 EI}{L^3}$$

Thus Rayleigh's method gives a value for p which is 0,6% in excess of the true value for a light beam.

It follows from the above, that Rayleigh's method can be used for both distributed and lumped systems.

D) Rayleigh's Quotient

Rayleigh's principle states that if a reasonable mode shape is assumed, satisfying at least the following end-conditions:

$y = 0$ for hinge-supported end

$$\left.\begin{matrix} y = 0 \\ y' = 0 \end{matrix}\right\} \text{ for a clamped end}$$

then a good approximation to the natural frequency will be obtained, using Rayleigh's quotient:

$$p^2 = \frac{\int_0^L EI\,(y'')^2\,dx}{\int_0^L m\,y^2\,dx}$$ for distributed systems with m = mass per unit length

or

$$p^2 = \frac{\int_0^L EI\,(y'')^2\,dx}{\int_0^L my^2 dx + \Sigma\, M_i Y_i^2}$$ for distributed systems with additional concentrated masses M_i

or

$$p^2 = \frac{X'\,K\,X}{X'\,M\,X}$$ for a system having only lumped masses.

The latter equation is given in matrix notation, where

K = stiffness matrix
M = mass matrix
X = displacement vector.

(see section 3.3.6 and Appendix B 4.1.3)

E) Comment on Rayleigh's Principle

The chief usefulness of Rayleigh's principle lies in the fact that it provides a quick method of estimating the lowest natural frequency of a vibrating system. It is only necessary to use a moderately accurate representation of the mode shape in order to obtain a good approximation of the fundamental frequency. However, by assuming the mode shape, constraints are imposed on the beam. This has the result that the calculated lowest natural frequency will always be <u>higher</u> than the true value, unless the exact mode shape is known, in which case, Rayleigh's quotient gives the exact value of the fundamental frequency.

The chief disadvantages of Rayleigh's method as an alternative to the direct calculation of the modes and frequencies from the frequency equation (see section 2.3.2) are:

(i) it can be expected to give accurate results only for the fundamental frequency of simple systems for which it is possible to guess the approximate mode shape;

(ii) it is difficult to apply to higher frequencies of the system;

(iii) it contains no "built-in" means of calculating an improved estimate of the mode shape from the calculated value of the frequency.

3.3.3 - The static deflection method

This method is a special case and differs from Rayleigh's method in that the potential energy is obtained in a different way and use is made of the static deflection curve.

In the examples given in the previous section, only harmonic deflection curves were assumed for the mode shape. However, employing static deflection curves for the assumed mode shapes leads to similar good approximation of the natural frequency and hence it seems justifiable to use the static deflection curve for evaluating the maximum potential energy of the system. Assume that the beam shown in fig. 3.9 is given a displacement upwards equal to the gravitational displacement due to uniformly distributed dead load.

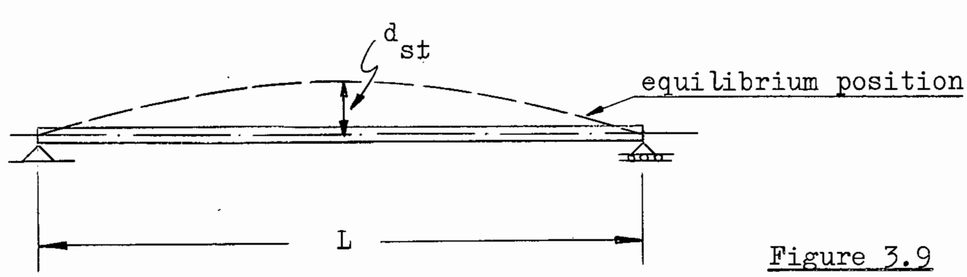


Figure 3.9

Consequently we can calculate the initial potential energy in the beam by summing or integrating, the work done against gravity. Denoting a concentrated mass by M and a distributed mass by m, the work done on these masses can be expressed by:

$$PE = \Sigma \tfrac{1}{2} M_i g d \quad \text{for } \underline{\text{concentrated}} \text{ masses}$$

and

$$PE = \int \tfrac{1}{2} mgy\,(dx) \text{ for } \underline{\text{distributed}} \text{ masses}$$

where d is the static deflection of the mass.

This procedure dispenses with the necessity of dealing with the second derivative or the curvature of the beam as shown in equation 3.13 and relates the maximum kinetic as well as the maximum potential energies to the static displacement curve. In general, we may write for any beam:

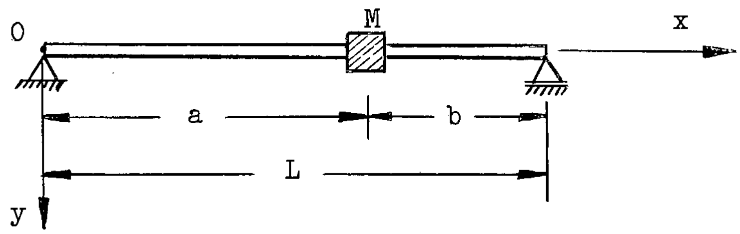
$$p^2 = g \frac{\int_0^L my(dx) + \sum_{i=1}^{n} M_i d_i}{\int_0^L my^2(dx) + \sum M_i d_i^2} \tag{3.18}$$

If the mass of the beam is neglected, or lumped at i discrete points in i equal masses M, equation (3.18) simplifies to

$$p^2 = \frac{g}{d} \tag{3.19}$$

Examples for a simply supported beam

(a) With one lumped mass



From Appendix A1, the static deflection d_{st} at $x = a$ is

$$y = \frac{M.g.a^2b^2}{3\ EI\ L} = d_{st}$$

Substituting in equation (3.19) gives for the fundamental frequency

$$p = \sqrt{\frac{g}{d_{st}}} = \frac{1}{ab}\sqrt{\frac{3\ EI\ L}{M}}$$

If $a = b = \frac{L}{2}$

$$p = 4\sqrt{\frac{3\ EI}{M\ L^3}}$$

$$p = 6{,}928\sqrt{\frac{EI}{M\ L^3}}$$

The value 6,928 is very close to that obtained above in example 1(ii), page 3.16, where

$$p = 6{,}98\sqrt{\frac{EI}{M\ L^3}}$$

But the amount of work is reduced considerably, when using the static deflection approach, since it was not necessary to make an intelligent guess of the mode shape and then obtain the expressions for the kinetic and potential energies.

(b) With two lumped masses

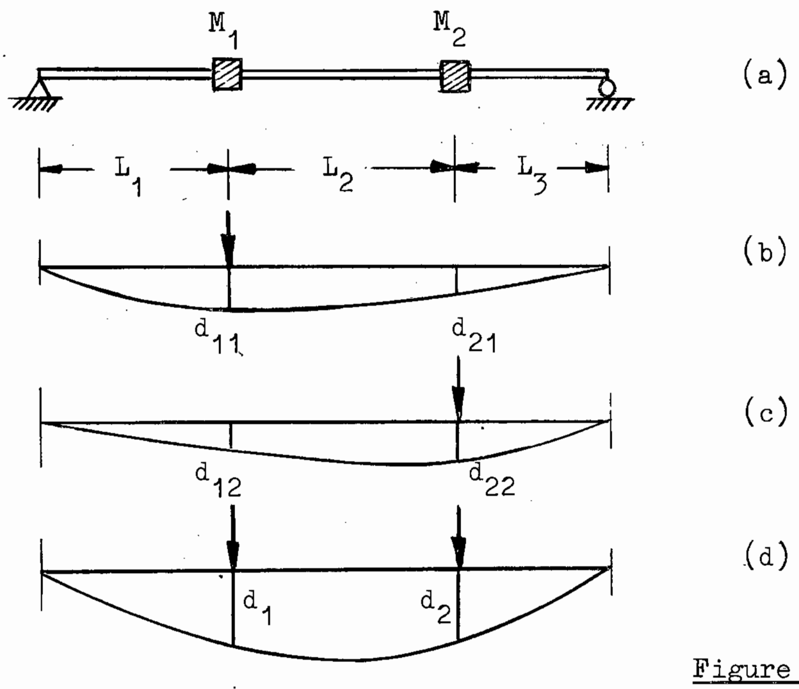


Figure 3.10

The principle of superposition may be applied to the system shown in fig 3.10. Considering M_1 first, we get from Appendix A1:

$$d_{11} = \frac{M_1\ g\ L_1^2}{3\ EI\ L}\ [L - L_1]^2 \qquad d_{21} = \frac{M_1 g\ L_1 L_3}{6\ EI\ L}\ [L^2 - L_1^2 - L_3^2]$$

Similarly for M_2, we get:

$$d_{22} = \frac{M_2 \ g \ L_3^2}{3 \ EI \ L} [L - L_3]^2 \qquad d_{12} = d_{21} = \frac{M_2 \ g \ L_1 \ L_3}{6 \ EI \ L} [L^2 - L_1^2 - L_3^2]$$

The total displacements are therefore given by:

$$d_1 = d_{11} + d_{12}$$

$$d_2 = d_{22} + d_{21}$$

and the expression for the frequency then becomes

$$p^2 = g \frac{M_1 \ d_1 + M_2 \ d_2}{M_1 \ d_1^2 + M_2 \ d_2^2} \tag{3.20}$$

Let us assume the following values:-

$$L_1 = L_2 = L_3 = \tfrac{1}{3} L$$

$$M_1 = M_2 = \tfrac{1}{2} M$$

then we get:

$$d_{11} = d_{22} = \frac{2}{243} \cdot \frac{M \ g \ L^3}{EI}$$

$$d_{12} = d_{21} = \frac{7}{972} \cdot \frac{M \ g \ L^3}{EI}$$

hence $d_1 = d_2 = d_{11} + d_{12}$

$$= \frac{15}{972} \cdot \frac{M \ g \ L^3}{EI}$$

$$\therefore \quad p^2 = g \frac{2 \left(\frac{M}{2} \cdot \frac{15}{972} \ \frac{M \ g \ L^3}{EI}\right)}{2 \cdot \frac{M}{2} \cdot \left(\frac{15}{972} \ \frac{M \ g \ L^3}{EI}\right)^2}$$

$$\underline{\underline{p = 8{,}95 \sqrt{\frac{EI}{M \ L^3}}}}$$

3.3.4 Dunkerley's Method

In this method several isolated systems are investigated and their natural frequencies are combined to a resultant fundamental frequency relating to the actual system of their combination. The method is developed as follows.

Assume that the potential energy is contained in one term, i.e. PE and that the kinetic energy is contributed by various inertia elements, i.e.

$$KE = k_1 p^2 + k_2 p^2 + \ldots\ldots + k_n p^2$$

where $k_i p^2$ is the kinetic energy of the i's element; and remembering that max PE = max KE, we get

$$\frac{1}{p^2} = \frac{k_1 + k_2 + \ldots\ldots + k_n}{PE}$$

A dynamic deflection curve may be estimated of the form:

$$y = y_1 + y_2 + \ldots\ldots + y_n$$

and the constants k_i of the kinetic energy contribution can be evaluated from such parts of y which are applicable.

If, however, the interdependent constraints between the inertia element be relaxed to such an extent that n fully isolated systems result, we may evaluate the isolated fundamental frequencies from the n hypothetical deflection curves

$$\frac{1}{p_1^2} = \left(\frac{k_1}{PE}\right)_1 \text{ with } y = y_1$$

$$\vdots \qquad \text{etc.}$$

$$\frac{1}{p_n^2} = \left(\frac{k_n}{PE}\right)_n \text{ with } y = y_n$$

The ratio $1/p_i^2$ will be a "true" but fictitious frequency if the isolated deflection curve y_i is the exact curve for the isolated system; if not, the ratio is smaller than the true one. It is, of course considerably <u>easier</u> to find the exact curve for an isolated system, i.e. a system having only one lumped mass, than for a system having several concentrated masses.

Assuming now that instead of the n isolated deflection curves we know the one exact composite deflection curve y and use it in calculating all the isolated approximate ratios $1/p_i^2$, we then have:

$$\frac{1}{p_1^2} = \left(\frac{k_1}{PE}\right)_1 \geqslant \frac{k_1}{PE}$$

$\vdots$ etc.

$$\frac{1}{p_n^2} = \left(\frac{k_n}{PE}\right)_n \geqslant \frac{k_n}{PE}$$

where the term in brackets is obtained from the isolated deflection curves y_i and the term k_1/PE without the brackets indicates the ratio which would be obtained from the exact composite deflection curve y.

If these inequalities are added, we get, on the right-hand side, the inverse of Rayleigh's approximation with zero error, i.e.

$$\frac{1}{p_1^2} + \frac{1}{p_2^2} + \ldots + \frac{1}{p_n^2} = \left(\frac{k_1}{PE}\right)_1 + \ldots + \left(\frac{k_n}{PE}\right)_n \geqslant \frac{k_1 + \ldots + k_n}{PE} = \frac{1}{p_{true}^2}$$

This shows that an upper limit of p^{-2} and consequently a lower limit of p^2 will be obtained. In other words, Dunkerley's approximation will be lower than, or equal to, the true value of the fundamental frequency of the combined system. The general equation for a system with n concentrated masses may be written as

$$\frac{1}{p^2} = \frac{1}{p_1^2} + \frac{1}{p_2^2} + \ldots\ldots + \frac{1}{p_n^2} \qquad (3.21)$$

where p is the fundamental frequency of the combined system, and p_i are the fundamental frequencies of the various masses acting alone.

The above derivation is taken from ref. (8) and no mention is made with regard to the application of Dunkerley's method for other frequencies than the fundamental frequency. Bishop and Johnson (9) state that Dunkerley's method is only applicable to problems where the second and higher natural frequencies of the combined system are considerably greater than the lowest natural frequency, as it is generally the case in beam structures. Furthermore, it is stated that Dunkerley's approach can be applied to higher frequencies under certain conditions which will not be discussed in this thesis.

(a) Example for a simply supported beam

Let us again consider the beam shown in fig. 3.10(a). This system may be imagined relaxed into two isolated systems, each involving the flexural properties of the beam and in turn each of the lumped masses. The two isolated frequencies corresponding to fig. 3.10(b) and fig. 3.10(c) on page 3.21 are

$$p_1 = \sqrt{\frac{g}{d_{11}}} \qquad p_2 = \sqrt{\frac{g}{d_{22}}}$$

therefore $$\frac{1}{p^2} = \frac{1}{p_1^2} + \frac{1}{p_2^2} = \frac{d_{11} + d_{22}}{g}$$

hence $$p = \sqrt{\frac{g}{d_{11} + d_{22}}}$$

We shall evaluate p for the same beam (fig. 3.10) investigated by the static deflection method. The values for d_{11} and d_{22} are already determined

$$d_{11} = d_{22} = \frac{2}{243}\frac{MgL^3}{EI}$$

hence $$p = 7,79\sqrt{\frac{EI}{ML^3}} \qquad (a)$$

compared with $$p = 8,95\sqrt{\frac{EI}{ML^3}} \qquad (b)$$

which was obtained using the static deflection method. Since Dunkerley's method gives an under estimate and Rayleigh's method gives an over estimate of the fundamental frequency, the true value for p of this system lies between the results (a) and (b).

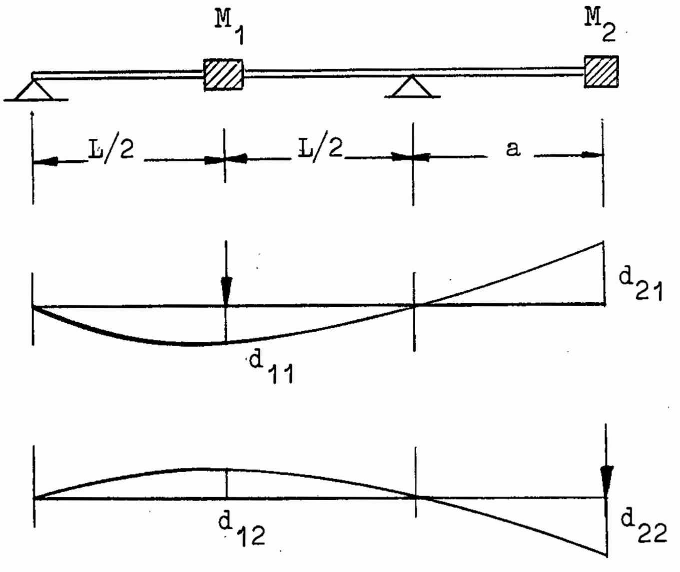
It is quite obvious that the fundamental frequency can be calculated more quickly using Dunkerley's method than by the static deflection method, since considerably less deflection calculations are required.

If, however, the beam with two lumped masses is assumed to be an approximation of a beam with continuous mass distribution, both results (a) and (b) are lower than the exact value, $p = 9,8696\ p_w$ and the result obtained from Dunkerley's formula has a greater percentage error than that obtained from the static deflection method.

Dunkerley's Method will be discussed again in connection with the flexibility matrix in sub-section 3.3.6.

(b) Simply supported beam with cantilever end

As a final example of this section, a beam with a cantilever end will be considered, which is shown below.



From Appendix A2:

$$d_{11} = \frac{M_1 g\ L^3}{48\ EI} \qquad d_{22} = \frac{M_2 g\ a^2}{3\ EI}(a + L)$$

$$d_{21} = \frac{M_1 g}{M_2 g} d_{12} \qquad d_{12} = -\frac{M_2 g\ a\ L^2}{16\ EI}$$

Let $M_1 = M_2 = \frac{M}{2}$ and $a = \frac{L}{2}$

Dunkerley's Method

$$p^2 = \frac{g}{d_{11} + d_{22}}$$

$$d_{11} = \frac{Mg\ L^3}{92\ EI} \qquad d_{22} = \frac{Mg\ L^3}{16\ EI}$$

$$\therefore\ p^2 = \frac{92\ EI}{7\ M\ L^3}$$

$$\therefore\ p = 3{,}62\sqrt{\frac{EI}{M\ L^3}}$$

Static Deflection Method

$$p^2 = g\,\frac{M_1 d_1 + M_2 d_2}{M_1 d_1^2 + M_2 d_2^2}$$

$$d_1 = d_{11} + d_{12} = \frac{Mg\ L^3}{92\ EI} + \frac{Mg\ L^3}{48\ EI} = \frac{3}{92}\,\frac{Mg\ L^3}{EI}$$

$$d_2 = d_{22} + d_{21} = \frac{Mg\ L^3}{16\ EI} + \frac{Mg\ L^3}{48\ EI} = \frac{8}{92}\,\frac{Mg\ L^3}{EI}$$

$$p^2 = \frac{\frac{11}{92}\left[\frac{ML^3}{EI}\right]}{\frac{73}{92^2}\left[\frac{ML^3}{EI}\right]^2} = \frac{1012\ EI}{73\ \ ML^3}$$

$$p = 3{,}73\sqrt{\frac{EI}{ML^3}}$$

(c) Comment on S.I. Units

It is necessary to comment briefly on the various units used for EI, M and L in the S.I. system. From Timoshenko's solution of the partial differential equation we get the following expression for any natural frequency:

$$p = \text{constant} \times \sqrt{\frac{EI}{M\ L^3}}$$

where EI : Nm^2 ✓

L : m ✓

M : kg = Ns^2/m ✓

hence $p = \text{constant} \times \sqrt{\frac{Nm^2\ .\ m}{m^3\ .\ Ns^2}}$

$= \text{constant}\ [1/\text{sec}]$

i.e. the above units lead to the correct units for p, which is 1/sec.

In Dunkerley's method, we evaluate deflection due to a force $W = M.g$, which acts on the beam. For the deflections, the following expressions are obtained:-

$$d = \text{constant} \cdot \frac{L^3}{EI} M.g$$

substituting $p = \sqrt{g/\Sigma d}$ again leads to

$$p = \text{constant} \times \sqrt{\frac{EI}{M\,L^3}}$$

with the units as given above.

3.3.5 - Rayleigh - Ritz Method

Another extension of Rayleigh's approximation is the Rayleigh-Ritz method (see ref. (10)), which will be discussed only very briefly. In this method, the system is assumed to vibrate only in combinations of a certain set of assumed modes, such as

$$Y(x) = s_1 y_1(x) + s_2 y_2(x) + \ldots\ldots\ldots + s_n y_n(x)$$

Each of the assumed functions $y_i(x)$ must satisfy the boundary conditions and they must impose as little constraint as possible on the motion of the system. The parameters s_i are arbitrary, but they are chosen in such a way as to make the frequencies, determined from Rayleigh's quotient, a minimum. After evaluating the natural frequency as in Rayleigh's Method, i.e. equating max PE to max KE, p^2 is obtained as a function of s_i, and now the set of equations

$$\frac{\partial(p^2)}{\partial\, s_i} = 0 \qquad\qquad i = 1, 2, 3, \ldots\ldots\ldots n$$

is formed. Elimination of the parameters s_i leads to an nth order determinant in p^2, of which the roots give approximate values of the first n natural frequencies of the system.

This method overcomes largely the drawbacks of Rayleigh's method and also of the special case of static deflection method as follows:-

(1) the method requires less guesswork than Rayleigh's method;

(2) it may be applied to frequencies other than the fundamental;

(3) it provides a means of improving the mode shape;

(4) it increases the accuracy in the determination of the fundamental frequency compared with Rayleigh's method.

3.3.6 - Methods Involving Flexibility and Stiffness Matrices

There are two elementary approaches by which we may write the differential equations of motion of n lumped masses, which represent a particular structural system. We can consider either of the two:

(i) the displacement of each mass in terms of all the forces acting on each mass,

(ii) the equilibrium of the forces acting on each mass.

In Chapter 2 we derived the equations of motion using the second approach. Obviously both methods are closely linked, as will be shown later.

In the following paragraphs, matrix notation and matrix computation will be used, since they provide an organized approach to the problem. We shall follow the above distinction between the two approaches. However, in either case we need some information about the mechanical properties of the structure, which is provided by the flexibility matrix F in the first approach and by the stiffness matrix K in the second approach.

The Stiffness Matrix

If an elastic body is subjected to static forces $W_1, W_2, \ldots W_n$ the body will deflect and the relation between the displacements and the forces causing them has the matrix form:-

$$W = K \cdot y \tag{3.23}$$

where K is the stiffness matrix for the system and y is the displacement vector. For a system, which vibrates freely, the applied forces W_i are the inertia forces $-M\ddot{y}$, where M denotes the mass matrix of the system, which is nearly always symmetrical and in most cases simply a diagonal matrix.* Hence we get the equation of motion in matrix form:-

* Only in the case, where there are say 3 masses, the movement of which can be fully described by two coordinates, is the mass matrix not a diagonal matrix.

$$M\ddot{y} + Ky = 0 \tag{3.24}$$

Again the differential equation for the free vibration is solved by assuming a solution of the form $y = Y \cos pt$ where Y is a vector containing the maximum displacements, and hence $\ddot{y} = -Y p^2 \cos pt = -p^2 y$ which leads, after substitution to:

$$Ky = p^2 My \tag{3.25}$$

Rewriting equation (3.25) leads to the homogeneous equation:

$$(K - p^2 M)\, y = 0 \tag{3.26}$$

In order to have a non-trivial solution, it is necessary that

$$\det (K - p^2 M) = 0$$

This determinant is known as the frequency determinant. Upon expansion it gives a polynomial in p^2, the roots of which are the natural frequencies of the system. For each value of p there exists an associated normal mode vector y.

The Flexibility Matrix

If an elastic system is subjected to dynamic displacements, forces will result and the relationship between forces and displacements has the matrix form:

$$y = F \cdot W \tag{3.27}$$

Where F is the flexibility matrix for the system. Since we consider free vibrations, the only forces acting on the system are the inertia forces $-M\ddot{y}$, and we have:

$$y = -F M\ddot{y}$$

Using the same solution for y as above, i.e. $y = Y \cos pt$, we obtain

$$y = p^2 F M y \tag{3.28}$$

Rewriting this equation leads again to a homogeneous equation:

$$\left(\frac{1}{p^2} I - FM\right) y = 0 \qquad (3.29)$$

and the frequency determinant has the form

$$\det \left(\frac{1}{p^2} I - FM\right) = 0$$

Premultiplying equation (3.28) by F^{-1}, we get

$$F^{-1} y = p^2 My$$

Comparing this equation with equation (3.25) it can be seen that

$$F^{-1} = K \quad \text{or} \quad F.K = I$$

that is, the stiffness matrix is the inverse of the flexibility matrix and vice versa, provided the corresponding displacements and forces used in the compilation of K and F are the same.

The System Matrix

On premultiplying equation (3.25) on both sides by M^{-1} we have

$$M^{-1} Ky = p^2 y \qquad (3.30)$$

and rewriting equation (3.28), we get

$$F\,My = \frac{1}{p^2} y \qquad (3.31)$$

The above two equations are alternative forms of the same basic equation which is known as the eigenvector equation.

For equation (3.30): $Ay = \lambda y$, with $A = M^{-1} K$ and $\lambda = p^2$

For equation (3.31): $A^{-1}y = \lambda^{-1}y$, with $A \cancel{=} F\,M$ and $\lambda \cancel{=} p^{-2}$

say $By = \lambda y$; $B = FM$ & $\lambda = \frac{1}{p^2}$

where A is the system matrix and A^{-1} its inverse.

The eigenvector equation has to be solved to obtain the eigenvalues and eigenvectors. Appendix B deals with the solution of the eigenvalue problem in greater detail.

It should be noted, that the maximum transverse displacements represented by the column matrix y correspond to the eigenvector X in the Appendix B.

It can be seen, that when the stiffness matrix K is used, the eigenvalues give directly the squares of the natural frequencies and the first or largest eigenvalue gives the fastest vibration. If, however, the F matrix is used then the eigenvalues give the reciprocals of p^2 and the largest eigenvalue gives the slowest vibration or fundamental mode, which is in many cases more relevant. With regard to the compilation of the F and the K matrix, the following needs to be mentioned:-

(i) The F matrix is much more easily obtained for statically determinate structures than for indeterminate ones.

(ii) The K matrix is more easily obtained for highly statically indeterminate structures than the F matrix, i.e. continuous beams, multistorey buildings.

Solution to the Eigenvalue Problem

There exist a variety of methods for determining the eigenvalues and eigenvectors of the eigenvalue problem and they are discussed in Appendix B.4. A very popular method of obtaining the fundamental frequency is the "Basic Iterative Method", which is given in Appendix B 4.1.2 and a flow-diagram for the computer program is shown in Appendix C.1. The method may be described by the following steps:-

(1) Guess a vector y and set one of the values of the vector equal to 1.

(2) Substitute the guessed vector in the left-hand side of the eigenvector equation and perform the multiplication.

(3) Take out a factor such that the same value in the vector y is again unity.

(4) If the resulting vector equals the guessed vector y, then the guess is correct, and the eigenvalue λ has the value of the common factor.

(5) If the resulting vector does not equal the guessed vector, repeat the process by substituting the resulting vector in the left-hand side of the eigenvector equation.

A numerical example is given in section 4.2. This method is a convergent iterative procedure and it was originally developed by Stodola for finding the natural frequencies of rotating shafts. The iterative method is in fact related to Rayleigh's method in so far as an arbitrary displacement vector y is assumed. However, after each cycle of matrix multiplication a better estimate of the displacement vector is obtained; this is an improvement of Rayleigh's method which has no built-in means of an improved estimate of the originally chosen mode shape.

Rayleigh's Quotient

If a quick assessment of the largest eigenvalue is required, Rayleigh's quotient can be used. Starting with equation

$$Ky = p^2 My \tag{3.32}$$

and premultiplying both sides by y' (where the symbol $'$ means the transpose of y) and dividing by 2, gives:

$$\tfrac{1}{2} y' Ky = \tfrac{1}{2} p^2 y' My \tag{3.33}$$

Equation (3.33) can be interpreted physically as (the maximum potential energy in the system) = (the maximum kinetic energy in the system) and this agrees with the findings in section 3.3.2. In order to obtain the natural frequency, we solve the matrix equation (3.33) for p^2, hence

$$p^2 = \frac{y' K y}{y' M y} \tag{3.34}$$

Equation (3.34) gives the highest natural frequency and the value will be greater than or equal to the true value.

Premultiplying equation (3.32) on both sides by $(y' MF)$, we have

$$y' My = p^2 y' MFMy$$

hence $$p^2 = \frac{y' My}{y' MFMy} \tag{3.35}$$

Equation (3.35) gives the fundamental natural frequency and the value is again an overestimate. Rayleigh's quotient in matrix form is also discussed in Appendix B.*

Dunkerley's Formula

Dunkerley's formula, which gives a lower bound solution of the lowest natural frequency is a useful counterpart to Rayleigh's quotient which gives an upper bound solution. Dunkerley's formula can be found from the following relationship (see Appendix B.2):

$$\sum_{i}^{n} \lambda_i^{-1} = \text{trace } A^{-1}$$

$$\text{hence} \quad \text{trace (FM)} = \frac{1}{p_1^2} + \frac{1}{p_2^2} + \dots + \frac{1}{p_n^2}$$

If p_1 which is the lowest natural frequency is appreciably less than p_2, then we can express the above equation in the form

$$\text{trace (FM)} \doteqdot p_1^{-2} \tag{3.36}$$

and the value thus determined must be less than the true value of the fundamental frequency. Equation (3.36) is of practical value because it is often possible to determine trace (FM) without the need to define the flexibility matrix F fully. In particular, if the mass matrix is diagonal we get

$$\text{trace (FM)} = f_{11}m_1 + f_{22}m_2 + \dots + f_{nn}m_n$$

Now f_{11} is the deflection at point 1 due to a unit force applied there and it equals $1/k_{11}$ where k_{11} is the stiffness of the structure measured correspondingly at point 1. If m_1 were the only concentrated mass on the structure then the fundamental frequency would be given by

$$p_1 = \frac{1}{\sqrt{\text{trace (FM)}}} = \frac{1}{\sqrt{f_{11}m_1}} = \sqrt{\frac{k_{11}}{m_1}}$$

*Rayleigh's method and Dunkerley's approach were already discussed in the previous sections. However, since it is possible to express the Rayleigh quotient in matrix form and since Dunkerley's formula can more easily be derived from the system matrix than from the energy concept, the writer felt that this repetition is justified.

If we can determine the fundamental frequency due to each mass when it alone is carried by the structure, we then can determine an approximate value for the lowest natural frequency when all masses are carried simultaneously from the following equation which is known as Dunkerley's formula:-

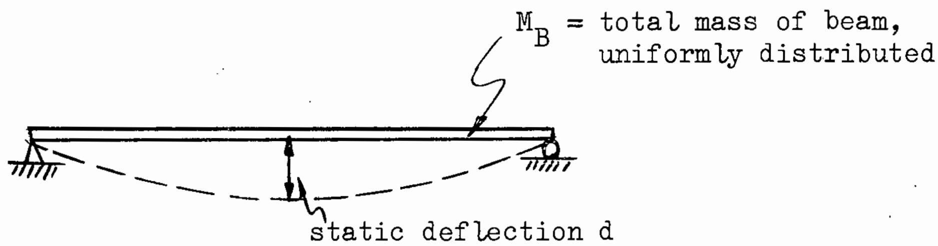
$$\frac{1}{p^2} \doteqdot f_{11}m_1 + f_{22}m_2 + \ldots + f_{nn}m_n \qquad (3.37)$$

However, it should be noted that the shape of the normal mode can not be obtained by either Rayleigh's quotient or Dunkerley's formula.

Numerical examples which use the flexibility matrix for determining the fundamental frequency and the corresponding normal mode will be given in Chapter 4. At this point a few simple numerical examples are given to illustrate Dunkerley's method.

A. Simply supported beam

(i) with distributed mass



Now $\quad p^2 = \frac{1}{f\,M_B} = \frac{k}{M_B}$

From static deflection consideration we know that

$$k \cdot d = M_B \cdot g$$

hence $\quad p^2 = \frac{g}{d} \qquad (3.38)$

where g is the gravitational constant

and d is the static deflection and is known to be equal to $5\,M_B g\,L^3/384\,EI$.

Substituting d in equation (3.38) gives

$$p = \sqrt{\frac{384\ EI}{5\ M_B\ L^3}}$$

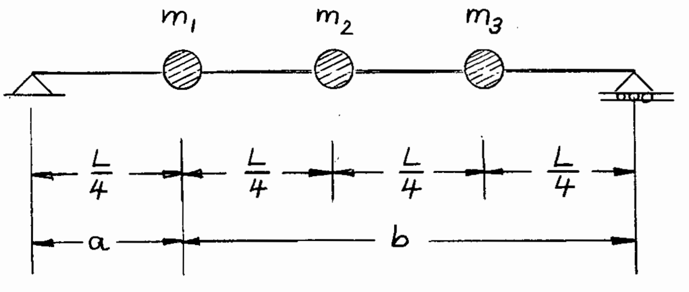
$$= 8{,}75\sqrt{\frac{EI}{M_B\ L^3}}$$

compared with

$$p = 9{,}87\sqrt{\frac{EI}{M_B\ L^3}}$$

which was obtained from Rayleigh's method, (see page 3.15)

(ii) <u>with lumped masses</u>



considering m_1:

$$f_{11} = \frac{a^2\ b^2}{3L\ EI} \qquad \text{where } a = \frac{L}{4},\quad b = \frac{3}{4}L$$

$$\therefore \quad f_{11}\ m_1 = \frac{3\ L^3}{256\ EI}\ m_1$$

considering m_2:

$$f_{22} = \frac{a^2\ b^2}{3L\ EI} \qquad \text{where } a = b = \frac{L}{2}$$

$$\therefore \quad f_{22}\ m_2 = \frac{L^3}{48\ EI}\ m_2$$

considering m_3:

from symmetry, $f_{33}\, m_3 \;=\; f_{11}\, m_1$

Hence:

$$\frac{1}{p^2} \div \sum_i^n f_{ii}\, m_i \;=\; \frac{L^3}{EI}\left[\frac{3m_1}{256} + \frac{m_2}{48} + \frac{3m_3}{256}\right]$$

Let $m_1 = m_2 = m_3 = M_B/3$ where M_B is the total mass of the beam, then

$$p \;=\; 8{,}32\,\sqrt{\frac{EI}{M_B\, L^3}}$$

From the two results above we can conclude, that if the continuous beam is approximated by lumped masses, the value for p obtained for the lumped case is lower than for the continuous one, using the simple relationship $p^2 = m/k = g/d$. When comparing the above results with the exact solution $p = 9{,}8696\, p_w$ (see section 3.1), we get the following percentage error:-

for case (i)	11,4
for case (ii)	15,7

B. Cantilever

(i) with distributed mass

$$p^2 \;=\; \frac{g}{d} \quad \text{with} \quad d \;=\; \frac{M_B\, g\, L^3}{8\, EI}$$

then $$p \;=\; \sqrt{\frac{8\, EI}{M_B\, L^3}} \;=\; 2{,}83\,.\sqrt{\frac{EI}{M_B\, L^3}}$$

This result is considerably lower than the exact value and this discrepancy can be overcome by assuming the total mass of the cantilever concentrated at a distance L/2 away from the fixed end. If the deflection is now determined at the tip of the cantilever, we get

$$d \;=\; \frac{M_B\, g\, L^3}{9{,}60\, EI}$$

hence $$p \;=\; 3{,}1\,\sqrt{\frac{EI}{M_B\, L^3}}$$

compared with $$p \;=\; 3{,}66\,\sqrt{\frac{EI}{M_B\, L^3}}$$

which was obtained from Rayleigh's method (page 3.17)

(ii) <u>with lumped masses</u>

When lumping the distributed mass of a cantilever, greater care has to be taken than in the case of a simply supported beam, with regard to the positioning of the lumped masses.

If the masses were lumped as shown in fig. 3.11, the value for the fundamental frequency would be considerably greater than the exact value for a uniform cantilever, since this type of lumping has the effect of shortening the actual beam and hence the frequency increases.

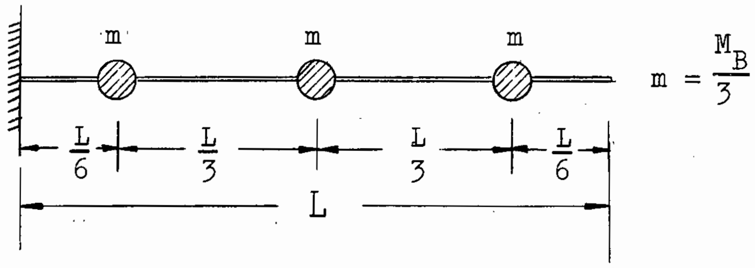


Figure 3.11

If the masses were lumped as shown in fig. 3.12 the value for the fundamental frequency would be lower than the exact value because this type of lumping has the effect of lengthening the cantilever and hence the frequency decreases.

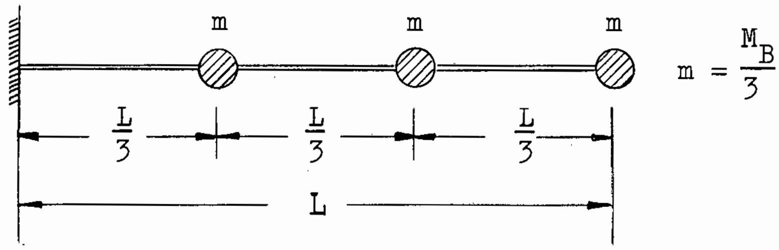


Figure 3.12

In the next chapter we shall come back to the question of how to approximate an actual continuous structure by lumped masses.

3.3.7 - Myklestad's Method

Myklestad's approach (1) is an extension of Holzer's method, which solves the torsional vibration problem of a shaft by first assuming a frequency and then calculating the deflection curve. Myklestad developed this method for tabular use and hence it lends itself very much to programming on digital computers. The method differs from the methods discussed so far, in that any particular natural frequency may be found independently of the others.

The beam under consideration is divided into a finite number of segments and the mass of each segment is concentrated at discrete points along the length of the beam. The elements of the beam connecting the masses are assumed weightless. Since the method uses a "lumped parameter" approximation, the accuracy of the result will depend on how many mass points are used. Myklestad gives the rough rule, that twice as many concentrated masses should be used as the number of natural frequencies desired.

1. Method of Analysis

The beam is subdivided as shown in fig. 3.13 and the equilibrium of any one section is considered. (Myklestad's notation is adopted). Fig. 3.14 shows the nth section of the beam between the stations n and n + 1, with the left end of the section clamped. In fig. 3.14(a) a unit shear force Q at the right end will cause a vertical deflection d_{Qn} at the left end and a rotation θ_{Qn} of the amounts:

$$d_{Qn} = \frac{L_n^3}{6\ EI_n} \qquad (3.39)$$

$$\theta_{Qn} = \frac{L_n^2}{2\ EI_n} \qquad (3.40)$$

where I_n is the moment of inertia of the nth section of the beam. Similarly in fig 3.14(b) a unit moment M will produce a deflection d_{Mn} at the left end and a rotation θ_{Mn} of the amounts:

$$d_{Mn} = \frac{L_n^2}{2\ EI_n} \qquad (3.41)$$

$$\theta_{Mn} = \frac{L_n}{EI} \qquad (3.42)$$

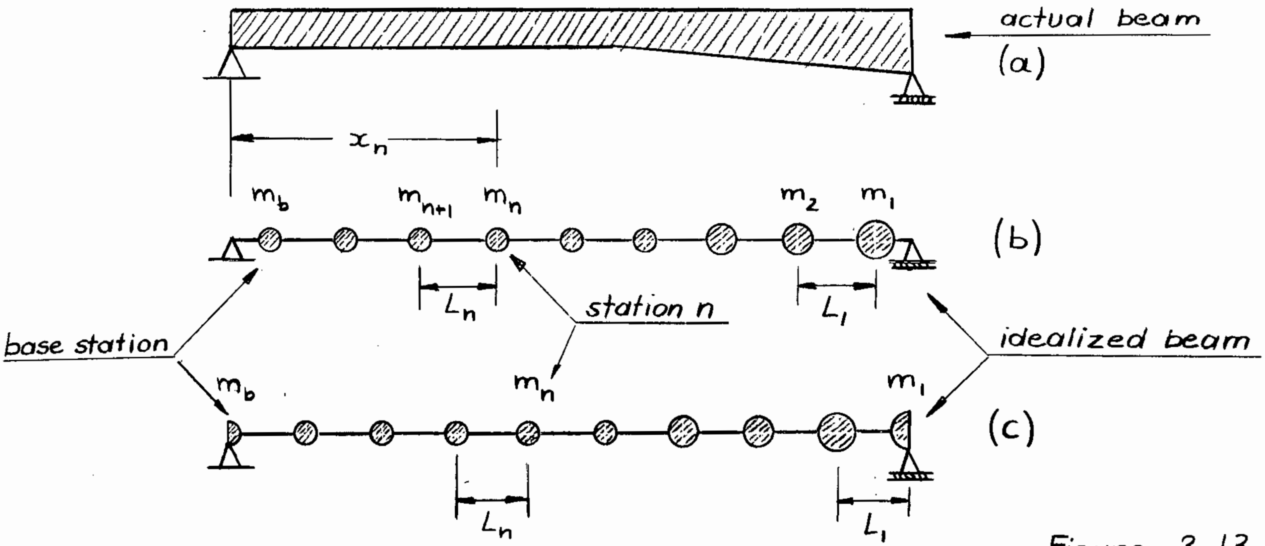


Figure 3.13

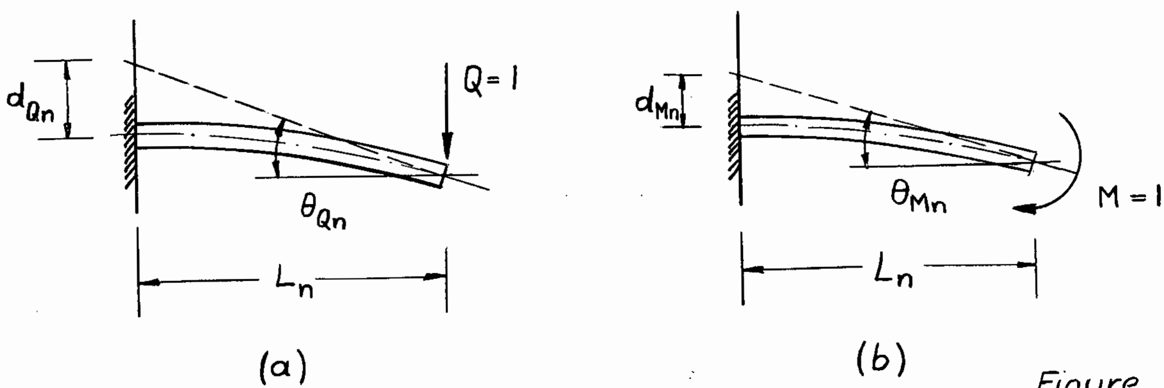


Figure 3.14

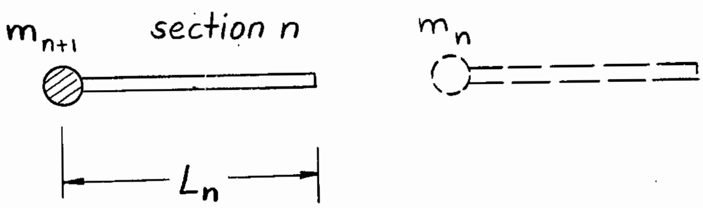


Figure 3.15

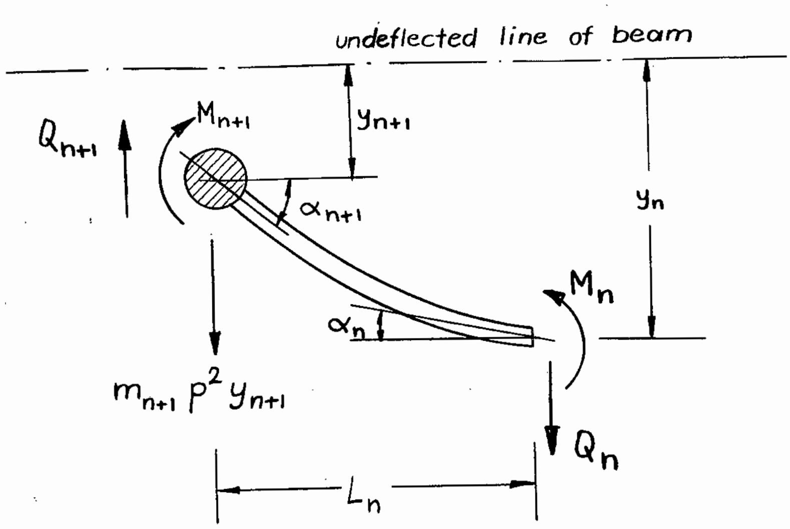


Figure 3.16

Each section contains the mass at the left-hand end and the element of the beam to the right as shown in fig. 3.15. The discrete mass m_{n+1} exerts an inertia force $m_{n+1}\, p^2 y_{n+1}$ when the element is at its maximum displacement position.* Considering the equilibrium of the forces of the element shown in fig 3.16 we get:

$$Q_{n+1} = Q_n + m_{n+1}\, p^2\, y_{n+1} \tag{3.43}$$

$$M_{n+1} = M_n - Q_n\, L_n \tag{3.44}$$

The deflections and rotations of the left-hand end can be written in terms of the deflection, rotation, shear force and moment existing at the right-hand end:-

$$y_{n+1} = y_n - L_n \alpha_n + Q_n d_{Qn} - M_n d_{Mn} \tag{3.45}$$

$$\alpha_{n+1} = \alpha_n - Q_n \theta_{Qn} + M_n \theta_{Mn} \tag{3.46}$$

Using this technique, one may proceed from one element to the next along the axis of the beam.

2. Procedure

Starting at station 1 with known values of the shear force Q_1, bending moment M_1, deflection y_1 and rotation α_1, equations (3.43) to (3.46) can be applied at each station, until the values at the base station Q_b, M_b, y_b, α_b are found for an assumed frequency value p. When the values obtained at the base station satisfy the boundary conditions for the beam, the value used for p in equation (3.43) will be the natural frequency of the idealized beam for all modes.

3. Examples

A. Simply supported beam

The end conditions are:

at station 1: $y_1 = 0$; $M_1 = 0$ $\alpha_1 + Q_1$ are unknown

at station 2: $y_2 = 0$; $M_2 = 0$ $\alpha_2 + Q_2$ are unknown

---

*The inertia force is $-m\ddot{y}$; assuming harmonic motion, we have $y = \cos pt$, hence $\ddot{y} = -p^2 \cos pt = -p^2 y$ and therefore $[-m\ddot{y} = mp^2 y]$

First assume: $\alpha_1 = 1 \quad Q_1 = 0$

For any value of p, the values at the base station can be found; let these be

$$Q_\alpha, M_\alpha, y_\alpha, \alpha_\alpha$$

Now assume: $\alpha_1 = 0 \quad Q_1 = 1$

new values at the base station for the same values of p can be found; let these be

$$Q_Q, M_Q, y_Q, \alpha_Q.$$

For any actual beam vibration, both α_1 and Q_1 will not be zero, but may be related to the form

$$\chi = +\alpha_1/Q_1$$

Then the actual values at the base station will be

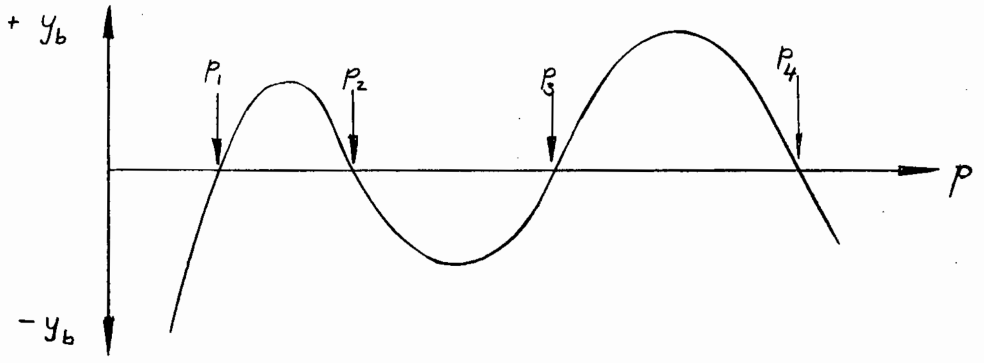
$$\begin{aligned} Q_b &= Q_Q + \chi Q_\alpha \\ M_b &= M_Q + \chi M_\alpha \\ y_b &= y_Q + \chi y_\alpha \\ \alpha_b &= \alpha_Q + \chi \alpha_\alpha \end{aligned}$$

Since at the base station $M_b = 0$, we get

$$\chi = -M_Q/M_\alpha$$

hence $$y_b = y_Q - \left(\frac{M_Q}{M_\alpha}\right) y_\alpha$$

can be computed for any chosen value of p. If y_b is calculated for several values of p and plotted against p, the boundary conditions are satisfied where $y_b = 0$ and the values of p at these points are the natural frequencies of the simply supported beam.



The mode shape is immediately available since the deflection y has been computed at every section for unit values of two parameters at all values of p. When the correct p is known, the appropriate proportions of each deflection can be added, e.g.

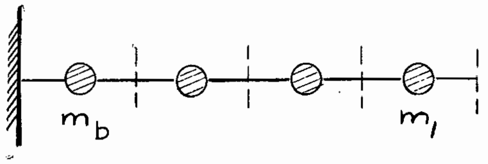
$$y_n = y_{nQ} + \chi\, y_{n\alpha}$$

B. Cantilever

The end conditions are

at station 1 : $Q_1 = m_1 p^2 y_1$; $M_1 = 0$ i.e. free, but y?, α?

at base station: $\alpha_b = 0$; $y_b = 0$ i.e. fixed, but Q?, M?



Assume: $y_1 = 1$, $\alpha_1 = 0$

For any value of p, the values at the base station can be found; let these be

$$Q_y,\ M_y,\ y_y,\ \alpha_y$$

Now assume: $y_1 = 0$, $\alpha_1 = 1$

New values at the base station for the same value of p can be found; let these be

$$Q_\alpha,\ M_\alpha,\ y_\alpha,\ \alpha_\alpha$$

For an actual beam vibration, both y_1 and α_1 will have real maximum displacements in some unknown proportion to each other, say

$$y_1 = 1, \quad \alpha_1 = \phi \quad \text{or} \quad \frac{\alpha_1}{y_1} = \phi .$$

Then the actual values at the base station will be

$$\begin{aligned} Q_b &= Q_y + \phi\, Q_\alpha \\ M_b &= M_y + \phi\, M_\alpha \\ y_b &= y_y + \phi\, y_\alpha \\ \alpha_b &= \alpha_y + \phi\, \alpha_\alpha \end{aligned}$$

Since the base station of the cantilever is built in,

therefore $y_b = \alpha_b = 0$

hence $y_y + \phi\, y_\alpha = 0$

or $\phi = -\, y_y/y_\alpha$

therefore $\alpha_b = \alpha_y - (y_y/y_\alpha)\, \alpha_\alpha$

Again α_b can be plotted against the natural frequency p. Fig. 3.17

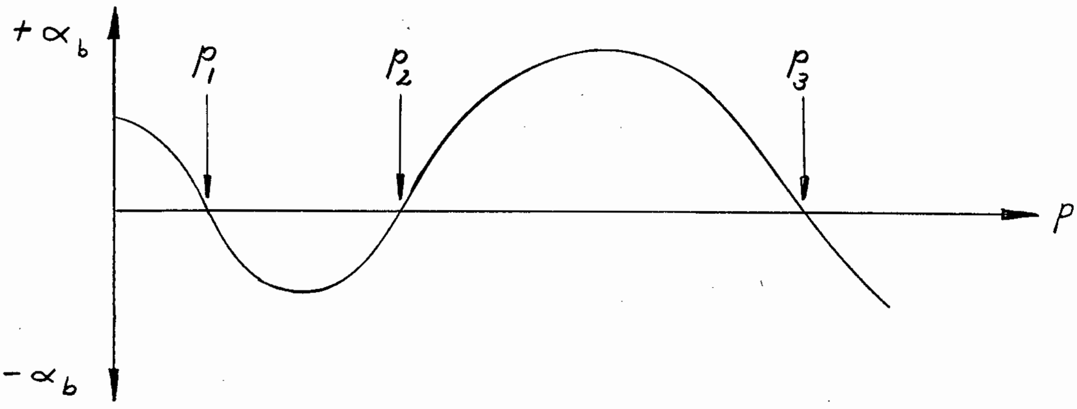


Fig. 3.17

indicates the shape of the curve for a cantilever and each natural frequency occurs when $\alpha_b = 0$, i.e. at p_1, p_2, etc.

In Tidbury's paper (2) on Myklestad's method, Q_1 was set equal to zero at the first station. The writer, however is of the opinion that much better results are obtained when Q_1 is assumed to be $m_1 p^2 y_1$, since the inertia of the first mass is known for an assumed y.

Tidbury also states that the natural frequencies for a lumped mass system are generally less than the exact value of the actual system. This however is not true, and the aspects of lumping the masses at certain discrete points will be discussed in Chapter 4. At this point it may be said, that if a cantilever is lumped as shown in fig. 3.18(a), then the frequencies of this approximation are greater than the exact values,

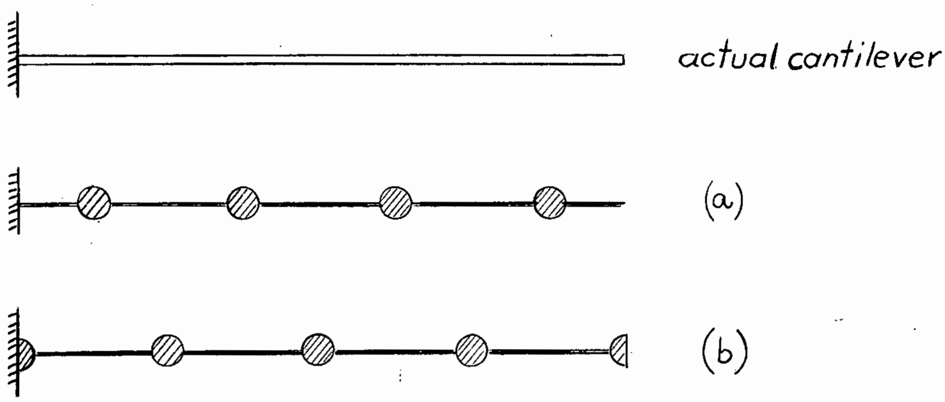


Fig. 3.18

and if the beam mass is lumped as shown in fig. 3.18(b), then the natural frequencies of this mass-approximation are smaller than the exact value (also see fig. 4.8).

4. Computer Program WMYKLE

The writer developed a computer program which can be used for both cantilever and simply supported beams. The flow-diagram is given in Appendix C and the computer program WMYKLE in FORTRAN IV can be seen in Appendix D. The program was used to investigate the effect of different types of lumping of the mass of the beam and how the number of lumped masses will affect the accuracy of the natural frequencies obtained. These results will be discussed in section 4.4.3.

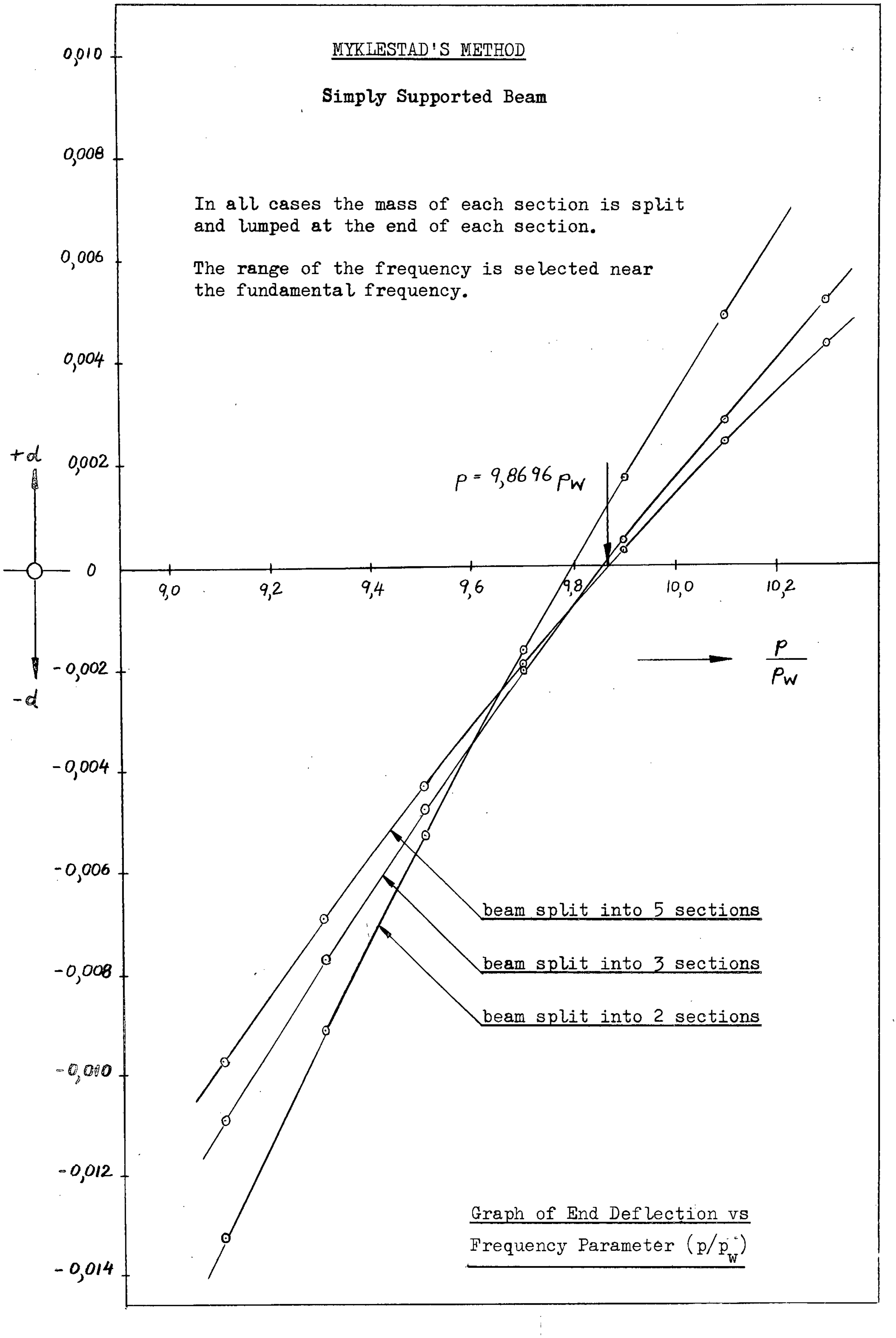
But at this point it should be mentioned that the results which were obtained for a simply supported beam were much more accurate than those for a cantilever, using the same number of lumped masses. This is due to the fact that a cantilever is a structure with completely unsymmetrical end-conditions whilst the supports of a simply supported beam (i.e. roller and hinge) make it a more symmetrical structure.

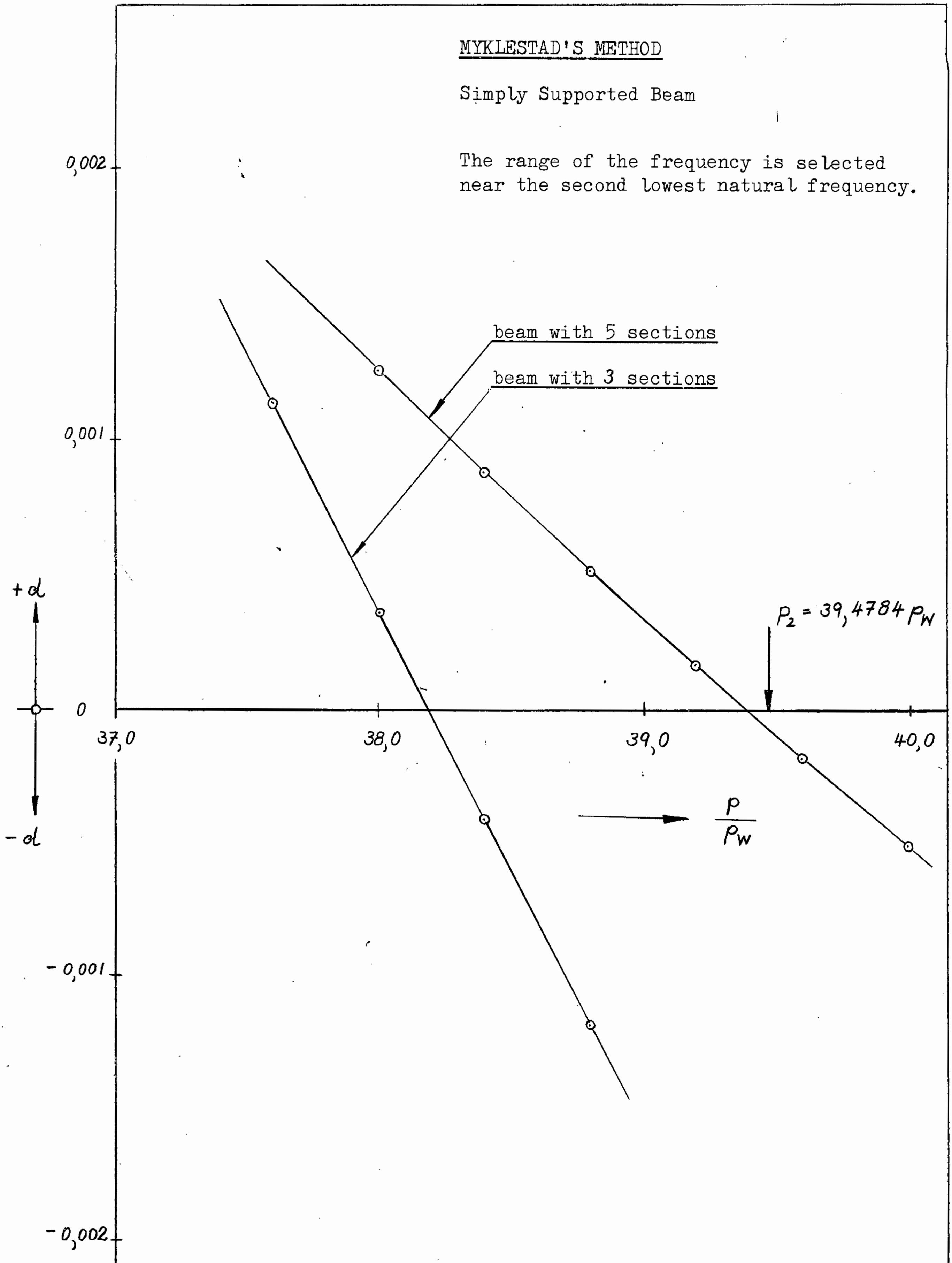
It can be seen that the program WMYKLE can easily be altered to allow for varying flexural rigidities along the beam.

The following two graphs are obtained from the program WMYKLE for a simply supported beam. The assumed frequency value p is plotted against the deflection of the left support.

3.3.8 - Transfer Matrix Method

Myklestad's method has been extended and generalized by Pestel and Associates (3) and is described as a method of transfer matrices. For this approach of determining the natural frequencies and normal modes of vibration the transfer matrices and state vectors have to be set up; the computation is conveniently carried out by initially inserting a trial value of the frequency into the transfer matrices. Since many such trial values may be necessary in order to find the frequencies, a great amount of computation is generally involved and therefore the method is usually only practical when electronic digital computing facilities are available.


MYKLESTAD'S METHOD
Simply Supported Beam
In all cases the mass of each section is split and lumped at the end of each section.
The range of the frequency is selected near the fundamental frequency.
p = 9,8696 pw
+d
-d
0,010
0,008
0,006
0,004
0,002
0
- 0,002
- 0,004
- 0,006
- 0,008
- 0,010
- 0,012
- 0,014
9,0
9,2
9,4
9,6
9,8
10,0
10,2
p/pw
beam split into 5 sections
beam split into 3 sections
beam split into 2 sections
Graph of End Deflection vs Frequency Parameter (p/pw)



Graph of End Deflection vs.
Frequency Parameter (p/p_w)

1. Definition

(a) The State Vector

A state vector defines the displacements and the associated internal forces at a point in a system. The state vector at the point i of a spring, fig. 3.19, is then defined as:

$$z_i = \begin{bmatrix} x \\ N \end{bmatrix}_i$$

where x is the displacement and N the spring force.

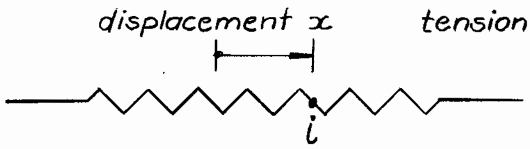


Figure 3.19

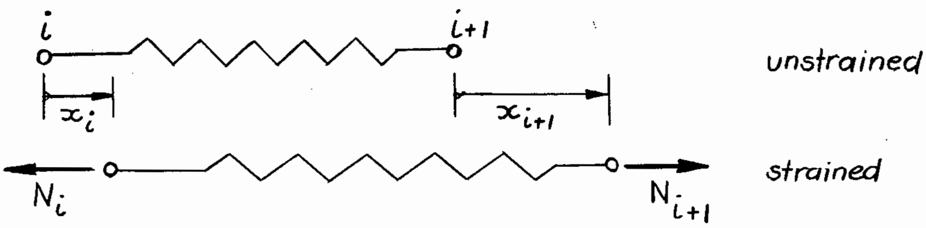


Figure 3.20

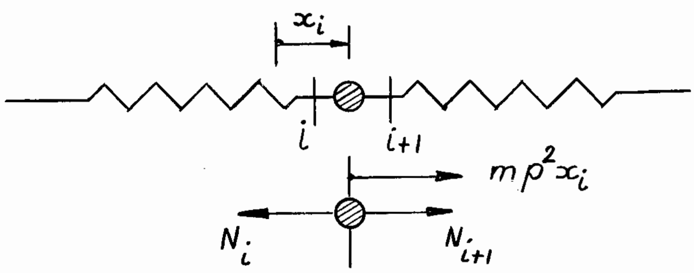


Figure 3.21

(b) Transfer matrix

A transfer matrix T relates the state vectors at two consecutive points. Consider the two points i and i + 1 on the spring of stiffness k shown in fig. 3.20. From equilibrium of the spring, we obtain

$$N_{i+1} = N_i$$

and from the stiffness property of the spring we have the further relation

$$N_i = k_i\,(x_{i+1} - x_i)$$

When written in matrix form these equations become

$$\begin{bmatrix} x \\ N \end{bmatrix}_{i+1} = \begin{bmatrix} 1 & 1/k \\ 0 & 1 \end{bmatrix}_i \begin{bmatrix} x \\ N \end{bmatrix}_i$$

or $$z_{i+1} = T_i \, z_i$$

in which T_i is defined as the transfer matrix of the element. When, as in this example, a transfer matrix relates the variables at two different points in a system it is called a field matrix and is denoted by T.

When a transfer matrix relates the variables on either side of a discontinuity it is referred to as a point matrix and is denoted by P. In fig 3.21 the points i and $(i+1)$ are immediately to the left and right of a concentrated mass m. If the mass executes simple harmonic motions, the maximum inertia force generated is $m_i p^2 x_i$ in the positive x direction. Since the mass is rigid, the deflections to the left and to the right are the same, so that

$$x_{i+1} = x_i$$

and from equilibrium of the forces we have

$$N_{i+1} = N_i - m_i p^2 x_i$$

In matrix form these equations are:

$$\begin{bmatrix} x \\ N \end{bmatrix}_{i+1} = \begin{bmatrix} 1 & 0 \\ -mp^2 & 1 \end{bmatrix}_i \begin{bmatrix} x \\ N \end{bmatrix}_i$$

Since the mass is concentrated, i and $(i+1)$ are coincident and it is convenient to write this equation as:

$$z_i^r = P_i \, z_i^l$$

where r and l denote left and right respectively.

2. Beam Systems

A mass which is rigidly attached to a flexible beam has two degrees of freedom since the mass rotates as the beam deflects transversely. The state vector therefore contains two displacement components. Associated with the displacements are two internal forces, a shear force Q and a bending moment M. The order in which these four variables are placed in the state vector is arbitrary but it is convenient to define the state vector as follows:-*

$$z_i = \begin{bmatrix} y \\ \theta \\ M \\ Q \end{bmatrix}_i$$

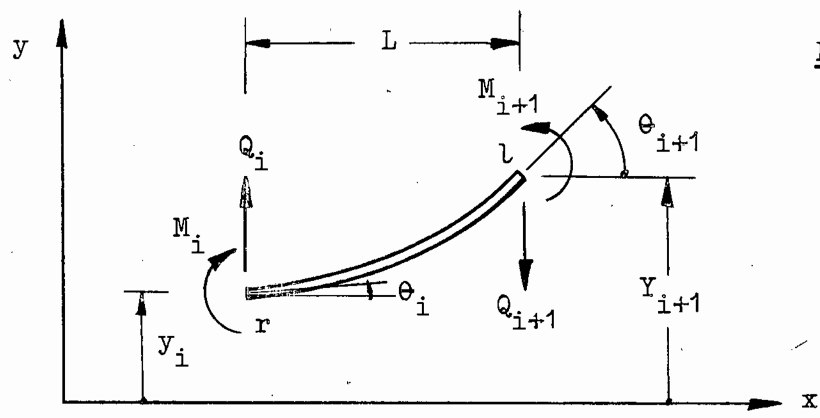
The sign convention used is given in fig 3.22. In this method, once more we follow the technique of replacing the actual beam by a beam of the same flexural stiffness which is massless between discrete points, where the mass is concentrated.

Considering the equilibrium of the beam element of fig. 3.22(a) we have

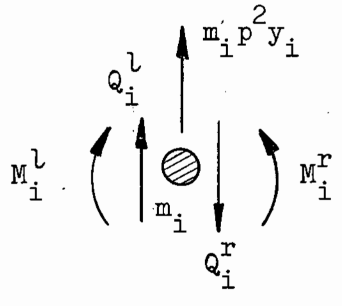
$$Q_{i+1} = Q_i$$

$$M_{i+1} = M_i + Q_i . L$$

Figure 3.22



(a)



(b)

Free-body diagram for mass m_i

---

*It can be seen later that the sequence of the variables as given above leads to a field matrix which is an upper triangular matrix.

and using the standard results of structural analysis (see previous section) the displacement equations are:

$$\begin{aligned}
\theta_{i+1} &= \theta_i - Q_{i+1}\, L^2/2\,EI + M_{i+1}\, L/EI \\
&= \theta_i + M_i\, L/EI + Q_i\, L^2/2\,EI \\
y_{i+1} &= y_i + \theta_i\, L + M_{i+1}\, L^2/2\,EI + Q_{i+1}\, L^3/3\,EI \\
&= y_i + \theta_i\, L + M_i\, L^2/2\,EI + Q_i\, L^3/6\,EI
\end{aligned}$$

Writing the above four equations in matrix form leads to the following relationship between the two state vectors:-

$$\begin{bmatrix} y \\ \theta \\ M \\ Q \end{bmatrix}_{i+1}^{l} = \begin{bmatrix} 1 & L & \dfrac{L^2}{2EI} & \dfrac{L^3}{6EI} \\ 0 & 1 & \dfrac{L}{EI} & \dfrac{L^2}{2EI} \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \theta \\ M \\ Q \end{bmatrix}_{i}^{r}$$

$$z_{i+1}^{l} = T_i\, z_i^{r} \qquad 3.47$$

The point matrix connecting $z_i^{\,r}$ and $z_i^{\,l}$ is found by noting that the deflection, slope and moment are continuous across the concentrated mass m_i, so that

$$y_i^{\,r} = y_i^{\,l}, \quad \theta_i^{\,r} = \theta_i^{\,l}, \quad M_i^{\,r} = M_i^{\,l}.$$

The vibrating mass, however, introduces an inertia force which causes a discontinuity in the shear. Equating the vertical forces in fig. 3.22(b), we have

$$Q_i^{\,r} = Q_i^{\,l} + m_i p^2 y_i$$

In matrix notation the above relationship becomes

$$\begin{bmatrix} y \\ \theta \\ M \\ Q \end{bmatrix}_{i}^{r} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ mp^2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \theta \\ M \\ Q \end{bmatrix}_{i}^{l}$$

$$z_i^{\,r} = P_i\, z_i^{\,l} \qquad (3.48)$$

3. The General State Vector Equation

Consider a beam which consists of several uniform massless elements with masses concentrated at discrete points. The transfer matrices have already been derived, i.e. equation (3.47) and equation (3.48), so that with the dimensions of the beam (shown in fig. 3.23) known, the following matrix relations exist between adjacent state vectors:

$$z_1^{\,l} = T_1\, z_0, \quad z_1^{\,r} = P_1\, z_1^{\,l}, \quad z_2^{\,l} = T_2\, z_1^{\,r}, \;\ldots\;, z_4 = T_4\, z_3^{\,r}.$$

Substitution leads to the following relation between the state vectors of the two ends of the beam:

$$z_4 = T_4\, P_3\, T_3\, P_2\, T_2\, P_1\, T_1\, z_0$$

$$z_4 = U\, z_0 \qquad (3.49)$$

where U is called the overall transfer matrix. In this manner all the intermediate state vectors have been eliminated. The matrix equation is equivalent to the following set of equations:

$$\begin{aligned}
y_4 &= u_{11}\, y_0 + u_{12}\, \theta_0 + u_{13}\, M_0 + u_{14}\, Q_0 && \text{(a)}\\
\theta_4 &= u_{21}\, y_0 + u_{22}\, \theta_0 + u_{23}\, M_0 + u_{24}\, Q_0 && \text{(b)}\\
M_4 &= u_{31}\, y_0 + u_{32}\, \theta_0 + u_{33}\, M_0 + u_{34}\, Q_0 && \text{(c)}\\
Q_4 &= u_{41}\, y_0 + u_{42}\, \theta_0 + u_{43}\, M_0 + u_{44}\, Q_0 && \text{(d)}
\end{aligned} \qquad (3.50)$$

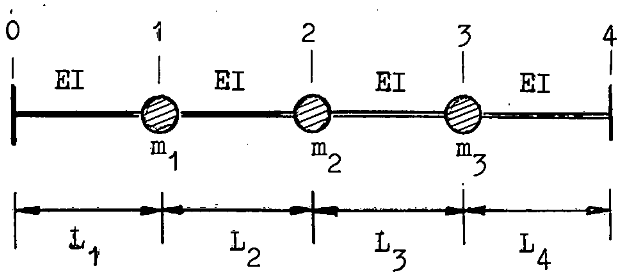


Figure 3.23

It is by applying the boundary conditions of a particular beam to these equations that the natural frequencies can be obtained.

(a) Simply supported beam

The boundary conditions are:

$$y_4 = 0 \qquad M_4 = 0 \qquad y_0 = 0 \qquad M_0 = 0$$

Substituting these in equations (3.50a) and (3.50c), we get:

$$0 = u_{12}\,\theta_o + u_{14}\,Q_o$$
$$0 = u_{32}\,\theta_o + u_{34}\,Q_o$$

For a non-trivial solution it is necessary that

$$u_{12}.u_{34} - u_{32}.u_{14} = 0 \tag{3.51}$$

Since the elements u_{ij} are known functions of the natural frequency p, the above equation serves to compute the natural frequencies. Equation (3.51) is in fact the natural frequency equation for the beam and since it possesses three lumped masses this equation is of third order in p^2. If the number of lumped masses is large, it is difficult to solve for p^2 from this equation and a numerical approach, given in the next paragraph, is more convenient.

(b) Cantilever

The boundary conditions are:

$$M_4 = 0 \qquad Q_4 = 0 \qquad y_o = 0 \qquad \theta_o = 0$$

Substitution in equations (3.50c) and (3.50d) yields:

$$0 = u_{33}\,M_o + u_{34}\,Q_o$$
$$0 = u_{43}\,M_o + u_{44}\,Q_o$$

which leads for a non-trivial solution to

$$u_{33}\,u_{44} - u_{43}\,u_{34} = 0 \tag{3.52}$$

from which the natural frequencies can be obtained.

Example:

Consider the cantilever in fig. 3.24. The beam is assumed massless, with flexural rigidity EI.

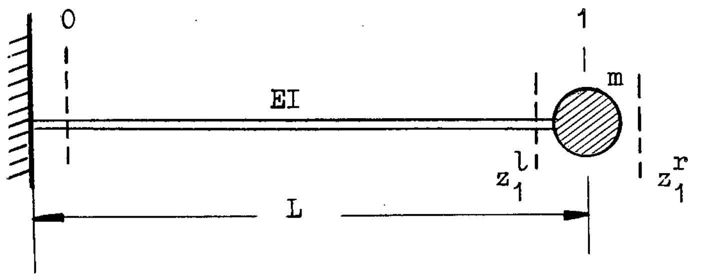


Figure 3.24

Using the results of the previous section, we obtain the following basic equation:

$$z_1^{\,r} = P_1\, T_1\, z_o^{\,r}$$

or

$$\begin{bmatrix} y \\ \theta \\ M \\ Q \end{bmatrix}_1^r = \begin{bmatrix} 1 & L & \frac{L^2}{2EI} & \frac{L^3}{6EI} \\ 0 & 1 & \frac{L}{EI} & \frac{L^2}{2EI} \\ 0 & 0 & 1 & L \\ mp^2 & mp^2 L & \frac{mp^2L^2}{2EI} & 1+\frac{mp^2L^3}{6EI} \end{bmatrix} \begin{bmatrix} y \\ \theta \\ M \\ Q \end{bmatrix}_o^r$$

Hence equation (3.52) becomes:

$$1 + \frac{mp^2L^3}{6EI} - \frac{mp^2L^3}{2EI} = 0$$

$$p^2 = \frac{3EI}{mL^3}$$

which is the same result as determined in the example in section 2.3.1 page 2.7, where we used the stiffness coefficient $k = 3EI/L^3$ for a cantilever.

Obviously the above example serves merely as an illustrative example of the transfer matrix method, since the result can be obtained more quickly by other methods, e.g. that of section 2.3.1. However, if several lumped masses are involved, the method is more simple than the use of the stiffness coefficients. In this method, only matrix multiplications are required and this can be done easily by means of a computer. This approach is described in the next paragraph.

4. <u>Numerical Procedures</u>

In the above example for a cantilever it was easily possible to carry out the matrix multiplication algebraically. However, as the number of lumped masses increases the algebraic labour would become prohibitive and the procedure adopted in practice is to choose certain values for the natural frequency p and compute the corresponding values Δ of the frequency equation (3.51) or (3.52). The value Δ of the equation is then plotted against the frequency p, the zero values of Δ occurring at each natural frequency of the system.

5. Example and Program WTRANS

The numerical procedure is demonstrated for the simply supported beam shown in fig. 3.25.

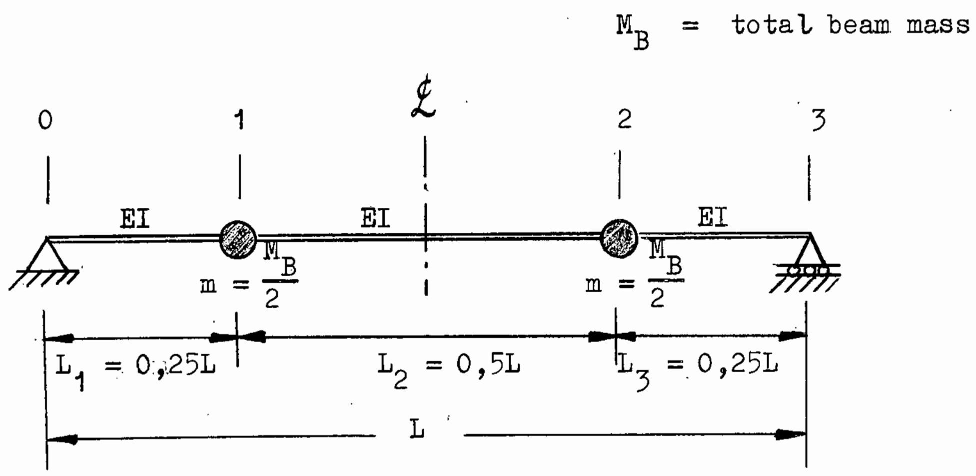


Figure 3.25

The substitution of the state vector relationships from point 0 to point 3 is carried out as follows:-

$$z_1^{l} = T_1 z_o^{r}$$

$$z_1^{r} = P_1 z_1^{l} = P_1 T_1 z_o^{r}$$

$$z_2^{l} = T_2 z_1^{r} = T_2 P_1 T_1 z_o^{r}$$

$$z_2^{r} = P_2 z_2^{l} = P_2 T_2 P_1 T_1 z_o^{r}$$

$$z_3^{l} = T_3 z_2^{r} = T_3 P_2 T_2 P_1 T_1 z_o^{r}$$

i.e. $$z_3^{l} = U z_o^{r}$$

Since $L_1 = L_3 = L/4$, and furthermore
$L = 1$ and $EI = \text{const.} = 1$ (i.e. EI is the same for all segments), the field matrices T_1 and T_3 are equal and take the form

$$T_1 = T_3 = \begin{bmatrix} 1,000 & 0,250 & 0,031 & 0,003 \\ 0 & 1,000 & 0,250 & 0,031 \\ 0 & 0 & 1,000 & 0,250 \\ 0 & 0 & 0 & 1,000 \end{bmatrix}$$

with $L_2 = \frac{1}{2}$

$$T_2 = \begin{bmatrix} 1{,}000 & 0{,}500 & 0{,}125 & 0{,}021 \\ 0 & 1{,}000 & 0{,}500 & 0{,}125 \\ 0 & 0 & 1{,}000 & 0{,}500 \\ 0 & 0 & 0 & 1{,}000 \end{bmatrix}$$

Since both lumped masses are equal, the point matrices P_1 and P_2 are equal.

Assuming $p = 10$ and $M_B = 1$, the point matrices become

$$P_1 = P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 50 & 0 & 0 & 1 \end{bmatrix}$$

Hence $U = T_3\, P_2\, T_2\, P_1\, T_1$ becomes

$$U = \begin{bmatrix} 4{,}78 & 2{,}01 & 0{,}65 & 0{,}19 \\ 17{,}25 & 6{,}09 & 1{,}93 & 0{,}65 \\ 63{,}02 & 22{,}01 & 6{,}09 & 2{,}01 \\ 152{,}08 & 63{,}02 & 17{,}25 & 4{,}78 \end{bmatrix}$$

Applying equation (3.51), we obtain:

$$\begin{aligned} \Delta &= 2{,}01 \,.\, 2{,}01 - 22{,}01 \,.\, 0{,}19 \\ &= -\,0{,}036 \end{aligned}$$

A computer program was written for the above computation which can be used for a simply supported beam having up to 9 lumped masses. The results of this program for the example given above are shown on the following page. The program WTRANS in FORTRAN IV is listed in Appendix D and the flow-diagram is given in Appendix C. This program is not discussed further, since its results are similar to Myklestad's method. It only confirms the findings of the effect of lumping the mass of the beam, which is discussed in the next chapter. It formed, however, a basic part of the

LUMPED MASSES	LENGTHS OF SEGMENTS	FEXURAL RIGIDITY
.500000+00	.250000+00	.100000+01
.500000+00	.500000+00	.100000+01
.000000	.250000+00	.100000+01

.100000+01	.250000+00	.312500-01	.260417-02	
.000000	.100000+01	.250000+00	.312500-01	$= T_1$
.000000	.000000	.100000+01	.250000+00	
.000000	.000000	.000000	.100000+01	

.100000+01	.250000+00	.312500-01	.260417-02	
.000000	.100000+01	.250000+00	.312500-01	$= P_1 T_1$
.000000	.000000	.100000+01	.250000+00	
.500000+02	.125000+02	.156250+01	.113021+01	

.204167+01	.101042+01	.313802+00	.730252-01	
.625000+01	.256250+01	.945313+00	.297526+00	$= T_2 P_1 T_1$
.250000+02	.625000+01	.178125+01	.815104+00	
.500000+02	.125000+02	.156250+01	.113021+01	

.204167+01	.101042+01	.313802+00	.730252-01	
.625000+01	.256250+01	.945313+00	.297526+00	$= P_2 T_2 P_1 T_1$
.250000+02	.625000+01	.178125+01	.815104+00	
.152083+03	.630208+02	.172526+02	.478147+01	

.478147+01	.201047+01	.650723+00	.185330+00	
.172526+02	.609440+01	.192977+01	.650723+00	$= T_3 P_2 T_2 P_1 T_1$
.630208+02	.220052+02	.609440+01	.201047+01	
.152083+03	.630208+02	.172526+02	.478147+01	

VALUE OF DETERMINANT	TRIAL VALUE FOR P
-.362414-01	.100000+02

program WTMCB for a two-span continuous beam, given in Chapter 5.

Finally the <u>mode shape</u> is to be determined. Assuming $p = 10$ is a good enough approximation of the fundamental frequency, i.e. $\Delta \doteq 0$, and noting that $y_3 = 0$, the following relationship between θ_0 and Q_0 is obtained:

$$2{,}01\ \theta_0 + 0{,}19\ Q_0 = y_3 = 0$$

where the numerical values are taken from the computer printout.

Hence $\quad Q_0 = -10{,}83\ \theta_0$

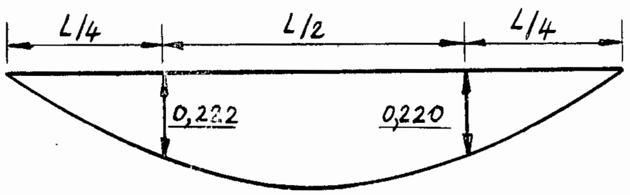
with $\theta_0 = 1$ the value of the deflections at the two internal points are computed as shown below:

$$z_1^{\,r} = P_1\, T_1\, z_0^{\,r}$$

$$\begin{bmatrix} y \\ \theta \end{bmatrix}_1^r = \begin{bmatrix} 0{,}250 & 0{,}003 \\ 1{,}000 & 0{,}031 \end{bmatrix} \begin{bmatrix} 1{,}00 \\ -10{,}83 \end{bmatrix} = \begin{bmatrix} 0{,}222 \\ 0{,}662 \end{bmatrix}$$

$$z_2^{\,r} = P_2\, T_2\, P_1\, T_1\, z_0^{\,r}$$

$$\begin{bmatrix} y \\ \theta \end{bmatrix}_2^r = \begin{bmatrix} 1{,}010 & 0{,}073 \\ 2{,}563 & 0{,}297 \end{bmatrix} \begin{bmatrix} 1{,}00 \\ -10{,}83 \end{bmatrix} = \begin{bmatrix} 0{,}220 \\ -0{,}657 \end{bmatrix}$$



SLOPE :- 0,662 -0,657

The first mode shape should be symmetrical about the centre and the slight discrepancy in the above results is due to the fact that the estimate which was used for the fundamental frequency, i.e. $p = 10$, was not quite correct. Fig. 3.26 is plotted from results of the program WTRANS and shows the variation of Δ with p for the beam shown in fig. 3.25. The determinant Δ assumes a zero value very close to the fundamental frequency value for a simply supported beam with distributed mass.

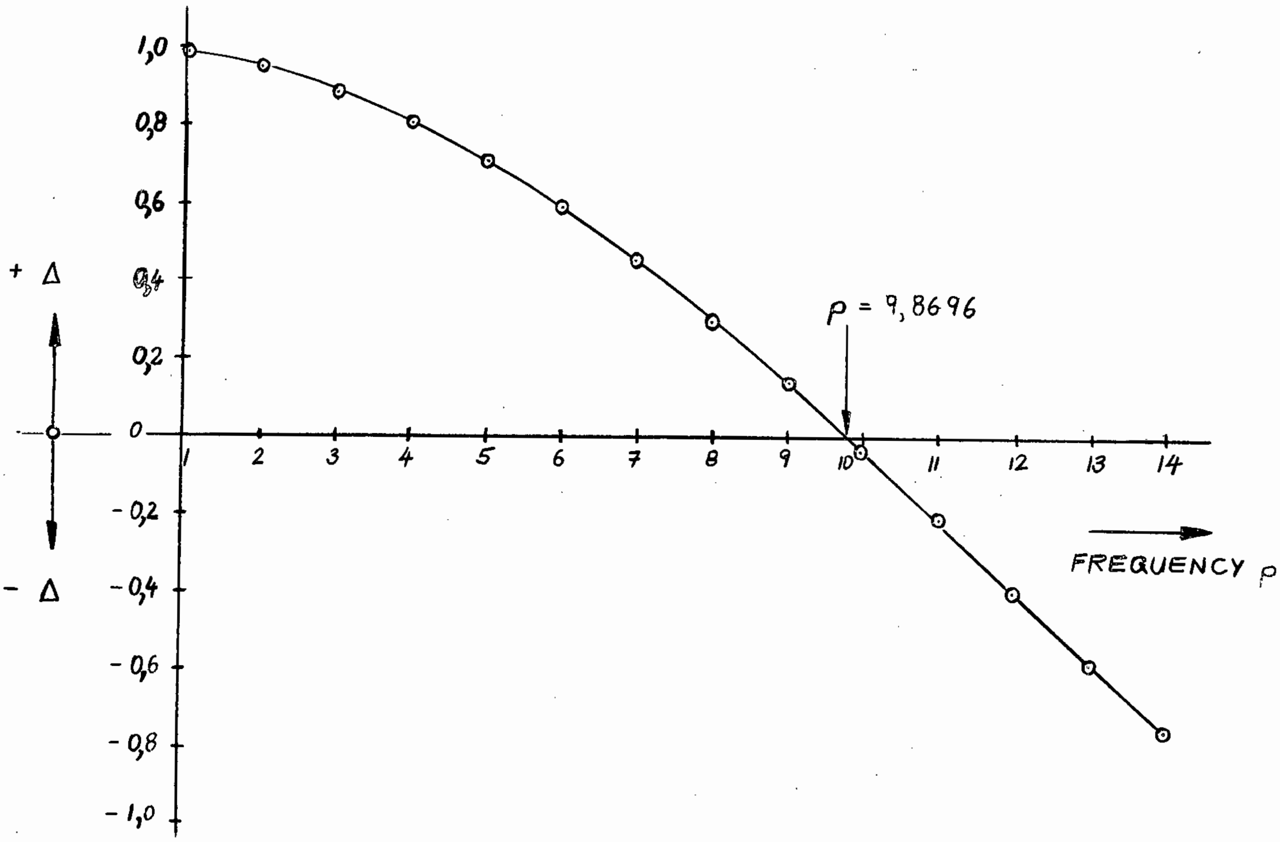


Figure 3.26

3.3.9 - Summary and comments

A. Energy methods

The methods presented in the first part of section 3.3 are all based on the principle that in a freely vibrating system a periodic exchange of kinetic and potential energies takes place. Since frictional effects are neglected, the law of conservation of energy requires that the instantaneously occurring maximum values of the two forms of energy must be equal.

In Rayleigh's method it is necessary that a proper estimate of the deflection curve is made. This is fairly easy for the fundamental mode, but Rayleigh's method is difficult to apply to higher frequencies. If the difference between the assumed mode shape and the exact one is small, then the difference between the fundamental frequency obtained from Rayleigh's method and the exact value will be very small. In other words the value of the fundamental frequency is not oversensitive to the shapes of the assumed curves as long as the curves are in reasonable agreement with the end-conditions of the system. This is the essence of Rayleigh's principle. Rayleigh's method gives an overestimate of the fundamental frequency.

The <u>static deflection method</u> is a special case of Rayleigh's method in that the static gravitational displacement (due to the suitable static loading) is used as a guess for the deflection curve, and again, in general, only the fundamental frequency is obtained. This approach is simpler than Rayleigh's method in so far as it is not necessary to guess the assumed mode shape and also to determine its derivative.

Another energy method - <u>Dunkerley's extension</u> of Rayleigh's method - is discussed, which may provide an alternative method to the static deflection method, since it gives a lower bound solution of the fundamental frequency. Dunkerley's method also uses the deflected shape due to gravity of the beam-structure, but because the mass of the beam is assumed to be lumped at discrete points and each lumped mass may be treated separately, <u>only</u> the deflection <u>under</u> the particular concentrated mass is required. Hence, from a practical point of view, the "isolated frequency method" of Dunkerley is often preferable to the static deflection method because it involves fewer calculations of the deflections - as was shown by examples. In fact, the writer considers Dunkerley's method to be a very easy and quick one, as long as the fundamental frequency only is required. In both the static deflection method and Dunkerley's method it is no problem to take a non-uniform mass distribution into account. However, a variation in bending stiffness results in a more complicated computation of the deflections since the elastic-line expression (in Appendix A) cannot be used.

Another extension of Rayleigh's method is the <u>Rayleigh-Ritz method</u> which uses specially constructed functions for the assumed mode shapes. The Rayleigh-Ritz method requires a considerable amount of calculation, but in spite of that fact, this type of the assumed mode method was often preferred to the lumped-mass methods because it can be used with far fewer degrees of freedom, i.e. it does not lead to a large number of equations, the solution of which used to present great difficulty. However, since digital computers are generally available the difficulty in problems of vibration analysis has been reduced, at least with regard to the solution of large matrix equations. For that reason the writer did not expand on the Rayleigh-Ritz method but investigated the lumped mass approach in greater detail.

B. Lumped mass methods

An alternative approach to the energy methods is the analysis of vibration problems is the setting up of the differential equations of motion of the structure, as was shown in section 2.3. In other words a beam is assumed to consist of lumped mass points which are interconnected by weightless segments, having only flexural properties. The differential equations of motion can easily be written in matrix form which leads to an organized method of computation and can readily be computerized.

If the equations of motion are obtained by considering the equilibrium of the dynamic forces which act on the freely vibrating lumped masses, we obtain a matrix equation of the form:

$$m\,\ddot{y} + K\,y = 0 \qquad \text{(a)}$$

where K is a matrix which contains the stiffness properties of the system. It is equally possible to use the flexibility matrix F, which contains the flexibility properties of the structure, since FK = I, which leads to

$$F\,M\,\ddot{y} + y = 0 \qquad \text{(b)}$$

The solution of either of the above equations leads to the classical eigenvalue problem, which can be solved by direct or indirect methods. However, in most cases indirect, i.e. iterative methods are used to solve the eigenvalue problem. It was mentioned that either the stiffness matrix or the flexibility matrix can be used, whichever is more convenient. If the stiffness matrix K is used, the eigenvalues give directly the squares of the natural frequencies and the first or largest eigenvalue gives the highest frequency vibration. However, in most practical problems the slowest vibration or fundamental frequency is usually of greatest importance. This is obtained when the eigenvalue problem contains the flexibility matrix F.

In Appendix B iterative methods are given, which may be used to determine the eigenvalues and corresponding eigenvectors, especially the largest and second-largest eigenvalues. Direct methods are not discussed since they lead to solutions giving all the eigenvalues and associated eigenvectors, which are seldom required in structural problems. Many computers have 'supplied programs' for the eigenvalue problem, which employ direct methods. However, it is worth mentioning that for instance the UNIVAC computer at UCT has only supplied programs for the eigenvalue problem of symmetrical matrices. On the other hand, the eigenvalue problem of an unsymmetrical matrix is considerably more complicated than that of a symmetrical matrix. Since it is likely that the systems matrix A is not symmetrical, the writer is of the opinion that in structural vibration analysis iterative methods should generally be used. For the fundamental frequency, the basic iterative method (Appendix B 4.1.2) can be employed for both symmetrical and unsymmetrical matrices and Wielandt's deflation (Appendix B 4.2.2) can be used for both types of matrices to determine the second largest eigenvalue.

Myklestad's method is not a matrix method but its extension and generalization led to the transfer matrix method which has certain advantages over the matrix methods using flexibility or stiffness matrices. Again the beam is represented by lumped masses which are interconnected by weightless segments which have flexural properties. The essence of both methods is that

(i) a trial value for the natural frequency is assumed,

(ii) the four quantities: displacement, slope, shear and moment (which are characteristic of bending) at the one end of the segment are expressed in terms of those at the previous end,

(iii) the boundary conditions of the structure are used to determine whether the trial value p for the particular natural frequency was correct.

The methods are based on a continuous substitution process of the end-conditions of the segments, from one support of the structure to the other, both of which have certain boundary conditions. However, at the supports usually only two of the four variables are known and the other two variables have to be guessed. In Myklestad's method the two quantities which were guessed at one support or boundary are used in the substitution process, and together with one of the known end-conditions at the other boundary (e.g. deflection = 0) the correct ratio of the two guessed quantities may be obtained. This ratio is now used to check whether the second known boundary condition (e.g. slope = 0) is also fulfilled at the end. If this is the case, the assumption for the natural frequency was correct.

In the transfer matrix method matrix notation is adopted, i.e. the four characteristic quantities for a beam in bending are listed in what is called a state vector, denoted by z. Again the four quantities at one end of the segment are expressed in terms of those at the previous end, with the only difference that the elastic, weightless segment is treated separately from the lumped mass. This leads to two matrices, the field matrix containing only the flexural properties of the segment and the point matrix considering only the inertia of the mass, equal to mp^2y, i.e. the assumption of the natural frequency enters into the point matrix. Substitution from segment to lumped mass, to segment, etc. leads to a string of matrix multiplications, the product of which is the overall matrix U, say. Hence an equation of the following form is obtained:-

$$z_n = U z_o$$

This is simply a set of four simultaneous equations, independent of the number of lumped masses. Substituting the known end-conditions at supports o and n, the four equations reduce usually to a homogeneous set of two equations, the determinant of which should be zero for a non-trivial solution. However, the determinant is only zero if the assumption for the natural frequency was correct. This trial and error method may be particularly useful, if it is required to investigate the vibration characteristics of a structure in a given frequency range.

In the following paragraph we conclude this section with a brief comparison of the two types of matrix methods:-

(i) Stiffness and Flexibility Matrices

Advantages: The stiffness or flexibility matrix might have been set up already in the process of statical analysis.

Disadvantages:

1) The setting up of the flexibility or stiffness matrix of the structure might be laborious.

2) The size of the matrices, involved, increases with the number of lumped masses, i.e. with the desired accuracy.

(ii) Transfer Matrices

Advantages:

1) Flexibility and stiffness matrices are not required.

2) The transfer matrices are very easily obtained.

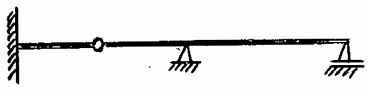
3) The calculation process gives all natural frequencies of the system.

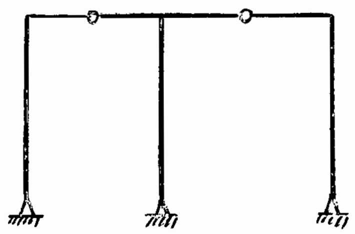
4) The sizes of the matrices involved are small, i.e. (4 × 4) matrices for a beam structure and do not increase with the number of lumped masses.

5) The programming of the calculation process is much easier and the program is also shorter (see Appendix D programs WCANT and WSBEAM c.f. WTRANS).

Disadvantages:

If the system is complicated, that is, if the structure has too many branches, the transfer matrix method becomes too complicated. In other words, the type of structure which can be analysed by means of transfer matrix method is one in which the joints are connected by a single chain of members as shown in the sketch as below.





CHAPTER 4

THE APPROXIMATION OF UNIFORM BEAMS BY LUMPED MASS-SPRING SYSTEMS

Section 4.1 - Introduction

In section 2.2 we mentioned that it is a common approach to lump the mass of a continuous system into a finite number of particles. This allows an approximate treatment of the system, which is simpler than an exact analysis of vibration of a continuous structure. The natural frequencies of a discrete system may be calculated in various ways as was shown in Chapter 3. Myklestad's method and the matrix methods are "exact" in the sense that they will yield the natural frequencies and normal modes to any desired accuracy; however, the frequencies that they give are those of the discrete approximating systems and not those of the original continuous structure. It is therefore important to know how the natural frequencies and normal modes of the discrete system are related to the original system. In particular it is of interest to know how the errors, incurred by calculating the frequencies from the discrete system are related to the number of degrees of freedom employed in the approximation.

The following discussion is based on papers by Duncan (4), Livesley (5) and Gladwell (6) and the present writer attempted to verify their findings by making use of her own computer programs.

Section 4.2 - Computer Program WCANT

Before giving the flow-diagram of the program WCANT and commenting on it, a complete solution for a cantilever beam is presented. The flexibility matrix is used in this analysis.

Consider the beam and its approximation shown in fig. 4.1.

Let

L_i be a decimal fraction of the total length L of the beam,

m_i be a decimal fraction of the total mass M_B of the beam.

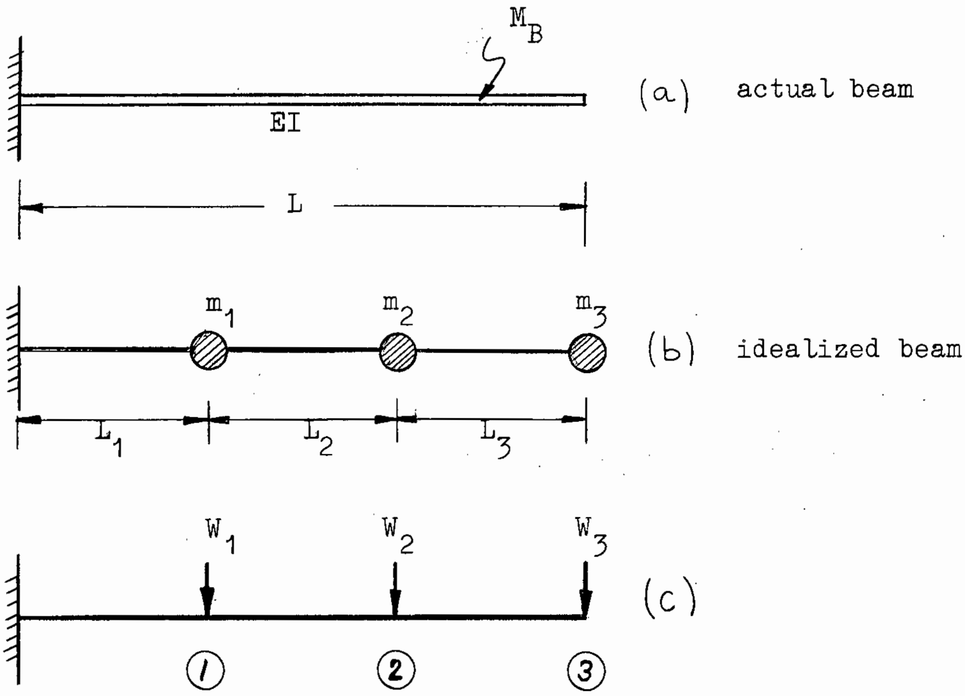


Figure 4.1

We make use of the general equation:

$$F\,M\,y = \frac{1}{p^2}\,y$$

$$A\,y = \lambda\,y$$

$\left\{ \begin{array}{l} Ay = \lambda . y \\ \lambda = \frac{1}{p^2} \end{array} \right.$ — see pg 4.5

The next step is to compile the matrices M and F. The mass matrix M is very easily obtained and has the form

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

In order to obtain the flexibility matrix F for the whole structure we must first set up the matrices B_o and F_m where:

B_o is the connection matrix which expresses the equilibrium of the member ends due to the external forces.

F_m is the flexibility matrix for the member ends.

Then $$F = B_o' \, F_m \, B_o$$

for a statically determinate structure. Now let

$$
f_i = \begin{array}{c c}
\begin{array}{cc} y & \theta \end{array} & \\
\begin{bmatrix} \dfrac{L_i^3}{3\cdot EI} & \dfrac{L_i^2}{2\ EI} \\[2ex] \dfrac{L_i^2}{2\cdot EI} & \dfrac{L_i}{EI} \end{bmatrix} & \begin{array}{c} Q \\[4ex] M \end{array}
\end{array}
$$

be the flexibility relationship for any section of the beam, we then may compile the matrix F_m for all member ends as follows:-

$$
F_m = \begin{bmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{bmatrix}
$$

It should be noted that the flexibility of a beam segment is defined in terms of vertical deflection and rotation at one end of the segment due to bending moment and shear force at that end; it would also have been possible to express the flexibility in terms of the rotations at each of the ends of the segment, caused by the bending moments.

The connection matrix B_o has the form (see fig. 4.1c):

$$
B_o = \begin{array}{cc}
\begin{array}{ccc} W_1 & W_2 & W_3 \end{array} & \\
\begin{bmatrix} 1 & 1 & 1 \\ 0 & L_2 & (L_2 + L_3) \\ 0 & 1 & 1 \\ 0 & 0 & L_2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} Q_1 \\ M_1 \\ Q_2 \\ M_2 \\ Q_3 \\ Q_3 \end{array}
\end{array}
$$

Hence,

$$
F = B_o' \, F_m \, B_o
$$

$$
F = \frac{1}{EI} \begin{array}{cc}
\begin{array}{ccc} d_1 & d_2 & d_3 \end{array} & \\
\left[\begin{array}{c|c|c}
\dfrac{L_1^3}{3} & \dfrac{L_1^3}{3} + \dfrac{L_1^2}{2} L_2 & \dfrac{L_1^3}{3} + \dfrac{L_1^2}{2}(L_2 + L_3) \\[3ex]
\dfrac{L_1^3}{3} + \dfrac{L_1^2}{2} L_2 & \dfrac{(L_1 + L_2)^3}{3} & \dfrac{(L_1 + L_2)^3}{3} + \dfrac{(L_1 + L_2)^2}{2} L_3 \\[3ex]
\dfrac{L_1^3}{3} + \dfrac{L_1^2}{2}(L_2 + L_3) & \dfrac{(L_1 + L_2)^3}{3} + \dfrac{(L_1 + L_2)^2}{2} L_3 & \dfrac{(L_1 + L_2 + L_3)^3}{3}
\end{array}\right] & \begin{array}{c} P_1 \\[6ex] P_2 \\[6ex] P_3 \end{array}
\end{array}
$$

The flexibility matrix F is reduced to a (3×3) matrix, giving only the deflection at points (1), (2) and (3) due to a unit load W at (1), (2) and (3).

Since the algebra of the following steps becomes too involved, the following numerical values will be inserted in the matrices:-

$$m_1 = 1/3\ M_B \qquad L_1 = 1/3\ L$$
$$m_2 = 1/3\ M_B \qquad L_2 = 1/3\ L$$
$$m_3 = 1/6\ M_B \qquad L_3 = 1/3\ L$$

hence

$$F = \frac{L^3 M_B}{EI}\frac{1}{162}\begin{bmatrix} 2 & 5 & 8 \\ 5 & 16 & 28 \\ 8 & 28 & 54 \end{bmatrix}$$

$$F.M = \frac{L^3\ M_B}{EI}\ \frac{1}{972}\begin{bmatrix} 4 & 10 & 8 \\ 10 & 32 & 28 \\ 16 & 56 & 54 \end{bmatrix}$$

In order to find the largest eigenvalue (which is in this case the fundamental frequency or slowest vibration) the basic iterative method (Appendix B 4.1.2) may be applied. An initial guess for the eigenvector corresponding to the largest eigenvalue has to be made and a good approximation is to use the relative deflections of the cantilever due to static loading, e.g. $x' = (1\ ;\ 4\ ;\ 7)$

First iteration:

$$\frac{L^3\ M_B}{972\ EI}\begin{bmatrix} 4 & 10 & 8 \\ 10 & 32 & 28 \\ 16 & 56 & 54 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \frac{L^3\ M_B}{972\ EI}\begin{bmatrix} 100 \\ 334 \\ 618 \end{bmatrix}$$

$$= \frac{L^3\ M_B}{EI}\ 0{,}103\begin{vmatrix} 1 \\ 3{,}34 \\ 6{,}18 \end{vmatrix}$$

Second iteration:

$$\frac{L^3\ M_B}{972\ EI}\begin{bmatrix} 4 & 10 & 8 \\ 10 & 32 & 28 \\ 16 & 56 & 54 \end{bmatrix}\begin{bmatrix} 1 \\ 3{,}34 \\ 6{,}18 \end{bmatrix} = \frac{L^3\ M_B}{EI}\ 0{,}0893\begin{bmatrix} 1 \\ 3{,}34 \\ 6{,}18 \end{bmatrix}$$

The second iteration gives an improved eigenvalue; the values of the eigenvector, however do not change any more, because they are calculated only to the second decimal place.

Now $\lambda_1^{-1} = p_1^2 = 0{,}0893$

$$= \frac{1}{\lambda_1} \frac{EI}{L^3 M_B}$$

$$p_1 = 3{,}34 \sqrt{\frac{EI}{L^3 M_B}}$$

The exact value of the fundamental frequency for a continuous cantilever is (see page 3.9):

$$p_1 = 3{,}516 \sqrt{\frac{EI}{L^3 M_B}}$$

The following flow-diagram indicates the major operations performed in the program. The program WCANT, in FORTRAN IV, is given in Appendix D. A number of subroutines are included and they are discussed briefly.

Subroutine MSMSW:

This subroutine calculates the largest eigenvalue only. It was added as a check for the basic iterative method. The method is described and illustrated in Appendix B 4.1.1.

Subroutine SWIT:

This subroutine calculates the largest eigenvalue and corresponding eigenvector by means of the basic iterative method which is described in great detail in Appendix B 4.1.2 and also in sub-section 3.3.6, page 3.32.

Subroutine LAMBDA:

This subroutine calculates the second largest eigenvalue and corresponding mode shape by means of Wielandt's deflation. This particular method for calculating the second largest eigenvalue is discussed in Appendix B 4.2.2.

This program is used to investigate

(i) the effect of different types of lumping,

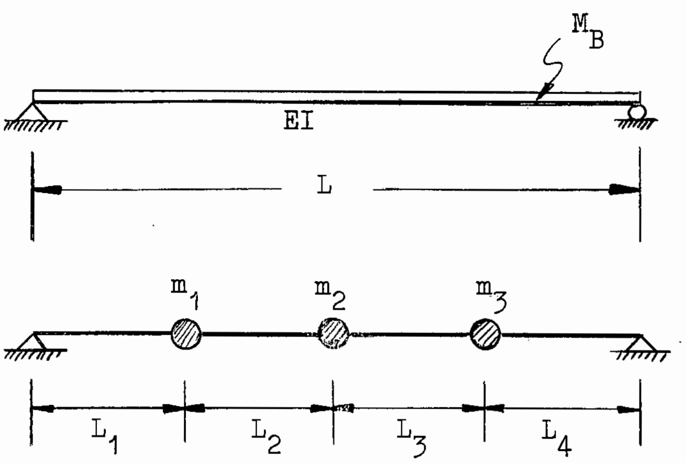
(ii) the effect of the number of lumped masses,

on the natural frequency. The results are discussed in section 4.4.

Section 4.3 - Computer Program WSBEAM

In the previous section it is mentioned that the flexibility of a beam may be expressed in terms of the rotations at each end of the beam segment, which are caused by the internal bending moment. The writer found this approach more convenient for computerizing the compilation of the required matrices. Again, a complete sample solution for a simply supported beam is given and thereafter the flow-diagram for the corresponding computer program.

Consider the beam and its approximation shown in fig. 4.2.



actual beam

idealized beam

Figure 4.2

Again we make use of the general equation

$$FM\ y = \frac{1}{p^2}\,y \qquad \text{or} \qquad Ay = \lambda . y$$

$$\cancel{A^{-1} y = \lambda^{-1} y}$$

The mass matrix is a diagonal matrix as before, and the overall flexibility matrix is given by $F = B_o' F_m B_o$.

However the F_m and B_o matrices are different from those in section 4.2. The flexibility relationship for any section of the beam is now given by (see fig. 4.3)

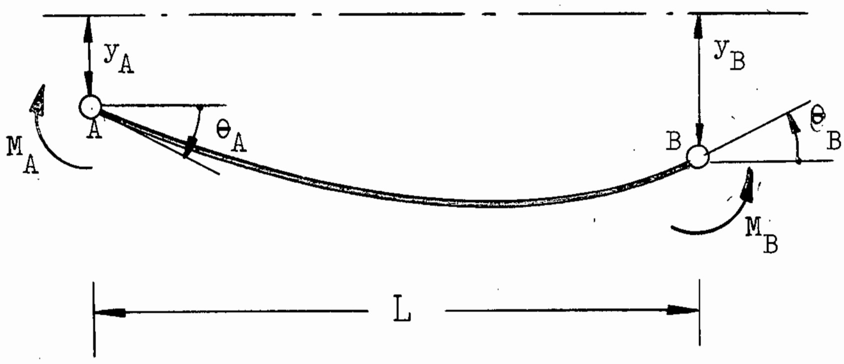


Figure 4.3

$$\begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} = \begin{bmatrix} \frac{L}{3EI} & \frac{L}{6EI} \\ \frac{L}{6EI} & \frac{L}{3EI} \end{bmatrix} \begin{bmatrix} M_A \\ M_B \end{bmatrix}$$

hence $$f_i = \frac{L_i}{6EI_i} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The matrix F_m for the member ends can now be compiled:

$$F_m = \begin{bmatrix} f_1 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & f_4 \end{bmatrix}$$

The connection matrix, expresses the relationship between the applied loads and internal member forces in terms of the end moments, and is found by computing the end moments as follows:-

Considering a unit load W at point (1):

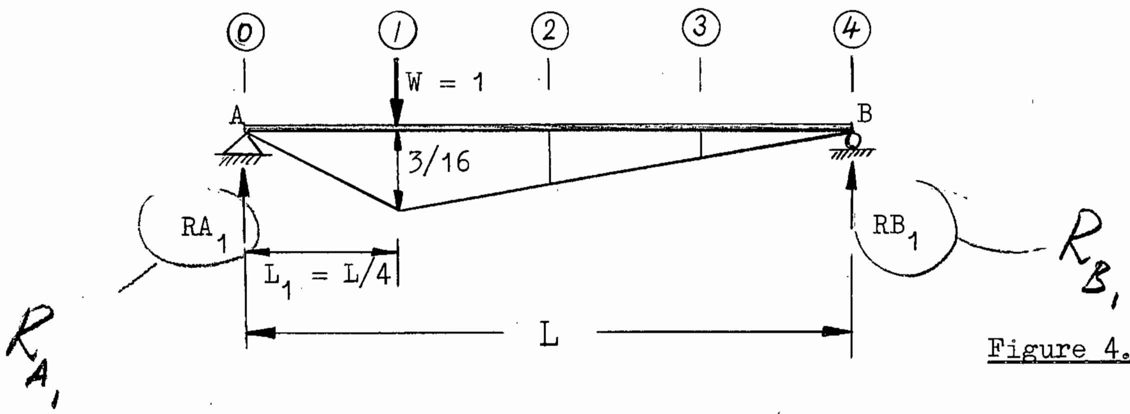


Figure 4.4

with $RA_1 = \frac{1}{L}(L - L_1)$ $\qquad RB_1 = \frac{L_1}{L}$

Knowing RA and RB, the bending moments at points 1, 2, 3 can easily be determined. Similarly the remaining two unit loads at points 2 and 3 are applied and the values for the bending moments at points 1 to 3, written in matrix form gives the B_o matrix

$$B_o = \begin{array}{c c} \begin{array}{ccc} W_1 & W_2 & W_3 \end{array} & \\ \begin{bmatrix} RA_1 \cdot L_1 & RA_2 \cdot L_1 & RA_3 \cdot L_1 \\ RB_1 \cdot (L_3 + L_2) & RB_2 \cdot (L_3 + L_2) & RA_3 \cdot (L_1 + L_2) \\ RB_1 \cdot L_3 & RB_2 \cdot L_3 & RB_3 \cdot L_3 \end{bmatrix} & \begin{array}{c} M_1 \\ M_2 \\ M_3 \end{array} \end{array}$$

In order to simplify the algebraic work, we assume:

$$L_1 = L_2 = L_3 = L_4 = \frac{L}{4}$$

then $$B_o = \frac{L}{4} \cdot \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = B_o'$$

Matrix B_o and F_m are not yet in a form so that they can be multiplied. The flexibility matrix F_m may be condensed into the following form by adding the rows (and columns) (2 + 3), (4 + 5) and (6 + 7) and omitting the first and last row (and column):

$$F_m = \frac{1}{6EI} \begin{bmatrix} 2(L_1 + L_2) & L_1 & 0 \\ L_1 & 2(L_2 + L_3) & L_2 \\ 0 & L_2 & 2(L_3 + L_4) \end{bmatrix}$$

or

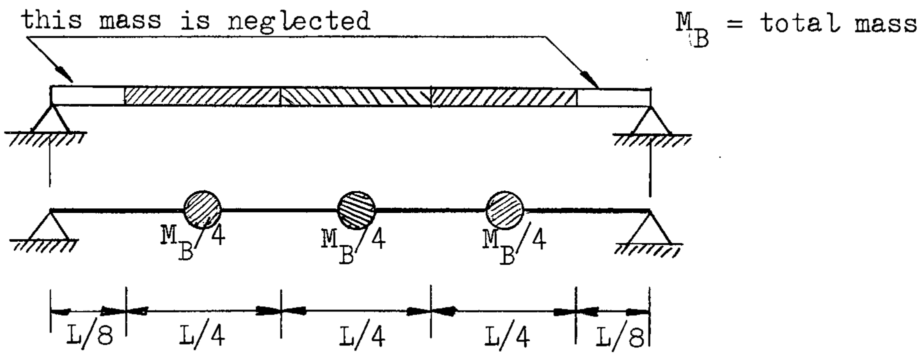
$$F_m = \frac{L}{24EI} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \quad \text{since all } L_i = \frac{L}{4}$$

This reduction of the matrix is possible because the moments at the supports are zero and at the intermediate points the values of the moments are the same just to the left and to the right of the point.

Hence:

$$F = B_o' \cdot F_m \cdot B_o = \frac{L^3}{EI} \cdot \frac{1}{768} \begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix}$$

Let $m_1 = m_2 = m_3 = M_B/4$, since we assume the following lumping:



hence $$M = \frac{M_B}{4}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

and $$F.M = \frac{M_B . L^3}{3072 . EI}\begin{bmatrix}9 & 11 & 7\\11 & 16 & 11\\7 & 11 & 9\end{bmatrix}$$

In order to find the fundamental frequency the basic iterative method may again be applied, with an initial guess for the eigenvector $y' = (1; 2; 1)$

First iteration:

$$\frac{M_B L^3}{3072 \cdot EI}\begin{bmatrix}9 & 11 & 7\\11 & 16 & 11\\7 & 11 & 9\end{bmatrix} \times \begin{bmatrix}1\\2\\1\end{bmatrix} = \frac{M_B L^3}{3072\ EI} \cdot 54 \begin{bmatrix}0,7\\1,0\\0,7\end{bmatrix}$$

$$= \frac{M_B L^3}{EI} 0,0176 \begin{bmatrix}0,7\\1,0\\0,7\end{bmatrix}$$

Third iteration:

$$\frac{M_B L^3}{3072\ EI}\begin{bmatrix}9 & 11 & 7\\11 & 16 & 11\\7 & 11 & 9\end{bmatrix} \times \begin{bmatrix}0,71\\1,00\\0,71\end{bmatrix} = \frac{M_B L^3}{EI} 0,0103 \begin{bmatrix}0,71\\1,00\\0,71\end{bmatrix}$$

The accuracy of the third iteration is quite adequate, and we obtain for the fundamental mode:

$$p_1^2 = \lambda^{\cancel{-2}} \quad \lambda^{-1}$$

hence

$$p_1 = \sqrt{\frac{EI}{0{,}0103\, M_B\, L^3}}$$

$$\underline{\underline{p_1 = 9{,}85 \sqrt{\frac{EI}{M_B\, L^3}}}}$$

The exact result for a continuous, simply supported beam is (see page 3.4)

$$\underline{\underline{p = 9{,}87\, p_W}}$$

This shows clearly that the approximation of the actual beam by a lumped mass system with three concentrated masses is a very good one.

Comment

The above calculation process has been programmed. The flow-diagram is given on the next page and the program WSBEAM in FORTRAN IV is listed in Appendix D.

The program WSBEAM was later extended and use was made of the UNIVAC supplied subroutine JACMX, which determines all the eigenvalues and eigenvectors of a symmetrical matrix. The result for the fundamental modes was identical with those obtained by means of the basic iterative method and provided a good check. However, there does not exist a UNIVAC supplied program which determines the eigenvalues and corresponding eigenvectors of a unsymmetrical matrix, which is a much more complicated problem. In other words, use can only be made of the subroutine JACMX if the matrix $A = F.M$, is a symmetrical matrix. The flexibility matrix F of course is symmetrical but the matrix A will only be symmetrical if the mass matrix M is a diagonal matrix with elements all having the same value. If the mass of the beam is uniform, then the system matrix A will be symmetrical provided that in the case of a cantilever the masses must not be lumped in a manner, known as Rayleigh's model (see section 4.4).

Figure 4.5 shows clearly that the mass matrix M will not have equal elements

FLOW-DIAGRAM FOR WSBEAM

Note:

$n = N$ = number of beam sections

m_i = MA(I) = lumped mass

L_i = L(I) = distance between lumped masses

READ: N, MA(I), L(I)

↓

Print out input

↓

Compilation of connection matrix: BO

↓

Compilation of flexibility matrix for member ends: FM

↓

Obtain transpose BOT of connection matrix BO

↓

Multiplication of BOT . FM . BO = F where F is the flexibility matrix of the structure

↓

Multiplication of F . MASS where MASS is the mass matrix

↓

CALL SUBROUTINE EVAL 1 - which calculates the largest eigenvalue and corresponding eigenvector by means of the basic iterative method

↓

Calculation of fundamental frequency p_1

↓

Output

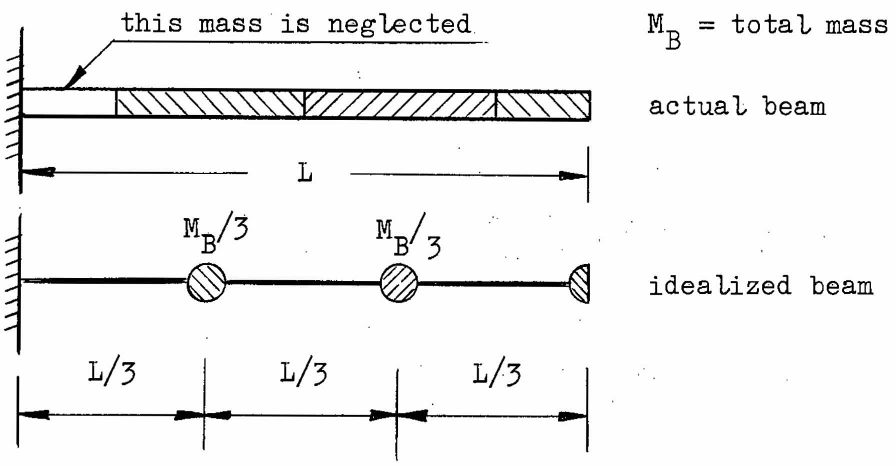


Figure 4.5

$$M = \frac{M_B}{3}\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \frac{1}{2} \end{bmatrix}$$

In the case of a beam having a non-uniform mass distribution, the mass matrix will not have equal elements, unless the length of the segments is not constant.

It should be pointed out, that the basic iterative method, although giving only the fundamental frequency and corresponding mode shape, is equally applicable to both symmetrical and non-symmetrical matrices.

The above two programs, WCANT and WSBEAM, can easily be extended for a beam with varying flexural stiffness. The symmetry of the flexibility matrix will not be affected.

Section 4.4 - Types of Approximations

There are essentially two ways of approximating a uniform beam. The first is analogous to the representation used by Rayleigh (7) for uniform strings, and the second is the model used by Duncan, as shown in fig. 4.6(c). In Rayleigh's representation the beam is assumed to have mass per unit length ρA and the beam is divided into n sections of length $a = L/n$. The mass of each section is replaced by two equal lumped masses $m/2 = \rho Aa/2$ at the ends of the beam segment. In this way we obtain a beam loaded by $u = n + 1$ masses, $1/2m$ at each end and m at each of the $(n - 1)$ intermediate points.

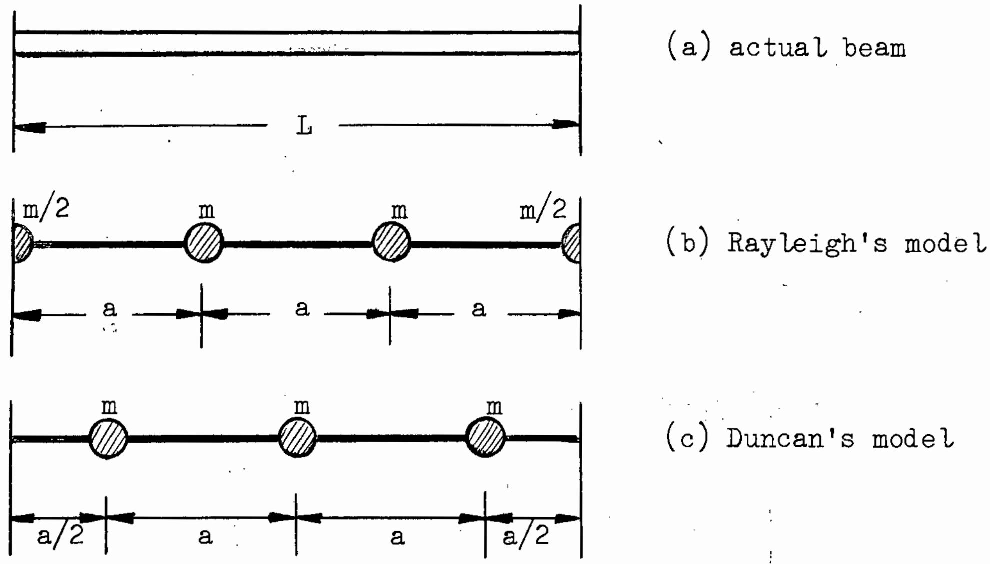


Figure 4.6

In Duncan's representation the mass of each section is replaced by a concentrated mass m at the centre. Now we obtain a beam loaded by n equal masses at the points a/2, 3a/2, (2n - 1)a/2, and the number of lumped masses u is equal to the number of sections n.

4.4.1 - Duncan's approach

Duncan (4), Livesley (5) and Gladwell (6) derived expressions for the frequency error by investigating cases in which both the continuous and the lumped mass system could be treated exactly. Duncan gave a complete answer to the problem as far as it concerns torsional systems. He showed that if an actual shaft is replaced by n flywheels on a light shaft then, provided that the positions of the flywheels are suitably chosen, the error involved in the natural frequencies will be inversely proportional to n^2, the square of the number of sections. Duncan also considered the problem of a uniform cantilever beam in transverse vibration. However, he did not know an exact solution for a segmented beam, and he showed from numerical calculations that the inverse square law for the frequency error applies also to a cantilever beam.

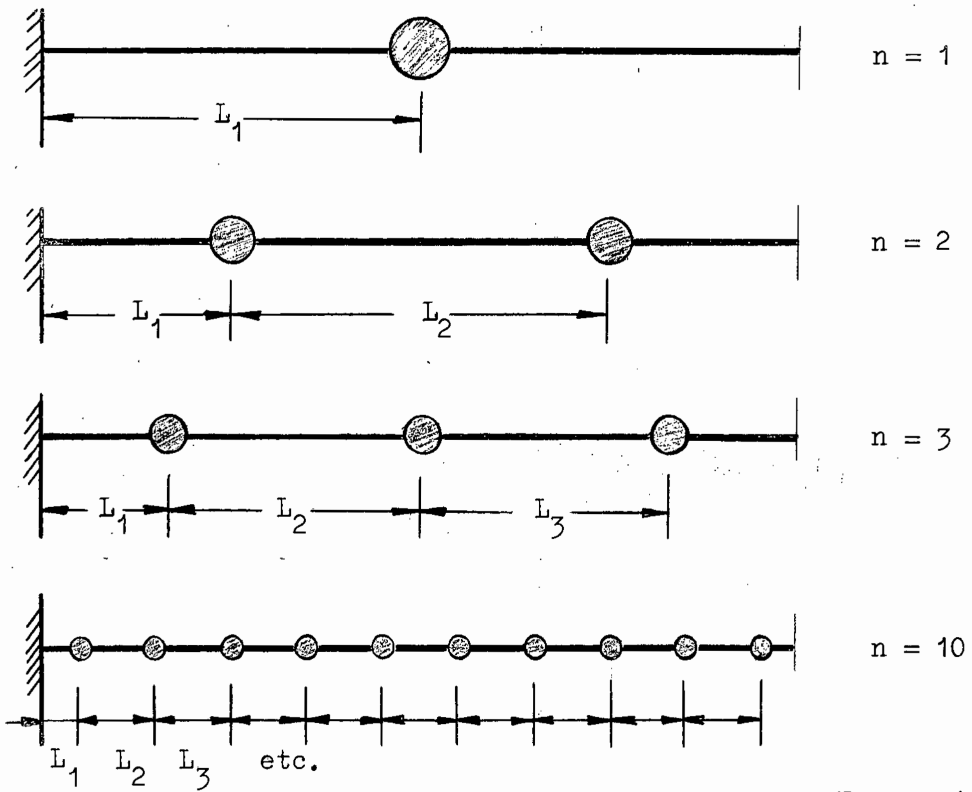


Figure 4.7

The computer program WCANT is used to demonstrate Duncan's empirical proof. The cantilever is divided into 1, 2, 3, ... to 10 lumped masses as shown in fig. 4.7, using Duncan's model. The fundamental frequency is calculated as a coefficient c of $p_w = \sqrt{EI/L^3M}$; for example (see Table 1):

$$\text{for } n = 1 \qquad p = c \cdot p_w = 4{,}89898\, p_w$$

$$\text{therefore} \qquad c = 4{,}89898$$

In Table 1, column C, a dimensionless frequency parameter β is given, which is the square of c:

$$\beta = \frac{L^3 M\, p^2}{EI} = c^2$$

Duncan used dimensionless frequency paramaters, i.e. β for his proofs of the inverse square law. Since β is proportional to p^2, the errors involved in the parameter β are related to the errors in the natural frequency as follows:-

$$\varepsilon(\beta) = \tfrac{1}{2}\,\varepsilon(p)$$

In Table 1, column D, the differences ε between the exact value β_∞ of the fundamental frequency for a continuous beam and the exact values β for the segmented beam are given; and multiplication of column D by n^2

leads to the values in column E in Table 1. It can be observed that these values ultimately assume a constant value as n gets large. Duncan found for a torsional shaft that:

$$\epsilon\ (\beta) \propto \frac{1}{n^2} + \text{higher inverse powers of } n.$$

Hence as n tends to infinity

$$\epsilon\ n^2 \longrightarrow \text{constant.}$$

From similar results as given in Table 1, column E, Duncan showed empirically that the inverse square law holds for a segmented beam with one end free and the other clamped.

With regard to torsional vibration, Duncan showed that if the flywheels are not placed at the centre but at the outer ends of the segments of the shaft, then the errors incurred by calculating the natural frequencies from these models are greatly increased. The errors vary ultimately proportional to the inverse first power of the number of sections n:

$$(\beta) = \beta_{\infty} - \beta \longrightarrow \frac{1}{n}\cdot\text{constant}$$

when n is large. Duncan mentioned that the law also holds for bending vibration of a uniform cantilever with masses placed at the outer end of the sections. The writer used the program WCANT to obtain the fundamental frequency for a cantilever, which is approximated in the manner just described. The values obtained for $c = p/p_w$ are given in Table 2, column B. In this case the values for ϵn are calculated and it can be observed that the convergence of ϵn is much slower than that of ϵn^2 in Table 1. The slow convergence and a comparison of the ϵ values in both Table 1 and Table 2, indicates the considerable loss of accuracy when the masses are moved away from the centre of the sections. The values of c from Table 1 and 2 are plotted in fig. 4.8 and the effect of the two types of lumping investigated by Duncan can be seen clearly.

In Table 3 the results from the program WCANT are used, when Rayleigh's model was adopted. Comparing columns E in Tables 1 and 3, it can be noticed that the convergence of the values ϵn^2 is slower in Table 3, which indicates that Rayleigh's model is not as good an approximation as Duncan's model. This can also be seen from fig. 4.8, where the c values of the various models are plotted versus the number of sections n or versus the number of lumped masses u, which are equal in this particular example.

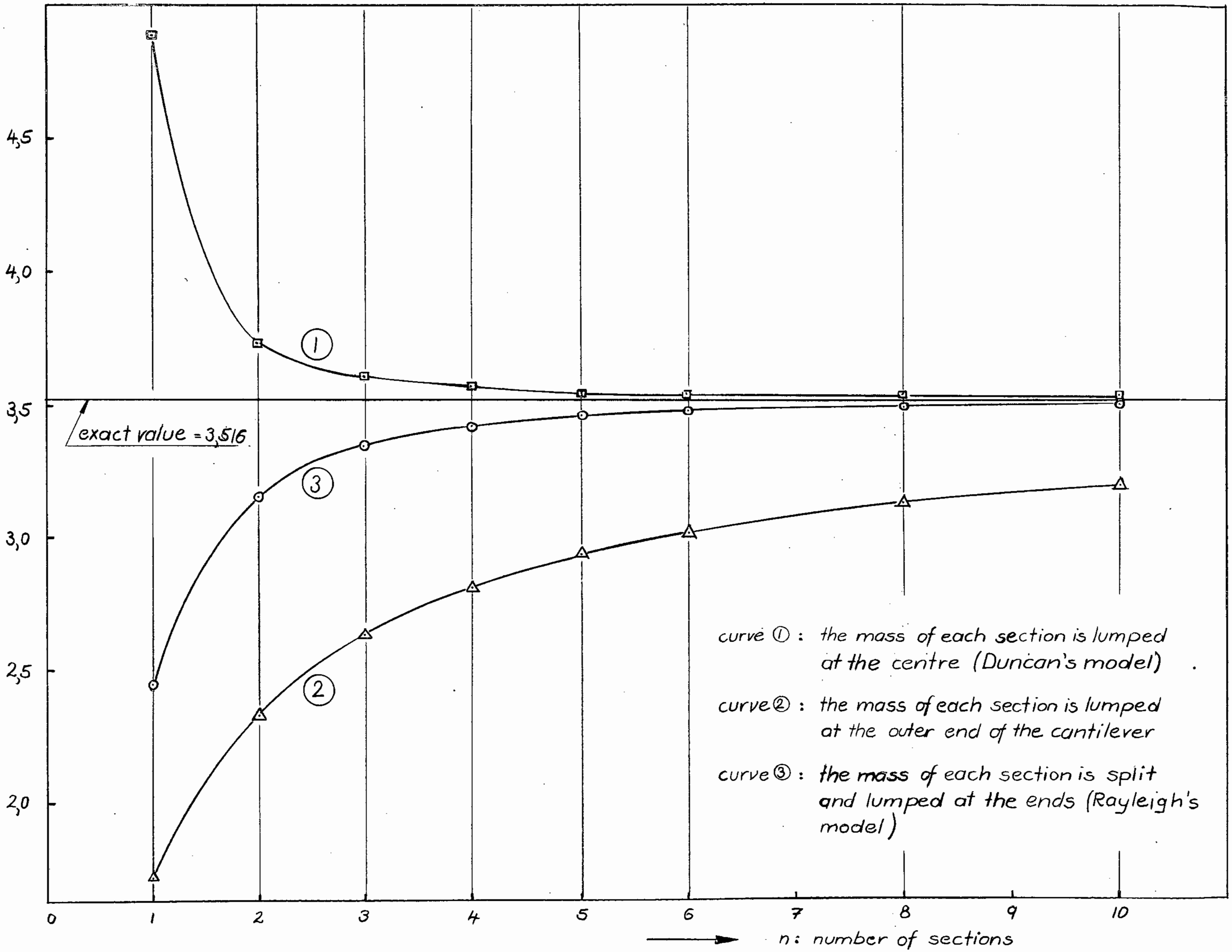


Figure 4.8

Effect of number of lumped masses and of types of lumping on the fundamental frequency for a cantilever.

A	B	C	D	E
n	c	β	$\epsilon = \beta - \beta_\infty$	$\epsilon\, n^2$
1	4,89898	24,0000	11,6377	11,6377
2	3,72992	13,9123	1,5500	6,2000
3	3,60784	13,0165	0,6542	5,8878
4	3,56709	12,7241	0,3618	5,7888
5	3,54855	12,5922	0,2299	5,7475
6	3,53850	12,5210	0,1587	5,7132
8	3,52866	12,4514	0,0891	5,7024
10	3,52410	12,4193	0,0570	5,7000

TABLE 1

Dependence of Error in the Frequency Parameter on the Number of Segments for Bending Vibration of a Uniform Cantilever, using Duncan's Model (with β_∞ = 12,3623 for the fundamental mode)

A	B	C	D	E
n	c	β	$\epsilon = \beta_\infty - \beta$	$\epsilon\, n$
1	1,73205	2,9999	9,3623	9,3623
2	2,33534	5,4538	6,9085	13,8170
3	2,63235	6,9293	5,4330	16,2991
4	2,80987	7,8954	4,4669	17,8678
5	2,92798	8,5731	3,7892	18,9462
6	3,01223	9,0735	3,2888	19,7326
8	3,12445	9,7622	2,6001	20,8008
10	3,19578	10,2130	2,1493	21,4929

TABLE 2

Convergence of ϵn for the Fundamental Frequency of a Uniform Cantilever with the Masses lumped at the Outer End of the Sections (β_∞ = 12,3623)

A	B	C	D	E
n	c	β	$\epsilon = \beta - \beta_\infty$	ϵn^2
1	2,44949	6,0000	6,3623	6,3623
2	3,15623	9,9618	2,4005	9,6020
3	3,34574	11,1940	1,1684	10,5156
4	3,41804	11,6830	0,6793	10,8688
5	3,45266	11,9209	0,4415	11,0375
6	3,47164	12,0523	0,3101	11,1636
8	3,49099	12,1870	0,1753	11,2192
10	3,49996	12,2497	0,1126	11,2580

TABLE 3

Dependence of Error in the Frequency Parameter on the Number of Sections for Bending Vibration of a Uniform Cantilever, using Rayleigh's Model. (β_∞ = 12,3623 for the fundamental frequency).

4.4.2 - Livesley's and Gladwell's approaches

Livesley (5) investigated the frequency error for the case of a uniform simply supported beam in transverse vibration. He showed that if the continuous mass distribution of the beam is replaced by a set of equally spaced lumped masses (using either Duncan's or Rayleigh's model) the errors in the natural frequencies are proportional to the inverse fourth power of the number of sections into which the beam is divided.

Gladwell pursued the problems of an exact analytical solution for segmented beams with various types of end-conditions. In his paper a mathematical proof is given of Duncan's empirical result for the frequency error in the case of a cantilever and furthermore Livesley's findings are confirmed.

The writer will not expand on the mathematical proofs given by Gladwell which are of great length and not considered relevant in the context of this thesis. Only some of the more important results are given. Livesley and Gladwell showed that if a simply supported beam is approximated by Rayleigh's model then the errors in the natural frequencies are ultimately proportional to $1/n^4$, and that

$$\frac{p_s - p_\infty}{p} \doteqdot \left(\frac{r_s L}{n}\right)^4 \cdot \frac{1}{1440} \qquad (4.1)$$

where r_sL is the value of the dimensionless parameter

$$rL = \left[\frac{\rho A L^4 p^2}{EI}\right]^{\frac{1}{4}}$$

in the sth mode of vibration of the original beam.* For the simply supported beam $r_sL = s\pi$ and hence:

$$\epsilon = \frac{p_s - p_\infty}{p} \doteqdot 0{,}0676 \left(\frac{s}{n}\right)^4$$

For instance, if a beam is divided into n = 3 sections, then the error in the fundamental frequency is given by:

$$\epsilon \doteqdot 0{,}0676 \left(\frac{1}{3}\right)^4 = 0{,}1\ \%$$

Gladwell showed that for beams with clamped, pinned or sliding ends the errors in the natural frequencies are proportional to $1/n^4$ and are always given by equation (4.1), where r_sL has the values appropriate to the mode and end-conditions. Table 4 is intended to show empirically that p is proportional to $1/n^4$, as was shown in the case of a cantilever. The c values in Table 4 are obtained from program WSBEAM.

n	c	$\beta = c^2$	$\epsilon = \beta - \beta_\infty$	ϵn^4
1	6,92820	47,99996	- 49,40914	- 49,40914
2	9,79796	96,00002	- 1,40907	- 22,54513
3	9,85901	97,20008	- 0,20901	- 16,93004
4	9,86659	97,34960	- 0,05949	- 14,23015
5	9,86843	97,38591	- 0,02318	- 14,48772
6	9,86904	97,39795	- 0,01114	- 14,43808
8	9,86943	97,40565	- 0,00344	- 14,10052

TABLE 4

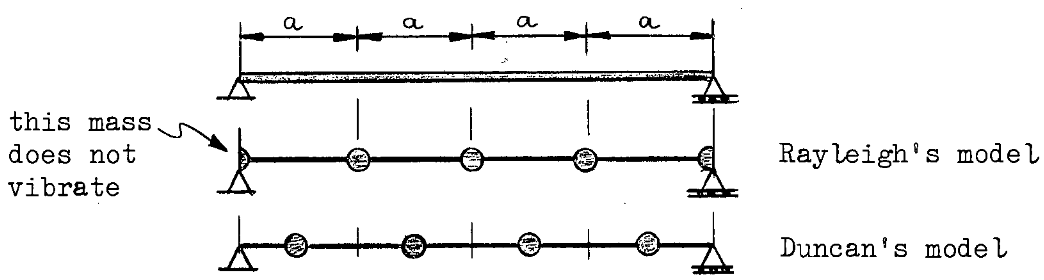
Dependence of Error in the Frequency Parameter on the Number of Sections for Bending Vibration of a Uniform Simply Supported Beam, using Duncan's Model (with β_∞= 97,40909, for the fundamental frequency)

---

* It should be noted that r is identical with that given in section 3.2.2. Also, the derivation of the equality $r_sL = s\pi$ is given in this section. Only the symbol for the mode has been changed from n to s, since in the present chapter n indicates the number of sections.

It will be noticed that the values of $\epsilon\, n^4$ converge towards a constant. The somewhat erratic convergence is attributed to a high sensitivity of the results, depending on the sixth decimal place of the c values. Furthermore both Livesley and Gladwell state that the errors involved by either type of mass representation, i.e. Rayleigh's and Duncan's models are practically identical in the case of a simply supported beam. In order to demonstrate this, use is made of program WSBEAM. The following four pages show a typical output of results of this program and it may be noted that the values for the fundamental frequency are identical to 6 significant figures, i.e.

$$p = 9{,}86843 \sqrt{\frac{EI}{M\ L^3}} .$$



It can be seen from the sketch above, that the two types of lumping do not lead to the same number of lumped masses in the case of a simply supported beam. It is therefore misleading to specify the number of lumped masses, since they vary with the lumping-model and end conditions and it is better to state the number of sections into which the beam is divided.

In the case of a <u>cantilever</u> the two types of approximation do not give identical results and the following relationship of the errors exists:

$$\epsilon(\text{Duncan}) = -\tfrac{1}{2}\,\epsilon(\text{Rayleigh}) \qquad (4.2)$$

In Table 3, column B, the results of the program WCANT are given for the case of Rayleigh's representation of the lumped masses and the values $c = p/p_W$ are plotted in fig. 4.8, curve 3. The curves 1 and 3 conform reasonably well with the above relationship. Since the errors in Duncan's approach are in fact negative, the minus sign in equation (4.2) is also explained. Also, by comparing the columns D in Table 1 and 3, the above relationship of the errors of the two models can be observed, i.e. the values in column D, Table 3, are roughly half of those in Table 1 Column D.

THE MASS OF EACH SECTION IS CONCENTRATED AT THE CENTRE OF EACH SECTION (DUNCANS MODEL)

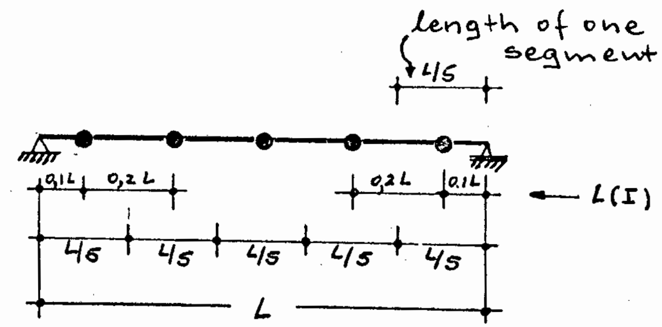
MASS MATRIX

.200000+00	.000000	.000000	.000000	.000000
.000000	.200000+00	.000000	.000000	.000000
.000000	.000000	.200000+00	.000000	.000000
.000000	.000000	.000000	.200000+00	.000000
.000000	.000000	.000000	.000000	.200000+00

THE DISTANCES BETWEEN THE LUMPED MASSES

L(I)

.100000+00
.200000+00
.200000+00
.200000+00
.200000+00
.100000+00



THE RIGIDITY VALUE 1.0000

CONNECTION MATRIX BO

.900000-01	.700000-01	.500000-01	.300000-01	.100000-01
.700000-01	.210000+00	.150000+00	.900000-01	.300000-01
.500000-01	.150000+00	.250000+00	.150000+00	.500000-01
.300000-01	.900000-01	.150000+00	.210000+00	.700000-01
.100000-01	.300000-01	.500000-01	.700000-01	.900000-01

THE FLEXIBILITY MATRIX

.270000-02	.583333-02	.616667-02	.450000-02	.163333-02
.583333-02	.147000-01	.165000-01	.123000-01	.450000-02
.616667-02	.165000-01	.208333-01	.165000-01	.616667-02
.450000-02	.123000-01	.165000-01	.147000-01	.583333-02
.163333-02	.450000-02	.616667-02	.583333-02	.270000-02

(F*M) MATRIX)

.540000-03	.116667-02	.123333-02	.900000-03	.326667-03
.116667-02	.294000-02	.330000-02	.246000-02	.900000-03
.123333-02	.330000-02	.416667-02	.330000-02	.123333-02
.900000-03	.246000-02	.330000-02	.294000-02	.116667-02
.326667-03	.900000-03	.123333-02	.116667-02	.540000-03

THE FIRST EIGENVALUE IS OBTAINED BY AN ITERATIVE METHOD

E-VALUE .102684-01
E-VECTOR
.309017+00
.809017+00
.100000+01
.809017+00
.309017+00

OMEGA .986843+01

PERIOD .636695+00

EXACT VALUE OF FUNDAMENTAL FREQUENCY FOR A
CONTINUOUS SIMPLY SUPPORTED BEAM .986960+01

FUNDAMENTAL FREQUENCY FOR A SEGMENTED SIMPLY
SUPPORTED BEAM HAVING 5 LUMPED MASSES $\boxed{.986843+01}$ ←

THE MASS OF EACH SECTION IS CONCENTRATED AT THE ENDS OF THE SECTION (RAYLEIGHS MODEL)

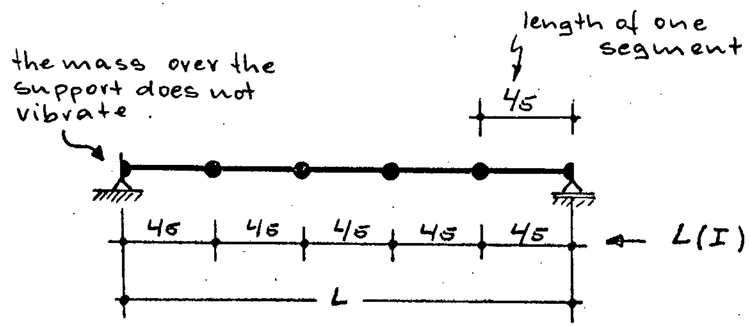
MASS MATRIX

.200000+00	.000000	.000000	.000000
.000000	.200000+00	.000000	.000000
.000000	.000000	.200000+00	.000000
.000000	.000000	.000000	.200000+00

THE DISTANCES BETWEEN THE LUMPED MASSES

L(I)
.200000+00
.200000+00
.200000+00
.200000+00
.200000+00

THE RIGIDITY VALUE   1.0000



CONNECTION MATRIX BO

.160000+00	.120000+00	.800000-01	.400000-01
.120000+00	.240000+00	.160000+00	.800000-01
.800000-01	.160000+00	.240000+00	.120000+00
.400000-01	.800000-01	.120000+00	.160000+00

THE FLEXIBILITY MATRIX

.853333-02	.120000-01	.106667-01	.613333-02
.120000-01	.192000-01	.181333-01	.106667-01
.106667-01	.181333-01	.192000-01	.120000-01
.613333-02	.106667-01	.120000-01	.853333-02

(F*M) MATRIX)

.170667-02	.240000-02	.213333-02	.122667-02
.240000-02	.384000-02	.362667-02	.213333-02
.213333-02	.362667-02	.384000-02	.240000-02
.122667-02	.213333-02	.240000-02	.170667-02

THE FIRST EIGENVALUE IS OBTAINED BY AN ITERATIVE METHOD

E-VALUE .102684-01
E-VECTOR
.618034+00
.100000+01
.100000+01
.618034+00

OMEGA .986843+01

PERIOD .636695+00

EXACT VALUE OF FUNDAMENTAL FREQUENCY FOR A
CONTINUOUS SIMPLY SUPPORTED BEAM .986960+01

FUNDAMENTAL FREQUENCY FOR A SEGMENTED SIMPLY
SUPPORTED BEAM HAVING 4 LUMPED MASSES $\boxed{.986843+01}$

FIN

RUNID: WSBEAM ACCOUNT: A0505-6005 PROJECT: QQQ$

4.4.3 - Differences between the two types of mass-lumping, as demonstrated by Myklestad's method

(1) Cantilevers

In the above sections the effects of different ways of lumping the continuous mass of a beam and different numbers of concentrated masses are investigated by employing the flexibility matrix of the system. This leads to the classical eigenvalue problem, which is solved by means of iterative methods. However, we also can use Myklestad's method to illustrate the effect of

(i) type of lumping,
(ii) number of sections,

on the accuracy of the natural frequencies. We considered a cantilever, approximated by:

(i) Duncan's model,
(ii) Rayleigh's model,

using three and six sections in both cases. The computer program WMYKLE (see page 3.45) was employed to plot fig. 4.9 and fig. 4.10, the latter being just an enlarged part of fig. 4.9 near the fundamental frequency. The rotation at the fixed end is plotted against the dimensionless parameter c which is introduced in the previous section. If c and therefore, p are correctly assumed, then the end-rotation must be zero, since the cantilever is clamped at the base station. Figures 4.9 and 4.10 illustrate clearly:

(i) Rayleigh's model leads to an under-estimate of the exact natural frequencies, for all modes,

(ii) Duncan's model leads to an over-estimate of the exact natural frequencies, for all modes,

(iii) the accuracy of the value obtained for the natural frequencies increases with the number of lumped masses. It can be seen from fig. 4.10 that the division of a cantilever into six sections leads to a value for the fundamental frequency which is very close to the exact one.

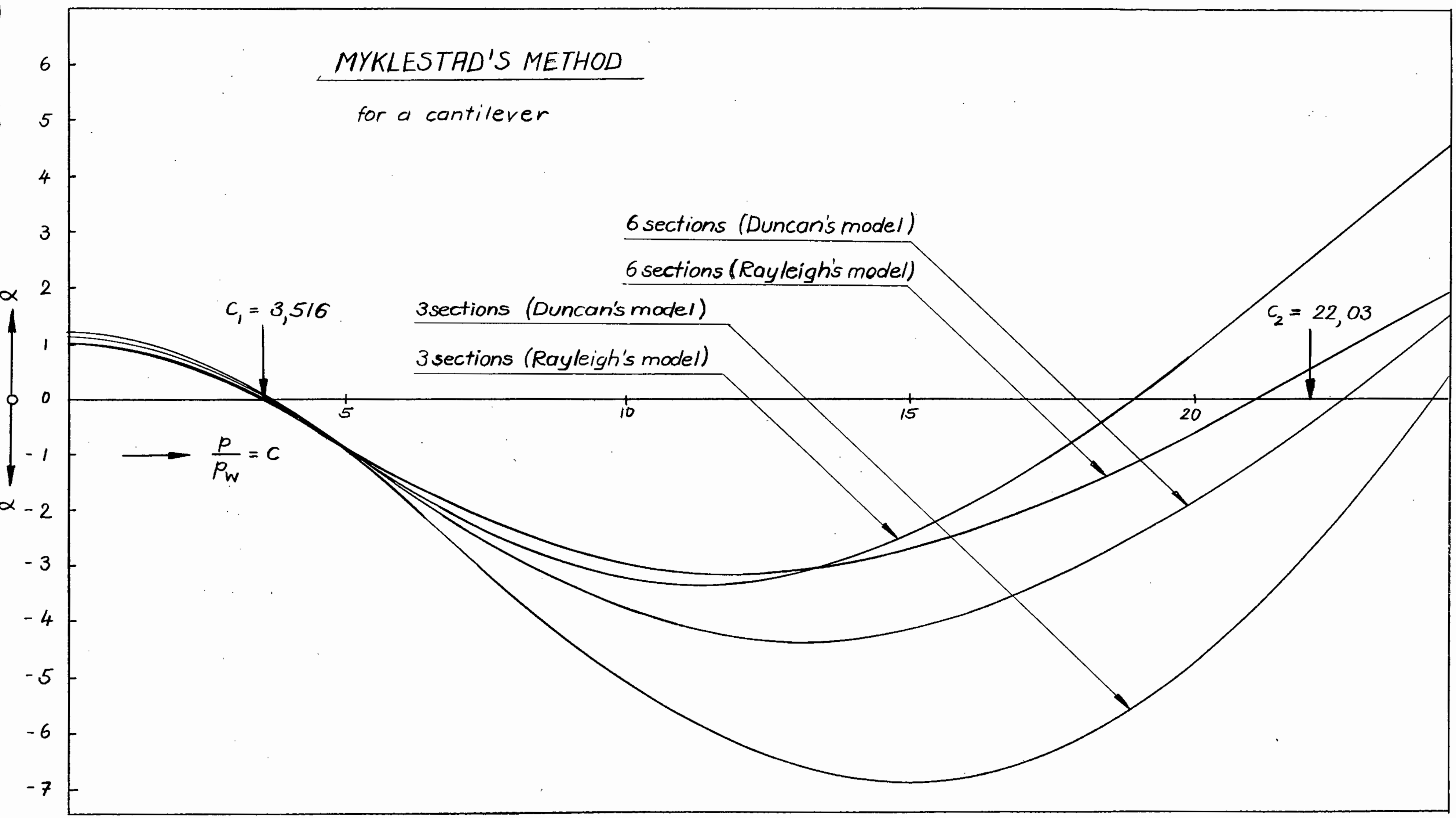


Figure 4.9

Plot of fixed end rotation vs. frequency coeff. c for various approximations of a cantilever

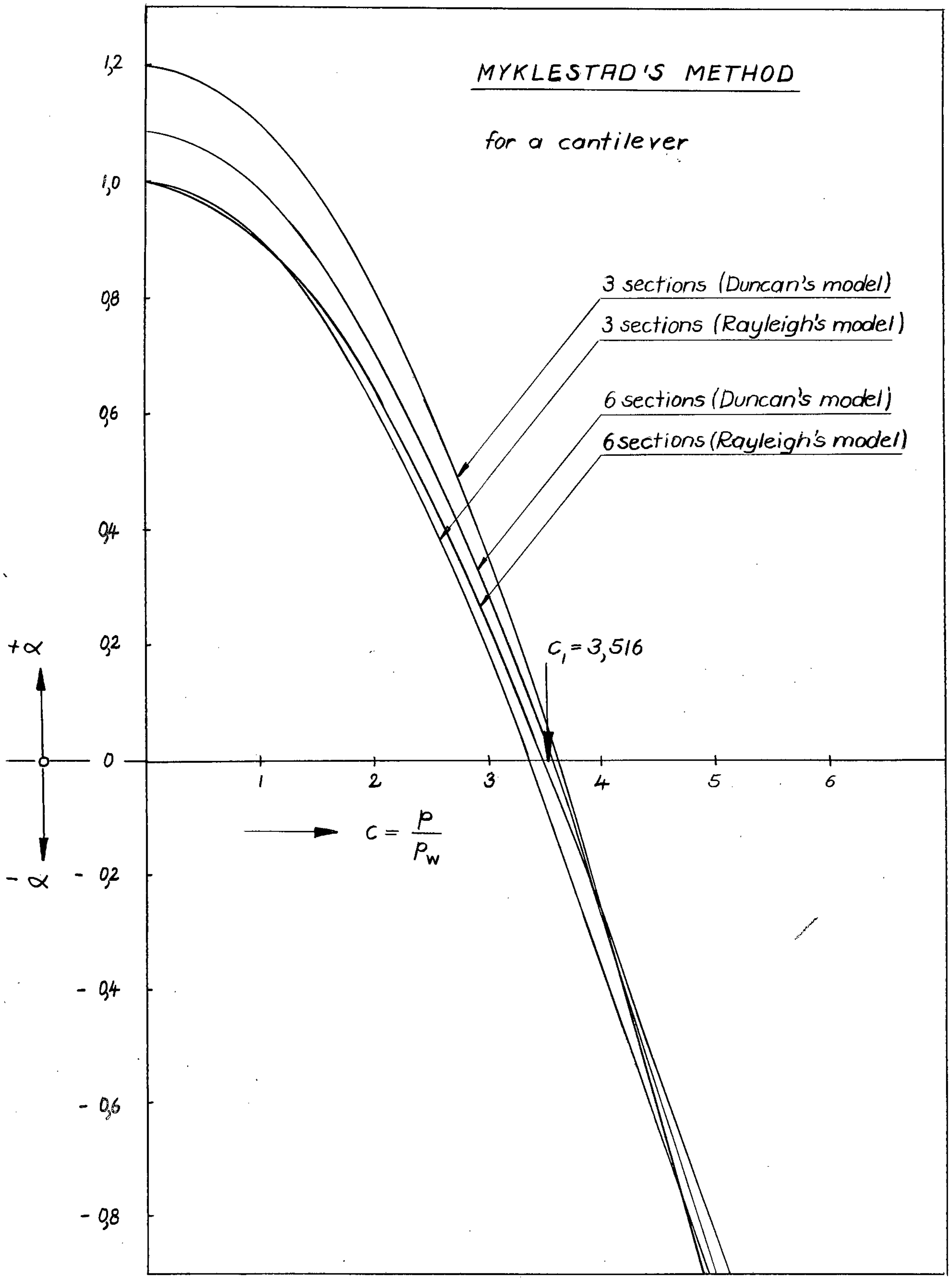


Fig. 4.10 Enlargement of Fig. 4.9 near fundamental frequency

(iv) The accuracy of the natural frequencies is best for the lowest frequency; it decreases considerably for the second natural frequency (see fig. 4.9) and will be even worse for the following frequencies.

(v) Point (iv) indicates that if a higher natural frequency is required with a certain accuracy, the beam has to be approximated by a greater number of sections.

(2) Simply supported beams

Most of the points listed above for cantilevers hold also for simply supported beams, with the exception of the following:-

(i) Rayleigh's model and Duncan's model give practically identical results for the same number of sections, for all modes.

(ii) Both models give an under-estimate of the exact natural frequencies, for all modes.

4.4.4 - Comments on the second lowest natural frequency

This chapter is concluded with a few remarks on the determination of the second lowest natural frequency.

In the program WCANT the second natural frequency was obtained by using a technique known as deflation (see Appendix B 4.2.2). This type of iterative method for evaluating eigenvalues other than the largest, is based on an important property of the normal modes, known as the 'orthogonality property'.

Table 5 gives the results from the program WCANT for a cantilever. The results are also shown graphically in fig. 4.11. Comparing fig. 4.8 and fig. 4.11 and noting the difference in scale of the y-axis, it follows that for a particular number of lumped masses, the results for the second lowest natural frequency are not at all as close to the exact value as was the case for the fundamental frequency. This means that a considerably greater number of sections is required if one wishes to determine the second lowest natural frequency to the same accuracy as the fundamental frequency.

Myklestad's method (as well as the transfer matrix method) does not require any additional calculation process for the second lowest frequency, only the range of c has to be changed. Fig. 4.9 confirms the findings mentioned above for the case of a <u>cantilever</u>:

(i) There is a loss in accuracy of the value for the second natural frequency compared with the fundamental frequency for the same number of sections.

(ii) Duncan's model gives an over-estimate and Rayleigh's model an under-estimate of the true value of the second lowest natural frequency.

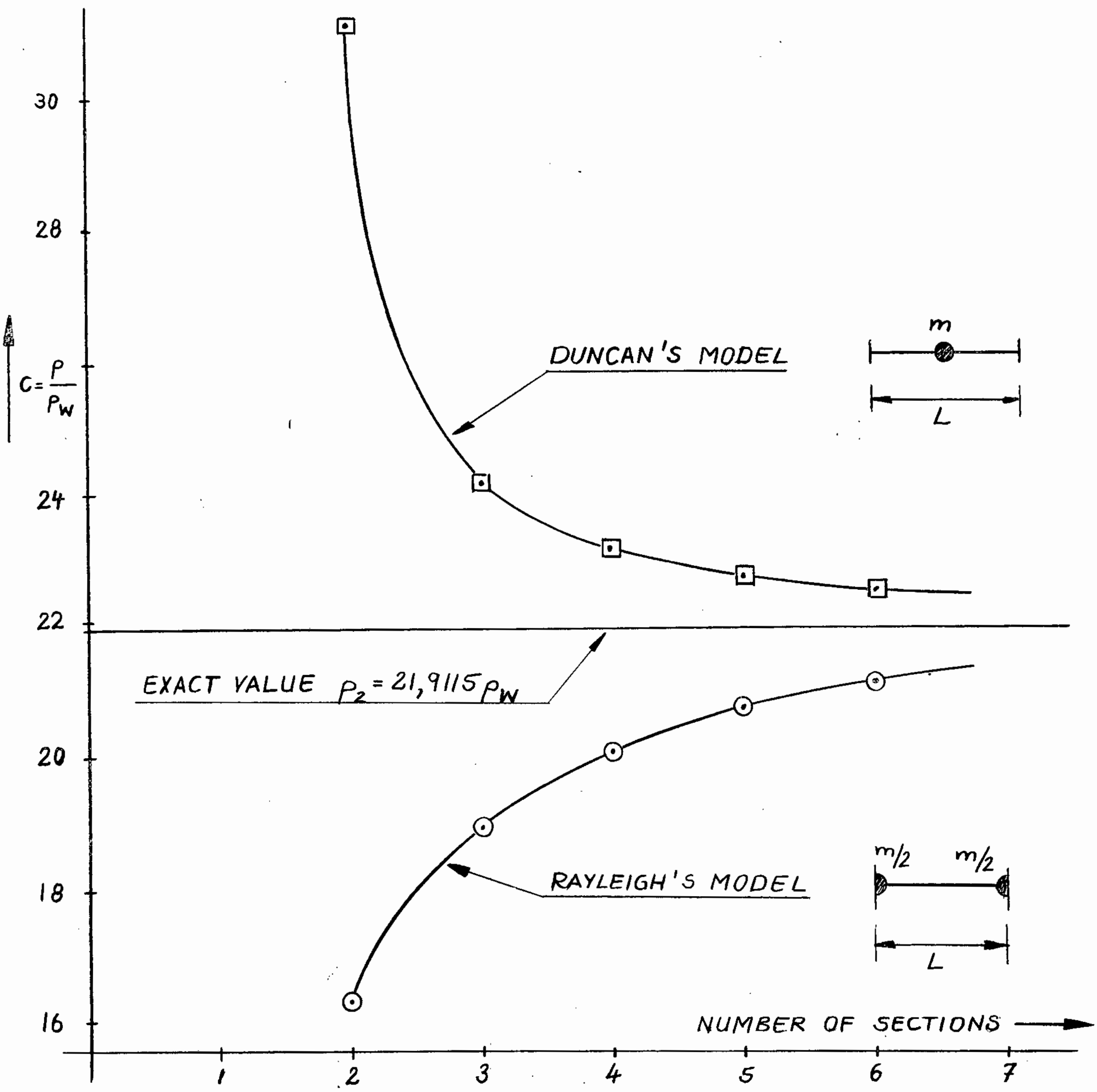


Figure 4.11 - Effect of number of sections and types of lumping on the values of the second natural frequency for a cantilever.

Number of Sections n	Rayleigh's model 	Duncan's model 
1	-	-
2	16,2580	31,0410
3	18,8855	24,1739
4	20,6903	23,1825
5	20,7335	22,7560
6	21,1083	22,5308

TABLE 5

Second Lowest Natural Frequency for a Segmented Cantilever (with $p_2 = 21{,}9115\ p_w$)

CHAPTER 5

VIBRATION OF STATICALLY INDETERMINATE BEAM-STRUCTURES

Section 5.1 - Introduction

The determination of the natural frequencies and corresponding normal modes for statically indeterminate structures is generally more complicated than for statically determinate structures.

The writer selected certain of the methods given in Chapter 3 and investigated their application to a two-span continuous beam. For beams on one or two intermediate supports, there exist exact solutions of the partial differential equations. With regard to the energy methods, the simplest (i.e. Dunkerley's method) is extended for the determination of the fundamental frequency. With respect to the lumped-mass approach, both the matrix methods are extended for a two-span continuous beam.

Section 5.2 - Timoshenko's Solution for a Continuous Beam with N spans

Timoshenko (11) gives a solution for a uniform continuous beam with N spans simply supported at the ends and at $(N - 1)$ intermediate supports. The flexural rigidity of the beam is the same for all spans and the lengths of the spans are denoted by $l_1, l_2, \dots l_N$.

The solution of the partial differential equation is not given and reference is made to (11). The solution leads to the following set of equations:-

$$\begin{aligned} - a_2 (\phi_1 + \phi_2) + a_3 \psi_2 &= 0 \\ a_2 \psi_2 - a_3 (\phi_2 + \phi_3) + a_4 \psi_3 &= 0 \\ a_{N-1} \psi_{N-1} - a_N (\phi_{N-1} + \phi_N) + 0 &= 0 \end{aligned} \tag{5.1}$$

where a_i is an unknown constant which is proportional to the bending moment at the support i, and

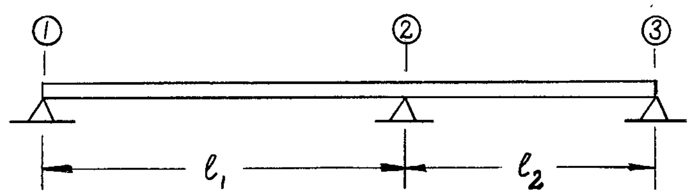
$$\phi_i = \coth (rl_i) - \cot (rl_i)$$

$$\psi_i = \operatorname{cosech} (rl_i) - \operatorname{cosec} (rl_i)$$

with $r^4 = \frac{\rho A\ p^2}{EI}$ as in section 3.2.

In order to obtain a non-trivial solution, the determinant of the above set of equations is put equal to zero. This leads to the frequency equation for the vibration of continuous beams.

For example, consider a beam with two spans. In this case only the first equation of (5.1) remains, with $a_3 = 0$ since a_3 is proportional to the moment, which is zero at the third support.



Hence $\phi_1 = -\phi_2$

The frequencies of the consecutive modes of vibration are obtained from the relationship

$$\phi\ (rl_1) = -\phi\ (rl_2)$$

For the solution of this transcendental equation it is convenient to draw a graph of the functions ϕ and $-\phi$. In fig. 5.1, ϕ and $-\phi$ are given as functions of the argument rl expressed in degrees. The problem then reduces to finding, by trial and error, a line parallel to the rl-axis which cuts the graphs of ϕ and $-\phi$ in points whose abscissae are in the ratio of the lengths of the spans. Fig. 5.1 is plotted from results of the computer program WTIM, which is given in Appendix D.

Taking, for instance, $l_1 : l_2 = 1 : 0{,}75$, we obtain from fig. 5.1:

$$rl_1 = 196^\circ = 3{,}416 \text{ rad}$$

from which the fundamental frequency can be calculated;

since $(rl_1)^4 = \frac{\rho A\ p^2}{EI}\ l_1^4$

hence $p^2 = (3{,}416)^4 \frac{EI}{\rho A\ l_1^4}$ with $M_1 = \rho A\ l_1$

$$p = 11{,}67 \sqrt{\frac{EI}{M_1\ l_1^3}} \qquad (5.2)$$

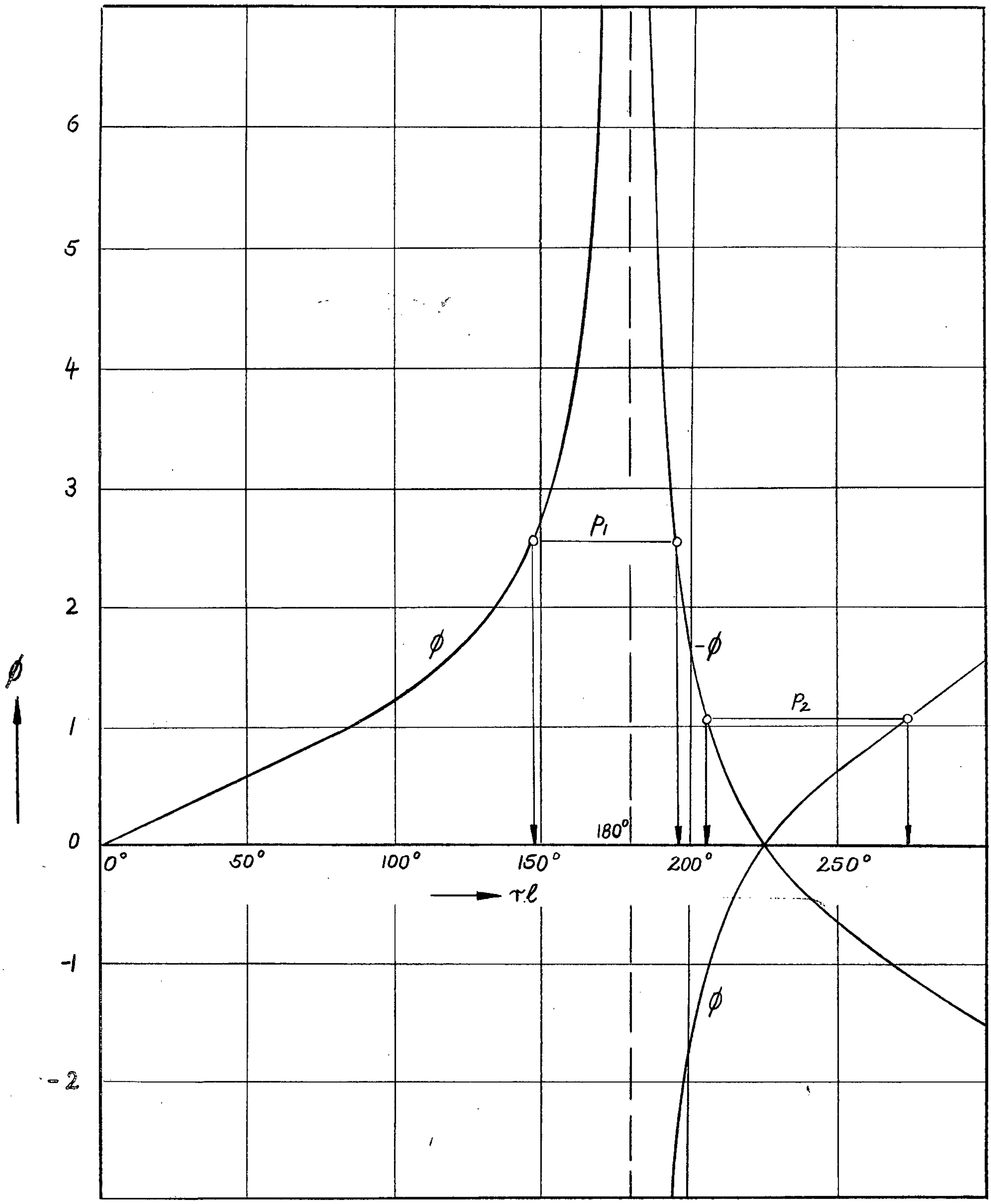


Figure 5.1

The same value for the lowest natural frequency is obtained, using the result for the second span, i.e.

$$rl_2 = 147^{\circ} = 2{,}562 \text{ rad}$$

Hence $$p^2 = (2{,}562)^4 \frac{EI}{\rho A\ l_2^4} \qquad \text{with } M_2 = \rho A\ l_2$$

$$p = (2{,}562)^2 \sqrt{\frac{EI}{M_2\ l_2^3}}$$

Substituting $M_2 = 0{,}75\ M_1$ and $l_2 = 0{,}75\ l_1$

gives $$p = 11{,}67 \sqrt{\frac{EI}{M_1\ l_1^3}} \qquad \text{as above.}$$

Obviously, the continuous beam must have the same natural frequency for both spans, as is shown above. It is only for convenience that we expressed the variable term $\sqrt{EI/M_1 l_1^3}$ in terms of the mass and length of the left-hand span in order to use this result as a basis for comparing results obtained later by other methods.

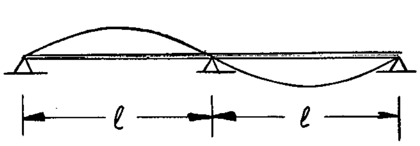
For the next higher frequency we obtain from fig. 5.1:

$$rl_1 = 274^{\circ} = 4{,}787 \text{ rad}$$

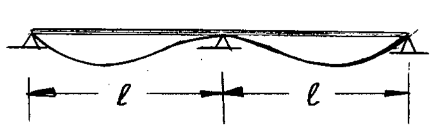
hence $$p_2 = 22{,}915 \sqrt{\frac{EI}{M_1\ l_1^3}} \tag{5.3}$$

As the lengths of the spans tend to become equal it is seen from fig. 5.1 that the lowest natural frequency tends to

$$rl_1 = rl_2 = \pi$$



(a) Fundamental mode shape.



(b) Mode shape of second lowest frequency.

Figure 5.3

As indicated in fig. 5.3(a), in the case of the fundamental mode of vibration each span will be in the condition of a beam with hinged ends, i.e. the frequency is the same as for a simply supported beam. The second mode of vibration is obtained by assuming the tangent of the intermediate support to be horizontal; then each span will be a propped cantilever, as shown in fig. 5.3(b).

Ayre and Jacobson (12) deal in great detail with continuous spans of equal lengths. A graphical network is developed for determining the natural frequencies of flexural vibration. In their paper, the following values are given for the first three natural frequencies of a beam with two equal spans:

$$p_1 = (\pi)^2 \sqrt{\frac{EI}{M\,l^3}}$$

$$p_2 = (1{,}25\ \pi)^2 \sqrt{\frac{EI}{M\,l^3}} \qquad (5.4)$$

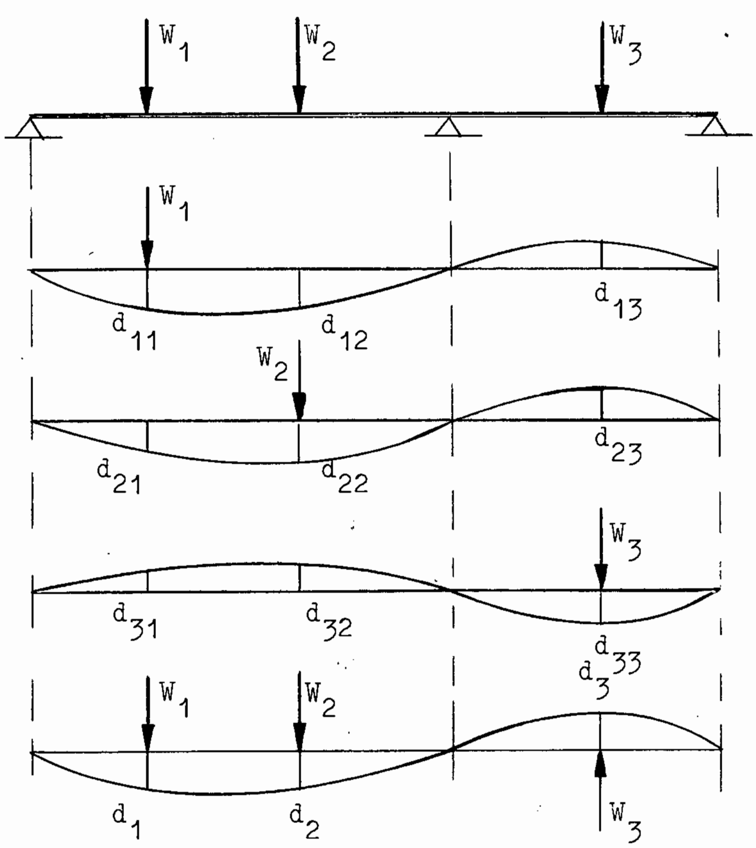
$$p_3 = (2{,}0\ \pi)^2 \sqrt{\frac{EI}{M\,l^3}}$$

where M = mass of one span

l = length of one span.

Section 5.3 - Dunkerley's Method

Dunkerley's method can be extended very easily for statically indeterminate beam structures. Consider again a two-span continuous beam, shown in fig. 5.4.



(a)

(b)

(c)

(d)

Figure 5.4

The deflection diagrams (a), (b) and (c) of fig. 5.4 enable us to obtain the three isolated frequencies by using only the deflections, d_{11}, d_{22} and d_{33}. Since the combined frequency is given by:

$$\frac{1}{p^2} = \frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \frac{1}{g}(d_{11} + d_{22} + d_{33})$$

hence $$p = \sqrt{\frac{g}{d_{11} + d_{22} + d_{33}}}$$

The result is again a lower-bound approximation of the fundamental frequency.

The effort in calculating the lowest natural frequency using Dunkerley's method can be compared with the effort in using the "static deflection curve" approximation. The latter makes use of diagram (d) in fig. 5.4 in which the deflections are given by:

$$d_1 = d_{11} + d_{12} + d_{13}$$

$$d_2 = d_{21} + d_{22} + d_{23}$$

$$d_3 = d_{31} + d_{32} + d_{33}$$

and $$p = \sqrt{\frac{\Sigma\ Wd_i}{\Sigma\ Wd_i^2}}$$

which gives an upper-bound approximation. But it is obvious, that for several loads W, considerably more deflections have to be evaluated for the "static deflection method" than in the case of Dunkerley's method.

In order to apply Dunkerley's formula, the deflections under the loads have to be determined. In Appendix A 3, the deflection equation for a two-span continuous beam is given. A computer program was written which uses the deflection equation for evaluating the required deflections for spans having different lengths and up to nine lumped masses per span. The program WDUNK is listed in Appendix D, and the flow-diagram is given in Appendix C. In the following examples the same two-span continuous beams are considered as in section 5.2

5.3.1 - <u>Continuous beam with two equal spans</u>

Consider the beam shown in fig. 5.5:

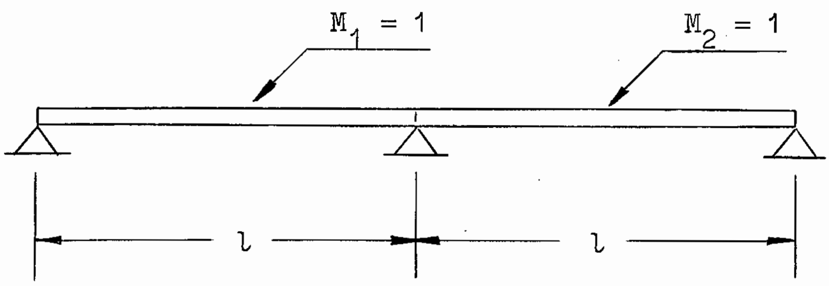


Figure 5.5

The beam is divided into $n = 2, 3, 4$ or 6 sections. As pointed out in section 4.2 it is better to specify the number of sections rather than the number of lumped masses. The results obtained for an equal-span continuous beam are plotted in fig. 5.6. Both types of mass-lumping are investigated and in both cases the same lower bound value of the fundamental frequency is obtained, equal to 7,938 p_w. The 'true'* value is 9,8696 p_w (see equation (5.4)); hence Dunkerley's formula under-estimates the exact value by 19,6 %. This percentage error seems to be larger in the case of statically indeterminate structures than for statically determinate ones. It can be seen from fig. 5.6 that the true value of the fundamental frequency is not approached with an increase in the number of lumped masses, but only the lower bound value. It should be noted that for a two-span continuous beam Rayleigh's model gives an over-estimate and Duncan's model an under-estimate of Dunkerley's approximate value for the fundamental mode.

5.3.2 - Continuous beam with two unequal spans

Consider the beam shown in fig. 5.8:

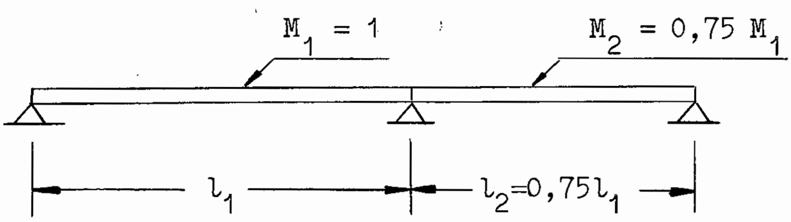


Figure 5.8

*The results obtained by Timoshenko's method are referred to as the true values of the natural frequencies.

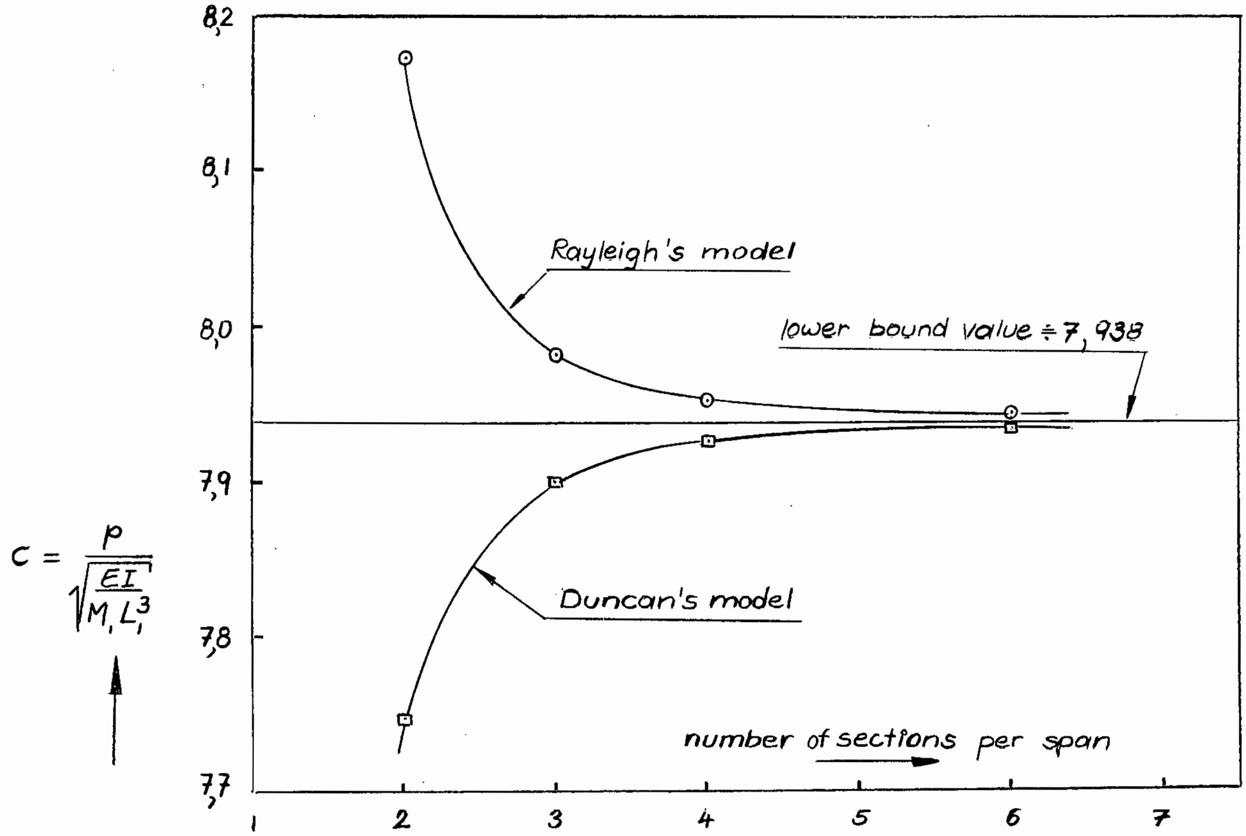


Figure 5.6 - Equal span continuous beam

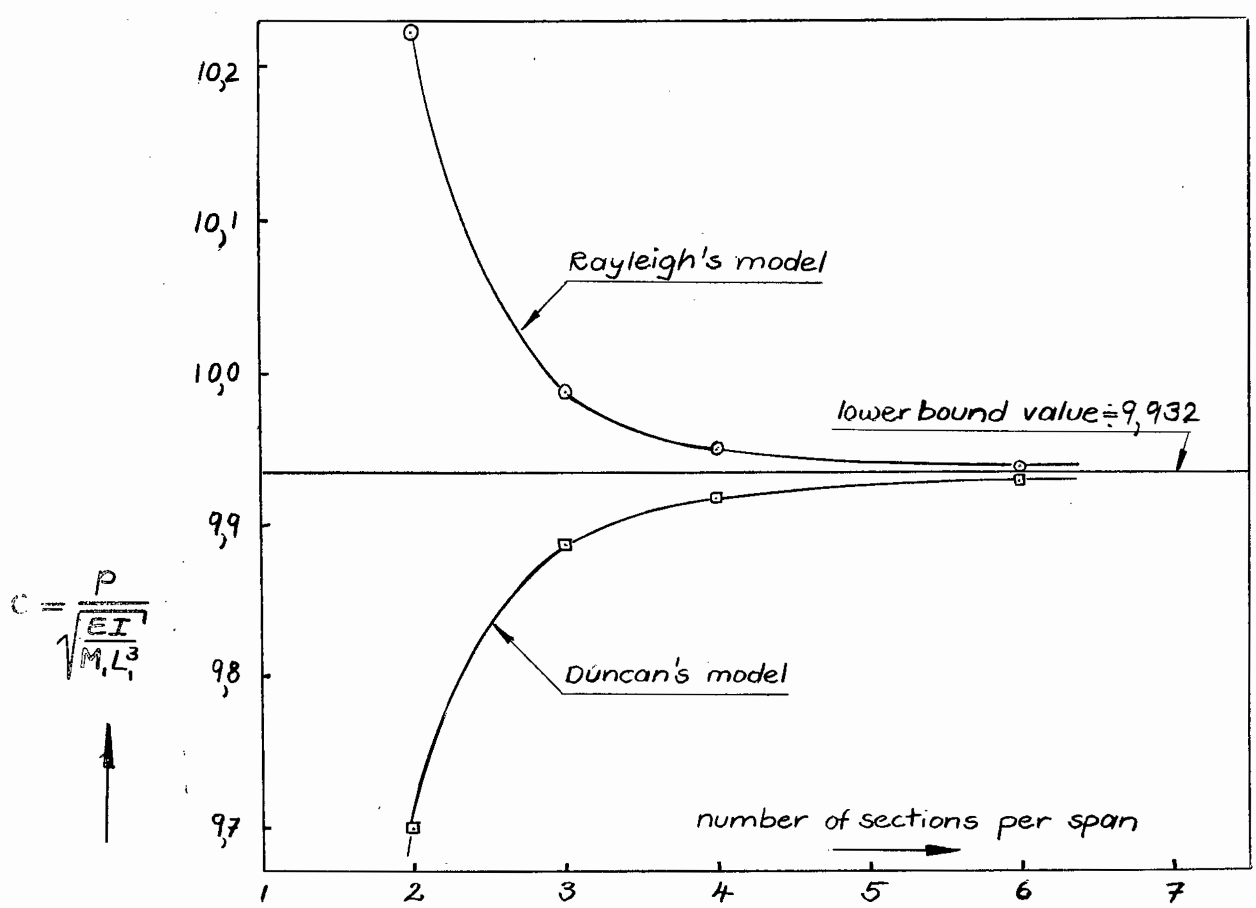


Figure 5.7 - Unequal span continuous beam.

Again the beam is divided into $n = 2, 3, 4$ or 6 sections and the results obtained from the program WDUNK are plotted in fig. 5.7. In this case, both types of mass-lumping lead to a lower-bound value of the fundamental frequency equal to 9,932 p_w. The true value is 11,67 p_w (see equation (5.2)) and hence Dunkerley's formula underestimates Timoshenko's value by 14.9 %. Again, it can be observed, that Rayleigh's and Duncan's model give an over-estimate and under-estimate respectively of Dunkerley's value.

5.3.3 - Comments on Dunkerley's method

There is a considerable difference in the percentage error for the above two beams. This may be due to the fact, that Dunkerley's method only gives good results if the second lowest natural frequency is numerically much larger than the fundamental frequency. However, for a beam with two equal spans the second lowest frequency is:

$$p_2 = 15{,}424\ p_w$$

whilst for a beam having two spans, with lengths in the ratio 1 : 0,75, the second lowest frequency becomes:

$$p_2 = 22{,}915\ p_w.$$

Since in the latter case the percentage difference between p_1 and p_2 is larger than for the equal span continuous beam, it may be assumed that the percentage error between the lower bound value and Timoshenko's value is smaller for this particular case.

Dunkerley's method does not seem very advantageous compared with the simple way the fundamental frequency can be obtained from Timoshenko's graph, because the determination of the deflections requires some effort. It is, however, possible to take a varying mass of the beam into consideration; but the flexural rigidity can not be varied along the beam in both Timoshenko's and Dunkerley's method.

If the "static deflection method" were employed for the same problem the approximate value would probably give a result with a smaller percentage error, but the approximate value would be an overestimate. The fact that Dunkerley's method gives an under-estimate is advantageous in so far as the designer can be sure that below this lower bound value of the fundamental frequency there is no possibility of resonance.

Section 5.4 - Transfer Matrix Method

The basic procedure described in section 3.3.8 can be applied to any simple continuous beam-structure, consisting of a set of lumped masses carried on a weightless beam. However, any intermediate support causes a rigidity which prevents a deflection and introduces a discontinuity in the shear force. As an example, we consider again a two-span continuous beam, shown in fig. 5.9.

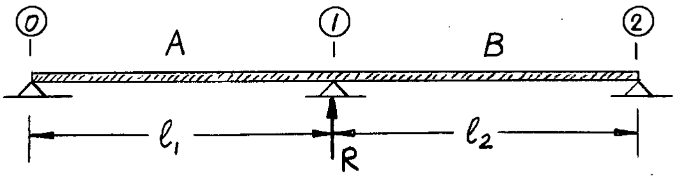


Figure 5.9

In the case of a two-span continuous beam the reaction R at the intermediate support is the only unknown discontinuity. Assuming that the overall transfer matrices A and B (see fig. 5.9) are already known, we have:

$$z_1^l = A\ z_o^r \tag{5.5}$$

$$z_2^l = B\ z_o^r \tag{5.6}$$

Since a relationship between z_2^l and z_2^r is missing, we make use of the end-condition $y_o = M_o = 0$, which after substitution into matrix equation (5.5) gives the following expression:-

$$y_1^l = a_{12}\ \theta_o + a_{14}\ Q_o$$

However, we know that $y_1^l = 0$, which leads to the following relation between θ_o and Q_o:

$$Q_o = -\frac{a_{12}}{a_{14}}\ \theta_o$$

This relationship makes it possible to express the state vector z_1^l in terms of θ_o only, so that

$$\begin{bmatrix} y \\ \theta \\ M \\ Q \end{bmatrix}_1^r = \begin{bmatrix} 0 \\ \frac{1}{a_{14}} \cdot (a_{14}\ a_{22} - a_{12}\ a_{24}) \\ \frac{1}{a_{14}} \cdot (a_{14}\ a_{32} - a_{12}\ a_{34}) \\ \frac{1}{a_{14}} \cdot (a_{14}\ a_{42} - a_{12}\ a_{44}) \end{bmatrix} \begin{bmatrix} \theta_o \end{bmatrix}$$

The deflection, slope and moment are continuous over the intermediate support 1, but the shear $Q_1^r = Q_1^l + R$ is discontinuous, with R as the unknown reaction. The state vector z_1^r may now be expressed by the relation

$$\begin{bmatrix} y \\ \theta \\ M \\ Q \end{bmatrix}_1^r = \begin{bmatrix} 0 & 0 \\ \frac{1}{a_{14}}(a_{14}\,a_{22} - a_{12}\,a_{24}) & 0 \\ \frac{1}{a_{14}}(a_{14}\,a_{32} - a_{12}\,a_{34}) & 0 \\ \frac{1}{a_{14}}(a_{14}\,a_{42} - a_{12}\,a_{44}) & 1 \end{bmatrix} \begin{bmatrix} \theta_o \\ R \end{bmatrix}$$

or

$$z_1^r = E \, . \begin{bmatrix} \theta_o \\ R \end{bmatrix}$$

The initial unknown Q_o is thereby eliminated and the new unknown R is introduced. We proceed now in the usual way to find the state vector z_2^l:

$$z_2^l = B\, z_1^r$$

$$= B \, . \, E \, . \begin{bmatrix} \theta_o \\ R \end{bmatrix} = D \begin{bmatrix} \theta_o \\ R \end{bmatrix}$$

or

$$\begin{bmatrix} y \\ \theta \\ M \\ Q \end{bmatrix}_2^l = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \\ d_{41} & d_{42} \end{bmatrix} . \begin{bmatrix} \theta_o \\ R \end{bmatrix}$$

Using the end-condition at support 2, $M_2 = y_2 = 0$, we obtain the following set of equations:-

$$0 = y_2 = d_{11}\,\theta_o + d_{12}\,R$$

$$0 = M_2 = d_{31}\,\theta_o + d_{32}\,R$$

Hence for a non-trivial solution, we have

$$\det = \Delta = d_{11}\,d_{32} - d_{31}\,d_{12} = 0 \qquad (5.7)$$

The same numerical procedure, outlined in section 3.3.8 may be used to determine any natural frequency p, i.e. certain trial values are chosen for p and the corresponding values for Δ of the frequency equation (5.7) are calculated. The value Δ is then plotted against the frequency p, the zero values of Δ occurring at each natural frequency of the system.

In order to investigate various types of beams and the effect of the two kinds of mass-lumping, a program was written, which may be used for any two-span continuous beam with up to 10 sections per span. The program WTMCB in FORTRAN IV is listed in Appendix D and the flow-diagram is given in Appendix C. The following page shows a typical output of the program.

Both types of mass-lumping are considered for a continuous beam with two equal spans, of which each is divided into 1, 2, 3 or 4 sections. However, the results obtained are not satisfactory as can be seen from fig. 5.10. The most crude lumping of the span masses gives reasonable values of the fundamental frequency, i.e. in the case of Duncan's model one section with one lumped mass at the centre, and in the case of Rayleigh's model two sections with three masses, of which only the one at the centre vibrates. With an increase in sections per span the results for the fundamental frequency do not improve, and this aspect requires further investigation. It is possible that the transfer matrix method is not entirely satisfactory for this type of beam-structure. Pestel and Leckie (3) mention that numerical difficulties may arise, when the stiffness of elastic supports is very large compared with the bending stiffness of the beam; and this is the case with a rigid intermediate support. Pestel and Leckie recommend for this type of problem the Delta-matrix method.

Section 5.5 - The Use of the Flexibility Matrix

In section 3.3.6 the application of the flexibility matrix to vibration problems is discussed and at the beginning of Chapter 4, a detailed description is given of the procedures involved when setting up the overall flexibility matrix. In the case of statically indeterminate structures more matrix operations are required to obtain the flexibility matrix of the whole system, because redundancies have to be considered. The following is a brief description of the steps involved.

The flexibility of a beam section of length L is defined in terms of the end-rotations due to unit moments applied at the ends, i.e.

$$f_i = \frac{L}{6EI}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

As before, the matrix F_m for all member ends is compiled as follows:-

FIRST SPAN

NUMBER OF SEGMENTS = 2

LUMPED MASSES	LENGTHS OF SEGMENTS	BENDING RIGIDITY
.500000+00	.500000+00	.100000+01
.000000	.500000+00	.100000+01

SECOND SPAN

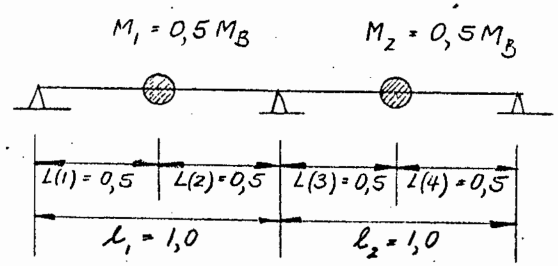
NUMBER OF SEGMENTS = 2

LUMPED MASSES	LENGTHS OF SEGMENTS	BENDING RIGIDITY
.500000+00	.500000+00	.100000+01
.000000	.500000+00	.100000+01

VALUE OF DETERMINANT	TRIAL VALUE FOR P
-.199627+01	.600000+01
-.171407+01	.650000+01
-.142980+01	.700000+01
-.114762+01	.750000+01
-.871795+00	.800000+01
-.606676+00	.850000+01
-.356670+00	.900000+01
-.126212+00	.950000+01
.802611-01	.100000+02
.258340+00	.105000+02
.403661+00	.110000+02
.511931+00	.115000+02
.578947+00	.120000+02
.600612+00	.125000+02
.572950+00	.130000+02
.492120+00	.135000+02
.354426+00	.140000+02

M_B = total mass of one span

$M_B = 1$



For the mass-lumping Rayleigh's model is used.

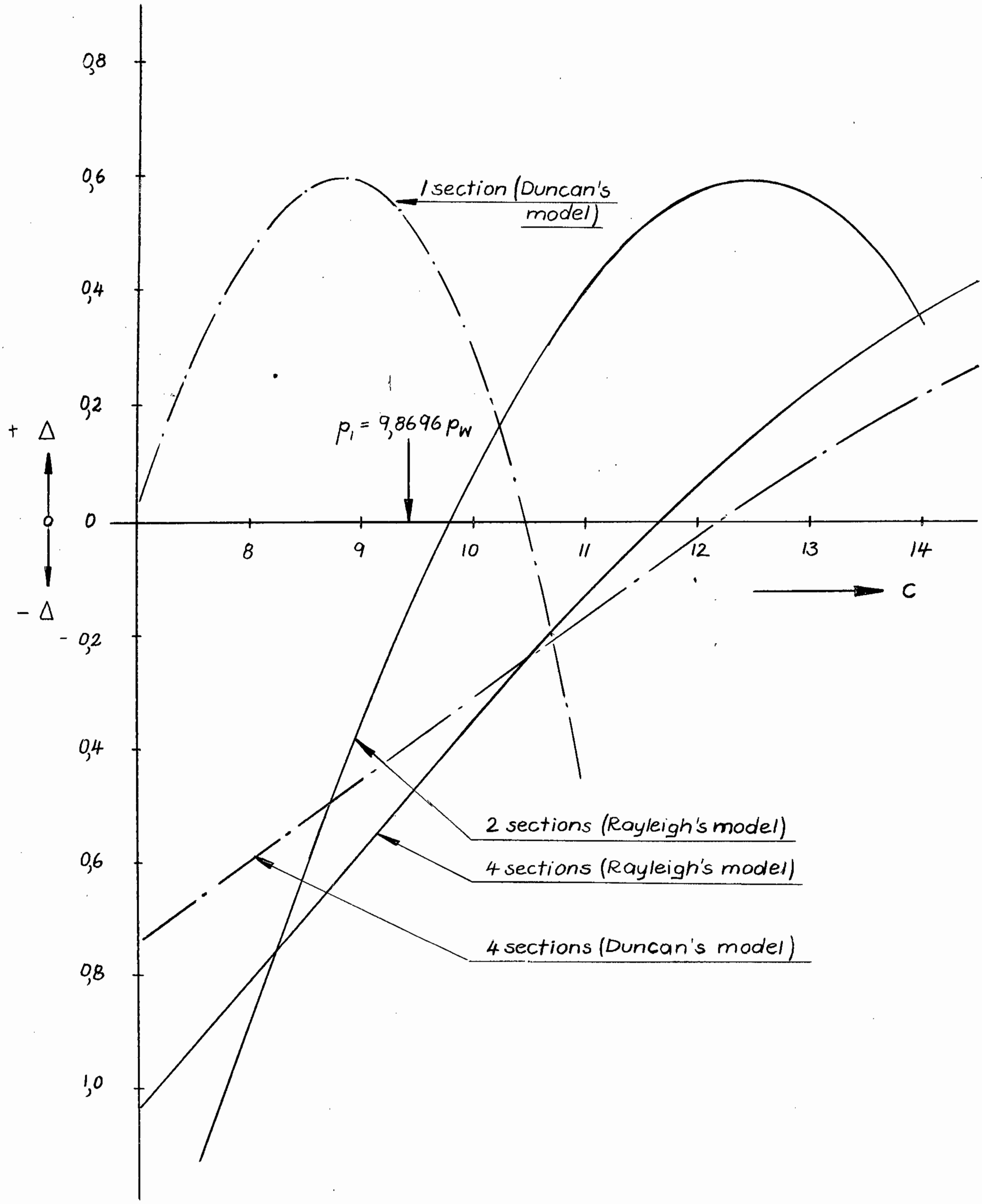


Figure 5.10 - Transfer Matrix Method
(for a continuous beam with two equal spans)

$$F_m = \begin{bmatrix} f_1 & & & & 0 \\ & f_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & f_n \end{bmatrix} \tag{5.8}$$

In order to set up the connection matrix B_o, which expresses the relationship between applied loads and internal member forces, it is first necessary to make the structure statically determinate. Then another connection matrix, denoted by B_1, is required which expresses the equilibrium relationship between the redundant forces and the internal member end forces.

To obtain the overall flexibility matrix F for the system, the following matrix operations have to be carried out:-

(i) Evaluate the displacements corresponding to the redundants

$$D = B_1' F_m B_1$$

(ii) Calculate the values of the redundants due to applied unit external loads

$$X = - D^{-1} B_1' F_m B_o$$

(iii) Calculate the B matrix, which relates the applied loads and redundancies to the internal forces

$$B = B_o + B_1 X$$

(iv) Calculate the flexibility matrix of the structure, i.e. the displacement corresponding to the external loads due to applied unit external loads

$$F = B_o' F_m B .$$

These matrix operations are demonstrated for a two-span continuous beam, shown in fig. 5.11.

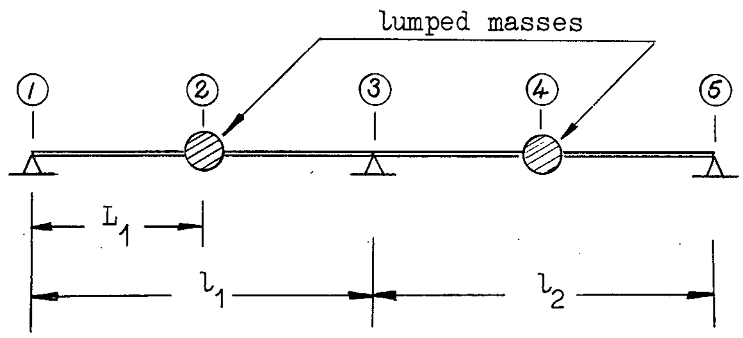


Figure 5.11

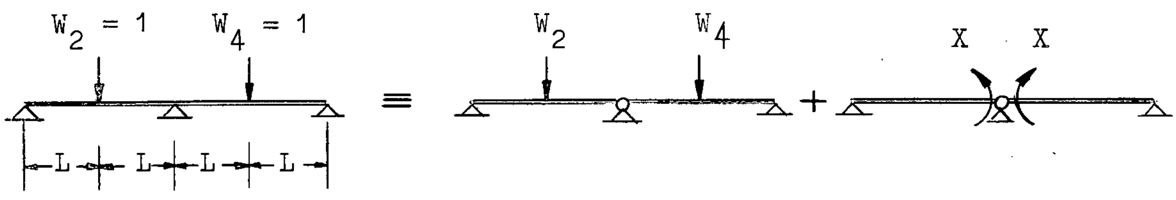
(1) The flexibility matrix for member ends

The flexibility matrix given in equation (5.8) above can be re-written in a reduced form, because the moments at the supports are zero and also because the moments at adjacent ends of the segments are equal, i.e.

$$F_m = \frac{L_i}{6\,EI}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

(2) The connection matrix B_o

The beam is made statically determinate by introducing a hinge at support 3 and the moment in the beam over the support 3 is taken as the redundant.



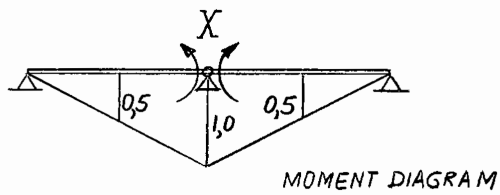
Hence

$$B_o = \frac{L}{2}\;\begin{array}{cc} W_2 & W_4 \\ \end{array}$$
$$\begin{array}{c} \\ \end{array}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{array}{c} M_2 \\ M_3 \\ M_4 \end{array}$$

(3) The connection matrix B_1

Now the redundant moment at support 3 is applied, which gives the B_1 matrix (see sketch)

$$B_1 = \begin{array}{c c} X & \\ \begin{bmatrix} 0{,}5 \\ 1 \\ 0{,}5 \end{bmatrix} & \begin{array}{c} M_1 \\ M_2 \\ M_3 \end{array} \end{array}$$



(4) Calculating the displacements

$$D = B_1' \, F_m \, B_1$$

$$D = \frac{4\ L}{3\ EI} \qquad \therefore\ D^{-1} = \frac{3\ EI}{4\ L}$$

(5) Calculating the redundant X

$$X = -\,D^{-1}\, B_1' \, F_m \, B_o$$

$$X = -\frac{3}{16}\, L \,[1 \quad 1]$$

(6) Calculating the connection matrix B

$$B = B_o + B_1 X$$

$$B_1 X = -\frac{3 \cdot L}{32} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{bmatrix}$$

$$B = \frac{L}{32} \begin{bmatrix} 12 & -3 \\ -6 & -6 \\ -3 & 13 \end{bmatrix}$$

(7) Calculating the flexibility matrix

$$F = B_o' \, F_m \, B$$

$$= \frac{L^3}{192\ EI} \begin{bmatrix} 23 & -9 \\ -9 & 23 \end{bmatrix}$$

(8) Calculating the system matrix

$$A = F.M \quad \text{where } M = M_1 \begin{bmatrix} 0,5 & 0 \\ 0 & 0,5 \end{bmatrix}$$

$$\text{hence } A = \frac{1}{2} M_1.F$$

(9) Applying the basic iterative method

$$\text{with } L = \frac{l_1}{2} \quad \text{and trial eigenvector } y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\text{then } A = \frac{l_1^3 M_1}{3072\ EI} \begin{bmatrix} 23 & -9 \\ -9 & 23 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{l_1^3 M_1}{3072\ EI} \begin{bmatrix} 32 \\ 32 \end{bmatrix}$$

$$= 0,0104 \frac{l_1^3 M_1}{EI} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The first iteration does not improve the eigenvector, because in the initial guess the ratio of the amplitudes is already correct, and hence the eigenvalue is obtained, and

$$p^2 = \frac{1}{\lambda} = \frac{EI}{0,0104\ M_1\ l_1^3}$$

$$\text{and} \quad p = 9,8 \sqrt{\frac{EI}{M_1\ l_1^3}}$$

which is very close to the value given in section 5.2, where

$$p = \pi^2 \sqrt{\frac{EI}{M_1\ l_1^3}}$$

The above calculation process was programmed. The flow-diagram is given in Appendix C and the program WCONT in FORTRAN IV is listed in Appendix D. The following two pages give the computer printout for the above example.

The program WCONT contains the UNIVAC supplied subroutine JACMX, which determines all the eigenvalues and eigenvectors of a symmetrical matrix. The results for the fundamental mode are identical with those obtained by means of the basic iterative method and provide a good check. It should be mentioned that the subroutine for the basic iterative method had to be altered slightly. This was because in the case of a two-span

```
SPAN  1        LUMPED MASSES
                                      .500000+00

               LENGTH OF SEGMENTS
                                      .500000+00
                                      .500000+00

               RIGIDITY VALUE         .100000+01

SPAN  2        LUMPED MASSES
                                      .500000+00

               LENGTH OF SEGMENTS
                                      .500000+00
                                      .500000+00

               RIGIDITY VALUE         .100000+01

MATRICES FOR THE
CONTINUOUS BEAM

               CONNECTION MATRIX 30
                                      .250000+00   .000000
                                      .000000      .000000
                                      .000000      .250000+00

               FLEXIBILITY MATRIX
               FOR MEMBER ENDS
                                      .333333+00   .833333-01   .000000
                                      .833333-01   .333333+00   .833333-01
                                      .000000      .833333-01   .333333+00

               CONNECTION MATRIX EI
                                      .500000+00
                                      .100000+01
                                      .500000+00

               D-MATRIX
                                      .666667+00
```

```
          INVERSE OF D
                                   .150000+01

          FLEXIBILITY MATRIX F
                                   .149740-01   -.585938-02
                                  -.585938-02    .149740-01

          MASS MATRIX
                                   .500000+00    .000000
                                   .000000       .500000+00

RESULTS FROM BASIC ITERATIVE METHOD, AFTER 20 ITERATIONS

          EIGENVALUE               .104167-01
          EIGENVECTOR
                                   .100000+01
                                  -.100000+01

          FUNDAMENTAL FREQUENCY    .979796+01

RESULTS FROM MATH-PACK PROGRAM JACMX WHICH DETERMINES EIGENVALUES AND EIGENVECTORS
(ONLY FOR SYMMETRICAL SYSTEMS MATRICES)

          THE MATRIX CONTAINS
          THE E-VALUES ALONG
          THE DIAGONAL
                                   .104167-01    .000000
                                   .000000       .455729-02

          THE MATRIX CONTAINS
          THE E-VECTORS
                                   .707107+00    .707107+00
                                  -.707107+00    .707107+00

@FIN
```

continuous beam the process did not always converge to the largest eigenvalue and eigenvector but to the second largest eigenvalue and its corresponding eigenvector. Again it should be noted that the program JACMX applies only to symmetrical matrices and hence its results cannot be used in the case of continuous beams with uniform mass but unequal spans if each span is divided into the same number of sections. To illustrate this, consider the beam-structure shown in fig. 5.12

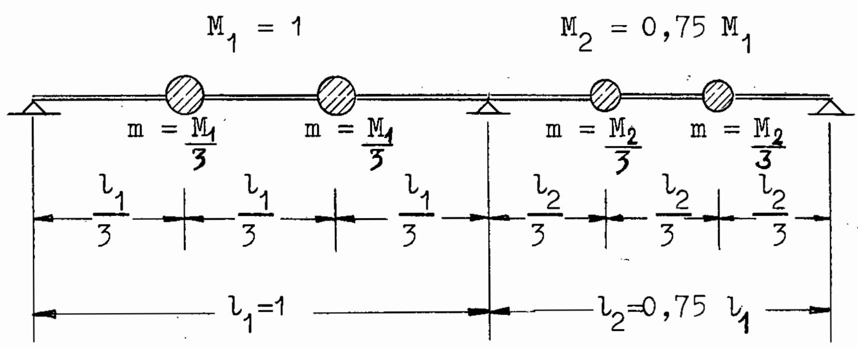


Figure 5.12

The overall mass matrix for the above structure has the form:

$$M = \begin{bmatrix} 0,333 & 0 & 0 & 0 \\ 0 & 0,333 & 0 & 0 \\ 0 & 0 & 0,25 & 0 \\ 0 & 0 & 0 & 0,25 \end{bmatrix}$$

And the system matrix F.M will not be symmetrical because the elements of the mass matrix along the diagonal do not all have the same values.

The program WCONT is used to analyse the two types of beam-structures discussed previously, i.e. continuous beams with two equal and two unequal spans. The results for the fundamental frequency of a continuous beam with two equal spans, where each span is subdivided into $n = 2, 3, 4$ or 6 sections, are plotted in fig. 5.13. In this case both types of mass-lumpings give identical results. In fig. 5.14 the results for a continuous beam having unequal spans in the ratio 1 : 0,75 are plotted, and it will be noted that the two types of mass-lumpings lead to different graphs. With increasing number of sections per span both models converge towards a value which is larger than Timoshenko's result. In the case of two equal spans the coefficient $p/\sqrt{EI/M_1L_1^3}$ is equal to $9,8696 = \pi^2$,

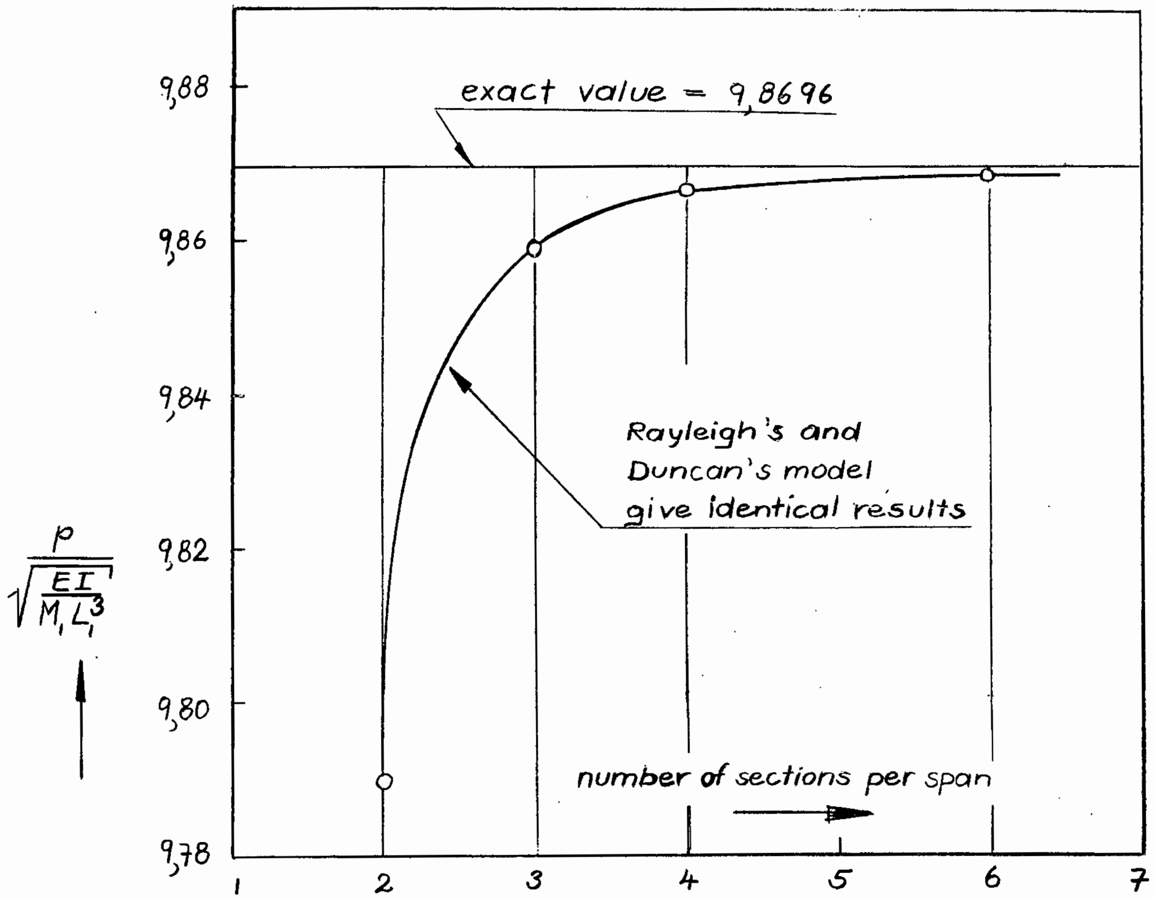


Figure 5.13 - Dependence of fundamental frequency on number of sections for continuous beam with two equal spans.

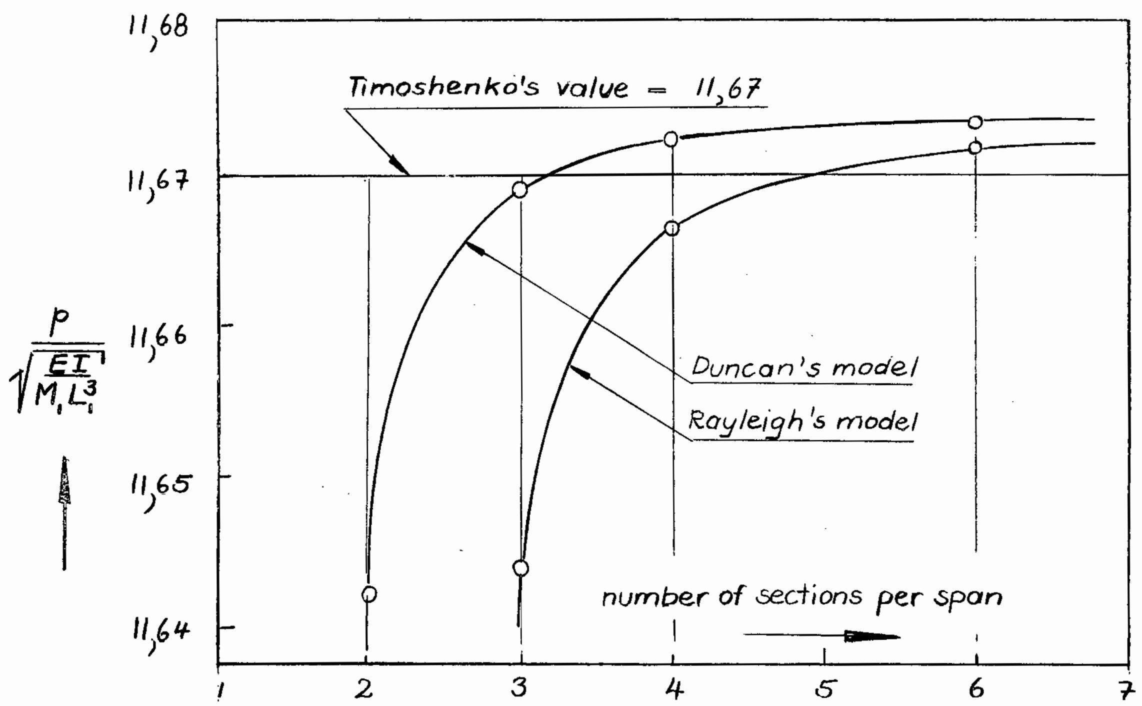


Figure 5.14 - Dependence of fundamental frequency on number of sections for continuous beam with two unequal spans (ratio of spans = 1: 0,75)

which is definitely an exact result. In the case of two unequal spans the coefficient, equal to 11,67, is obtained from Timoshenko's graph (fig. 5.1) and hence is probably subject to inaccuracies of the graph. The error, however, is not greater.

Section 5.6 - Summary and Comments

Comparing the methods given in this chapter, which are applied only to two-span, continuous beam structures, the following can be said:-

(i) Timoshenko's graphical solution is the easiest and quickest. It can be applied to equal and unequal spans and provided the graph (fig. 5.1) is extended, higher frequencies may also be obtained. Timoshenko's theoretical solution can be adopted for any number of spans but the trial and error approach becomes more complicated. A disadvantage is that a variation in mass or flexural rigidity cannot be considered.

(ii) Dunkerley's method is also very easy; only the determination of the required deflection may be lengthy. It can be applied to any type of continuous beam including cantilever ends. The results are, however, not very close to the exact values, but they constitute a lower bound solution. As in Timoshenko's method a variation in mass or flexural rigidity cannot be included.

(iii) The transfer matrix method does not lead to satisfactory results and further investigation is necessary.

(iv) The use of the flexibility matrix is very laborious especially if the spans are subdivided into several sections. This leads to large matrices and since many matrix operations are required the evaluation of the fundamental frequency is a lengthy procedure. Nevertheless, this approach has several advantages:

1) the method can be used for continuous beams with varying mass and flexural rigidity;

2) for more than three sections per span the results for the

fundamental frequency is very good in the case of a two-span continuous beam (see figs. 5.13 and 5.14);

4) the method gives the mode shape of the fundamental frequency, and provided appropriate numerical methods are applied, higher frequencies and their associate mode shapes may be obtained.

The compilation of the program WCONT required a considerable amount of time. The program, however, has the advantage of allowing for variations of mass, span lengths, number of spans, and it can easily be extended to take into account a variation in flexural rigidity.

Hence, the use of the flexibility matrix constitutes a method which leads to very good results though the effort required to produce a computer program might be prohibitive.

CHAPTER 6

DYNAMIC BEHAVIOUR OF HIGHWAY BRIDGES

Section 6.1 - Introduction

The main object of the thesis is the investigation of methods used to determine free vibrations of beam structures. Estimating the natural frequencies is, however, only the first step in the design calculations. The knowledge of the fundamental and higher frequencies is valuable in so far as it indicates which external frequency or frequency range must be avoided in order not to cause resonance. The next step in the vibration analysis is to consider the effect of a dynamic loading on the response of a structure. Many more factors, however, also influence the dynamic behaviour of the structure, such as:-

(i) bridge characteristics:

- (a) geometry,
- (b) material (steel, concrete, composite),
- (c) dynamic state (at rest or already vibrating);

(ii) vehicle characteristics:

- (a) type (axle-mass distribution),
- (b) speed,
- (c) dynamic state (at rest or vibrating);

(iii) road characteristics:

- (a) roughness on approach pavement,
- (b) roughness on bridge pavement,
- (c) irregularities (settlement near abutments).

In the case of railway bridges the dynamic forces, which are mainly periodic, can be defined accurately, whilst in the case of highway bridges the assessment of the dynamic forces is more problematic. A modern truck on its springs and tyres, and with its shock absorbers, is a very complex elastic system. Joints, bumps and other forms of pavement roughness may cause it to move in any one mode of vibration, or simultaneously in several modes.

In Chapter 2 we dealt briefly with general types of dynamic loading.

In this chapter the live loads are discussed which produce dynamic effects in highway bridges. Furthermore, other important factors influencing the dynamic responses are considered.

Section 6.2 - Impact Factors

A moving load causes deflections and stresses in a bridge which differ from those which would result from the same load applied statically. In order to ensure structural safety and freedom from vibrations which might cause discomfort and apprehension to pedestrians or static motorists, highway bridge designers took account of the dynamic stresses by adding to the dead and (static) live-load stresses an "impact" fraction of the latter. With the advent of new forms of construction, higher strength steels, welding and prestressed concrete, it became necessary to set more rational rules to ensure the economical use of these developments. The impact factors have been reduced appreciably since 1950. British Standard 153, now includes a factor I of 1,25 to be applied to H.A. static loading whilst the heavy H.B. loading is applied with no allowance for impact and using increased permissible stresses. The American Specification for highway bridges (AASHO 1961) requires an impact allowance for dynamic, vibratory and impact effects and gives the formula:

$$I = 1 + \frac{50}{L + 125}$$

where L is the length (in feet) of the span which is loaded to produce maximum stress in the member. The impact factor must not exceed 30 percent. The German Specification for the calculation of steel bridges (DIN 1073) adopts an impact factor

$$I = 1 + \frac{4,5}{L + 5} \qquad (1,04 \leqslant I \leqslant 1,64)$$

where L is in metres. The Hungarian Specification for highway bridges uses

$$I = 1,05 + \frac{5}{L + 5} \qquad (I \leqslant 1,50)$$

where L is in metres. The Swedish Specification gives an impact factor of 1,40 which is applied to point loads only and does not depend on the span length. Uniformly distributed lane loading is applied without impact allowance. For bridges designed to carry any non-standard loading the percentage increase applied for impact is not only dependent on the length of the span but also on the speed of the vehicle.

Considerable attention is paid to the provision of road surfaces free of major corrugations or depressions on highway bridges but no variation in impact factor has been allowed with differing qualities of running surface.

Section 6.3 - An Alternative Approach

Considering the variety of standard loads to which these impact factors apply, no simple comparison of the factors is possible. These crude guides for impact are not related to the frequency of a bridge but only to conditions that arise with vehicle~~s~~ of the type and with speeds currently used when moving over bridges ~~with~~ of average proportions. A more rational approach might be to design a bridge in such a way as to ensure that its natural frequencies are well in excess of the forcing frequencies of the vehicles. Highway vehicles have no large out-of-balance reciprocating components; vibrations are mainly caused by vertical oscillation of the vehicle on its springs, by impact due to irregularities of the road surface or by successive application of axle loads of a multi-axle truck. As a design guide, it has been commonly assumed that provided a bridge has a fundamental frequency of at least 4,5 cps, resonance is unlikely, although 6,5 cps has also been recommended (see ref. 27). Research on British military vehicles showed that a complete vehicle vibrates on the suspensions at between 1 and 3 cps.

Repeated application of axle loads may also augment the bridge motion considerably if the frequency of passage of truck axles is close to a natural frequency of the bridge. This particular frequency depends on the speed v of the truck and the distance d between the axles, i.e.

$$f = \frac{v}{2\pi d} \quad \text{(cps)}$$

Section 6.4 - Limitation of Deflection

The standard method of ensuring high natural frequencies of the structure is to restrict the "static" deflection under live load, usually including impact. The British Ministry of Transport imposes no limit but the AASHO Specification restricts live load deflection to L/800. A more severe limit, recently proposed (AASHO) to allow for the possibility of vehicles bouncing on to short spans due to surface irregularities at the bridge abutments, restricts the deflection to

$$\Delta/L = 0{,}0075\ (2 - e^{L/25}).$$

The German Specification (DIN 1037) permits a value of $\Delta = L/500$ under live load applied without impact.

It has also been suggested to limit the span/depth ratio for highway bridges in order to conform to a lower frequency limit of 3 cps. Due to half the H.A. loading these span/depth ratios range from 53 to 20 for spans varying from 46 to 100 feet. Ref. ?

Section 6.5 - Limitation of Vertical Acceleration

"9/64" only

In order to avoid discomfort to pedestrians, vertical accelerations should be restricted. During the passage of a vehicle at 40 mph a limiting acceleration of 0,5 ft/sec^2 has been adopted as an acceptable acceleration (see ref. 19). For a bridge traversed by a uniformly distributed load the vertical acceleration based on the theory due to Inglis (20) is given by:

$$a = \frac{168\ v^2\ \Delta}{5\ \pi^3\ L^3} \qquad (\text{ft/sec}^2)$$

where v = velocity of uniform load (ft/sec)

Δ = static deflection due to the same uniformly distributed live load (ft)

L = span (ft)

Section 6.6 - Literature Review

During the last 20 years the behaviour of highway bridges under the passage of heavy vehicles has been the subject of numerous investigations. A partial list of recent studies is given in the bibliography from (22) to (38); which were available to the writer. The most systematic and comprehensive investigations have probably been conducted at the University of Illinois since 1950. In this section a brief survey of some of the papers is given in chronological order.

6.6.1 - Highway Research Board

The Highway Research Board Bulletin 124 contains six papers under the heading "Vibration and Stresses in Girder Bridges". These reports contain a substantial amount of impact data of an exploratory nature and call attention to some important factors. The paper by Biggs and Suer (22) presents results of field measurements of dynamic midspan deflections of simply supported, single-span bridges due to the passage of a heavy vehicle. The measurements are limited to deflections, because once it is possible to predict dynamic deflections the increase in stress may be determined. The tests were carried out on five bridges, two of which were RSJ bridges and the remaining were plate-girder bridges. The following classification with regard to the causes of vibration is given:

(1) the simple passage of an unsprung mass load on a smooth road surface.- In this case the free vibrations are superimposed on the static or crawl deflection. The frequency of this vibration is essentially the same as the natural frequency of the bridge. However, due to the mass of the load, the natural frequency changes continually as the vehicle crosses the span; but generally the maximum variation in natural frequency does not exceed 20 percent. This type of excitation causes only small amplitudes;

(2) the vertical motion of the vehicle on its springs which is induced on approach. - This cause of vibration is very important. The vertical oscillation of the vehicle mass on its suspension system results in a periodic variation in the applied force. Theoretical solutions for this case indicate that the predominant frequency of the vibration is the same as the frequency of the alternating force rather than the fundamental frequency of the bridge. The amplitudes of these vibrations depend on

(a) the magnitude of the alternating force,

(b) the ratio between the frequency of the force and the frequency of the structure. (Resonance occurs if the ratio approaches one).

Bridge amplitudes resulting from vehicle springing can be much larger than those caused by a smoothly running load, case (1) above;

(3) irregularities in the road surface of the bridge. - In the case of an unsprung vehicle this results in a large and suddenly excited vibration. Comparing this with section 2.4.4.B it can be assumed that the amplitudes depend on both the duration of the impact and the period of the

structure. The presence of springs generally reduces the magnitude of vibration. The authors (22) believe that this cause of vibration is not very important for modern highway bridges.

The natural frequencies of the five bridges were obtained from the residual vibration which continues after the vehicle has left the span and it was found that for all five bridges the first mode of vibration was predominant. When testing the stationary vehicle, which was used in the test series, its natural frequency was found to be 3,1 cps. However, the frequency of the moving vehicle varies and depends upon the roughness of the roadway. From the various tests carried out is was concluded that the frequency of vibration of the bridge, at least during the first part of the crossing, is related to the natural frequency of the vehicle. Furthermore, it was found, that the type of suspension system has a marked influence on the amplitude of bridge vibration; the vibration due to an unsprung vehicle being much more severe than that of a sprung vehicle. Since the amplitude of vibration was found to be much larger than that which would be caused by a smoothly running load, it was concluded that vertical oscillation of the vehicle as it approaches the span is of primary importance. - The repetition of axle passages as a cause of vibration is not mentioned; also, the speed of the vehicle is not reported to have any influence on the response.

<u>Hayes and Sbarounis</u> (24) present a report on a vibration study of a three-span continuous RSJ highway bridge with a composite concrete deck slab. Considerable vibrations had been observed in some structures of this type. Data was collected in order to investigate the seriousness of the possible vibrations in such structures. The test results show that the natural frequency of vibration of the bridge varies with the amount of interaction between the concrete slab and the steel girders, which depends upon the capacity for horizontal shear transfer between the slab and the girders. If no shear connectors are present, this capacity for shear transfer depends on bond and/or friction.

The frequency of application of axle loads at a point was considered to be the frequency of a repetitive dynamic force at that point on the bridge. This is a function of vehicle speed and axle spacing, as was mentioned in section 6.3. The authors found that resonance occurs if the frequency of application of axle loads at a point coincides with the natural frequency of the bridge with the mass at that point. It should be remembered that the natural frequency of the bridge varies as the load moves across the structure. It is suggested that in order to prevent undesirable vibrations, the fundamental frequency of the bridge, including the effect of the mass of the live load, should be greater than the frequency of applications of axle loads based on a reasonable maximum speed and axle spacing.

Because the fundamental frequency is proportional to the square root of the moment of inertia, it is of great importance to assess the latter value correctly. A simplified analysis of the computation of moment of inertia of the transformed section due to partial interaction of the concrete slab with the steel girders is presented.

Foster and Oehler (25) report on the vibration and deflection characteristics of an eight-span plate-girder bridge and a six-span RSJ bridge. The plate-girder bridge consists of five simple spans and a three-span continuous beam, whilst the RSJ bridge has only simply supported spans. It was found that the observed amplitudes and duration of vibration increase with span flexibility, as might be expected. An increase in dynamic deflection was observed when the time interval between axles passing a given point on the span was close to the fundamental period of vibration of the span. This finding is in accordance with that of the paper by Hayes (24), i.e. the type of truck and its axle spacing, in conjunction with its speed, do have an effect on bridge vibration. It is concluded that in order to control bridge vibration, the fundamental frequency of the bridge span should be limited and on the basis of the test data a lower limit of 6,5 cps is suggested.

The fundamental frequencies of the simple spans of the highway bridges were calculated, using the equation(3.5) of Chapter 3. In calculating the theoretical frequency, an effective cross-section was used which included the girders and 50 percent of the area of the concrete deck, which was considered to act compositely with the girders. The values obtained were in good agreement with the experimental data.

The paper by Tung et al (26) is a review of analytical and experimental research on the highway bridge impact problem which was carried out at the University of Illinois. It was intended to assess the relative importance of the various factors that influence the dynamic stresses in a highway bridge.

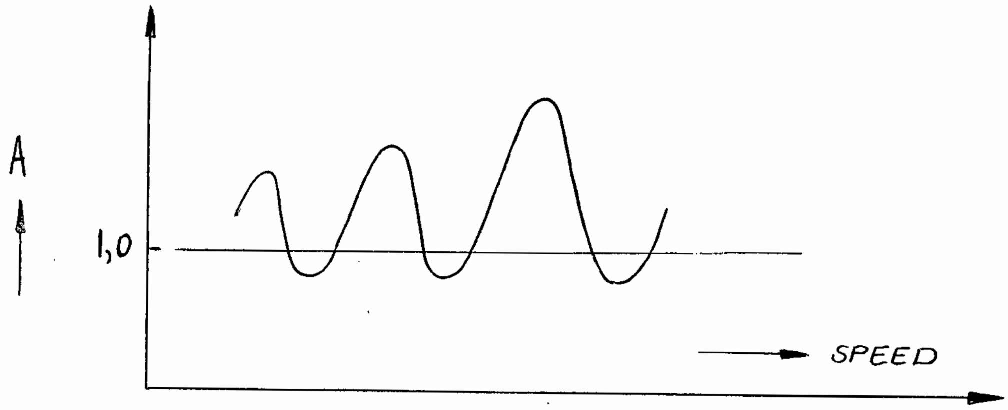
Because of the very complex dynamic behaviour of the actual bridge-vehicle system, assumptions and simplifying approximations were necessary to make the solution of the problem possible. Three idealized types of loading were considered. In all cases the load moved at a constant speed across a simply supported flexible beam and the three types of loading are given in the order of increasing importance:

(1) a single unsprung mass,
(2) a single sprung mass, with zero deflection and bouncing velocity, when it enters the bridge,
(3) a sprung mass, oscillating with some definite amplitude such that it produces the worst possible condition in the beam.

The derivation and solution of the equations of motion will not be discussed, but the graphs presented in the paper seem to indicate a good agreement between most of the theoretical and experimental results which were carried out on model structures. It should however be mentioned that the authors did not find the analytical treatment given by Inglis (20) and Timoshenko (11) adequate for the problem at hand.

Owing to the presence of pavement joints, irregularities in the approaches and settlement at the bridge abutments, etc. the sprung part of the vehicle will not be motionless as it enters the span. It is much more realistic to assume that the vehicle has an initial displacement and bouncing velocity. Because there is some difficulty in selecting appropriate values for these variables the authors chose the initial condition for the sprung mass such as to cause approximately the worst possible condition with regard to the resulting stresses in the beam.

It was found that for both sprung and unsprung vehicles with zero initial conditions there is a good deal of variation in bending moment with increasing speed and graphs of the following form were obtained:-



where A = ratio of maximum dynamic moment at midspan based on maximum moment at midspan.

However, in the case of a sprung load with initial displacement and bouncing velocity as it enters the bridge, the increase in bending moment was found to be much larger than for an unsprung or sprung mass with no initial vibratory motion. This applied for all speeds.

The studies reported in the paper (26) did not take into account a possible motion of the bridge at the time the vehicle enters the span, which may have arisen from the passage of previous vehicles. Such oscillation can cause a further increase in bending moment and stresses.

6.6.2 - American Society of Civil Engineers

In 1957 Biggs et al (27) published an analytical method by which the magnitude and character of highway bridge vibration due to the passage of heavy vehicles can be predicted. Certain assumptions were made, such as:

(i) the bridge is represented as a simple beam, of which only the fundamental mode is considered. Thus, a single-degree-of-freedom system is assumed for the bridge;

(ii) the vehicle is treated as a single-degree-of-freedom system, and the entire mass is spring supported.

The differential equations of motion for the sprung mass and for the bridge were solved by numerical integration.

Laboratory tests on model bridges and field tests were carried out in order to investigate both the accuracy of the analytical methods and the validity of the assumptions made. The proposed method was found to be satisfactory and hence the authors concluded that for the computation of midspan deflections the bridge may be approximated by a single-degree-of-freedom system. Damping in both the bridge and the vehicle were neglected, because damping is generally small in highway bridges and vehicle damping appears to complicate the analysis unnecessarily, with the exception of long spans.

It is concluded that the initial oscillation of the vehicle on its suspension system is the major cause of vibration and that for design purposes the effect of surface roughness on the bridge can be neglected. This is in agreement with the findings by Tung (26). Furthermore, it was pointed out that the ratio of vehicle mass to bridge mass is of importance for the case when the natural frequency of the vehicle is close to the natural frequency of the bridge, i.e. an increase in the mass ratio causes an increase in bridge amplitude.

Oehler (28) investigated the vibration susceptibility of various highway bridge types and the following grouping was used:-

(1) single-span RSJ bridges, designed with and without composite action,

(2) continuous-span bridges of steel girder and reinforced concrete type,

(3) cantilever girder bridges, designed with and without composite action.

Measurements on all bridge types seemed to indicate that bridges with lower natural frequencies are subject to larger amplitudes of vibration and vice versa. Test results of bridges falling in group (2) showed:

(a) reinforced concrete bridges vibrated with smaller amplitudes than steel girder bridges;

(b) the duration of vibration was considerably larger for the girder bridges than the reinforced concrete bridges;

(c) on average, reinforced concrete bridges had high values for the fundamental frequency, i.e. 7,6 cps (compared with 5,5 cps for ~~girder~~ Steel bridges;

(d) the deflecti~~ons of~~ reinforced concrete bridges were less than those of other types.

In comparison with the other two bridge groups it was concluded that cantilever bridges

(a) were the most flexible type,

(b) had the smallest values for the fundamental frequency,

(c) had the longest duration of vibration,

(d) had the largest amplitudes of vibration.

i.e. cantilever bridges are most susceptible to vibrations.

Oehler is of the opinion that blanket deflection limitations, when applied to all three bridge groups, do not result in satisfactory stiffnesses for each type. A thorough study of the cantilever bridges is necessary in order to propose a method which controls vibration susceptibility of this type of highway bridge.

A fairly complete bibliography on the investigation of the vibration problem in highway bridges was given in a report by the Committee on Deflection Limitations of Bridges of the Structural Division of the ASCE (29).

Looney (30) developed a method which is essentially the same as Biggs's method (27), i.e. the differential equation is written for the fundamental mode of the simply supported bridge and is solved numerically. The equations include the mass of both the load and the bridge. The main differences between the two procedures are:

(a) Looney assumed a smoothly running unsprung load whereas Biggs included the effect of vehicle springing,

(b) Looney considered multi-axle loads whereas Biggs approximated the vehicle only by a single-degree-of-freedom system.

Fleming and Rumualdi (32) determined the dynamic response for both single-span and multi-span continuous bridges. Factors influencing the response, such as the mass of the load, the mass of the bridge and vehicle springing were included in the analysis. The mass of the single-span bridge was approximated by three concentrated masses and Rayleigh's model was adopted for the mass-lumping. The authors were of the opinion that the assumption of a single-axle load is an over-simplification of the general type of loading which causes vibration. It was concluded that one of the most important considerations is the springing of the vehicle and the condition of the bridge approach. Extremely large impacts were observed due to initial vibration of the load which was caused by a settlement in the approaching roadway.

Wen and Veletsos (33) presented an analytical study which included most of the important factors regarding the dynamic response of a bridge which so far were only treated separately by the various authors. The factors considered were:

1) speed of the vehicle
2) spacing of the vehicle axles,
3) dynamic state of the vehicle,
4) dynamic state of the bridge,
5) roughness of the bridge surface.

In the analysis, the bridge is idealized as a single-degree-of-freedom system and the vehicle as a two-axle sprung load. It was found that:

1) the magnitude of the dynamic effects in a bridge increases with the vehicle speed,

2) the magnitude of the dynamic response is larger for short spans than for long spans,

3) relatively large dynamic effects occur when the time interval between the application of the two-axle load unit at a point is equal to the fundamental period of vibration of the bridge,

4) if the bridge is already in a state of oscillation when the vehicle enters the span, the dynamic response of the bridge can be amplified depending on the time of entry of the vehicle,

5) roughness of the bridge deck can cause large dynamic effects in highway bridges.

Damping of both the vehicle and the bridge was not considered in the analysis, and only single-span bridges were investigated.

Wen and Toridis (34) confirmed the findings by Oehler (28) with regard to cantilever bridges, i.e. this type of bridge is subject to considerably larger dynamic effects than simply supported or continuous span bridges. Whilst Oehler's paper was mainly a report on field observations, Wen and Toridis gave an analytical investigation of the dynamic behaviour of cantilever bridges.

Cantilever bridges were found to vibrate not only in the fundamental mode but also in the second and third modes. Hence the representation of the bridge as a single-degree-of-freedom system (which was possible for a simply supported beam), is not adequate and a multi-degree-of-freedom system should be used to represent cantilever bridges. Wen and Toridis recommend a five-degree-of-freedom system. The authors pointed out that vibration of the vehicle on its springs and surface roughness of the pavement were not included in the study and hence it is possible that the already considerable maximum dynamic effects would be even increased.

6.6.3 - University of Illinois

Walker and Veletsos (35) presented a very detailed study of simple-span highway bridges due to moving vehicles. Most factors affecting the dynamic behaviour of this type of bridge, which are mentioned so far in this Chapter were considered and in addition the effect of 'interleaf friction' in the suspension system of the vehicle was included. The most significant parameters in vibration problems were given as:

1) speed of the vehicle,
2) axle spacing,
3) weight and frequency ratios of bridge and vehicle,
4) vehicle suspension (damping),
5) dynamic state of vehicle.

The effect of surface irregularities was, however, not included. Mention was made that other investigators (32) found that bumps in the bridge pavement (idealized by sinusoidal shapes) can lead to large dynamic effects, but the authors (35) felt that this was not a very realistic approximation of such irregularities.

The study by Nieto-Ramirez and Veletsos (36) of the response of three-span highway bridges due to moving vehicles is as comprehensive as the one mentioned above (35). The bridge was idealized as a continuous beam and the vehicle was represented as a multi-axle sprung unit and again interleaf friction was taken into account. The effects of uncontrollable parameters such as the phase difference between the motions of the individual axles and the initial values of the interleaf friction in the suspension system were pointed out. The ratio of vehicle mass to bridge mass did not seem to have much influence on the response of a continuous bridge.

The interleaf friction in the suspension system of the vehicle was found to be an important source of energy dissipation and hence should not be neglected in the analysis. The dynamic effects computed on the assumption that the vehicle is a linearly elastic, undamped system may be extremely conservative.

Section 6.7 - Final Comments

This brief survey of some of the publications on the dynamic behaviour of highway bridges is intended to indicate the development of vibration analysis. These investigations consisted initially only of field measurements and subsequently the experimental data was compared with newly developed analytical approaches. The studies were aimed at a better understanding of the many factors affecting the dynamic response of highway bridges and the analyses become increasingly comprehensive but also more complicated. It is apparent that there is considerable scope for further research in this field.

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ASME	:	American Society of Mechanical Engineers
HRB	:	Highway Research Board

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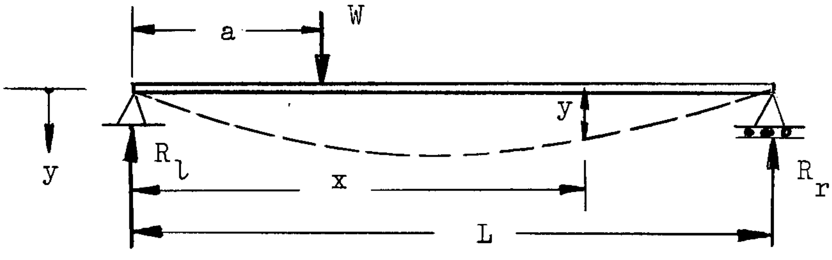
APPENDIX A

A.1 THE DEFLECTION EQUATION

In order to find the flexibility coefficients for a simply supported beam, we have to find the relationship between load and deflection of the beam shown below



$$R_l = W(1 - \frac{a}{L}) \qquad\qquad R_r = W\frac{a}{L}$$

From the condition $\Sigma M = 0$ we get

$$EI\frac{d^2y}{dx^2} = -W(1 - \frac{a}{L})\,x + [W(x-a)]_{x>a}$$

$$EI\frac{dy}{dx} = -W(1 - \frac{a}{L})\frac{x^2}{2} + [W\frac{(x-a)^2}{2}]_{x>a} + A$$

$$EI\;y = -W(1 - \frac{a}{L})\frac{x^3}{6} + [W\frac{(x-a)^3}{6}]_{x>a} + Ax + B$$

The constants A and B are found from the two end conditions $y = 0$ for $x = 0$ and $x = L$

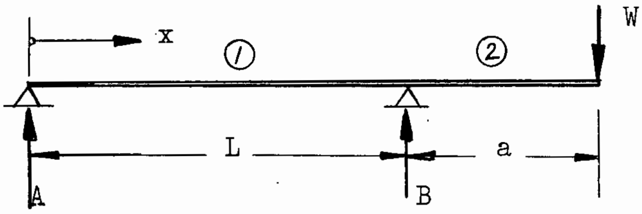
therefore: $B = 0$ and $A = \frac{Wa}{6L}(L - a)(2L - a)$

Hence

$$y = \frac{W}{6EI}\frac{1}{L}x(L - a)(2aL - a^2 - x^2) + \frac{W}{6EI}[x - a]^3$$

This is the equation for the deflected curve of the beam. The term in square brackets will be rejected if the quantity inside is negative.

A.2 DEFLECTION EQUATION FOR BEAM WITH CANTILEVER END



$$A = \frac{Wa}{L} \qquad B = W(1 + \frac{a}{L})$$

$$EI\, y_1'' = -M = -(-\frac{Wa}{L}x) = \frac{Wa}{L}x$$

$$EI\, y_1' = \frac{Wa}{L}\frac{x^2}{2} + C_1$$

$$EI\, y_1 = \frac{Wa}{L}\frac{x^3}{6} + C_1 x + C_2$$

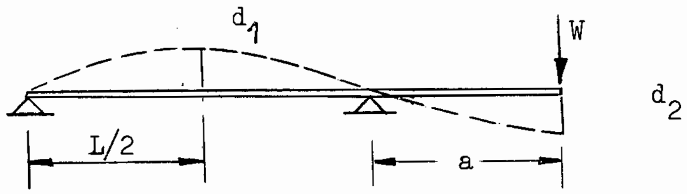
$$EI\, y_2'' = -[-\frac{Wa}{L}x + W(1 + \frac{a}{L})(x - L)]$$

$$EI\, y_2' = \frac{Wa}{L}\frac{x^2}{2} - W(1 + \frac{a}{L})\frac{(x - L)^2}{2} + C_3$$

$$EI\, y_2 = \frac{Wa}{L}\frac{x^2}{6} - W(1 + \frac{a}{L})\frac{(x - L)^3}{6} + C_3 x + C_4$$

Four boundary conditions are needed to determine the four constants.

at	$x = 0$	$y_1 = 0$	$\therefore C_2 = 0$
at	$x = L$	$y_2 = 0$	$\therefore C_4 = 0$
at	$x = L$	$y_1 = 0$	$\therefore C_1 = -\frac{WaL}{6}$
at	$x = L$	$y_1' = y_2'$	$\therefore C_3 = -\frac{WaL}{6}$



$$d_1 = -\frac{W}{EI}\frac{aL^2}{16}$$

$$d_2 = \frac{W}{EI}\frac{a^2}{3}(a + L)$$

A was obtained after substituting for B the following value, which follows from the two equilibrium conditions $\Sigma V = 0$, $\Sigma M = 0$:

$$B = P - A\left(1 + \frac{L_1}{L_2}\right) + \frac{W}{L_2}(L_1 - a)$$

Substituting B into the equation for y_3 and using boundary condition (7), A is obtained.

APPENDIX B

1. THE EIGENVECTOR EQUATION

A set of n simultaneous equations can be represented by the equation

$$AX = Y \tag{1}$$

where A is a square matrix of order n,
X is a column matrix having n unknowns,
Y is a column matrix of order n.

In general the column matrix Y will not have the same direction or magnitude as the vector X. However, if the vector Y has the same direction as the vector X, i.e. if $Y = \lambda.X$, where λ is a scalar quantity, then the equation (1) becomes

$$AX = \lambda.X \tag{2}$$

Equation (2) is called the "eigenvector equation". It represents the eigenvalue problem, in which the solution of the simultaneous equations is a vector X, such that the product A.X is a scalar multiple of X itself. Equation (2) can be rewritten as a homogeneous equation.

$$(A - \lambda I)\, X = 0 \tag{3}$$

This set of linear homogeneous equations only has a non-trivial solution if the determinant of its coefficients is equal to zero; if

$$\det\,(A - \lambda I) = 0 \tag{4}$$

When this determinant is expanded, an algebraic equation in λ is obtained, which is known as the "characteristic equation" of the matrix A or as "frequency equation" in vibration problems.

2. THE EIGENVALUES AND EIGENVECTORS OF A nTH ORDER SQUARE MATRIX

The roots $\lambda_1, \lambda_2 \ldots\ldots \lambda_n$ of the eigenvector equation are known as the eigenvalues of the matrix A.

To each eigenvalue corresponds a solution vector which is called eigenvector. In general, to find the associated eigenvector X, the eigenvalue is substituted in the equation (3) which leads to a set of n linear simultaneous equations. However, because the determinant

$(A - \lambda I)$ is equal to zero, this homogeneous system of n linear simultaneous equations has only $(n - 1)$ independent equations with n unknowns. Hence one of the equations can be discarded and one of the remaining variables may be given an arbitrary value. It is therefore obvious that the elements in the eigenvector have only relative values.

Frequently the first element or the largest element of the eigenvector is chosen to be equal to unity. Alternatively the value of one of the elements can be chosen such that the vector has unit length, e.g.

$$x_1 = \frac{1}{\sqrt{x_1^2 + \dots + x_n^2}}$$

The eigenvector is said to be normalized.

The solution of the polynomial, i.e. the characteristic equation presents no theoretical difficulties. However, if n is even moderately large, the amount of work becomes excessive and an alternative method not requiring the expansion of the determinant is necessary and will be given later.

Before continuing, the following numerical example is given to illustrate what is stated above. Consider the equations

$$\begin{aligned} 3\,x_1 + 2\,x_2 &= \lambda . x_1 \\ 1\,x_1 + 2\,x_2 &= \lambda . x_2 \end{aligned} \qquad (a)$$

This is the eigenvector equation $A.X = \lambda.X$ where A is the matrix:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

This set of equations has a non-zero solution only if the determinant of its coefficients is equal to zero, i.e.

$$\det \begin{vmatrix} 3 - \lambda & 2 \\ 1 & 2 - \lambda \end{vmatrix} = 0$$

The expansion of the determinant results in the characteristic equation

$$\lambda^2 - 5\,\lambda + 4 = 0$$

The roots or eigenvalues of this equation are

$$\lambda_1 = 4$$

$$\lambda_2 = 1$$

If $\underline{\lambda = 4}$ is substituted in equations (a) they become

$$-x_1 + 2x_2 = 0$$

$$x_1 - 2x_2 = 0$$

Either x_1 or x_2 can be fixed arbitrarily. If $x_1 = 1$, then $x_2 = 0{,}5$

For $\underline{\lambda = 1}$, the equations become

$$2x_1 + 2x_2 = 0$$

$$x_1 + x_2 = 0$$

Assuming $x_1 = 1$, then $x_2 = -1$.

The vectors $X_1 = \begin{bmatrix} 1 \\ 0{,}5 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

are the eigenvectors corresponding to the eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 1$ respectively.

Normalising the eigenvectors such that each vector has unit length, we calculate the first elements as follows:-

for λ_1 $$x_1 = \frac{1}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{\sqrt{1 + 0{,}25}} = \frac{1}{\sqrt{5}}$$

for λ_2 $$x_1 = \frac{1}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{\sqrt{1 + 1}} = \frac{1}{\sqrt{2}}$$

Hence the normalised vectors are

$$X_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \qquad X_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

The eigenvalues fulfill two conditions that can serve as valuable checks:

1) $(\lambda_1 + \lambda_2 + \ldots\ldots + \lambda_n) = \text{tr } A$ (5)

where tr A is the trace of the matrix A which is defined as $\sum_{i=1}^{n} a_{ii}$.

In the above numerical example we have

$$3 + 2 = 4 + 1$$

and the check is satisfied.

2) $(\lambda_1 \times \lambda_2 \times \ldots\ldots \times \lambda_n) = \det(A)$ (6)

In the above numerical example we have

$$(4) \times (1) = (3) \times (2) - (1) \times (2)$$

and the check is satisfied.

From check (2) it follows that if det (A) = 0 so that A is a singular matrix, then at least one of the eigenvalues is zero.

3. PROPERTIES OF THE EIGENVALUES AND EIGENVECTORS OF A SQUARE MATRIX

In this section several of the most important properties of the eigenvalues and eigenvectors of a square matrix A are discussed. These properties are of theoretical and practical importance and will be needed for the following section.

When the proof of a theorem is either relatively unimportant or too long for inclusion in the script, adequate reference will be made to the literature.

THEOREM 1

If all eigenvalues of a matrix A are different, then the eigenvectors are different and linearly independent. That is, it is not possible to obtain any one of the eigenvectors by adding together simple multiples of the other eigenvectors in the form

$$X_i = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_n X_n$$

Where there is such a set of linearly independent vectors, any arbitrary vector V_o may be expressed as a unique combination of the set, such that

$$V_o = \alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n.$$

Use of this theorem will be made in section B 4.1.2.

THEOREM 2.

If a matrix A is symmetric and has eigenvectors X_i and X_j that correspond to eigenvalues λ_i and λ_j, where $\lambda_i \neq \lambda_j$, then X_i and X_j are orthogonal vectors, i.e.

$$X_i' X_j = 0$$

The proof can be found in ref. (13).

THEOREM 3.

If an unsymmetric matrix A has eigenvalues λ_i and eigenvectors X_i, then its transpose has the same eigenvalues λ_i and eigenvectors U_i which are orthogonal to X_i. That is

$$U_j' X_i = 0 \qquad \text{when } i \neq j$$

The proof is given in ref. (14).

This is an important property when the eigenvalues of an unsymmetric matrix are required.

THEOREM 4.

If a matrix A is symmetric and all its elements are real (not complex) numbers, then all its eigenvalues and eigenvectors are real.

This is not an obvious fact for the roots of an nth order equation with real coefficients are, in general, complex numbers.

The proof of this theorem is given in ref. (13).

THEOREM 5.

If the matrix A is a real, skew-symmetric matrix, so that $A' = -A$, all its eigenvalues are purely imaginary. This can be shown similarly as above.

THEOREM 6.

If the eigenvalues of a matrix A are $\lambda_1, \lambda_2, \ldots \lambda_n$ then the eigenvalues of $\underline{A^m}$ are $\lambda_1^m, \lambda_2^m, \ldots \lambda_n^m$ and its eigenvectors are the same as for A.

Proof: as before $A X_i = \lambda_i X_i$

premultiplying both sides by A gives

$$A^2 X_i = \lambda_i A X_i = \lambda_i \lambda_i X_i$$
$$= \lambda_i^2 X_i$$

This process can be repeated to give

$$A^m X_i = \lambda_i^m X_i$$

The theorem is true for all integer values of n, positive and negative, provided λ_i is not zero.

Note: This proof is important for the basic iterative method and the matrix squaring method in B 4.1

THEOREM 7.

If the eigenvalues of a matrix A are λ_i and its eigenvectors are X_i, then the eigenvalues of the matrix $\underline{B = T^{-1}AT}$ are the same λ_i and its eigenvectors are $T^{-1}X_i$, T being any non-singular square matrix.

Such a transformation is also called a "<u>similarity transformation</u>". The matrices A and B are said to be similar matrices, and similar matrices always have the same set of eigenvalues. This theorem is important for finding the eigenvalues other than the first.

THEOREM 8.

The <u>reciprocals</u> of the eigenvalues of a matrix A are the eigenvalues of A^{-1}, and the eigenvectors of A are the eigenvectors of A^{-1}.

Proof: Consider the eigenvector equation

$$AX = \lambda X$$

premultiplying by A^{-1} gives

$$A^{-1} AX = IX = \lambda A^{-1} X$$

this equation may be written in the form

$$A^{-1}X = \lambda^{-1}X. \tag{7}$$

Equation (7) is the eigenvector equation of A^{-1}; therefore λ^{-1} is the eigenvalue of A^{-1}, and the eigenvectors of A or the eigenvectors of A^{-1}.

THEOREM 9.

Every matrix A satisfies its own characteristic equation. (Cayley-Hamilton Theorem). For the proof see ref. (14); a numerical example is given, to demonstrate the theorem.

Consider the matrix used earlier $\quad A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

The characteristic equation is $\quad \lambda^2 - 5\lambda + 4 = 0,$

therefore $\quad A^2 - 5A + 4I = 0.$

Summing $\quad A^2 = \begin{bmatrix} 11 & 10 \\ 5 & 6 \end{bmatrix} \quad 5A = \begin{bmatrix} 15 & 10 \\ 5 & 10 \end{bmatrix} \quad 4I = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

gives a total $= 0$

4. ITERATIVE METHODS FOR EIGENVALUES AND EIGENVECTORS.

In many engineering applications the matrix A is a real matrix, and frequently also a symmetric matrix. This section will deal with iterative methods for determining the eigenvalues and their associated eigenvectors for a real matrix. However, it is interesting to note that the theory is much simpler in the case of a symmetric matrix than in that of a non-symmetric matrix.

Similar to the solution of linear equations the methods of obtaining the eigenvalues fall into two groups:

(a) direct method,
(b) indirect method.

The decision of which to use, does not depend so much on the form of the matrix, as on what solutions are required. If only one or a few solutions are needed, an iterative (or indirect) method may well be applied, whereas if all or most of the eigenvalues and eigenvectors are required, one of the direct methods is more suited.

In engineering problems such as the vibration of a beam, the size of the matrix A will depend on the approximation of the system containing a finite degree of freedom (lumped system) to the real structure. The order of A will increase with the desired accuracy for any of the eigenvalues, i.e. with the number of degrees of freedom. However, in general, only the first three eigenvalues are of practical interest for structural work. Hence in most cases iterative methods are best suited, and only they are discussed in the following paragraphs.

4.1 DETERMINATION OF THE LARGEST EIGENVALUE

4.1.1 - Matrix Squaring Method

This is a fairly quick method to obtain the largest eigenvalue, but it is not suited for hand calculation. The proof can be found in ref. (14) and an example is given in the following paragraph, to illustrate the method.

If $B = A^m$ and $C = A^{m+1}$ then

$$\lambda_1 = \frac{c_{ij}}{b_{ij}}$$

where c_{ij} is any element of the C matrix and
b_{ij} is the corresponding element of the B matrix.

Numerical example:

Consider the same matrix A as in the previous example:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} 43691 & 43690 \\ 21845 & 21846 \end{bmatrix}$$

$$A^9 = \begin{bmatrix} 174763 & 174762 \\ 87381 & 87382 \end{bmatrix}$$

$$\lambda_1 = \frac{c_{11}}{b_{11}} = \frac{174763}{43691} = 3{,}999\ 977$$

$$\lambda_1 = \frac{c_{12}}{b_{12}} = \frac{174762}{43690} = 4{,}000\ 046$$

$$\lambda_1 = \frac{c_{21}}{b_{21}} = \frac{87381}{21845} = 4{,}000\ 046$$

$$\lambda_1 = \frac{c_{22}}{b_{22}} = \frac{87382}{21846} = 3{,}999\ 908$$

The exact value is $\lambda_1 = 4$.

However, as mentioned earlier, this method is not suitable for sliderule calculation with respect to both the amount of arithmetic and the accuracy.

The following method is a modification of the matrix squaring method and reduces the amount of work considerably. Furthermore, the corresponding eigenvector is determined by this method.

4.1.2 - Basic iteration method (power method)

If the matrix A of order n is real and symmetric, all eigenvectors are linearly independent (see Theorem 1.) and any arbitrary vector V_o can be expressed as a linear combination of the n eigenvectors

$$V_o = \alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n$$

i.e. V_o can be regarded as being composed of contributions from each of the eigenvectors and with appropriate selection of the α's any V_o can be formed.

We assume that V_o is the starting vector for the iterative process and substitute it for X in the left hand side of eigenvector equation (2). This gives

$$A.V_o = \lambda.X = V_1.$$

A second vector V_1 is obtained which is again substituted, giving a third vector

$$V_2 = A\ V_1 = A^2 V_o$$

in general $V_m = A^m V_o$

$$= A^m\ (\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n)$$

Using the result of Theorem 6, i.e.

$$A^m X_i = \lambda_i^m\ X_i \qquad \text{for } i = 1,\ 2,\ \dots\ n$$

we get $V_m = \lambda_1^m \alpha_1 X_1 + \lambda_2^m \alpha_2 X_2 + \dots + \lambda_n^m \alpha_n X_n$

Since we know that all eigenvalues are real for real symmetric matrices (see Theorem 4) and furthermore we assume that no two eigenvalues have the same absolute value, we can divide by λ_1^m, so that

$$V_m = \lambda_1^m \left[\alpha_1 X_1 + \left[\frac{\lambda 2}{\lambda 1}\right]^m \alpha_2 X_2 + \ldots + \left[\frac{\lambda n}{\lambda 1}\right]^m \alpha_n X_n \right]$$

If $\lambda_1 > \lambda_2 > \lambda_n$ we see that the contributions of X_2 to X_n become of less and less importance as the number of iterations is increased and ultimately X_1 is completely dominant.

The rate of convergence depends partly on the constants α_1 to α_n, but more effectively on the ratios $|\lambda_2/\lambda_1|$, $|\lambda_3/\lambda_1|$ ... The smaller the ratios the faster will be the convergence.

It can happen that $\alpha_1 = 0$ in which case convergence is to X_2. However, provided that $\alpha_1 \neq 0$ then, no matter how small α_1 is, convergence will ultimately be to X_1, though it may appear otherwise in the initial stages of the iteration process. In practice we work with a fixed number of figures and successive iterations will introduce rounding errors. Due to these rounding errors α_1 will never be exactly zero and convergence will be towards X_1. For this reason the iterative process is less valuable for the computation of any but the most dominant eigenvalue and corresponding vector. In a later section (p. B.15) a method is given, referred to as "Rinsing Technique" which makes use of setting α_1 to zero whilst keeping the rounding errors small, thus evaluating the second eigenvalue and eigenvector.

The writer felt justified in explaining the method and its mechanism in greater detail because of the importance of this method.

<u>Example</u>:

Consider the same matrix A, and assume an arbitrary vector V_o. (The choice of the starting vector will only effect the rate of convergence).

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \qquad V_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{then } V_1 = A.V_o = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$V_2 = A.V_1 = A.A.V_o = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ 11 \end{bmatrix}$$

$$V_3 = A.V_2 = A.A.A.V_o = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 21 \\ 11 \end{bmatrix} = \begin{bmatrix} 85 \\ 43 \end{bmatrix}$$

The numbers of this calculation become larger and larger at each step and it is usual to divide each vector by its first, or its largest, element and to use the result for the next iteration. This does not only keep the numbers small but each successive divisor is an approximation to λ_1 and the convergence of the process can be watched.

In this form the calculation of the above becomes:

$$V_1 = A.V_o = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 5 \times \begin{bmatrix} 1 \\ 0,6 \end{bmatrix}$$

$$V_2 = A.V_1 = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0,6 \end{bmatrix} = \begin{bmatrix} 4,2 \\ 2,2 \end{bmatrix} = 4,2 \times \begin{bmatrix} 1 \\ 0,52 \end{bmatrix}$$

$$\vdots$$

$$V_5 = A.V_4 = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0,5002 \end{bmatrix} = \begin{bmatrix} 4,0004 \\ 2,0004 \end{bmatrix} = 4,0004 \times \begin{bmatrix} 1 \\ 0,50005 \end{bmatrix}$$

It will be seen that the calculation is converging on $\lambda_1 = 4$

and $X_1' = (1; \; 0,5)$

It should be noted that as the order of the matrix A (i.e. as the number of degrees of freedom) is increased, the rate of convergence to the highest eigenvalue is slowed down. In addition it is often necessary to carry a large number of significant figures in the elements of the matrix A.

If the eigenvalues are very close, the rate of convergence is slowed down, since it was shown previously that the rate of convergence depends on the ratios $|\lambda_i/\lambda_1|$ with $i = 2, 3, \ldots n$.

4.1.3 - Rayleigh's quotient

When the iteration method is used it is generally necessary to expend a fair amount of work to obtain an accurate assessment of the largest eigenvalue from an initial guess of the eigenvector X_o. A method which enables a good estimate of the largest eigenvalue using the same guess for X_o as in the basic iterative method is given by the Rayleigh quotient.

Since $$AX = \lambda X$$

then $$X'AX = \lambda X'X$$

and $$\lambda = \frac{X'AX}{X'X}$$

Example:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \qquad X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X'AX = 8$$

$$X'X = 2$$

$$\therefore \lambda = \frac{8}{2} = \underline{4}$$

It can be observed, that the correct eigenvalue is obtained though the guess of the eigenvector was not correct.

This method needs considerably less calculation effort, but it does not give the corresponding eigenvector.

4.2 DETERMINATION OF THE SUBDOMINANT EIGENVALUES

4.2.1 - Use of the Orthogonality Relationship

In order to determine the second largest eigenvalue λ_2 and its corresponding eigenvector X_2, the order of the matrix A must be reduced by an elimination process. This process uses the orthogonality property (see Theorem 2) of the eigenvectors when the matrix is symmetrical. For unsymmetrical matrices it is necessary to consider the eigenvectors of the transpose (see Theorem 3) because the eigenvectors of an unsymmetrical matrix A are not necessarily orthogonal to each other.

The iteration procedure is applied to the new matrix of order $(n - 1)$ to obtain λ_2 and X_2. This process of matrix reduction followed by iteration is continued until all eigenvalues and eigenvectors are determined.

When the matrix A has been reduced to a 2×2 matrix, the remaining eigenvalues can be found either by iteration or by solving the characteristic equation (which is quadratic).

The process of matrix reduction can be accomplished in the following way. The orthogonality condition between the first eigenvector X_i is given by:

$$X_1' X_i = 0$$

or, when the product is formed:

$$a_1 x_1 + a_2 x_2 + \ldots . \; a_n x_n = 0 \tag{8}$$

where: a_1 to a_n are the known elements of the eigenvector X_1

and: x_1 to x_n are the unknown elements of the eigenvector X_i.

Equation (8) can be solved for x_1 in terms of x_2, x_3, etc., thus

$$x_1 = -\frac{a_2}{a_1} x_2 - \frac{a_3}{a_1} x_3 - \ldots . - \frac{a_n}{a_1} x_n \tag{9}$$

Then, this expression for x_1 can be substituted into the original equations of the eigenvalue problem $AX = \lambda X$, to produce n equations in $(n - 1)$ unknowns.

The first of these equations will be a linear combination of the remaining $(n - 1)$ equations and can be discarded, leaving the reduced set of $(n - 1)$ equations in $(n - 1)$ unknowns:

$$BX = \lambda X \tag{10}$$

Equation (10) constitutes an eigenvalue problem involving only the $(n - 1)$ remaining eigenvalues of the original system of equations. Now the iterative method can be used for the new matrix B of order $(n - 1)$. The result will be the numerically highest eigenvalue of the reduced matrix (which is the second eigenvalue of the original matrix A) and the corresponding eigenvector for the reduced matrix. This reduced eigenvector contains only the values $x_2, x_3, \ldots, x_n$ of the eigenvector for the original matrix. The associated value of x_1 can be found from equation (9). Thus, the second eigenvector of the original matrix is determined.

<u>Example 1</u>: <u>symmetrical</u> matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Using the basic iteration method we obtain the following values for the largest eigenvalue and its corresponding eigenvector.

$$\lambda = 3,247 \qquad X_1 = \begin{bmatrix} 1,0 \\ -1,247 \\ 0,555 \end{bmatrix}$$

We want to find the second largest eigenvalue and eigenvector.

The orthogonality condition of equation (9) results in

$$\underline{x_1 = + 1,247x_2 - 0,555x_3} \qquad (11)$$

The values of x_1, x_2 and x_3 should also satisfy the equation $AX_2 = \lambda_2 X_2$, i.e.

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda_2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Substituting x_1 into the second and third equations and discarding the first equation gives the following set of equations:

$$\begin{aligned} 0,753x_2 - 0,445x_3 &= \lambda_2 x_2 \\ - x_2 + \quad x_3 &= \lambda_2 x_3 \end{aligned} \qquad (12)$$

hence

$$\begin{vmatrix} 0,753 - \lambda_2 & - 0,445 \\ - 1 & 1 - \lambda_2 \end{vmatrix} = 0$$

which leads to the characteristic equation

$$\lambda^2 - 1,753\lambda + 0,308 = 0$$

with the roots:

$$\lambda_2 = 1,555$$
$$\lambda_3 = 0,198$$

Substituting λ_2 in equation (12) and assuming $x_3 = 1$, gives $x_2 = -0{,}555$, which in turn must be substituted in equation (11) to give the second eigenvector

$$X_2 = \begin{bmatrix} -1{,}247 \\ -0{,}555 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -0{,}445 \\ +0{,}802 \end{bmatrix}$$

and X_3 can be obtained in the same way by substituting λ_3.

<u>Example 2</u>: <u>unsymmetrical</u> matrix

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \qquad \lambda_1 = 4 \qquad X_1 = \begin{bmatrix} 1 \\ 0{,}5 \end{bmatrix} \qquad U_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where U_1 is the eigenvector of the transpose of A.

Now, according to Theorem 3, U_1 is orthogonal to X_2, hence

$$1.x_1 + 1.x_2 = 0$$

therefore $$x_1 = -x_2$$

and substituting x_1 in equation $AX = \lambda X$ and discarding the first equation leads to

$$1(-x_2) + 2(x_2) = \lambda x_2$$

therefore $$\lambda_2 = 1$$

choosing $$x_2 = -1 \qquad \text{then } x_1 = +1$$

and hence $$X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ as before in section B.2

<u>Note</u>: In the case of an unsymmetrical matrix, use has to be made of the eigenvector U_1 of A', since for a unsymmetrical matrix the eigenvectors are not necessarily orthogonal to each other. However, the eigenvectors U of the transpose of A are orthogonal to the eigenvectors X of A.

"<u>Rinsing technique</u>"

This is another way of obtaining the second largest eigenvalue which uses the orthogonality properties.

It has already been stated that any guessed vector V can be expressed as the sum of the eigenvectors, i.e.

$$V = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_n X_n \tag{17}$$

Using the basic iteration method leads ultimately to the suppression of all terms on the right hand side except the first. This assumes that all the eigenvalues are <u>positive</u> and $\lambda_1 > \lambda_2 > \ldots > \lambda_n$.

It is evident that if $\alpha_1 = 0$ but $\alpha_2 \neq 0$ the process will converge to the second largest eigenvalue λ_2 and the associated eigenvector X_2. To find these it is therefore necessary to devise a procedure for obtaining a starting vector V in whose expansion, equation (17), the coefficient of X_1 will be zero. This is done in the following way:-

Premultiplying equation (17) by X_1' which has already been determined, gives on account of the orthogonality conditions:

$$X_1' V = \alpha_1 X_1' X_1 + 0 + 0 + \ldots + 0.$$

Hence

$$\alpha_1 = \frac{X_1' V}{X_1' X_1} \tag{18}$$

Then obviously the difference

$$V - \alpha_1 X_1 = \alpha_2 X_2 + \ldots + \alpha_n X_n$$

is a vector in whose expansion the coefficient α_1 is zero. The iteration method for finding λ_2 and X_2 can then be employed using $V_1 = V - \alpha_1 X_1$ as the starting vector.

In practical problems the components of X_1 cannot in general be found with perfect accuracy and rounding errors are introduced as mentioned in section 4.1.2. The value of α_1 given in the equation (18) will be only approximate and consequently the coefficient of X_1, while it is presumably very small, will not be zero. Now the effect of each iteration is to increase the term containing X_1 relative to the other terms. Hence, to prevent the process converging very slowly, but none the less inevitably, to λ_1 and X_1 rather than to λ_2 and X_2, it is necessary after each iteration, or at least after every few iterations, to remove the accumulated component of X_1. This method of preventing the convergence to X_1 is referred to as rinsing technique.

In order to remove the accumulated component of X_1, the value of α_1 in the expansion vector V_i is computed and the iterative process is continued with the rinsed vector $(V_i - \alpha_1 X_1)$ instead of with V_i. The "rinsing technique" applies only to <u>symmetrical</u> matrices.

4.2.2 - Deflations

The term deflation is used to describe the process of determining a new matrix which has the same latent roots as the original matrix except for one which is replaced by zero.

Hotelling's Deflation

Hotelling found that the matrix

$$H = A - \lambda_1 X_1 U_1'$$

has certain desirable properties, where

A is any square matrix,
X_1 is the eigenvector of A corresponding to λ_1,
U_1 is the eigenvector of the transpose of A.

These properties are:

(a) H has the same eigenvalues as the matrix A except for λ_1 which is zero.

Proof: postmultiplying by X_1 gives

$$H X_1 = A X_1 - \lambda_1 X_1 U_1' X_1$$

from Theorem 3 $U_1' X_1 = 1$ and noting that $A X_1 = \lambda_1 X_1$ we get

$$H X_1 = 0$$

Therefore X_1 is a eigenvector of H corresponding to an eigenvalue which is zero.

(b) H has the same eigenvectors as the matrix A.

Proof: postmultiplying by another eigenvector X_i and noting that $U_1' X_i = 0$ (from Theorem 3) gives

$$H X_i = A X_i - \lambda_1 X_1 U_1' X_i$$
$$= A X_i$$
$$= \lambda_i X_i$$

Therefore λ_i is an eigenvalue of the matrix H as it is of A, with the same eigenvector X.

Hotelling's theorem, however, is used almost exclusively for the determination of only the second eigenvalue.

Wielandt's Deflation

Hotelling's deflation requires knowledge of the eigenvectors of both matrix A and its transpose A', except where the matrix is symmetrical. Wielandt found that it is not necessary to use U_1, but that an arbitrary vector Z could be chosen to obtain the matrix W. The deflated matrix

$$W = A - \lambda_1 \frac{X_1 \, Z'}{Z' \, X_1} \tag{19}$$

has similar properties to the matrix A:-

(a) The eigenvalues of W are the same as those of A except that the first eigenvalue is replaced by zero.

(b) The eigenvectors of A, X_i and of W, Y_i are related by the formulae

$$\begin{aligned} X_1 &= Y_1 \\ X_i &= (A - \lambda_1 I) \, Y_i \end{aligned}$$

A proof of this can be found in ref. (14). Obviously if Z is arbitrary it can be chosen to simplify the work, and one way of doing this is to choose it to make the first row of W all zero.

If we denote the first row of A by the symbol A_1, then putting

$$Z' = \frac{A_1}{\lambda_1}$$

has the desired result

$$\begin{aligned} Z'.X_1 &= A_1.X_1/\lambda_1 \\ &= 1 \end{aligned}$$

and so equation (19) becomes

$$\begin{aligned} W &= A - \lambda_1 \, X_1 \frac{A_1}{\lambda_1} \\ &= A - X_1 \, A_1 \end{aligned}$$

This deflation has been used in the program WCANT, to obtain the second eigenvalue and eigenvector.

Comparison of both deflations

1) To compute Wielandt's matrix W only, the eigenvector X_1 must be known. To compute H, the eigenvector U_1 of A' must also be known. So, in the case where A is unsymmetrical, the computation of Wielandt's matrix W is simpler.

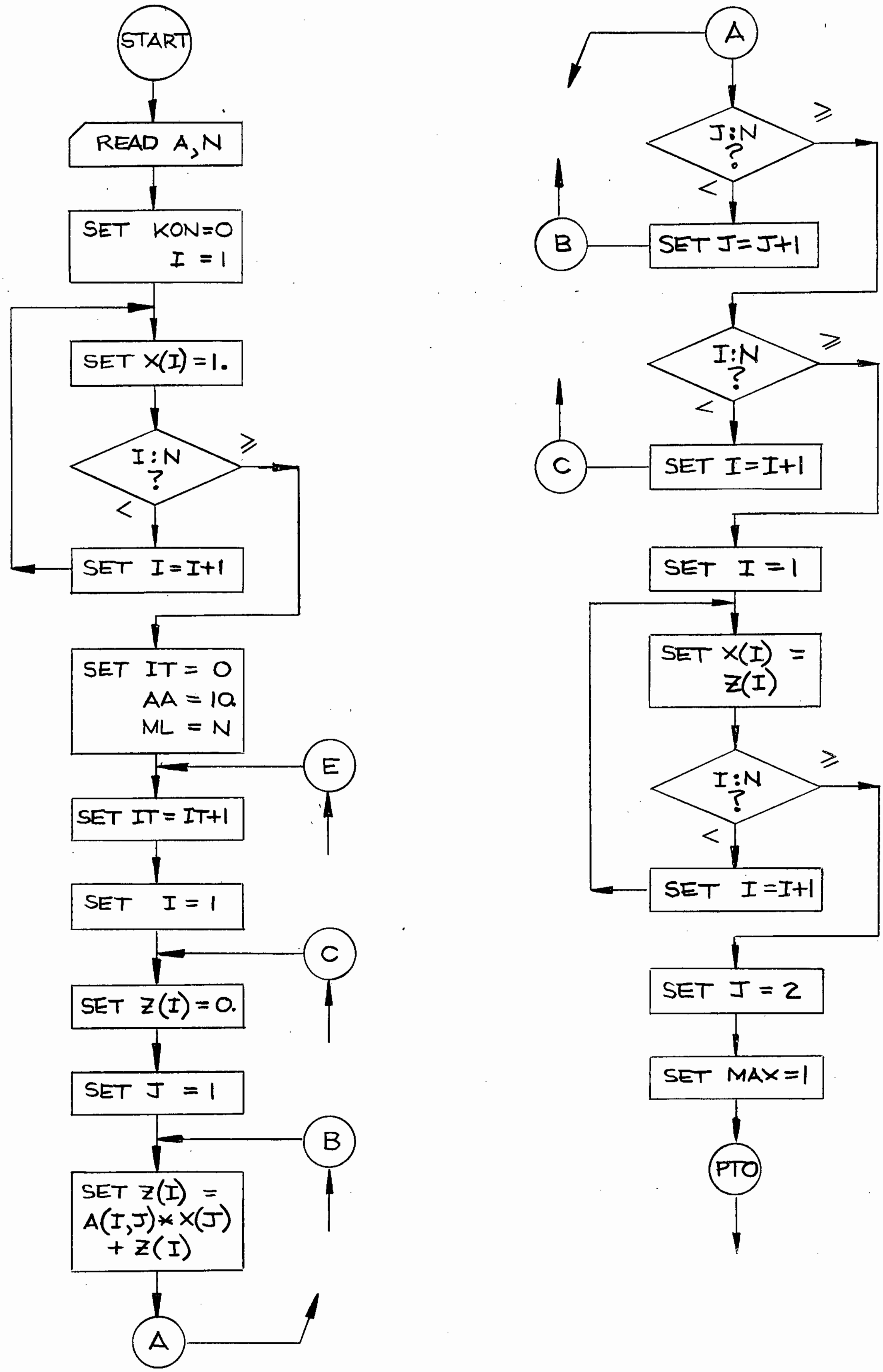
2) Hotelling's deflation does not change the order of the matrix, whereas Wielandt's deflation reduces it by one. This, however, is only significant if several eigenvalues are to be determined since then the iterations require fewer and fewer operations.

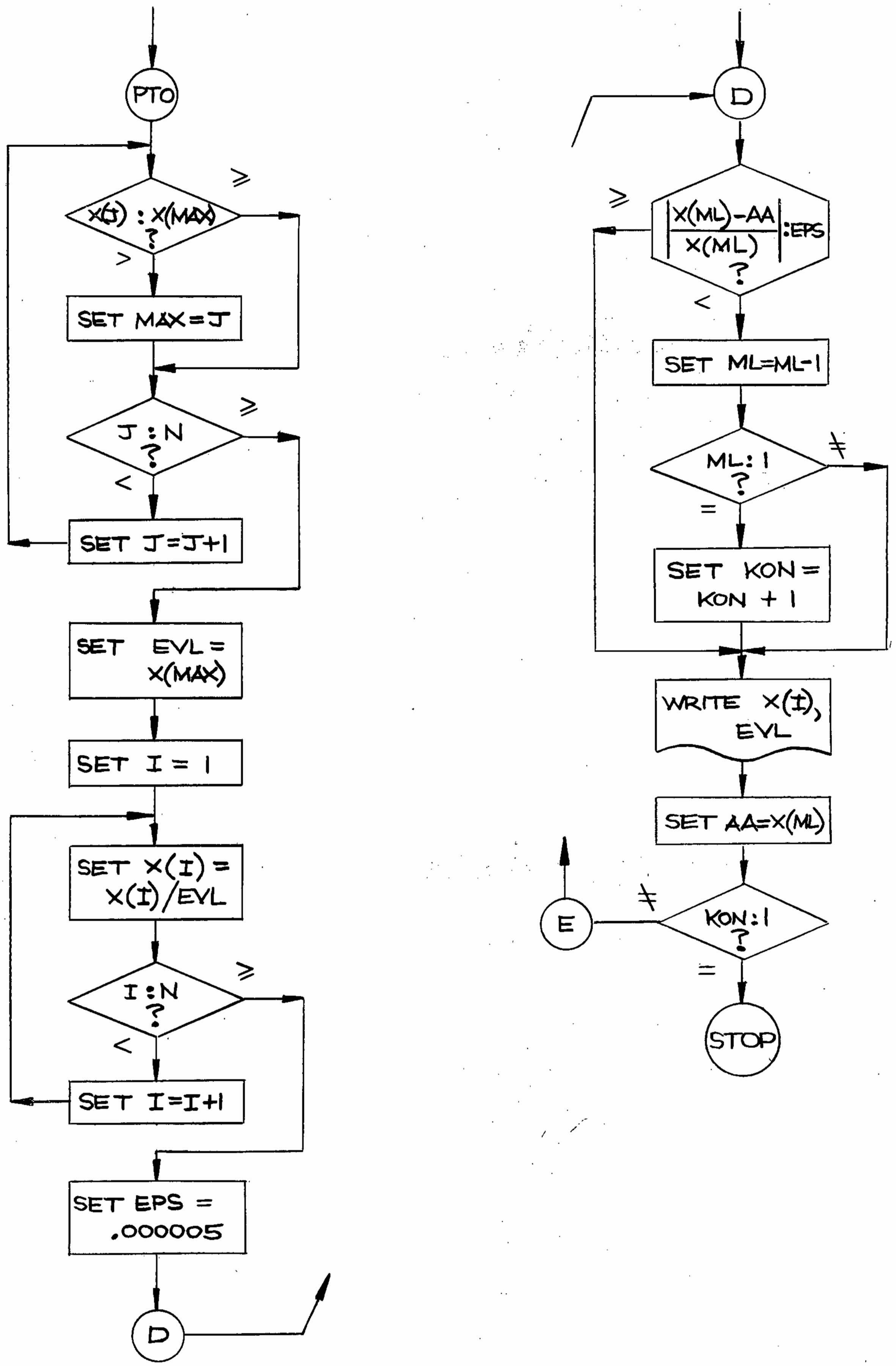
3) In general, therefore, Wielandt's deflation is preferable.

4.2.3 - Final remarks

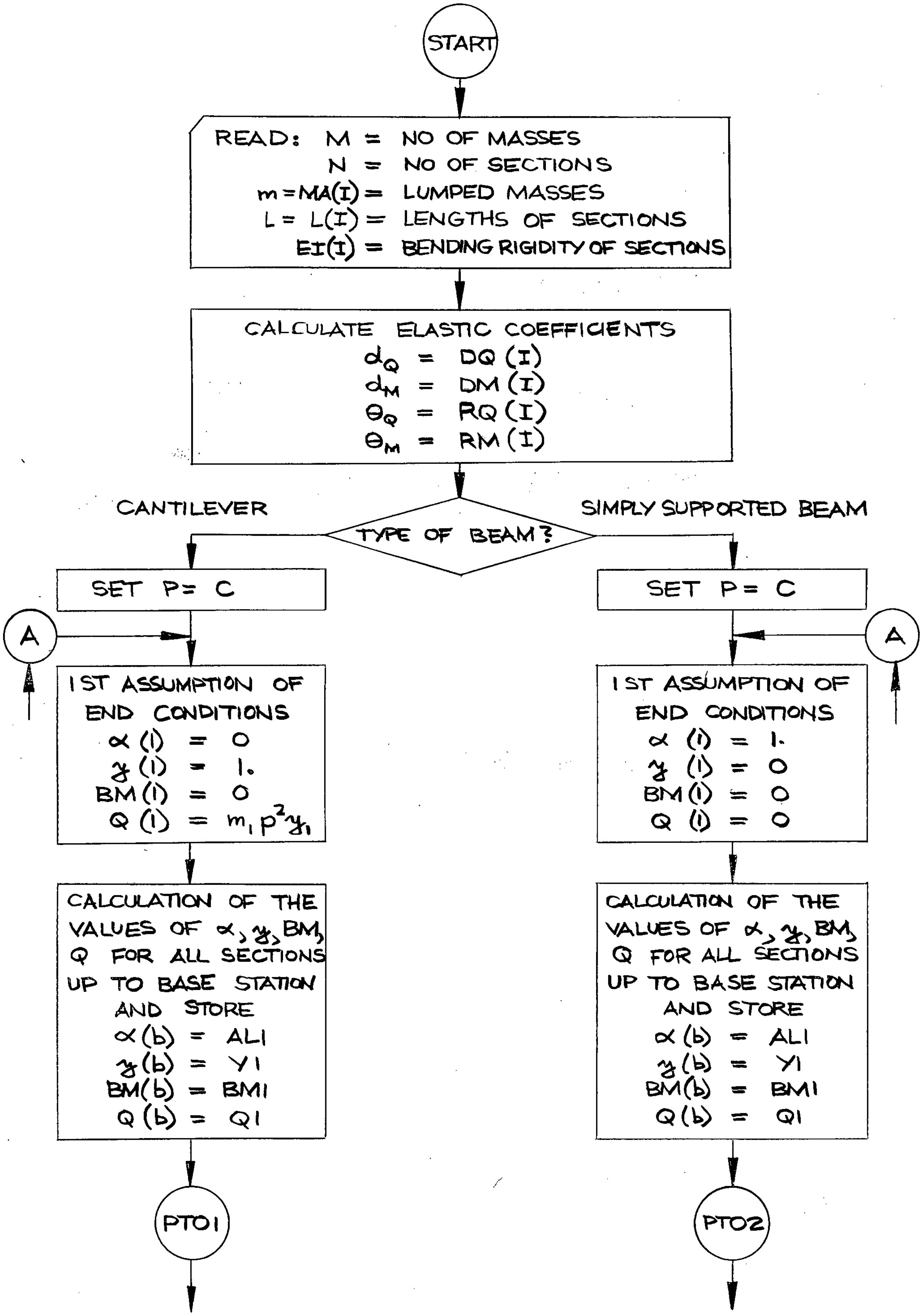
There exist many more methods which determine eigenvalues other than the first. Similarity transformations (see Theorem 7) can be used to eliminate the first eigenvalue, ref.(15). Bodwig (16) gives a method known as annihilation and Fox (17) favours "inverse iteration" besides "deflations" to determine the subdominant eigenvalues.

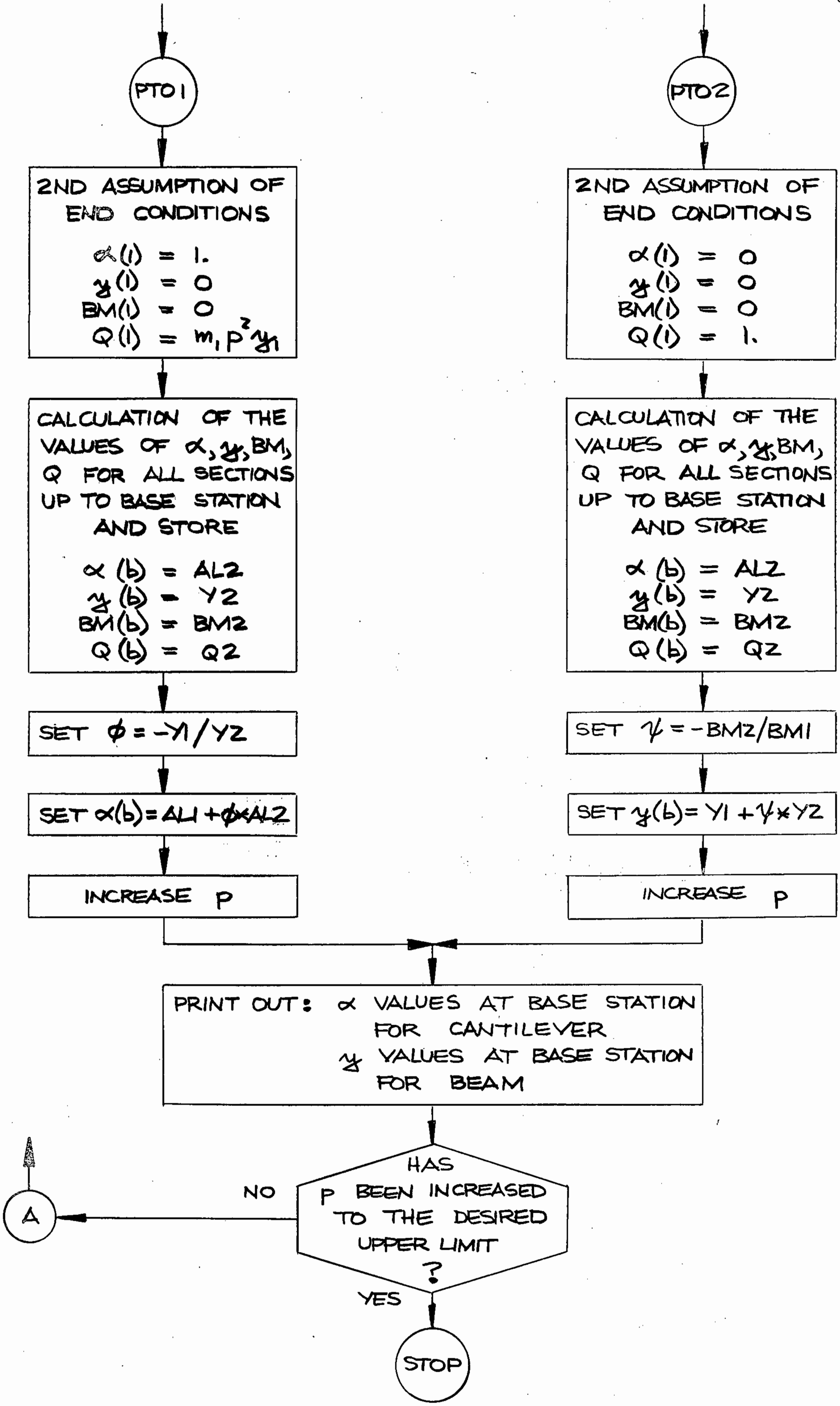
FLOWDIAGRAM FOR PROGRAM WITER



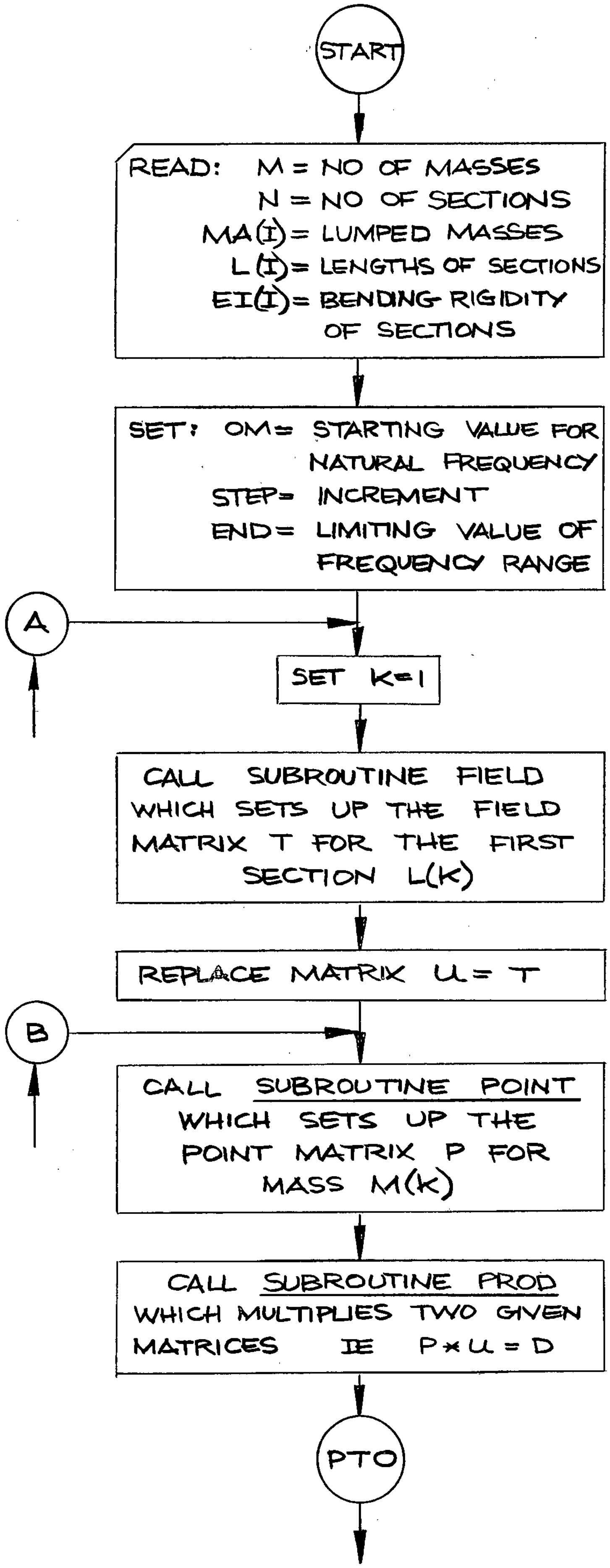

PTO
X(J) : X(MAX) ?
≥
>
SET MAX=J
J : N ?
≥
<
SET J=J+1
SET EVL = X(MAX)
SET I = 1
SET X(I) = X(I)/EVL
I : N ?
≥
<
SET I=I+1
SET EPS = .000005
D
D
|(X(ML)-AA)/X(ML)| : EPS ?
≥
<
SET ML=ML-1
ML: 1 ?
≠
=
SET KON = KON + 1
WRITE X(I), EVL
SET AA=X(ML)
KON : 1 ?
≠
=
E
STOP

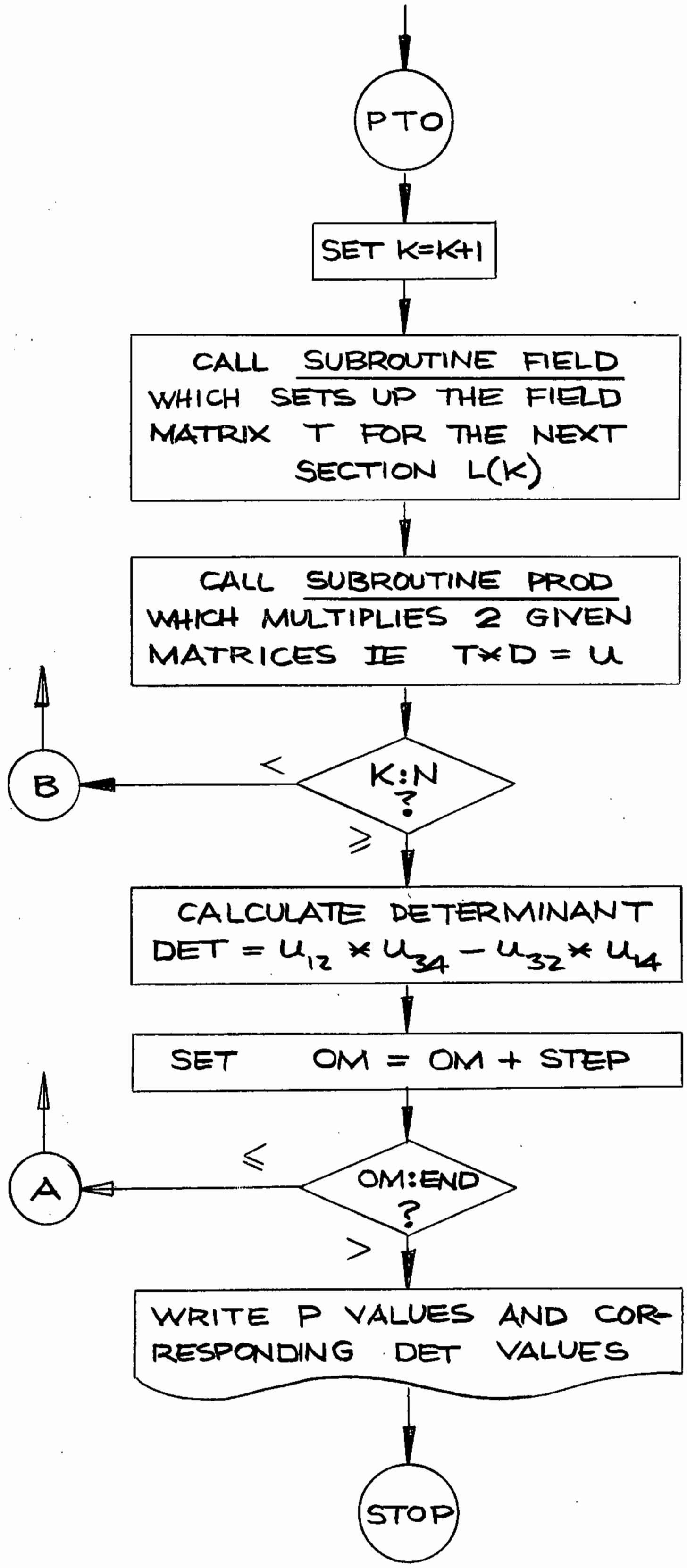
FLOWDIAGRAM FOR PROGRAM WMYKLE



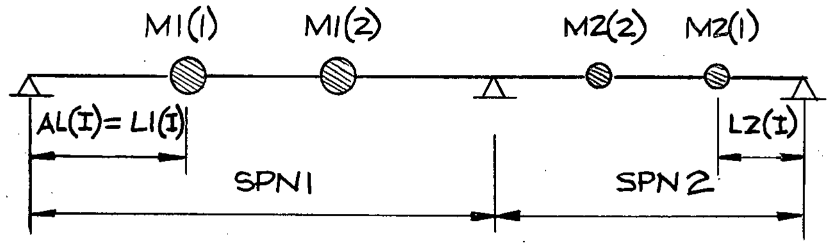

PTO1
2ND ASSUMPTION OF END CONDITIONS
α(1) = 1.
y(1) = 0
BM(1) = 0
Q(1) = m₁p²y₁
CALCULATION OF THE VALUES OF α, y, BM, Q FOR ALL SECTIONS UP TO BASE STATION AND STORE
α(b) = AL2
y(b) = Y2
BM(b) = BM2
Q(b) = Q2
SET φ = -Y1/Y2
SET α(b) = AL1 + φ×AL2
INCREASE p
PTO2
2ND ASSUMPTION OF END CONDITIONS
α(1) = 0
y(1) = 0
BM(1) = 0
Q(1) = 1.
CALCULATION OF THE VALUES OF α, y, BM, Q FOR ALL SECTIONS UP TO BASE STATION AND STORE
α(b) = AL2
y(b) = Y2
BM(b) = BM2
Q(b) = Q2
SET ψ = -BM2/BM1
SET y(b) = Y1 + ψ×Y2
INCREASE p
PRINT OUT: α VALUES AT BASE STATION FOR CANTILEVER
y VALUES AT BASE STATION FOR BEAM
HAS p BEEN INCREASED TO THE DESIRED UPPER LIMIT ?
NO
A
YES
STOP

FLOWDIAGRAM FOR PROGRAM W TRANS




PTO
SET K=K+1
CALL SUBROUTINE FIELD WHICH SETS UP THE FIELD MATRIX T FOR THE NEXT SECTION L(K)
CALL SUBROUTINE PROD WHICH MULTIPLIES 2 GIVEN MATRICES IE T*D = U
K:N ?
<
≥
B
CALCULATE DETERMINANT DET = $U_{12} * U_{34} - U_{32} * U_{14}$
SET OM = OM + STEP
OM:END ?
≤
>
A
WRITE P VALUES AND CORRESPONDING DET VALUES
STOP

FLOWDIAGRAM FOR PROGRAM WDUNK



START

READ: N1, N2
SPN1, SPN2
L1(I), L2(I)
M1(I), M2(I)
EI

SET: AL(I) = L1(I)
AM(I) = M1(I)
LL = SPN1
LR = SPN2

SET: G = 9.81
K = 0

A

SET: I = 1
SUM = 0

B

SET REACTION A

$$A = \frac{AM(I)*(LL-AL(I))}{LL} - \frac{AM(I)*AL(I)*[LL^2-(AL(I))^2]}{2*LL^2*(LL+LR)}$$

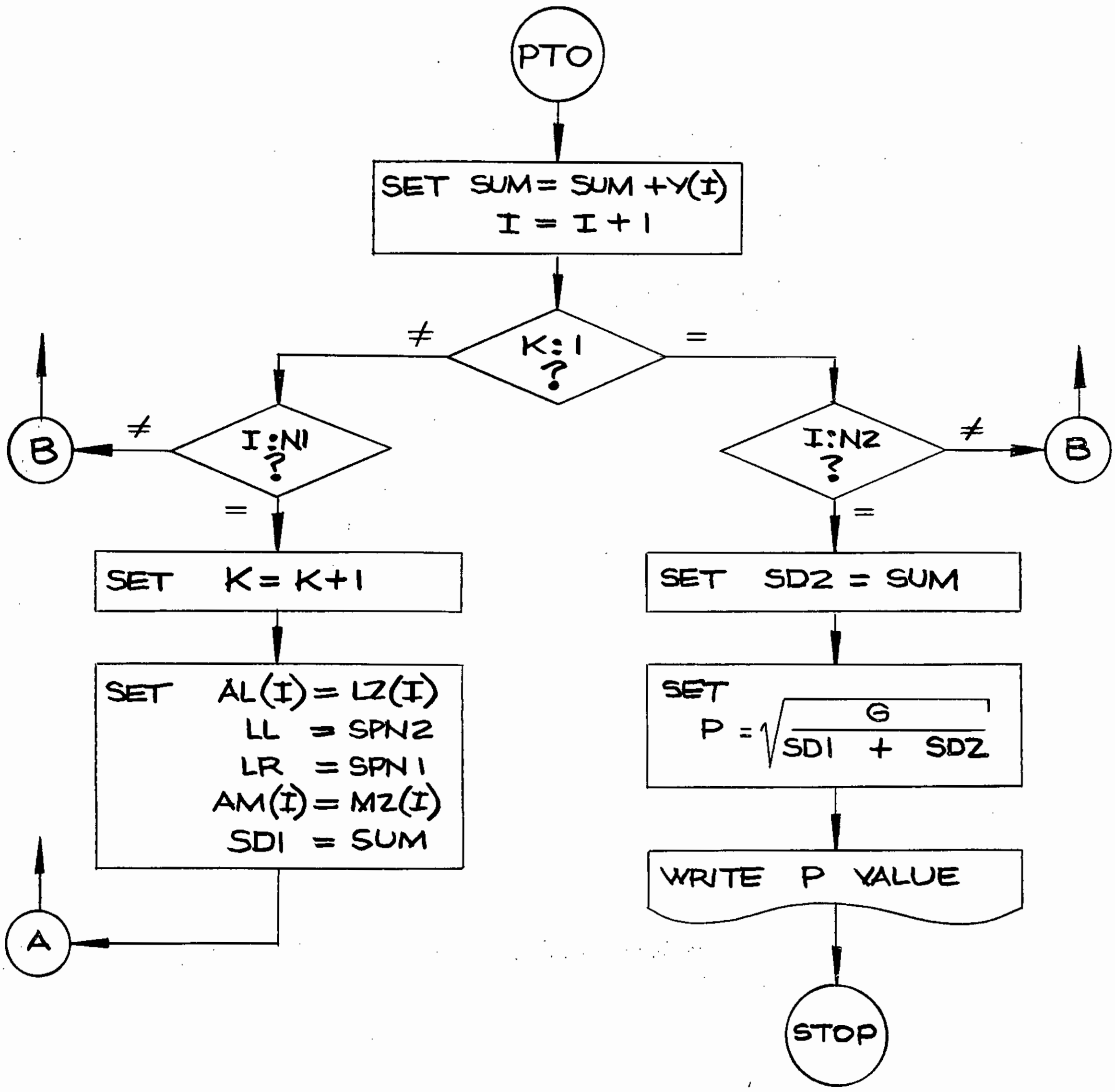
SET CONSTANT C

$$C = \frac{1}{6*LL}*[A*LL^3-AM(I)*(LL-AL(I))^3]$$

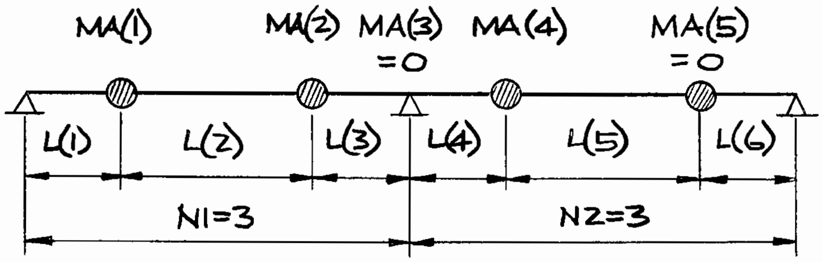
SET DEFLECTION

$$Y(I) = \frac{G}{EI}*\left[\frac{A*AL(I)^3}{6}+C*AL(I)\right]$$

PTO


PTO
SET SUM= SUM +Y(I)
I = I + 1
K:1
?
≠
=
I:N1
?
I:N2
?
B
B
SET K= K+1
SET AL(I) = LZ(I)
LL = SPN2
LR = SPN1
AM(I) = MZ(I)
SD1 = SUM
A
SET SD2 = SUM
SET
P = √(G / SD1 + SD2)
WRITE P VALUE
STOP

FLOWDIAGRAM FOR PROGRAM WTMCB



START

READ: N1 = NO OF SECTIONS OF 1ST SPAN
N2 = NO OF SECTIONS OF 2ND SPAN
MA(I) = LUMPED MASSES ON BOTH SPANS
L(I) = LENGTH OF SECTIONS OF BOTH SPANS
EI(I) = BENDING STIFFNESSES OF THE SECTIONS OF BOTH SPANS

SET N = N1 + N2

SET: OM = STARTING VALUE FOR NAT. FREQ.
STEP = INCREMENT
END = LIMITING VALUE OF FREQ. RANGE

A

SET K = 1

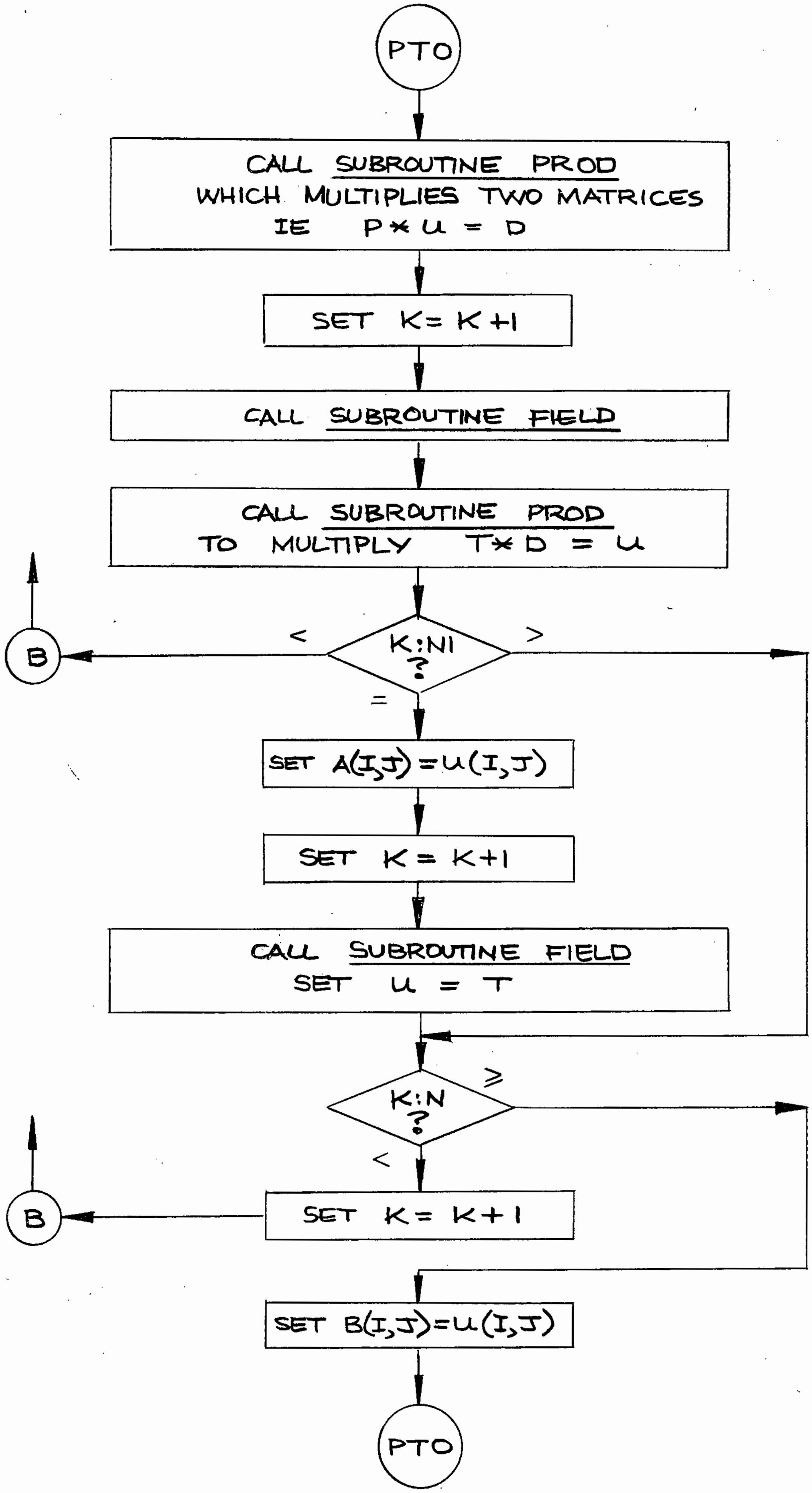
CALL SUBROUTINE FIELD
WHICH SETS UP THE FIELD MATRIX T FOR SECTION K

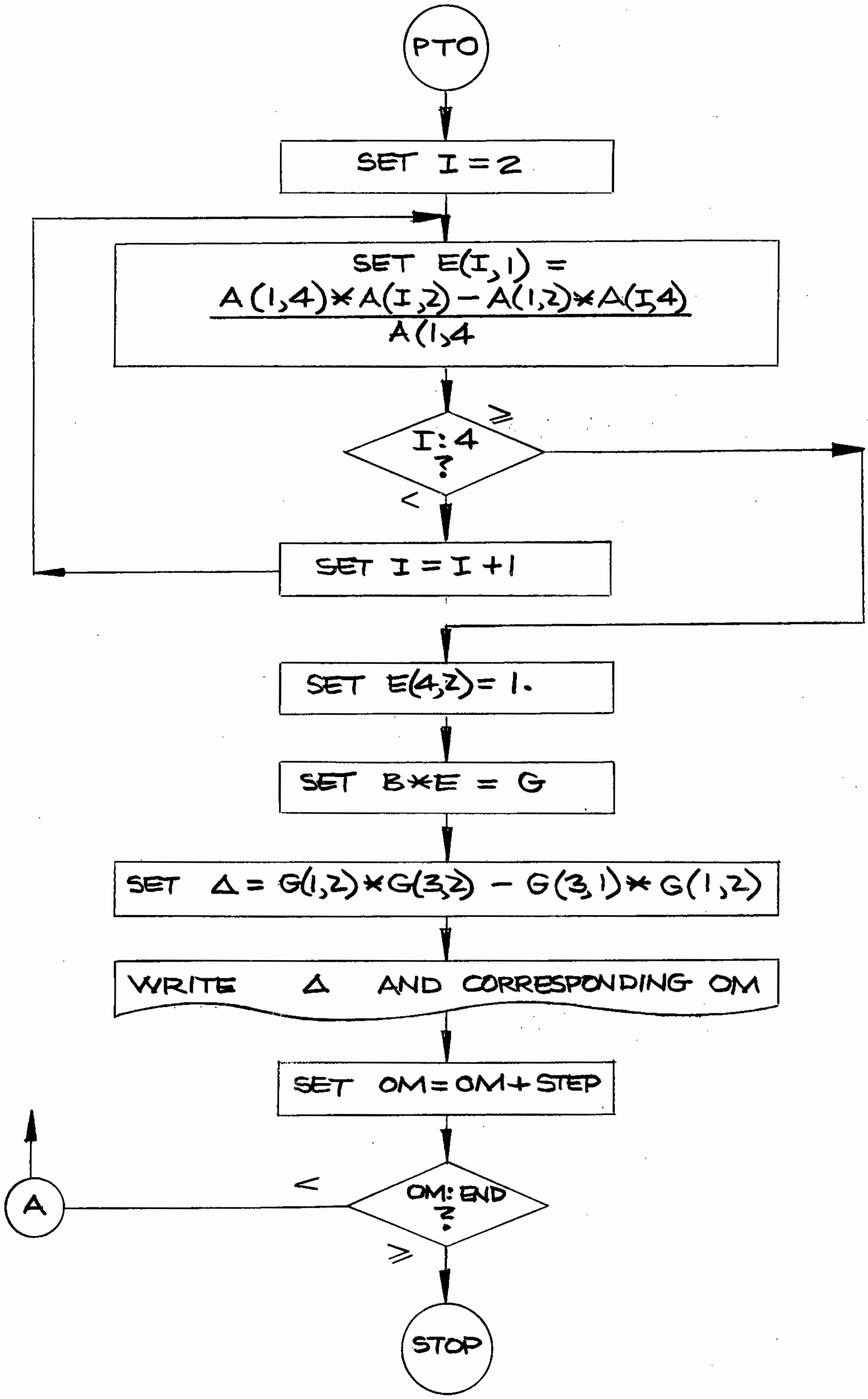
SET U(I,J) = T(I,J)

B

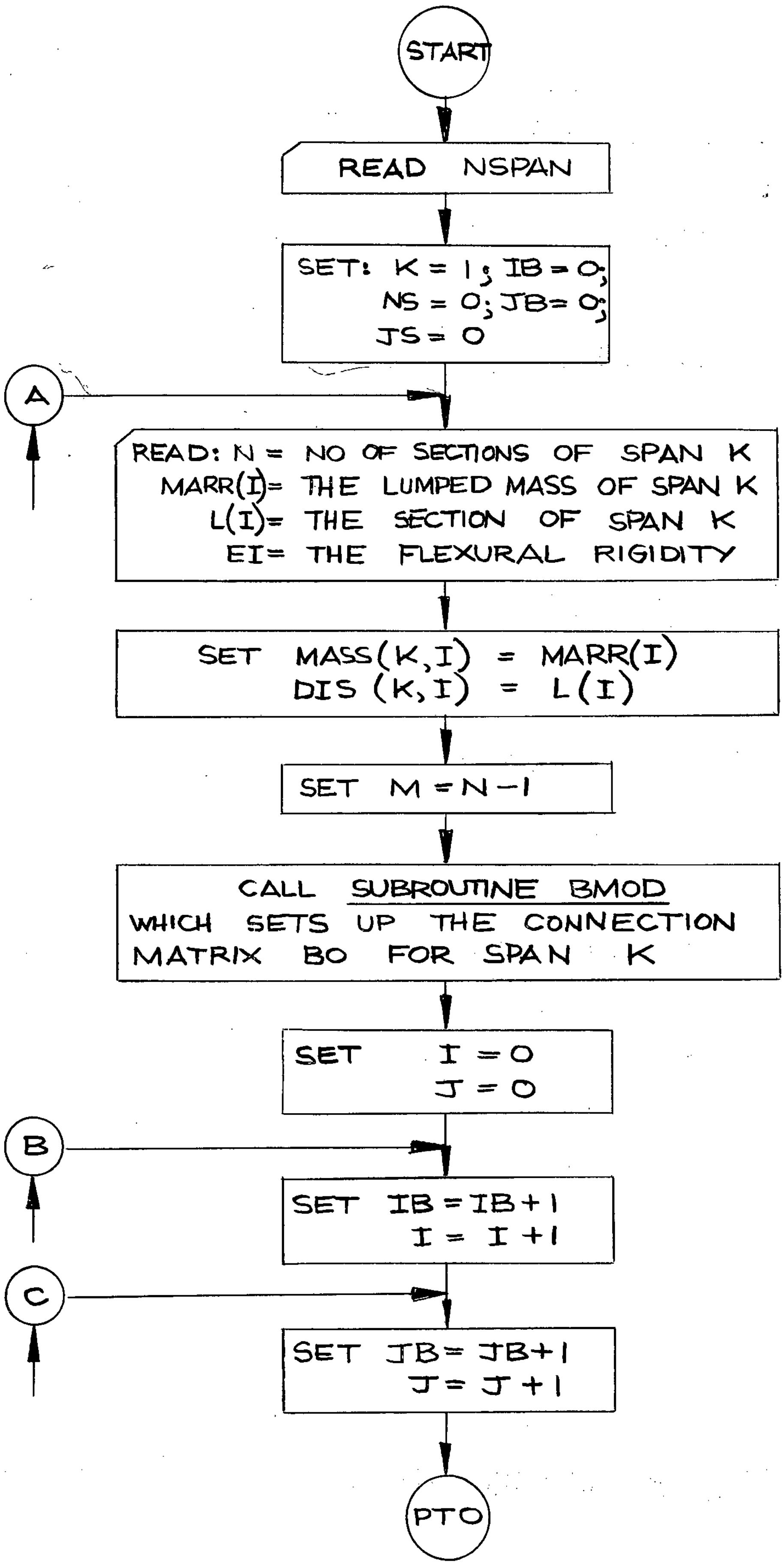
CALL SUBROUTINE POINT
WHICH SETS UP THE POINT MATRIX P FOR MASS K

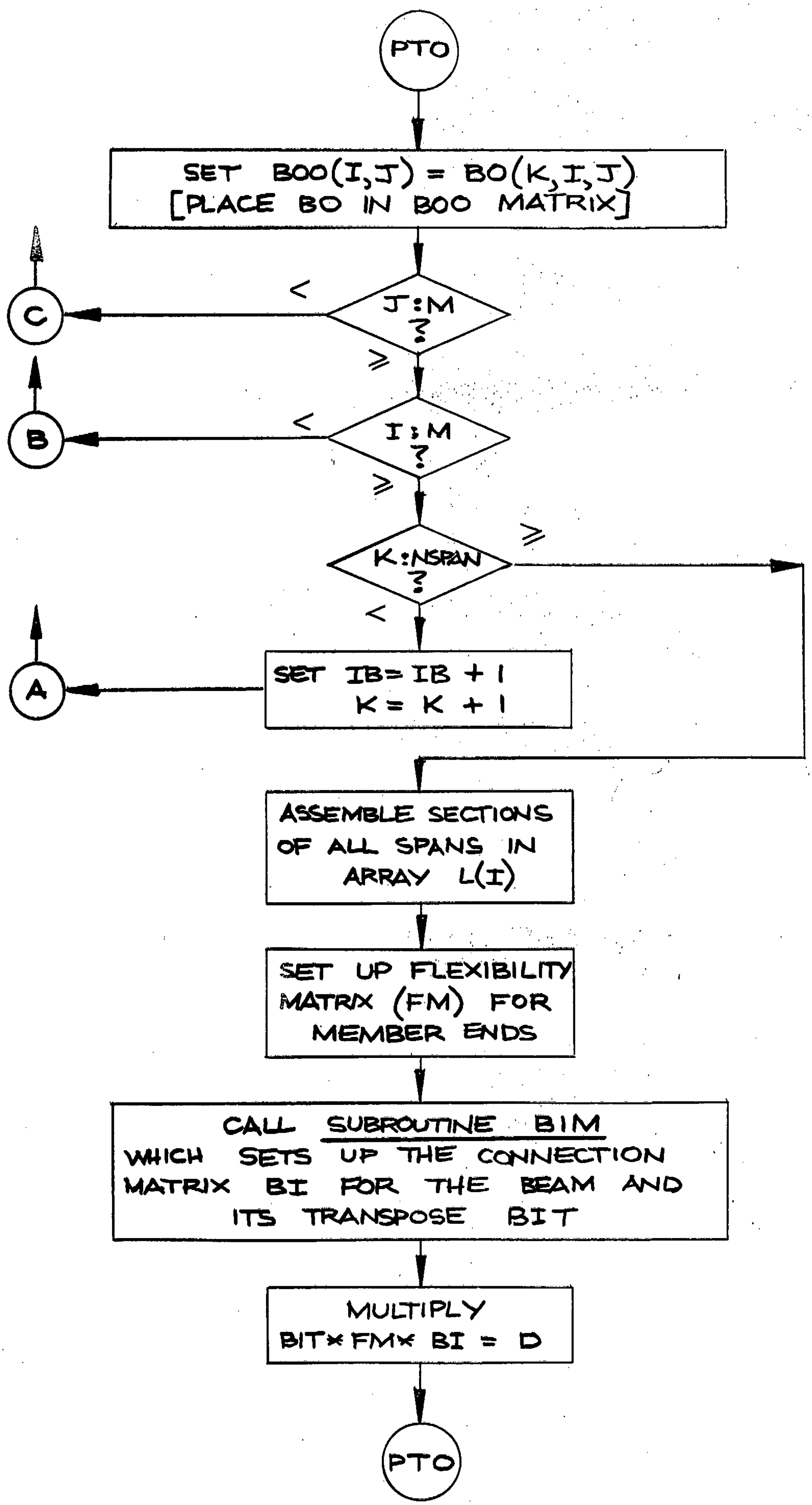
PTO


PTO
CALL SUBROUTINE PROD
WHICH MULTIPLIES TWO MATRICES
IE P*U = D
SET K = K+1
CALL SUBROUTINE FIELD
CALL SUBROUTINE PROD
TO MULTIPLY T*D = U
K:N1
?
<
>
=
B
SET A(I,J)=U(I,J)
SET K = K+1
CALL SUBROUTINE FIELD
SET U = T
K:N
?
≥
<
SET K = K+1
B
SET B(I,J)=U(I,J)
PTO


PTO
SET I=2
SET E(I,1) = A(1,4)*A(I,2)-A(1,2)*A(I,4) / A(1,4)
I:4 ?
≥
<
SET I=I+1
SET E(4,2)= 1.
SET B*E = G
SET Δ= G(1,2)*G(3,2) - G(3,1)*G(1,2)
WRITE Δ AND CORRESPONDING OM
SET OM=OM+STEP
OM: END ?
<
≥
A
STOP

FLOWDIAGRAM FOR PROGRAM WCONT




PTO
SET BOO(I,J) = BO(K,I,J)
[PLACE BO IN BOO MATRIX]
J:M ?
<
≥
C
I:M ?
<
≥
B
K:NSPAN ?
≥
<
SET IB= IB + 1
K = K + 1
A
ASSEMBLE SECTIONS OF ALL SPANS IN ARRAY L(I)
SET UP FLEXIBILITY MATRIX (FM) FOR MEMBER ENDS
CALL SUBROUTINE BIM WHICH SETS UP THE CONNECTION MATRIX BI FOR THE BEAM AND ITS TRANSPOSE BIT
MULTIPLY BIT* FM* BI = D
PTO

PTO

↓

CALL SUBROUTINE GJR
WHICH IS A UNIVAC SUPPLIED PROGRAM WHICH INVERTS MATRIX (D)

↓

SET G = BIT * FM * BOO
X = D^{-1} * G
BIX = BI * X
B = BOO − BIX

↓

SET BOT = TRANSPOSE OF BO

↓

SET F = BOT * FM * B
F IS THE FLEXIBILITY MATRIX FOR THE CONTINUOUS BEAM

↓

ASSEMBLE THE MASS MATRIX ARRAYS INTO AN OVERALL MASS MATRIX FOR THE CONTINUOUS BEAM : ALLM (I, J)

↓

SET S = F * ALLM
WHERE S IS THE SYSTEMS MATRIX

↓

CALL SUBROUTINE ITERW
WHICH DETERMINES THE LARGEST EIGENVALUE

↓

CALL SUBROUTINE JACMX
WHICH IS A UNIVAC SUPPLIED PROGRAM TO DETERMINE ALL EIGENVALUES AND EIGENVECTORS OF MATRIX S

↓

STOP

APPENDIX D

The following programs are listed:-

WITER
WMYKLE
WTRANS
WCANT
WSBEAM
WTIM
WDUNK
WTMCB
WCONT

WITER

WITER
PROGRAM FOR THE BASIC ITERATIVE METHOD

```
      DIMENSION C(20,20),X(20),Z(20)                                  WEINHEIM
63    READ(8,68)N
68    FORMAT(I2)                                                      WEINH
      DO 80 I=1,N
80    READ(8,50)(C(I,J),J=1,N)
50    FORMAT( )
      WRITE(5,65)                                                     WEINH
65    FORMAT(1H1,' GIVEN MATRIX',/)                                   WEINH
      DO 66 I=1,N                                                     WEINH
66    WRITE(5,67)(C(I,J),J=1,N)                                       WEINH
67    FORMAT(12E13.6)                                                 WEINH
      WRITE(5,60)
60    FORMAT(///,' NO OF ITER.  EIGENVALUE  EIGENVECTOR')             WEINH
      KON=0                                                           WEINH
      M=N
      A=10.
      DO 51 I=1,N                                                     WEINH
51    X(I)=1.0
      K=0                                                             WEINH
52    K=K+1                                                           WEINH
       DO 53 I=1,N                                                    WEINH
      Z(I)=0.                                                         WEINH
       DO 53 J=1,N                                                    WEINH
      Z(I)=C(I,J)*X(J)+Z(I)                                           WEINH
53    CONTINUE                                                        WEINH
      DO 54 I=1,N
54    X(I)=Z(I)                                                       WEINH
      MAX=1                                                           WEINH
      DO 71 J=2,N
      IF(X(J)-X(MAX))71,71,70
70    MAX=J                                                           WEINH
71    CONTINUE                                                        WEINH
      EVL=X(MAX)                                                      WEINH
      WRITE(5,61)K,EVL
61    FORMAT(/,I7,6X,E13.6)                                           WEINH
      DO 57 I=1,N                                                     WEINH
      X(I)=X(I)/EVL                                                   WEINH
      EPS=.000005                                                     WEINH
      IF(ABS((X(M)-A)/X(M))-EPS)56,57,57
56    M=M-1
      IF(M)55,55,57
55    KON=KON+1                                                       WEINH
57    WRITE(5,58)X(I)                                                 WEINH
58    FORMAT(27X,E13.6)                                               WEINH
      A=X(M)
      IF(KON)52,52,59                                                 WEINH
C INSERT ACARD WITHO AFTER EACH SET OF DATA AND AT THE END A CARD WITH 1WEINH
```

```
59     READ(8,62)NEND                                                     WEINH
62     FORMAT(I1)                                                         WEINH
       IF(NEND)63,63,64
64     CALL EXIT                                                          WEINH
       END
```

WMYKLE

WMYKLE - (MYKLESTAD'S METHOD)
THIS PROGRAM WAS USED IN SECTION 3.3.7

```
       REAL L(12),MA(12)                                                  WEINH
       DIMENSION DQ(12),DM(12),RQ(12),RM(12),BM(12),Q(12),Y(12),AL(12)    WEINH
       IR=8                                                               WEINH
       IW=5                                                               WEINH
C-----READ   NTYP = 0 FOR A CANTILEVER BEAM
C            NTYP = 1 FOR A SIMPLY SUPPORTED BEAM
C            N    =   NUMBER OF BEAM SEGMENTS
C            M    =   NUMBER OF LUMPED MASSES
1      READ(IR,100)NTYP,N,M                                               WEINH
100    FORMAT( )                                                          WEINH
       IF(NTYP)99,2,3                                                     WEINH
2      WRITE(IW,101)N                                                     WEINH
101    FORMAT(1H1,' EVALUATION OF A CANTILEVER DIVIDED INTO ',I3,'  SECTIWEINH
      1ONS',///)                                                          WEINH
       GO TO 4                                                            WEINH
3      WRITE(IW,102)N                                                     WEINH
102    FORMAT(1H1,' EVALUATION OF A SIMPLY SUPPORTED BEAM DIVIDED INTO ',WEINH
      2I3,' SECTIONS',///)                                                WEINH
4      WRITE(IW,103)                                                      WEINH
103    FORMAT('  STATION     LUMPED MASS (T)',8X,'LENGTH OF SECTION(M)',/,/
      U)
C-----READ   MA(I) =  THE LUMPED MASSES
C            L(I)  =  THE LENGTHS OF THE SECTIONS
       READ(IR,100)(MA(I),I=1,M)                                          WEINH
       READ(IR,100)(L(I),I=1,N)                                           WEINH
       DO 5 I=1,M                                                         WEINH
5      WRITE(IW,104)I,MA(I),L(I)                                          WEINH
104    FORMAT(4X,I2,5X,E13.6,/,35X,E13.6)                                 WEINH
       READ(IR,100)EI                                                     WEINH
       WRITE(IW,105)EI                                                    WEINH
105    FORMAT(//,' THE RIGIDITY VALUE   ',E13.6,//)
C-----CALCULATION OF THE FLEXIBILITY COEFFICIENTS
       WRITE(IW,106)                                                      WEINH
```

```
106    FORMAT(/,' THE FLEXIBILITY COEFFICIENTS',/)                    WEINH
       DO 6 K=1,N                                                     WEINH
       DQ(K)=L(K)**3./(6.*EI)
       DM(K)=L(K)**2./(2.*EI)                                         WEINH
       RQ(K)=DM(K)                                                    WEINH
6      RM(K)=L(K)/EI                                                  WEINH
       WRITE(IW,107)(DQ(I),I=1,N)                                     WEINH
107    FORMAT(5X,'DQ(I)',5X,7E13.6)                                   WEINH
       WRITE(IW,108)(DM(I),I=1,N)                                     WEINH
108    FORMAT(5X,'DM(I)',5X,7E13.6)                                   WEINH
       WRITE(IW,109)(RQ(I),I=1,N)                                     WEINH
109    FORMAT(5X,'RQ(I)',5X,7E13.6)                                   WEINH
       WRITE(IW,110)(RM(I),I=1,N)                                     WEINH
110    FORMAT(5X,'RM(I)',5X,7E13.6)                                   WEINH
       IF(NTYP)99,7,12                                                WEINH
7      OM=0.                                                          WEINH
       OMEND=40.
       STEP=1
       WRITE(IW,111)                                                  WEINH

C-----THE FOLLOWING APPLIES TO A CANTILEVER ONLY

C-----FIRST ASSUMPTION FOR STARTING VALUES AT STATION 1

10     Y(1)=1.
       AL(1)=0.                                                       WEINH

C-----BOUNDARY CONDITIONS AT STATION 1

       BM(1)=0.                                                       WEINH
       Q(1)=MA(1)*OM**2*Y(1)
       CALL TBURY                                                     WEINH
       BM1=BM(M)                                                      WEINH
       Y1=Y(M)                                                        WEINH
       Q1=Q(M)                                                        WEINH
       AL1=AL(M)                                                      WEINH

C-----SECOND ASSUMPTIONS FOR STARTING VALUES AT STATION 1

       AL(1)=1.                                                       WEINH
       Y(1)=0.                                                        WEINH
       BM(1)=0.                                                       WEINH
       Q(1)=MA(1)*OM**2*Y(1)
       CALL TBURY                                                     WEINH
       BM2=BM(M)                                                      WEINH
       Y2=Y(M)                                                        WEINH
       Q2=Q(M)                                                        WEINH
       AL2=AL(M)                                                      WEINH
       PHI=-Y1/Y2                                                     WEINH
       ALPHA=AL1+PHI*AL2                                              WEINH
```

```
111   FORMAT(/,61X,'NAT. FREQUENCIES',3X,'ALPHA AT BASE STATION',/)
      WRITE(IW,112)OM,ALPHA                                                  WEINH
112   FORMAT(60X,2(E13.6,5X))
      OM=OM+STEP
      IF(OM-OMEND)10,10,11
C-----THE FOLLOWING APPLIES FOR A SIMPLY SUPPORTED BEAM ONLY                  WEINH
12    OM=1.0                                                                 WEINH
      OMEND=40.                                                              WEINH
      STEP=.1
      WRITE(IW,120)                                                          WEINH
C-----FIRST ASSUMPTIONS FOR STARTING VALUES AT STATION 1                      WEINH
13    AL(1)=1.                                                               WEINH
      Q(1)=0.                                                                WEINH
      BM(1)=0.                                                               WEINH
      Y(1)=0.                                                                WEINH
      CALL TBURY                                                             WEINH
      BM1=BM(M)                                                              WEINH
      Y1=Y(M)                                                                WEINH
      Q1=Q(M)                                                                WEINH
      AL1=AL(M)                                                              WEINH
C-----SECOND ASSUMPTIONS FOR STARTING VALUES AT STATION 1                     WEINH
      AL(1)=0.                                                               WEINH
      Q(1)=1.                                                                WEINH
      BM(1)=0.                                                               WEINH
      Y(1)=0.                                                                WEINH
      CALL TBURY                                                             WEINH
      BM2=BM(M)                                                              WEINH
      Y2=Y(M)                                                                WEINH
      Q2=Q(M)                                                                WEINH
      AL2=AL(M)                                                              WEINH
      XI=-BM2/BM1                                                            WEINH
      YDFL=Y2+XI*Y1
120   FORMAT(/,T79,'DISPLACEMENT',/,T57,'NAT. FREQUENCIES',T79,'AT BASE
     1STATION',/)
      WRITE(IW,121)OM,YDFL                                                   WEINH
121   FORMAT(56X,2(E13.6,7X))                                                WEINH
      OM=OM+STEP                                                             WEINH
      IF(OM-OMEND)13,13,11
11    READ(IR,100)NO
      IF(NO),,99
      DO 90 I=1,12
      MA(I)=0.
90    L(I)=0.
      IF(NO)99,1,99
```

```
99      CALL EXIT                                                      WEINH
C-----SUBROUTINE TBURY                                                 WEINH
        SUBROUTINE TBURY                                               WEINH
        DO 1 K=1,N
        BM(K+1)=BM(K)-Q(K)*L(K)                                        WEINH
        Y(K+1)=Y(K)-L(K)*AL(K)+Q(K)*DQ(K)-BM(K)*DM(K)                  WEINH
        Q(K+1)=Q(K)+MA(K+1)*(OM**2.)*Y(K+1)                            WEINH
1       AL(K+1)=AL(K)-Q(K)*RQ(K)+BM(K)*RM(K)
        RETURN                                                         WEINH
        END                                                            WEINH
```

WTRANS

WTRANS - (TRANSFER MATRIX METHOD)
THIS PROGRAM WAS USED IN SECTION 3.3.8

```
        REAL MA(10),L(10)                                              WEINH
        DIMENSION EI(10),T(4,4),P(4,4),D(4,4),U(4,4)                   WEINH
        IR=8                                                           WEINH
        IW=5                                                           WEINH
C          N = NUMBER OF BEAM SEGMENTS
C          M = NUMBER OF LUMPED MASSES
1       READ(IR,100)N                                                  WEINH
        M=N-1                                                          WEINH
        READ(IR,100)(MA(I),I=1,M)                                      WEINH
        READ(IR,100)(L(I),I=1,N)                                       WEINH
199     FORMAT(/)
        READ(IR,100)(EI(I),I=1,N)
100     FORMAT( )                                                      WEINH
        WRITE(IW,101)                                                  WEINH
101     FORMAT(1H1,'      LUMPED MASSES',11X,'LENGTH OF SEGMENTS',4X,'FLEXURA
       1XURAL RIGIDITY'/)
        DO 2 I=1,N                                                     WEINH
2       WRITE(IW,102)MA(I),L(I),EI(I)                                  WEINH
102     FORMAT(4X,3(E13.6,13X))                                        WEINH
        CM=10.
        STEP=1.
        END=20.
        WRITE(IW,105)
105     FORMAT(//,T36,'VALUE OF DETERMINANT',T61,'TRIAL VALUE FOR P')
        WRITE(IW,106)
106     FORMAT(T35,44('-'),/)
3       K=1                                                            WEINH
        CALL FIELD                                                     WEINH
        DO 4 I=1,4                                                     WEINH
        DO 4 J=1,4
```

```
4     U(I,J)=T(I,J)                                          WEINH
5     CALL POINT                                             WEINH
      CALL PROD (P,U,D)                                      WEINH
      K=K+1                                                  WEINH
      CALL FIELD                                             WEINH
      CALL PROD(T,D,U)                                       WEINH
      IF(K-N)5,10,5                                          WEINH
10    DET=U(1,2)*U(3,4)-U(3,2)*U(1,4)                        WEINH
      WRITE(IW,107)DET,OM
107   FORMAT(T30,2(10X,E13.6),/)
      OM=OM+STEP
      IF(OM.LE.END)GO TO 3
      DO 11 I=1,10
      MA(I)=0.
      L(I)=0.
11    EI(I)=0.
      READ(IR,100)NO
      IF(NO)1,1,
      CALL EXIT                                              WEINH

      SUBROUTINE PROD(A,B,PR)                                WEINH
      DIMENSION A(4,4),B(4,4),PR(4,4),C(4)                   WEINH
      DO 1 I=1,4                                             WEINH
      DO 2 J=1,4                                             WEINH
      C(J)=0.                                                WEINH
      DO 2 KT=1,4                                            WEINH
2     C(J)=C(J)+A(I,KT)*B(KT,J)                              WEINH
      DO 1 J=1,4                                             WEINH
1     PR(I,J)=C(J)                                           WEINH
      RETURN                                                 WEINH
C
C-----SUBROUTINE SETS UP THE FIELD MATRIX

      SUBROUTINE FIELD                                       WEINH
      DO 1 I=1,4
1     T(I,I)=1.                                              WEINH
      T(1,2)=L(K)                                            WEINH
      T(1,3)=L(K)**2./(2.*EI(K))                             WEINH
      T(1,4)=L(K)**3./(6.*EI(K))                             WEINH
      T(2,3)=L(K)/EI(K)                                      WEINH
      T(2,4)=T(1,3)                                          WEINH
      T(3,4)=L(K)                                            WEINH
      RETURN                                                 WEINH

C-----SUBROUTINE SETS UP THE POINT MATRIX

      SUBROUTINE POINT                                       WEINH
      DO 1 I=1,4
1     P(I,I)=1.                                              WEINH
      P(4,1)=OM*OM*MA(K)
```

```
      RETURN                                                    WEINH
C
      END                                                       WEINH
```

WCANT

```
          WCANT - (METHOD USING FLEXIBILITY MATRIX FOR A CANTILEVER)
          THIS PROGRAM WAS USED IN SECTION 4.2

      REAL MASS(10,10),L(10),MA(10)
      DIMENSION F(10,10),FM(20,20),B(20,20),BT(20,20),C(20),FMB(20,20),F
     UMS(10,10),X(10),FIRST(5),TT(5),Y(10)
C-----READ THE NUMBER OF LUMPED MASSES N
99    READ(8,100)N,LM
100   FORMAT(2I2)
C-----LM=-1 INDICATES THAT THE MASSES ARE CONCENTRATED AT EACH
C     MEMBER END (HALF THE MASS)
C-----LM=0 INDICATES THAT THE MASSES ARE CONCENTRATED AT THE CENTRE OF
C     EACH SECTION
C-----LM=+1 INDICATES THAT THE MASSES ARE CONCENTRATED AT ONE
C     MEMBER END ONLY
C-----READ THE MASS MATRIX
      DO 103 I=1,N
      READ(8,101)MA(I)
101   FORMAT(F10.5)
103   MASS(I,I)=MA(I)
C-----READ THE LENGTHS OF THE SEGMENTS
      DO 104 I=1,N                                              WEINH
104   READ(8,102)L(I)                                           WEINH
102   FORMAT(F10.5)
C-----READ THE RIGIDITY VALUE EI OF THE BEAM
      READ(8,224)EI
224   FORMAT(F10.3)
      IF(LM)25,26,27
25    WRITE(5,105)
105   FORMAT(1H1,' THE MASS OF EACH SECTION IS SPLIT AND LUMPED AT BOTH
     UMEMBER ENDS',/)
      GO TO 23
26    WRITE(5,106)
```

```
106   FORMAT(1H1,' THE MASS OF EACH SECTION IS CONCENTRATED AT THE CENTR
     UE OF THE SECTICN',/)
      GO TO 23
27    WRITE(5,107)
107   FORMAT(1H1,' THE MASS OF EACH SECTION IS CONTRATED AT ONE MEMBER
     UEND ONLY',/)
23    WRITE(5,200)
200   FORMAT(//,' MASS MATRIX',/)
      DO 20 I=1,N                                                       WEINH
20    WRITE(5,201)(MASS(I,J),J=1,N)                                     WEINH
201   FORMAT(12F10.6)
      WRITE(5,202)                                                      WEINH
202   FORMAT(//,' THE DISTANCES BETWEEN THE LUMPED MASSES',/)
      DO 21 I=1,N                                                       WEINH
21    WRITE(5,203)L(I)                                                  WEINH
203   FORMAT(1H ,F10.5)
      WRITE(5,204)EI
204   FORMAT(//,' THE RIGIDITY VALUE ',E13.6,//)

C-----THE FLEXIBILITY MATRIX FOR THE MEMBER ENDS

      NN=2*N                                                            WEINH
      K=1                                                               WEINH
      I=1                                                               WEINH
      J=1                                                               WEINH
      DO 1 K=1,N
      FM(I,J)=(L(K)**3.)/(3.*EI)                                        WEINH
      J=J+1                                                             WEINH
      FM(I,J)=(L(K)**2.)/(2.*EI)                                        WEINH
      FM(J,I)=FM(I,J)                                                   WEINH
      I=I+1                                                             WEINH
      FM(I,J)=L(K)/EI
      I=I+1                                                             WEINH
1     J=J+1                                                             WEINH

C-----THE CONNECTION MATRIX BO

      I=1                                                               WEINH
      J=1                                                               WEINH
      K=1                                                               WEINH
      DO 4 J=K,N                                                        WEINH
4     B(I,J)=1.                                                         WEINH
5     I=I+1                                                             WEINH
      SUM=0.                                                            WEINH
      KN=K+1                                                            WEINH
      DO 6 J=KN,N                                                       WEINH
      B(I,J)=SUM+L(J)                                                   WEINH
6     SUM=B(I,J)
      I=I+1                                                             WEINH
      K=K+1                                                             WEINH
```

```
      IF(I-2*N)7,7,10                                                   WEINH
7     DO 8 J=K,N                                                        WEINH
8     B(I,J)=1.
      IF(KN-N)5,10,10

C-----MULTIPLICATION OF BT*FM*B, WHICH GIVES THE OVERALL FLEXIBILITY MATRIX

10    DO 12 I=1,NN
      DO 13 J=1,N                                                       WEINH
      C(J)=0.                                                           WEINH
      DO 13 K=1,NN
13    C(J)=C(J)+FM(I,K)*B(K,J)                                          WEINH
      DO 12 J=1,N                                                       WEINH
12    FMB(I,J)=C(J)                                                     WEINH
      DO 14 I=1,N
      DO 14 J=1,NN
14    BT(I,J)=B(J,I)                                                    WEINH
      DO 15 I=1,N                                                       WEINH
      DO 16 J=1,N                                                       WEINH
      C(J)=0.                                                           WEINH
      DO 16 K=1,NN
16    C(J)=C(J)+BT(I,K)*FMB(K,J)                                        WEINH
      DO 15 J=1,N                                                       WEINH
15    F(I,J)=C(J)                                                       WEINH
      WRITE(5,209)                                                      WEINH
209   FORMAT(//,' THE FLEXIBILITY MATRIX',/)
      DO 17 I=1,N                                                       WEINH
17    WRITE(5,210)(F(I,J),J=1,N)                                        WEINH
210   FORMAT(1H ,9E13.6)

C-----MULTIPLICATION OF F TIMES M

      DO 18 I=1,N                                                       WEINH
      DO 19 J=1,N                                                       WEINH
      C(J)=0.                                                           WEINH
      DO 19 K=1,N
19    C(J)=C(J)+F(I,K)*MASS(K,J)                                        WEINH
      DO 18 J=1,N                                                       WEINH
18    FMS(I,J)=C(J)                                                     WEINH
      WRITE(5,211)                                                      WEINH
211   FORMAT(///,' (F*M) MATRIX',/)
      DO 29 I=1,N
29    WRITE(5,210)(FMS(I,J),J=1,N)
      CALL MSMSW(FMS,N,EAVG)
      CALL SWIT(FMS,N,EVL,X)                                            WEINH
      EV=ABS(1./EVL)
      OMEGA=SQRT(EV)
      WRITE(5,220)OMEGA
220   FORMAT(/,' OMEGA',8X,E13.6)
      T=2.*3.1415927/OMEGA                                              WEINH
```

```
      WRITE(5,213)T                                                     WEINH
213   FORMAT(/,' PERIOD',7X,E13.6,'  SEC')
      PI=3.1415927
      SM=1.
      SL=1.
      CNST=SQRT(EI/(SL**3.*SM))
32    FIRST(1)=(.597*PI)**2.*CNST
      FIRST(2)=(1.49*PI)**2.*CNST
      FIRST(3)=(2.5*PI)**2.*CNST
      DO 33 I=1,3                                                       WEINH
33    TT(I)=2.*PI/FIRST(I)                                              WEINH
      IF(N-1)38,252,38                                                  WEINH
38    CALL LAMBDA(FMS,N,EVL,X,EVL2,Y)
      OM2=SQRT(ABS(1./EVL2))
      WRITE(5,220)OM2
      T2=2.*PI/OM2
      WRITE(5,213)T2
252   WRITE(5,214)
214   FORMAT(1H1,' THE EXACT VALUES FOR A CANTILEVER ARE:',T61,'THE CALCULATED
     UULATED VALUES ARE (USING FORCE METHOD OF ANALYSIS',//)
     WRITE(5,212)                                                       WEINH
212   FORMAT(56('-'),T61,56('-'),/)
      WRITE(5,215)
215   FORMAT(' MODE',T16,'1',T32,'2',T48,'3',T61,'MODE',T76,'1',T92,'2',
     IT108,'3')
      WRITE(5,212)                                                      WEINH
      WRITE(5,250)(FIRST(I),I=1,3),OMEGA,OM2
250   FORMAT(/,' OMEGA ',3(E13.6,3X),T61,'OMEGA ',2(E13.6,3X))
      WRITE(5,251)(TT(I),I=1,3),T,T2
251   FORMAT(/,' PERIOD',3(E13.6,3X,),T61,'PERIOD',2(E13.6,3X))
      WRITE(5,212)

C INSERT AFTER EACH SET OF DATA A CARD WITH 0 AND AFTER THE LAST SET ONE WITH 1

      READ(8,221)NEND
221   FORMAT(I1)
      IF(NEND)99,99,999
999   CALL EXIT

C-----SUBROUTINE SWIT TO FIND THE LARGEST EIGENVALUE
C     AND ITS CORRESPONDING EIGENVECTOR

      SUBROUTINE SWIT(A,N,EVL,X)                                        WEINH
      INTEGER L
      DIMENSION A(10,10),X(10),Z(10)
      DO 301 I=1,N                                                      WEINH
301   X(I)=1.0                                                          WEINH
      K=0                                                               WEINH
302   K=K+1
      DO 303 I=1,N                                                      WEINH
```

```
      Z(I)=0.                                                           WEINH
      DO 303 J=1,N                                                      WEINH
      Z(I)=A(I,J)*X(J)+Z(I)                                             WEINH
303   CONTINUE                                                          WEINH
      DO 304 I=1,N                                                      WEINH
304   X(I)=Z(I)                                                         WEINH
      L=1                                                               WEINH
      EVL=X(L)                                                          WEINH
      DO 306 I=1,N                                                      WEINH
306   X(I)=X(I)/EVL                                                     WEINH
      CHECK=10.
      IF(K-CHECK)302,308,308                                            WEINH
308   WRITE(5,300)K                                                     WEINH
300   FORMAT(//,' RESULTS FROM BASIC ITERATIVE METHOD, AFTER',I3,'  ITERWEINH
     1ATIONS',/)                                                        WEINH
      WRITE(5,310)EVL
310   FORMAT(/,' E-VALUE',6X,E13.6,/,' E-VECTOR')
      DO 312 I=1,N                                                      WEINH
312   WRITE(5,311)X(I)                                                  WEINH
311   FORMAT(14X,E13.6)
      RETURN                                                            WEINH

C-----SUBROUTINE MSMSW, TO FIND THE LARGEST EIGENVALUE (ONLY)
C     BY MATRIX SQUARING METHOD

      SUBROUTINE MSMSW(A,N,EAVG)                                        WEINH
      INTEGER L
      DIMENSION A(10,10),S(10,10),Z(10),EIG(10,10),P(10,10)
      KON=10
      NO=0                                                              WEINH
      DO 400 I=1,N                                                      WEINH
      DO 400 J=1,N                                                      WEINH
400   S(I,J)=A(I,J)
401   DO 408 I=1,N
      DO 408 J=1,N
408   P(I,J)=S(I,J)
      DO 402 I=1,N
      DO 403 J=1,N                                                      WEINH
      Z(J)=0.                                                           WEINH
      DO 403 K=1,N                                                      WEINH
403   Z(J)=Z(J)+S(I,K)*A(K,J)
      DO 402 J=1,N                                                      WEINH
402   S(I,J)=Z(J)                                                       WEINH
      NO=NO+1                                                           WEINH
      IF(NO-KON)401,401,404                                             WEINH
404   L=0                                                               WEINH
      SUM=0.                                                            WEINH
      DO 405 I=1,N                                                      WEINH
      DO 405 J=1,N                                                      WEINH
      EIG(I,J)=S(I,J)/P(I,J)
```

```
      L=L+1                                                                  WEINH
405   SUM=EIG(I,J)+SUM                                                       WEINH
      EAVG=SUM/L                                                             WEINH
      WRITE(5,406)                                                           WEINH
406   FORMAT(//,' CALCULATION OF LARGEST E-VALUE BY MEANS OF MATRIX SQUA
     URING METHOD',/)
      WRITE(5,407)EAVG                                                       WEINH
407   FORMAT(' AVERAGED EIGENVALUE  ',E13.6)                                 WEINH
      RETURN

C-----SUBROUTINE LAMBDA, TO FIND THE SECOND EIGENVALUE AND ITS CORRESPONDING
C     EIGENVECTOR USING WIELANDT'S DEFLATION

      SUBROUTINE LAMBDA(A,N,EVL1,X,EVL2,Y)
      INTEGER L
      REAL LI(10,10)                                                         WEINH
      DIMENSION A(10,10),X(10),XA(10,10),B(10,10),BCO(10,10),Z(10),Y(10)WEINH
     1,Y2(10),ALI(10,10)                                                     WEINH
      WRITE(5,85)                                                            WEINH
85    FORMAT(///,' EVALUATION OF SECOND EIGENVALUE USING WIELANDT S DEFLA
     UATION',/)
      L=1                                                                    WEINH
      DO 84 I=1,N                                                            WEINH
      DO 84 K=1,N                                                            WEINH
84    XA(I,K)=X(I)*A(L,K)                                                    WEINH
      DO 88 I=1,N                                                            WEINH
      DO 88 J=1,N                                                            WEINH
88    B(I,J)=A(I,J)-XA(I,J)                                                  WEINH
      K=N-1                                                                  WEINH
      DO 91 I=1,K                                                            WEINH
      DO 91 J=1,K                                                            WEINH
91    BCO(I,J)=B(I+1,J+1)                                                    WEINH
      KON=0                                                                  WEINH
      DO 92 I=1,K                                                            WEINH
92    Y(I)=1.                                                                WEINH
      M=K
      IT=0                                                                   WEINH
      AA=10.                                                                 WEINH
93    IT=IT+1                                                                WEINH
      DO 94 I=1,K
      Z(I)=0.                                                                WEINH
      DO 94 J=1,K
94    Z(I)=BCO(I,J)*Y(J)+Z(I)                                                WEINH
      DO 95 I=1,K
95    Y(I)=Z(I)                                                              WEINH
      L=1                                                                    WEINH
      EVL=Y(L)                                                               WEINH
      DO 97 I=1,K
      Y(I)=Y(I)/EVL                                                          WEINH
110   IF(ABS((Y(M)-AA)/Y(M))-.00005)111,97,97
```

```
111     M=M-1
        IF(M-1)99,99,97
99      KON=KON+1
97      CONTINUE
        AA=Y(M)
        IF(KON)93,93,60                                          WEINH
60      EVL2=EVL                                                 WEINH
        I=1                                                      WEINH
        Y2(I)=0.                                                 WEINH
        DO 62 I=2,N                                              WEINH
62      Y2(I)=Y(I-1)
        DO 64 I=1,N                                              WEINH
        DO 64 J=1,N                                              WEINH
64      LI(I,J)=0.                                               WEINH
        DO 65 I=1,N                                              WEINH
65      LI(I,I)=EVL1                                             WEINH
        DO 66 I=1,N                                              WEINH
        DO 66 J=1,N                                              WEINH
66      ALI(I,J)=A(I,J)-LI(I,J)                                  WEINH
        DO 69 I=1,N                                              WEINH
        Z(I)=0                                                   WEINH
        DO 69 J=1,N                                              WEINH
69      Z(I)=ALI(I,J)*Y2(J)+Z(I)
        DO 71 I=1,N                                              WEINH
71      X(I)=Z(I)
        L=1
        EVL=X(L)
        DO 107 I=1,N
107     Y(I)=X(I)/EVL
        WRITE(5,108)EVL2
108     FORMAT(//,' E-VALUE',6X,E13.6,/,' E-VECTOR')
        DO 112 I=1,N
112     WRITE(5,113)Y(I)
113     FORMAT(14X,E13.6)
        RETURN
        END                                                      WEINH
```

WSBEAM

WSBEAM - (METHOD USING FLEXIBILITY MATRIX FOR A SIMPLY SUPPORTED BEAM)

THIS PROGRAM WAS USED IN SECTION 4.3

```
C-----THIS PROGRAM APPLIES ONLY FOR A SIMPLY SUPPORTED BEAM,
C     WITH VERTICAL LOADING

      REAL L(30),MS(30),MASS(30,30)
      DIMENSION BO(30,30),FM(30,30),X(10),FB(30,30),F(30,30),BOT(30,30),
     UFMM(30,30)
      IW=5                                                       WEINH
```

```
      IR=8                                                                WEINH
C-----READ THE NUMBER OF THE BEAM SECTION                                 WEINH
1     READ(IR,100)M,LM
      N=M+1
C-----  LM =  0   THE MASS OF EACH SECTION IS LUMPED AT
C                 THE CENTRE OF THE SECTION
C       LM = +1   THE MASS OF EACH SECTION IS LUMPED AT
C                 THE ENDS OF THE SECTION
C       LM = -1   THE MASS IS LUMPED IN A DIFFERENT WAY
      IF(LM)20,21,22
20    WRITE(IW,133)
       GO TO 2
21     WRITE(IW,131)
      GO TO 2
22    WRITE(IW,132)
100    FORMAT( )
131   FORMAT(1H1,15(/),' THE MASS OF EACH SECTION IS CONCENTRATED AT THE
     U CENTRE OF EACH SECTION (DUNCANS MODEL)',/)
132   FORMAT(1H1,15(/),' THE MASS OF EACH SECTION IS CONCENTRATED AT THE ENDS
     U ENDS OF THE SECTION (RAYLEIGHS MODEL)',/)
133   FORMAT(1H1)
C-----READ THE LUMPED MASSES                                              WEINH
2     WRITE(IW,101)
101   FORMAT(//,' MASS MATRIX',/)
      READ(IR,100)(MS(I),I=1,M)
      DO 3 I=1,M                                                          WEINH
      MASS(I,I)=MS(I)                                                     WEINH
3     WRITE(IW,110)(MASS(I,J),J=1,M)
C-----READ THE DISTANCES BETWEEN THE LUMPED MASSES                        WEINH
      WRITE(IW,105)                                                       WEINH
105   FORMAT(/,' THE DISTANCES BETWEEN THE LUMPED MASSES',/)              WEINH
      TL=0.0                                                              WEINH
      READ(IR,100)(L(I),I=1,N)
      DO 4 I=1,N                                                          WEINH
      WRITE(IW,106)L(I)                                                   WEINH
106   FORMAT(2X,E13.6)                                                    WEINH
4     TL=TL+L(I)                                                          WEINH
      READ(IR,107)EI                                                      WEINH
107   FORMAT(F10.5)                                                       WEINH
      WRITE(IW,108)EI                                                     WEINH
108   FORMAT(/,' THE RIGIDITY VALUE',F10.4,/)
      CALL BOMSW
```

```
C-----SET UP OF MEMBER FLEXIBILITY MATRIX FM                            WEINH
      I=1                                                               WEINH
      J=1                                                               WEINH
6     K=I+1
      FM(I,J)=(L(I)+L(K))/(3.*EI)
      IF(I-M)13,12,12
13    J=J+1
      FM(I,J)=L(J)/(6.*EI)
      FM(J,I)=FM(I,J)                                                   WEINH
      I=I+1                                                             WEINH
      GO TO 6
C-----MULTIPLICATION OF BOT*FM*BO                                       WEINH
12    WRITE(IW,109)
109   FORMAT(/,' THE FLEXIBILITY MATRIX',/)                             WEINH
110   FORMAT(2X,8E13.6)                                                 WEINH
      CALL SWMUL (FM,BO,FB,M,M,M)
      DO 8 I=1,M                                                        WEINH
      DO 8 J=1,M
8     BOT(I,J)=BO(J,I)
      CALL SWMUL (BOT,FB,F,M,M,M)
      DO 9 I=1,M                                                        WEINH
9     WRITE(IW,110)(F(I,J),J=1,M)
C-----MULTIPLICATION OF  F*MASS                                         WEINH
      CALL SWMUL (F,MASS,FMM,M,M,M)
      WRITE(IW,130)
130   FORMAT(/,' (F*M) MATRIX)',/)
      DO 10 I=1,M                                                       WEINH
10    WRITE(IW,110)(FMM(I,J),J=1,M)
      PI=3.1415927                                                       WEINH
      CALL EVAL1(FMM,N,EV1,X)
      WRITE(IW,116)                                                     WEINH
116   FORMAT(///,' THE FIRST EIGENVALUE IS OBTAINED BY AN ITERATIVE METHWEINH
     1OD',//)                                                           WEINH
      WRITE(IW,117)EV1                                                  WEINH
117   FORMAT(/,' E-VALUE',T12,E13.6,/,' E-VECTOR',)                     WEINH
      DO 16 I=1,M
16    WRITE(IW,118)X(I)                                                 WEINH
118   FORMAT(T12,E13.6)                                                 WEINH
      O=SQRT(1./EV1)                                                    WEINH
      T=2.*PI/O                                                         WEINH
      WRITE(IW,119)O                                                    WEINH
119   FORMAT(/,' OMEGA ',T12,E13.6)                                     WEINH
      WRITE(IW,120)T                                                    WEINH
120   FORMAT(/,' PERIOD',T12,E13.6)                                     WEINH
      SL=1.
```

```
      SM=1.
      CNST=SQRT(EI/(SL**3.*SM))
      OM=PI**2.*CNST
      WRITE(IW,121)OM
121   FORMAT(//,'   EXACT VALUE OF FUNDAMENTAL FREQUENCY FOR A ',/,'    CON
     UONTINUOUS SIMPLY SUPPORTED BEAM',T50,E13.6)
      WRITE(IW,122)M,O
122   FORMAT(//,'   FUNDAMENTAL FREQUENCY FOR A SEGMENTED SIMPLY',/,'    SUPPORTE
     USUPPORTED BEAM HAVING ',I2,' LUMPED MASSES',T50,E13.6)
      READ(IR,99)NO
99    FORMAT(I1)
      IF(NO)999,1,999
999   CALL EXIT

C-----SUBROUTINE TO SET UP THE CONNECTION MATRIX BO

      SUBROUTINE BOMSW                                              WEINH
      DIMENSION RA(10),RB(10)                                       WEINH

C-----SUPPORT REACTIONS                                             WEINH

      SL=0.0                                                        WEINH
      DO 200 I=1,M
      SL=SL+L(I)                                                    WEINH
      RA(I)=(TL-SL)/TL                                              WEINH
200   RB(I)=1.-RA(I)
      J=1
201   SL=0.0
      I=M                                                           6
      K=N
202   SL=SL+L(K)
      BO(I,J)=RB(J)*SL
      IF(K-J-1)205,205,204                                          WEINH
204   I=I-1
      K=K-1
      GO TO 202
205   IF(J-1)206,207,206
206   I=I-1
      K=K-1
      SL=SL+L(K)
      BO(I,J)=RA(J)*(TL-SL)
207   IF(I-1)206,208,206
208   J=J+1
      IF(J-1-M)201,210,201                                          WEINH
210   WRITE(IW,211)                                                 WEINH
211   FORMAT(/,' CONNECTION MATRIX BO',/)                           WEINH
      DO 212 I=1,M
212   WRITE(IW,213)(BO(I,J),J=1,M)
213   FORMAT(2X,8E13.6)                                             WEINH
      RETURN                                                        WEINH
```

```
C-----SUBROUTINE FOR MULTIPLICATION OF 2 MATRICES                       WEINH

      SUBROUTINE SWMUL (A,B,PR,NA,NB,NC)
      DIMENSION A(30,30),B(30,30),C(30),PR(30,30)                      WEINH
      DO 401 I=1,NA
      DO 402 J=1,NC                                                    W¢47¢H
      C(J)=0.0                                                         WEINH
      DO 402 K=1,NB                                                    W¢47¢H
402   C(J)=C(J)+A(I,K)*B(K,J)
      DO 401 J=1,NC
401   PR(I,J)=C(J)
      RETURN                                                           WEINH

C-----SUBROUTINE FOR BASIC ITERATION PROCES

      SUBROUTINE EVAL1(A,N,EV1,X)
      DIMENSION A(30,30),X(30),Z(30)
      ML=M                                                             WEINH
      KON=0                                                            WEINH
      AA=10.                                                           WEINH
      DO 300 I=1,M                                                     WEINH
300   X(I)=1.0                                                         WEINH
      K=0                                                              WEINH
301   K=K+1                                                            WEINH
      DO 302 I=1,M                                                     WEINH
      Z(I)=0.                                                          WEINH
      DO 302 J=1,M                                                     WEINH
302   Z(I)=A(I,J)*X(J)+Z(I)                                            WEINH
      DO 303 I=1,M                                                     W¢47¢H
303   X(I)=Z(I)
      MAX=1                                                            W¢47¢H
      DO 304 J=2,M                                                     WEINH
      IF(X(J)-X(MAX))304,304,305                                       W¢47¢H
305   MAX=J                                                            WEINH
304   CONTINUE                                                         WEINH
      EV1=X(MAX)                                                       WEINH
      DO 306 I=1,M                                                     WEINH
306   X(I)=X(I)/EV1                                                    WEINH
      EPS=.000005                                                      WEINH
      IF(ABS((X(ML)-AA)/X(ML))-EPS)307,309,309
307   ML=ML-1
      IF(ML)308,308,309
308   KON=KON+1                                                        WEINH
309   AA=X(ML)
      IF(KON)301,301,310                                               WEINH
310   RETURN                                                           WEINH
      END                                                              WEINH
```

WTIM

```
                WTIM - (PROGRAM TO OBTAIN A GRAPH SHOWN IN TIMOSHENKO)
                THIS PROGRAM WAS USED IN SECTION 5.2

      IR=8                                                          WEINH
      IW=5                                                          WEINH
      RAD=3.1415927/180.                                            WEINH
      WRITE(IW,100)                                                 WEINH
100   FORMAT(1H ,T6,'X',T20,'XRAD',T36,'COT',T50,'COTH',T66,'PHI',/) WEINH
      X=0.                                                          WEINH
1     X=X+10.
      XRAD=X*RAD                                                    WEINH
      A=TAN(XRAD)                                                   WEINH
      COT=1./A                                                      WEINH
      B=TANH(XRAD)                                                  WEINH
      COTH=1./B                                                     WEINH
      PHI=COTH-COT
      WRITE(IW,101)X,XRAD,COT,COTH,PHI                              WEINH
101   FORMAT(1H ,5(E13.6,2X))
      IF(X-300)1,,                                                  WEI5H
      CALL EXIT                                                     WEI5H
      END                                                           WEI5H
```

WDUNK

```
                WDUNK - (DUNKERLEY'S METHOD)
                THIS PROGRAM WAS USED IN SECTION 5.3

      REAL L1(10),L2(10),M1(10),M2(10),LL,LR
      DIMENSION AL(10),AM(10),Y(10)
      IR=8                                                          WEINH
      IW=5                                                          WEINH
1     READ(IR,100)N1,N2                                             WEINH
      READ(IR,100)SPN1,SPN2
      READ(IR,100)(L1(I),I=1,N1)                                    WEINH
      READ(IR,100)(L2(I),I=1,N2)                                    WEINH
      MM1=N1-1                                                      WEINH
      MM2=N2-1
      READ(IR,100)(M1(I),I=1,MM1)                                   WEINH
      READ(IR,100)(M2(I),I=1,MM2)                                   WEINH
      READ(IR,100)EI                                                WEINH
100   FORMAT( )                                                     WEINH
C        N1      =  NUMBER OF SEGMENTS IN THE FIRST SPAN
C        N2      =  NUMBER OF SEGMENTS IN THE SECOND SPAN
C        SPN1    =  LENGTH OF FIRST SPAN
C        SPN2    =  LENGTH OF SECOND SPAN
C        L1(I)   =  LENGTH OF SEGMENTS IN THE FIRST SPAN
C        L2(I)   =  LENGTH OF SEGMENTS IN THE SECOND SPAN
C        M1(I)   =  LUMPED MASSES IN THE FIRST SPAN
```

```
C         M2(I)    =  LUMPED MASSES IN THE SECOND SPAN
C         EI       =  RIGIDITY VALUE

      WRITE(IW,102)
102   FORMAT(1H1,'FIRST SPAN',/)
      WRITE(IW,103)
103   FORMAT(T36,'LUMPED',T51,'DISTANCE FROM',/,T36,'MASSES',T52,'END SUPPORT')
     UPPORT')
      WRITE(IW,104)MM1,SPN1
104   FORMAT(5X,'NUMBER OF MASSES  =',I3,T72,'LENGTH OF SPAN',F10.6)
      DO 8 I=1,N1
8     WRITE(IW,105)M1(I),L1(I)
105   FORMAT(32X,3(E13.6,3X))
      WRITE(IW,106)
106   FORMAT(//,' SECOND SPAN',/)
      WRITE(IW,103)
      WRITE(IW,104)MM2,SPN2
      DO 9 I=1,N2
9     WRITE(IW,105)M2(I),L2(I)
      K=0                                                                 WEINH
      WRITE(IW,200)                                                       WEINH
200   FORMAT(///,I5,'I',T19,'A',T39,'C',T59,'Y',/)                        WEINH
      DO 2 I=1,N1                                                         WEINH
      AL(I)=L1(I)                                                         WEINH
2     AM(I)=M1(I)                                                         WEINH
      LL=SPN1                                                             WEINH
      LR=SPN2                                                             WEINH
      G=9.81                                                              WEINH
3     I=1                                                                 WEINH
      SUM=0.                                                              WEINH
4     A=AM(I)*(LL-AL(I))/LL-AM(I)*AL(I)*(LL**2.-(AL(I))**2.)/(2.*LL**2.*(LL+LR
     I(LL+LR))
      C=1./(6.*LL)*(A*LL**3.-AM(I)*(LL-AL(I))**3.)                        WEINH
      Y(I)=(1./EI)*(-A*(AL(I))**3./6.+C*AL(I))                            WEINH
      Y(I)=Y(I)*G

C--------Y IS MULTIPLIED BY G BECAUSE  A AND C WERE IN MASS UNITS
C         INSTEAD OF IN FORCE UNITS

      SUM=SUM+Y(I)                                                        WEINH
      WRITE(IW,201)I,A,C,Y(I)                                             WEINH
201   FORMAT(I5,3(7X,E13.6),/)                                            WEINH
      I=I+1                                                               WEINH
      IF(K),,6                                                            WEINH
      IF(I-N1)4,,                                                         WEINH
      K=K+1                                                               WEINH
      DO 5 I=1,10
      AL(I)=L2(I)                                                         WEINH
5     AM(I)=M2(I)
      LL=SPN2                                                             WEINH
```

```
      LR=SPN1                                                           WEINH
      SD1=SUM                                                           WEINH
      WRITE(IW,202)                                                     WEINH
202   FORMAT(//)                                                        WEINH
      GO TO 3                                                           WEINH
6     IF(I-N2)4,,                                                       WEINH
      SD2=SUM                                                           WEINH
      P=SQRT(G/(SD1+SD2))
      WRITE(IW,101)P                                                    WEINH
101   FORMAT(/,' THE FUNDAMENTAL FREQUENCY IS',F10.4)
      DO 7 I=1,10
      L1(I)=0.
      L2(I)=0.
      M1(I)=0.
      M2(I)=0.
      AM(I)=0.
7     AL(I)=0.
      READ(IR,100)NO
      IF(NO)1,1,
      CALL EXIT                                                         WEINH
      END                                                               WEINH
```

WTMCB

WTMCB - (TRANSFER MATRIX METHOD FOR A TWO-SPAN CONTINUOUS BEAM)
THIS PROGRAM WAS USED IN SECTION 5.4

```
      REAL MA(20),L(20)
      DIMENSION EI(20),T(4,4),P(4,4),D(4,4),U(4,4),A(4,4),E(4,2),G(4,2) WEINH
      DIMENSION B(4,4),C(4)
      IR=8                                                              WEINH
      IW=5                                                              WEINH
C        N1 = NUMBER OF SEGMENTS OF FIRST SPAN                          WEINH
C        N2 = NUMBER OF SEGMENTS OF SECOND SPAN                         WEINH
C        MA = LUMPED MASSES ON BOTH SPANS                               WEINH
C        L  = LENGTHS OF SEGMENTS OF BOTH SPANS                         WEINH
C        EI = BENDING STIFFNESSES OF BOTH SPANS                         WEINH
1     READ(IR,100)N1,N2                                                 WEINH
      N=N1+N2                                                           WEINH
      M=N-1
      READ(IR,100)(MA(I),I=1,M)
      READ(IR,100)(L(I),I=1,N)                                          WEINH
      READ(IR,100)(EI(I),I=1,N)                                         WEINH
100   FORMAT( )                                                         WEINH
      WRITE(IW,101)                                                     WEINH
101   FORMAT(1H1,'FIRST SPAN')
      WRITE(IW,102)                                                     WEINH
102   FORMAT(T36,'LUMPED',T51,'LENGTHS OF',T71,'BENDING',/,T36,'MASSES',WEINH
```

```
     2T52,'SEGMENTS',T71,'RIGIDITY')                                   WEINH
      WRITE(IW,103)N1                                                  WEINH
103   FORMAT(5X,'NUMBER OF SEGMENTS =',I3)                             WEINH
      DO 2 I=1,N1                                                      WEINH
2     WRITE(IW,104)MA(I),L(I),EI(I)                                    WEINH
104   FORMAT(32X,3(E13.6,3X))                                          WEINH
      WRITE(IW,105)                                                    WEINH
105   FORMAT(///,' SECOND SPAN',/)                                     WEINH
      WRITE(IW,102)                                                    WEINH
      WRITE(IW,103)N2                                                  WEINH
      NN=N1+1                                                          WEINH
      DO 3 I=NN,N
3     WRITE(IW,104)MA(I),L(I),EI(I)                                    WEINH
200   FORMAT(/)                                                        WEINH
      OM=6.
      STEP=.5
      END=14.
      WRITE(IW,106)                                                    WEINH
106   FORMAT(//,T36,'VALUE OF DETERMINANT',T61,'TRIAL VALUE FOR P',/,T35WEINH
     1,40('-'))                                                        WEINH
11    K=1                                                              WEINH
10    CALL FIELD                                                       WEINH
      DO 4 I=1,4
      DO 4 J=1,4                                                       WEINH
4     U(I,J)=T(I,J)                                                    WEINH
5     CALL POINT                                                       WEINH
      CALL PROD(P,U,D)                                                 WEINH
      K=K+1                                                            WEINH
      CALL FIELD                                                       WEINH
      CALL PROD(T,D,U)                                                 WEINH
      IF(K-N1)5,6,8                                                    WEINH
6     DO 7 I=1,4                                                       WEINH
      DO 7 J=1,4                                                       WEINH
7     A(I,J)=U(I,J)                                                    WEINH

C        ' A '  IS THE OVERALL TRANSFER MATRIX FOR THE FIRST SPAN

      K=K+1                                                            WEINH
      GO TO 10                                                         WEINH
8     K=K+1
      IF(K-N)5,5,
      DO 9 I=1,4                                                       WEINH
      DO 9 J=1,4                                                       WEINH
9     B(I,J)=U(I,J)                                                    WEINH

C--------' B '  IS THE OVERALL TRANSFER MATRIX FOR THE SECOND SPAN

C--------IN THE FOLLOWING PART THE MATRIX IS SET UP WHICH CONNECTS
C        THE TWO STATE VECTORS AT THE INTERMEDIATE SUPPORT
```

```
      DO 12 I=2,4                                                       WEINH
12    E(I,1)=(A(1,4)*A(I,2)-A(1,2)*A(I,4))/A(1,4)                       WEINH
      E(4,2)=1.

C--------MULTIPLY B * E

      DO 14 I=1,4
      DO 15 J=1,2
      C(J)=0.
      DO 15 KN=1,4
15    C(J)=C(J)+B(I,KN)*E(KN,J)
      DO 14 J=1,2
14    G(I,J)=C(J)
      DET=G(1,1)*G(3,2)-G(3,1)*G(1,2)
      WRITE(IW,107)DET,OM                                               WEINH
107   FORMAT(T30,2(10X,E13.6))
      OM=OM+STEP                                                        WEINH
      IF(OM.LE.END)GO TO 11
      DO 13 I=1,20                                                      WEINH
      MA(I)=0.                                                          WEINH
      L(I)=0.                                                           WEINH
13    EI(I)=0.                                                          WEINH
      READ(IR,100)NO
      IF(NO)1,1,
      CALL EXIT                                                         WEINH

      SUBROUTINE PROD(A,B,PR)                                           WEINH
      DIMENSION A(4,4),B(4,4),PR(4,4),C(4)                              WEINH
      DO 1 I=1,4                                                        WEINH
      DO 2 J=1,4                                                        WEINH
      C(J)=0.                                                           WEINH
      DO 2 KT=1,4                                                       WEINH
2     C(J)=C(J)+A(I,KT)*B(KT,J)                                         WEINH
      DO 1 J=1,4                                                        WEINH
1     PR(I,J)=C(J)                                                      WEINH
      RETURN                                                            WEINH

      SUBROUTINE FIELD                                                  WEINH
      DO 1 I=1,4                                                        WEINH
1     T(I,I)=1.                                                         WEINH
      T(1,2)=L(K)                                                       WEINH
      T(1,3)=L(K)**2./(2.*EI(K))                                        WEINH
      T(1,4)=L(K)**3./(6.*EI(K))                                        WEINH
      T(2,3)=L(K)/EI(K)                                                 WEINH
      T(2,4)=T(1,3)                                                     WEINH
      T(3,4)=L(K)                                                       WEINH
      RETURN                                                            WEINH

      SUBROUTINE POINT                                                  WEINH
      DO 1 I=1,4                                                        WEINH
```

```
1     P(I,I)=1.                                                         WEINH
      P(4,1)=OM*OM*MA(K)                                                WEINH
      RETURN
      END                                                               WEINH
```

WCONT

```
         WCONT - (METHOD USING FLEXIBILITY MATRIX FOR MULTI-SPAN CON-
            TINUOUS BEAM)
         THIS PROGRAM WAS USED IN SECTION 5.5

      REAL L(10),MARR(10),MALL(5,30)
      DIMENSION FM(30,30),B00(30,30),B0(10,10,10),NBEAM(10),MBEAM(10),DI WEINH
     1S(10,10),F(30,30),BIX(30,30),BOT(30,30),B(30,30),BIT(30,30),D(30,3W¢4?¢H
     10),G(30,30),X(30,30),JC(30),V(7),BI(30,30),E(30,30),ALLM(30,30),S(WEINH
     230,30)                                                            WEINH
      DIMENSION Y(30)
      IR=8
      IW=5
1     READ(IR,100)NSPAN
      WRITE(IW,110)
110   FORMAT(1H1,11(/))
      DO 50 I=1,30
      DO 50 J=1,30
50    B00(I,J)=.0
      K=1                                                               WEINH
      NS=0                                                              WEINH
      MS=0                                                              WEINH
      IB=0                                                              WEINH
      JB=0                                                              WEINH
C-----READ THE NUMBER OF BEAM SECTIONS N AND LUMPED MASSES M             WEINH
2     READ(IR,100)N
      M=N-1                                                             WEINH
      NBEAM(K)=N
      NS=NS+NBEAM(K)
      MBEAM(K)=NBEAM(K)-1
      MS=MS+MBEAM(K)
C-----READ THE LUMPED MASSES
      WRITE(IW,101)K                                                    WEINH
101   FORMAT(//,' SPAN',I3,T18,'LUMPED MASSES')
      READ(IR,100)(MARR(I),I=1,M)
100   FORMAT()
      DO 3 I=1,M
      MALL(K,I)=MARR(I)
3     WRITE(IW,103)MARR(I)
103   FORMAT(T41,E13.6)
```

```
C-----READ THE DISTANCES BETWEEN THE LUMPED MASSES

      WRITE(IW,105)
105   FORMAT(/,T18,'LENGTH OF SEGMENTS')
      TL=0.0
      READ(IR,100)(L(I),I=1,N)
      DO 4 I=1,N
      DIS(K,I)=L(I)                                          WEINH
      WRITE(IW,103)L(I)
4     TL=TL+L(I)
      READ(IR,100)EI
      WRITE(IW,106)EI
106   FORMAT(//,T18,'RIGIDITY VALUE',T41,E13.6)
      CALL BMOD
      I=0                                                    WEINH
      J=0
9     IB=IB+1
      I=I+1
11    JB=JB+1
      J=J+1
      BOO(IB,JB)=BO(K,I,J)
      IF(J.LT.M)GO TO 11                                     WEINH
      IF(I-M)12,13,13
12    J=J-M                                                  WEINH
      JB=JB-M                                                WEINH
      GO TO 9
13    IF(K-NSPAN)14,21,21                                    WEINH
14    K=K+1
      IB=IB+1
      GO TO 2
21    WRITE(IW,107)
107   FORMAT(//,' MATRICES FOR THE'/' CONTINUOUS BEAM')
      WRITE(IW,109)
109   FORMAT(/,T18,'CONNECTION MATRIX BO')
      JBO=MS
      IBO=NS-1                                               WEINH
      DO 5 IB=1,IBO                                          WEINH
5     WRITE(IW,108)(BOO(IB,JB),JB=1,JBO)
108   FORMAT(T41,6(E13.6))

C-----ASSEMBLING ALL DISTANCES IN ONE ARRAY                  WEINH

      KT=0                                                   WEINH
      K=0                                                    WEINH
15    I=0                                                    WEINH
      K=K+1                                                  WEINH
16    KT=KT+1                                                WEINH
      I=I+1                                                  WEINH
      L(KT)=DIS(K,I)                                         WEINH
```

```
      IF(I.LT.NBEAM(K))GO TO 16                                         WEINH
      IF(K.LT.NSPAN)GO TO 15                                            WEINH
C-----SET UP OF FLEXIBILITY MATRIX FOR MEMBER ENDS                      WEINH
      NF=NS-1
      J=1                                                               WEINH
      I=1                                                               WEINH
6     K=I+1                                                             WEINH
      FM(I,J)=(L(I)+L(K))/(3.*EI)                                       WEINH
      IF(I-NF)10,7,7                                                    WEINH
10    J=J+1
      FM(I,J)=L(J)/(6.*EI)                                              WEINH
      FM(J,I)=FM(I,J)                                                   WEINH
      I=I+1                                                             WEINH
      GO TO 6
7     WRITE(IW,111)                                                     WEINH
111   FORMAT(/,T18,'FLEXIBILITY MATRIX',/,T18,'FOR MEMBER ENDS')
      DO 8 I=1,NF                                                       WEINH
8     WRITE(IW,108)(FM(I,J),J=1,NF)
      CALL BIM(NBEAM)
      CALL TRIPM(BIT,FM,BI,D,NP,NF,NF,NP)                               WEINH
      WRITE(IW,112)                                                     WEINH
112   FORMAT(/,T18,'D-MATRIX')
      DO 17 I=1,NP                                                      WEINH
17    WRITE(IW,108)(D(I,J),J=1,NP)
C-----INVERSE FROM MATH PACK                                            WEINH
                                                                        WEINH
      WRITE(IW,110)
      V(1)=3                                                            WEINH
      CALL GJR(D,30,30,NP,NP,$999,JC,V)                                 WEINH
      WRITE(IW,113)                                                     WEINH
113   FORMAT(/,T18,'INVERSE OF D')
      DO 18 I=1,NP                                                      WEINH
18    WRITE(IW,108)(D(I,J),J=1,NP)
      CALL TRIPM(BIT,FM,BOO,G,NP,NF,NF,MS)                              WEINH
      CALL TWICE(D,G,X,NP,NP,MS)                                        WEINH
      CALL TWICE(BI,X,BIX,NF,NP,MS)                                     WEINH
      DO 19 I=1,IBO                                                     WEINH
      DO 19 J=1,JBO                                                     WEINH
      B(I,J)=BOO(I,J)-BIX(I,J)
19    BOT(J,I)=BOO(I,J)                                                 WEINH
      CALL TRIPM(BOT,FM,B,F,JBO,NF,NF,JBO)                              WEINH
      WRITE(IW,114)                                                     WEINH
114   FORMAT(/,T18,'FLEXIBILITY MATRIX F')
      DO 20 I=1,JBO                                                     WEINH
20    WRITE(IW,108)(F(I,J),J=1,JBO)
      KT=0                                                              WEINH
      K=0                                                               WEINH
```

```
25     I=0                                                          WEINH
       K=K+1                                                        WEINH
26     KT=KT+1                                                      WEINH
       I=I+1                                                        WEINH
       ALLM(KT,KT)=MALL(K,I)                                        WEINH
       IF(I.LT.MBEAM(K))GO TO 26                                    WEINH
       IF(K.LT.NSPAN)GO TO 25                                       WEINH
       WRITE(IW,118)                                                WEINH
118    FORMAT(/,T18,'MASS MATRIX')
       DO 27 I=1,JBO                                                WEINH
27     WRITE(IW,108)(ALLM(I,J),J=1,JBO)
       CALL TWICE(F,ALLM,S,JBO,JBO,JBO)

       CALL ITERW(S,JBO,EVL,Y)                                      WEINH

C-----USING JACMX FROM MATH-PACK TO CALCULATE EIGENVALUES AND EIGENVECTORS

       WRITE(IW,115)
115    FORMAT(//,' RESULTS FROM MATH-PACK PROGRAM JACMX WHICH DETERMINES WEINH
      1EIGENVALUES AND EIGENVECTORS',/,' (ONLY FOR SYMMETRICAL SYSTEMS MAWEINH
      2TRICES)',//)                                                 WEINH
       ACC=.000001
       IT=400
       CALL JACMX(S,E,30,JBO,ACC,IT,$22,1)                          WEINH
22     WRITE(IW,116)
116    FORMAT(T18,'THE MATRIX CONTAINS',/,T18,'THE E-VALUES ALONG',/,T18,
      U'THE DIAGONAL')
       DO 23 I=1,JBO
23     WRITE(IW,108)(S(I,J),J=1,JBO)
       WRITE(IW,117)
117    FORMAT(/,T18,'THE MATRIX CONTAINS',/,T18,'THE E-VECTORS',)
       DO 24 I=1,JBO
24     WRITE(IW,108)(E(I,J),J=1,JBO)
       READ(IR,99)NO                                                WEINH
99     FORMAT(I1)                                                   WEINH
       IF(NO)999,1,999
999    CALL EXIT

C-----SUBROUTINE TO SET UP THE CONNECTION MATRIX BO

       SUBROUTINE BMOD                                              WEINH
       DIMENSION RA(10),RB(10)                                      WEINH

C-----SUPPORT REACTIONS                                             WEINH

       SL=0.0                                                       WEINH
       DO 520 I=1,M                                                 WEINH
       SL=SL+L(I)                                                   WEINH
       RA(I)=(TL-SL)/TL                                             WEINH
520    RB(I)=1.-RA(I)                                               WEINH
```

```
      J=1                                                               WEINH
530   SL=0.0                                                            WEINH
      I=M                                                               WEINH
      KNT=N
532   SL=SL+L(KNT)
      BO(K,I,J)=RB(J)*SL
      IF(KNT-J-1)534,534,533
533   I=I-1                                                             WEINH
      KNT=KNT-1
      GO TO 532                                                         WEINH
534   IF(J-1)535,536,535                                                WEINH
535   I=I-1                                                             WEINH
      KNT=KNT-1
      SL=SL+L(KNT)
      BO(K,I,J)=RA(J)*(TL-SL)
536   IF(I-1)535,537,535                                                WEINH
537   J=J+1                                                             WEINH
      IF(J-1-M)530,538,530                                              WEINH
538   RETURN                                                            WEINH
C-----SUBROUTINE TO SET UP THE CONNECTION MATRIX BI FOR THE REDUNDANCIES WEINH
      SUBROUTINE BIM(N)                                                 WEINH
      DIMENSION N(30)
      DO 1 I=1,30
       DO 1 J=1,30
1     BI(I,J)=.0
      J=0                                                               WEINH
      NEND=N(J+1)
2     J=J+1                                                              EINH
      TL1=0.
      TL2=0.                                                             WEINH
      NSUM=NEND
      KT=NEND-N(J)
5     KT=KT+1                                                           WEINH
      TL1=TL1+L(KT)                                                     WEINH
      IF(KT-NEND)5,4,4
4     NEND=NEND+N(J+1)
6     KT=KT+1                                                           WEINH
      TL2=TL2+L(KT)                                                     WEINH
      IF(KT-NEND)6,7,7
7     SL=0.                                                             WEINH
      I=NSUM-N(J)
      RA=1./TL1                                                         WEINH
      RB=1./TL2                                                         WEINH
8     I=I+1
      SL=SL+L(I)                                                        WEINH
      BI(I,J)=RA*SL                                                     WEINH
      IF(I-NSUM)8,9,9
9     SL=0.
```

```
      NEND=NSUM+N(J+1)
10    I=I+1                                                         WEINH
      SL=SL+L(I)                                                    WEINH
      BI(I,J)=RB*(TL2-SL)                                           WEINH
      IF(I-NEND+1)10,11,11
11    IF(J-NSPAN+1)2,12,12
12    WRITE(IW,100)                                                 WEINH
100   FORMAT(/,T18,'CONNECTION MATRIX BI')
      NP=NSPAN-1
      DO 14 I=1,NF
14    WRITE(IW,101)(BI(I,J),J=1,NP)                                 WEINH
101   FORMAT(T41,6(E13.6))
      DO 15 I=1,NF
      DO 15 J=1,NP                                                  WEINH
15    BIT(J,I)=BI(I,J)                                              WEINH
      RETURN                                                        WEINH

C-----SUBROUTINE TO MULTIPLY 2 MATRICES

      SUBROUTINE TWICE(A,B,PR,NA,NB,NC)                             WEINH
      DIMENSION A(30,30),B(30,30),PR(30,30),C(30)                   WEINH
      DO 601 I=1,NA                                                 WEINH
      DO 602 J=1,NC                                                 WEINH
      C(J)=0.                                                       WEINH
      DO 602 K=1,NB                                                 WEINH
602   C(J)=C(J)+A(I,K)*B(K,J)                                       WEINH
      DO 601 J=1,NC                                                 WEINH
601   PR(I,J)=C(J)                                                  WEINH
      RETURN                                                        WEINH

C-----SUBROUTINE TO MULTIPLY 3 MATRICES

      SUBROUTINE TRIPM(A,B,C,PR,NA,NB,NC,ND)                        WEINH
      DIMENSION A(30,30),B(30,30),C(30,30),PR(30,30),Z(30),DUM(30,30)
      DO 601 I=1,NA                                                 WEINH
      DO 602 J=1,NC                                                 WEINH
      Z(J)=0.                                                       WEINH
      DO 602 K=1,NB                                                 WEINH
602   Z(J)=Z(J)+A(I,K)*B(K,J)                                       WEINH
      DO 601 J=1,NC                                                 WEINH
601   DUM(I,J)=Z(J)
      DO 603 I=1,NB                                                 WEINH
      DO 604 J=1,ND                                                 WEINH
      Z(J)=0.                                                       WEINH
      DO 604 K=1,NC                                                 WEINH
604   Z(J)=Z(J)+DUM(I,K)*C(K,J)
      DO 603 J=1,ND
603   PR(I,J)=Z(J)
      RETURN                                                        WEINH
```

```
      SUBROUTINE ITERW(A,N,EVL,X)                                       WEINH
      INTEGER L                                                         WEINH
      DIMENSION A(30,30),X(30),Z(30)
      K=0
      IE=0
      IS=1
      K=K+1
12    IE=IE+MBEAM(K)
      DO 13 I=IS,IE
13    X(I)=1.
      IF(K-NSPAN),15,15
      K=K+1
      IS=IS+MBEAM(K)
      IE=IE+MBEAM(K)
      DO 14 I=IS,IE
14    X(I)=-1.
      IF(K-NSPAN),15,15
      K=K+1
      IS=IS+MBEAM(K)
      GO TO 12
15    K=0
2     K=K+1                                                             WEINH
      DO 3 I=1,N                                                        WEINH
      Z(I)=0.                                                           WEINH
      DO 3 J=1,N                                                        WEINH
3     Z(I)=A(I,J)*X(J)+Z(I)                                             WEINH
      DO 4 I=1,N                                                        WEINH
4     X(I)=Z(I)                                                         WEINH
      L=1                                                               WEINH
      EVL=X(L)                                                          WEINH
      DO 6 I=1,N                                                        WEINH
6     X(I)=X(I)/EVL                                                     WEINH
      CHECK=20.
      IF(K-CHECK)2,,                                                    WEINH
      WRITE(IW,7)K                                                      WEINH
7     FORMAT(//,' RESULTS FROM BASIC ITERATIVE METHOD, AFTER',I3,' ITERA
     UTIONS',/)
      WRITE(IW,8)EVL                                                    WEINH
8     FORMAT(/,T18,'EIGENVALUE',T41,E13.6,/,T18,'EIGENVECTOR')
      DO 9 I=1,N                                                        WEINH
9     WRITE(IW,10)X(I)                                                  WEINH
10    FORMAT(T41,E13.6)
      P=SQRT(1./EVL)

C-----THIS IS CORRECT AS LONG AS EI,M AND L OF THE SECOND SPAN ARE
C     EXPRESSED IN TERMS OF THE FIRST SPAN, WITH EI,M AND L EQUAL TO UNITY

      WRITE(IW,11)P
11    FORMAT(/,T18,'FUNDAMENTAL FREQUENCY',T41,E13.6)
      RETURN                                                            WEINH
```