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Geometric realisation of representations of complex semi-simple
Lie groups

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I dedicate this dissertation to Dr Japie Vermeulen.

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Contents

1	Introduction	1
2	Lie groups and Lie algebras	2
2.1	Semi-simple Lie algebras	4
2.2	Borel subalgebras	11
2.3	Verma modules	12
3	Weyl's Construction	15
3.1	Semi-simple algebras and commutators	15
3.2	Young diagrams and representations of \mathfrak{S}_n	18
3.3	Representations of $SL(V)$	20
3.4	Representations of $SP(V)$	23
3.5	Representations of $SO(V)$	25
4	Induced Representations on Flag Manifolds	30
4.1	Flag manifolds	30
4.2	Vector bundles on Flag manifolds	37
5	The Bernstein-Gel'fand-Gel'fand Resolution	41
5.1	The Casimir Operator	41
5.2	The Bruhat order and the Weyl group	43
5.3	Homology of Lie algebras	47
5.4	The weak Bernstein-Gel'fand-Gel'fand Resolution	51
5.5	The Bernstein-Gel'fand-Gel'fand Resolution	53
6	The Borel-Weil-Bott Theorem	58
7	Cohomology of Flag manifolds	66
7.1	Borel's theorem	66
7.2	Bruhat cells	70
7.3	The ring of polynomials on \mathfrak{h}	73
7.4	Schubert varieties	77

1 Introduction

This dissertation looks at some aspects of the representation theory of complex semi-simple Lie groups.

Section 2 gives a brief overview of the theory of Lie groups and their corresponding Lie algebras. We concentrate on Lie algebras since these are more tractable, describing their root structures and fleshing out the examples of the three classical cases in considerable detail. We then look at certain distinguished sub-algebras of semi-simple Lie algebras before moving onto the key theorem in this section, the Theorem of Highest Weight. This provides a 1 – 1 correspondence between certain roots, and irreducible representations of Lie algebras, and will be used repeatedly in the sequel.

Section 3 provides a method of constructing representations of the classical groups “by hand”. It is highly combinatorial, using a duality between \mathfrak{S}_k -modules and $GL(V)$ -modules. We start by using a combinatorial object known as a Young diagram to construct irreducible representations of the symmetric group. Then using the duality and after some work we are able to produce representations of certain subgroups of $GL(V)$ – the classical groups. We parametrise these representations with classes of Young diagrams.

Section 4 provides a second method of producing the irreducible representations of the classical groups, this time more geometrically and less explicitly. Representations are constructed as the spaces of sections of line bundles of smooth algebraic varieties known as flag manifolds. The Young diagram corresponding to the representation is shown to encode information regarding the geometry of the flag manifold.

In Section 5 we change tack radically. This section looks at the structure of infinite-dimensional highest weight modules known as Verma modules. These are in some sense the simplest highest weight modules from which all others can be constructed. We use the Verma modules to construct free resolutions of irreducible representations parametrised by the Weyl group and the Bruhat order.

Section 6 generalises the results of §4 using §5. Using an easy result on the cohomology of the complex projective space \mathbb{P}^1 , we are able to deduce the Bott vanishing theorem, and then we derive the Borel-Weil theorem, the climax of the dissertation.

We finish off in Section 7 by taking a closer look at the geometry of the flag manifolds. We find that the cohomology ring can be described as the space of coinvariants of the Weyl group acting on the space of polynomials on the Cartan subalgebra. The Weyl group also parametrises a decomposition of the flag manifold into disjoint cells – Bruhat cells. Their closures provide a basis for the cohomology ring, and we finish by finding an algebraic construction of this basis.

2 Lie groups and Lie algebras

We give a brief summary of results and notation on Lie groups and algebras. For more detail see [33], [34] or [28]. A Lie group is a topological group structure on a smooth manifold such that the operations of multiplication and inversion are smooth. Standard examples of Lie groups include

1. \mathbb{R}^n and \mathbb{C}^n with their normal topology and normal addition.
2. S^1 , the unit circle in the complex plane, and \mathbb{C}^* , the complex plane with the origin deleted are Lie groups under complex multiplication defined normally. Their cartesian products $(S^1)^n$ and $(\mathbb{C}^*)^n$ are the real and complex tori.
3. Let V be a complex n -dimensional vector space equipped with a Hermitian form (\cdot, \cdot) . Given a linear operator $x : V \rightarrow V$ we define its adjoint to be the operator x^* such that $(x(v), w) = (v, x^*(w))$ for all v and w in V . We now define the Lie group

$$U(V) = \{x \in \text{Aut}(V) \mid x \circ x^* = e\}.$$

If we let e_1, \dots, e_n be a basis for V such that $(e_i, e_j) = \delta_{ij}$ then it follows that

$$U(V) = U_n\mathbb{C} = \left\{ x \in \text{Aut}(\mathbb{C}^n) \mid x \circ x^h = e \right\},$$

where $\text{Aut}(\mathbb{C}^n)$ is the set of invertible complex $n \times n$ matrices and x^h is the conjugate transpose of x . This is known as the unitary group. If we impose the additional condition that the matrices have determinant one we have the special unitary group $SU(V)$.

We can also define $SL_n\mathbb{C} = \{x \in \text{Aut}(\mathbb{C}^n) \mid \det(x) = 1\}$, the complex special linear group.

4. Let V be a real n -dimensional vector space with a non-degenerate symmetric form Q . As in the above example we can define the adjoint to a linear operator on V . We then have

$$O(V) = \{x \in \text{Aut}(V) \mid x \circ x^* = e\},$$

the orthogonal group. Complexifying V to $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ and extending the form Q complex linearly we have the complex orthogonal group

$$O(V^{\mathbb{C}}) = \left\{ x \in \text{Aut}(V^{\mathbb{C}}) \mid x \circ x^* = e \right\}.$$

As above we can construct a nice basis with respect to the form in which case it follows that we are looking at matrices which result in the identity after acting on their transpose. These groups are not connected topologically. For the most part we will concentrate on the connected component containing the identity: the special orthogonal group $SO(V)$ of matrices with determinant one.

5. Let V be a $2n$ -dimensional vector space with non-degenerate skew-symmetric (symplectic) form ω . It is necessary that V be even dimensional for the form to be non-degenerate. As before

we define

$$\begin{aligned} \text{SP}(V) &= \{x \in \text{Aut}(V) \mid x \circ x^* = e\}, \text{ and} \\ \text{SP}(V^{\mathbb{C}}) &= \left\{x \in \text{Aut}(V^{\mathbb{C}}) \mid x \circ x^* = e\right\}. \end{aligned}$$

Let e_1, \dots, e_{2n} be a basis for V such that $\omega(e_i, e_{n+j}) = \delta_{ij}$ and $\omega(e_i, e_j) = 0$ for i and j between 1 and n . Then we can write the symplectic form in terms of Q as

$$\omega(v, w) = Q(v, Jw), \text{ where } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

From this it follows that

$$\begin{aligned} \text{SP}(V) &= \text{SP}_{2n}\mathbb{R} = \left\{x \in \text{Aut}(\mathbb{R}^{2n}) \mid x^t \circ J \circ x = e\right\}, \text{ and we also have} \\ \text{SP}(V^{\mathbb{C}}) &= \text{SP}_{2n}\mathbb{C} = \left\{x \in \text{Aut}(\mathbb{C}^{2n}) \mid x^t \circ J \circ x = e\right\}. \end{aligned}$$

The Lie groups in the last three examples are the classical groups and their complexifications, three families of compact Lie groups named and intensively studied by H. Weyl. We rewrite the groups in terms of canonical bases since this is useful in describing the structure of the group in more detail later. There is a close relationship between the representations of a compact Lie group and its complexification, which comes out of Weyl's unitary trick, which implies that their representations are in 1-1 correspondence. So by studying the complexifications we obtain a lot of information about the classical Lie groups. We will thus concentrate on complex Lie groups, those where the underlying manifold has a complex structure and the operations of multiplication and inversion are holomorphic.

Given a Lie group G , we choose an $x \in G$ and define the following natural maps:

$$L_x : G \rightarrow G : g \mapsto x \cdot g \text{ and } C_x : G \rightarrow G : g \mapsto x \cdot g \cdot x^{-1}.$$

Denote by TG the tangent bundle over G , and refer to sections of TG as vector fields on G . A vector field X on G is left-invariant if the following diagram commutes for all $x \in G$

$$\begin{array}{ccc} TG & \xrightarrow{dL_x} & TG \\ x \uparrow & & x \uparrow \\ G & \xrightarrow{L_x} & G \end{array}$$

We have a representation of G on the complex vector space of holomorphic functions from G to \mathbb{C} known as the regular representation. The G -action is given by

$$g \cdot f(x) = f(g^{-1} \cdot x)$$

Using this we construct an isomorphism between $T_e G$, the tangent space to G at the identity, and LTG , the space of left-invariant vector-fields on G . Define maps

$$\begin{aligned} \psi : LTG &\rightarrow T_e G : X \mapsto X_e, \text{ and} \\ \phi : T_e G &\rightarrow LTG : X_e \mapsto X \text{ where } Xf(x) = X_e(L_{x^{-1}}f). \end{aligned}$$

These are clearly inverse to each other. We refer to either of these vector spaces as \mathfrak{g} . It is the Lie algebra associated to the Lie group G . Given an $x \in G$ we have the map $dC_x : TG \rightarrow TG$. This preserves left-invariant vector fields and so it pushes down to \mathfrak{g} . We then have a map

$$\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g}) : x \mapsto dC_x.$$

This is a representation of G on \mathfrak{g} , the adjoint representation. Taking the differential of this map we obtain

$$\text{ad} := d\text{Ad} : \mathfrak{g} \rightarrow \mathcal{T}\text{Aut}(\mathfrak{g}) = \mathfrak{der}(\mathfrak{g}).$$

This gives us our Lie bracket on \mathfrak{g} : $[X, Y] = \text{ad}_X(Y)$ for $X, Y \in \mathfrak{g}$.

2.1 Semi-simple Lie algebras

As one would expect, the complexifications of the Lie algebras of the classical groups are the Lie algebras of their complexifications. Now that we have given a brief description of the relationship between a Lie group and its Lie algebra we specialise drastically to the semi-simple case.

Before describing the semi-simple Lie algebras, there is another important class of Lie algebras which we will encounter,

Definition 2.1 Let $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k]$. Then we have a decreasing sequence, the commutator series,

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \dots$$

\mathfrak{g} is solvable if $\mathfrak{g}^k = 0$ for some k .

There is a subset of the solvable algebras subject to a stronger condition,

Definition 2.2 Let $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_k]$. Then we have a decreasing sequence, the lower central series,

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \dots$$

\mathfrak{g} is nilpotent if $\mathfrak{g}_k = 0$ for some k . Clearly $\mathfrak{g}^k \subset \mathfrak{g}_k$ so all nilpotent algebras are solvable.

A subalgebra of a Lie algebra is a subspace $\mathfrak{a} \subset \mathfrak{g}$ such that

$$[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}.$$

An ideal is a subspace $\mathfrak{b} \subset \mathfrak{g}$ such that

$$[\mathfrak{g}, \mathfrak{b}] \subset \mathfrak{b}.$$

Definition 2.3 A Lie algebra is simple if it contains no non-zero ideals and is non-abelian (to exclude trivial one-dimensional algebras).

Definition 2.4 A Lie algebra is semi-simple iff it has no non-zero solvable ideals.

This is a fairly opaque characterisation, but there are better ones. Given any Lie algebra define the Killing form

$$B(X, Y) = \text{tr}(\text{ad}_X \circ \text{ad}_Y)$$

on \mathfrak{g} . This satisfies $B([X, Y], Z) = B(X, [Y, Z])$ for any X, Y and Z in \mathfrak{g} and so is an invariant bilinear form. We have

Theorem 2.5 Cartan's criterion. *A Lie algebra \mathfrak{g} is semi-simple iff the Killing form for \mathfrak{g} is non-degenerate. ■*

As a consequence of this we have that for any ideal \mathfrak{a} in \mathfrak{g} , $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ where \mathfrak{a}^\perp is the ideal which is the orthogonal complement of \mathfrak{a} . Thus every semi-simple Lie algebra splits into a direct sum of simple Lie algebras.

Simple complex Lie algebras have been classified and this classification consists of five exceptional Lie algebras together with three infinite families:

1. $\mathfrak{sl}_n \mathbb{C} = \{X \in \text{end}(\mathbb{C}^n) \mid \text{tr}(X) = 0\}$
2. $\mathfrak{so}_n \mathbb{C} = \{X \in \text{end}(\mathbb{C}^n) \mid X + X^t = 0\}$ for $n \neq 2$ or 4 , and
3. $\mathfrak{sp}_{2n} \mathbb{C} = \{X \in \text{end}(\mathbb{C}^{2n}) \mid X^t J + JX = 0\}$ with J as above.

These are the complex Lie algebras corresponding to the classical groups (where we exclude $\mathfrak{so}_2 \mathbb{C}$ and $\mathfrak{so}_4 \mathbb{C}$ since these algebras are not simple). Since our classification is so neat it would seem as though our job is almost done, but this is not the case. There is a tremendous richness of structure present and an elaborate general theory. Our emphasis will be on concrete examples in the beginning with general results creeping in gradually.

We turn to analysing the structure of these Lie algebras. First we state without proof some basic results on representations of $\mathfrak{sl}_2 \mathbb{C}$. This is the simplest semi-simple Lie algebra, and all other semi-simple Lie algebra can be thought of as built up out of representations of this simplest case.

First we construct a basis for $\mathfrak{sl}_2 \mathbb{C}$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with multiplication table

$$[h, x] = 2x \quad [h, y] = -2y \quad [x, y] = h.$$

Theorem 2.6 [28] *Up to isomorphism there exists a unique irreducible $\mathfrak{sl}_2 \mathbb{C}$ -module of each positive dimension. Let X be an irreducible module of dimension $m + 1$.*

1. *Relative to h , X is the direct sum of one dimensional weight spaces with weights X_μ for $\mu = m, m - 2, \dots, -(m - 2), -m$.*
2. *X has a unique (up to scalar multiplication) highest weight vector, say $v_0 \in V_m$, with weight m .*
3. *Set $v_{-1} = 0$ and $v_i = (1/i!)y^i.v_0$ for $i \geq 0$. Then we have*

$$h.v_i = (m - 2i)v_i, \tag{1}$$

$$y.v_i = (i + 1)v_{i+1}, \tag{2}$$

$$x.v_i = (m - i + 1)v_{i-1}. \blacksquare \tag{3}$$

Working with a arbitrary semi-simple Lie algebra we generalise h , x and y . A Cartan subalgebra \mathfrak{h} of \mathfrak{g} is a maximal subalgebra of \mathfrak{g} consisting of semisimple (diagonalisable) elements. These must exist by dimensional considerations and it turns out that they are maximal abelian.

We have considerable freedom in our choice of \mathfrak{h} ; however once we have chosen an \mathfrak{h} we consider it fixed.

Let $H_1 \dots H_n$ be a basis for \mathfrak{h} , and let $\psi : \mathfrak{g} \rightarrow \text{end}(V)$ be a representation of \mathfrak{g} . Since the H_i 's commute and are semisimple we can simultaneously diagonalise V into eigenspaces with respect to the action of \mathfrak{h} . So

$$V = \bigoplus_{\text{weights}} V_\lambda, \text{ where } V_\lambda = \{v \in V \mid \psi(H) \cdot v = \lambda(H) \cdot v \text{ for all } H \in \mathfrak{h}\}$$

where weights are linear functionals $\lambda \in \mathfrak{h}^*$ such that $V_\lambda \neq \{0\}$.

In the special case of the adjoint representation of \mathfrak{g} on itself, we refer to the weight space decomposition as the root space decomposition and define

$$\begin{aligned} \Delta &= \{ \text{roots of the adjoint representation} \} \\ &= \{ \lambda \in \mathfrak{h}^* \mid \mathfrak{g}_\lambda \neq 0 \} \text{ where } \mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [h, x] = \lambda(h)x \text{ for all } h \in \mathfrak{h}\} \end{aligned}$$

and so

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

It turns out that $\mathfrak{g}_0 = \mathfrak{h}$ since any 0-weight spaces clearly commutes with \mathfrak{h} and so must be contained in \mathfrak{h} if \mathfrak{h} is to be maximal. The root space decomposition is important because it contains large amounts of structural information. If λ is a weight of ψ with weight space V_λ and if α is a weight or 0 then

$$\psi(\mathfrak{g}_\alpha)V_\lambda \subset \begin{cases} V_{\lambda+\alpha} & \text{if } \lambda + \alpha \text{ is a weight} \\ 0 & \text{if not} \end{cases} \quad (4)$$

One easy consequence of (4) is that $B|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate (by invariance). From this it follows that for all $\alpha \in \Delta$ there exists an H_α in \mathfrak{h} such that $\alpha(H) = B(H, H_\alpha)$ for all $H \in \mathfrak{h}$. We can also construct a form on \mathfrak{h}^* using the Killing form. Define

$$\langle \lambda, \mu \rangle := B(H_\lambda, H_\mu). \quad (5)$$

This gives a bilinear form on \mathfrak{h} . We set $|\lambda| = \sqrt{\langle \lambda, \lambda \rangle}$. It is a fact that the roots span \mathfrak{h}^* . However we can also look at the real span of the roots, which we denote $\mathfrak{h}_\mathbb{R}$. We have $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$.

Since we know that the roots span $\mathfrak{h}_\mathbb{R}$ it is natural to ask whether or not they form a basis. The answer is no, however we can fairly naturally (though not uniquely) choose subsets which do form a basis. To do this we first pick a generic element of $\mathfrak{h}_\mathbb{R}$. By generic we mean one which is not perpendicular to any of the finite number of roots. This acts as a linear functional on $\mathfrak{h}_\mathbb{R}^*$, and so we can classify roots as positive or negative according to their value. We then define a (positive) root as indecomposable if it cannot be written as a sum of other positive roots. We denote by Π the set of indecomposable roots, which we also refer to as the simple roots. These form a basis of $\mathfrak{h}_\mathbb{R}$ under \mathbb{R} and span the root system of \mathfrak{g} under \mathbb{Z} . Note that the simple roots determine and are determined by the positive roots.

We have said that any semi-simple Lie algebra can be thought of as a collection of representations of $\mathfrak{sl}_2\mathbb{C}$ strung together. We flesh out this claim as follows,

Definition 2.7 Given $\alpha \in \Delta$ and $\beta \in \Delta \cup \{0\}$, define the α -string containing β to be the set of all members of $\Delta \cup \{0\}$ of the form $\beta + n\alpha$ for n in \mathbb{Z} .

Proposition 2.8 The following hold:

1. all roots and weights are real-valued on $\mathfrak{h}_{\mathbb{R}}$.
2. If α is a root then so is $-\alpha$ and $n\alpha$ is not a root for $n \geq 2$. $\dim \mathfrak{g}_{\alpha} = 1$.
3. If α is a root and x_{α} and y_{α} are representatives of \mathfrak{g}_{α} and $\mathfrak{g}_{-\alpha}$ respectively then

$$[x_{\alpha}, y_{\alpha}] = B(x_{\alpha}, y_{\alpha})H_{\alpha}. \quad (6)$$

4. Given α and β as in Definition 2.7, the α -string containing β has the form $\beta + n\alpha$ for $-p \leq n \leq q$ where $p, q \geq 0$ and

$$p - q = 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = 2 \frac{\langle \alpha, \beta \rangle}{\alpha(H_{\alpha})} \in \mathbb{Z}. \blacksquare \quad (7)$$

Let \mathfrak{g}^{α} be the subalgebra of \mathfrak{g} generated by x_{α} , y_{α} and H_{α} . We normalise so as to create a basis which can be mapped isomorphically to our basis for $\mathfrak{sl}_2\mathbb{C}$.

$$h_{\alpha} = \frac{2}{\alpha(H_{\alpha})}H_{\alpha} \quad x_{\alpha} = x_{\alpha} \quad y_{\alpha} = y_{\alpha} \quad (8)$$

Mapping $h_{\alpha} \mapsto h$, $x_{\alpha} \mapsto x$ and $y_{\alpha} \mapsto y$ provides the required isomorphism. Equation (4) together with the above theorem then implies that each α -string is a representation of \mathfrak{g}^{α} . Replacing β by $\beta + n\alpha$ for choice of n such that $\beta(h_{\alpha})$ is maximised, (7) can be reinterpreted as saying

the α -string containing β is isomorphic to the representation of $\mathfrak{sl}_2\mathbb{C}$ of dimension $\beta(h_{\alpha})$.

We now introduce the Weyl group \mathbb{W} . This is the group generated by reflections in hyperplanes in $\mathfrak{h}_{\mathbb{R}}^*$ perpendicular to the roots. So given a root $\alpha \in \Delta$ and an arbitrary element λ of \mathfrak{h}^* we define

$$\begin{aligned} \sigma_{\alpha}(\lambda) &= \lambda - 2 \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \alpha \\ &= \lambda - \langle \alpha^{\vee}, \lambda \rangle \alpha \end{aligned}$$

where $\alpha^{\vee} = 2 \frac{\alpha}{\alpha(H_{\alpha})}$. By definition the Weyl group is a subset of the orthogonal group for the Killing form. It turns out to be finite and in addition leaves the set Δ stable.

For future reference we include the definitions,

Definition 2.9 Define $\phi \in \mathfrak{h}^*$ to be algebraically integral if $\langle \phi, \alpha_i^{\vee} \rangle \in \mathbb{Z}$ for all $\alpha_i \in \Pi$. We say ϕ is dominant if $\langle \phi, \alpha_i \rangle \geq 0$ for all $\alpha_i \in \Pi$. Denote by P the semigroup of dominant algebraically integral weights.

In the case of the classical groups, the root systems and Weyl groups are as follows:

$$\mathbf{A}_n = \mathfrak{sl}_{n+1}\mathbb{C}$$

We set e_1, \dots, e_{n+1} as a basis for \mathbb{R}^{n+1} and find that $\mathfrak{h}_{\mathbb{R}}$ is isomorphic to $V = \{v \in \mathbb{R}^{n+1} | \langle v, e_1 + \dots + e_{n+1} \rangle = 0\}$. This follows from choosing \mathfrak{h} to be the space of diagonal matrices, on which we have the condition that the trace is zero.

As a basis for \mathfrak{h} we use $H_i = E_{i,i} - E_{i+1,i+1}$ for $i = 1 \dots n$ where $E_{i,j}$ has a 1 in the $(i, j)^{\text{th}}$ entry and zeroes elsewhere. We then calculate

$$[H, E_{i,j}] = (e_i(H) - e_j(H))E_{i,j}$$

where

$$e_i \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix} = h_i$$

Thus if we write $\mathfrak{g}_{i,j}$ for $\mathbb{C}E_{i,j}$ we have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{i \neq j} \mathfrak{g}_{i,j}$$

with root system $\Delta = \{e_i - e_j | i \neq j\}$.

From this we calculate the Weyl group as follows,

$$\begin{aligned} s_{e_i - e_j}(e_i - e_j) &= e_j - e_i \\ s_{e_i - e_j}(e_i - e_l) &= e_j - e_l \text{ where } l \text{ is assumed not equal to } i \text{ or } j \\ s_{e_i - e_j}(e_k - e_l) &= e_k - e_l \text{ where } i \text{ and } j \text{ do not equal } k \text{ and } l. \end{aligned}$$

So reflection in the root $e_i - e_j$ simply permutes e_i and e_j , and the Weyl group of $\mathfrak{sl}_{n+1}\mathbb{C}$ is \mathfrak{S}_n . Our standard choice of simple roots is $\Pi = \{e_1 - e_2, \dots, e_n - e_{n+1}\}$.

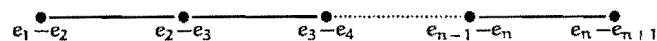
One consequence of the classification of complex semi-simple Lie algebras is that a knowledge of the simple roots Π , and the conformal structure of the vector space \mathfrak{h} are enough to determine \mathfrak{g} . Thus each semi-simple Lie algebra is uniquely determined by its Cartan matrix

$$(c_{ij}), \text{ where } c_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle,$$

where α_i ranges over all of Π . Using the Cartan matrix we can construct the more interesting Dynkin diagram. This is a graph which has as its nodes the simple roots, and whose edges are determined by the c_{ij} according to the rules,

1. $\alpha_i \neq \alpha_j$ are connected if and only if $\langle \alpha_i, \alpha_j^\vee \rangle \neq 0$
2. $\alpha \text{ --- } \beta$ iff $\langle \alpha, \beta^\vee \rangle = -1$
- (9)
- $\alpha \rightrightarrows \beta$ iff $\langle \alpha, \beta^\vee \rangle = -2$ and $\langle \beta, \alpha^\vee \rangle = -1$
- $\alpha \rightleftarrows \beta$ iff $\langle \alpha, \beta^\vee \rangle = -3$ and $\langle \beta, \alpha^\vee \rangle = -1$

Considering the case at hand, we have $\langle e_i - e_{i+1}, (e_{j+1} - e_{j+2})^\vee \rangle = -\delta_{ij}$ for $j \geq i$. Thus the Dynkin diagram for A_n is



$B_n = \mathfrak{so}_{2n+1}\mathbb{C}$:

This Lie algebra can be defined in two ways; corresponding to the two canonical bases that can be defined with respect to Q :

$$Q(e_i, e_{n+j}) = \delta_{ij} \text{ for } 1 \leq i, j \leq n; \quad Q(e_{2n+1}, e_{2n+1}) = 1, \text{ and}$$

$$Q(f_i, f_j) = \delta_{ij} \text{ for all } i, j \text{ between } 1 \text{ and } 2n + 1.$$

Thus we have

$$\mathfrak{so}_{2n+1}\mathbb{C} = \{X \in \text{end}(2n+1, \mathbb{C}) \mid X + X^t = 0\} \approx \{X \in \text{end}(2n+1, \mathbb{C}) \mid X^t M + M X = 0\}$$

where

$$M = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using the second form of $\mathfrak{so}_{2n+1}\mathbb{C}$ we obtain $H_i = E_{i,i} - E_{n+i,n+i}$ for $i = 1 \dots n$ for $\mathfrak{h}_{\mathbb{R}}$. By analysing the adjoint action of \mathfrak{h} on \mathfrak{g} we obtain $\Delta = \{\pm e_i \pm e_j \mid i < j\} \cup \{\pm e_i\}$. The resulting decomposition of \mathfrak{g} into weight spaces is

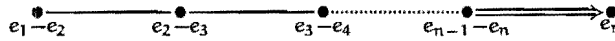
$$\begin{aligned} \mathfrak{g}_{e_i - e_j} &= \mathbb{C}(E_{i,j} - E_{n+j,n+i}) \\ \mathfrak{g}_{e_i + e_j} &= \mathbb{C}(E_{i,n+j} - E_{j,n+i}) \\ \mathfrak{g}_{-e_i - e_j} &= \mathbb{C}(E_{n+i,j} - E_{n+j,i}) \\ \mathfrak{g}_{e_i} &= \mathbb{C}(E_{i,2n+1} - E_{2n+1,n+i}) \\ \mathfrak{g}_{-e_i} &= \mathbb{C}(E_{n+i,2n+1} - E_{2n+1,i}) \end{aligned}$$

The simple roots of $\mathfrak{so}_{2n+1}\mathbb{C}$ are $\Pi = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_n\}$. If we think of the Weyl group as acting on the alphabet of signed numbers from 1 to n then it is generated by permuting the numbers as before, and the new roots of the form $\pm e_i$ have the effect of changing their sign. One way of describing this formally is

$$\mathbb{W} = \{w \in \mathfrak{S}_{2n} \mid w(i) + w(2n+1-i) = 2n+1 \text{ for all } i\}.$$

We interpret this by noting that any w in \mathbb{W} is determined by its values on $1 \dots n$. The characters $n+1, \dots, 2n$ are the negatively signed values.

Drawing the Dynkin diagram for B_n is identical to the previous case except for the last simple root where we have $\langle e_{n-1} - e_n, e_n^\vee \rangle = -2$. So we have



$C_n = \mathfrak{sp}_{2n}\mathbb{C}$:

This case is very similar to that above. The H_i 's take the same form as for B_n and the roots differ only slightly:

$$\Delta = \{\pm e_i \pm e_j \mid i < j\} \cup \{\pm 2e_i\}.$$

The simple roots are as for B_n with $2e_n$ replacing e_n . The weight spaces differ subtly...

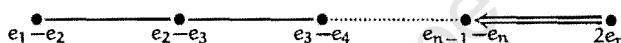
$$\begin{aligned} \mathfrak{g}_{e_i - e_j} &= \mathbb{C}(E_{i,j} - E_{n+j,n+i}) \\ \mathfrak{g}_{e_i + e_j} &= \mathbb{C}(E_{i,n+j} + E_{j,n+i}) \\ \mathfrak{g}_{-e_i - e_j} &= \mathbb{C}(E_{n+i,j} + E_{n+j,i}) \\ \mathfrak{g}_{2e_i} &= \mathbb{C}(E_{i,n+i}) \\ \mathfrak{g}_{-2e_i} &= \mathbb{C}(E_{n+i,i}) \end{aligned}$$

and the Weyl group is exactly the same by the same reasoning as above. However we can interpret it slightly differently to better reflect the root structure:

$$W = \{w \in \mathfrak{S}_{2n+1} \mid w(i) + w(2n+2-i) = 2n+2 \text{ for all } i\}.$$

Since $w(n+1) = n+1$ we have again that w is determined by its values on the first n characters.

Once again since the simple roots are mostly identical to the two previous cases, the Dynkin diagram looks similar. In this case the final root is $2e_n$ and $\langle e_{n-1} - e_n, (2e_n)^\vee \rangle = -1$, $\langle 2e_n, (e_{n-1} - e_n)^\vee \rangle = -2$, giving



$D_n = \mathfrak{so}_{2n} \mathbb{C}$:

We write this in the form $\{X \in \mathfrak{end}(2n+1, \mathbb{C}) \mid X^t M + M X = 0\}$ where in this case

$$M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

Although this looks similar to the two previous cases we have looked at, it is different from them in a number of ways, and has a considerably more complicated representation theory.

The roots are $\Delta = \{\pm e_i \pm e_j \mid i < j\}$ and the simple roots are $\Pi = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}$. The roots don't appear unusual, but already from the simple roots we start to realise that D_n differs more from B_n and C_n than they do from each other. The Weyl group is again an extension of \mathfrak{S}_n but this time we are only able to do even sign changes - ie change the sign of two letters at once - since there are no roots of the form $\pm e_i$. The Weyl group is thus

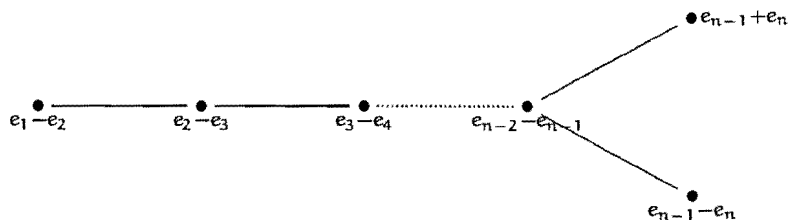
$$W = \{w \in \mathfrak{S}_{2n} \mid w(i) + w(2n+1-i) = 2n+1 \text{ for all } i, \\ \text{and the number of } i \leq n \text{ such that } w(i) > n \text{ is even}\}.$$

This is as in the previous case, with the additional condition imposed to ensure only even sign changes are allowed.

Calculating the Dynkin diagram is much the same as before with the difference that here we have two strange roots tacked on at the end. Looking at the relevant Cartan matrix entries,

$$\begin{aligned} \langle e_{n-2} - e_{n-1}, (e_{n-1} - e_n)^\vee \rangle &= -1 & \langle e_{n-2} - e_{n-1}, (e_{n-1} + e_n)^\vee \rangle &= -1 \quad \text{and} \\ \langle e_{n-1} - e_n, (e_{n-1} + e_n)^\vee \rangle &= 0 \end{aligned}$$

so that the Dynkin diagram is



2.2 Borel subalgebras

We define a Borel subalgebra \mathfrak{b} of \mathfrak{g} to be a maximal solvable subalgebra of \mathfrak{g} . Once we have chosen a Cartan subalgebra \mathfrak{h} we can add the condition that the Borel subalgebra contain \mathfrak{h} , in which case choosing a Borel subalgebra is equivalent to choosing a system of positive roots. We then have

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \text{ where } \mathfrak{n} = \bigoplus_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_\alpha$$

A parabolic subalgebra is an algebra containing a Borel subalgebra. Once we have fixed a Cartan subalgebra and a positive root system, the choice of parabolic subalgebra is equivalent to a choice of a subset of Π . This can be seen by thinking of the simple roots as a collection of generators.

Given \mathfrak{g} , \mathfrak{h} , Π , let Π_p be a subset of Π . We then have

$$\begin{aligned} \Delta(\mathfrak{l}, \mathfrak{h}) &= \text{span}(\Pi_p) \cap \Delta(\mathfrak{g}, \mathfrak{h}) \\ \text{with } \mathfrak{l} &= \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l}, \mathfrak{h})} \mathfrak{g}_\alpha \text{ a reductive subalgebra of } \mathfrak{g} \end{aligned}$$

A reductive Lie algebra is one in which any ideal has an orthogonal complement. This is more general than a semi-simple Lie algebra since the decomposition of the algebra now possibly contains trivial one-dimensional ideals – thus a reductive Lie algebra may have a non-trivial centre which is impossible in the semi-simple case.

$$\begin{aligned} \mathfrak{l} &= [\mathfrak{l}, \mathfrak{l}] + \mathfrak{l}_Z \text{ where } \mathfrak{l}_Z \text{ is the centre} \\ &= \mathfrak{l}_S + \mathfrak{l}_Z \text{ where } \mathfrak{l}_S \text{ is semi-simple of rank } |\Pi_p| \end{aligned}$$

$$\text{We now finally arrive upon } \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} = \mathfrak{l} \oplus \left[\bigoplus_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}) - \Delta(\mathfrak{l}, \mathfrak{h})} \mathfrak{g}_\alpha \right]$$

So a parabolic algebra breaks down into a reductive Lie algebra summed with a nilpotent Lie algebra. In the extreme case of a Borel algebra, the reductive part is the Cartan subalgebra, and the nilpotent part consists of the upper-triangular matrices.

Since parabolic subalgebras correspond to subsets of Π , it is possible to depict them using Dynkin diagrams. We put \times 's for simple roots which are excluded and \bullet 's for simple roots which are included. We reserve the notation \mathfrak{p}^α for the parabolic subalgebra corresponding to the single simple root α .

Some examples are,

$$\begin{aligned}
 \begin{array}{c} \times \\ \text{---} \\ \times \\ \text{---} \\ \times \end{array} \begin{array}{c} e_1 - e_2 \\ e_2 - e_3 \\ e_3 - e_4 \end{array} &= \left\{ \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & \bullet \end{pmatrix} \in \mathfrak{sl}(4, \mathbb{C}) \right\} = \mathfrak{b}, \text{ the Borel subalgebra.} \\
 \begin{array}{c} \bullet \\ \text{---} \\ \times \\ \text{---} \\ \times \end{array} \begin{array}{c} e_1 - e_2 \\ e_2 - e_3 \\ e_3 - e_4 \end{array} &= \left\{ \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & \bullet \end{pmatrix} \in \mathfrak{sl}(4, \mathbb{C}) \right\} = \mathfrak{p}^{e_1 - e_2}, \\
 \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \times \end{array} \begin{array}{c} e_1 - e_2 \\ e_2 - e_3 \\ e_3 - e_4 \end{array} &= \left\{ \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet \end{pmatrix} \in \mathfrak{sl}(4, \mathbb{C}) \right\}.
 \end{aligned}$$

2.3 Verma modules

We now wish to look at the representation theory of semi-simple Lie algebras. Our basic tool for later use will be the Theorem of Highest Weight; which is proved using the theory of Verma modules.

Definition 2.10 Let $\rho := 1/2(\sum_{\alpha \in \Delta^+} \alpha) \in \mathfrak{h}^*$.

Given $\lambda \in \mathfrak{h}^*$, let \mathbb{C}_λ be the 1-dimensional irreducible \mathfrak{b} -module where \mathfrak{h} acts via λ and \mathfrak{n} acts trivially.

Theorem 2.11 Engel. [34]. Let V be a finite dimensional vector space and \mathfrak{n} a Lie algebra of nilpotent endomorphisms of V . Then

- (a) \mathfrak{n} is a nilpotent Lie algebra,
- (b) there exists a non-zero $v \in V$ such that $x(v) = 0$ for all $x \in \mathfrak{n}$, and
- (c) in a suitable basis of V , all x in \mathfrak{n} are upper triangular with 0 on the diagonal. ■

Theorem 2.12 Lie. [34]. Let \mathfrak{b} be solvable, let $X \neq 0$ be a finite dimensional vector space and let $\rho : \mathfrak{b} \rightarrow \text{end}(V)$ be a representation of \mathfrak{b} . Then there is a simultaneous eigenvector $x \neq 0$ for all elements of $\rho(\mathfrak{b})$. ■

By Engel's theorem, given any representation of a nilpotent Lie algebra \mathfrak{n} , we can form a flag of subspace of V ,

$$0 \subset U_1 \subset \dots \subset U_n = V$$

where $\dim U_i = i$. We do this inductively as follows. Let U_1 be the line generated by a $v \in V$ such that $x(v) = 0$ for all x in \mathfrak{n} . Then V/U_1 is a representation of \mathfrak{n} and we can find a line in V/U_1 sent into zero – ie into U_1 – by \mathfrak{n} . Similarly for all $k < n$. Thus all irreducible representations of \mathfrak{n} are trivial. Similarly Lie's theorem shows that all irreducible representations of solvable Lie algebras are one dimensional and so all irreducible representations of \mathfrak{b} are of the form \mathbb{C}_λ for some λ .

We now wish to demonstrate a useful class of infinite dimensional \mathfrak{g} -modules, but before we can do this we need to construct the universal enveloping algebra of a Lie algebra.

Definition 2.13 Let \mathfrak{g} be a finite dimensional Lie algebra and let

$$T(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \dots = \sum_{k=0}^{\infty} \left(\bigotimes^k \mathfrak{g} \right)$$

be the tensor algebra of \mathfrak{g} with multiplication given by the tensor product. Now consider the two-sided ideal J generated by all $(x \otimes y - y \otimes x - [x, y])$ for x and y in \mathfrak{g} . The quotient $U(\mathfrak{g}) = T(\mathfrak{g})/J$ is the universal enveloping algebra of \mathfrak{g} .

There is a canonical embedding of \mathfrak{g} into the tensor algebra $T(\mathfrak{g})$ and this pushes down to the embedding

$$\iota : \mathfrak{g} \rightarrow U(\mathfrak{g}).$$

This is an embedding by the Poincaré-Birkhoff-Witt theorem, see §5.3, the fundamental result on the universal enveloping algebra.

Definition 2.14 The Verma module $V(\lambda)$ is then defined to be

$$V(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{C}_{\lambda-\rho}$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} . If $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$ then $V(\lambda) \approx U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda-\rho}$ as a \mathfrak{n}^- or \mathfrak{h} -module. As a \mathfrak{n}^- -module it is free on one generator.

Verma modules are useful because they are universal highest weight modules and can be used to construct irreducible representations. Let V be a \mathfrak{g} -module. A weight vector $v \in V$ is a highest weight vector if $\mathfrak{n} \cdot v = 0$. V is a highest weight module if it is generated by a highest weight vector. It follows that in this case the highest weight vector is unique up to scalar multiple; we say its weight is the highest weight of V . We have in this case that all the weight spaces of V are finite dimensional.

Verma modules are characterised amongst highest weight modules by the universal property:

$V(\lambda)$ is the universal highest weight module with highest weight $\lambda - \rho$ in the sense that if X is a highest weight \mathfrak{g} -module and $x \in X$ is a highest weight vector with weight $\lambda - \rho$ then there exists a unique surjective \mathfrak{g} -module map $V(\lambda) \rightarrow X : 1 \otimes 1 \mapsto x$.

Often, and especially when working with Verma modules it is convenient to change perspective, and to think of representations as \mathfrak{g} or $U(\mathfrak{g})$ -modules. An irreducible representation is then a simple \mathfrak{g} -module; and the property that all finite dimensional representations of \mathfrak{g} are completely reducible merely states that all finite-dimensional quotients of $U(\mathfrak{g})$ are semi-simple algebras (see §3.1).

One constructs irreducible highest weight representations by quotienting out all the proper submodules of $V(\lambda + \rho)$ giving (for certain choice of λ) a simple finite dimensional \mathfrak{g} -module $R(\lambda + \rho)$. Using Verma modules one can prove the

Theorem 2.15 Highest Weight. *Up to isomorphism the simple finite dimensional \mathfrak{g} -modules are in 1-1 correspondence with dominant algebraically integral linear functionals $\lambda \in \mathfrak{h}^*$. The linear functionals in \mathfrak{h} correspond to highest weight vectors in the irreducible representations.*

- (a) λ depends on the simple system Π and not on the ordering used
- (b) the weight space $R(\lambda + \rho)_\lambda$ is one dimensional.

(c) each root vector x_α , $\alpha \in \Delta^+$ annihilates the members of $R(\lambda + \rho)_\lambda$, and the elements of $R(\lambda + \rho)_\lambda$ are characterised by this property.

(d) every weight of $R(\lambda + \rho)$ is of the form $\lambda - \sum_{i=1}^l n_i \alpha_i$ where $n_i \geq 0$ and $\alpha_i \in \Pi$.

(e) each weight space $R(\lambda + \rho)_\mu$ has $\dim R(\lambda + \rho)_{w\mu} = \dim R(\lambda + \rho)_\mu$ where $w \in \mathbb{W}$ and each weight μ of $R(\lambda + \rho)$ has $|\mu| \leq |\lambda|$; with equality only if $\mu \in \mathbb{W} \cdot \lambda$. ■

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3 Weyl's Construction

In this section we explicitly construct representations of the classical groups as quotients or subspaces of the n -fold tensor product of the standard representation following [15] and [16]. We do this using a duality between representations of the classical groups and \mathfrak{S}_k , the symmetric group on k letters.

3.1 Semi-simple algebras and commutators

First some basic facts about associative algebras, for proofs see [20].

Definition 3.1 *By an associative algebra \mathcal{A} we refer to a vector space over \mathbb{C} equipped with a bilinear multiplication map such that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all x, y and z in \mathcal{A} . In addition we require that there exists an identity element e such that $x \cdot e = e \cdot x = x$ for all $x \in \mathcal{A}$.*

Definition 3.2 *An associative algebra \mathcal{A} is simple if it has 0 and \mathcal{A} as its only two-sided ideals.*

We have the following theorem,

Theorem 3.3 Wedderburn. *The associative algebra $\text{End}(V)$ is simple for all finite dimensional complex vector spaces V . Conversely given any simple finite dimensional associative algebra then there exists a complex vector space V such that $\mathcal{A} \approx \text{End}(V)$. ■*

(Note that by $\text{End}(V)$ we refer to the associative algebra with multiplication defined as composition, not the Lie algebra $\mathfrak{end}(V)$ with multiplication defined as the Lie bracket.)

A semi-simple associative algebra is an associative algebra which is isomorphic to a direct sum of simple algebras. Thus by Wedderburn's theorem

$$\Phi : \mathcal{A} \approx \bigoplus_{i \in I} \text{End}(V_i)$$

The maps $\pi_i \Phi$ where π_i is projection onto the i^{th} matrix algebra give irreducible representations of \mathcal{A} . Let

$$E_i = 0 \oplus \dots \oplus 1_{V_i} \oplus \dots \oplus 0$$

and set $e_i = \Phi^{-1}(E_i)$. These act as operators on \mathcal{A} . Clearly they are in the center of \mathcal{A} ; in addition we can see that they are idempotent. We have $\mathcal{A} \cdot e_i \approx \pi_i \Phi(\mathcal{A}) = \text{End}(V_i)$ and so the irreducible representations of \mathcal{A} are obtained by multiplying \mathcal{A} by distinguished elements.

Proposition 3.4 *The representations $(\pi_i \Phi, V_i)$ are distinct, irreducible, and exhaust the set of irreducible representations of \mathcal{A} . ■*

Any representation of a semi-simple algebra breaks down into irreducible representations. If we consider \mathcal{A} as an \mathcal{A} -module, then the submodules of \mathcal{A} are sums of irreducible representations, corresponding to

$$\mathcal{A} \cdot \left(\sum_{i \in I} \delta_i e_i \right) \text{ where } \delta_i \text{ is equal to } 0 \text{ or } 1.$$

Lemma 3.5 Schur. *If (ρ, V) and (τ, W) are irreducible finite dimensional representations of an associative algebra A then*

$$\text{Hom}_A(V, W) \approx \begin{cases} \mathbb{C} & \text{if } V \text{ and } W \text{ are isomorphic,} \\ 0 & \text{otherwise.} \end{cases} \blacksquare$$

Proposition 3.6 *Given an associative algebra A , if a finite-dimensional representation (ρ, W) of A is completely reducible then $\rho(A)$ is a semi-simple algebra. \blacksquare*

Given a group G , we can form the group algebra $\mathbb{C}[G]$ as follows. Let $\mathbb{C}[G]$ be the set of complex valued functions with finite support. This space has basis $\{\delta_g \mid g \in G\}$ where

$$\delta_g(x) = \begin{cases} 1 & \text{if } x = g, \\ 0 & \text{otherwise.} \end{cases}$$

We define multiplication on the basis as $\delta_g \cdot \delta_h = \delta_{g \cdot h}$ and extend linearly. It is clear that the $\mathbb{C}[G]$ -modules and representations of G are equivalent. Since we know that representations of finite groups are completely reducible, we have that the group algebra of a finite group is semi-simple by Proposition 3.6.

Now given a finite group G – which in practice will be the symmetric group – and a right $A = \mathbb{C}[G]$ -module U , let $B = \text{Hom}_G(U, U)$. B is the algebra of all operations on U commuting with G and is referred to as the commutant of A . It has a canonical left-action on U which by construction commutes with the A action. Since A is semi-simple, U breaks down into a direct sum of irreducible A -modules, $U = \bigoplus_i U_i^{n_i}$. Schur's lemma implies that $\text{Hom}_G(U_i^{n_i}, U_j^{n_j}) \approx \text{End}(\mathbb{C}^{n_i})$ if $i = j$ and that it is trivial otherwise. Thus

$$B = \bigoplus_i \text{Hom}_G(U_i^{n_i}, U_i^{n_i}) = \bigoplus_i \text{End}(\mathbb{C}^{n_i}).$$

Theorem 3.7 *Let U be a finite dimensional A -module.*

1. *For any $c \in A$, the canonical map $U \otimes_A A \cdot c \rightarrow U \cdot c$ is an isomorphism of left B -modules.*
2. *If $W = A \cdot c$ is an irreducible left A -module, then $U \otimes_A W = U \cdot c$ is an irreducible left B -module.*
3. *If $W_i = A \cdot c_i$ are the distinct irreducible left A -modules, with n_i the dimension of W_i then,*

$$U \approx \bigoplus_i (U \otimes_A W_i)^{\oplus n_i} \approx \bigoplus_i (U \cdot c_i)^{\oplus n_i}$$

is the decomposition of U into irreducible left B -modules.

Proof.

1. Since A is semi-simple, $A \cdot c$ can be thought of as a representation of A and so is a direct summand of A . Consider the following commutative diagram

$$\begin{array}{ccccc} U \otimes_A A & \xrightarrow{\cdot c} & U \otimes_A A \cdot c & \longrightarrow & U \otimes_A A \\ \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow{\cdot c} & U \cdot c & \longrightarrow & U \end{array}$$

where the vertical map take $u \otimes a \mapsto u \cdot a$; the left horizontal maps are surjective; and the right horizontal maps are injective. Then since the left and right vertical maps are isomorphisms, it follows that the middle vertical maps is also.

2. First consider the special case where U is an irreducible A -module. Then $B = \mathbb{C}$ and it is sufficient to show that $\dim U \otimes_A W = 0$ or 1 . Since A is semi-simple we have an isomorphism $A \xrightarrow{\phi} \sum_i \text{End}(\mathbb{C}^{m_i})$ between A and a sum of matrix algebras. W is a minimal left ideal of A . So $\phi(W)$ is a minimal left ideal in a sum of matrix algebras. This implies $\phi(W)$ is zero in all except one matrix algebra; and in that algebra it is zero everywhere except for one column. Similarly U is a minimal right ideal, and so can be identified as a single row within one of the matrix algebras comprising A . So $U \otimes_A A$ can either be zero, or consist of those matrices which are zero everywhere except in one row and one column of a single summand – in which case it will have dimension one.

In general U decomposes into a sum of irreducible right A -modules, so

$$U \otimes_A W = \bigoplus_i (U_i \otimes_A W)^{n_i} = \mathbb{C}^{\oplus n_k}$$

for some k where the summands coincide as above. This is clearly irreducible over $B = \bigoplus_i \text{End}(\mathbb{C}^{n_i})$.

3. Break down A into irreducibles as $A \approx \bigoplus_i W_i^{\oplus m_i}$. This gives the chain of isomorphisms

$$U \approx U \otimes_A A \approx U \otimes_A \left(\bigoplus_i W_i^{m_i} \right) \approx \bigoplus_i (U \otimes_A W_i)^{\oplus m_i} \blacksquare$$

We apply this theorem to find the irreducible representations of $SL(V)$, $SO(V)$ and $SP(V)$. The first case is the easiest, since $SL(V)$ is just the subgroup of $GL(V)$ consisting of matrices with determinant one. We will show that the categories of $GL(V)$ and \mathfrak{S}_n modules are dual to one another (in the sense that their group algebras are commutants).

First we need to find a module which supports the action of \mathfrak{S}_n and the classical groups. We form the tensor product

$$V^{\otimes n} = V \otimes \dots \otimes V.$$

This carries a left action of $GL(V)$,

$$g \cdot (v_1 \otimes \dots \otimes v_n) = (g \cdot v_1) \otimes \dots \otimes (g \cdot v_n)$$

as well as a right action of \mathfrak{S}_n

$$(v_1 \otimes \dots \otimes v_n) \cdot \sigma = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}.$$

These clearly commute since they are left and right actions, however more is true:

Proposition 3.8 *Let U be the right $\mathbb{C}[\mathfrak{S}_n]$ -module $V^{\otimes n}$. Then the commutant $B = \text{Hom}_{\mathfrak{S}_n}(U, U)$ is the linear subspace of $\text{End}(V^{\otimes n})$ spanned by $\text{End}(V)$. A subspace of $V^{\otimes n}$ is a B -submodule iff it is invariant under $GL(V)$.*

Proof. Given a finite dimensional space W , $\text{Sym}^n(W)$ is the subspace of $W^{\otimes n}$ spanned by all $w \otimes \dots \otimes w$ for $w \in W$. Setting $W = \text{End}(V) = V^* \otimes V$ it follows that

$$\text{End}(V^{\otimes n}) = (V^*)^{\otimes n} \otimes V^{\otimes n} = W^{\otimes n}.$$

Thus we see immediately that the algebra of commutators for \mathfrak{S}_n is equal to $\text{Sym}^n(\text{End } V)$. This is spanned by n^{th} tensor products of elements of $\text{End}(V)$. $\text{GL}(V)$ is dense in $\text{End}(V)$, and so the condition on invariance follows. ■

Thus Theorem 3.7 is applicable to this situation and using the decomposition of $V^{\otimes n}$ into irreducible \mathfrak{S}_n -modules it should be possible to find its decomposition into $\text{GL}(V)$ -modules. From this we can easily obtain irreducible representations of $\text{SL}(V)$.

The symplectic and orthogonal groups are more difficult, since they are not commutators for \mathfrak{S}_n . To get around this introduce additional operators extending the group algebra as follows:

Given a pair $l = \{p < q\}$, define the map

$$\Phi_l: V^{\otimes n} \rightarrow V^{\otimes(n-2)}: v_1 \otimes \dots \otimes v_n \mapsto \langle v_p, v_q \rangle v_1 \otimes \dots \otimes \hat{v}_p \otimes \dots \otimes \hat{v}_q \otimes \dots \otimes v_n. \quad (10)$$

for $\langle v_p, v_q \rangle$ equal to either $Q(v_p, v_q)$ or $\omega(v_p, v_q)$ depending on the case. Now given either a quadratic or symplectic form, there is a canonical basis consisting of either e_1, \dots, e_{2m} or e_1, \dots, e_{2m+1} . Define

$$\psi = \sum (e_i \otimes e_{m+i} + e_{m+i} \otimes e_i) + (e_{2m+1} \otimes e_{2m+1}) \quad (11)$$

in the orthogonal case (dropping the last term for $\text{SO}_{2n} \mathbb{C}$), and

$$\psi = \sum (e_i \otimes e_{m+i} - e_{m+i} \otimes e_i) \quad (12)$$

in the symplectic case. Now define

$$\Psi_l: V^{\otimes(n-2)} \rightarrow V^{\otimes n} \quad (13)$$

to be the map inserting ψ in the $p^{\text{th}}, q^{\text{th}}$ positions; and

$$\vartheta_l = \Psi_l \circ \Phi_l: V^{\otimes n} \rightarrow V^{\otimes n}. \quad (14)$$

We will study these operators in more detail later on in this section, but for now we highlight their importance with

any endomorphism of $V^{\otimes n}$ that commutes with all permutations in \mathfrak{S}_n and all the operators ϑ_l is a finite linear combination of operators of the form $A \otimes \dots \otimes A$ for $A \in \text{SP}(V)$ or $\text{SO}(V)$.

3.2 Young diagrams and representations of \mathfrak{S}_n

So before looking at representations of the classical groups, we need to first find a way of producing representations of \mathfrak{S}_n . We do this using Young diagrams. These are combinatorial objects consisting of a collection of boxes arranged in left-justified rows, where the lengths of the rows are weakly decreasing.

The collection of Young diagrams with n boxes is in 1 – 1 correspondence with partitions of n : given a partition

$$\lambda_1 + \dots + \lambda_k = n \text{ where } \lambda_i \in \mathbb{N}^+$$

we may as well assume that $\lambda_1 \geq \dots \geq \lambda_k > 0$ and so we write $\lambda = (\lambda_1, \dots, \lambda_k)$ for the corresponding Young diagram consisting of k rows of lengths given by the k -tuple. Write $\lambda \vdash n$ when λ partitions n , and denote by $|\lambda|$ the sum of the lengths of the rows in λ . Given a Young diagram λ , we have the conjugate $\tilde{\lambda}$ given by swapping the rows and columns. In Figure 1, the 1st and 3rd diagrams are conjugate.

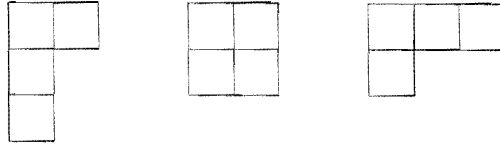


Figure 1: Young diagrams

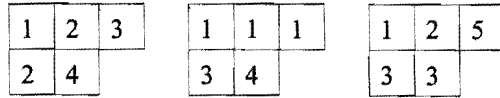


Figure 2: Young tableau on $\lambda = (3, 2)$

Definition 3.9 A Young tableau is a filling of the boxes with natural numbers such that the filling is

1. weakly increasing along each row, and
2. strongly increasing down each column.

A standard tableau is a filling in which the numbers are chosen from $1, \dots, |\lambda|$ with each number occurring once.

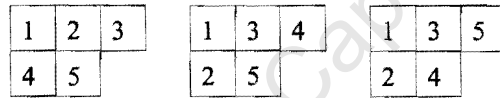


Figure 3: Standard fillings of $\lambda = (3, 2)$

Given a Young diagram $\lambda \vdash n$ with standard filling T , define two subgroups of \mathfrak{S}_n :

$$R(T) = \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ preserves each row}\}$$

$$C(T) = \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ preserves each column}\}$$

We now introduce three distinguished elements of the group algebra $\mathbb{C}[\mathfrak{S}_n]$. These are the Young symmetrizers; defined as

$$a_T = \sum_{p \in R(T)} e_p, \quad b_T = \sum_{q \in C(T)} \text{sgn}(q) e_q,$$

$$\text{and } c_T = b_T \cdot a_T.$$

The image of c_T on $\mathbb{C}[\mathfrak{S}_n]$ by right multiplication is a representation of $\mathbb{C}[\mathfrak{S}_n]$. We claim that it is an irreducible representation, and that all the irreducible representations of \mathfrak{S}_n are obtained in this way.

The Young symmetrizer c_T is used to construct irreducible representations of \mathfrak{S}_n known as Specht modules. Set $S^\lambda = \mathbb{C}[\mathfrak{S}_n] \cdot c_T$. We can do this since $\mathbb{C}[\mathfrak{S}_n] \cdot c_T \approx \mathbb{C}[\mathfrak{S}_n] \cdot c_{T'}$ for any two standard fillings T and T' of λ . First note that we can always find a $\sigma \in \mathfrak{S}_n$ such that $T' = \sigma T$. Then $R(T') = \sigma R(T) \sigma^{-1}$ and so $a_{T'} = \sigma a_T \sigma^{-1}$. Similarly for $b_{T'}$ and $c_{T'}$. Now to

construct the isomorphism, given $x \in \mathbb{C}[\mathfrak{S}_n]$, map $x \cdot c_T \mapsto \sigma x \cdot c_T \sigma^{-1} = \sigma x \sigma^{-1} c_{\sigma T} = x \cdot c_{T'}$. This respects the $\mathbb{C}[\mathfrak{S}_n]$ -module structure and is easily seen to be an isomorphism since it has as inverse conjugation by σ^{-1} ,

So for each partition λ of n there is up to isomorphism a representation \mathbb{S}^λ ; since these partitions are in 1-1 correspondence with conjugacy classes of \mathfrak{S}_n , it is sufficient to show that the \mathbb{S}^λ 's are distinct and irreducible. From here on we write c_λ ignoring the choice of filling, since we are concerned only with the isomorphism classes of the representations generated.

We introduce the lexicographic ordering on Young diagrams:

$$\lambda > \mu \text{ iff the first nonvanishing } \lambda_i - \mu_i \text{ is positive.}$$

This is a linear ordering. The following lemma is proved in [16].

Lemma 3.10 *If $\lambda > \mu$ and $x \in \mathbb{C}[\mathfrak{S}_n]$ then $c_\lambda \cdot x \cdot c_\mu = 0$, and $c_\lambda \cdot x \cdot c_\lambda = k c_\lambda$ for some $k \in \mathbb{C}$ - depending on x . In particular $c_\lambda \cdot c_\lambda$ is equal to some nonzero multiple of c_λ . ■*

Theorem 3.11 *Each \mathbb{S}^λ is a distinct irreducible representation of \mathfrak{S}_n .*

Proof. By the lemma, $c_\lambda \mathbb{S}^\lambda \subset \mathbb{C} c_\lambda$. If $W \subset \mathbb{S}^\lambda$ is a submodule, then $c_\lambda W$ is either $\mathbb{C} c_\lambda$ or 0. In the first case, $\mathbb{S}^\lambda = \mathbb{C}[\mathfrak{S}_n] \cdot c_\lambda \subset \mathbb{C}[\mathfrak{S}_n] \cdot c_\lambda W \subset W$. Otherwise $W \cdot W \subset \mathbb{C}[\mathfrak{S}_n] \cdot c_\lambda W = 0$. By the theory of semi-simple algebras, there exists an element ϕ of W such that multiplication by ϕ is projection onto W . Then $\phi = \phi^2 \in W \cdot W = 0$, implying $W = 0$.

The lexicographic order is total, so assume $\lambda > \mu$. Then $c_\lambda \mathbb{S}^\lambda = \mathbb{C} c_\lambda \neq 0$ and $c_\lambda \mathbb{S}^\mu = c_\lambda \cdot \mathbb{C}[\mathfrak{S}_n] \cdot c_\mu = 0$ by the lemma, and clearly the two representations are distinct. ■

3.3 Representations of $SL(V)$

Let V be an $n + 1$ dimensional complex vector space. Using the results of the previous two sections, we will construct the irreducible representations of $SL(V)$ as quotients of the d -fold tensor product $V^{\otimes d}$. Following the suggestion of Lemma 3.8 we start by looking at the general linear group.

Theorem 3.12 *Let m_λ be the dimension of the irreducible representation \mathbb{S}^λ of \mathfrak{S}_d corresponding to λ . Then*

$$V^{\otimes d} \approx \bigoplus_{\lambda \vdash d} \left(V^{(\lambda)} \right)^{\oplus m_\lambda}$$

where $V^{(\lambda)}$ is the irreducible representation of $GL(V)$ given by $V^{\otimes d} \cdot c_\lambda$.

Remark. Given a Young diagram λ , we will construct representations of $SL(V)$, $SP(V)$ and $SO(V)$. We will denote these representations by $V^{(\lambda)}$, $V^{(\lambda)}$ and $V^{[\lambda]}$ respectively. If we wish to refer to representations corresponding to Young diagrams without limiting to a specific Lie group, we will write V^λ .

Proof. Again we can write $V^{(\lambda)}$ since different fillings of λ produce isomorphic representations. The theorem is an immediate consequence of Theorem 3.7 and Proposition 3.8. ■

Let us consider a few examples of these representations. The two simplest cases are $\lambda = (d)$ and $\lambda = (1, \dots, 1)$. These are already familiar to us.

$\lambda = (d)$: Clearly $C(T) = \{\text{id}\}$ and $R(T) = \mathfrak{S}_d$. Thus $c_\lambda = \sum_{\sigma \in \mathfrak{S}_d} e_\sigma$. The vector space generated by this c_λ is thus the subspace of $V^{\otimes d}$ consisting of symmetric tensors – $\text{Sym}^d V$. In this case we typically replace the tensor product \otimes with a dot. So for example in $\text{Sym}^3 V$ we would have $e_1 \cdot e_1 \cdot e_4 = e_1 \cdot e_4 \cdot e_1$.

$\lambda = (1, \dots, 1)$: Here the positions are reversed and we have $c_\lambda = \sum_{\sigma \in \mathfrak{S}_d} \text{sgn}(\sigma) e_\sigma$. This gives us the subspace of all antisymmetric tensors, $\bigwedge^d V$. Here we use wedges and have the following antisymmetry relation: $e_1 \wedge e_2 = -e_2 \wedge e_1$.



Figure 4: T and T'

$\lambda = (2, 1)$: Let T and T' be as in Figure 4. Then $a_T = e_{\text{id}} + e_{(12)}$ and $b_T = e_{\text{id}} - e_{(13)}$. Thus $c_T = e_{\text{id}} + e_{(12)} - e_{(13)} - e_{(132)}$. By $V^{((2,1))}$ we are then referring to the space

$$V^{\otimes d} \cdot c_T \approx V^{\otimes d} \cdot c_{T'},$$

where we know the isomorphism exists by §3.2. Taking an arbitrary tensor,

$$\begin{aligned} a \otimes b \otimes c &\mapsto a \otimes b \otimes c + b \otimes a \otimes c - c \otimes b \otimes a - c \otimes a \otimes b, \text{ we see that} \\ (a \otimes b \otimes c) \cdot c_T &= (b \otimes a \otimes c) \cdot c_T = -(c \otimes b \otimes a) \cdot c_T \text{ and} \\ (a \otimes b \otimes c) \cdot c_T &+ (b \otimes c \otimes a) \cdot c_T + (a \otimes c \otimes b) \cdot c_T = 0. \end{aligned}$$

So we have the expected symmetries and anti-symmetries, together with an additional relation which results from the multiplication of a_T and b_T .

There is an alternative method of constructing these representations. Given a Young diagram $\lambda \vdash d$, form the vector space $V^{\otimes d}$. Let e_1, \dots, e_n is a basis for V . Now to each filling of λ with the alphabet $1, \dots, n$ we associate a tensor by reading off from top to bottom and from left to right. For example, we associate with Figure 5 the tensor $e_1 \otimes e_3 \otimes e_5 \otimes e_2 \otimes e_4 \otimes e_6$. We

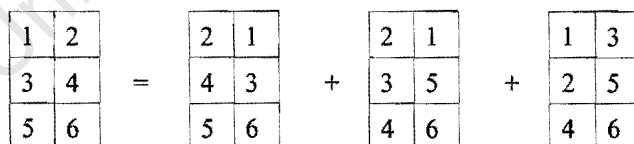


Figure 5:

then impose the follow relations (by quotienting):

Q1. set the tensor space alternating in its columns

Q2. $v - \sum w = 0$ where the sum is over all w obtained from v by an exchange between two columns of a given subset of the rightmost column with the elements of the other.

Q2 needs some clarification; continuing the above example, choosing the entire first and second column, we see that we impose relations such that we are considering a quotient space of

$\text{Sym}^2(\bigwedge^3 V)$. Choosing the first two columns, and then the top two boxes of the second column we have the additional relation,

$$(e_1 \wedge e_3 \wedge e_5) \otimes (e_2 \wedge e_4 \wedge e_6) = (e_2 \wedge e_4 \wedge e_5) \otimes (e_1 \wedge e_3 \wedge e_6) \\ + (e_2 \wedge e_3 \wedge e_4) \otimes (e_1 \wedge e_5 \wedge e_6) + (e_1 \wedge e_2 \wedge e_4) \otimes (e_3 \wedge e_5 \otimes e_6).$$

Further relations between the tensors are obtained by choosing different collections of boxes according to the rules prescribed. As another example consider the case of $V^{((2,1))}$. Here Q2 imposes the relation already given previously as the third symmetry of $V^{((2,1))}$.

In general $GL(V)$ acts on this space in the obvious manner and it can be shown, see [15], that it is isomorphic to $V^{(\lambda)}$. This isomorphism is a generalisation of the fact that $\text{Sym}^d V$ and $\bigwedge^d V$ can be constructed as either subspaces or quotients of $V^{\otimes d}$. The complicated second relation captures the result of ‘multiplying’ a_τ and b_τ .

1	1	1
2	2	
3	3	

Figure 6: $U(\lambda)$ for $\lambda = (3, 2, 2)$

This second method of obtaining $V^{(\lambda)}$ is particularly useful for constructing bases. Our goal is to find the basis element corresponding to the highest weight vector of the representation. We can do this since any irreducible representation of $GL(V)$ is also an irreducible representation of $SL(V)$. The difference is that in the case of $SL(V)$ the action of the determinant is trivial.

Theorem 3.13 $V^{(\lambda)}$ is an irreducible representation of $SL(V)$ with highest weight vector $e_{U(\lambda)}$; where $e_{U(\lambda)}$ is the vector corresponding to filling the i^{th} row of λ with i 's. Representations of $SL(V)$ are uniquely parametrised by diagrams λ with $\lambda_{n+1} = 0$. Restricting to λ 's of this form, $e_{U(\lambda)}$ has weight $(\lambda_1, \dots, \lambda_n)$.

Proof. Since we know $V^{(\lambda)}$ is an irreducible representation we know that there is a unique (up to scalar multiplication) highest weight vector; thus it suffices to show $\mathfrak{n} \cdot e_{U(\lambda)} = 0$. This follows since $E_{e_i - e_j}$ sends $e_j \mapsto e_i$, and if $i < j$ then this sends $e_{U(\lambda)}$ to zero – so \mathfrak{n} has a null action.

If we have $\lambda_{n+1} = s$ then $V^{(\lambda)}$ is a quotient of $\text{Sym}^s \bigwedge^{n+1} V \otimes \{ \text{other terms} \}$. The $GL(V)$ action on $\bigwedge^{n+1} V$ is to multiply by the determinant of the given group element ($\dim V = n+1$). So $SL(V)$ has trivial action on $\text{Sym}^s \bigwedge^{n+1} V$ and so we restrict our attention to diagrams with $\lambda_{n+1} = 0$.

$e_{U(\lambda)}$ is the image under quotienting of the tensor $e_1 \otimes e_1 \otimes \dots \otimes e_1 \otimes e_2 \otimes \dots \otimes e_n$ where e_i is repeated λ_i times. Given an element $t = \text{diag}(t_1, \dots, t_{n+1}) \in SL(V)$ we obviously have

$$t \cdot e_{U(\lambda)} = t_1^{\lambda_1} \dots t_{n+1}^{\lambda_{n+1}} e_{U(\lambda)} \\ = t_1^{\lambda_1} \dots t_n^{\lambda_n} e_{U(\lambda)},$$

since $\lambda_{n+1} = 0$. ■

3.4 Representations of $SP(V)$

In this section we let V be a $2n$ dimensional vector space equipped with a symplectic form ω and a canonical basis e_1, \dots, e_{2n} such that

$$\omega(e_i, e_{n+j}) = \delta_{ij} \quad \omega(e_i, e_j) = 0$$

for i, j between 1 and n . We then impose a Hermitian metric $(,)$ with orthogonal basis $\{e_i\}$. We can extend this metric to tensor powers of V by setting $(u_1 \otimes \dots \otimes u_n, v_1 \otimes \dots \otimes v_n) = (u_1, v_1) \cdots (u_n, v_n)$ and extending linearly.

Lemma 3.14 *Let ψ be as defined in (12); then $(\psi, u \otimes v) = \omega(u, v)$.*

Proof.

$$(\psi, u \otimes v) = \sum_{i=1}^n (e_i, u)(e_{i+n}, v) - (e_{i+n}, u)(e_i, v)$$

The claim then follows after expanding u, v into linear combinations of the basis vectors. ■

From this it follows that

$$\begin{aligned} (g \cdot \psi, u \otimes v) &= (\psi, g^T \cdot u \otimes g^T \cdot v) \\ &= \omega(g^T u, g^T v) \\ &= \omega(u, v) \\ &= (\psi, u \otimes v), \end{aligned}$$

implying that Ψ_I , see (13), commutes with the action of $SP(V)$.

Definition 3.15

$$\text{Let } V^{(d)} = \bigcap_{I \subset \{1, \dots, d\}} \ker(\Phi_I) \subset V^{\otimes d}.$$

By definition of Φ_I , see (10), this is invariant under $SP(V)$, and since the intersection is over all subsets of $1, \dots, d$ it is also invariant under \mathfrak{S}_d .

Proposition 3.16

$$V^{\otimes d} = V^{(d)} \bigoplus \sum_I \Psi_I(V^{\otimes(d-2)})$$

where the sum is orthogonal with respect to $(,)$.

Proof. By Lemma 3.14 $(\psi, u \otimes v) = \omega(u, v)$, so that $\ker \Phi_I = (\text{im } \Psi_I)^\perp$ for all I . The claim now follows immediately. ■

$V^{(d)}$ is referred to as the space of harmonic tensors, and the aim of this section is to show

$$V^{(\lambda)} := V^{(d)} \bigcap V^{(\lambda)}$$

is an irreducible representation of $SP(V)$ iff $\lambda_{n+1} = 0$.

Definition 3.17

$$V_{d-2r}^{(d)} := \sum \Psi_{I_1} \circ \dots \circ \Psi_{I_r} \left(V^{(d-2r)} \right)$$

$$\text{and } F_r^d := \bigcap \ker \Phi_{I_1} \circ \dots \circ \Phi_{I_r}.$$

Lemma 3.18

$$V^{\otimes d} = V^{(d)} \oplus V_{d-2}^{(d)} \oplus \dots \oplus V_{d-2p}^{(d)} \text{ where } p = \left\lfloor \frac{d}{2} \right\rfloor = \text{the biggest integer } \leq \frac{d}{2},$$

$$\text{and } F_r^d = V^{(d)} \oplus \dots \oplus V_{d-2r+2}^{(d)}.$$

Proof. As in Proposition 3.16, we have $\ker \Phi_{I_1} \circ \dots \circ \Phi_{I_r} = (\text{im } \Psi_{I_1} \circ \dots \circ \Psi_{I_r})^\perp$ which implies that

$$V^{\otimes d} = F_r^d \oplus \sum \Psi_{I_1} \circ \dots \circ \Psi_{I_r} (V^{\otimes d-2r}).$$

The first part of the Lemma follows by induction: by simple calculation, $V^{\otimes 1} = V^{(1)}$ and $V^{\otimes 2} = V^{(2)} \oplus \mathbb{C}\psi$. Lemma 3.16 states that

$$\begin{aligned} V^{\otimes d} &= F_1^d \oplus \sum \Psi_{I_1} (V^{\otimes d-2}) \\ &= V^{(d)} \oplus \sum \Psi_{I_1} \left(V^{(d-2)} \oplus \dots \oplus V_{d-2p}^{(d-2)} \right). \Psi_{I_1} \text{ respects } (,), \text{ so} \\ &= V^{(d)} \oplus \sum \Psi_{I_1} (V^{(d-2)}) \oplus \dots \oplus \Psi_{I_1} (V_{d-2p}^{(d-2)}) \\ &= V^{(d)} \oplus V_{d-2}^{(d)} \oplus \dots \oplus V_{d-2p}^{(d)}. \end{aligned}$$

For the second part,

$$\begin{aligned} V^{\otimes d} &= F_r^d \oplus \sum \Psi_{I_1} \circ \dots \circ \Psi_{I_r} \left(V^{(d-2r)} \oplus V_{d-2(r+1)}^{(d-2r)} \oplus V_{d-2p}^{(d-2r)} \right) \\ &= F_r^d \oplus V_{d-2r}^{(d)} \oplus \dots \oplus V_{d-2p}^{(d)}. \end{aligned}$$

Since this is an orthogonal direct sum it must be the case that

$$F_r^d = V^{(d)} \oplus \dots \oplus V_{d-2(r-1)}^{(d)}. \blacksquare$$

So $V^{\otimes d}$ decomposes orthogonally into spaces invariant under $SP(V)$ and \mathfrak{S}_d . Thus

$$V^{(\lambda)} = V^{(\lambda)} \cap V^{(d)} = V^{(d)} \cdot c_\lambda.$$

Lemma 3.19 $V^{(\lambda)} \neq 0$ if and only if $\lambda_{n+1} = 0$.

Proof. $e_{\mathbf{u}(\lambda)}$ is a vector of highest weight λ for $V^{(\lambda)}$. If $\lambda_{n+1} = 0$, then since $\omega(e_i, e_j) = 0$ for all $i, j \leq n$ it follows that $e_{\mathbf{u}(\lambda)} \in V^{(d)}$. Thus $e_{\mathbf{u}(\lambda)}$ is a highest weight vector for $SP(V)$ in $V^{(\lambda)}$ of weight λ .

We show that $\lambda_{n+1} \neq 0$ implies $V^{(\lambda)} = 0$ by showing $s > m$ implies $\bigwedge^s V \otimes V^{\otimes d-s} \subset \sum \Psi_{I_1} (V^{\otimes d-2})$. $s > m$ implies the map

$$\bigwedge^{s-2} V \xrightarrow{\Psi_{\{1,2\}}} \bigwedge^s V$$

is a surjection since given any basis element

$$e_{i_1} \wedge \dots \wedge e_{i_s} \in \bigwedge^s V, \text{ which is equal to} \\ \pm e_k \wedge e_{k+n} \wedge e_{i_1} \wedge \dots \wedge \hat{e}_{i'} \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_{i_s}$$

for some k , and so is in the image of $\Psi_{\{1,2\}}$. ■

Recalling the definition (14), $\vartheta_{\{p,q\}} = \Psi_{\{p,q\}} \circ \Phi_{\{p,q\}} : V^{\otimes d} \rightarrow V^{\otimes d}$, we see that this map has the following effect,

$$v_1 \otimes \dots \otimes v_d \mapsto \sum_{i=1}^n \omega(v_p, v_q) v_1 \otimes \dots \otimes e_i \otimes \dots \otimes e_{i+n} \otimes \dots \otimes v_d$$

where e_i is in the p^{th} position and e_{i+n} is in the q^{th} position. As a result ϑ_I commutes with $SP(V)$ and $\bigcap \ker \vartheta_I = V^{\langle d \rangle}$.

Finally it follows that $V^{\langle \lambda \rangle}$ is an irreducible representation for all λ by the Double Commutant Theorem 3.7 and the following fact from invariant theory, see [16] or [26].

Theorem 3.20 *Any endomorphism of $V^{\otimes d}$ that commutes with all permutations in \mathfrak{S}_d and all the operators ϑ_1 is a finite linear combination of operators of the form $A \otimes \dots \otimes A$ for $A \in SP(V)$.* ■

Finally we have

Theorem 3.21 *$V^{\langle \lambda \rangle}$ is a non-zero irreducible representation of $SP(V)$ if and only if $\lambda_{n+1} = 0$. It has highest weight $(\lambda_1, \dots, \lambda_n)$.*

Proof. Everything is done except for the final step. This follows by the same reasoning as Theorem 3.13. ■

3.5 Representations of $SO(V)$

There are actually two cases here, the even dimensional and odd dimensional cases, but we try to deal with them identically as far as is possible. We begin with the orthogonal group, and then restrict to its index 2 subgroup, the special orthogonal group.

Start by letting V be an m dimensional vector space (where m is either $2n$ or $2n + 1$) with non-degenerate symmetric form Q , and canonical basis f_1, \dots, f_m such that

$$Q(f_i, f_j) = \delta_{ij}$$

for i, j between 1 and m . Then set the distinguished element ψ from Equation (11) to be

$$\psi = \sum_{i=1}^m (e_i \otimes e_i).$$

Following the same reasoning as in the previous section, we define a Hermitian form on V with orthogonal basis given by the f_i 's. Then, as required, $(\psi, u \otimes v) = Q(u, v)$.

Definition 3.22 *Let $V^{\langle d \rangle} := \bigcap_I \ker \Phi_I$ be the space of harmonic tensors under Q .*

This is invariant under the orthogonal group $O(V)$ and \mathfrak{S}_d . All the decompositions proved in the symplectic case carry through to the orthogonal and so it follows that

$$V^{[\lambda]} = V^{[d]} \cdot c_\lambda = V^{(\lambda)} \cap V^{[d]}$$

is a representation of $O(V)$. It is an irreducible representation as a result of the following fact from invariant theory, see [16] or [26]:

Theorem 3.23 *Any endomorphism of $V^{\otimes d}$ that commutes with all permutations in \mathfrak{S}_d and all the operators ϑ_1 is a finite linear combination of operators of the form $A \otimes \dots \otimes A$ for $A \in O(V)$.*

■

and the Double Commutant Theorem 3.7. All that remains to be done is to check when this representation is trivial.

Lemma 3.24 $V^{[\lambda]} = 0$ iff the sum of the lengths of the first two columns of λ is greater than m .

Proof. First we show that $a + b > m$ implies

$$\bigwedge^{a-1} V \otimes \bigwedge^{b-1} V \xrightarrow{\Psi_{(a,b)}} \bigwedge^a V \otimes \bigwedge^b V$$

is onto. Since $a + b > m$, we know there is a basis for $\bigwedge^a V \otimes \bigwedge^b V$ of the form

$$f_{i_1} \wedge \dots \wedge f_{i_k} \wedge \dots \wedge f_{i_a} \otimes f_{i_1} \wedge \dots \wedge f_{i_k} \wedge f_{j_{k+1}} \wedge \dots \wedge f_{j_b}. \quad (15)$$

In other words, the two “terms” have an overlap of exactly k basis elements where $k \geq 1$.

We want to explicitly find an element of $\bigwedge^{a-1} V \otimes \bigwedge^{b-1} V$ which is mapped to the basis element (15), which we denote as x_0 . We do this by induction on k .

Let I be the set of all indices labelled i_x and similarly for J . Set $K = \{1, \dots, m\} - (I \cup J)$. Then K has $m + k - (a + b)$ elements. Consider

$$\begin{aligned} & \Psi \left(\sum_{\rho=1}^k f_{i_1} \wedge \dots \wedge \hat{f}_{i_\rho} \wedge \dots \wedge f_{i_k} \wedge \dots \wedge f_{i_a} \otimes f_{i_1} \wedge \dots \wedge \hat{f}_{i_\rho} \wedge \dots \wedge f_{i_k} \wedge \dots \wedge f_{j_b} \right) \\ &= k \cdot x_0 + \sum (\text{over all substitutions of an } i_k \text{ by an element of } K). \end{aligned}$$

Now

$$\begin{aligned} & \Psi \left(\sum_{1 \leq \rho < \tau \leq k} \sum_{\kappa \in K} f_\kappa \wedge f_{i_1} \wedge \dots \wedge \hat{f}_{i_\rho} \wedge \dots \wedge \hat{f}_{i_\tau} \wedge \dots \wedge f_{i_a} \otimes f_\kappa \wedge f_{i_1} \wedge \dots \wedge \hat{f}_{i_\rho} \wedge \dots \wedge f_{i_\tau} \wedge \dots \wedge f_{j_b} \right) \\ &= \binom{2}{1} \sum (\text{over all substitutions of an } i_k \text{ by an element of } K) \\ & \quad + \binom{|K|}{1} \sum (\text{over all substitutions of two } i_k\text{'s by elements of } K) \end{aligned}$$

and

$$\begin{aligned} \Psi \left(\sum_{P \subset \{1, \dots, k\}} \sum_{J \subset K} f_J \wedge f_{i_1} \wedge \dots \wedge \hat{f}_{i_p} \wedge \dots \wedge f_{i_a} \otimes f_J \wedge f_{i_1} \wedge \dots \wedge \hat{f}_{i_p} \wedge \dots \wedge f_{j_b} \right) \\ = \binom{\alpha + 1}{1} \sum (\text{over all substitutions of } \alpha \text{ } i_k \text{'s by elements of } K) \\ + \binom{|K|}{\alpha} \sum (\text{over all substitutions of } \alpha + 1 \text{ } i_k \text{'s by elements of } K) \end{aligned}$$

where $|J| = \alpha = |P| - 1$. The sum which we are taking the image of should be interpreted as summing over the following:

- subsets P of $1, \dots, k$ which are removed from the basis element, and
- subsets J of K which are inserted at the front.

When $\alpha > |K|$, this recursion ends, and so keeping track of coefficients we are done.

Now we wish to prove the converse; that the sum of the lengths of the first two columns is less than or equal to m implies that $V^{[\lambda]} \neq 0$. Restricting to $SO(V)$ there is an isomorphism $\wedge^\alpha V \xrightarrow{\tau} \wedge^{m-\alpha} V$ which takes basis elements

$$e_{i_1} \wedge \dots \wedge e_{i_a} \mapsto e_J \text{ where } J = \{1, \dots, m\} - I.$$

τ respects $SO(V)$ since $g \in SO(V)$ maps one orthogonal basis to another, and we simply use this basis to reconstruct the map τ . Lifting to $O(V)$ has the effect of allowing changing orientation, ie multiplying by ± 1 .

So if we set

$$\bar{\lambda} = \begin{cases} (\lambda_1, \dots, \lambda_k) & \text{if } \lambda_1 \leq n, \\ (m - \lambda_1, \dots, \lambda_k) & \text{if not.} \end{cases}$$

Then this implies that given a highest weight vector in $V^{[\lambda]}$ we can up to sign force it into $V^{[\bar{\lambda}]}$ where it is a highest weight vector which is clearly in $V^{[d]}$. So $V^{[\bar{\lambda}]}$ is nonzero and thus $V^{[\lambda]}$ is nonzero. ■

$O(V)/SO(V) \approx \mathbb{Z}_2$ and this implies a close relationship between their representations. We will now modify a proof on index 2 subgroups of finite groups, so as to apply it to the compact case. In order to do this we need the following two theorems, see [8]:

Theorem 3.25 *Let G be a compact Lie group and let $\mathcal{C}(G)$ be the real vector space of continuous functions on G . The invariant integral,*

$$\mathcal{C}(G) \rightarrow \mathbb{R} : f \mapsto \int f(g) dg$$

is uniquely determined by the following properties:

1. It is linear, monotone and normalised ($\int 1 = 1$).
2. It is right-invariant: $\int f(gh) dg = \int f(g) dg$ for any $h \in G$. ■

Theorem 3.26 *Let χ_V and χ_W be the characters corresponding to the representations V and W of the compact Lie group G . Then*

$$\langle \chi_W, \chi_V \rangle := \int \bar{\chi}_V(g) \chi_W(g) dg = \dim \text{Hom}_G(V, W). \blacksquare$$

Now let G be a compact Lie group, and H be a subgroup of G of index 2. Then $G/H \approx \mathbb{Z}_2$. This has two representations: one trivial and one nontrivial. These in turn give us two representations of G ; U corresponding to the trivial representation of G/H , and U' corresponding to the nontrivial representation.

Given any representation V of G , let $V' = V \otimes U'$. If we denote by $\text{Res}_H^G V$ the restriction of a representation of G to a subgroup H then clearly $\text{Res}_H^G V = \text{Res}_H^G V'$. However the two representations behave differently on elements not in H .

Given a representation W of H , we define a conjugate representation as follows: let χ be the character of W , and let $g \in G$ be any element not in H . Then the character of the conjugate is $h \mapsto \chi(ghg^{-1})$. g is unique up to multiplication by an element of H , so the conjugate representation is unique up to isomorphism.

We have the following general result,

Proposition 3.27 *Let V be an irreducible representation of G and let $W = \text{Res}_H^G V$ be the restriction of V to H . Then exactly one of the following holds:*

1. V is not isomorphic to V' ; W is irreducible and isomorphic to its conjugate, or
2. $V \approx V'$; $W = W' \oplus W''$, where W' and W'' are irreducible and conjugate but not isomorphic.

Each irreducible representation of H arises uniquely in this way, noting that in the first case, V and V' determine the same representation.

Proof. Note first that W is self-conjugate since it is the restriction of a representation of G . Let \int be an invariant integral on G , normalised so that $\int_{g \in G} 1 dg = 2$. Then

$$\int_{g \in G} 1 dg = \int_{h \in H} 1 dh + \int_{h \notin H} 1 dh = 2 \left(\int_{h \in H} 1 dh \right)$$

by left-invariance, since the left action of some $t \notin H$ on 1 is trivial. Now let χ be the character of V . Then given $t \notin H$

$$\int_{h \in H} |\chi(h)|^2 dh + \int_{h \in H} |\chi(t \cdot h)|^2 dh = 2 \left(\int_{h \in H} |\chi(h)|^2 dh \right) = \int_{g \in G} |\chi(g)|^2 dg = 2$$

by Theorems 3.25 and 3.26. From this it follows that $\int_{h \in H} |\chi(h)|^2 dh$ is equal to either 1 or 2 – ie $\dim_H(W, W) = 1$ and W irreducible or $\dim_H(W, W) = 2$ and $W = W' \oplus W''$ where these are irreducible. These are the two cases of the theorem. The first case is done.

In the second case we have $\int_{t \notin H} |\chi(t)|^2 dt = 0$ which implies that V and V' are isomorphic. Since W is self-conjugate W' and W'' must be conjugate representations of H as the alternative would be for them to be self-conjugate – implying that they are representations of G – contradicting the irreducibility of V . ■

We are not working with compact Lie groups, but rather with their complexifications. To convert between the two we use Weyl's unitary trick, see [33] or [43]:

Theorem 3.28 *Given a linear semi-simple Lie group G , let $G^{\mathbb{C}}$ be the analytic group of matrices with Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$. Suppose $G^{\mathbb{C}}$ simply-connected. Then if X is any finite dimensional complex vector space, a representation of any of the following kinds on X leads to a representation of each of the other kinds;*

1. a representation of G on X
2. a holomorphic representation of $G^{\mathbb{C}}$ on X
3. a representation of \mathfrak{g} on X
4. a complex linear representation of $\mathfrak{g}^{\mathbb{C}}$ on X . ■

Theorem 3.28 allows us to apply Proposition 3.27 to $SO_m \mathbb{C}$ using the case $SO_m \mathbb{R}$ – all that remains to be done is to find out which Young diagrams correspond to representations of $O_m \mathbb{C}$ that split under restriction.

We refer to two Young diagrams as associated if the sums of the lengths of their first columns is m , and all the rest of their columns have the same length. During the course of the proof of Lemma 3.24 we showed that associated Young diagrams produce isomorphic representations of $SO(V)$; and so, since elements of $O(V)$ merely change orientations of our basis,

$$V^{[\lambda]} \approx V^{[\bar{\lambda}]} \otimes (\text{determinant representation of } O(V))$$

as representations of $O(V)$. Thus if $\lambda \neq \bar{\lambda}$ then $V^{[\lambda]}$ is not isomorphic to $V^{[\bar{\lambda}]}$ and the restriction to $SO(V)$ of these representations are isomorphic and self-conjugate. Note in particular that we can always choose λ so that the length of the first column is less than or equal to n .

In the case of the even orthogonal group $SO_{2n} \mathbb{C}$, it is possible for $\lambda = \bar{\lambda}$ when the length of the first column is n . Then by Proposition 3.27 it follows that the restriction of $V^{[\lambda]}$ to $SO(V)$ breaks up into two conjugate irreducible representations. It is easy to check that the two representations have highest weight vectors $e_{\mathcal{U}(\lambda)}$ and $e_{\mathcal{U}'(\lambda)}$ where $\mathcal{U}'(\lambda)$ is an identical filling of λ to $\mathcal{U}(\lambda)$ except the n^{th} row is filled with $2n$'s instead of n 's.

So to summarise

Theorem 3.29 • if $\dim V = 2n + 1$, then $V^{[\lambda]}$ is a non-zero irreducible representation of $SO(V)$ iff $\lambda_{n+1} = 0$. It has highest weight $(\lambda_1, \dots, \lambda_n)$.

• If $\dim V = 2n$ and $\lambda_n = 0$ then $V^{[\lambda]}$ is a non-zero irreducible representation of $SO(V)$ with highest weight $(\lambda_1, \dots, \lambda_{n-1}, 0)$.

• If $\dim V = 2n$ and $\lambda_n > 0$ then $V^{[\lambda]}$ is a sum of two irreducible representations, with highest weights $(\lambda_1, \dots, \lambda_{n-1}, \lambda_n)$ and $(\lambda_1, \dots, \lambda_{n-1}, -\lambda_n)$. ■

4 Induced Representations on Flag Manifolds

In this section we look at representations of the classical groups from a more geometric perspective. We construct a smooth algebraic variety called a flag variety within the projectivisation of a representation. There is a strong link between the geometry of the flag variety and the Young diagram corresponding to the representation.

In addition the equations cutting out the flag variety in projective space are closely linked to the equations cutting out the representations as subspaces of a tensor product. We then obtain irreducible representations of G in each of the three classical cases by looking at sections of line bundles on the flag variety \mathcal{F} . For the most part we will follow [15].

4.1 Flag manifolds

We start by looking at complex projective space and Grassmannian varieties. Given a vector space V , we have an action of \mathbb{C}^* on $V - \{0\}$. The quotient of this action is complex projective space over the vector space V , $\mathbb{P}(V)$. Each point of $\mathbb{P}(V)$ is a line through the origin in V . For our purposes it is more convenient to work with $\mathbb{P}^*(V)$, the dual space of hyperplanes in V .

Suppose V has dimension m . For any $0 < d \leq m$ we form the Grassmannian $\text{Gr}^d V$ of subspaces of V of codimension $m-d$. An alternative way of thinking of the Grassmannian is to take a subspace E of V of codimension d . We define $\text{Stab}(E) = \{g \in \text{GL}(V) \mid g \cdot E \subset E\}$ and then form the quotient space $\text{GL}(V)/\text{Stab}(E)$. By definition $\mathbb{P}^*(V) = \text{Gr}^1 V$ and $\mathbb{P}(V) = \text{Gr}^{m-1} V$.

We now wish to embed $\text{Gr}^d V$ into some projective space as an algebraic subvariety. Given a subspace E of codimension d , the kernel of the map

$$\bigwedge^d(V) \rightarrow \bigwedge^d(V/E)$$

is a hyperplane in $\bigwedge^d(V)$. Assigning E to this kernel results in the Plücker embedding

$$\text{Gr}^d(V) \hookrightarrow \mathbb{P}^*(\bigwedge^d V).$$

We need to show that this map is an embedding and to find the equations that cut it out as a subvariety of $\mathbb{P}^*(\bigwedge^d V)$. To do this we look at the Plücker embedding more explicitly using coordinates.

Set a basis v_1, \dots, v_n for $V \approx \mathbb{C}^n$. Then define linear forms $X_{i_1, \dots, i_d} = v_{i_1} \wedge \dots \wedge v_{i_d}$ on $\mathbb{P}^*(\bigwedge^d V)$. These forms are skew-commutative in the subscripts. Points of $\mathbb{P}^*(\bigwedge^d V)$ are given homogeneous coordinates x_{i_1, \dots, i_d} , skew-commutative in the subscripts.

Given a subspace E of V , we find the coordinates of its Plücker embedding as follows. Find a $d \times n$ matrix $A : \mathbb{C}^n \rightarrow \mathbb{C}^d$ of rank d with kernel E . $\bigwedge^d A$ maps $\bigwedge^d(\mathbb{C}^n)$ to $\bigwedge^d(\mathbb{C}^d) = \mathbb{C}$ taking $v_{i_1} \wedge \dots \wedge v_{i_d}$ to the determinant of the minor of A obtained by selecting the columns numbered i_1, \dots, i_d . Thus the Plücker coordinate x_{i_1, \dots, i_d} of E in $\mathbb{P}^*(\bigwedge^d V)$ is this determinant.

Lemma 4.1 *The Plücker embedding is a bijection between $\text{Gr}^d(V)$ and the subvariety of $\mathbb{P}^*(\bigwedge^d V)$ defined by the quadratic equations*

$$x_{i_1, \dots, i_d} x_{j_1, \dots, j_d} - \sum X_{i'_1, \dots, i'_d} X_{j'_1, \dots, j'_d} = 0, \quad (16)$$

with the sum over all pairs obtained by exchanging a fixed set of k of the subscripts j_1, \dots, j_d with k of the subscripts i_1, \dots, i_d maintaining the order.

To prove (16) cuts out the Grassmannian we need the following lemma of Sylvester on the determinants of matrices.

Lemma 4.2 *Given any two $d \times d$ matrices M and N and any number k such that $1 \leq k \leq d$,*

$$\det(M) \cdot \det(N) = \sum \det(M') \cdot \det(N')$$

where the sum is over all pairs (M', N') of matrices obtained from M and N by swapping a fixed set of k columns of N with any k columns of M , preserving the ordering of columns.

Proof. We can assume that the first k columns of N are to be interchanged without loss of generality by the alternating property of determinants under interchange of column vectors. Split the matrices up into column vectors and write $|v_1 \dots v_d|$ for the determinant of the matrix formed by these vectors. It suffices to show

$$|v_1 \dots v_d| \cdot |w_1 \dots w_d| = \sum_{i_1 < \dots < i_k} |v_1 \dots w_1 \dots w_k \dots v_d| \cdot |v_{i_1} \dots v_{i_k} w_{k+1} \dots w_d|.$$

This is equivalent to showing that the difference of the two sides is an alternating function of the $d+1$ vectors v_1, \dots, v_d, w_1 since the vectors themselves are only d -dimensional, and so an alternating function of $d+1$ of them must vanish.

If two vectors v_i and v_{i+1} are equal then both sides of the equation vanish and we are done. If $v_d = w_1$ then we will show the difference of the two sides is an alternating function of v_1, \dots, v_d, w_2 . The case $v_i = v_{i+1}$ is immediate, and the case $v_d = w_2$ follows since then $w_1 = w_2$. ■

Proof of Lemma 4.1 We apply Sylvester's Lemma to the matrices M and N the minors of A obtaining by selecting columns numbered i_1, \dots, i_d and j_1, \dots, j_d respectively. This shows that the coordinates arising from the Plücker embedding satisfy the quadratic relation. To show the converse assume a point with coordinates x_{j_1, \dots, j_d} satisfies the quadratic relations (16). Fix some i_1, \dots, i_d such that $x_{i_1, \dots, i_d} \neq 0$. Since we are working in projective space set $x_{i_1, \dots, i_d} = 1$. Define a $d \times n$ matrix $A = (a_{s,t})$ by

$$a_{s,t} = x_{i_1, \dots, i_{s-1}, t, i_{s+1}, \dots, i_d}, \quad 1 \leq s \leq d, \quad 1 \leq t \leq n. \quad (17)$$

We claim $A: \mathbb{C}^n \rightarrow \mathbb{C}^d$ has kernel a subspace of codimension d with Plücker coordinates given by the x_{j_1, \dots, j_d} . Let $I = (i_1, \dots, i_d)$ and consider the determinants of minors corresponding to all possible $J = (j_1, \dots, j_d)$. For $J = I$ the minor is the identity matrix with determinant one as expected. This also shows A has rank d .

When I and J have $d-1$ entries in common, say J is obtained by replacing i_s with t , then the corresponding minor looks like the identity matrix except in the s^{th} column which will have entry $a_{s,t}$ on the diagonal giving the required determinant.

For other J we use induction on the number of differing entries. If j_r does not occur in I , then using relation (16) with one exchange: j_r , we are able to rewrite x_{j_1, \dots, j_d} as a linear combination of products of known coordinates – coordinates differing from I by less than J . We can thus express these coordinates as determinants of minors of A , and so by Sylvester's lemma x_{j_1, \dots, j_d} is the determinant of the corresponding minor of A .

The map is an embedding since given two distinct subspaces E and F we can choose a basis e_1, \dots, e_n so that $E = \langle e_{d+1}, \dots, e_n \rangle$ and $F = \langle e_1, \dots, e_r, e_{d+r+1}, \dots, e_n \rangle$ for some $r \geq 1$. Then

it is clear these subspaces have different Plücker coordinates. ■

An immediate consequence is that Grassmannians are compact. In addition they are smooth algebraic varieties and thus are complex manifolds. We now look at the quotient G/B for B the Borel subgroup of G corresponding to \mathfrak{b} .

Proposition 4.3 *B is a closed subgroup of G . G/B is compact.*

Proof. Consider the adjoint representation of \mathfrak{g} on itself. The Borel subalgebra \mathfrak{b} obviously preserves the subspace $\mathfrak{b} \subset \mathfrak{g}$ and in fact

$$\mathfrak{b} = \{x \in \mathfrak{g} \mid [x, b] \in \mathfrak{b} \text{ for all } b \in \mathfrak{b}\}.$$

This follows since given any $x \in \mathfrak{g} - \mathfrak{b}$ we can write x in the form $x = b + \sum_{\alpha \in \Delta^-} y_\alpha$ where $b \in \mathfrak{b}$ and $y_\alpha \in \mathfrak{g}_\alpha$ with at least one $y_\alpha \neq 0$. Then given any $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$ it follows $[x, h] \cap \mathfrak{g}_\alpha \neq 0$ and so $[x, h] \notin \mathfrak{b}$.

If we now lift to the group G we see that B is the connected component of the identity in

$$\text{Ad}^{-1}(\{\text{subgroup of } GL(\mathfrak{g}) \text{ holding } \mathfrak{b} \text{ invariant}\}).$$

Since this subgroup is closed it follows that B is closed. The map $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ is an embedding and so we have an embedding of G/B into the Grassmannian $GL(\mathfrak{g})/\text{Stab}(\mathfrak{b})$. Since the image of G is closed in $GL(\mathfrak{g})$ and the image of B is $\text{Stab}(\mathfrak{b})$, we have a closed subspace of the Grassmannian, and hence G/B is compact since the Grassmannian is compact. ■

Flag manifolds can be realised more concretely as the orbits of a distinguished point in the projective space obtained from an irreducible representation. Let (W, ϕ) be an irreducible representation of \mathfrak{g} generated by a highest weight vector. Since we prefer working with $\mathbb{P}^*(W)$ to $\mathbb{P}(W)$, we look at the dual representation on W^* with lowest weight vector v_ϕ . Consider the space $\mathbb{P}(W^*) = \mathbb{P}^*(W)$. The line of lowest weight vectors is mapped to a point in $\mathbb{P}^*(W)$ which we denote by $[v_\phi]$. Set

$$P = \{g \in G \mid g \cdot [v_\phi] = [v_\phi]\}.$$

We claim that P is a parabolic subgroup of G and so that the G -orbit of $[v_\phi]$ in $\mathbb{P}^*(W)$ is isomorphic to the flag manifold G/P . We will prove this case by case, at the same time looking at the structure of the flag manifold in detail.

$\mathfrak{sl}_{n+1}\mathbb{C}$:

Let V be an $n+1$ dimensional vector space. There is a 1-1 correspondence between irreducible representations and Young diagrams $\lambda = (\lambda_1, \dots, \lambda_m)$ with at most n rows. The highest weight vector in $V^{(\lambda)}$ is $e_{\mathbb{U}(\lambda)}$, and we have a basis for $V^{(\lambda)}$ consisting of e_T for T a standard filling of λ .

From this we obtain the dual basis e_{T^*} for $(V^{(\lambda)})^*$. To find the structure of \mathfrak{p} , we look at the action of the root vectors $E_{e_i - e_j}$ on $e_{\mathbb{U}(\lambda)}^*$. First we have

$$E_{e_i - e_j}(e_{\mathbb{U}(\lambda)}^*) = 0 \text{ iff } E_{e_j - e_i}(e_{\mathbb{U}(\lambda)}) = 0$$

since $E_{e_i - e_j}$ and $E_{e_j - e_i}$ are adjoint. $E_{e_j - e_i}(e_T) = \sum E_{T'}$ where the sum is over all $e_{T'}$ obtained from E_T by interchanging an e_i with a e_j . Thus it immediately follows that for $i > j$ $E_{e_j - e_i}(e_{\mathbb{U}(\lambda)}) = 0$ by virtue of the fact that the Young diagram is antisymmetric in its columns.

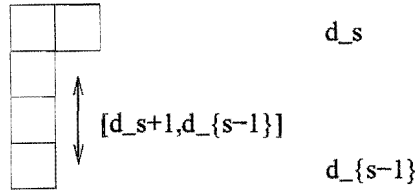


Figure 7:

Let $\tilde{\lambda} = (d_1^{\alpha_1}, \dots, d_s^{\alpha_s})$. Then if i and j are both in one of the intervals $[1, d_s]$, $[d_s + 1, d_{s-1}]$, \dots , $[d_2 + 1, d_1]$, $[d_1 + 1, n + 1]$ it follows that $E_{e_i - e_j}(e_{\mathcal{U}(\lambda)}) = 0$ also by antisymmetry of columns, since i 's and j 's always appear in the same columns.

So \mathfrak{p} is the Lie algebra of lower triangular matrices together with the roots generated by $E_{e_i - e_{i+1}}$ for i and $i + 1$ in one of the intervals listed above. So we have a parabolic subalgebra containing all the negative roots and generated by a subset of Π . Thus this is parabolic containing $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^-$.

If a column of length k occurs in λ , then the root $E_{e_k - e_{k+1}}$ will not be in \mathfrak{p} , so if we define a flag $V^{d_1} \subset V^{d_2} \subset \dots \subset V^{d_s} \subset V$ by $V^{d_k} = \langle e_{d_k+1}, e_{d_k+2}, \dots, e_{n+1} \rangle$, then \mathcal{P} can be characterised by

$$\mathcal{P} = \{g \in \text{SL}(V) \mid g(V^{d_k}) \subset V^{d_k} \text{ for } 1 \leq k \leq s\}$$

From this it follows that $\text{SL}_{n+1} \mathbb{C} / \mathcal{P}$ equals

$$\mathcal{F}^{(d_1, \dots, d_s)} := \{0 \subset V^{d_1} \subset \dots \subset V^{d_s} \subset \mathbb{C}^{n+1} \mid \text{codim } V^{d_k} = d_k\}$$

for $n \geq d_1 > \dots > d_1 \geq 0$.

We now wish to find the equations which cut out these flag manifolds as subvarieties of projective space. We know the Plücker embedding takes $\text{Gr}^d V$ into $\mathbb{P}^*(\wedge^d V)$. Thus $\mathcal{F}^{(d_1, \dots, d_k)}$ embeds in

$$\mathbb{P}^*(\wedge^{d_1} V) \times \mathbb{P}^*(\wedge^{d_2} V) \times \dots \times \mathbb{P}^*(\wedge^{d_k} V)$$

as a product of Plücker embeddings. The flag manifold is characterised by the additional incidence relations which demand that each k -tuple of spaces is an increasing chain of subspaces. The following holds

Proposition 4.4 *The flag variety $\mathcal{F}^{(d_1, \dots, d_k)}(V) \subset \prod_{i=1}^k \mathbb{P}^*(\wedge^{d_i} V)$ is cut out by the quadratic equations*

$$X_{i_1, \dots, i_p} X_{j_1, \dots, j_q} - \sum X_{i'_1, \dots, i'_p} X_{j'_1, \dots, j'_q} \quad (18)$$

where the sum is over all pairs obtain by interchanging the first k of the j subscripts with any k of the i subscripts, maintaining the order, and where $p \geq q$ are in $\{d_1, \dots, d_k\}$.

Remark. Equation (18) and relation Q2 from §3.3 are different ways of expressing the same basic relationship – algebraically in §3.3 and geometrically here.

Proof. The flag variety is clearly preserved by the action of $\text{GL}(E)$ and equation (18) is preserved by this action since the Plücker embedding and thus the homogeneous coordinates are basis independent. Thus we may pick any convenient basis for V . So given $V^i \subset V^j$ assume

$V^i = \langle e_{i+1}, \dots, e_n \rangle \subset \langle e_{j+1}, \dots, e_n \rangle = V^j$. These subspaces then each have one nonzero coordinate: $x_{1, \dots, i}$ and $x_{1, \dots, j}$ respectively. (18) is clearly satisfied since interchanging subscripts either makes no difference (in one case) or results in repeated subscripts in every other case and the coordinates are antisymmetric with respect to the subscripts.

On the other hand if $V^i \not\subset V^j$ take

$$V^i = \langle e_1, \dots, e_r, e_{i+r+1}, \dots, e_n \rangle \text{ and } V^j = \langle e_{j+1}, \dots, e_n \rangle.$$

We then have

$$\begin{aligned} 1 &= X_{r+1, \dots, r+i} X_{1, \dots, j} \neq X_{1, r+2, \dots, r+i} X_{r+1, \dots, j} \\ &\quad + X_{r+1, 1, \dots, r+i} X_{r+2, 2, \dots, j} \\ &\quad \vdots \\ &\quad + X_{r+1, \dots, r+i-1, 1} X_{r+i, 2, \dots, j} \\ &= 0. \blacksquare \end{aligned}$$

$\mathfrak{sp}_{2n} \mathbb{C}$:

Now we adapt the working above to the symplectic case, and later on the orthogonal. Let V be a $2n$ -dimensional vector space with symplectic form ω . $SP(V)$ is the group of all automorphisms of V preserving a non-degenerate antisymmetric bilinear form ω . This means that $SP(V)$ preserves isotropic subspaces and maps any two isotropic subspaces of the same dimension onto each other.

As with $\mathfrak{sl}_{n+1} \mathbb{C}$ there is a 1-1 correspondence between Young diagrams with at most n rows and irreducible representations. As before we choose an irreducible representation $(V^{(\lambda)}, \phi)$ with highest weight vector $e_{U(\lambda)}$. We wish to know the structure of \mathfrak{p} , the subalgebra annihilating the lowest weight vector in the dual representation.

Following the same reasoning as for the previous case, it is clear that $E_{-2e_i}(e_{U(\lambda)})^* = 0$ since E_{2e_i} sends e_{n+i} to e_i and there are no e_{n+i} 's in $e_{U(\lambda)}$. Similarly $E_{-e_i - e_j}$ is in \mathfrak{p} as are $E_{e_i - e_j}$ for $i > j$. Thus as before \mathfrak{p} contains all the negative roots.

Also arguing as before $E_{e_i - e_{i+1}}$ annihilates $e_{U(\lambda)}^*$ iff i and $i+1$ are in one of the intervals $[1, d_s], [d_s + 1, d_{s-1}], \dots, [d_2 + 1, d_1], [d_1 + 1, n]$. E_{2e_n} is in \mathfrak{p} iff $\lambda_n = 0$ as E_{-2e_n} maps e_n to e_{2n} and this is only zero if e_n does not occur.

Thus \mathfrak{p} holds invariant the following flag

$$V^{d_i} = \langle e_{d_i+1}, \dots, e_{2n} \rangle$$

It immediately follows that

$$SP(V)/P = \left\{ 0 \subset V^{d_1} \subset \dots \subset V^{d_s} \subset V \mid \text{codim } V^{d_k} = d_k \text{ and } (V^{d_k})^\perp \text{ isotropic for all } k \right\}$$

where $W^\perp = \{v \in V \mid \omega(w, v) = 0 \text{ for all } w \in W\}$. It suffices to have $(V^{d_1})^\perp$ isotropic; this is clearly equivalent to

$$\{0 \subset V_{d_s} \subset \dots \subset V_{d_1} \subset V \mid \dim V_k = k \text{ and } \omega(V_{d_1}, V_{d_1}) = 0\}.$$

We use the notation $\mathcal{F}^{(d_1, \dots, d_s)}$ for symplectic flags. These are cut out by the same equations as standard flags, but have additional relations imposed to reflect the requirement that all

subspaces are isotropic. Suppose we are given a matrix $\mathbb{C}^{2n} \xrightarrow{A} \mathbb{C}^d$ acting on the left of the form

$$A = \left(\begin{array}{c|ccc} I_{d \times d} & \alpha_1^1 & \dots & \alpha_1^{2n-d} \\ & \vdots & & \vdots \\ & \alpha_d^1 & \dots & \alpha_d^{2n-d} \end{array} \right)$$

then the transposes of the basis vectors for the kernel of this map are of the form

$$u_k = (-\alpha_1^k, \dots, -\alpha_d^k, 0, \dots, 0, 1, 0, \dots, 0)$$

where the 1 follows $(k-1)$ 0's for $1 \leq k \leq 2n-d$. Let K be the kernel. We wish to find K^\perp and impose conditions which make this isotropic. K^\perp has basis

$$v_j = (-\alpha_j^{n-d+1}, \dots, -\alpha_j^{2n-d}, 0, \dots, 1, \dots, 0, \alpha_j^1, \dots, \alpha_j^{n-d})$$

where the 1 follows $j-1$ zeros and $1 \leq j \leq d$. This is isotropic iff $\omega(v_k, v_j) = 0$ for all $1 \leq k, j \leq d$, so

$$\alpha_k^{n+1} \alpha_j^1 + \dots + \alpha_k^{2n-d} \alpha_j^{n-d} + \alpha_k^{n-d+j} = \alpha_j^{n+1} \alpha_k^1 + \dots + \alpha_j^{2n-d} \alpha_k^{n-d} + \alpha_j^{n-d+k}$$

taking determinants of minors it follows that

$$X_{1, \dots, d} = 1 \text{ and } X_{1, \dots, j-1, (d+k), j+1, \dots, d} = \alpha_j^k \text{ so}$$

$$\begin{aligned} & X_{1, \dots, k-1, (d+n+1), k+1, \dots, d} X_{1, \dots, j-1, (d+1), j+1, \dots, d} + \dots + \\ & X_{1, \dots, k-1, (d+2n-d), k+1, \dots, d} X_{1, \dots, j-1, (n), j+1, \dots, d} + X_{1, \dots, k-1, (n-j), k+1, \dots, d} X_{1, \dots, d} \\ & = X_{1, \dots, j-1, (d+n+1), j+1, \dots, d} X_{1, \dots, k-1, (d+1), k+1, \dots, d} + \dots + \\ & X_{1, \dots, j-1, (d+2n-d), j+1, \dots, d} X_{1, \dots, k-1, (n), k+1, \dots, d} + X_{1, \dots, j-1, (n-k), j+1, \dots, d} X_{1, \dots, d} \end{aligned}$$

for $k, j \in 1, \dots, d$ ensure the kernel of A is symplectic. Since homogeneous coordinates are basis independent, our choice of positioning for $I_{d \times d}$ is arbitrary.

Set $J = (j_1, \dots, j_d)$ and let $K = \{\text{the remaining elements of } 1, \dots, 2n\}$. Let J_i^j denote J with $K(i)$ in the j^{th} position. The equations are then

$$X_{J_k^{n+1}} X_{j_1} + \dots + X_{j_k^{2n-d}} X_{j_1^{n-d}} + X_{j_k^{n-d+j}} X_j = X_{j_1^{n+1}} X_{j_k^1} + \dots + X_{j_1^{2n-d}} X_{j_k^{n-d}} + X_{j_1^{n-d+k}} X_j$$

as J varies over ordered subsets of $(1, \dots, 2n)$, with $k, j \in 1, \dots, d$.

These equations characterise isotropic subspaces of \mathbb{C}^{2n} of dimension $2n-d$. This is clear since by changing the position of the identity matrix in A , we cover the Grassmannian $\text{Gr}^d(V)$ with open affine subspaces on which the above equations hold.

If we take $d = d_1$ then we have the additional equation needed for the symplectic flag variety $\mathcal{F}^{(d_1, \dots, d_s)}$, since

$$\mathcal{F}^{(d_1, \dots, d_s)} \subset \text{Gr}^{d_1}(V) \times \dots \times \text{Gr}^{d_s}(V).$$

$\mathfrak{so}_{2n+1}\mathbb{C}$:

There is a 1-1 correspondence between representations of $O_n\mathbb{C}$ and Young diagrams with the sum of the lengths of the first two columns at most n . When restricting to the special

orthogonal group, it suffices to consider Young diagrams with the added condition that the length of the first column is at most $\frac{1}{2}n$. As before we consider the link between the shape of a Young diagram λ , and the algebra \mathfrak{p} holding invariant the lowest weight vector of the representation $(V^{[\lambda]})^*$.

The negative roots are clearly in \mathfrak{p} . Concentrating on the simple roots, $E_{e_i - e_{i+1}} \in \mathfrak{p}$ iff i and $i+1$ are in one of the intervals $[1, d_s], [d_s + 1, d_{s-1}], \dots, [d_2 + 1, d_1], [d_1 + 1, n]$. E_{e_n} is in \mathfrak{p} iff $\lambda_n = 0$ as E_{-e_n} maps e_n to e_{2n+1} and this is only zero if e_n does not occur. So almost identically to the symplectic case,

$$P = \left\{ g \in SO(V) \mid g(V^{d_k}) \subset V^{d_k} \text{ for all } k \right\}$$

for $V^{d_k} = \langle e_{d_k+1}, \dots, e_{2n+1} \rangle$.

$SO(V)/P = \mathcal{F}^{[d_1, \dots, d_s]}$ is the flag manifold of isotropic subspaces under Q of the given codimensions. So we follow the same line of reasoning as in the symplectic case to find the equations which cut out this subvariety.

Here we have a matrix A acting on the left, $\mathbb{C}^{2n+1} \xrightarrow{A} \mathbb{C}^d$.

$$A = \left(\begin{array}{c|ccc} & \alpha_1^1 & \dots & \alpha_1^{2n+1-d} \\ I_{d \times d} & \vdots & & \vdots \\ & \alpha_d^1 & \dots & \alpha_d^{2n+1-d} \end{array} \right)$$

The kernel K of this matrix has basis consisting of (transposed) vectors of the form

$$u_k = (-\alpha_1^k, \dots, -\alpha_d^k, 0, \dots, 1, \dots, 0)$$

for k between 1 and $2n+1-d$. The orthogonal complement, K^\perp has basis

$$v_j = (\alpha_j^{n-d+1}, \dots, \alpha_j^{2n-d}, 0, \dots, 1, \dots, 0, \alpha_j^1, \dots, \alpha_j^{n-d}, \alpha_j^{2n+1-d}).$$

for j between 1 and d . The requirement $Q(v_k, v_j) = 0$ is equivalent to

$$\alpha_j^{n-d+k} + \alpha_k^{n+1} \alpha_j^1 + \dots + \alpha_k^{2n-d} \alpha_j^{n-d} + \alpha_k^{2n+1-d} \alpha_j^{2n+1-d} + \alpha_k^{n-d+j} + \alpha_j^{n+1} \alpha_k^1 + \dots + \alpha_j^{2n-d} \alpha_k^{n-d} = 0$$

Setting $I = (i_1, \dots, i_d)$ and J to be any $2n+1$ of the remaining $n+1$ indices, with the last element e_{2n+1} in J satisfying $Q(e_{2n+1}, e_{2n+1}) = 1$. Then the equations are

$$X_I X_{I_j^{n-d+k}} + \sum_{l=1}^n X_{I_k^{n+1} X_{I_l^1}} + X_{I_k^{2n-d} X_{I_j^{n-d}}} + X_{I_k^{2n+1-d} X_{I_j^{2n+1-d}}} + X_I X_{I_k^{n-d+j}} + \sum_{l=1}^n X_{I_j^{n+1} X_{I_l^1}} = 0$$

summing over the tuples I and J , and the indices j and k . Letting $d = d_1$ results in the desired equations.

$\mathfrak{so}_{2n}\mathbb{C}$:

This case is much the same as the others except for unusual behaviour when $\lambda_n \neq 0$. Then $V^{[\lambda]}$ splits into two irreducible representations of $\mathfrak{so}_{2n}\mathbb{C}$. To see how this is reflected in terms of flags, we take a representation $V^{[\lambda]}$ and consider the stabiliser \mathfrak{p} of the lowest weight vector of

its dual. As usual \mathfrak{p} contains all the negative roots. The link between the Young diagram and the corresponding flag manifold is as usual, with \mathfrak{p} holding

$$V^{d_k} = \langle e_{d_k+1}, \dots, e_{2n} \rangle.$$

invariant. The difference comes in when $d_1 = n$. In this case \mathfrak{p} holds invariant two different flags, the chain described above with either

$$V^{d_1} = V^n = \langle e_{n+1}, \dots, e_{2n} \rangle \text{ or } V^{d'_1} = V^{n'} = \langle e_n, \dots, e_{2n-1} \rangle$$

at the end of the chain. Elements of $SO_{2n} \mathbb{C}$ interchange two elements at a time with e_1, \dots, e_n and so the dimension of the intersection of V^{d_1} and $E = \langle e_1, \dots, e_n \rangle$ is preserved mod 2. So each Young diagrams corresponds to two flag manifolds,

$$\begin{aligned} SO_{2n} \mathbb{C}/P = \{0 \subset V^{d_1} \subset \dots \subset V^{d_s} \subset \mathbb{C}^{2n} \mid Q(V^{d_1}, V^{d_1}) = 0 \\ \text{and } d_1 = n \text{ implies } \dim V^{d_1} \cap E \equiv 0 \pmod{2}\}, \end{aligned}$$

and

$$\begin{aligned} SO_{2n} \mathbb{C}/P = \{0 \subset V^{d_1} \subset \dots \subset V^{d_s} \subset \mathbb{C}^{2n} \mid Q(V^{d_1}, V^{d_1}) = 0 \\ \text{and } d_1 = n \text{ implies } \dim V^{d_1} \cap E \equiv 1 \pmod{2}\}, \end{aligned}$$

where the two coincide if $\lambda_n = 0$. It turns out that the equations which cut out the isotropic subspaces in $Gr^d(\mathbb{C}^{2n})$ are

$$X_I X_{I_j^{n-d+k}} + \sum_{l=1}^n X_{I_k^{n+l}} X_{I_j^l} + X_I X_{I_k^{n-d+l}} + \sum_{l=1}^n X_{I_j^{n+l}} X_{I_k^l} = 0$$

For $d = n$ we need to impose an additional relation to distinguish the two possible flag manifolds. We may as well take a basis for E to be $e_k = (0, \dots, 0, 0, \dots, 1, \dots, 0)$, ie n zeros followed by $k-1$ zeros and then a 1. The dimension of the intersection of E and $\ker A$ is equal to $\text{rank}(\alpha_{i,j})$, and equations cutting out the flags can be derived from this.

4.2 Vector bundles on Flag manifolds

We start with a quick description of complex vector bundles, before specialising to the flag manifolds and (for the most part) line bundles.

Definition 4.5 [38]. *A complex vector bundle \mathcal{E} of rank n over a manifold X consists of*

- a topological space \mathcal{E} , the total space,*
- a map $\pi: \mathcal{E} \rightarrow X$ called the projection map, and*
- for each $x \in X$ the fiber $\mathcal{E}_x = \pi^{-1}(x)$ has the structure of a complex vector space of rank n .*

$$\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathcal{E} \\ & & \downarrow \pi \\ & & X \end{array}$$

This structure is subject to the requirement of local triviality:

Each point of x has a (not necessarily unique) neighbourhood U such that there is a homeomorphism

$$U \times \mathbb{C}^n \xrightarrow{\phi} \pi^{-1}(U) := \mathcal{E}_U.$$

ϕ has the property that for each $x \in U$ the map $x \mapsto \phi(x, v)$ defines a linear isomorphism between \mathbb{C}^n and $\pi^{-1}(x)$.

The pairing (U, ϕ) is referred to as a local system for \mathcal{E} at x .

Given an open cover $\{U_\alpha\}$ of X with trivialisations $\phi_\alpha : \mathcal{E}_{U_\alpha} \rightarrow U_\alpha \times \mathbb{C}^n$ we define the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL_n \mathbb{C}$ by

$$g_{\alpha\beta}(z) = \left(\phi_\alpha \circ \phi_\beta^{-1} \right) |_{\mathcal{E}_z} \in GL_n \mathbb{C}.$$

These satisfy the condition

$$\begin{cases} g_{\alpha\beta} \cdot g_{\beta\alpha} = \text{id}, \\ g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \text{id}. \end{cases} \quad (19)$$

Conversely given any collection of local functions satisfying these identities we can construct the corresponding vector bundle by taking the union of $U_\alpha \times \mathbb{C}^n$ over all α and identifying $\{z\} \times \mathbb{C}^n$ in $U_\alpha \times \mathbb{C}^n$ and $U_\beta \times \mathbb{C}^n$ via $g_{\alpha\beta}(z)$.

We are interesting in looking at homogeneous vector bundles constructed over the flag manifolds.

Definition 4.6 Given a group G and a manifold X with a left G -action, $G \times X \rightarrow X : (g, x) \mapsto g \cdot x$, a homogeneous vector bundle \mathcal{E} over X is a vector bundle with left G -action satisfying

- (a) $g \cdot \mathcal{E}_x = \mathcal{E}_{g \cdot x}$ for $x \in X$ and $g \in G$.
- (b) The mapping $\mathcal{E}_x \xrightarrow{g \cdot} \mathcal{E}_{g \cdot x}$ induced by g is linear.

If we assume that X is of the form G/H for H a subgroup of the Lie group G then we can construct any homogeneous vector bundle over X as follows,

Let (ρ, E) be a finite dimensional representation of H and define a right H -action on $G \times E$,

$$(g, v) \cdot h = (g \cdot h, \rho(h)^{-1}v) \text{ for } g \in G, v \in E \text{ and } h \in H.$$

Let $\mathcal{E} = G \times_\rho E = (G \times E)/H$. Then let $[g, v] = (g, v) \cdot H$ and define $\pi([g, v]) = g \cdot H$. π is a well-defined map $\mathcal{E} \rightarrow X$. It can be shown that \mathcal{E} is locally trivial and that all homogeneous vector bundles arise in this manner, see [41].

Now we specialise to the case of flag manifolds. Each point of the flag has a collection of vector spaces attached to it, we expect to find many naturally defined vector bundles which sit over the flag. We start by looking at complex projective space.

$\mathbb{P}^*(V)$ has a natural line bundle associated to it. At each point x in $\mathbb{P}^*(V)$, associate the line formed by taking the quotient of V by the hyperplane corresponding to x . It can be shown that this construction trivialises when restricted to the affine opens

$$D(X_i) = \{x \in \mathbb{P}(V) \mid X_i(x) \neq 0\},$$

for $X_i \in \text{Sym}^1(V)$. We denote this line bundle by $\mathcal{O}_{\mathbb{P}^*(V)}(1)$ and refer to it as the canonical line bundle. Set $\mathcal{O}_{\mathbb{P}^*(V)}(n) = \mathcal{O}_{\mathbb{P}^*(V)}(1)^{\otimes n}$. Given a subvariety X of $\mathbb{P}^*(V)$ write $\mathcal{O}_X(n)$ for the restriction of $\mathcal{O}_{\mathbb{P}^*(V)}(n)$ to X . Elaborating, we have an imbedding $\iota : X \hookrightarrow \mathbb{P}^*(V)$, and we set $\mathcal{O}_X(n) = \iota^*(\mathcal{O}_{\mathbb{P}^*(V)}(n))$, the pullback of the canonical line bundle.

Let \mathcal{F} refer to $\mathcal{F}^{d_1, \dots, d_s}(V) = G/P$. We construct the following commutative diagram.

$$\begin{array}{ccccc}
 \mathcal{F} & \longrightarrow & \prod_{i=1}^s \text{Gr}^{d_i}(V) & \longrightarrow & \prod_{i=1}^s \mathbb{P}^*(\wedge^{d_i} V) \\
 \downarrow & & & & \downarrow \\
 & & & & \prod_{i=1}^s \mathbb{P}^*(\text{Sym}^{a_i}(\wedge^{d_i} V)) \\
 & & & & \downarrow \\
 \mathbb{P}^*(V^\lambda) & \longrightarrow & & \longrightarrow & \mathbb{P}^*(\otimes_{i=1}^s \text{Sym}^{a_i}(\wedge^{d_i} V)).
 \end{array}$$

The diagram commutes since the equations which define the flag manifold $\mathcal{F}^{d_1, \dots, d_s}(V)$ in $\prod_{i=1}^s \mathbb{P}^*(\text{Sym}^{a_i}(\wedge^{d_i} V))$ also define $\mathbb{P}^*(V^\lambda)$ in $\mathbb{P}^*(\otimes_{i=1}^s \text{Sym}^{a_i}(\wedge^{d_i} V))$ by construction of the vector space V^λ . In the symplectic and orthogonal cases the additional conditions that the flag be of isotropic subspaces corresponds to the condition on $V^{(\lambda)}$ and $V^{[\lambda]}$ that we restrict to harmonic tensors (Definitions 3.15 and 3.22).

Definition 4.7 Let $\mathcal{L}^\lambda = \mathcal{O}_{\mathcal{F}}(1)$ for the embedding of \mathcal{F} in $\mathbb{P}^*(V^\lambda)$. In other words \mathcal{L}^λ is the pullback of the canonical line bundle on $\mathbb{P}^*(V^\lambda)$.

By commutativity of the above diagram and since the maps used behave well under pullbacks it follows that

$$\begin{aligned}
 \mathcal{L}^\lambda &= \mathcal{O}_{\mathcal{F}}(a_1, \dots, a_s) \\
 &:= \pi_1^* \mathcal{O}_{\mathbb{P}^*(V_1)}(a_1) \otimes \dots \otimes \pi_s^* \mathcal{O}_{\mathbb{P}^*(V_s)}(a_s)
 \end{aligned}$$

since \mathcal{F} embeds in $\prod_{i=1}^s \mathbb{P}^*(\text{Sym}^{a_i}(\wedge^{d_i} V))$.

There is an alternative construction of this line bundle more closely related to the geometry of the flag. \mathcal{F} has a canonical flag of vector bundles

$$\mathcal{F} \times \{0\} = \mathcal{O}_{\mathcal{F}} \subset \mathcal{V}^{d_1} \subset \dots \subset \mathcal{V}^{d_s} \subset \mathcal{V}_{\mathcal{F}} = \mathcal{F} \times V, \quad \text{rank}(\mathcal{V}^{d_k}) = \dim V - d_k.$$

Here the fiber above a point is clearly just the flag of vector spaces which the point corresponds to. In the case $\mathcal{F} = \mathbb{P}^*(V)$, $\mathcal{O}_{\mathbb{P}^*(V)}(1)$ is the quotient of $\mathcal{V}_{\mathcal{F}}$ by the canonical flag.

The Plücker embedding $\text{Gr}^d V \hookrightarrow \mathbb{P}^*(\wedge^d V)$ maps $\mathcal{U} \subset V$ to the hyperplane that is the kernel of the map $\wedge^d V \rightarrow \wedge^d(V/\mathcal{U})$. So pulling back the line bundle $\mathcal{O}_{\mathbb{P}^*(\wedge^d V)}(1)$ to $\text{Gr}^d V$ results in the bundle $\wedge^d(\mathcal{V}_{\mathcal{F}}/\mathcal{U})$, the where $\mathcal{U} \subset \mathcal{V}_{\mathcal{F}}$ is the canonical bundle. From this it follows

$$\mathcal{L}^\lambda = \mathcal{O}_{\mathcal{F}}(a_1, \dots, a_s) = \bigwedge^{d_1}(\mathcal{V}/\mathcal{U}_1)^{\otimes a_1} \otimes \dots \otimes \bigwedge^{d_s}(\mathcal{V}/\mathcal{U}_s)^{\otimes a_s}.$$

There is a unique fixed point for P on $\mathcal{F} = G/P$. This is the flag $\mathfrak{x} = V^{d_1} \subset \dots \subset V^{d_s} \subset V$ which was constructed above in each case. So the fiber of \mathcal{L}^λ at \mathfrak{x} is $\wedge^{d_1}(V/\mathcal{U}^{d_1})^{\otimes a_1} \otimes \dots \otimes \wedge^{d_s}(V/\mathcal{U}^{d_s})$.

$\bigwedge^{d_k}(V/V^{d_k})$ is generated by the image of $e_1 \wedge \dots \wedge e_{d_k}$ and so the action of $p \in P$ on this is multiplication by the determinant of the upper-left $d_k \times d_k$ corner of p . If we let A_k be the upper-left $d_k \times d_k$ corner of p , then the action of p on \mathcal{L}^λ is to multiply by $\det(A_1)^{\alpha_1} \dots \det(A_s)^{\alpha_s}$.

If we are given a character $\chi : P \rightarrow \mathbb{C}^*$, let $\mathbb{C}_{\chi, P}$ be the one dimensional P -module with P -action given by $p : z \mapsto \chi(p) \cdot z$. We can then form a line bundle $\mathcal{L}(\chi)$ as a quotient

$$\mathcal{L}(\chi) = G \times_P \mathbb{C}_{\chi, P} = G \times \mathbb{C} / (g \cdot p \times z) \sim (g \times \chi(p)z)$$

This is an equivariant line bundle over G/P , indeed it is a special case of the our general technique for constructing homogeneous vector bundles. Set $\chi_\lambda(g) = \det(A_1)^{\alpha_1} \dots \det(A_s)^{\alpha_s}$. Then we have shown above that $\mathcal{L}^\lambda = \mathcal{L}(\chi_\lambda)$.

Proposition 4.8 *The space $\Gamma(G/P, \mathcal{L}^\lambda)$ of sections of \mathcal{L}^λ is isomorphic to V^λ .*

Remark. Before proving this we define a section. A section of G/P is a function $f : G \rightarrow \mathbb{C}$ that satisfies

$$\chi(p)f(g \cdot p) = f(g) \text{ for all } g \in G, p \in P.$$

$\Gamma(G/P, \mathcal{L}^\lambda)$ then denotes the vector space of all section of \mathcal{L}^λ . G acts on this space on the left by the formula $(g \cdot f)(h) = f(g^{-1} \cdot h)$ for $g, h \in G$.

Proof. Firstly it is a standard fact of algebraic geometry, see [24] that the space of algebraic sections of a projective variety is finite dimensional. Since G has a representation on the space of sections, all that is required is to show that there is only one highest weight vector up to scalar multiplication, and that it has weight λ .

Let U be the subgroup of upper triangular unipotent matrices, and B' the group of lower triangular matrices. Any highest weight vector satisfies $f(g \cdot u) = f(g)$ for $u \in U$. We have that $U \cdot B'$ is dense in G , see [19], implying that the highest weight vector f is determined by its value at the identity e in G . So there is at most one highest weight vector with $f(e) = 1$. The formula $f(g) = \chi_\lambda(g^{-1})$ determines this vector. This has weight λ since writing $x = \text{diag}(x_1, \dots, x_m)$ it follows $x \cdot f(1) = f(x^{-1}) = \chi_\lambda(x) \cdot f(1) = x_1^{\lambda_1} \dots x_m^{\lambda_m} \cdot f(1)$. ■

5 The Bernstein-Gel'fand-Gel'fand Resolution

In this section we construct the Bernstein-Gel'fand-Gel'fand resolution and show its exactness. We do this following the original article of Bernstein-Gel'fand-Gel'fand [31] and a subsequent article by Garland and Lepowsky [18] which simplifies and generalises parts of the argument. The most notable simplification is the use of the Casimir operator as a distinguished member of the centre of the universal enveloping algebra in place of the Harish-Chandra isomorphism.

This section is considerably more abstract than the previous two. It provides detailed structural information about $U(\mathfrak{g})$ -modules, and combined with Bott's vanishing theorem (§6) gives a proof of the Borel-Weil theorem.

5.1 The Casimir Operator

First we define our category – we work in a category of \mathfrak{g} -modules which is reasonably well behaved. Bernstein-Gel'fand-Gel'fand work in a category \mathcal{O} which we enlarge slightly following Garland and Lepowsky.

Definition 5.1 *Given $\nu \in \mathfrak{h}^*$, define $D(\nu) = \{\nu - \sum_{i=1}^l n_i \alpha_i \mid n_i \in \mathbb{Z}\} \subset \mathfrak{h}^*$. Then let \mathcal{C} be the full subcategory of \mathfrak{g} -modules X such that*

1. X has a weight space decomposition.
2. Each weight space is finite dimensional.
3. The weights of X lie in a finite union of sets of the form $D(\nu)$ for $\nu \in \mathfrak{h}^*$.

Clearly \mathcal{C} is closed under taking quotients and submodules. It is also clear that \mathcal{C} contains all highest weight modules and all simple \mathfrak{g} -modules. Following Garland and Lepowsky, we define a (generalised) Casimir element Γ in $U(\mathfrak{g})$. Then for each \mathfrak{g} -module X there is a Casimir operator Γ_X in $\text{End } X$.

Pick any basis e_α for \mathfrak{n} where $e_\alpha \in \mathfrak{g}_\alpha$ and $\alpha \in \Delta^+$. Using the Killing form a dual basis f_α for \mathfrak{n}^- can be found. Set $\Gamma_1 = 2 \sum_{\alpha \in \Delta^+} f_\alpha e_\alpha$. It is clear that Γ_1 is independent of the choice of basis from the following alternative construction: first notice the isomorphism

$$(\mathfrak{g}_\alpha)^* \otimes \mathfrak{g}_\alpha \xrightarrow{\approx} \text{End } \mathfrak{g}_\alpha : f \otimes v \mapsto f_v$$

where $f_v : \mathfrak{g}_\alpha \rightarrow \mathfrak{g}_\alpha : u \mapsto f(u)v$. This is an isomorphism since the two spaces have the same dimension, and $f_v = 0$ implies either f or v equal zero – so it is an injection. Set $\iota_\alpha \in (\mathfrak{g}_\alpha)^* \otimes \mathfrak{g}_\alpha \approx \text{End } \mathfrak{g}_\alpha$ as the element corresponding to $1_{\mathfrak{g}_\alpha} \in \text{End } \mathfrak{g}_\alpha$. If we set $B_\alpha : \mathfrak{g}_\alpha^* \rightarrow \mathfrak{g}_{-\alpha}$ as the isomorphism induced by the Killing form and have $m : \mathfrak{g} \otimes \mathfrak{g} \rightarrow U(\mathfrak{g})$ as multiplication then we claim

$$\Gamma_1 = 2 \sum_{\alpha \in \Delta^+} m \circ (B_\alpha \otimes 1)(\iota_\alpha).$$

This is equivalent to claiming that $e_\alpha f_\alpha = m \circ (B_\alpha \otimes 1)(\iota_\alpha)$. This is clear since $v^* \otimes v \mapsto 1_\alpha$ under the isomorphism described above, and $B(e_\alpha, f_\alpha) = 1$ by construction.

For X a \mathfrak{g} -module we have Γ_1 an operator on X . Define a second operator Γ_2 on X by having Γ_2 act on X_ϕ as scalar multiplication by $\langle \phi + \rho, \phi + \rho \rangle$. The Casimir operator $\Gamma_X \in \text{End } X$ is defined to be $\Gamma_1 + \Gamma_2$.

Proposition 5.2 *Given $f : X \rightarrow Y$ a \mathfrak{g} -module map then $f \circ \Gamma_X = \Gamma_Y \circ f : X \rightarrow Y$. ■*

Proposition 5.3 For $X \in \mathcal{C}$ the Casimir operator Γ_X commutes with the action of \mathfrak{g} on X . ■

The first result is clear since Γ_X is defined using elements of \mathfrak{g} and so commutes with \mathfrak{g} -module maps. The second is a standard computational result, see [18]. Now using the notation $\lambda \sim \delta$ when $\lambda = w \cdot \delta$ for some $w \in \mathbb{W}$, where λ and δ are weights, we have the following

Corollary 5.4 Let X be a highest weight module for \mathfrak{g} of weight λ . The Casimir operator Γ_X acts on X as scalar multiplication by $\langle \lambda + \rho, \lambda + \rho \rangle$. Moreover, in the case of a Verma module $V(\lambda)$ we have $\Gamma_{V(\lambda)}$ acting as scalar multiplication by $c_\lambda = \langle \lambda, \lambda \rangle$. Thus since the Killing form is invariant under \mathbb{W} , we have $c_\lambda = c_\delta$ if $\lambda \sim \delta$. ■

This corollary suggests that it would be interesting to study how a \mathfrak{g} -module splits up into eigenspaces under the action of the Casimir operator. For a start we know that each eigenspace will contain entire highest weight modules – for example entire Verma modules or irreducible representations – and this is useful since it provides a means of digging into the structure of a \mathfrak{g} -module.

So with this in mind we collect together all the eigenvalues of a \mathfrak{g} -module X as follows: for $X \in \mathcal{C}$ let

$$\Theta(X) = \{c \in \mathbb{C} \mid \Gamma_X x = cx \text{ for some } x \in X, x \neq 0\}.$$

$$\text{We also let } X_{(c)} = \{x \in X \mid (\Gamma_X - c)^n x = 0 \text{ for some } n > 0\}.$$

Thus $\Theta(X)$ is the set of eigenvalues of the Casimir operator, with $X_{(c)}$ the space of generalized eigenvectors with eigenvalue c . From these definitions it follows that $\Theta(X) = \{c \in \mathbb{C} \mid X_{(c)} \neq 0\}$. Also from the Corollary we have $\Theta(X) = \{\langle \lambda + \rho, \lambda + \rho \rangle\}$ for X a highest weight module generated by a vector of weight λ .

Since the Casimir operator collects highest weight modules together, it would be useful to see how any given \mathfrak{g} -module breaks down into highest weight modules. A first step towards this is the following

Lemma 5.5 Let $X \in \mathcal{C}$. Then X has a (possibly finite) \mathfrak{g} -module filtration $0 = X_0 \subset X_1 \subset \dots$ such that $X = \bigcup X_i$ and each \mathfrak{g} -module X_{i+1}/X_i is a highest weight module. In particular if $X \neq 0$ then X contains a highest weight vector.

Proof. Let $\nu_1, \dots, \nu_k \in \mathfrak{h}^*$ be a set of weights such that the weights of X are in $D(\nu_1) \cup \dots \cup D(\nu_k)$. We say two weights λ and ϕ are compatible if $\lambda - \phi = \sum_{i=1}^l \mathbb{Z}\alpha_i$. Given λ and ϕ compatible we can find a κ so that $D(\lambda) \cup D(\phi) \subset D(\kappa)$. Thus we may as well take ν_1, \dots, ν_k to be mutually incompatible.

As a result of this assumption we have that every weight of X lies in a unique $D(\nu_j)$. So given a weight μ of X which lies in ν_j , write $\nu_j - \mu = \sum_{i=1}^l n_i \alpha_i$ for n_i in \mathbb{Z}^+ . We can then define $N(\mu) = \sum_{i=1}^l n_i$ a positive integer. We use these integers to set up our filtration.

Define $X(n)$ to be the sum of all weight spaces X_μ such that $N(\mu) = n$. Then $X(n)$ is finite dimensional since the weight spaces are finite dimensional and there are a finite number of ways that l positive integers can sum to n .

Let n_X be the minimal nonnegative integer such that $X(n_X) \neq 0$ with μ a weight with $N(\mu) = n_X$. Choose a vector x in X_μ . Then x is n_X -invariant by minimality so the \mathfrak{g} -module X_1 generated by x is a highest weight module. Consider the quotient module X/X_1 . It follows that $n_{X/X_1} \geq n_X$ with equality iff $\dim(X/X_1)(n_X) < \dim X(n_X)$.

Applying this procedure inductively we obtain the desired filtration of X . ■

Proposition 5.6 For all $X \in \mathcal{C}$,

$$X = \bigoplus_{c \in \Theta(X)} X_{(c)}.$$

Let $0 = X_0 \subset X_1 \subset \dots$ be any filtration with the properties of the previous lemma with $\lambda_i \in \mathfrak{h}^*$ the highest weight of X_{i+1}/X_i for each i . Then

$$\Theta(X) = \{(\lambda_i + \rho, \lambda_i + \rho)\}_i.$$

Proof. Given a subspace Y of X , denote by $[Y]$ the subspace generated by Y under the action of Γ_X . Let $c_1, c_2, \dots \in \mathcal{C}$ be the distinct elements of the set $\{(\lambda_i + \rho, \lambda_i + \rho)\}_i$. Any finite dimensional subspace of X is contained in some X_i ; and each X_i is generated by a finite number of highest weight vectors, so if Y is finite dimensional subspace of X_i then so is $[Y]$. It also follows that $[Y]$ will be annihilated by products of powers of a finite number of $\Gamma_X - c_j$. So we have $[Y] = \bigoplus_j ([Y] \cap X_{(c_j)})$.

Set $Y_n = \bigoplus_{k < n} X_{(c_k)}$. This is a filtration $0 = Y_0 \subset Y_1 \subset \dots$ of X where each Y_k is finite dimensional and $X = \bigcup Y_k$. From this we see that $X = \bigoplus_j X_{(c_j)}$ and so $X_{(c)} \neq 0$ implies that $c = c_j$ for some j . For each i we can choose an element $x \in X_{i+1} - X_i$. Let $c_i = (\lambda_i + \rho, \lambda_i + \rho)$. Then for any $c \neq c_j$, $\Gamma_X - c$ multiplies x by a nonzero scalar so $x \in X_{(c_j)}$ and $X_{(c_j)}$ is not empty.

Now $\Theta(X) = \{c \in \mathcal{C} \mid X_{(c)} \neq 0\}$ so $\Theta(X) = \{c_1, c_2, \dots\}$. ■

If a filtration exists with the properties described above then we say that X is of type Ψ where Ψ is the collection of weights ψ_i (not necessarily distinct) such that X_{i+1}/X_i has highest weight ψ_i . Ψ simply lists the weights which occur in the Jordan-Hölder decomposition of X .

Proposition 5.7 Let $c \in \mathcal{C}$. $X \mapsto X_{(c)}$ is an exact functor from \mathcal{C} to \mathcal{C} . In particular if $Y \in \mathcal{C}$ is a \mathfrak{g} -submodule of X then $(X/Y)_{(c)} = X_{(c)}/Y_{(c)}$.

Proof. $X_{(c)}$ is a \mathfrak{g} -submodule of X since Γ_X commutes with \mathfrak{g} . Also from Proposition 5.2 we have that the map $X \mapsto X_{(c)}$ is functorial. From the above splitting we see that the functor is exact. ■

Corollary 5.8 Let $X \in \mathcal{C}$ and $0 = X_0 \subset X_1 \subset \dots$ be a filtration with the properties stated in Lemma 5.5 with $\lambda_i \in \mathfrak{h}^*$ the highest weight of X_{i+1}/X_i . Then $X_{(c)}$ has a \mathfrak{g} -module filtration $0 = Y_0 \subset Y_1 \subset \dots$ such that $X_{(c)} = \bigcup Y_j$ and the family of \mathfrak{g} -modules Y_{i+1}/Y_i corresponds up to isomorphism with the family of \mathfrak{g} -modules X_{i+1}/X_i which satisfy $(\lambda_i + \rho, \lambda_i + \rho) = c$. ■

5.2 The Bruhat order and the Weyl group

In this subsection we collect together results on the Bruhat order and the Weyl group which will be of use later on in this section as well as in §7. We also collect together without proof some results on Verma modules, see [29]. References for this subsection are [7], [30] and [31]. The last two have different (dual) definitions of the Bruhat order. We follow [31], altering the results of [30] when necessary.

Theorem 5.9 Let $\chi, \psi \in \mathfrak{h}^*$. Then either

1. $\text{Hom}_{\mathbf{U}(\mathfrak{g})}(V(\chi), V(\psi)) = 0$, or,
2. $\text{Hom}_{\mathbf{U}(\mathfrak{g})}(V(\chi), V(\psi)) = \mathbb{C}$, and all nonzero maps are injections. ■

Given $w, w' \in \mathbb{W}$, we write $w \xrightarrow{\alpha} w'$ if $w = \sigma_{\alpha} w'$ and $l(w) = l(w') + 1$. We say that $w < w'$ if there exists a sequence w_1, \dots, w_k of elements of \mathbb{W} such that

$$w \rightarrow w_1 \rightarrow \dots \rightarrow w_k \rightarrow w'.$$

This defines a partial ordering on \mathbb{W} , the Bruhat order. The Bruhat order is important because of the following two theorems (amongst other things)

Theorem 5.10 *Let $\chi \in \mathbb{P}$, $w, w' \in \mathbb{W}$. Then $\text{Hom}_{\mathbb{U}(\mathfrak{g})}(V(w\chi), V(w'\chi)) = \mathbb{C}$ iff $w \leq w'$. ■*

Theorem 5.11 *Let $\chi \in \mathbb{P}$, $w, w' \in \mathbb{W}$. Then $R(w\chi) \in \text{JH}(V(w'\chi))$ iff $w \leq w'$. ■*

These theorems also provide a partial explanation for the strange definitions of Verma modules. $V(\lambda)$ has highest weight $\lambda - \rho$ since this is what is required to ensure the above theorems are true. This is useful and important since the theorems provide a strong link between the (combinatorial) structure of the Weyl group and the way in which highest weight modules embed in one another.

$$\text{Set } \Phi_w = \{\gamma \in \Delta^+ \mid w^{-1}\gamma \in \Delta^-\} = \Delta^+ \cap w\Delta^-. \quad (20)$$

Lemma 5.12 *Let $w = \sigma_{\alpha_1} \dots \sigma_{\alpha_l}$ be a reduced decomposition of $w \in \mathbb{W}$. Let $\gamma_i = \sigma_{\alpha_1} \dots \sigma_{\alpha_{i-1}}(\alpha_i)$. Then the roots $\gamma_1, \dots, \gamma_l$ are distinct and the set $\{\gamma_1, \dots, \gamma_l\}$ coincides with $\Phi_w = \Delta^+ \cap w\Delta^-$.*

Proof. See [7]. ■

Lemma 5.13 *The following hold:*

(a) *Let $w = \sigma_{\alpha_1} \dots \sigma_{\alpha_l}$ be a reduced composition and let $\gamma \in \Delta^+$ be a root such that $w^{-1}\gamma \in \Delta^-$. Then for some i ,*

$$\sigma_{\gamma} \sigma_{\alpha_1} \dots \sigma_{\alpha_l} = \sigma_{\alpha_1} \dots \sigma_{\alpha_{i-1}} \quad (21)$$

(b) *Let $w \in \mathbb{W}$, $\gamma \in \Delta^+$. Then $l(w) < l(\sigma_{\gamma}w)$ if and only if $w^{-1}\gamma \in \Delta^+$.*

Proof. For (a), the condition on γ is that it is an element of $\Delta^+ \cap w\Delta^-$. Then Lemma 5.12 implies that $\gamma = \sigma_{\alpha_1} \dots \sigma_{\alpha_{i-1}}(\alpha_i)$ for some i , implying (21).

(b) If $w^{-1}\gamma \in \Delta^-$ then by (21), $\sigma_{\gamma}w = \sigma_{\alpha_1} \dots \sigma_{\alpha_{i-1}} \sigma_{\alpha_i} \dots \sigma_{\alpha_l}$ implying $l(\sigma_{\gamma}w) < l(w)$. Interchanging w and $\sigma_{\gamma}w$ it follows that if $w^{-1}\gamma \in \Delta^+$ then $l(w) < l(\sigma_{\gamma}w)$. ■

Lemma 5.14 *The following hold:*

(a) *If $\sigma_{\alpha}w \xrightarrow{\alpha} w$ for all $\alpha \in \Pi$, then $w = e$, the identity.*

(b) *There exists a unique element $s \in \mathbb{W}$ such that $s \rightarrow \sigma_{\alpha}s$ for all $\alpha \in \Pi$.*

(c) *Let $w \in \mathbb{W}$, $\alpha \in \Pi$. Then $\alpha \in \Phi_w$ implies $w \xrightarrow{\alpha} \sigma_{\alpha}w$, and $\alpha \notin \Phi_w$ implies $\sigma_{\alpha}w \xrightarrow{\alpha} w$.*

(d) *$|\Phi_w| = k$ when $l(w) = k$.*

Discussion. This lemma is proved in [7]. It confirms what we intuitively expect to happen. At the top of the Bruhat order we have the identity; and at the bottom an element s .

For the last two statements one thinks of elements of the Weyl group as decomposing into strings of reflections in simple root vectors. (c) then states that Φ_w is the set of all roots “flipped” over from negative to positive by w . So if $\alpha \in \Phi_w$ we have that w flips α and so σ_{α} has the effect of “stripping” α out of w . (d) states that the number of roots “flipped” is equal to the length of w . ■

Lemma 5.15 Let $w_1, w_2 \in \mathbb{W}$, $\alpha \in \Pi$, $\gamma \in \Delta^+$ and $\gamma \neq \alpha$. Set $\gamma' = \sigma_\alpha \gamma$. Under these assumptions the following two diagrams are equivalent,

$$\begin{array}{ccc} \sigma_\alpha w_1 & \xrightarrow{\gamma} & w_2 \\ \downarrow \alpha & & \\ w_1 & & \end{array} \quad (22)$$

and

$$\begin{array}{ccc} & & w_2 \\ & & \downarrow \alpha \\ w_1 & \xrightarrow{\gamma'} & \sigma_\alpha w_2 \end{array} \quad (23)$$

Proof. We show (22) implies (23); the converse is similar. $\alpha \in \Pi$ and $\gamma \neq \alpha$, so since we know that σ_α permutes the positive simple roots aside from α it follows that $\gamma' = \sigma_\alpha \gamma \in \Delta^+$. Thus it is sufficient to show $l(\sigma_\alpha w_2) < l(w_2) = l(w_1)$. We know $\sigma_\alpha w_2 = \sigma_\alpha \sigma_\gamma \sigma_\alpha w_1 = \sigma_{\gamma'} w_1$ and $(\sigma_\alpha w_2)^{-1} \gamma' = w_2^{-1} \sigma_\alpha \gamma' = w_2^{-1} \gamma \in \Delta^+$ by Lemma 5.13 since (22) states $l(\sigma_\gamma w_2) > l(w_2)$.

Thus $\gamma' \in w_1 \Delta^- \cap \Delta^+$ and we are done. ■

Lemma 5.16 Let $w, w' \in \mathbb{W}$, $\alpha \in \Pi$ and assume $w < w'$. Then

1. either $\sigma_\alpha w \leq w'$ or $\sigma_\alpha w < \sigma_\alpha w'$.
2. either $w \leq \sigma_\alpha w'$ or $\sigma_\alpha w < \sigma_\alpha w'$.

Proof. We prove the first part. The second is similar. Let

$$w = w_1 \rightarrow w_2 \rightarrow \dots \rightarrow w_k = w'.$$

We use induction on k . If $\sigma_\alpha w < w$ or $\sigma_\alpha w = w_2$ then the assertion is obvious. Let $w < \sigma_\alpha w$, $\sigma_\alpha w \neq w_2$. Then by Lemma 5.15 $\sigma_\alpha w < \sigma_\alpha w_2$. Now apply the inductive hypothesis to the pair (w_2, w') . ■

Corollary 5.17 Let $\alpha \in \Pi$, $w_1 \xrightarrow{\alpha} w'_1$ and $w_2 \xrightarrow{\alpha} w'_2$. If one of the elements w_1, w'_1 is smaller in the ordering than one of w_2, w'_2 then $w_1 \leq w_2 < w'_2$ and $w_1 \leq w'_1 \leq w_2$. ■

Proposition 5.18 Suppose we are given a partial ordering $w \dashv w'$ on \mathbb{W} with the following properties:

- P1. If $\alpha \in \Pi$, $w \in \mathbb{W}$ with $l(\sigma_\alpha w) = l(w) - 1$ then $w \dashv \sigma_\alpha w$.
- P2. If $w \dashv w'$, $\alpha \in \Pi$ then either $\sigma_\alpha w \dashv w'$ or $\sigma_\alpha w \dashv \sigma_\alpha w'$.

Then $w \dashv w'$ if and only if $w \leq w'$. In other words these two properties characterise the Bruhat order.

Proof. From P1 it follows that $s \dashv w \dashv e$ for all $w \in \mathbb{W}$, where as in Lemma 5.14 s is the element of maximal length in \mathbb{W} .

We show $w \leq w'$ implies $w \dashv w'$. We do this by reverse induction on $l(w')$. If $l(w') = r = l(s)$ then $w' = s$, $w = s$ and thus $w \dashv w'$. Now let $l(w') < r$ and let $\alpha \in \Pi$ be a root such that $l(\sigma_\alpha w') = l(w') + 1$. Then by Lemma 5.16, either $\sigma_\alpha w \leq \sigma_\alpha w'$ or $w \leq \sigma_\alpha w'$. Considering each case in turn,

- $w \leq \sigma_\alpha w' \Rightarrow w \dashv \sigma_\alpha w'$ (by the inductive hypothesis), and P1 implies $w \dashv w'$.

• $\sigma_\alpha w \leq \sigma_\alpha w' \Rightarrow \sigma_\alpha w \dashv \sigma_\alpha w'$ (by the inductive hypothesis) which by P2 implies either $w \dashv \sigma_\alpha w'$ (see the previous case) or $w \dashv w'$.

We show $w \dashv w'$ implies $w \leq w'$. We use induction on $l(w)$. If $l(w) = 0$ then $w = e = w'$ and so $w \leq w'$. Now let $l(w) > 0$ and let α be an element of Π such that $l(\sigma_\alpha w) = l(w) - 1$. Then by P2 either $\sigma_\alpha w \dashv w'$ or $\sigma_\alpha w \dashv \sigma_\alpha w'$. Considering each in turn,

- $\sigma_\alpha w \dashv w' \Rightarrow \sigma_\alpha w \leq w'$ (by the inductive hypothesis), implying $w \leq w'$.
- $\sigma_\alpha w \dashv \sigma_\alpha w' \Rightarrow \sigma_\alpha w \leq \sigma_\alpha w' \Rightarrow w \leq w'$ by Corollary 5.17. ■

Proposition 5.19 Let $w \in \mathbb{W}$ with reduced decomposition $\sigma_{\alpha_1} \dots \sigma_{\alpha_l}$.

(a) If $1 \leq i_1 < i_2 < \dots < i_k \leq l$ and

$$w' = \sigma_{\alpha_{i_1}} \dots \sigma_{\alpha_{i_k}}, \quad (24)$$

then $w \leq w'$.

(b) If $w < w'$, then w' can be represented in the form (24) for some indexing set $\{i_j\}$.

(c) If $w \rightarrow w'$, then there is a unique index i , $1 \leq i \leq l$, such that

$$w' = \sigma_{\alpha_1} \dots \sigma_{\alpha_{i-1}} \sigma_{\alpha_{i+1}} \dots \sigma_{\alpha_l}. \quad (25)$$

Proof. First we prove (c). Let $w \rightarrow w'$. Then by Lemma 5.12 there is at least one index i for which (25) holds. Now suppose (25) holds for two indices i, j , $i < j$. Then $\sigma_{\alpha_{i+1}} \dots \sigma_{\alpha_j} = \sigma_{\alpha_i} \dots \sigma_{\alpha_{j-1}}$. Then

$$\sigma_{\alpha_i} \dots \sigma_{\alpha_j} = \sigma_{\alpha_{i+1}} \dots \sigma_{\alpha_{j-1}},$$

contradicting the assumption that the decomposition $w = \sigma_{\alpha_1} \dots \sigma_{\alpha_l}$ is reduced.

(b) follows immediately from (c) after noting that the decomposition (25) is reduced.

We prove (a) by induction on l . There are two cases:

- $i_1 > 1$. Then by the inductive hypothesis $w' \geq \sigma_{\alpha_2} \dots \sigma_{\alpha_l}$, that is, $w' \geq \sigma_{\alpha_1} w > w$.
- $i_1 = 1$. Then again by inductive hypothesis, $\sigma_{\alpha_1} w' = \sigma_{\alpha_2} \dots \sigma_{\alpha_l} \geq \sigma_{\alpha_1} w = \sigma_{\alpha_2} \dots \sigma_{\alpha_l}$.

So by Corollary 5.17, $w \leq w'$. ■

Definition 5.20 Given a quadruple (w_1, w_2, w_3, w_4) of elements of \mathbb{W} we say we have a square if we have

$$\begin{array}{ccc} w_1 & \longrightarrow & w_2 \\ \downarrow & & \downarrow \\ w_3 & \longrightarrow & w_4 \end{array}$$

We state the following two lemmas, proofs can be found in [31].

Lemma 5.21 Let $w_1, w_2 \in \mathbb{W}$ and $l(w_1) - 2 = l(w_2)$. Then the number of elements $w' \in \mathbb{W}$ such that $w_1 \rightarrow w' \rightarrow w_2$ is either two or zero. ■

Lemma 5.22 Given any $w_1, w_2 \in \mathbb{W}$ such that $w_1 \rightarrow w_2$ we can assign numbers $s(w_1, w_2) = \pm 1$ in such a way that for any square (w_1, w_2, w_3, w_4) the product of the assigned numbers is -1 . ■

5.3 Homology of Lie algebras

In this section we construct a resolution dual to the de Rham complex on the homogeneous space G/B , where G is a Lie group with Lie algebra \mathfrak{g} and B is the subgroup corresponding to the subalgebra \mathfrak{b} . Our notation is highly suggestive, however in the beginning \mathfrak{g} and \mathfrak{b} are arbitrary, although we will specialise to a semi-simple \mathfrak{g} with Borel subalgebra \mathfrak{b} later on.

Before going any further, we state without proof a theorem and one of its corollaries which we will use repeatedly in this section, see [34]

Theorem 5.23 Poincaré-Birkhoff-Witt. *Let $\{X_i\}_{i \in A}$ be a basis for \mathfrak{g} , and suppose a linear ordering has been imposed on A . Then the set of monomials*

$$(\iota X_{i_1})^{j_1} \dots (\iota X_{i_n})^{j_n}$$

for $i_1 < \dots < i_n$ and $j_k \geq 0$ form a basis for $U(\mathfrak{g})$. In particular $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$ is an imbedding. ■

Corollary 5.24 $U(\mathfrak{g})$ has a natural filtration which it inherits from the tensor algebra $T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$. The associated graded algebra, $\text{gr } U(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} U(\mathfrak{g})_n / U(\mathfrak{g})_{n-1}$ is canonically isomorphic to $\text{Sym}(\mathfrak{g})$, the symmetric algebra on the vector space \mathfrak{g} , as a \mathfrak{g} -module. ■

Now we give a brief summary of Lie algebra homology loosely following [42] interspersed with results from [31]. Assume throughout that \mathfrak{g} is finite dimensional. Define

$$E_k(\mathfrak{g}) = U(\mathfrak{g}) \otimes \bigwedge^k \mathfrak{g}.$$

and use this to construct the Chevalley-Eilenberg complex (see [10]),

$$0 \leftarrow \mathbb{C} \xleftarrow{\epsilon_0} E_0(\mathfrak{g}) \xleftarrow{d} E_1(\mathfrak{g}) \xleftarrow{d} \dots$$

where ϵ_0 is the augmentation map sending $\iota(\mathfrak{g})$ to zero – ie sending elements of $U(\mathfrak{g})$ to their constant term – and

$$\begin{aligned} d(u \otimes x_1 \wedge \dots \wedge x_k) &= \sum_{i=1}^k (-1)^{i+1} u x_i \otimes x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_k \\ &+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_k. \end{aligned}$$

Now given a finite dimensional right \mathfrak{g} -module X , we construct the following chain complex

$$X \otimes_{U(\mathfrak{g})} E_{\bullet}(\mathfrak{g}) = X \otimes_{U(\mathfrak{g})} \bigwedge^{\bullet} \mathfrak{g},$$

and define the homology of \mathfrak{g} with coefficients in the module X , $H_{\bullet}(\mathfrak{g}, X)$, to be the homology groups of this chain complex. There are two special cases of particular importance. The first is when X is the trivial representation \mathbb{C} . In this case we write $H_{\bullet}(\mathfrak{g})$ and refer to this as the homology of \mathfrak{g} .

The second case is the Koszul complex. Notice that the symmetric algebra $\text{Sym}(\mathfrak{g})$ inherits a bilinear map, the Poisson map. This is denoted by

$$\{-, -\} : \text{Sym}(\mathfrak{g}) \otimes \text{Sym}(\mathfrak{g}) \rightarrow \text{Sym}(\mathfrak{g})$$

and is defined inductively by the condition that it be a derivation, and that $\{x, y\} = [x, y]$ for x and y in \mathfrak{g} . We then have a \mathfrak{g} -module structure on the symmetric algebra given by $x \cdot f = \{x, f\}$. The Koszul complex is defined as

$$\text{Sym}^\bullet(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{g})} E_\bullet(\mathfrak{g}).$$

It is proven in Loday [36] that

$$H_n(\mathfrak{g}, \text{Sym}^\bullet(\mathfrak{g})) = \begin{cases} \mathbb{C} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

There is an alternate construction which can be used to calculate the homology of a Lie algebra \mathfrak{g} , with coefficients in the module X . We need to be able to find an exact sequence of \mathfrak{g} -modules C_i free over $\mathbf{U}(\mathfrak{g})$ as follows

$$0 \leftarrow X \leftarrow C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \dots$$

Now we take any other \mathfrak{g} -module Y , and construct the complex

$$0 \rightarrow \text{Hom}_{\mathfrak{g}}(C_0, Y) \xrightarrow{d'_1} \text{Hom}_{\mathfrak{g}}(C_1, Y) \xrightarrow{d'_2} \dots,$$

then define $\text{Ext}^i(X, Y) = \ker d'_{i+1} / \text{im } d'_i$. Now let $t : \mathbf{U}(\mathfrak{g}) \rightarrow \mathbf{U}(\mathfrak{g})$ be the anti-automorphism characterised by $t_{\mathfrak{g}} : u \mapsto -u$. Denote by X^t the right $\mathbf{U}(\mathfrak{g})$ -module with underlying space X and right $\mathbf{U}(\mathfrak{g})$ action given by

$$x \cdot u = t(u) \cdot x \text{ for } x \in X \text{ and } u \in \mathbf{U}(\mathfrak{g}).$$

We now construct a complex

$$0 \leftarrow Y^t \otimes_{\mathfrak{g}} C_0 \xleftarrow{d''_1} Y^t \otimes_{\mathfrak{g}} C_1 \xleftarrow{d''_2} \dots$$

and define $\text{Tor}_i(Y^t, X) = \ker d''_i / \text{im } d''_{i+1}$. The following facts are proved in [42]

1. The groups $\text{Tor}_i(Y^t, X)$ and $\text{Ext}^i(X, Y)$ are independent of the choice of resolution.
2. If we define X^* to be the vector space dual of X , ie $\text{Hom}_{\mathbb{C}}(X, \mathbb{C})$ then

$$\begin{aligned} \left[\text{Ext}^i(X, Y) \right]^* &= \text{Tor}_i(Y^*, X) \\ \text{Tor}_i(Y^*, X) &= \text{Tor}_i(X^t, (Y^t)^*). \end{aligned}$$

As shown in [42], $H^i(\mathfrak{g}, X) = \text{Ext}^i(\mathbb{C}, X)$.

The Chevalley-Eilenberg complex is generalized in [31]. Take a subalgebra \mathfrak{b} of \mathfrak{g} . Then restrict the adjoint representation of \mathfrak{g} on itself to \mathfrak{b} and form the quotient, obtaining a natural

action of \mathfrak{b} on $\mathfrak{g}/\mathfrak{b}$. This then extends uniquely to the exterior algebra $\bigwedge(\mathfrak{g}/\mathfrak{b})$. $\bigwedge^j(\mathfrak{g}/\mathfrak{b})$ is a \mathfrak{b} -submodule of $\bigwedge(\mathfrak{g}/\mathfrak{b})$, and from it induce the \mathfrak{g} -module

$$D_k = \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{b})} \bigwedge^k(\mathfrak{g}/\mathfrak{b}).$$

For $k > 0$ define a linear map $D_k \xrightarrow{d_k} D_{k-1}$ generalising d above as follows: given $x_1, \dots, x_k \in \mathfrak{g}/\mathfrak{b}$ fix representatives $y_1 \dots y_k \in \mathfrak{g}$. Then for all $x \in \mathbf{U}(\mathfrak{g})$,

$$\begin{aligned} d_k(x \otimes x_1 \wedge \dots \wedge x_k) &= \sum_{i=1}^k (-1)^{i+1} (xy_i) \otimes x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_k \\ &+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} x \otimes \pi([y_i, y_j]) \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_k, \end{aligned}$$

where $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{b}$ is projection. d_k is a well-defined \mathfrak{g} -module map. $\epsilon_0: D_0 \rightarrow \mathbb{C}$ is defined as before.

We have constructed a sequence $V(\mathfrak{g}, \mathfrak{b})$ of \mathfrak{g} -modules and \mathfrak{g} -morphisms

$$\mathbb{C} \xleftarrow{\epsilon_0} D_0 \xleftarrow{d_1} D_1 \xleftarrow{d_2} \dots$$

We claim

Theorem 5.25 *The sequence $V(\mathfrak{g}, \mathfrak{b})$ is exact.*

Proof. We show this by first defining a filtration in $V(\mathfrak{g}, \mathfrak{b})$. We write $A \in D_k^{(l)}$ if $A \in D_k$ can be written

$$A = \sum_i c_i (x_i \otimes x_{i1} \wedge \dots \wedge x_{ik}),$$

where $c_i \in \mathbb{C}$, $x_i \in \mathbf{U}(\mathfrak{g})$, $x_{ij} \in \mathfrak{g}/\mathfrak{b}$ and $\deg(x_i) \leq l - k$ for all i .

It is clear that this filtration is preserved by the d_k 's and so to prove the theorem it is sufficient to prove that for all l

$$0 \longleftarrow M^{(l)} \longleftarrow D_0^{(l)}/D_0^{(l-1)} \xleftarrow{d_1^{(l)}} D_1^{(l)}/D_1^{(l-1)} \xleftarrow{d_2^{(l)}} \dots$$

is exact. We define $M^{(0)} := \mathbb{C}$ and $M^{(l)} := 0$ for $l > 0$. It is a consequence of Corollary 5.24 of the Poincaré-Birkhoff-Witt Theorem that the universal enveloping algebra is a symmetric algebra up to grading and so we have $D_k^{(l)}/D_k^{(l-1)} \approx \text{Sym}^{l-k}(\mathfrak{g}/\mathfrak{b}) \otimes \bigwedge^k(\mathfrak{g}/\mathfrak{b})$. The operator

$$d_k^{(l)}: D_k^{(l)}/D_k^{(l-1)} \rightarrow D_{k-1}^{(l)}/D_{k-1}^{(l-1)}$$

is given by the formula

$$d_k^{(l)}(x \otimes x_1 \wedge \dots \wedge x_k) = \sum_{i=1}^k (-1)^{i+1} (xx_i \otimes x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_k),$$

where we have stripped out the second term as a result of quotienting out the grading. This implies that the complex $\text{Gr } V(\mathfrak{g}, \mathfrak{b})$

$$0 \longleftarrow \mathbb{C} \longleftarrow \bigoplus_{\mathfrak{l}} D_0^{(\mathfrak{l})}/D_0^{(\mathfrak{l}-1)} \longleftarrow \bigoplus_{\mathfrak{l}} D_1^{(\mathfrak{l})}/D_1^{(\mathfrak{l}-1)} \longleftarrow \dots$$

is isomorphic to an augmentation of the Koszul complex of the vector space $\mathfrak{g}/\mathfrak{b}$. So we have that $\text{Gr } V(\mathfrak{g}/\mathfrak{b})$ is exact everywhere, implying that each of the direct summands is an exact complex and so $V(\mathfrak{g}, \mathfrak{b})$ is exact. ■

Proposition 5.26 *Let \mathfrak{b} and \mathfrak{n}^- be subalgebras of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$ as a vector space. Then $V(\mathfrak{g}, \mathfrak{b}) \approx V(\mathfrak{n}^-)$ as a complex of $U(\mathfrak{g})$ -modules.*

Proof. We define a morphism of complexes $\phi : V(\mathfrak{n}^-) \rightarrow V(\mathfrak{g}, \mathfrak{b})$ by

$$\phi_k(x \otimes x_1 \wedge x_1 \wedge \dots \wedge x_k) = x \otimes \bar{x}_1 \wedge \dots \wedge \bar{x}_k,$$

for $x \in U(\mathfrak{n}^-)$, $x_i \in \mathfrak{n}^-$ and \bar{x}_i the image of x_i in $\mathfrak{g}/\mathfrak{b}$. By the Poincaré-Birkhoff-Witt theorem we have an isomorphism. ■

The proposition is important because it implies the following: given a subalgebra \mathfrak{b} of \mathfrak{g} ; if there exists a complementary subalgebra then the action of \mathfrak{b} on $V(\mathfrak{b})$ can be extended to the whole of \mathfrak{g} . The extension depends on the choice of complementary subalgebra.

In the semi-simple case to which we now specialise we set \mathfrak{g} a semi-simple Lie algebra. We pick out the distinguished subalgebra \mathfrak{b} , the Borel subalgebra.

Lemma 5.27 *Let X be a \mathfrak{b} -module and define $X^{\mathfrak{g}} = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} X$. The mapping $X \mapsto X^{\mathfrak{g}}$ is an exact functor from the category of \mathfrak{b} -modules to the category of \mathfrak{g} -modules.*

Proof. By the Poincaré-Birkhoff-Witt theorem $U(\mathfrak{g})$ and $\text{Sym}(\mathfrak{g})$ are isomorphic as vector spaces. Thus we have a vector space isomorphism

$$U(\mathfrak{b}) \otimes U(\mathfrak{n}^-) \rightarrow U(\mathfrak{g})$$

given by multiplication. This shows that $U(\mathfrak{g})$ is a free $U(\mathfrak{b})$ -module, and so the functor is exact. ■

Corollary 5.28 *Let $X \in \mathcal{C}$ be a finite dimensional \mathfrak{b} -module, and set $\Psi(X) = \{\phi + \rho\}$ where ϕ runs through the weights of X with multiplicities. Then $X^{\mathfrak{g}}$ is of type $\Psi(X)$.*

Proof. By Lie's theorem any irreducible \mathfrak{b} -module is one dimensional. Let V be a one dimensional \mathfrak{b} -module with $Hv = \phi(H)v$ and \mathfrak{n} acting trivially. Then $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V$ is the Verma module $V(\phi + \rho)$ and so the corollary follows after breaking X down into its irreducible components. ■

We now wish to study the modules D_k . Firstly from the above Corollary, D_k is of type $\Psi(\wedge^k(\mathfrak{g}/\mathfrak{b}))$. Now specialise to the subcomplex of $V(\mathfrak{g}, \mathfrak{b})$ consisting of the zero eigenvalues of the action of Γ_X . Clearly have $(D_k)_{(0)} \subset D_k$ and we study the exact complex $V_{(0)}(\mathfrak{g}, \mathfrak{b})$. It

follows from Corollary 5.8 that $(D_k)_{(0)}$ is of type $[\Psi(\bigwedge^k(\mathfrak{g}, \mathfrak{b}))]_{(0)}$. We need to understand the structure of this set.

Let Φ be a subset of Δ . Set $\langle \Phi \rangle = \sum_{\gamma \in \Phi} \gamma$. Now the weights of $\mathfrak{g}/\mathfrak{b}$ are Δ^- so the weights of $\bigwedge^k(\mathfrak{g}, \mathfrak{b})$ (with multiplicities) coincide with the collection of weights of the form $-\langle \Phi \rangle$ for all $\Phi \subset \Delta^+$ with $|\Phi| = k$. Therefore (see Corollary 5.4),

$$\begin{aligned} [\Psi(\bigwedge^k(\mathfrak{g}, \mathfrak{b}))]_{(0)} &= \{\delta \in \Delta \mid \delta \sim \rho\} \cap \{\rho - \langle \Phi \rangle \mid \Phi \subset \Delta^+, |\Phi| = k\} \\ &= \{\rho - \langle \Phi \rangle \mid \Phi \subset \Delta^+, |\Phi| = k, (\rho - |\Phi|) \sim \rho\} \end{aligned}$$

Remembering Equation (20), we prove the following lemma,

Lemma 5.29 *Let $w \in \mathbb{W}$, $\Phi \subset \Delta^+$. Then $\rho - w\rho = \langle \Phi \rangle$ iff $\Phi = \Phi_w$.*

Proof. Assume that $\rho - w\rho = \langle \Phi \rangle$. We will show that $\Phi = \Phi_w$. The converse follows automatically. Using induction on $l(w)$ we have for $l(w) = 0$ that $w = e$ and so the lemma is obvious. Now let $l(w) = k > 0$. Choose $\alpha \in \Pi$ so that $w \xrightarrow{\alpha} \sigma_\alpha w$ or in other words (Lemma 5.14) $\alpha \in \Phi_w$. Then

$$\langle \sigma_\alpha \Phi_w \rangle = \sigma_\alpha \rho - \sigma_\alpha w\rho = \rho - \sigma_\alpha w\rho - \alpha.$$

$\Phi \subset \Delta^+$ implies that $\alpha \notin \sigma_\alpha \Phi$ since either $\alpha \in \Phi$, in which case it is sent to $-\alpha$; or it was not in the set to begin with, and the only element that σ_α sends to α is $-\alpha$.

Therefore we have

$$\rho - (\sigma_\alpha w)\rho = \langle \sigma_\alpha \Phi \rangle + \alpha = \langle \sigma_\alpha \Phi \cup \{\alpha\} \rangle.$$

Assume $\alpha \notin \Phi$. Then $\sigma_\alpha \Phi \cup \{\alpha\} \subset \Delta^+$ and by the inductive hypothesis $\Phi_{\sigma_\alpha w} = \sigma_\alpha \Phi \cup \{\alpha\}$ and so $\alpha \in \Phi_{\sigma_\alpha w}$. Thus (Lemma 5.14) $\sigma_\alpha w \rightarrow w$ - a contradiction and thus $\alpha \in \Phi$.

If we set $\Phi - \{\alpha\} = \Phi'$ then $\rho - \sigma_\alpha w\rho = \langle \sigma_\alpha \Phi' \rangle$ and $\sigma_\alpha \Phi' \subset \Delta^+$. By the inductive hypothesis $\Phi_{\sigma_\alpha w} = \sigma_\alpha \Phi'$, so $\Phi = \sigma_\alpha \Phi_{\sigma_\alpha w} \cup \{\alpha\}$. By definition $\sigma_\alpha \Phi_{\sigma_\alpha w} \cup \{\alpha\} = \sigma_\alpha(\Delta^+ \cap \sigma_\alpha w \Delta^-) \cup \{\alpha\} = \Phi_w$ and we are done. ■

An immediate consequence of Lemma 5.29 is

$$\begin{aligned} [\Psi(\bigwedge^k(\mathfrak{g}, \mathfrak{b}))]_{(0)} &= \{\rho - \langle \Phi \rangle \mid \Phi \subset \Delta^+, |\Phi| = k, (\rho - |\Phi|) \sim \rho\} \\ &= \{w\rho \mid l(w) = k\} \end{aligned}$$

or in other words

Proposition 5.30 *Let $\Psi_k = \{w\rho \mid l(w) = k\}$. Then $(D_k)_{(0)}$ is of type Ψ_k . ■*

5.4 The weak Bernstein-Gel'fand-Gel'fand Resolution

Now we come to one of our main theorems, the construction and proof of exactness of the weak BGG resolution. The weak BGG resolution provides a free resolution of a simple \mathfrak{g} -module by modules which have nice filtrations. The strong BGG resolution - the proof of which uses the weak BGG resolution - provides a free resolution by a direct sum of Verma modules.

Theorem 5.31 Weak Bernstein-Gel'fand-Gel'fand Resolution.

Let X be a simple finite dimensional \mathfrak{g} -module with highest weight λ . Then there exists an exact sequence of $U(\mathfrak{g})$ -modules

$$0 \leftarrow X \leftarrow D_0^X \leftarrow \dots \leftarrow D_s^X \leftarrow 0$$

where $s = \dim \mathfrak{n}^-$ and D_k^X is a module of type $\Psi_k(\lambda) = \{w(\lambda + \rho) \mid l(w) = k\}$.

Remark. The weak BGG resolution is a generalisation of Proposition 5.30. This gives a resolution of the trivial \mathfrak{g} -module \mathbb{C} by the modules $(D_k)_{(0)}$, which are of type $\{w\rho \mid l(w) = k\}$.

Proof. By applying the Casimir operator to the complex $V(\mathfrak{g}, \mathfrak{b})$ and focusing on the zero eigenvalues we obtain the complex $V(\mathfrak{g}, \mathfrak{b})_{(0)}$ which is the required exact sequence for the case $X = \mathbb{C}$, the trivial representation.

In general consider the exact sequence $D_*^{\mathbb{C}} \otimes_{\mathbb{C}} X$ and define

$$D_k^X = (D_k^{\mathbb{C}} \otimes X)_{(c\lambda)}.$$

We shall prove that the sequence

$$0 \leftarrow X \leftarrow D_0^X \leftarrow D_1^X \leftarrow \dots$$

satisfies the conditions of the theorem. Exactness follows since the tensor product and the map $X \mapsto X_{(c)}$ are exact functors since vector spaces are projective and by Proposition 5.7. The proof then comes down to the following two lemmas.

Lemma 5.32 Let $\chi \in \mathfrak{h}^*$, X be a finite dimensional \mathfrak{g} -module. Set $\Psi = \{\lambda + \chi\}$ where λ runs through the weights of X with multiplicities. Then $V(\chi) \otimes X$ is of type Ψ .

Proof. Let e_1, \dots, e_k be a basis in X of weight vectors with weights $\lambda_1, \dots, \lambda_k$. Number the vectors so that $\lambda_i < \lambda_j$ implies $i > j$. Set $a_i = f_{\chi} \otimes e_i \in V(\chi) \otimes X$ and define $X^{(i)} = U(\mathfrak{g})(a_1, \dots, a_i)$. This is a filtration of X . It will suffice to show that

$$X^{(i)}/X^{(i-1)} = V(\lambda_i + \chi) \text{ and } X^{(k)} = V(\chi) \otimes X.$$

We denote by \bar{a}_i the image of a_i in $X^{(i)}/X^{(i-1)}$. By construction this generates $X^{(i)}/X^{(i-1)}$ with weight $\chi + \lambda_i - \rho$ and is a highest weight vector. Thus $X^{(i)} = U(\mathfrak{n}^-)(a_1, \dots, a_i)$. We will complete the proof by showing $X^{(i)}$ is a free $U(\mathfrak{n}^-)$ -module generated by a_1, \dots, a_i .

Given $x_j \in U(\mathfrak{n}^-)$ for each $j = 1, \dots, i$ consider $\sum_{j=1}^i x_j a_j = \sum_{j=1}^i x_j f_{\chi} \otimes e_j$. Since $V(\chi)$ is free on $U(\mathfrak{n}^-)$ we obtain $x_j f_{\chi} = f_j$ for some $f_j \in V(\chi)$ and so

$$\sum_{j=1}^i x_j a_j = \sum_{j=1}^i f_j \otimes e_j \neq 0$$

and so $X^{(i)}/X^{(i-1)}$ is a free $U(\mathfrak{n}^-)$ -module with generator \bar{a}_i or in other words $X^{(i)}/X^{(i-1)} \approx V(\chi + \lambda_i)$. From this it follows that $X^{(i)}$ is a free $U(\mathfrak{n}^-)$ -module generated by a_1, \dots, a_i . It is also clear that $X^{(k)} = V(\chi) \otimes X$. ■

From this it follows that $D_k^{\mathbb{C}} \otimes X$ is of type

$$\Psi = \{\lambda_i + w\rho \mid \lambda_i \text{ are the weights of } X \text{ with multiplicities, } l(w) = k\}.$$

So to complete the proof we need

Lemma 5.33 *Let X be a finite dimensional simple \mathfrak{g} -module with highest weight λ . Then for each $w \in \mathbb{W}$ there exists exactly one weight μ for X such that $\mu + w\rho \sim \lambda + \rho$. μ has multiplicity one.*

Proof. Existence is obvious. Now given μ a weight of X with $w, w_1 \in \mathbb{W}$ such that $w_1(\mu + w\rho) = \lambda + \rho$ we will show uniqueness.

Firstly $w_1\mu \leq \lambda$ since λ is a highest weight vector for X and the roots of X are closed under the action of \mathbb{W} . Also by definition of ρ it follows that $w_1w\rho \leq \rho$ so $w_1\mu = \lambda$ and $w_1w\rho = \rho$. Thus $w_1 = w^{-1}$ and $\mu = w\lambda$.

μ has multiplicity one since this is the multiplicity of λ . ■

We finish off this section with an application of the weak BGG resolution. We prove a theorem of Bott's [5] which we will use later to prove the exactness of the BGG resolution.

Corollary 5.34 Bott. *Let X be a simple finite dimensional \mathfrak{g} -module. Then*

$$\dim H^i(\mathfrak{n}^-, X) = \text{card } \mathbb{W}^{(i)}.$$

Proof. We know

$$H^i(\mathfrak{n}^-, X) = \text{Ext}_{\mathfrak{n}^-}^i(\mathbb{C}, X) = \text{Tor}_{\mathfrak{t}}^{i, \mathfrak{n}^-}(X^*, \mathbb{C})^* = \text{Tor}_{\mathfrak{t}}^{i, \mathfrak{n}^-}(\mathbb{C}, (X^t)^*)^*.$$

We now construct the resolution for the module $X_1 = (X^t)^*$. We have proven the exactness of the resolution

$$0 \leftarrow X_1 \leftarrow (D_0^{\mathbb{C}} \otimes X_1)_{(\mathfrak{c}_\lambda)} \xleftarrow{d_1} (D_1^{\mathbb{C}} \otimes X_1)_{(\mathfrak{c}_\lambda)} \xleftarrow{d_2} \dots$$

and we know that it is a free resolution over $U(\mathfrak{n}^-)$ since each $D_k^{\mathbb{C}} = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \wedge^k(\mathfrak{g}/\mathfrak{b})$ is free over $U(\mathfrak{n}^-)$ by construction. $\text{Tor}_{\mathfrak{t}}^{i, \mathfrak{n}^-}(\mathbb{C}, X_1)$ will be the i^{th} homology group of the complex

$$0 \leftarrow \mathbb{C} \otimes_{\mathfrak{n}^-} D_0^{X_1} \xleftarrow{d_1''} \mathbb{C} \otimes_{\mathfrak{n}^-} D_1^{X_1} \xleftarrow{d_2''} \dots$$

It is clear that $\mathbb{C} \otimes_{\mathfrak{n}^-} D_k^{X_1} = D_k^{X_1} / \mathfrak{n}^- D_k^{X_1}$. $D_k^{X_1}$ is of type $\{w(\lambda + \rho) \mid l(w) = k\}$ and so $D_k^{X_1} / \mathfrak{n}^- D_k^{X_1}$ is a finite dimensional vector space with weight space decomposition under \mathfrak{h} given by the weights $w(\lambda + \rho)$ for $w \in \mathbb{W}^{(k)}$, where each weight occurs with multiplicity one.

Thus $\dim \mathbb{C} \otimes_{\mathfrak{n}^-} D_k^{X_1} = \text{card } \mathbb{W}^{(k)}$. The maps d_k'' are null maps since they are \mathfrak{g} -maps, and so map weight spaces onto each other. ■

5.5 The Bernstein-Gel'fand-Gel'fand Resolution

We move immediately onto the proof of the strong Bernstein-Gel'fand-Gel'fand resolution. We start by describing the modules and the maps involved. Given a simple finite dimensional \mathfrak{g} -module X of highest weight λ construct the following exact sequence of \mathfrak{g} -modules:

$$0 \leftarrow X \xleftarrow{\varepsilon_0} V(\lambda + \rho) \xleftarrow{d_1} \bigoplus_{w \in \mathbb{W}^{(1)}} V(w(\lambda + \rho)) \xleftarrow{d_2} \dots \xleftarrow{d_s} \bigoplus_{w \in \mathbb{W}^{(s)}} V(w(\lambda + \rho)) \leftarrow 0 \quad (26)$$

where $s = \dim \mathfrak{n}^-$ and $\mathbb{W}^{(k)} = \{w \in \mathbb{W} \mid l(w) = k\}$.

This is clearly a strengthening of the weak BGG resolution, since these modules are of the required type, and have the additional property that instead of having a mere Jordan-Hölder decomposition into the required highest weight modules, they have a direct sum decomposition into Verma modules.

The map ϵ_0 is defined to be the natural surjection of $V(\lambda + \rho)$ onto an irreducible representation of highest weight λ . We now construct the maps d_k . By Theorem 5.10 we can think of $V(w(\lambda + \rho))$ as a submodule of $V(\lambda + \rho)$ and any mapping $V(w_1(\lambda + \rho)) \rightarrow V(w_2(\lambda + \rho))$ is a multiple of the canonical embedding for $w_1 < w_2$, and so can be represented as a complex number $c_{w_1 w_2}$. So any map

$$\bigoplus_{w \in \mathbb{W}^{(k)}} V(w(\lambda + \rho)) \rightarrow \bigoplus_{w \in \mathbb{W}^{(k-1)}} V(w(\lambda + \rho))$$

can be represented by a matrix $(c_{w_1 w_2})$, $w_1 \in \mathbb{W}^{(k)}$ and $w_2 \in \mathbb{W}^{(k-1)}$. Thus we need to define matrices $(d_{w_1 w_2}^{(k)})$ so that the sequence is exact.

To do this we use Definition 5.20 and Lemma 5.22. We assign numbers to each quadruple as the Lemma shows us we can, and then we define the matrices $(d_{w_1 w_2}^{(k)})$ as

$$d_{w_1 w_2}^{(k)} = \begin{cases} s(w_1, w_2) & \text{if } w_1 \rightarrow w_2 \\ 0 & \text{otherwise.} \end{cases} \quad (27)$$

Theorem 5.35 Bernstein-Gel'fand-Gel'fand Resolution.

The resolution constructed in (26) with maps given by (27) is exact.

Proof. We start by showing $d_i \circ d_{i+1} = 0$. We restrict to individual summands, and by Lemma 5.21 there are two cases. Either there is no map, or there are two maps which cancel by Lemma 5.22.

Exactness at X is equivalent to surjectivity and is clear.

The kernel of the surjection $V(\lambda + \rho) \rightarrow X$ is generated by highest weight vectors of the form $f_{\alpha}^{(\lambda + \rho)(H_i) + 1} \cdot f_{\lambda + \rho}$ for $\alpha \in \Pi$, [28]. This has weight $s_{\alpha}(\lambda + \rho) - \rho$ and so the kernel of this map is generated by weight vectors of this form, ie the kernel is $\bigoplus_{w \in \mathbb{W}^{(1)}} V(w(\lambda + \rho))$. Thus we have exactness at $V(\lambda + \rho)$.

All that remains is to prove that $K = \ker d_i = \text{im } d_{i+1}$. To do this we use three lemmas.

Lemma 5.36 *Let X be a free $U(\mathfrak{n}^-)$ -module with generators $v_1 \dots v_n$ and $\eta : X \rightarrow K$ a $U(\mathfrak{n}^-)$ -map such that $\eta(v_i)$ is a weight vector in K with respect to \mathfrak{h} . Then η is a surjection iff the induced map $\bar{\eta} : X/\mathfrak{n}^-X \rightarrow K/\mathfrak{n}^-K$ is surjective.*

Proof. The forward implication is clear. For the converse assume $\bar{\eta}$ is surjective and η is not. Since $K \in \mathcal{C}$ it follows that given any weight $\psi \in \mathfrak{h}^*$ there are only finitely many weights $\psi' > \psi$ with $K_{\psi'} \neq \{0\}$ and so we can find a weight vector f such that

1. $f \notin \text{im } \eta$.
2. any vector f' with weight $\psi' > \psi$ belongs to $\text{im } \eta$.

Let \bar{f} be the image of f in K/\mathfrak{n}^-K . Then $\bar{f} = \sum c_i \bar{\eta}(v_i)$. \mathfrak{n}^-K is invariant under \mathfrak{h} so there is a natural action of \mathfrak{h} on \mathfrak{n}^-K . From this it follows we can assume $c_i = 0$ for all i such that the weight of $\bar{\eta}(v_i)$ is greater than ψ by choice of f , and also when the weight is less than ψ , since we have control over this when choosing f .

Now $g = f - \sum c_i \eta(v_i)$ is a weight vector in \mathfrak{n}^-K , so $g = \sum_{\gamma \in \Delta^+} E_{-\gamma} g_{\gamma}$ where g_{γ} has weight $\psi + \gamma > \psi$. Thus by construction of f it follows that $g_{\gamma} \in \text{im } \eta$ and so $f \in \text{im } \eta$. ■

Lemma 5.37 *The map*

$$\left[\bigoplus_{w \in \mathbb{W}^{(i+1)}} V(w(\lambda + \rho)) \right] / \mathfrak{n}^- \left[\bigoplus_{w \in \mathbb{W}^{(i+1)}} V(w(\lambda + \rho)) \right] \longrightarrow K / \mathfrak{n}^- K$$

is an injection.

Proof. The source of the map is a vector space with basis $\{\bar{f}_{w\lambda} | w \in \mathbb{W}^{(i+1)}\}$. Since \bar{d}_{i+1} commutes with \mathfrak{h} and the basis vectors all have different weights it suffices to show that $\bar{d}_{i+1}(\bar{f}_{w\lambda}) \neq 0$ for all $w \in \mathbb{W}^{(i+1)}$.

We start by looking at the structure of K . Recalling the weak BGG resolution we have

$$\text{JH}(D_i^X) = \bigcup_{w \in \mathbb{W}^{(i)}} \text{JH}[V(w(\lambda + \rho))] = \text{JH} \left(\bigoplus_{w \in \mathbb{W}^{(i)}} V(w(\lambda + \rho)) \right).$$

Since at each term the two sequences have identical Jordan-Hölder decompositions and by exactness of both resolutions up to $i-1$ it follows that $\text{JH}(K) = \text{JH}(\ker D_i^X \rightarrow D_{i-1}^X)$. By exactness of the weak BGG resolution the second kernel is equal to the image of D_{i+1}^X and so

$$\text{JH}(\ker D_i^X \rightarrow D_{i-1}^X) \subset \text{JH}(D_{i+1}^X) = \bigcup_{w \in \mathbb{W}^{(i)}} \text{JH}[V(w(\lambda + \rho))]$$

which implies the irreducible modules which arise in the Jordan-Hölder decomposition of K are of the form $R(w(\lambda + \rho))$ for $l(w) > i$.

To complete the lemma we show that given a \mathfrak{g} -module X in \mathcal{C} with $l(w) \geq l(w_0)$ for all $R(w\lambda)$ in $\text{JH}(X)$, it follows that for a map $V(w_0\lambda) \xrightarrow{\tau} X$ with $\tau(f_{w_0\lambda}) \neq 0$ that the image of $\tau(f_{w_0\lambda}) \neq 0$ in X/\mathfrak{n}^-X is not zero.

Applying this to the case $X = K$ completes the proof.

We use induction on the number of elements in $\text{JH}(X)$.

Let v_ψ be an element of X of maximal weight $\psi - \rho$, and let Y be the submodule of X generated by v_ψ . There are two cases:

- $\tau(f_{w_0\lambda}) \in Y$. This implies

$$R(w_0\lambda) \subset \text{JH}(Y) \subset \text{JH}(V(\psi)),$$

the first inclusion by the condition on the case, and the second by construction of Y . So by Theorem 5.11, $\psi = w_1\lambda$ for some $w_1 \geq w_0$. But we also know that

$$R(\psi) \subset \text{JH}(Y) \subset \text{JH}(X),$$

since Y is a submodule of X generated by a vector of weight $\psi - \rho$. But now we have $w_1 \geq w_0$ and $l(w_1) \geq l(w_0)$, so that $w_0 = w_1$. Thus $\psi - \rho = w_0\lambda - \rho$ is the maximal weight of X and $\tau(f_{w_0\lambda}) \notin \mathfrak{n}^-X$ since it has weight $\psi - \rho$.

• $\tau(f_{w_0\lambda}) \notin Y$. $\text{JH}(X/Y) \subset \text{JH}(X)$ is a proper inclusion and so we apply the inductive hypothesis. ■

Lemma 5.38

$$\dim \left[\bigoplus_{w \in \mathbb{W}^{(1)}} V(w(\lambda + \rho)) \right] / \mathfrak{n}^- \left[\bigoplus_{w \in \mathbb{W}^{(1)}} V(w(\lambda + \rho)) \right] = \dim K/\mathfrak{n}^-K < \infty$$

Proof. Since K is a module in our category \mathcal{C} , we know that K/\mathfrak{n}^-K is finite dimensional. Let f_1, \dots, f_n be elements of K such that their images form a basis for K/\mathfrak{n}^-K . Let C be the free $U(\mathfrak{n}^-)$ -module generated by n elements, g_1, \dots, g_n and define a $U(\mathfrak{n}^-)$ -map $\vartheta : C \rightarrow K : g_i \mapsto f_i$. By Lemma 5.36 this is surjective.

Now consider the exact sequence

$$0 \leftarrow X \leftarrow V(\lambda + \rho) \leftarrow \dots \leftarrow \bigoplus_{w \in \mathbb{W}^{(1)}} V(w(\lambda + \rho)) \xleftarrow{\vartheta} C$$

of $U(\mathfrak{n}^-)$ -modules. Since all the terms in the sequence (excepting X) are free $U(\mathfrak{n}^-)$ -modules this sequence can be augmented to a free resolution of X :

$$0 \leftarrow X \leftarrow V(\lambda + \rho) \leftarrow \dots \leftarrow \bigoplus_{w \in \mathbb{W}^{(1)}} V(w(\lambda + \rho)) \xleftarrow{\vartheta} C \xleftarrow{\eta} D_1 \leftarrow D_2 \leftarrow \dots$$

Given a $U(\mathfrak{n}^-)$ -module M , let \bar{M} denote $1 \otimes_{U(\mathfrak{n}^-)} M = M/\mathfrak{n}^-M$. Now consider the sequence

$$\bar{D}_1 \xrightarrow{\bar{\eta}} \bar{C} \xrightarrow{\bar{\vartheta}} \overline{\bigoplus_{w \in \mathbb{W}^{(1)}} V(w(\lambda + \rho))} \xrightarrow{\bar{d}_1} \overline{\bigoplus_{w \in \mathbb{W}^{(i-1)}} V(w(\lambda + \rho))}$$

By definition, $\text{Tor}_i^{\mathfrak{n}^-}(\mathbb{C}, X) = \ker \bar{\vartheta} / \text{im } \bar{\eta}$. If we can show that $\bar{\vartheta}$ and $\bar{\eta}$ are null maps then as an immediate consequence we have

$$\dim \text{Tor}_i^{\mathfrak{n}^-}(\mathbb{C}, X) = \dim \bar{C} = \dim K/\mathfrak{n}^-K$$

and by Bott's Theorem, Corollary 5.34, we know

$$\dim \text{Tor}_i^{\mathfrak{n}^-} = \text{card } \mathbb{W}^{(i)} = \dim \left[\bigoplus_{w \in \mathbb{W}^{(1)}} V(w(\lambda + \rho)) / \mathfrak{n}^- \bigoplus_{w \in \mathbb{W}^{(1)}} V(w(\lambda + \rho)) \right]$$

and we are done.

We have the exact sequence

$$D \xrightarrow{\eta} C \xrightarrow{\vartheta} K \rightarrow 0$$

which after quotienting results in

$$\bar{D} \xrightarrow{\bar{\eta}} \bar{C} \xrightarrow{\bar{\vartheta}} \bar{K} \rightarrow 0.$$

By construction $\bar{\vartheta}$ is an isomorphism forcing $\bar{\eta}$ to be zero.

We also have the exact sequence

$$C \xrightarrow{\vartheta} \bigoplus_{w \in \mathbb{W}^{(1)}} V(w(\lambda + \rho)) \xrightarrow{d_1} K_{i-1} \rightarrow 0,$$

and quotienting again we obtain

$$\bar{C} \xrightarrow{\bar{\vartheta}} \overline{\bigoplus_{w \in W^{(i)}} V(w(\lambda + \rho))} \xrightarrow{\bar{d}_i} \bar{K}_{i-1} \rightarrow 0.$$

Where by K_{i-1} we mean $\ker d_{i-1} = \text{im } d_i$ by the inductive hypothesis. By Lemma 5.37 \bar{d}_i is an injection, forcing $\bar{\vartheta}$ null. ■

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6 The Borel-Weil-Bott Theorem

The Borel-Weil-Bott theorem was originally proved by Borel and Weil [39] (with exposition by Serre), and then extended by Bott, [5]. This section proves the theorem for an arbitrary semisimple Lie group, generalising the results of §3 – in particular Proposition 4.8. We follow [2], and [14] – a drastic simplification of an earlier paper [11].

The plan of the proof is to study the special case of complex projective space, $\mathrm{SL}_2\mathbb{C}/\mathrm{B}$, and then extend this to other semisimple algebras using their structure theory – which tells us they can be thought of as a collection of representations of $\mathrm{SL}_2\mathbb{C}$ strung together. This proves the Bott Vanishing theorem. For the purposes of this section \mathcal{F} will refer to some sheaf, rather than the flag manifold G/P .

We start by looking at the sheaf cohomology of \mathbb{P}^1 following [32] (see also [21]). First we need some elementary facts from algebraic geometry. A space is affine if it is of the form $\mathrm{Spec} A$ for A a finitely generated k -algebra with no nilpotents. In our case $k = \mathbb{C}$. For example \mathbb{C}^n has corresponding ring $\mathbb{C}[X_1, \dots, X_n]$ and $D(X_1) = \{x \in \mathbb{C}^n \mid X_1(x) \neq 0\}$ has ring $\mathbb{C}[X_1, \dots, X_n]_{(X_1)} = \mathbb{C}[X_1, X_1^{-1}, X_2, \dots, X_n]$. A map $f : X \rightarrow Y$ between algebraic varieties is affine if there is an open cover of Y such that the inverse images of the opens are affines.

We are mostly interested in sheaves defined on algebraic varieties, specifically quasi-coherent sheaves. These are sheaves which are given locally by generators and relations. We say a sheaf is coherent if the generators and relations can be finitely listed. A locally free coherent sheaf (where there are no relations) corresponds to a finite rank vector bundle. We have two results regarding sheaves on affines which we will use, see [32]:

1. If X is an affine variety and \mathcal{F} is a quasi-coherent sheaf then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.
2. If $f : X \rightarrow Y$ is an affine map, and \mathcal{F} a quasi-coherent \mathcal{O}_X -module then for all i there is an isomorphism

$$H^i(X, \mathcal{F}) \rightarrow H^i(Y, f_*\mathcal{F})$$

Lemma 6.1 *The following hold:*

1. $H^0(\mathbb{C}^2 - \{0\}, \mathcal{O}_{\mathbb{C}^2}) = \mathbb{C}[X, Y]$
2. $H^1(\mathbb{C}^2 - \{0\}, \mathcal{O}_{\mathbb{C}^2}) = \bigoplus_{\alpha, \beta \leq -1} \mathbb{C}X^\alpha Y^\beta$ as a $\mathbb{C}[X, Y]$ -module.

Proof. Given sheaf \mathcal{F} and an open set U , define $U\mathcal{F}$ by $U\mathcal{F}(V) = \mathcal{F}(U \cap V)$. $U\mathcal{F}$ is then a sheaf, and using this we construct the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{C}^2 - \{0\}} \rightarrow_{D(Y)} \mathcal{O}_{\mathbb{C}^2 - \{0\}} \rightarrow \bigoplus_{\beta \leq -1} \mathcal{O}_{\mathbb{C} - \{0\}} Y^\beta \rightarrow 0$$

This is exact, since the sheaf restricted to $D(Y)$ is localised, and quotienting out by the original sheaf leaves only the terms with Y to a negative power. $D(Y)$ is affine and the inclusion of $D(Y)$ in $\mathbb{C}^2 - \{0\}$ is affine, so we have

$$H^i(D(Y), \mathcal{O}_{D(Y)}) = H^i(\mathbb{C}^2 - \{0\}, i_*\mathcal{O}_{D(Y)}) = H^i(\mathbb{C}^2 - \{0\}, \mathcal{O}_{\mathbb{C}^2 - \{0\}}) = 0$$

for $i > 0$. Thus the middle sheaf has vanishing higher cohomology, and clearly its section is $\mathbb{C}[X, Y, Y^{-1}]$. Now consider the long exact sequence

$$0 \rightarrow H^0(\mathbb{C}^2 - \{0\}, \mathcal{O}_{\mathbb{C}^2}) \rightarrow \mathbb{C}[X, Y, Y^{-1}] \xrightarrow{\epsilon} \mathbb{C}[X, Y^{-1}]Y^{-1} \rightarrow H^1(\mathbb{C}^2 - \{0\}, \mathcal{O}_{\mathbb{C}^2}) \rightarrow 0.$$

$H^0(\mathbb{C}^2 - \{0\}, \mathcal{O}_{\mathbb{C}^2})$ is the kernel of the map ϵ , which sends non-negative powers of Y to zero. $H^1(\mathbb{C}^2 - \{0\}, \mathcal{O}_{\mathbb{C}^2})$ is the cokernel of this map. ■

Vector bundles correspond to locally free coherent sheaves, and we need to know which sheaves correspond to the line bundles $\mathcal{O}_{\mathbb{P}^n}(d)$ before continuing. Let $p = (0, \dots, 0, 1) \in \mathbb{P}^{n+1}$ and consider the projection

$$\pi: \mathbb{P}^{n+1} - \{p\} \longrightarrow \mathbb{P}^n: (X_0, \dots, X_{n+1}) \mapsto (X_0, \dots, X_n)$$

The inverse image of the affine open $D(X_i)$ is $D(X_i) \times \mathbb{C}$ with transition functions $v_{ij} = \frac{X_j}{X_i}$. This thus determines a line bundle. Now the inverse image of a point (X_0, \dots, X_n) is the line $\{(\lambda X_1, \dots, \lambda X_n, 1) \mid \lambda \in \mathbb{C}\}$. Thus this line bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$. From this it is clear that d^{th} tensor power of this will be the line bundle with transition functions $v_{ij} = \left(\frac{X_j}{X_i}\right)^d$.

Define a sheaf of $\mathcal{O}_{\mathbb{P}^n}$ -modules $\mathcal{F}(d)$ by

$$\Gamma(U, \mathcal{F}(d)) = \left\{ \text{algebraic functions } f \text{ on } \pi^{-1}(U) \text{ such that } f(\lambda X) = \lambda^d f(X) \text{ for all } X \in \pi^{-1}(U) \right\}.$$

Restricting to affine patches $D(X_i)$ there is an isomorphism between $\Gamma(D(X_i), \mathcal{F}(d))$ and $\Gamma(D(X_i), \mathcal{O}_{\mathbb{P}^n})$ which we now construct. The map $\pi: \pi^{-1}D(X_i) \rightarrow D(X_i)$ can be written in coordinates as

$$(X_0, \dots, X_n) \mapsto \left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0} \right).$$

Given $f \in \Gamma(D(X_i), \mathcal{O}_{\mathbb{P}^n})$ define \tilde{f} on $\pi^{-1}(D(X_i))$ by

$$\tilde{f}(X_0, \dots, X_n) = X_0^d f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right),$$

and going the other way, given an \tilde{f} , define f as

$$f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right) = \tilde{f}\left(1, \frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right).$$

This defines an isomorphism. The transition functions for $\mathcal{F}(d)$ are clearly $v_{ij} = \left(\frac{X_j}{X_i}\right)^d$ and so $\mathcal{F}(d)$ is the sheaf of algebraic sections of $\mathcal{O}_{\mathbb{P}^n}(d)$.

Proposition 6.2 *The following hold,*

1. $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = \mathbb{C}[X, Y]_{\deg d}$
2. $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = \mathbb{C}[X^{-1}, Y^{-1}]_{\deg d}$

Proof. Consider the affine projection $\phi: \mathbb{C}^2 - \{0\} \longrightarrow \mathbb{P}^1$. The image $\phi_* \mathcal{O}_{\mathbb{C}^2 - \{0\}}$ will be all homogeneous functions defined on \mathbb{P}^1 . This can be broken down into the direct sum $\bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(d)$. So $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ is the term of degree d in $H^1(\mathbb{C}^2 - \{0\}, \mathcal{O}_{\mathbb{C}^2})$. ■

Theorem 6.3 Leray-Hirsch. [6]. *Let \mathcal{E} be a fiber bundle over B with fiber F . Suppose B is compact (this can be weakened). If there are global cohomology classes e_1, \dots, e_r on \mathcal{E} which when restricted to each fiber freely generate the cohomology of the fiber, then $H^*(\mathcal{E})$ is a free module over $H^*(B)$ with basis $\{e_1, \dots, e_r\}$, thus*

$$H^*(\mathcal{E}) \approx H^*(B) \otimes \mathbb{R}\langle e_1, \dots, e_r \rangle \approx H^*(B) \otimes H^*(F). \blacksquare$$

Now we are ready to prove Bott's theorem. Let G be a simply connected complex semi-simple Lie group. Since G is simply connected, characters of the maximal torus are in 1-1 correspondence with algebraically integral elements of \mathfrak{h}^* . Since characters on B have trivial restriction to the nilpotent subgroup N , we thus have that characters on B are in 1-1 correspondence with integral elements of \mathfrak{h}^* .

Given a simple root $\alpha \in \Pi$ let \mathfrak{g}^α be the subalgebra of \mathfrak{g} generated by x_α, y_α and h_α as in §2.1. Then let \mathfrak{p}^α denote the parabolic subalgebra of \mathfrak{g} generated by \mathfrak{b} and \mathfrak{g}^α . We denote by P^α the parabolic subgroup of G corresponding to \mathfrak{p}^α .

If (τ, V) is a representation of B , then let \mathcal{V} denote the corresponding homogeneous vector bundle over G/B , as constructed in §4.2.

Lemma 6.4 *Let $\tau : B \rightarrow GL(V)$ be a representation of B and μ a character of B . If τ can be extended to a representation of P^α in V and if $\langle \alpha^\vee, \mu \rangle = -1$ then*

$$H^i(G/B, \mathcal{V} \otimes \mathcal{L}(\mu)) = 0 \text{ for all } i$$

Proof. The fibers of the projection $G/B \rightarrow G/P^\alpha$ are copies of the projective line \mathbb{P}^1 . We will show in the next section that the cohomology of the fibers is generated freely by the restriction of cohomology classes of G/B . By a generalisation of the Leray-Hirsch theorem to sheaf cohomology it follows that

$$H^*(G/B, \mathcal{V} \otimes \mathcal{L}(\mu)) = H^*(G/P^\alpha, \pi_* \mathcal{V} \otimes \mathcal{L}(\mu)) \otimes H^*(B/P^\alpha, i^* \mathcal{V} \otimes \mathcal{L}(\mu))$$

and so by Proposition 6.2, $H^*(B/P^\alpha, i^* \mathcal{V} \otimes \mathcal{L}(\mu))$ is zero; or alternatively we can use [23] Corollaire 7.9.9 for the same result. ■

Let α be a simple root, and $\lambda \in \mathfrak{h}^*$ such that $\langle \alpha^\vee, \lambda \rangle \geq 0$. Then there is a representation $P^\alpha \rightarrow GL(V_{\lambda, \alpha})$ which has the following weight space decomposition: $V_{\lambda, \alpha}$ is a direct sum of one dimensional weight spaces of weight $\lambda, \lambda - \alpha, \dots, s_\alpha(\lambda)$. Let L^λ be the B -module given by the weight λ ; then L^λ and $L^{s_\alpha(\lambda)}$ are respectively quotient and subobjects of $V_{\lambda, \alpha}$. Depending on $\langle \alpha^\vee, \lambda \rangle$ there is an exact sequence

$$\begin{aligned} 0 \rightarrow 0 \rightarrow V_{\lambda, \alpha} \rightarrow L^\lambda \rightarrow 0 & \text{ if } \langle \alpha^\vee, \lambda \rangle = 0, \\ 0 \rightarrow L^{s_\alpha(\lambda)} \rightarrow V_{\lambda, \alpha} \rightarrow L^\lambda \rightarrow 0 & \text{ if } \langle \alpha^\vee, \lambda \rangle = 1, \\ 0 \rightarrow L^{s_\alpha(\lambda)} \rightarrow K \rightarrow V_{\lambda - \alpha, \alpha} \rightarrow 0 & \text{ if } \langle \alpha^\vee, \lambda \rangle \geq 2. \end{aligned}$$

In the last exact sequence, the B -module K has weight spaces with weights $\lambda - \alpha, \dots, s_\alpha(\lambda) + \alpha, s_\alpha(\lambda)$. We deduce the

Theorem 6.5 *Let α be a simple root and $\lambda \in \mathfrak{h}^*$ such that $\langle \alpha^\vee, \lambda + \rho \rangle \geq 0$. Then there exist G -module isomorphisms*

$$H^i(G/B, \mathcal{L}^\lambda) \cong H^{i+1}(G/B, \mathcal{L}^{s_\alpha(\lambda + \rho) - \rho}) \text{ for all } i. \quad (28)$$

Proof. There are three cases. First if $\langle \alpha^\vee, \lambda + \rho \rangle \geq 2$ then construct exact sequences of B -modules

$$\begin{aligned} 0 \rightarrow K \rightarrow V_{\lambda + \rho, \alpha} \rightarrow L^{\lambda + \rho} \rightarrow 0 & \text{ and} \\ 0 \rightarrow L^{s_\alpha(\lambda + \rho)} \rightarrow K \rightarrow V_{\lambda + \rho - \alpha, \alpha} \rightarrow 0, \end{aligned}$$

where we can see \mathcal{K} occurs in both sequence by looking at corresponding weight structures. From this we get exact sequences of G -bundles on G/B taking the corresponding sheaves and tensoring with the line bundle $\mathcal{L}^{-\rho}$:

$$\begin{aligned} 0 \rightarrow \mathcal{K} \rightarrow \mathcal{V}_{\lambda+\rho, \alpha} \otimes \mathcal{L}^{-\rho} \rightarrow \mathcal{L}^{\lambda} \rightarrow 0 \\ 0 \rightarrow \mathcal{L}^{s_{\alpha}(\lambda+\rho)-\rho} \rightarrow \mathcal{K} \rightarrow \mathcal{V}_{\lambda+\rho-\alpha, \alpha} \otimes \mathcal{L}^{-\rho} \rightarrow 0. \end{aligned}$$

Since the cohomology of $\mathcal{V}_{\lambda+\rho, \alpha} \otimes \mathcal{L}^{-\rho}$ vanishes, looking at the long exact sequences we have

$$\begin{aligned} 0 \rightarrow H^i(G/B, \mathcal{L}^{\lambda}) \rightarrow H^{i+1}(G/B, \mathcal{K}) \rightarrow 0 \\ 0 \rightarrow H^i(G/B, \mathcal{L}^{s_{\alpha}(\lambda+\rho)-\rho}) \rightarrow H^i(G/B, \mathcal{K}) \rightarrow 0 \end{aligned}$$

and so the desired isomorphism follows. In the cases where $\langle \alpha^{\vee}, \lambda + \rho \rangle = 0$ or 1 , the result follows easily from the exact sequences given before the theorem. ■

Corollary 6.6 *Let $\lambda \in \mathfrak{h}^*$ with $\lambda + \rho$ dominant integral. If $w \in \mathbb{W}$ has length $n = l(w)$, then $H^i(G/B, \mathcal{L}^{\lambda})$ and $H^{n+i}(G/B, \mathcal{L}^{w(\lambda+\rho)-\rho})$ are isomorphic as G -modules. ■*

Theorem 6.7 Bott.

- (a) *If λ dominant integral then $H^i(G/B, \mathcal{L}^{\lambda}) = 0$ for $i > 0$.*
- (b) *If there exists a root α with $\langle \alpha^{\vee}, \lambda + \rho \rangle = 0$ then $H^i(G/B, \mathcal{L}^{\lambda}) = 0$ for all i .*
- (c) *If not, $H^i(G/B, \mathcal{L}^{\lambda}) \neq 0$ for exactly one i . Write $\lambda = w(\mu + \rho) - \rho$ with $w \in \mathbb{W}$ and μ dominant integral; then $H^i(G/B, \mathcal{L}^{\lambda}) = 0$ for $i \neq l(w)$ and $H^{l(w)}(G/B, \mathcal{L}^{\lambda})$ is isomorphic to $H^0(G/B, \mathcal{L}^{\mu})$ as a G -module.*

Proof. Let s be the unique bottom element of the Weyl group, with maximal length. Then $l(s) = \dim G/B$ and so by Corollary 6.6 $H^i(G/B, \mathcal{L}^{\lambda})$ is isomorphic to $H^{l(s)+i}(G/B, \mathcal{L}^{s(\lambda+\rho)-\rho})$ and this is equal to zero for $i > 0$.

For (b), write $\alpha = w \cdot \beta$ for β simple. Then

$$\langle \alpha^{\vee}, \lambda + \rho \rangle = \langle (w \cdot \beta)^{\vee}, \lambda + \rho \rangle = \langle \beta^{\vee}, w(\lambda + \rho) \rangle = 0$$

and so $\langle \beta^{\vee}, w(\lambda + \rho) - \rho \rangle = -1$. Then from the Lemma 6.4

$$H^i(G/B, \mathcal{L}^{\lambda}) = H^{i+l(w)}(G/B, \mathcal{L}^{w(\lambda+\rho)-\rho}) = 0.$$

The final part is an immediate consequence of the first and Corollary 6.6. ■

Now we reach our goal,

Theorem 6.8 Borel-Weil. *Let G be a simply-connected complex semi-simple Lie group, and B a Borel subgroup. Suppose $\lambda \in \mathfrak{h}^*$ is an integral weight for G .*

- (a) *If there exists a root α such that $\langle \alpha^{\vee}, \lambda + \rho \rangle = 0$ then*

$$H^i(G/B, \mathcal{L}^{\lambda}) = 0 \text{ for all } i.$$

- (b) *Otherwise we have the following G -module isomorphism,*

$$H^{l(w)}(G/B, \mathcal{L}^{\lambda}) \approx R(w(\lambda + \rho))$$

for w the unique element such that $w(\lambda + \rho)$ is dominant.

Proof. Part (a) is already proven. For part (b), all that is required is to show that for a dominant integral weight λ , $H^0(G/B, \mathcal{L}^\lambda) \approx R(\lambda + \rho)$, recalling that $R(\lambda + \rho)$ is the irreducible representation of G of highest weight λ . The rest follows from Bott's theorem.

We previously constructed the line bundle \mathcal{L}^λ in §4.1 as a pullback of the canonical bundle on a projective space. In more detail, we showed that G/P embeds in $\mathbb{P}^*(W)$ for W an irreducible representation with highest weight vector λ , W^* an irreducible representation with corresponding lowest weight ν_λ . In the case we are considering, where $\langle \alpha^\vee, \lambda \rangle \neq 0$ for all roots α , we have that the stabiliser of ν_λ under the action of G on W^* is B , and so we have an embedding of G/B into $\mathbb{P}^*(W)$. \mathcal{L}^λ is then the pullback of the canonical bundle $\mathcal{O}(1)$ on $\mathbb{P}^*(W)$.

For the purposes of proving Borel-Weil, it is convenient to give a more conceptual construction of \mathcal{L}^λ , which we do following [19] and [37]. Consider the "tautological" bundle \mathcal{B} on G/B , where the fiber of \mathcal{B} at x , $\mathcal{B}_x = \{z \in \mathfrak{g} \mid z \in \mathfrak{b}_x\}$ where by \mathfrak{b}_x we mean the Borel subalgebra of \mathfrak{g} corresponding to the point x . This is clearly a subbundle of the trivial bundle $G/B \times \mathfrak{g}$. Moreover under the adjoint action of G on \mathfrak{g} , it is a G -homogeneous space.

\mathcal{B} has a G -subbundle \mathcal{N} with fibers given by $\mathcal{N}_x = \mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$. Form the quotient bundle $\mathcal{H} = \mathcal{B}/\mathcal{N}$. \mathcal{H} is a trivial bundle on G/B : it is a G -bundle, such that the stabiliser $B_x \subset G$ of a point $x \in G/B$ acts trivially on \mathcal{H}_x . Since G/B is a projective variety, sections of \mathcal{H} are constant, and we obtain the finite dimensional vector space \mathfrak{h} of global sections of \mathcal{H} . This is the abstract Cartan algebra of \mathfrak{g} . Given a Cartan subalgebra \mathfrak{h}' of \mathfrak{g} and a choice of positive roots, there is a canonical isomorphism with \mathfrak{h} given by the composition $\mathfrak{h}' \rightarrow \mathfrak{b}_x \rightarrow \mathfrak{b}_x/\mathfrak{n}_x = \mathcal{H}_x \rightarrow \mathfrak{h}$ where \mathfrak{b}_x is the Borel subalgebra of \mathfrak{g} spanned by \mathfrak{h} and the positive roots, and the final map is the inverse of evaluation at a point.

We also have the dual map $\mathfrak{h}^* \rightarrow (\mathfrak{h}')^*$. The line bundle \mathcal{L}^λ is then the line bundle with fiber λ_x at $x \in G/B$, where λ_x is the element of $(\mathfrak{h}')^*$ corresponding to $\lambda \in \mathfrak{h}^*$. This is G -homogeneous since the G -action is to give isomorphisms between the fibers over points in G/B , and these isomorphisms are how we identify λ_x in each fiber.

Let F^λ be the irreducible representation of G with highest weight λ . We form the trivial vector bundle $G/B \times F^\lambda$ and let \mathcal{F}^λ denote the sheaf of sections of this bundle. Then

$$H^i(G/B, \mathcal{F}^\lambda) = H^i(G/B, \mathcal{O}_{G/B}) \otimes_{\mathbb{C}} F^\lambda \text{ for } i \in \mathbb{Z}^+. \quad (29)$$

In general, given a G -module X , we can consider it as a B -module. It then has a Jordan-Hölder filtration which translates into a filtration of \mathcal{X} - the corresponding (trivial) sheaf - by locally free G -homogeneous $\mathcal{O}_{G/B}$ -modules of the form $F_p \mathcal{X}/F_{p-1} \mathcal{X}$ isomorphic to the line bundle \mathcal{L}^{ν_p} for some weight ν_p of X . To understand this isomorphism, recall that the quotients obtained from the Jordan-Hölder filtration are 1-dimensional weight spaces. So the quotient considered globally is simply picking a weight at each point of G/B . But this is how we constructed \mathcal{L}^ν .

Consider the Jordan-Hölder decomposition of \mathcal{F}^λ . The Casimir element acts on sections of \mathcal{L}^{ν_p} as scalar multiplication by $c_{\nu_p} = \langle \nu_p + \rho, \nu_p + \rho \rangle$. (Here we have altered notation from §5 to simplify).

It can easily be checked that the scalar c_λ can only occur for sections of \mathcal{L}^λ since it is a highest weight. Thus we have a direct sum decomposition of \mathcal{F}^λ into Γ invariant subspaces - the Γ -eigensheaf \mathcal{L}^λ , and the rest, denoted \mathcal{E} . Cohomology commutes with direct sums, so it remains to see that \mathcal{E} has no global sections.

Since $H^0(\mathcal{F}^\lambda)$ is an irreducible representation of \mathfrak{g} with highest weight λ , Γ will act as scalar multiplication by c_λ on it. Elements of this representation correspond to global sections of \mathcal{F}^λ . \mathcal{E} is subsheaf of \mathcal{F}^λ by construction of the Jordan-Hölder filtration, so global sections of \mathcal{E}

are global sections of \mathcal{F}^λ . But sections of \mathcal{E} have the “wrong” action of Γ , hence cannot exist globally. ■

We have not shown our two constructions of \mathcal{L}^λ coincide. We can now do this with the Borel-Weil theorem under our belts. Let $E = H^0(G/B, \mathcal{L}^\lambda)$, the space of global sections of \mathcal{L}^λ . We construct a map

$$\iota_E : G/B \rightarrow \mathbb{P}^*(E)$$

such that $\iota_E^*(\mathcal{O}(1)) = \mathcal{L}^\lambda$. This suffices since by Borel-Weil E is an irreducible representation of highest weight λ . For each point $x \in G/B$, let $E_x = \{s \in E \mid s(x) = 0\}$; E_x is a hyperplane in E . ι_E is then the map $x \mapsto E_x$. Sections of $\mathcal{O}(1)$ are of the form $\sum a_i Z_i$ where the Z_i are affine coordinates on $\mathbb{P}^*(E)$. These pullback to sections of the form $\sum a_i s_i$ where the s_i are sections in E . This gives a correspondence between sections of $\mathcal{O}(1)$ and \mathcal{L}^λ , showing we have a pullback.

We briefly consider some of the consequences of the Borel-Weil Theorem. We start by recalling the theory of the first Chern class following [35] and [22]. This theory states that given a manifold X , there is an isomorphism between $\mathcal{L}(X)$, the group of line bundles on X , and $H^2(X, \mathbb{Z})$. We construct this isomorphism using Čech cohomology.

A covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X is called contractible if $U_{i_0 \dots i_k} = U_{i_0} \cap \dots \cap U_{i_k}$ is contractible for every k and for every $k+1$ -tuple (i_0, \dots, i_k) such that $U_{i_0, \dots, i_k} \neq \emptyset$ – in other words for every k -simplex. It is known that every covering has a contractible refinement.

We consider cohomology with coefficients in an arbitrary quasi-coherent sheaf \mathcal{F} . Define

$$\begin{aligned} C^0(\mathcal{U}, \mathcal{F}) &= \prod_{\alpha} \mathcal{F}(U_{\alpha}), \\ C^1(\mathcal{U}, \mathcal{F}) &= \prod_{\alpha < \beta} \mathcal{F}(U_{\alpha} \cap U_{\beta}), \\ &\vdots \\ C^k(\mathcal{U}, \mathcal{F}) &= \prod_{\alpha_0 < \dots < \alpha_k} \mathcal{F}(U_{\alpha_0} \cap \dots \cap U_{\alpha_k}). \end{aligned}$$

An element $\sigma \in C^k(\mathcal{U}, \mathcal{F})$ is referred to as a k -cochain. Define the coboundary operator,

$$\begin{aligned} \delta : C^k(\mathcal{U}, \mathcal{F}) &\rightarrow C^{k+1}(\mathcal{U}, \mathcal{F}) \text{ by the formula} \\ (\delta\sigma)_{i_0, \dots, i_{k+1}} &= \sum_{j=0}^{k+1} (-1)^j \sigma_{i_0, \dots, \hat{i}_j, \dots, i_{k+1}}|_{U_{i_0} \cap \dots \cap U_{i_k}}. \end{aligned}$$

$\delta^2 = 0$ so we can define the cohomology groups $H^k(\mathcal{U}, \mathcal{F})$. If \mathcal{V} is a refinement of \mathcal{U} then there is a homomorphism $H^k(\mathcal{U}, \mathcal{F}) \rightarrow H^k(\mathcal{V}, \mathcal{F})$. Define the k^{th} Čech cohomology group with coefficients in \mathcal{F} of the space X , $H^k(X, \mathcal{F})$ to be the inductive limit over all coverings of the cohomology groups arising from these coverings.

Thus there is a homomorphism $H^k(\mathcal{U}, \mathcal{F}) \rightarrow H^k(X, \mathcal{F})$ for any covering \mathcal{U} . In the special case where \mathcal{U} is acyclic it turns out (Leray’s theorem) that this map is an isomorphism.

We can re-interpret the group of line bundles on X sheaf-theoretically as follows. If we denote by \mathcal{O}^* the sheaf of nowhere vanishing holomorphic functions then the transition functions $\{g_{\alpha\beta}\}$ for a line bundle $\mathcal{L} \rightarrow X$ are in $\mathcal{O}^*(U_{\alpha} \cap U_{\beta})$. By (19) we know that $\delta(\{g_{\alpha\beta}\}) = 0$ – where we

are writing the abelian sheaf \mathcal{O}^* multiplicatively instead of additively. Thus $\{g_{\alpha\beta}\}$ is a Čech cocycle.

A line bundle \mathcal{L} does not trivialise uniquely. Given a collection of nowhere zero holomorphic functions $f_\alpha \in \mathcal{O}^*(U_\alpha)$ we can define trivialisations over $\{U_\alpha\}$ by $\phi'_\alpha = f_\alpha \cdot \phi_\alpha$, which results in transition functions

$$g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \cdot g_{\alpha\beta}. \quad (30)$$

This method allows us to produce all alternative trivialisations of \mathcal{L} . So two collections of transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ result in the same line bundle iff (30) is satisfied.

$$\text{Thus } H^1(X, \mathcal{O}^*) \approx \mathcal{L}(X).$$

We can construct the following exact sequence of sheaves over an arbitrary space X ,

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0.$$

If we look at the long exact sequence in the cohomology groups we obtain

$$\dots \rightarrow H^1(X, \mathcal{O}) \rightarrow H^1(X, \mathcal{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}) \rightarrow \dots$$

where δ is the boundary operator which we rename ch_1 . This map can be explicitly constructed as follows, [35].

Given any $a \in C^2(\mathcal{U}, \mathbb{Z})$, it is a cocycle if and only if it satisfies

$$a_{kjl} - a_{ikl} + a_{ijl} - a_{ijk} = 0 \quad (31)$$

for all 3-simplices U_{ijkl} . This defines an element $[a] \in H^2(X, \mathbb{Z})$ which vanishes if and only if there is a $b \in C^2(\mathcal{U}, \mathbb{Z})$ such that

$$a_{ijk} = b_{ij} + b_{jk} - b_{ik}.$$

Given a line bundle \mathcal{L} over X , let $(U_i, s_i)_{i \in I}$ be a local system for \mathcal{L} where \mathcal{U} is a contractible covering. Since $U_i \cap U_j$ is simply connected when not empty we can define a function $f_{ij} : U_i \cap U_j \rightarrow \mathbb{C}$ given by

$$f_{ij} = \frac{1}{2\pi i} \log c_{ij}$$

where the c_{ij} are the transition functions. If $U_i \cap U_j \cap U_k \neq \emptyset$ then since $c_{ij}c_{jk} = c_{ik}$ we have $\exp 2\pi i a_{ijk} = 1$ where

$$a_{ijk} = f_{ij} + f_{jk} - f_{ik}.$$

Now a_{ijk} is \mathbb{Z} -valued and continuous – so it is constant on U_{ijk} thus defining an element of $C^2(\mathcal{U}, \mathbb{Z})$. It is also a cocycle and so defines an element $[a] \in H^2(X, \mathbb{Z})$ which is independent of the choice of logarithm. It turns out that the cohomology class $[a]$ is also independent from the choice of local system and depends only on the isomorphism class of \mathcal{L} in $\mathcal{L}(X)$. Thus we have explicitly constructed

$$\text{ch}_1 : \mathcal{L}(X) \rightarrow H^2(X, \mathbb{Z}).$$

This can be shown, [35], to be a bijection for any paracompact space X . In the case of flag variety this is a consequence of the Borel-Weil theorem. The trivial line bundle $G/P \times \mathbb{C}$

corresponds to the sheaf \mathcal{O} . By the Borel-Weil theorem this has a trivial higher cohomology groups since ρ is a dominant root, forcing ch_1 an isomorphism by exactness.

Given any vector bundle $\mathcal{E} \rightarrow G/P$ we can form a representation of G in the space of sections as in the remark after Proposition 4.8. Since G is semi-simple this representation splits into irreducibles, and each of these irreducibles corresponds to a unique line bundle.

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7 Cohomology of Flag manifolds

There is a beautiful and surprising connection between the Weyl group of a complex semi-simple Lie group G and the cohomology of the flag manifold \mathcal{F} . Namely the quotient of the algebra $\text{Sym } \mathfrak{h}$ by the additive subgroup of W -invariant polynomials is isomorphic to $H^*(\mathcal{F}, \mathbb{Q})$. This was originally proven by Borel [3]. More modern K-theoretic proofs can be found in [1] and [19]. We will prove it case by case without resorting to high-powered machinery.

In addition \mathcal{F} can be partitioned into open cells – the Bruhat cells – such that the homology classes of their closures provide a basis for $H^*(\mathcal{F}, \mathbb{Z})$. Thus we have two different approaches to studying the cohomology of \mathcal{F} . The aim of this section is to reconcile the them.

7.1 Borel's theorem

The main tool in the proof of Borel's theorem is the theory of Chern classes. References for this theory are [6] and [15], which we draw on here, or alternatively [38]. We first list some basic properties of the Chern classes, and then give a brief sketch of how they are constructed.

To each complex vector bundle $\mathcal{E} \xrightarrow{\rho} X$ there corresponds a cohomology class $c(\mathcal{E}) \in H^*(X, \mathbb{Z})$ with the following properties:

C1. $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \dots + c_n(\mathcal{E})$, where $c_i(\mathcal{E}) \in H^{2i}(X, \mathbb{Z})$ and $c_i(\mathcal{E}) = 0$ for $i > \dim \mathcal{E}$.

C2. (Naturality) Given a map $Y \xrightarrow{f} X$, it follows that $f^*(c(\mathcal{E})) = c(f^*(\mathcal{E}))$.

C3. (Whitney formula) Suppose \mathcal{E} can be written as a quotient \mathcal{F}/\mathcal{G} . Then we have that $c(\mathcal{F}) = c(\mathcal{E}) \cup c(\mathcal{G})$ – where \cup denotes the cup product.

C4. If we denote by \mathcal{L} the canonical line bundle on $\mathbb{P}(\mathbb{C}^n)$, then $c_1(\mathcal{L})$ freely generates $H^*(\mathbb{P}(\mathbb{C}^n), \mathbb{Z})$ as an algebra.

C5. If \mathcal{L} and \mathcal{L}' are arbitrary line bundles then $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$.

C6. Given a section μ of a line bundle \mathcal{L} , this determines a divisor $D = \sum n_i D_i$ where each D_i is an irreducible hypersurface and n_i is the order of vanishing of μ along D_i . We then have

$$c_1(\mathcal{L}) = \sum n_i \langle D_i \rangle \text{ in } H^2(X, \mathbb{Z}),$$

where $\langle D_i \rangle$ is the cohomology class of the subvariety D_i .

We will typically write the cup product using a dot, since Borel's theorem will show that we can think of the cohomology of G/P as a polynomial algebra.

Proposition 7.1 [27] *Let \mathcal{E} and \mathcal{F} be vector bundles over a space X .*

(a) *If \mathcal{E} and \mathcal{F} are isomorphic then $c_i(\mathcal{E}) = c_i(\mathcal{F})$ for all i .*

(b) *If \mathcal{E} is a trivial bundle then $c_i(\mathcal{E}) = 0$ for all $i > 0$.*

(c) *If \mathcal{E} has rank n and possesses a nowhere zero global section then $c_n(\mathcal{E}) = 0$.*

Proof. (a) follows by functoriality. For (b) consider the bundle map f ,

$$\begin{array}{ccc} \mathcal{E} = X \times \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & \{\text{point } x\}. \end{array}$$

$H^i(\{x\}, \mathbb{Z}) = 0$ for $i \geq 1$ so that $f^*H^i(\{x\}, \mathbb{Z}) = 0$, implying $c_i(\mathcal{E}) = 0$ for $i \geq 1$.

Finally to prove (c), impose a Hermitian metric on the bundle \mathcal{E} . The global section s picks out a trivial subbundle \mathcal{F} of \mathcal{E} . Using the metric we obtain a perpendicular subbundle \mathcal{F}^\perp . We then have

$$c(\mathcal{E}) = c(\mathcal{F}) \cdot c(\mathcal{F}^\perp) = c(\mathcal{F}^\perp)$$

by the Whitney formula and (b). Since \mathcal{F}^\perp has a lower rank than \mathcal{E} , it follows that the top Chern class of \mathcal{E} is zero. ■

We constructed the first Chern class in §6. Now using the first Chern class we go on to construct the higher Chern classes. Again let $\mathcal{E} \xrightarrow{\rho} X$ be a vector bundle; with transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL_m \mathbb{C}$. We can form the projective bundle $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$ as follows:

at each point $x \in X$ let \mathcal{E}_x denote the fiber over x . The projectivisation has fiber $\mathbb{P}(\mathcal{E}_x)$ at the point x , and transition maps $\bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow PGL_m \mathbb{C}$ induced by the $g_{\alpha\beta}$'s. Each point in $\mathbb{P}(\mathcal{E})$ is a line l_x in the fiber \mathcal{E}_x .

The projective bundle $\mathbb{P}(\mathcal{E})$ is a manifold in its own right – a generalisation of complex projective space – and it comes equipped with its own canonical vector bundles. We form the exact sequence,

$$0 \rightarrow \mathcal{L} \rightarrow \pi^{-1}\mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0$$

of vector bundles over $\mathbb{P}(\mathcal{E})$. We describe each element of the sequence in turn. $\pi^{-1}\mathcal{E}$ is the pullback of the vector bundle \mathcal{E} under the map π ,

$$\begin{array}{ccc} & \mathcal{E} & \\ & \downarrow \rho & \\ \mathbb{P}(\mathcal{E}) & \xrightarrow{\pi} & M \end{array}$$

and can be described more explicitly as having fiber \mathcal{E}_x at the point l_x . If we restrict to the fiber $\pi^{-1}(x)$, which we denote by $\mathbb{P}(\mathcal{E})_x$, the bundle trivialises

$$\pi^{-1}\mathcal{E}|_{\mathbb{P}(\mathcal{E})_x} = \mathbb{P}(\mathcal{E})_x \times \mathcal{E}_x$$

since $\mathcal{E}_x \xrightarrow{\rho} \{x\}$ is a trivial bundle – ie we are pulling back a trivial bundle.

\mathcal{L} , the canonical line bundle, is defined by

$$\mathcal{L} = \{ (l_x, v) \in \pi^{-1}\mathcal{E} \mid v \in l_x \}$$

and is a straightforward generalisation of the universal bundle over projective space. The bundle \mathcal{Q} is the quotient of the pullback bundle by the canonical line bundle.

Set $y = -c_1(\mathcal{L}) = c_1(\mathcal{L}^\vee)$ where by \mathcal{L}^\vee we mean the dual line bundle to \mathcal{L} satisfying $\mathcal{L} \otimes \mathcal{L}^\vee \approx \mathbb{P}(\mathcal{E}) \times \mathbb{C}$, the trivial bundle. Then y is a cohomology class in $H^2(\mathbb{P}(\mathcal{E}), \mathbb{Z})$. Restricting \mathcal{L} to a fiber $\mathbb{P}(\mathcal{E}_x)$ results in the canonical line bundle on the complex projective space $\mathbb{P}(\mathcal{E}_x)$ and so by property C2 it follows that the restriction of $-y$ is the Chern class of the canonical line bundle on $\mathbb{P}(\mathcal{E}_x)$. So the cohomology classes $1, y, \dots, y^{n-1}$ are global classes whose restrictions to the fibers freely generated their cohomology – property C4. So by the Leray-Hirsch theorem $H^\bullet(\mathbb{P}(\mathcal{E}))$ is a free module over $H^\bullet(X)$ with basis $\{1, y, \dots, y^{n-1}\}$.

Thus we can uniquely write

$$y^n + c_1(\mathcal{E})y^{n-1} + \dots + c_n(\mathcal{E}) = 0$$

where $c_i(\mathcal{E}) \in H^{2i}(X)$. $c_i(\mathcal{E})$ is defined to be the i^{th} Chern class of \mathcal{E} . In the special case where $n = 1$, we have that $\mathbb{P}(\mathcal{E}) = X$ and $y = -c_1(\mathcal{E}) = -c_1(\mathcal{L})$ so the definition works.

Definition 7.2 We define the following,

1. Let $R = \text{Sym } \mathfrak{h}_{\mathbb{Q}}^* = \bigoplus R_i$ be the graded algebra of polynomial functions on $\mathfrak{h}_{\mathbb{Q}}$ with rational coefficients - with R_i the space of homogeneous polynomials of degree i . \mathbb{W} acts on R according to the rule $w \cdot f(\mathfrak{h}) = f(w^{-1}\mathfrak{h})$.

2. I is the subring of \mathbb{W} -invariant elements in R , and

$$I^+ = \{f \in I \mid f(0) = 0\}.$$

3. J is the ideal in R generated by I^+ .

4. $\bar{R} = R/J$.

Theorem 7.3 Borel. [17] We construct a homomorphism $\alpha : R \rightarrow H^*(G/B, \mathbb{Q})$. For each $\chi \in \mathfrak{h}_{\mathbb{Z}}^*$ there is a character $\theta \in \text{Mor}(H, \mathbb{C}^*)$ such that $\theta(\exp \mathfrak{h}) = \exp \chi(\mathfrak{h})$ for all $\mathfrak{h} \in \mathfrak{h}$. Extend this to a character on B by setting $\theta(\mathfrak{n}) = 1$ for all $\mathfrak{n} \in \mathfrak{N}$. θ defines a line bundle \mathcal{L}^{θ} . Finally we define $\alpha(\chi) = c_1(\mathcal{L}^{\theta}) \in H^2(G/B, \mathbb{Z})$. Thus α is a homomorphism of $\mathfrak{h}_{\mathbb{Z}}^*$ into $H^2(G/B, \mathbb{Z})$ which extends naturally.

It then turns out that $\ker \alpha = J$, so that

$$\bar{\alpha} : R/J \rightarrow H^*(G/B, \mathbb{Q}) \text{ is an isomorphism.}$$

Proof. We prove this case by case.

(\mathbf{A}_n). Let E be a n dimensional complex vector space. To prove the theorem we construct the flag $\mathcal{F}^{n, \dots, 1}(E) = \mathcal{F}(E)$ as a sequence of projective bundles starting with $\mathbb{P}(E)$. Points in this space are lines in E . Let \mathcal{U}_1 denote the canonical line bundle over $\mathbb{P}(E)$ and \mathbf{E} the product bundle $\mathbb{P}(E) \times E$. Take the quotient \mathbf{E}/\mathcal{U}_1 and form the projective bundle $\mathbb{P}(\mathbf{E}/\mathcal{U}_1) \rightarrow \mathbb{P}(E)$.

Consider the fiber over a point l in $\mathbb{P}(E)$. This fiber consists of all the lines in E/l - in other words all the planes in E containing the line l .

Now construct the canonical line bundle on $\mathbb{P}(\mathbf{E}/\mathcal{U}_1)$. This bundle can be represented as $\mathcal{U}_2/\mathcal{U}_1$ for a rank-2 bundle \mathcal{U}_2 such that $\mathcal{U}'_1 \subset \mathcal{U}_2 \subset \mathbf{E}'$ as vector bundles over $\mathbb{P}(\mathbf{E}/\mathcal{U}_1)$. So in the inclusions above we have that \mathcal{U}'_1 is in fact the pullback

$$\begin{array}{ccc} & \mathcal{U}_1 & \\ & \downarrow & \\ \mathbb{P}(\mathbf{E}/\mathcal{U}_1) & \longrightarrow & \mathbb{P}(E) \end{array}$$

and \mathbf{E}' is the vector bundle $\mathbb{P}(\mathbf{E}/\mathcal{U}_1) \times E$. Now construct $\mathbb{P}(\mathbf{E}'/\mathcal{U}_2)$. Each point in this space is a line contained in a plane contained in a 3-space. We continue with this construction until we arrive at the flag manifold $\mathbb{P}(\mathbf{E}/\mathcal{U}_{n-2})$ with canonical line bundle $\mathcal{U}_{n-1}/\mathcal{U}_{n-2}$.

Now let $\mathcal{L}_1 = \mathcal{U}_1$ and $\mathcal{L}_i = \mathcal{U}_i/\mathcal{U}_{i-1}$. Then \mathcal{L}_i is the line bundle equal to $\mathcal{L}(\theta_i)$ where $\theta_i : B \rightarrow \mathbb{C}^*$ is the character taking $b \in B$ to the i^{th} element of its diagonal. We set $x_i = -c_1(\mathcal{L}(\theta_i))$.

This is an iteration of the construction used to define the higher Chern classes and so we can see that $x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ where $i_k \leq n - k$ form an additive basis for $H^*(\mathcal{F})$.

We have a filtration of vector bundles,

$$\{0\} \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \dots \subset \mathcal{U}_{n-1} \subset \mathbf{E}$$

By Whitney's formula we have

$$\begin{aligned} 1 &= c(\mathbf{E}) = c(\mathcal{U}_{n-1}) \cdot c(\mathcal{L}_n) = c(\mathcal{U}_{n-2}) \cdot c(\mathcal{L}_{n-1}) \cdot c(\mathcal{L}_n) \\ &= \dots = c(\mathcal{L}_1) \cdot c(\mathcal{L}_2) \dots c(\mathcal{L}_n) \\ &= (1 - x_1)(1 - x_2) \dots (1 - x_n) \end{aligned}$$

and so it follows that $e_1(x_1, \dots, x_n) = \dots = e_n(x_1, \dots, x_n) = 0$ where $e_k(x_1, \dots, x_n)$ is the k^{th} elementary symmetric polynomial, given by the sum of all monomials $x_{i_1} \dots x_{i_k}$ where the subscripts satisfy $1 \leq i_1 < \dots < i_k \leq n$. These form a basis under \mathbb{Z} for the symmetric polynomials.

Thus we have

$$H^*(\mathcal{F}, \mathbb{Z}) \approx \mathbb{Z}[x_1, \dots, x_n] / (e_1(x), \dots, e_n(x)),$$

or in other words it follows that $H^*(\mathcal{F}, \mathbb{Z})$ is isomorphic to the space of co-invariants of $\text{Sym } \mathfrak{h}^*$ under the symmetric group as required.

(C_n): We still want to build up the flag manifold as a sequence of projective bundles – but we have to deal with the complication that this must be done in such a way as to ensure the subspaces are all isotropic.

Since any line is isotropic, choosing a line is equivalent to simply picking a point in $\mathbb{P}(E)$. Having done this we choose an isotropic plane containing this line by choosing a line L_2/L_1 in $\mathbb{P}(E/L_1)$. We continue, iterating this construction until we obtain the symplectic flag,

$$\{0\} \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \dots \subset \mathcal{U}_n = \mathcal{U} \subset E$$

where $\dim E = 2n$. Following the strategy above we set $x_i = -c_1(\mathcal{U}_i/\mathcal{U}_{i-1})$ for i between 1 and n . The cohomology ring of the flag $\mathcal{F}(E)^{(n, \dots, 1)}$ is generated by the x_i 's, but now with an additional relation. The symplectic form forces \mathcal{U} and E/\mathcal{U} dual vector bundles, implying

$$\begin{aligned} 1 &= c(\mathcal{U}_n) \cdot c(E/\mathcal{U}_n) = c(\mathcal{L}_n) \cdot c(\mathcal{U}_{n-1}) \cdot c(E/\mathcal{U}_{n-1}) \cdot c(\mathcal{L}_n^\vee) \\ &= \dots = (1 + x_1) \dots (1 + x_n)(1 - x_1) \dots (1 - x_n) \\ &= (1 - x_1^2) \dots (1 - x_n^2). \end{aligned}$$

So following the reasoning above we have

$$H^*(\mathcal{F}^{\text{sym}}, \mathbb{Z}) \approx \mathbb{Z}[x_1, \dots, x_n] / \left[(e_1(x_1^2, \dots, x_n^2), \dots, e_n(x_1^2, \dots, x_n^2)) \right].$$

(B_n): Since the Weyl groups for B_n and C_n are the same we expect to obtain the same end results. In this case not all lines are isotropic, for example the vector $e_1 + e_{n+1}$ defines a non-isotropic line. Thus instead of choosing any point in $\mathbb{P}(E)$, we choose points in a subvariety defined by a single quadratic equation (see §3). Thus the flag manifold $\mathcal{F}(E)^{(n, \dots, 1)}$ is constructed as a sequence of quadric projective bundles

$$\{0\} \subset \mathcal{U}_1 \subset \dots \subset \mathcal{U}_n \subset E$$

which carries the same information as

$$\{0\} \subset U_1 \subset \dots \subset U_n \subset U_n^\perp \subset \dots \subset U_1^\perp \subset E$$

The quadratic form implies that U_n and E/U_n^\perp are dual bundles, giving the relation

$$1 = c(U_n) \cdot c(E/U_n^\perp).$$

We can break this down into its components and in the end we obtain exactly as in the case for C_n ,

$$H^*(\mathcal{F}^{\text{orth-odd}}, \mathbb{Z}) \approx \mathbb{Z}[x_1, \dots, x_n] / \left[(e_1(x_1^2, \dots, x_n^2), \dots, e_n(x_1^2, \dots, x_n^2)) \right].$$

(D_n): This case is identical to the previous one, with an additional relation. Choose any non-isotropic vector v . For example with our canonical basis let $v = e_1 + e_{n+1}$. Then v cannot be contained in any isotropic subspace of E , and so v defines an everywhere nonvanishing section of the bundle E/U_n .

Before going any further let us see why this behaviour does not occur in the previous two cases. C_n does not allow non-isotropic vectors. In the case of B_n we are considering the vector bundle E/U_n^\perp . U_n^\perp is an $n + 1$ dimensional vector space and thus $Q(U_n^\perp, U_n^\perp) \neq 0$ and we are not guaranteed an everywhere non-zero section.

Continuing, this implies this bundle has vanishing top Chern class, and so

$$0 = c_n(E/U_n) = x_1 \dots x_n.$$

The Weyl group for D_n has only even sign changes, so we have this additional relation which is unaffected by even changes in sign. Thus

$$H^*(\mathcal{F}^{\text{orth-even}}, \mathbb{Z}) \approx \mathbb{Z}[x_1, \dots, x_n] / \left[e_1(x_1^2, \dots, x_n^2), \dots, e_n(x_1^2, \dots, x_n^2), (x_1 \dots x_n) \right]. \blacksquare$$

Suppose we are given the map $G/B \rightarrow G/P^\alpha$ with fibers copies of the projective line. Then since G/B can be built out of projective bundles it follows by property C4 of the Chern classes that the cohomology of the fibers is freely generated by the restriction of cohomology classes of G/B as required in Lemma 6.4.

7.2 Bruhat cells

There is a strong link between the Bruhat order on the Weyl group and the topology of the flag manifold G/B , which we look at in this section. We draw heavily on results proven in §5.2.

We start by describing the Bruhat decomposition of the flag manifold following [19]. Let G be a complex semi-simple Lie group, with fixed Borel subgroup B and corresponding Borel subalgebra \mathfrak{b} . Let N be the unipotent radical of B with Lie algebra \mathfrak{n} , and set $N^- = sNs^{-1}$ – the subgroup of G corresponding to the algebra \mathfrak{n}^- of negative roots. Denote by $\mathcal{F} = G/B$ the flag manifold consisting of all Borel subgroups of G . We form a series of set-theoretic bijections,

$$\mathbb{W} \rightarrow \{N\text{-orbits on } \mathcal{F}\} \rightarrow \{N_w \mid w \in \mathbb{W}\}. \quad (32)$$

• Let $N_G(T)$ be the normaliser of T in G . It is a standard fact, see [34] or [8], that $N_G(T)/T$ is isomorphic to the Weyl group. Define a map $\mathbb{W} \rightarrow \{N\text{-orbits on } \mathcal{F}\}$ taking $w \in \mathbb{W}$ to the coset $\mathcal{F}_w = N \cdot \bar{w} \cdot B$ where \bar{w} is a representative of w in $N(T)$.

• Let $N_w = w \cdot N^- \cdot w^{-1} \cap N$. Define a map

$$N_w \rightarrow \mathcal{F}_w : n \mapsto n \cdot w \cdot B. \quad (33)$$

This is an isomorphism of algebraic varieties and provides the second map.

This has the immediate consequence that we can parametrise the N -orbits on \mathcal{F} by the Weyl group. We refer to these N -orbits as Bruhat cells, and refer to the decomposition of \mathcal{F} into these cells as the Bruhat decomposition, [9],

$$\mathcal{F} = \bigcup_{w \in W} \mathcal{F}_w = \bigcup_{w \in W} N \cdot \bar{w} \cdot B.$$

The flag variety \mathcal{F} can be embedded into a projective space. This is described in §4.1 for explicit cases – we now consider a general semi-simple group. Let X be a finite dimensional G -module with regular highest weight λ . By regular we mean $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Pi$, or equivalently that the $w\lambda$ are distinct.

Choose for each $w \in W$ a non-zero vector $f_w \in X$ of weight $w\lambda$. This uniquely determines a point $[f_w] \in \mathbb{P}(X)$. By the regularity of λ , the stabiliser of the point $[f_e]$ under the natural induced action of G is B , the Borel subgroup. Thus the orbit of $[f_e]$ in $\mathbb{P}(X)$ is naturally isomorphic to \mathcal{F} . Denote this embedding by

$$\iota : \mathcal{F} \hookrightarrow \mathbb{P}(X).$$

In addition the Bruhat cells have a natural description in terms of this embedding. Let $\bar{\mathcal{F}}_w$ be the closure of \mathcal{F}_w in \mathcal{F} . This is an irreducible algebraic variety which we refer to as a Schubert variety.

Definition 7.4 For each $w \in W$ let ϕ_w be the linear function on X satisfying

$$\phi_w(f_w) = 1, \quad \phi_w(f) = 0 \text{ if } f \in X \text{ is a vector of weight not equal to } w\lambda.$$

Lemma 7.5

$$\bar{\mathcal{F}}_w = \iota^{-1}(\mathbb{P}(\mathbf{U}(\mathfrak{n}) \cdot w \cdot [f_e])). \quad (34)$$

Proof. We will show that given $f \in X$ such that $[f] \in \mathcal{F}$ it follows that

$$[f] \in \bar{\mathcal{F}}_w \text{ if and only if } f \in \mathbf{U}(\mathfrak{n}) \cdot f_w \text{ and } \phi_w(f) \neq 0. \quad (35)$$

Since $[f] \in \mathcal{F}$ we can assume that $f = g \cdot f_e$ for some $g \in G$. First assume $[f] \in \bar{\mathcal{F}}_w$. Thus $g \in N \cdot w \cdot B$ and so $f = c_1 \exp(Y) \cdot w \cdot f_e$ for some $Y \in \mathbf{U}(\mathfrak{n})$, thus $f \in \mathbf{U}(\mathfrak{n}) \cdot f_w$ and $\phi_w(f) \neq 0$.

It is also clear that for each $f \in X$ there is at most one $w \in W$ such that $f \in \mathbf{U}(\mathfrak{n})f_w$ and $\phi_w(f) \neq 0$. (35) follows since $\mathcal{F} = \bigcup_{w \in W} \mathcal{F}_w$.

$\mathbb{P}(\mathbf{U}(\mathfrak{n}) \cdot [f_w])$ is the closure of $\mathbf{U}(\mathfrak{n}) \cdot [f_w]$ in \mathcal{F} , immediately implying (34). ■

\mathcal{F}_w has dimension $l(w)$. We can see this since the annihilator of $w \cdot [f_e] = [f_w]$ in N corresponds to the root spaces in \mathfrak{n} conjugate to positive root spaces under w . This is the case since under this conjugation $[f_w]$ is a highest weight space. Thus \mathcal{F}_w is parametrised by $w \cdot \mathfrak{n} \cap s \cdot \mathfrak{n}$ (where s is the bottom of the Bruhat order) and is an affine cell.

For each root $\gamma \in \Delta$ we fix root vectors x_γ and y_γ in \mathfrak{g} such that $[x_\gamma, y_\gamma] = h_\gamma$. Denote by \mathfrak{g}^γ the subalgebra of \mathfrak{g} generated by x_γ, y_γ and h_γ . \mathfrak{g}^γ is isomorphic to $\mathfrak{sl}_2 \mathbb{C}$, see §2.1. Let $w \xrightarrow{\gamma} w'$ and let \tilde{X} be the smallest \mathfrak{g}^γ -invariant subspace of X containing $f_{w'}$.

Lemma 7.6 Let $n = \langle w'\lambda, \gamma^\vee \rangle \in \mathbb{Z}$, $n > 0$. The elements $\{y_\gamma^i \cdot f_{w'} \mid i = 0, \dots, n\}$ form a basis for \tilde{X} . Put $\tilde{f} = y_\gamma^n \cdot f_{w'}$. Then $y_\gamma \cdot \tilde{f} = 0$, $x_\gamma^n \cdot \tilde{f} = c' f_{w'}$, ($c' \neq 0$) and $f_w = c\tilde{f}$ ($c \neq 0$).

Proof. By Lemma 5.12, $w'^{-1}\gamma \in \Delta^+$, so $x_\gamma \cdot f_{w'} = cx_\gamma \cdot w' \cdot f_e = cw' \cdot x_{w'^{-1}\gamma} \cdot f_e = 0$, in other words $f_{w'}$ is a highest weight with respect to \mathfrak{g}^\vee . It then follows from the theory of representations of $\mathfrak{sl}_2\mathbb{C}$, see §2.1, that the $\{y_\gamma^i \cdot f_{w'}\}_{i=0, \dots, n}$ form a basis for \tilde{X} and that

$$x_\gamma^n \cdot \tilde{f} = x_\gamma^n \cdot y_\gamma^n \cdot f_{w'} = c' f_{w'}.$$

Since $w = \sigma_\gamma w'$, we have $w\lambda = \sigma_\gamma(w'\lambda) = w'\lambda - w'\lambda(h_\gamma)\gamma$ and thus $y_\gamma^n \cdot f_{w'}$ has weight $w\lambda$. Since the weight space $w\lambda$ has multiplicity one, it follows that $y_\gamma^n \cdot f_{w'} = \tilde{f} = cf_w$ for some $c \neq 0$. ■

Theorem 7.7 Let X be a finite dimensional representation of \mathfrak{g} with highest weight λ . Assume that the weights $w\lambda$ for $w \in \mathbb{W}$ are distinct and select for each w a non-zero $f_w \in X$ with weight $w\lambda$. Then

$$w' \geq w \text{ if and only if } f_{w'} \in \mathbf{U}(\mathfrak{n})f_w \quad (36)$$

Proof. Our strategy is to introduce a partial ordering on \mathbb{W} :

$$w \dashv w' \text{ if and only if } f_{w'} \in \mathbf{U}(\mathfrak{n})f_w.$$

This is indeed a partial ordering since the weights $w\lambda$ are distinct. We now show that it coincides with the Bruhat order using Proposition 5.18. Thus the ordering must satisfy P1 and P2,

P1. Let $\alpha \in \Pi$ and $l(\sigma_\alpha w) = l(w) - 1$. Then $w \xrightarrow{\alpha} \sigma_\alpha w$, and by Lemma 7.6, $f_{\sigma_\alpha w} \in \mathbf{U}(\mathfrak{n})f_w$, that is, $w \dashv \sigma_\alpha w$.

P2. Let $w \dashv w'$. We choose an $\alpha \in \Pi$ such that $w \xrightarrow{\alpha} \sigma_\alpha w$. We choose w' so that $\sigma_\alpha w' \rightarrow w'$. We will prove that $\sigma_\alpha w \dashv w'$, that is $f_{w'} \in \mathbf{U}(\mathfrak{n})f_{\sigma_\alpha w}$. This is equivalent to showing $f_{\sigma_\alpha w} \in \mathbf{U}(\mathfrak{n}^-)f_{w'}$. By Lemma 7.6 we have $x_\alpha \cdot f_{w'} = 0$ and $x_\alpha^n \cdot f_w = cf_{\sigma_\alpha w}$ where $n = \langle \sigma_\alpha w \cdot \lambda, \alpha^\vee \rangle$. By assumption $f_{w'} \in \mathbf{U}(\mathfrak{n})f_w$ and thus $f_w \in \mathbf{U}(\mathfrak{n}^-)f_{w'}$. Letting $\mathfrak{p}^{-\alpha}$ denote the subalgebra of \mathfrak{g} generated by \mathfrak{h} , \mathfrak{n}^- and \mathfrak{g}^α it follows that

$$f_{\sigma_\alpha w} = cx_\alpha^n \cdot f_w = X \cdot f_{w'} \text{ for some } X \text{ in } \mathbf{U}(\mathfrak{p}^{-\alpha}).$$

X can be written in the form

$$X = \sum_{i=1}^l Y_i Y'_i + \tilde{Y} x_\alpha$$

for $Y_i \in \mathbf{U}(\mathfrak{n}^-)$, $Y'_i \in \mathbf{U}(\mathfrak{h})$ and $\tilde{Y} \in \mathbf{U}(\mathfrak{g}^\alpha)$. Thus

$$f_{\sigma_\alpha w} = \sum Y_i Y'_i \cdot f_{w'} = \sum c_i Y_i \cdot f_{w'} \in \mathbf{U}(\mathfrak{n}^-)f_{w'}. \blacksquare$$

Finally we reach the theorem linking the topology of the flag variety with the Bruhat order. We find that the Bruhat order is a schematic of the Bruhat decomposition.

Theorem 7.8 Let $w \in \mathbb{W}$, $\mathcal{F}_w \subset \mathcal{F}$ a Bruhat cell, and $\bar{\mathcal{F}}_w$ its closure. Then

$$\mathcal{F}_{w'} \subset \bar{\mathcal{F}}_w \text{ if and only if } w \leq w'.$$

Proof.

• Let $\mathcal{F}_{w'} \subset \bar{\mathcal{F}}_w$. Then $[f_{w'}] \in \bar{\mathcal{F}}_w$, and by Lemma 7.5, $f_{w'} \in U(\mathfrak{n})f_w$. So by Theorem 7.7 $w \leq w'$.

• For the converse consider the case $w \xrightarrow{\gamma} w'$. Let $n = \langle w'\lambda, \gamma^\vee \rangle \in \mathbb{Z}$. By Lemma 7.6 it follows that $n > 0$, $x_\gamma^n \cdot f_w = cf_{w'}$ and $x_\gamma^{n+1} \cdot f_w = 0$. Thus

$$\lim_{t \rightarrow \infty} t^{-n} \exp(tx_\gamma) f_w = \frac{c}{n!} f_{w'},$$

so that $[f_{w'}] \in \bar{\mathcal{F}}_w$ or in other words $\mathcal{F}_{w'} \subset \bar{\mathcal{F}}_w$. ■

7.3 The ring of polynomials on \mathfrak{h}

In this section we study the rings R and \bar{R} constructed in Definition 7.2 following [30]. See also [12], [13] and [25]. Since we know by Borel's Theorem that \bar{R} and $H^*(\mathcal{F}, \mathbb{Q})$ are isomorphic it is interesting to try and find an algebraic construction of the cohomology classes of the Schubert varieties. This is the goal of the rest of this section.

Definition 7.9 Let $\gamma \in \Delta$. Let $\Delta_\gamma : R \rightarrow R$ be the operator defined by

$$\Delta_\gamma f = \frac{f - \sigma_\gamma f}{\gamma}.$$

This maps R to itself since $f - \sigma_\gamma f = 0$ on the hyperplane $\gamma = 0$ in $\mathfrak{h}_\mathbb{Q}$ – in other words γ divides into the numerator.

It is clear that the operator Δ_γ reduce the grading by one.

Lemma 7.10 Let $\gamma \in \Delta$, f and g in R . Then

- (a) $\Delta_{-\gamma} = -\Delta_\gamma$, $\Delta_\gamma^2 = 0$.
- (b) $w \cdot \Delta_\gamma \cdot w^{-1} = \Delta_{w\gamma}$.
- (c) $\sigma_\gamma \cdot \Delta_\gamma = -\Delta_\gamma \cdot \sigma_\gamma = \Delta_\gamma$. $\sigma_\gamma = -\gamma \Delta_\gamma + 1 = \Delta_\gamma \gamma - 1$.
- (d) $\Delta_\gamma f = 0$ if and only if $\sigma_\gamma \cdot f = f$.
- (e) $\Delta_\gamma J \subset J$.
- (f) Let $\chi \in \mathfrak{h}_\mathbb{Q}^*$. Then the commutator of Δ_γ with the operator of multiplication by χ has the form $[\Delta_\gamma, \chi] = \chi(h_\gamma) \sigma_\gamma$.

Proof.

(a) through (d) follows immediately from the definition of Δ_γ .

To prove (e), let $f = f_1 f_2$, where $f_1 \in I^+$ and $f_2 \in R$. Then $\Delta_\gamma f = f_1 \Delta_\gamma f_2 \in J$.

For (f), since $\sigma_\gamma \chi = \chi - \langle \chi, \gamma^\vee \rangle \gamma = \chi - \chi(h_\gamma) \gamma$ it follows that

$$[\Delta_\gamma, \chi] = \Delta_\gamma(\chi f) - \chi \Delta_\gamma(f) = \frac{1}{\gamma} (\chi f - \sigma_\gamma \chi \cdot \sigma_\gamma f - \chi f + \chi \sigma_\gamma f) = \frac{\chi - \sigma_\gamma \chi}{\gamma} \cdot \sigma_\gamma f = \chi(h_\gamma) \cdot \sigma_\gamma f \quad \blacksquare$$

We wish to show that given $w \in W$ with reduced decomposition $\sigma_{\alpha_1} \dots \sigma_{\alpha_k}$, we can legitimately define $\Delta_w = \Delta_{\alpha_1} \dots \Delta_{\alpha_k}$. This follows from

Theorem 7.11 Let $\alpha_1, \dots, \alpha_k \in \Pi$ and set $w = \sigma_{\alpha_1} \dots \sigma_{\alpha_k}$. Let $\Delta_{(\alpha_1, \dots, \alpha_k)} = \Delta_{\alpha_1} \dots \Delta_{\alpha_k}$.

(a) If $l(w) < l$ then $\Delta_{(\alpha_1, \dots, \alpha_k)} = 0$.

(b) If $l(w) = l$ then $\Delta_{(\alpha_1, \dots, \alpha_k)}$ depends only on w and we define this operator to be Δ_w .

Proof. We use induction on k ; the case $k = 1$ is trivial.

(a). Assume by inductive hypothesis that $l(\sigma_{\alpha_1} \dots \sigma_{\alpha_{k-1}}) = k-1$, so that $l(\sigma_{\alpha_1} \dots \sigma_{\alpha_{k-1}} \sigma_{\alpha_k}) = l-2$. Then $\sigma_{\alpha_i} \sigma_{\alpha_{i+1}} \dots \sigma_{\alpha_{k-1}} = \sigma_{\alpha_{i+1}} \dots \sigma_{\alpha_{k-1}} \sigma_{\alpha_k}$ for some i by Lemma 5.13. Since $k-i < k$, the inductive hypothesis implies $\Delta_{\alpha_i} \Delta_{\alpha_{i+1}} \dots \Delta_{\alpha_{k-1}} = \Delta_{\alpha_{i+1}} \dots \Delta_{\alpha_{k-1}} \Delta_{\alpha_k}$ and so by Lemma 7.10(a) we have

$$\Delta_{\alpha_i} \dots \Delta_{\alpha_k} = \Delta_{\alpha_{i+1}} \dots \Delta_{\alpha_k} \Delta_{\alpha_k} = 0.$$

To prove (b), we introduce the operators

$$\Gamma_{(\alpha_1, \dots, \alpha_k)} = \sigma_{\alpha_k} \dots \sigma_{\alpha_1} \Delta_{(\alpha_1, \dots, \alpha_k)}.$$

We then have the following

Lemma 7.12 See [30]. Let $\chi \in \mathfrak{h}_{\mathbb{Q}}^*$. The commutator of $\Gamma_{(\alpha_1, \dots, \alpha_k)}$ with the operator of multiplication by χ is given by the formula

$$[\Gamma_{(\alpha_1, \dots, \alpha_k)}, \chi] = \sum_{i=1}^k \chi(w_{i+1} h_{\alpha_i}) w_{i+1} w_i^{-1} \Gamma_{(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_k)}. \quad (37)$$

where $w_i = \sigma_{\alpha_k} \dots \sigma_{\alpha_i}$. ■

This is a messy calculation which we avoid.

If $l(\sigma_{\alpha_1} \dots \hat{\sigma}_{\alpha_i} \dots \sigma_{\alpha_k}) < k-1$ then we are in case (a). So assume this has length $k-1$ and let $w' = \sigma_{\alpha_1} \dots \hat{\sigma}_{\alpha_i} \dots \sigma_{\alpha_k}$ and $\gamma = \sigma_{\alpha_1} \dots \sigma_{\alpha_{i-1}}(\alpha_i)$. By Lemma 5.12 it follows that $w \xrightarrow{\gamma} w'$ and

$$\chi(w_{i+1} h_{\alpha_i}) = w' \chi(w' w_{i+1} h_{\alpha_i}) = w' \chi(\sigma_{\alpha_1} \dots \sigma_{\alpha_{i-1}} h_{\alpha_i}) = w' \chi(h_{\gamma})$$

and

$$w_{i+1} w_i^{-1} \Gamma_{(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_k)} = w_{i+1} w_i^{-1} w'^{-1} \Delta_{(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_k)} = w^{-1} \Delta_{(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_k)}.$$

Using Proposition 5.19 and the inductive hypothesis, (37) can be rewritten as

$$[\Gamma_{(\alpha_1, \dots, \alpha_k)}, \chi] = \sum_{w \xrightarrow{\gamma} w'} w' \chi(h_{\gamma}) w^{-1} \Delta_{w'}. \quad (38)$$

The right hand side of this equation does not depend on the decomposition of w . The proof of the Theorem thus comes down to the following fact,

if Γ be an operator in \mathcal{R} such that $\Gamma(1) = 0$ and $[\Gamma, \chi] = 0$ for all $\chi \in \mathfrak{h}_{\mathbb{Q}}^*$ then $\Gamma = 0$;

which is clear since Γ commutes with all the non-constant terms and is zero on the constants. Thus our description of the behaviour of $\Gamma_{(\alpha_1, \dots, \alpha_k)}$ characterises it. We can recover Δ_w by the formula $\Delta_w = w \Gamma_w$. ■

Lemma 7.12 can be rewritten using the working of the above theorem as follows,

Corollary 7.13 *The operators Δ_w satisfy the commutator relation*

$$[w^{-1}\Delta_w, \chi] = \sum_{w \xrightarrow{\gamma} w'} w'\chi(h_\gamma)w^{-1}\Delta_{w'}. \quad \blacksquare \quad (39)$$

Definition 7.14 *We define the duals to the above,*

1. Let $S_i = R_i^*$ and $S = \bigoplus S_i$. Denote by $(,)$ the pairing $S \times R \rightarrow \mathbb{Q}$. S inherits a \mathbb{W} -action from R .

2. For any $\chi \in \mathfrak{h}_{\mathbb{Q}}^*$ let χ^* denote the transformation of S adjoint to the operator of multiplication by χ in R . Thus for all $f \in R$ and $D \in S$ χ^* satisfies

$$(D, \chi f) = (\chi^* D, f)$$

3. Similarly denote by $F_\gamma : S \rightarrow S$ the linear transformation adjoint to $\Delta_\gamma : R \rightarrow R$.

S is equivalent to the algebra of differential operators on \mathfrak{h} with constant coefficients. Under this equivalence, the pairing $(,)$ is given by $(D, f) = (Df)(0)$. Under this correspondence we also have $\chi^*(D) = [D, \chi]$ as operators on R . This follows from the equations

$$\begin{aligned} (\chi^* D, f) &= (D, \chi f) = \chi(Df)(0) + f(0) = f(0) \\ [D, \chi] f &= (D\chi)f(0) - \chi(Df)(0) = f(0). \end{aligned}$$

We prove some basic results on F_γ .

Lemma 7.15 *Let $\gamma \in \Delta$. For any $D \in S$ there is a $\tilde{D} \in S$ such that $\gamma^*(\tilde{D}) = D$. If \tilde{D} is any such operator, then $\tilde{D} - \sigma_\gamma \tilde{D} = F_\gamma(D)$. So the left hand side of the equation does not depend on the choice of \tilde{D} .*

Proof. The existence of \tilde{D} follows from the fact multiplication by γ is a monomorphism in R , implying γ^* is surjective. Also, for any $f \in R$ we have

$$(\tilde{D} - \sigma_\gamma \tilde{D}, f) = (\tilde{D}, f - \sigma_\gamma f) = (\tilde{D}, \Delta_\gamma f \cdot \gamma) = (\gamma^*(\tilde{D}), \Delta_\gamma f) = (D, \Delta_\gamma f),$$

thus $\tilde{D} - \sigma_\gamma \tilde{D} = F_\gamma$. \blacksquare

Theorem 7.16 *Let $\alpha_1, \dots, \alpha_k \in \Pi$, $w = \sigma_{\alpha_1} \dots \sigma_{\alpha_k}$.*

(a) *If $l(w) < k$ then $F_{\alpha_k} \dots F_{\alpha_1} = 0$.*

(b) *If $l(w) = k$ then $F_{\alpha_k} \dots F_{\alpha_1}$ depends only on w ; denote the transformation by $F_w = \Delta_w^*$.*

(c)

$$[\chi^*, F_w w] = \sum_{w \xrightarrow{\gamma} w'} w'\chi(h_\gamma)F_{w'} w. \quad (40)$$

Proof. This is a direct consequence of Theorem 7.11 and Corollary 7.13. \blacksquare

Definition 7.17 *Let $D_w = F_w(1)$.*

Theorem 7.18 *The following hold,*

(a) $D_w \in S_{l(w)}$.

(b) Let $w \in \mathbb{W}$, $\alpha \in \Pi$. Then

$$F_\alpha D_w = \begin{cases} 0 & \text{if } l(w\sigma_\alpha) = l(w) - 1, \\ D_{w\sigma_\alpha} & \text{if } l(w\sigma_\alpha) = l(w) + 1. \end{cases}$$

(c) Let $\chi \in \mathfrak{h}_{\mathbb{Q}}^*$. Then

$$\chi^*(D_w) = \sum_{w \xrightarrow{\gamma} w'} w' \chi(h_\gamma) D_{w'}.$$

(d) Let $\alpha \in \Pi$. Then

$$\sigma_\alpha D_w = \begin{cases} -D_w & \text{if } l(w\sigma_\alpha) = l(w) - 1, \\ -D_w + \sum_{w\sigma_\alpha \xrightarrow{\gamma} w'} w' \alpha(h_\gamma) D_{w'} & \text{if } l(w\sigma_\alpha) = l(w) + 1. \end{cases}$$

(e) Let $w \in \mathbb{W}$, $l(w) = k$, $\chi_1, \dots, \chi_k \in \mathfrak{h}_{\mathbb{Q}}^*$. Then $(D_w, \chi_1 \dots \chi_k) = \sum \chi_1(h_\gamma) \dots \chi_k(h_\gamma)$, where the sum is over all chains

$$w^{-1} = w_k \xrightarrow{\gamma_k} \dots \rightarrow w_2 \xrightarrow{\gamma_2} w_1 \xrightarrow{\gamma_1} e.$$

Proof.

(a) Δ_w drops the grading by $l(w)$ and so its adjoint will increase the grading in S .

(b) follow from Theorem 7.16.

(c) $\chi^*(D_w) = \chi^* F_w w(1) = [\chi^*, F_w w](1)$ since $\chi^*(1) = 0$ and so (c) follows from (40).

(d) By Lemma 7.10(c) we have $\sigma_\alpha = \alpha^* F_\alpha - 1$. Thus (d) is an immediate consequence of (b) and (c).

(e) Define $\tilde{D}_w = D_{w^{-1}}$. Then these satisfy the relation

$$\chi^*(\tilde{D}_w) = \sum_{w^{-1} \xrightarrow{\gamma} w'^{-1}} w'^{-1} \chi(h_\gamma) D_{w'^{-1}} = \sum_{w \xrightarrow{\gamma} w'} \chi(h_\gamma) \tilde{D}_{w'}. \quad (41)$$

Now since $(D, \chi f) = (\chi^*(D), f)$, (e) is a consequence of (41) applied inductively on k . ■

Definition 7.19 Let \mathcal{H} be the subspace of S orthogonal to $J \subset R$ under $(,)$.

By Lemma 7.10(f), \mathcal{H} is invariant under the F_γ 's. We have $1 \in \mathcal{H}$ and so $D_w \in \mathcal{H}$ for all $w \in \mathbb{W}$.

Theorem 7.20 $\{D_w \mid w \in \mathbb{W}\}$ is a basis for \mathcal{H} .

Proof. *linear independence:* Let $s \in \mathbb{W}$ be the element of maximal length and $r = l(s)$. By Theorem 7.18(e), $D_s(\rho^r) > 0$ so that $D_s \neq 0$. Now let $\sum c_w D_w = 0$ with \tilde{w} one of the elements of maximal length such that $c_{\tilde{w}} \neq 0$. Set $k = l(\tilde{w})$. There is a sequence $\alpha_1, \dots, \alpha_{r-k}$ satisfying $\tilde{w}\sigma_{\alpha_1} \dots \sigma_{\alpha_{r-k}} = s$. Let $F = F_{\alpha_{r-k}} \dots F_{\alpha_1}$. Then by Theorem 7.16 $FD_{\tilde{w}} = D_s$ and $FD_w = 0$ if $l(w) \geq k$, $w \neq \tilde{w}$. Thus

$$F \left(\sum c_w D_w \right) = c_{\tilde{w}} D_s + \text{terms of lower degree} \neq 0.$$

spanning: It suffices to show $f \in R$ and $(D_w, f) = 0$ for all $w \in \mathbb{W}$ imply $f \in J$. Assume f is a homogeneous element of degree k . This is clear for $k = 0$.

Let $k > 0$; we use induction assuming the result for all homogeneous polynomials of degree $< k$. Thus given any $\alpha \in \Pi$ and $w \in \mathbb{W}$ we have by assumption $(F_\alpha D_w, f) = 0$. But $(F_\alpha D_w, f) = (D_w, \Delta_\alpha f)$ and so by the inductive hypothesis $\Delta_\alpha f \in J$, so $\alpha \Delta_\alpha f = f - \sigma_\alpha f \in J$. Therefore for all $w \in \mathbb{W}$, $f \equiv wf \pmod{J}$ and so averaging over the Weyl group we obtain

$$|\mathbb{W}|^{-1} \sum_{w \in \mathbb{W}} wf \equiv f \pmod{J}.$$

The left hand side belongs to l^+ so $f \in J$. ■

7.4 Schubert varieties

Finally we prove that the functionals D_w defined in the previous section correspond to the cohomology classes of the Schubert varieties $\bar{\mathcal{F}}_w$. Given a Bruhat cell \mathcal{F}_w with closure $\bar{\mathcal{F}}_w$ let

$$[\bar{\mathcal{F}}_w] \in H_{2l(w)}(\bar{\mathcal{F}}_w, \mathbb{Z})$$

be the fundamental class of the algebraic variety $\bar{\mathcal{F}}_w$, and let

$$x_w = i_*([\bar{\mathcal{F}}_w]) \in H_{2l(w)}(\mathcal{F}, \mathbb{Z})$$

be the image under the map induced by the embedding $\bar{\mathcal{F}}_w \hookrightarrow \mathcal{F}$. We know, see [15], that given a non-singular projective variety X with filtration $X = X_s \supset X_{s-1} \supset \dots \supset X_0 = \emptyset$ by closed algebraic subsets with $X_i \setminus X_{i-1}$ a disjoint union of irreducible varieties $U_{i,j}$ each isomorphic to an affine space $\mathbb{C}^{n(i,j)}$ then the cohomology classes $[U_{i,j}]$ of the closures provide an additive basis for $H^*(X, \mathbb{Z})$ over the integers. Thus the Schubert varieties form an additive basis for $H(\mathcal{F}, \mathbb{Z})$.

Each Schubert variety gives rise to a linear functional \hat{D}_w on R as follows,

$$\hat{D}_w(f) = \langle x_w, \alpha(f) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the usual pairing between homology and cohomology, and $\alpha : R \rightarrow H^*(\mathcal{F}, \mathbb{Q})$ is the map from Borel's Theorem.

Theorem 7.21 $\hat{D}_w = D_w$.

This is our main theorem. It shows that we have found a natural algebraic construction in R for the cohomology classes of the Schubert varieties. The theorem is a consequence of the proposition:

Proposition 7.22 (a) $\hat{D}_e = 1$ and for any $\chi \in h_{\mathbb{Z}}^*$

$$\chi^*(\hat{D}_w) = \sum_{w \xrightarrow{\gamma} w'} w' \chi(h_\gamma) \hat{D}_{w'}. \quad (42)$$

(b) Suppose that for each $w \in \mathbb{W}$ we are given an element $\hat{D}_w \in S_{l(w)}$ with $\hat{D}_e = 1$ and such that (42) holds for all $\chi \in h_{\mathbb{Z}}^*$. Then $\hat{D}_w = D_w$.

Proof. The second part follows immediately by induction on $l(w)$ along with Theorem 7.18(c). For the first part we need to look more closely at the geometric structure.

Given any topological space Y there is a bilinear mapping known as the cap-product,

$$H^i(Y, \mathbb{Q}) \times H_j(Y, \mathbb{Q}) \xrightarrow{\cap} H_{j-i}(Y, \mathbb{Q})$$

This has the properties,

1. For all $y \in H_j(Y, \mathbb{Q})$, $z \in H^{j-i}(Y, \mathbb{Q})$ and $c \in H^i(Y, \mathbb{Q})$,

$$\langle c \cap y, z \rangle = \langle y, c \cup z \rangle. \quad (43)$$

2. Let $f: Y_1 \rightarrow Y_2$ be a continuous mapping. Then for all $y \in H_j(Y_1, \mathbb{Q})$ and $c \in H^i(Y_2, \mathbb{Q})$,

$$f_*(f^*c \cap y) = c \cap f_*y. \quad (44)$$

By (44) for any $\chi \in \mathfrak{h}_{\mathbb{Z}}^*$, $f \in R$

$$\langle \chi^*(\hat{D}_w), f \rangle = \langle \hat{D}_w, \chi f \rangle = \langle x_w, \alpha_1(\chi) \cdot \alpha_2(f) \rangle = \langle \alpha_1(\chi) \cap x_w, \alpha(f) \rangle.$$

Thus (42) is equivalent to showing that for all $\chi \in \mathfrak{h}_{\mathbb{Z}}^*$,

$$\alpha_1(\chi) \cap x_w = \sum_{w \xrightarrow{Y} w'} w' \chi(h_Y) x_{w'}. \quad (45)$$

We restrict the line bundle $\mathcal{L}(\chi)$ to $\bar{\mathcal{F}}_w \subset \mathcal{F}$ and let $c_\chi \in H^2(\bar{\mathcal{F}}_w, \mathbb{Q})$ be the first Chern class of $\mathcal{L}(\chi)$. By (44) and the definition of $\alpha: \mathfrak{h}_{\mathbb{Z}}^* \rightarrow H^2(\mathcal{F}, \mathbb{Q})$ it is sufficient to prove that

$$c_\chi \cap x_w = \sum_{w \xrightarrow{Y} w'} w' \chi(h_Y) x_{w'}. \quad (46)$$

in $H_{2l(w)-2}(\bar{\mathcal{F}}_w, \mathbb{Q})$. To do this we use the following lemma

Lemma 7.23 *Let Y be a compact analytic space of dimension n such that the codimension of the space of singularities of Y is greater than one. Let \mathcal{L} be a line bundle on Y with first Chern class $c \in H^2(Y, \mathbb{Q})$. Let μ be a non-zero analytic section of \mathcal{L} and $\sum m_i Y_i = \text{div } \mu$ the divisor of μ . Then $c \cap [Y] = \sum m_i [Y_i] \in H_{2n-2}(Y, \mathbb{Q})$, where $[Y]$ and $[Y_i]$ are the fundamental classes of Y and Y_i .*

Proof. If Y is smooth then by Poincaré duality the map $_ \cap [Y]: H^i(Y, \mathbb{Q}) \rightarrow H_{2n-i}(Y, \mathbb{Q})$ is an isomorphism taking cohomology classes to homology classes. Thus by property C6, we have

$$c \cap [Y] = \sum m_i (Y_i) \cap [Y] = \sum m_i [Y_i].$$

In the case at hand where the space of singularities has codimension greater than one, we use relative Poincaré duality, see [40], excising the subvariety containing the singularities. ■

Proposition 7.22 follows from the lemma and the next two propositions. The first shows that the space of singularities has codimension two or more, since the dimension of \mathcal{F}_w is $l(w)$. The second calculates the divisor of a section. ■

Proposition 7.24 *Let $w \xrightarrow{Y} w'$. Then $\bar{\mathcal{F}}_{w'}$ is non-singular at points $x \in \mathcal{F}_{w'}$.*

Proposition 7.25 *There is a section μ of the vector bundle $\mathcal{L}(\chi)$ over $\tilde{\mathcal{F}}_w$ such that*

$$\operatorname{div} \mu = \sum_{w \xrightarrow{\gamma} w'} w' \chi(h_\gamma) \tilde{\mathcal{F}}_{w'}.$$

To prove these propositions we use the representation theoretic description of the Bruhat cells given in the last section. So as before we let X be a finite dimensional representation of G with regular highest weight λ .

Proof of 7.24. Given a root $\gamma \in \Delta^+$ let \mathfrak{g}^γ denote the subalgebra of \mathfrak{g} generated by h_γ , x_γ and y_γ . Then set $\iota : \mathrm{SL}_2\mathbb{C} \rightarrow G$ as the homomorphism corresponding to the embedding $\mathfrak{g}^\gamma \rightarrow \mathfrak{g}$. Within $\mathrm{SL}_2\mathbb{C}$ consider the subgroups

$$B' = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}, \quad H' = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}, \quad N'^- = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right\}$$

and the element $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We can assume $\iota(H') \subset H$ and $\iota(B') \subset B$. Let \tilde{X} be the smallest \mathfrak{g}^γ -invariant subspace of X containing $f_{w'}$. By definition \tilde{X} is invariant under $\iota(\mathrm{SL}_2\mathbb{C})$, and the stabiliser of the line $[f_{w'}]$ is B' . This then gives a mapping $\mathrm{SL}_2\mathbb{C}/B' \xrightarrow{\delta} \mathcal{F}$. $\mathrm{SL}_2\mathbb{C}/B' \approx \mathbb{P}^1$. Let o and $\infty \in \mathbb{P}^1$ be the images of e and σ under ι . Define a mapping

$$\xi : N_{w'} \times \mathbb{P}^1 \rightarrow \mathcal{F} : (n, z) \mapsto n \cdot \delta(z).$$

Lemma 7.26 ξ has the following properties,

- (a) $\xi(N_{w'} \times \{o\}) = \mathcal{F}_{w'}$, $\xi(N_{w'} \times (\mathbb{P}^1 - \{o\})) \subset \mathcal{F}_w$.
- (b) The restriction of ξ to $N_{w'} \times (\mathbb{P}^1 - \{o\})$ is an isomorphism onto an open subset of $\tilde{\mathcal{F}}_{w'}$.

Proof. The first part of (a) follows immediately from the isomorphism (33). Since \mathcal{F}_w is invariant under N , the second part reduces to showing $\delta(z) \in \mathcal{F}_w$ for all $z \in \mathbb{P}^1 - \{o\}$. Let $h \in \mathrm{SL}_2\mathbb{C}$ be the inverse image of z . Then h can be written $b_1\sigma b_2$ with $b_1, b_2 \in B'$. Clearly $\iota(b_2)f_{w'} = c_1 f_{w'}$ and $\iota(\sigma)f_{w'} = c_2 f_w$ where c_1 and c_2 are constant. So $\iota(h)f_{w'} = c_1 c_2 \iota(b_1)f_w$ so that $\delta(z) \in \mathcal{F}_w$ as required.

(b) $\mathbb{P}^1 - \{o\}$ is naturally isomorphic to $N'^- \subset \mathrm{SL}_2\mathbb{C}$. Consider the map

$$w^{-1} \circ \xi : N_{w'} \times (\mathbb{P}^1 - \{o\}) \rightarrow \mathcal{F}.$$

$\xi : N_{w'} \times N'^- \rightarrow \mathcal{F}$ can be explicitly written as

$$\xi(n, n_1) = n \cdot \iota(n_1)[f_{w'}], \text{ for } n \in N_{w'} \text{ and } n_1 \in N'^-. \quad (47)$$

$$\text{Thus } w'^{-1} \circ \xi(n, n_1) = (w'^{-1} n w') (w'^{-1} \cdot \iota(n_1) w') [f_e]. \quad (48)$$

By definition of $N_{w'}$, $w'^{-1} N_{w'} w' \subset N^-$ and since $w'^{-1} \gamma \in \Delta^+$ it follows that $w'^{-1} \iota(N'^-) w' \subset N^-$. In addition the intersection of the tangent spaces of these two subgroups is zero since $N_{w'} \subset N$ and $\iota(N'^-) \subset N^-$. The mapping $N^- \rightarrow \mathcal{F} : n \mapsto n[f_e]$ is an isomorphism onto an open subset of \mathcal{F} . Now we use the lemma,

Lemma 7.27 [4]. *Let N_1 and N_2 be two closed algebraic subgroups of a unipotent group N whose tangent spaces at the identity have zero intersection. Then the product mapping $N_1 \times N_2 \rightarrow N$ gives an isomorphism of $N_1 \times N_2$ with a closed subvariety of N . ■*

We know the tangent spaces of $N_{w'}$ and $\iota(N'^{-})$ have zero intersection at the identity. (48) shows that we have a product embedding into N^- and so by Lemma 7.27 we are done. ■

So the image under ξ of $N_{w'} \times \{o\}$ is $\mathcal{F}_{w'}$ and also $N_{w'} \times (\mathbb{P}^1 - \{\infty\}) \supset N_{w'} \times \{o\}$ is mapped isomorphically onto an open subset of \mathcal{F}_w . Thus the points of $\mathcal{F}_{w'}$ are non-singular in $\bar{\mathcal{F}}_w$ and Proposition 7.24 is proven. ■

Proof of 7.25. Any element of $\mathfrak{h}_{\mathbb{Z}}^*$ can be written $\chi = \lambda - \lambda'$ where λ and λ' are regular highest weights. So $\mathcal{L}^\chi = \mathcal{L}^\lambda \otimes \mathcal{L}^{\lambda'^{-1}}$ and it is sufficient to find a section μ with the required properties for the special case $\chi = \lambda$.

Let \mathcal{E}_X be the vector bundle on $\mathbb{P}(X)$ consisting of pairs (l, ϕ) where ϕ is a linear functional on the line $l \subset X$. Then, see Definition 4.7, $\mathcal{L}^\lambda = i^*(\mathcal{E}_X)$ where i is the embedding $\mathcal{F} \hookrightarrow \mathbb{P}(X)$. In Definition 4.7 we used the dual projective space, and the ordinary canonical bundle, whereas here we use normal projective space and the dual of the canonical bundle.

The linear functional ϕ_w , Definition 7.4 on X gives a section of the bundle \mathcal{E} . We will prove that the restriction μ of to this section to $\bar{\mathcal{F}}_w$ is a section of the line bundle \mathcal{L}^λ with the required properties.

By Lemma 7.5 $\mu(x) \neq 0$ for all $x \in \mathcal{F}_w$. Thus the support of the divisor $\text{div } \mu$ is contained in

$$\bar{\mathcal{F}}_w - \mathcal{F}_w = \bigcup_{w \xrightarrow{\gamma} w'} \bar{\mathcal{F}}_{w'}.$$

Since $\bar{\mathcal{F}}_{w'}$ is an irreducible variety we have $\text{div } \mu = \sum_{w \xrightarrow{\gamma} w'} a_\gamma \bar{\mathcal{F}}_{w'}$, where $a_\gamma \in \mathbb{Z}_0^+$. The final step is to show $a_\gamma = w'\chi(h_\gamma)$.

As a result of Lemma 7.26 the coefficient a_γ is equal to the multiplicity of zero of the section $\delta^*(\mu)$ of the line bundle $\delta^*(\mathcal{L}^\lambda)$ on \mathbb{P}^1 at the point o - that is the multiplicity of zero of the function

$$\psi(t) = \phi_{w'}((\exp t\gamma)f_{w'}) \text{ for } t = 0.$$

By Lemma 7.6 $\psi(t) = ct^n$ so that $a_\gamma = n = w'\chi(h_\gamma)$. ■

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