E-Compactness in Pointfree Topology

Nizar Marcus

A thesis prepared under the supervision
of Professor C.R.A. Gilmour for the degree of
Doctor of Philosophy in Mathematics

The copyright of this thesis vests in the author. No quotation from it or information derived from it is to be published without full acknowledgement of the source. The thesis is to be used for private study or non-commercial research purposes only.

Published by the University of Cape Town (UCT) in terms of the non-exclusive license granted to UCT by the author.

University of Cape Town

Acknowledgements

I would like to thank my supervisor, Professor C.R.A. Gilmour for all his help and encouragement he has given me during the preparation of this thesis, and for the guidance and support he has given me throughout the major part of my university studies.

I am especially grateful to Professor B. Banaschewski for many fruitful discussions. He is an inspiring example to me and is always willing to offer his assistance.

I thank Professor G.C.L. Brümmer and the members of his Topology and Category Theory Research Group for all their assistance and the use of their facilities.

I acknowledge financial support from the Deutscher Akademischer Austauschdienst for 1995-1996 and from the Foundation for Research and Development for 1997-1998. I further thank the University of the Western Cape for awarding me the Mellon University Fellowship (funded by the Andrew Mellon Foundation) in the form of lecture relief for the first semester of 1997.

I am also indebted to Professor A.W. Hager for helping me maintain momentum whilst visiting Wesleyan University. I have learnt much from the brief time I visited there.

Finally I wish to thank my wife Yumnah for all her support and encouragement.

Contents

Index of selected definitions

Selected categories

Selected functors

Selected reflection and coreflection maps

Chapter 1

Introduction

1.1 Historical introduction

Frame (locale) theory is concerned with the study of topological spaces from a lattice-theoretic point of view. The earliest work of this nature dates back to the late 30's in the form of two classical papers by Stone [85] and Wallman [89]. An account of the influence of Stone's work on various branches of mathematics is given in Johnstone's book [54]. A brief history of the development of point-free topology appears in the notes of Chapter 2 of the same book, and recent developments are discussed by Banaschewski in [10]. See also [55] for a motivation for this point-free perspective.

Our thesis is primarily concerned with a frame-theoretic version of E-compact spaces. E-regular and E-compact spaces were first introduced by Engelking and Mrówka [36] as a common generalisation of compact and realcompact spaces. The more general theory of $\mathcal{E}$-regular and $\mathcal{E}$-compact spaces was developed by Herrlich [44] - originally unaware of the work of Engelking and Mrówka (See [46].) - in order to obtain a common generalisation of compactness, realcompactness and κ-compactness. The κ-compact spaces were introduced by Herrlich [43] to measure the 'degree of compactness' of Hausdorff spaces. These classes of $\mathcal{E}$-regular and $\mathcal{E}$-compact spaces occupy a special place in topology since they are exactly the full isomorphism-closed epireflective subcategories of **Top** and **Haus** respectively.

The first notion of frame-theoretic realcompactness is due to Madden and Vermeer [60]. The authors showed that those frames that are closed quotients of a copower of the open set lattice of the reals are precisely the regular Lindelöf frames. This, together with the fact that the spectra of such frames are realcompact spaces motivated the authors to call such frames 'realcompact frames'. At this stage the prevailing point of view was that "regular σ-frames are the correct lattice-theoretic setting for the topological notion of realcompactness" [15]. Schlitt [77] later on provided a conservative notion of realcompactness of a frame L in terms of the frame of completely regular ideals on L. This definition, and the construction of the point-free version of the Hewitt-realcompactification obtained by Schlitt was expressed in terms of ideals on the σ-frame $CozL$ of all cozero elements of L [62]. It was with the recent paper by Banaschewski and Gilmour [17] that the precise role that regular σ-frames play in connection with realcompactness had become evident. The authors showed that a frame L is realcompact iff every maximal σ-ideal $I \in CozL$ is principal, illustrating the distinction with Lindelöf frames: A completely regular frame is Lindelöf iff every σ-ideal $I \in CozL$ is contained in a principal ideal. In the same paper, Banaschewski and Gilmour showed that the regular Lindelöf frames are those frames that are complete with respect to their real uniformity, and the realcompact frames are the frames that are Cauchy complete with respect to the same uniformity.

The study of $\mathbb{N}$-compact frames has followed roughly the same course as that of realcompactness. Paseka [71] defined a frame L to be $\mathbb{N}$-compact if L is a closed quotient of a copower of the frame $\mathcal{O}\mathbb{N}$. He then showed that the $\mathbb{N}$-compact frames are precisely the zero dimensional Lindelöf frames. This same result was observed earlier on by Schlitt [77] who furthermore obtained the equivalence of this with the Axiom of Countable Choice [79]. As with realcompactness Schlitt formulated a conservative point-free notion of $\mathbb{N}$-compactness, and in [17] Banaschewski and

Gilmour showed that the zero dimensional Lindelöf frames are the frames that are complete with respect to the uniformity generated by all countable covers of complemented elements, whereas the $\mathbb{N}$-compact frames are the Cauchy complete frames with respect to the same uniformity.

1.2 Synopsis

The main purpose of this thesis is to develop a point-free notion of E-compactness. Our approach follows that of Banascheski and Gilmour in [17]. Any regular frame E has a fine nearness and hence induces a nearness on an E-regular frame L. We show that the frame L is complete with respect this nearness iff L is a closed quotient of a copower of E. This resembles the classical definition, but it is not a conservative definition: There are spaces that may be embedded as closed subspaces of powers of a space E, but their frame of opens are not closed quotients of copowers of the frame of opens of E. A conservative definition of E-compactness is obtained by considering Cauchy completeness with respect to this nearness.

Another central notion in the thesis is that of κ-Lindelöf frames, a generalisation of Lindelöf frames introduced by J.J. Madden [59]. In the last chapter we investigate the interesting relationship between the completely regular κ-Lindelöf frames and the κ-compact frames.

Outline of the thesis

Chapter two deals with κ-Lindelöf frames. In particular we discuss the result of J.J. Madden that this category is a coreflective subcategory of the category of all completely regular frames. We give a slightly different presentation of the coreflection obtained by Madden.

In chapter three we characterise all those quotients of frames that are spatial. In particular, we use this characterisation to obtain alternative proofs of some familiar results. In this chapter we also consider the maximal spectrum of a frame and "T_1" frames. Finally, we discuss the relatively spatial quotients of Banaschewski and Hong [18]. Here an extension (called the relatively spatial hull) of a strict coreflective subcategory of the category of regular frames is obtained, with coreflection maps given by the relatively spatial reflections of the original coreflections.

Chapter four deals with frame-theoretic realcompactness. We review the existing theory of realcompactness in frames and obtain the realcompact coreflection as the relatively spatial quotient with respect to the real points of βL.

In chapter five we consider strongly zero dimensional frames. In particular we characterise these frames as those for which the cozero part is zero dimensional. As with realcompactness, we obtain the $\mathbb{N}$-compact coreflection of a zero dimensional frame as the relatively spatial quotient with respect to the natural points of ζL. We close this chaper with a discussion of $\mathbb{N}$-pseudocompact frames.

In the final chapter we study E-compact frames. We firstly show that the frames that are closed quotients of copowers of a regular frame E are precisely the frames that are complete with respect to the nearness induced by the fine nearness on E. We call such frames E-complete. Spaces that may be embedded as closed subspaces of powers of a space E are called E-compact spaces and were initially introduced as a common generalisation of compact, zero dimensional compact, realcompact and $\mathbb{N}$-compact spaces. (The realcompact spaces are precisely the $\mathbb{R}$-compact spaces.) It has already been established that the frames of open sets of realcompact spaces are not necessarily $\mathcal{O}\mathbb{R}$-complete, and similarly the frames of open sets of $\mathbb{N}$-compact spaces are not necessarily $\mathcal{O}\mathbb{N}$-complete. In fact it was shown by Madden and Vermeer [60] that the $\mathcal{O}\mathbb{R}$-complete frames are exactly the regular Lindelöf frames and later on by Schlitt [77] and Pašeka [71] that the $\mathcal{O}\mathbb{N}$-complete frames are exactly the zero dimensional Lindelöf frames. This chapter deals essentially with more general forms of these two results. It was shown by Hušek [51], [52] and Hong [49] that the categories of k-compact spaces (introduced by Herrlich [43]) and the zero dimensionally k-compact spaces (introduced by Hong [49]) are simple in **Haus**. (A subcategory is called simple if it is the epireflective hull of a single space. In the case of subcategories of **Haus** this is equivalent to each space in the subcategory being E-compact for some space E.) We consider the frame of opens of the spaces of Hušek and Hong (denoted by

H_κ and H_κ^0 respectively) and show that the H_κ-complete frames are exactly the completely regular κ-Lindelöf frames and the H_κ^0-complete frames are precisely the zero dimensional κ-Lindelöf frames.

In exact analogy to [17], we define the E-compact frames as those frames that are Cauchy complete with respect to their E-nearness. Thus κ-compact frames are defined as those frames that are Cauchy complete with respect to their H_κ-nearness, and the zero dimensionally κ-compact frames are those frames that are Cauchy complete with respect to their H_κ^0-nearness. These notions are shown to be conservative, i.e. they are the exact frame-theoretic analogues of classical κ-compactness, zero dimensional κ-compactness and E-compactness.

We close the chapter with a discussion of the frame-theoretic analogue of Herrlich's $\mathcal{E}$-compact spaces.

1.3 Preliminaries

In this section we provide the background material that are necessary in order to read this thesis. Other general introductions to frame theory may be found in [7] (The lecture notes for this series of lectures presented by Professor Banaschewski also appears in a typed yet unpublished manuscript.) as well as [54] and [87]. We have tried as far as possible to be consistent with the notation and nomenclature of [7].

1.3.1 Frames and κ-frames

Let κ be any regular cardinal. By a κ-set we shall mean a set having cardinality strictly less than κ. A frame L (respectively κ-frame) is a bounded lattice (with bottom $\mathbf{0}$ and top $\mathbf{e}$) for which every subset (respectively κ-subset) S has a join $\bigvee_L S$ satisfying the infinite distributive law:

$$a \wedge \bigvee_L S = \bigvee_L \{a \wedge s | s \in S\}$$

where $a \in L$. We shall suppress the subscript if it is obvious from the context. Frame (respectively κ-frame) homomorphisms preserve binary meets and infinite joins (respectively joins of κ-sets) as well as the bottom $\mathbf{0}$ and top $\mathbf{e}$, forming a category denoted by **Frm** (repectively κ**Frm**.)
If $\kappa = \omega_1$, a κ-frame is called a σ-frame. Typical examples of σ-frames are the cozero-set lattices of topological spaces. The topologies of spaces are typical examples of frames. Such frames are called *spatial frames.* The dual of the category of frames is denoted by **Loc**, the objects being called locales.

An element a of a frame or κ-frame L is said to be *rather below* b in L (written $a \prec b$) if there exists a *separating* element $s \in L$ such that $a \wedge s = \mathbf{0}$ and $b \vee s = \mathbf{e}$. A frame L with the property that every $a \in L$ may be written as a join of elements rather below it is called *regular.* A κ-frame L is called regular if every $a \in L$ may be written as a join of a κ-set of elements, each of which is rather below a. The

L, $CozL$ is a sub-σ-frame of L and furthermore $CozL$ join generates L iff L is completely regular. See [16] and [12] for a detailed discussion on the cozero part of a frame.

1.3.2 Frames and topological spaces

The lattice $\mathcal{O}X$ of opens of a topological space X is a frame. Furthermore, for any continuous map $f : X \to Y$ the map $h : \mathcal{O}Y \to \mathcal{O}X$ defined by $h(U) = f^{-1}(U)$ is a frame homomorphism. This gives a contravariant functor $\mathcal{O}$ from the category **Top** of topological spaces to **Frm**.

On the other hand, given a frame L, we denote by ΣL the set of all prime elements of L. This set endowed with the topology consisting of the open sets $\Sigma_a = \{p \in \Sigma L | a \not\leq p\}$ for $a \in L$ is called the *spectrum* of L. We shall not discriminate between the space ΣL and the underlying set. Any frame homomorphism $h : M \to L$ has a right adjoint h_*, where $h_*(y) = \bigvee\{x \in M | h(x) = y\}$. It is easy to see that $h_*(p)$ is prime whenever p is prime. Furthermore, for $a \in L$, $h_*^{-1}(\Sigma_a) = \{p \in \Sigma L | h_*(p) \in \Sigma_a\} = \{p \in \Sigma L | a \not\leq h_*(p)\} = \{p \in \Sigma L | h(a) \not\leq p\} = \Sigma_{h(a)}$. Thus, for any frame homomorphism $h : M \to L$, the map $f : \Sigma L \to \Sigma M$ defined by $f(p) = h_*(p)$ is a continuous function. Hence Σ is a contravariant functor from **Frm** to **Top**. The spectrum ΣL of a frame L is also described as the space of all frame homomorphisms $\xi : L \to 2$ with open sets $\Sigma_a = \{\xi | \xi(a) = 1\}$ for $a \in L$. A third way of describing the spectrum ΣL is as the space of *completely prime filters* $\mathcal{F}$ with open sets of the form $\Sigma_a = \{F \in \Sigma L | a \in F\}$ for $a \in L$. A filter $\mathcal{F}$ is said to be completely prime if it is the complement of a principal prime ideal. Unless otherwise stated, we shall always use the description of ΣL in terms of the prime elements of L.

The functors Σ and $\mathcal{O}$ are adjoint on the right, restricting to a dual equivalence between the full subcategories of *sober spaces* and *spatial frames*. The adjunction maps $\eta_L : L \to \mathcal{O}\Sigma L : a \mapsto \Sigma_a$ and $\epsilon_X : X \to \Sigma\mathcal{O}X : x \mapsto X - cl_X(\{x\})$ are respectively the spatial reflection and sobrification. We shall denote the spatial reflection $\mathcal{O}\Sigma L$ of L be SL and the spatial reflection map by $s_L : L \to SL$ and the full subcategory of all spatial frames by **SpFrm**.

1.3.3 Quotients of frames

A surjective frame homomorphism $h : L \to M$ is called a quotient map. We say that the frame M is a *quotient* (frame) of L and that L is an *extension* of M. There exists a natural ordering on the set of quotients of a frame L, namely, $h \leq k$ if there exists a map $\varphi : M \to N$ (necessarily also a quotient map) such that $\varphi \cdot k = h$, where M and N are the codomains of h and k respectively. Note that the ordering $\leq$ defined above is a pre-order. We form the poset reflection in the usual way, by calling h and k *equivalent* if $h \leq k$ and $k \leq h$, and then forming the congruence modulo this equivalence relation. The resulting poset is in fact a frame.

A *congruence* on a frame is an equivalence relation θ on L which is a subframe of $L \times L$. The lattice of congruences on a frame L is a frame with bottom $\mathbf{0} = \{(x, x) | x \in L\}$ and top $\mathbf{e} = L \times L$.

For any frame map $h : L \to M$ we call the set $\{(x, y) | h(x) = h(y)\}$ the *kernel* of h. The kernel of any frame map $h : L \to M$ is a congruence, and conversely, any congruence is a kernel. This gives an isomorphism between the frame of all congruences on L and the frame of all quotients on L.

A *nucleus* is a map $n : L \to L$ satisfying:

N1: $x \leq n(x)$ for all $x \in L$.

N2: $x \leq y \Rightarrow n(x) \leq n(y)$.

N3: $n^2(x) = n(x)$ for all $x \in L$.

N4: $n(x \wedge y) = n(x) \wedge n(y)$

For any nucleus n on a frame L, the closure system $\text{Fix}(n) = \{a \in L | n(a) = a\}$ is a map such that the map $n : L \to \text{Fix}(n)$ is a frame quotient. On the other hand, given a congruence θ on a frame L, the map $n : L \to L$ defined by $n(a) =$

$\bigvee\{x \in L|(a,x) \in \theta\}$ is a nucleus. This gives an isomorphism between the frame of quotients of L and the the frame of nuclei on L, with the pointwise order. Note that for any nucleus n on a frame L, the map $n : L \to Fix(n)$ is a quotient map, and for any quotient map $h : L \to M$, the map $h_*h : L \to L$ is the corresponding nucleus. In this case $Fix(h_*h) \cong M$. We shall denote by $\mathfrak{C}L$ all three equivalent descriptions of the frame of all quotients of L.

A *prenucleus* ([5]) on a frame L is a map $n_0 : L \to L$ such that:

PN1: $x \leq n_0(x)$ for all $x \in L$.

PN2: $x \leq y \Rightarrow n_0(x) \leq n_0(y)$.

PN3: $n_0(x) \wedge y \leq n_0(x \wedge y)$.

Given a prenucleus n_0 on a frame L, $Fix(n_0)$ is a closure system, and the associated closure operator n is a nucleus. Hence $Fix(n_0) = Fix(n)$ is a quotient frame of L. (See [5], [8].)

Let L be a frame. A nucleus n on L is called an *open* (respectively *closed*) nucleus if it is of the form $n(x) = a \to x$ (respectively $n(x) = x \vee a$) for some $a \in L$. (For spatial frames these correspond exactly to the open and closed subspaces of the corresponding spaces.) The corresponding quotients are called open (respectively closed) quotients and the resulting quotient frames are isomorphic to $\downarrow a$ (respectively $\uparrow a$.)

1.3.4 Strong inclusions and compactification of frames

Let L be any frame. An element a in L is said to be *compact* if $a = \bigvee S$ implies $a = \bigvee F$ where F is a finite subset of S. The frame L is called compact if its top element $\mathbf{e}$ is compact. Given any $a, b \in L$, we say a is *way below* b, written $a \ll b$ if $b = \bigvee S$ implies $a = \bigvee F$ for some finite $F \subseteq S$.

A *compactification* of a completely regular frame L is a dense extension $h : M \to L$ where M is a compact regular frame. A *strong inclusion* $\triangleleft$ on a completely regular frame L is binary relation satisfying the following conditions:

SI1: $x \leq a \triangleleft b \leq y \Rightarrow x \triangleleft y$

SI2: $\triangleleft \subseteq L \times L$ is a sublattice.

SI3: $x \triangleleft a \Rightarrow x \prec a$.

SI4: $a \triangleleft b \Rightarrow \exists x \in L$ such that $a \triangleleft x \triangleleft b$, i.e. $\triangleleft$ interpolates.

SI5: For each $a \in L$, $a = \bigvee\{x \in L | x \triangleleft a\}$.

Given a strong inclusion $\triangleleft$ on L, and ideal I is called $\triangleleft$-regular if for each $a \in I$ there exists $b \in I$ such that $a \triangleleft b$. The frame $\triangleleft\mathfrak{J}L$ of all $\triangleleft$-regular ideals on L is compact regular, and the join map $j_L : \triangleleft\mathfrak{J}L \to L$ is a compactification of L. Given a compactification $h : K \to L$ of L, then the relation $\triangleleft$ defined by $a \triangleleft b$ iff $h_*(a) \prec h_*(b)$ is a strong inclusion on L. Thus ([6]) the poset of all strong inclusions on L is isomorphic to the poset of all compactifications of L. The completely below relation $\prec\!\prec$ is the largest strong inclusion on L, giving rise to the largest compactification of L. This compactification, called the Stone-Čech compactification and denoted by βL, is the compact regular coreflection of L to the category **KRFrm** of compact regular frames. (The reader is referred to [20], [21] and [6] for an in depth discussion of this compactification.) A $\prec\!\prec$-regular ideal is called a *completely regular ideal.* Thus βL is the frame of all completely regular ideals in L. Another usefull description of βL is that of the frame of all regular ideals in $CozL$. An ideal I is said to be *regular* if $a \in I$ implies there exists $b \in I$ such that $a \prec b$. We shall always assume this description of βL, unless otherwise stated.

A zero dimensional frame has a universal zero dimensional compactification, called the Banaschewski compactification. The Banaschewski compactification of a zero dimensional frame L may be described as the frame of all ideals on $\mathbb{B}L$, and is denoted by ζL. The corresponding strong inclusion $\triangleleft_{\mathbb{B}L}$ is as follows: $x \triangleleft_{\mathbb{B}L} y$ iff there exists $b \in \mathbb{B}L$ such that $x \leq b \leq y$. Thus ζL may equivalently be described as the frame of all $\triangleleft_{\mathbb{B}L}$-regular frames in L.

A frame L is called *continuous* if each $a \in L$ is a join of elements way below it. In [6] it is shown that a frame L has a smallest strong inclusion, and hence a smallest compactification, iff L is regular continuous. The description of the smallest strong inclusion $\lhd$ on a regular continuous frame L is as follows: $a \lhd b$ iff $a \prec b$ and either $\uparrow(a^*)$ or $\uparrow b$ is a compact frame. (See [6].)

1.3.5 Nearness Frames

Let L be a frame. Then a *cover* of L is a subset $A \subseteq L$ such that $\bigvee A = \mathbf{e}$. The set of all covers of L is denoted by $Cov(L)$. Let $A, B \in Cov(L)$. We say A *refines* B (written $A \leq B$) if for each $a \in A$ there exists $b \in B$ such that $a \leq b$. Let A be a cover of L. For any $x \in L$, the *star* of x relative to A is $Ax = \bigvee\{a \in A | a \wedge x \neq \mathbf{0}\}$. For $\mathcal{A} \subseteq Cov(L)$, $x \lhd_{\mathcal{A}} y$ means that $Ax \leq y$ for some $A \in \mathcal{A}$. $\mathcal{A}$ is called *admissible* whenever $a = \bigvee\{x \in L | x \lhd_{\mathcal{A}} a\}$ for all $a \in L$. A *nearness* on L is an admissible filter $\mathcal{N}$ in $Cov(L)$. A *nearness frame* is a frame together with a specified nearness. The relation $\lhd_{\mathcal{A}}$ is called the uniformly below relation. For any nearness frame L, $\mathcal{N}L$ will denote its nearness. Given nearness frames M and L, a frame homomorphism $h : M \to L$ of the underlying frames is called *uniform* if $h[C] \in \mathcal{N}L$ for any $C \in \mathcal{N}M$.

Given any $A \in Cov(L)$. Then $B \in Cov(L)$ is called a *star refinement* of A (written $B \leq^* A$) if the cover $\{Bx | x \in B\}$ refines A. A nearness $\mathcal{N}$ on L is called a *uniformity* if for each $A \in \mathcal{N}$ there exists $B \in \mathcal{N}$ such that $B \leq^* A$. If $\mathcal{N}$ is a uniformity on L then the uniformly below relation is a strong inclusion on L. The nearness $\mathcal{N}$ is called *strong* if for each $A \in \mathcal{N}$, the cover $\check{A} = \{x \in L | x \lhd_{\mathcal{N}} a$ for some $a \in A\}$ also belongs to $\mathcal{N}$.

A uniform map $h : M \to L$ is called a surjection if it is both onto on the underlying frames and the nearnesses. A nearness frame L is said to be *complete* if every dense surjection $h : M \to L$ is an isomorphism. Any nearness frame has a unique completion [24] and furthermore, completion is a coreflection for strong nearness frames [19]. A subset F of a nearness frame L is called a (proper) filter if $\mathbf{0} \notin F$, $\mathbf{e} \in F, a \wedge b \in F$ whenever $a, b \in F$, and $a \in F$ for any $a \geq b$ where $b \in F$. A filter

F in L is called a Cauchy filter if $C \cap F \neq \emptyset$ for all $C \in \mathcal{N}L$. The filter F is said to *converge* if it meets every cover of L. (See [11].) For any frame T, a *T-valued Cauchy filter* on a nearness frame L is a $(\mathbf{0}, \wedge, \mathbf{e})$-homomorphism $\varphi : L \to T$ such that $\varphi[C]$ is a cover for each $C \in \mathcal{N}L$. If in addition $\varphi(a) = \bigvee\{\varphi(x) | x \lhd a\}$ for each $a \in L$ (where $\lhd$ is the uniformly below relation) then L is called a T-valued *regular* Cauchy filter. If φ is a T-valued Cauchy filter on L, and L is strong, then φ° is a T-valued regular Cauchy filter on L, where $\varphi^\circ = \bigvee\{\varphi(x) | x \lhd a\}$ [19]. A nearness frame L is complete iff each T-valued regular Cuachy filter on L is a T-valued point, i.e. a frame homomorphism $\psi : L \to T$. See [19].

Chapter 2

κ-Lindelöf Frames

2.1 Introduction

In [59] Madden presents some basic aspects of the general theory of κ-frames. In this paper he proves in particular the equivalence between the categories of completely regular κ-frames and completely regular κ-Lindelöf frames. (Actually, the result in [59] is more general than this, but it is this particular special case that we shall consider.) Furthermore, the author also shows that the category of completely regular κ-Lindelöf frames is coreflective in the category of completely regular frames. It is this property that is of particular interest to us, since we shall later show how this relates to Herrlich's notion of κ-compactness.

The present chapter will deal with this very important result. Madden describes the completely regular κ-Lindelöf coreflection of a completely regular frame L as the largest completely regular subframe of the free frame generated by L, considered as a κ-frame. We shall present a slightly different approach to that of the author, following rather the original approach by Reynolds [75] in his description of the 'realcompact reflection' in locales. We also discuss the relation between strong inclusions on a frame L and κ-Lindelöfications of L, thereby obtaining the completely regular κ-Lindelöf coreflection as an intermediate quotient between the frame L and its compact regular coreflection.

2.2 κ-Bases and κ-Lindelöfication

As in [59] the cardinal κ we consider is always assumed to be regular. Recall that a κ-set is a set of cardinality strictly less than κ.

Definition 2.1 *Let L be a frame. An element $a \in L$ is called a κ-element if for all subsets $S \subseteq L$ such that $a \leq \bigvee S$ there exists a κ-subset $S' \subseteq S$ such that $a \leq \bigvee S'$. L is called κ-Lindelöf if the top element* $\mathbf{e}$ *is a κ-Lindelöf element.*

Remark 2.1 As remarked by Madden, the ω-Lindelöf frames are the compact frames, and the ω_1-Lindelöf frames are the Lindelöf frames in the usual sense.

We shall denote the full subcategory of all completely regular κ-Lindelöf frames by **CRκLindFrm**.

Definition 2.2 *Let L be a frame. Then we call a κ-subframe A of L a κ-basis for L if A join generates L.*

Examples

1. For any completely regular frame L, $CozL$ is a regular ω_1-basis for L.
2. For any zero dimensional frame L, $\mathbb{B}L$ is a regular ω-basis for L

Proposition 2.3 *A frame L is completely regular iff L has a completely regular κ-basis for some uncountable cardinal κ.*

PROOF: If L is completely regular, then $CozL$ is a basis for L ([16]). On the other hand, if L has a completely regular κ-basis A then each $a \in L$ is a join of elements in A. But since A is completely regular, it follows that each $a \in L$ is a join of elements completely below it. □

Remark 2.3 The restriction to uncountable cardinals is necessary since a frame L is zero dimensional iff L has a completely regular ω-basis.

Let κ be an infinite cardinal. We denote by $Coz_\kappa L$ the set $\{s \in L | s = \bigvee K$, K a κ-set and $k \prec\!\prec s$ for each $k \in K\}$. Note that for any frame L, $Coz_\omega L$ is the lattice of all complemented elements of L and $Coz_{\omega_1} L$ is the lattice of all cozero elements of L. Also, whenever $\kappa > \lambda$, then $Coz_\kappa L \supseteq Coz_\lambda L$.

If $k \prec\!\prec a$ in a completely regular κ-subframe A of a frame L, then there exists a cozero element c of L such that $k \prec\!\prec c \prec\!\prec a$. Thus, for any uncountable cardinal κ, $Coz_\kappa L$ consists of exactly those elements that may be expressed as the join of a κ-set of cozero elements of L.

Proposition 2.4 *Let L be a frame. Then $Coz_\kappa L$ is the largest completely regular κ-subframe of L.*

PROOF: It is clear that $\mathbb{B}L$ is the largest completely regular ω-subframe of L. We now prove the proposition for $\kappa > \omega$. We know that each $a \in Coz_\kappa L$ is a join of a κ-set of cozero elements. Since $CozL$ is a completely regular σ-frame, it follows that each $a \in Coz_\kappa L$ is a join of a κ-set of cozero elements completely below it. Hence $Coz_\kappa L$ is completely regular. We now show that $Coz_\kappa L$ is a κ-frame. Let S be a κ-subset of $Coz_\kappa L$. Then each $s \in S$ is a join of a κ-set T_s such that $t \prec\!\prec s$ for each $t \in T_s$. Now, κ is a regular cardinal, and so $T = \bigcup_{s \in S} T_s$ is a κ-set. Furthermore, $t \prec\!\prec \bigvee S$ for each $t \in T$ and $\bigvee S = \bigvee T$. Thus, $\bigvee S \in Coz_\kappa L$.
The fact that $Coz_\kappa L$ is the largest completely regular κ-subframe of L follows immediately from the definition of $Coz_\kappa L$. □

Remark 2.4 If L has a completely regular *κ-basis*, then $Coz_\kappa L$ is the largest one. Thus, for an uncountable cardinal κ, $Coz_\kappa L$ is the largest completely regular κ-basis for a completely regular frame L. Similarly, $\mathbb{B}L$ is the largest completely regular ω-basis for a zero dimensional frame L.

Given any frame homomorphism $h : L \to M$, then $h[Coz_\kappa L] \subseteq Coz_\kappa M$ since $a \prec\!\!\prec b$ in L implies $h(a) \prec\!\!\prec h(b)$ in M. Thus, the restriction of h to $Coz_\kappa L$ is a κ-frame homomorphism. Hence, Coz_κ is a functor assigning to a completely regular frame, its largest completely regular κ-basis.

Let A be a κ-frame. A κ-ideal on A is an ideal which is closed under joins of κ-sets. We denote by $\mathcal{H}_\kappa A$ the frame of all κ-ideals of A.

For any κ-frame homomorphism $\varphi : A \to B$, the map $\bar{\varphi} : \mathcal{H}_\kappa A \to \mathcal{H}_\kappa B$ defined by $\bar{\varphi}(I) = [\varphi[I]]$, the ideal generated by $\varphi[I]$, is a frame homomorphism. Thus, $\mathcal{H}_\kappa$ is a functor from the category of κ-frames to **Frm**. We shall consider the restriction of $\mathcal{H}_\kappa$ to the subcategory of all completely regular κ-frames, denoted also by $\mathcal{H}_\kappa$.

Proposition 2.5 *Let A be a completely regular κ-frame. Then $\mathcal{H}_\kappa A$ is a completely regular κ-Lindelöf frame.*

PROOF: If $a \prec\!\!\prec b$ in A, then $\downarrow a \prec\!\!\prec \downarrow b$ in $\mathcal{H}_\kappa A$ and hence every $I \in \mathcal{H}_\kappa A$ is a join of principal ideals completely below it. If $\bigvee \mathcal{I} = \mathcal{H}_\kappa A$ for $\mathcal{I} \subseteq \mathcal{H}_\kappa A$, then there is a κ-set $S \subseteq \bigcup \mathcal{I}$ such that $\bigvee S = \mathbf{e}$. But this means that there is a κ-set $\mathcal{S} \subseteq \mathcal{I}$ such that $\bigvee \mathcal{S} = \mathcal{H}_\kappa A$. Thus, $\mathcal{H}_\kappa A$ is completely regular κ-Lindelöf. □

Proposition 2.6 *The functor Coz_κ is right adjoint $\mathcal{H}_\kappa$.*

The above proposition is proved in exactly the same way as the well-known result that the functor Coz $(= Coz_{\omega_1})$ is right adjoint to $\mathcal{H} = (\mathcal{H}_{\omega_1})$, originally proved by Reynolds [75] (see also Johnstone [54]). The counit $\epsilon_L : \mathcal{H}_\kappa Coz_\kappa L \to L$ is given by join and the unit $\eta_A : A \to Coz_\kappa \mathcal{H}_\kappa A$, given by taking the downset, is an isomorphism. This too may be proved in the same way as in the special case of Coz and $\mathcal{H}$. Note that a simpler version of this proof appears in [16], which we outline here for the sake of completeness:

Let A be a completely regular κ-frame, and suppose $J \in Coz_\kappa \mathcal{H}_\kappa A$. Then $J = \bigvee \mathcal{J}$ where $\mathcal{J}$ is a κ-set and $J_i \prec\!\!\prec J$ for each $J_i \in \mathcal{J}$. Thus, for each i there are elements $a_i, b_i \in A$ such that $J_i \subseteq \downarrow a_i$, $\downarrow b_i \subseteq J$ and $a_i \prec\!\!\prec b_i$. Since J is a κ-ideal, it follows that $a = \bigvee a_i \in J$, and since $J_i \subseteq \downarrow a$ for all i, it follows that $J = \downarrow a$. On the other hand, if $a \in A$, then $a = \bigvee S$, where S is a κ-set and $s \prec\!\!\prec a$ for each $s \in S$. Thus, $\downarrow a = \bigvee \{\downarrow s | s \in S\}$ and $\downarrow s \prec\!\!\prec \downarrow a$ for each $s \in S$, in $\mathcal{H}_\kappa A$.

Proposition 2.7 *Let κ be an uncountable cardinal. Then a completely regular frame L is κ-Lindelöf iff $L \cong \mathcal{H}_\kappa Coz_\kappa L$.*

PROOF: Since $Coz_\kappa L$ join generates L we know that the join map is surjective. It now suffices to show that it is codense. Suppose $\bigvee I = \mathbf{e}$ for some $I \in \mathcal{H}_\kappa Coz_\kappa L$. Then since L is κ-Lindelöf, there exists a κ-set $S \subseteq I$ such that $\bigvee S = \mathbf{e}$. Since I is a κ-ideal, it follows that $I = L$. □

Thus, we may conclude

Proposition 2.8 (Madden [59]) *For $\omega < \kappa$, the functors $\mathcal{H}_\kappa$ and Coz_κ restrict to an equivalence between the categories* **CRκFrm** *and* **CRκLindFrm** *of completely regular κ-Lindelöf frames.*

Corollary 2.9 *For $\omega < \kappa$, the category* **CRκLindFrm** *is coreflective in* **CRFrm** *with coreflection given by the join map $j_L : \mathcal{H}_\kappa Coz_\kappa L \to L$.*

Remark 2.9 In the case where $\kappa = \omega$, we consider the zero dimensional frames, and the map $j_L : \mathcal{H}_\omega Coz_\omega L \to L$ gives the universal zero dimensional compactification, or the Banaschewski compactification of L.

Corollary 2.10 *The category* **CRκLindFrm** *is closed under the formation of coproducts and closed quotients.*

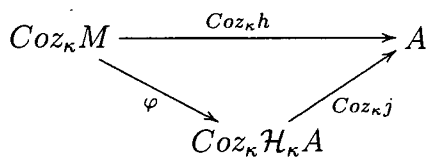
PROOF: That **CRκLindFrm** is closed under coproducts follows from the general fact that isomorphism closed full coreflective subcategories are closed under colimits. Now, suppose L is a κ-Lindelöf frame and $k : L \to M$ is a closed quotient. Then $M \cong \uparrow s$ for some $s \in L$ and so we may consider the equivalent map $(-) \vee s : L \to \uparrow s$. Let A be a cover of $\uparrow s$. Then A is a cover of L and hence there exists a κ-set $S \subseteq A$ with $\bigvee S = \mathbf{e}$, and hence trivially S covers $\uparrow s$. Now, since all quotients of completely regular frames are completely regular, the result follows.□

Definition 2.11 *Let L be a completely regular frame. A κ-Lindelöfication of L is a dense surjection $h : M \to L$ where M is a completely regular κ-Lindelöf frame.*

Definition 2.12 *Let A be a completely regular κ-basis for a completely regular frame L. A κ-Lindelöfication $h : M \to L$ is over A if $h[Coz_\kappa M] \subseteq A$.*

Proposition 2.13 *Let A be a completely regular κ-basis for a completely regular frame L. Then the join map $j_L : \mathcal{H}_\kappa A \to L$ is the universal κ-Lindelöfication over A.*

PROOF: Suppose $h : M \to L$ is a κ-Lindelöfication over A. Then $Coz_\kappa M$ is mapped into A by the map $Coz_\kappa h$. Define the map $\varphi : Coz_\kappa M \to Coz_\kappa \mathcal{H}_\kappa A$ as $\varphi(a) = \downarrow Coz_\kappa h(a)$.



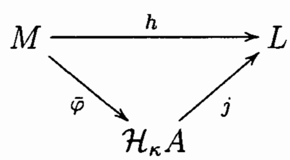
Trivially, φ preserves finite meets. Now, let S be a κ-subset of $Coz_\kappa M$. Then

$$\begin{aligned} \varphi(\bigvee S) &= \downarrow Coz_\kappa h(\bigvee S) \\ &= \downarrow \bigvee Coz_\kappa h[S] \end{aligned}$$

$$= \bigvee_{s \in S} \downarrow Coz_\kappa h(s)$$

$$= \bigvee_{s \in S} \varphi(s)$$

Thus φ is a κ-frame homomorphism. Applying the functor $\mathcal{H}_\kappa$ and noting that $\mathcal{H}_\kappa Coz_\kappa M \cong M$ and $\mathcal{H}_\kappa Coz_\kappa \mathcal{H}_\kappa A \cong \mathcal{H}_\kappa A$ one obtains a frame homomorphism $\bar{\varphi} : M \to \mathcal{H}_\kappa A$ such that the triangle



commutes. Uniqueness of $\bar{\varphi}$ follows from the fact that j is dense and hence monic.□

2.3 Strong inclusions and κ-Lindelöfications

We now turn our attention to the role of strong inclusions to κ-Lindelöfication of a frame L.

Let L be a completely regular frame and let $h : M \to L$ be a κ-Lindelöfication of L. Then the relation $\triangleleft$ defined as $x \triangleleft y$ iff $h_*(x) \prec\prec h_*(y)$, where h_* is the right adjoint of h, is a strong inclusion on L. The conditions of the strong inclusion are easily verified using the same proof as given in [6].

We now obtain a κ-Lindelöf quotient of the compact regular frame $\triangleleft\mathfrak{J}L$:
Let κ be an uncountable cardinal. Define a map $n_\kappa : \triangleleft\mathfrak{J}L \to \triangleleft\mathfrak{J}L$ as

$$n_\kappa(I) = \{a \in L | a \triangleleft \bigvee S, \text{ for some } \kappa\text{-set } S \subseteq I\}$$

Lemma 2.14 *The map n_κ defined above is a nucleus.*

PROOF:

N1: It is clear that $I \leq n_\kappa(I)$.

N2: Suppose $I \leq J$ and $a \in n_\kappa(I)$. Then $a \triangleleft \bigvee S$ for some κ-set $S \subseteq I$. But then $S \subseteq J$, and so $a \in n_\kappa(J)$.

N3: Suppose $a \in n_\kappa^2(J)$. Then $a \triangleleft \bigvee S$ for some κ-set $S \subseteq n_\kappa(J)$. Now, for each $s \in S$, there exists a κ-set $T_s \subseteq J$ such that $s \triangleleft T_s$. Let $T = \bigcup_{s \in S} T_s$. Then T is a κ-subset of J since κ is regular, and $a \triangleleft \bigvee S \leq \bigvee T$. Thus $a \in n_\kappa(J)$.

N4: It suffices to show that $n_\kappa(I) \cap n_\kappa(J) \subseteq n_\kappa(I \cap J)$. Suppose $a \in n_\kappa(I) \cap n_\kappa(J)$. Then there exists a κ-set $S_1 \subseteq I$ such that $a \triangleleft \bigvee S_1$ and a κ-set $S_2 \subseteq J$ such that $a \triangleleft \bigvee S_2$. Thus $a \triangleleft \bigvee S_1 \wedge \bigvee S_2 = \bigvee \{s_1 \wedge s_2 | s_1 \in S_1, s_2 \in S_2\}$. Thus $a \in n_\kappa(I \cap J)$ since $\{s_1 \wedge s_2 | s_1 \in S_1, s_2 \in S_2\}$ is a κ-subset of $I \cap J$.

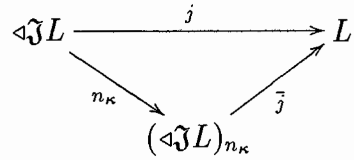
□

Proposition 2.15 *The quotient frame $(\triangleleft\mathfrak{I}L)_{n_\kappa} = Fix(n_\kappa)$ is a completely regular κ-Lindelöf frame and the map $j_L : (\triangleleft\mathfrak{I}L)_{n_\kappa} \to L$ given by join is a κ-Lindelöfication.*

PROOF: Suppose $\mathcal{T} \subseteq (\triangleleft\mathfrak{I}L)_{n_\kappa}$ such that $\bigvee_{(\triangleleft\mathfrak{I}L)_{n_\kappa}} \mathcal{T} = \mathbf{e}_{(\triangleleft\mathfrak{I}L)_{n_\kappa}}$. Then $n_\kappa \bigvee_{\triangleleft\mathfrak{I}L} \mathcal{T} = \mathbf{e}_{\triangleleft\mathfrak{I}L}$, and hence $\mathbf{e}_L \leq \bigvee_L S$ for some κ-set $S \subseteq \bigvee_{\triangleleft\mathcal{J}L} \mathcal{T}$. Now, each $s \in S$ is contained in some $T_s \in \mathcal{T}$. Thus $\bigvee_{\triangleleft\mathfrak{I}L} \{T_s | s \in S\} = \mathbf{e}_{\triangleleft\mathfrak{I}L}$ and hence $\bigvee_{(\triangleleft\mathfrak{I}L)_{n_\kappa}} \{T_s | s \in S\} = \mathbf{e}_{(\triangleleft\mathfrak{I}L)_{n_\kappa}}$
That $j_L : (\triangleleft\mathfrak{I}L)_{n_\kappa} \to L$ is a κ-Lindelöfication follows from the fact that

$$\bigvee_L \bar{n}_\kappa(I) = \bigvee_L \{a \in L | a \leq \bigvee_L S, S \text{ a } \kappa\text{-subset of } I\} \leq \bigvee_L I$$

Hence $\bigvee_L \bar{n}_\kappa(I) = \bigvee_L(I)$, from which it follows that the join map j_L factors through the map n_κ.



Thus $\bar{j}$ is dense and onto since j is. The quotient $(\lhd\mathfrak{J}L)_{n_\kappa}$ is completely regular since quotients of completely regular frames are completely regular. □

Corollary 2.16 *A frame L has a κ-Lindelöfication iff it has a strong inclusion.*

Lemma 2.17 *Let L be a completely regular frame, and suppose $\lhd$ is a strong inclusion on L. Then the following are equivalent:*

1) $a = \bigvee S$, S a κ-set and $s \lhd a$ for all $s \in S$.

2) $a = \bigvee I$, where I is a $\lhd$-regular ideal in L generated by a κ-set.

PROOF: (1) $\Rightarrow$ (2) : Suppose $a = \bigvee S$, where S is a κ-set and $s \lhd a$ for each $s \in S$. Let I be the ideal generated by S. It suffices to show that I is $\lhd$-regular. If $x \in I$. Then $x \leq s \lhd a$ for some $s \in S$. But $\lhd$ interpolates, and so $x \lhd y$ for some $y \in I$.
(2) $\Rightarrow$ (1) : Let I be generated by the κ-set S. Then for each $s \in S$ there exists $\bar{s} \in I$ such that $s \lhd \bar{s}$ since I is $\lhd$-regular. Thus $s \lhd a = \bigvee I$ for each $s \in S$. Furthermore $a = \bigvee S$ since S generates I. □

We denote by $\kappa_\lhd L$ the set of all $a \in L$ satisfying the above conditions. We claim that $\kappa_\lhd L$ is a completely regular κ-basis for L: Since κ is a regular cardinal, it is clear that $\kappa_\lhd L$ is a κ-frame. The fact that $\kappa_\lhd L$ join generates L follows from the property (SI5) of the strong inclusion $\lhd$.

Note that $Coz_\kappa(\lhd\mathfrak{J}L)$ is the lattice of all $\lhd$-regular ideals that are generated by κ-sets. It follows therefore that $\kappa_\lhd L$ is the image under the join map of the κ-frame $Coz_\kappa(\lhd\mathfrak{J}L)$. Thus a strong inclusion $\lhd$ on L has a naturally associated completely regular κ-basis of L. On the other hand, let A be a completely regular κ-basis for L. Then it is easily checked that the relation $\lhd_A$ defined as $x \lhd_A y$ iff there exist elements $a, b \in A$ such that $x \leq a \prec_A b \leq y$ is a strong inclusion on L. (The relation $\prec_A$ is the rather below relation in A, i.e. the separating element is in A.)

Examples

1. If $\lhd$ is the completely below relation on a frame L, then $\kappa_{\lhd}L = Coz_{\kappa}L$.

2. If L is a uniform frame, and $\lhd$ is the uniformly below relation, then $(\omega_1)_{\lhd}$ is the σ-frame of all uniformly cozero elements. (See [90].)

3. If L is zero-dimensional, and $\lhd_{\mathbb{B}L}$ is the strong inclusion corresponding to the Banaschewski compactification of L. Then $\kappa_{\lhd_{\mathbb{B}L}}$ is the collection of all elements of L that are joins of κ-sets of complemented elements. We denote this κ-frame by $\mathbb{B}_{\kappa}L$.

Proposition 2.18 *Let L be a frame and let $\lhd$ be a strong inclusion on L. Then $(\lhd\mathfrak{J}L)_{n_\kappa} \cong \mathcal{H}_{\kappa}\kappa_{\lhd}L$ (where n_κ is the nucleus defined in the first part of this section).*

PROOF: Define $\varphi : (\lhd\mathfrak{J}L)_{n_\kappa} \to \mathcal{H}_{\kappa}\kappa_{\lhd}L$ as $\varphi(I) = I \cap \kappa_{\lhd}L$. Then φ is clearly a well-defined frame homomorphism. It is also clear that φ is codense, so we need to show that φ is onto. Suppose $J \in \mathcal{H}_{\kappa}\kappa_{\lhd}L$. Let $[J]$ be the ideal in L generated by J. Then $[J]$ is closed under joins of κ-sets and hence $\bar{n}_{\kappa}[J] = [J]$, i.e. $[J] \in (\lhd\mathfrak{J}L)_{n_\kappa}$. Furthermore, $\varphi([J]) = J$. □

Corollary 2.19 *Let L be a completely regular frame, and κ an uncountable cardinal. Then $(\beta L)_{n_\kappa}$ is the κ-Lindelöf coreflection of L.*

Chapter 3

Spatial quotients of Frames

3.1 Introduction

Frames or locales are often referred to as generalised spaces. Though this description may not be entirely precise (since not all spaces may be recovered from their open-set lattices), it is true for Hausdorff spaces. Furthermore, frame quotients are more general than embeddings of such spaces: Given a Hausdorff space X, then there are in general more frame quotients of $\mathcal{O}X$ than there are subspaces of X. This poses two immediate questions:

1. Is it possible to characterise those quotients of a frame L that are spatial?

2. Which frames have only spatial quotients?

An answer to the second question was provided by Niefield and Rosenthal [70]. The frame of quotients of a frame was first studied by Dowker and Papert [31]. It was in this paper that Dowker and Papert established a frame congruence θA of $\mathcal{O}X$ corresponding to every subspace A of X. Moreover, the authors showed that $\theta A = \theta B$ for subspaces A and B of a space X iff A and B have the same b-closure in X. (Recall that the b-closure of a subspace A of X is the closure of A in the Skula modification of X.) Perhaps the most remarkable difference between quotients of $\mathcal{O}X$ and subspaces of X is the fact that $\mathcal{O}X$ has a largest dense quotient (or, equivalently, a smallest dense sublocale of X). Although Dowker and Papert did not mention this fact explicitly, they proved that the join of all dense spatial quotients of $\mathcal{O}X$ is the quotient M_ρ, the lattice of regular open sets of X. Thus, the smallest dense sublocale of the reals, for example, has no points - which is not surprising since the reals has two disjoint dense subspaces ($\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$). The proof that every frame has a largest dense quotient may be found in [54]. (Note that although Dowker and Papert's approach is algebraic, they have inverted the ordering on the lattice of quotients of a frame in order to be consistent with the ordering on the lattice of subspaces of a space. Their original result is thus in terms of *meets* of dense quotients.)

Frame quotients are frequently described in terms of *nuclei*, an invention of Macnab and Beazer [58], [26] who describe frame quotients in the form of 'modal extensions of Heyting algebras'. In [26] the authors provide an algebraic characterisation for those frames L for which the corresponding frame of quotients is Boolean. Simmons [81] provided the topological significance of this result, showing that a T_0 space has a Boolean assembly iff it is scattered.

This chapter will be largely devoted to the first of the questions listed above.

3.2 Spatial nuclei

Recall that the spectrum ΣL of a frame L has various equivalent descriptions. For the purpose of this chapter we will assume the description in terms of the prime elements of L, i.e., the underlying set will consist of the prime elements of L.

Lemma 3.1 *Let L be a spatial frame, and let $a, b \in L$. Then $\bigvee\{x | x \wedge b \leq a\} = \bigwedge\{p \in \Sigma L | p \not\geq b$ and $p \geq a\}$.*

PROOF: Since L is spatial, it suffices to show that for a prime p, $p \geq \bigvee\{x|x \wedge b \leq a\}$ iff $p \geq \bigwedge\{p \in \Sigma L | p \not\geq b$ and $p \geq a\}$

Suppose $p \not\geq b$ and $p \geq a$. Then $p \geq x \wedge b$ for each x such that $x \wedge b \leq a$. Since p is prime and $p \not\geq b$, it follows that $p \geq x$ for each x such that $x \wedge b \leq a$. Hence

$p \geq \bigvee\{x | x \wedge b \leq a\}$.

Conversely, suppose $p \geq \bigvee\{x | x \wedge b \leq a\}$. Then $p \geq b^*$ and $p \geq a$. Now,

$$\begin{aligned} b^* \vee a &= \bigwedge\{q | q \geq b^* \vee a\} \\ &= \bigwedge\{q \geq b^* \vee a | q \not\geq b\} \wedge \bigwedge\{q \geq b^* \vee a | q \geq b\} \\ &= \bigwedge\{q \geq b^* \vee a | q \not\geq b\} \wedge \bigwedge\{q | q \geq b \vee b^* \vee a\} \\ &= \bigwedge\{q \geq b^* \vee a | q \not\geq b\} \end{aligned}$$

Since p is prime, it follows that $p \not\geq b \Rightarrow p \geq b^*$ and so

$$b^* \vee a = \bigwedge\{p \geq a | p \not\geq b\} = \bigwedge\{p \in \Sigma L | p \geq a \text{ and } p \not\geq b\}$$

Hence

$$p \geq b^* \vee a = \bigwedge\{p \in \Sigma L | p \geq a \text{ and } p \not\geq b\}$$

□

Let $A_b = \{p \in \Sigma L | p \not\geq b\}$ and denote by $s_{A_b}(a)$ the set $\bigwedge\{p \in A_b | p \geq a\}$.

Corollary 3.2 *Let L be a spatial frame and let $b \in L$. Then the open nucleus $b \to (-)$ is equivalent to the map s_{A_b}.*

PROOF: This follows from the fact that $b \to a = \bigvee\{x | x \wedge b \leq a\}$. □

Thus the open nuclei of a spatial frame L may be described equivalently as certain nuclei associated with particular subsets of the set of all primes of L. More generally, let L be any frame and let $A \subseteq \Sigma L$. Define $s_A : L \to L$ as

$$s_A(a) = \bigwedge\{p \in A | p \geq a\}$$

Lemma 3.3 *The map s_A defined above is a nucleus on L.*

Proof: It is clear that $a \leq s_A(a)$. Now,

$$\begin{aligned} s_A(a) \wedge s_A(b) &= \bigwedge\{p \in A | p \geq a\} \wedge \bigwedge\{p \in A | p \geq b\} \\ &= \bigwedge\{p \in A | p \geq a \text{ or } p \geq b\} \\ &= \bigwedge\{p \in A | p \geq a \wedge b\} \quad \text{since } p \text{ is prime} \\ &= s_A(a \wedge b) \end{aligned}$$

For the idempotency of s_A note that $s_A(p) = p$ for each $p \in A$. Hence $p \geq a \Rightarrow p = s_A(p) \geq s_A(a)$ for each $p \in A$. But this means that

$$\begin{aligned} s_A(a) &= \bigwedge\{p \in A | p \geq a\} \\ &\geq \bigwedge\{p \in A | p \geq s_A(a)\} \\ &= s_A^2(a) \end{aligned}$$

□

Remark 3.3 The spatial reflection SL of a frame L is a quotient of L given by the nucleus $s_L(a) = \bigwedge\{p \in \Sigma L | p \geq a\}$. From the terminology above, $s_L = s_{\Sigma L}$.

Definition 3.4 *A congruence and the corresponding nucleus n of a frame are called spatial if the quotient frame L_n is a spatial frame.*

Proposition 3.5 *Let L be a frame. Then the spatial nuclei of L are precisely those of the form s_A where $A \subseteq \Sigma L$.*

PROOF: Note that $s_A(p) = p$ for each $p \in A$ and furthermore, if $p \geq a \wedge b$ in L_{s_A} then $p \geq a \wedge b$ in L from which it follows that $p \geq a$ or $p \geq b$ in L and hence also in L_{s_A}. Thus $A \subseteq \Sigma L_{s_A}$. Since each element of L_{s_A} is a meet (in L_{s_A}) of elements of A, it follows that L_{s_A} is spatial.

On the other hand, suppose M is a spatial quotient of L. Let $A = \{h_*(p) | p \in \Sigma M\}$. Our claim is that $L_{s_A} = M$, i.e. $s_A = h_*h$. Suppose $a \in L$. Then $h(a) \in M$ and since M is spatial, $h(a) = \bigwedge\{p \in \Sigma M | p \geq h(a)\}$ and hence $h_*h(a) = \bigwedge\{h_*(p) | p \in \Sigma M, p \geq h(a)\}$. Note also that $p \geq h(a)$ iff $h_*(p) \geq a$. So we get

$$\begin{aligned} s_A(a) &= \bigwedge\{p \in A | p \geq a\} \\ &= \bigwedge\{h_*(p) | p \in \Sigma M, h_*(p) \geq a\} \\ &= \bigwedge\{h_*(p) | p \in \Sigma M, p \geq h(a)\} \\ &= h_*h(a) \end{aligned}$$

□

Examples

1. Open quotients of spatial frames are spatial. (Corollary 3.2.) To see that closed quotients of spatial frames are spatial, note that the closed nucleus $b \vee (-)$ is equivalent to the nucleus s_{A_b} where $A_b = \{p \in \Sigma L | p \geq b\}$.

2. One-point quotients of spatial frames are spatial since they are open. (A quotient $L \to M$ is called a one-point quotient if $M \cong \downarrow a$ for some maximal $a \in L$.) Note that the one-point quotient of a spatial frame is obtained by considering the set $\Sigma L \backslash \{p\}$ for some maximal p.

Proposition 3.6 *The collection of all spatial nuclei form a frame with* $\bigwedge s_{A_i} = s_{\bigcup A_i}$.

Proof: We prove the more general fact that $A_i \subseteq L \Rightarrow \bigwedge_i \bigwedge A_i = \bigwedge \bigcup_i A_i$. Firstly, $\bigwedge \bigcup_i A_i \leq \bigwedge A_j$ for each j. Hence $\bigwedge \bigcup_i A_i \leq \bigwedge_i \bigwedge A_i$. Conversely, $\bigwedge A_i \leq x$ for each $x \in A_i$. Thus each $x \in \bigcup A_i$ dominates some $\bigwedge A_i$ and hence $\bigwedge_i \bigwedge A_i \leq \bigwedge \bigcup A_i$. □

For $p \in \Sigma L$ we denote the nucleus $s_{\{p\}}$ by s_p.

Proposition 3.7 *The nuclei of the form s_p for $p \in \Sigma L$ are precisely maximal in the frame $\mathfrak{C}L$ of all nuclei on L.*

PROOF: Suppose $n > s_p$. Then $n(p) > p$: If $n(p) = p$, then $n = s_p$. Thus $n(p) = n^2(p) \geq s_p n(p) = \mathbf{e}$. Thus $n(x) = \mathbf{e}$ for all $x \in L$, i.e. $n = \mathbf{e}$.
Conversely, suppose $n \neq s_p$ for any $p \in \Sigma L$. If n fixes a prime q, then $n < s_q$ in which case n is not maximal. If n does not fix any prime, then n fixes some non-prime $c \neq \mathbf{e}$. Since c is not prime, there exists $a, b \in L$ such that $c \geq a \wedge b$ and $c \not\geq a$ and $c \not\geq b$. Since c is fixed by n, it follows that $c \geq n(a) \wedge n(b)$ and hence either $n(a) \neq \mathbf{e}$ or $n(b) \neq \mathbf{e}$. We may assume $n(a) \neq \mathbf{e}$. Note that $c \not\geq n(a)$.
Define $k : L \to L$ by $k(x) = n(n(a) \vee n(x))$. We show that k is a nucleus:

N1: It is clear that $x \leq k(x)$

N2:

$$\begin{aligned} k(x \wedge y) &= n(n(a) \vee n(x \wedge y)) \\ &= n(n(a) \vee (n(x) \wedge n(y))) \\ &= n((n(a) \vee n(x)) \wedge (n(a) \wedge n(y))) \\ &= k(x) \wedge k(y) \end{aligned}$$

N3: Suppose $x \leq y$. Then $k(x) \leq k(y)$ since $n(x) \leq n(y)$.

N4:

$$\begin{aligned} k^2 &= n(n(a) \vee n(k(x))) \\ &= n(n(a) \vee n(n(a) \vee n(x))) \\ &= n(n(n(a) \vee n(x)) \\ &= n(n(a) \vee n(x)) \\ &= k(x) \end{aligned}$$

Note that $k \neq \mathbf{e}$ since $k(a) = n(a) \neq \mathbf{e}$. It is clear that $n \leq k$. Also, $k(c) = n(n(a) \vee c) > c$ since $c \not\geq n(a)$. Hence $n < k$, i.e. n is not maximal.□

Since $\mathfrak{C}L$ is zero dimensional and hence regular, it follows that the prime elements are all maximal.

Corollary 3.8 *The frame of all spatial quotients of L is isomorphic to $S\mathfrak{C}L$, the spatial reflection of $\mathfrak{C}L$.*

PROOF: Let $A \subseteq \Sigma L$. Then from Proposition 3.6 above, $s_A = \bigwedge\{s_p | p \in A\}$. Thus, a nucleus is of the form s_A for some $A \subseteq \Sigma L$ iff it is a meet of primes. It follows that $n \in Fix(s_{\mathfrak{C}L})$ iff $n = s_A$ for some $A \subseteq \Sigma L$, i.e. the frame $\mathfrak{C}L$ consists precisely of the spatial quotients of L. □

Remark 3.8

1. Corollary 3.8 appears in [70]. However the proof presented there is different to ours.

2. A frame has only spatial quotients iff $\mathfrak{C}L \cong S\mathfrak{C}L$ iff $\mathfrak{C}L$ is spatial. This result also appears in [70].

3. From Proposition 3.7 we may deduce that there is a one-to-one correspondence between the primes of L and the primes of $\mathfrak{C}L$. This fact is observed in [80].

Proposition 3.9 *Let $\mathbb{P}\Sigma L$ denote the power set of ΣL. The map $\varphi : \mathbb{P}\Sigma L^{opp} \to S\mathfrak{C}L$ given by $\varphi(A) = s_A$ is a frame quotient. Furthermore, φ is an isomorphism iff no prime p may be expressed as a meet of primes not equal to p.*

PROOF: Suppose p may be expressed as a meet of a set A of primes not containing p. Then $\bigwedge\{q \in A | q \geq x\} = \bigwedge\{q \in A \cup \{p\} | q \geq x\}$. Thus $\varphi(A) = \varphi(A \cup \{p\})$, i.e. φ is not injective.
On the other hand, suppose $A \neq B$. Then there exists a $p \in A$ (say) such that $p \notin B$. Now $s_A(p) = p$ and $s_B(p) > p$ by our assumption. This means that $\varphi(A) \neq \varphi(B)$ and hence φ is injective. □

It is well-known that the space $\Sigma\mathfrak{C}L$ is homeomorphic to $Sk\Sigma L$, the Skula modification of ΣL. (See [80], [81], [14].)

Corollary 3.10 *$Sk\Sigma L$ is discrete iff no prime p of L may be expressed as a meet of primes other than p.*

Corollary 3.11 *If a prime p of L can be expressed as a meet of other primes, then $L \not\cong \mathfrak{C}L$.*

PROOF: If $L \cong \mathfrak{C}L$, then L is Boolean [80] and thus $SL \cong S\mathfrak{C}L$ is atomic Boolean. Since the spaces ΣL and $\Sigma\mathfrak{C}L$ have the same underlying sets, it follows that $S\mathfrak{C}L \cong \mathbb{P}\Sigma L$. From Proposition 3.9 it follows that no prime p may be expressed as a meet of primes. □

Example Let L be the totally ordered frame $[0, 1]$. Then every element other than 0 and $\mathbf{e}$ is prime, and each of these primes is a meet of all primes above it. The space ΣL is the real line endowed with the upper topology and $Sk\Sigma L(\cong \Sigma\mathfrak{C}L)$ is the Sorgenfrey line.

Proposition 3.12 *Let L be a frame, and let $h : L \to M$ be a quotient of L. If $A = \{h_*(p) | p \in \Sigma M\}$, then the frame L_{s_A} is isomorphic to SM.*

PROOF: Define the map $\bar{h} : L_{s_A} \to SM$ by $\bar{h}(s_A(x)) = s_M \cdot h(x)$.

$$\begin{array}{ccc} L & \xrightarrow{h} & M \\ {\scriptstyle s_A}\downarrow & & \downarrow{\scriptstyle s_M} \\ L_{s_A} & \xrightarrow[\bar{h}]{} & SM \end{array}$$

Firstly note that $\bar{h}$ is well-defined:

$$\begin{aligned} s_A(x) = s_A(y) &\Rightarrow \bigwedge\{p \in A | p \geq a\} = \bigwedge\{p \in A | p \geq y\} \\ &\Rightarrow \bigwedge\{q \in \Sigma M | h_*(q) \geq x\} = \bigwedge\{q \in \Sigma M | h_*(q) \geq y\} \\ &\Rightarrow h_* \bigwedge\{q \in \Sigma M | q \geq h(x)\} = h_* \bigwedge\{q \in \Sigma M | q \geq h(y)\} \\ &\Rightarrow s_M(h(x)) = s_M(h(y)) \\ &\Rightarrow \bar{h}(s_A(x)) = \bar{h}(s_A(y)) \end{aligned}$$

Furthermore, $\bar{h}$ is clearly a frame homomorphism onto SM since $s_M \cdot h$ is onto. It therefore suffices to show that $\bar{h}$ is one-to-one. Suppose $s_A(x) \neq s_A(y)$. Then $\bigwedge\{p \in A | p \geq x\} \neq \bigwedge\{p \in A | p \geq y\}$. But this means that $h_* s_M \cdot h(x) \neq h_* s_M \cdot h(y)$. Thus $s_M \cdot h(x) \neq s_M \cdot h(y)$ and hence $\bar{h}(s_A(x)) \neq \bar{h}(s_A(y))$. $\square$

3.3 The Maximal Spectrum

In this section we depart briefly from our main theme in order to consider an example of spatial quotients discussed in the previous section. A special example of a subset of the set of primes of a frame L, is the set of all maximal elements of L, which we denote by $\mathfrak{M}L$. Now, from Section 3.2 the map $s_{\mathfrak{M}} : L \to L$ defined by

$$s_{\mathfrak{M}}(a) = \bigwedge\{p \in \mathfrak{M}L | p \geq a\}$$

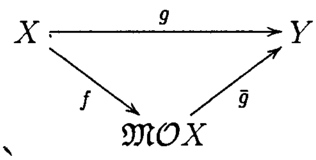
is a nucleus. The assumed topology on $\mathfrak{M}L$ will be the collection of all sets of the form $\mathfrak{M}_a = \{m \in \mathfrak{M}L | a \not\leq m\}$ for each $a \in L$.

Proposition 3.13 *For any frame L, $\mathfrak{M}L$ is a T_1 space. Moreover, $f : X \to \mathfrak{M}\mathcal{O}X$ defined by $f(x) = X \backslash cl\{x\}$ is the T_1 reflection of a space X.*

PROOF: Suppose x, y are two distinct elements of $\mathfrak{M}L$. Then x and y are distinct maximal elements of L. Now, $\mathfrak{M}_x = \{m \in \mathfrak{M}L | x \not\leq m\} = \{m \in \mathfrak{M}L | x \neq m\}$. Hence $y \in \mathfrak{M}_x$ and $x \notin \mathfrak{M}_x$.
Now, consider any space X. Then the map $f : X \to \mathfrak{M}\mathcal{O}X$ defined above is continuous: Consider any open set U in $\mathfrak{M}\mathcal{O}X$. Then U is of the form $\mathfrak{M}_V = \{M \in \mathfrak{M}\mathcal{O}X | V \not\subseteq M\}$. Now,

$$\begin{aligned} x \in f^{-1}(\mathfrak{M}_V) &\Leftrightarrow f(x) \in \mathfrak{M}_V \\ &\Leftrightarrow V \not\subseteq X \backslash cl\{x\} \\ &\Leftrightarrow x \in V \end{aligned}$$

Thus $f^{-1}(\mathfrak{M}_V) = V$. Suppose Y is a T_1 space and $g : X \to Y$ is a continuous funtion. Define $\bar{g}$ as follows:



$\bar{g}(U) = g(x)$ where $x \in X \backslash U$. We show that $\bar{g}$ is well-defined. Consider two distinct elements x, y in $X \backslash U$. Suppose $g(x) \neq g(y)$. Since Y is T_1 there exists an open set V in Y such that $g(x) \in V$ and $g(y) \notin V$. But this means that $x \in g^{-1}(V)$ and $y \notin g^{-1}(V)$ and hence $x \notin cl\{y\}$. This contradicts the fact that x and y are both in the complement of a maximal open set.
Furthermore, $\bar{g}$ is continuous. Suppose V is an open set in Y. Then

$$\begin{aligned} U \in \bar{g}^{-1}(V) &\Leftrightarrow \bar{g}(U) \in V \\ &\Leftrightarrow g(x) \in V, \text{ for some } x \notin U \\ &\Leftrightarrow g^{-1}(V) \not\subseteq U \end{aligned}$$

Thus, $\bar{g}^{-1}(V) = \{U \in \mathfrak{M}\mathcal{O}X | g^{-1}(V) \not\subseteq U\}$, which is open in $\mathfrak{M}\mathcal{O}X$. □

Definition 3.14 *A frame L is called T_1 if $\mathfrak{M}L = \Sigma L$.*

Remark 3.14

1. This definition is due to Rosický and Šmarda[74] who also showed that the T_1 frames are precisely those that are quotients of open-set lattices of sober T_1 spaces.

2. If X is a sober T_1 space, then $\mathcal{O}X$ is a spatial T_1 frame: If X is sober T_1 then the irreducible closed sets are precisely the singletons, and hence the prime elements of $\mathcal{O}X$ are precisely the maximal elements.

Proposition 3.15 *A frame L is spatial and T_1 iff each $a \in L$ is a meet of maximal elements.*

PROOF: It is clear from the definition that a spatial T_1 frame has this property. Also, if each $a \in L$ is a meet of maximals, then L is spatial. We now show that each prime is maximal. Let $p \in \Sigma L$. If p is not maximal, then p is a meet of maximal elements, i.e. $p = \bigwedge m_i$. Now, $p \geq m_j \wedge \bigwedge_{i \neq j} m_i$ for each j, and since p is prime, it follows that $p \geq \bigwedge_{i \neq j} m_i$ for each j. It follows that $m_j \leq \bigwedge_{i \neq j} m_i$ which is impossible since m_j is maximal. □

Proposition 3.16 *A compact frame L is T_1 iff ΣL is T_1.*

PROOF: If L is T_1 then $\Sigma L = \mathfrak{M} L$ and hence ΣL is T_1. On the other hand, suppose ΣL is T_1. Consider a prime p in L which is not maximal. Then, since L is compact there exists another prime $q \in L$ such that $p < q < \mathbf{e}$. Now, suppose $p \in \Sigma_a$ for some $a \in L$. Then $p \not\leq a$ from which it follows that $q \not\leq a$ and hence $q \in \Sigma_a$. This contradicts the fact that ΣL is T_1. □

Proposition 3.17 *The map $f : X \to \Sigma(\mathcal{O}X)_{s_{\mathfrak{M}}}$ defined by $f(x) = X \backslash cl\{x\}$ is the sober T_1 reflection of X.*

We omit the proof since it is essentially the same as the proof of Proposition 3.13.

Recall that the saturation nucleus s is defined as

$$s(a) = \bigvee \{x \in L | x \vee y = \mathbf{e} \Rightarrow a \vee y = \mathbf{e}\}$$

Proposition 3.18 *Let L be a compact frame. Then L_s is compact, spatial and T_1. Furthermore, $L_{s_{\mathfrak{M}}} = L_s$.*

PROOF: Note that for any maximal element $m \in L$, $s(m) = m$ and hence $a \leq m \Rightarrow s(a) \leq m$. Thus $s_{\mathfrak{M}}(a) \geq s_{\mathfrak{M}} s(a) \geq s(a)$.
On the other hand, if $a \vee y \neq \mathbf{e}$ then $a \vee y \leq m$ for some maximal $m \in L$ since every compact frame has a maximal element. Hence $(x \vee y = \mathbf{e} \Rightarrow n(a) \vee y = \mathbf{e}) \Rightarrow (x \vee y = \mathbf{e} \Rightarrow a \vee y = \mathbf{e})$, i.e. $s s_{\mathfrak{M}}(a) \leq s(a)$. Thus, $s_{\mathfrak{M}}(a) \leq s(a)$. □

Definition 3.19 *A frame L is called subfit if for each $a, b \in L$*

$$a < b \Rightarrow \exists c \in L \text{ such that } a \vee c \neq \mathbf{e} \text{ and } b \vee c = \mathbf{e}$$

Compact subfit frames are called Wallman frames.

Proposition 3.20 *Every compact spatial T_1 frame is Wallman.*

PROOF: Suppose L is a compact spatial T_1 frame. Then $a < b \Rightarrow \exists$ a maximal $m \in L$ such that $a \leq m$ and $b \not\leq m$. But this means that $a \vee m \neq$ and $b \vee m = \mathbf{e}$. □

Denote by $\mathbf{Sob_1}$ and $\mathbf{Frm_1}$ the full subcategories of sober topological spaces and T_1 frames. Note that $\mathfrak{M} : \mathbf{Sob_1} \to \mathbf{Frm_1}$ is simply the restriction of the spectrum functor. We may also restrict the functor $\mathcal{O}$ to $\mathbf{Sob_1}$. Since $\mathcal{O}X$ is a T_1 frame for any sober T_1 space, the map $\mathcal{O} : \mathbf{Sob_1} \to \mathbf{Frm_1}$ is functorial.

Proposition 3.21 *The functors $\mathfrak{M}$ and $\mathcal{O}$ are adjoint on the right. The unit of the adjunction $\eta_L : L \to \mathcal{O}\mathfrak{M}L : a \mapsto M_a$ is the spatial reflection and the counit $\epsilon_X : X \to \mathfrak{M}\mathcal{O}X : x \mapsto X \backslash clx$ is a homeomorphism.*

Denote by $\mathbf{SpFrm_1}$ the full subcategory of all spatial T_1 frames.

Proposition 3.22 *$\mathfrak{M}$ and $\mathcal{O}$ induce a dual equivalence between $\mathbf{Sob_1}$ and $\mathbf{SpFrm_1}$.*

Corollary 3.23 *The functors $\mathfrak{M}$ and $\mathcal{O}$ reduce to an equivalence between the categories $\mathbf{WFrm}$ of Wallman frames and $\mathbf{KSob_1}$ of compact sober T_1 spaces.*

3.4 Relatively Spatial Extensions

In [18] Banaschewski and Hong introduces the notion of strict extensions $h : M \to L$ that are spatial over L and showed that these are precisely the strict extensions that are determined by sets of filters in L. Their construction is analogous to the classical procedure of extending a topological space X by a suitable set $\mathcal{A}$ of filters.

Such extensions of a space X are called strict extensions and were first considered by Banaschewski [4].

In this section we express some definitions given in [18] in terms of prime elements in order to use our results in Section 3.2 to obtain a characterisation of the strict extensions $h : M \to L$ that are spatial over L in terms of spatial quotients of M.

Definition 3.24 *A dense extension $h : M \to L$ is called a strict extension if M is generated by the image of the right adjoint $h_* : L \to M$.*

Definition 3.25 (Banaschewski-Hong, [18]) *A strict extension $h : M \to L$ is called relatively spatial over L if, in each fibre $h^{-1}\{a\}$, $a \in L$, distinct elements are separated by a completely prime filter of M.*

Remark 3.25 Note that $h : M \to L$ is spatial over L iff, in each fibre $h^{-1}\{a\}, a \in L$ distinct elements x and y are separated by a prime element in M, i.e. there exists a $p \in \Sigma M$ such that $x \leq p$ and $y \not\leq p$ (or conversely). This follows immediately from the fact that the complement of a completely prime filter is a prime principal ideal.

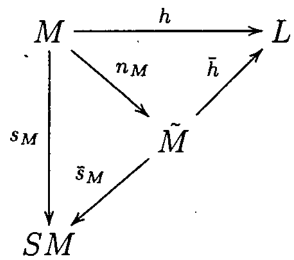
Proposition 3.26 (Banaschewski-Hong, [18]) *The relatively spatial strict extensions form a reflective subcategory of the comma category of all strict extensions of a frame L.*

Remark 3.26 In [18] the relatively spatial reflection of a strict extension $h : M \to L$ is contructed by way of the *trace filters* of L, i.e. the images of completely prime filters in M which are not completely prime in L. In [17] Banaschewski and Gilmour present an alternative description of this reflection. Here the relatively spatial reflection of h is given as a quotient of M determined by the nucleus which is the meet of the nucleus h_*h and the nucleus s_M corresponding to the spatial reflection of M:

Proposition 3.27 *(Banaschewski-Gilmour, [17]) The relatively spatial reflection of a strict extension $h : M \to L$ is given by the factorization*

$$M \xrightarrow{h} L = M \xrightarrow{n_M} Fix(n_M) = \tilde{M} \xrightarrow{\bar{h}} L$$

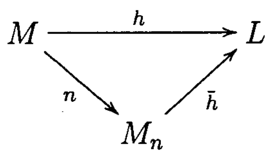
for the nucleus $n_M = (h_*h) \wedge s_M$



In [18] Banaschewski and Hong show that up to isomorphism, the relatively spatial extensions $M \to L$ over L are exactly the spatial extensions $\tau_X L \to L$ determined by sets of filters in L

Proposition 3.28 *Up to isomorphism, the relatively spatial extensions $M \to L$ over L are exactly those that are the meet of a strict extension $N \to L$ and a spatial quotient of N (in the frame of quotients of N).*

PROOF: Suppose $h : M \to L$ is a strict extension, and let $A \subseteq \Sigma M$. We show that the quotient corresponding to the meet of the two nuclei h_*h and s_A is a strict extension spatial over L. Let n be the quotient involved. Then M_n is clearly an extension of L, since $n \leq h_*h$.



Furthermore, $\bar{h}$ is dense (since h is dense) and strict, since n is onto and h is strict. (See Lemma 1 in [18].) Now, consider any two distinct $n(x)$ and $n(y)$ in a fibre $\bar{h}^{-1}\{a\}, a \in L$. Then $h_*h(x) \wedge \bigwedge\{p \in A | p \geq x\} \neq h_*h(y) \wedge \bigwedge\{p \in A | p \geq y\}$. Note

that $h(x) = h(y) = a$ and hence $h_*h(x) = h_*h(y)$, and thus $\bigwedge\{p \in A | p \geq x\} \neq \bigwedge\{p \in A | p \geq y\}$. Thus, there is a prime $p \in \Sigma(M_n)$ separating x and y.
For the converse, we firstly note that the relatively spatial extensions of $h : M \to L$ are reflective in the comma category of all strict extensions of L. (See Banaschewski-Hong [18].) Secondly, the reflection is given by the meet of the spatial reflection of M and the quotient map h. Thus, any relatively spatial extension $h : M \to L$ is a meet of the spatial reflection of M and the quotient h. □

Definition 3.29 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be subcategories of* **Frm**. *Then we call $\mathfrak{A}$ a strict coreflective subcategory of $\mathfrak{B}$ if $\mathfrak{A}$ is coreflective in $\mathfrak{B}$ and the coreflection maps are strict extensions.*

Examples:

1. **KRFrm** is a strict coreflective subcategory of **RFrm**. Similarly the category of compact zero dimensional frames is a strict coreflective subcategory of **0DFrm**.

2. **CRκLindFrm** is a strict coreflective subcategory of **CRFrm** for every regular cardinal κ.

We quote the following lemma which appears in [17] which is used to prove a more general version of Proposition 3 in the same paper.

Lemma 3.30 *Any commuting square*

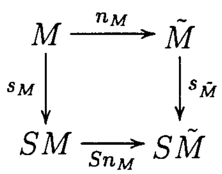
$$\begin{array}{ccc} M & \xrightarrow{h} & L \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ N & \xrightarrow[k]{} & K \end{array}$$

with strict extensions $h : M \to L$ and $k : N \to K$ determines a commuting diagram

$$\begin{array}{ccccc} M & \xrightarrow{n_M} & \tilde{M} & \xrightarrow{\bar{h}} & L \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ N & \xrightarrow[n_N]{} & \tilde{N} & \xrightarrow[\bar{k}]{} & K \end{array}$$

Lemma 3.31 *Let n_M be the relatively spatial reflection of a strict extension $h : M \to L$, for regular L and M. Then Sn_M is an isomorphism.*

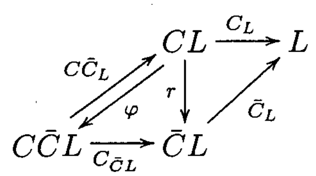
PROOF: Consider the commuting square:



The map Sn_M is onto since $s_{\tilde{M}} \cdot n_M$ is onto. Since both SM and $S\tilde{M}$ are regular, it suffices to show that Sn_M is codense. Suppose $Sn_M(s_M(a)) = \mathbf{e}$. Then $s_{\tilde{M}}(n_M(a)) = \mathbf{e}$. Note that n_M fixes all primes, since $n_M = (h_* h) \wedge s_M$ and s_M fixes all primes. Furthermore, $p = n_M(p)$ is prime in $\tilde{M}$ for any prime p in M since n_M is onto and primes in $\tilde{M}$ are maximal. Thus $s_{\tilde{M}} \cdot n_M$ fixes all primes in M. Hence, if $s_{\tilde{M}}(n_M(a)) = \mathbf{e}$ then there is no prime $p \geq a$. Thus $s_M(a) = \mathbf{e}$. □

Proposition 3.32 *Let $\mathfrak{B}$ be a subcategory of* **RFrm** *and let $\mathfrak{A}$ be a strict coreflective subcategory of $\mathfrak{B}$, with coreflection $C_L : CL \to L$. Then $\bar{\mathfrak{A}} = \{L \in Ob(\mathfrak{B}) | SC_L : SCL \to SL$ is an isomorphism $\}$ is also coreflective in $\mathfrak{B}$, with coreflection map $\bar{C}_L : \bar{C}L \to L$ given by the relatively spatial reflection of $C_L : CL \to L$. Furthermore, $Ob(\mathfrak{A}) \subseteq Ob(\bar{\mathfrak{A}})$.*

PROOF: The correspondence $L \rightsquigarrow \bar{C}L$ is functorial with natural $\bar{C}_L : \bar{C}L \to L$ by the functoriality of C and Lemma 3.30. We now show that $\bar{C}L \in Ob(\bar{\mathfrak{A}})$. Consider the diagram below:

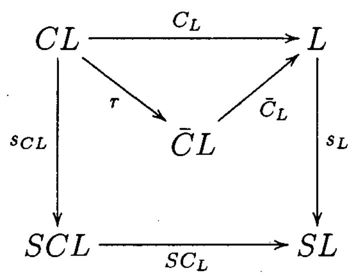


The map r is the quotient corresponding to the relatively spatial reflection of C_L. By the coreflective property of C, there exists a unique frame homomorphism

$\varphi : CL \to C\bar{C}L$ such that $C_{\bar{C}L} \cdot \varphi = r$. Also, by the functoriality of C, there exists a frame homomorphism $C\bar{C}_L : C\bar{C}L \to CL$ such that $C_L \cdot C\bar{C}_L = \bar{C}_L \cdot C_{\bar{C}L}$. Now,

$$\begin{aligned} C_L \cdot C\bar{C}_L \cdot \varphi &= \bar{C}_L C_{\bar{C}L} \cdot \varphi \\ &= \bar{C}_L \cdot r \\ &= C_L \end{aligned}$$

Since L is regular, it follows that C_L is monic, since it is strict and hence $C\bar{C}_L \cdot \varphi = id_{C\bar{C}L}$. Also, $C_{\bar{C}L}$ is monic and hence $C\bar{C}_L$ is monic from which it follows that $C\bar{C}_L$ is an isomorphism. Thus $SC\bar{C}_L : SC\bar{C}L \to SCL$ is an isomorphism. Now, $r \cdot C\bar{C}_L = C_{\bar{C}L}$ and hence $Sr \cdot SC\bar{C}_L = SC_{\bar{C}L}$. From Lemma 3.31 the map Sr is an isomorphism, and since $SC\bar{C}_L$ is also an isomorphism it follows that $SC_{\bar{C}L} : SC\bar{C}L \to S\bar{C}L$ is an isomorphism, i.e. $\bar{C}L \in Ob(\bar{\mathfrak{A}})$.
It now remains to show that $\bar{C}_L : \bar{C}L \to L$ is an isomorphism whenever $L \in Ob(\bar{\mathfrak{A}})$. Suppose $L \in Ob(\bar{\mathfrak{A}})$. Then $SC_L : SCL \to SL$ is an isomorphism.



We show that $\bar{C}_L$ is an isomorphism. We know that $\bar{C}_L$ is onto, so it suffices to show it is codense. Suppose $\bar{C}_L \cdot r(a) = \mathbf{e}$. Then $C_L(a) = \mathbf{e}$ and thus $r(a) = s_{CL}(a)$ since $r = (C_L)_* C_L \wedge s_{CL}$. Thus, $SC_L \cdot r(a) = SC_L \cdot s_{CL}(a) = s_L \cdot C_L(a) = \mathbf{e}$ and hence $r(a) = \mathbf{e}$ since SC_L is an isomorphism.
Finally, it is clear that $Ob(\mathfrak{A}) \subseteq Ob(\bar{\mathfrak{A}})$ since $L \cong CL \Rightarrow SL \cong SCL$. □

Definition 3.33 *A coreflective subcategory* $\mathfrak{A}$ *of a category* $\mathfrak{B} \subseteq$ **RFrm** *is called relatively spatial if the coreflection maps* $C_L : CL \to L$ *are relatively spatial over* L.

Remark 3.33

1. It is clear from Proposition 3.27 that for any strict coreflective subcategory $\mathfrak{A}$ of $\mathfrak{B} \subseteq$ **RFrm**, the category $\bar{\mathfrak{A}}$ is a relatively spatial subcategory of $\mathfrak{B}$. We shall call $\bar{\mathfrak{A}}$ the *relatively spatial hull* of $\mathfrak{A}$.

2. If $\mathfrak{A}$ is a strict subcategory of some $\mathfrak{B} \subseteq$ **RFrm**, then $\bar{\bar{\mathfrak{A}}} = \bar{\mathfrak{A}}$: Suppose $L \in \bar{\bar{\mathfrak{A}}}$. Then $S\bar{C}_L$ is an isomorphism.

$$\begin{array}{ccccc} CL & \xrightarrow{r} & \bar{C}L & \xrightarrow{\bar{C}_L} & L \\ \downarrow{\scriptstyle s_{CL}} & & \downarrow{\scriptstyle s_{\bar{C}L}} & & \downarrow{\scriptstyle s_L} \\ SCL & \xrightarrow[Sr]{} & S\bar{C}L & \xrightarrow[S\bar{C}_L]{} & SL \end{array}$$

From Lemma 3.31 the map Sr is an isomorphism, and hence $SC_L = S\bar{C}_L \cdot Sr$ is an isomorphism. Thus, $L \in Ob(\bar{\mathfrak{A}})$.

3. In general, for any coreflective subcategory $\mathfrak{A}$ of $\mathfrak{B}$ in **RFrm**, it does not necessarily imply that $\mathfrak{A}\cap$**SpFrm** is coreflective in $\mathfrak{B}\cap$**SpFrm**. For example, **RLindFrm** is a coreflective subcategory of **CRFrm**, but the spatial regular Lindelöf frames do not form a coreflective subcategory of the category of spatial completely regular frames. It is however true that for any strict subcategory $\mathfrak{A}$ of $\mathfrak{B}$ in **RFrm**, $\bar{\mathfrak{A}} \cap$ **SpFrm** is coreflective in $\mathfrak{B} \cap$ **SpFrm**. This of course follows from the fact that the coreflection of a spatial frame in $\mathfrak{B}$ to $\bar{\mathfrak{A}}$ is spatial. (See [17])

Chapter 4

Realcompact Frames

4.1 Introduction

As pointed out by J.R. Isbell [53], the 'inclusion functor' from (sober) spaces to locales does not preserve limits. In particular the product of spaces taken in the category of locales does not coincide with their Tychonoff product. Indeed, this product need not even be spatial, as is the case with uncountable powers of the localic reals.

In [60] Madden and Vermeer shows that a completely regular locale L is regular Lindelöf iff L is a closed sublocale of a localic product of the reals. Thus, there are realcompact spaces that are not realcompact in the sense of Madden and Vermeer as locales..

It was G. Schlitt [77], a student of B. Banaschewski who first provided a conservative definition of realcompactness in the pointfree setting. 'Conservative' here means that a space X is realcompact iff its frame of opens $\mathcal{O}X$ is realcompact. Schlitt also showed that the full subcategory of realcompact frames are coreflective in the category **CRFrm** of completely regular frames, and obtained the realcompact coreflection of a frame L as a quotient of βL. In [62] this realcompact coreflection was obtained as a quotient of $\mathcal{H}CozL$. Finally, in [17] a neat description of the realcompact coreflection of a frame L was obtained as the relatively spatial reflection of the strict extenstion given by the join map $j : \mathcal{H}CozL \to L$.

In this chapter we present some basic properties of realcompact frames. In particular we show that the category **ℝKFrm** of realcompact frames is the relatively spatial envelope of the category **RLindFrm** of regular Lindelöf frames. We also show how Schlitt's original description of the realcompact coreflection may be obtained as a relatively spatial quotient from the familiar real points of the Stone-Čech compactification.

4.2 Realcompact frames

Definition 4.1 *Let L be a frame. An ideal I in L is called*

i) σ-proper if $\bigvee S \neq \mathsf{e}$ for every countable $S \subseteq I$.

ii) completely proper if $\bigvee I \neq \mathsf{e}$.

Proposition 4.2 *The following are equivalent for a completely regular frame L:*

1) Every σ-proper maximal completely regular ideal in L is completely proper.

2) Every σ-proper maximal regular ideal in $CozL$ is completely proper.

3) Every σ-proper maximal ideal in $CozL$ is completely proper.

4) Every maximal σ-ideal in $CozL$ is completely proper.

PROOF: (1) $\Leftrightarrow$ (2): This follows from the fact that there is a one-to-one correspondence between the completely regular ideals of L and the regular ideals of $CozL$.
(2) $\Rightarrow$ (3): Let I be a σ-proper maximal ideal in $CozL$. Then $J = \{a \in CozL | a \prec b \in I\}$ is a σ-proper maximal regular ideal in $CozL$: Suppose $K \not\subseteq J$ is a regular

ideal in $CozL$. Then there exists $s \in K - J$. Thus, $s \not\prec x$ for any $x \in I$. But, K is regular, and so $s \prec t \in K$, and $t \notin I$. Thus, there exists $a \in I$ such that $a \vee t = \mathbf{e}$. Now, $CozL$ is regular and hence normal [15], and so there exists $b \in J$ such that $b \vee t = \mathbf{e}$. Thus, $J \vee K = CozL$. That J is regular follows from the fact that $CozL$ is normal and hence the rather below relation interpolates. Now, $\bigvee J = \bigvee\{a \in CozL | a \prec b \in I\} = \bigvee\{b \in CozL | b \in I\} = \bigvee I$. Thus, I is completely proper.

(3) $\Rightarrow$ (2): Let J be a σ-proper maximal regular ideal in $CozL$. Then the ideal $\bar{J} = \bigvee\{\downarrow a | a \in J\}$ is a σ-proper maximal ideal in $CozL$: To show maximality, suppose $a \notin \bar{J}$. Then $r(a) \not\subseteq J$, where $r(a) = \{s \in CozL | s \prec a\}$. Thus, $r(a) \vee J = CozL$, i.e. there exist $s \prec a, b \in J$ such that $s \vee b = \mathbf{e}$. Hence $\downarrow s \vee \bar{J} = CozL$ and so $\downarrow a \vee \bar{J} = CozL$.

Now, suppose S is a countable subset of $\bar{J}$. Then each $s \in S$ is a join of a countable set T_s of elements rather below s. Now, $T = \bigcup_{s \in S} T_s$ is a countable set and $T \subseteq J$. Since J is σ-proper, it follows that $\bigvee T \neq \mathbf{e}$ and hence $\bigvee S \neq \mathbf{e}$, i.e. $\bar{J}$ is σ-proper. It follows from our assumption that $\bar{J}$ is completely proper. But $\bigvee \bar{J} = \bigvee J$ and hence $\bar{J}$ is completely proper.

(3) $\Leftrightarrow$ (4): This follows from the fact that the σ-proper maximal ideals in $CozL$ are exactly the maximal σ-ideals in $CozL$. (See [17].) □

Definition 4.3 *A completely regular frame satisfying the above equivalent conditions is called realcompact.*

Proposition 4.4 *A completely regular frame L is realcompact iff the map $Sj_L : S\mathcal{H}CozL \to SL$ is an isomorphism, where $j_L : \mathcal{H}CozL \to L$ is the join map, i.e. the coreflection to regular Lindelöf frames.*

PROOF: Consider the commuting square

$$\begin{array}{ccc} \mathcal{H}CozL & \xrightarrow{j_L} & L \\ {\scriptstyle s_{\mathcal{H}CozL}} \downarrow & & \downarrow {\scriptstyle s_L} \\ S\mathcal{H}CozL & \xrightarrow[Sj_L]{} & SL \end{array}$$

Since Sj_L is onto, it suffices to show that L is realcompact iff Sj_L is codense. Suppose L is realcompact. Let J be a maximal element of $\mathcal{H}CozL$. Since L is realcompact, $j_L(J) \neq \mathbf{e}$ and since j_L is onto, $j_L(J)$ is maximal. Thus, $s_L \cdot j_L(J) \neq \mathbf{e}$. Now, towards showing that Sj_L is codense, suppose $Sj_L \cdot s_{\mathcal{H}CozL}(I) = \mathbf{e}$. Then $s_L \cdot j_L(I) = \mathbf{e}$. From the preceding argument it follows that I is not contained in a maximal ideal. But this means that $s_{\mathcal{H}cozL}(I) = \mathbf{e}$.
Conversely, suppose Sj_L is codense. Let J be a maximal element of $\mathcal{H}CozL$. Then $s_L \cdot j_L(J) = Sj_L \cdot s_{\mathcal{H}CozL}(J) \neq \mathbf{e}_{sL}$. Thus, $j_L(J) \neq \mathbf{e}_L$, i.e. L is realcompact. □

Remark 4.4 From Proposition 3.32 it follows that the category $\mathbb{R}$**KFrm** of realcompact frames is coreflective in **CRFrm** with coreflection maps given by the relatively spatial reflection of the Lindelöf coreflection. This neat description of the realcompact coreflection is due to Banaschewski and Gilmour [17].

Corollary 4.5 *The category* $\mathbb{R}$**KFrm** *is the relatively spatial hull of* **RLindFrm**.

Corollary 4.6 *A completely regular frame L is spatial and realcompact iff L is the spatial reflection of a regular Lindelöf frame.*

Remark 4.6 This result was known already by Madden and Vermeer [60] and was one of their main motivations for calling the regular Lindelöf frames realcompact.

4.3 The real points of βL

The original description of the realcompact coreflection by Schlitt [77] was as a quotient of the Stone-Čech compactification. In this section we show that this quotient can be expressed as the relatively spatial quotient of the Stone-Čech compactification with respect to its real points.

Let X be a Tychonoff space and suppose f is a real-valued continuous function on X. Let $\mathbb{R}^*$ be the one-point compactification of $\mathbb{R}$. Then, by the universal property of βX, f as a mapping into $\mathbb{R}^*$ extends uniquely to $\bar{f} : \beta X \to \mathbb{R}^*$, i.e the square:

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{R} \\ \downarrow & & \downarrow \\ \beta X & \xrightarrow[\bar{f}]{} & \mathbb{R}^* \end{array}$$

commutes. A point $x \in \beta X$ is called *real* if $\bar{f}(x) \neq \infty$ for all $f \in C(X)$.

The set of all real points of βX endowed with the subspace topology is a realcompact space. Moreover this space is the smallest realcompact space containing X. Thus, the set $\bigcap_{f \in C(X)} \bar{f}^{-1}(\mathbb{R}) \subseteq \beta X$ endowed with the subspace topology is the universal realcompactification vX of X. This description of the realcompact reflection of a Tychonoff space is given in [39].

In order to develop a frame-theoretic analogue of the real points of βL for a frame L, we first need to describe the *frame of reals.* The description we use is that given by Banaschewski and Mulvey [21]. A different but equivalent description appears in Johnstone's book [54].

The frame of reals is the frame $\mathcal{L}(\mathbb{R})$ generated by all $(p, q) \in \mathbb{Q} \times \mathbb{Q}$ subject to the relations:

(R1) $(p, q) \wedge (s, t) = (p \vee s, q \wedge t)$.

(R2) $(p, q) \vee (s, t) = (p, t)$ whenever $p \leq s < q \leq t$.

(R3) $(p, q) = \bigvee\{(s, t) | p < s < t < q\}$.

(R4) $\mathbf{e} = \bigvee\{(p, q) | p, q \in \mathbb{Q}\}$.

Since $\mathcal{L}(\mathbb{R})$ is regular continuous [12], it follows that it has a smallest strong inclusion $\triangleleft$. Recall that $u \triangleleft v$ iff $u \prec v$ and either $\uparrow u$ is compact, or $\uparrow v^*$ is compact. But this is precisely the condition that $u \prec v$, and there exists $p, q \in \mathbb{Q}$ such that either $u \leq (p, q)$, or $v^* \leq (p, q)$.

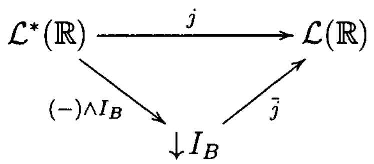
Denote by $\mathcal{L}^*(\mathbb{R})$ the frame of all $\lhd$-regular ideals in $\mathcal{L}(\mathbb{R})$.

Now consider the ideal I_B generated by all (p,q) where $p,q \in \mathbb{Q}$.

Proposition 4.7 *I_B is a maximal element of the frame $\mathcal{L}^*(\mathbb{R})$.*

PROOF: It is clear that $I_B \in \mathcal{L}^*(\mathbb{R})$. Suppose $J \supset I_B$. Then there exists $u \in J$ such that $u \not\leq (p,q)$, for all $(p,q) \in \mathbb{Q} \times \mathbb{Q}$. Now, since J is $\lhd$-regular, there exists $v \in J$ such that $u \lhd v$. But this implies that $u \prec v$ and $v^* \leq (p,q)$, for some $(p,q) \in \mathbb{Q} \times \mathbb{Q}$. Since $v \in J$, there exists $w \in J$ such that $v^* \vee w = \mathbf{e}$. But this implies that $w \vee (p,q) = \mathbf{e}$, and hence $J = \downarrow \mathbf{e}$. □

Consider that map $j : \mathcal{L}^*(\mathbb{R}) \to \mathcal{L}(\mathbb{R})$ given by join. Now j factors through the map $(-) \wedge I_B$.



Note that if $I \in \mathcal{L}^*(\mathbb{R})$ such that $\bigvee I = \mathbf{e}$ then $I \supseteq I_B$ and thus the map $\bar{j}$ is codense. Also, $\bar{j}$ is onto since j is onto and hence $\downarrow I_B \cong \mathcal{L}(\mathbb{R})$.

Remark 4.7

1. It is clear that the ideal I_B in the previous proposition is just the ideal of all $a \in \mathcal{L}(\mathbb{R})$ such that $a \ll \mathbf{e}$. In general, for any regular continuous frame L, $a \ll \mathbf{e}$ iff $\uparrow a^*$ is compact. Thus the set I_B of all elements a such that $a \ll \mathbf{e}$ is a $\lhd$-regular ideal (where $\lhd$ is the smallest strong inclusion on L). It is easy to check that I_B is maximal in the frame of all $\lhd$-regular ideals in L and that $\downarrow I_B \cong L$. This result is well-known and appears in [29]

2. $\mathcal{L}^*(\mathbb{R}) \cong \mathcal{O}\mathbb{R}^*$ via the isomorphism $\phi : \mathcal{L}^*(\mathbb{R}) \to \mathcal{O}\mathbb{R}^*$ given by

$$\phi(I) = \begin{cases} \bigcup I & \text{if } I \in \downarrow I_B \\ \bigcup I \cup \{\infty\} & \text{otherwise} \end{cases}$$

Let $h : \mathcal{L}(\mathbb{R}) \to L$ be a frame homomorphism, and suppose $j : \mathcal{L}^*(\mathbb{R}) \to \mathcal{L}(\mathbb{R})$ be the join map. Then since $j_L : \beta L \to L$ is the compact regular coreflection of L, it follows that there is a unique extension $\bar{h} : \mathcal{L}^*(\mathbb{R}) \to \beta L$ such that the square

$$\begin{array}{ccc} \mathcal{L}(\mathbb{R}) & \xrightarrow{h} & L \\ {\scriptstyle j_{\mathcal{L}(\mathbb{R})}}\uparrow & & \uparrow{\scriptstyle j_L} \\ \mathcal{L}^*(\mathbb{R}) & \xrightarrow[\bar{h}]{} & \beta L \end{array}$$

commutes.

Definition 4.8 *A prime ideal $I \in \beta L$ is called real if $I \not\supseteq \bar{h}(I_B)$ for any frame homomorphism $h : \mathcal{L}(\mathbb{R}) \to L$.*

Proposition 4.9 *An ideal $I \in \beta L$ is real iff I is σ-proper maximal.*

PROOF: Firstly note that since βL is regular, all the prime elements of βL are maximal. It remains to show that a maximal ideal $I \in \beta L$ is σ-proper iff I does not contain any of the $\bar{h}(I_B)$.

Let I be a maximal ideal in βL. If I is σ-proper, then I cannot contain any $\bar{h}(I_B)$ since none of these ideals are σ-proper.

Conversely, suppose I is not σ-proper. Then I contains a regular sequence $a_1 \prec a_2 \prec a_3 \prec \cdots$ such that $\bigvee_L a_n = e$.

Let $\{c_{nq} | q \in \mathbb{Q} \cap [0,1]\}$ be a scale between a_n and a_{n+1}. For each n, put

$$c_r = \begin{cases} 0 & \text{if } r < 0 \\ c_{nr-n} & n \leq r \leq n+1 \end{cases}$$

and define

$$\varphi(p,q) = \bigvee \{c^*_{p'} \wedge c_{q'} | p < p' < q' < q\}$$

To see that φ extends to a frame homomorphism it remains to show that the relations (R1)-(R4) are transformed into identities.

Re (R1):

$$\begin{aligned}\varphi(p,q)\wedge\varphi(r,s) &= \bigvee\{c^*_{p'}\wedge c_{q'}|p<p'<q'<q\}\wedge\bigvee\{c^*_{r'}\wedge c_{s'}|r<r'<s'<s\}\\ &= \bigvee\{c^*_{p'}\wedge c_{q'}\wedge c^*_{r'}\wedge c_{s'}|p<p'<q'<q \text{ and } r<r'<s'<s\}\\ &= \bigvee\{c^*_{p'\vee r'}\wedge c_{q'\wedge s'}|p\vee r<p'\vee r'<q'\wedge s'<q\wedge s\}\\ &= \varphi(p\vee r,q\wedge s)\end{aligned}$$

Re (R2):

Given $p\le s<q\le t$ then $\varphi(p,q)\le\varphi(p,t)$ and $\varphi(s,t)\le\varphi(p,t)$ and hence $\varphi(p,q)\vee\varphi(s,t)\le\varphi(p,t)$.

For the reverse inequality, we consider three cases: If $s<p'$, then $c^*_{p'}\wedge c_{t'}\le c^*_{s'}\wedge c_{t'}$ for $s<s'<p'$. Hence $c^*_{p'}\wedge c_{t'}\le\varphi(p,q)$.

Similarly, if $t'<q$, then $c^*_{p'}\wedge c_{t'}\le c^*_{p'}\wedge c_{q'}$ for $t'<q'<q$. Hence $c^*_{p'}\wedge c_{t'}\le\varphi(p,q)$.

If $p'\le s$ and $q\le t'$ and suppose s' and q' are such that $s\le s'\le q'\le q$, then

$$\begin{aligned}(c^*_{p'}\wedge c_{t'})\wedge((c^*_{p'}\wedge c_{q'})\vee(c^*_{s'}\wedge c_{t'})) &= (c^*_{p'}\wedge c_{t'}\wedge c_{q'})\vee(c^*_{p'}\wedge c_{t'}\wedge c^*_{s'})\\ &= (c^*_{p'}\wedge c_{q'})\vee(c^*_{p'}\wedge c_{t'})\\ &= (c^*_{p'}\vee c^*_{p'})\wedge(c_q\vee c^*_{p'})\wedge(c^*_{p'}\vee c_{t'})\wedge(c_{q'}\vee c_{t'})\\ &= c^*_{p'}\wedge \mathbf{e}\wedge\mathbf{e}\wedge c_{t'}\\ &= c_{p'}\wedge c_{t'}\end{aligned}$$

Thus, $c^*_{p'}\wedge c_{t'}\le(c^*_{p'}\wedge c_{q'})\vee(c^*_{s'}\wedge c_{t'})$ and hence $c^*_{p'}\wedge c_{t'}\le\varphi(p,q)\vee\varphi(s,t)$.

In all three cases $\varphi(p,t)\le\varphi(p,q)\vee\varphi(s,t)$.

Re (R3):

$$\begin{aligned}\bigvee\{\varphi(r,s)|p<r<s<q\} &= \bigvee\{c^*_{r'}\wedge c_{s'}|p<r<r'<s'<s<q\}\\ &= \varphi(p,q)\end{aligned}$$

Re (R4):

$$\begin{aligned}\bigvee\{\varphi(p,q)|p,q\in\mathbb{Q}\} &= \bigvee\{c^*_p|p\in\mathbb{Q}\}\wedge\bigvee\{c_q|q\in\mathbb{Q}\}\\ &= \mathbf{e}\end{aligned}$$

since $c^*_{-1} = \mathbf{e}$ and $\bigvee\{c_q | q \in \mathbb{Q}\} = \bigvee\{a_n | n \in \mathbb{N}\}$.
Now, $\varphi(p,q) \leq c_q$ and hence $\varphi(p,q) \in I$, from which it follows that $\bar{\varphi}(I_B) \subseteq I$. Thus I is not real. □

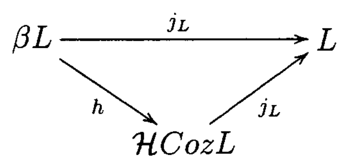
Remark 4.9 The description of the Urysohn-type function φ as well as the proof that it determines a frame homomorphism appears in [16] where the authors use it to characterise pseudocompactness in frames. We have quoted the details here for the sake of completeness.

Definition 4.10 *A frame L is called pseudocompact if every frame homomorphism $h : \mathcal{L}(\mathbb{R}) \to L$ is bounded, i.e. $h((p,q)) = \mathbf{e}$ for some $p, q \in \mathbb{Q}$.*

Proposition 4.11 *A frame L is pseudocompact iff every maximal $I \in \beta L$ is real.*

PROOF: Note that L is pseudocompact iff $\bar{h}(I_B) = \mathbf{e}$ for any frame homomorphism $h : \mathcal{L}(\mathbb{R}) \to L$, where $\bar{h} : \mathcal{L}^*(\mathbb{R}) \to \beta L$ is the extension of L. Thus, if L is pseudocompact, $I \not\supseteq \bar{h}(I_B)$ for every maximal $I \in \beta L$. On the other hand, suppose L is not pseudocompact. Then there exists a frame homomorphism $\varphi : \mathcal{L}(\mathbb{R}) \to L$ such that $\bar{\varphi}(I_B) \neq \mathbf{e}$. But then $\bar{\varphi}(I_B)$ maximal, since I_B is maximal. Clearly $\bar{\varphi}(I_B)$ is not real. □

Since βL is a regular Lindelöf frame, it follows that the join map $j_L : \beta L \to L$ factors uniquely through the join map $j_L : \mathcal{H}CozL \to L$.



If βL denotes the frame of all regular ideals in $CozL$, then $h(I) = \{a \in CozL | a \leq \bigvee S$, for some countable $S \subseteq I\}$.

Proposition 4.12 *The real points of βL are precisely the images of the prime elements of $\mathcal{H}CozL$ under the right adjoint h_* of h.*

PROOF: $\mathcal{H}CozL$ is regular, so the prime elements are precisely the maximal elements. Let J be a maximal element of $\mathcal{H}CozL$. Then $h_*(J)$ is prime and hence maximal in βL. Also, $h_*(J)$ is σ-proper, for suppose $S \subseteq h_*(J)$ is countable. Then $h[S] \subseteq J$ and hence $\bigvee h[S] \in J$. But then $h(\bigvee S) \in J$ from which it follows that $\bigvee S \neq \mathbf{e}$. Hence by Proposition 4.9 $h_*(J)$ is real.
On the other hand, suppose J is a σ-proper maximal regular ideal in $CozL$. Then $h(J) \neq \mathbf{e}$ and since h is onto it follows that $h(J)$ is maximal. Thus, $h_*h(J)$ is maximal in βL, i.e. $h_*h(J) = J$. □

Corollary 4.13 *Let L be a frame, then the spatial quotient of βL corresponding to the real points is isomorphic to $S\mathcal{H}CozL$.*

PROOF: This follows immediately from Proposition 3.12 □

Corollary 4.14 *The relatively spatial quotient of $j_L : \beta L \to L$ corresponding to the real points of βL is the universal realcompactification of L.*

Chapter 5

Zero dimensional frames

5.1 Introduction

As with realcompactness, there are two frame-theoretic notions of $\mathbb{N}$-compactness. Both notions are discussed by Schlitt [77], [78] in the form of Stone-$\mathbb{N}$-compactness (S-$\mathbb{N}$-compactness) and Herrlich-$\mathbb{N}$-compactness (H-$\mathbb{N}$-compactness). Schlitt furthermore shows that the S-$\mathbb{N}$-compact frames are precisely the zero dimensional Lindelöf frames iff the Axiom of Countable Choice holds. It is also to be noted that Paseka [71] had independently, and at about the same time as Schlitt, observed that this notion of $\mathbb{N}$-compactness corresponds to the zero dimensional Lindelöf property (but he did not mention the equivalence of this fact to the Axiom of Countable Choice).

Schlitt's second notion of $\mathbb{N}$-compactness, viz. H-$\mathbb{N}$-compactness, turned out to be more compatable with the classical notion. He showed that a space X is $\mathbb{N}$-compact iff its frame of opens $\mathcal{O}X$ is H-$\mathbb{N}$-compact. In this chapter we investigate Schlitt's H-$\mathbb{N}$-compactness which we will refer to as the frame-theoretic version of $\mathbb{N}$-compactness.

5.2 Strongly zero dimensional frames

Definition 5.1 *A completely regular frame L is called strongly zero dimensional if βL is zero dimensional.*

Remark 5.1 Since compact regular frames are spatial (assuming the Boolean ultrafilter theorem), it follows that a space X is strongly zero dimensional iff $\mathcal{O}X$ is strongly zero dimensional.

Proposition 5.2 *The following are equivalent for a frame L:*

1) L is strongly zero-dimensional.

2) $\beta L \cong \zeta L$.

3) If $a \prec\!\!\prec b$ in L then there exists $c \in \mathbb{B}L$ such that $a \leq c \leq b$.

4) If $a \prec b$ in $CozL$ then there exists $c \in \mathbb{B}L$ such that $a \leq c \leq b$.

5) $CozL$ is zero dimensional (as a σ-frame).

6) $CozL = \mathbb{B}_{\omega_1}L$.

7) $Coz_\kappa L = \mathbb{B}_\kappa L$ for every cardinal $\kappa > \omega_1$.

8) $Coz_\kappa L$ is zero dimensional for every cardinal $\kappa > \omega$.

9) $\mathcal{H}_\kappa Coz_\kappa L$ is zero-dimensional for every cardinal $\kappa > \omega$.

PROOF: (1) $\Rightarrow$ (2): This follows from the fact that ζL is the universal zero-dimensional compactification of L.
(2) $\Rightarrow$ (3): If $\beta L \cong \zeta L$ then the strong inclusions corresponding to the two compactifications coincide. But the strong inclusion on L corresponding to βL is the completely below relation, and the strong inclusion corresponding to ζL is $\lhd_{\mathbb{B}L}$ where $a \lhd_{\mathbb{B}L} b$ iff there exists $c \in \mathbb{B}L$ such that $a \leq c \leq b$.
(3) $\Rightarrow$ (4): If $a \prec b$ in $CozL$, then $a \prec\!\!\prec b$ in L.

(4) $\Rightarrow$ (5): Each $a \in CozL$ is a countable join of cozero elements rather below it. Thus, each $a \in CozL$ is a countable join of complemented elements.
(5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (9): Trivial.
(9) $\Rightarrow$ (5): If each $\mathcal{H}_\kappa Coz_\kappa L$ is zero dimensional, then in particular $\mathcal{H}CozL$ is zero dimensional. Note that the complemented elements of $\mathcal{H}CozL$ are all principal ideals generated by complemented elements of L. Since the principal ideals of $\mathcal{H}CozL$ are exactly the Lindelöf elements, it follows that these are countable joins of complemented elements of $\mathcal{H}CozL$. Thus, $CozL$ is a zero dimensional σ-frame.
(5) $\Rightarrow$ (1): Suppose $I \in \beta L$, i.e. I is a regular ideal in $CozL$. Let $s \in I$. Then $s \prec t$ for some $t \in I$, i.e. there exists $\bar{s} \in CozL$ such that $\bar{s} \wedge s = \mathbf{0}$ and $\bar{s} \vee t = \mathbf{e}$. Since $CozL$ is zero-dimensional, there exists a countable set $\{b_n \in \mathbb{B}L | n \in \mathbb{N}\}$ such that $\bigvee b_n = \mathbf{e}$ and $b_i \leq \bar{s}$ or $b_i \leq t$ for each $i \in \mathbb{N}$. Let $z_1 = b_1$ and $b_n = z_n \wedge (\bigvee_{i<n} b_i)^*$. Then $\bigvee z_n = \mathbf{e}$ and $z_i \wedge z_j = \mathbf{0}$ if $i \neq j$. Let $b = \bigvee\{z_i | z_i \leq t\}$. Note that $b \in \mathbb{B}L$ since $b^* = \bigvee\{z_i | z_i \not\leq t\} = \bigvee\{z_i | z_i \leq \bar{s}\}$. Thus $b^* \leq \bar{s}$ and hence $s \wedge b^* = \mathbf{0}$. Also, $b \leq t$ and so $s \leq b \leq t$ and hence $b \in I$. Thus I may be expressed as a join of principal ideals generated by complemented elements. □

Remark 5.2

1. Every strongly zero dimensional frame is zero dimensional: Let $a \in L$. Then since L is completely regular, $a = \bigvee a_i$, $a_i \prec\prec a$. But then for each i there exists $b_i \in \mathbb{B}L$ such that $a_i \leq b_i \leq a$. Thus $a = \bigvee b_i$.

2. Since $CozL \cong Coz\,vL$, it follows that L is strongly zero dimensional iff vL is strongly zero dimensional.

Corollary 5.3 *Let $\kappa > \omega$. A completely regular κ-Lindelöf frame L is zero-dimensional iff L is strongly zero-dimensional.*

PROOF: If L is zero-dimensional κ-Lindelöf, then $\mathcal{H}_\kappa Coz_\kappa L$ is zero-dimensional, since $L \cong \mathcal{H}_\kappa Coz_\kappa L$. But from Proposition 5.2 this means that L is strongly zero dimensional. □

Corollary 5.4 *Let $\kappa > \omega$. Then $j_L : \mathcal{H}_\kappa \mathbb{B}_\kappa L \to L$ is the universal zero-dimensional κ-Lindelöfication.*

PROOF: Let $h : M \to L$ be a zero-dimensional Lindelöfication. Then $h[\mathbb{B}M] \subseteq \mathbb{B}L$ and hence $h[\mathbb{B}_\kappa M] \subseteq \mathbb{B}_\kappa L$. But M is strongly zero-dimensional, so $\mathbb{B}_\kappa M = Coz_\kappa M$. Thus h is over $\mathbb{B}_\kappa L$. From Proposition 2.13, $\mathcal{H}_\kappa \mathbb{B}_\kappa L$ is the universal κ-Lindelöfication over $\mathbb{B}_\kappa L$. Hence h factors uniquely through $\mathcal{H}_\kappa \mathbb{B}_\kappa L$. □

Remark 5.4 A description of the universal zero dimensional Lindelöfication of a zero dimensional frame L first appeared in [71]. Here it is described as the smallest subframe of the frame σ-$\mathfrak{J}L$ of all σ-ideals of L containing $\mathbb{B}\sigma$-$\mathfrak{J}L$. In [78] and [17] the universal zero dimensional Lindelöfication is obtained as a quotient of ζL by way of a special case of the nucleus described in Proposition 2.15. Proposition 2.18 may be applied to obtain the equivalence between $\mathcal{H}\mathbb{B}_{\omega_1} L$ and the frame $\mathfrak{H}\mathbb{B}L$ described in [17].

5.3 $\mathbb{N}$-Compact Frames

Definition 5.5 *A frame L is called $\mathbb{N}$-compact if every σ-proper maximal ideal in $\mathbb{B}L$ is completely proper.*

Remark 5.5 Schlitt [78] also showed that the above definition is conservative, i.e. a zero dimensional space X is $\mathbb{N}$-compact iff $\mathcal{O}X$ is an $\mathbb{N}$-compact frame.

Lemma 5.6 (Banaschewski-Gilmour [17]) *A frame L is $\mathbb{N}$-compact iff every maximal ideal $J \in \mathcal{H}\mathbb{B}_{\omega_1} L$ is principal.*

PROOF: We show that there is a one-to-one correspondence between the σ-proper maximal ideals in $\mathbb{B}L$ and the maximal σ-ideals in $\mathbb{B}_{\omega_1} L$.
Let J be a σ-proper maximal ideal in $\mathbb{B}L$. Then the ideal $[J] = \{\bigvee S | S \subseteq J,\ \text{countable}\ \}$ is a proper ideal in $\mathbb{B}_{\omega_1} L$. Now, if $K \supseteq [J]$ for some $K \in \mathcal{H}\mathbb{B}_{\omega_1} L$,

then $K \cap \mathbb{B}L = J$ since J is maximal. Hence $K = [J]$, i.e. $[J]$ is a maximal σ-ideal in $\mathbb{B}_{\omega_1} L$.
On the other hand, let P be a maximal σ-ideal in $\mathbb{B}_{\omega_1} L$. Then $P \cap \mathbb{B}L$ is σ-proper. We claim that $P \cap \mathbb{B}L$ is maximal. For, suppose $b \notin P \cap \mathbb{B}L$. Then $\downarrow b \cap \downarrow b^* \subseteq P$. Now P is maximal and hence prime and thus $\downarrow b^* \subseteq P$. It follows that $b^* \in P \cap \mathbb{B}L$, and hence $\downarrow b \vee P \cap \mathbb{B}L = \mathbf{e}$. $\square$

Remark 5.6 Since $\mathbb{B}_{\omega_1} L \subseteq CozL$ it follows from Proposition 4.2 that all $\mathbb{N}$-compact frames are realcompact.

Corollary 5.7 *If L is strongly zero dimensional, then L is realcompact iff L is $\mathbb{N}$-compact.*

PROOF: This follows from the fact that $CozL = \mathbb{B}_{\omega_1} L$. $\square$

Proposition 5.8 *A frame L is $\mathbb{N}$-compact iff the map $Sj : S\mathcal{H}\mathbb{B}_{\omega_1} L \to SL$ is an isomorphism.*

PROOF: Consider the following commuting square:

$$\begin{array}{ccc} \mathcal{H}\mathbb{B}_{\omega_1} L & \xrightarrow{\ j\ } & L \\ {\scriptstyle s_{\mathcal{H}\mathbb{B}_{\omega_1} L}} \downarrow & & \downarrow {\scriptstyle s_L} \\ S\mathcal{H}\mathbb{B}_{\omega_1} L & \xrightarrow[Sj]{} & SL \end{array}$$

Now, $Sj \cdot s_{\mathcal{H}\mathbb{B}_{\omega_1} L} = s_L \cdot j$ is onto and hence Sj is onto. Thus it suffices to show that L is $\mathbb{N}$-compact iff Sj is codense.
Suppose L is $\mathbb{N}$-compact. Let J be maximal in $\mathcal{H}\mathbb{B}_{\omega_1} L$. Then $j(J) \neq \mathbf{e}$ and since j is onto, it follows that $j(J)$ is maximal. Thus $s_L \cdot j(J) \neq \mathbf{e}$. Now, suppose $Sj \cdot s_{\mathcal{H}\mathbb{B}_{\omega_1} L}(I) = \mathbf{e}$. Then $s_L \cdot j(I) = \mathbf{e}$. Thus I is not contained in a maximal ideal, and hence $s_{\mathcal{H}\mathbb{B}L_{\omega 1}}(I) = \mathbf{e}$.
Conversely, suppose Sj is codense. Let K be a maximal element of $\mathcal{H}\mathbb{B}_{\omega_1} L$. Then $s_L \cdot j(K) = Sj \cdot s_{\mathcal{H}\mathbb{B}_{\omega_1} L}(K) = Sj(K) \neq \mathbf{e}$, since Sj is codense. Thus, $j(K) \neq \mathbf{e}$,

and hence L is $\mathbb{N}$-compact. $\square$

Corollary 5.9 *The category of $\mathbb{N}$-compact frames is coreflective in the category of zero dimensional frames with coreflection maps given by the relatively spatial reflection of $j_L : \mathcal{H}\mathbb{B}_{\omega_1} L \to L$.*

PROOF: This follows immediately from Proposition 3.32. $\square$

Corollary 5.10 *The category* **NKFrm** *of $\mathbb{N}$-compact frames is the relatively spatial hull of the category of zero dimensional Lindelöf frames.*

Remark 5.10 The description of the $\mathbb{N}$-compact coreflection as the relatively spatial reflection of the zero dimensional Lindelöf coreflection appears in [17].

Corollary 5.11 *A frame L is spatial and $\mathbb{N}$-compact iff L is the spatial reflection of a zero dimensional Lindelöf frame.*

5.4 κ-Type Stone dualities

Denote by **0DFrm** and **0DκFrm** the categories of zero dimensional frames and zero dimensional κ-frames. We define a covariant functor $\mathfrak{S}_\kappa$: **0DFrm** $\to$ **0DκFrm** by $\mathfrak{S}_\kappa(L) = \mathbb{B}_\kappa L$ and homomorphism restriction. Note that for every frame homomorphism $h : M \to L$, $h[\mathbb{B}M] \subseteq \mathbb{B}L$ and thus $h[\mathbb{B}_\kappa M] \subseteq \mathbb{B}_\kappa L$. Hence $\mathfrak{S}_\kappa h : \mathfrak{S}_\kappa M \to \mathfrak{S}_\kappa L$ defined by $\mathfrak{S}_\kappa h(a) = h(a)$ is a well-defined κ-frame homomorphism.

Recall that the functor $\mathcal{H}_\kappa$: **RκFrm** $\to$ **CRFrm** takes a κ-frame to its frame of κ-ideals. Now for any zero dimensional κ-frame A, $\mathcal{H}_\kappa A$ is a (strongly) zero dimensional frame. Thus, the restriction $\mathcal{H}_\kappa$: **0DκFrm** $\to$ **0DFrm** is well-defined.

Proposition 5.12 *The functor $\mathcal{H}_\kappa$ is left adjoint to $\mathfrak{S}_\kappa$. The unit of the adjunction $\eta_A : A \to \mathfrak{S}_\kappa\mathcal{H}_\kappa A$ given by $\eta_L(a) =\downarrow a$ is an isomorphism and the counit $\varepsilon_L : \mathcal{H}_\kappa\mathfrak{S}_\kappa L \to L$ is given by $\varepsilon_L(J) = \bigvee J$.*

PROOF: For the naturality of η and ε, let $h : A \to B$ be a κ-frame homomorphism and $k : L \to M$ a frame homomorphism. Now, consider the two squares:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathfrak{S}_\kappa\mathcal{H}_\kappa A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathfrak{S}_\kappa\mathcal{H}_\kappa h} \\ B & \xrightarrow[\eta_B]{} & \mathfrak{S}_\kappa\mathcal{H}_\kappa B \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{H}_\kappa\mathfrak{S}_\kappa L & \xrightarrow{\varepsilon_L} & L \\ {\scriptstyle \mathcal{H}_\kappa\mathfrak{S}_\kappa k}\downarrow & & \downarrow{\scriptstyle k} \\ \mathcal{H}_\kappa\mathfrak{S}_\kappa M & \xrightarrow[\varepsilon_M]{} & M \end{array}$$

The first square commutes since $\mathfrak{S}_\kappa\mathcal{H}h \cdot \eta_A(a) = \mathfrak{S}_\kappa\mathcal{H}_\kappa h[\downarrow a] =\downarrow h(a) = \eta_B \cdot h(a)$. Towards showing the second square commutes, let $J \in \mathcal{H}_\kappa\mathfrak{S}_\kappa L$. Then $\varepsilon_M \cdot \mathcal{H}_\kappa\mathfrak{S}_\kappa k(J) = \varepsilon_M K$, where K is the κ-ideal in $\mathbb{B}_\kappa M$ generated by $k[J]$. Now, $\varepsilon_M K = \bigvee_M k[J] = k(\bigvee_L J) = k \cdot \varepsilon_L(J)$. For the adjunction identities:
$\mathfrak{S}_\kappa L \xrightarrow{\eta_{\mathfrak{S}_\kappa L}} \mathfrak{S}_\kappa\mathcal{H}\mathfrak{S}_\kappa L \xrightarrow{\mathfrak{S}_\kappa\varepsilon_L} \mathfrak{S}_\kappa L$ is the identity $a \mapsto \downarrow a \mapsto \bigvee \downarrow a = a$. And $\mathcal{H}_\kappa A \xrightarrow{\mathcal{H}_\kappa\eta_A} \mathcal{H}_\kappa\mathfrak{S}_\kappa\mathcal{H}_\kappa A \xrightarrow{\varepsilon_{\mathcal{H}_\kappa A}} \mathcal{H}_\kappa A$ is the identity $J \mapsto \{K \in \mathcal{H}_\kappa A | K \subseteq \downarrow a$, for some $a \in J\} \mapsto \bigcup\{\downarrow a | a \in J\} = J$. □

Proposition 5.13 *The functors $\mathcal{H}_\kappa$ and $\mathfrak{S}_\kappa$ induce an equivalence between the full subcategories of (strongly) zero dimensional κ-Lindelöf frames and zero dimensional κ-frames.*

PROOF: This follows immediately from the fact that the unit η of the adjunction above is an isomorphism, and the counit yields the zero dimensional κ-Lindelöf coreflection. □

Remark 5.13 In [92] *P-frames* are defined analogously to P-spaces [39], viz. a frame L is called a P-frame iff $CozL$ is a Boolean σ-algebra. Clearly all P-frames are strongly zero dimensional, and the functors $\mathcal{H}_{\omega_1}$ and $\mathfrak{S}_{\omega_1}$ induce an equivalence between the categories of Lindelöf P-frames and Boolean σ-algebras.

5.5 The natural points of ζL

We denote by $\mathcal{N}$ the complete atomic Boolean algebra with countably many atoms. The frame $\mathcal{N}$ is called the frame of naturals. Since $\mathcal{N}$ is regular continuous, it has a smallest compactification $\mathcal{N}^*$,where $\mathcal{N}^*$ is the frame of all ideals I in $\mathcal{N}$ such that for every $a \in I$, a is either a join of finitely many atoms, or $a \leq b \in I$, where b^* is a join of finitely many atoms.

Let L be a zero-dimensional frame. Then every frame homomorphism $h : \mathcal{N} \to L$ extends to a frame homomorphism $\bar{h} : \mathcal{N}^* \to \zeta L$ such that the square

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{h} & L \\ \uparrow & & \uparrow \\ \mathcal{N}^* & \xrightarrow[\bar{h}]{} & \zeta L \end{array}$$

commutes.

Definition 5.14 *A prime element $J \in \Sigma\zeta L$ is called a natural prime or a natural point if $J \not\supseteq \bar{h}(I_F)$ for any frame homomorphism $h : \mathcal{N} \to L$, where I_F is the ideal of all finite elements of $\mathcal{N}$.*

Proposition 5.15 *Let L be a zero-dimensional frame. Then the natural points of ζL are precisely the σ-proper maximal ideals of $\mathbb{B}L$.*

PROOF: Note that $\bar{h}(I_F)$ is the ideal in $\mathbb{B}L$ generated by $h[I_F]$, and since I_F is not σ-proper, it follows that $\bar{h}(I_F)$ is not σ-proper. Thus, if $J \supseteq \bar{h}(I_F)$ then J is not σ-proper.
Conversely, suppose J is not σ-proper. Then there exists a countable $S \subseteq J$ such that $\bigvee S = \mathbf{e}$. Let $z_1 = s_1$, and $z_n = s_n \wedge (\bigvee_{i<n} s_i)^*$, for $n > 1$. Then $\{z_n | n \in \mathbb{N}\}$ is a set of pairwise disjoint complemented elements whose join is $\mathbf{e}$. Suppose $\{a_n | n \in \mathbb{N}\}$ is the set of atoms of $\mathcal{N}$. Consider the map $\varphi : \mathcal{N} \to L$ where $\varphi(a_n) = z_n$. It is clear that φ defines a frame homomorphism. Furthermore $\varphi[I_F] \subseteq J$. Thus J is not a natural point of ζL. □

Let $h : \zeta L \to \mathcal{H}\mathbb{B}_{\omega_1} L$ be the frame homomorphisms determined by the nucleus in Proposition 2.15. If one considers ζL as the frame $\mathfrak{J}\mathbb{B}L$ of all ideals in $\mathbb{B}L$, then $h(I) = \{a \in \mathbb{B}_{\omega_1} L | a \leq \bigvee S, \text{ for some countable } S \subseteq I\}$.

Proposition 5.16 *The natural points of ζL are precisely the images of the prime (maximal) elements of $\mathcal{H}_\kappa \mathbb{B}_{\omega_1} L$ under the right adjoint h_* of the map h described above.*

PROOF: Let J be a maximal σ-ideal in $\Sigma_{\mathbb{B}B}$. Then $h_*(J)$ is prime and hence maximal in ζL. Now, suppose $S \subseteq h_*(J)$ is countable. Then $h[S] \subseteq J$ and since J is a σ-ideal, it follows that $\bigvee h[S] \in J$. Thus $h(\bigvee S) \in J$ and since J is proper it follows that $\bigvee S \neq \mathbf{e}$. Thus, $h_*(J)$ is σ-proper.
On the other hand, suppose I is a σ-proper maximal ideal in $\mathbb{B}L$. Then $h(I) \neq \mathbf{e}$ and since h is onto, it follows that $h(I)$ is maximal. Thus $h_*h(I)$ is proper. But since $h_*h(I) \supseteq I$, and I is maximal it follows that $I = h_*h(I)$. □

Corollary 5.17 *Let L be a zero dimensional frame. Then the spatial quotient of ζL corresponding to the natural points is isomorphic to $S\mathcal{H}\mathbb{B}_{\omega_1} L$.*

Corollary 5.18 *Let L be a zero dimensional frame. Then the relatively spatial quotient of $j_L : \zeta L \to L$ corresponding to the natural points of ζL is νL, the $\mathbb{N}$-compact coreflection of L.*

5.6 $\mathbb{N}$-Pseudocompact Frames

Definition 5.19 *Let M be a regular continuous frame. We call $a \in M$ bounded if $a << \mathbf{e}$. A frame homomorphism $h : M \to L$ is called bounded if $h(a) = \mathbf{e}$ for some bounded $a \in M$.*

Definition 5.20 *Let M be a regular continuous frame. A frame L is called M-pseudocompact if every frame homomorphism $h : M \to L$ is bounded.*

Remark 5.20 The set of all bounded elements of a regular continuous frame M form an ideal denoted by I_B. It is clear from our definition that a frame L is M-pseudocompact iff the ideal generated by $h[I_B]$ is L (or, simply $h[I_B] = L$).

Proposition 5.21 *Every compact regular frame L is M-pseudocompact for every regular continuous frame M.*

PROOF: Let $\varphi : M \to L$ be a frame homomorphism. Since M is regular continuous, it follows that $\bigvee I_B = \mathbf{e}$. Thus, $\bigvee \varphi[I_B] = \mathbf{e}$. Now, since L is compact, there exists $a \in I_B$ such that $\varphi(a) = \mathbf{e}$. Thus, $\varphi[I_B] = L$. □

Proposition 5.22 *Let L be a frame. Then the following are equivalent:*

1) L is $\mathcal{N}$-pseudocompact

2) Every increasing sequence of complemented elements in L terminates.

3) $\mathbb{B}_{\omega_1} L$ is compact.

4) $\mathcal{H}\mathbb{B}_{\omega_1} L$ is compact.

5) νL is compact.

6) $\nu L \cong \zeta L$

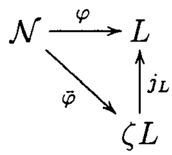
PROOF: (1) ⇒ (2): Let $s_1 \leq s_2 \leq s_3 \leq \cdots$ be an increasing sequence of complemented elements in L. Let $\{a_n | n \in \mathbb{N}\}$ be the set of all atoms of $\mathcal{N}$. Define a map $\varphi : \mathcal{N} \to L$ by $\varphi(a_1) = s_1$ and $\varphi(a_i) = (\bigvee_{i<n} s_i)^* \wedge s_n$. Then φ extends to a frame homomorphism which, by our hypothesis is bounded. Thus, $\mathbf{e} = \varphi(s_{k_1} \vee \cdots s_{k_n}) \leq s_k$, where $k = \max\{k_1, \ldots, k_n\}$.
(2) ⇒ (3): Let $\{a_n | n \in \mathbb{N}\} \subseteq \mathbb{B}_{\omega_1} L$ with $\bigvee a_i = \mathbf{e}$. Let $x_1 = a_1$ and $x_n = \bigvee_{i \leq n} a_i$. Then (x_n) is an increasing sequence with $\bigvee x_i = \mathbf{e}$. Since (x_n) terminates, $x_k = \mathbf{e}$ for some $k \in \mathbb{N}$ and hence $\bigvee_{i \leq k} a_i = \mathbf{e}$.
(3) ⇒ (4): This follows from the fact that the functor $\mathcal{H}$ preserves compactness

[90].

(4) $\Rightarrow$ (5): $\mathcal{H}\mathbb{B}_{\omega_1}L$ is compact regular and hence spatial. Thus, it is isomorphic to the relatively spatial reflection of $j_L : \mathcal{H}\mathbb{B}_{\omega_1}L \to L$ [17]. Thus, $\nu L \cong \mathcal{H}\mathbb{B}_{\omega_1}L$.

(5) $\Rightarrow$ (6): Since νL is zero dimensional compact, and ζL is $\mathbb{N}$-compact, it follows from the universal properties of νL and ζL that $\nu L \cong \zeta L$.

(6) $\Rightarrow$ (1): Since $\mathcal{N}$ is $\mathbb{N}$-compact, it follows that every frame homomoprhism $\varphi : \mathcal{N} \to L$ factors through $j_L : \nu L \to L$. But, $\nu L \cong \zeta L$ and hence φ factors through the map $j_L : \zeta L \to L$.



Now, ζL is compact and hence $\mathcal{N}$-pseudocompact, and thus $\bar{\varphi}$ is bounded. Hence φ is bounded. $\square$

Remark 5.22 Since $\mathbb{B}_{\omega_1}L \subseteq CozL$ it follows that every pseudocompact frame is $\mathbb{N}$-pseudocompact. The converse holds if L is strongly zero-dimensional.

Chapter 6

E-compact Frames

6.1 Introduction

The notion of E-compact spaces was introduced by Engelking and Mrówka [36], [65] as a common generalisation of compact Hausdorff spaces and realcompact spaces (originally called Q-spaces). The definition of E-complete regularity of a space X given by the authors in [36] was that points and closed sets are separated by continuous functions from X to some finite power of E. This definition was earlier shown to be equivalent (by Mrówka himself) to the more popular characterisation that X may be embedded into some power of E.

In [43] H. Herrlich introduced the notion of a k-compact space, for an infinite nonmeasurable cardinal k. M. Hušek later showed that the class of k-compact spaces is simple in **Haus**, i.e. there exists a k-compact space P_k such that every k-compact space can be embedded as a closed subspace of a power of P_k. Independently of Engelking and Mrówka, Herrlich introduced the notions of $\mathcal{E}$-regular and $\mathcal{E}$-compact spaces, where $\mathcal{E}$ may be an arbitrary class of topological spaces. The idea was to develop a common generalisation of compact, realcompact as well as k-compact spaces. A more complete account of the development of these notions is provided in [46]. In [49] Hong introduced the notion of a zero dimensionally k-compact space and showed that the category of zero dimensionally k-compact spaces is simple in **Haus**.

In this chapter we shall discuss the frame-theoretic analogues of k-compact spaces, zero dimensionally k-compact spaces and E-compact spaces. We also investigate the role of the frames of open sets of Hušek and Hong's spaces.

6.2 E-regular frames

Definition 6.1 *Let E be a frame. A frame L is called E-regular if it is a quotient of a copower of E.*

Remark 6.1

1. It is clear from the definition that E-regular frames are closed under coproducts and quotients.

2. In [36] Engelking and Mrówka defined a space X to be E-regular (E-completely regular in their terminology) for some space E if for any closed subspace $A \subseteq X$ and any $p \notin A$ there exists a continuous function $f : X \to E^n$ for some natural number n, such that $f(p) \notin Cl_{E^n} f[A]$. This condition was shown to be equivalent to the condition that X be embedded into some power E^m of E by Mrówka [64].

Proposition 6.2 *Let E be a sober space. Then a T_0 space X is E-regular iff $\mathcal{O}X$ is $\mathcal{O}E$-regular.*

PROOF: Suppose X is E-regular. Then there exists an embedding $e : X \hookrightarrow \prod_I E$ for some index set I. Applying the functor $\mathcal{O}$, we obtain:

$$\mathcal{O}X \xleftarrow{\mathcal{O}e} \mathcal{O}\prod_I E \cong S\bigoplus_I \mathcal{O}E \xleftarrow{s_{\bigoplus \mathcal{O}E}} \bigoplus \mathcal{O}E$$

Both $\mathcal{O}e$ and $s_{\bigoplus \mathcal{O}E}$ are quotients and hence $\mathcal{O}X$ is an $\mathcal{O}E$-regular frame. Conversely, suppose $\mathcal{O}X$ is $\mathcal{O}E$-regular. Then there exists a quotient

$q : \bigoplus \mathcal{O}E \to \mathcal{O}X$. Applying the functor Σ, we obtain:

$$X \hookrightarrow \Sigma\mathcal{O}X \xrightarrow{\Sigma q} \Sigma \bigoplus \mathcal{O}E$$

Now, $\mathcal{O}\Sigma(\bigoplus \mathcal{O}E) \cong \mathcal{O}\prod E$ and hence $\Sigma(\bigoplus \mathcal{O}E) \cong \Sigma\mathcal{O}(\prod E) \cong \prod E$ since E is sober. Since X is T_0, the sobrification map $X \hookrightarrow \Sigma\mathcal{O}X$ is an embedding, and since $q : \bigoplus \mathcal{O}E \to \mathcal{O}X$ is a quotient between spatial frames, it follows that Σq is an embedding. Hence X is E-regular. □

Examples

1. All frames are $\mathbb{3}$-regular, where $\mathbb{3}$ is the 3-chain.
2. The $\mathcal{L}(\mathbb{R})$-regular frames are precisely the completely regular frames. See [54].
3. Let $\mathbb{B}_2$ denote the four-element Boolean algebra. Then the $\mathbb{B}_2$-regular frames are precisely the zero dimensional frames. See [71].
4. Let A be any zero dimensional frame with at least two atoms. Then $\mathbb{B}_2$ is clearly a quotient of A. Thus, $\mathbb{B}_2$ is A-regular, and hence from Example 3 above, every zero dimensional frame is A-regular. On the other hand, A is zero dimensional, and hence $\mathbb{B}_2$-regular, from which it follows that the A-regular frames are precisely the zero dimensional frames. In particular, for any complete atomic Boolean algebra (except for the 2-chain) A, the A-regular frames are exactly the zero dimensional frames.

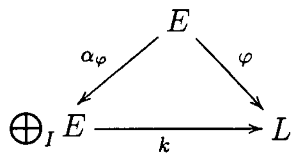
Definition 6.3 *Let E be a regular frame. Then a dense extension $h : M \to L$ of a frame L is called a C_E-extension if for any frame homomorphism $\varphi : E \to L$ there exists a frame homomorphism $\bar{\varphi} : E \to M$ such that the triangle*

$$\begin{array}{ccc} M & \xrightarrow{\;h\;} & L \\ & {}_{\bar{\varphi}}\nwarrow \quad \nearrow_{\varphi} & \\ & E & \end{array}$$

commutes.

Recall that for a regular frame L, the collection $Cov(L)$ of all covers of L forms a nearness, called the *fine* nearness of L. (See [11].) Let E be a regular frame endowed with its fine nearness. Let L be an E-regular frame. Then the nearness generated by the covers $\varphi_1[C] \wedge \cdots \wedge \varphi_k[C]$ where $\varphi_i : E \to L$ are frame homomorphisms and $C \in Cov(E)$ is called the E-nearness of L.

Let E be a fine nearness frame. Consider all frame homomorphisms $\varphi : E \to L$, and form the copower of E indexed by the set I of all such frame homomorphisms. If L is E-regular let $k : \bigoplus_I E \to L$ be the canonical frame homomorphism such that for any $\varphi : E \to L$ the diagram



commutes. Now, take any $C \in \mathcal{N} \bigoplus_I E$. Then $C = \alpha_{\varphi_1}[\bar{C}] \wedge \cdots \wedge \alpha_{\varphi_k}[\bar{C}]$, for some $\bar{C} \in Cov(E)$ and frame homomorphisms $\varphi_i : E \to L$. Now $k[C] = k[\alpha_{\varphi_1}[\bar{C}] \wedge \cdots \wedge \alpha_{\varphi_k}[\bar{C}]] = k \cdot \alpha_{\varphi_1}[\bar{C}] \wedge \cdots \wedge k \cdot \alpha_{\varphi_k}[\bar{C}] = \varphi_1[\bar{C}] \wedge \cdots \wedge \varphi_k[\bar{C}]$. Thus, the members of the E-nearness on L are precisely the images of the members of $\mathcal{N} \bigoplus_I E$. In particular, since $\bigoplus_I E$ is strong, it follows that L with the E-nearness is a strong nearness frame.

Proposition 6.4 *Let E be a regular frame, and L an E-regular frame. Then the following are equivalent:*

1) L is a closed quotient of a copower of E.

2) Every C_E-extension of L is an isomorphism.

3) L is complete in its E-nearness.

PROOF: (1) $\Rightarrow$ (2): Suppose $L \cong \uparrow s$ for $s \in \bigoplus_I E$ for some indexing set I. Let M be a C_E-extension of L. Then, for each of the coproduct maps $\alpha_i : E \to \bigoplus_I E$

there exists $\bar{\alpha}_i : E \to M$ such that the square

$$\begin{array}{ccc} \bigoplus_I E & \xrightarrow{\nu=(-)\vee s} & \uparrow s \\ {\scriptstyle \alpha_i}\uparrow & & \uparrow{\scriptstyle h} \\ E & \xrightarrow[\bar{\alpha}_i]{} & M \end{array}$$

commutes. By the property of the coproduct, there exists a unique $\bar{\alpha} : \bigoplus_I E \to M$ such that $\bar{\alpha}_i = \bar{\alpha} \cdot \alpha_i$ for each $i \in I$. Then $\nu \cdot \alpha_i = h \cdot \bar{\alpha}_i = h \cdot \bar{\alpha} \cdot \alpha_i$, and hence $\nu = h \cdot \bar{\alpha}$. Now, $0_L = \nu(s) = h \cdot \bar{\alpha}(s)$. Since h is dense, it follows that $\bar{\alpha}(s) = 0_M$. Thus the restriction k of ν to $\uparrow s$ is a frame homomorphism such that $\bar{\alpha} = k \cdot \nu$. Thus, $\nu = h \cdot k \cdot \nu$ and since ν is an epimorphism, it follows that $h \cdot k = id_M$. Now, h is dense and since it is a homomorphism between regular frames, it is monic. Hence h is a retraction and a monomorphism, i.e. h is an isomorphism.
(2) $\Rightarrow$ (3): Any frame homomorphism $\varphi : E \to L$ is clearly a nearness map. Since E is a complete nearness frame, it factors through the completion $\gamma_L L$ of L, since L is a strong nearness. (See [19] for the proof that completion is a coreflection for strong nearness frames.) Thus, the completion $\gamma_L L$ of L is a C_E-extension, from which it follows that L is complete.
(3) $\Rightarrow$ (1): Consider the dense-closed quotient factorisation of the quotient map $k : \bigoplus E \to L$:

$$\begin{array}{ccc} \bigoplus E & \xrightarrow{k} & L \\ & \searrow \quad \nearrow & \\ & \uparrow s & \end{array}$$

We endow $\uparrow s$ with the E-nearness, i.e. the nearness consisting of covers of the form $s \vee [C] = \{s \vee c | c \in C\}$ for a nearness cover C in $\bigoplus E$. Then $\uparrow s$ is a nearness frame and $h : \uparrow s \to L$ is a nearness map. But since L is complete, it follows that $L \cong \uparrow s$ and hence L is E-complete. $\square$

Definition 6.5 *Let E be a regular frame. An E-regular frame satisfying the above equivalent conditions is called an E-complete frame.*

Examples

1. If $E = \mathcal{O}\mathbb{R}$ then the E-complete frames are precisely the regular Lindelöf frames [60].

2. If $E = \mathcal{O}\mathbb{N}$ then the E-complete frames are precisely the zero dimensional Lindelöf frames [77], [71].

Unless otherwise stated, from now onwards the frame E shall be assumed to be regular.

Let E be a regular frame. We denote by E**RFrm** and by E**CFrm** the categories of E-regular and E-complete frames respectively.

Proposition 6.6 *For any frame E the category* E**CFrm** *is coreflective in* E**RFrm**.

PROOF: The result follows immediately from the fact that every E-regular frame is a strong nearness frame, and completion is coreflective for strong nearness frames.□

We denote the E-completion of an E-regular frame L by $\gamma_L : C_E L \to L$.

6.3 Strongly κ-Lindelöf frames

We know that the $\mathcal{N}$-complete frames are the zero dimensional Lindelöf frames ([77],[71]). In this section we investigate the E-complete frames for an arbitrary complete atomic Boolean algebra E.

Let $A_{\kappa+1}$ denote the complete atomic Boolean algebra with κ atoms.

Remark It is clear from our notation that we wish to consider only A_κ where κ is a non-limit cardinal. Since our study is restricted to regular cardinals, it means that the cardinals we consider are those that are not weakly inaccessible. Note that the existence of weakly inaccessible cardinals cannot be proven in ZF without the assumption of the existence of inaccessible cardinals.

Proposition 6.7 *The A_κ-nearness on an A_κ-regular (i.e. zero dimensional) frame L is exactly the nearness generated by all κ-partitions (i.e. partitions that are κ-sets).*

PROOF: Any frame homomorphism $\varphi : A_\kappa \to L$ induces a κ-partition $\mathcal{P} = \{\varphi(a) | a \in \text{Atom}(A_\kappa)\}$, where $\text{Atom}(A_\kappa)$ is the set of all atoms of A_κ. Conversely, for any κ-partition $\mathcal{P}$ of L, one may define a frame homomorphism $\varphi : A_\kappa \to L$ mapping the atoms of A_κ to the members of $\mathcal{P}$. In the case where $|\mathcal{P}| < \text{Atom}(A_\kappa)$, simply map the remaining atoms to the bottom. □

Definition 6.8 *A zero dimensional frame L is called strongly κ-Lindelöf if every cover of L has a κ-partition refinement.*

We denote the full subcategory of **0DFrm** consisting of the strong κ-Lindelöf frames by **SκLindFrm**. We now show that **SκLindFrm** is a coreflective subcategory of **0DFrm**. We do this by forming a quotient of ζL by way of a nucleus suggested to us by B. Banaschewski.

Consider the map $k : \zeta L \to \zeta L$ defined by:

$$k(I) = \{a \in \mathbb{B}L | \{a\} \wedge S \subseteq I, \text{ for some } \kappa\text{-partition } S\}$$

Lemma 6.9 *The map k defined above is a nucleus.*

PROOF:

N1: For each $a \in I$, $\{a\} \wedge \{\mathbf{e}\} \subseteq I$ and hence $I \subseteq k(I)$.

N2: If $I \subseteq J$ and if $\{a\} \wedge S \subseteq I$ for some κ-partition S, then $\{a\} \wedge S \subseteq J$ and thus $k(I) \subseteq k(J)$.

N3: Suppose $\{a\} \wedge S \subseteq k(I)$ for some κ-partition S of L. Then, for each $t \in S$, there exists a κ partition S_t such that $\{a \wedge t\} \wedge S_t \subseteq I$. Put $T = \bigcup\{\{t\} \wedge S_t | t \in S\}$. Then T is a κ-partition of L, and $\{a\} \wedge T \subseteq I$. It follows that $a \in k(I)$.

N4: If $a \in k(I) \cap k(J)$, then there exist κ-partitions S and T such that $\{a\} \wedge S \subseteq I$ and $\{a\} \wedge T \subseteq J$. But, $S \wedge T$ is also a κ-partition and $\{a\} \wedge S \wedge T \subseteq I \cap J$. Thus, $k(I) \cap k(J) \subseteq k(I \cap J)$. □

Lemma 6.10 *Let k be the nucleus defined above. Then the quotient frame $(\zeta L)_k$ is strongly κ-Lindelöf.*

PROOF Let $\mathfrak{X}$ be any cover of $(\zeta L)_k$. Then $\mathbf{e} \in k(\bigvee \mathfrak{X})$, from which it follows that there is a κ-partition S of L such that $S \subseteq \bigvee \mathfrak{X}$. Thus, for each $t \in S$, $t \in I_{1t} \vee \cdots \vee I_{nt}$ where $I_{it} \in \mathfrak{X}$ for $i = 1, \ldots, n$. Hence $t = c_1 \vee \cdots \vee c_n$ where $c_i \in I_{it}, i = 1, \ldots, n$. Put $t_1 = c_1$ and $t_i = c_i \wedge c_1^* \wedge \cdots \wedge c_{i-1}^*$ and let $[t_i]$ be the ideal in $\mathbb{B}L$ generated by t_i. Then $\{[t_i] | t \in S\}$ is a pairwise disjoint κ-set refining $\mathfrak{X}$. Also, since $t = t_1 \vee \cdots \vee t_n$, it follows that $S \subseteq \bigvee\{[t_i] | t \in S\}$ and so $\mathbf{e} \in k \bigvee\{[t_i] | t \in S\}$. Thus, $\{[t_i] | t \in S\}$ is a κ-partition refinement of $\mathfrak{X}$. □

Lemma 6.11 *Let L be a strongly κ-Lindelöf frame and let k be the above nucleus. Then $L \cong (\zeta L)_k$.*

PROOF: We show that the join map $j : (\zeta L)_k \to L$ is a frame isomorphism. Firstly note that j is a frame homomorphism since it clearly preserves $\mathbf{0}, \mathbf{e}$ and binary meets. Also, if $a \in k(I)$ then $\{a\} \wedge S \subseteq I$ for some κ-partition S and hence $a = \bigvee\{a\} \wedge S \leq \bigvee I$. Thus, $j(k(\bigvee I_i)) \leq j(\bigvee I_i)$. On the other hand, $j(k(\bigvee I_i) \geq j(\bigvee I_i) = \bigvee j(I_i)$, and so $j(k(\bigvee I_i)) = \bigvee j(I_i)$.
Secondly, j is onto: If $a \in k(\downarrow x)$ then $\{a\} \wedge S \subseteq \downarrow x$ for some κ-partition S and hence $a \leq x$. Thus $\downarrow x \in (\zeta L)_k$ for each $x \in L$, and $j(\downarrow x) = x$.
It now suffices to show that j is codense. Suppose $j(k(I)) = \mathbf{e}$. Then since L is strongly κ-Lindelöf, the cover $k(I)$ is refined by some κ-partition T. Thus $T \subseteq k(I)$ and hence $\mathbf{e} \in k^2(I) = k(I)$. □

h is codense. Suppose $h(a) = \mathbf{e}$. Then a is a join of complemented elements of M, i.e. $a = \bigvee\{b_i \in \mathbb{B}M | b_i \leq a\}$. Now, $\mathbf{e} = h(a) = h(\bigvee\{b_i \in \mathbb{B}M | b_i \leq a\}) = \bigvee\{h(b_i) | b_i \leq a, b_i \in \mathbb{B}M\}$. Thus, $\{h(b_i) | b_i \leq a, b_i \in \mathbb{B}M\}$ is a cover of L and therefore has a κ-partition refinement $\mathcal{A} = \{a_i | i \in I\}$. From Proposition 6.7 we know that $\mathcal{A}$ is in the A_κ-nearness of L, and since h is a uniform surjection, it follows that $\{h_*(a_i) | i \in I\} \in \mathcal{N}M$. In particular, $\bigvee\{h_*(a_i) | i \in I\} = \mathbf{e}$. But for each $i \in I$ there exists $b \in \mathbb{B}M, b \leq a$ such that $a_i \leq h(b)$, or $h_*(a_i) \leq b$. (h is one-to-one on the Boolean part of M.) Thus, $\bigvee\{h_*(a_i) | i \in I\} \leq a$ from which it follows that $= \mathbf{e}$. □

Corollary 6.15 *A zero dimensional frame L is strongly κ-Lindelöf iff L is complete with respect to the nearness generated by all κ-partitions of L. Furthermore, the strong κ-Lindelöfication may be obtained by completing with respect to this nearness.*

6.4 Zero dimensional κ-Lindelöf frames

We recall the definition of a κ-Lindelöf frame, given in Chapter 2.

Definition 6.16 *Let κ be an infinite cardinal. Then a frame is called κ-Lindelöf if every cover has a κ-subcover, i.e. a cover which is a κ-set.*

Remark 6.16

1. Every strongly κ-Lindelöf frame is κ-Lindelöf.
2. Every κ-Lindelöf Boolean frame is strongly κ-Lindelöf.
3. The ω_1-Lindelöf frames are precisely the strongly ω_1-Lindelöf frames.

We shall now show that for every non-limit regular cardinal κ, there exists a zero dimensional frame H^0_κ which is κ-Lindelöf but not strongly κ-Lindelöf. The example is the frame of opens of the space used by Hong [49] to show that the class of zero dimensionally k-compact spaces is simple.

Let $\mathbb{2}$ be the two point discrete space and let $H^0_{\lambda+1}$ be the frame of opens of the space $Y_{\lambda+1} = \mathbb{2}^X - \{\mathbf{0}\}$, for any set X with cardinality λ and $\mathbf{0}_x = 0$ for each $x \in X$.

Proposition 6.17 *For $\kappa = \lambda + 1$, the frame H^0_κ is κ-Lindelöf, but not strongly κ-Lindelöf.*

PROOF: Firstly note that the subbasic opens of Y_κ are of the form $\mathbb{1}_x = \{a \in Y_\kappa | a_x = 1\}$ and $\mathbb{0}_x = \{a \in Y_\kappa | a_x = 0\}$. Thus H^0_κ has a basis of cardinality strictly less than κ and hence every cover has a κ-cover refinement.
We shall now show that the cover $\mathcal{U} = \{\mathbb{1}_x | x \in X\}$ has no κ-partition refinement. Suppose $\mathcal{P}$ is a κ-partition refining $\mathcal{U}$. For each $x \in X$, let χ_x be the characteristic function at x, i.e. $(\chi_x)_x = 1$ and $(\chi_x)_y = 0$ for $y \neq x$. Now, each χ_x is in some unique $P_x \in \mathcal{P}$. Since $\mathcal{P}$ refines $\mathcal{U}$ and $\chi_x \in \mathbb{1}_y$ iff $x = y$, it follows that $\chi_x \in P_x \subseteq \mathbb{1}_x$. Now, $\chi_x \in B_x \subseteq P_x$ for some basic open set B_x in Y_κ. Since $B_x \subseteq \mathbb{1}_x$ and $\mathbb{0}_y \not\subseteq \mathbb{1}_x$ for any $y \in X$, it follows that $B_x = \mathbb{1}_x \cap \mathbb{0}_{F_x}$, where F_x is finite, $x \notin F_x$ and $\mathbb{0}_{F_x} = \bigwedge\{\mathbb{0}_z | z \in F_x\}$. Now, consider any $y \notin F_x \cup \{x\}$. We now claim that $x \in F_y$ where $\chi_y \in \mathbb{1}_y \cap \mathbb{0}_{F_y} \subseteq P_y \subseteq \mathbb{1}_y$. For, if $x \notin F_y$, then $\mathbb{1}_x \cap \mathbb{0}_{F_x} \cap \mathbb{1}_y \cap \mathbb{0}_{F_y} \neq \emptyset$. (Consider a with $a_x = a_y = 1$ and $a_z = 0$ for all $z \notin \{x, y\}$.) But, $\mathbb{1}_x \cap \mathbb{0}_{F_x} \cap \mathbb{1}_y \cap \mathbb{0}_{F_y} \subseteq P_x \cap P_y$ and hence $P_x = P_y$. Thus $\chi_x \in \mathbb{1}_y$ ($\mathcal{P}$ refines $\mathcal{U}$) which contradicts the fact that $x \neq y$. Let S be any countably infinite subset of X. Now the set $\bigcup\{F_x | x \in S\} \cup S$ is countable, and since X is uncountable, we may choose $y \notin \bigcup\{F_x | x \in S\} \cup S$. From the above argument, it follows that $S \subseteq F_y$. This clearly contradicts the fact that F_y is finite. Hence it is not possible to refine $\mathcal{U}$ with a κ-partition. □

Recall that the category **0DκFrm** is coreflective in **0DFrm**, with the coreflection given by the join map $j_L : \mathcal{H}_\kappa \mathbb{B}_\kappa L \to L$. The frame $\mathcal{H}_\kappa \mathbb{B}_\kappa L$ may also be obtained as a quotient of the Banaschewski compactification ζL using the nucleus n_κ defined by $n_\kappa(I) = \{a \in \mathbb{B}L | a \leq \bigvee S, S \text{ a } \kappa\text{-subset of } I\}$. Note that this is a special case of Proposition 2.18. Thus, we may use the familiar arguments to show:

Corollary 6.18 *The category of zero dimensional κ-Lindelöf frames is closed under formation of coproducts and closed quotients.*

The following lemma was suggested to me by Bernhard Banaschewski as a means to prove the proposition that follows:

Lemma 6.19 *Let M be any regular frame and $\varphi : M \to L$ a $(\mathbf{0}, \wedge, \mathbf{e})$-homomorphism taking covers to covers. Then there exists a frame homomorphism $h : M \to L$ such that $h(c) = \varphi(c)$ for all complemented $c \in M$.*

PROOF: The map φ is an L-valued Cauchy filter for the fine nearness of M. Hence $h = \varphi^\circ : M \to L$, where $\varphi^\circ(a) = \bigvee\{\varphi(x) | x \prec a\}$, is a frame homomorphism because M is complete and strong [19]. Further, $h(c) = \varphi(c)$ for complemented $c \in M$ since $c \prec c$. □

Proposition 6.20 *Let L be a zero dimensional frame. Then the H^0_κ-nearness on L is exactly the nearness generated by all κ-covers of complemented elements of L.*

PROOF: The frame H^0_κ is zero dimensional κ-Lindelöf and hence every uniform cover in the H^0_κ-nearness of L is refined by a κ-cover of complemented elements. On the other hand, suppose A is a κ-cover of L. We need to find a frame homomorphism $h : H^0_\kappa \to L$ such that h takes a cover of H^0_κ to A. Let $Y_\kappa = \mathbb{2}^A - \{\mathbf{0}\}$, and $H^0_\kappa = \mathcal{O}Y_\kappa$. Define $\varphi : H^0_\kappa \to L$ as follows:

$$\varphi(U) = \bigvee\{a_1 \wedge \cdots \wedge a_{n'} \wedge a^*_{n'+1} \wedge \cdots \wedge a^*_n | \mathbb{1}_{a_1} \cap \cdots \mathbb{1}_{a_{n'}} \cap \mathbb{0}_{a_{n'+1}} \cap \cdots \cap \mathbb{0}_{a_n} \subseteq U\}$$

where $\mathbb{1}_a = \{z \in Y_\kappa | z_a = 1\}$ and $\mathbb{0}_a = \{z \in Y_\kappa | z_a = 0\}$. We claim that φ is a $(\mathbf{0}, \wedge, \mathbf{e})$-homomorphism that takes covers to covers:

Firstly note that $\varphi(\emptyset) = \bigvee\{a_1 \wedge \cdots \wedge a_{n'} \wedge a^*_{n'+1} \wedge \cdots \wedge a^*_n | \mathbb{1}_{a_1} \cap \cdots \mathbb{1}_{a_{n'}} \cap \mathbb{0}_{a_{n'+1}} \cap \cdots \cap \mathbb{0}_{a_n} \subseteq \emptyset\}$. Now, if $\mathbb{1}_{a_1} \cap \cdots \cap \mathbb{1}_{a_{n'}} \cap \mathbb{0}_{a_{n'+1}} \cap \cdots \cap \mathbb{0}_{a_n} \subseteq \emptyset$, then $a_i = a_j$ for some $i \in \{1, \dots, n'\}$ and some $j \in \{n'+1, \dots, n\}$, and hence $a_i \wedge a^*_j = 0$. Thus $\varphi(\emptyset) = 0$. Also, $\mathbb{1}_a \subseteq Y_\kappa$ for all $a \in A$, and since A is a cover of L, it follows that $\varphi(Y_\kappa) = \mathbf{e}$.

It is clear that φ is order-preserving.

Suppose $a \leq \varphi(U) \cap \varphi(V)$. Then $a \leq a_1 \wedge \cdots \wedge a_{n'} \wedge a^*_{n'+1} \wedge \cdots \wedge a^*_n$, where $\mathbb{1}_{a_1} \cap \cdots \cap \mathbb{1}_{a_{n'}} \cap \mathbb{0}_{a_{n'+1}} \cap \cdots \cap \mathbb{0}_{a_n} \subseteq U$ and $a \leq b_1 \wedge \cdots \wedge b_{m'} \wedge b^*_{m'+1} \wedge \cdots \wedge b^*_m$, where $\mathbb{1}_{b_1} \cap \cdots \cap \mathbb{1}_{b_{m'}} \cap \mathbb{0}_{b_{m'+1}} \cap \cdots \cap \mathbb{0}_{b_m} \subseteq V$. But then $a \leq a_1 \wedge \cdots \wedge a_{n'} \wedge b_1 \wedge \cdots \wedge b'_m \wedge a^*_{n'+1} \wedge \cdots \wedge a^*_n \wedge b^*_{m'+1} \wedge \cdots \wedge b^*_m$, and $\mathbb{1}_{a_1} \cap \cdots \cap \mathbb{1}_{a_{n'}} \cap \mathbb{1}_{b_1} \cap \cdots \cap \mathbb{1}_{b_{m'}} \cap \mathbb{0}_{a_{n'+1}} \cap \cdots \mathbb{0}_{a_n} \cap \cdots \mathbb{0}_{b_{m'+1}} \cap \cdots \cap \mathbb{0}_{b_m} \subseteq U \cap V$. Thus, $a \leq \varphi(U \cap V)$.

It now suffices to show that φ takes covers to covers. Suppose $\bigcup_{i \in I} W_i = Y_\kappa$. Now, since each $\mathbb{1}_a$ is compact, it follows that $\mathbb{1}_a \subseteq W_{i_1} \cup \cdots \cup W_{i_t}$. We may assume each W_i is a basic open, i.e. $W_{i_k} = \mathbb{1}_{a_{k1}} \cap \cdots \cap \mathbb{1}_{a_{kn'_k}} \cap \mathbb{0}_{a_{kn'_k+1}} \cap \cdots \cap \mathbb{0}_{a_{kn_k}}$. Thus,

$$\begin{aligned}
\mathbb{1}_a &\subseteq \bigcup_{1 \leq k \leq t} W_{i_k} \\
&= \bigcup_{1 \leq k \leq t} \left(\bigcap_{1 \leq r \leq n_k} X_{a_{kr}} \right) \quad \text{where } X = \mathbb{1} \text{ if } 1 \leq r \leq n'_k \text{ and } \mathbb{0} \text{ if } n'_k < r \leq n_k \\
&= \bigcap_{\mathbf{r} \in \mathcal{S}} \left(\bigcup_{1 \leq k \leq t} X_{a_{kr_k}} \right)
\end{aligned}$$

where $\mathcal{S} = \{(r_1, \dots, r_t) | 1 \leq r_k \leq n_k\}$ and $X = \mathbb{1}$ if $1 \leq r_k \leq n'_k$ and $\mathbb{0}$ if $n'_k < r \leq n_k$.

It follows that $\mathbb{1}_a \subseteq \bigcup_{1 \leq k \leq t} X_{a_{kr_k}}$ for all $\mathbf{r} \in \mathcal{S}$.

Now, note that if $\mathbb{1}_a \subseteq \mathbb{1}_{a_1} \cup \cdots \cup \mathbb{1}_{a_{n'}} \cup \mathbb{0}_{a_{n'+1}} \cup \cdots \cup \mathbb{0}_{a_n}$, then $a \in \{a_1, \dots, a_{n'}\}$ or $\{a_1, \dots, a_{n'}\} \cap \{a_{n'+1}, \dots, a_n\} \neq \emptyset$. For, if $a \notin \{a_1, \dots, a_{n'}\}$ and $\{a_1, \dots, a_{n'}\} \cap \{a_{n'+1}, \dots, a_n\} = \emptyset$, then we may take $z \in Y_\kappa$ such that $z_a = 1 = z_{a_j}$ for $n'+1 \leq j \leq n$, and $z_{a_i} = 0$ for $1 \leq i \leq n'$. It is clear that $z \in \mathbb{1}_a$ and $z \notin \mathbb{1}_{a_1} \cup \cdots \cup \mathbb{1}_{a_{n'}} \cup \mathbb{0}_{a_{n'+1}} \cup \cdots \cup \mathbb{0}_{a_n}$, contradicting our hypothesis. Thus, if $\mathbb{1}_a \subseteq \mathbb{1}_{a_1} \cup \cdots \cup \mathbb{1}_{a_{n'}} \cup \mathbb{0}_{a_{n'+1}} \cup \cdots \cup \mathbb{0}_{a_n}$ then $a \leq a_1 \vee \cdots \vee a_{n'} \vee a^*_{n'+1} \vee \cdots \vee a^*_n$.

In our situation, we therefore have that $a \leq \bigvee_{1 \leq r_k \leq n'_k} a_{kr_k} \vee \bigvee_{n'_k < r_k \leq n_k} a^*_{kr_k}$ for all $\mathbf{r} \in \mathcal{S}$, i.e.

$$\begin{aligned} a &\leq \bigwedge_{\mathbf{r} \in \mathcal{S}} \left(\bigvee_{1 \leq r_k \leq n'_k} a_{kr_k} \vee \bigvee_{n'_k < r_k \leq n_k} a^*_{kr_k} \right) \\ &= \bigvee_{1 \leq k \leq t} a_{k_1} \wedge \cdots \wedge a_{k_{n'_k}} \wedge a^*_{k_{n'_k+1}} \wedge \cdots \wedge a^*_{kn_k} \\ &= \bigvee_{1 \leq k \leq t} \varphi(W_{i_k}) \end{aligned}$$

The last equality holds because $\varphi(\mathbb{1}_a) = a$ and $\varphi(\mathbb{0}_a) = a^*$ and hence $\varphi(W_{ik}) = \varphi(\mathbb{1}_{a_{k1}} \cap \cdots \cap \mathbb{1}_{a_{kn'_k}} \cap \mathbb{0}_{a_{kn'_k+1}} \cap \cdots \cap \mathbb{0}_{a_{kn_k}}) = a_{k_1} \wedge \cdots \wedge a_{k_{n'_k}} \wedge a^*_{k_{n'_k+1}} \wedge \cdots \wedge a^*_{kn_k}$.
Thus, $a \leq \bigvee \varphi(W_i)$ for all $a \in A$ and hence $\varphi(W_i)$ covers L.
Applying Lemma 6.19, we obtain a frame homomorphism $h = \varphi^\circ : H^0_\kappa \to L$, with $h(\mathbb{1}_a) = \varphi(\mathbb{1}_a) = a$. This completes the proof since φ takes the cover $\{\mathbb{1}_a | a \in A\}$ to A. □

Proposition 6.21 *The following are equivalent for a zero dimensional frame L:*

1) L is H^0_κ-complete.

2) L is zero dimensional κ-Lindelöf.

PROOF: The proof is similar to that of Proposition 6.14 and is therefore omitted.□

Corollary 6.22 *A zero dimensional frame L is κ-Lindelöf iff it is complete with respect to the nearness generated by all κ-covers of complemented elements of L. Furthermore the zero dimensional κ-Lindelöfication may be obtained by completing with respect to this nearness.*

We know that the nearness generated by all κ-partitions of a zero dimensional frame L is a uniformity, and from Proposition 6.17 this is in general different to the nearness generated by all κ-covers of complemented elements of L. It is therefore natural to enquire when the latter nearness is a uniformity.

Proposition 6.23 *For every non-limit cardinal $\kappa > \omega_1$, there exists a zero dimensional frame L_κ such that the nearness generated by all κ-covers of complemented elements in L_κ is not a uniformity.*

PROOF: Our candidate for L_κ is the frame $H_\kappa^0 = \mathcal{O}Y_\kappa, Y_\kappa = 2^X - \{\mathbf{0}\}, \mathbf{X}$ a κ-set and the nearness in question is the fine nearness. We show that the cover $\mathcal{U} = \{\mathbb{1}_x | x \in X\}$ has no star refinement. Let $\mathcal{A} \leq^* \mathcal{U}$. Then for each $x \in X$, there exists $A_x \in \mathcal{A}$ such that $\chi_x \in F_x$, and $\mathcal{A}A_x \subseteq \mathbb{1}_x$. (Recall that $\mathcal{A}A_x$ is the star of A_x in $\mathcal{A}$.) Let B_x be a basic neighbourhood of χ_x in A_x. Then $B_x = \mathbb{1}_x \cap \mathbb{0}_{F_x}$ for some finite set F_x. Now, if $y \notin \mathbb{0}_{F_x} \cup \{x\}$, then $x \in F_y$: For, if $x \notin F_y$, then $\mathbb{1}_x \cap \mathbb{0}_{F_x} \cap \mathbb{1}_y \cap \mathbb{0}_{F_y} \neq \emptyset$ (by the same argument as that of Proposition 6.17). Thus $B_x \cap B_y \neq \emptyset$ and hence $B_y \subseteq \mathcal{A}A_x \subseteq \mathbb{1}_x$, which is not possible since $\chi_y \in B_y$ and $\chi_y \notin \mathbb{1}_x$. Now take any countably infinite subset S of X. Then the set $\bigcup\{F_x | x \in S\} \cup S$ is countable. If $y \notin \bigcup\{F_x | x \in S\} \cup S$, then $S \subseteq F_y$ contradicting the fact that F_y is finite. □

6.5 Completely regular κ-Lindelöf frames

In Chapter 2 we discussed Madden's [59] result that the category **CRκLindFrm** of all completely regular κ Lindelöf frames is coreflective in **CRFrm**. In this section we shall show that the completely regular κ-Lindelöf frames are exactly the H_κ-complete frames for some frame H_κ.

Recall (Corollary 2.10) that the category **CRκLindFrm** is closed under the formation of coproducts and closed quotients.

Proposition 6.24 *Assuming no measurable cardinal exists, then there is no frame E such that the realcompact frames are precisely those frames that are E-complete.*

PROOF: Suppose such a frame E exists. Then E is completely regular κ-Lindelöf for some cardinal κ. Thus an E-complete frame is completely regular κ-Lindelöf. Now, any complete atomic Boolean algebra A_λ, for (regular) $\lambda > \kappa$ is realcompact since λ is nonmeasurable. However, A_λ is clearly not E-complete. □

Remark 6.24

1. This answers a personal question of Hušek to the author.

2. Since all complete atomic Boolean algebras are also $\mathbb{N}$-compact (under the assumption that no measurable cardinal exists), the proposition above also applies to $\mathbb{N}$-compact frames, i.e. there is no frame E such that the $\mathbb{N}$-compact frames are precisely the E-complete frames.

In [52] Hušek showed that the class of all κ-compact spaces is simple in **Haus**. We now show that the H_κ-complete frames are exactly the κ-Lindelöf frames, where H_κ is the frame of opens of Hušek's space.

Let $Z_{\lambda+1}$ be the space $\mathbb{I}^X - \{\mathbf{0}\}$ where $|X| = \lambda$, $\mathbb{I}$ is the unit interval and $\mathbf{0}_x = 0$ for all $x \in X$. Let $H_{\lambda+1}$ be the frame of opens of $Z_{\lambda+1}$. Note that $\mathcal{O}\mathbb{I}$ is a quotient of $H_{\lambda+1}$ and hence the $H_{\lambda+1}$-regular frames are precisely the completely regular frames.

Proposition 6.25 *Let L be a completely regular frame and κ a non-limit cardinal. Then the H_κ-nearness on a completely regular frame is exactly the nearness generated by all κ-covers of cozero elements of L.*

PROOF: The subbasic opens of $Z_\kappa = Z_{\lambda+1}$ are of the form $\pi_a^{-1}(U) = \{z \in Z_\kappa | z_a \in U\}$ for some basic open $U \in \mathcal{O}\mathbb{I}$. (The map π_a is the a^{th} projection map.) Thus Z_κ has a basis of cardinality strictly less than κ, and hence H_κ is κ-Lindelöf. Hence every uniform cover in the H_κ-nearness of L is refined by a κ-cover of cozero elements.

On the other hand, suppose A is a κ-cover of cozero elements of L. Then for each

$a \in A$, there exists a frame homomorphism $h_a : \mathcal{O}\mathbb{I} \to L$ such that $h_a((0,1]) = a$. Define φ as follows:

$$\varphi(U) = \bigvee\{ \bigwedge_{1\le i\le n} h_{a_i}(U_i) | \bigcap_{1\le i\le n} \pi_{a_i}^{-1}(U_i) \subseteq U\}$$

where $a_i \in A$ and $U_i \in \mathcal{O}\mathbb{I}$.

Note that $\varphi(\pi_a^{-1}(U)) = h_a(U)$ and hence $\varphi(\pi_a^{-1}((0,1])) = a$. We show that φ is a $(\mathbf{0}, \wedge, \mathbf{e})$-homomorphism. If $\bigcap_{1\le i\le n} \pi_{a_i}^{-1}(U_i) = \emptyset$, then there exist $s, t \in \{1, \ldots, n\}$ such that $a_s = a_t$ and $U_s \cap U_t = \emptyset$ for $s \neq t$. Thus $h_{a_s}(U_s \cap U_t) = \mathbf{0}$ and hence $\varphi(\emptyset) = 0$. Also, $h_{a_i}((0,1]) = a_i$, and $\pi_{a_i}^{-1}((0,1]) \subseteq Z_\kappa$, and so $\varphi(Z_\kappa) = \mathbf{e}$ since A is a cover.

It is clear that φ is order-preserving, and thus $\varphi(U \cap V) \subseteq \varphi(U) \wedge \varphi(V)$.
Suppose $a \le \varphi(U) \wedge \varphi(V)$. Then $a \le h_{a_1}(U_1) \wedge \cdots \wedge h_{a_n}(U_n)$, where $\bigcap \pi_{a_i}^{-1}(U_i) \subseteq U$ and $a \le h_{b_1}(V_1) \wedge \cdots \wedge h_{b_n}(V_m)$, where $\bigcap \pi_{b_i}^{-1}(V_i) \subseteq V$. Thus $\bigcap \pi_{a_i}^{-1}(U_i) \cap \bigcap \pi_{a_i}^{-1}(V_i) \subseteq U \cap V$, and hence $a \in \varphi(U \cap V)$.
We now show that φ takes covers to covers. Let $\{W_i | i \in I\}$ be a cover of Z_κ. Then, since $\pi_a^{-1}((0,1]) << Z_\kappa$, it follows that $\pi_a^{-1}((0,1]) \subseteq W_{i_1} \cup \cdots \cup W_{i_n}$. We may assume that these are basic opens,

i.e. $W_{i_k} = \bigcap_{1\le j\le n_k} \pi_{a_{k_j}}^{-1}(U_{k_j})$, and so

$$\begin{aligned} \pi_a^{-1}((0,1]) &\subseteq \bigcup_{1\le k\le n} \left(\bigcap_{1\le j\le n_k} \pi_{a_{k_j}}^{-1}(U_{k_j}) \right) \\ &= \bigcap_{\mathbf{r}\in\mathcal{S}} \left(\bigcup_{1\le k\le n} \pi_{a_{kr_k}}^{-1}(U_{kr_k}) \right) \end{aligned}$$

where $\mathcal{S} = \{(r_1, \ldots, r_n) | 1 \le r_k \le n_k\}$. Hence $\pi_a^{-1}((0,1]) \subseteq \bigcup_{1\le k\le n} \pi_{a_{kr_k}}^{-1}(U_{kr_k})$ for all $\mathbf{r} \in \mathcal{S}$. Thus either $a = a_{jk_j}$ and $U_{jk_j} = (0,1]$, or $a = a_{l_1 r_{l_1}} = \cdots = a_{l_t r_{l_t}}$ and $U_{l_1 r_{l_1}} \cup \cdots \cup U_{l_t r_{l_t}} = (0,1]$. Thus $a \le \bigvee_{1\le k\le n} h_{a_{kr_k}}(U_{kr_k})$ for all $\mathbf{r} \in \mathcal{S}$. Hence

$$a \le \bigwedge_{\mathbf{r}\in\mathcal{S}} \left(\bigvee_{1\le k\le n} h_{a_{kr_k}}(U_{kr_k}) \right)$$

$$
\begin{aligned}
&= \bigvee_{1\le k\le n}\left(\bigwedge_{1\le j\le n_k} h_{a_{k_j}}(U_{k_j})\right)\\
&= \bigvee_{1\le k\le n}\left(\bigwedge_{1\le j\le n_k} \varphi(\pi^{-1}_{a_{k_j}}(U_{k_j}))\right)\\
&= \bigvee_{1\le k\le n}\varphi\left(\bigcap_{1\le j\le n_k} \pi^{-1}_{a_{k_j}}(U_{k_j})\right)\\
&= \bigvee_{1\le k\le n}\varphi(W_{i_k})
\end{aligned}
$$

It follows that $a \le \bigvee_{i\in I}\varphi(W_i)$ for all $a \in A$ and hence $\varphi(W_i)$ covers L. Thus φ is an L-valued Cauchy filter for the fine nearness on H_κ and hence the map $h = \varphi^\circ : H_\kappa \to L$ where $\varphi^\circ(a) = \bigvee\{\varphi(x)|x \prec a\}$ is a frame homomorphism. Furthermore,

$$
\begin{aligned}
h(\pi_a^{-1}((0,1]) &= \bigvee\{\varphi(\mathcal{V})|\mathcal{V} \prec \pi_a^{-1}((0,1])\}\\
&\ge \bigvee\{\varphi(\pi_a^{-1}(U))|U \prec (0,1]\}\\
&= \bigvee\{h_a(U)|U \prec (0,1]\}\\
&= h_a((0,1])\\
&= a
\end{aligned}
$$

Thus, the frame homomorphism h takes the uniform cover $\mathcal{U} = \{\pi_a^{-1}((0,1])|a \in A\}$ to the cover A from which it follows that A is in the H_κ-nearness of L. □

Proposition 6.26 *Let L be a completely regular frame and let κ be a regular non-limit cardinal. Then the following are equivalent:*

1) L is κ-Lindelöf.

2) L is H_κ-complete.

PROOF:(1) $\Rightarrow$ (2) : The frame H_κ is κ-Lindelöf, and hence closed quotients of copowers of H_κ are κ-Lindelöf.

(2) $\Rightarrow$ (1): Let M be a completely regular frame and suppose $h : M \to L$ is a dense uniform surjection. It suffices to show that h is codense (i.e. one-to-one). Suppose $h(a) = \mathbf{e}$. Then $a = \bigvee\{c | c \in CozM, c \prec\prec a\}$. Thus, $\{h(c) | c \in CozM, c \prec\prec a\}$ is a cover of L and hence has a κ-subcover $\{h(c_i) | i \in I\}$. Applying the Axiom of Choice, we have d_i such that $c_i \prec d_i$ in $CozM$ and $d_i \leq a$ for each $i \in I$. Since h is a uniform surjection, $\{h_*h(c_i) | i \in I\}$ is a uniform cover of M. Now, since h is dense, it follows that $h_*h(c_i) \leq d_i$, and thus, $a = \mathbf{e}$. □

Corollary 6.27 *A completely regular frame L is κ-Lindelöf iff it is complete with respect to the nearness generated by all κ-covers of cozero elements. Furthermore, the completely regular κ-Lindelöfication may be obtained by completing with respect to this nearness.*

6.6 κ-Compact frames

In [43] Herrlich introduced the notion of a κ-compact space for an infinite cardinal κ, as a common generalisation of compact and realcompact spaces. We present here the point-free analogue of this notion, as well as Hong's zero dimensionally κ-compactness.

Consider the frame $H_\kappa = \mathcal{O}Z_\kappa$ introduced in the previous section, i.e. $Z_\kappa = \mathbb{I}^X - \{\mathbf{0}\}$ where X is a κ-set and $\mathbf{0}_a = 0$ for all $a \in X$. Let $\bar{H}_\kappa$ be the one-point compactification of H_κ, i.e. $\bar{H}_\kappa$ is the frame of all $\triangleleft$-regular ideals in H_κ where $\triangleleft$ is the smallest strong inclusion on H_κ. Note that $\bar{H}_\kappa \cong \mathcal{O}\mathbb{I}^X$.

Now, for any frame homomorphism $h : H_\kappa \to L$, for a completely regular frame

L, there exists a frame homomorphism $\bar{h} : \bar{H}_\kappa \to \beta L$ such that the square

$$\begin{array}{ccc} H_\kappa & \xrightarrow{h} & L \\ {\scriptstyle j_{H_\kappa}} \uparrow & & \uparrow {\scriptstyle j_L} \\ \bar{H}_\kappa & \xrightarrow[\bar{h}]{} & \beta L \end{array}$$

commutes. The maps j_{H_κ} and j_L are join maps. Let J_κ be the ideal generated by the set $\{\pi_a^{-1}((0,1])|a \in X\}$. It is clear that J_κ is maximal in $\bar{H}_\kappa$. We call a maximal ideal $I \in \beta L$ a *κ-point* if $I \not\supseteq \bar{h}(J_\kappa)$ for any frame homomorphism $h : H_\kappa \to L$.

Definition 6.28 *Let L be any bounded distributive lattice. An ideal I in L is said to be κ-proper if $\bigvee S \neq \mathbf{e}$ for any κ-subset S of I.*

Proposition 6.29 *The κ-points of βL are precisely the κ-proper maximal ideals $I \in \beta L$.*

PROOF: Firstly note that the ideal J_κ is not κ-proper since it is generated by a κ-cover. Thus $\bar{h}(J)$ is not κ-proper and hence any ideal containing $\bar{h}(J)$ is not κ-proper.
On the other hand, if I is not κ-proper then there exist a κ cover $\mathcal{A} \subseteq I$. Thus there exists a frame homomorphism $h : H_\kappa \to L$ such that $h[\pi_a^{-1}((0,1])|a \in X\}] = \mathcal{A}$, and hence $h[H_\kappa] \subseteq I$. □

Remark 6.29 The ω_1-points of βL are exactly the real points of βL.

The following proposition is a generalisation of Lemma 1 in [17] and is proved similarly.

Proposition 6.30 *Let L be a completely regular frame. Then the κ-proper maximal ideals in $CozL$ are exactly the maximal κ-ideals in $Coz_\kappa L$.*

PROOF: Let J be a k-proper maximal ideal in $CozL$. Then for any κ-set A, $\bigvee A \neq \mathbf{e}$, and hence the ideal K generated by J and $\downarrow \bigvee A$ is proper. Since J is maximal, it follows that $J = K$, i.e. $\bigvee A \in J$.
Conversely, let I be a maximal κ-ideal, and suppose $a \notin I$. Then, since $Coz_\kappa L$ is regular, there exists $b \in Coz_\kappa L - I$ such that $b \prec a$. Thus there exists $s \in Coz_\kappa L$ such that $s \wedge b = \mathbf{0}$ and $s \vee a = \mathbf{e}$. It follows that $\downarrow s \cap \downarrow b \subseteq I$ and hence $\downarrow s \subseteq I$ or $\downarrow b \subseteq I$ since I is prime in $\mathcal{H}_\kappa Coz_\kappa L$. Thus $\downarrow s \subseteq I$ since $b \notin I$ and hence the ideal generated by I and a is $\mathbf{e}$. Thus I is a maximal κ-proper ideal in $Coz_\kappa L$.□

Definition 6.31 *A completely regular frame is called κ-compact if every maximal κ-ideal in $Coz_\kappa L$ is completely proper. We denote the full subcategory of κ-compact frames by* κ**KFrm**

Remark 6.31 It is clear that a Tychonoff space X is κ-compact iff $\mathcal{O}X$ is κ-compact.

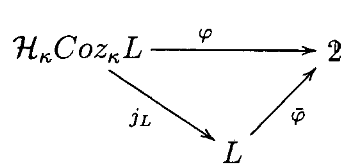
Proposition 6.32 *The following are equivalent for a completely regular frame L:*

1) L is κ-compact.

2) Every frame homomorphism $\varphi : \mathcal{H}_\kappa Coz_\kappa L \to 2$ factors through the join map $j_L : \mathcal{H}_\kappa Coz_\kappa L \to L$.

3) Every κ-frame homomorphism $Coz_\kappa L \to 2$ extends to a frame homomorphism $L \to 2$.

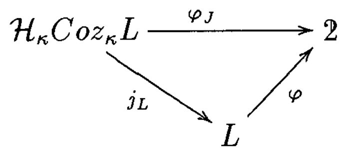
PROOF: (1) $\Rightarrow$ (2): Let $\varphi : \mathcal{H}_\kappa Coz_\kappa L \to 2$ be a frame homomorphism. Then $\varphi^{-1}(0)$ is a maximal κ-ideal in $Coz_\kappa L$ and hence $j_L(\varphi^{-1}(0)) \neq \mathbf{e}$



Since j_L is onto, it follows that $j_L(\varphi^{-1}(0))$ is maximal. Thus there exists a frame homomorphism $\bar{\varphi} : L \to 2$ such that $\bar{\varphi} \cdot j_L = \varphi$.

(2) $\Rightarrow$ (3): Let $\psi : Coz_\kappa L \to 2$ be a κ-frame homomorphism. Then $\mathcal{H}_\kappa \psi : \mathcal{H}_\kappa Coz_\kappa L \to 2$ is a frame homomorphism, which by our hypothesis factors through $j_L : \mathcal{H}_\kappa Coz_\kappa L \to L$, i.e. there exists a frame homomorphism $\varphi : L \to 2$ such that $j_L \cdot \varphi = \mathcal{H}_\kappa \psi$. Now, if $a \in Coz_\kappa L$, then $\downarrow a \in Coz_\kappa \mathcal{H}_\kappa Coz_\kappa L$, and $\varphi(a) = \varphi \cdot j_L(\downarrow a) = \mathcal{H}_\kappa \psi(\downarrow a) = \psi(a)$.

(3) $\Rightarrow$ (1): Suppose J is maximal in $\mathcal{H}_\kappa Coz_\kappa L$. Then there exists a frame homomorphism $\varphi_J : \mathcal{H}_\kappa Coz_\kappa L \to 2$ such that $\varphi_J(J) = 0$. This restricts to a κ-frame homomorphism $\bar{\varphi}_J : Coz_\kappa L \to L$, which by our hypothesis extends to a frame homomorphism $\varphi : L \to 2$.



Now, $\varphi(\bigvee J) = j_L \cdot \varphi(J) = \varphi_J(J) = \mathbf{0}$. Thus $\bigvee J \neq \mathbf{e}$. □

Corollary 6.33 *A Boolean frame L is κ-compact iff every κ-frame homomorphism $L \to 2$ is a frame homomorhism.*

PROOF: If L is Boolean, then $Coz_\kappa L = L$. □

Remark 6.33 The special case of the above result where $\kappa = \omega_1$ appears in [17].

Proposition 6.34 *Let L be a completely regular frame. Then the following are equivalent:*

1) L is κ-compact.

2) The map $Sj_L : S\mathcal{H}_\kappa Coz_\kappa L \to SL$ is an isomorphism.

PROOF: (1) $\Rightarrow$ (2): Consider the commuting square

$$\begin{array}{ccc} L & \xrightarrow{\quad s_L \quad} & SL \\ {\scriptstyle j_L}\uparrow & & \uparrow{\scriptstyle Sj_L} \\ \mathcal{H}_\kappa Coz_\kappa L & \xrightarrow[s_{\mathcal{H}_\kappa Coz_\kappa L}]{} & S\mathcal{H}_\kappa Coz_\kappa L \end{array}$$

Since $s_L \cdot j_L$ is onto, it follows that Sj_L is onto. It now suffices to show that sj_L is codense. Suppose $sj_L(s_{\mathcal{H}_\kappa Coz_\kappa L}(I)) = \mathbf{e}$. Then $s_L \cdot j_L(I) = \mathbf{e}$ and thus I is not contained in a maximal ideal. Thus $s_{\mathcal{H}_\kappa Coz_\kappa L}(I) = \mathbf{e}$.
(2) $\Rightarrow$ (1): Suppose I is maximal in $\mathcal{H}_\kappa Coz_\kappa L$. Then $s_{\mathcal{H}_\kappa Coz_\kappa L}(I) \neq \mathbf{e}$. Since $Sj_L$is an isomorphism, it follows that $s_L \cdot j_L(I) \neq \mathbf{e}$ and hence $j_L(I) \neq \mathbf{e}$. $\square$

Corollary 6.35 *Every κ-compact frame is κ-Lindelöf.*

PROOF: This follows directly from the last part of Proposition 3.32. $\square$

Corollary 6.36 *The category κ***KFrm** *is a coreflective subcategory of* **CRκLindFrm** *with coreflection maps given by the relatively spatial reflection of $j_L : \mathcal{H}_\kappa Coz_\kappa L \to L$.*

PROOF: This follows immediately from Proposition 6.34 and Proposition 3.32 since **CRκLindFrm** is a strict coreflective subcategory of **CRFrm**. $\square$

We shall denote the κ-compactification of a completely regular frame L by $\bar{\gamma}_L : \upsilon_\kappa L \to L$.

Corollary 6.37 *The category κ***KFrm** *is the relatively spatial hull of* **CRκLindFrm** *in* **CRFrm**.

We may obtain the analogous results for the zero dimensional case. As with the case for completely regular frames, let $H^0_{\kappa+1} = \mathcal{O}Y_{\kappa+1}$, where $Y_{\kappa+1} = \mathbf{2}^X - \{\mathbf{0}\}$,

$|X| = \kappa$, and $\mathbf{0}_a = 0$ for each $a \in X$. Denote by $\bar{H}^0_{\kappa+1}$ the one-point compactification of $H^0_{\kappa+1}$. Let J^0_κ be the ideal in $H^0_{\kappa+1}$ generated by the cover $\mathcal{C} = \{\mathbb{1}_a | a \in X\}$.

For any zero dimensional frame L and any frame homomorphism $h : H^0_\kappa \to L$, there exists a frame homomorphism $\bar{h} : \bar{H}^0_\kappa \to \zeta L$ such that the square

$$\begin{array}{ccc} H^0_\kappa & \xrightarrow{h} & L \\ {\scriptstyle j_{H^0_\kappa}}\uparrow & & \uparrow{\scriptstyle j_L} \\ \bar{H}^0_\kappa & \xrightarrow[\bar{h}]{} & \zeta L \end{array}$$

commutes. We call a maximal ideal $I \in \zeta L$ a *zero dimensional κ-point* if $I \not\supseteq \bar{h}(J^0_\kappa)$ for any frame homomorphism $h : H^0_\kappa \to L$.

Proposition 6.38 *The zero dimensional κ-points of ζL are precisely the κ-proper maximal ideals $I \in \zeta L$.*

PROOF: The proof is similar to that of Proposition 6.29 and is therefore omitted.□

Remark 6.38 The zero dimensional ω_1-points are exactly the natural points of ζL.

The following results are analogous to 6.30 to 6.36 and are therefore stated without proof

Proposition 6.39 *Let L be a zero dimensional frame. Then the κ-proper maximal ideals in $\mathbb{B}L$ are exactly the maximal κ-ideals in $\mathbb{B}L$.*

Definition 6.40 *A zero dimensional frame is called zero dimensionally κ-compact if every maximal κ-ideal in $\mathbb{B}L$ is completely proper. The full subcategory of zero dimensionally κ-compact frames is denoted by* **0DκKFrm**.

Remark 6.40 A zero dimensional space X is zero dimensionally κ-compact iff $\mathcal{O}X$ is zero dimensionally κ-compact.

Proposition 6.41 *The following are equivalent for a zero dimensional frame L:*

1) L is zero dimensionally κ-compact.

2) Every frame homomorphism $\varphi : \mathcal{H}_\kappa \mathbb{B}_\kappa L \to 2$ factors through the join map $j_L : \mathcal{H}_\kappa \mathbb{B}_\kappa L \to L$.

3) Every κ-frame homomorphism $\mathbb{B}_\kappa L \to 2$ extends to a frame homomorphism $L \to 2$.

Corollary 6.42 *A Boolean frame L is zero dimensionally κ-compact iff every κ-frame homomorphism $L \to 2$ is a frame homomorphism.*

Proposition 6.43 *Let L be a zero dimensional frame. Then the following are equivalent:*

1) L is zero dimensionally κ-compact.

2) The map $Sj_L : S\mathcal{H}_\kappa \mathbb{B}_\kappa L \to SL$ is an isomorphism.

Proposition 6.44 *The category* **0DκKFrm** *is a coreflective subcategory of the category* **0DκLindFrm** *with coreflection maps given by the relatively spatial reflection of $j_L : \mathcal{H}_\kappa \mathbb{B}_\kappa L \to L$.*

We shall denote the zero dimensionally κ-compact coreflection of a zero dimensional frame L by $\bar{\gamma}_L : \nu_\kappa L \to L$.

It is true that all zero dimensional κ-compact frames are zero dimensionally κ-compact. This follows from the fact that $\mathbb{B}_\kappa L \subseteq Coz_\kappa L$. The converse is however not in general true. Some well-known spatial counterexamples exist. The following result is due to Hong and appears in [49]. Our proof makes use of our characterisation in Proposition 5.2

Proposition 6.45 *A strongly zero dimensional frame L is κ-compact iff L is zero dimensionally κ-compact.*

PROOF: If L is strongly zero dimensional, then $Coz_\kappa L \cong \mathbb{B}_\kappa L$ for $\kappa > \omega$. It follows that every maximal ideal I in $Coz_\kappa L$ is completely proper iff every maximal ideal in $\mathbb{B}_\kappa L$ is completely proper. □

6.7 κ-Pseudocompact frames

It is well known that for spaces realcompactness coincides with compactness in the presence of pseudocompactness. This was shown to be true for frames as well [62], [17]. In this section we discuss a generalisation of the notion of pseudocompactness as well as how this relates to κ-compactness and compactness.

Definition 6.46 *Let $\kappa > \omega$. A completely regular frame L is κ-pseudocompact if $Coz_\kappa L$ is a compact κ-frame.*

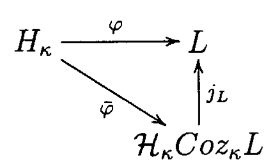
Remark 6.46 A frame L is pseudocompact iff L is ω_1-pseudocompact.

Proposition 6.47 *Let $\kappa > \omega$. Then a completely regular frame L is κ-pseudocompact iff $\mathcal{H}_\kappa Coz_\kappa L$ is compact.*

PROOF: This follows from the fact that the functors $\mathcal{H}_\kappa$ and Coz_κ preserve compactness. □

Proposition 6.48 *Let $\kappa > \omega$. Then a completely regular frame L is κ-pseudocompact iff L is H_κ-pseudocompact. (c.f. Definition 5.18.)*

PROOF: ($\Rightarrow$): Suppose $Coz_\kappa L$ is compact. Note that H_κ is κ-Lindelöf and hence every frame homomorphism $\varphi : H_\kappa \to L$ factors through $\mathcal{H}_\kappa Coz_\kappa L$:



Now, $\mathcal{H}_\kappa Coz_\kappa L$ is compact, and hence $\bar{\varphi}$ is bounded. Thus φ is bounded.
($\Leftarrow$): Let A be a κ-cover of L consisting of elements of $Coz_\kappa L$. Let H_κ be the frame of opens of the space $Z_\kappa = 2^A - \{\mathbf{0}\}$. Now, the frame homomorphism $\varphi : H_\kappa \to L$ defined in Proposition 6.25 is bounded. Thus $\mathbf{e} = \varphi(\bigvee_{1\leq i\leq n} \pi_{a_i}^{-1}((0,1])) = \bigvee_{1\leq i\leq n} \varphi(\pi_{a_i}^{-1}((0,1])) = \bigvee_{1\leq i\leq n} a_i$. Hence $Coz_\kappa L$ is compact. □

Proposition 6.49 *For a κ-pseudocompact frame L, the following are equivalent:*

1) L is compact.

2) L is spatial and κ-Lindelöf.

3) L is spatial and κ-compact.

PROOF: (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial. Note that a compact regular frame is spatial by the Boolean Ultrafilter Theorem.
(3) $\Rightarrow$ (1): Since L is κ-pseudocompact, it follows that $Coz_\kappa L$ is compact and hence $\mathcal{H}_\kappa Coz_\kappa L$ is compact and consequently spatial. Thus, since L is spatial, we have $L \cong SL \cong S\mathcal{H}_\kappa Coz_\kappa L \cong \mathcal{H}_\kappa Coz_\kappa L$. Hence L is compact. □

The analogous results for the zero dimensional case are proved similarly. We shall omit the proofs here.

Definition 6.50 *A zero dimensional frame L is called zero dimensionally κ-pseudocompact if $\mathbb{B}_\kappa L$ is a compact κ-frame.*

Remark 6.50

1. It is clear that a zero dimensionally κ-pseudocompact frame is zero dimensionally κ-pseudocompact. The two notions coincide for strongly zero dimensional frames.

2. The zero dimensionally ω_1-pseudocompact frames are exactly the $\mathbb{N}$-pseudocompact frames.

Proposition 6.51 *A zero dimensional frame is zero dimensionally κ-pseudocompact iff $\mathcal{H}_\kappa\mathbb{B}_\kappa L$ is compact.*

Proposition 6.52 *Let $\kappa > \omega$. Then a zero dimensional frame L is zero dimensionally κ-pseudocompact iff L is H^0_κ-pseudocompact.*

Proposition 6.53 *For a zero dimensionally κ-pseudocompact frame L, the following are equivalent:*

1) L is compact.

2) L is spatial and κ-Lindelöf.

3) L is spatial and zero dimensionally κ-compact.

6.8 E-compact frames

We state the following result which appears as Lemma 8 in [17] for uniform frames. It is easily checked that the result also holds for nearness frames:

Lemma 6.54 *For any completion $h : M \to L$ of nearness frames $h(s) \neq \mathsf{e}$ for each maximal $s \in M$ iff each Cauchy filter in L converges.*

If a frame L is endowed with its E-nearness, we shall call a Cauchy filter in L an E-Cauchy filter in L.

Proposition 6.55 *The following are equivalent for an E-regular frame L:*

1) Every E-Cauchy filter in L converges.

2) $\gamma_L(m) \neq \mathsf{e}$ for every maximal $m \in C_E L$.

3) The map $S\gamma_L : SC_E L \to SL$ is an isomorphism.

Corollary 6.58 *Every E-complete frame is E-compact.*

Corollary 6.59 *The category E**KFrm** is the relatively spatial hull of E**CFrm** in E**RFrm**.*

Proposition 6.60 *The E-compactification of a spatial frame is spatial.*

PROOF: This follows from the general fact that the relatively spatial extension of a spatial frame is spatial. (See the remark after Lemma 3 in [16].) □

Proposition 6.61 *The spatial reflection of an E-compact frame is E-compact.*

PROOF: Suppose L is E-compact. Let $s_L : L \to SL$ be the spatial reflection of L. Since L is E-compact, there is a unique $\bar{s}_L : L \to v_E L$ such that $\bar{\gamma}_{SL} \cdot \bar{s}_L = s_L$

$$\begin{array}{ccc} L & \xrightarrow{s_L} & SL \\ & \searrow^{\bar{s}_L} & \uparrow \bar{\gamma}_{SL} \\ & & v_E SL \end{array}$$

Now, we know that $v_E SL$ is spatial, and so there exists a map $h : SL \to v_E SL$ such that $h \cdot s_L = \bar{s}_L$. Now, $\bar{\gamma}_{SL} \cdot h \cdot s_L = \bar{\gamma}_{SL} \cdot \bar{s}_L = s_L$ and since s_L is epic, it follows that $\bar{\gamma}_{SL} \cdot h = id_{SL}$. Thus $\bar{\gamma}_{SL}$ is epic section and hence an isomorphism.□

Corollary 6.62 *The spatial E-compact frames are precisely those frames that are spatial reflections of E-complete frames.*

PROOF: Suppose L is spatial E-compact. Then $L \cong SL \cong SC_E L$. Thus L is the spatial reflection of $C_E L$.
The converse follows from the fact that E-complete frames are E-compact and the spatial reflection of E-compact frames are E-compact. □

Corollary 6.63 *i) The spatial κ-compact frames are exactly those frames that are spatial reflections of κ-Lindelöf frames.*

ii) The spatial zero dimensionally κ-compact frames are exactly those frames that are spatial reflections of zero dimensional κ-Lindelöf frames.

PROOF: This follows from the fact that the κ-compact (respectively zero dimensionally κ-compact) frames are exactly the H_κ-compact (respectively H_κ^0-compact) frames. $\square$

Proposition 6.64 *Let E be a Hausdorff space. Then an E-regular space X is E-compact iff $\mathcal{O}X$ is $\mathcal{O}E$-compact.*

PROOF: Suppose $\mathcal{O}X$ is $\mathcal{O}E$-compact. Then $\mathcal{O}X \cong S\mathcal{O}X \cong S\gamma_{\mathcal{O}E}\mathcal{O}X$. Consider the diagram:

$$\begin{array}{ccc} S(\bigoplus \mathcal{O}E) & \xleftarrow{s_{\bigoplus \mathcal{O}E}} & \bigoplus \mathcal{O}E \\ {\scriptstyle s\nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ SC_{\mathcal{O}E}\mathcal{O}X & \xleftarrow[s_{C_{\mathcal{O}E}\mathcal{O}X}]{} & C_{\mathcal{O}E}\mathcal{O}X \end{array}$$

where the coproduct is taken over all frame homomorphisms $\varphi : \mathcal{O}E \to \mathcal{O}X$ and the map ν is a closed quotient. Now, X is a closed subspace of a power of E since $\mathcal{O}(\prod E) \cong S(\bigoplus \mathcal{O}E)$ and $\mathcal{O}X \cong S\gamma_{\mathcal{O}E}\mathcal{O}X$. Thus X is E-compact.
Conversly, suppose X is E-compact. Then X is (isomorphic to) a closed subspace of $\prod E$, the product being taken over all continuous functions $f : X \to E$. Thus $\mathcal{O}X$ is a closed quotient of $\mathcal{O}(\prod E) \cong S(\bigoplus \mathcal{O}E)$. It follows that $SC_{\mathcal{O}E}\mathcal{O}X \cong \mathcal{O}X$ and hence from Proposition 6.61 $\mathcal{O}X$ is E-compact. $\square$

Corollary 6.65 *Let E be a Hausdorff space, and let X be an E-regular space. If $v_E X$ is the E-compactification of X then $\mathcal{O}v_E X \cong v_{\mathcal{O}E}\mathcal{O}X$.*

PROOF: Since $\upsilon_{\mathcal{O}E}\mathcal{O}X$ is spatial, the result follows from the co-universal properties of $\upsilon_{\mathcal{O}E}\mathcal{O}X \to \mathcal{O}X$ and $\mathcal{O}\upsilon_E X \to \mathcal{O}X$. □

6.9 $\mathcal{E}$-Compact frames

Definition 6.66 *Let $\mathcal{E}$ be any class of frames. We call a frame L $\mathcal{E}$-regular if L is a quotient of a copower of members of $\mathcal{E}$. An $\mathcal{E}$-regular frame L is called $\mathcal{E}$-complete if L is a closed quotient of a copower of members of $\mathcal{E}$.*

Examples

1. Let $\mathcal{E}$ be the class of all Boolean algebras. Then the $\mathcal{E}$-regular frames are the zero dimensional frames.
2. (Rosický and Šmarda) Let $\mathcal{E}$ be the class of all open-set lattices of T_1 spaces. Then the $\mathcal{E}$-regular frames are the T_1 frames as defined in Section 3.3.

Definition 6.67 *Let $\mathcal{E}$ be a class of regular frames. Then a dense extension $h : M \to L$ is called a $C_{\mathcal{E}}$-extension if for any frame homomorphism $\varphi : E \to L$, where $E \in \mathcal{E}$ there exists a frame homomorphism $\bar{\varphi} : E \to M$ such that $h \cdot \bar{\varphi} = \varphi$.*

Definition 6.68 *Let L be a regular frame, and let $\mathcal{E}$ be a class of regular frames. Then the nearness generated by the covers $\varphi_1[C_1] \wedge \cdots \wedge \varphi_k[C_k]$, where $\varphi_i : E_i \to L$, $E_i \in \mathcal{E}$ are frame homomorphisms and $C_i \in Cov(E_i)$ is called the $\mathcal{E}$-nearness of L.*

Note that the members of the $\mathcal{E}$-nearness of L are the images of uniform covers of coproducts of members of $\mathcal{E}$. In particular, the $\mathcal{E}$-nearness of L is a strong nearness.

Proposition 6.69 *Let $\mathcal{E}$ be a class of regular frames, and L an $\mathcal{E}$-regular frames. Then the following are equivalent:*

1) L is $\mathcal{E}$-complete.

2) Every $C_{\mathcal{E}}$-extension of L is an isomorphism.

3) L is complete in its $\mathcal{E}$-nearness.

PROOF: The proof is similar to that of Proposition 6.4 and is therefore omitted.□

Since completion is coreflective for strong nearness frames, it follows that the category of all $\mathcal{E}$-complete frames, denoted by $\mathcal{E}$**CFrm** is coreflective in the category $\mathcal{E}$**RFrm** of all $\mathcal{E}$-regular frames. In fact, since every frame homomorphism has a unique regular mono-epi factorisation, it follows that the cateogries $\mathcal{E}$**KFrm** for some class $\mathcal{E}$ of frames are exactly the monocoreflective subcategories of **Frm**.

Proposition 6.70 *Let $\mathcal{E}$ be the class of all complete Boolean algebras. Then the $\mathcal{E}$-complete frames are those zero dimensional frames for which every cover is refined by a partition.*

PROOF: We know that for any single complete atomic Boolean algebra A_κ with κ atoms, the A_κ-complete frames are the zero dimensional frames for which every cover is refined by a κ-partition. Taking the class of all atomic Boolean algebras, we obtain the zero dimensional frames for which every cover is refined by a partition. Now, every complete Boolean algebra satisfies this condition, and so the result follows. □

Remark 6.70

1. A zero dimensional frame is strongly zero dimensional if every finite cozero cover is refined by a finite partition. Thus the $\mathcal{E}$-complete frames (where $\mathcal{E}$ is the class of all complete Boolean algebras) are all strongly zero dimensional:

If every cover is refined by a partition, then in particular every finite cozero cover is refined by a partition. But this means that every finite cozero cover is refined by a finite partition.

2. We arrived at this result as well as the preceding proposition after some discussion with A. Hager.

Proposition 6.71 *Let $\mathcal{E}$ be a class of regular frames. Then the following are equivalent for an $\mathcal{E}$-regular frame L:*

1) Every $\mathcal{E}$-Cauchy filter in L converges.

2) $\gamma_L(a) \neq \mathbf{e}$ for every maximal $a \in C_{\mathcal{E}}L$.

3) The map $s\gamma_L : SC_{\mathcal{E}}L \to SL$ is an isomorphism.

4) Every frame homomorphism $\varphi : C_{\mathcal{E}}L \to 2$ factors through the completion map $\gamma_L : C_{\mathcal{E}}L \to L$.

PROOF: The proof is similar to that of Proposition 6.55 and is therefore omitted.□

Definition 6.72 *$\mathcal{E}$-regular frames satisfying the equivalent conditions in Proposition 6.71 are called $\mathcal{E}$-compact frames.*

Corollary 6.73 *The category $\mathcal{E}$**KFrm** is the relatively spatial hull of $\mathcal{E}$**CFrm** in $\mathcal{E}$**RFrm** and is therefore coreflective in $\mathcal{E}$**RFrm** with coreflection given by the relatively spatial reflection of the completion map $\gamma_L : C_{\mathcal{E}}L \to L$.*

Proposition 6.74 *Let $\mathcal{E}$ be a class of Hausdorff spaces, and let $\mathcal{O}\mathcal{E}$ denote the class $\{\mathcal{O}Y | Y \in \mathcal{E}\}$. Then an $\mathcal{E}$-regular space X is $\mathcal{E}$-compact iff $\mathcal{O}X$ is $\mathcal{O}E$-compact.*

Corollary 6.75 *Let $\mathcal{P}$ be the class of all zero dimensional frames L such that every cover of L is refined by a partition, and let $\bar{\mathcal{P}}$ be the relatively spatial hull of $\mathcal{P}$ in* **ODFrm**. *Then $\bar{\mathcal{P}} \cap Ob(\mathbf{SpFrm})$ is exactly the class of all spatial $\mathbb{N}$-compact frames, assuming no measurable cardinal exists.*

PROOF: Let $\mathcal{E}$ the the class of all discrete spaces. Now, $\mathcal{O}\mathcal{E}$ is the class of all complete atomic Boolean algebras, and hence from Proposition 6.70, the class $\mathcal{P}$ is the class of all $\mathcal{O}\mathcal{E}$-complete frames. Hence $\bar{\mathcal{P}}$ is the class of all $\mathcal{O}\mathcal{E}$-compact frames. Now, a space X is $\mathcal{E}$-compact iff $\mathcal{O}X$ is $\mathcal{O}\mathcal{E}$-compact. If no measurable cardinal exists then the class of all $\mathcal{E}$-compact spaces is precisely the class of all $\mathbb{N}$-compact spaces. It follows that the class of all spatial frames in $\bar{\mathcal{P}}$ is exactly the class of all spatial $\mathbb{N}$-compact spaces. □

Remark 6.75 Since $\mathcal{N} \in \mathcal{P} \subseteq \bar{\mathcal{P}}$, it follows that the class of all $\mathbb{N}$-compact frames is contained in $\bar{\mathcal{P}}$. We do not know if the two classes are the same.

6.10 Remarks and unsolved problems

1. The frame-theoretic versions of Hušek's and Hong's spaces, H_κ and H_κ^0 respectively are defined only for non-limit regular cardinals. Hušek's [52] original results covered all infinite cardinals by way of the following result:

   Theorem 1 *(Hušek [52]) Let κ be a limit cardinal number greater than $\aleph_0$ and let M be a cofinal set in the set of infinite cardinals less than κ. Assume that for each $m \in M$ there is a topological space P_m such that (P_m)-compact spaces are just m-compact spaces. Then $(\prod_{m\in M} P_m)$-compact spaces are just κ-compact spaces.*

   A natural way to extend our frame-theoretic results to cover the limit cardinals as well would be by way of a theorem similiar to that of Hušek. Is it true that for any limit cardinal $\kappa > \aleph_0$ and cofinal set M of cardinals less than κ the κ-Lindelöf frames are simply the $(\bigoplus_{\lambda\in M} H_\lambda)$-complete frames? (Note that the frame $\bigoplus_{\lambda\in M} H_\lambda$ need not be spatial.)

2. The Axiom of Countable Choice (ACC) states that every countable family of nonempty sets has a choice function. In [79] G. Schlitt proved that ACC holds iff the coproduct of regular Lindelöf frames is Lindelöf and the discrete space of integers is Lindelöf, and consequently ACC holds iff $\mathcal{N}$-complete implies Lindelöf iff $\mathcal{L}(\mathbb{R})$-complete implies Lindelöf. See also [12].
   It is conceivable that a variant of this axiom by replacing 'countable family' with 'κ-family' may be equivalent to each of the statements H_κ-complete implies κ-Lindelöf, H_κ^0-complete implies κ-Lindelöf and A_κ-complete implies strongly κ-Lindelöf. This question of the equivalence was raised by Bernhard Banaschewski.

Bibliography

[1] J. Adámek, H. Herrlich and G. Strecker. Abstract and Concrete Categories. *Wiley Interscience. 1990.*

[2] R.A. Alo and H.L. Shapiro. Normal topological spaces. *Cambridge University Press. 1974.*

[3] D. Baboolal. *Fan-Gottesman type compactifications of frames.* Comment. Math. Univ. Carolinae **31, 2** (1990) 373–381.

[4] B. Banaschewski. *Extensions of topological spaces.* Can. Math. Bull. **7** (1964) 1–22.

[5] B. Banaschewski. *Another look at the localic Tychonoff theorem.* Comment. Math. Univ. Carolinae **29,4** (1988) 647–656.

[6] B. Banaschewski. *Compactification of frames.* Math. Nach. **149** (1990) 105–116.

[7] B. Banaschewski. *Frames.* Seminar, University of Cape Town, 1991.

[8] B. Banaschewski. *The frame envelope of a σ-frame.* Quaest. Math. **16** (1991) 51–60.

[9] B. Banaschewski. Wallman revisited. *Seminar lectures, University of Cape Town, 1993.*

[10] B. Banaschewski. *Recent results in pointfree topology.* Annals New York Academy of Sciences. **659** (Sept) (1992) 29–41.

[11] B. Banaschewski. Completion in pointfree topology. *Lecture notes in Math. and Appl. Math. 2/96, University of Cape Town, 1996.*

[12] B. Banaschewski. The real numbers in pointfree topology. *Seminar Notes, University of Coimbra, 1997.*

[13] B. Banaschewski. *The Axiom of Countable Choice and pointfree topology.* Mathematik-Arbeitspapiere **48** Universitat Bremen (1997) 27–38.

[14] B. Banaschewski, J.L Frith and C.R.A. Gilmour. *On the congruence lattice of a frame* Pacific J. Math. **130, 2** (1987) 209–213

[15] B. Banaschewski and C.R.A. Gilmour. *Stone-Čech compactification and dimension theory for regular σ-frames.* J. London Math. Soc. **(2) 39** (1989) 1–8.

[16] B. Banaschewski and C.R.A. Gilmour. *Pseudocompactness and the cozero part of a frame.* Comment. Math. Univ. Carolinae **37, 3** (1996) 577–587.

[17] B. Banaschewski and C.R.A. Gilmour. *Realcompactness and the cozero part of a frame. (Submitted.)*

[18] B. Banaschewski and S.S. Hong. *Filters and strict extensions of frames.* Preprint 1995

[19] B. Banaschewski, S.S. Hong and A. Pultr. *On the completion of nearness frames.* Preprint.

[20] B. Banaschewski, C.J. Mulvey. *Stone-Čech compactification of localesI.* Houston J. Math. **6** (1980) 301–312.

[21] B. Banaschewski, C.J. Mulvey. *Stone-Čech compactification of locales II.* Journal of Pure and Applied Algebra. **33** (1984) 107–122.

[22] B. Banaschewski and A. Pultr. *Samuel compactification and completion of uniform frames.* Math. Proc. Camb. Phil. Soc. (1990) **108**, 63–78.

[23] B. Banaschewski and A. Pultr. *Paracompactness revisited.* Appl. Cat. Struct. **1** (1993) 181–190

[24] B. Banaschewski and A. Pultr. *Cauchy points of uniform and nearness frames.* Quaest. Math. **19** (1996) 101–127..

[25] B. Banaschewski and A Pultr. *On Cauchy homomorphisms of nearness frames.* Math. Nachr. **183** (1997) 5–18.

[26] R. Beazer and D.S. Macnab. *Modal extensions of Heyting algebras.* Colloq. Math. **41** (1979) 1–12

[27] L. Bentley and H. Herrlich. *Completion as reflection.* Comment. Math. Univ. Carolinae. **19** (1978), 541–568.

[28] S. Broverman. *Another realcompact, 0-dimensional, non-ℕ-compact space.* Proc. Amer. Math. Soc. **69** (1978) 156–158.

[29] X. Chen. Closed frame homomorphisms. *PhD thesis, McMaster University, 1991.*

[30] G.D. Dimov and G. Tironi. *Compactifications and A-compactifications of frames. Proximal frames.* Math. Proc. Camb. Phil. Soc. **119, 2** (1996) 321–339

[31] C.H. Dowker and D. Papert. *Quotient frames and subspaces.* J. London Math. Soc. **(3) 16)** (1966) 275–269.

[32] C.H. Dowker and D. Papert. *On Uryshon's lemma.* In General Topology and its Relations to Modern Analysis and Algebra II, Proc. of the Second Prague Top. Symp. (1966) 111–115.

[33] T.A. Dube. Structures in Frames. *PhD thesis, University of Durban-Westville, 1992.*

[34] T.A. Dube. *Tamano-Dowker type theorems for nearness frames.* J. Pure Appl. Alg. **99** (1995) 1–7.

[35] T.A. Dube. *A note on complete regularity and normality.* Quaest. Math. **19** (1996) 467–478.

[36] R. Engelking and S. Mrówka. *On E-compact spaces.* Bull. Acad. Pol. Sci. Ser. Sci. Math. Astr. Phys. **6**,7 (1958) 429–436.

[37] K. Eda, T. Kiyosawa and H. Ohta. ℕ-Compactness and its applications. In: *Topics in General Topology.* K. Morita, J. Nagata, Eds. Elsevier (1989) 459–521.

[38] J.L. Frith. Structured Frames. *PhD thesis University of Cape Town. 1986.*

[39] L. Gillman and M. Jerrison. Rings of continuous functions. *Van Nostrand, Princeton, 1960.*

[40] C.R.A. Gilmour. *Realcompactifications through zero-set spaces.* Quaest. Math. **6** (1983) 73–95.

[41] C.R.A. Gilmour. *Realcompact Alexandroff spaces and regular σ-frames.* Math. Proc. Camb. Phil. Soc. **96** (1984) 73–79.

[42] C.R.A. Gilmour. Realcompact Alexandroff Spaces and Regular σ-Frames. *Mathematical Monographs of the University of Cape Town. 1985.*

[43] H. Herrlich. *Fortsetzbarkeit stetiger Abbilungen und Kompaktheitsgrad topologischer Räume.* Math. Zeitschr. **96** (1967) 64–72.

[44] H. Herrlich. *$\mathfrak{E}$-kompakte Räume.* Math. Zeitschr. **96** (1967) 228–255.

[45] H. Herrlich. *A concept of nearness.* Gen. Top. Appl. **4** (1974), 191–212.

[46] H. Herrlich. *Compactness and the Axiom of Choice.* Appl. Cat. Struct. **4** (1996) 1–14.

[47] H. Herrlich and G. Strecker. *Categorical topology - Its origins.* in C.E. Aull and R. Lowen (eds.) Handbook of the History of General Topology, Vol. I, 255–341, Kluwer Academic Publishers, 1997.

[48] R.E. Hoffmann. *On the sobrification remainder* $sX\backslash X$. Pacific J. Math. **83, 1** (1979) 145–156.

[49] S.S. Hong. *On k-compactlike spaces and reflective subcategories.* Gen. Top. Appl., **3** (1973) 319–330.

[50] S.S. Hong and Y.K. Kim. *Cauchy completions of nearness frames.* Appl. Cat. Struct. **3** (1995), 371–377.

[51] M. Hušek. *On a problem of H. Herrlich.* Comment. Math. Univ. Carolinae. **9** (4) (1968) 571–572.

[52] M. Hušek. *The Class of k-compact Spaces is Simple.* Math. Z. **110** (1969) 123–126.

[53] J. Isbell. *Atomless parts of spaces.* Math. Scand. **31** (1972) 5–32.

[54] P.T. Johnstone. Stone Spaces. *Cambridge studies in advanced mathematics 3, Cambridge University Press. 1982.*

[55] P.T. Johnstone. *The point of pointless topology.* Bull. Amer. Math. Soc. **8** (1983) 41–53

[56] I. Kříž. *A direct description of uniform completion and characterization of LT groups.* Cah. Top. Géom. Diff. Cat. **27** (1986), 19–34.

[57] S. Mac Lane. Categories for the Working Mathematician. *Graduate Texts in Mathematics 5, Springer-Verlag, 1971.*

[58] D.S. Macnab. *Modal operators in Heyting algebras.* Algebra Universalis, **12** (1981) 5–29

[59] J.J. Madden. *κ-frames.* J. Pure and App. Alg. **70** (1991) 107–127.

[60] J. Madden and J. Vermeer. *Lindelöf locales and realcompactness.* Math. Proc. Camb. Phil. Soc. **99** (1986) 473–480.

[61] J. Madden and J. Vermeer. *Epicomplete archemedian l-groups via a localic Yosida theorem.* J. Pure Appl. Algebra **68** (1990) 243–252.

[62] N. Marcus. Realcompactifications of frames. *Masters Thesis, University of Cape Town, 1993.*

[63] N. Marcus. *Realcompactification of frames.* Comment. Math. Univ. Carolinae **36, 2** (1995) 347–356.

[64] S. Mrówka. *On universal spaces.* Bull. Acad. Polon. Sci., Cl. III **8** (1956) 479–481.

[65] S. Mrówka. *A property of the Hewitt-extension υL of topological spaces.* Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys. **6** 95–96.

[66] S. Mrówka. *Further results on E-compact spaces. I.* Acta Math. **20** (1968) 161–185

[67] S. Mrówka. *E-complete regularity and E-compactness.* General Topology and it Relation to Modern Analysis and Algebra, III (Proc. Conf., Kanpur, 1968), 207–214. Prague: Academia 1971.

[68] S. Mrówka. *Recent results on E-compact spaces.* TOPO 72, Lecture Notes in Math., vol. 378, Springer-Verlag, 1974, 298–301.

[69] N. Mysior. *Two easy examples of zerodimensional spaces.* Proc. Amer. Math. Soc. **92**, 4 (1984) 615–617

[70] S.B. Niefield and K.I. Rosenthal. *Spatial sublocales and essential primes.* Topology and its Applications. **26** (1987) 263–269.

[71] J. Paseka. *Lindelöf locales and* $\mathbb{N}$*-compactness.* Math. Proc. Camb. Phil. Soc. **109** (1991) 187–191.

[72] J. Paseka. *Paracompact locales and metric spaces.* Math. Proc. Camb. Phil. Soc. **110** (1991) 251–256.

[73] T. Przymusiński. *On the dimension of product spaces and an example of M. Wage.* Proc. Amer. Math. Soc. **76** (1979) 315–321.

[74] J. Rosický and B. Šmarda. T_1*-locales.* Math. Proc. Camb. Phil. Soc. **98** (1985) 81–86.

[75] G. Reynolds. *On the spectrum of a real representable ring.* In Applications of Sheaves, Springer-Verlag Lecture Notes in Mathematics, vol. 753, 595–611

[76] A. Robertson. Normal Bases and Compactifications of Frames. *Masters Thesis, University of Cape Town, 1995.*

[77] G. Schlitt. $\mathbb{N}$-Compact frames and Applications. *PhD Thesis, McMaster University, 1990.*

[78] G. Schlitt. $\mathbb{N}$*-Compact frames.* Comment. Math. Univ. Carolinae **32,1** (1991) 173–187.

[79] G. Schlitt. *The Lindelöf Tychonoff Theorem and choice principles.* Math. Proc. Camb. Phil. Soc. **110** (1991) 57–65.

[80] H. Simmons. *A framework for topology.* Logic Colloquium 77, Studies in Logic 96, North-Holland 1978, 239–251.

[81] H. Simmons. *Spaces with Boolean assemblies.* Colloq. Math. **43** (1980) 23–39.

[82] H. Simmons. *The Cantor-Bendixson analysis of a frame.* Univ. Cath. de Louvain, Sém. de math. pure. Rapport no. 92, 1980.

[83] H. Simmons. *A chapter on some functorial aspects of frame theory.* Univ. Cath. de Louvain, Sém. de math. pure. Rapport no. 106, 1980.

[84] M.H. Stone. *The theory of representations for Boolean algebras.* Trans. Amer. Math. Soc. **40** (1936) 37–111.

[85] M.H. Stone. *Applications of the theory of Boolean rings to general topology.* Trans. Amer. Math. Soc. **41** (1937) 375–481.

[86] M.H. Stone. *The representation of Boolean algebras.* Bull. Amer. Math. Soc. **44** (1938) 807–816.

[87] S. Vickers. Topology via Logic. *Cambridge Tracts in Theoretical Computer Science. No. 5, Cambridge University Press, Cambridge 1985.*

[88] R.C. Walker. The Stone-Čech compactification. *Springer-Verlag 1974.*

[89] H. Wallman. *Lattices and topological spaces.* Ann. of Math. **39** (1938) 112–126.

[90] J.L. Walters. Uniform sigma frames and the cozero part of uniform frames. *Masters Thesis, University of Cape Town, 1989.*

[91] J.L. Walters. *Compactifications and uniformities on σ-frames.* Comment. Math. Univ. Carolinae **32** (1) (1991) 189–198.

[92] J.L. Walters-Wayland. Completeness and nearly fine uniform frames. *PhD Thesis, Universite Catholique de Louvain, 1995.*

[93] M.D. Weir. Hewitt-Nachbin Spaces. *Mathematical Studies no. 17, North Holland, 1975.*

[94] S. Willard. General Topology. *Addison-Wesley. 1970.*