On Models of Kirchhoff Equations with Damping Terms: Existence Results and Asymptotic Behaviour of Solutions

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Declaration

I declare that all the work in this thesis, except for that which is properly acknowledged is my own unaided work and it is this day submitted to the University of Cape Town for Masters in Mathematics. It has not been submitted before in any University for a degree or examination.

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With a grateful heart to God Almighty for allowing this thesis to see the light of day.

Abstract

In this thesis we consider a Cauchy problem in a Hilbert space H for the equation of the Kirchhoff-type in its abstract form

$$u''(t) + \delta u'(t) + M(\|A^{1/2}u(t)\|^2)Au(t) = 0,$$

with initial conditions

$$u(0) = u_0, \qquad u'(0) = u_1,$$

where $\delta > 0$, M is a suitable continuous function and A is some unbounded operator on the Hilbert space H.

Under some smallness assumption on the initial data and the non-degeneracy condition, we establish results of local/global existence together with the regularity and the asymptotic behaviour of the global solution. The existence results are based on two main approaches developed in the literature, namely the approach via the Schauder fixed point theorem and the Galerkin approximation method. For both methods, suitable a priori estimates are established.

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Contents

Introduction

In 1883, Gustav Kirchhoff [73] proposed as a generalization of the well-known D'Alembert's wave equation, a partial differential equation describing the oscillations of an elastic streched string particularly, taking into account the subsequent change in string length caused by oscillations. Since then, this equation became the subject of intense research attracting mathematicians, physists and engineers. Precisely, Kirchhoff derived the nonlinear integro-differential equation:

$$\rho\, h\, \frac{\partial^2 u}{\partial t^2} - \left\{ p_0 + \frac{E\, h}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right\} \frac{\partial^2 u}{\partial x^2} = 0 \quad 0 \le x \le L, \quad t \ge 0\,, \tag{1}$$

where

- $u = u(x,t)$ is the lateral displacement at the space coordinate x and time t;
- E is the Young modulus;
- ρ is the mass density;
- h is the cross-section area;
- L is the length;
- p_0 is the initial axial tension.

The model including linear damping and external force is given by

$$\rho\, h\, \frac{\partial^2 u}{\partial t^2} + \delta\, \frac{\partial u}{\partial t} - \left\{ p_0 + \frac{E\, h}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right\} \frac{\partial^2 u}{\partial x^2} = f\,, \tag{2}$$

where δ is the resistance modulus and f is the external force.

In general, the model (1) is extended in higher dimension $N \ge 2$ into the second-order hyperbolic partial differential equation (PDE):

$$\frac{\partial^2 u}{\partial t^2} - M\Big(\int_\Omega \|\nabla u\|^2\, dx \Big)\, \Delta u = f(u)\,, \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta = \sum_{i=1}^N \partial^2/\partial x_i^2$ is the Laplace operator, f is a nonlinear function (e.g., $f(u) = \pm|u|^\alpha \mathrm{u}$, $\alpha > 0$) while $M(r)$ is a non-negative locally Lipschitz function for $r \ge 0$ like

$$M(r) = a + b\, r^\gamma \qquad \text{with } a \ge 0,\ \ b \ge 0,\ \ a + b > 0,\ \ \gamma > 0\,. \tag{4}$$

In the case of (4), the equation (3) is called *non-degenerate* when $a > 0$ and $b > 0$, *degenerate* when $a = 0$ and $b > 0$, and is the usual semilinear wave equation when $a > 0$ and $b = 0$. In general, the equation (3) is called non-degenerate if $M(r) \ge m_0 > 0$ for $r \ge 0$ and degenerate if $M(r) \ge 0$ for $r \ge 0$. In the case where M may vanish but not at the initial time, i.e., $M\big(\int_\Omega \|\nabla u(x,0)\|^2\, dx\big) > 0$, we say that the equation (3) is *midly degenerate*.

For the derivation of (1), besides the original source [73] as mentioned above, Carrier [26] gave later a somewhat different description of the motion of an elastic string. This explains why the equation (1) is also called the Kirchhoff-Carrier equation by some authors. We can also mention [8, 70, 98].

While results of local existence, for the initial boundary problem involving the Kirchhoff equation (3) were obtained by several authors like Bernstein [20], Dickey [41, 42], Menzala [90] and even in the context of Sobolev spaces (see e.g., [114, 115, 65]), the first global solvability result was established by Bernstein [20] in dimension $N = 1$ for analytic initial data. Bernstein's result was then extended later in higher dimensions ($N \geq 2$), by Pohozaev [104], Arosio-Spagnolo [10], Kajatani-Yamaguti [68] . Throughout the years, these results on the global solvability for analytical initial data were extended and refined by several authors (see for instance, Nishihara [99], Ghisi-Gobbino [55]).

Several results on the global solvability for small non-analytical (mainly of class $\mathcal{C}^\infty$ with compact support, Gevrey class) initial data are well established in the literature (see for instance,[25, 33, 34, 57, 89, 105, 116, 117]). We also mention that, for non-analytical initial data, some authors like Pohozaev [103] and Menzala-Pereira [91] for instance, have obtained some global existence results, using non-physical functions $M(r)$ behaving like $(\alpha r + \beta)^{-2}$, α and β being positive constants.

Regarding the global solvability of the initial boundary value problem for the equation (3) in the context of Sobolev spaces, the question is still open for bounded domains while some results are available for unbounded domains (see e.g., [16])

In order to obtain results of global existence in bounded domains, some extra terms have been added to the equation. Some authors modified the equation (3) in order to introduce some dissipative effects. For instance, initial boundary problems with damping terms of the type

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - M\Big(\displaystyle\int_\Omega \|\nabla u\|^2\, dx\Big)\, \Delta u + g(u_t) = f(u) \quad \text{in} \quad \Omega \times [0, \infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times [0, \infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega \end{cases} \tag{5}$$

were proposed with g being an increasing continuous function with some growth assumptions.

Our focus in this thesis will be on the initial boundary value problem (5). Let us mention that for $g(u_t) = \delta u_t$ and $f = 0$, Brito [36] proved the existence and uniqueness of global solutions for sufficiently small initial data by using the Galerkin approximation method. For $g(u_t) = \delta u_t$ and $f = f(t)$ then Nishihara [100] obtained the existence of global solutions of the problem (5) for sufficiently smooth initial data. In the case where the Dirichlet boundary condition in (5) is replaced by the Neuman boundary condition, Ikehata [60] has shown the existence of global solutions using Galerkin approximation method as well. Later, Ikehata and Okazawa [63] used the Yosida approximation method together with some compactness argument to obtain global solutions to the problem (5).

Ghisi and Gobbino [54] rather proposed an approach based on the Schauder fixed theorem to derive the existence of global solution of (5) together with its asymptotic behaviour in the case where $g(u_t) = \delta u_t$ and $f = 0$ under an assumption of smallness of the initial data.

We also recall that when $M \equiv 1$ and $g(u_t) = \delta u_t$, then (5) is the wave equation with linear interior damping and a source term, for which Ikehata and Suzuki [64] investigated the dynamics and have shown that, for sufficiently small initial data (u_0, u_1), the trajectory of the solution (u, u_t) tends to $(0, 0)$ in some Sobolev-L^p norm as $t \to \infty$. Also, Georgiev and Todorova [53] have shown for $g(u_t) = \delta\, |u_t|^{m-1} u_t$ $(m > 1)$ and $f(u) = \mu\, |u|^{p-1} u$ $(\mu > 0$ and $p \geq 1)$ that, if the damping term dominates over the source, then a global solution to (5) with $M \equiv 1$ (wave equation), exists for any initial data. Many other results were established in the literature for the wave equation with nonlinear interior damping and a source term (see e.g., [60, 112, 96, 1, 3]).

Several authors discussed the existence of a global decaying solution for (5) in the case where $M(r)$ is a non-constant function and $g(u_t)$ is nonlinear. For instance, Ikehata and Matsuyama [61] obtained a global decaying solution of (5) in the case where

$$\begin{aligned} &M(r) = m_0 + r^{\frac{\gamma}{2}} \quad (\gamma \geq 2), \\ &g(u_t) = \delta |u_t|^r u_t, \quad \text{with } \delta > 0,\ 0 \leq r \leq 2/(N-2) \quad (0 \leq r \leq \infty \text{ if } N = 1,\, 2) \\ &f(u) = \mu\, |u|^p u,\ 0 < p \leq 4/(N-2)\ (0 < p < \infty \text{ if } N = 1,\, 2),\ \mu \geq 0). \end{aligned}$$

Other global solvability results for (5) in the case where $M(r)$ is a non-constant function and $g(u_t)$ is nonlinear have been established for instance in [62, 1, 3, 5, 72]. Let us also mention that models with nonlocal interior damping such as $g(u_t) := \alpha\,\Delta u_t$ (also called strong damping), were considered (see e.g., [100, 81, 120]). Also, since viscoelastic materials are known to provide a natural damping, which is due to the fact that those materials keep a certain memory, models of Kirchhoff equation with the visco-elastic additional term of the form $\int_0^t k(t-s)\Delta u\,dt$, with k being a kernel, were considered in [118, 79, 28, 94].

Other types of boundary conditions for the Kirchhoff equation have been considered in [123] as well as boundary damping terms in [122, 81, 123]. Also, higher-order Kirchhoff-type equation have been studied for instance in [80, 93]. Inspired by results on wave equations involving the p-Laplacian operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ (see e.g., [51, 43]), some authors studied the Kirchhoff-type equations with the p-Laplacian operator (see e.g., [44, 13, 59, 69, 14]).

The Kirchhoff-type equations enjoyed such a popularity that they were studied even in noncylindrical domains in [17, 27, 15, 16, 77]. Also, stationary Kirchhoff-type equations were studied through the use of well-known techniques of nonlinear analysis such as the Palais-Smale compactness condition, the mountain pass theorem (see e.g., [30, 85, 7, 86, 87, 31]). Notice that the nonlinearity $M\big(\int_\Omega \|\nabla u\|^2\,dx\big)$ in (5) occurs quite frequently. Arosio [9] calls the term $M\big(\int_\Omega \|\nabla u\|^2\,dx\big)$ the Kirchhoff-correction (briefly, the K-correction) and makes a reasonable statement that the K-correction is inherent in a lot of physical phenomena. For example, the K-correction is present in various models of beams [24, 47, 108] and plates [97, 113]. Let us also mention that models involving nonlinear coefficients depending on some average appear in the description of several other real-life phenomena. For instance, parabolic equations of the type

$$u_t - M\Big(\int_\Omega u(x,t)\,dx\Big)\Delta u = f \tag{6}$$

are studied in [29] as models for the reaction-diffusion describing the density of a population where the diffusion of individuals is guided by the global state of the population in the medium. Inspired by (6), some authors (like [56, 35, 50] studied the so-called parabolic Kirchhoff type equation

$$u_t - M\Big(\int_\Omega \|\nabla u\|^2_{L^2}\,dx\Big)\Delta u = f\,.$$

In this thesis, our focus is mainly on the initial boundary value problem with the hyperbolic Kirchhoff type equation (5) for both linear and nonlinear damping. We present here the contributions by Brito [36] and Ghisi and Gobbino [54] for linear damping $g(u_t) = \delta\,u_t$ and also Kim et al. [72] for nonlinear damping.

A well-known strategy to study the initial boundary value problem (5) as well as many other evolution problems is to reformulate the problem as an abstract Cauchy problem on a suitable Banach (functions) space. Then one solves that Cauchy problem using tools like the Galerkin approximation method (see e.g., [36, 72]), the Schauder's fixed theorem (see e.g., [54]), semi-groups (see e.g., [119]) and the Yosida approximation (see e.g., [2]) followed by suitable a priori estimates. Precisely, to a possible solution u of (5), we associate a mapping that we will still denote by u from $[0,\infty)$ to some Banach space V of functions from Ω to $\mathbb{R}$ i.e., $u : [0,\infty) \to V$ is such that $t \to u(t) : \Omega \to \mathbb{R}$ defined as follows,

$$u(t)(x) := u(x,t) \quad \forall x \in \Omega,\ t \geq 0\,.$$

The problem (5) is then transformed into the Cauchy problem

$$\begin{cases} u'' + M\Big(\big\|A^{1/2}u\big\|^2\Big)\,Au + g(u') = f(u) & \text{in} \quad [0,\infty), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{7}$$

Some authors refer to the formulation (7) as the Kirchhoff type equation and the setting is often presented as follows. Let H be a Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $A : H \to H$ be a self-adjoint linear non-negative operator with domain $D(A) := V$ (i.e., $(Au,u) \geq 0 \quad \forall u \in D(A)$) and $D(A^{1/2}) := W$. Given $f : V \to H$ we consider the problem of finding the global solution of the Cauchy problem (7) and study its asymptotic behaviour.

Structure of the thesis

This thesis is organized as follows. In Chapter 1, we first derive (for the reader's convenience the Kirchhoff equation(1) and we collect all the preliminaries results that are used in the thesis. Chapter 2 is based on the work of Brito [36]. We show how Brito obtained his global existence result for linear damping i.e., $g(u') = \delta\, u'$ and $f \equiv 0$, following the standard approach that combines the Galerkin approximation method with the a priori estimates. Another global existence result due to Ghisi and Gobbino [54] presented in Chapter 3 is obtained through a combination of the Schauder fixed point theorem and some suitable energy inequalities. In Chapter 4, we present a global existence result for a Kirchhoff model with a nonlinear damping by Kim *et al.* [72], where the nonlinearity satisfies some growth assumptions. Once again the global existence result is obtained using the Galerkin approximation method and some integral inequalities due to Haraux [58]. Having understood the methods and techniques used in the literature for the study of the Kirchhoff equation, we propose in the last chapter on conclusion and ongoing research, to study the problem in [72] (studied in Chapter 4) through the approach in [54]. We also present some new problems to be considered.

Chapter 1

Preliminaries

In this chapter, we recall some essential notions and concepts from abstract analysis that will be used throughout this thesis, as well as some results on L^p-spaces ($1 \leq p \leq \infty$), Sobolev spaces and some methods used in the study of ordinary differential equation or evolution problems. Let $x = (x_1, x_2, \ldots, x_n)$ be a generic point in an open subset Ω of $\mathbb{R}^n$. Given a vector $a = (a_1, a_2, \ldots, a_n)$ in $\mathbb{R}^n$, we denote by $\|a\|^2 = \sum_{i=1}^n |a_i|^2$ the Euclidean norm of the vector a. Let $u : \Omega \to \mathbb{R}$ be a function. We recall that u_{x_i} is the partial derivative of u with respect to the ith variable x_i and the gradient and Laplacian of u are $\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})^T$ and $\Delta u = \sum_{i=1}^n u^2_{x_i}$, respectively.

$C(\Omega)$ denotes the space of continuous functions from Ω to $\mathbb{R}$ while $C(\Omega)^m$ is the space of continuous functions from Ω to $\mathbb{R}^m$, $C_b(\overline{\Omega})$ is the space of bounded continuous functions from the closure $\overline{\Omega}$ of Ω to $\mathbb{R}$ equipped with the norm $\|u\|_\infty := \sup\limits_{x\in\overline{\Omega}} |u(x)|$.
For every $k \geq 1$ integer, $C^k(\Omega)$ is the space of continuous functions together with their partial derivatives up to order k and $C^k_c(\Omega)$ is the space of function in $C^k(\Omega)$ having a compact support contained in Ω. We also recall that $C^k(\overline{\Omega})$ denotes the space of the restrictions to $\overline{\Omega}$ of functions in $C^k(\mathbb{R}^n)$. We recall also that $C^\infty_c(\Omega) = \mathcal{D}(\Omega)$ is the space of infinitely differentiable functions with compact supports in Ω and is also called the space of *test functions*.

Theorem 1.1 (Ascoli Arzela).
Let (X, d_X) and (Y, d_Y) be two metric spaces with (X, d_X) compact and (Y, d_Y) complete. Let F be a family of continuous functions from X into Y. Then F is relatively compact in $C(X, Y)$ if and only if

(i) F is an equicontinuous family;

(ii) for every $x \in X$, the set $F(x) := \{f(x) : f \in F\}$ is relatively compact in Y.

Definition 1.2 (Convex set)
Let V be a vector space over the scalar field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The set $C \subset V$ is convex if the line segment

$$[x, y] := \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$$

joining any two points x and y of C lies entirely in C.

Definition 1.3 (Bounded bilinear form, coercive bilinear form)
Let $b(.,.) : V \times V \to \mathbb{R}$ be a bilinear form on the Banach space V.Then it is bounded if

$$|b(u, v)| \leq M\|u\|_V\|v\|_V \qquad \forall u, v \in V, \quad M > 0,$$

where the constant M is independent of u and v. The bilinear form is coercive or V-elliptic if

$$b(u, v) \geq m\|u\|^2_V \quad \forall u \in V, m > 0,$$

where the constant m is independent of u.

Definition 1.4 (Gronwall's Inequality (integral form).
Let $\xi(t)$ be a nonnegative, summable function on $[0,T]$ which satisfies for a.e. t the integral inequality

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds + C_2$$

for constants $C_1, C_2 \geq 0$. Then

$$\xi(t) \leq C_2(1 + C_1 t e^{C_1 t})$$

for a.e. $0 \leq t \leq T$.
In particular, if

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds$$

for a.e. $0 \leq t \leq T$, then

$$\xi(t) = 0 \qquad a.e.$$

1.1 Strong and Weak Convergence

Definition 1.5 Let $(V, \|\cdot\|)$ be a normed space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

- (Strong Convergence). A sequence (u_n) is said to be strongly convergent (or convergent in the norm) if there is an $u \in V$ such that

$$\lim_{n\to\infty} \|u_n - u\| = 0,$$

written

$$u_n \to u,$$

u is called the strong limit of (u_n), and we say that (u_n) converges to u.

- (Weak Convergence). We say a sequence $\{u_k\}_{k=1}^{\infty} \subset V$ converges weakly to $u \in V$, written

$$u_k \rightharpoonup u,$$

if

$$\langle u^*, u_k\rangle \to \langle u^*, u\rangle.$$

For each bounded linear functional $u^* \in V'$.
It is easy to check that if $u_k \to u$, then $u_k \rightharpoonup u$. It is also true that any weakly convergent sequence is bounded. In addition, if $u_k \rightharpoonup u$, then

$$\|u\| \leq \liminf_{k\to\infty} \|u_k\|.$$

- (Weak Star Convergence). A sequence $(f_n) \subseteq V'$ is weak star convergent to $f \in F^*$ if $f_n(u)$ converges to $f(u)$ for all $u \in V$.
Note that, the weak star limit of $(f_n) \subseteq V$ is unique and if the space V is reflexive, then we can replace $\hat{u} \in V'$ with $u \in V$ to show that weak star convergence implies weak convergence. Therefore weak and weak star convergence are equivalent on reflexive Banach spaces.

Definition 1.6 (Dual space) The dual space of V, denoted by V', is the space of all linear functionals on V; i.e $V' := \mathcal{L}(V, \mathbb{K})$.

Definition 1.7 Let V be a normed space and $V'' = (V')'$ denote the bidual space of V. The canonical map $u \in V \longmapsto \hat{u} \in V''$ defined by $\langle \hat{u}, f\rangle_{V'',V'} = \langle f, u\rangle_{V',V}$, for every $f \in V'$, the space V is called reflexive if the canonical map is surjective, in that case V can identified to V''.

Theorem 1.8 (Weak compactness). *Let V be a reflexive Banach space and suppose the sequence $\{u_k\}_{k=1}^{\infty} \subset V$ is bounded. Then there exists a subsequence $\{u_{kj}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and $u \in V$ such that*

$$u_{kj} \rightharpoonup u.$$

In other words, bounded sequences in a reflexive Banach space are weakly precompact. In particular, a bounded sequence in a Hilbert space contains a weakly convergent subsequence.

1.2 Spectral analysis for compact and self-adjoint operators

The Galerkin approximation method used in this thesis relies on the existence of the so-called *Hilbert basis* in the main Hilbert space. This is often obtained through the spectral analysis of some operator. In this section, we present briefly the key aspects of the spectral analysis of an operator from the definition of a spectrum of an operator to the spectral decomposition of compact self-adjoint operators.

1.2.1 The spectrum

Definition 1.9 Let $(V, \|\cdot\|_V)$ be a Banach space and let $L \in \mathcal{L}(V)$. The *resolvent set* of L is defined by

$$\rho(L) := \left\{\lambda \in \mathbb{C}: \ \ \lambda I - L \text{ is bijective from } V \text{ to } V\right\}.$$

The *spectrum* of L

$$\sigma(L) := \mathbb{C} \setminus \rho(L) = \left\{\lambda \in \mathbb{C} : \begin{cases} \bullet \ \lambda I - L \text{ is not injective or} \\ \bullet \ \lambda I - L \text{ is injective but not surjective} \end{cases}\right\}.$$

A number $\lambda \in \mathbb{C}$ such that the mapping $\lambda I - L$ is not injective, i.e., $N(\lambda I - L) \neq \{0_V\}$ is called an *eigenvalue* of L and the set of all eigenvalues of L is denoted by $\sigma_p(L)$.
Notice that since $L \in \mathcal{L}(V)$, whenever $\lambda \in \rho(L)$ it follows from the open mapping theorem that $(\lambda I - L)^{-1} \in \mathcal{L}(V)$.

1.2.2 The adjoint operator

Definition 1.10 Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed spaces with dual spaces V' and W' respectively. Let $L \in \mathcal{L}(V, W)$. Then the *adjoint operator* $T^* \in \mathcal{L}(W', V')$ of T defined by

$$\langle T^* w^*, v\rangle_{V',V} = \langle w^*, Tv\rangle_{W',W} \qquad \forall w^* \in W', \quad \forall v \in V.$$

Remark 1.11 Whenever V and W are Hilbert spaces with scalar products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$, then identifying the spaces to their duals we get $L^* \in \mathcal{L}(W, V)$ and

$$(L^* w, v)_V = (w, Lv)_W \qquad \forall v \in V, \ \forall w \in W.$$

Definition 1.12 Let $(H, (\cdot,\cdot))$ be a Hilbert space. The operator $L \in \mathcal{L}(H)$ is said to be *self-adjoint* if $L^* = L$. That is

$$(Lu, v) = (u, Lv) \qquad \forall u, \, v \in H.$$

1.2.3 Compact operators

Now we introduce a class of operators having interesting spectral properties even infinite dimensional spaces and which play an important role in the study of ordinary and partial differential equations as well as integral equations.

Definition 1.13 Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two Banach spaces. An operator $T \in \mathcal{L}(V, W)$ is called a compact operator if $T(B_V)$ has compact closure in W. Equivalently, T is compact if for every norm-bounded sequence $(u_n) \subseteq V$, $(T(u_n))$ has a strongly convergent subsequence in W. We denote the set of compact operators from V to W by $\mathcal{K}(V, W)$ and write $\mathcal{K}(V) = \mathcal{K}(V, V)$.

Theorem 1.14 (Spectral theorem for Compact, Self adjoint Operator). *Let $A : H \to H$ be a compact, self adjoint operator on a Hilbert space H. There is an orthonormal basis of H consisting of eigenvectors of A. The non-zero eigenvalues of A form a finite or countably infinite set (λ_k) of real numbers, and*

$$A = \sum_k \lambda_k P_k, \tag{1.1}$$

where P_k is the orthogonal projection onto the finite dimensional eigenspace of eigenvectors with eigenvalue λ_k. If the number of non-zero eigenvalues is countably infinite, then the series in (1.1) converges to A in the operator norm.

The initial boundary value problems (IBVPs) considered in this thesis are transformed into Cauchy problems governed by ODEs on Banach spaces. We will be having an abstract framework with the following structure: Let $(H, (\cdot,\cdot)_H)$ be a real Hilbert space with $\|\cdot\|_H$ being the norm induced by the scalar product $(\cdot,\cdot)_H$. Let $A : H \to H$ be a linear operator with domain $V = D(A)$ dense in H. We assume that the operator A is self-adjoint and (strictly) positive. This implies that there exists $\alpha > 0$ such that $(Au, u) \geq \alpha \|u\|_H^2$. Hence, we obtain a scalar product on V defined by

$$(u, v)_A := (Au, v)_H \qquad \forall u, v \in V$$

with norm induced

$$\|u\|_A := (Au, u)_H^{1/2} \quad \forall u \in V\,.$$

equivalent on V to the norm $\|\cdot\|_H$. Hence, $(V, (\cdot,\cdot)_A)$ is a Hilbert space. Under these assumptions, we get the so-called Hilbert triple: $V = D(A) \subset H \subset V'$ with H identified to its dual H'.
The following spectral theorem holds.

Theorem 1.15 *Assume that the operator A is such that the space V is compactly embedded in H. Then, under the assumptions above, there exists a sequence (λ_n) of eigenvalues of the operator A such that*

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \quad \textit{with} \quad \lambda_n \to \infty$$

and an orthonormal system of eigenvectors (w_n) with Span$(w_n$: $n \geq 1)$ dense in H.

Proof: The idea is to first prove the operator A is invertible and that the inverse operator A^{-1} is compact. First of all let $f \in H$. We seek a unique $u \in D(A)$ such that $A(u) = f$. Let us define $F \in V'$ by $\langle F, v\rangle_{V'V} := (f, v)_H \quad \forall v \in V$. Then, by the Riesz's representation theorem, it follows that there exists a unique $u \in V$ such that

$$\langle F, v\rangle_{V'V} = (A(u), v) \quad \Leftrightarrow \quad (f, v) = (A(u), v) \quad \forall v \in V\,.$$

Which, since V is assumed dense in H, gives $A(u) = f$. Also, from Cauchy-Schwartz inequality and the coercivity inequality we get

$$\|u\|_H \leq \frac{1}{\alpha}\, \|f\|_H$$

which implies that the operator A is invertible and its inverse A^{-1} is continuous. Hence, every bounded subset B of H is mapped to $A^{-1}(B)$ a bounded subset of V. Now since, V is compactly embedded in H, we have that $A^{-1}(B)$ is a relatively compact subset of H. Therefore, A^{-1} is a compact operator.

1.2.4 Spectral Resolution of Identity

Definition 1.16 (Spectral resolution of identity)
A family $\{P_\lambda : \lambda \in \mathbb{R}\}$ of orthogonal projections on a Hilbert space V is called a spectral resolution of the identity if it possesses the following properties:

(i) $P_\lambda P_\lambda' = P_{min(\lambda,\lambda')}$;

(ii) there exist real numbers α and β such that

$$P_\lambda = 0 \qquad \forall \lambda \in (-\infty, \alpha) \quad \textit{and} \quad P_\lambda = I \quad \forall \lambda \in [\beta, \infty);$$

(iii) for every $v \in V$, $\lim_{\lambda \downarrow \lambda_0} P_\lambda v = P_{\lambda_0} v$. Thus, $\lambda \mapsto P_\lambda$ is an "increasing" projection valued function on the real line.

Definition 1.17 For every λ in $\mathbb{R}$, define the operator P_λ by

$$P_\lambda := \sum_{\zeta \in \sigma(A), \zeta \leq \lambda} E_\zeta,$$

where E_ζ are the spectral projections. The family $\{P_\lambda\}_{\lambda \in \mathbb{R}}$ is called the spectral resolution of the identity associated to the self ad-joint operator A

Proposition 1.18 *Let $\{P_\lambda\}$ be the spectral resolution of the identity associated to the self adjoint operator A, the following hold:*

(i) for every λ in $\mathbb{R}$, the operator P_λ is the orthogonal projection onto

$$\bigoplus_{\zeta \in \sigma(A): \zeta \leq \lambda} N_\zeta^1;$$

(ii) if $\lambda < \zeta_{min}$, then $P_\lambda = 0$, and if $\lambda \geq \zeta_{max}$, then $P_\lambda = I$;

(iii) $P_\lambda P_\mu = P_{min(\lambda,\mu)}$;

(iv) for each λ_0 in $\mathbb{R}$, the map $\lambda \mapsto P_\lambda$ is constant in a right neighbourhood of λ_0. Hence

$$\lim_{\lambda \downarrow \lambda_0} P_\lambda v = P_{\lambda_0} v \quad \forall v \in V;$$

(v) for each pair v and w in V, the function $\lambda \mapsto (P_\lambda v, w)$ is a step function(hence of bounded variation). In particular, if $\zeta_1 < \cdots < \zeta_{N_1}$ are the distinct eigenvalues of A, then

$$(P_\lambda v, w) = \sum_{j=1}^{N_1} (E_{\zeta j} v, w) 1_{[\zeta j, \infty)}(\lambda);$$

(vi) for each v and w in V

$$(v, w) = \int_{-\infty}^{\infty} d(P_\lambda v, w) \quad and \quad \|v\|^2 = \int_{-\infty}^{\infty} d(P_\lambda v, v),$$

where these integrals are (Lebesgue-) Stieltjes integrals with respect to the measures associated to the distribution functions $\lambda \mapsto (P_\lambda v, w)$ and $\lambda \mapsto (P_\lambda v, v)$ respectively.

1.3 The L^p-spaces for real valued functions

We assume that the open subset Ω is equipped with the Lebesgue measure that we denote by dx. For $1 \leq p < \infty$ we define the space of classes of functions $L^p(\Omega)$ by

$$L^p(\Omega) := \left\{ \text{classes of measurable functions } u : \Omega \to \mathbb{R} \text{ such that } \int_\Omega |u(x)|^p dx < \infty \right\}$$

equipped with the norm

$$\|u\|_p := \left(\int_\Omega |u(x)|^p dx \right)^{1/p}.$$

We say that a measurable function $u : \Omega \to \mathbb{R}$ is essentially bounded if there exists $\alpha \geq 0$ such that $|u(x)| \leq \alpha$ for a.e. $x \in \Omega$ and the space $L^\infty(\Omega)$ is defined as

$$L^\infty(\Omega) := \{\text{classes of measurable functions } u : \Omega \to \mathbb{R} \text{ essentially bounded}\}$$

and is equipped with norm

$$\|u\|_\infty := \text{ess-}\sup_{x\in\Omega} |u(x)| := \inf\{\alpha \geq 0\colon\ |u(x)| \leq \alpha \text{ for a.e. } x \in \Omega\}\,.$$

Also, the space $L^p_{\text{loc}}(\Omega)$ is defined by

$$L^p_{\text{loc}}(\Omega) := \big\{\text{classes of } u : \Omega \to \mathbb{R} \text{ measurable such that } f1_K \in L^p(\Omega) \quad \forall K \subset \Omega \text{ compact}\big\},$$

where 1_K denotes the characteristic function of the set K.

We recall that $(L^p(\Omega), \|\cdot\|_p)$ $(1 \leq p \leq \infty)$ is a Banach space and $L^2(\Omega)$ is a Hilbert space equipped with the scalar product

$$(u,v) := \int_\Omega u(x)v(x)\,dx \quad u,v \in L^2(\Omega)\,.$$

Theorem 1.19 (Dunford-Pettis). *Let F be a bounded set in $L^1(\Omega)$. Then F has compact closure in the weak topology $\sigma(L^1, L^\infty)$ if and only if F is equi-integrable, that is*

(a)

$$\begin{cases} \forall \epsilon > 0 \quad \exists \quad \delta > 0 \quad \textit{such} \quad \textit{that} \\ \int_A |f| < \epsilon \quad \forall A \subset \Omega, \quad \textit{measurable} \quad \textit{with} \quad |A| < \delta, \quad \forall f \in F \end{cases}$$

(b)

$$\begin{cases} \forall \epsilon > 0 \quad \exists \quad \omega \subset \Omega \quad \textit{measurable} \quad \textit{with} \quad |\omega| < \infty \quad \textit{such} \quad \textit{that} \\ \int_{\Omega\setminus\omega} |f| < \epsilon \quad \forall f \in F. \end{cases}$$

1.4 Sobolev spaces for real valued functions

Sobolev spaces are natural generalizations of the Lebesgue spaces L^p. Simply stated, if $\Omega \subseteq \mathbb{R}^N$ is an open set, $1 \leq p \leq \infty$, and k is a positive integer, the Sobolev space $W^{k,p}(\Omega)$ consists of functions (or equivalence classes of functions) in $L^p(\Omega)$ whose partial derivatives up to order k are in $L^p(\Omega)$. We will start by giving a more precise definition of what we mean by 'partial derivatives', since these are not defined in the classical sense. Throughout these section, dx refers to Lebesgue measure on $\mathbb{R}^N$ and we assume that all functions are real-valued. We will also identify elements of L^p with functions. Before discussing weak derivative, we introduce some notation. Let $C_c(\Omega)$ denote the class of continuous real-valued functions on Ω with compact support and denote C_c^∞ as the subset of smooth functions. We also, use the definition of $L^p_{loc}(\Omega)$ in section (1.3) and note that $L^p_{loc}(\Omega) \subseteq L^1_{loc}(\Omega)$ whenever $1 \leq p \leq \infty$.
If u and φ are C^1 functions on Ω and φ has compact support, then

$$\int_\Omega u\partial_k\varphi dx = \int_\Omega \partial_k u\varphi dx,$$

where $\partial_k = \partial/\partial_{x_k}$. This can e.g. be proved using Fubini's theorem and the usual integration by parts formula in one dimension. We take this as our definition of weak derivatives.

Definition 1.20 Let $u \in L^1_{loc}(\Omega)$. We say that f is weakly partially differentiable with respect to x_k if there exists a function $v \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u\partial_k\varphi dx = -\int_\Omega v\varphi dx \qquad \textit{for} \quad \textit{every} \quad \varphi \in C_c^\infty.$$

If u is weakly partially differentiable with respect to x_k, we call v the weak partial derivative of u with respect to x_k and denote it by $\partial_k u$. More generally, if α is a multi-index, we say that $v \in L^1_{loc}$ is the α^{th} weak derivative of u if

$$\int_\Omega u D^\alpha \varphi dx = -(1)^{|\alpha|} \int_\Omega v\varphi dx \qquad for \quad every \quad \varphi \in C_c^\infty.$$

Where $D^\alpha \varphi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}} \varphi$. For the above definition to make sense then the weak derivative v must be unique up to a set of measure zero. This is a consequence of the following:

Lemma 1.21 (Uniqueness of weak derivatives). *Let $v, \bar{v} \in L^1_{loc}(\Omega)$ be such that*

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v\varphi dx = (-1)^{|\alpha|} \int_\Omega \bar{v}\varphi dx \quad for \quad every \quad \varphi \in C_c^\infty(\Omega).$$

Then

$$\int_\Omega (v - \bar{v})\varphi dx = 0 \quad for \quad all \quad \phi \in C_c^\infty(\Omega),$$

where $v - \bar{v} = 0$ a.e.

We remark that a weak α^{th}-partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.
Note that, if u is k-times continuously differentiable on Ω, then, for each α with $|\alpha| \leq k$, the classical partial derivative $D^\alpha u$ is also the α^{th} weak derivative of u. For this reason, we use the notation $D^\alpha u$ also for the α^{th} weak derivative of u.
The Sobolev space are indispensable tools in the study of boundary value problems. We are now going to define the Sobolev space.

Definition 1.22 Let k be a non-negative integer, $k \in \mathbb{Z}_+$, and let $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ is defined as the set of all functions $u \in L^1_{loc}(\Omega)$ such that for each multi-index α with $|\alpha| \leq k$, the α^{th}-partial derivative $D^\alpha u$ exists and belongs to $L^p(\Omega)$.

The norm in the space $W^{k,p}(\Omega)$ is defined as

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} (\sum\limits_{|\alpha|\leq k} \int_\Omega \|D^\alpha u\|^p_{L^p(\Omega)} dx)^{1/p} & when \quad 1 \leq p < \infty \\ \sum\limits_{|\alpha|\leq k} ess\sup_\Omega \|D^\alpha u\|_{L^\infty(\Omega)} & when \quad p = \infty. \end{cases}$$

We note that $W^{0,p}(\Omega) = L^p(\Omega)$, and when p=2, we write $W^{k,2}(\Omega) := H^k(\Omega)$. It is not difficult to see that $W^{k,p}(\Omega)$ is a normed space. Moreover, we have the following results, which summarize the basic properties of Sobolev spaces.

Theorem 1.23 *Let Ω be an open bounded set in $\mathbb{R}^N$, $k \in \mathbb{Z}_+$ and $1 \leq p \leq \infty$. Then:*

(i) $W^{k,p}(\Omega)$ is a Banach space;

(ii) $W^{k,p}(\Omega)$ is reflexive if $1 < p < \infty$;

(iii) $W^{k,p}(\Omega)$ is separable if $1 \leq p < \infty$.

Corollary 1.24 *The Sobolev space $H^k(\Omega)$ is a separable Hilbert space with the inner product*

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} (D^\alpha u, D^\alpha v)_2 = \int_\Omega \sum_{|\alpha|\leq k} D^\alpha u, D^\alpha v dx \quad \forall \quad u, v \in H^k(\Omega).$$

The closure of the space $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}}(\Omega)$ gives a closed subspace of $W^{k,p}(\Omega)$, denoted $W_0^{k,p}(\Omega)$. When $p=2$, we use the notation $H_0^k(\Omega) \equiv W_0^{k,2}(\Omega)$. It follows from Theorem 1.23 and Corollary 1.24 that $W_0^{k,p}(\Omega)$ is a Banach space and $H_0^k(\Omega)$ is a Hilbert space. It can be shown that the seminorm $|\cdot|_{W^{k,p}(\Omega)}$ is a norm on $W_0^{k,p}(\Omega)$ and there exists a constant $c>0$ such that

$$|u|_{W^{k,p}(\Omega)} \leq \|u\|_{W^{k,p}(\Omega)} \leq c|u|_{W^{k,p}(\Omega)} \quad \forall u \in W_0^{k,p}(\Omega).$$

Theorem 1.25 *Assume that Ω is bounded and $\partial\Omega$ is C^1. Then there exists a bounded linear operator $T: W^{1,p}(\Omega) \to L^p(\partial\Omega)$, called the trace operator, such that $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and $\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}}$ for each $u \in W^{1,p}(\Omega)$, with the constant C that depends on p and q only.*

We state that Tu is the trace of u on $\partial\Omega$. Now, we need to examine what it means for a function to have zero trace.

Theorem 1.26 *Assume that Ω is bounded and $\partial\Omega$ is C^1, and $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $Tu = 0$ on $\partial\Omega$.*

Theorem 1.27 (Gagliardo-Nirenberg-Sobolev inequality).
Let $1 \leq p < N$. Then there exists a constant C, depending only on p and N, such that

$$\|u\|_{L^{p*}(\Omega)} \leq C\|Du\|_{L^p(\Omega)},$$

for all $u \in C_c^1(\Omega)$ and u have a compact support.

Definition 1.28 If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, we say that X is compactly embedded in Y, denoted $X \subset\subset Y$ provided $\|x\|_Y \leq C\|x\|_X$, $(x \in X)$ for some constant C, and each bounded sequence in X is precompact in Y.

The idea is to show that if $u \in W^{k,p}(\Omega)$, then, in addition to $u \in L^p(\Omega)$, $W^{k,p}(\Omega)$ is also embedded into other L^q-spaces. Let $N \geq 3$, for $1 \leq p \leq N$ and define the Sobolev conjugate of p to be

$$p^* := \frac{Np}{N-p}.$$

Theorem 1.29 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary and $1 \leq p < N$. Then $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and there exists a constant $C>0$, dependent only on p, N and Ω, such that*

$$\|u\|_{p^*} \leq C\|u\|_{W^{1,p}} \quad for \quad every \quad u \in W^{1,p}(\Omega).$$

We will use alternative notation and write

$$\|\nabla u\|_p = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}$$

for $u \in W^{1,p}(\Omega)$. In $W_0^{1,p}(\Omega)$ we have even better results.

Theorem 1.30 (Rellich-Kondrachov Compactness Theorem).
Assume Ω is a bounded open subset of $\mathbb{R}^N$, $\partial\Omega$ is a C^1 and $1 \leq p < N$. Then $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for every $1 \leq q < p^*$.

Remark 1.31 *It follows that $W_0^{1,p}(\Omega)$ is also compactly embedded into $L^q(\Omega)$ for every $1 \leq q < p^*$.*

Remark 1.32 *The embedding is not compact when $q = p^*$ $(for \quad N \geq 3)$.*

Theorem 1.33 (Poincare's Inequality)
Let Ω be a bounded, connected, open subset of $\mathbb{R}^N$ with a boundary $\partial\Omega$ in C^1. Assume $1 \leq p \leq \infty$, then there exist a constant C, depending only on p, q, N and Ω, such that

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$$

for each $u \in W^{1,p}(\Omega)$.

Remark 1.34 The above theorem implies that when Ω is bounded, the norm $\|u\|_{W_0^{1,p}} := \|Du\|_{L^p(\Omega)}$, for each $u \in W_0^{1,p}(\Omega)$.

Theorem 1.35 (Rademacher theorem).
Let Ω be an open subset of $\mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}$ be a Lipschitz map. Then f is differentiable almost everywhere.

1.5 Functions taking values in a Banach space

Bochner spaces

In this section, we deal with functions defined on a bounded interval on $(0, T)$ with $0 < T < \infty$ and taking values in a Banach space $(V, \|\cdot\|_V)$. Let V' be the dual space of V with $\langle \cdot, \cdot \rangle_{V',V}$ denoting the duality brackets.

Definition 1.36

(i) A function $u : (0, T) \to V$ is called *simple* if it takes only a finite number of values $v_i \in V$ and $E_i = u^{-1}(v_i)$ is Lebesgue measurable and in that case, the integral of the simple function u is defined as $\int_0^T u(t)\, dt = \sum_{\text{finite}} \text{meas}(A_i) v_i$.

(ii) We say that $u : (0, T) \to V$ is *strongly measurable* if u is a pointwise limit (in the norm on V) of a sequence (u_k) of simple functions, i.e., $\|u_k(t) - u(t)\|_V \to 0$ for a.e. $t \in (0, T)$.

(iii) We say that $u : (0, T) \to V$ is *weakly measurable* if $t \in (0, T) \to \langle u^*, u(t) \rangle$ is measurable for every $u^* \in V'$.

We recall the following theorem by Pettis [102].

Theorem 1.37 *If the Banach space $(V, \|\cdot\|_V)$ is separable, then u is strongly measurable if and only if it is weakly measurable.*

Definition 1.38 (Bochner integral)
A function $u : (0, T) \to V$ is Bochner integrable if there exists a sequence of simple functions $u_n : (0, T) \to V$ such that the following two conditions are met:

(1) $\lim_{n\to\infty} \|u_n(t) - u(t)\|_V = 0$ for a.e. $t \in I$;

(2) $\lim_{n\to\infty} \int_0^T \|u_n(t) - u(t)\|_V \, dt = 0$.

If u is Bochner integrable, then the limit

$$\int_0^T u(t)\, dt := \lim_{n\to\infty} \int_0^T u_n(t)\, dt$$

called the Bochner integral of u, exists in V and is independent of the sequence (u_n).

Theorem 1.39 *Let $u : (0, T) \to V$ be Bochner integrable function. Then* (*i*) $\left\| \int_0^T u(t)\,dt \right\|_V \leq \int_0^T \|u(t)\|_V \,dt.$

(*ii*) *Let* $(W, \|\cdot\|_W)$ *be another Banach space and* $L : V \to W$ *be a linear continuous mapping, then the function* $Lu : t \in (0,T) \to Lu(t) \in W$ *is also Bochner inegrable and*

$$L\left(\int_0^T u(t)\,dt\right) = \int_0^T L(u(t))\,dt\,.$$

In particular, for every $u^* \in V'$, *the function* $\langle u^*, u(\cdot)\rangle$ *is integrable and*

$$\left\langle u^*, \int_0^T u(t)\,dt \right\rangle = \int_0^T \langle u^*, u(t)\rangle\,dt.$$

The following theorem by Bochner gives a characterization of Bochner integrability of a function $u : (0,T) \to V$ in terms of the summability of the real valued function $t \in [0,T] \to \|u(t)\|_V$.

Theorem 1.40 (Bochner)
A function $u : (0,T) \to V$ *is Bochner integrable if and only if it is strongly measurable and* $\int_0^T \|u(t)\|_V \,dt < \infty$.

Remark 1.41 As in the case of the Lebesgue integral for real valued functions, we also regard here functions that are equal a.e. as equivalent and identified them. They have the same Bochner integral.

The dominated convergence theorem also holds for Bochner inegrals.

Theorem 1.42 *Let* $(f_n) : (0,T) \to V$ *be a sequence of Bochner integrable functions such that* $\lim_{n\to\infty} \|u_n(t) - u(t)\|_V = 0$ *as* $n \to \infty$ *for a.e.* $t \in (0,T)$ *and there exists a summable function* $g : (0,T) \to \mathbb{R}$ *such that*

$$\|u_n(t)\|_V \leq g(t) \quad \textit{for a.e. } t \in (0,T) \textit{ and every } n \in \mathbb{N}.$$

Then $u : (0,T) \to V$ *is Bochner integrabe and*

$$\lim_{n\to\infty} \int_0^T u_n(t)\,dt = \int_0^T u(t)\,dt \quad \textit{and} \quad \lim_{n\to\infty} \int_0^T \|u_n(t) - u(t)\|_V = 0\,.$$

We now introduce the L^p-spaces of V-valued functions that are used for the study of evolution problems.

Definition 1.43 For every $1 \leq p < \infty$, the spaces $L^p(0,T;V)$ consists of all classes of strongly measurable functions $u : [0,T] \to V$ such that $\int_0^T \|u(t)\|_V^p \,dt < \infty$ and is equipped with the norm

$$\|u\|_{L^p(0,T;V)} := \left(\int_0^T \|u(t)\|_V^p \,dt\right)^{1/p} \quad \forall u \in L^p(0,T;V)\,.$$

The space $L^\infty(0,T;V)$ consists of classes of strongly measurable functions $u : [0,T] \to V$ such that $\text{ess-}\sup_{t\in[0,T]} \|u(t)\|_V < \infty$ with norm

$$\|u\|_{L^\infty(0,T;V)} = \text{ess-}\sup_{t\in[0,T]} \|u(t)\|_V \quad \forall u \in L^\infty(0,T;V)\,.$$

We collect in the next theorems some key properties of the L^p-spaces of V-valued functions.

Theorem 1.44

(*i*) **(Banach)** *If* $(V, \|\cdot\|_V)$ *is a Banach space, then the normed space* $(L^p(0,T;V), \|\cdot\|_{L^p(0,T;V)})$ $(1 \leq p \leq \infty)$ *is also a Banach space.*

(*ii*) **(Density)** *The vector subspace of simple functions*

$$\mathcal{S} := \Big\{ u = \sum_{\text{finite}} v_i 1_{E_i} \colon v_i \in V \text{ and } E_i \text{ measurable subsets of } [0,T] \Big\}$$

is dense in $L^p(0,T;V)$*. Moreover, by mollification w.r.t.* t *also the vector subspace*

$$\mathcal{S}_c := \{ u = \sum_{\text{finite}} v_i \varphi_i \colon v_i \in V \text{ and } \varphi_i \in C_c^\infty(0,T) \}$$

is dense in $L^p(0,T;V)$*.*

(*iii*) **(Duality)** *Let* $1 \leq p < \infty$ *and* $(V, \|\cdot\|_V)$ *be a reflexive Banach space with dual space* V'*, then the dual space* $L^p(0,T;V')$ *of* $L^p(0,T;V)$ *is isomorphic to* $L^{p'}(0,T;V')$ *for* $1/p + 1/p'$ *and the duality brackets between* $L^{p'}(0,T;V')$ *and* $L^p(0,T;V)$ *is defined by*

$$\langle f, u \rangle_{L^p, L^{p'}} := \int_0^T \langle f(t), u(t) \rangle \, dt \qquad \forall f \in L^{p'}(0,T;V'), \quad \forall u \in L^p(0,T;V) \, .$$

We are now in a position to introduce Sobolev spaces of V-valued functions.

1.6 Sobolev spaces of V-valued functions

We will first start as in the case of real valued functions, with the concept of weak-differentiability. Since we deal with V-valued functions with V having the strong topology (induced by the norm $\|\cdot\|_V$ and some weak topology), the notion of weak differentiability in this case might be confusing. Therefore, it is more important to be clearer as possible in this section.

Definition 1.45 (Strong continuity - strong differentiability)

(i) We say that $u : (0,T) \to V$ is strongly continuous at $t \in (0,T)$ if $\lim_{s \to t} u(s) = u(t)$ in $(V, \|\cdot\|_V)$.

(ii) We say that $u : (0,T) \to V$ is strongly differentiable at $t \in (0,T)$ if

$$\lim_{s \to t} \frac{u(s) - u(t)}{s - t} \text{ exists in } (V, \|\cdot\|_V)$$

and the vector

$$u'(t) := \lim_{s \to t} \frac{u(s) - u(t)}{s - t}$$

is called the strong derivative of u at $t \in (0,T)$.
Moreover, a function $u : (0,T) \to V$ is continuously differentiable in $(0,T)$ if u is strongly differentiable at every $t \in (0,T)$ and its strong derivative $u' : (0,T) \to V$ is a strongly continuous function.

As in the case of real valued functions, the above-mentioned notion of strong differentiability excludes classes of functions which naturally arise from the study of various physical phenomena. Therefore, the notion of distributional or weak derivative has been introduced in order to accommodate such classes of functions.

Definition 1.46 (Weak differentiability of V-valued functions)
A function $u \in L^1_{\text{loc}}(0,T;V)$ is *weakly differentiable* with *weak derivative* $v = D_t u \in L^1_{\text{loc}}(0,T;V)$ if the following identity holds

$$\int_0^T u \varphi' \, dt = - \int_0^T v \varphi \, dt \qquad \forall \varphi \in C_c^\infty(0,T) \, , \tag{1.2}$$

where the integrals in (1.2) are Bochner integrals.
In general, for every integer $j \geq 1$, if there exists a function $v_j \in L^1_{\mathrm{loc}}(0, T; V)$ such that

$$\int_0^T u\varphi^{(j)}\, dt = (-1)^j \int_0^T v_j \varphi\, dt \qquad \forall \varphi \in C_c^\infty(0, T)\,. \tag{1.3}$$

Then, we say that the function v_j is the weak derivative of the function u of order j and is written $v_j = D_t^{(j)} u$.

Remark 1.47 Quite often, the study of initial boundary value problems presents situations involving two banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ with $J : V \hookrightarrow W$ a continuous embedding such that $u \in L^p(0, T, V)$ and $v = D_t u \in L^q(0, T, W)$. In that case, one needs to define a W-valued weak derivative of a V-valued function. This often done through the identities

$$J\left(\int_0^T u\varphi'\, dt\right) = \int_0^T (Ju)\varphi'\, dt = -\int_0^T \varphi D_t(Ju)\, dt\,.$$

Thus, we can identify u and Ju and use (1.2) to define the W-valued weak derivative of a V-valued function.

Theorem 1.48 (Lebesgue differentiation)
Let $(V, \|\cdot\|_V)$ be a Banach space that can be assumed w.l.g. separable and let $u \in L^1(0, T; V)$. Then

$$u(t) = \frac{1}{h}\int_t^{t+h} u(s)\, ds \quad \text{for a.e. } t \in (0, T)\,.$$

Corollary 1.49 *Let $u \in L^1_{loc}(0, T; V)$ be such that*

$$\int_0^T u\varphi\, dt = 0 \qquad \forall \varphi \in C_c^\infty(0, T)\,,$$

then $u(t) = 0$ for a.e. $t \in (0, T)$.

Corollary 1.50 *If the function $u : (0, T) \to V$ is weakly differentiable and $D_t u = 0$, then f is constant a.e. in $(0, T)$.*

As a consequence of the Lebesgue differentiation theorem (1.48) and the corollaries (1.49) and (1.50) we have the characterization of a weakly differentiable function as the Bochner integral of a Bochner integrable function.

Theorem 1.51 *Let $(V, \|\cdot\|_V)$ be a Banach space and $u \in L^1(0, T; V)$. Then u is weakly differentiable with Bochner integrable derivative $D_t u = v \in L^1(0, T; V)$ if and only if*

$$u(t) = v_0 + \int_0^t v(s)\, ds\,.$$

In that case, the function u is strongly differentiable a.e. and its strong derivative coincides with its weak derivative.

We also have in the following theorem, a characterization of weak derivative of a V-valued function in terms of weak derivatives of the real-valued functions obtained by duality.

Theorem 1.52 *Let $(V, \|\cdot\|_V)$ be a Banach space with dual space V'. The function $u \in L^1(0, T; V)$ is weakly differentiable with weak derivative $D_t u = v \in L^1(0, T; V)$ if and only if for every $u^* \in V'$, the function $\langle u^*, u\rangle : (0, T) \to \mathbb{R}$ is weakly differentiable with weak derivative*

$$\frac{d}{dt}\langle u^*, u\rangle = \langle u^*, v\rangle$$

that is,

$$\int_0^T \langle u^*, u\rangle\, \varphi'\, dt = -\int_0^T \langle u^*, v\rangle\, \varphi\, dt \qquad \forall \varphi \in C_c^\infty(0, T)\,.$$

We are now in a position to define Sobolev spaces of V-valued functions also called *Bochner Sobolev spaces*.

Definition 1.53 Let $(V, \|\cdot\|_V)$ be a Banach space. Let $m \in \mathbb{N}$ and $1 \geq p \leq \infty$. The V-valued Sobolev space $W^{m,p}(0,T;V)$ consists of all (equivalent classes of) strongly measurable functions $u : (0,T) \to V$ whose weak derivatives $D_t^{(j)} u$ of order $0 \leq j \leq m$ belong to $L^p(0,T;V)$. For $1 \leq p < \infty$, the space $W^{m,p}(0,T;V)$ is a Banach space when equipped with the norm

$$\|u\|_{W^{m,p}(0,T;V)} := \left(\sum_{j=1}^{m} \int_0^T \left\| D_t^{(j)} u \right\|_V^p \, dt \right)^{1/p} = \left(\sum_{j=1}^{m} \left\| D_t^{(j)} u \right\|_{L^p(0,T;V)}^p \right)^{1/p} .$$

For $p = \infty$, the space $W^{m,\infty}(0,T;V)$ is a Banach space when equipped with the norm

$$\|u\|_{W^{m,\infty}(0,T;V)} := \sup_{1 \leq j \leq m} \left\| D_t^{(j)} u \right\|_{L^\infty(0,T;T)} .$$

When $p = 2$ and $V = H$ a Hilbert space with scalar product $(\cdot,\cdot)_H$ then $W^{m,2}(0,T;V) = \mathbb{H}^m(0,T;H)$ is a Hilbert space with scalar product defined by

$$(u,v)_{\mathbb{H}^m(0,T;H)} := \sum_{1 \leq j \leq m} \int_0^T (D_t^{(j)} u, D_t^{(j)} v)_H \, dt \qquad \forall u,\, v \in \mathbb{H}^m(0,T;H) .$$

We also have the following embedding theorem known for real valued Sobolev spaces.

Theorem 1.54 *Let $1 \leq p \leq \infty$. Then $W^{1,p}(0,T;V) \hookrightarrow C(0,T;V)$ (the space $W^{1,p}(0,T;V)$ is continuously embedded in $C(0,T;V)$), i.e., There exists a constant $C = C(p,V) > 0$ such that*

$$\|u\|_{L^\infty(0,T;V)} \leq C \, \|u\|_{W^{1,p}(0,T;V)} \qquad \forall u \in W^{1,p}(0,T;V) .$$

1.7 The Aubin-Lion lemma

The Aubin-Lions lemma states criteria under which a set of functions is relatively compact in $L^p(0,T;V)$. Typically, when studying an evolution problem infinite-dimensional space, one introduces a sequence of approximate problems on suitable finite-dimensional subspaces whose solutions are established through the existence theorems (Cauchy-Lipschitz or Carathéodory) for systems of ordinary differential equations (ODEs). One obtains then a sequence of functions which, through *a priori estimates*, is bounded in some $L^p(0,T;X)$ and using the equation in the approximate problem, one also gets that the sequence of time derivatives is bounded in some $L^q(0,T;Y)$. The Aubin-Lions Lemma establishes conditions on X, V, Y and p, q under which the sequence of the sequence of solutions to the approximate problem is relatively compact in $L^p(0,T;V)$.

Theorem 1.55 (Aubin-Lions [11, 82])
Let X, V, Y be banach spaces with the embeddings: $X \hookrightarrow V$ compact and $V \hookrightarrow Y$ continuous. If $1 \leq p < \infty$ and $r = 1$ or $p = \infty$ and $r > 1$, then the set U bounded in $L^p(0,T;X)$ with $\partial U/\partial t := \left\{ D_t u \colon u \in U \right\}$ bounded in $L^q(0,T;Y)$ is relatively compact in $L^p(0,T;V)$.

1.8 The Galerkin approximation method

Several methods for studying nonlinear partial differential equations were developed over the years. We refer the interested reader for instance to the master piece by J.L. Lions [83] published in 1959, in which a large collection of different methods allowing to study various examples coming from applications has been presented. We explain in this section how the Galerkin approximation method works through a simple example.

Let V and H be two Hilbert spaces with $V \subset H$, V dense in H and continuously embedded in H. We will identify H to its dual and have

$$V \subset H \subset V' \quad \text{with continuous embeddings}\,.$$

Let $A \in \mathcal{L}(V, V')$. We consider the following linear problem

$$\begin{cases} \dfrac{du}{dt} + A(u) = f \ \text{ in } \ (0,T), \\ \\ u(0) = u_0. \end{cases} \tag{1.4}$$

For instance, the case $A = -\Delta$ with Dirichlet boundary conditions corresponds to the spaces $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. We assume that $u_0 \in H$ and $f \in L^2(0,T;V')$ and thus the problem (1.4) reads as follows

$$\begin{cases} D_t u + A(u) = f \ \text{ in } \ (0,T), \\ \\ u(0) = u_0. \end{cases} \tag{1.5}$$

where $D_t u$ is the weak derivative of u in the sense of Definition 1.46. We are looking for $u \in L^2(0,T;V) \cap C(0,T;H)$ such that $D_t u \in L^2(0,T;V')$. The following existence theorem is formulated in [111].

Theorem 1.56 *Under the assumptions given above, the problem (1.5) has a unique solution*

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap L^2(0,T;V').$$

Proof: Assuming that V is sparable, let $(w_n)_{n=0}^\infty$ be a basis in V such that $\mathrm{span}(w_n) : n \in \mathbb{N}$ is dense in V. For each $m \geq 1$ integer, define the approximate solution u_m of (1.5):

$$\begin{aligned} & u_m = \sum_{i=1}^m g_{im} w_i, \\ & \frac{d}{dt}(u_m, w_i) + \langle A(u_m), w_i \rangle = \langle f, w_i \rangle, \quad i = 1, \ldots, m, \\ & u_m(0) = u_{0m}, \end{aligned} \tag{1.6}$$

where u_{0m} is the projection in H of u_0 onto the subspace $\mathrm{span}(w_1, \ldots, w_m)$. The functions $\{w_n \colon n \geq 1 \text{ integer}\}$ are often chosen as orthonormal eigenfunctions of the operator A:

$$A(w_j) = \lambda_j\, w_j \quad j \geq 1 \text{ integer}\,.$$

Notice that such a basis in H exists provided that A is self-adjoint i.e., $(A(u), v)_H = (u, A(v))_H$ for every $u, v \in V$ and the embedding $V \hookrightarrow H$ is compact.

In the case where (w_n) is an orthonormal system of eigenfunctions of A, the equation (1.6) then becomes

$$\begin{aligned} & u_m = \sum_{i=1}^m g_{im} w_i, \\ & \frac{d}{dt} g_{im}(t)(w_i, w_i) + \lambda_i g_{im}(w_i, w_i) = \langle f, w_i \rangle, \quad i = 1, \ldots, m, \\ & g_{im}(0) = (u_0, w_i), \end{aligned} \tag{1.7}$$

(1.7) is a system of linear differential equation and hence has a unique global solution and $u \in C(0,T;V)$ and $D_t u_m \in L^2(0,T;V)$. Multiply the equation (1.6) by $g_{im}(t)$ and adding up $i = 1, \ldots, m$ and get

$$(D_t u_m, u_m) + (A(u_m), u_m) = (f, u_m) \quad \Rightarrow \quad \frac{1}{2}\frac{d}{dt}(u_m, u_m)_H + \alpha \left\|u_m\right\|_V^2 \leq \left\|f\right\|_{V'} \left\|u_m\right\|_V.$$

Integrating over $(0,t)$, we get

$$\|u_m\|_H^2 + 2\alpha \int_0^t \|u_m(s)\|_V^2 \leq \|u_m\|_H^2 + \int_0^2 \left[\|u_m(s)\|_V^2 + \|f(s)\|_{V'}^2\right] ds \tag{1.8}$$

or

$$\|u_m\|_H^2 + \alpha \int_0^t \|u_m(s)\|_V^2 \leq \|u_m(0)\|_H^2 + \int_0^2 \|f(s)\|_{V'}^2 \, ds \,. \tag{1.9}$$

Hence, (u_m) is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$. Now, $L^2(0,T;V)$ is a Hilbert space and hence, reflexive while the space $L^\infty(0,T;H)$ is the dual space of $L^1(0,T;H)$ which is separable. Therefore we can extract from (u_m) a subsequence (u_{m_k}) such that

$$\left.\begin{array}{l} u_{m_k} \to u \text{ weakly,} \\ u_{m_k} \to u \text{ weakly}^* \end{array}\right\} \tag{1.10}$$

Now, passing to the limit in (1.6), we obtain

$$\frac{d}{dt}(u, w_i) + (A(u), w_i) = (f, w_i) \quad i = 1, \ldots, m \,.$$

Hence, since (w_n) is a basis in V, we get

$$\frac{d}{dt}(u, v) + (A(u), v) = (f, v) \quad \forall v \in V \,. \tag{1.11}$$

Using Theorem 1.52, we find that $\frac{d}{dt}(u,v) = (D_t u, v)$ and hence,

$$D_t u + A(u) = f \,.$$

Now, since $f \in L^2(0,T;V')$ and $A(u) \in L^2(0,T;V')$ we get that $D_t u \in L^2(0,T;V')$. Since $L^2(0,T;V') \subset L^1(0,T;V')$, it follows from Theorem 1.54 that u coincides a.e. with a continuous form $[0,T]$ to V'. Hence, we will also pass to the limit in the initial condition to get that $u(0) = u_0$.

Remark 1.57 Notice that when the mapping A is not linear, the only estimates on u_m are not sufficient to pass to the limit in nonlinear terms in (1.6). We will need a uniform in m estimate

$$\left\|u_m'\right\|_{L^2(0,T;V')} \leq C \,.$$

In that case, we get

$$\|u_m\|_{L^2(0,T;V)} \leq C \quad \text{and} \quad \left\|u_m'\right\|_{L^2(0,T;V')} \leq C \,.$$

Therefore, based on the Aubin-Lions compactness theorem 1.55 we are able to extract a subsequence which converges strongly and thus pass to the limit in the nonlinear term.

1.9 Derivation of Kirchhoff Equation

We are now in a convenient space to derive the Gustav Kirchhoff equation. This is done to give the reader some basic concept of the Kirchhoff equation.
Let τ_0 be the initial tension of the string at the rest position $[\alpha_0, \beta_0]$; $\hat{\tau}(t)$ be the tension of the string in the position $[\alpha(t), \beta(t)]$ which is the deformation of $[\alpha_0, \beta_0]$ and $\tau(t)$ the tension of the curve deformation $u(x,t)$ of $[\alpha(t), \beta(t)]$. The tension at each point of the curve $u(x,t)$ is a vector that has the direction of the tangent vector of this curve at this point and has modulus $\tau(t)$. Thus its vertical component is

$$\tau(t) \sin\theta,$$

where θ is the angle of the direction $0x$ with the tangent vector. From the hypothesis of small deformations we don't consider the horizontal component of the tension. We have $\sin\theta \approx \tan\theta = \frac{\partial u}{\partial x}$. Thus the vertical component $\overrightarrow{\tau(t)}$ is

$$\tau(t) \sin\theta = \tau(t) \tan\theta = \tau(t) \frac{\partial u}{\partial x}.$$

The variation of the tension generates a force and by Newton's second Law, we obtain

$$\frac{\partial}{\partial x}(\tau(t)\sin\theta) = m\frac{\partial^2 u}{\partial t^2},$$

where $\frac{\partial^2 u}{\partial t^2}$ is the acceleration of the deformation $u(x,t)$ and m is the mass of the string. We have

$$m\frac{\partial^2 u}{\partial t^2} = \tau(t)\frac{\partial^2 u}{\partial x^2}. \tag{1.12}$$

Calculation of the Tension $\tau(t)$.
We calculate the tension $\tau(t)$ in function of $\alpha(t), \beta(t), u(x,t)$. In fact, by Hooke's Law, we have

$$\hat{\tau}(t) - \tau_0 = k\frac{\gamma(t) - \gamma_0}{\gamma_0}, \tag{1.13}$$

where $\gamma(t) = \beta(t) - \alpha(t)$ is the length of $[\alpha(t), \beta(t)]$ and $\gamma_0 = \gamma(0) = \beta_0 - \alpha_0$ the length of $[\alpha_0, \beta_0]$.

If we represent by $S(t)$ the length of the arc of the curve $u(x,t)$, deformation of $[\alpha(t), \beta(t)]$, we obtain:

$$S(t) = \int_{\alpha(t)}^{(\beta(t)} \sqrt{1 + \Big(\frac{\partial u}{\partial x}\Big)^2 dx} \approx \int_{\alpha(t)}^{\beta(t)} \Big(1 + \frac{1}{2}\Big(\frac{\partial u}{\partial x}\Big)^2\Big)dx.$$

By hypothesis of small deformation $|\frac{\partial u}{\partial x}| \ll 1$. Thus

$$S(t) - \gamma(t) = \frac{1}{2}\int_{\alpha(t)}^{(\beta(t)} \Big(\frac{\partial u}{\partial x}\Big)^2 dx.$$

By Hooke's Law we obtain

$$\tau(t) - \hat{\tau}(t) = k\frac{S(t) - \gamma(t)}{\gamma(t)}$$

or

$$\tau(t) - \hat{\tau}(t) = \frac{k}{2\gamma(t)}\int_{\alpha(t)}^{(\beta(t)} \Big(\frac{\partial u}{\partial x}\Big)^2 dx. \tag{1.14}$$

From (1.13) and (1.14) we obtain the tension $\tau(t)$

$$\tau(t) = \tau_0 + k\frac{\gamma(t) - \gamma_0}{\gamma_0} + \frac{1}{2\gamma(t)}\int_{\alpha(t)}^{\beta(t)} \Big(\frac{\partial u}{\partial x}\Big)^2 dx. \tag{1.15}$$

Substituting $\tau(t)$ given by (1.15) in (1.12) and dividing by m, we obtain

$$\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\tau_0}{m} + \frac{k}{m}\frac{\gamma(t) - \gamma_0}{\gamma_0} + \frac{k}{2m\gamma(t)}\int_{\alpha(t)}^{(\beta(t)} \Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0. \tag{1.16}$$

This is a model for small vertical deformations $u(x,t)$ when the ends of the strings are not fixed, that is it has small displacements $\alpha(t) < \alpha_0$ and $\beta(t) > \beta_0$.

Remark 1.58 If we suppose the extremes of the string fixed, that is $\alpha(t) = \alpha_0$, $\beta(t) = \beta_0$ for all $t > 0$.

The model (1.16) reduces to

$$\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\tau_0}{m} + \frac{k}{2m\gamma_0}\int_{\alpha(t)}^{\beta(t)} \Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0. \tag{1.17}$$

This model was proposed by Kirchhoff in 1883.

Chapter 2

A Global Existence Result For The Kirchhoff Type Equation With Linear Damping: Galerkin Approximation Method

In this chapter, for suitable initial conditions (u_0, u_1) and sufficiently large δ, we prove using Galerkin approximation method, local and global existence, uniqueness, regularity and stability of solutions of the Cauchy problem for (2.3).
Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We consider the initial boundary value problem (IBVP)

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} - M\left(\int_\Omega \|\nabla u\|^2 dx\right) \triangle u = f, \\ &u(x,t) = 0 \qquad (x,t) \in \partial\Omega \times [0, \infty), \\ &u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x). \end{aligned}$$

The function M is a C^1 real function satisfying convenient hypothesis such that $M(s) \geq \beta + ks$, where β and k are positive constants and $M'(s) \geq 0$ for every $s \geq 0$.
As mentioned in the introduction, this is the model of the damped small amplitude vibrations of an elastic, stretched string of natural length Ω, whose ends are held a fixed distance apart, that was proposed by Kirchhoff in 1883.

A well known strategy to study initial boundary value problem (IBVP) above is to rewrite it as a Cauchy problem on a suitable Banach space. Precisely, to a possible solution u of IBVP, we associate a mapping that we will still denote by u from $[0, \infty)$ which takes values in some Banach space V of function from Ω to $\mathbb{R}$ as follows:
$u : [0, \infty) \to V$ is such that $t \to u(t) : \Omega \to \mathbb{R}$ defined as

$$u(t)(x) := u(x,t) \quad for \quad every \quad x \in \Omega, \quad \forall t \in [0, \infty).$$

The initial boundary value problem (IBVP) above is then transformed into the Cauchy problem with $f = 0$, as earlier stated in the introduction.

$$\begin{aligned} u''(t) + &\delta u'(t) + M(\|A^{1/2} u(t)\|) A u(t) = 0, \\ u(0) = &u_0, \quad u'(0) = u_1, \end{aligned}$$

where the derivatives $u'(t)$, $u''(t)$ are obtained in a suitable sense and the unbounded linear operator A with domain $D(A) = V$ is obtained from the laplacian operator $-\triangle$.
We are interested in the well posedness of the equation, by well-posed we mean we are interested in the existence, uniqueness, regularity and possible stability of the solution of the equation.

2.1 The Abstract Model

Let H be a real Hilbert space, with scalar product $(\cdot,\cdot)$ and norm $\|.\|$ induced by the scalar product $(\cdot,\cdot)$.
Let A be a linear operator on H with domain $D(A) = V$ dense in H.
We consider on $V = D(A)$, the scalar product

$$(u, v)_A = (u, v) + (Au, Av) \qquad \forall u, v \in V,$$

which induces the norm

$$\|u\|_A^2 = \|u\|^2 + \|Au\|^2$$

called the graph norm of the operator A.
The space $(V, (.,.)_A)$ is clearly a Hilbert space which is continuously embedded in $(H, (.,.))$.
We will further assume that this embedding is compact. We will also suppose that the operator A is self-adjoint and positive. Therefore,

$$(Av, v) \geq c\|v\|_H^2 \qquad \forall v \in V \tag{2.1}$$

for some positive constant c.
Now set $W := D(A^{1/2})$. Since A is self adjoint, it follows

$$(Au, v) = (A^{1/2}u, A^{1/2}v) \qquad u \in V \quad \textit{and} \quad \forall v \in W. \tag{2.2}$$

For the change of the initial boundary value problem(IBVP) into an abstract problem, we take $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and rewrite the (IBVP) into the abstract Cauchy problem in the Hilbert space H as:

$$\begin{aligned} &u''(t) + \delta u'(t) + M(\|A^{1/2}u(t)\|^2)Au(t) = 0, \qquad t \geq 0, \\ &u(0) = u_0, \quad u'(0) = u_1, \end{aligned} \tag{2.3}$$

and take

$$M(s) \geq \beta + ks, \tag{2.4}$$

where β and k are positive constants, and

$$M'(s) \geq 0. \tag{2.5}$$

Let $\overline{M}(\sigma)$ and $E(t)$ be defined as follows:

$$\overline{M}(\sigma) := \int_0^\sigma M(s)ds, \tag{2.6}$$

$$E(t) := \frac{1}{2}[\|u'(t)\|^2 + \overline{M}\|A^{1/2}u(t)\|^2]. \tag{2.7}$$

$$B = \max_{0 \leq s \leq s_0} M'(s) \quad \textit{and} \quad B_1 = \max_{0 \leq s \leq s_0} M(s), \quad \textit{with} \quad s_o = 2E(0)/\beta. \tag{2.8}$$

We are now in a position to formulate the first existence result obtained by **Brito** and **Hale**[36].

Theorem 2.1 *(Existence, Uniqueness and Stability). Assume that the initial data u_0 and u_1 satisfy the conditions:*

$$u_0 \in D(A) = V, \quad u_1 \in D(A^{1/2}) = W\,,$$

with

$$\frac{B}{\sqrt{k}}\left(\frac{\|A^{1/2}u_1\|}{\beta} + \|Au_0\|^2\right)^{\frac{1}{2}} < \delta. \tag{2.9}$$

Then there exists a unique function u such that, for any $T > 0$,
a) $u \in L^\infty(0,T;V)$; $u' \in L^\infty(0,T;W)$; $u'' \in L^\infty(0,T;H)$
b) u is a solution in $L^\infty(0,T;H)$ of the initial value problem (IVP)

$$u''(t) + \delta u'(t) + M(\|A^{1/2}u(t)\|^2)Au(t) = 0, \tag{2.10}$$

and u satisfies the initial conditions

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2.11}$$

c) u satisfies

$$E(t) + \delta \int_0^t \|u'(s)\|^2 ds = E(0), \quad for \quad all \quad t \geq 0. \tag{2.12}$$

of the energy identity
d) $E(t) \to 0$ as $t \to \infty$, hence $\|u(t)\|_W + \|u'(t)\|_H$ goes to zero as $t \to \infty$ (stability).

In the next sections we present the proof of the theorem which is based on the standard Galerkin method to follow with a priori estimates.

2.2 Existence of the solution

We apply the Galerkin method. Assume that, for simplicity, we take $V = D(A)$ to be separable. Let $(w^j)_{j\geq 1}$ be a sequence of eigenfunctions of A that correspond to positive real eigenvalues λ_j that tends to infinity, so that $Aw^j = \lambda_j w^j, \quad j \geq 1$. Let V_m denote the linear hull of $w^1, w^2, \cdots, w^m$. Note that $(w^j)_{j\geq 1}$ is a basis of H, V and W and as a result it is dense in H,V and W.
We find

$$u_m(t) = \sum_{j=1}^{m} g_j^m(t) w^j \tag{2.13}$$

for all v in V_m, $u_m(t)$ satisfies the approximate equation

$$(u_m''(t) + \delta u_m'(t) + M(\|A^{1/2}u_m(t)\|^2)Au_m(t), v) = 0 \quad \forall v \in V_m, \tag{2.14}$$

with the following initial projections

$$u_m(0) = u_{0m} = \sum_{j=1}^{m} (u_0, w^j)w^j \to u_0 \quad in \quad V, \tag{2.15}$$

$$u_m'(0) = u_{1m} = \sum_{j=1}^{m} (u_1, w^j)w^j \to u_1 \quad in \quad W. \tag{2.16}$$

For $v = w^j$, $j = 1, 2, \cdots, m$, the system (2.14) $-$ (2.16) of ordinary differential equations of variable t has a solution $u_m(t)$ in an interval $[0, t_m)$. Now we obtain a priori estimates for the solution $u_m(t)$ and it can be extended to $t \geq 0$.

To show that $u_m(t)$ can be extended for all $t \geq 0$, we have the following a priori estimate:

A priori estimate I
Taking now $v = 2u_m'(t)$ in (2.14), we get

$$(u_m''(t), 2u_m'(t)) + \delta(u_m'(t), 2u_m'(t)) + M(\|A^{1/2}u_m(t)\|^2)(Au_m(t), 2u_m'(t)) = 0.$$

That is,

$$\frac{d}{dt}\|u_m'(t)\|^2 + 2\delta\|u_m'(t)\|^2 + M(\|A^{1/2}u_m(t)\|^2 \frac{d}{dt}\|A^{1/2}u_m(t)\|^2 = 0.$$

Integrate in $[0, t]$, with $t < t_m$, and use (2.6) − (2.7) and (2.15) − (2.16), to get

$$\|u_m'(t)\|^2 + 2\delta \int_0^t \|u_m'(s)\|^2 ds + \overline{M}\|A^{1/2}u_m(t)\|^2 \leq 2E(0).$$

But $\overline{M}(\sigma) \geq \beta\sigma + (\frac{k}{2})\sigma^2$, by (2.4), (2.6). Hence,

$$\|u_m'(t)\|^2 + 2\delta \int_0^t \|u_m'(s)\|^2 ds + \beta\|A^{1/2}u_m(t)\|^2 + \frac{k}{2}[\|A^{1/2}u_m(t)\|^2]^2 \leq 2E(0). \tag{2.17}$$

Because of (2.1) and (2.17), it follows that $u_m(t)$ is bounded, and hence can be extended to $[0, T]$, for any $0 < T < \infty$. Therefore, (2.17) holds, for all $t \geq 0$.

A priori estimate II

Taking now $v = 2Au_m'(t)$ in (2.14), we get

$$(u_m''(t), 2Au_m'(t)) + \delta(u_m'(t), 2Au_m'(t)) + M(\|A^{1/2}u_m(t)\|^2)(Au_m(t), 2Au_m'(t)) = 0.$$

That is,

$$\frac{d}{dt}[\|A^{1/2}u_m'(t)\|^2 + M\|A^{1/2}u_m(t)\|^2\|Au_m(t)\|^2] + 2\delta\|A^{1/2}u_m'(t)\|^2 = \|Au_m(t)\|^2\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2. \tag{2.18}$$

Now, we set

$$z_m(t) := \frac{\|A^{1/2}u_m'(t)\|^2}{M\|A^{1/2}u_m(t)\|^2} + \|Au_m(t)\|^2. \tag{2.19}$$

This implies that

$$\|A^{1/2}u_m'(t)\| + M\|A^{1/2}u_m(t)\|^2\|Au_m(t)\|^2 = M\|A^{1/2}u_m(t)\|^2 z_m(t).$$

The preceding equality (2.18), becomes

$$\frac{d}{dt}[M\|A^{1/2}u_m(t)\|^2 z_m(t)] + 2\delta\|A^{1/2}u_m'(t)\|^2 = \|Au_m(t)\|^2\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2.$$

That is

$$\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2 z_m(t) + M\|A^{1/2}u_m(t)\|^2 z_m'(t) + 2\delta\|A^{1/2}u_m'(t)\|^2 = \|Au_m(t)\|^2\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2.$$

Substituting (2.19) we get

$$\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2\frac{\|A^{1/2}u'(t)\|^2}{M\|A^{1/2}u_m(t)\|^2} + \frac{d}{dt}M\|A^{1/2}u_m(t)\|^2\|Au_m(t)\|^2 + M\|A^{1/2}u_m(t)\|^2 z_m'(t)$$
$$+2\delta\|A^{1/2}u_m'(t)\|^2 = \|Au_m(t)\|^2\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2,$$

which after simplification, becomes

$$\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2\frac{\|A^{1/2}u_m'(t)\|^2}{M\|A^{1/2}u_m(t)\|^2} + M\|A^{1/2}u_m(t)\|^2 z_m'(t) + 2\delta\|A^{1/2}u_m'(t)\|^2 = 0.$$

Dividing through by $M\|A^{1/2}u_m(t)\|^2$, we get

$$\frac{\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2}{M\|A^{1/2}u_m(t)\|^2}\frac{\|A^{1/2}u'(t)\|^2}{M\|A^{1/2}u_m(t)\|^2} + z_m'(t) = \frac{-2\delta\|A^{1/2}u_m'(t)\|^2}{M\|A^{1/2}u_m(t)\|^2},$$

that is,

$$z_m'(t) \leq (-2\delta + \alpha_m(t)) \frac{\|A^{1/2}u'(t)\|^2}{M\|A^{1/2}u_m(t)\|^2}, \tag{2.20}$$

where

$$\alpha_m(t) = \frac{\left|\frac{d}{dt} M\|A^{1/2}u_m(t)\|^2\right|}{M\|A^{1/2}u_m(t)\|^2}. \tag{2.21}$$

As

$$\sqrt{\|A^{1/2}u_m(t)\|^2} \leq \left[\frac{\beta + k\|A^{1/2}u_m(t)\|^2}{k}\right]^{\frac{1}{2}} \leq \left[\frac{M\|A^{1/2}u_m(t)\|^2}{k}\right]^{\frac{1}{2}},$$

$(see (2.4))$, we find using (2.8), (2.17) and (2.19).That, for all $t \geq 0$,

$$\begin{aligned}|M'\|A^{1/2}u_m(t)\|^2 \frac{d}{dt}\|A^{1/2}u_m(t)\|^2| \leq B|\frac{d}{dt}\|A^{1/2}u_m(t)\|^2| &= 2B\|(A^{1/2}u_m(t), A^{1/2}u_m'(t))\| \\ &\leq 2B\|A^{1/2}u_m(t)\|\|A^{1/2}u_m'(t)\|,\end{aligned}$$

we get

$$\left|\frac{d}{dt} M\|A^{1/2}u_m(t)\|^2\right| \leq 2B\left[\frac{M\|A^{1/2}u_m(t)\|^2}{k}\right]^{\frac{1}{2}} \left[M\|A^{1/2}u_m(t)\|^2 z_m(t)\right]^{\frac{1}{2}}.$$

Hence, see (2.21)

$$\alpha_m(t) = \frac{\left|\frac{d}{dt} M\|A^{1/2}u_m(t)\|^2\right|}{M\|A^{1/2}u_m(t)\|^2} \leq \frac{2B\left[\frac{M\|A^{1/2}u_m(t)\|^2}{k}\right]^{\frac{1}{2}} \left[M\|A^{1/2}u_m(t)\|^2 z_m(t)\right]^{\frac{1}{2}}}{M\|A^{1/2}u_m(t)\|^2},$$

which implies that

$$\alpha_m(t) \leq 2B\left[\frac{z_m(t)}{k}\right]^{\frac{1}{2}}. \tag{2.22}$$

Now we claim that

$$-2\delta + \alpha_m(t) < 0 \qquad for \quad all \quad t \geq 0. \tag{2.23}$$

In fact, for $t = 0$, we have, with (2.4), (2.9), (2.15), (2.16), (2.19), that

$$2B\left[\frac{z_m(0)}{k}\right]^{\frac{1}{2}} \leq \frac{2B}{k^{\frac{1}{2}}}\left[\frac{\|A^{1/2}u_1\|^2}{\beta} + \|Au_0\|^2\right]^{\frac{1}{2}} < 2\delta. \tag{2.24}$$

From (2.22) and (2.24) we have

$$\alpha_m(0) \leq 2B\left[\frac{z_m(0)}{k}\right]^{\frac{1}{2}} < 2\delta,$$

which implies that,

$$\alpha_m(0) < 2\delta. \tag{2.25}$$

Suppose, then, that (2.23) does not hold for all $t \geq 0$. Because of (2.25) and the continuity of $\alpha_m(t)$, there is $t^* > 0$ such that

$$\alpha_m(t) < 2\delta \qquad for \quad 0 \leq t \leq t^* \quad and \qquad \alpha_m(t^*) = 2\delta. \tag{2.26}$$

(2.20) and (2.26) give

$$0 > \int_0^{t^*} z'_m(s)ds = z_m(t^*) - z_m(0). \tag{2.27}$$

This inequality, with (2.22) and (2.24), yields

$$\alpha_m(t^*) \leq 2B\left[\frac{z_m(0)}{k}\right]^{\frac{1}{2}} < 2\delta,$$

which contradicts (2.26). Hence (2.23) holds.
Because of (2.20), (2.23), (2.24) we then get

$$2B\left[\frac{z_m(t)}{k}\right]^{\frac{1}{2}} < 2B\left[\frac{z_m(0)}{k}\right]^{\frac{1}{2}} \leq \frac{2B}{k^{\frac{1}{2}}}\left[\frac{\|A^{1/2}u_1\|^2}{\beta} + \|Au_0\|^2\right]^{\frac{1}{2}} < 2\delta,$$

and hence

$$2B\left[\frac{z_m(t)}{k}\right]^{\frac{1}{2}} < 2B\left[\frac{z_m(0)}{k}\right]^{\frac{1}{2}} < 2\delta.$$

Dividing through by $2B$ and squaring, we get

$$\frac{z_m(t)}{k} < \frac{z_m(0)}{k} < \left(\frac{\delta}{B}\right)^2,$$

which, multiplied by k, gives

$$z_m(t) < z_m(0) < k\left(\frac{\delta}{B}\right)^2 \qquad for \quad all \quad t \geq 0. \tag{2.28}$$

On the other hand, it follows from (2.17) that

$$\|A^{1/2}u_m(t)\|^2 \leq \frac{2E(0)}{\beta} = s_0,$$

which from (2.8) gives

$$M\|A^{1/2}u_m(t)\|^2 \leq B_1 \qquad for \quad all \quad t \geq 0. \tag{2.29}$$

Therefore we obtain from (2.19),(2.28), (2.29), we have then, for all $t \geq 0$ that,

$$\|A^{1/2}u'_m(t)\|^2 < B_1 k\left(\frac{\delta}{B}\right)^2 \tag{2.30}$$

and

$$\|Au_m(t)\|^2 < k\left(\frac{\delta}{B}\right)^2. \tag{2.31}$$

A priori estimate III

Taking now $v = u_m''(t)$ in (2.14) we get

$$(u_m''(t), u_m''(t)) + (\delta u_m'(t), u_m''(t)) + M\|A^{1/2}u_m(t)\|^2(Au_m(t), u_m''(t)) = 0.$$

That is,

$$\|u_m''(t)\|^2 + \delta\|u_m'(t)\|^2 + M\|A^{1/2}u_m(t)\|^2(Au_m(t), u_m''(t)) = 0.$$

Hence, using Cauchy-Schwartz inequality (2.17), (2.29), (2.30), we obtain for all $t \geq 0$ that

$$\|u_m''(t)\| \leq 2\delta E(0) + B_1 k\left(\frac{\delta}{B}\right)^2. \tag{2.32}$$

Passing to the limit

From the a priori estimates $(I) - (III)$ we would like extract subsequences which converges in some suitable sense in some Banach spaces.

We can imply from (2.14), (2.30), (2.31), (2.32) that

$$\begin{cases} (u_m) \quad is \quad bounded \quad in \quad L^\infty(0,T;V), \\ (u_m') \quad is \quad bounded \quad in \quad L^\infty(0,T;W), \\ (u_m'') \quad is \quad bounded \quad in \quad L^\infty(0,T;H); \end{cases}$$

hence there is a subsequence that we still denote by u_m such that

$$u_m \to u \quad weak \quad star \quad in \quad L^\infty(0,T;V), \tag{2.33}$$

$$u_m' \to u' \quad weak \quad star \quad in \quad L^\infty(0,T;W), \tag{2.34}$$

$$u_m'' \to u'' \quad weak \quad star \quad in \quad L^\infty(0,T;H). \tag{2.35}$$

Besides, from (2.29), we have

$$M\|A^{1/2}u_m(t)\|^2 Au_m(t) \to \psi \quad weak \quad star \quad in \quad L^\infty(0,T;H). \tag{2.36}$$

As (u_m) is bounded in $L^\infty(0,T;V)$ and, by hypothesis, the injection of V in H is compact, we can further suppose that

$$u_m \to u \quad strong \quad in \quad L^2(0,T;H). \tag{2.37}$$

We claim that,

$$M\|A^{1/2}u_m\|^2 Au_m \to M\|A^{1/2}u\|^2 Au \quad weak \quad star \quad in \quad L^\infty(0,T;H). \tag{2.38}$$

To prove the claim (2.38),

in fact, let ϕ in $L^2(0,T;H)$, we write

$$\begin{aligned}\int_0^T(\psi - M\|A^{1/2}u\|^2)Au,\phi)dt &= \int_0^T(\psi - M\|A^{1/2}u_m\|^2Au_m,\phi)dt \\ &+ \int_0^T M\|A^{1/2}u\|^2(Au_m - Au,\phi)dt + \int_0^T(M\|A^{1/2}u_m\|^2 - M\|A^{1/2}u\|^2)(Au_m,\phi)dt.\end{aligned} \tag{2.39}$$

It is obvious that the first and second terms in the right-hand side of (2.39) tend to zero as $m \to \infty$, by (2.33),(2.36). For the third term we can write with C_1 and C positive constants, by (2.33):

$$\begin{aligned}&\int_0^T(M\|A^{1/2}u_m\|^2 - M\|A^{1/2}u\|^2)(Au_m,\phi)dt \\ &\qquad \le C_1\int_0^T(\|A^{1/2}u_m\|^2 - \|A^{1/2}u\|^2)(Au_m,\phi)dt \\ &\qquad = C_1\int_0^T(A(u_m+u),u_m-u)(Au_m,\phi)dt \le C\Big[\int_0^T\|u_m-u\|^2dt\Big]^{\frac{1}{2}},\end{aligned} \tag{2.40}$$

(2.40), with (2.37), shows that the third term in (2.39) also goes to zero as $m \to \infty$. Hence (2.38) is true.
We now want to prove that u satisfies (2.10). We take $v = w_j$ in (2.14). Then, for fixed j and $m > j$

$$(u_m^{''}(t) + \delta u_m'(t) + M\|A^{1/2}u\|^2Au_m(t), w_j) = 0.$$

Hence because of (2.34), (2.35), (2.38), as $m \to \infty$, we have

$$(u^{''}(t) + \delta u'(t) + M\|A^{1/2}u\|^2Au(t), w_j) = 0 \quad for \quad all \quad j.$$

As (w_j) is a basis of H, this implies (2.10).

2.2.1 Initial conditions of the solution

We now want to prove (2.11) for the theorem 1. It follows from (2.33) and (2.34) that for v in H and θ in $C^{'}([0,T])$ such that $\theta(0) = 1$, $\theta(T) = 0$

$$\int_0^T(u_m(t),v)\theta'(t)dt \to \int_0^T(u(t),v)\theta'(t)dt$$

and

$$\int_0^T(u_m'(t),v)\theta(t)dt \to \int_0^T(u'(t),v)\theta(t)dt.$$

Adding up, we get $(u_m(0),v) \to (u(0),v)$. This and the fact that $u_m(0) \to u_0$ strong in V imply $u(0) = u_0$. As $u'(0) = u_1$ follows similarly from (2.34) and (2.35).
This complete the proof of existence.

2.3 The Energy Equation and Stability

We obtained (2.12) by taking the scalar product of (2.10) with $2u'(t)$.

$$(u^{''}(t), 2u'(t)) + \delta(u'(t), 2u'(t)) + M(\|A^{1/2}u(t)\|^2)(Au(t), 2u'(t)) = 0,$$

that is,

$$\frac{d}{dt}\|u'(t)\|^2 + 2\delta\|u'(t)\|^2 + M\|A^{1/2}u(t)\|^2\frac{d}{dt}\|A^{1/2}u(t)\|^2 = 0.$$

Integrating in $[0,t]$, we have

$$\|u'(t)\|^2 + 2\delta\int_0^t \|u'(s)\|^2 ds + \overline{M}\|A^{1/2}u(t)\|^2 = E(0) \quad for \quad all \quad t \geq 0,$$

it follows that,

$$E(t) + \delta\int_0^t \|u'(s)\|^2 ds = E(0).$$

In order to prove the stability property of Theorem 1, we take the scalar product of (2.10) with $u(t)$ that is,

$$(u''(t), u(t)) + \delta(u'(t), u(t)) + M\|A^{1/2}u(t)\|^2(Au(t), u(t)) = 0,$$

integrating in $[0,t]$, it becomes

$$\int_0^t (u''(s), u(s))ds + \delta\int_0^t (u'(s), u(s))ds + \int_0^t M\|A^{1/2}u(s)\|^2\|A^{1/2}u(s)\|^2 ds = 0,$$

and note that

$$\int_0^t (u'', u)ds = (u', u)\Big|_0^t - \int_0^t \|u'\|^2 ds; \qquad \int_0^t (u', u)ds = \frac{1}{2}\|u\|^2\Big|_0^t.$$

We obtain

$$(u', u)\Big|_0^t - \int_0^t \|u'\|^2 ds + \frac{1}{2}\delta\|u\|^2\Big|_0^t + \int_0^t M\|A^{1/2}u(s)\|^2\|A^{1/2}u(s)\|^2 ds = 0,$$

and so we have

$$\begin{aligned}\int_0^t [M\|A^{1/2}u(s)\|^2\|A^{1/2}u(s)\|^2 ds + \|u'(s)\|^2]ds \leq 2\int_0^t \|u'(s)\|^2 ds + \frac{1}{2}[\|u'(t)\|^2 \\ +(1+\delta)\|u(t)\|^2] + \frac{1}{2}[\|u_1\|^2 + (1+\delta)\|u_0\|^2].\end{aligned} \tag{2.41}$$

From the energy equation (2.7), with (2.4), (2.6), we obtain for all $t \geq 0$

$$\frac{1}{2}[\|u'(t)\|^2 + \overline{M}\|A^{1/2}u(t)\|^2] + \delta\int_0^t \|u'(s)\|^2 ds \leq E(0).$$

That is,

$$\frac{1}{2}\|u'(t)\|^2 + \frac{\beta}{2}\|A^{1/2}u(t)\|^2 + \frac{k}{4}[\|A^{1/2}u(t)\|^2]^2 + \delta\int_0^t \|u'(s)\|^2 ds \leq E(0). \tag{2.42}$$

As $\overline{M}(\sigma) \leq \sigma M(\sigma)$, because M is increasing, it follows from (2.41), (2.42) that for $t > 0$

$$\int_0^t E(s)ds = \int_0^t \frac{1}{2}\Big[\|u'(s)\|^2 + \overline{M}\|A^{1/2}u(s)\|^2\Big]ds < C \tag{2.43}$$

where C is a positive constant.
From (2.43) and $E'(t) = -\delta\|u'(t)\|^2 < 0$, see (2.12), we have that $E(t) \to 0$ as $t \to \infty$. In particular, as

$$E(t) \geq \frac{1}{2}\Big[\|u'(t)\|^2 + \beta\|A^{1/2}u(t)\|^2 + \frac{k}{2}[\|A^{1/2}u(t)\|^2]^2\Big],$$

also $u(t) \to 0$ and $u'(t) \to 0$, in the norms of W and H respectively.

2.4 Uniqueness of solution

Let u and v be two solutions of (2.10),(2.11) in the conditions of Theorem 2.1. That is,

$$(u''(t)-v''(t))+\delta(u'(t)-v'(t))+M(\|A^{1/2}u(t)\|^2)Au(t)-M(\|A^{1/2}v(t)\|^2)Au(t)=0.$$

Then $w=u-v$ is a solution of

$$w''(t)+\delta w'(t)+M\|A^{1/2}u(t)\|^2Aw(t)+\Big(M\|A^{1/2}u(t)\|^2-M\|A^{1/2}v(t)\|^2\Big)Av(t)=0, \tag{2.44}$$

$$w(0)=0; \quad w'(0)=0. \tag{2.45}$$

Taking the scalar product of (2.44) with $2w'(t)$, we have

$$\begin{aligned}(w''(t),2w'(t))+\delta(w'(t),2w'(t))+M\|A^{1/2}u(t)\|^2(Aw(t),2w'(t))\\ +(M\|A^{1/2}u(t)\|^2-M\|A^{1/2}v(t)\|^2)(Av(t),2w'(t))=0,\end{aligned}$$

becomes,

$$\begin{aligned}\frac{d}{dt}[\|w'(t)\|^2+M\|A^{1/2}u(t)\|^2\|A^{1/2}w(t)\|^2]+2\delta\|w'(t)\|^2=\|A^{1/2}w(t)\|^2\frac{d}{dt}M\|A^{1/2}u(t)\|^2\\ +\Big[M\|A^{1/2}v(t)\|^2-M\|A^{1/2}u(t)\|^2\Big](Av(t),2w'(t)).\end{aligned}$$

Integrating with respect to t, using (2.45), hence $\|A^{1/2}w(0)\|^2=0$, and (2.4), we find, compare with (2.40)

$$\|w'(t)\|^2+\beta\|A^{1/2}w(t)\|^2+2\delta\int_0^t\|w'(s)\|^2\le C\int_0^t\Big[\|w'(s)\|^2+\beta\|A^{1/2}w(t)\|^2\Big]ds$$

with C a positive constant. Hence uniqueness follows from Gronwall lemma.

2.5 Regularity of the solution

Theorem 2.2 *Now, we want to show that the hypothesis in Theorem 2.1 holds true with additional assumption that*

$$u_0\in D(A^p), \quad u_1\in D(A^{p-1/2}), \quad for \quad 2p=3,4,5,\cdots,$$

then

$$u\in L^\infty(0,T;D(A^p)), \quad u'\in L^\infty(0,T;D(A^{p-1/2})),$$

$$u''\in L^\infty(0,T;D(A^{p-1})), \quad u'''\in L^\infty(0,T;D(A^{p-3/2})).$$

Proof of Theorem 2.2
We want to prove for $p=k+1/2$, $k=1,2,3,\cdots$. We build additional estimates to the Theorem 2.1 as we now have $u_0\in D(A^{k+1/2})$, $u_1\in D(A^k)$. It is important to note that the proof when $p=k+1$ is similar and shall be omitted.
A priori estimates IV
Taking now $v=2A^{2k}u_m'(t)$ in (2.14), we get

$$(u_m''(t)+\delta u_m'(t)+M(\|A^{1/2}u_m(t)\|^2)Au_m(t),2A^{2k}u_m'(t))=0,$$

which implies,

$$(u''_m(t), 2A^{2k}u'_m(t)) + \delta(u'_m(t), 2A^{2k}u'_m(t)) + M\|A^{1/2}u_m(t)\|^2(Au_m(t), 2A^{2k}u'_m(t)) = 0.$$

It follows that

$$\frac{d}{dt}[\|A^k u'_m(t)\|^2 + M\|A^{1/2}u_m(t)\|^2\|A^{k+1/2}u_m(t)\|^2] + 2\delta\|A^k u'_m(t)\|^2 = \|A^{k+1/2}u_m(t)\|^2\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2. \tag{2.46}$$

Set

$$x_m(t) = \frac{\|A^k u'_m(t)\|^2}{M\|A^{1/2}u_m(t)\|^2} + \|A^{k+1/2}u_m(t)\|^2. \tag{2.47}$$

It implies that

$$M\|A^{1/2}u_m(t)\|^2 x_m(t) = |A^k u'_m(t)|^2 + M\|A^{1/2}u_m(t)\|^2\|A^{k+1/2}u_m(t)\|^2,$$

it then follows from (2.46), that

$$\frac{d}{dt}[M\|A^{1/2}u_m(t)\|^2 x_m(t)] + 2\delta\|A^k u'_m(t)\|^2 = \|A^{k+1/2}u_m(t)\|^2\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2,$$

that is,

$$\begin{aligned}\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2(x_m(t)) + M\|A^{1/2}u_m(t)\|^2 x'_m(t) + 2\delta\|A^k u'_m(t)\|^2 \\ = \|A^{k+1/2}u_m(t)\|^2\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2,\end{aligned}$$

substitute $x_m(t)$ from (2.47) and after simplification, becomes

$$\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2\frac{\|A^k u'_m(t)\|^2}{M\|A^{1/2}u_m(t)\|^2} + M\|A^{1/2}u_m(t)\|^2 x'_m(t) + 2\delta\|A^k u'_m(t)\|^2 = 0,$$

dividing through by $M\|A^{1/2}u_m(t)\|^2$, we have

$$\frac{\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2}{M\|A^{1/2}u_m(t)\|^2}\frac{\|A^k u'_m(t)\|^2}{M\|A^{1/2}u_m(t)\|^2} + x'_m(t) + 2\delta\frac{\|A^k u'_m(t)\|^2}{M\|A^{1/2}u_m(t)\|^2} = 0.$$

Hence,

$$x'_m(t) \leq (-2\delta + \alpha_m(t))\frac{\|A^k u'(t)\|^2}{M\|A^{1/2}u_m(t)\|^2},$$

where

$$\alpha_m(t) = \frac{\left|\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2\right|}{M\|A^{1/2}u_m(t)\|^2}.$$

This together with (2.23) and the assumed regularity of u_0 and u_1 show that

$$x_m(t) < \frac{\|A^k u_1\|^2}{\beta} + \|A^{k+1/2}u_0\|^2. \tag{2.48}$$

The inequality (2.48) together with (2.47) and (2.29) show that there is a constant $C > 0$ independent of m such that

$$\|A^{k+\frac{1}{2}}u_m(t)\|^2 < C \tag{2.49}$$

and

$$\|A^k u'_m(t)\|^2 < C. \tag{2.50}$$

A priori estimate V
Taking now, $v = A^{2(k-1/2)}u''_m(t)$ in (2.14), we get

$$(u''_m(t) + \delta u'_m(t) + M(\|A^{1/2}u_m(t)\|^2)Au_m(t), A^{2(k-1/2)}u''_m(t)) = 0,$$

which implies,

$$(u''_m(t), A^{2(k-1/2)}u''_m(t)) + \delta(u'_m(t), A^{2(k-1/2)}u''_m(t)) + M(\|A^{1/2}u_m(t)\|^2)(Au_m(t), A^{2(k-1/2)}u''_m(t)) = 0,$$

and hence, we get

$$\|A^{k-\frac{1}{2}}u''_m(t)\| \leq \delta\|A^{k-\frac{1}{2}}u'_m(t)\| + M\|A^{1/2}u_m(t)\|^2\|A^{k+1/2}u_m(t)\|.$$

This with (2.29),(2.49),(2.50), use (2.1), proves the existence of a positive constant C independent of m such that for all $t \geq 0$

$$\|A^{k-\frac{1}{2}}u''_m(t)\| < C. \tag{2.51}$$

A priori estimate VI
We have from (2.14) that

$$\Big(u''_m(t) + \delta u'_m(t) + M(\|A^{1/2}u_m(t)\|^2)Au_m(t), v\Big) = 0.$$

By differentiating (2.14) with respect to t, we have

$$\Big(u'''_m(t) + \delta u''_m(t) + M\|A^{1/2}u_m(t)\|^2Au'_m(t) + Au_m(t)\frac{d}{dt}M\|A^{1/2}u_m(t)\|^2, v\Big) = 0.$$

Taking now, $v = A^{2(k-1)}u'''_m(t)$ we get, using (2.8), that

$$\|A^{k-1}u'''_m(t)\| \leq \delta\|A^{k-1}u''_m(t)\| + B_1\|A^ku'_m(t)\| + 2B\|A^ku_m(t)\|\|u'_m(t)\|\|Au_m(t)\|. \tag{2.52}$$

Hence, because of (2.50) and (2.49), (2.51) , using (2.1), there is a positive constant C, independent of m such that for all $t \geq 0$

$$\|A^{k-1}u'''_m(t)\| < C. \tag{2.53}$$

Remark 2.3 We conclude here that the proof of Theorem 2.2 now runs the same way as Theorem 2.1.

2.6 Application

Now we discuss existence for all $t \geq 0$, uniqueness and stability of classical solutions of an initial-boundary value problem for a nonlinear hyperbolic partial differential equation in $\mathbb{R}^n$.
Let Ω an open bounded set in $\mathbb{R}^n$ and $\partial\Omega$ a regular boundary. Take $V = D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, for $A = -\triangle$ (i.e the laplace operator in $\mathbb{R}^n$). This satisfies the assumption on A. It is crucial to note here that the compactness of V in H is Rellich theorem and the equation (2.1) is the Fredrichs-Poincare inequality as defined in the preliminary.
Also, given that $W = D(A^{\frac{1}{2}}) = H_0^1(\Omega)$ and ∇ is the gradient in $\mathbb{R}^n$, we have $\|A^{\frac{1}{2}}w\| = \|\nabla w\|$, for $w \in W$.
Let $M(s) = \beta + ks$, where β, k positive constants. Take $u_0 \in V = H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in W = H_0^1(\Omega)$, where

$$\sqrt{k}\Big(\frac{\|\nabla u_1\|^2}{\beta} + \|\triangle u_0\|^2\Big)^{\frac{1}{2}} < \delta.$$

Theorem 1 establishes existence, for all $t \geq 0$, of a unique solution u of the nonlinear hyperbolic equation.

$$u^{''}(t) + \delta u'(t) - (\beta + k\|\nabla u(t)\|^2)\triangle u(t) = 0,$$

with the initial conditions $u(0) = u_0$, $u'(0) = u_1$ such that, for any $T > 0$, $u \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega))$, $u' \in L^\infty(0, T; H_0^1)$, $u^{''} \in L^\infty(0, T; L^2(\Omega))$. In addition, if

$$E(t) = \frac{1}{2}[\|u'(t)\|^2 + \beta\|Du(t)\|^2 + \frac{k}{2}\|Du(t)\|^4],$$

then u satisfies that $E(t) + 2\delta \int_0^t |u'(s)|^2 ds = E(0)$, of the energy equation, for every $t \geq 0$ and as t goes to infinity, $E(t)$ goes to zero, hence also $\|u(t)\|_W + \|u'(t)\|_H \to 0$.
In particular, Theorem 2.2 gives a more regular initial data and solution to u. If we take p to be sufficiently large in Theorem 2.2 depending on the dimension n of Ω, and by applying the Sobolev Embedding Theorem, we obtain a classical solution of the initial-value problem (IVP) above, that satisfies the boundary condition $u|_{\partial\Omega} = 0$.
Furthermore, in the case when $n = 1$, by the application of both theorem and take $p = 3/2$, we have the

2.6.1 Regularity theorem for the damped elastic stretched string

Given $u_0 \in H_0^1(\Omega) \cap H^3(\Omega)$, $u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$, for

$$\sqrt{k}\left(\frac{\|Du_1\|^2}{\beta} + \|D^2u_0\|^2\right)^{\frac{1}{2}} < \delta$$

there is a unique function u such that

(i) $u \in C^2(\bar{\Omega}) \times C^2([0, T])$ i.e. for fixed t, $\quad u(0, t) \in C^2(\bar{\Omega})$; for x fixed, we have $u(x, 0) \in C^2([0, T])$,

(ii) $u \in C^1(\bar{\Omega})$, $\quad$ for $\bar{\Omega} = \Omega \times (0, T)$,

(iii) u is a solution of (2.10) and satisfies (2.11),

(iv) if
$$E(t) = \frac{1}{2}[\|u'(t)\|^2 + \beta\|Du(t)\|^2 + \frac{k}{2}\|Du(t)\|^4]$$
then u satisfies $E(t) + 2\delta \int_0^t \|u'(s)\|^2 ds = E(0)$ of the energy equation,

(v) $E(t) \to 0$, as $t \to \infty$, hence $\|u(t)\|_W + \|u'\|_H \to 0$ (stability).

Proof
We know from theorem 2 that when $k = 1$ with the hypotheses, $u \in L^\infty(0, T; H^3(\Omega) \cap H_0^1(\Omega))$, $\quad D^2u \in L^\infty(0, T; H_0^1(\Omega))$, $\quad u' \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ $\quad u^{''} \in L^\infty(0, T; H_0^1(\Omega))$, $\quad u^{'''} \in L^\infty(0, T; L^2(\Omega))$. Then $u \in H^3(\Omega)$.
The proof then follows from the Sobolev Embedding in $\mathbb{R}$, i.e., if $f \in H^k(\Omega)$ then $f \in C^{k-1}(\bar{\Omega})$ and of Sobolev Embedding in $\mathbb{R}^2$, i.e., if $f \in H^3(\Omega)$ then $f \in C^1(\bar{\Omega})$.

Chapter 3

A Global Existence Result For The Kirchhoff Type Equation With Linear Damping: The Schauder Fixed Point Method

In this chapter, we prove the local existence using the Shauders fixed point method and show that there exists a unique global solution provided that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy the suitable smallness assumption (cf. Theorem (3.3)) and the non-degeneracy condition $M(\|A^{1/2}u_0\|^2) > 0$. In the general case this solution may not decay to zero as $t \to +\infty$. However we prove that $(u(t), u'(t), u''(t)) \to (u_\infty, 0, 0) \quad in \quad D(A) \times D(A^{1/2}) \times H$ as $t \to +\infty$, and the asymptotic behaviour that $\|A^{1/2}u_\infty\|.M(\|A^{1/2}u_\infty\|^2) = 0$ (cf. Theorem (3.4)).

3.1 The Abstract Model

Let H be a real Hilbert space, with norm $\|.\|$ and scalar product $\langle ., . \rangle$. Let A be a self adjoint linear non-negative operator on H with domain $D(A)$ dense in H. That is $\langle Au, u \rangle \geq 0$ for all $u \in D(A)$.
We consider the Cauchy problem

$$\begin{aligned} &u''(t) + \delta u'(t) + M(\|A^{1/2}u(t)\|^2)Au(t) = 0, \qquad t \geq 0, \\ &u(0) = u_0, \qquad u'(0) = u_1, \end{aligned} \tag{3.1}$$

where $M : [0, +\infty[\to [0, +\infty[$ is a locally Lipschitz continuous function and $\delta > 0$.
We establish in the previous chapter that equation (3.1) is an abstract setting of the initial-boundary value problem (IBVP) for the hyperbolic PDE with a non-local non-linearity of Kirchhoff type

$$u_{tt} + \delta u_t - M\bigg(\int_\Omega \|\nabla u\|^2 dx\bigg)\triangle u = 0, \quad in \qquad \Omega \times [0, +\infty[, \tag{3.2}$$

where $\Omega \subseteq \mathbb{R}^n$ is a (non-necessarily bounded) open set, ∇u is the gradient of u with respect to space variables x, and $\triangle$ is the Laplace operator.

3.2 Local existence of the solution

The following theorems by **Ghisi** and **Gobbino** [54] help us to study the local and global existence results of the Cauchy problem. We begin by stating the local existence theorem.

Theorem 3.1 ***(Local Existence)****. Let $\delta \geq 0$, let $M : [0, +\infty[\to [0, +\infty[$ be a locally Lipschitz continuous function, and let $(u_0, u_1) \in D(A) \times D(A^{1/2})$ with $M(\|A^{1/2}u_0\|^2) > 0$.*

Then there exists $T > 0$ such that problem (3.1) has a unique solution

$$u \in C^2([0,T[;H) \cap C^1([0,T[;D(A^{1/2})) \cap C^0([0,T[;D(A)).$$

Moreover, u can be uniquely continued to a maximal solution defined in an interval $[0,T_*[$, and at least one of the following statements is valid:

(i) $T_* = +\infty$*;*

(ii) $\limsup_{t\to T_*^-} \|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2 = +\infty$*;*

(iii) $\liminf_{t\to T_*^-} M(\|A^{1/2}u(t)\|^2) = 0$*.*

Proof of Theorem (3.1).
Step 1.
The idea is to solve equation (3.1), as we saw in the previous chapter above. The difficulty lies essentially in the presence of the nonlinear term $M(\|A^{1/2}u(t)\|^2)$. Unlike the previous chapter where the Galerkin approximation method was used to obtain the local solution with suitable a priori estimates which led to the global existence result, in this chapter we present another mathematically well sounded approach based on the use of the Schauder fixed point theorem.
The idea here is to start from a suitable linear problem, namely, given a function $a(t)$ in a suitable Banach space X

Let us set
$m_0 := M(\|A^{1/2}u_0\|^2)$, $\quad m_* := \max\left\{1, \frac{2}{m_0}\right\}$, $E_0 := \|u_1\|^2 + m_0\|A^{1/2}u_0\|^2$, $\quad F_0 := \|A^{1/2}u_1\|^2 + m_0\|Au_0\|^2$, $\quad c := \|m'\|_{L^\infty([0,2m_*E_0])}$.

We consider the set of function $X_{L,T}$ defined by

$$X_{L,T} := \{a \in Lip([0,T];\mathbb{R}) : a(0) = m_0, \|a'\|_{L^\infty([0,T])} \leq L\},$$

with

$$L := 2cm_*(E_0 + F_0) \quad \textit{and} \quad T := \frac{m_0.\ln 2}{2L}. \tag{3.3}$$

Notice that since $LT < \frac{m_0}{2}$, it follows that

$$a(t) > \frac{m_0}{2} \qquad \forall a \in X_{L,T}, \qquad \forall t \in [0,T]. \tag{3.4}$$

We now consider the mapping defined by

$$\phi : X_{L,T} \to X_{L,T}$$

and

$$[\phi(a)](t) := M(\|A^{1/2}u(t)\|^2), \quad \forall a \in X_{L,T},$$

where $u \in C^2([0,T];H) \cap C^1([0,T];D(A^{1/2}) \cap C^0([0,T];D(A)))$ is the unique solution of the linear Cauchy problem

$$\begin{aligned} u''(t) + &\delta u'(t) + a(t)Au(t) = 0, \qquad t \geq 0, \\ u(0) = &u_0, \qquad u'(0) = u_1. \end{aligned} \tag{3.5}$$

step 2. We show that the mapping ϕ is well-defined.

$$\phi : X_{L,T} \to X_{L,T}$$

we know that $[\phi(a)](0) = M(\|A^{\frac{1}{2}}u(0)\|^2) = m_0$ and so, it is enough to show that for all $a \in X_{LT}$ we have that

$$|[\phi(a)]'(t)| \leq L \qquad for \quad a.e. \quad t \in [0,T]. \tag{3.6}$$

Since,

$$[\phi(a)]'(t) = 2M'(\|A^{1/2}u(t)\|^2)\langle Au(t).u'(t)\rangle,$$

we will need some uniform bounds on $\|A^{1/2}u(t)\|^2$, $\|u'(t)\|$ and $\|Au(t)\|$. To this aim we introduce the function

$$E(t) := \|u'(t)\|^2 + a(t)\|A^{1/2}u(t)\|^2, \qquad F(t) := \|A^{1/2}u'(t)\|^2 + a(t)\|Au(t)\|^2.$$

We claim that $E(t) \leq E_0 e^{\frac{2LT}{m_0}}$.

In fact, by multiplying (3.1) by $u^{'}(t)$, we have

$$(u^{''}(t), u'(t)) + \delta(u'(t), u'(t)) + a(t)(Au(t), u'(t)) = 0,$$

that is,

$$\frac{d}{dt}\left[\frac{1}{2}\|u'(t)\|^2\right] + \delta\|u'(t)\|^2 + \frac{1}{2}\frac{d}{dt}\left[a(t)\|A^{1/2}u(t)\|^2\right] = \frac{1}{2}a'(t)\|A^{1/2}u(t)\|^2.$$

By simplification, we get

$$\frac{1}{2}\frac{d}{dt}\left[\|u'(t)\|^2 + a(t)\|A^{1/2}u(t)\|^2\right] \leq \frac{L}{m_0}E(t),$$

thus

$$\frac{1}{2}E'(t) \leq \frac{L}{m_0}E(t),$$

which implies that

$$E(t) \leq E_0 e^{\frac{2LT}{m_0}}.$$

Similarly, by multiplying (3.1) by $Au^{'}(t)$, we have ,

$$(u^{''}(t), Au'(t)) + \delta(u'(t), Au'(t)) + a(t)(Au(t), Au'(t)) = 0,$$

that is,

$$\frac{1}{2}\frac{d}{dt}\left[\|A^{1/2}u'(t)\|^2\right] + \delta\|A^{1/2}u'(t)\|^2 + \frac{1}{2}\frac{d}{dt}\left[a(t)\|Au(t)\|^2\right] = \frac{1}{2}a'(t)\|A^{1/2}u(t)\|^2,$$

by simplification, we get

$$\frac{1}{2}\frac{d}{dt}\left[\|A^{1/2}u'(t)\|^2 + a(t)\|Au(t)\|^2\right] \leq \frac{L}{m_0}F(t),$$

thus

$$F'(t) \leq \frac{2L}{m_0}F(t),$$

which implies that

$$F(t) \leq F_0 e^{\frac{2LT}{m_0}}.$$

Hence by (3.4) and (3.3), it follows that

$$\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2 \leq m_* E_0 e^{\frac{2LT}{m_0}} = 2m_* E_0,$$

and

$$\|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2 \leq m_* F_0 e^{\frac{2LT}{m_0}} = 2m_* F_0.$$

Therefore

$$|[\phi(a)]'(t)| = |2M'(\|A^{1/2}u(t)\|^2)\langle Au(t).u'(t)\rangle| \leq c(\|Au(t)\|^2 + \|u'(t)\|^2) \leq 2cm_*(E_0 + F_0) = L$$

and (3.6) is proved.
Step 3.
By the Rademacher theorem on the differentiability of Lipschitz function, it follows that

$$X_{L,T} := \{a \in Lip([0,T];\mathbb{R}) : a(0) = m_0, \quad with \quad |a(s) - a(t)| \leq L|s-t| \quad \forall s,t \in [0,T]\}.$$

In this form it is easier to show that $X_{L,T}$ is a closed subset of $C([0,T];\mathbb{R})$. In fact, let $(a_n) \subset X_{L,T}$ be a sequence which converges uniformly to a, i.e $\lim\limits_{n\to\infty} \|a_n - a\|_\infty = 0$ on one hand and $a(0) = \lim\limits_{n\to\infty} a_n = m_0$ on the other hand, for every $s,t \in [0,T]$, it follows that

$$|a(s) - a(t)| \leq |a(s) - a_n(s)| + |a_n(s) - a_n(t)| + |a_n(t) - a(t)| \leq 2\|a - a_n\|_\infty + L|s-t| \quad \forall n.$$

Taking $n \to \infty$, we get

$$|a(s) - a(t)| \leq L|s-t| \Rightarrow a \in X_{L,T}.$$

This shows that the set $X_{L,T}$ is closed. Also $X_{L,T}$ is trivially convex.

Now let us show that $X_{L,T} = \bar{X}_{L,T}$ is compact in $\Big(C([0,T];\mathbb{R}), \|.\|_\infty\Big)$. This follows easily from the Ascoli-Arzela's theorem. In fact,

(i) $X_{L,T}$ is equicontinuous at every $t \in [0,T]$ being a family of equi-lipschitz functions i.e.,
$|a(s) - a(t)| \leq L|s-t| \qquad \forall s,t \in [0,T]$ and $\quad \forall a \in X_{L,T}$.

(ii) $\forall t \in [0,T]$, $|a(t) - a(0)| \leq LT \Rightarrow |a(t)| \leq |m_0| + LT$,
implies that, $X_{L,T} = \{a(t) : a \in X_{L,T}\} \subset \{-|m_0| + LT, |m_0| + LT\}$ which is compact in $\mathbb{R}$.
Hence $X_{L,T}$ is compact in $\mathbb{R}$.

Now let us show that the mapping $\phi : X_{L,T} \to X_{L,T}$ is continuous.
Let $(a_n) \subset X_{L,T}$ converges uniformly to a, i.e $\lim\limits_{n\to\infty} \|a_n - a\|_\infty = 0$.

We want to show that $\lim\limits_{n\to\infty} \|\phi(a_n) - \phi(a)\|_\infty = 0$.
That is,

$$\|\phi(a_n) - \phi(a)\|_\infty = |M(\|A^{1/2}u_{an}(t)\|^2) - M(\|A^{1/2}u_a(t)\|^2| \leq K|\|A^{1/2}u_{an}(t))\|^2 - \|A^{1/2}u_a(t)\|^2|$$

for some K because the mapping is assumed locally Lipschitz.

We know that u_{a_n} and u_a are solution of the linear equation

$$u''_{a_n}(t) + \delta u'_{a_n}(t) + a_n(t)Au_{a_n}(t) = 0, \tag{3.7}$$

$$u''_a(t) + \delta u'_a(t) + a(t)Au_a(t) = 0, \tag{3.8}$$

with initial conditions

$$
\begin{aligned}
u_{a_n}(0) &= u_0 = u_a(0), \\
u'_{a_n}(0) &= u_1 = u'_a(0).
\end{aligned}
$$

Subtract (3.8) from (3.7) and by scalar multiplication, multiply by $\langle u'_{a_n}(t) - u'_a(t)\rangle$ gives

$$
(u''_{a_n}(t) - u''_a(t), u'_{a_n} - u'_a) + \delta(u'_{a_n}(t) - u'_a(t), u'_{a_n} - u'_a) + \langle a_n(t)Au_{a_n} - a(t)Au_a, u'_{a_n} - u'_a\rangle = 0,
$$

that is,

$$
\begin{aligned}
\frac{d}{dt}\left[\frac{1}{2}\|u'_{a_n} - u'_a\|^2\right] + \delta\|u'_{a_n} - u'_a\|^2 + \langle a_n(t)Au_{a_n}(t) - a_n(t)Au_a(t) \\
+ a_n(t)Au_a(t) - a(t)Au_a, u'_{a_n}(t) - u'_a(t)\rangle = 0,
\end{aligned}
$$

it follows that

$$
\frac{d}{dt}\left[\frac{1}{2}\|u'_{a_n} - u'_a\|^2\right] + \delta\|u'_{a_n} - u'_a\|^2 + D_n(t) = 0, \tag{3.9}
$$

where $D_n(t) = \langle a_n(t)Au_{a_n}(t) - a_n(t)Au_a(t) + a_n(t)Au_a(t) - a(t)Au_a, u'_{a_n}(t) - u'_a(t)\rangle$.
Now, we rewrite $D_n(t)$ as

$$
\begin{aligned}
D_n(t) = \frac{d}{dt}\left[\frac{1}{2}a_n(t)\|A^{1/2}u_{a_n}(t) - A^{1/2}u_a(t)\|^2\right] - a'_n(t)\frac{1}{2}\|A^{1/2}u_{a_n}(t) - A^{1/2}u_a(t)\|^2 \\
+ (a_n(t) + a(t))\langle Au_a(t), u'_{a_n}(t) - u'_a(t)\rangle,
\end{aligned}
$$

putting the value of $D_n(t)$ into (3.9), we get

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|u'_{a_n}(t) - u'_a(t)\|^2 + \frac{1}{2}a_n(t)\|A^{1/2}u_{a_n}(t) - A^{1/2}u_a(t)\|\right] + \delta\|u'_{a_n}(t) - u'_a(t)\|^2 \\
&\qquad = a'_n(t)\frac{1}{2}\|A^{1/2}u_{a_n}(t) - A^{1/2}u_a(t)\|^2 - (a_n(t) - a(t))\langle Au_a(t), u'_{a_n}(t) - u'_a(t)\rangle \\
&\qquad \leq \frac{L}{2}\|A^{1/2}u_{a_n}(t) - A^{1/2}u_a(t)\|^2 + \frac{2L}{c}\|a_n(t) - a(t)\|_\infty.
\end{aligned}
$$

Set
$F_n(t) := [\frac{1}{2}\|u'_{a_n}(t) - u'_a(t)\|^2 + \frac{1}{2}a(t)\|A^{1/2}u_{a_n}(t) - A^{1/2}u_a(t)\|]$, and then rewrite the above as

$$
\begin{aligned}
\frac{d}{dt}F_n(t) + \delta\|u'_{a_n}(t) - u'_a(t)\|^2 &\leq \frac{L}{2}\|A^{1/2}u_{a_n}(t) - A^{1/2}u_a(t)\|^2 + \frac{2L}{c}\|a_n - a\|_\infty \\
&\leq \frac{2L}{m_0}F_n(t) + \frac{2L}{c}\|a_n - a\|_\infty.
\end{aligned}
$$

It follows that

$$
F'_n(t) + \delta\|u'_{a_n}(t) - u'_a(t)\|^2 \leq \frac{2L}{m_0}F_n(t) + \frac{2L}{c}\|a_n - a\|_\infty,
$$

that is,

$$
\frac{d}{dt}F_n(t) \leq \frac{2L}{m_0}F_n(t) + \frac{2L}{c}\|a_n - a\|_\infty,
$$

becomes

$$
\frac{d}{dt}F_n(t) - \frac{2L}{m_0}F_n(t) \leq \frac{2L}{c}\|a_n - a\|_\infty,
$$

from which it follows that

$$F_n(t) \leq \frac{2LT}{c} e^{\frac{-2Lt}{m_0}} \|a_n - a\|_\infty,$$

substitute back the value of $F_n(t)$ we have

$$\frac{1}{2}\|u'_{a_n}(t) - u'_a(t)\|^2 + \frac{1}{2}a(t)\|A^{1/2}u_{a_n}(t) - A^{1/2}u_a(t)\|^2 \leq \frac{2LT}{c} e^{\frac{-2Lt}{m_0}} \|a_n - a\|_\infty,$$

becomes

$$\begin{aligned}\frac{1}{2}\|u'_{a_n}(t) - u'_a(t)\|^2 + \frac{m_0}{2}\|A^{1/2}u_{a_n}(t) - A^{1/2}u_a(t)\|^2 &\leq \frac{2LT}{c} e^{\frac{-2Lt}{m_0}} \|a_n - a\|_\infty \\ &\leq \frac{m_0}{c} e\|a_n - a\|_\infty,\end{aligned}$$

it follows that

$$\frac{1}{2}\|u'_{an}(t) - u'_a(t)\|^2 + \frac{m_0}{2}\|A^{1/2}u_{an}(t) - A^{1/2}u_a(t)\|^2 \leq \frac{m_0}{c} e\|a_n - a\|_\infty,$$

implies that

$$\|A^{1/2}u_{an}(t) - A^{1/2}u_a(t)\|^2 \leq \frac{2e}{c}\|a_n - a\|_\infty. \tag{3.10}$$

Also

$$\begin{aligned}|\phi(a_n(t)) - \phi(a(t))| &= |M(\|A^{1/2}u_{a_n}(t)\|^2) - M(\|A^{1/2}u_a(t)\|^2)| \\ &\leq K|\|A^{1/2}u_a(t)\|^2\| - \|A^{1/2}u_a(t)\|^2| \\ &\leq K\|A^{1/2}u_{a_n}(t) - A^{1/2}u_a(t)\|2\sqrt{\frac{L}{c}},\end{aligned}$$

it follows that

$$|M(\|A^{\frac{1}{2}}u_{an}(t)\|^2) - M(\|A^{\frac{1}{2}}u_a(t)\|^2)| \leq 2K\sqrt{\frac{L}{c}}\|A^{1/2}u_{an}(t) - A^{1/2}u_a(t)\|, \tag{3.11}$$

putting (3.10) and (3.11) together, we obtain that

$$\begin{aligned}|\phi(a_n(t)) - \phi(a(t))| &\leq 2K\sqrt{\frac{L}{c}}\sqrt{\frac{2e}{c}}\|a_n - a\|_\infty^{1/2} \\ &\leq \frac{2\sqrt{2}}{c}\sqrt{Le}\|a_n - a\|_\infty^{1/2}.\end{aligned}$$

This shows that the mapping ϕ is continuous and Hölderian of exponent $\frac{1}{2}$. Since $X_{L,T}$ is compact and convex subset of $C^0([0,T];\mathbb{R})$, by Schauder's fixed point theorem it follows that ϕ has at least one fixed point a. The corresponding solution u of (3.5) is a solution of (3.1) .
It remains now to prove the uniqueness of the solution.
Let $u_1(t)$ and $u_2(t)$ be two solutions of equation (3.1). Then $u(t) = u_2(t) - u_1(t)$ is a solution of

$$u''(t) + \delta u'(t) + M(\|A^{1/2}u_2\|^2)Au(t) + (M(\|A^{1/2}u_2(t)\|^2) - M(\|A^{1/2}u_1(t)\|^2))Au_1(t) = 0, \tag{3.12}$$

$$u(0) = 0, \quad u'(0) = 0. \tag{3.13}$$

Taking the scalar product of (3.12) with $2u'$, we obtain that,

$$\frac{d}{dt}\Big[\|u'(t)\|^2 + M(\|A^{1/2}u_2(t)\|^2)\|A^{1/2}u(t)\|^2\Big] + 2\delta\|u'(t)\|^2 = \|A^{1/2}u(t)\|^2\frac{d}{dt}M(\|A^{1/2}u_2(t)\|^2)$$
$$+\Big[M(\|A^{1/2}u_1(t)\|^2) - M(\|A^{1/2}u_2(t)\|^2)\Big](Au_1(t), 2u'(t)).$$

Integrating over the interval $[0,t]$, we get

$$\int_0^t \frac{d}{dt}\Big[\|u'(t)\|^2 + M(\|A^{1/2}u_2(t)\|^2)\|A^{1/2}u(t)\|^2\Big] + \int_0^t 2\delta\|u'(t)\|^2 = \int_0^t \|A^{1/2}u(t)\|^2\frac{d}{dt}M(\|A^{1/2}u_2(t)\|^2)$$
$$+\int_0^t\Big[M(\|A^{1/2}u_1(t)\|^2) - M(\|A^{1/2}u_2(t)\|^2)\Big](Au_1(t), 2u'(t)).$$

Using (3.13), hence $\|A^{1/2}u(0)\|^2 = 0$, and the fact that M is locally Lipschitz continuous, we find, compare with (3.11), we have,

$$\|u'(t)\|^2 + \beta\|A^{1/2}u(t)\|^2 + 2\delta\int_0^t \|u'(s)\|^2 ds \le K\int_0^t [\|u'(s)\|^2 + \beta\|A^{1/2}u(s)\|^2]ds$$

with K a positive constant. Hence the uniqueness of the solution follows from the Gronwall lemma.
Step 4. We want to prove the last part of the statement in the theorem of the local existence that u can be uniquely extended to a maximal solution defined in an interval $[0, T_*[$, and at least one of the following statement is valid:

(i) $T_* = +\infty$;

(ii) $\limsup_{t\to T_*^-} \|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2 = +\infty$;

(iii) $\liminf_{t\to T_*^-} M\Big(\|A^{1/2}u(t)\|^2\Big) = 0.$

Let $[0, T_*[$ be the maximal interval where the solution exists, let us assume by contradiction that (i),(ii) and (iii) are false.Then, it simply means that:

(i) $T_* < \infty$;

(ii) there exists a constant M such that $\|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2 \le M$ for every $t \in [0, T_*[$;

(iii) there exists a constant ν such that $M(\|A^{\frac{1}{2}}u(t)\|^2) \ge \nu > 0$ in a left neighborhood of T_*.

From the above contradiction, it follows that

$$\int_0^t A^{1/2}u'(\tau)d\tau = A^{1/2}u(\tau)\Big|_0^t = A^{1/2}u(t) - A^{1/2}u(0),$$
$$\int_0^t A^{1/2}u'(\tau)d\tau = A^{1/2}u(t) - A^{1/2}u(0),$$

$$|A^{1/2}u(t)| = \left|A^{1/2}u_0 + \int_0^t A^{1/2}u'(\tau)d\tau\right| \le |A^{1/2}u_0| + M^{1/2}T_*. \tag{3.14}$$

Moreover, since the function

$$H(t) = \|u'(t)\|^2 + \int_0^{\|A^{1/2}u(t)\|^2} M(s)ds$$

is a non-increasing, we have that $\|u'(t)\|^2 \le H(0)$, hence by (3.14) there exist a constant $N > 0$ such that

$$\|u'(t)\|^2 + M(\|A^{1/2}u(t)\|^2)\|A^{1/2}u(t)\|^2 \le N \quad \forall t \in [0, T_*[.$$

By (3.1) it follows that, for all S in a left neighborhood of T_*, the life span of the solution of

$$\begin{aligned} w'' + \delta w' + M(\|A^{1/2}w(t)\|^2)Aw(t) = 0, \qquad t \geq S, \\ w(S) = u(S), \qquad w'(S) = u'(S), \end{aligned}$$

is larger than a strictly positive quantity independent of S. This contradicts the maximality of T_*.
We are now in a position to state and prove the lemma that happens to be very useful while solving the global existence.

Lemma 3.2 *Let $T > 0$, and let $f \in C^1([0, T[; \mathbb{R})$. Let us assume that $f(t) \geq 0$ in $[0, T[$, and that there exists two constants $c_1 > 0$, $c_2 \geq 0$ such that*

$$f'(t) \leq -\sqrt{f(t)}(c_1\sqrt{f(t)} - c_2) \qquad \forall t \in [0, T[.$$

Then

$$\sqrt{f(t)} \leq \max\left\{\sqrt{f(0)}, \frac{c_2}{c_1}\right\}$$

for all $t \in [0, T[$.

Proof of Lemma 3.2. The proof can be put into two cases.
Case 1. Assume that there exist t_1 such that $f(t_1) = 0$ then $f'(t) \leq 0$, which implies that the function is decreasing and so the lemma holds.
Case 2. Assume that for every $t \in [0, T[$, $f(t) > 0$ then

$$f'(t) \leq -\sqrt{f(t)}(c_1\sqrt{f(t)} - c_2) \qquad t \in [0, T[.$$

Dividing through by $2\sqrt{f(t)}$,

$$\frac{f'(t)}{2\sqrt{f(t)}} \leq \frac{-c_1\sqrt{f(t)}}{2} + \frac{c_2}{2},$$

we can rewrite as,

$$(\sqrt{f(t)})' + \frac{c_1\sqrt{f(t)}}{2} \leq \frac{c_2}{2},$$

which implies that

$$\sqrt{f(t)} \leq \frac{c_2}{c_1} + e^{-\frac{c_1}{2}t}\left(\sqrt{f(0)} - \frac{c_2}{c_1}\right),$$

- if $\sqrt{f(0)} \leq \frac{c_2}{c_1}$,

we get,

$$\begin{aligned} f(t) &\leq \frac{c_2}{c_1} \\ &\leq \max\left(\sqrt{f(0)}, \frac{c_2}{c_1}\right), \end{aligned}$$

- if $\sqrt{f(0)} > \frac{c_2}{c_1}$, then

$$\begin{aligned} \sqrt{f(t)} &\leq \frac{c_2}{c_1} + \sqrt{f(0)} - \frac{c_2}{c_1} \\ &\leq \max\left(\sqrt{f(0)}, \frac{c_2}{c_1}\right), \end{aligned}$$

finally, from case 1 and case 2, we have that,

$$\sqrt{f(t)} \leq \max\left(\sqrt{f(0)}, \frac{c_2}{c_1}\right) \qquad for \quad all \quad t \in [0, T[.$$

3.3 Global existence result.

We now want to prove that given the initial data $(u_0, u_1) \in D(A) \times D(A^{\frac{1}{2}})$ which satisfy the non-degeneracy condition $M(\|A^{1/2}u_0\|^2) > 0$, then the Cauchy problem (3.1) has a global solution for every large enough δ. We start by introducing the theorem that we want to prove.

Theorem 3.3 ***(Global existence)****. Let $\delta > 0$, and let $M : [0, +\infty[\to [0, +\infty[$ be a locally Lipschitz continuous function. Let us assume that the initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy the non-degeneracy condition*

$$M(\|A^{1/2}u_0\|^2) > 0, \tag{3.15}$$

and the smallness assumption

$$\|M'\|_{L^\infty([0,E(0)])} \max\left\{\frac{\|u_1\|}{M(\|A^{1/2}u_0\|^2)}, \frac{2}{\delta}\sqrt{F(0)}\right\}\sqrt{F(0)} < \frac{\delta}{4}, \tag{3.16}$$

where $E(0) = \frac{\|u_1\|^2}{M(\|A^{1/2}u_0\|)} + \|A^{1/2}u_0\|^2$, $\qquad F(0) = \frac{\|A^{1/2}u_1\|^2}{M(\|A^{1/2}u_0\|^2)} + \|Au_0\|^2$.

Then problem (3.1) admits a unique global solution

$$u \in C^2([0, \infty[; H) \cap C^1([0, \infty[; D(A^{1/2}) \cap C^0([0, \infty[; D(A))).$$

Proof of Theorem 3.3. The proof of the global existence is divided into four (4) steps as follows:
Step 1. Let us assume that $M \in C^1([0, +\infty]; \mathbb{R})$, and let $[0, T_*[$ be the maximal interval where the solution exists. Let us set

$$c(t) := M(\|A^{1/2}u(t)\|^2),$$

and

$$T := \sup\left\{\tau \in [0, T_*[: \left|\frac{c'(t)}{c(t)}\right| \leq \frac{\delta}{2}, c(t) > 0 \quad \forall t \in [0, \tau]\right\}.$$

Let us consider the functions

$$E(t) := \frac{\|u'(t)\|^2}{c(t)} + \|A^{1/2}u(t)\|^2, \qquad F(t) := \frac{\|A^{1/2}u'(t)\|^2}{c(t)} + \|Au(t)\|^2, \qquad G(t) := \frac{\|u'(t)\|}{c(t)}.$$

With simple computations it follows that

$$\begin{aligned} E'(t) &= \frac{d}{dt}\|u'(t)\|^2 \times \frac{1}{c(t)} - \frac{c'(t)}{(c(t))^2}\|u'(t)\|^2 + \frac{d}{dt}\|A^{1/2}u(t)\|^2 \\ &= \frac{2(u''(t), u'(t))}{c(t)} - \frac{c'(t)}{(c(t))^2}\|u'(t)\|^2 + 2(u'(t), Au(t)). \end{aligned} \tag{3.17}$$

Note that

$$c(t) := M(\|A^{1/2}u(t)\|^2) = a(t).$$

Now, recalling from (3.5) that,

$$u''(t) + \delta u'(t) + c(t)Au(t) = 0.$$

and taking the scalar product with $u'(t)$, we substitute the term $(u''(t), u'(t))$ in (3.17) and get that

$$E'(t) = -\frac{1}{c(t)}\left(2\delta + \frac{c'(t)}{c(t)}\right)\|u'(t)\|^2. \tag{3.18}$$

Similarly, from

$$\begin{aligned} F'(t) &= \frac{d}{dt}\|A^{1/2}u'(t)\|^2 \times \frac{1}{c(t)} - \frac{c'(t)}{(c(t))^2}\|A^{1/2}u'(t)\|^2 + \frac{d}{dt}\|Au(t)\|^2 \\ &= \frac{2(u''(t), Au'(t))}{c(t)} - \frac{c'(t)}{(c(t))^2}\|A^{1/2}u'(t)\|^2 + 2(Au'(t), Au(t)), \end{aligned} \tag{3.19}$$

we use the equation

$$u''(t) + \delta u'(t) + c(t)Au(t) = 0.$$

We take the scalar product with $Au'(t)$ and substitute the term $(u''(t), Au'(t))$ in (3.19) to finally obtain that

$$F'(t) = -\frac{1}{c(t)}\left(2\delta + \frac{c'(t)}{c(t)}\right)\|A^{1/2}u'(t)\|^2. \tag{3.20}$$

Also,

$$G(t) := \frac{\|u'(t)\|}{c(t)} \Longrightarrow (G^2)(t) = \frac{\|u'(t)\|^2}{(c(t))^2},$$

from which we get that

$$\begin{aligned} (G^2)'(t) &= \frac{\frac{d}{dt}\|u'(t)\|^2}{(c(t))^2} - \frac{\|u'(t)\|^2 2c(t)c'}{(c(t))^4} \\ &= \frac{2(u''(t)), u'(t)}{(c(t))^2} - \frac{2\|u'(t)\|^2 c'(t)}{(c(t))^3}. \end{aligned} \tag{3.21}$$

We use the equation

$$u''(t) + \delta u'(t) + c(t)Au(t) = 0,$$

take the scalar product with $u'(t)$ and substitute the term $(u''(t), u'(t))$ in (3.21) to finally obtain that

$$(G^2)'(t) \leq -G(t)\left\{2\left(\delta + \frac{c'(t)}{c(t)}\right)G(t) - 2\|Au(t)\|\right\} \tag{3.22}$$

for every $t \in [0, T[$.

Step 2. We show that $T = T_*$.

Let us assume by contradiction that $T < T_*$. Since $|c'(t)| \leq (\frac{\delta}{2})c(t)$ in $[0, T[$, it implies that

$$\begin{cases} c'(t) \leq \frac{\delta}{2}c(t) \\ -c'(t) \leq \frac{\delta}{2}c(t). \end{cases}$$

The first case gives after integration over $[0, T]$,

$$c(T) \leq c(0)e^{\frac{\delta}{2}T},$$

while the second case gives

$$c(T) \geq c(0)e^{-\frac{\delta}{2}T}.$$

We therefore obtain

$$0 < c(0)e^{\frac{-\delta T}{2}} \leq c(T) \leq c(0)e^{\frac{\delta T}{2}}. \tag{3.23}$$

Since $c(t)$ and $c'(t)$ are continuous functions, by the maximality of T we have that necessarily

$$\liminf_{t \to T_*^-}\left|\frac{c'(t)}{c(t)}\right| \geq \frac{\delta}{2}$$

and

$$\limsup_{t\to T_*^-}\left|\frac{c'(t)}{c(t)}\right|\leq\frac{\delta}{2}\qquad\forall t\in[0,T],$$

implies that,

$$\left|\frac{c'(T)}{c(T)}\right|=\frac{\delta}{2}. \tag{3.24}$$

From (3.18) and (3.20), we have that their derivatives are negatives, it follows that E and F are non-increasing functions, hence

$$\|A^{1/2}u(t)\|^2\leq E(t)\leq E(0), \tag{3.25}$$

$$\|Au(t)\|^2\leq F(t)\leq F(0) \tag{3.26}$$

$\forall t\in[0,T]$.

Moreover by (3.22) and lemma (3.2) we have that

$$(G^2)'(t)\leq -G(t)(\delta G(t)-2\sqrt{F(0)}),$$

hence, by Lemma (3.2) and (3.16) with $f=G^2$,

$$G(t)\leq\max\left\{G(0),\frac{2}{\delta}\sqrt{F(0)}\right\},\qquad\forall t\in[0,T]. \tag{3.27}$$

By (3.25)-(3.27), and the smallness assumption (3.16), we have that

$$\begin{aligned}\left|\frac{c'(T)}{c(T)}\right|=\left|\frac{2M'(\|A^{\frac{1}{2}}u(T)\|^2)\langle u'(T),Au(T)\rangle}{c(T)}\right| &\leq 2\max_{0\leq r\leq E(0)}|M'(r)|\frac{|u'(T)|}{c(T)}|Au(T)|,\\ &\leq 2\max_{0\leq r\leq E(0)}|M'(r)|\max\left\{G(0),\frac{2}{\delta}\sqrt{F(0)}\right\}\sqrt{F(0)}<\frac{\delta}{2}.\end{aligned}$$

This contradicts (3.24).

Step 3. Let us assume by contradiction that $T_*<+\infty$. By (3.23) and (3.26), it follows that

$$\liminf_{t\to T_*^-}M(\|A^{1/2}u(t)\|^2)\geq M(\|A^{1/2}u_0\|^2)e^{\frac{-\delta T_*}{2}}>0,$$

this is true from the non-degeneracy condition of (3.15), also,

$$\limsup_{t\to T_*^-}\|A^{1/2}u'(t)\|^2+\|Au(t)\|^2\leq\max\left\{1,c(0)e^{\frac{\delta T_*}{2}}\right\}F(0)<+\infty.$$

This contradicts the last statement of the theorem (3.3). If M' is continuous, this completes the proof.

Step 4. In step 1-3, we proved that if the function M is of class C^1, then the equation (3.1) has a global solution. Now, we want to show the global existence still holds under the assumption that M is locally Lipschitz.

For this purpose we consider a sequence (M_ϵ) of class C^1 function which converges locally uniformly to M.

$$\sup_{0\leq s\leq k}|M_\epsilon(s)-M(s)|\to 0,$$

that is, for every $k>0$, as $\epsilon\to 0^+$. Precisely, fix $k>0$, to be chosen appropriately later.

Set

$$b(s)=\begin{cases}0 & if\quad s\leq 0\\ M'(s) & if\quad s\in(0,k)\\ 0 & if\quad s\geq k\end{cases}\qquad b\in L^\infty(\mathbb{R})$$

and

$$b_\epsilon(s)=\int_{-\infty}^{\infty}\rho_\epsilon(s-t)b(s)ds$$

where (ρ_ϵ) is a family of mollifiers. Notice that $b_\epsilon\in C^\infty(\mathbb{R})$.
Now set,

$$M_\epsilon(s)=M(0)+\int_0^s b_\epsilon(\tau)d\tau.$$

Then, $(M_\epsilon)\subset C^1(0,\infty)$, M_ϵ converges locally uniformly to M and $\|M_\epsilon\|_{L^\infty(0,k)}\leq\|M\|_{L^\infty(0,k)}$.
Now, we consider this equation

$$v''_\epsilon(t)+\delta v'_\epsilon(t)+M_\epsilon(|A^{1/2}v_\epsilon(t)|^2)Av_\epsilon=0. \tag{3.28}$$

We want to pass to the limit in (3.28) as $\epsilon>0^+$ in order to obtain a global solution of

$$v''(t)+\delta v'(t)+M(|A^{1/2}v(t)|^2)Av(t)=0.$$

This approximating process by know classical requires the establishment of some estimates for the sequence (v_ϵ). For that purpose we choose $k\geq|A^{1/2}v_0|^2$.
Take the scalar product of the equation in (3.28) with $w=2v'_\epsilon(t)$ and set

$$2(v''_\epsilon(t),v'(t))+2\delta|v'_\epsilon(t)|^2+2M_\epsilon(|A^{1/2}v_\epsilon(t)|^2)(Av_\epsilon(t),v'_\epsilon(t))=0,$$

which implies that,

$$\frac{d}{dt}[|v'_\epsilon(t)|^2+\hat{M}_\epsilon(|A^{1/2}v_\epsilon(t)|^2)]+2\delta|v'_\epsilon(t)|^2=0,$$

where

$$\hat{M}_\epsilon(s)=\int_0^s M_\epsilon(\tau)d\tau.$$

Setting

$$\hat{E}_\epsilon(t)=|v'_\epsilon(t)|^2+\hat{M}_\epsilon(|A^{1/2}v_\epsilon(t)|^2),$$

we get,

$$\frac{d}{dt}\hat{E}_\epsilon(t)\leq 0\Rightarrow\hat{E}_\epsilon(t)\leq\hat{E}_\epsilon(0),$$

which implies that,

$$\begin{aligned}\hat{E}_\epsilon(t)=|v'_\epsilon(t)|^2+\hat{M}_\epsilon(|A^{1/2}v_\epsilon|^2)\leq E(0)=|v_1|^2+\int_0^{|A^{1/2}v_0|^2}M_\epsilon(s)ds&\leq|v_1|^2+\int_0^k M_\epsilon(s)ds\\ &\leq|v_1|^2+k\|M_\epsilon\|_{L^\infty(0,k)}\leq C.\end{aligned}$$

From which we get that the sequence $(v'_\epsilon(t))$ is bounded in $C(0,\infty;H)$.
Let $T>0$, we obtain that,

$$\begin{cases}(v'_\epsilon(t)) \quad is \quad bounded \quad in \quad C(0,T;H)\\ v_\epsilon(0)=v_0.\end{cases}$$

Hence, by the Ascoli theorem there exists $v_T\in C(0,T;H)$ such that

$$\|v_\epsilon(t)-v\|_{\infty,T}=\sup_{[0,T]}|v_\epsilon-v_T|\to 0 \quad up \quad to \quad subsequence.$$

Thus,
$$\sup_{t\in[0,T]}|Av_\epsilon(t)-Av_T|=0 \qquad and \qquad \sup_{t\in[0,T]}|A^{1/2}v_\epsilon(t)-A^{1/2}v_T(t)|\to 0,$$
from which it follows that

$$|A^{1/2}v_\epsilon(t)|\to|A^{1/2}v_T| \quad uniformly \quad in \quad [0,T].$$

Therefore there exists K_T such that
$$\max(\|A^{1/2}v_\epsilon\|_\infty^2,\|A^{1/2}v_T\|_\infty^2)\le K_T,$$
and hence,
$$\sup_{[0,T]}|M_\epsilon(|A^{1/2}v_\epsilon(t)|^2)-M(|A^{1/2}v_T(t)|^2)|\le\sup_{s\in[0,K_T]}|M_\epsilon(s)-M(s)|\to 0.$$
On the other hand,

$$\begin{aligned}\|M_\epsilon(|A^{1/2}v_\epsilon(t)|^2)Av_\epsilon(t)-M(|A^{1/2}v_T(t)|^2)Av_T(t)\|&\le\|[M_\epsilon(|A^{1/2}v_\epsilon(t)|^2)-M(|A^{1/2}v_\epsilon(t)|^2)]\|\|Av_\epsilon(t)\|\\&+\|[M(|A^{1/2}v_\epsilon(t)|^2)-M(|A^{1/2}v_T(t)|^2)]Av_\epsilon(t)\|+|M(|A^{1/2}v_T(t)|^2)|\|(Av_\epsilon(t)-Av_T(t))\|.\end{aligned}\tag{3.29}$$

Since M is locally lipschitz and hence locally bounded, we get constant $M_T>0$ such that
$$\begin{aligned}\|M(|A^{1/2}v_\epsilon|^2)-M(|A^{1/2}v_T|^2)\|_{\infty,T}&=\sup_{t\in[0,T]}|M(|A^{1/2}v_\epsilon(t)|^2)-M(|A^{1/2}v_T(t)|^2)|\\&\le M_T\||A^{1/2}v_\epsilon(t)|^2-|A^{1/2}v_\epsilon(t)|^2\|_{\infty,T}\to 0\end{aligned}$$
and
$$\|M(|A^{1/2}v_T|^2)\|_{\infty,T}\le M_T.$$
Therefore, we obtain from (3.29) that
$$\begin{aligned}\|M_\epsilon(|A^{1/2}v_\epsilon)|^2-M(|A^{1/2}v_T|^2)\|\|Av_T\|_{\infty,T}&\le k_T^{1/2}\|M_\epsilon(|A^{1/2}v_\epsilon|^2)-M(|A^{1/2}v_T|^2)\|_{\infty,T}\\&+k^{1/2}\|M(|A^{1/2}v_\epsilon|^2)-M(|A^{1/2}v_T|)^2\|_{\infty,T}+M_T\|Av_\epsilon-Av_T\|_{\infty,T}.\end{aligned}$$
Note that the second, third and fourth terms of the right side of the inequality above tends to zero, as $\epsilon\to 0^+$.
Now going back to the equation (3.28), that we write in the form

$$(v'_\epsilon(t)+\delta v_\epsilon(t))'=-M_\epsilon(|A^{1/2}v_\epsilon|^2)Av_\epsilon(t),$$
we obtain that the sequence $(v'_\epsilon(t)+\delta v_\epsilon(t))'$ is bounded in $C(0,T;H)$ and for every $\epsilon>0$
$$v'_\epsilon(0)+\delta v_\epsilon(0)=v_1+\delta v_0,$$
thus, apply the Ascoli theorem to $(v'_\epsilon(t)+\delta v_\epsilon(t))$, we get that $(v'_\epsilon(t)+\delta v_\epsilon(t))$ converges to some function w_T in $C(0,T;H)$, up to a subsequence.
Since v_ϵ converges already to v_T in $C(0,T;H)$, it follows that $v'_\epsilon=(v'_\epsilon+\delta v_\epsilon)-\delta v_\epsilon$ converges to some function u_T in $C(0,T;H)$.
Therefore, we obtain that
$$\begin{cases}(v_\epsilon)\to v_T \quad in \quad C(0,T;H)\\(v'_\epsilon)'\to u_T \quad in \quad C(0,T;H).\end{cases}$$

Hence, v_T is continuously differentiable in $[0, T]$ and $v'_T(t) = u_T$.
Now, once again we go back to the equation (3.28), that we rewrite this time in the form

$$v''_\epsilon(t) = -\delta v'_\epsilon(t) + M_\epsilon(|A^{1/2}v_\epsilon|^2)Av_\epsilon.$$

Hence, it follows that (v''_ϵ) converges to some function h in $C(0, T; H)$.
We obtain therefore that

$$\begin{cases} (v'_\epsilon(t)) \to v'_T \quad in \quad C(0, T; H) \\ (v''_\epsilon(t)) \to h \quad in \quad C(0, T; H). \end{cases}$$

Hence, v'_T is continuously differentiable in $[0, T]$ and $(v'_T)' = v''_T = h$ in $[0, T]$.
We are now in a position to pass the limit in the equation (3.28) and get that v_T is the solution of

$$\begin{cases} v''_T(t) + \delta v'_T + M(|A^{1/2}v_T|^2)Av_T = 0, \\ v'_T(0) = v_1, \quad v_T(0) = v_0. \end{cases}$$

$$Set \quad v(t) = v_T(t) \quad if \quad t \in [0, T], \quad for \quad some \quad T > 0. \tag{3.30}$$

Notice that the function v is well defined . In fact, let $T_1, T_2 > 0$ with $T_1 \leq T_2$, therefore by the uniqueness of the local solution, we get

$$v_T = v_{T_2}|_{[0,T_1]}.$$

In fact, by definition v_{T_i} is the unique solution of

$$\begin{aligned} &v''(t) + \delta v'(t) + M(|A^{1/2}v(t)|^2)Av(t) = 0 \qquad \forall t \in (0, T_i), \quad i = 1, 2, \cdots \\ &v(0) = v_0, \quad v'(0) = v_1. \end{aligned}$$

In particular, v_{T_2} satisfies the equation also for every $t \in [0, T_1]$, which implies that $v_{T_2}|_{[0,T_1]}$ is also a solution to the equation in $[0, T_1]$. From the uniqueness of v_{T_1} as solution in $[0, T_1]$, we get that

$$v_{T_1} = v_{T_2}|_{[0,T_1]}.$$

This complete the proof of the global existence.
We are now going to study the asymptotic behaviour of the global solution obtained above.

3.4 The Asymptotic Behaviour of the Global solution

We now want to prove that $(u(t), u'(t), u''(t)) \to (u_\infty, 0, 0)$ in $D(A) \times D(A^{1/2}) \times H$ as $t \to +\infty$, and that $\|A^{1/2}u_\infty\| \times M(\|A^{1/2}u_\infty\|^2) = 0$. In particular, if the operator A is coercive and $m(r) > 0$ for every $r > 0$, then necessarily $u_\infty = 0$. We start by introducing the theorem that we want to prove.

Theorem 3.4 (Asymptotic behaviour). *Let us assume that all the hypotheses of Theorem 3.3 are satisfied.*
Then we have that:
(i) $M(\|A^{1/2}u(t)\|^2) > 0$ for all $t \geq 0$;
(ii) there exists $u_\infty \in D(A)$ such that

$$u(t) \to u_\infty \quad in \quad D(A), \tag{3.31}$$

$$u'(t) \to 0 \quad in \quad D(A^{1/2}), \tag{3.32}$$

$$u''(t) \to 0 \quad in \quad H \tag{3.33}$$

as $t \to +\infty$. Moreover $\|A^{1/2}u_\infty\| \times M(\|A^{1/2}u_\infty\|^2) = 0$.

Remark 3.5 We remark here that the proof of Theorem 3.4 relies on a result about the asymptotic behaviour of solutions of the linearization of (3.1).

Let v be the global solution obtained in the theorem 3.3. In order to study the asymptotic behaviour of the solutions of (3.1), we consider the linearized problem

$$\begin{aligned} v''(t)+&\delta v'(t) + c(t)Av(t) = 0, \qquad t \geq 0, \\ v(0) =&v_0, \qquad v'(0) = v_1. \end{aligned} \tag{3.34}$$

In the following lemma we examine the asymptotic behaviour of the solutions of (3.34).

Lemma 3.6 *Let $\delta > 0$, and let $c : [0, +\infty[\to]0, +\infty[$ be a Lipschitz continuous bounded function such that*

$$\left|\frac{c'(t)}{c(t)}\right| \leq \frac{\delta}{2} \qquad for \quad a.e. \quad t \geq 0. \tag{3.35}$$

Let v be the unique global solution of (3.34) with $(v_0, v_1) \in D(A) \times D(A^{1/2})$.
Then there exists $v_\infty \in D(A)$ such that

$$v(t) \to v_\infty \qquad in \quad D(A), \tag{3.36}$$

$$v'(t) \to 0 \qquad in \quad D(A^{1/2}), \tag{3.37}$$

$$v''(t) \to 0 \qquad in \quad H \tag{3.38}$$

as $t \to +\infty$. Furthermore, if $Av_\infty \neq 0$ then necessarily $c(t) \to 0$ as $t \to +\infty$.

Proof of Lemma 3.6. We divide the proof in several steps, according to the following strategy. In Steps $1 - 6$ we prove that

$$Av(t) \quad has \quad a \quad limit \quad w_\infty \quad in \quad H \quad as \quad t \to +\infty, \tag{3.39}$$

$$A^{1/2}v'(t) \to 0 \quad in \quad H \quad as \quad t \to +\infty, \tag{3.40}$$

$$c(t)\|Av(t)\| \to 0 \quad as \quad t \to +\infty. \tag{3.41}$$

If the operator A is coercive (i.e there exists $\nu > 0$ such that $\langle Au, u\rangle \geq \nu\|u\|^2$ for all $u \in D(A)$), then Lemma 3.6 is proved.
If A is not coercive, it remains to prove (Step 7) that

$$v(t) \quad has \quad a \quad limit \quad v_\infty \quad in \quad H \quad as \quad t \to +\infty, \tag{3.42}$$

$$v'(t) \to 0 \quad in \quad H \quad as \quad t \to +\infty. \tag{3.43}$$

Indeed, since A is closed, from (3.39) and (3.42) it follows that $v_\infty \in D(A)$, $w_\infty = Av_\infty$, and $v(t) \to v_\infty$ in $D(A)$.
Step 1. Let us consider the function

$$F(t) := \frac{\|A^{1/2}v'\|^2}{c(t)} + \|Av(t)\|^2.$$

A simple computation shows that

$$F'(t) = \frac{2(v''(t), Av'(t))}{c(t)} - \frac{c'(t)}{(c(t))^2}\|A^{1/2}v'(t)\|^2 + 2(Av'(t), Av(t)). \tag{3.44}$$

Now, recalling from (3.34) that $v''(t) + \delta v'(t) + c(t)Av(t) = 0$ and taking the scalar product with $Av'(t)$, we substitute the term $(v''(t), Av'(t))$ in (3.44) and get that

$$F'(t) = -\left(2\delta + \frac{c'(t)}{c(t)}\right)\frac{\|A^{1/2}v'(t)\|}{c(t)},$$

hence

$$F'(t) = -\left(2\delta + \frac{c'(t)}{c(t)}\right)\frac{\|A^{1/2}v'(t)\|}{c(t)} \leq -\frac{3}{2}\delta\frac{\|A^{1/2}v'(t)\|^2}{c(t)}. \tag{3.45}$$

In particular, the function $F(t)$ is non-increasing, hence there exists

$$F_\infty := \lim_{t\to+\infty} F(t).$$

If $F_\infty = 0$, then (3.39) holds true with $w_\infty = 0$, while (3.40) and (3.41) follow from the boundedness of c.

Now in steps $2-6$ we deal with the case $F_\infty > 0$ (in particular in step 4 we show that in this case necessarily $c(t) \to 0$).

Step 2. We show that

$$\int_0^\infty \|A^{1/2}v'(t)\|^2 dt < +\infty. \tag{3.46}$$

Indeed, integrating (3.45) we have that

$$\int_0^\infty \frac{\|A^{1/2}v'(t)\|^2}{c(t)}dt \leq -\frac{2}{3\delta}\int_0^\infty F'(t)dt = \frac{2}{3\delta}(F(0) - F_\infty).$$

Since the function c is bounded, (3.46) follows from this inequality.
Step 3. We show that

$$\int_0^\infty c(t)\|Av(t)\|^2 dt < +\infty. \tag{3.47}$$

Indeed, taking the scalar product of the equation (3.34) with Av, we obtain that

$$\begin{aligned}
0 &= \langle v''(t), Av(t)\rangle + \delta\langle v'(t), Av(t)\rangle + c(t)\|Av(t)\|^2 \\
&= (\langle v'(t), Av(t)\rangle + \frac{\delta}{2}\|A^{1/2}v(t)\|^2)' - \|A^{1/2}v'(t)\|^2 + c(t)\|Av(t)\|^2 \\
c(t)\|Av(t)\|^2 &= -(\langle v'(t), Av(t)\rangle + \frac{\delta}{2}\|A^{1/2}v(t)\|^2)' + \|A^{1/2}v'(t)\|^2.
\end{aligned}$$

Integrating in $[0, T]$ it follows that

$$\begin{aligned}
\int_0^T c(t)\|Av(t)\|^2 dt &= \langle v_1, Av_0\rangle + \frac{\delta}{2}\|A^{1/2}v_0\|^2 - \langle v'(T), Av(T)\rangle - \frac{\delta}{2}\|A^{1/2}v(T)\|^2 + \int_0^T \|A^{1/2}v'(t)\|^2 dt \\
&\leq \langle v_1, Av_0\rangle + \frac{\delta}{2}\|A^{1/2}v_0\|^2 + \frac{1}{2\delta}\|c\|_\infty F(0) + \int_0^T \|A^{1/2}v'(t)\|^2 dt.
\end{aligned}$$

Passing to the limit as $T \to \infty$, (3.47) follows from (3.46).
Step 4. We know from the definition of $F(t)$ that,

$$F(t) = \frac{\|A^{1/2}v'(t)\|^2}{c(t)} + \|Av(t)\|^2,$$

multiply through by $c(t)$

$$c(t)F(t) = \|A^{1/2}v'(t)\|^2 + c(t)\|Av(t)\|^2,$$

integrating in $[0, \infty]$

$$\int_0^\infty c(t)F(t)dt = \int_0^\infty \|A^{1/2}v'(t)\|^2 + \int_0^\infty c(t)\|Av(t)\|^2.$$

From (3.46),(3.47) and it follows that

$$\int_0^{\infty} c(t)F(t)dt < +\infty.$$

Since $F(t) \geq F_\infty > 0$, then also

$$\int_0^{\infty} c(t)dt < +\infty. \tag{3.48}$$

Since c is Lipschitz continuous, it follows that $c(t) \to 0$ as $t \to +\infty$, hence (3.41) is proved. Since $\|A^{1/2}v'(t)\|^2 \leq c(t)F(0)$, then also (3.40) is proved.
Step 5. We show that (3.39) holds true with the additional assumption that $(v_0, v_1) \in D(A^2)\times D(A^{3/2})$. To this end, let us introduce the function

$$\hat{F}(t) := \frac{\|A^{3/2}v'(t)\|^2}{c(t)} + \|A^2 v(t)\|^2.$$

As in step 1, it is possible to prove that $\hat{F}$ is non-increasing,

$$\hat{F}'(t) = \frac{2(A^3 v'(t), v''(t))}{c(t)} - \frac{c'(t)}{(c(t))^2}\|A^{3/2}v'(t)\|^2 + 2(Av(t), A^3 v'(t)), \tag{3.49}$$

recall that,

$$v''(t) + \delta v'(t) + c(t)Av(t) = 0,$$

we multiply through by $A^3 v'(t)$ and write the resulting equation in the form

$$(A^3 v'(t), v''(t)) = -\delta\|A^{3/2}v'(t)\|^2 - c(t)(Av(t), A^3 v'(t)) = 0. \tag{3.50}$$

Now, substituting $(A^3 v'(t), v''(t))$ into (3.49), we get after simplification that

$$\hat{F}'(t) = \frac{-2\delta\|A^{3/2}v'(t)\|^2}{c(t)} - \frac{c'(t)}{(c(t))^2}\|A^{3/2}v'(t)\|^2,$$

hence which is written in the form

$$\hat{F}'(t) = -\frac{1}{c(t)}\left(2\delta + \frac{c'(t)}{c(t)}\right)\|A^{3/2}v'(t)\|^2 \quad \forall t \geq 0.$$

From the definition of $\hat{F}(t)$ and the fact $\hat{F}'(t) < 0$ (see(3.45)) it follows that

$$\|A^2 v(t)\|^2 \leq \hat{F}(t) \leq \hat{F}(0) \quad \forall t \geq 0.$$

Now let us consider the function

$$\hat{G}(t) := \frac{\|Av'(t)\|}{c(t)}.$$

Then, we have that

$$(\hat{G})^2(t) = \frac{\|Av'(t)\|^2}{(c(t))^2},$$

and hence,

$$(\hat{G}^2)'(t) = \frac{2(v''(t), A^2 v'(t))}{(c(t))^2} - \frac{\|Av'(t)\|^2 2c'(t)}{(c(t))^3}, \tag{3.51}$$

recall that,

$$v''(t) + \delta v'(t) + c(t)Av(t) = 0,$$

we multiply through by $A^2v'(t)$ and write the resulting equation in the form

$$(v''(t), A^2v'(t)) = -\delta\|Av'(t)\|^2 - c(t)(Av(t), A^2v'(t)). \tag{3.52}$$

Now, substituting $(v''(t), A^2v'(t))$ into (3.51), we get after simplification

$$(\hat{G}^2)'(t) \leq \frac{-2\delta\|Av'(t)\|^2}{(c(t))^2} - \frac{-2\|Av'(t)\|\|A^2v(t)\|}{c(t)} - \frac{\|Av'(t)\|^2 2c'(t)}{(c(t))^3},$$

thus

$$(\hat{G}^2)(t) \leq -\hat{G}(t)\left\{2\left(\delta + \frac{c'(t)}{c(t)}\right)\hat{G}(t) - 2\|A^2v(t)\|\right\} \leq -\hat{G}(t)\left\{\delta\hat{G}(t) - 2\sqrt{\hat{F}(0)}\right\},$$

hence, by Lemma 3.1 with $f = \hat{G}^2$, it follows that

$$\hat{G}(t) \leq \max\left\{\hat{G}(0), \frac{2}{\delta}\sqrt{\hat{F}(0)}\right\} \quad \forall t \in [0, T[.$$

By (3.48), this implies that

$$\int_0^\infty \|Av'(t)\|dt < +\infty$$

and

$$|Av(t) - Av(s)| \leq \int_s^t |Av'(\tau)|d\tau = \left[\int_0^t |Av'(\tau)|d\tau - \int_0^s |Av'(\tau)|d\tau\right]$$

as s and t goes to ∞. Therefore $Av(t)$ has a limit as $t \to +\infty$.

Step 6. We show that (3.39) hold true for every initial data $(v_0, v_1) \in D(A) \times D(A^{1/2})$.

To this end, let us consider a sequence $\{(v_{0n}, v_{1n})\} \subseteq D(A^2) \times D(A^{3/2})$ converging to (v_0, v_1) in $D(A) \times D(A^{1/2})$. Let $\{v_n\}$ be the corresponding solutions of (3.34), and let us set $w_n := v - v_n$. Since w_n is a solution of (3.34) we have that

$$|Aw_n(t)|^2 \leq \frac{|A^{1/2}w_n'(t)|^2}{c(t)} + |Aw_n(t)|^2 \leq \frac{|A^{1/2}(v_{1,n} - v_1)|}{c(0)} + |A(v_{0,n} - v_0)|^2.$$

This proves that $\{Av_n\} \to Av$ uniformly in $[0, +\infty[$. Since $Av_n(t)$ has a limit as $t \to +\infty$ for every $n \in \mathbb{N}$ (see step 5), then necessarily $Av(t)$ has a limit as $t \to +\infty$.
This completes the proof of (3.39).
Step 7. If A is not coercive, it remain to prove (3.42) and (3.43) . Let us consider the function

$$E(t) := \frac{\|v'(t)\|^2}{c(t)} + \|A^{1/2}v(t)\|^2.$$

Arguing as in step 1, we have that, for every $T \geq 0$,

$$E'(t) = -\left(2\delta + \frac{c'(t)}{c(t)}\right)\frac{\|v'(t)\|^2}{c(t)} \leq -\frac{3}{2}\delta\frac{\|v'(t)\|^2}{c(t)}. \tag{3.53}$$

In particular, the function $E(t)$ is non-increasing, hence there exists

$$E_\infty := \lim_{t\to+\infty} E(t).$$

If $E_\infty = 0$, then (3.42) holds true with $v_\infty = 0$, while (3.43) follows from the boundedness of c.
By integrating (3.53) over $[0,T]$, we get

$$\int_0^T \frac{\|v'(t)\|^2}{c(t)}dt \leq -\frac{2}{3\delta}\int_0^T E'(t)dt = \frac{2}{3\delta}\big(E(0)-E(T)\big).$$

Since the function c is bounded, it implies that for every $T \geq 0$,

$$\int_0^T \|v'(t)\|^2 dt \leq \frac{2}{3\delta}\|c\|_\infty E(0), \qquad \|v'(T)\|^2 \leq \|c\|_\infty E(0).$$

Moreover, taking the scalar product of (3.34) with v, and arguing as in step 3, it follows that

$$\langle v''(t), v(t)\rangle + \delta\langle v'(t), v(t)\rangle + c(t)\langle Av(t), v(t)\rangle = 0,$$
$$\left(\langle v'(t), v(t)\rangle + \frac{\delta}{2}\|v(t)\|^2\right)' - \|v'(t)\|^2 + c(t)\|A^{1/2}v(t)\|^2 = 0.$$

Integrating in $[0,T]$ it follows that

$$\begin{aligned}
&\int_0^T c(t)\|A^{1/2}v(t)\|^2 dt + \frac{\delta}{2}\|v(T)\|^2 = \int_0^T \|v'(t)\|^2 + \frac{\delta}{2}\|v_0\|^2 + \langle v_1, v_0\rangle - \langle v'(T), v(T)\rangle \\
&\leq \int_0^T \|v'(t)\|^2 + \frac{\delta}{2}\|v_0\|^2 + \frac{1}{2}\left(\delta\|v_0\|^2 + \frac{1}{\delta}\|v_1\|^2\right) + \frac{1}{2}\left(\frac{\delta}{2}\|v'(T)\|^2 + \frac{2}{\delta}\|v(T)\|^2\right) \\
&\leq \int_0^T \|v'(t)\|^2 + \frac{\delta}{2}\|v_0\|^2 + \frac{\delta}{2}\|v_0\|^2 + \frac{1}{2\delta}\|v_1\|^2 + \frac{\delta}{4}\|v(T)\|^2 + \frac{1}{\delta}\|v'(T)\|^2,
\end{aligned}$$

we can rewrite as,

$$\begin{aligned}
\|v(T)\|^2 &\leq \frac{4}{\delta}\left\{\int_0^T \|v'(t)\|^2 + \frac{1}{\delta}\|v'(T)\|^2 + \delta\|v_0\|^2 + \frac{1}{2\delta}\|v_1\|^2\right\} \\
&\leq \frac{4}{\delta}\left\{\frac{2}{3\delta}\|c\|_\infty E(0) + \frac{1}{\delta}\|c\|_\infty E(0) + \delta\|v_0\|^2 + \frac{1}{2\delta}\|v_1\|^2\right\} \\
&\leq \frac{4}{\delta}\left\{\frac{5\|c\|_\infty}{3\delta}E(0) + \delta\|v_0\|^2 + \frac{1}{2\delta}\|v_1\|^2\right\},
\end{aligned}$$

hence

$$\|v(T)\|^2 \leq \frac{4}{\delta}\left\{\frac{5\|c\|_\infty}{3\delta}\left(\frac{\|v_1\|^2}{c(0)} + \|A^{1/2}v_0\|^2\right) + \delta\|v_0\|^2 + \frac{1}{2\delta}\|v_1\|^2\right\}. \tag{3.54}$$

Now we consider the spectral decomposition of self adjoint operator A.
We fix $\sigma > 0$. Let E be the spectral resolution of the identity associated to the self adjoint operator A, and let H be the orthogonal decomposition such that,

$$H = N_A \oplus N_B^\sigma \oplus N_C^\sigma,$$

where

$$N_A = \Re(E(\{0\})), \qquad N_B^\sigma = \Re(E(]0,\sigma[)), \qquad N_C^\sigma = \Re(E([0,+\infty[)).$$

Let $v_A(t)$, $v_B(t)$,$v_C(t)$ be the components of $v(t)$. Since N_A is the kernel of A, then $v_A(t)$ solves the ODE

$$v_A''(t) + \delta v_A'(t) = 0,$$

hence by a direct computation we see that $v_A(t)$ has a limit as $t \to +\infty$.
Since the restriction of A to N_C^σ is coercive, by the results of Steps $1-6$ it follows that also $v_C(t)$ has

a limit as $t \to +\infty$.
Moreover, applying (3.54) to the function v_B, we have that

$$|v_B(t)|^2 \leq \frac{4}{\delta}\left\{\frac{5\|c\|_\infty}{3\delta}\left(\frac{|(v_1)_B|^2}{c(0)} + |A^{1/2}(v_0)_B|^2\right) + \delta|(v_0)_B|^2 + \frac{1}{2\delta}(v_1)_B^2\right\}$$

for every $t \geq 0$.
This proves that $|v_B(t)|$ is small, uniformly in t, provided that σ is small enough. Since v_A and v_C have a limit for every $\sigma > 0$, then (3.42) follows by a standard argument. The proof of (3.43) is completely analogous.
Proof of Theorem 3.4. We use the same notations as in the proof of Theorem 3.3. Let us first remark that u is the solution of (3.34) with

$$c(t) = M(\|A^{1/2}u(t)\|^2), \qquad (v_0, v_1) = (u_0, u_1).$$

In step 2 of the proof of theorem 3.3, we showed that $c(t) > 0$ for every $t \geq 0$ (this proves statement (i)), and

$$\left|\frac{c'(t)}{c(t)}\right| \leq \frac{\delta}{2}, \qquad for \quad all \quad t \geq 0.$$

Since (3.25) holds true for every $t \geq 0$, it follows that c is bounded. Since M is locally Lipschitz continuous, and $\|A^{1/2}u'(t)\|^2 \leq F(t)c(t) \leq F(0)c(t)$ (see (3.26), then it turns out that c is globally Lipschitz continuous.
By the Lemma 3.6, there exists $u_\infty \in D(A)$ such that (3.31)-(3.33) are satisfied. If $A^{1/2}u_\infty \neq 0$, hence $Au_\infty \neq 0$, then by the last statement of Lemma 3.6 we have that $c(t) \to 0$ as $t \to +\infty$, and therefore

$$0 = \lim_{t \to +\infty} M(\|A^{1/2}u(t)\|^2) = M(\|A^{1/2}u_\infty\|^2).$$

Chapter 4

A Global Existence Result For The Kirchhoff Type Equation With Non Linear Dissipation

In this chapter, we consider the initial boundary value problem in abstract form

$$\begin{cases} u'' + M\left(\left\|A^{1/2}u\right\|^2\right) Au + g(u') = 0 \quad \text{in} \quad [0, \infty), \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \tag{4.1}$$

for a Kirchhoff-type equation with a nonlinear local damping term. Under suitable assumptions on A, M and g, we proved following a method based on the multiplier techniques, the Galerkin approximation method and some integral inequality techniques due to Harraux [58], the existence and uniqueness of global solution and the energy decay rates.

4.1 The Abstract Model

We take Ω to be an open, bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Take $H = L^2(\Omega)$ that is equipped with a norm $\|.,.\|$, and a scaler product denoted by $(.,.)$. Let A be a linear, nonnegative and self adjoint operator on H and $c > 0$, is a constant, such that

$$(Au, u) \geq c\|u\|^2_{L^2(\Omega)}, \quad \forall u \in H. \tag{4.2}$$

With dense domain in H, denoted by $V := D(A)$. We write

$$\|u\|_A = \|u\|^2 + \|Au\|^2$$

as the graph norm of the operator A. We take the imbedding $V \subset H$ to be compact and identify H' as the dual of H. Also we have that $V \subset H \subset V'$, where the dual of V is denoted by V'. Take $W := D(A^{1/2})$ and let $\langle .,. \rangle_{V',V}$ stands for the duality pairing between V' and V.
Now, we state certain assumptions that proves very useful in this section.
Assumption 1: Let $M(s)$ be a $C^1[0, \infty)$ real function and $M^{'}(s) \geq 0$. Take β and γ_0 to be two positive constants, such that $M(s) \geq \beta > 0$ for all $s \geq 0$ and $|M'(s)s|/M(s) \leq \gamma_0$.
Assumption 2: Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function and $g(0) = 0$ such that the constant $k > 0$ and $q \geq 1$, then

$$|g(x)| \leq k(1 + |x|^q) \quad \forall x \in \mathbb{R}. \tag{4.3}$$

We note that for all $u \in D(A) \cap D(A^{1/2})$, then $g((u), Au) \geq 0$. This is true if $A = -\triangle$ and $g(u) = |u|^\alpha u$, $\alpha \geq 1$.
Assumption 3: Take $M'(s) > M(s)|g(x)|$, $s \in [0, \infty)$, $\forall\, x \in \mathbb{R}$.
Assumption 4: Let $V \subset L^{q+1}(\Omega)$ for some $q \geq 1$.
Next, we define $\overline{M}(t)$ and $E(t)$ as follows:

$$\overline{M}(t) = \int_0^t M(s)ds, \tag{4.4}$$

$$E(t) = \frac{1}{2}\left[\|u'(t)\|^2 + \overline{M}(\|A^{1/2}u(t)\|^2)\right]. \tag{4.5}$$

We are now going to state the theorem that we want to prove.

Theorem 4.1 *Assume that the initial conditions* $(u_0, u_1) \in W \times L^{2q}(\Omega)$ *satisfy the smallness assumption*

$$\|M'\|_{L^\infty([0,X(0)])} B(u_0, u_1)\sqrt{Y(0)} < \frac{1}{4}, \tag{4.6}$$

where

$$B(u_0, u_1) = \max\left\{\frac{\|u_1\|}{M(\|A^{1/2}u_0\|^2)}, M(\|A^{1/2}u_0\|^2)\right.$$
$$\left.\times\left((M(\|A^{1/2}u_0\|^2))' - g(u_1) \times M(\|A^{1/2}u_0\|)\right)^{-1}\sqrt{Y(0)}\right\}.$$

Then there is a unique function $u \in L^\infty(0, T; D(A^{1/2})) \cap W^{1,\infty}(0, T; D(A)) \cap W^{2,\infty}(0, T; L^2(\Omega))$ *such that for every* $T > 0$,

$$u''(t) + M\left(\|A^{1/2}u(t)\|^2\right)Au(t) + g(u'(t)) = 0 \quad in \quad L^{(q+1)/q}(0, T; V'), \tag{4.7}$$

$$u(0) = u_0 \quad (u')(0) = u_1. \tag{4.8}$$

Remark 4.2 It is important to note here that the smallness assumption together with the lemma (3.2) and the fact that $|K'(t)/K(t)| = 1/2$ based on the maximality of T, help us to establish with appropriate a priori estimate that $u_m(t)$ can be extended to $[0, T)$ for any $T \in (0, \infty)$.

4.2 Existence of the solution

It will follow from the Galerkin approximation method. Suppose that, for simplicity, we take $V = D(A)$ to be separable. Let $(w^k)_{k\geq 1}$ be a sequence of eigenfunctions of the operator A that correspond to positive real eigenvalues λ_k that tends to $+\infty$, so that $Aw^k = \lambda_k w^k$, $k \geq 1$.
Let V_m denote the linear subspace of $w^1, w^2, \cdots w^m$. Note that $(w^k)_{k\geq 1}$ is a basis of $L^2(\Omega)$, $D(A)$ and $D(A^{1/2})$ and hence it is dense in H, V, and W.
We find

$$u_m(t) = \sum_{k=1}^{m} g_{km}(t)w^k,$$

for any $v \in V_m, u_m(t)$ is a solution of the approximate equation.

$$\left(u''_m(t) + M(\|A^{1/2}u_m(t)\|^2)Au_m(t) + g(u'_m(t)), v\right) = 0, \tag{4.9}$$

with the initial conditions u_0 and u_1 as a projections over V_m that satisfy

$$u_m(0) = u_{0m} = \sum_{k=1}^{m}(u_m(0), w^k)w^k \to u_0 \quad in \quad D(A^{1/2}), \tag{4.10}$$

$$(u'_m)(0) = u_{1m} = \sum_{k=1}^{m}(u'_m(0), w^k)w^k \to u_1 \quad in \quad L^{2q}(\Omega). \tag{4.11}$$

Note that we can solve (4.9)-(4.11). In fact the problems (4.9)-(4.11) have a unique continuous solution u_m on some interval $[0, t_m)$. The extension of the solution to the whole interval $[0, \infty)$ is a consequence of the A priori estimate which we are going to solve below. Now, we show that $u_m(t)$ can be extended to $[0, T)$ for all $T > 0$. To show that $u_m(t)$ can be extended to $[0, T)$ for all $T > 0$, we have the

following A Priori estimates.
The First A priori Estimate. Taking now $v = u'_m(t)$ in (4.9), we get

$$(u''_m(t), u'_m(t)) + M(\|A^{1/2}u_m(t)\|^2)(Au_m(t), u'_m(t)) + g(u'_m(t), u'_m(t)) = 0.$$

That is,

$$\frac{1}{2}\frac{d}{dt}\|u'_m(t)\|^2 + M(\|A^{1/2}u_m(t)\|^2)\frac{1}{2}\frac{d}{dt}\|A^{1/2}u_m(t)\|^2 + (g(u'_m(t), u'_m(t))) = 0.$$

Multiply through by 2, and use (4.4) to get

$$\frac{d}{dt}\Big(\|u'_m(t)\|^2 + \overline{M}(\|A^{1/2}u_m(t)\|^2)\Big) + 2\Big(g(u'_m(t)), u'_m(t)\Big) = 0. \tag{4.12}$$

Integrating (4.12) over $(0,t)$, $t \le t_m$, and using (4.5), we get

$$2E(0) = \Big[\Big\|u'_m(t)\Big\|^2 + \overline{M}\Big(\Big\|A^{1/2}u_m(t)\Big\|^2\Big)\Big] + 2\int_0^t \Big(g(u'_m(s)), u'_m(s)\Big)ds. \tag{4.13}$$

By using (4.2) and (4.4), we deduce that

$$2E(0) \ge \Big\|u'_m(t)\Big\|^2 + \beta\Big\|A^{1/2}u_m(t)\Big\|^2 + 2\int_0^t\int_\Omega u'_m(s)g(u'_m(s))dxds, \tag{4.14}$$

where the left hand side is constant independent of m and t. Thus estimation (4.14) yields, for any $0 < T < \infty$,

$$u'_m \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \tag{4.15}$$

$$A^{1/2}u_m \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \tag{4.16}$$

$$u'_m g(u'_m) \quad \text{is bounded in} \quad L^1([0,T]\times\Omega). \tag{4.17}$$

This conclude that $u_m(t)$ can be extended to $[0,T)$ for all $T > 0$.
Now we show that $u_m(t)$ can be extended to $[0,\infty)$. We recall now the following smallness assumption:

$$\begin{aligned}\|M'\|_{L^\infty([0,X(0)])} \times \max\Bigg\{&\frac{\|u_{m1}\|}{M(\|A^{1/2}u_{m0}\|^2)}, (M(\|A^{1/2}u_{m0}\|^2)) \times \Big((M(\|A^{1/2}u_{m0}\|^2))' - g(u_{m1})\\ &\times M(\|A^{1/2}u_{m0}\|^2)\Big)^{-1}\sqrt{Y(0)}\Bigg\} \times \sqrt{Y(0)} < \frac{1}{4},\end{aligned} \tag{4.18}$$

where $X(0) = (\|u_{m1}\|^2/M(\|A^{1/2}u_{m0}\|^2)) + \|A^{1/2}u_{m0}\|^2$, $\quad Y(0) = \|A^{1/2}u_{m1}\|^2/M(\|A^{1/2}u_{m0}\|^2)) + \|Au_{m0}\|^2$.

Let $[0,T^*)$ be the maximal interval where the solution exists.
We set

$$K(t) := M(\|A^{1/2}u_m(t)\|^2)$$

and

$$T := \sup\left\{\tau \in [0,T^*)| \left|\frac{K'(t)}{K(t)}\right| \le \frac{1}{2}, K(t) > 0, \forall t \in [0,\tau)\right\}. \tag{4.19}$$

Let us consider the functions

$$X(t) := \tfrac{\|u'_m(t)\|^2}{K(t)} + \|A^{1/2}u_m(t)\|^2, \quad Y(t) := \tfrac{\|A^{1/2}u'_m(t)\|^2}{K(t)} + \|Au_m(t)\|^2, \quad J(t) := \tfrac{\|u'_m(t)\|^2}{K(t)}.$$

With simple computations it follows that

$$\begin{aligned}X'(t) &= \frac{d}{dt}\|u'_m(t)\|^2 \times \frac{1}{K(t)} - \frac{K'(t)}{(K(t))^2}\|u'_m(t)\|^2 + \frac{d}{dt}\|A^{1/2}u_m(t)\|^2\\ &= \frac{2(u''_m(t), u'_m(t))}{K(t)} - \frac{K'(t)}{(K(t))^2}\|u'_m(t)\|^2 + 2(u'_m(t), Au_m(t)).\end{aligned} \tag{4.20}$$

Now, recalling from (4.9), substitute $K(t)$, we get

$$u''_m(t) + K(t)Au_m(t) + g(u'_m(t)) = 0,$$

and taking the scalar product with $u'_m(t)$, we substitute the term $(u''_m(t), u'_m(t))$ in (4.20) and get that

$$\begin{aligned} X'(t) &= -\frac{1}{K(t)}\Big(2(g(u'_m(t), u'_m(t)) + \frac{K'(t)}{K(t)}\|u'_m(t)\|^2\Big) \\ &\leq 0. \end{aligned} \tag{4.21}$$

Remark 4.3 This is true since $g(u'_m(t), u'_m(t))$ and $\frac{K'(t)}{K(t)}\|u'_m(t)\|^2$ are positive.

Similarly, from

$$\begin{aligned} Y'(t) &= \frac{d}{dt}\|A^{1/2}u'_m(t)\|^2 \times \frac{1}{K(t)} - \frac{K'(t)}{(K(t))^2}\|A^{1/2}u'_m(t)\|^2 + \frac{d}{dt}\|Au_m(t)\|^2 \\ &= \frac{2(u''_m(t), Au'_m(t))}{K(t)} - \frac{K'(t)}{(K(t))^2}\|A^{1/2}u'_m(t)\|^2 + 2(Au'_m(t), Au_m(t)), \end{aligned} \tag{4.22}$$

we use the equation

$$u''_m(t) + K(t)Au_m(t) + g(u'_m(t)) = 0.$$

Take the scalar product with $Au'(t)$ and substitute the term $(u''_m(t), Au'_m(t))$ in (4.22) to finally obtain that

$$\begin{aligned} Y'(t) &= -\frac{1}{K(t)}\Big(2(g(u'_m(t)), Au'_m(t)) + \frac{K'(t)}{K(t)}\|A^{1/2}u'_m(t)\|^2)\Big) \\ &\leq 0. \end{aligned} \tag{4.23}$$

Also,

$$(J^2)(t) = \frac{\|u'_m(t)\|^2}{(K(t))^2},$$

from which we get that

$$\begin{aligned} (J^2)'(t) &= \frac{\frac{d}{dt}\|u'_m(t)\|^2}{(K(t))^2} - \frac{\|u'_m(t)\|^2 2K(t)K'}{(K(t))^4} \\ &= \frac{2(u''_m(t)), u'_m(t)}{(K(t))^2} - \frac{2\|u'_m(t)\|^2 K'(t)}{(K(t))^3}. \end{aligned} \tag{4.24}$$

We use the equation

$$u''_m(t) + K(t)Au_m(t) + g(u'_m(t)) = 0.$$

Take the scalar product with $u'_m(t)$ and substitute the term $(u''_m(t), u'_m(t))$ in (4.24) to finally obtain that

$$(J^2)'(t) \leq -J(t)\Big\{2\Big(\frac{K'(t)}{K(t)} - \|g(u'_m(t))\|\Big)J(t) - 2\|Au(t)\|\Big\}, \tag{4.25}$$

for every $t \in [0, T[$.
Next, we want to show that $T = T^*$.
Let us assume by contradiction that $T < T^*$. Since $|K'(t)| \leq (\frac{1}{2})K(t)$ in $[0, T[$, it implies that

$$\begin{cases} K'(t) \leq \frac{1}{2}K(t) \\ -K'(t) \leq \frac{1}{2}K(t). \end{cases}$$

The first case gives after integration over $[0,T]$,

$$K(T) \leq K(0)e^{\frac{1}{2}T},$$

while the second case gives

$$K(T) \geq K(0)e^{-\frac{1}{2}T}.$$

We therefore obtain

$$0 < K(0)e^{\frac{-T}{2}} \leq K(T) \leq K(0)e^{\frac{T}{2}}. \tag{4.26}$$

Since $K(t)$ and $K'(t)$ are continuous functions, by the maximality of T we have that necessarily

$$\liminf_{t\to T_*^-} \left|\frac{K'(t)}{K(t)}\right| \geq \frac{1}{2}$$

and

$$\limsup_{t\to T_*^-} \left|\frac{K'(t)}{K(t)}\right| \leq \frac{1}{2} \qquad \forall t \in [0,T],$$

implies that,

$$\left|\frac{K'(T)}{K(T)}\right| = \frac{1}{2}. \tag{4.27}$$

From (4.21) and (4.23), we have that their derivatives are negatives, it follows that X and Y are non-increasing functions, hence

$$\|A^{1/2}u_m(t)\|^2 \leq X(t) \leq X(0), \tag{4.28}$$

$$\|Au_m(t)\|^2 \leq Y(t) \leq Y(0) \tag{4.29}$$

$\forall t \in [0,T]$.

Moreover by Lemma (4.3) we have that

$$J(t) \leq \max\left\{J(0), \frac{K(0)}{K'(0)-g(u_{m1})K(0)}\sqrt{Q(0)}\right\}, \qquad \forall t \in [0,T]. \tag{4.30}$$

By (4.27)-(4.29), and the smallness assumption (4.18), we have that

$$\begin{aligned}
\left|\frac{K'(t)}{K(t)}\right| &= \left|\frac{2M'(\|A^{1/2}u_m(t)\|^2)(u_m'(T), Au_m(T)}{K(t)}\right| \\
&\leq 2\max_{0\leq r\leq X(0)}\|M'(r)\|\frac{\|u_m'(T)\|}{K(T)}\|Au_m(T)\| \\
&\leq 2\max_{0\leq r\leq X(0)}\|M'(r)\| \times \max\left\{J(0), \frac{K(0)}{K'(0)-g(u_{m1})K(0)}\sqrt{Y(0)}\right\} \times \sqrt{Y(0)} \\
&< \frac{1}{2}.
\end{aligned}$$

This contradicts (4.27). Therefore it follows that $u_m(t)$ can be extended to $[0,T)$ for any $T \in (0,\infty)$. Furthermore, taking now $v = Au_m'$ in (4.9), we get

$$(u_m''(t), Au_m') + M(\|A^{1/2}u_m\|^2)(Au_m, Au_m') + g(u_m', Au_m') = 0.$$

Divide through by $M(\|A^{1/2}u_m\|^2)$, we get

$$\frac{(u_m'', Au_m')}{M(\|A^{1/2}u_m\|^2)} + (Au_m, Au_m') + \frac{g(u_m'), Au_m')}{M(\|A^{1/2}u_m\|^2)} = 0. \tag{4.31}$$

By simplification we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\frac{u_m', Au_m'}{M(\|A^{1/2}u_m\|^2} + \|Au_m\|^2\right) + \frac{(g(u_m'), Au_m')}{M(\|A^{1/2}u_m\|^2)}\\ = \frac{-(u_m', Au_m)M'(\|A^{1/2}u_m\|^2)(A^{1/2}u_m', A^{1/2}u_m)}{(M(\|A^{1/2}u_m\|^2))^2}.\end{aligned} \tag{4.32}$$

Integrating (4.32) over $(0,t)$ and taking into account assumption (1) and (2), and applying Gronwall's inequality, we get

$$Au_m \text{ is bounded in } L^\infty(0,T;L^2(\Omega)). \tag{4.33}$$

From (4.17) and (4.3), we have that

$$g(u_m')\text{is bounded in } \quad L^{(q+1)/q}([0,T]\times\Omega). \tag{4.34}$$

The Second A Priori Estimate. Taking now $v = u_m''(t)$ in (4.9) and choosing $t = 0$, we get

$$\|u_m''(0)\|^2 + \Big(M(\|A^{1/2}u_{0m}\|^2)Au_{0m} + g(u_{1m}), u_m''(0)\Big) = 0.$$

Hence

$$\begin{aligned}\|u_m''(0)\|^2 &\le \Big(\|g(u_{1m})\| + |M(\|A^{1/2}u_{0m}\|^2)Au_{0m}|\Big)\|u_m''(0)\|\\ &\le \Big(\|g(u_{1m})\| + |M(\|A^{1/2}u_0\|^2)Au_0|\Big) \times \|u_m''(0)\|.\end{aligned}$$

From the assumption (4.3), we infer from (4.11) that $(g(u_{1m}))$ is bounded in $L^2(\Omega)$. Therefore we conclude that the right hand side is bounded; that is

$$u_m''(0) \text{ is bounded } \quad \text{in} \quad H. \tag{4.35}$$

The Third A Priori Estimate. We make use of (4.9) at the points t and $t+\varphi$. For $t < T$, then $0 < \varphi < T - t$. Taking now the difference $v = u_m'(t+\varphi) - u_m'(t)$ in (4.9) and the assumption (2), we get

$$\begin{aligned}&\Big(u_m''(t+\varphi) - u_m''(t), u_m'(t+\varphi) - u_m'(t)\Big) + \Big(M(\|A^{1/2}u_m(t+\varphi)\|^2)Au_m(t+\varphi)\\ &- M(\|A^{1/2}u_m(t)\|^2)Au_m(t), u_m'(t+\varphi) - u_m'(t)\Big) \le 0.\end{aligned} \tag{4.36}$$

Thus we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\|(u_m'(t+\varphi) - u_m'(t))\|^2\Big] + M(\|A^{1/2}u_m(t+\varphi)^2\|)\\ &\times (Au_m(t+\varphi) - Au_m(t), u_m'(t+\varphi) - u_m'(t))\\ &+ [M(\|A^{1/2}u_m(t+\varphi)\|^2) - M(\|A^{1/2}u_m(t)\|^2)] \times (Au_m(t), u_m'(t+\varphi) - u_m'(t)) \le 0.\end{aligned} \tag{4.37}$$

Set

$$\phi_{\varphi m(t)} = \|(u_m'(t+\varphi) - u_m'(t))\|^2. \tag{4.38}$$

We apply the Young's inequality on (4.37), and use the assumption (1) and the fact that A is non negative self-adjoint operator, we see that $\phi_{\varphi_m}'(t) \le c\phi_{\varphi_m}(t)$. Therefore we deduce that

$$\phi_{\varphi_m}(t) \le \phi_{\varphi_m}(0)\exp(cT) \quad \forall t \in [0,T]. \tag{4.39}$$

By dividing both sides of (4.39) by φ^2, letting $\varphi \to 0$, and using (4.38), we get

$$c\|u_m''\|^2 \le \|u_m''(0)\|^2. \tag{4.40}$$

From (4.35), it follows that $\|u''_m\|^2 \leq C$.
We know that $u_m \in C^2[0,T]$, for all $t \in [0,T]$. This verified the previous inequality and conclude that

$$u''_m \text{ bounded } \quad in \quad L^\infty(0,T;H). \tag{4.41}$$

Furthermore, using (4.15) and (4.41), we get that

$$u'_m \quad \text{is bounded} \quad in \quad L^2(0,T;H), \tag{4.42}$$
$$u''_m \quad \text{is bounded} \quad in \quad L^2(0,T;H). \tag{4.43}$$

Applying a compactness theorem stated in the preliminary chapter, we get

$$u'_m \text{ precompact } in \ L^2(0,T;H). \tag{4.44}$$

We are know in a convenient place to pass the limit.
Passing to the limit.
By using the Dunford-Pettis theorem, we conclude from (4.15), (4.16), (4.34) and (4.41) - (4.44), replacing the sequence u_m with a subsequence if needed, that

$$u_m \to u \quad \text{weak-star in} \quad L^\infty(0,T;V), \tag{4.45}$$

$$u'_m \to u' \quad \text{weak-star in} \quad L^\infty(0,T;H), \tag{4.46}$$

$$u''_m \to u'' \quad \text{weak-star in} \quad L^\infty(0,T;H), \tag{4.47}$$

$$u'_m \to u' \quad \text{a.e in} \quad \Omega \times [0,T], \tag{4.48}$$

$$g(u'_m) \to \psi \quad \text{weak-star in} \quad L^{(q+1)/q}(0,T;H), \tag{4.49}$$

$$M(\|A^{1/2}u_m\|^2)Au_m \to \chi \quad \text{weak-star in} \quad L^\infty(0,T;H) \tag{4.50}$$

for appropriate function $u \in L^\infty(0,T;V), \chi \in L^\infty(0,T;H)$ and $\psi \in L^{(q+1)/q}(\Omega \times [0,T])$.
Now we show that u is a solution of the problem (4.7) - (4.8). In fact, from (4.45) to (4.47), we get

$$\begin{aligned} \int_\Omega u_m(0)w^k dx &\to \int_\Omega u(0)w^k dx, \\ \int_\Omega u'_m(0)w^k dx &\to \int_\Omega u'(0)w^k dx, \end{aligned} \tag{4.51}$$

for any fixed $k \geq 1$. So we conclude that, for any $k \geq 1$,

$$\int_\Omega (u_m(0) - u_0)w^k dx = \int_\Omega (u'(0) - u_1)w^k dx = 0 \quad \text{ as } m \to \infty, \tag{4.52}$$

it follows that (4.8) holds.
Now, we will prove that, indeed, $\chi = M(\|A^{1/2}u\|^2)Au$; that is

$$M(\|A^{1/2}u_m\|^2)Au_m \to M(\|A^{1/2}u\|^2)Au \ \text{ weak-star in } L^\infty(0,\infty;H). \tag{4.53}$$

For $v \in L^2(0,T;H)$, we have

$$\begin{aligned} \int_0^T (\chi - M(\|A^{1/2}u\|^2)Au, v)dt &= \int_0^T (\chi - M(\|A^{1/2}u_m\|^2)Au_m, v)dt \\ &+ \int_0^T M(\|A^{1/2}u\|^2)(Au_m - Au, v)dt \\ &+ \int_0^T (M(\|A^{1/2}u_m\|^2) - M(\|A^{1/2}u\|^2)) \text{ x } (Au_m, v)dt. \end{aligned} \tag{4.54}$$

We deduce from (4.45) and (4.50) that both the first and second terms in (4.54) tends to zero as $m \to \infty$. For the last term, using the fact that $M \in C'$ and (4.16), we can obtain (with some positive constants C_1, C_2).

$$\begin{aligned}\int_0^T (M(\|A^{1/2}u_m\|^2) - M(\|A^{1/2}u\|^2))(Au_m, v)dt &\leq C_1 \int_0^T \|A(u_m+u), u_m - u\|dt \\ &\leq C_2\Big(\int_0^T \|u_m - u\|^2 dt\Big)^{1/2}.\end{aligned} \tag{4.55}$$

Since u_m is bounded in $L^\infty(0,T;V)$ and the injection of V in H is compact, we have

$$u_m \to u \text{ strongly in } \quad L^2(0,T;H). \tag{4.56}$$

From (4.54) to (4.56), we deduce (4.53). It follows from (4.45), (4.47) and (4.53) that for each fixed $v \in L^{q+1}(0,T;V)$,

$$\int_0^T (u_m'' + M(\|A^{1/2}u_m\|^2)Au_m, v)dt \to \int_0^T (u'' + M(\|A^{1/2}u\|^2)Au, v)dt \quad \text{as} \quad m \to +\infty. \tag{4.57}$$

For the nonlinear term $g(u')$, it remains to show that for any fixed $v \in L^{q+1}(0,T;V)$,

$$\int_0^T \int_\Omega vg(u_m')dxdt \to \int_0^T \int_\Omega vg(u')dxdt \quad \text{as} \quad m \to \infty. \tag{4.58}$$

At this point we introduce the following lemma by Jung and Choi [67].

Lemma 4.4 *Assume that $\Omega \times [0,T]$ is an open, bounded domain of $\mathbb{R}^n \times \mathbb{R}$; g_m and g are in $L^q(\Omega \times [0,T])$, $1 < q < \infty$, such that $g_m \to g$ a.e., in $\Omega \times [0,T]$. Then $g_m \to g$ weakly in $L^q(\Omega \times [0,T])$.*

By (4.49), we have that $g(u_m') \to g(u')$ a.e. in $(\Omega \times [0,T])$. From (4.34), we can use the above lemma and so we have $\psi = g(u')$; that is,

$$g(u_m) \to g(u) \quad \text{weak in } L^{(q+1)/q}(\Omega \times (0,T)), \tag{4.59}$$

which implies (4.58). Therefore we obtain

$$\int_0^T (u'' + M(\|A^{1/2}u\|^2)Au + g(u'), v)dt = 0, \quad \forall v \in L^{q+1}(0,T;V). \tag{4.60}$$

4.3 Uniqueness of the solution

Let u and v be two solutions of (4.7) and (4.8) in the conditions of Theorem (5.1). Then $w = u - v$ is a solution of

$$w'' + gw' + M(\|A^{1/2}u\|^2)Aw + (M(\|A^{1/2}u\|^2)) - M(\|A^{1/2}v\|^2))Av = 0, \tag{4.61}$$

$$w(0) = 0; \quad w'(0) = 0. \tag{4.62}$$

Taking the scalar product of (4.61) with $2w'$, we get

$$\begin{aligned}&\frac{d}{dt}[\|w'(t)\|^2 + M(\|A^{1/2}u(t)\|^2)\|A^{1/2}w(t)\|^2] + 2g\|w'(t)\|^2 \\ &= \|A^{1/2}w(t)\|^2 \frac{d}{dt}M(\|A^{1/2}u(t)\|^2) + [M\|A^{\frac{1}{2}}v(t)\|^2 - M(\|A^{\frac{1}{2}}u(t)\|^2)](Av(t), 2w'(t)).\end{aligned}$$

By integrating with respect to t, using (4.62), hence $\|A^{1/2}w(0)\|^2 = 0$ and (4.55), we find, compare with (4.55)

$$\|w'(t)\|^2 + \beta\|A^{1/2}w(t)\|^2 + 2g\int_0^t \|w'(s)\|^2 ds \leq C\int_0^t [\|w'(s)\|^2 + \beta\|A^{1/2}w(s)\|^2]ds$$

where C is a positive constant. Hence uniqueness follows from the Gronwall lemma.

4.4 Decay Energy Estimates

In this section we want to study the energy estimate under suitable growth conditions on g. Let us assume that there exist a number $p \geq 1$ and positive constants $c_1, i = 1, 2$, such that

$$c_1 \min\{|x|, |x|^p\} \leq |g(x)| \leq c_2 \max\{|x|, |x|^{1/p}\} \text{ for all } x \in \mathbb{R}. \tag{4.63}$$

Theorem 4.5 *Assume that (4.63) holds. Then one obtains the following energy decay:*

$$E(t) \leq \begin{cases} c_0 E(0) e^{-\omega t} & \forall t \geq 0, \quad \text{if } p = 1, \\ \tilde{c}_0 (1+t)^{-2/(p-1)} & \forall t \geq 0, \quad \text{if } p > 1, \end{cases} \tag{4.64}$$

where c_0, ω and $\tilde{c}_0$ are some positive constants.

Proof.
Let $T > 0$ be arbitrary and fixed, and let $u \in L^\infty(0,T;V) \cap W^{2,\infty}(0,T;H)$ be a solution of (4.7) and (4.8). Multipying (4.7) by $u'(t)$, we get

$$(u''(t), u'(t)) + M(\|A^{1/2}u(t)\|^2)(Au(t), u'(t)) + g(u'(t), u'(t)) = 0.$$

By simplification, we get

$$\frac{d}{dt}\Big(\|u'(t)\|^2 + \overline{M}(\|A^{1/2}u(t)\|^2)\Big) + 2(g(u'(t)), u'(t)) = 0.$$

Integrating by parts in $\Omega \times (s,T)(0 \leq s < T)$, we obtain that

$$E(T) - E(s) = -\int_s^T (g(u'(t)), u'(t))dt. \tag{4.65}$$

By $(g(u'(t), u'(t))) \geq 0$ and being the primitive of an integrable function, it follows that the energy E is nonincreasing, locally absolutely continuous and

$$E'(t) = -(g(u'(t)), u'(t)) \quad a.e \quad in \quad [0, \infty). \tag{4.66}$$

Here and in what follows we will denote by c diverse positive constants. We are going to show that the energy of this solution satisfies

$$\int_s^T E(t)^{(p+1)/2} \leq cE(s) \quad \forall 0 \leq s \leq T < \infty. \tag{4.67}$$

Once (4.67) is satisfied, the integral inequalities given in Komornik [74] and Haraux [58] will establish (4.64).
Now, multiplying (4.7) by $E(t)^{(p-1)/2}u$ and integrating by parts, we have

$$\begin{aligned} 0 =& \int_s^T E(T)^{(p-1)/2}(u'' + M(\|A^{1/2}u\|^2)Au + g(u'), u)dt \\ =& \Big[E(t)^{(p-1)/2}(u', u)\Big]_s^T - \frac{p-1}{2}\int_s^T E(t)^{(p-3)/2}E'(t)(u', u)dt \\ &- \int_s^T E(t)^{(p-1)/2}\|u'\|^2 dt + \int_s^T E(t)^{(p-1)/2}(M(\|A^{1/2}u\|^2)\|A^{1/2}u\|^2, u)dt \\ &+ \int_s^T E(t)^{(p-1)/2}(g(u'), u)dt. \end{aligned} \tag{4.68}$$

Note that by the assumption (1) and (4.16). We can choose some positive number

$$\alpha = \max_{s \in [0, \|A^{1/2}u\|^2]} \{M(s)\} < \infty, \tag{4.69}$$

so that $2E(t) \leq \|u'\|^2 + \alpha\|A^{1/2}u\|^2$. Thus we deduce that

$$\begin{aligned}\frac{2\beta}{\alpha}\int_s^T E(t)^{(p+1)/2}dt \leq &-\Big[E(t)^{(p-1)/2}(u',u)dt\Big]_s^T + \frac{p-1}{2}\int_s^T E(t)^{(p-3)/2}E'(t)(u',u)dt \\ &+ \int_s^T E(t)^{(p-1)/2}((1+\alpha^{-1})\|u'\|^2 - (g(u'),u))dt \\ &\equiv I_1 + I_2 + I_3.\end{aligned} \tag{4.70}$$

Using the continuity of the imbedding $V \subset H$, the Cauchy-Schwartz and the Young inequalities, we obtain

$$|(u',u)| \leq c|u'|\|u\| \leq cE(t). \tag{4.71}$$

Hence, since $E(t)$ is nonincreasing, we obtain

$$I_1 \leq cE^{(p-1)/2}(0)E(s), \quad I_2 \leq \frac{(p-1)}{2}\int_s^T cE(t)^{(p-1)/2}E'(t)dt \leq cE^{(p-1)/2}(0)E(s). \tag{4.72}$$

In order to estimate the last term I_3 of (4.70), we set

$$\Omega_1 = \{x \in \Omega : |u'(t,x)| \leq 1\}, \quad \Omega_2 = \{x \in \Omega : |u'(t,x)| > 1\}. \tag{4.73}$$

Then we have

$$\int_\Omega |u'(t,x)|^2 dx = \int_{\Omega_1} |u'(t,x)|^2 dx + \int_{\Omega_2} |u'(t,x)|^2 dx. \tag{4.74}$$

The Hölder inequality yields

$$\begin{aligned}\int_\Omega |u'(t,x)|^2 dx &\leq c\Big(\int_{\Omega_1} |u'(t,x)|^{p+1}dx\Big)^{2/(p+1)} + \int_{\Omega_2} |u'(t,x)|^2 dx \\ &\equiv J_1 + J_2.\end{aligned} \tag{4.75}$$

Using (4.63) and (4.66), we deduce that

$$\begin{aligned}J_1 &\leq c\Big(\int_{\Omega_1} u'g(u'dx)\Big)^{2/p+1} \leq c|E'(t)|^{2/(p+1)}, \\ J_2 &\leq c\int_{\Omega_2} |u'g(u')|dx \leq c(-E'(t)).\end{aligned} \tag{4.76}$$

By putting together the inequalities in 4.76 with (4.75), we get

$$\int_\Omega |u'(t,x)|^2 dx \leq c(-E'(t))^{2/(p+1)} + c(-E'(t)). \tag{4.77}$$

Applying Young's inequality, it follows that, for any $\epsilon > 0$,

$$\int_s^T E(t)^{(p-1)/2}|u'|^2 dt \leq \epsilon c\int_s^T E(t)^{(p+1)/2}dt + c\Big(\epsilon^{(1-p)/2} + E^{(p-1)/2}(0)\Big)E(s). \tag{4.78}$$

It remains to estimate the second term of I_3. Using (4.5) we have

$$\begin{aligned}\Big|\int_\Omega ug(u')dx\Big| &\leq c\|u\|_L^{(p+1)/p}(\Omega_1)\|g(u')\|_L^{p+1}(\Omega_1) \\ &\leq c\|u\|_L^{(p+1)/p}(\Omega_1)\Big(\int_{\Omega_1}(u'g(u')\Big)^{1/(p+1)} \\ &\leq cE(t)^{1/2}(-E'(t))^{1/(p+1)}.\end{aligned} \tag{4.79}$$

Similarly, using (4.8), we obtain

$$\begin{aligned}\left|\int_{\Omega_2} ug(u')dx\right| &\le c\|u\|_{L^2(\Omega_2)}\|g(u')\|_{L^2(\Omega_2)} \\ &\le c\|u\|_{L^2(\Omega_2)}\|u'g(u')\|^{1/2}_{L^1(\Omega_2)} \\ &\le cE(t)^{1/2}(-E'(t))^{1/2}.\end{aligned} \tag{4.80}$$

From (4.79) and (4.80), we deduce

$$\left|\int_{\Omega} ug(u')dx\right| \le cE(t)^{1/2}(-E'(t))^{1/(p+1)} + cE(t)^{1/2}(-E'(t))^{1/2}. \tag{4.81}$$

Using Young's inequality and

$$E(t)^{p/2}(-E'(t))^{1/2} = E(t)^{(p+1)/4}\Big(E(t)^{(p-1)/4}(-E'(t))^{1/2}\Big), \tag{4.82}$$

it follows from (4.80) that, for any $\epsilon > 0$,

$$\begin{aligned}-\int_s^T E(t)^{(p-1)/2}(g(u'),u)dt &= -\int_s^T E(t)^{(p-1)/2}\int_\Omega ug(u')dxdt \\ &\le c\int_s^T E(t)^{p/2}(-E'(t))^{1/(p+1)}dt + c\int_s^T E(t)^{\frac{p}{2}}(-E'(t))^{\frac{1}{2}}dt \\ &\le 2\epsilon c\int_s^T E(t)^{(p+1)/2}dt + c(\epsilon^{-p}+\epsilon^{-1}E(0)^{(p-1)/2})\times\int_s^T(-E'(t))dt \\ &\le 2\epsilon c\int_s^T E(t)^{(p+1)/2}dt + c(\epsilon^{-p}+\epsilon^{-1}E(0)^{(p-1)/2})E(s).\end{aligned} \tag{4.83}$$

Combining (4.78) with (4.83) and setting $\tilde\alpha = 1+\alpha^{-1}$, we obtain

$$\begin{aligned}I_3 &\le \int_s^T E(t)^{(p-1)/2}2\|u'\|^2dt + \int_s^T E(t)^{(p-1)/2}\int_\Omega ug(u')dxdt \\ &\le (\tilde\alpha+2)\epsilon c\int_s^T E(t)^{(p+1)/2}dt + c\Big(\tilde\alpha\epsilon^{(1-p)/2}+\epsilon^{-p}+(\tilde\alpha+\epsilon^{-1})E(0)^{(p-1)/2}\Big)E(s).\end{aligned} \tag{4.84}$$

Therefore we conclude that

$$\left(\frac{2\beta}{\alpha}-(\tilde\alpha+2)\epsilon c\right)\int_s^T E^{(p+1)/2} \le c\Big(\tilde\alpha\epsilon^{(1-p)/2}+\epsilon^{-p}+(\tilde\alpha+\epsilon^{-1})E(0)^{(p-1)/2}\Big)E(s). \tag{4.85}$$

If we take ϵ as $\epsilon \in (0, \frac{2\beta}{(3\alpha+1)c})$; then (4.67) follows. This conclude the proof.

Chapter 5

Conclusion and Ongoing Research

In this thesis, we studied the initial boundary value problem governed by the hyperbolic Kirchhoff-Carrier equation with some local damping terms, which, written as an abstract Cauchy problem in a suitable Banach space, takes the form

$$\begin{cases} u''(t) + M(\left\|A^{1/2}u\right\|^2)Au(t) + g(u') = 0 \quad \text{in} \quad [0,\infty), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{5.1}$$

For the linear damping case $g(u') = \delta u'$ $(\delta > 0)$, we first discussed in Chapter 2 results of local/global existence and asymptotic behaviour using the classic approach of combining the Galerkin approximation method with some a priori estimates. We also establish the stability as $t \to \infty$, the energy $E(t) \to 0$, and hence $\|u(t)\|_W + \|u'(t)\|_H \to 0$ as $t \to \infty$.

Still in the linear damping case, we followed Ghisi-Gobbino [54] by showing in Chapter 3 that a global existence result to (5.1) can still be obtained as a result of combining the Schauder fixed point theorem with some suitable differential inequalities, under an assumption of smallnesss of the initial data (u_0, u_1). Precisely, we determine that there exist a unique global solution provided that $(u_0, u_1) \in D(A) \cap D(A^{1/2})$ satisfy a suitable smallness assumption (see Theorem 3.3) and the non-degeneracy condition $M(\|A^{1/2}u_0\|^2) > 0$. We observe that in general case the solution to (5.1) may not decay to zero as $t \to +\infty$. However, as $t \to +\infty$ we show that $(u(t), u'(t), u''(t)) \to (u_\infty, 0, 0)$ in $D(A) \times D(A^{1/2}) \times H$ and that $\|A^{1/2}u_\infty\| \times M(\|A^{1/2}u_\infty\|^2) = 0$ (see. Theorem 3.4). In particular, if the operator A is coercive and $M(r) > 0$ for every $r > 0$, then necessarily $u_\infty = 0$.

For the nonlinear local damping case, we followed Kim et al. [72] to obtain in Chapter 4 a result of global existence for (5.1) under an assumption of smallness of the initial data (u_0, u_1) together with suitable growth assumptions on the nonlinear damping function g, using also a combination of Galerkin approximation and some a priori estimates as in Chapter 2. The main challenge here is as one could expect on the passing to the limit in the nonlinear local damping part in the approximate problem. Also a decay energy estimate is obtained.

A natural question that arises from the study above is whether one can still obtain the global existence result in Chapter 4 using rather the approach in Chapter 3 based on the combination of the Schauder fixed point theorem with some suitable differential inequalities. Considering the problem with the function $g(s) := \mu|s|^{m-1}s$ $(m > 1)$, following Chapter 3. Precisely, we consider the Cauchy problem

$$\begin{cases} u''(t) + M(\left\|A^{1/2}u\right\|^2)Au(t) + \mu|u'|^{m-1}u' = 0 \quad \text{in} \quad [0,\infty), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{5.2}$$

Following Chapter 3, this will require a global existence result for the Cauchy problem for the wave equation with a variable coefficient $a(t)$

$$\begin{cases} u''(t) + a(t)Au(t) + \mu|u'|^{m-1}u' = 0 \quad \text{in} \quad [0,\infty), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{5.3}$$

This problem has indeed a global solution as a result of Galerkin approximation method and a suitable a priori estimates (see e.g., Lions [84]). As an ongoing research, we are currently busy with adapting in this case, the remaining steps in Chapter 3 for the global existence result for (5.2).

Another proposal is to consider models with strong damping, i.e., we replace $g(u')$ with $g(\Delta u_t)$ or a combination of both. Such models are extensively studied for classical wave equations (e.g. [120, 52, 101]). For instance, the following model for strongly damped wave equation with nonlinear damping and some source terms was proposed in [120]

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \mu \, |u_t|^{m-1} u_t = |u|^{p-1} u & x \in \Omega, \ t \geq 0, \\ u(x,t) = 0, & x \in \partial\Omega, \ t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x) = u_1(x), & x \in \Omega \end{cases} \tag{5.4}$$

where $\omega \geq 0, \ \mu \geq 0, \ m \geq 1, \ p > 1$. The author established results of global existence, blow-up and exponential decay. Our next task is to see whether one can extend such model to the Kirchhoff type equation. Precisely, we intend to study the following initial boundary value problem

$$\begin{cases} u_{tt} - M(\|\nabla u\|_{L^2}^2) \Delta u - \omega \Delta u_t + \mu \, |u_t|^{m-1} u_t = |u|^{p-1} u & x \in \Omega, \ t \geq 0, \\ u(x,t) = 0, & x \in \partial\Omega, \ t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x) = u_1(x), & x \in \Omega. \end{cases} \tag{5.5}$$

Other types of models under consideration are the ones involving the p-Laplacian operator with the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ or higher other models with bi-Laplacian operator Δ^2 under the Navier boundary conditions $u = \Delta u = 0$ on $\partial\Omega$.

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