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Regular and Generalized Regular Rings

by

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INTRODUCTION

During his study of continuous geometries, J. von Neumann found that any complemented modular lattice satisfying certain mild conditions arises from a specific type of ring, which he termed a regular ring. In the first place, these rings turn out to be a generalization of artinian semisimple rings, and subsequent studies have shown that the concepts of regularity and some of its weaker and stronger forms are characterized by specific ideal structures which give these rings an important place in the general theory.

In the first three chapters of this thesis (and in section one of chapter four), many of the known results on regularity and some of its generalizations are collected and arranged. The results are not always presented in chronological order, but are so organised as to show some development of and relationships amongst the various properties of regular and generalized regular rings. Some results which are well-known or less relevant to this development have been stated without proof. Occasionally, proofs have been supplied for results which seem only to be stated in the literature, and at times the original proofs have been modified in the interests of directness or increased generality. In all cases, the proofs of known results carry a reference in brackets, ( ).

In the final two sections of the thesis, the author examines some of the properties of weakly regular rings, and defines generalizations of these rings parallel to some of those which have previously been defined for regularity. These new types of weak regularity are found to have several properties analogous to those known for the corresponding types of regularity, and in particular the ws-regular ring (defined on page 51) is characterized by an ideal structure of the same type as those found for strongly regular, regular, and weakly regular rings (as detailed on pages 53 and 54). Finally, on page 56 the author defines weak π-regularity analogously to the definition of π-regularity, and the diagram on page 59 shows how ws-regular and weakly π-regular rings fit into a pattern formed by the above mentioned known types of regularities.

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PRELIMINARIES AND NOTATION

Unless otherwise indicated, "ring" shall mean an associative ring, not necessarily with identity. Also, "ideal" shall mean two-sided ideal, and a ring R will be called artinian if it satisfies the descending chain condition on right ideals.

We shall use the concept of the Jacobson radical $J(R)$ of a ring R as defined in [13]. This possesses the following properties : if R is any ring, and I an ideal of R, then:

(1) $J(R/_{J(R)}) = \{0\}$.

(2) $J(I) = I \cap J(R)$

(3) $J(R_n) = (J(R))_n$, where R_n is the $n \times n$ matrix ring over R, for any integer $n \geq 1$.

$J(R)$ contains no non-zero idempotents, and in the case where R has an identity, it coincides with the intersection of the maximal right (or left) ideals of R.

An ideal I of a ring R is called prime if $AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$, for any ideals A,B of R. Further, I is called completely prime if $ab \in I$ implies that $a \in I$ or $b \in I$, for any $a,b \in R$. In particular, in a commutative ring, the concepts of prime and completely prime coincide.

The right (left) annihilator of a subset A of a ring R will be denoted by $(A)r$ $((A)\ell)$, and the two-sided annihilator of A by A'.

If M is a subset of R, then the complement of M in R will be denoted by $R-M$.

Elements of the direct product $\prod_{\alpha\in A} R_\alpha$ of rings R_α, where A is some indexing set, will be denoted by (a_α), $a_\alpha \in R_\alpha$ for all $\alpha \in A$.

Finally, all other standard notation may be found in (20).

CHAPTER 1

REGULAR RINGS

1.1 Regular Rings

The concept of regularity was introduced by J. von Neumann during his study of continuous geometries. The property is a generalization of invertibility and, as will be shown below, it is characterized by a structure which relates the regular ring to other well-known types.

Definition 1.1.1:

An element a of an associative ring R is said to be regular if there is an element x of R such that $a = axa$. A one or two-sided ideal of R is called regular if all its elements are regular, and in particular a ring R is regular if all its elements are regular.

Included among regular rings are division rings, Boolean rings, and as will be seen, the class of all semisimple artinian rings. In order to obtain more classes of regular rings, we require:

Lemma 1.1.2 (McCoy):

If a and x are elements of a ring R such that $axa-a$ (or $a-axa$) is regular, then a is regular.

Proof:

Let $y \in R$ be such that $axa-a = (axa-a)y(axa-a)$, then clearly $a = a(x-y-xayax + xay + yax)a$, and hence a is regular. Finally, we note that $z \in R$ is regular if and only if $-z$ is regular.

Theorem 1.1.3 (4):

If a ring R is regular, then so is R_n, the $n \times n$ matrix ring over R.

Proof:

We first prove the statement for $n = 2$. For any $r \in R$, we shall denote by r' an element of R such that $r = rr'r$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any element of R_2, and let $X = \begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix}$, then clearly $A - AXA = \begin{pmatrix} g & 0 \\ h & i \end{pmatrix}$ for some g, h, i of R. Let $B = A - AXA$ and $Y = \begin{pmatrix} g' & 0 \\ 0 & i' \end{pmatrix}$, then $C = B - BYB = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}$ for some $k \in R$. Finally, if we let $Z = \begin{pmatrix} 0 & k' \\ 0 & 0 \end{pmatrix}$, then $C - CZC = 0$, and so C is regular. By lemma 1.1.2, B and hence A is regular. Thus R_2 is regular.

Next, since R_{2^k} is isomorphic to $\left(R_{2^{k-1}}\right)_2$ for any positive integer k, it follows by induction that R_{2^k} is regular.

Finally, if n is any positive integer, choose any integer k such that $2^k \geq n$. Then R_n can be imbedded in R_{2^k} by letting $A \in R_n$ be mapped into A_1 in R_{2^k}, where A_1 has A in the upper left-hand corner, and zeros elsewhere. Since R_{2^k} is regular, there is an $X = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$ of R_{2^k} such that $A_1 = A_1 X A_1$, where $B \in R_n$ and C, D, E

are some matrices over R. It is then clear that $A = ABA$, and hence R_n is regular.

A further class of regular rings is obtained from:

Theorem 1.1.4 (17):

The ring $\mathrm{Hom}_D(V,V)$ of all linear endomorphisms of a vector space V over a division ring D is a regular ring.

Proof:

Let A be a linear endomorphism of V, and let $\{v_\alpha\}$, $\alpha \in I$, be a basis for the range space of A, where I is some indexing set. Choose $u_\alpha \in V$ such that $A(u_\alpha) = v_\alpha$ for all $\alpha \in I$. Then $\{u_\alpha\}$ together with the null space of A span V.

Since the v_α's are linearly independent, we can extend $\{v_\alpha\}$ to a basis $\{v_\alpha\} \cup \{v_\beta\}$ of V, where $\beta \in I'$, some other indexing set. Thus we can define $X \in \mathrm{Hom}_D(V,V)$ by $X(v_\alpha) = u_\alpha$ for $\alpha \in I$, and $X(v_\beta)$ arbitrarily for $\beta \in I'$. Then clearly $A = AXA$, and so $\mathrm{Hom}_D(V,V)$ is regular.

It is also clear that a direct product of regular rings is regular, and that a homomorphic image of a regular ring is regular. As a partial converse to the latter statement, we have:

Theorem 1.1.5 (17):

If I is an ideal of a ring R, then R is regular if and only if both I and $^R/I$ are regular.

Proof:

If R is regular, then by the above remark $^R/I$ is regular. Further, if $a \in I$, then there is an x in R such that $a = axa$. Thus $a = a.xax.a$, where $xax \in I$, and so I is a regular ring.

Conversely, suppose I and $^R/I$ are regular, and let $a \in R$. Then there is an x in R such that $a+I = (a+I)(x+I)(a+I)$, and so $axa - a \in I$. Thus $axa-a$, and hence a, is regular.

1.2 Ideal Structure and Characterizations of a Regular Ring with Identity.

We begin by examining right ideals in a regular ring with identity. Unless otherwise indicated, "right" may be replaced by "left" in all statements and theorems in this section.

The following theorem also provides a first characterization:

Theorem 1.2.1 (29):

A ring R with identity is regular if and only if every principal right ideal of R is generated by an idempotent.

Proof:

Let R be regular and let $a \in R$. Then there is an x in R such that $a = axa$. Thus $(ax)^2 = ax$, and ax is idempotent. Further, $aR = axR$, for if $y \in aR$, then $y = as$ for some $s \in R$. It follows that $y = axas \in axR$, and since always $axR \subseteq aR$, we have $aR = axR$.

Conversely, suppose every principal right ideal of R is generated by an idempotent. Let $a \in R$, then there is an idempotent e of R such that $aR = eR$. Thus $a = es$ and $e = ar$ for some $r,s \in R$. It follows that $ea = es = a$

and $a = ara$, and so R is regular.

We make use of the following general lemma:

Lemma 1.2.2 (29):

If A and B are right ideals of a ring R with identity, then R is a direct sum of A and B if and only if $A = eR$ and $B = (1-e)R$ for some idempotent e.

We now find another characterization:

Theorem 1.2.3 (29):

A ring R with identity is regular if and only if for each $a \in R$, there is a right ideal B of R such that $R = aR \oplus B$. Further, aR and B are necessarily of the form eR and $(1-e)R$ for some idempotent e of R.

Proof:

Suppose R is regular and let $a \in R$. Then there is an idempotent e in R such that $aR = eR$ by theorem 1.2.1. By the lemma, we have $R = eR \oplus (1-e)R$, and so $(1-e)R$ is the required right ideal B.

Conversely, suppose that for each $a \in R$ there is a right ideal B of R such that $R = aR \oplus B$. Then again by the lemma, $aR = eR$ and $B = (1-e)R$ for some idempotent e. Thus every principal right ideal is generated by an idempotent, and so R is regular by theorem 1.2.1.

In the next theorem, some relationships between principal right and left ideals are proved:

Theorem 1.2.4 (29):

If A is a principal right ideal of a regular ring R with identity, then:

(1) There exists a principal left ideal B of R such that $A = (B)r$.

(2) $A = ((A)\ell)r$.

(3) $(A)\ell$ is a principal left ideal of R.

Proof:

(1) There is an idempotent e of R such that $A = eR$. Thus if $x \in A$, then $x = es$ for some $s \in R$, and so $ex = x$. It follows that $z(1-e)x = 0$ for all $z \in R$, and so $x \in (R(1-e))r$.
Conversely, if $x \in (R(1-e))r$, then in particular $1(1-e)x = 0$, and so $x \in A$. Thus $R(1-e)$ is the required ideal B.

(2) For any left ideal L of R it is true that $(L)r = (((L)r)\ell)r$, and in particular, by (1), $A = (B)r = (((B)r)\ell)r = ((A)\ell)r$.

(3) Again, by (1), $A = (B)r$ for some principal left ideal B of R. Applying the left-right analogue of (2) to B, we obtain $B = ((B)r)\ell = (A)\ell$, and so $(A)\ell$ is a principal left ideal.

As an immediate consequence of this theorem, we have:

Corollary 1.2.5 (29):

If R is a regular ring with identity, then there exists a one-to-one correspondence between the set of all principal right ideals of R and the set of all principal left ideals; the correspondence being given by $A \longrightarrow (A)\ell$, where A is any principal right ideal.

Using the above results, we now prove some theorems on principal ideals:

Theorem 1.2.6 (23):

If R is a regular ring with identity, then every finitely generated right ideal is principal.

Proof:

It is sufficient to prove the theorem for a right ideal $aR + bR$ generated by elements $a,b \in R$.

Now there is an idempotent e in R such that $aR = eR$, and clearly $bR \subseteq ebR + (1-e)bR$. Thus $aR + bR = eR + bR \subseteq eR + ebR + (1-e)bR \subseteq eR + (1-e)bR$ and since $eR + (1-e)bR \subseteq eR + bR$, we have $aR + bR = eR + (1-e)bR$. By theorem 1.2.1, $(1-e)bR = fR$ for some idempotent $f = (1-e)bx$, where $x \in R$. Thus $ef = (e-e)bx = 0$.

If we let $g = f(1-e)$, then we have the following equalities:

(1) $gf = f(1-e)f = f(f-ef) = f^2 = f$

(2) $g^2 = g.g = gf(1-e) = f(1-e) = g$, and

(3) $eg = ef(1-e) = 0 = f(1-e)e = ge$.

From the above we see that $g \in fR$ and $f \in gR$, and so $fR = gR$. Thus $aR + bR = eR + gR$, and we now show that $eR + gR = (e+g)R$. Let $r,r' \in R$, then by (2) and (3) we have $er + gr' = (e+g)(er+gr')$, and so $eR + gR \subseteq (e+g)R$. The converse inclusion being clear, the theorem is now proved.

We also have

Theorem 1.2.7 [23]:

In a regular ring R with identity, the intersection of finitely many principal right ideals is again principal.

Proof:

By theorem 1.2.1 and induction, it is sufficient to prove the theorem for principal right ideals eR and fR generated by idempotents $e,f \in R$. Now $eR \cap fR = \{R(1-e) + R(1-f)\}r$, for if $x \in eR \cap fR$, then $x = er_1 = fr_2$ for some $r_1, r_2 \in R$. Thus $(s_1(1-e) + s_2(1-f))x = s_1(1-e)er_1 + s_2(1-f)fr_2 = 0$ for any $s_1, s_2 \in R$.

Conversely, if $x \in \{R(1-e) + R(1-f)\}r$, then in particular $(1(1-e) + 0(1-f))x = 0$, i.e. $x \in eR$; and similarly $x \in fR$. Then by theorem 1.2.6, $eR \cap fR = \{R(1-e) + R(1-f)\}r = \{Rg\}r$ for some idempotent g in R, and by theorem 1.2.4 part (3), $\{Rg\}r$ is a principal right ideal of R.

Next, we turn to maximal, prime, and completely prime ideals of a regular ring. We first require:

Definition 1.2.8:

An m-system in a ring R is a subset M of R such that if $r,s \in M$, then there exists an $x \in R$ with $rxs \in M$.

It is known that in a ring R, any ideal not intersecting an m-system M, is contained in a prime ideal not intersecting M.

We may now prove:

Theorem 1.2.9 (31):

If R is a regular ring, then:

(1) Each ideal of R is the intersection of the prime ideals containing it.

(2) Each maximal ideal is prime.

(3) If R has an identity, then each completely prime ideal is maximal.

Proof:

(1) Let I be an ideal of R, and let P_α, $\alpha \in A$, be the family of all prime ideals of R containing I — this is non-empty since R itself is such a prime ideal. If $\bigcap_{\alpha \in A} P_\alpha$ strictly contains I, then there is an a belonging to each P_α such that $a \notin I$. Now there is an x in R such that $a = axa$, so $\{a\}$ is an m-system. Thus I is contained in a prime ideal P_β for some $\beta \in A$ such that $a \notin P_\beta$. This is a contradiction, and so $\bigcap_{\alpha \in A} P_\alpha = I$.

(2) If M is a maximal ideal of R and is not prime, then the only prime ideal containing it is R, and so by (1), $M = R$. This contradiction shows that M is prime.

(3) Let P be a completely prime ideal of R. Then $^R/P$ has no zero-divisors, for if $(r+P)(s+P) = P$, for any $r,s \in R$ then $rs \in P$. Thus $r \in P$ or $s \in P$, so either $r+P = P$ or $s+P = P$.

Now $^R/P$ is a regular ring with identity and no zero-divisors, and so is clearly a division ring. It follows that P must be a maximal ideal.

We note that the presence of an identity is only required in part (3) of the above theorem.

In order to obtain some further characterizations, we require:

Definition 1.2.10:

If A and L are subsets of a ring R, then we define $A.L = \{ \sum_{i=1}^{n} a_i \ell_i : a_i \in A, \ell_i \in L,$ n some positive integer$\}$.

We now come to a result which is not right-left symmetric:

Theorem 1.2.11 [27]:

The following conditions on a ring R with identity are equivalent:

(1) R is regular.

(2) $A.L = A \cap L$ for any right ideal A and any left ideal L of R.

(3) $aR.Rb = aR \cap Rb$ for any $a,b \in R$.

(4) $aR.Ra = aR \cap Ra$ for any $a \in R$.

Proof:

(1) implies (2): Clearly $A.L \subseteq A \cap L$. If $a \in A \cap L$, then there is an x in R such that $a = a.xa$, where $xa \in L$ and $a \in A$. Thus $a \in A.L$.

That (2) implies (3) and (3) implies (4) is immediate.

(4) implies (1) : Let $a \in R$, then $a \in aR \cap Ra = aR.Ra$. Thus $a = \sum_{i=1}^{n} ar_i s_i a$ for some $r_i, s_i \in R$ and some positive integer n. It follows that $a = a\left(\sum_{i=1}^{n} r_i s_i\right)a$, and R is regular.

We remark that if "right" and "left" are interchanged in condition (2) of the above theorem, then we obtain a stronger form of regularity, as will be shown in the next chapter.

Our next characterization is:

Theorem 1.2.12 [21]:

Let R be a ring with identity, then the following statements are equivalent:

(1) R is regular.

(2) For every multiplicatively closed subgroup M of R, $MRM \subseteq M$ implies that $MRM = M$.

(3) For every multiplicatively closed subgroup M of R such that $MR \cap RM \subseteq M$, we have $MRM = M$.

Proof:

(1) implies (2) : Let M be a multiplicatively closed subgroup of R such that $MRM \subseteq M$. If $a \in M$, then there is an x in R such that $a = axa$, and so $a \in MRM$. Thus $MRM = M$.

(2) implies (3) : Let M be a multiplicatively closed subgroup of R such that $MR \cap RM \subseteq M$, then clearly $MRM \subseteq M$, and so $MRM = M$.

(3) implies (1) : Let $a \in R$ and let $M = aR \cap Ra$, then clearly M is a subgroup of R; and if $x, y \in M$, then $x = ar$ and $y = sa$ for some $r, s \in R$. Thus $xy = arsa \in aR \cap Ra$, and M is multiplicatively closed. Further, $MR \cap RM \subseteq M$, since if $y \in MR$, then $y = \sum_{i=1}^{n} w_i s_i$, where $w_i \in M$ and $s_i \in R$ for all $i = 1, 2, \ldots n$. It follows that $y \in aR$, and similarly if $y \in RM$ as well, then $y \in Ra$. By assumption, therefore, we have

$M = MRM$. Finally, $a \in M = MRM \subseteq aR.R.Ra$, and so a is regular.

Our final characterization is due to Ikeda and Nakayama (see [12]):

Theorem 1.2.13:

A ring R with identity is regular if and only if it satisfies the following condition : for every left R-homomorphism f of a finitely generated left ideal L of R into a left quotient module ${}^{R}/I$, where I is a left ideal of R, there exists an $x \in R$ such that $f(a) = ax+I$, for all $a \in L$.

Proof:

Suppose the condition holds, and let $a \in R$. Define $f : Ra \longrightarrow R/_{(a)\ell}$ by $f(ya) = y + (a)\ell$ for any $y \in R$. Then clearly f is a left R-homomorphism, and so there is an $x \in R$ such that $f(ya) = yax + (a)\ell$. Thus $y + (a)\ell = yax + (a)\ell$, and hence $ya = yaxa$ for all $y \in R$. In particular if $y = 1$, then a is regular.

Conversely if R is regular, then every finitely generated right ideal is principal (theorem 1.2.6). Let $a \in R$, then aR is generated by an idempotent e, say. Then clearly $e = ab$ for some $b \in R$ and $(a)\ell = (e)\ell$. Now let I be any left ideal of R, and let $f : Ra \longrightarrow {}^{R}/I$ be a left R-homomorphism. Further, let $f(a) = a'+I$, where $a' \in R$. Then for any $y \in R$, $0 = f(0) = f((y-yab)e) = f((y-yab)a) = (y-yab)f(a) = (y-yab)(a'+I)$, and thus $f(ya) = (ya)(ba') + I$, and the condition is proved.

1.3 The Wedderburn-Artin Theorem

In this section we relate the property of regularity to semisimplicity and the descending chain condition.

Theorem 1.3.1 (17):

A regular ring R is semisimple.

Proof:

Let $a \in J(R)$, the Jacobson radical of R. Then there exists an x in R such that $a = axa$, and ax is an idempotent in $J(R)$. Thus ax, and hence a, is zero.

Since a semisimple ring R is a subdirect sum of primitive rings which are homomorphic images of R, we have:

Theorem 1.3.2:

A regular ring R is a subdirect sum of regular primitive rings. In particular, since a commutative primitive ring is a field, a commutative regular ring is a subdirect sum of fields.

One form of the Wedderburn-Artin theorem states that a semisimple artinian ring is a finite direct sum of finite-order matrix rings over division rings. By theorem 1.1.3, these rings are regular, and hence a semisimple artinian ring is a regular ring with identity. It follows that semisimplicity is a generalization of regularity, which in turn is a generalization of artinian semisimplicity.

In particular, a finite regular ring is a direct sum of matrix rings over fields.

Finally, we note that since a regular ring is semisimple, it follows that every ideal in a regular ring with identity is the intersection of all maximal right (or left) ideals containing it.

1.4 The Maximal Regular Ideal of a Ring

In this section we prove that an arbitrary ring R possesses a unique maximal regular ideal $M(R)$, satisfying certain "radical-like" properties. Moreover, we show that if R is artinian, then it is a direct sum of $M(R)$ and $M(R)'$, where $M(R)'$ is the two-sided annihilator of $M(R)$.

We first show the existence of a unique maximal regular ideal:

Theorem 1.4.1 (4):

If R is an arbitrary ring, then the set $M(R)$ of all elements a of R such that (a) is a regular ideal, is the unique maximal regular ideal of R.

Proof:

If $a \in M(R)$, and $r \in R$, then $(ar) \subseteq (a)$, and so (ar) is regular. Thus $ar \in M(R)$, and similarly $ra \in M(R)$. If $z, w \in M(R)$, and $a \in (z-w)$, then $a = u-v$, where $u \in (z)$ and $v \in (w)$. Since (z) is regular, there is an r in R such that $u = uru$. Thus $ara-a = (u-v)r(u-v) - (u-v) = v - urv - vru + vrv$, and this is regular since $v \in (w)$. Thus by lemma 1.1.2, a is regular, and so $z-w \in M(R)$.

Finally, if I is a regular ideal of R, and $a \in I$, then $(a) \subseteq I$, and so $a \in M(R)$. Thus $I \subseteq M(R)$, and clearly $M(R)$ is the unique maximal regular ideal of R containing all regular ideals.

We now proceed to prove some properties of $M(R)$:

Theorem 1.4.2 (4):

For any ring R,

(1) $M\left(R/M(R)\right) = \{0\}$.

(2) If I is an ideal of R, then $M(I) = I \cap M(R)$.

(3) $M(R_n) = (M(R))_n$, where R_n is the $n \times n$ matrix ring over R.

Proof:

(1) Let $b+M(R) \in M\left(R/M(R)\right)$, where $b \in R$, and let $a \in (b)$. Then $a+M(R)$ is an element of the ideal generated by $b+M(R)$, which is regular. Hence there is an $x \in R$ such that $a-axa \in M(R)$. By lemma 1.1.2 a is regular, and so (b) is a regular ideal. It follows that $b \in M(R)$ and $M\left(R/M(R)\right) = \{0\}$.

(2) Let $b \in M(I)$, so that the ideal $(b)_I$ generated by b in I is regular. We show that $(b)_I = (b)$, the ideal generated by b in R. Since b is regular, there is an x in I such that $b = bxb$. Let $c \in (b)$, then for some $r,s,r_i,s_i \in R$ and some positive integer n, we have $c = nb+rb+bs+\sum_{i=1}^{n} r_i bs_i = nb+(rbx)b+b(xbs)+\sum_{i=1}^{n}(r_i bx)b(xbs_i)$, which is an element of $(b)_I$. Since clearly $(b)_I \subseteq (b)$, we have $(b)_I = (b)$, and (b) is regular. It follows that $M(I) \subseteq I \cap M(R)$, and the reverse inclusion being clear, we have that $M(I) = I \cap M(R)$.

(3) By theorem 1.1.3, $(M(R))_n$ is a regular ideal of R_n, and so $(M(R))_n \subseteq M(R_n)$. Conversely, let A be a matrix in $M(R_n)$, and let a_{ij} be an entry of A for

some $1 \leq i,j \leq n$. Since (A) is a regular ideal, there is an element X of R_n such that $A = AXA = AX.A.XA$, and so $a_{ij} = \sum_{k,\ell} t_{ik}a_{k\ell}s_{\ell j}$ for suitable elements $t_{ik}, s_{\ell j}$ of R, where $1 \leq i,k,\ell,j \leq n$. But it is clear that there is a matrix of (A) with $t_{ik}a_{k\ell}s_{\ell j}$ in the $(1,1)$th position and zeros elsewhere, and hence an element of (A) with a_{ij} in the $(1,1)$th position and zeros elsewhere. Thus if $b \in (a_{ij})$, it follows that there is an element B of (A) with b in the $(1,1)$th position and zeros elsewhere. Furthermore, we have $BYB = B$ for some Y in R_n, since (A) is regular. This implies that $by_{11}b = b$, and b is regular. Finally, it follows that $a_{ij} \in M(R)$, and so $M(R_n) \subseteq (M(R))_n$.

The following results show some relationships between the Jacobson radical and the maximal regular ideal of a ring:

Theorem 1.4.3 (4):

If R is any ring, with maximal regular ideal M and Jacobson radical J, then:

(1) $M \cap J = \{0\}$.

(2) $J \subseteq M'$ and $M \subseteq J'$.

(3) $M \cap M' = \{0\}$.

(4) J is the radical of the ring M', and M is the maximal regular ideal of the ring J'.

Proof:

(1) Let $a \in M \cap J$, then there is an $x \in M$ such that $a = axa$. Thus ax is an idempotent in J, and so $ax = 0$. Finally, $a = ax.a = 0$.

(2) Clearly $MJ \cup JM \subseteq J \cap M = \{0\}$, and so $MJ = JM = \{0\}$. It follows that $J \subseteq M'$ and $M \subseteq J'$.

(3) If $a \in M \cap M'$, then $a = axa$ for some x in M, and so $a = 0$.

(4) By the properties of the Jacobson radical mentioned in the introduction, we have $J(M') = M' \cap J$. By (2) above, we have $M' \cap J = J$, and so the radical of M' is J. Further, by the properties of the maximal regular ideal, we have $M(J') = J' \cap M$. By (2) again, we have $J' \cap M = M$.

Following (4), we now proceed to a decomposition theorem for artinian rings. We require:

Lemma 1.4.4:

If R is a ring with an ideal B such that B has an identity element, then $R = B \oplus B'$.

Lemma 1.4.5:

If a ring R is artinian, then so is $M(R)$, which thus has an identity.

Proof:

We need only show that a right ideal I of $M(R)$ is a right ideal of R. Let $a \in I$ and $r \in R$, then $ar \in M(R)$, and so there is a y in R such that $aryar = ar$. Since $ryar \in M(R)$, $a.ryar \in I$, i.e. $ar \in I$.

The above results lead immediately to:

Theorem 1.4.6:

If R is an artinian ring, then $R = M(R) \oplus M(R)'$.

1.5 Imbedding a Regular Ring as an Ideal in a Regular Ring with Identity.

It is well known that any associative ring can be imbedded as an ideal in a ring with identity. In this section we show that a regular ring can be imbedded as an ideal in a regular ring with identity.

We first define the centroid of an arbitrary ring R:

Definition 1.5.1:

Given a ring R, let $E(R)$ be the ring of additive endomorphisms of R, with multiplication in $E(R)$ being composition of functions. Then the centroid $C(R)$ of R is that subring of $E(R)$ whose elements commute with right and left multiplication by elements of R. Thus for any $f \in C(R)$, and any $r,s \in R$, we have $f(rs) = f(r)s = rf(s)$.

If $R^2 = R$ (and in particular if R is regular), then it is clear that $C(R)$ is a commutative ring with identity.

We proceed to the main theorem by a series of lemmas. For convenience, we shall denote the kernel and image of any $f \in C(R)$ by Ker f and Im f respectively.

Lemma 1.5.2 (10):

If R is a ring and $f \in C(R)$, then $R = \text{Ker } f \oplus \text{Im } f$ if and only if both the following conditions hold:

(1) $f^2(x) = 0$ implies that $f(x) = 0$ for any $x \in R$.

(2) For any $x \in R$, there exists an element $y \in R$ such that $f(x) = f^2(y)$.

Further, the "y" in (2) may be chosen in Im f, and is unique in Im f.

We use this lemma to prove:

Lemma 1.5.3 [10]:

If R is a ring, and $f \in C(R)$ is such that $R = \operatorname{Ker} f \oplus \operatorname{Im} f$, then there exists a $g \in C(R)$ such that $f = fgf$.

Proof:

Given any $x \in R$, there is a unique y in $\operatorname{Im} f$ such that $f(x) = f^2(y)$. Define $g : R \longrightarrow R$ by $g(x) = y$, then clearly g is an additive endomorphism of R. If $r \in R$, then $f(xr) = f(x)r = f^2(y)r = f^2(yr)$, and since $yr \in \operatorname{Im} f$, it follows that $g(xr) = yr = g(x)r$. Similarly $g(rx) = rg(x)$, and $g \in C(R)$.

Now for any $x \in R$, let $g(f(x)) = y$, where $y \in \operatorname{Im} f$. Then by the definition of g, $f(f(x)) = f^2(y)$, and so $f^2(x-y) = 0$, and thus by the previous lemma $f(x-y) = 0$, i.e. $f(x) = f(y) = f(g(f(x)))$. Thus $f = fgf$.

Using the above two lemmas, we can now prove:

Lemma 1.5.4 [10]:

If R is a regular ring, then its centroid $C(R)$ is a commutative regular ring with identity.

Proof:

It is clear that we need only show that each $f \in C(R)$ satisfies conditions (1) and (2) of lemma 1.5.2. Let $x \in R$, and suppose $f^2(x) = 0$. Then there is a y in R such that $f(x) = f(x)yf(x)$, so that $f(x) = f^2(x)yx = 0$. Further, $f(x) = f(x)yf(x) = f^2(xyx)$, and the lemma is proved.

We require one final lemma:

Lemma 1.5.5 (8):

If a is an element of a regular ring R, then there exists an idempotent e in R such that $a = ae = ea$.

Proof:

There is an x in R such that $a = axa$. Let $r = ax$, then r is idempotent and $ra = a$. Then there is a $y \in R$ such that $a-ar = (a-ar)y(a-ar)$. Again, $s = y(a-ar)$ is idempotent. Let $e = r+s-rs$, then clearly e is idempotent, and $a = ae = ea$.

Given an arbitrary regular ring R, we can now form the ring $\overline{R}$ as follows: Let $\overline{R}$ be the set of all ordered pairs (a,f), where $a \in R$ and $f \in C(R)$. $\overline{R}$ can be made into a ring by defining equality and addition componentwise, and multiplication by $(a,f)(b,g) = (ab+f(b) + g(a), gf)$, where $b \in R$ and $g \in C(R)$. Clearly $\overline{R}$ is a ring with identity $(0,1)$, and the mapping $a \longrightarrow (a,0)$, where $a \in R$, is a ring monomorphism of R into $\overline{R}$. Further, the image of R under this map is an ideal of $\overline{R}$.

To show that $\overline{R}$ is regular, we need to show that, for each $(a,f) \in \overline{R}$, there exists a $(b,g) \in \overline{R}$ such that $(a,f) = (a,f)(b,g)(a,f)$, i.e. (1): $f = fgf$ and (2): $aba + f(ab) + f(ba) + g(a^2) + f^2(b) + fg(a) + gf(a) = a$. Now (1) is clear since $C(R)$ is regular. By the above lemma, there is an idempotent $e \in R$ such that $a = ae = ea$. By the regularity of R there exists an $x \in R$ such that $(a+f(e))x(a+f(e)) = a + f(e)$. Let $b = exe - g(e)$, then clearly b satisfies (2), and $\overline{R}$ is regular. We thus have:

Theorem 1.5.6 (10):

An arbitrary regular ring R can be imbedded as an ideal in a regular ring $\overline{R}$ with identity.

CHAPTER 2

STRONGLY REGULAR RINGS AND UNIT-REGULAR RINGS

2.1 Strongly Regular Rings

Strongly regular rings were first introduced by Arens and Kaplansky in their study of topological representations of algebras, [2]. We shall, however, use the slightly different definitions of Azumaya (see [3]), which allow a rather more general approach to these rings.

Definition 2.1.1:

An element a of a ring R is called right (or left) regular if there is an x in R such that $a = a^2x$ (or $a = xa^2$). The element x is sometimes called a quasi-inverse of a. If a is both left and right regular, then a is called strongly regular. An ideal of R is said to be strongly regular (or right or left regular) if all its elements are strongly regular (or right or left regular).

The following theorem shows, in particular, that we need make no distinction between right, left, and strongly regular rings.

Theorem 2.1.2 [19]:

Let R be a right regular ring. Then:

(1) R has no non-zero nilpotent elements, and hence every idempotent of R is central.

(2) R is regular.

(3) R is left (whence strongly) regular.

Proof:

(1) Let $a \in R$, then there is an x in R such that $a = a^2x$. It follows by induction that $a = a^n x^{n-1}$ for any integer $n > 1$, so that if $a^m = 0$ for some $m > 1$, then $a = 0$. Further, in any ring without nilpotent elements, idempotents are central.

(2) Again, let $a \in R$, and let $x \in R$ be such that $a = a^2x$. Then $(a-axa)^2 = a^2 + axa^2xa - a^2xa - axa^2 = 0$, and so by (1) $a = axa$, and thus R is regular.

(3) For a and x as above, $(a-xa^2)^2 = a^2 + xa^2xa^2 - axa^2 - xa^3 = 0$, and so $a = xa^2$. Thus R is strongly regular.

In virtue of the above theorem, we need speak only of strongly regular rings.

Included among strongly regular rings are commutative regular rings and artinian rings without nilpotent elements. It is also clear that homomorphic images and direct products of strongly regular rings are again strongly regular.

In general, a strongly regular element of a ring R does not commute with all of its quasi-inverses, but we can show:

Theorem 2.1.3 (3):

Let R be a ring, and let $a \in R$ be a strongly regular element. Then there is a unique $z \in R$ such that $az = za$, $a^2z = a$, and $az^2 = z$. Further, z commutes with all elements of R which commute with a.

Proof:

Let $x, y \in R$ be such that $a = a^2x$ and $a = ya^2$. Then

(1) $ax = ya^2x = ya$, and so

(2) $ax^2 = yax = y^2a$.

From (1), we also have that

(3) $axa = ya^2 = a = a^2x = aya$.

Let $z = ax^2$, then it follows from (1), (2) and (3) that $az = a^2x.x = ayax = ax = ya = yaxa = za$, and $a^2z = a.az = a.za = a^2x.xa = axa = a$. Further, $az^2 = az.z = zaz = yaz = yax = ax^2 = z$.

Suppose now that z' is an element of R also satisfying $az' = z'a$, $a^2z' = a$, and $a(z')^2 = z'$. Then we may replace x by z' in (2), and also y by z (since $a = a^2z = za^2$). Then $z = az^2 = z^2a = a(z')^2 = z'$, and so z is unique.

Finally, let $c \in R$ be such that $ca = ac$. Then $zac = zca = zca^2z = za^2cz = acz = caz$, and so $zc = z^2ac = zcaz = zcza = zcaz = zac.z = c.azz = cz$, and the theorem is proved.

Definition 2.1.4: Let a be a nilpotent element of a ring R, and n the least positive integer such that $a^n = 0$. Then n is called the index of nilpotency of the element a.

If there is an upper bound to the indices of nilpotency of the nilpotent elements of a ring R, then R is said to be of bounded index, in which case the least upper bound of these indices is called the index of R.

We state without proof the following theorem, which gives a sufficient condition for a right regular element to be strongly regular:

Theorem 2.1.5 [3]:

Let R be a ring of bounded index. Then every right regular element of R is left (whence strongly) regular.

In the next theorem we mention some equivalent conditions for a regular ring R to be strongly regular:

Theorem 2.1.6 [31]:

If a ring R is regular, then the following statements are equivalent:

(1) R is strongly regular.

(2) R has no non-zero nilpotent elements.

(3) Every idempotent of R is central.

(4) Every right or left ideal is two-sided.

(5) Every ideal is the intersection of all completely prime ideals containing it.

(6) Every prime ideal is completely prime.

By theorem 1.2.9 and the above result, it follows that in a strongly regular ring with identity, each completely prime ideal is a maximal right ideal. In order to establish a partial converse to this, we require the following definition and series of lemmas involving completely and semi-completely prime ideals:

Definition 2.1.7:

Let R be a ring, and let $0 \neq x \in R$. Then $r(xR) =$

$= \{y \in R$: there is some $s \in R$ such that $xs \neq 0$ and $(xs)y = 0\}$. The complement in R of $r(xR)$ is clearly multiplicatively closed. Further, an ideal Q of R is said to be <u>semicompletely prime</u> if for any $x \in R$ and any positive integer n, $x^n \in Q$ implies that $x \in Q$.

<u>Lemma 2.1.8 (30)</u>:

Let R be a ring with no non-zero nilpotent elements, and let $0 \neq x \in R$. Then $(x)r$ is a two-sided semicompletely prime ideal of R, and $ab \in (x)r$ implies that $ba \in (x)r$, for any $a,b \in R$.

<u>Proof</u>:

Let $s \in R$ and $a \in (x)r$. Then $xa = 0$, and so $(ax)^2 = a.xa.x = 0$. It follows that $ax = 0$, and thus $(xsa)^2 = xs.ax.sa = 0$. Hence $xsa = 0$, and $sa \in (x)r$, which is thus a two-sided ideal.

Next, suppose $b \in R$ and $b^n \in (x)r$ for some positive integer n. If $n = 1$, then $b \in (x)r$ already, and if $n > 1$, then $0 = xb^n = xb^{n-2}b^2 = bxb^{n-2}b$. Thus $(bxb^{n-2})^2 = bxb^{n-2}bxb^{n-2} = 0$, and so $bxb^{n-2} = 0$. It follows that $xb^{n-1} = 0$, and continuing this way, we eventually have $xb = 0$. Thus $b \in (x)r$, which is semicompletely prime.

Finally, if $a,b \in R$ are such that $ab \in (x)r$, then $(ba)^2 = b.ab.a \in (x)r$, and so $ba \in (x)r$.

The following lemma is stated without proof:

<u>Lemma 2.1.9 (1)</u>:

Let Q be a semicompletely prime ideal of a ring R, then every m-system of R not intersecting Q is contained in a maximal m-system $\overline{M}$ of R such that $\overline{M}\cap Q = \phi$, and $R-\overline{M}$ is a completely prime ideal.

Using this result, we may now prove:

Lemma 2.1.10 (30):

Let R be a ring with identity and no non-zero nilpotent elements, and let $0\neq x\in R$. If P is a completely prime ideal, minimal with respect to the property of containing $(x)r$, then $P \subseteq r(xR)$.

Proof:

Let $M = \{(a_1m_1)(a_2m_2) \ldots (a_nm_n) : a_i\notin P, m_i\notin r(xR)$, and n some positive integer$\}$. Then clearly M is a multiplicatively closed m-system, and since $1\notin P$ and $1\notin r(xR)$, M contains the complements in R of P and $r(xR)$.

Next, we show that $M\cap(x)r = \phi$. Suppose $(a_1m_1)(a_2m_2) \ldots (a_{n-1}m_{n-1})(a_nm_n)\in M\cap(x)r$, for some $a_i\notin P$, $m_i\notin r(xR)$, $i = 1,2,\ldots n$. If $n = 1$, then $a_1m_1\in(x)r$, so $xa_1m_1 = 0$, and thus $xa_1 = 0$. It follows that $a_1\in(x)r \subseteq P$, a contradiction. If $n > 1$, let $(a_1m_1) \ldots (a_{n-1}m_{n-1}) = a$, say, then $xaa_nm_n = 0$, and thus $xaa_n = 0$. By lemma 2.1.8, $a_na\in(x)r$, i.e. $(a_na_1m_1)(a_2m_2) \ldots (a_{n-1}m_{n-1})\in(x)r$, where $a_na_1\notin P$ since P is completely prime. Continuing in this way, we will eventually have $a_2a_3 \ldots a_na_1m_1\in M\cap(x)r$, with $a_2a_3 \ldots a_na_1\notin P$ and $m_1\notin r(xR)$. By the reasoning in the case $n = 1$, this is a contradiction, and so $M\cap(x)r = \phi$.

Next, since a completely prime ideal is semicompletely prime, we have by the previous lemma a maximal m-system $\overline{M}$ such that $M \subseteq \overline{M}$, $\overline{M}\cap(x)r = \phi$, and the complement of $\overline{M}$ in

R, say $\overline{P}$, is a completely prime ideal. Then clearly $(x)r \subseteq \overline{P} \subseteq P$, and $\overline{P} \subseteq r(xR)$. By the minimality of P, either $\overline{P} = P$ or $\overline{P} = (x)r$. If $\overline{P} = P$, then $P \subseteq r(xR)$, while if $\overline{P} = (x)r$, then $(x)r$ is semicompletely prime, and so $(x)r = P = \overline{P} \subseteq r(xR)$.

As a final lemma, we require:

Lemma 2.1.11 (30):

Let R be a ring with identity and no non-zero nilpotent elements. If every completely prime ideal of R is a maximal right ideal, then every non-unit of R is contained in a completely prime ideal.

Proof:

If x is not a unit in R, then $xR \neq R$, otherwise there is a y in R such that $xy = 1$. Thus $(1-yx)^2 = 1-yx$, and since idempotents are central in R, we have $(1-yx)x = x(1-yx)$. It follows that $yx^2 = x$, and so $yx = yx.xy = xy = 1$, a contradiction.

Let $M = \{xr+1 : r \in R\}$, then clearly M is a multiplicatively closed m-system and $0 \notin M$. Since R has no non-zero nilpotent elements, $\{0\}$ is a semicompletely prime ideal, and so by lemma 2.1.9, there is a completely prime ideal P of R such that $P \cap M = \phi$. By assumption P is a maximal right ideal, and so if $xR \not\subseteq P$, then $xR+P = R$. Thus $xr+p = 1$ for some $r \in R$ and $p \in P$, and so $p = x(-r)+1 \in P \cap M$, a contradiction. It follows that $xR \subseteq P$, i.e. $x \in P$.

Let R be a ring with identity and no non-zero nilpotent elements, in which every completely prime ideal is a maximal right ideal. Then it is clear from lemmas 2.1.10 and 2.1.11 that any element $x \in R$ which is not a zero-divisor (i.e. $(x)r = \{0\}$) is a unit.

Using the above results we may now prove:

Theorem 2.1.12 (30):

Let R be a ring with identity and no non-zero nilpotent elements. If every completely prime ideal of R is a maximal right ideal, then R is strongly regular.

Proof:

Let $0 \neq x \in R$, then it is clear that $R/_{(x)r}$ is again a ring without non-zero nilpotent elements in which every completely prime ideal is a maximal right ideal.

Now $x + (x)r$ is not a zero divisor in $R/_{(x)r}$, for suppose $(x+(x)r)(a+(x)r) = 0$ for some $a \in R$. Then $xa \in (x)r$, and so $ax \in (x)r$ by lemma 2.1.8. Thus $(xa)^2 = 0$, and so $xa = 0$, and $a \in (x)r$. By the above remark, it follows that there is an $a \in R$ such that $(x+(x)r)(a+(x)r) = 1 + (x)r$, i.e. $xa-1 \in (x)r$. Thus $x(xa-1) = 0$, and x is strongly regular.

As a consequence of the above theorem, we have

Corollary 2.1.13(30):

Let R be a ring with identity and no non-zero nilpotent elements. If $^R/P$ is regular for every prime ideal P in R, then R is strongly regular.

Next, we turn to some characterizations of strongly regular rings with identity.

In theorem 1.2.1 we showed that a ring with identity is regular if and only if every principal right (left) ideal is generated by an idempotent, and we now have:

Theorem 2.1.14:

A ring R with identity is strongly regular if and only if every principal right (left) ideal is generated by a central idempotent.

Proof:

If R is strongly regular, then R is regular and all idempotents are central, and thus every principal right ideal is generated by a central idempotent.

Conversely, suppose every principal right ideal is generated by a central idempotent, and let $a \in R$. Then there is a central idempotent e in R such that $aR = eR$, and $a = ey$ and $e = ax$ for some $x, y \in R$. Thus $a = ye$, $ae = ye = a$, and so $a.ax = a$. Thus R is strongly regular.

The following characterization is not right-left symmetric, as may be seen by comparison with theorem 1.2.11:

Theorem 2.1.15 (18):

A ring R with identity is strongly regular if and only if $L.A = L \cap A$ for every left ideal L and right ideal A of R.

Proof:

Suppose R is strongly regular, then since R has only two-sided ideals, it follows that $L.A \subseteq L\cap A$ for any left ideal L and right ideal A. Further, if $a\in L\cap A$, then there is an $x\in R$ such that $a = ax.a\in L.A$. Thus $L.A = L\cap A$.

Conversely, suppose $L.A = L\cap A$ for any left ideal L and right ideal A of R. Then in particular, $L.R = L\cap R = L$, and so L is also a right ideal. Similarly, every right ideal is also a left ideal. By theorems 1.2.11 and 2.1.6, it follows that R is strongly regular.

Using the above result, we may prove the following characterization:

Theorem 2.1.16 (24):

A ring R with identity is strongly regular if and only if $L\cap A = LRA$ for any left ideal L and right ideal A of R.

Proof:

Suppose R is strongly regular, and that L and A are left and right ideals respectively of R. By theorem 2.1.6, L and A are two-sided ideals. If $a\in L\cap A$, then there is an $x\in R$ such that $a = axa$, so $a\in LRA$. Thus $L\cap A \subseteq LRA \subseteq LA$, and by the previous theorem $LA = L\cap A$. Thus $L\cap A = LRA$.

Conversely, suppose $L\cap A = LRA$ for every left ideal L and right ideal A of R. Then, in particular, $L = L\cap R = LRR = LR$, and so every left ideal of R is two-sided. Similarly, every right ideal is two-sided, and so $L\cap A = LRA \subseteq LA \subseteq L\cap A$. Thus $L\cap A = L.A$, and by theorem 2.1.15, R is strongly regular.

We have seen (theorem 1.3.2) that a regular ring is a subdirect sum of regular primitive rings. A stronger result for strongly regular rings is obtained as a consequence of our last characterization.

Definition 2.1.17:

A ring R is subdirectly irreducible if the intersection of all its non-zero ideals is a non-zero ideal.

Lemma 2.1.18 (7):

If R is a subdirectly irreducible ring without non-zero nilpotent elements, then the only idempotents in R are the zero and the identity (in case R has an identity).

Proof:

Suppose e is an idempotent in R, $e \neq 0$ (and $e \neq 1$, if R has an identity). We show that R is not subdirectly irreducible. Since e is central, $A = \{x-ex : x \in R\}$ is a two-sided ideal in R. A is non-zero, otherwise e would be an identity. Let $B = \{ex : x \in R\}$, then B is also a non-zero two-sided ideal.

Now if $y \in A \cap B$, then $y = x-ex = er$ for some $x,r \in R$. Thus $ye = ex-ex = e^2r = er = y$, i.e. $y = 0$, a contradiction.

Theorem 2.1.19 (7):

A ring R is strongly regular if and only if it is regular and a subdirect sum of division rings.

Proof:

If R is a subdirect sum of division rings, then there is a monomorphism of R into their direct product, which clearly contains no non-zero nilpotent elements. Thus R contains no non-zero nilpotent elements, and if it is regular as well, then it is strongly regular.

Conversely, suppose R is strongly regular. It is well-known that R is a subdirect sum of subdirectly irreducible rings R_α, $\alpha \in A$. Being homomorphic images of R, each R_α has no non-zero nilpotent elements and is regular. If $0 \neq a_\alpha \in R_\alpha$, $\alpha \in A$, then there is an $x_\alpha \in R_\alpha$ such that $a_\alpha = a_\alpha x_\alpha a_\alpha$. Thus $a_\alpha x_\alpha$ is a non-zero idempotent, and so by lemma 2.1.18, it is the identity of R_α. Similarly $x_\alpha a_\alpha$ is the identity of R_α, and thus R_α is a division ring for all $\alpha \in A$.

We have already seen (section 1.4) that any ring R has a unique maximal regular ideal M(R), and we now have a corresponding result for strong regularity:

Theorem 2.1.20 (14):

If R is any ring, then the set N(R) of all elements a of R such that (a) is a strongly regular ideal, is the unique maximal strongly regular ideal of R.

It is clear that $N(R) \subseteq M(R)$ for any ring R, and that analogous to M(R), N(R) has the following properties (see (14)):

(1) $N(R/N(R)) = \{0\}$.

(2) If I is an ideal of R, then $N(I) = I \cap N(R)$.

The analogy, however, breaks down with matrix rings over a strongly regular ring. Whereas $M(R_n) = (M(R))_n$, we now have that $N(R_n) = \{0\}$ for all integers $n > 1$. Thus R_n is not strongly regular for any ring R and any integer $n > 1$.

Finally, in this section, we extend the proof of theorem 1.5.6 to show:

Proposition 2.1.21:

Any strongly regular ring R can be imbedded as an ideal in a strongly regular ring $\overline{R}$ with identity.

Proof:

Since R is strongly regular, it is also regular, and thus may be imbedded as an ideal in the regular ring $\overline{R}$ with identity discussed in section 2.5. We prove that $\overline{R}$ is strongly regular by showing that it has no non-zero nilpotent elements : let $(a,f) \in \overline{R}$, where $a \in R$, $f \in C(R)$. Then by induction, it follows for all positive integers n that $(a,f)^n = (a^n+f(b),f^n)$ for some $b \in R$. Thus if $(a,f)^n = (0,0)$, then $f^n = 0$ and $a^n + f(b) = 0$. Since $C(R)$ is strongly regular, $f = 0$, and so $a^n = 0$, whence $a = 0$ by the strong regularity of R.

2.2 Unit-Regular Rings

Unit-regular rings were first introduced by G. Ehrlich in [6] as a stronger form of regularity in rings with identity.

Definition 2.2.1:

An element a of a ring R with identity is said to be *unit-regular* if there is a unit $u \in R$ such that $a = aua$. A ring R is called unit-regular if all its elements are unit-regular.

A class of unit-regular rings is given by:

Theorem 2.2.2 [6]:

Every strongly regular ring R with identity is unit-regular.

Proof:

Let $a \in R$, and let $x \in R$ be such that $a = a^2x$ and $ax = xa$ is a central idempotent. Let $u = a^2x^3 - ax+1$, then clearly $u(a^3x^2-ax+1) = (a^3x^2-ax+1)u = 1$. Thus u is a unit, and $a = aua$. It follows that R is a unit-regular ring.

Further examples of unit-regular rings are given by:

Theorem 2.2.3 [6]:

An artinian semisimple ring is unit-regular.

Proof:

By the Wedderburn-Artin theorem, R is a finite direct sum of matrix rings over division rings. Since clearly a direct sum of unit-regular rings is again unit-regular, we need only prove that D_n, the $n \times n$ matrix ring over a division ring D, is unit-regular.

Now given any matrix $A \in D_n$, there exist invertible matrices P and Q in D_n such that PAQ is an idempotent (diagonal) matrix. Then $X = QP$ is a unit, and $A = AXA$. Thus D_n, and hence R, is unit-regular.

We remark that the matrix ring D_n over a division ring D is an example of an artinian semisimple ring which is not strongly regular.

Further, a regular ring with identity need not be unit-regular. The ring of all linear endomorphisms of an infinite-dimensional vector space over a division ring is an example of such (see [6]).

We now prove some characterizations of unit-regularity:

Theorem 2.2.4 [6,11]:

Let R be a ring with identity, and a an element of R. Then the following statements are equivalent:

(1) There is a unit u such that $a = aua$.

(2) There is a unit u such that au or ua is idempotent.

(3) There are units u,v in R such that uav is idempotent.

(4) There is a unit w and an idempotent e in R such that $a = we$ or $a = ew$.

Proof:

Clearly (1) implies (2); and since R has an identity, (2) implies (3). If (3) holds, then $(uav)(uav) = uav$. Thus $a = a(vu)a$, where vu is a unit, and so (1) holds.

Assume (1) again, then $a = u^{-1}(ua)$, where u^{-1} is a unit and ua is an idempotent, and so (4) holds.

Conversely, suppose (4) holds. Then $a = we$ for a unit w and an idempotent e. Then $w^{-1}a = e = e^2 = w^{-1}aw^{-1}a$, and so $a = aw^{-1}a$. Thus (1) holds.

We conclude this section by proving some properties of unit-regular rings:

Theorem 2.2.5 (6,11):

If R is a unit-regular ring, then:

(1) Every one-sided inverse of an element of R is two-sided and hence unique.

(2) For each $a,b \in R$, there exists $s,t \in R$, not both zero, such that $sa = tb$.

(3) If $1+1$ is a unit in R, then every element of R is the sum of two units.

Proof:

(1) Let $a \in R$, and suppose $b \in R$ is such that $ab = 1$. Now, there is a unit $u \in R$ such that $a = aua$, and so $au = au.ab = ab = 1$. Thus $ua = 1$, whence $b = u.ab = u.1 = u$ is the two-sided inverse of a. Similarly, $ba = 1$ implies that $ab = 1$.

(2) If both a and b have left inverses s and t respectively, then $sa = tb = 1$. If at least one of a and b has no left inverse, say a, then by theorem 2.2.4 $a = ue$ for some unit u and idempotent e in R. Then $(1-e)u^{-1}a = 0$, and for $s = (1-e)u^{-1}$ and $t = 0$, we have $sa = tb$. Further, $s \neq 0$, otherwise $e = 1$ and a is a unit.

(3) Let $a \in R$, and suppose $a = eu$ for some idempotent e and unit u of R. Then $e = 2^{-1}(2e-1) + 2^{-1}$ and $a = (2^{-1}(2e-1))u + 2^{-1}u$. Both summands are units, with their inverses being, respectively, $u^{-1}(2e-1)2$ and $2u^{-1}$.

CHAPTER 3

π- REGULAR RINGS

π-Regular rings were first introduced by N.McCoy in (22) as a generalization of regular rings. We also discuss analogous generalizations of right, left, and strongly regular rings.

Definition 3.1:

An element a of a ring R is called π-regular if some power of a is regular, i.e. if there is a positive integer n and an $x \in R$ such that $a^n = a^n x a^n$. Further, a is called right (left) π-regular if some power of a is right (left) regular, and a is called strongly π-regular if it is both right and left π-regular.

An ideal of a ring R is called (right, left, strongly) π-regular if every element of it has that property.

We remark immediately that homomorphic images and finite direct products of such rings have the same property; and the centres of such rings are again rings of the same type.

We first investigate some relationships between π-regular and strongly π-regular rings. We require:

Lemma 3.2 (3):

An element a of a ring R is right (left) π-regular if and only if there is some positive integer n and some $x \in R$ such that $a^{n+1}x = a^n$ $(a^n = xa^{n+1})$.

Proof:

Suppose $a \in R$ is right π-regular, then there is a $y \in R$ and a positive integer n such that $a^n = (a^n)^2 y$. Thus $a^n = a^{n+1}(a^{n-1}y)$, and $a^{n-1}y$ is the required x.

Conversely, if there is a positive integer n and an $x \in R$ such that $a^{n+1}x = a^n$, then clearly $a^n = a^{n+1}x = a^{n+2}x^2 = \ldots = a^{n+n}x^n = (a^n)^2x^n$, and a is right π-regular.

Lemma 3.3 (3):

Let a be an element of a ring R such that $a^n = a^{n+1}x$ and $a^m = ya^{m+1}$ for some $x,y \in R$ and some positive integers m,n. Then $a^m = a^{m+1}x$ and $a^n = ya^{n+1}$.

Proof

If $m \geq n$, then $a^n = a^{n+1}x$ immediately implies that $a^m = a^{m+1}x$. If $m < n$, then $a^m = ya^{m+1}$ implies $a^m = ya^{m+1} = y^2a^{m+2} = \ldots = y^{n-m}a^n$. Thus $a^{m+1}x = y^{n-m}a^{n+1}x = y^{n-m}a^n = a^m$. Similarly, it follows that $a^n = ya^{n+1}$.

Using the above lemmas, we may prove:

Theorem 3.4 (3):

Let a be a strongly π-regular element of a ring R. Then a is π-regular.

Proof:

By lemmas 3.2 and 3.3, there are $x,y \in R$ and a positive integer n such that $a^n = a^{n+1}x$ and $a^n = ya^{n+1}$. By lemma 3.2 again, $a^n = (a^n)^2x^n$ and $a^n = y^n(a^n)^2$, and so

a^n is strongly regular. By theorem 2.1.3, there is a $z \in R$ such that $a^{2n}z = a^n$, and z commutes with all elements of R which commute with a^n. But a commutes with a^n, and thus $az = za$. Finally, $a^n = a^{2n}z = a^n z a^n$, and a is π-regular.

The next result contains a partial converse to the above theorem:

Theorem 3.5 (3):

Let R be a ring of bounded index, then the following conditions on R are equivalent:

(1) R is π-regular.
(2) R is right π-regular.
(3) R is left π-regular.
(4) R is strongly π-regular.

It is true in general that a left or right artinian ring R is π-regular, (9). The above theorem enables us to prove this in the case where R has an identity.

Theorem 3.6:

An artinian ring R with identity is π-regular and of bounded index.

Proof:

Since R_R has an identity, it is noetherian as well, and hence has a composition series of length m, say. Then if $a \in R$ is a nilpotent element of index n, then $n \leq m$, otherwise $aR \supseteq a^2R \supseteq \ldots \supseteq a^mR = a^{m+1}R = \ldots = a^nR = \{0\}$, i.e. $a^m = 0$. This is a contradiction of the definition of the index of a, and so $n \not> m$. Thus R is of bounded index. Finally, for any $x \in R$, it is clear that there is some $r \in R$ and positive integer j such that $x^j = x^{j+1}r$, and so by the previous theorem R is π-regular (in fact, strongly π-regular).

We remark further that under the assumption that a ring R is of bounded index, it can be proved (see [28]) that R possesses a unique maximal π-regular ideal $P(R)$ satisfying:

(1) $P({}^{R}/P(R)) = \{0\}$.

(2) If I is an ideal of R, then $P(I) = I \cap P(R)$.

(3) $P(R_n) = (P(R))_n$, and is of bounded index.

The following theorem gives an alternative way of defining an element of a ring to be strongly π-regular:

Theorem 3.7 [5]:

An element a of a ring R is strongly π-regular if and only if there is a positive integer m and an element $c \in R$ such that $ac = ca$, $a^m = a^{m+1}c$, and $c = c^2a$. Further, if such an element c exists, then it is unique, and commutes with all elements of R which commute with a.

Proof:

Suppose $a \in R$ is strongly π-regular. Then there are positive integers p,q such that $a^p = a^{p+1}x$ and $a^q = ya^{q+1}$. Let $m = \max\{p,q\}$, then $a^{m+1}x = a^m = ya^{m+1}$, so that $a^m x = ya^{m+1}x = ya^m$. By induction, it follows that $a^m x^k = y^k a^m$ (for $k = 1,2,\ldots$). Let $c = a^m x^{m+1}$, then also $c = y^{m+1}a^m$. We thus have $ac = a^{m+1}x^{m+1} = a^{m+1}x.x^m = a^m x^m = y^m a^m = y^m y a^{m+1} = y^{m+1}a^{m+1} = y^{m+1}a^m.a = ca$, i.e.

(1) $ac = ca$.

Further, by induction, $a^m = a^{m+k}x^k$ (for $k = 1,2,\ldots$), and so $a^{m+1}c = a^{m+1}a^m x^{m+1} = a^{m+(m+1)}x^{m+1} = a^m$, i.e.

(2) $a^{m+1}c = a^m$.

Finally, by (1) and (2) above, we have $c^2a = c.ac = ca^{m+1}x^{m+1} = a^{m+1}cx^{m+1} = a^m x^{m+1} = c$, i.e.

(3) $c^2a = c$.

Conversely, if $a \in R$ satisfies (1), (2) and (3), then it is clearly strongly π-regular.

Next, we show the uniqueness of c. Suppose $c_1, c_2 \in R$ satisfy the conditions imposed on c by (1), (2) and (3), for some positive integers m_1, m_2. Let $m = \max\{m_1, m_2\}$, then by (1) and (2) we have:

(4) $c_1a^{m+1} = a^m = a^{m+1}c_2$,

while by (1) and (3) we have:

(5) $c_1 = c_1^2a \qquad c_2 = ac_2^2$.

By induction on (5), $c_1 = c_1^{k+1}a^k$ and $c_2 = a^kc_2^{k+1}$ (for $k = 1,2,\ldots$), and in particular $c_1 = c_1^{m+1}a^m$ and $c_2 = a^mc_2^{m+1}$. Then, by (4), it follows that $c_1 = c_1^{m+1}a^m = c_1^{m+1}a^{m+1}c_2 = c_1ac_2 = c_1aa^mc_2^{m+1} = a^mc_2^{m+1} = c_2$, and c is unique.

Finally, if r is an element of R satisfying $ar = ra$, then by (2) and (1) we find $ca^mr = cra^m = cra^{m+1}c = ca^{m+1}rc = a^mrc$, so that $c^{m+1}a^mr = a^mrc^{m+1}$. However, from (3), we have $c = c^{m+1}a^m$, and thus by (1) it follows that $cr = c^{m+1}a^mr = a^mrc^{m+1} = rc^{m+1}a^m = rc$.

Next, we turn to commutative π-regular rings with identity. We first require:

Definition 3.8:

A ring R with identity is called irreducible if it cannot be expressed as the direct sum of two proper two-sided ideals.

Theorem 3.9 (22):

A commutative π-regular ring R with identity is irreducible if and only if each element of it which is not nilpotent has an inverse.

Proof:

Suppose that each element of R which is not nilpotent has an inverse, and that R can be expressed as the direct sum of two proper ideals. Let $1 = e_1+e_2$ be the decomposition of the identity, where $e_1, e_2 \in R$. Then clearly e_1 and e_2 are non-zero, $e_1^2 = e_1$ and $e_2^2 = e_2$, and $e_1e_2 = 0$. Thus e_1 can neither be nilpotent nor have an inverse, a contradiction.

Conversely, suppose R is irreducible and that $a \in R$ is not nilpotent. Then for some positive integer n and some $x \in R$, we have $a^n = xa^{2n}$, where $xa^n \neq 0$. Let $e_1 = xa^n$ and $e_2 = 1-e_1$, then clearly $e_1^2 = e_1$, $e_2^2 = e_2$, and $e_1e_2 = 0$; and hence $R = e_1R \oplus e_2R$. Since $e_1R \neq \{0\}$, it follows that $1-xa^n = 0$, i.e. $a.a^{n-1}x = 1$, and a has an inverse.

It is well-known that a ring R with identity is irreducible if and only if its centre is irreducible. We thus have:

Corollary 3.10 (22):

A π-regular ring R with identity is irreducible if and only if its centre is a ring in which each element which is not nilpotent has an inverse.

We have remarked that homomorphic images of π-regular rings are again π-regular, and we now state a partial converse to this:

Theorem 3.11 (9):

If R is a commutative ring with identity and I is an ideal of R, then R is π-regular if and only if both I and $R/_I$ are π-regular.

We now proceed to some characterizations of π-regularity in commutative rings with identity:

Theorem 3.12 (25):

Let R be a commutative ring with identity, then the following statements are equivalent for each $a \in R$:

(1) a is π-regular, with positive integer n such that a^n is regular.

(2) There is a unit u, idempotent e, and $r \in R$ with $r^n = 0$, such that $a = u(e+r)$.

(3) There is a unit u and an idempotent $f \in R$ such that $a^n = uf$.

Theorem 3.13 (26):

The following conditions on a commutative ring R with identity are equivalent:

(1) R is π-regular.

(2) For each $a \in R$, there is a $y \in R$ and a positive integer n such that $a^n = a^{n+1}y$.

(3) The Jacobson radical $J(R)$ of R is a nil ideal, and $^R/J(R)$ is regular.

(4) $^R/P$ is regular, where P is the intersection of all prime ideals of R.

(5) In each homomorphic image of R, including R itself, each element is either a zero-divisor or a unit.

Proof:

(1) implies (2): For $a \in R$, there exists an $x \in R$ and a positive integer n such that $a^n = a^n x a^n$. Let $y = a^{n-1}x$, then $a^n = a^{n+1}y$.

(2) implies (1): If $a^n = a^{n+1}y$, then $a^n = a^{n+1}y = a^{n+2}y^2 = \ldots = a^{2n}y^n = a^n y^n a^n$, and a is π-regular.

(1), (2) imply (3): Let $r \in J(R)$, then for some $x \in R$ and positive integer n, $r^n x r^n = r^n$. Thus $r^n x$ is an idempotent in $J(R)$, whence $r^n = 0$. Further, $^R/J(R)$ has no non-zero nilpotent elements, for suppose $(r+J(R))^n = 0$ for some n and some $r \in R$. Then $r^n \in J(R)$, and so r^n (and thus r) is nilpotent. Finally, suppose $a \in R$, with $y \in R$ and positive integer n such that $a^n = a^{n+1}y$. If $n = 1$, then clearly $a+J(R)$ is regular in $^R/J(R)$. If $n > 1$, then $(a^n y - a^{n-1})^2 = a^{n-1}ya^{n+1}y - a^{2n-1}y + a^{2n-2} - a^{n+1}ya^{n-2} = 0$, and so $a^n y - a^{n-1} \in J(R)$. Continuing this way, we eventually have that $a^2y - a \in J(R)$, and again $a + J(R)$ is regular.

(3) implies (4): If $J(R)$ is a nil ideal, then $J(R) = P$.

(4) implies (2): Let $a \in R$, then there is an $x \in R$ such that $a-a^2x \in P$. Since R is commutative, P is a nil ideal, and so there is a positive integer n such that $(a-a^2x)^n = a^n(1-ax)^n = 0$. Writing $a^n(1-ax)^n = a^n(1-pax)$, for some polynomial p in ax, we have that $a^n = a^{n+1}px$, and so (2) holds.

Finally, the equivalence of (1) and (5) is clear.

CHAPTER 4

WEAKLY AND GENERALIZED WEAKLY REGULAR RINGS

4.1 Weakly Regular Rings

Weakly regular rings were first suggested by I.E. Segal, although it was I. Kaplansky who initiated their study in (15). The first section consists of results from this paper.

Definition 4.1.1:

An element a of a ring R is called weakly regular if there is an $x \in R$ such that $x = xax$, where x can be chosen non-zero if a is non-zero. An ideal I (and in particular R itself) is called weakly regular if all its elements are weakly regular.

The following theorem gives a characterization of weakly regular rings with identity:

Theorem 4.1.2 (15):

A ring R with identity is weakly regular if and only if every non-zero right (left) ideal contains a non-zero idempotent.

Proof:

Let I be a non-zero right ideal of R, and let $a \in I$, $a \neq 0$. Then if R is weakly regular, there is an $0 \neq x \in R$ such that $x = xax$. Then ax is a non-zero idempotent in I.

Conversely, let $0 \neq a \in R$. Then aR is a non-zero right ideal of R, and there is a y in R such that ay is a non-zero idempotent. Let $x = yay$, then $x = xax$. Further, $x \neq 0$, otherwise $ay = (ay)^2 = ax = 0$, a contradiction. Thus a, and hence R, is weakly regular.

Kaplansky used the above theorem to prove that a regular ring R with identity is weakly regular. We give a direct proof avoiding the assumption of an identity:

Theorem 4.1.3

A regular ring R is weakly regular.

Proof:

Let $0 \neq a \in R$. Then there is an $0 \neq x \in R$ such that $a = axa$. Then $xax = xax.a.xax$, and if $xax = 0$, it follows that $a = 0$, a contradiction. Thus R is weakly regular.

That there are weakly regular rings which are not regular, even in the commutative case, can be seen from:

Example 1:

Let X be a compact Hausdorff space in which every open set contains a (non-empty) open and closed subset, and let $C(X)$ be the ring of all real continuous functions on X. Then $C(X)$ is an infinite-dimensional real Banach algebra. Kaplansky has proved (see [15]) that a regular real Banach algebra is finite-dimensional, and so $C(X)$ is not regular.

$C(X)$, however, is weakly regular. To see this, let $0 \neq a \in C(X)$, and let U be the (open) subset of X where

a is non-zero. Then by assumption U contains an open and closed subset V, say. Define a function x from X into the reals by $x(r) = \frac{1}{a(r)}$ for $r \in V$ and $x(r) = 0$ for r in the complement of V. Then clearly x is continuous, $x \neq 0$, and $x = xax$. Thus $C(X)$ is weakly regular but not regular.

Returning to weakly regular rings, we have:

Lemma 4.1.4 [15]:

A weakly regular ring R is semisimple.

Proof:

Let $a \in J(R)$, the Jacobson radical of R. If $a \neq 0$, there is an $0 \neq x \in R$ such that $x = xax$. Then ax is a non-zero idempotent in $J(R)$, which is impossible. Thus $a = 0$, and R is semisimple.

Using this lemma we obtain a sufficient condition for a weakly regular ring to be regular:

Theorem 4.1.5 [15]:

An artinian weakly regular ring R is regular.

Proof:

Since R is weakly regular, it is semisimple; being artinian as well, it is thus regular.

4.2 Weakly and Generalized Weakly Regular Rings.

We now continue to study weak regularity, and examine the effects of similar generalizations on other forms of regularity.

We remark first that an ideal of a weakly regular ring is weakly regular, and the two-sided annihilator R' of a weakly regular ring R is zero. To see the latter, take $0 \neq a \in R'$. Then there is an $0 \neq x \in R$ such that $x = xax$. But $xax = 0$, a contradiction. Thus $a = 0$ and $R' = \{0\}$.

Proposition 4.2.1:

A direct product $\prod_\alpha R_\alpha$ of rings R_α, $\alpha \in A$, is weakly regular if and only if each R_α is weakly regular.

Proof:

Suppose $\prod_\alpha R_\alpha$ is weakly regular. For any $\beta \in A$, let $0 \neq r \in R_\beta$. Now define an element $(r_\alpha) \in \prod_\alpha R_\alpha$ by $r_\alpha = 0$ for $\alpha \neq \beta$, and $r_\beta = r$. Then $(r_\alpha) \neq 0$, and so there is an $0 \neq (x_\alpha) \in \prod_\alpha R_\alpha$ such that $(x_\alpha) = (x_\alpha)(r_\alpha)(x_\alpha)$. Thus $x_\alpha = x_\alpha . 0 . x_\alpha = 0$ for $\alpha \neq \beta$, and $x_\beta = x_\beta r x_\beta$. Since $(x_\alpha) \neq 0$, $x_\beta \neq 0$, and hence R_β is weakly regular for all $\beta \in A$.

Conversely, suppose each R_α is weakly regular, and let $0 \neq (r_\alpha) \in \prod_\alpha R_\alpha$. Then for each α such that $r_\alpha \neq 0$, there is an $0 \neq x_\alpha \in R_\alpha$ such that $x_\alpha = x_\alpha r_\alpha x_\alpha$. Now define $(y_\alpha) \in \prod_\alpha R_\alpha$ by $y_\alpha = x_\alpha$ if $r_\alpha \neq 0$, and $y_\alpha = 0$ if $r_\alpha = 0$. Then clearly $(y_\alpha) = (y_\alpha)(r_\alpha)(y_\alpha)$, and $(y_\alpha) \neq 0$. Thus $\prod_\alpha R_\alpha$ is weakly regular.

In general, a homomorphic image of a weakly regular ring is not weakly regular. This is seen by the following example (suggested by S. Salbany):

Example 2:

Let $N = \{1,2,3,...\}$ with the usual subspace topology and let $X = N\cup\{\infty\}$ be the one-point compactification of N. Then clearly X is compact Hausdorff, and every open set of X contains an open and closed subset. Thus $C(X)$ is weakly regular.

Let I be the ideal of $C(X)$ whose elements are zero off a compact subset of X (i.e. those functions which vanish on a neighbourhood of ∞). Then the function $a\in C(X)$ defined by $a(n) = \frac{1}{n}$ for $n\in N$ and $a(\infty) = 0$, is not an element of I (but is in the uniform closure of I).

Suppose $^R/I$ is weakly regular. Since $a\notin I$, there is an $x\notin I$ such that $(x+I)(a+I)(x+I) = x+I$. Thus $x^2a = x$ on some neighbourhood of ∞. However, since $x\notin I$, there is an increasing sequence $n_1 < n_2 < \ldots < n_r < \ldots$ of positive integers n_r, $r\in N$ such that $x(n_r) \neq 0$, and $(x(n_r))^2.a(n_r) = x(n_r)$ for all $r\in N$. Thus $x(n_r).\frac{1}{n_r} = 1$, and hence $x(n_r) = n_r$. This is a contradiction since x is an unbounded function on a compact space. Thus $^R/I$ is <u>not</u> weakly regular, although R is.

Analogous to the definition of strong regularity, we now define:

Definition 4.2.2:

An element a of a ring R is called ws-regular (for "weakly strongly regular") if there is an x in R such that $x = x^2a$, where x may be chosen non-zero if a is non-zero. An ideal of R is said to be ws-regular if all its elements are ws-regular.

By theorem 2.1.3, it is clear that any strongly regular ring is ws-regular.

The following results show that, as in the case of strongly regular rings, the definition of ws-regularity is left-right symmetric.

Lemma 4.2.3:

A ws-regular ring R has no non-zero nilpotent elements.

Proof:

Suppose $a \in R$, and $a^n = 0$ for some integer $n > 1$. If $a \neq 0$, then there is an $0 \neq x \in R$ such that $x = x^2 a$. Hence, by induction, $x = x^{r+1} a^r$ for any integer $r \geq 1$. In particular, $x = x^{n+1} a^n = 0$, a contradiction. Thus $a = 0$.

It follows that all idempotents of a ws-regular ring are central.

Lemma 4.2.4:

A ws-regular ring R is weakly regular.

Proof:

Let $0 \neq a \in R$, then there is an $0 \neq x \in R$ such that $x = x^2 a$. Then $(x - xax)^2 = x^2 + xax^2ax - xax^2 - x^2ax = x^2 + xax^2 - xax^2 - x^2 = 0$, and so by the previous lemma $x = xax$, and R is weakly regular.

Proposition 4.2.5:

If R is a ws-regular ring, then for each $0 \neq a \in R$ there is an $0 \neq x \in R$ such that $x = x^2 a = ax^2 = xax$, and $ax = xa$. Further, x commutes with all elements of R which commute with a.

Proof:

Let $0 \neq a \in R$, then there is an $0 \neq x \in R$ such that $x = x^2a$. From lemma 4.2.4 it follows that $x = xax$. Now $(x-ax^2)^2 = x^2 + ax^2ax^2 - xax^2 - ax^3 = x^2 + ax^3 - x^2 - ax^3 = 0$, and so $x = ax^2$. Further, $xa = ax^2a = ax$. Finally, suppose $r \in R$ and $ar = ra$. Then since idempotents are central in R, we have $xr = x^2a.r = x.xa.r = x.r.xa = x.ra.x = xa.r.x = r.xax = rx$.

The writer does not know whether the centre of a weakly regular ring is again weakly regular. We have, however, from the above result:

Corollary 4.2.6:

The centre of a ws-regular ring is again ws-regular.

We show now that the definition of ws-regularity leads to a characterization along the lines of those found for regular, strongly regular, and weakly regular rings.

Theorem 4.2.7:

A ring R with identity is ws-regular if and only if every non-zero right (left) ideal of R contains a non-zero central idempotent.

Proof:

If R is ws-regular and A is a non-zero right ideal of R, then choose any $0 \neq a \in A$. Then there is an $0 \neq x \in R$ such that $x = x^2a = xax$. Then ax is an idempotent in A, and is clearly non-zero. Since R has no non-zero nilpotent elements, ax is also central.

Conversely, suppose every non-zero right ideal of R contains a non-zero central idempotent. Let $0 \neq a \in R$, then aR is non-zero, and there is an s in R such that as is a non-zero central idempotent. Thus $a(sas)^2 = asas.sas = as.sas = sas.as = sas$, and since $sas \neq 0$, a is ws-regular.

By the symmetry of the definition of ws-regularity, as proved in proposition 4.2.5, the result is also true with "right" replaced by "left".

The following summary shows a pattern which is completed by the above definition and characterization of ws-regularity:

A ring R with identity is:

(1) Regular if and only if every principal one-sided ideal is <u>generated</u> by some idempotent.

(2) Strongly regular if and only if every principal one-sided ideal is <u>generated</u> by a central idempotent.

(3) Weakly regular if and only if every non-zero one-sided ideal <u>contains</u> some non-zero idempotent.

(4) Ws-regular if and only if every non-zero one-sided ideal <u>contains</u> a non-zero central idempotent.

We have mentioned that a regular ring without nilpotent elements is strongly regular. We now have a corresponding siutation with weakly regular rings:

<u>Proposition 4.2.8</u>:

A weakly ring R with no non-zero nilpotent elements is ws-regular.

Proof:

Let $0 \neq a \in R$, then there is an $0 \neq x \in R$ such that $x = xax$. Then $(x-x^2a)^2 = x^2 + x^2ax^2a - x^3a - x^2ax = x^2 + x^3a - x^3a - x^2 = 0$, and so $x = x^2a$ and R is ws-regular.

That even a commutative ws-regular ring need not be strongly regular is seen from example 1. We now give two sufficient conditions for a ws-regular ring to be strongly regular:

Proposition 4.2.9:

A commutative ws-regular ring R which is also π-regular, is strongly regular.

Proof:

Since R is ws-regular, it is also weakly regular and hence semisimple. Thus R is isomorphic to $^R/J(R)$, which is regular by theorem 3.13. Since R has no non-zero nilpotent elements, it is strongly regular.

Proposition 4.2.10:

An artinian ws-regular ring R is a strongly regular ring with identity.

Proof:

Being ws-regular R is semisimple. By section 1.3, it is regular with identity, and having no non-zero nilpotent elements, it is strongly regular.

Finally, in this section, we note that the proof of proposition 4.2.1 carries over directly to ws-regular rings, and we obtain:

Proposition 4.2.11:

A direct product of rings is ws-regular if and only if each component is ws-regular.

4.3 Weakly π-Regular Rings

In this section, weakly π-regular rings are defined analogously to weakly regular and ws-regular rings.

Definition 4.3.1:

An element a of a ring R is said to be weakly π-regular if some power of a is weakly regular. Thus there is an integer $n \geq 1$ and an $x \in R$ such that $x = xa^n x$, where x may be chosen non-zero if $a^n \neq 0$. A ring R is called weakly π-regular if all its elements are weakly π-regular.

It is immediate that a π-regular ring R is weakly π-regular, since if some power of an element is regular, it is also weakly regular by theorem 4.1.3.

The next result is useful when working with weakly π-regular rings:

Lemma 4.3.2:

Let R be a ring, and let a be an element of R which is not nilpotent. Then if a is weakly π-regular, it is also weakly regular.

Proof:

Since a is weakly π-regular and is not nilpotent, there exists an integer $n \geq 1$ and an $0 \neq x \in R$ such that $x = xa^n x$. If $n = 1$, then a is weakly regular. If $n > 1$, then $xa^{n-1} = xa^{n-1}.a.xa^{n-1}$; and if $xa^{n-1} = 0$, then $x = xa^n x = 0$, a contradiction. Thus $xa^{n-1} \neq 0$, and a is weakly regular.

From the above result it is immediate that a weakly π-regular ring with no non-zero nilpotent elements is ws-regular.

Next, we turn to the Jacobson radical of a weakly π-regular ring. The following result is a generalization of theorem 3.13 part (3):

Lemma 4.3.3:

The Jacobson radical $J(R)$ of a weakly π-regular ring R, is a nil ideal.

Proof:

Let $a \in J(R)$, and suppose a is not nilpotent. Then there exists a positive integer n and an $0 \neq x \in R$ such that $x = xa^n x$. Then $(a^n x)^2 = a^n x$, and $a^n x \in J(R)$. Thus $a^n x = 0$, and hence $x = 0$, a contradiction. Thus a is nilpotent, and $J(R)$ is a nil ideal.

Proposition 4.3.4:

Let R be a weakly π-regular ring with identity, then the following conditions on R are equivalent:

(1) R is weakly regular

(2) R is semisimple

(3) R has no nil right or left ideals, other than the zero ideal.

Proof:

(1) implies (2) : proved in lemma 4.1.4.

(2) implies (3) : this is true in any semisimple ring.

(3) implies (1) : Let A be a non-zero right (left) ideal of R. Then A cannot be nil by assumption, and there is some $a \in A$ which is not nilpotent. Then there is a positive integer n and an $0 \neq x \in R$ such that $x = xa^n x$. Thus $a^n x$ (xa^n) is a non-zero idempotent in A, and so by theorem 4.1.2 R is weakly regular.

We proved in theorem 3.13 that if R is a commutative π-regular ring, then ${}^R/J(R)$ is regular. Our last result is analogous to this:

Proposition 4.3.5:

Let R be a commutative weakly π-regular ring. Then ${}^R/J(R)$ is weakly regular.

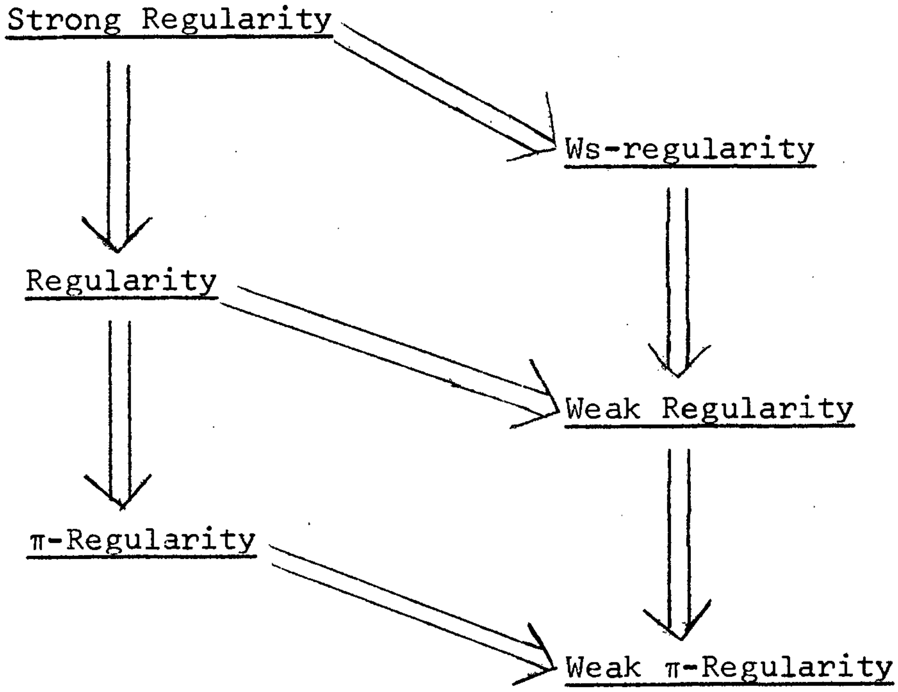
Proof:

${}^R/J(R)$ has no non-zero nilpotent elements, for suppose $(r+J(R))^n = 0$ for some $r \in R$ and some integer $n > 1$. Then $r^n \in J(R)$, and so by lemma 4.3.3 r^n is nilpotent. Thus r is nilpotent, and in a commutative ring R, all nilpotent elements are in the radical $J(R)$ and so $r+J(R) = 0$.

Now let $r+J(R) \in {}^R/J(R)$, where $r \in R$ and $r \notin J(R)$. Thus

r is not nilpotent, and by lemma 4.3.2 r is weakly regular. Then there exists an $0 \neq x \in R$ such that $x = x^2r$. Thus $x = x^nr^{n-1}$ for all integers $n > 1$, so that if x is nilpotent, then $x = 0$. It follows that x is not nilpotent and so $x \notin J(R)$. Finally, $x+J(R) = (x+J(R))(r+J(R))(x+J(R))$ where $x \notin J(R)$, and so $^R/J(R)$ is weakly regular.

In conclusion, we give a diagram showing relationships amongst the various types of rings studied in this thesis.



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