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Nearness and Convergence in Pointfree Topology

by
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Abstract

We introduce and investigate the concept of a nearness structure on a σ -frame. Analogues of the Samuel Compactification, Uniform Coreflection and Completion in the nearness σ -frame setting are obtained. Convergence in uniform frames is also a subject of this thesis integrating compactness, precompactness and paracompactness. Finally, the notion of uniform paracompactness is introduced and its relation with convergence is investigated.

ERRATA

Page	Line	Where is	Replace by
16	-5	B	A
17	5	nearnes	nearness
51	-9	ThE	The
53	-7	f is uniform, $f(B)$	f is uniform and ν is strong, $f(\check{B})$
53	-6, -4	$f(B)$	$f(\check{B})$
53	-2	$\leq B$	$\leq (s \circ f)(\check{B}) \leq B$

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Introduction

*We lay down a fundamental principle of generalization by abstraction:
"The existence of analogies between central features of various theories
implies the existence of a general theory which underlies the particular
theories and unifies them with respect to those central features..."*

- E. H. Moore (1862 - 1932)

Historical preview

Frames and structured frames

The study of topological concepts from a lattice theoretical approach was initiated in the late 1930's as a consequence of M. H. Stone's celebrated *Representation Theorem for Boolean Algebras* in [78] linking topology and lattice theory. The 1957 *Séminaire Ehresmann* in Paris benchmarked a novel theory of *generalized* topological spaces. Ehresmann and his student Bénéabou considered complete lattices in which finite meets distributed over arbitrary joins. They termed such lattices *local lattices*. Dowker and Papert who attended the Paris seminar then developed this lattice theoretic generalization of topological spaces in a series of joint papers [28], [29], [30] and [31] calling the local lattices of Ehresmann and Bénéabou *frames*. Frame theory or pointfree topology uses the principal notion of the lattices of open sets as the approach to topology as the open subsets of any topological space form a frame in which the join of a family $\{T_\alpha\}_{\alpha \in \Lambda}$ of open subsets is the union

$$\bigvee_{\alpha \in \Lambda} T_\alpha = \bigcup_{\alpha \in \Lambda} T_\alpha$$

and meet, the interior of the intersection,

$$\bigwedge_{\alpha \in \Lambda} T_\alpha = \left(\bigcap_{\alpha \in \Lambda} T_\alpha \right)^\circ$$

Various classical topological theorems generalize to theorems about frames frequently in a more elegant form. One benefit of frames is that many classical results are proved constructively without the appeal to the Axiom of Choice *e.g.* the localic version of Tychonoff's theorem (see [57]). Another is that (closely related) properties such as Lindelöf and realcompact, for topological spaces, may be analysed in the frame setting, giving new insights to their topological nature (see [10]). Johnstone's *Stone Spaces* [52] provides an exposition of and to a large extent motivation for pointfree topology and serves as the basic reference for the theory of frames.

Isbell's classic paper *Atomless parts of spaces* (see [56]) defined uniform structures on frames via a system of covers as the exact translation into frames of the cover approach to uniform spaces given by Tukey in [86]. Pultr expounded on the cover approach to uniformity on frames in [68] and later resumed the study of metric locales, originally appearing in the paper [56] by Isbell. In [69], Pultr introduced metric diameters for frames and proved metrization theorems for pointless uniformities. Frith then considered uniform type structures from a categorical perspective in [40] introducing quasi-uniformities (uniformity without symmetry) and proximities into the pointfree context. Tukey's exposition on uniformity on spaces defined in terms of collections of covers (which first appeared in the paper [90] by Weil in 1936) provided an alternative to the original approach to uniformity given by Weil in terms of entourages. The localic adaptation of Weil's entourage approach has been developed by Fletcher and Hunsaker in [38] and more recently by Picado in [66] for structures on frames. In the language of frames, just as in spaces, the cover approach to uniformity is more predominant as it is used more extensively than its entourage equivalent.

In 1974, Herrlich introduced into spaces the notion of a nearness and in [45] and [46] developed a rich theory of nearness spaces and their applications as a unifying concept for the theories of topological, uniform and proximity spaces. Nearness spaces

originated particularly out of interest in the study of extensions of topological spaces, where they have proved to be effective amongst their applications to homology and connectedness. Nearness was introduced into frames by Banaschewski at a seminar series at the University of Cape Town culminating in the paper [17]. Dube who attended this seminar series then established a theory of nearness frames in [32] as a pointfree generalization of Herrlich's classical nearness spaces adopting the cover approach developed by Banaschewski in [17]. In [4] the completion of a nearness frame unique up to isomorphism is constructed and completion is shown to be a coreflection for uniform and metric frames. However, [19] establishes that completion is a coreflection on a substantially larger class of nearness frames, namely, the strong nearness frames.

σ -Frames and Structured σ -frames

In 1974, Charalambous in [24] introduced σ -frames as a generalization of the frames of Dowker and Papert, where finite meets distribute over countable joins. Reynolds presented a lattice theoretic characterization of complete regularity in [74] and showed that the Alexandroff algebras are the cozero-set lattices of arbitrary frames. Gilmour in [42] provided an adjunction between regular σ -frames (the Alexandroff algebras of Reynold's) and Alexandroff spaces. The manuscript [2] by Banaschewski provides a detailed account of σ -frames.

Uniformities and proximities for σ -frames were first investigated by Walters in [87] as a generalization of the results in [41] of Frith together with those in [74] of Reynold's. In [89] the completion of a uniform σ -frame is established via the cozero-part of a uniform frame. In August 2001, at the International Conference on Applicable General Topology hosted by Hacettepe University in Ankara, [81] and [82] was presented by Tabatabaee and Mahmoudi respectively. The authors in [81], [82] and [85] defined a metric diameter for a σ -frame (similar to that in [69] by Pultr for frames) and proved

metrization theorems for σ -frames.

Convergence

The 1915 paper *Definition of limit in general integral analysis* by Moore in [62] introduced the notion of convergence in spaces. This led to a theory of convergence by way of nets in 1922 by Moore and his student Smith, in the paper *A General Theory of Limits* (see [63]). In 1937, Birkhoff in [21] applied the Moore-Smith theory to general topology and in the same year Cartan in [23] introduced filters, the concept dating back as early as 1908, as a more natural generalization of sequences. This was further developed by Bourbaki in their 1940 treatise *Topologie generale*. Tukey at the same time worked comprehensively with phalanxes, objects that generalized sequences. He also studied various modifications of nets and filters and extensively applied the Moore-Smith theory of convergence in his acclaimed monograph *Convergence and uniformity in Topology*. In 1948, Samuel defined ultrafilters and in [75] reviewed the principal applications of filters and ultrafilters in general topology. Bartle in 1955 first noted a natural equivalence between filters and nets (see [20]) which was also discussed, in the same year, by Bruns and Schmidt in [22]. Due to their algebraic finesse the filter approach is the convergence theory of choice for many authors.

In the pointfree setting, filters have been used to provide completeness criteria for structured frames (see [4], [13] and [17]). Hong in [50] introduced the notions of convergent and clustering filters in a frame and together with Banaschewski, in [11], considered strict extensions in the pointfree context by means of sets of filters.

Synopsis

This thesis may be classified in two parts. In Chapters 2, 3 and 4 we generalize uniformity on σ -frames (see [87]) to nearness σ -frames. In particular, we characterize the Samuel compactification and the uniform coreflection of a nearness sigma-frame.

It is also shown that the separable strong Lindelöf nearness frames and the strong nearness σ -frames are equivalent as categories. We exhibit a completion of a strong nearness σ -frame unique up to isomorphism akin to the constructions in [17], [88] and [84].

The second part involves the notion of clustering and convergence of filters in uniform frames. In 1995, Hong introduced a concept of convergence of filters in a frame by using covers of a frame and characterized compact regular frames by convergence of maximal filters in [50]. Howes describes the notion of cofinal completeness for uniform spaces in [51] by way of nets. We explore this concept in the setting of uniform frames by using the elegant filter approach. We conceptualize *weakly Cauchy* filters together with the notion of *strongly Cauchy complete* and establish a necessary and sufficient condition for which a uniform frame has a paracompact completion. We also introduce *uniform paracompactness* in the pointfree setting and investigate its correspondence with strong Cauchy completeness.

Throughout this thesis choice principles such as the Axiom of Choice or the Countable Dependent Axiom of Choice are used, and generally without mention. The treatment of category theory is not entirely self-contained but used prolifically throughout. Known results will be referenced, the proofs of which will be omitted. We next describe each chapter of this dissertation in more detail.

Chapter 1

This chapter provides the basic general theory required in the ensuing chapters. First, the language of category theory is that of the book [67] by Preuss. σ -frames as surveyed by Banaschewski in [2] are introduced. The reference for frames is the book [58] by Johnstone. Pertinent reminders of aspects of uniform and nearness spaces are given. Basic results and definitions of structured frames and uniform σ -frames also form part of this chapter.

Chapter 2

The definition of a nearness structure on a sigma-frame is given and the category $\mathbf{N}\sigma\mathbf{Frm}$, of nearness σ -frames and uniform σ -frame homomorphisms, is introduced. The compact regular coreflection of a nearness sigma-frame is described via its countably generated uniformly normally regular ideals. For a nearness sigma-frame we show that the cozero part of the uniform coreflection of the nearness frame of all sigma ideals, which is a uniform sigma-frame, is its uniform coreflection.

Chapter 3

This chapter focusses on the completeness of a nearness sigma-frame. In particular, we devote our study to the category $\mathbf{SN}\sigma\mathbf{Frm}$ of strong nearness σ -frames which encapsulates uniform σ -frames. We show that the category $\mathbf{SepSLNFrm}$, of separable strong Lindelöf nearness frames, and the category $\mathbf{SN}\sigma\mathbf{Frm}$ are equivalent. This is in analogy to the uniform frame case (see [87]). This also provides a structured version of the functors $\mathcal{C}oz$ and \mathcal{H} between regular σ -frames and regular Lindelöf frames elucidated by Madden and Vermeer in [61] which initially appeared in Reynold's paper [74]. The complete strong nearness σ -frames are shown to be exactly the cozero parts of complete separable strong Lindelöf nearness frames (cf. the uniform case in [87]). The functors $\mathcal{C}oz$ and \mathcal{H} thus induce an equivalence between the categories of complete strong nearness σ -frames ($\mathbf{CSN}\sigma\mathbf{Frm}$) and complete separable strong Lindelöf nearness frames ($\mathbf{CSepSLNFrm}$). This amounts to a subequivalence of that between $\mathbf{SN}\sigma\mathbf{Frm}$ and $\mathbf{SepSLNFrm}$. We also prove the existence of a completion of a strong nearness sigma-frame unique up to isomorphism (cf. the uniform σ -frame case in [88] and the metric σ -frame case in [83]). We also show that completion is a coreflection on strong nearness σ -frames. We then associate a nearness space with a nearness σ -frame and define a contravariant functor from the category of nearness σ -frames to the category of nearness spaces.

Chapter 4

This chapter is devoted to coreflective subcategories of nearness frames. In particular, the category SepNFrm , of separable nearness frames, is introduced and is shown to be coreflective in NFrm .

Dube in [36] introduced localic analogues of complete regularity and normality to nearness frames. Also in [36], a *uniformly normal* nearness frame is defined as a strong nearness frame with strong totally bounded coreflection and it is shown that a uniformly normal nearness frame has the same underlying frame as its uniform coreflection (Proposition 3.7 in [36]). In [1] the Samuel compactification of a nearness frame is described using special ideals of the underlying frame, namely, the normally regular ideals. Using the above-mentioned result of Dube in [36], we show that the Samuel compactification of a uniformly normal nearness frame can be described as the completion of its totally bounded coreflection (cf. [13] for the uniform case).

Chapter 5

We continue with the study of convergence in uniform frames by way of filters adopting the cover approach conceived by Hong in [50]. We formulate *weakly Cauchy filters* and introduce the concept of *strongly Cauchy complete* in the setting of uniform frames. This chapter also focusses on compactness, precompactness (total boundedness) and paracompactness in frames.

Similar to compactness, a precompact uniform frame is usually distinguished by way of covers where a uniform frame is precompact if each uniform cover has a finite uniform refinement. However, in [50], an alternate representation of compact regular frames in terms of convergence of filters is given. As precompactness is a generalization of compactness, we investigate whether precompact uniform frames can be characterized in terms of filters. To this end, we introduce into uniform frames the

concept of *uniformly almost compact* by confining the notion of almost compactness (see [49] and [65]) to uniform covers. Subsequently, this provides further descriptions of precompactness in the pointfree setting. In particular, filter adaptations of classical topological results concerning precompact and compact uniform frames are proved, providing the desired approach to pointfree precompactness in terms of filters. In addition, we introduce *preparcompactness*, a filter generalization of precompact uniform frames.

In 1944, J. Dieudonné in [27] introduced *paracompact* spaces, a generalization of both compact and metrizable spaces. The pointfree definition of paracompactness can be attributed to Isbell in [56]. The paper by Dowker and Papert in [30] provides a comprehensive discussion of paracompactness in frames. In 1986, at the Sixth Prague Topological Symposium, Pultr presented a review of the developments of the theory of frames and further considered paracompactness in metric frames leaving the paracompactness of metrizable locales (the pointfree version of the famous theorem by Stone in [79]) an open question (see [70]). Shu-Hao resumed the study of paracompactness in (metric) frames and in 1989 proved that every metric frame is paracompact (see [77]) which subsequently resolved the question posed by Pultr in [70]. In the same year, Pultr and Ůlehla in [72] formulated an alternative characterization of paracompactness for frames by way of *quasi-covers*. In 1993, Banaschewski and Pultr in [14], by using the description of the completion of a uniform frame as a certain quotient of its Samuel compactification, characterized paracompact frames as those frames that admit a complete uniformity, a result originally due to Isbell in [56].

Using weakly Cauchy filters, the paracompact characterization of Pultr and Ůlehla in [72] and metric frames, we present equivalent conditions for a frame to be paracompact. We also show that preparcompactness provides a pointfree analogue of the result by Howes in [54] where a sufficient condition for a uniform space to have

a paracompact completion is given.

Since every metric space is paracompact, and associated with each metric space we have the natural (metric) uniformity, Rice in 1977 considered paracompactness of the associated uniform structure in any uniform space and introduced the notion of *uniform paracompactness* in [73]. Motivated by the associated uniformity of a metric space, Rice investigated whether metric spaces are uniformly paracompact. In [73] it is shown that the set of points of a uniformly paracompact metric space that admit no compact neighbourhood is compact and hence a metric topological group is uniformly paracompact if and only if it is locally compact.

Identical to the classical definition given by Rice in [73], we define a cover A in a uniform frame (L, μ) to be *uniformly locally finite* if there is a uniform cover B such that each member of B meets at most finitely many elements of A . Consequently, we define a *uniformly paracompact* uniform frame as one in which each cover has a uniformly locally finite refinement. These notions and their correspondence with paracompactness and strong Cauchy completeness are investigated. Included are discussions on frame-theoretic analogues of the results of Rice in [73] and Howes in [54] on uniformly paracompact frames. We conclude this chapter by showing that for Boolean uniform frames, uniform paracompactness and strong Cauchy completeness are equivalent.

Selected categories

Name	Objects	Morphisms	Page
Alex	Alexandroff spaces	Coz-maps	8
CRegFrm	Completely regular frames	Frame homomorphisms	5
Frm	Frames	Frame homomorphisms	1
Frm ^{op} (Loc)	Locales	Continuous maps	1
σ Frm	σ -frames	σ -frame homomorphisms	1
KRegFrm	Compact regular frames	Frame homomorphisms	5
KReg σ Frm	Compact regular σ -frames	σ -frame homomorphisms	4
LN σ Frm	Lindelöf Nearness frames	Uniform frame homomorphisms	12
LRegFrm	Regular Lindelöf frames	Frame homomorphisms	6
Near	Nearness spaces	Uniformly continuous maps	17
NFrm	Nearness frames	Uniform frame homomorphisms	12
N σ Frm	Nearness σ -frames	Uniform σ -frame homomorphisms	20
RegFrm	Regular frames	Frame homomorphisms	5
Reg σ Frm	Regular σ -frames	σ -frame homomorphisms	4
SepNFrm	Separable nearness frames	Uniform frame homomorphisms	12
SepSLNFrm	Separable strong Lindelöf nearness frames	Uniform frame homomorphisms	12
SNFrm	Strong nearness frames	Uniform frame homomorphisms	12
SN σ Frm	Strong nearness σ -frames	Uniform σ -frame homomorphisms	23
Sob	Sober spaces	Continuous maps	3
SpFrm	Spatial frames	Frame homomorphisms	3
Top	Topological spaces	Continuous maps	2
Unif	Uniform spaces	Uniformly continuous maps	19
UFrm	Uniform frames	Uniform frame homomorphisms	16
U σ Frm	Uniform σ -frames	Uniform σ -frame homomorphisms	10

Selected functors

Functor	Page
$\mathcal{D} : \mathbf{Top} \rightarrow \mathbf{Frm}$	2
$\Sigma : \mathbf{Frm} \rightarrow \mathbf{Top}$	2
$\mathit{Coz} : \mathbf{Frm} \rightarrow \mathbf{Reg}\sigma\mathbf{Frm}$	6
$\mathcal{A} : \mathbf{Alex} \rightarrow \mathbf{Reg}\sigma\mathbf{Frm}$	8
$\Psi : \mathbf{Reg}\sigma\mathbf{Frm} \rightarrow \mathbf{Alex}$	9
$\mathfrak{N}\mathfrak{A}_\sigma : \mathbf{N}\sigma\mathbf{Frm} \rightarrow \mathbf{KReg}\sigma\mathbf{Frm}$	30
$\mathcal{H} : \mathbf{N}\sigma\mathbf{Frm} \rightarrow \mathbf{LN}\mathbf{Frm}$	33
$\mathcal{H} : \mathbf{SN}\sigma\mathbf{Frm} \rightarrow \mathbf{SepSLN}\mathbf{Frm}$	39
$\mathit{Coz} : \mathbf{SepSLN}\mathbf{Frm} \rightarrow \mathbf{SN}\sigma\mathbf{Frm}$	39
$\Psi : \mathbf{N}\sigma\mathbf{Frm} \rightarrow \mathbf{Near}$	64
$e : \mathbf{N}\mathbf{Frm} \rightarrow \mathbf{SepN}\mathbf{Frm}$	69

Selected symbols

Symbol	Meaning	Page
0	zero or bottom of a σ -frame	1
1	unit or top of a σ -frame	1
2	frame with two elements	2
\Subset	finite subset	3
$\text{cov}L$	the collection of all covers of the σ -frame L	3
\prec	rather below	4
x^*	pseudocomplement of x	4
\leq	refines	4
$\prec\prec$	completely below	5
$\downarrow x$	principal ideal generated by x	7
\leq^*	star-refines	9
Ax	the star of x relative to A	9
\triangleleft_μ	μ -strongly below, uniformly below	9
\check{A}	the collection of elements in a frame that are uniformly below a member of A	11
h_*	right Galois adjoint of the homomorphism h	12
\blacktriangleleft	uniformly normally below	25
$\langle h(x) \rangle$	ideal generated by $h(x)$	33
$\text{sec}F$	the collection of elements in a frame that meets every member of the filter F	78
S^-	the collection of pseudocomplements of S	81
A^{\prec}	the collection of elements in a frame that are rather below a member of A	83
μ_F	the collection of all normal covers in a frame, the fine uniformity	87

Chapter 1

General theory

1.1 σ -frames and frames

A *sigma-frame* L is a bounded lattice (with 0 (the zero) and 1 (the unit)) with all countable joins, satisfying the distributive law

$$x \wedge \bigvee T = \bigvee_{t \in T} (x \wedge t)$$

for all $x \in L$ and any countable $T \subseteq L$. σ -Frame *homomorphisms* are maps between σ -frames preserving all finite meets and countable joins (including the unit and the zero). A *sub- σ -frame* of the σ -frame L is any subset of L which is closed under the operations and distinguished elements of L . $\sigma\mathbf{Frm}$ is the category of σ -frames and σ -frame homomorphisms.

A *frame* is a σ -frame which has arbitrary joins and satisfies the above distributive law but for arbitrary subsets T of L . A *frame homomorphism* is a map between frames preserving finite meets and all arbitrary joins. The resulting category of frames and frame homomorphisms is denoted by \mathbf{Frm} . The opposite category \mathbf{Frm}^{op} , usually denoted by \mathbf{Loc} is the category of *locales* and *continuous maps*. The book by Johnstone [58] is the standard reference and provides an extensive treatment of frames and locales.

The adjunction between frames and topological spaces

The lattice of open sets $\mathcal{O}X$ of a topological space X , with intersection for meet and union for join, is a frame and frames isomorphic to such are called *spatial frames*.

We have the contravariant *open set* functor

$$\begin{array}{ccc} \mathcal{O} & : & \mathbf{Top} \longrightarrow \mathbf{Frm} \\ & & X \dashrightarrow \mathcal{O}X \end{array}$$

$$\begin{array}{ccccc} X & & \mathcal{O}X & & \mathcal{O}h(U) = h^{-1}(U) \\ \downarrow h & \dashrightarrow & \uparrow \mathcal{O}h & & \uparrow \\ Y & & \mathcal{O}Y & & U \end{array}$$

which takes every topological space to the corresponding frame of open sets and any continuous map $h : X \rightarrow Y$ between spaces to the frame homomorphism $\mathcal{O}h : \mathcal{O}Y \rightarrow \mathcal{O}X$ taking each open set U to $h^{-1}(U)$. For any frame L , the *spectrum* of L is the set ΣL of all frame homomorphisms $\xi : L \rightarrow 2$, with $2 = \{0, 1\}$ the chain with just the unit and the zero. With $\Sigma_x = \{\xi \in \Sigma L : \xi(x) = 1\}$ for each $x \in L$, $\mathcal{O}\Sigma L = \{\Sigma_x : x \in L\}$ is a topology on ΣL . We then have the contravariant *spectrum* functor

$$\begin{array}{ccc} \Sigma & : & \mathbf{Frm} \longrightarrow \mathbf{Top} \\ & & L \dashrightarrow \Sigma L \end{array}$$

$$\begin{array}{ccccc} L & & \Sigma L & & \Sigma f(\xi) = \xi \circ f \\ \downarrow f & \dashrightarrow & \uparrow \Sigma f & & \uparrow \\ M & & \Sigma M & & \xi \end{array}$$

which takes every frame L to the topological space $(\Sigma L, \mathcal{O}\Sigma L)$ and any frame homomorphism $f : L \rightarrow M$ to the continuous map $\Sigma f : \Sigma M \rightarrow \Sigma L$ taking any $\xi \in \Sigma M$

to the composite ξf . \mathfrak{D} and Σ are adjoint on the right with natural transformations φ and ψ as follows:

- $\varphi : \mathbf{1}_{\mathbf{Top}} \longrightarrow \Sigma\mathfrak{D}$ where for any topological space X , $\varphi_X : X \longrightarrow \Sigma\mathfrak{D}X$ is a continuous map taking any $x \in X$ to $\tilde{x} : \mathfrak{D}X \longrightarrow \mathbf{2}$ defined by

$$\tilde{x}(U) = \begin{cases} 0 & \text{if } x \notin U \\ 1 & \text{if } x \in U \end{cases}$$

- $\psi : \mathbf{1}_{\mathbf{Frm}} \longrightarrow \mathfrak{D}\Sigma$ where for any frame L , $\psi_L : L \longrightarrow \mathfrak{D}\Sigma L$ is a frame homomorphism taking each $x \in L$ to Σ_x .

The details of the adjunction appear in [28] and [58]. L is spatial if and only if ψ_L is an isomorphism and we have the full reflective subcategory \mathbf{SpFrm} of spatial frames and frame homomorphisms with unit of the reflection given by ψ . Also, the topological space X is called *sober* if φ_X is an homeomorphism. \mathbf{Sob} is then the full reflective subcategory of \mathbf{Top} of sober spaces and continuous maps with unit of the reflection given by φ . ΣL is sober for each frame L and for each space X , $\mathfrak{D}X$ is spatial. The functors Σ and \mathfrak{D} define a dual adjunction between \mathbf{SpFrm} and \mathbf{Sob} , \mathfrak{D} being a left adjoint of Σ . The sobrification may also be described by the map $\epsilon_X : X \longrightarrow \Sigma\mathfrak{D}X$ given by $x \rightsquigarrow X - cl_X(\{x\})$ and the sober spaces are characterized as those for which each irreducible closed set is the closure of a unique point [58].

Covers and homomorphisms

Let L be a σ -frame with subsets A and B . For elements a, b, x and y in L we use the following notations and terminologies.

- $A \Subset B$ would mean that A is a finite subset of B .
- A *cover* on L is any countable subset whose join is the unit. The collection of all (countable) covers on L will be $covL$.

- Let S and T be covers on L . We say that S *refines* T , $S \leq T$, provided that for each $s \in S$ there exists $t \in T$ such that $s \leq t$.
- For covers S and T of L , $S \wedge T = \{s \wedge t : s \in S \text{ and } t \in T\}$.
- y is *rather below* x (written as $y \prec x$) if there is $t \in L$ (called a *separating element*) such that $y \wedge t = 0$ and $t \vee x = 1$. If L is a frame, then each element x of L has a *pseudocomplement*

$$x^* = \bigvee \{y \in L : y \wedge x = 0\}$$

Clearly, if y has a pseudocomplement then $y \prec x$ if and only if $y^* \vee x = 1$. Generally, elements of σ -frames do not have pseudocomplements. L is called a *regular σ -frame* if for each $x \in L$,

$$x = \bigvee Y \text{ for some countable } Y \subseteq \{y \in L : y \prec x\}$$

i.e. each of the elements of L is a countable join of elements rather below it. Consequently, we have the full subcategory **Reg σ Frm** of regular σ -frames. In [2] Banaschewski shows that **Reg σ Frm** is coreflective in **σ Frm** with $\mathfrak{R}L$ the regular coreflection, the largest regular sub- σ -frame of L and coreflection map $\mathfrak{R}L \rightarrow L$ given by join.

- The cover $S = (s_n)$ is *shrinkable* if there exists a cover $T = (t_n)$ such that $t_n \prec s_n$ for each n . In [8] it is shown that every cover in a regular σ -frame is shrinkable.
- $S \subseteq L$ is *locally finite* if there exists $T \in \text{cov}L$ such that $T_s = \{t \in T : t \wedge s \neq 0\} \in T$ for each $s \in S$. L is a *paracompact σ -frame* provided that every cover on L has a locally finite refinement. It is known that every regular σ -frame L is paracompact (see [8]).
- L is a *compact σ -frame* if whenever X is a cover of L , we have $\bigvee F = 1$ for some $F \in X$. **KReg σ Frm** denotes the corresponding full coreflective subcategory

of **Reg σ Frm**. An ideal in a σ -frame L is any nonempty subset I in L such that $x \vee y \in I$ whenever $x, y \in I$ and $a \leq b$ implies $a \in I$ whenever $b \in I$. The ideal I is *regular* if for each $x \in I$, $x \prec y$ for some $y \in I$ and I is *countably generated* if there is a countable sequence $(x_n) \subseteq I$ such that for each $a \in I$, $a \leq x_n$ for some n . The set of all countably generated regular ideals of L , $\mathfrak{K}L$, is the compact regular coreflection of L with coreflection map $\kappa_L : \mathfrak{K}L \rightarrow L$ given by join (see [8]).

- L is a *normal* σ -frame if for each pair $a, b \in L$ with $a \vee b = 1$ there exists $s, t \in L$ with $a \vee s = 1 = t \vee b$ and $s \wedge t = 0$. Normality can also be expressed using \prec . L is normal in case for each pair $a, b \in L$ with $a \vee b = 1$ there exists $s \in L$ such that $s \prec b$ and $a \vee s = 1$. In [8] it is shown that every regular σ -frame is normal so that if $x \prec z$ in L then $x \prec y \prec z$ for some $y \in L$ i.e. \prec interpolates in this special case.

With the obvious modifications to the corresponding definitions above we have for frames the subcategory **RegFrm** of regular frames. In contrast with regular σ -frames, regular frames need not be normal. **KRegFrm** denotes the category of compact regular frames and frame homomorphisms. The frame L is *completely regular* if for each $x \in L$,

$$x = \bigvee \{y \in L : y \prec\prec x\}$$

where $y \prec\prec x$ (y *completely below* x) means that there is a *scale* $\{c_\alpha : \alpha \in \mathbb{Q} \cap [0, 1]\}$ in L with $c_0 = x, c_1 = y$ and $c_\alpha \prec c_\beta$ whenever $\alpha < \beta$. **CRegFrm** is then the full subcategory of completely regular frames. Given a frame L , $x \in L$ is called a *cozero* element of L if $x = h(\mathbb{R} - \{0\})$ for some frame homomorphism $h : \mathfrak{D}\mathbb{R} \rightarrow L$. The *cozero part* of L is the set of all cozero elements of L denoted by $Coz L$. Reynolds in [74] as well as Banaschewski and Gilmour in [9] show that a completely regular frame is join generated by its *cozero part* which is a regular- σ -frame. The following important properties of $Coz L$ appear in [9] which we will focus on in Chapter 3.

Proposition 1.1.1

Let L be a frame. The following are equivalent for any $x \in L$:

1. $x \in \text{Coz } L$.
2. $x = \bigvee T$ for some countable $T \subseteq \{y \in L : y \prec\prec x\}$.
3. $x = \bigvee y_n$ where $y_n \prec\prec y_{n+1}$ for all $n = 1, 2, \dots$

An element x in a frame L is *Lindelöf* or *countable* if $x \leq \bigvee S$ implies that $x \leq \bigvee T$ for some countable $T \subseteq S$. We denote by $\sigma(L)$ the set of all the Lindelöf elements in a frame L . The frame L is a *Lindelöf frame* if the unit in L belongs to $\sigma(L)$. **LRegFrm** is the full coreflective subcategory of **Frm** of regular Lindelöf frames and frame homomorphisms (see [76]).

We also have the following significant results from [9] and [87].

Lemma 1.1.1

For a frame L we have that

1. $\text{Coz } L$ is a regular sub- σ -frame of L .
2. $\text{Coz } L$ generates L as a frame for $L \in \mathbf{CRegFrm}$.
3. if L is regular and Lindelöf then $\sigma(L) = \text{Coz } L$.

As frame maps preserve cozero elements, the correspondence $L \longrightarrow \text{Coz } L$ is functorial providing the functor $\text{Coz} : \mathbf{Frm} \longrightarrow \mathbf{Reg}\sigma\mathbf{Frm}$.

The adjunction between σ -frames and frames

For any σ -frame L , an ideal I is a σ -ideal in case I is closed under countable joins *i.e.* if $X \subseteq I$ and X is countable, then $\bigvee X \in I$. Let $\mathcal{H}L$ be the collection of all σ -ideals of L . Then $\mathcal{H}L$ is a Lindelöf frame called the *frame envelope* of L (see [3]) and $\mathcal{H}L$

is regular whenever L is regular. Given any σ -frame homomorphism $h : L \longrightarrow M$, $\mathcal{H}h : \mathcal{H}L \longrightarrow \mathcal{H}M$ defines a frame homomorphism where $\mathcal{H}h(I) = \langle h(I) \rangle$, the σ -ideal generated by $h(I)$. Then \mathcal{H} is functorial and is left adjoint to the functor $Coz : \mathbf{Frm} \longrightarrow \mathbf{Reg}\sigma\mathbf{Frm}$ with unit $\downarrow : L \longrightarrow Coz(\mathcal{H}L)$ an isomorphism taking any $x \in L$ to the principal ideal generated by x , $\downarrow x = \{y \in L : y \leq x\}$. The counit, $\bigvee : \mathcal{H}(CozL) \longrightarrow L$ is the join map, which is an isomorphism provided that L is regular and Lindelöf. Thus the functors \mathcal{H} and Coz induce an equivalence between the categories $\mathbf{LRegFrm}$ and $\mathbf{Reg}\sigma\mathbf{Frm}$ (see [61]).

For a σ -frame (or frame) homomorphism $h : L \longrightarrow M$ we say that h is *dense* (*codense*) in case $x = 0_L$ ($x = 1_L$) whenever $h(x) = 0_M$ ($h(x) = 1_M$). The well known proofs of the following results for regular frames carry over easily to the regular σ -frames.

Lemma 1.1.2

For any regular σ -frame L with σ -frame homomorphism $h : L \longrightarrow M$

1. *if h is dense, then h is monic.*
2. *if h is codense, then h is injective.*
3. *if M is compact and h is dense, then h is injective.*

The adjunction between regular σ -frames and Alexandroff spaces

An *Alexandroff space* is a pair (X, \mathcal{A}) with X any set and \mathcal{A} , called the *Alexandroff structure* on X , any collection of subsets of X satisfying the following criteria:

1. \mathcal{A} is closed under finite intersections and countable unions; in particular \emptyset and X belong to \mathcal{A} .

2. If $A, B \in \mathcal{A}$ and $A \cup B = X$, then there exist $C, D \in \mathcal{A}$ such that $A \cup C = X = B \cup D$ and $C \cap D = \phi$.
3. If $A \in \mathcal{A}$, then there is a sequence $\{A_n\}$ in \mathcal{A} such that $A = \bigcup(X - A_n)$.
4. Each pair of distinct points in X are contained in disjoint members of \mathcal{A} .

The elements of \mathcal{A} are called *cozero sets* and their complements *zero-sets*. A map $f : (X, \mathcal{A}) \rightarrow (Y, \mathcal{A}')$ is a *coz-map* if $f^{-1}(A) \in \mathcal{A}$ for each $A \in \mathcal{A}'$ i.e. preimages of cozero-sets are cozero-sets. This terminology is justified by the results of Gordon in [44] who shows that each $A \in \mathcal{A}$ is the cozero set of a coz-map $f : X \rightarrow \mathbb{R}_{\text{Alex}}$ where \mathbb{R}_{Alex} is the real line with cozero sets precisely the open subsets in the usual topology. The category of all Alexandroff spaces and coz-maps is denoted **Alex**. Given any Alexandroff space X , $\mathfrak{A}X$ will denote the lattice of cozero sets which is a σ -frame. Gilmour in [42] shows that the Alexandroff structures on a set X are precisely the regular sub- σ -frames of $\mathcal{P}X$, the power set of X . Then $\mathfrak{A} : \mathbf{Alex} \rightarrow \mathbf{Reg}\sigma\mathbf{Frm}$ defines a contravariant functor

$$\mathfrak{A} : \mathbf{Alex} \longrightarrow \mathbf{Reg}\sigma\mathbf{Frm}$$

$$\begin{array}{ccccc}
 X & \dashrightarrow & \mathfrak{A}X & & \\
 \downarrow h & & \uparrow \mathfrak{A}h & & \mathfrak{A}h(A) = h^{-1}(A) \\
 Y & \dashrightarrow & \mathfrak{A}Y & & \uparrow \\
 & & & & A
 \end{array}$$

which takes any Alexandroff space (X, \mathcal{A}) to the σ -frame $\mathfrak{A}X$. Any coz-map $h : (X, \mathcal{A}) \rightarrow (Y, \mathcal{A}')$ is taken to the σ -frame homomorphism $\mathfrak{A}h = h^{-1} : \mathfrak{A}Y \rightarrow \mathfrak{A}X$.

Let F be a filter (dual ideal) of a σ -frame L . If for each countable $X \subseteq L$, $\bigvee X \in F$ implies $X \cap F \neq \phi$, then F is called σ -prime. For any regular σ -frame L ,

$\Psi(L) = \{F \subseteq L : F \text{ is a } \sigma\text{-prime filter}\}$ is an Alexandroff space with cozero sets of the form $\Psi_x = \{F \in \Psi(L) : x \in F\}$ for each $x \in L$. This defines a contravariant functor

$$\Psi : \mathbf{Reg}\sigma\mathbf{Frm} \longrightarrow \mathbf{Alex}$$

$$L \dashrightarrow \Psi(L)$$

$$\begin{array}{ccccc} L & & \Psi(L) & & \Psi h(F) = h^{-1}(F) \\ \downarrow h & \dashrightarrow & \uparrow \Psi h & & \uparrow \\ M & & \Psi(M) & & F \end{array}$$

taking any σ -frame L to its *spectrum* $\Psi(L)$ and any σ -frame homomorphism $h : L \rightarrow M$ to the *coz-map* $\Psi h = h^{-1} : \Psi(M) \rightarrow \Psi(L)$. Ψ and \mathcal{U} define a dual adjunction between realcompact Alexandroff spaces and Alexandroff σ -frames (see [42]).

1.2 Structures

Uniform σ -frames

For the σ -frame L let A be a countable subset of L with $\bigvee A = 1$ i.e. $A \in \text{cov}L$. For any $x \in L$, the *star* of x relative to A is denoted by

$$Ax = \bigvee \{a \in A : a \wedge x \neq 0\}.$$

The *star* of A is the set

$$A^* = AA = \{Aa : a \in A\}.$$

Since $A \leq A^*$, A^* is a cover. The cover A *star-refines* the cover B if $A^* \leq B$. This is often expressed as $A \leq^* B$. For $\mu \subseteq \text{cov}L$ and $x, y \in L$ we write

$$x \triangleleft_\mu y \text{ iff } Ax \leq y \text{ for some } A \in \mu.$$

If μ is understood we merely write $x \triangleleft y$. The pair (L, μ) is called a *uniform σ -frame* if the following conditions are satisfied:

U_σ -I For any $A, B \in \mu$, $A \wedge B \in \mu$. Also if $C \in \mu$ and $C \leq D$, then $D \in \mu$

i.e. μ is a filter of covers.

U_σ -II For each $A \in \mu$ there is $B \in \mu$ such that $B^* \leq A$ and

U_σ -III μ is an *admissible* system of covers *i.e.* for each $x \in L$, there is a

countable $T \subseteq \{y \in L : y \triangleleft x\}$ such that $x = \bigvee T$.

The relation \triangleleft is called the *uniformly below* relation and is a *strong inclusion* (see [88]) *i.e.* \triangleleft satisfies

SI-I $x \leq a \triangleleft b \leq y \Rightarrow x \triangleleft y$.

SI-II $\triangleleft \subseteq L \times L$ is a sublattice *i.e.* $0 \triangleleft 0$, $1 \triangleleft 1$ and if $x \triangleleft a$ and $y \triangleleft b$ then $x \wedge y \triangleleft a \wedge b$ and $x \vee y \triangleleft a \vee b$.

SI-III $x \triangleleft y \Rightarrow x \prec y$.

SI-IV $x \triangleleft z \Rightarrow x \triangleleft y \triangleleft z$ for some $y \in L$ *i.e.* \triangleleft interpolates.

SI-V $x \triangleleft y \Rightarrow$ there exist $a, b \in L$ with $b \triangleleft a$, $b \vee y = 1$ and $a \wedge x = 0$.

Given uniform σ -frames (L, μ) and (N, ν) , a σ -frame homomorphism $h : L \rightarrow N$ between the underlying σ -frames is called *uniform* if $h(A) \in \nu$ for each $A \in \mu$. $\mathbf{U}\sigma\mathbf{Frm}$ is the category of uniform σ -frames and uniform homomorphisms. A uniform homomorphism $h : (L, \mu) \rightarrow (N, \nu)$ is called a *surjection* if it is onto and, for each $A \in \nu$, there exists $B \in \mu$ such that $h[B] \leq A$. A uniform σ -frame is *complete* if every dense surjection $h : (L, \mu) \rightarrow (N, \nu)$ is an isomorphism. A *completion* of a uniform σ -frame (N, ν) is a pair $((L, \mu), h)$ with (L, μ) a complete uniform σ -frame and $h : (N, \nu) \rightarrow (L, \mu)$ a dense uniform surjection. Walters in [89] shows the

existence of a unique completion up to a unique isomorphism for any uniform σ -frame. The dissertations [87] and [89] together with the paper [88] provide further analysis of uniform σ -frames.

Nearness frames

Let L be a frame. Any $A \subseteq L$ is a cover on L if $\bigvee A = 1$. $\text{cov}L$ will denote the collection of all covers on the frame L .

Let $A, B \in \text{cov}L$. We say that A *refines* B (written as $A \leq B$) if for each $a \in A$, $a \leq b$ for some $b \in B$. The *meet* of A and B is the set $A \wedge B = \{a \wedge b : a \in A \text{ and } b \in B\}$. For any element $x \in L$ the set $Ax = \bigvee\{a \in A : a \wedge x \neq 0\}$ is the *star* of x with respect to the cover A .

Let $\mu \subseteq \text{cov}L$ and $x, y \in L$. The element x is said to be *uniformly below* the element $y \in L$ (expressed as $x \triangleleft_{\mu} y$ or simply $x \triangleleft y$ if μ is understood) if $Ax \leq y$ for some $A \in \mu$. A *nearness* on the frame L is any non-empty collection μ of covers of L satisfying:

N-I For any $A, B \in \mu$, $A \wedge B \in \mu$.

N-II If $C \in \mu$ and $C \leq D$, then $D \in \mu$.

N-III μ is an *admissible* system of covers

i.e. for each $x \in L$, $x = \bigvee\{y \in L : y \triangleleft x\}$.

The members of the nearness μ on the frame L are called *nearness covers*. The nearness μ is called *strong* if for each $A \in \mu$ the cover $\check{A} = \{b \in L : b \triangleleft a \text{ for some } a \in A\}$ is uniform and μ is *almost uniform* if it is strong and \triangleleft interpolates.

The frame L together with the nearness structure μ (written as (L, μ)) is called a *nearness frame*. For nearness frames (L, μ) and (N, ν) a frame homomorphism $h : (L, \mu) \rightarrow (N, \nu)$ is called a *uniform* homomorphism if $h(A) = \{h(a) : a \in A\} \in \nu$

whenever $A \in \mu$. Nearness frames and uniform homomorphisms are the objects and maps in the category \mathbf{NFrm} treated in [17] and [32]. We will focus on the following subcategories in the sequel.

- **SNFrm**: the category of strong nearness frames.
- **LNfFrm**: the category of Lindelöf nearness frames.
- **SepNFrm**: the category of separable nearness frames. A nearness frame is *separable* if the nearness is generated by its countable uniform covers. We will discuss this category in more detail in Chapter 4.
- **SepSLNFrm**: the category of separable strong Lindelöf nearness frames.

For nearness frames (L, μ) and (N, ν) , a uniform homomorphism $h : (L, \mu) \rightarrow (N, \nu)$ is a *surjection* if h is onto such that $\{h_*(A) : A \in \nu\}$ generates μ , where $h_* : N \rightarrow L$ is the *right Galois adjoint* to h , which is a map preserving all meets (see [31]) such that

$$h(x) \leq y \text{ iff } x \leq h_*(y)$$

and is explicitly given by

$$h_*(y) = \bigvee \{a \in L : h(a) \leq y\}.$$

The following Lemma about the right Galois adjoint is required in Chapter 3 and Chapter 5.

Lemma 1.2.1

Let $h : L \rightarrow M$ be a dense onto frame homomorphism. Then

1. $(h_*(x))^* = h_*(x^*)$ for each $x \in M$.
2. $h(x^*) = (h(x))^*$ for each $x \in L$.

3. If L and M are regular frames then

(a) $h : (L, \text{cov}L) \longrightarrow (M, \text{cov}M)$ is a nearness surjection.

(b) $a \prec b$ in M implies that $h_*(a) \prec h_*(b)$ in L .

(c) $a \prec\prec b$ in M implies that $h_*(a) \prec\prec h_*(b)$ in L .

Proof :

(1) Let $x \in M$. Since h_* preserves \wedge and h is dense we have

$$h_*(x^*) \wedge h_*(x) = h_*(x^* \wedge x) = h_*(0) = 0.$$

Thus $h_*(x^*) \leq (h_*(x))^*$. Since $h_*(x) \wedge (h_*(x))^* = 0$, we have

$$\begin{aligned} & h(h_*(x) \wedge (h_*(x))^*) = 0 \\ \Rightarrow & hh_*(x) \wedge h[(h_*(x))^*] = 0 \\ \Rightarrow & x \wedge h[(h_*(x))^*] = 0 \quad (\text{since } h \text{ is onto, } hh_* = 1) \\ \Rightarrow & h[(h_*(x))^*] \leq x^* \\ \Rightarrow & (h_*(x))^* \leq h_*(x^*). \end{aligned}$$

Hence $h_*(x^*) = (h_*(x))^*$.

(2) Let $x \in L$. Then $h(x^*) \wedge h(x) = h(x^* \wedge x) = h(0) = 0$. Thus $h(x^*) \leq [h(x)]^*$.

Also

$$\begin{aligned} & [h(x)]^* \wedge h(x) = 0 \\ \Rightarrow & hh_*([h(x)]^*) \wedge h(x) = 0 \quad (\text{since } h \text{ is onto}) \\ \Rightarrow & h(h_*([h(x)]^*) \wedge x) = 0 \\ \Rightarrow & h_*([h(x)]^*) \wedge x = 0 \quad (\text{since } h \text{ is dense}) \\ \Rightarrow & h_*([h(x)]^*) \leq x^* \\ \Rightarrow & [h(x)]^* \leq h(x^*). \end{aligned}$$

Hence $h(x^*) = [h(x)]^*$.

(3a) We need to show that $\{h_*(T) : T \in \text{cov}M\}$ generates $\text{cov}L$. Let $A \in \text{cov}L$. Then $\check{A} = \{x \in L : x \prec a \text{ for some } a \in A\} \in \text{cov}L$. Since h is a frame homomorphism, $h(\check{A}) \in \text{cov}M$. Since $\check{A} \leq h_*h(\check{A})$, $h_*h(\check{A}) \in \text{cov}M$. We next show that $h_*h(\check{A}) \leq A$. To this end, let $h_*h(x) \in h_*h(\check{A})$. Then $x^* \vee a = 1_L$ for some $a \in A$. Since h is onto, $hh_* = id_M$ and

$$h(h_*h(x) \wedge x^*) = h(x) \wedge h(x^*) = h(x \wedge x^*) = h(0_L) = 0_M.$$

Since h is dense, $h_*h(x) \wedge x^* = 0_L$. Since $x^* \vee a = 1_L$, $h_*h(x) \prec a$. Hence $h_*h(x) \leq a$ so that $h_*h(\check{A}) \leq A$.

(3b) Let $a \prec b$ in M . Then $a^* \vee b = 1_M$ i.e. $\{a^*, b\} \in \text{cov}M$. By (3a), $h_*({a^*, b}) = \{h_*(a^*), h_*(b)\} \in \text{cov}L$. By (1), $(h_*(a))^* = h_*(a^*)$. Thus $(h_*(a))^* \vee h_*(b) = 1_L$. Hence, $h_*(a) \prec h_*(b)$ in L .

(3c) Let $a \prec\prec b$ in M . Then there exists a scale $\{x_\alpha : \alpha \in \mathbb{Q} \cap [0, 1]\}$ such that $x_0 = a, x_1 = b$ and $x_\alpha \prec x_\beta$ whenever $\alpha < \beta$. Then $h_*(x_0) = h_*(a)$, $h_*(x_1) = h_*(b)$ and, by (3b), $h_*(x_\alpha) \prec h_*(x_\beta)$ whenever $\alpha < \beta$. Thus $\{h_*(x_\alpha) : \alpha \in \mathbb{Q} \cap [0, 1]\}$ is a scale in L with the desired property so that $h_*(a) \prec\prec h_*(b)$ in L . \square

The nearness frame (L, μ) is called *complete* if any dense uniform surjection $h : (N, \nu) \rightarrow (L, \mu)$ is an isomorphism. A *completion* of $(L, \mu) \in \mathbf{NFrm}$ is any pair $((N, \nu), h)$ with (N, ν) a complete nearness frame and $h : (N, \nu) \rightarrow (L, \mu)$ a dense uniform surjection. The following description of the completion of a nearness frame is to be found in [4] and [17] where it is shown that the completion of a nearness frame is unique up to isomorphism.

Let $\mathfrak{D}L$ be the lattice of all non-empty down-sets in L i.e. all $U \subseteq L$ such that $0 \in U$ and $x \in U$ whenever $x \leq y$ and $y \in U$. Then $\mathfrak{D}L$ is a frame and the join map $\bigvee : \mathfrak{D}L \rightarrow L, U \rightsquigarrow \bigvee U$, is a frame homomorphism with right Galois adjoint $\downarrow : L \rightarrow \mathfrak{D}L$ taking each $x \in L$ to $\downarrow x$. Now let (L, μ) be a nearness frame. For $x \in L$ let $k(x) = \{y \in L : y \triangleleft x\}$. For $A \in \mu$, $x \wedge A = \{x \wedge a : a \in A\}$. Let CL be

the system of all $U \in \mathfrak{D}L$ such that

1. $x \in U$ whenever $k(x) \subseteq U$ and
2. $x \in U$ whenever $x \wedge A \subseteq U$ for some $A \in \mu$.

Then CL is a frame with intersection for meet and $\gamma_L : CL \rightarrow L$ given by \bigvee is a dense homomorphism with right Galois adjoint \downarrow . Then $\{\downarrow A : A \in \mu\}$ generates an admissible nearness $C\mu$ on CL and $(CL, C\mu) = C(L, \mu)$ is complete. Finally $(C(L, \mu), \gamma_L)$ is the completion of the nearness frame (L, μ) unique up to isomorphism *i.e.* for any completion $((M, \nu), h)$ there is an isomorphism $g : CL \rightarrow M$ such that the following triangle commutes

$$\begin{array}{ccc}
 (L, \mu) & & \\
 \uparrow h & \swarrow \gamma_L & \\
 (M, \nu) & \xleftarrow[\simeq]{g} & C(L, \mu)
 \end{array}$$

i.e. $h \circ g = \gamma_L$. An alternate description of the completion of a nearness frame is given by Dube in [32] as in the sense of Kritz in [60]. In [19] it is shown that, in general, completion is not a coreflection on all nearness frames but it is in the category **SNFrm**.

Uniform frames

If A is a cover of the frame L then the *star* of A is defined as the set $A^* = AA = \{Aa : a \in A\}$ which is also a cover of L as $A \leq A^*$. We say that A *star refines* B (written as $A \leq^* B$) if $A^* \leq B$. A *uniform frame* is a nearness frame (L, μ) where μ satisfies the additional condition

N-IV For each $A \in \mu$ there is $B \in \mu$ such that $B \leq^* A$.

A frame map $h : (L, \mu) \longrightarrow (N, \nu)$ between uniform frames (L, μ) and (N, ν) is also called a *uniform homomorphism* if $h(A) \in \nu$ whenever $A \in \mu$. The category **UFrm** denotes the category of uniform frames and uniform homomorphisms studied in [56] and [13]. For uniform frames (L, μ) and (N, ν) , a uniform homomorphism $h : (L, \mu) \longrightarrow (N, \nu)$ is a *surjection* if h is onto such that the covers $h(A)$ with $A \in \mu$ generate ν . A uniform frame (L, μ) is *complete* if every dense uniform surjection to (L, μ) is an isomorphism. A *completion* of the uniform frame (L, μ) is a pair $((N, \nu), h)$ with (N, ν) a complete uniform frame and $h : (N, \nu) \longrightarrow (L, \mu)$ a dense uniform surjection. For different constructions of the completion of a uniform frame see [13] or [60]. In contrast with nearness frames, completion is a coreflection for all uniform frames as is shown in [56], also [13] and [60].

The concept of surjections between nearness frames and that of complete nearness frames, illustrated on page 14, together with the corresponding concepts for uniform frames given above are related. The following Lemma provides the details of this relationship.

Lemma 1.2.2 *If $h : (L, \mu) \longrightarrow (M, \nu)$ is a dense surjection between strong nearness frames (L, μ) and (M, ν) , then $\{h_*(T) \mid T \in \nu\}$ generates $\mu \Leftrightarrow \{h(S) \mid S \in \mu\}$ generates ν , where $h_* : M \longrightarrow N$ is the right adjoint of h .*

Proof :

Suppose that $\{h_*(T) \mid T \in \nu\}$ generates μ . Let $A \in \nu$. Then $h_*(A) \in \mu$ and since h is onto, $hh_*(A) = id_M(A) = A$. Thus $\{h(S) \mid S \in \mu\}$ generates ν .

The proof of the converse is similar to that of Lemma 1.2.1(3a). Suppose that $\{h(S) \mid S \in \mu\}$ generates ν . Let $B \in \mu$. Since (L, μ) is a strong nearness, $\check{A} \in \mu$. Since h is uniform, $h(\check{A}) = \{h(x) \mid x \in \check{A}\} = \{h(x) \mid x \triangleleft_\mu a \text{ for some } a \in A\} \in \nu$. Since $\check{A} \leq h_*h(\check{A})$, $h_*h(\check{A}) \in \mu$. If $h_*h(x) \in h_*h(\check{A})$, then $x \triangleleft_\mu a$ for some $a \in A$. Thus $x \prec a$. Then $x^* \vee a = 1_L$. Since h is onto,

$$h[h_*h(x) \wedge x^*] = hh_*h(x) \wedge h(x^*) = h(x) \wedge h(x^*) = h(x \wedge x^*) = h(0_L) = 0_M$$

Since h is dense, $h_*h(x) \wedge x^* = 0_L$. Since $x^* \vee a = 1_L$, $h_*h(x) \prec a$. Consequently, $h_*h(x) \leq a$. Thus $h_*h(\check{A}) \leq A$ and hence $\{h_*(T) \mid T \in \nu\}$ generates μ . \square

By the above Lemma the notion of surjections between strong nearness frames coincides with that of surjections between uniform frames. Indeed, uniform frames are strong nearness frames.

Nearness and uniform spaces

The following notations and details on nearness spaces can be found in [45].

Let X be a set and $\xi = \{\mathcal{A} : \mathcal{A} \subseteq \mathcal{P}X\}$ a collection of covers of X i.e. $\bigcup\{A : A \in \mathcal{A}\} = X$. The pair (X, ξ) is called a *nearness space* if the following conditions are satisfied:

N-I $\mathcal{A} \in \xi$ and $\mathcal{A} \leq \mathcal{B} \Rightarrow \mathcal{B} \in \xi$ where $\mathcal{A} \leq \mathcal{B}$ means that for each $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$.

N-II $\{X\} \in \xi$.

N-III $\mathcal{A}, \mathcal{B} \in \xi \Rightarrow \mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\} \in \xi$.

N-IV $\mathcal{A} \in \xi \Rightarrow \text{int}(\mathcal{A}) = \{\text{int}(A) : A \in \mathcal{A}\} \in \xi$ where
 $\text{int}(A) = \{x \in X : \{A, X - \{x\}\} \in \xi\}$.

The members of ξ are called *nearness covers*. A *uniformly continuous map* $f : (X, \xi) \rightarrow (Y, \nu)$ between nearness spaces is a function such that for each $\mathcal{A} \in \nu$, $f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\} \in \xi$. **Near** will be the category of nearness spaces and uniformly continuous maps. For the nearness space (X, ξ) a subset β of ξ is a *base* for ξ if and only if

$$\xi = \{\mathcal{A} : \exists \mathcal{B} \in \beta \text{ such that } \mathcal{B} \leq \mathcal{A}\}.$$

A *subbase* for ξ is any subset τ of ξ such that all finite intersections of elements of τ form a base for ξ . Given a set X , a map $A \rightarrow \text{int}(A)$, of $\mathcal{P}X$ into $\mathcal{P}X$ is called an *interior operator* in X if the following conditions are satisfied:

- (1) $\text{int}(A) \subseteq A$.
- (2) $\text{int}(\text{int}(A)) = \text{int}(A)$.
- (3) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$.
- (4) $\text{int}(X) = X$.

Then every nearness space (X, ξ) has an underlying topological space whose structure is determined by the interior operator int as follows

$$\begin{array}{ccc} \text{int} & : & \mathcal{P}X \longrightarrow \mathcal{P}X \\ & & U \longmapsto \text{int}(U) \end{array}$$

with $\text{int}(U) = \{x \in X : st(x, \mathcal{V}) \subseteq U \text{ for some } \mathcal{V} \in \xi\}$ where $st(x, \mathcal{V}) = \bigcup\{V \in \mathcal{V} : x \in V\}$. Then, a set A is open in X if and only if $\text{int}(A) = A$. We then say that the nearness ξ on X generates or induces the topology Γ_ξ . Then $U \in \Gamma_\xi$ if and only if for each $x \in U$ there is $\mathcal{A} \in \xi$ such that $st(x, \mathcal{A}) \subseteq U$. int then defines a symmetric topology on X determining a functor $T : \mathbf{Near} \rightarrow \mathbf{STop}$ with \mathbf{STop} the category of symmetric topological spaces *i.e.* those spaces satisfying $x \in cl\{y\}$ if and only if $y \in cl\{x\}$. T has a right inverse $\mathbf{STop} \rightarrow \mathbf{Near}$ which is a full embedding of \mathbf{STop} as a bireflective subcategory of \mathbf{Near} .

The nearness space (X, ξ) is called a *uniform space* if ξ satisfies the additional condition

$$\mathcal{A} \in \xi \Rightarrow \mathcal{B} \leq^* \mathcal{A} \text{ for some } \mathcal{B} \in \xi$$

where

$$\mathcal{B} \leq^* \mathcal{A} \text{ means that } \mathcal{B}^* = \{st(B, \mathcal{B}) : B \in \mathcal{B}\} \leq \mathcal{A}$$

with

$$st(B, \mathcal{B}) = \bigcup\{D \in \mathcal{B} : D \cap B \neq \emptyset\}.$$

We then have the category \mathbf{Unif} of uniform spaces and uniformly continuous maps. Isbell's book [55] provides an extensive treatment of the theory of uniform spaces, using the above cover approach.

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Chapter 2

The category of nearness σ -frames

2.1 $\mathbf{N}\sigma\mathbf{Frm}$ and $\mathbf{SN}\sigma\mathbf{Frm}$

We introduce the concept of a nearness σ -frame as a generalization of a uniform σ -frame defined by Walters in [87]. This is in analogy to the nearness structure on a frame established by Banaschewski and Pultr in [13] and [17].

Let L be a σ -frame and let $\mu \subseteq \text{cov}L = \{A \subseteq L : \bigvee A = 1 \text{ and } A \text{ is countable}\}$. For $a, b \in L$ we say that a is μ -strongly below b , $a \triangleleft_{\mu} b$ (or simply $a \triangleleft b$ for brevity) provided that $Aa \leq b$ for some μ -cover A . If for each $a \in L$,

$$a = \bigvee T \text{ for some countable } T \subseteq \{b \in L : b \triangleleft a\}$$

then μ is *admissible*. A *nearness* on L is any admissible filter $\mu \subseteq \text{cov}L$. The couple (L, μ) is then a *nearness σ -frame*. The members of μ are called *uniform* or *nearness covers*. A map $h : (L, \mu) \rightarrow (M, \nu)$ between nearness σ -frames (L, μ) and (M, ν) is a *uniform* or *nearness homomorphism* if h is a σ -frame homomorphism on the underlying σ -frames preserving uniform covers *i.e.* $h(A) \in \nu$ whenever $A \in \mu$. We then have the category $\mathbf{N}\sigma\mathbf{Frm}$ of nearness σ -frames and uniform homomorphisms. As in the case for \mathbf{NFrm} , the following Lemma asserts that regularity is of particular importance in the theory of nearness σ -frames, as it is in $\mathbf{Reg}\sigma\mathbf{Frm}$ that a nearness

lives.

Lemma 2.1.1

L has a nearness if and only if L is regular.

Proof :

(\Rightarrow) Suppose that μ is a nearness on L . Let $a \in L$. Then $a = \bigvee T$ for some countable $T \subseteq \{b \in L : b \triangleleft a\}$. However, if $t \in T$ then $At \leq a$ for some μ -cover A . Then $s = \bigvee \{y \in A : y \wedge t = 0\}$ separates t and a . Thus $t \prec a$ and T is then a countable subset of $\{b \in L : b \prec a\}$. So L is regular.

(\Leftarrow) Let L be regular and $a \in L$. Let μ be the filter generated by all countable covers on L . By regularity, $a = \bigvee S$ for some countable $S \subseteq \{x \in L : x \prec a\}$. If $x \prec a$, then there exists $s \in L$ which separates x and a . Then $A = \{s, a\} \in \mu$ with $Ax = a$. So, $x \triangleleft a$ and thus S is a countable subset of $\{x \in L : x \triangleleft a\}$ with $a = \bigvee S$ rendering μ admissible and hence a nearness on L . \square

Consequently the filter in $covL$ generated by all finite covers is a nearness on L . Moreover, $covL$ is a nearness on L , called the *fine* nearness and any filter on $covL$ containing all finite covers is a nearness on L . A nearness σ -frame (L, μ) is called *fine* if $\mu = covL$.

Lemma 2.1.2

A compact regular σ -frame L has a unique nearness, namely covL, which is a uniformity.

Proof :

We already have that $covL$ is a nearness. For uniqueness, let ν be any nearness on L . We show that ν contains all finite covers and hence all covers of L . Let

$A = \{a_1, a_2, \dots, a_n\}$ be any finite cover on L . By admissibility, $a_i = \bigvee_m \{x_{im} \in L : x_{im} \triangleleft_\nu a_i\}$ for each $1 \leq i \leq n$. Then $\bigvee_i \bigvee_m \{x_{im} \in L : x_{im} \triangleleft_\nu a_i\} = 1$. By compactness $\bigvee_i \bigvee_j \{x_{ij} \in L : x_{ij} \triangleleft_\nu a_i\} = 1$ for some $\{x_{ij}\} \in \{x_{im}\}$. Then for each i there exists $B_i \in \nu$ such that $B_i x_{ij} \leq a_i$. Then $B = \bigwedge_i B_i \in \nu$ and $B x_{ij} \leq a_i$ for each i . Then for each $b \in B$, some $b \wedge x_{ij} \neq 0$ so that $b \leq B x_{ij} \leq a_i$. Thus $B \leq A$ and so $A \in \nu$.

The proof that $\text{cov}L$ is a uniformity essentially follows the proof in [4]. We need only show that any finite cover on L has a finite star refinement. Let $A = \{a_1, a_2, \dots, a_n\}$ be any finite cover on L . By regularity, $a_i = \bigvee_m \{x_{im} \in L : x_{im} \prec a_i\}$ for each $1 \leq i \leq n$. Then $\bigvee_i \bigvee_m \{x_{im} \in L : x_{im} \prec a_i\} = 1$. Again, by compactness $\bigvee_i \bigvee_j \{x_{ij} \in L : x_{ij} \prec a_i\} = 1$ for some $\{x_{ij}\} \in \{x_{im}\}$. Then for each i there exists $t_i \in L$ such that $x_{ij} \wedge t_i = 0$ and $t_i \vee a_i = 1$. Since $\{x_{ij}\}$ is a cover on L , $\bigwedge_i t_i = 0$. Let $B = \bigwedge_i \{a_i, t_i\}$. Then B is a cover on L and each element in B can be expressed for $E \subseteq \{1, 2, \dots, n\}$ as

$$a_E = \bigwedge_i \{a_i : i \in E\} \wedge \bigwedge_j \{t_j : j \notin E\}$$

Then $a_E \leq a_i$ for each i and $B \leq A$. Hence, if one obtains a finite cover $C_i \leq^* \{a_i, t_i\}$ for each i , then

$$C = \bigwedge_i C_i \leq^* B \leq A$$

It then suffices to show that each two-cover $\{a, b\}$ has a finite star-refinement. Since L is normal from [6], there exists $u \prec a$ and $v \prec b$ such that $u \vee v = 1$. Then there exists $s, t \in L$ such that

$$u \wedge s = 0, \quad s \vee a = 1, \quad v \wedge t = 0, \quad t \vee b = 1$$

Let $D = \{a, s\} \wedge \{b, t\} \wedge \{u, v\} = \{a \wedge b \wedge u, a \wedge b \wedge v, a \wedge t \wedge u, b \wedge s \wedge v\}$. Then $D \in \text{cov}L$ and $D(s \wedge b \wedge v) \leq b$, $D(a \wedge b \wedge u) \leq a$, $D(a \wedge b \wedge v) \leq b$ and $D(a \wedge t \wedge u) \leq a$. Thus $D \leq^* \{a, b\}$. \square

Thus for compact regular σ -frames, $\text{cov}L$ is its unique nearness which in fact is a uniformity. Also, by the above Lemma, every compact nearness σ -frame is fine.

The nearness μ is *strong* if for each $A \in \mu$ there exists $B \in \mu$ such that for each $b \in B$, $b \triangleleft a$ for some $a \in A$. For any regular σ -frame L , if $A \in \text{cov}L$, then for each $a \in A$, $a = \bigvee T_a$ for some countable $T_a \subseteq \{y \in L : y \prec a\}$. Let $a \in A$, then for each $y \in T_a$, $y \prec a$. Since L is a regular σ -frame \prec interpolates (see page 5). Thus for each $y \in T_a$, $y \prec b_y \prec a$ for some $b_y \in L$. Let $B_y^a = \{b_y : y \in T_a\}$. Then for each $y \in T_a$, B_y^a is countable. Consequently, $B = \bigcup_{a \in A} B_y^a$ is countable such that $\bigvee B = \bigvee_{a \in A} \bigvee_{y \in T_a} B_y^a = \bigvee_{a \in A} \bigvee_{y \in T_a} b_y = \bigvee_{a \in A} a = \bigvee A = 1$. Hence $B \in \text{cov}L$ such that for each $b \in B$, $b \prec a$ for some $a \in A$. All this shows is that $\text{cov}L$ is a strong nearness on L . We will denote the category of strong nearness σ -frames by **SN σ Frm**.

A filter ν is a *preuniformity* on L if for each $A \in \nu$ there exists $B \in \nu$ such that $B \leq^* A$. A nearness μ is a *uniformity* if for each $A \in \mu$ there exists $B \in \mu$ such that $B \leq^* A$ i.e. every μ -cover has a μ -star refinement. So, a uniformity on L is then a preuniformity with the additional admissibility criterion. We call the nearness μ *almost uniform* if μ is strong and \triangleleft interpolates.

Lemma 2.1.3

If μ is a uniformity on the σ -frame L , then μ is almost uniform.

Proof:

Let $A \in \mu$. Find $B \in \mu$ such that $B \leq^* A$. Then for each $b \in B$, $Bb \leq a$ for some $a \in A$. Thus $b \triangleleft a$ and so μ is strong.

Now let $x \triangleleft y$ in L . Then $Cx \leq y$ for some $C \in \mu$. Again as μ is a uniformity we can find $D \in \mu$ such that $D \leq^* C$. If $d \in D$ and $d \wedge Dx \neq 0$, then

$$0 \neq d \wedge Dx = d \wedge \bigvee \{t \in D : t \wedge x \neq 0\} = \bigvee \{d \wedge t : t \in D, t \wedge x \neq 0\}.$$

Thus $d \wedge \tilde{d} \neq 0$ for some $\tilde{d} \in D$ such that $\tilde{d} \wedge x \neq 0$. Then $d \leq D\tilde{d}$ with $\tilde{d} \wedge x \neq 0$. As $D \leq^* C$, $D\tilde{d} \leq c$ for some $c \in C$. If $c \wedge x = 0$, then $x \wedge \tilde{d} \leq x \wedge D\tilde{d} \leq c \wedge x$. Thus $\tilde{d} \wedge x = 0$, a contradiction. So we have that $c \wedge x \neq 0$. Then $d \leq D\tilde{d} \leq c \leq Cx$

and hence, $\bigvee\{d \in D : d \wedge Dx \neq 0\} = D(Dx) \leq Cx$. Since $x \triangleleft Dx$ we have $x \triangleleft Dx \triangleleft D(Dx) \leq Cx \leq y$. Thus $x \triangleleft Dx \triangleleft y$ and so \triangleleft interpolates. Hence, μ is almost uniform. \square

Lemma 2.1.4

For any nearness σ -frame (L, μ) with $h : L \rightarrow M$ any onto σ -frame homomorphism, $\nu = \{B \in \text{cov}M : h(A) \leq B \text{ for some } A \in \mu\}$ is a nearness on M . Furthermore, ν is strong or a uniformity whenever μ is strong or a uniformity respectively.

Proof:

If $A, B \in \nu$ then $h(C) \leq A$ and $h(D) \leq B$ for some $C, D \in \mu$. Then $C \wedge D \in \mu$ and as h is a σ -frame map we have $h(C \wedge D) = h(C) \wedge h(D) \leq A \wedge B$. Thus $A \wedge B \in \nu$. Clearly, if $S \in \nu$ and $S \leq T$ then $h(E) \leq S \leq T$ for some $E \in \mu$. Thus $T \in \nu$. Hence, ν is a filter of M -covers.

For admissibility, let $y \in M$. Since h is onto, $y = h(x)$ for some $x \in L$. By the admissibility of μ , $x = \bigvee T$ for some countable $T \subseteq \{t \in L : t \triangleleft_\mu x\}$. If $t \in T$ then $At \leq x$ for some $A \in \mu$. Since h is a σ -frame homomorphism we have

$$\bigvee h(A) = h(\bigvee A) = h(1_L) = 1_M.$$

Thus $h(A) \in \nu$. Further, if $h(a) \wedge h(t) \neq 0_M$ for $a \in A$, then $h(a \wedge t) \neq 0_M$ and hence $a \wedge t \neq 0_L$. Thus $a \leq At$. Then $h(A)h(t) \leq h(At) \leq h(x) = y$. Thus $t \triangleleft_\mu x$ implies that $h(t) \triangleleft_\nu y$. Let $S = \{h(t) : t \in T\}$. Then S is a countable subset of $\{s \in M : s \triangleleft_\nu y\}$ and since $x = \bigvee T$, $y = \bigvee S$ showing the admissibility of ν .

Now suppose that μ is strong. Let $B \in \nu$ and find $C \in \mu$ such that $h(C) \leq B$. Since μ is strong there exists $A \in \mu$ such that for each $a \in A$, $a \triangleleft_\mu c$ for some $c \in C$. Then $h(A) \in \nu$ and $h(c) \triangleleft_\nu h(a)$. But $h(a) \leq b$ for some $b \in B$. Thus $h(c) \triangleleft_\nu b$. Thus $h(A) \in \nu$ such that for each $h(a) \in h(A)$, $h(a) \triangleleft_\nu b$ for some $b \in B$ showing that ν

is strong.

Now let $(L, \mu) \in \mathbf{U}\sigma\mathbf{Frm}$ and $B \in \nu$. Then $h(C) \leq B$ as above. Since μ is a uniformity, $A \leq^* C$ for some $A \in \mu$. Then for each $a \in A$, $Aa \leq c$ for some $c \in C$.

Then

$$\begin{aligned}
h(A)h(a) &= \bigvee_{y \in A} \{h(y) : h(y) \wedge h(a) \neq 0_M\} \\
&= \bigvee_{y \in A} \{h(y) : h(y \wedge a) \neq 0_M\} \quad (\text{since } h \text{ is a } \sigma\text{-frame map}) \\
&\leq \bigvee_{y \in A} \{h(y) : y \wedge a \neq 0_L\} \quad (\text{since } h(y \wedge a) \neq 0_M \text{ implies } y \wedge a \neq 0_L) \\
&= h(\bigvee \{y \in A : y \wedge a \neq 0_L\}) \quad (\text{since } h \text{ is a } \sigma\text{-frame map}) \\
&= h(Aa) \\
&\leq h(c).
\end{aligned}$$

Thus $h(A) \leq^* h(C) \leq B$. Thus $h(A) \in \nu$ such that $h(A) \leq^* B$. Hence ν is a uniformity. \square

For a nearness μ on L , $A \in \mu$ is *normal* if there is some sequence of uniform covers $(A_n) \subseteq \mu$ such that $A = A_1$ and $A_{n+1} \leq^* A_n$ for each n . We denote by μ_N the normal covers of μ . Clearly μ_N is a preuniformity on L . If $x \triangleleft_{\mu_N} a$ in L , we say that x is *uniformly (strongly) normally below* a and in keeping with [1], we write this as $x \triangleleft a$. The following results for the relation \triangleleft are in analogy with those in [1] and [87] respectively.

Proposition 2.1.1

For $(L, \mu) \in \mathbf{N}\sigma\mathbf{Frm}$

1. \triangleleft is a sublattice of $L \times L$.
2. for any $a, b, x, y \in L$ with $a \leq x \triangleleft y \leq b$, $a \triangleleft b$.

3. $x \triangleleft y \Rightarrow x \triangleleft_{\mu} y \Rightarrow x \prec y$.

4. \triangleleft interpolates.

5. \triangleleft is preserved by uniform σ -frame homomorphisms.

Proof:

1. If $a \triangleleft b$ and $c \triangleleft d$ in (L, μ) , then $Aa \leq b$ and $Cc \leq d$ for some $A, C \in \mu_N$.
As μ_N is a preuniformity, $A \wedge C \in \mu_N$ and $(A \wedge C)(a \vee c) \leq b \vee d$. Also $(A \wedge C)(a \wedge c) \leq b \wedge d$. Thus $a \vee c \triangleleft b \vee d$ and $a \wedge c \triangleleft b \wedge d$. As for any $B \in \mu_N$, $B0 \leq 0$ and $B1 \leq 1$ thus $0 \triangleleft 0$ and $1 \triangleleft 1$.
2. If $a \leq x \triangleleft y \leq b$ in L , then $Ax \leq y$ for some $A \in \mu_N$ and $Aa \leq Ax \leq y \leq b$.
So, $a \triangleleft b$.
3. Obvious.
4. If $x \triangleleft y$, then $Ax \leq y$ for some $A \in \mu_N$. As μ_N is a preuniformity $B \leq^* A$ for some $B \in \mu_N$. Then $B(Bx) \leq Ax \leq y$ and $x \triangleleft Ax \triangleleft y$. Thus \triangleleft interpolates.
5. Let $h : (L, \mu) \rightarrow (M, \nu)$ be any uniform σ -frame homomorphism in $\mathbf{N}\sigma\mathbf{Frm}$ with $x \triangleleft y$ in (L, μ) . Then $Ax \leq y$ for some μ_N -cover A . Then $A = A_1$ and $A_{n+1} \leq^* A_n$ for each n for some sequence $\{A_n\} \subseteq \mu$. Since h is uniform $h(A) = h(A_1)$ and $h(A_{n+1}) \leq^* h(A_n)$ for each n . So, $\{h(A_n)\} \subseteq \nu_N$. Thus $h(A) \in \nu_N$ and $Ax \leq y$ implies that $h(A)h(x) \leq h(Ax) \leq h(y)$. Thus $h(x) \triangleleft h(y)$. \square

Lemma 2.1.5

For $a \triangleleft b \triangleleft c$ in $(L, \mu) \in \mathbf{N}\sigma\mathbf{Frm}$ there exists $s, t \in L$ such that s separates $a \prec b$, $t \vee c = 1$ and $t \triangleleft s$.

Proof:

Let $a \triangleleft b$ in L . Then there exists $A \in \mu_N$ such that $Aa \leq b$. Since μ_N is a

preuniformity, $B \leq^* A$ for some $B \in \mu_N$. But $B(Ba) \leq Aa$. Thus $a \triangleleft Ba \triangleleft Aa \leq b \triangleleft c$. Let $s = \bigvee\{b \in B : b \wedge a = 0\}$. Then $a \wedge s = 0$ and $s \vee Ba = 1$. So, $a \prec Ba$. Since $Ba \leq b$, also $a \prec b$. Set $t = \bigvee\{b \in B : b \wedge Ba = 0\}$. Then $t \vee Ba = 1$ and as $Ba \triangleleft c$, $t \vee c = 1$. Now $Bt = \bigvee\{b \in B : b \wedge t \neq 0\} \leq s$ as, if $b \in B$ with $b \wedge t \neq 0$, then $b \wedge b_m \neq 0$ for some $b_m \in B$ with $b_m \wedge Ba = 0$. Then $b_m \wedge x = 0$ for each $x \in B$ with $x \wedge a \neq 0$. In particular, if $b \wedge a \neq 0$, then $b \wedge b_m = 0$ which contradicts $b \wedge b_m \neq 0$. Thus $b \wedge a = 0$ and so $b \leq s$. Hence, $Bt \leq s$ and thus $t \triangleleft s$. \square

2.2 The Samuel compactification

Baboolal and Ori in [1] gave a description of the Samuel compactification of a nearness frame in terms of the normally regular ideals of a nearness frame itself. Walters in [87] employed the techniques developed by Banaschewski and Pultr in [13] in constructing the Samuel compactification of a uniform σ -frame. In this section, we present the compact regular coreflection of a nearness σ -frame as an adaptation of the corresponding results in [1] and [87].

An ideal I in any nearness σ -frame (L, μ) is

1. *uniformly regular* if for each $x \in I$, $x \triangleleft y$ for some $y \in I$.
2. *uniformly normally regular* if for each $x \in I$, $x \triangleleft y$ for some $y \in I$ and
3. *countably generated* if there is a sequence (y_n) in I such that for each $x \in I$, $x \leq y_m$ for some m .

It should be noted that if μ is a uniformity on L then there is no distinction between \triangleleft and \triangleleft . So in uniform σ -frames there is no distinction between the uniformly regular ideals and the uniformly normally regular ones. However, in $\mathbf{N}\sigma\mathbf{Frm}$ this may not be the case. Every uniformly normally regular ideal is uniformly regular

but the converse may not be true. Let $\mathfrak{NR}_\sigma L$ be the set of all countably generated uniformly normally regular ideals of (L, μ) . Clearly any $J \in \mathfrak{NR}_\sigma L$ may be generated by a sequence $a_1 \triangleleft a_2 \dots$ and any ideal generated by any such sequence belongs to $\mathfrak{NR}_\sigma L$. Using the same method as that in [13], in analogy with [87], we show that $\mathfrak{NR}_\sigma L$ is a compact regular σ -frame (L, μ) .

Proposition 2.2.1

$\mathfrak{NR}_\sigma L$ is a compact regular σ -frame.

Proof :

Suppose that $I, J \in \mathfrak{NR}_\sigma L$. If $x \in I \cap J$, then $x \triangleleft s$ and $x \triangleleft t$ for some $s \in I$ and $t \in J$ as I and J are normally regular. Then $x \triangleleft s \wedge t \in I \cap J$. Thus $I \cap J \in \mathfrak{NR}_\sigma L$. Again by the properties of \triangleleft , $I \vee J \in \mathfrak{NR}_\sigma L$. As any updirected join of normally regular ideals is again normally regular $\mathfrak{NR}_\sigma L$ is closed under finite \wedge and (countable) \vee . Since $\triangleleft \Rightarrow \prec$, $\mathfrak{NR}_\sigma L \subseteq \mathfrak{KL}$, where \mathfrak{KL} is the compact regular coreflection of the σ -frame L (see [8]). Thus $\mathfrak{NR}_\sigma L$ is compact.

For regularity, consider any $J \in \mathfrak{NR}_\sigma L$ with generating sequence a_1, a_2, \dots . By repeated interpolation of \triangleleft , for each n let J_n be the ideal generated by a sequence

$$a_n = a_{n_0} \triangleleft a_{n_1} \triangleleft a_{n_2} \triangleleft \dots \triangleleft a_{n+1}.$$

Then $J_n \in \mathfrak{NR}_\sigma L$ and $J = \bigvee J_n$. Also for each n , $a_n \triangleleft a_{n+1} \triangleleft a_{n+2}$ and by Lemma 2.1.5 we can find x_n, y_n such that $a_n \wedge x_n = 0$, $x_n \vee a_{n+1} = 1$, $y_n \vee a_{n+2} = 1$ and $y_n \triangleleft x_n$. Let I_n be the ideal generated by the sequence

$$y_{n_0} = y_n \triangleleft y_{n_1} \triangleleft y_{n_2} \triangleleft \dots \triangleleft x_n.$$

Then $I_n \in \mathfrak{NR}_\sigma L$ and $I_n \cap J_{n+1} = \{0\}$ (as $a_n \wedge x_n = 0$) and $I_n \vee J_{n+2} = L$ (as $a_{n+2} \vee y_n = 1$). Thus $J_{n+1} \prec J_n$ in $\mathfrak{NR}_\sigma L$. Hence $\mathfrak{NR}_\sigma L$ is regular. \square

As a compact regular σ -frame has a unique nearness (the fine nearness), $\mathbf{KReg}\sigma\mathbf{Frm}$ may be seen as a full subcategory of $\mathbf{N}\sigma\mathbf{Frm}$.

Lemma 2.2.1

For $(L, \mu) \in \mathbf{N}\sigma\mathbf{Frm}$, $\rho_L : \mathfrak{NA}_\sigma L \rightarrow (L, \mu)$ given by join is a uniform σ -frame homomorphism.

Proof :

ρ_L is a σ -frame homomorphism being the restriction of $\kappa_L : \mathfrak{KL} \rightarrow L$ (see [8] and [87]) to the countably generated uniformly normally regular ideals. For uniformity, take any finite cover $\{J_1, J_2, \dots, J_n\}$ of $\mathfrak{NA}_\sigma L$. Then there exists $a_i \in J_i$ such that $a_1 \vee a_2 \vee \dots \vee a_n = 1$. Let $c_i = \rho_L(J_i)$. By the uniformly normal regularity of J_i , $a_i \triangleleft c_i$. Then $B_i a_i \leq c_i$ for some $B_i \in \mu_N$ for each $i = 1, 2, \dots, n$. Thus $B = \bigwedge_{i=1}^n B_i \in \mu$ and $B a_i \leq c_i$ for each i . As $\bigvee_{i=1}^n a_i = 1$, for each $t \in B$, $t \wedge a_i \neq 0$ for some i . Thus $t \leq B a_i \leq c_i$. Hence, $B \leq \{c_1, c_2, \dots, c_n\} = C$. Thus $C \in \mu$ i.e. $\rho_L(\{J_1, J_2, \dots, J_n\}) \in \mu$ and so ρ_L is uniform. \square

Using Lemma 2.1.2 we have the following result which corresponds to Lemma 3.19 in [87].

Lemma 2.2.2

If $M \in \mathbf{KReg}\sigma\mathbf{Frm}$, then $\rho_M : \mathfrak{NA}_\sigma M \rightarrow M$ is an isomorphism.

Proof :

The proof is immediate since in $\mathbf{KReg}\sigma\mathbf{Frm}$, $\triangleleft = \triangleleft = \prec$ so that $\mathfrak{NA}_\sigma M = \mathfrak{KM}$ and $\rho_M = \kappa_L : \mathfrak{KM} \rightarrow M$ (see [8]) is the coreflection. Hence, if M is a compact regular σ -frame, then ρ_M is an isomorphism. \square

Proposition 2.2.2

$\mathfrak{NR}_\sigma L$ is the compact regular coreflection of the nearness σ -frame (L, μ) with coreflection $\rho_L : \mathfrak{NR}_\sigma L \rightarrow (L, \mu)$ and coreflection functor \mathfrak{NR}_σ .

Proof :

Let $h : (M, \nu) \rightarrow (L, \mu)$ be any uniform σ -frame morphism with M compact. Then $\mathfrak{NR}_\sigma h$ is the map taking each $I \in \mathfrak{NR}_\sigma M$ to the ideal generated by $h(I)$, $\langle h(I) \rangle$. By Proposition 2.1.1(5), h preserves \triangleleft and so $\mathfrak{NR}_\sigma h$ is a well-defined σ -frame homomorphism to $\mathfrak{NR}_\sigma L$. We then have the following

$$\begin{array}{ccc} \mathfrak{NR}_\sigma L & \xleftarrow{\mathfrak{NR}_\sigma h} & \mathfrak{NR}_\sigma M \\ \downarrow & & \downarrow \simeq \rho_M \\ (L, \mu) & \xleftarrow{h} & (M, \nu) \end{array}$$

As M is compact regular, by the previous result, ρ_M is an isomorphism. Put $\bar{h} = \mathfrak{NR}_\sigma h \rho_M^{-1}$. We then have that $\rho_L \bar{h} = \rho_L \mathfrak{NR}_\sigma h \rho_M^{-1} = h$. Since ρ_L is dense and monic, the uniqueness of \bar{h} follows. \square

The above establishes $\mathfrak{NR}_\sigma L$ as the Samuel compactification of a nearness σ -frame via its countably generated uniformly normally regular ideals.

2.3 The uniform coreflection

For the nearness frame (L, μ) , the family μ_N of normal uniform covers is a preuniformity on L . Let $k : L \rightarrow L$ be the interior operator given by

$$k(a) = \bigvee \{x \in L : x \triangleleft a\}$$

where as before $x \triangleleft a$ means that there exists $A \in \mu_N$ such that $Ax \leq a$. Then $\mathcal{U}L = \text{Fix } k$ is a subframe of L (see [13]), and $\mathcal{U}\mu = \{k(A) : A \in \mu_N\}$ is a uniformity

on $\mathcal{U}L$. Then $(\mathcal{U}L, \mathcal{U}\mu)$ is the uniform coreflection of the nearness frame (L, μ) with coreflection map given by the inclusion $j : \mathcal{U}L \longrightarrow L$ (see [1]).

Now, let (M, ν) be any uniform frame. As in [87], the *cozero part* of (M, ν) is the set

$$\text{Coz}_u M = \{a \in M : a = h((0, 1]) \text{ for some } h : \mathfrak{D}[0, 1] \longrightarrow (M, \nu) \in \mathbf{UFrm}\}.$$

The members of $\text{Coz}_u M$ are called *uniformly cozero elements*. Also let

$$\text{Coz}_u \nu = \{(a_n) = A \in \text{cov}M : a_n \in \text{Coz}_u M, \text{ for each } n\}$$

i.e. $\text{Coz}_u \nu$ is the collection of all countable uniform covers consisting of uniformly cozero elements. It is shown in [87] (see Proposition 3.27) that $(\text{Coz}_u M, \text{Coz}_u \nu)$ is a uniform sigma-frame. Given any nearness sigma-frame (L, μ) we show that $(\text{Coz}_u \mathcal{U}(\mathcal{H}L), \text{Coz}_u \mathcal{U}(\mathcal{H}\mu))$ is the uniform coreflection of (L, μ) , with $(\mathcal{H}L, \mathcal{H}\mu)$ the nearness frame of all sigma ideals of L and $(\mathcal{U}(\mathcal{H}L), \mathcal{U}(\mathcal{H}\mu))$ its uniform coreflection.

Let L be any σ -frame. Consider the frame envelope of L , $\mathcal{H}L$, the Lindelöf frame of all σ -ideals of L . Then $\downarrow : L \longrightarrow \mathcal{H}L$ taking each $a \in L$ to the principal ideal generated by a , $\downarrow a = \{y \in L : y \leq a\}$, is the universal homomorphism from σ -frames to frames (see [3]).

Lemma 2.3.1

For each countable collection (a_n) in L , $\bigvee_{\mathcal{H}L} \downarrow a_n = \downarrow \bigvee a_n$.

Proof :

Indeed $\downarrow \bigvee a_n$ is a σ -ideal. So $\downarrow \bigvee a_n \in \mathcal{H}L$. Certainly, $\downarrow \bigvee a_n \supseteq \bigvee_{\mathcal{H}L} \downarrow a_n$. Thus, $\downarrow \bigvee a_n$ is an upper bound for $\downarrow a_n$ for each n . Moreover, it is the least for if $J \in \mathcal{H}L$ such that $\downarrow a_n \subseteq J$ for each n , then $a_n \in J$ for each n . Since J is a σ -ideal, $\bigvee a_n \in J$. Thus $\downarrow \bigvee a_n \subseteq J$. Hence, $\downarrow \bigvee a_n = \bigvee_{\mathcal{H}L} \downarrow a_n$. \square

Now, let (L, μ) be any nearness σ -frame and $\mathcal{H}\mu$ be the filter on L generated by $\downarrow A = \{\downarrow a : a \in A\}$ where $A \in \mu$. Then by the above Lemma for each $A = (a_n) \in \mu$, $\bigvee_n \downarrow a_n = \downarrow \bigvee_n a_n = \downarrow 1_L = L = 1_{\mathcal{H}L}$. Thus $\downarrow A \in \text{cov}\mathcal{H}L$ for each $A \in \mu$. We then have the following result for any nearness σ -frame (L, μ) .

Lemma 2.3.2

$\mathcal{H}\mu$ is a nearness on the frame $\mathcal{H}L$.

Proof :

We need only show admissibility. So let $J \in \mathcal{H}L$. Then $J = \bigvee \{\downarrow a : a \in J\}$. But for each $a \in J$, by the admissibility of μ , $a = \bigvee \{b_n : b_n \triangleleft_\mu a\}$. Let $I_n = \downarrow b_n$ for each n with $b_n \triangleleft_\mu a$. Then $I_n \in \mathcal{H}L$ for each n . Also, $Bb_n \leq a$ for some $B \in \mu$ whenever $b_n \triangleleft_\mu a$. Then $\downarrow B \in \mathcal{H}\mu$ and $(\downarrow B)(\downarrow b_n) = \downarrow (Bb_n) \subseteq \downarrow a \subseteq J$. So, $(\downarrow B)I_n \subseteq J$ and thus $I_n \triangleleft_{\mathcal{H}\mu} J$ for each n . Moreover, $J = \bigvee \{I_n : I_n \triangleleft_{\mathcal{H}\mu} J\}$. Hence $\mathcal{H}\mu$ is a nearness on $\mathcal{H}L$. \square

Lemma 2.3.3

If μ is a strong nearness on the σ -frame L , then $(\mathcal{H}L, \mathcal{H}\mu)$ is a strong nearness frame.

Proof:

Let $\downarrow A \in \mathcal{H}\mu$ for any $A \in \mu$. We must show that

$$(\downarrow \check{A}) = \{J \in \mathcal{H}L : J \triangleleft_{\mathcal{H}\mu} \downarrow a \text{ for some } a \in A\} \in \mathcal{H}\mu$$

Since $A \in \mu$ and μ is strong there exists $B \in \mu$ such that for each $b \in B$, $b \triangleleft_\mu a$ for some $a \in A$. Then $\downarrow B \in \mathcal{H}\mu$ and for each $\downarrow b \in \downarrow B$ we have that $(\downarrow B)(\downarrow b) \subseteq \downarrow a$ for some $a \in A$. Then $\downarrow b \triangleleft_{\mathcal{H}\mu} \downarrow a$ and thus $\downarrow B \leq (\downarrow \check{A})$. Hence $(\downarrow \check{A}) \in \mathcal{H}\mu$. \square

Lemma 2.3.4

$\mathcal{H} : \mathbf{N}\sigma\mathbf{Frm} \longrightarrow \mathbf{LN}\mathbf{Frm}$ is functorial.

Proof :

We have the following

$$\begin{array}{c}
 \mathcal{H} \quad : \quad \mathbf{N}\sigma\mathbf{Frm} \longrightarrow \mathbf{LN}\mathbf{Frm} \\
 \\
 \text{objects} \quad \quad L \longrightarrow \mathcal{H}L \\
 \\
 \text{morphisms} \quad \begin{array}{ccc}
 \begin{array}{c} L \\ \downarrow h \\ M \end{array} & \begin{array}{c} \mathcal{H}L \\ \downarrow \mathcal{H}h \\ \mathcal{H}M \end{array} & \begin{array}{c} I \\ \downarrow \\ \langle h(I) \rangle \end{array} & \begin{array}{c} A \in \mathcal{H}\mu \\ \downarrow \\ \downarrow h(A) \in \mathcal{H}\nu \end{array}
 \end{array}
 \end{array}$$

For any nearness σ -frame (L, μ) , $(\mathcal{H}L, \mathcal{H}\mu)$ is a nearness frame by Lemma 2.3.2. That $\mathcal{H}L$ is Lindelöf is well known (see [3]). So for objects any $(L, \mu) \in \mathbf{N}\sigma\mathbf{Frm}$ is taken by \mathcal{H} to $(\mathcal{H}L, \mathcal{H}\mu) \in \mathbf{LN}\mathbf{Frm}$. For morphisms, let $h : (L, \mu) \longrightarrow (M, \nu)$ be any uniform σ -frame homomorphism. Then $\mathcal{H}h : (\mathcal{H}L, \mathcal{H}\mu) \longrightarrow (\mathcal{H}M, \mathcal{H}\nu)$ takes any σ -ideal I of L to the σ -ideal $\langle h(I) \rangle$ in M generated by $h(I)$. $\mathcal{H}h$ is a σ -frame homomorphism since for any $J \in \mathcal{H}L$ we have that

$$\begin{aligned}
 \mathcal{H}h\left(\bigvee_{\mathcal{H}M} J\right) &= \left\langle h\left(\bigvee_{\mathcal{H}M} J\right)\right\rangle \\
 &= \left\langle \bigvee_{\mathcal{H}M} h(J)\right\rangle \quad (\text{since } h \text{ is a } \sigma\text{-frame homomorphism}) \\
 &= \bigvee_{\mathcal{H}M} \mathcal{H}h(J).
 \end{aligned}$$

$\mathcal{H}h$ is also uniform for if $A \in \mu$, then $\mathcal{H}h(A) = \downarrow h(A)$. But h is uniform so $h(A) \in \nu$ and $\downarrow h(A) \in \mathcal{H}\nu$. Also, we have $\mathcal{H}id_L = id_{\mathcal{H}L}$ and if $h : (L, \mu) \longrightarrow (M, \nu)$ and $g : (M, \nu) \longrightarrow (N, \tau)$ are uniform σ -frame homomorphisms, then $\mathcal{H}(g \circ h) = \mathcal{H}g \circ \mathcal{H}h$. Hence, we have that \mathcal{H} is functorial. \square

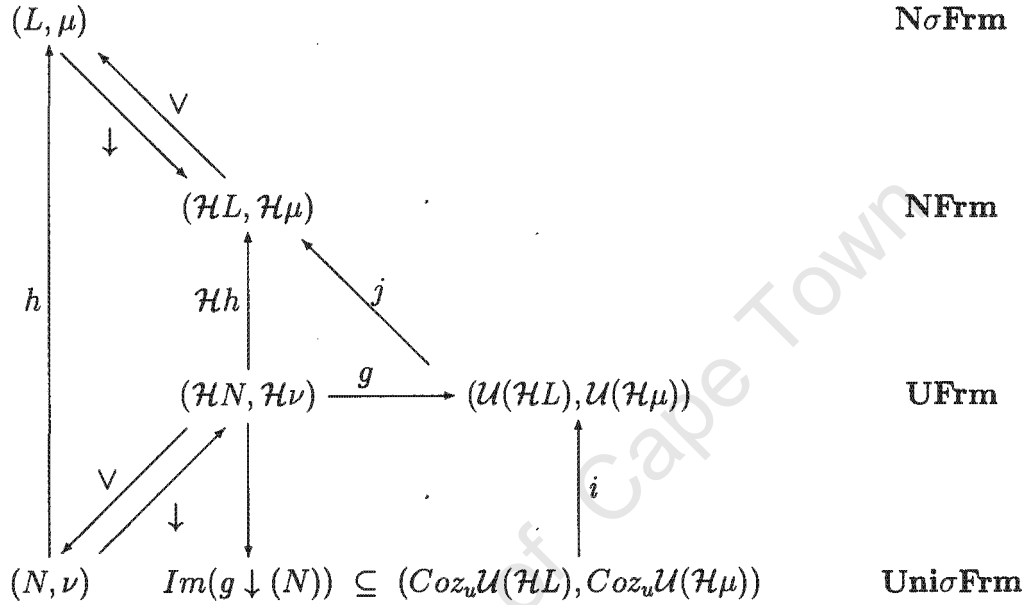
Theorem 2.3.1

$((Coz_u\mathcal{U}(\mathcal{H}L), Coz_u\mathcal{U}(\mathcal{H}\mu)))$ is the uniform coreflection of the nearness σ -frame (L, μ) .

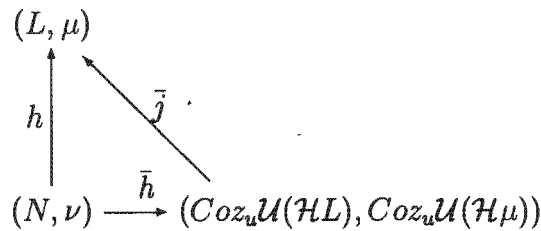
Proof :

Let (N, ν) be any uniform σ -frame with uniform homomorphism

$h : (N, \nu) \longrightarrow (L, \mu)$. We then have the following,



The map $\downarrow : (N, \nu) \longrightarrow (\mathcal{H}N, \mathcal{H}\nu)$ is a σ -frame homomorphism. We claim that $\bar{j} = \bigvee \circ j \circ i$ is the coreflection map, where $\bigvee : (\mathcal{H}L, \mathcal{H}\mu) \longrightarrow (L, \mu)$, j (the inclusion) is the uniform coreflection map of the nearness frame $(\mathcal{H}L, \mathcal{H}\mu)$ and i the inclusion. We need to find $\bar{h} : (N, \nu) \longrightarrow (Coz_u\mathcal{U}(\mathcal{H}L), Coz_u\mathcal{U}(\mathcal{H}\mu))$ such that the triangle below commutes *i.e.* $\bar{j}\bar{h} = h$.



Since j is the uniform coreflection map and $\mathcal{H}h : (\mathcal{H}N, \mathcal{H}\nu) \longrightarrow (\mathcal{H}L, \mathcal{H}\mu)$ with

$(\mathcal{H}N, \mathcal{H}\nu)$ a uniform frame (by Proposition 3.25 in [87]) there exists a unique uniform homomorphism $g : (\mathcal{H}N, \mathcal{H}\nu) \longrightarrow (\mathcal{U}(\mathcal{H}L), \mathcal{U}(\mathcal{H}\mu))$ such that $fg = \mathcal{H}h$. But $(\mathcal{H}N, \mathcal{H}\nu)$ is a Lindelöf uniform frame. The Lindelöf elements, however, are precisely the uniformly cozero elements (see [87]) which are precisely the principle ideals $\downarrow x$ for each $x \in N$ (see [3]). Since g is uniform and uniform homomorphisms preserve uniform cozero elements, g maps cozero elements to cozero elements. Thus $Im(g \downarrow N) \subseteq (Coz_u \mathcal{U}(\mathcal{H}L), Coz_u \mathcal{U}(\mathcal{H}\mu))$. Let $\bar{h} = g \downarrow$. Since $\mathcal{H}h(\downarrow x) = \langle h(\downarrow x) \rangle = \downarrow h(x)$ implies

$$\bigvee \mathcal{H}h \downarrow (x) = \bigvee (\mathcal{H}h(\downarrow x)) = \bigvee \downarrow h(x) = h(x),$$

we have $\bigvee \mathcal{H}h \downarrow = h$. Then the desired triangle above commutes since

$$\bar{j}\bar{h} = \bigvee j i g \downarrow = \bigvee (j g) \downarrow = \bigvee \mathcal{H}h \downarrow = h.$$

It now remains to show that \bar{h} is unique. To this end, suppose that $h' : (N, \nu) \longrightarrow (Coz_u \mathcal{U}(\mathcal{H}L), Coz_u \mathcal{U}(\mathcal{H}\mu))$ with $\bar{j}h' = h$. But for any $I \in Coz_u \mathcal{U}(\mathcal{H}L)$, if $\bar{j}(I) = 0$, then $\bigvee j i(I) = 0$. Thus $\bigvee I = 0$ and hence $I = \{0\}$. Thus \bar{j} is dense and hence monic. Since $\bar{j}h' = h = \bar{j}\bar{h}$, $h = h'$. Hence \bar{h} is unique with the property that $\bar{j}\bar{h} = h$ completing the proof. \square

It is a noteworthy observation that the composition of the uniform coreflection $\mathbf{N}\sigma\mathbf{Frm} \longrightarrow \mathbf{U}\sigma\mathbf{Frm}$ of Theorem 2.3.1 with the Samuel compactification $\mathbf{U}\sigma\mathbf{Frm} \longrightarrow \mathbf{KReg}\sigma\mathbf{Frm}$ of [87] gives us, by basic category theory, an alternative way of getting the Samuel compactification of nearness σ -frames.

Chapter 3

Completeness

3.1 Separable strong Lindelöf nearness frames and strong nearness σ -frames

Lindelöf frames have been recalled in Chapter 1. [76] provides further details into the study of Lindelöf frames. In this section we show that \mathcal{H} and Coz are also functors between $SepSLNFrm$ and $SN\sigma Frm$ defining an adjoint equivalence. Let $(L, \mu) \in LNFrm$ and set

$$Coz \mu = \{A \in \mu : A \subseteq Coz L \text{ and } A \text{ is countable}\}.$$

Kaiser in [59] defines *powers* of covers inductively for a frame L . For the cover A in a frame L let

$$\begin{aligned} A^1 &= A, \\ A^2 &= A^* = AA = \{Aa : a \in A\}, \\ A^3 &= A(A^2), \\ &\vdots \\ A^{n+1} &= A(A^n). \end{aligned}$$

For a system μ of covers in the frame L , μ is *down-directed* if every pair of μ -covers has a common refinement. It is then shown in [59] that for any admissible down-directed system of covers μ , the system of covers $\mu^k = \{A^k : A \in \mu\}$ for $k \geq 1$ is also admissible and down-directed as well. As a special case of Proposition 2.6 in [59] we subsequently prove that $(\text{Coz}L, \text{Coz} \mu)$ is a nearness σ -frame for any Lindelöf nearness frame (L, μ) .

Lemma 3.1.1

If $A \in \mu$ then there exists $D \in \text{Coz} \mu$ such that $D \leq A^3$.

Proof :

Let $A \in \mu$. Since L is Lindelöf, we can find a countable cover $B \subseteq A$. Consider $AB = \{Ab : b \in B\}$. If $0 \neq a \in A$, then $0 \neq a = a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$. Thus $\exists b \in B$ such that $a \wedge b \neq 0$. Then $a \leq Ab$ and hence $A \leq AB \leq A^*$. Thus $C = AB$ is a countable uniform cover such that $A \leq C$.

Since $A^* \in \mu$ and $A^* \leq AC$, $AC \in \mu$ is countable such that $AC \leq A(A^*) = A^3$. Then for each $c \in C$, $Ac \leq A(Aa)$ for some $a \in A$. Thus $c \triangleleft_\mu A(Aa)$ and hence $c \prec A(Aa)$. Since L is regular and Lindelöf, L is normal and so \prec interpolates. As a result there exists a sequence $\{x_n\}$ in L such that $c = x_1 \prec x_2 \prec \dots \prec A(Aa)$. Let $d_c = \bigvee x_n$. By Proposition 1.1.1(3), $d_c \in \text{Coz}L$. Thus $c \leq d_c \leq A(Aa)$ and so $C \leq D = \{d_c : c \in C\}$. Then $D \in \mu$ and $D \subseteq \text{Coz}L$ is countable. Hence, $D \in \text{Coz} \mu$ such that $D \leq A^3$. \square

Theorem 3.1.1 $(\text{Coz}L, \text{Coz} \mu)$ is a nearness σ -frame for any Lindelöf nearness frame (L, μ) .

Proof :

We show admissibility. Let $x \in \text{Coz}L$. Since μ is a nearness, the uniform covers form a down-directed admissible system and by Proposition 2.6 of [59], $\mu^3 = \{A^3 : A \in \mu\}$ is also admissible on L . As L is regular and Lindelöf, by Lemma 1.1.1.(3) $\sigma(L) =$

$\text{Coz}L$. Thus for $x \in \text{Coz}L$, $x = \bigvee T$ for some countable $T \subseteq \{y \in \text{Coz}L : y \triangleleft_{\mu^3} x\}$. Then for $t \in T$, $A^3 t \leq x$ for some $A \in \mu$. By the above Lemma there exists $B \in \text{Coz} \mu$ such that $B \leq A^3$. Hence $Bt \leq A^3 t \leq x$. Consequently, $t \triangleleft_{\text{Coz} \mu} x$. Thus T is a countable subset of $\{y \in \text{Coz}L : y \triangleleft_{\text{Coz} \mu} x\}$ such that $x = \bigvee T$. Hence, $\text{Coz} \mu$ is admissible on the σ -frame $\text{Coz} L$. \square

Lemma 3.1.2

Let $(L, \mu) \in \text{SepSLNFrm}$. If $A \in \mu$, then there is $B \in \text{Coz} \mu$ such that $B \leq A$.

Proof :

Let $A \in \mu$. Since μ is strong, $\check{A} \in \mu$. Since μ is separable, there exists a countable $C \in \mu$ such that $C \leq \check{A}$. Then there is a countable $D \subseteq \check{A}$ such that $C \leq D$. Then D is a countable uniform cover such that $D \leq A$. If $d \in D$ then $d \triangleleft_{\mu} a$ for some $a \in A$. Then $d \prec a$. Since L is regular and Lindelöf, \prec interpolates so that $d \prec x_1 \prec x_2 \prec x_3 \prec \dots \prec a$. For each $d \in D$ let $b_d = \bigvee_n x_n$. Then $B = \{b_d : d \in D\} \subseteq \text{Coz} L$. Since for each $d \in D$, $d \leq b_d \leq a$ for some $a \in A$, $D \leq B \leq A$. Hence $B \in \text{Coz} \mu$ such that $B \leq A$. \square

Lemma 3.1.3 If $(L, \mu) \in \text{SepSLNFrm}$ then $(\text{Coz} L, \text{Coz} \mu) \in \text{SN}\sigma\text{Frm}$.

Proof :

Let $(L, \mu) \in \text{SepSLNFrm}$. By Theorem 3.1.1, $(\text{Coz} L, \text{Coz} \mu)$ is indeed a nearness σ -frame. To show that $\text{Coz} \mu$ is strong, let $A \in \text{Coz} \mu$. Then A is countable, $A \subseteq \text{Coz} L$ and $A \in \mu$. Since μ is separable and strong, by the above Lemma, there is a countable $C \in \mu$ such that for each $c \in C$, $c \triangleleft_{\mu} a$ for some $a \in A$. Again, by the above Lemma, there is $B \in \text{Coz} \mu$ such that $B \leq C$. Let $b \in B$. Then $b \leq c \triangleleft_{\mu} a$ for some $c \in C$ and $a \in A$. Then there exists $D \in \mu$ such that $Dc \leq a$. Again, by the above Lemma, there exists $E \in \text{Coz} \mu$ such that $E \leq D$. Then $Eb \leq Ec \leq Dc \leq a$ and thus $b \triangleleft_{\text{Coz} \mu} a$. Hence $B \in \text{Coz} \mu$ such that for each $b \in B$, $b \triangleleft_{\text{Coz} \mu} a$ for some $a \in A$ showing that $\text{Coz} \mu$ is a strong nearness on the σ -frame $\text{Coz} L$. \square

Lemma 3.1.4

$Coz : \mathbf{SepSLNFrm} \longrightarrow \mathbf{SN}\sigma\mathbf{Frm}$ is functorial.

Proof :

We have the following

$$\begin{array}{c}
 Coz : \mathbf{SepSLNFrm} \longrightarrow \mathbf{SN}\sigma\mathbf{Frm} \\
 \\
 \text{objects} \quad L \longrightarrow Coz L \\
 \\
 \text{morphisms} \quad \begin{array}{ccc}
 \begin{array}{c} L \\ \downarrow g \\ M \end{array} & \begin{array}{c} Coz L \\ \downarrow Cozg = g|_{Coz L} \\ Coz M \end{array} & \begin{array}{c} a \\ \downarrow \\ g(a) \end{array} \quad \begin{array}{c} A \in Coz \mu \\ \downarrow \\ g(A) \in Coz \nu \end{array}
 \end{array}
 \end{array}$$

By Lemma 3.1.3, for objects, any separable strong Lindelöf nearness frame (L, μ) is taken by Coz to the strong nearness σ -frame $(Coz L, Coz \mu)$. For any uniform homomorphism $h : (L, \mu) \longrightarrow (M, \nu)$ between separable strong Lindelöf nearness frames, $Coz h : (Coz L, Coz \mu) \longrightarrow (Coz M, Coz \nu)$ is given by the restriction of h to $Coz L$ i.e. $Coz h = h|_{Coz L}$. Since uniform homomorphisms preserve cozero elements, $Coz h$ is a well defined mapping into $Coz M$. Also, for any $A \in Coz \mu$, $h(A) \in \nu$ is countable and $h(A) \subseteq Coz M$. Thus $h(A) \in Coz \nu$. So, $Coz h$ is uniform. Also, $Coz(id_L) = id_{L|_{Coz L}} = id_{Coz L}$. If $h : (L, \mu) \longrightarrow (N, \nu)$ and $g : (M, \nu) \longrightarrow (N, \tau)$ are uniform homomorphisms, then $Coz(g \circ h) = (g \circ h)|_{Coz L} = (g|_{Coz L}) \circ (h|_{Coz L})$. Hence Coz is functorial. \square

By Lemma 2.3.3 for any strong nearness σ -frame (L, μ) , $(\mathcal{H}L, \mathcal{H}\mu)$ is a strong nearness frame. Moreover, for any strong nearness σ -frame (L, μ) , $(\mathcal{H}L, \mathcal{H}\mu) \in \mathbf{SepSLNFrm}$ and as in Lemma 2.3.4, $\mathcal{H} : \mathbf{SN}\sigma\mathbf{Frm} \longrightarrow \mathbf{SepSLNFrm}$ is also functorial.

Lemma 3.1.5

$\eta_L : L \longrightarrow \text{Coz}\mathcal{H}L$ given by $\eta_L(x) = \downarrow x$ for each $x \in L$ is a uniform isomorphism in $\text{SN}\sigma\text{Frm}$.

Proof:

The countable elements of $\mathcal{H}L$ are precisely the Lindelöf elements which are exactly the principal ideals (see [3]). But $\mathcal{H}L$ is a regular Lindelöf frame, hence normal. Thus $\mathcal{H}L$ is a completely regular Lindelöf frame. By Corollary 4 in [8], the principal ideals $\downarrow x$ for each $x \in L$ are indeed cozero elements of $\mathcal{H}L$. Thus η_L is well defined. At the unstructured level, by Proposition 1 in [3], η_L is a σ -frame homomorphism. For any $A \in \mu$, η_L maps A into $\downarrow A = \{\downarrow a : a \in A\}$ which is a countable uniform cover of $\mathcal{H}L$ consisting of cozero elements. Thus $\eta_L(A) \in \text{Coz}\mathcal{H}\mu$ and so η_L is uniform. As η_L is an isomorphism (see [9]), the result follows. \square

Theorem 3.1.2

η is a natural isomorphism.

Proof:

Since η_L is an isomorphism for each $L \in \text{SN}\sigma\text{Frm}$ by the above Lemma, we need only show naturality *i.e.* for any uniform σ -frame homomorphism $h : (L, \mu) \longrightarrow (M, \nu)$, the following is a commutative square

$$\begin{array}{ccc}
 (L, \mu) & \xrightarrow{\eta_L} & (\text{Coz}\mathcal{H}L, \text{Coz}\mathcal{H}\mu) & & a & \longmapsto & \downarrow a \\
 \downarrow h & & \downarrow \text{Coz}\mathcal{H}h & & \downarrow & & \\
 (M, \nu) & \xrightarrow{\eta_M} & (\text{Coz}\mathcal{H}M, \text{Coz}\mathcal{H}\nu) & & h(a) & \longmapsto & \downarrow h(a)
 \end{array}$$

Let h be such and let $a \in L$. Then

$$\begin{aligned}
 (\text{Coz}\mathcal{H}h \circ \eta_L)(a) &= \text{Coz}\mathcal{H}h(\downarrow a) \\
 &= \mathcal{H}h|_{\text{Coz}\mathcal{H}L}(\downarrow a)
 \end{aligned}$$

$$\begin{aligned}
&= \mathcal{H}h(\downarrow a) \quad (\text{as } \downarrow a \in \text{Coz}\mathcal{H}L) \\
&= \langle h(\downarrow a) \rangle \\
&= \langle \downarrow h(a) \rangle \\
&= \downarrow h(a) \\
&= (\eta_M \circ h)(a).
\end{aligned}$$

Thus $\text{Coz}\mathcal{H}h \circ \eta_L = \eta_M \circ h$. Hence, η is a natural isomorphism. \square

Lemma 3.1.6

If $(L, \mu) \in \text{SepSLNFrm}$ then $\epsilon_L : \mathcal{H}\text{Coz} L \rightarrow L$ given by join is an uniform isomorphism.

Proof :

At the unstructured level, ϵ_L is a frame homomorphism. Moreover ϵ_L is an isomorphism as L is completely regular since L is a regular Lindelöf frame (see [8]). If $A = \{\downarrow a_n : a_n \in \text{Coz} L\} \in \text{Coz} \mu$, $\{a_n\} \in \mu$ then

$$\epsilon_L(A) = \bigvee A = \bigvee \{\downarrow a_n : a_n \in \text{Coz} L\} = \{a_n\}.$$

But $\{a_n\} \in \mu$. Hence ϵ_L is uniform. Together with Lemma 3.1.2 we have that $\epsilon_L : (\mathcal{H}\text{Coz} L, \mathcal{H}\text{Coz} \mu) \rightarrow (L, \mu)$ is an uniform isomorphism. \square

Theorem 3.1.3

ϵ is a natural isomorphism.

Proof :

As ϵ_L is an isomorphism by the above Lemma, we need only show naturality *i.e.* for any uniform frame homomorphism $g : (L, \mu) \rightarrow (M, \nu)$, the following is a

commutative square

$$\begin{array}{ccc}
 (\mathcal{H}Coz L, \mathcal{H}Coz \mu) & \xrightarrow{\epsilon_L} & (L, \mu) \\
 \mathcal{H}Cozg \downarrow & & \downarrow g \\
 (\mathcal{H}Coz M, \mathcal{H}Coz \nu) & \xrightarrow{\epsilon_M} & (M, \nu)
 \end{array}
 \qquad
 \begin{array}{ccc}
 J & \longmapsto & \bigvee J \\
 \downarrow & & \downarrow \\
 \langle g(J) \rangle & \longmapsto & \bigvee \langle g(J) \rangle
 \end{array}$$

Let g be such and let $J \in \mathcal{H}Coz L$. Then

$$\begin{aligned}
 (\epsilon_M \circ \mathcal{H}Cozg)(J) &= (\epsilon_M \circ \mathcal{H}g|_{Coz L})(J) \\
 &= \epsilon_M(\mathcal{H}g(J)) \\
 &= \epsilon_M(\langle g(J) \rangle) \\
 &= \bigvee \langle g(J) \rangle \\
 &= \bigvee g(J) \\
 &= g\left(\bigvee J\right) \quad (\text{as } g \text{ is a frame homomorphism}) \\
 &= (g \circ \epsilon_L)(J).
 \end{aligned}$$

Thus $\epsilon_M \circ \mathcal{H}Cozg = g \circ \epsilon_L$. Hence, ϵ_L is a natural transformation. \square

Theorem 3.1.4

For each $(L, \mu) \in \text{SepSLNFrm}$, ϵ_L is a universal arrow.

Proof :

Let $(M, \nu) \in \text{SN}\sigma\text{Frm}$ with $h : \mathcal{H}M \rightarrow L$ any uniform frame homomorphism. We need to find a unique $g : M \rightarrow Coz L$ such that the following triangle commutes

$$\begin{array}{ccc}
 (L, \mu) & & \\
 \uparrow h & \swarrow \epsilon_L & \\
 (\mathcal{H}M, \mathcal{H}\nu) & \xrightarrow{\mathcal{H}g} & (\mathcal{H}Coz L, \mathcal{H}Coz \mu)
 \end{array}$$

i.e. $\epsilon_L \circ \mathcal{H}g = h$. Consider the following

$$\begin{array}{ccc}
 \mathcal{H}M & (\text{Coz}\mathcal{H}M, \text{Coz}\mathcal{H}\nu) & \xleftarrow{\eta_M} (M, \nu) \\
 \downarrow h & \downarrow \text{Coz } h & \searrow g \\
 L & (\text{Coz}L, \text{Coz } \mu) &
 \end{array}$$

Let $g = \text{Coz } h \circ \eta_M$. As η_M and $\text{Coz } h$ are uniform σ -frame homomorphisms, g is also a uniform σ -frame homomorphism. Then $g : (M, \nu) \rightarrow (\text{Coz } L, \text{Coz } \mu)$ and

$$\begin{array}{ccc}
 \text{Coz } L & (\mathcal{H}\text{Coz } L, \mathcal{H}\text{Coz } \mu) & \xrightarrow{\epsilon_L} (L, \mu) \\
 \uparrow g & \uparrow \mathcal{H}g & \nearrow h \\
 M & (\mathcal{H}M, \mathcal{H}\nu) &
 \end{array}$$

Then for any $J \in \mathcal{H}M$,

$$\begin{aligned}
 (\epsilon_L \circ \mathcal{H}g)(J) &= \epsilon_L(\langle g(J) \rangle) \\
 &= \epsilon_L(\langle \text{Coz } h \circ \eta_M(J) \rangle) \\
 &= \epsilon_L(\langle \text{Coz } h(\{\downarrow j : j \in J\}) \rangle) \\
 &= \epsilon_L(\langle \{h(\downarrow j) : j \in J\} \rangle) \quad (\text{since for each } j \in J, \downarrow j \in \text{Coz}\mathcal{H}M) \\
 &= \bigvee_L (\langle \{h(\downarrow j) : j \in J\} \rangle) \\
 &= h(J).
 \end{aligned}$$

Certainly $h(\downarrow j) \leq h(J)$ for each $j \in J$. If $a \geq h(\downarrow j)$ for each $j \in J$, then $a \geq h(J)$ since $J = \bigvee_{\mathcal{H}M} \downarrow j$ and h preserves joins. Thus $\epsilon_L \circ \mathcal{H}g = h$. Also g is unique such that $\epsilon_L \circ \mathcal{H}g = h$ for if $g' : M \rightarrow \text{Coz } L$ such that $\epsilon_L \circ \mathcal{H}g' = h$, then for any $x \in M$

$$\begin{aligned}
 g'(x) &= \bigvee_L \downarrow g'(x) \\
 &= \epsilon_L(\downarrow g'(x)) \\
 &= (\epsilon_L \circ \mathcal{H}g')(\downarrow x)
 \end{aligned}$$

$$\begin{aligned}
&= h(\downarrow x) \\
&= \epsilon_L(\mathcal{H}g(\downarrow x)) \\
&= \epsilon_L(\downarrow g(x)) \\
&= \bigvee_L \downarrow g(x) \\
&= g(x).
\end{aligned}$$

Thus g is unique such that $\epsilon_L \circ \mathcal{H}g = h$ making ϵ_L a universal map from \mathcal{H} to L . \square

Proposition 3.1.1

Coz and \mathcal{H} induce an equivalence between $\mathbf{SepSLNFrm}$ and $\mathbf{SN}\sigma\mathbf{Frm}$.

Proof :

Since ϵ_L is a universal map for each $(L, \mu) \in \mathbf{SepSLNFrm}$, \mathcal{H} is a left adjoint to Coz with unit η_M and counit ϵ_L for $L \in \mathbf{SepSLNFrm}$ and $M \in \mathbf{SN}\sigma\mathbf{Frm}$. Since ϵ_L and η_M are natural isomorphisms, Coz and \mathcal{H} induce an equivalence between these categories. \square

At the unstructured level, Coz and \mathcal{H} give an equivalence between the category $\mathbf{Reg}\sigma\mathbf{Frm}$ and the category $\mathbf{LRegFrm}$ (see [61]). Thus the equivalence of Proposition 3.1.1 is a structured version of the adjointness between Coz and \mathcal{H} given in [61]. Also, since every Lindelöf uniform frame is a separable strong nearness frame and every uniform σ -frame is a strong nearness σ -frame (see Lemma 2.1.3), the equivalence above is a generalization of Proposition 3.31 given in [87].

3.2 Complete nearness σ -frames

A uniform σ -frame homomorphism $h : (M, \nu) \rightarrow (L, \mu)$ between nearness σ -frames (M, ν) and (L, μ) is a *surjection* if h is onto and for each $A \in \mu \exists B \in \nu$ such that $h(B) \leq A$. The nearness σ -frame (L, μ) is *complete* if every dense surjection

$h : (M, \nu) \longrightarrow (L, \mu)$ is an isomorphism. A *completion* of a nearness σ -frame (L, μ) is a pair $((M, \nu), h)$ where (M, ν) is a complete nearness σ -frame and $h : (M, \nu) \longrightarrow (L, \mu)$ is a dense surjection.

Lemma 3.2.1

Any fine nearness σ -frame is complete.

Proof:

Let $h : (M, \mu) \longrightarrow (L, \nu)$ be any dense surjection between nearness σ -frames M and L with L fine. We show that h is codense. Let $h(x) = 1_L$. By the admissibility of μ , $x = \bigvee S$ for some countable $S \subseteq \{s \in M : s \triangleleft_{\mu} x\}$. As h is a σ -frame homomorphism we have

$$1_L = h(x) = h(\bigvee S) = \bigvee \{h(s) : s \in S\}.$$

Since h is surjective, there exists $A \in \mu$ such that $h(A) \leq \{h(s) : s \in S\}$. Let $a \in A$. Then $h(a) \leq h(s)$ for some $s \in S$. Since $s \in S$, $Bs \leq x$ for some $B \in \mu$. Set $\tilde{b} = \bigvee \{b \in B : b \wedge s = 0_M\}$. Then $\tilde{b} \vee x = 1_M$ as $1_M = \tilde{b} \vee Bs \leq \tilde{b} \vee x$. Since $h(a) \leq h(s)$,

$$h(a \wedge \tilde{b}) = h(a) \wedge h(\tilde{b}) \leq h(s) \wedge h(\tilde{b}) = h(s \wedge \tilde{b}) = h(0_M) = 0_L.$$

Thus $h(a \wedge \tilde{b}) = 0_L$. Since h is dense, $a \wedge \tilde{b} = 0_M$. Then

$$\begin{aligned} a &\leq a \vee x \\ &= (a \vee x) \wedge (\tilde{b} \vee x) \\ &= (a \wedge \tilde{b}) \vee x \\ &= 0_M \vee x = x. \end{aligned}$$

Hence, $a \leq x$ for each $a \in A$. Thus $1_M = \bigvee A \leq x$ and $x = 1_M$ showing that h is codense. By Lemma 1.1.2 h is injective and hence an isomorphism. Thus L is complete. \square

We thus conclude that a compact nearness σ -frame is complete as a compact nearness σ -frame is fine (see Lemma 2.1.2). We now show that the complete strong nearness σ -frames are exactly the cozero parts of complete separable strong Lindelöf nearness frames (cf. the uniform σ -frame case in [87] together with [89] and the metric σ -frame case in [85]).

Lemma 3.2.2 $\mathcal{H} : \text{SN}\sigma\text{Frm} \longrightarrow \text{SepSLNFrm}$ *preserves dense surjections.*

Proof :

Let $h : (L, \mu) \longrightarrow (M, \nu)$ be any dense surjection between strong nearness σ -frames (L, μ) and (M, ν) . If $J \in \mathcal{H}L$ and $\mathcal{H}h(J) = \{0_M\}$, then the ideal generated by $h(J)$, $\langle h(J) \rangle$ is the zero ideal in M i.e. $\langle h(J) \rangle = \{0_M\}$. Thus $h(x) = 0_L$ for each $x \in J$. As h is dense $x = 0$ for each $x \in J$. Thus $J = \{0_L\}$ and hence $\mathcal{H}h$ is dense.

Now let $J \in \mathcal{H}M$. Since h is onto, for each $y \in J$ there exists $x \in L$ such that $h(x) = y$. Let $I = h^{-1}(J) = \{x \in L : h(x) = y \text{ for some } y \in J\}$. If $t \in I$ and $s \leq t$, then $h(t) = u$ for some $u \in J$. Then $h(s) \leq h(t) = u \in J$. As J is an ideal, $h(s) \in J$. Thus $h(s) = v$ for some $v \in J$ and hence $s \in I$. Also, if $X = \{x_n\}$ is any countable subcollection of I , then $h(x_n) = y_n$ for some $y_n \in J$ for each n . As J is a σ -ideal in M , $\bigvee y_n \in J$. Then $h(\bigvee X) = \bigvee y_n$ and thus $\bigvee X \in I$. Hence $I \in \mathcal{H}L$. Furthermore, since h is onto, for $J \in \mathcal{H}M$, $I \in \mathcal{H}L$ such that $\mathcal{H}h(I) = h(h^{-1}(J)) = J$. Thus $\mathcal{H}h$ is onto.

Now let $\mathcal{A} \in \mathcal{H}\nu$. Then there exists $A \in \nu$ such that $\downarrow A \leq \mathcal{A}$. Since h is surjective, there exists $B \in \mu$ such that $h(B) \leq A$. Since $\downarrow B \in \mathcal{H}\mu$ and

$$\mathcal{H}h(\downarrow B) = \langle \downarrow h(B) \rangle = \downarrow h(B) \leq \downarrow A \leq \mathcal{A},$$

$\{\mathcal{H}h(\mathcal{B}) \mid \mathcal{B} \in \mathcal{H}\mu\}$ generates $\mathcal{H}\nu$. Since $(L, \mu) \in \text{SN}\sigma\text{Frm}$, by Lemma 2.3.3 $(\mathcal{H}L, \mathcal{H}\mu) \in \text{SepSLNFrm}$. Hence by Lemma 1.2.2, $\mathcal{H}h$ is a surjective nearness frame homomorphism. \square

Lemma 3.2.3

$Coz : \text{SepSLNFrm} \longrightarrow \text{SN}\sigma\text{Frm}$ preserves dense surjections.

Proof :

Let (L, μ) and (M, ν) be separable strong Lindelöf nearness frames with $h : (L, \mu) \longrightarrow (M, \nu)$ a dense surjection. If $x \in Coz L$ with $Coz h(x) = 0$, then $0 = Coz h(x) = h|_{Coz L}(x) = h(x)$. Since h is dense, $x = 0$. Thus $Coz h$ is also dense.

Now let $a \in Coz M$. By Proposition 1.1.1, $a = \bigvee_n b_n$ where $b_n \prec\prec b_{n+1}$ for each $n = 1, 2, \dots$. By Lemma 1.2.1(3c), for each n , $h_*(b_n) \prec\prec h_*(b_{n+1})$. Again, by Proposition 1.1.1, $x = \bigvee_n h_*(b_n) \in Coz L$. Since h is onto, $hh_* = id_M$ and thus $Coz h(x) = h|_{Coz L}(x) = h(x) = h\left(\bigvee_n h_*(b_n)\right) = \bigvee_n hh_*(b_n) = \bigvee_n b_n = a$. Hence $Coz h$ is onto.

Now let $A \in Coz \nu$. Then A is countable, $A \subseteq Coz M$ and $A \in \nu$. Since h is surjective, $h_*(A) \in \mu$. Since (L, μ) is separable and strong, by Lemma 3.1.2 there exists $B \in Coz \mu$ such that $B \leq h_*(A)$. Since h is onto,

$$Coz h(B) = h|_{Coz L}(B) = h(B) \leq hh_*(A) = id_M(A) = A.$$

Hence $Coz h$ is surjective. \square

Lemma 3.2.4

If $(L, \mu) \in \text{SepSLNFrm}$ is complete, then $(Coz L, Coz \mu) \in \text{SN}\sigma\text{Frm}$ is also complete.

Proof :

Let $(L, \mu) \in \text{SepSLNFrm}$ with $h : (M, \nu) \longrightarrow (Coz L, Coz \mu)$ any dense uniform surjection in $\text{SN}\sigma\text{Frm}$. Applying \mathcal{H} gives

$$\begin{array}{ccc} (\mathcal{H}M, \mathcal{H}\nu) & \xrightarrow{\mathcal{H}h} & (\mathcal{H}Coz L, \mathcal{H}Coz \mu) \\ & & \downarrow \simeq \epsilon_L \\ & & (L, \mu) \end{array}$$

Since \mathcal{H} preserves dense surjections, $\mathcal{H}h$ is a dense surjection. As (L, μ) is complete, $\mathcal{H}h$ is an isomorphism. Applying Coz produces the following

$$\begin{array}{ccc}
 (Coz\mathcal{H}M, Coz\mathcal{H}\nu) & \xrightarrow[Coz\mathcal{H}h]{\simeq} & (Coz\mathcal{H}CozL, Coz\mathcal{H}Coz\mu) \\
 \eta_M \uparrow \simeq & & \simeq \uparrow \eta_{CozL} \\
 (M, \nu) & \xrightarrow{h} & (CozL, Coz\mu)
 \end{array}$$

Note that $Coz\mathcal{H}h$ is an isomorphism. Using the naturality of η we then have $Coz\mathcal{H}h \circ \eta_M = \eta_{CozL} \circ h$. Since $\eta_{CozL}, Coz\mathcal{H}h$ and η_M are isomorphisms, $h = (\eta_{CozL})^{-1} \circ Coz\mathcal{H}h \circ \eta_M$ is an isomorphism. Thus $(CozL, Coz\mu)$ is a complete strong nearness σ -frame. \square

Similarly, using the naturality of ϵ we also have:

Lemma 3.2.5

If $(L, \mu) \in \mathbf{SN}\sigma\mathbf{Frm}$ is complete, then $(\mathcal{H}L, \mathcal{H}\mu) \in \mathbf{SepSLNFrm}$ is also complete.

Proof :

Let (L, μ) be any complete strong nearness σ -frame and (M, ν) be any separable strong Lindelöf nearness frame with $h : (M, \nu) \rightarrow (\mathcal{H}L, \mathcal{H}\mu)$ any dense uniform surjection in $\mathbf{SepSLNFrm}$. Applying Coz gives

$$\begin{array}{ccc}
 (CozM, Coz\nu) & \xrightarrow{Cozh} & (Coz\mathcal{H}L, coz\mathcal{H}\mu) \\
 & & \eta_L \uparrow \simeq \\
 & & (L, \mu)
 \end{array}$$

Since Coz preserves dense surjections, $Coz h$ is a dense surjection. As (L, μ) is complete, $Coz h$ is an isomorphism. Applying \mathcal{H} produces

$$\begin{array}{ccc}
 (\mathcal{H}Coz M, \mathcal{H}Coz \nu) & \xrightarrow[\simeq]{\mathcal{H}Coz h} & (\mathcal{H}Coz \mathcal{H}L, \mathcal{H}Coz \mathcal{H}\mu) \\
 \epsilon_M \downarrow \simeq & & \simeq \downarrow \epsilon_{\mathcal{H}L} \\
 (M, \nu) & \xrightarrow{h} & (\mathcal{H}L, \mathcal{H}\mu)
 \end{array}$$

Since \mathcal{H} preserves isomorphisms, $\mathcal{H}Coz h$ is also an isomorphism. Using the naturality of ϵ we then have $h \circ \epsilon_M = \epsilon_{\mathcal{H}L} \circ \mathcal{H}Coz h$. Since $\epsilon_{\mathcal{H}L}$, $\mathcal{H}Coz h$ and ϵ_M are isomorphisms, $h = \epsilon_{\mathcal{H}L} \circ \mathcal{H}Coz h \circ (\epsilon_M)^{-1}$ is an isomorphism. Thus $(\mathcal{H}L, \mathcal{H}\mu)$ is a complete separable strong Lindelöf nearness frame. \square

Theorem 3.2.1

The complete strong nearness σ -frames are exactly the cozero parts of complete separable strong Lindelöf nearness frames.

Proof :

If (L, μ) is any complete strong nearness σ -frame, then by the previous Lemma $(\mathcal{H}L, \mathcal{H}\mu)$ is a complete separable strong Lindelöf nearness frame. But $\eta_L : (L, \mu) \rightarrow (Coz \mathcal{H}L, Coz \mathcal{H}\mu)$ is an isomorphism. Thus $L \simeq Coz \mathcal{H}L$. Hence (L, μ) is isomorphic to the cozero part of the complete separable strong Lindelöf nearness frame $(\mathcal{H}L, \mathcal{H}\mu)$. \square

As a result of Lemma 3.2.4, Lemma 3.2.5 and Theorem 3.2.1, we have the following subequivalence of that in Proposition 3.1.1, between the category $\mathbf{CSN}_\sigma\mathbf{Frm}$, of complete strong nearness σ -frames and the category $\mathbf{CSepSLNFrm}$, of complete separable strong Lindelöf nearness frames.

Theorem 3.2.2

Coz and \mathcal{H} induce an equivalence between $\mathbf{CSN}_\sigma\mathbf{Frm}$ and $\mathbf{CSepSLNFrm}$.

3.3 The completion of a strong nearness σ -frame

In this section we prove the existence of a completion, unique up to isomorphism, for any strong nearness σ -frame in the spirit of [17] following the uniform σ -frame case in [89] and the metric σ -frame case in [84].

Regarding frame quotients, a *nucleus* on the frame L is a map $j : L \rightarrow L$ satisfying for all $a \in L$

$$(i) \quad a \leq j(a),$$

$$(ii) \quad j(a) \wedge j(b) = j(a \wedge b) \text{ and}$$

$$(iii) \quad j^2(a) = j(j(a)) = j(a).$$

So a nucleus on L is a closure operator on L preserving binary meet. For any nucleus $j : L \rightarrow L$, the closure system $Fix(j) = \{x \in L : j(x) = x\}$ is a frame such that $j : L \rightarrow Fix(j)$ is a frame homomorphism (see Johnstone [58], page 49).

A *prenucleus* on L is a preclosure operator $j_0 : L \rightarrow L$ satisfying for any $a, b \in L$

$$(i) \quad a \leq j_0(a),$$

$$(ii) \quad a \leq b \rightarrow j_0(a) \leq j_0(b) \text{ and}$$

$$(iii) \quad j_0(a) \wedge b \leq j_0(a \wedge b).$$

Given a prenucleus j_0 on a frame L , $Fix(j_0)$ is a closure system in L and the associated closure operator j is a nucleus. Then $Fix(j_0) = Fix(j)$ is a quotient frame of L (see [17]).

Let $(L, \mu) \in \mathbf{N}\sigma\mathbf{Frm}$, and $\mathfrak{D}L$ be the frame of all non-empty down-sets of L (i.e. consisting of all $U \subseteq L$ such that $0 \in U$ and $y \in U$ and $x \leq y \Rightarrow x \in U$) partially ordered by inclusion so that meet is intersection and join is union. Also, let $\mathfrak{R}L$ be the closure system in $\mathfrak{D}L$ determined by the conditions:

($\mathfrak{R}1$) For $k(a) = \{x \in L : x \triangleleft a\}$, if $k(a) \subseteq U$, then $a \in U$ and

($\mathfrak{R}2$) If $\{a\} \wedge C \subseteq U$ for some $C \in \mu$, then $a \in U$.

Then as in [17], $U \in \mathfrak{X}L$ may be expressed in the form $\ell_0(U) = U$ for the prenucleus $\ell_0 : \mathfrak{D}L \rightarrow \mathfrak{D}L$ defined such that

$$\ell_0(U) = \{a \in L : k(a) \subseteq U\} \cup \{a \in L : \{a\} \wedge C \subseteq U \text{ for some } C \in \mu\}.$$

Then the closure operator ℓ on $\mathfrak{D}L$ determined by $\mathfrak{X}L$ is a nucleus so that $\mathfrak{X}L$ is a frame. Now put $\mu' = \{\mathcal{A} \in \text{cov}(\mathfrak{X}L) : \downarrow B \leq \mathcal{A} \text{ for some } B \in \mu\}$.

The proofs of the ensuing results are in analogy to those of the corresponding results in [84] adapted to the nearness setting. We present the proofs of the results particular and significant to the nearness case.

Lemma 3.3.1

For $(L, \mu) \in \text{SN}\sigma\text{Frm}$, μ' is a strong nearness on the frame $\mathfrak{X}L$.

Proof :

If $\mathcal{A} \in \mu'$ and $\mathcal{A} \leq \mathcal{B}$, then $\downarrow B \leq \mathcal{A}$ for some $B \in \mu$. Then also $\downarrow B \leq \mathcal{B}$ and so $\mathcal{B} \in \mu'$. If $\mathcal{A}, \mathcal{B} \in \mu'$, then $\downarrow A \leq \mathcal{A}$ and $\downarrow B \leq \mathcal{B}$ for some $A, B \in \mu$. Then $A \wedge B \in \mu$ and $\downarrow(A \wedge B) = (\downarrow A) \wedge (\downarrow B) \leq \mathcal{A} \wedge \mathcal{B}$. So $\mathcal{A} \wedge \mathcal{B} \in \mu'$ making μ' a filter in $\text{cov}\mathfrak{X}L$.

For admissibility, let $U \in \mathfrak{X}L$ and $W = \bigvee \{T \in \mathfrak{X}L : T \triangleleft_{\mu'} U\}$. We must show that $U = W$. The fact that $W \subseteq U$ is trivial since $T \triangleleft_{\mu'} U$ implies $T \subseteq U$; so one only has to show $U \subseteq W$. Now if $a \in U$, by admissibility of μ find $x \in L$ such that $x \triangleleft_{\mu} a$. Then $Ax \leq a$ for some $A \in \mu$. Then $(\downarrow A)(\downarrow x) = \downarrow(Ax) \subseteq U$. So $\downarrow x \triangleleft_{\mu'} U$ as $\downarrow A \in \mu'$. Thus $\downarrow x \in \{T \in \mathfrak{X}L : T \triangleleft_{\mu'} U\}$. So $x \in W$ and thus $\{x \in L : x \triangleleft_{\mu} a\} = k(a) \in W$. Since $W \in \mathfrak{X}L$, $a \in W$. Hence $U \subseteq W$ and so

$$U = \bigvee \{T \in \mathfrak{X}L : T \triangleleft_{\mu'} U\}$$

showing that μ' is admissible.

We next show that μ' is strong. To this end let $\mathcal{A} \in \mu'$. Then there exists $B \in \mu$ such that $\downarrow B \leq \mathcal{A}$. Since μ is strong, there is $C \in \mu$ such that for each $c \in C$, $c \triangleleft_{\mu} b$

for some $b \in B$. Let $x \in C$. Then $x \triangleleft_{\mu} b$ for some $b \in B$. Then there is $D \in \mu$ such that $Dx \leq b$. Consequently, $(\downarrow D)(\downarrow x) \leq \downarrow b$. Since $\downarrow B \leq \mathcal{A}$, $\downarrow b \leq V$ for some $V \in \mathcal{A}$. Then $(\downarrow D)(\downarrow x) \leq V$. Since $\downarrow D \in \mu'$, $\downarrow x \triangleleft_{\mu'} V$. Hence $\downarrow x \in \check{\mathcal{A}}$. Thus $\downarrow C \leq \check{\mathcal{A}}$ and so $\check{\mathcal{A}} \in \mu'$. \square

Lemma 3.3.2

For a strong nearness σ -frame (L, μ) , define

$$h : (\mathfrak{R}L, \mu') \longrightarrow (\mathcal{H}L, \mathcal{H}\mu) \quad \text{by}$$

$$h(U) = \bigvee \{\downarrow x : x \in U\} \text{ for each } U \in \mathfrak{R}L$$

Then h is a dense nearness surjection.

Proof :

Suppose that $h(U) = \bigvee \{\downarrow x : x \in U\} = \{0\}$. Then $\downarrow x = \{0\}$ for each $x \in U$. Consequently, $x = 0$ for each $x \in U$ and so $U = \{0\}$. Hence h is dense.

Let $J \in \mathcal{H}L$. We show that $J \in \mathfrak{R}L$. Suppose that $k(x) = \{y \in L : y \triangleleft_{\mu} x\} \subseteq J$. However, since μ is admissible there exists a countable $T \subseteq k(x)$ such that $x = \bigvee T$. Then $T \subseteq J$ and as T is countable and J is a σ -ideal, $\bigvee T = x \in J$. Thus $k(x) \subseteq J$ implies that $x \in J$.

If $\{x\} \wedge C \subseteq J$ for some $C \in \mu$, then $X = \{x \wedge c : c \in C\}$ is countable as C is countable. Moreover $X \subseteq J$. Again, as J is a σ -ideal

$$x = x \wedge \bigvee_{c \in C} c = \bigvee_{c \in C} (x \wedge c) = \bigvee X \in J$$

Thus $\{x\} \wedge C \subseteq J$ for $C \in \mu$ implies $x \in J$. Thus $J \in \mathfrak{R}L$. But $h(J) = \bigvee \{\downarrow x : x \in J\} = J$. Thus h is onto.

Let $\downarrow A \in \mathcal{H}\mu$ for any $A \in \mu$. Since $\downarrow A \in \mu'$ and $h(\downarrow A) = \downarrow A$, $\{h(\mathcal{A}) : \mathcal{A} \in \mu'\}$ generates $\mathcal{H}\mu$. Since $(\mathcal{H}L, \mathcal{H}\mu)$ is a strong nearness frame (Lemma 2.3.3), we have by Lemma 1.2.2 that h is a nearness surjection. \square

Lemma 3.3.3

If (M, ν) is a nearness frame with (L, μ) a nearness σ -frame such that $f : (M, \nu) \longrightarrow (\mathcal{H}L, \mathcal{H}\mu)$ is a dense nearness surjection, then there exists a dense onto frame homomorphism $\bar{f} : (\mathfrak{R}L, \mu') \longrightarrow (M, \nu)$ such that the following triangle commutes

$$\begin{array}{ccc} (\mathcal{H}L, \mathcal{H}\mu) & \xleftarrow{f} & (M, \nu) \\ \uparrow h & \nearrow \bar{f} & \\ (\mathfrak{R}L, \mu') & & \end{array}$$

where h is as defined in the previous Lemma.

The proof is given in [17] and [83] with \bar{f} defined as the restriction of $\bar{s} : \mathcal{D}L \longrightarrow M$ to $\mathfrak{R}L$ where \bar{s} is the dense frame homomorphism defined by $\bar{s}(U) = \bigvee \{s(\downarrow x) : x \in U\}$, $s : (\mathcal{H}L, \mathcal{H}\mu) \longrightarrow (M, \nu)$ being the right adjoint of $f : (M, \nu) \longrightarrow (\mathcal{H}L, \mathcal{H}\mu)$.

Lemma 3.3.4

If (M, ν) is a strong nearness frame, then $\bar{f} : (\mathfrak{R}L, \mu') \longrightarrow (M, \nu)$ is a dense nearness surjection for a strong nearness σ -frame (L, μ) .

Proof :

\bar{f} is dense and onto by the previous Lemma. Since $(\mathfrak{R}L, \mu')$ is a strong nearness frame by Lemma 3.3.1, we will use Lemma 1.2.2 to show that \bar{f} is a nearness surjection i.e. $\{\bar{f}(\mathcal{A}) : \mathcal{A} \in \mu'\}$ generates ν . Let $B \in \nu$. Since f is uniform, $f(B) \in \mathcal{H}\mu$. Thus there is $A \in \mu$ such that $\downarrow A \leq f(B)$. Since h is a nearness surjection between the strong nearness frames $(\mathfrak{R}L, \mu')$ and $(\mathcal{H}L, \mathcal{H}\mu)$, by Lemma 1.2.2 $\{h(\mathcal{A}) : \mathcal{A} \in \mu'\}$ generates $\mathcal{H}\mu$. Thus $h(\mathcal{A}) \leq \downarrow A \leq f(B)$ for some $\mathcal{A} \in \mu'$. Since $f \circ \bar{f} = h$ and s is the right adjoint of f we have

$$\bar{f}(\mathcal{A}) \leq (s \circ f)(\bar{f}(\mathcal{A})) = s \circ ((f \circ \bar{f})(\mathcal{A})) = s \circ (h(\mathcal{A})) \leq B.$$

Hence $\mathcal{A} \in \mu'$ such that $\bar{f}(\mathcal{A}) \leq B$. \square

Theorem 3.3.1

If (L, μ) is a strong nearness σ -frame, then $(\mathfrak{R}'L, \mu')$ is a complete separable strong Lindelöf nearness frame.

Proof :

$\mathfrak{R}'L$ is Lindelöf by [83]. By Lemma 3.3.1, $(\mathfrak{R}'L, \mu')$ is strong. By definition it is clear that μ' is separable. We now show that $(\mathfrak{R}'L, \mu')$ is complete. Let $z : (N, \xi) \longrightarrow (\mathfrak{R}'L, \mu')$ be any dense nearness surjection for a strong nearness frame (N, ξ) . Since $h : (\mathfrak{R}'L, \mu') \longrightarrow (\mathcal{H}L, \mathcal{H}\mu)$ is a dense nearness surjection, $h \circ z : (N, \xi) \longrightarrow (\mathcal{H}L, \mathcal{H}\mu)$ is also a dense nearness surjection. We then have the following

$$\begin{array}{ccc}
 (\mathfrak{R}'L, \mu') & \xleftarrow{z} & (N, \xi) \\
 \downarrow h & \nearrow h \circ z & \uparrow \bar{f} \\
 (\mathcal{H}L, \mathcal{H}\mu) & \xleftarrow{h} & (\mathfrak{R}'L, \mu')
 \end{array}$$

where \bar{f} is a dense nearness surjective frame homomorphism that can be found by Lemma 3.3.3 such that $(h \circ z) \circ \bar{f} = h$. Since h is dense, h is monic by Lemma 1.1.2. Thus $z \circ \bar{f} = id_{\mathfrak{R}'L}$. Now let $z(x) = 1_{\mathfrak{R}'L}$. Since \bar{f} is onto, $\bar{f}(U) = x$ for some $U \in \mathfrak{R}'L$. Then

$$\begin{aligned}
 z(x) = 1_{\mathfrak{R}'L} &\Rightarrow (z \circ \bar{f})(U) = 1_{\mathfrak{R}'L} \\
 &\Rightarrow id_{\mathfrak{R}'L}(U) = L \\
 &\Rightarrow U = L \\
 &\Rightarrow \bar{f}(U) = \bar{f}(L) \\
 &\Rightarrow x = 1_N \quad (\text{since } \bar{f} \text{ is a frame homomorphism}).
 \end{aligned}$$

Thus z is codense and hence injective. Consequently, z is an isomorphism. \square

Theorem 3.3.2

For a nearness σ -frame (L, μ) the dense nearness surjection $h : (\mathfrak{R}'L, \mu') \longrightarrow (\mathcal{H}L, \mathcal{H}\mu)$ is a completion of $(\mathcal{H}L, \mathcal{H}\mu)$ in **SepSLNFrm** unique up to isomorphism.

Proof :

By the previous Theorem $(\mathfrak{R}'L, \mu')$ is a complete separable strong Lindelöf nearness frame. Since h is a dense surjection it is indeed a completion of $(\mathcal{H}L, \mathcal{H}\mu)$ in **SepSLNFrm**. If $z : (N, \xi) \longrightarrow (\mathcal{H}L, \mathcal{H}\mu)$ is any completion of $(\mathcal{H}L, \mathcal{H}\mu)$ in **SepSLNFrm**, find a dense surjective frame homomorphism $\bar{f} : (\mathfrak{R}'L, \mu') \longrightarrow (N, \xi)$ by Lemma 3.3.3 such that $z \circ \bar{f} = h$. However, since (N, ξ) is a complete separable strong Lindelöf nearness frame, \bar{f} is an isomorphism. Hence, h is unique up to isomorphism. \square

Now let (L, μ) be a strong nearness σ -frame and $h : (\mathfrak{R}'L, \mu') \longrightarrow (\mathcal{H}L, \mathcal{H}\mu)$ be the dense nearness surjection given by $h(U) = \bigvee \{\downarrow x : x \in U\}$. Since $(\mathfrak{R}'L, \mu') \in \mathbf{SepsLNFrms}$, by Lemma 3.2.3 $Coz h : (Coz \mathfrak{R}'L, Coz \mu') \longrightarrow (Coz \mathcal{H}L, Coz \mathcal{H}\mu)$ is a dense nearness surjection between the strong nearness σ -frames $(Coz \mathfrak{R}'L, Coz \mu')$ and $(Coz \mathcal{H}L, Coz \mathcal{H}\mu)$. By Theorem 3.1.2 $\eta_L : (L, \mu) \longrightarrow (Coz \mathcal{H}L, Coz \mathcal{H}\mu)$ is a natural isomorphism in **SN σ Frm**. Then $\eta_L^{-1} \circ Coz h : Coz \mathfrak{R}'L \longrightarrow L$.

Theorem 3.3.3

For a strong nearness σ -frame (L, μ) , $((Coz \mathfrak{R}'L, Coz \mu'), \eta_L^{-1} \circ Coz h)$ is the completion of (L, μ) unique up to isomorphism.

Proof :

h was shown to be a completion of $\mathcal{H}L$ by the previous Theorem and by Theorem 3.3.1 $\mathfrak{R}'L$ is a complete separable strong Lindelöf nearness frame. By Lemma 3.2.4 $Coz \mathfrak{R}'L$ is then a complete strong nearness σ -frame. As h is a dense surjection, by Lemma 3.2.3 $Coz h$ is a dense surjection. As η_L is an isomorphism by Lemma 3.1.5, $\eta_L^{-1} \circ Coz h$ is a dense surjection. Since $Coz \mathfrak{R}'L$ is complete, $((Coz \mathfrak{R}'L, Coz \mu'), \eta_L^{-1} \circ Coz h)$ is

a completion of (L, μ) .

For uniqueness, let $g : (N, \xi) \longrightarrow (L, \mu)$ be any completion of (L, μ) in $\mathbf{SN}\sigma\mathbf{Frm}$. Then $\mathcal{H}g : (\mathcal{H}N, \mathcal{H}\xi) \longrightarrow (\mathcal{H}L, \mathcal{H}\mu)$ is a dense surjection by Lemma 3.2.2. Since (N, ξ) is a complete strong nearness σ -frame, by Lemma 3.2.5 $(\mathcal{H}N, \mathcal{H}\xi)$ is a complete separable strong Lindelöf nearness frame. Find a dense nearness surjection $\overline{\mathcal{H}g} : (\mathcal{R}L, \mu') \longrightarrow (\mathcal{H}N, \mathcal{H}\xi)$ such that $\mathcal{H}g \circ \overline{\mathcal{H}g} = h$. Then $\text{Coz}\mathcal{H}g \circ \text{Coz}\overline{\mathcal{H}g} = \text{Coz}h$. Since $\mathcal{H}N$ is complete, $\overline{\mathcal{H}g}$ is an isomorphism. Since η is natural the following is a commutative diagram

$$\begin{array}{ccc} (L, \mu) & \xrightarrow{\eta_L} & (\text{Coz}\mathcal{H}L, \text{Coz}\mathcal{H}\mu) \\ \uparrow g & & \uparrow \text{Coz}\mathcal{H}g \\ (N, \xi) & \xrightarrow{\eta_N} & (\text{Coz}\mathcal{H}N, \text{Coz}\mathcal{H}\xi) \end{array}$$

Then $\text{Coz}\mathcal{H}g \circ \eta_N = \eta_L \circ g$. Thus $\eta_L^{-1} \circ \text{Coz}\mathcal{H}g = g \circ \eta_N^{-1}$ and we have the following

$$\begin{array}{ccc} (\text{Coz}\mathcal{R}L, \text{Coz}\mu') & & \\ \downarrow \text{Coz}h & \searrow \text{Coz}\overline{\mathcal{H}g} & \\ (\text{Coz}\mathcal{H}L, \text{Coz}\mathcal{H}\mu) & \xleftarrow{\text{Coz}\mathcal{H}g} & (\text{Coz}\mathcal{H}N, \text{Coz}\mathcal{H}\xi) \\ \downarrow \eta_L^{-1} & & \downarrow \eta_N^{-1} \\ (L, \mu) & \xleftarrow{g} & (N, \xi) \end{array}$$

Then

$$g \circ \eta_N^{-1} \circ \text{Coz}\overline{\mathcal{H}g} = \eta_L^{-1} \circ \text{Coz}\mathcal{H}g \circ \text{Coz}\overline{\mathcal{H}g} = \eta_L^{-1} \circ \text{Coz}h.$$

Since $\overline{\mathcal{H}g}$ is an isomorphism, $\text{Coz}\overline{\mathcal{H}g}$ is an isomorphism. Consequently, $\eta_N^{-1} \circ \text{Coz}\overline{\mathcal{H}g}$ is an isomorphism. So we have shown that $\eta_L^{-1} \circ \text{Coz}h$ is unique up to isomorphism since given any completion $g : (N, \xi) \longrightarrow (L, \mu)$ we can find an isomorphism $\bar{g} :$

$(\text{Coz}\mathfrak{R}'L, \text{Coz}\mu') \rightarrow (N, \xi)$ such that the following is commutative

$$\begin{array}{ccc}
 (N, \xi) & \xrightarrow{g} & (L, \mu) \\
 \bar{g} \uparrow & \nearrow \eta_L^{-1} \circ \text{Coz} h & \\
 (\text{Coz}\mathfrak{R}'L, \text{Coz}\mu') & &
 \end{array}$$

i.e. $g \circ \bar{g} = \eta_L^{-1} \circ \text{Coz} h$. \square

If (L, μ) is a uniform σ -frame, then $((\text{Coz}\mathfrak{R}'L, \text{Coz}\mu'), \eta_L^{-1} \circ \text{Coz} h)$ is also its uniform completion. We will denote by h_L the dense nearness surjection given in Lemma 3.3.2. Also let $C_\sigma(L, \mu)$ denote the completion $(\text{Coz}\mathfrak{R}'L, \text{Coz}\mu')$, and γ_L the completion map $\eta_L^{-1} \circ \text{Coz} h_L$.

Theorem 3.3.4

Completion is a coreflection for strong nearness σ -frames.

Proof:

Let (L, μ) and (M, ν) be strong nearness σ -frames and let $t : (M, \nu) \rightarrow (L, \mu)$ with (M, ν) complete. We must show that there is a unique $g : (M, \nu) \rightarrow C_\sigma(L, \mu)$ such that $\gamma_L \circ g = t$ i.e. the following triangle commutes:

$$\begin{array}{ccc}
 (M, \nu) & \xrightarrow{t} & (L, \mu) \\
 g \downarrow & \nearrow \gamma_L & \\
 C_\sigma(L, \mu) & &
 \end{array}$$

Consider the completion $C(\mathcal{H}L, \mathcal{H}\mu)$ of the nearness frame $(\mathcal{H}L, \mathcal{H}\mu)$ with the completion map $\gamma_{\mathcal{H}L} : C(\mathcal{H}L, \mathcal{H}\mu) \rightarrow (\mathcal{H}L, \mathcal{H}\mu)$ illustrated in [4]. We then have the following diagram:

$$\begin{array}{ccccc}
 (\mathcal{H}M, \mathcal{H}\nu) & \xrightarrow{\mathcal{H}t} & (\mathcal{H}L, \mathcal{H}\mu) & & \\
 \uparrow \simeq h_M & \searrow f & \uparrow \gamma_{\mathcal{H}L} & \swarrow h_L & \\
 (\mathfrak{R}'M, \nu') & & C(\mathcal{H}L, \mathcal{H}\mu) & \xrightarrow[\simeq]{q} & (\mathfrak{R}'L, \mu')
 \end{array}$$

Since \mathcal{H} preserves completeness by Lemma 3.2.5 $(\mathcal{H}M, \mathcal{H}\nu)$ is a complete separable strong Lindelöf nearness frame. Since $h_M : (\mathfrak{R}'M, \nu') \rightarrow (\mathcal{H}M, \mathcal{H}\nu)$ is a completion of $(\mathcal{H}M, \mathcal{H}\nu)$ by Theorem 3.3.2 and $(\mathcal{H}M, \mathcal{H}\nu)$ is complete, we have that h_M is an isomorphism. Since (L, μ) is a strong nearness σ -frame Lemma 2.3.3 implies that $(\mathcal{H}L, \mathcal{H}\mu)$ is a strong nearness frame. Since completion is coreflection on strong nearness frames (see [19]) there exists a unique $f : (\mathcal{H}M, \mathcal{H}\nu) \rightarrow C(\mathcal{H}L, \mathcal{H}\mu)$ such that $\gamma_{\mathcal{H}L} \circ f = \mathcal{H}t$. Since $h_L : (\mathfrak{R}'L, \mu') \rightarrow (\mathcal{H}L, \mathcal{H}\mu)$ is a completion of $(\mathcal{H}L, \mathcal{H}\mu)$ and $\gamma_{\mathcal{H}L}$ is unique up to an isomorphism (see [4]) there exists a unique isomorphism $q : C(\mathcal{H}L, \mathcal{H}\mu) \rightarrow (\mathfrak{R}'L, \mu')$ such that $h_L \circ q = \gamma_{\mathcal{H}L}$. Now consider

$$(\mathfrak{R}'M, \nu') \xrightarrow[\simeq]{h_M} (\mathcal{H}M, \mathcal{H}\nu) \xrightarrow{f} C(\mathcal{H}L, \mathcal{H}\mu) \xrightarrow[\simeq]{q} (\mathfrak{R}'L, \mu')$$

Applying Coz gives

$$\begin{array}{ccccc} C_\sigma(M, \nu) & \xrightarrow[\simeq]{\text{Coz } h_M} & (C_\sigma \mathcal{H}M, C_\sigma \mathcal{H}\nu) & \xrightarrow{\text{Coz } f} & C_\sigma C(\mathcal{H}L, \mathcal{H}\mu) \\ \gamma_M \downarrow \simeq & & & & \downarrow \text{Coz } q \simeq \\ (M, \nu) & \xrightarrow{t} & (L, \mu) & \xleftarrow{\gamma_L} & C_\sigma(L, \mu) \end{array}$$

Note that $\text{Coz } h_M$ and $\text{Coz } q$ are isomorphisms. Furthermore, as (M, ν) is complete γ_M is an isomorphism. We then have

$$\begin{aligned} \gamma_L \circ (\text{Coz } q \circ \text{Coz } f \circ \text{Coz } h_M) &= \eta_L^{-1} \circ (\text{Coz } h_L \circ \text{Coz } q \circ \text{Coz } f \circ \text{Coz } h_M) \\ &= \eta_L^{-1} \circ \text{Coz}(h_L \circ q \circ f \circ h_M) \\ &= \eta_L^{-1} \circ \text{Coz}(\gamma_{\mathcal{H}L} \circ f \circ h_M) \quad (\because h_L \circ q = \gamma_{\mathcal{H}L}) \\ &= \eta_L^{-1} \circ \text{Coz}(\mathcal{H}t \circ h_M) \quad (\because \gamma_{\mathcal{H}L} \circ f = \mathcal{H}t) \\ &= \eta_L^{-1} \circ \text{Coz } \mathcal{H}t \circ \text{Coz } h_M \\ &= (\eta_L^{-1} \circ \text{Coz } \mathcal{H}t \circ \eta_M) \circ \eta_M^{-1} \circ \text{Coz } h_M \\ &= (\eta_L^{-1} \circ \text{Coz } \mathcal{H}t \circ \eta_M) \circ \gamma_M. \end{aligned}$$

By Theorem 3.1.2 η_L is a natural isomorphism. Thus the following diagram is commutative:

$$\begin{array}{ccc} (M, \nu) & \xrightarrow{\eta_M} & (\text{Coz}\mathcal{H}M, \text{Coz}\mathcal{H}\nu) \\ \downarrow t & & \downarrow \text{Coz}\mathcal{H}t \\ (L, \mu) & \xrightarrow{\eta_L} & (\text{Coz}\mathcal{H}L, \text{Coz}\mathcal{H}\mu) \end{array}$$

Then $\eta_L \circ t = \text{Coz}\mathcal{H}t \circ \eta_M$ and thus $t = \eta_L^{-1} \circ \text{Coz}\mathcal{H}t \circ \eta_M$.

Let $g = (\text{Coz}q \circ \text{Coz}f \circ \text{Coz}h_M) \circ \gamma_M^{-1}$. We then have

$$\begin{aligned} \gamma_L \circ (\text{Coz}q \circ \text{Coz}f \circ \text{Coz}h_M) &= (\eta_L^{-1} \circ \text{Coz}\mathcal{H}t \circ \eta_M) \circ \gamma_M \\ &= t \circ \gamma_M \\ \therefore \gamma_L \circ (\text{Coz}q \circ \text{Coz}f \circ \text{Coz}h_M) \circ \gamma_M^{-1} &= t \\ \text{i.e. } \gamma_L \circ g &= t. \end{aligned}$$

Now if $g' : (M, \nu) \rightarrow C_\sigma(L, \mu)$ such that $\gamma_L \circ g' = t$ then $\gamma_L \circ g = \gamma_L \circ g'$. Since γ_L is dense by Lemma 1.1.3 it is monic. Thus $g = g'$. Hence g is unique such that $\gamma_L \circ g = t$ proving our result. \square

3.4 The spectrum of a nearness σ -frame

We define a functor Ψ from the category $\text{N}\sigma\text{Frm}$ to the category Near .

A filter F on a σ -frame L is σ -prime if whenever $S \subseteq L$ is at most countable with $\bigvee S \in F$, we have $S \cap F \neq \phi$. The spectrum of a regular σ -frame L is

$$\Psi(L) = \{F \subseteq L : F \text{ is a } \sigma\text{-prime filter}\}.$$

For a nearness σ -frame (L, μ) with $\Psi(L)$ the spectrum of all σ -prime filters on L , let

$$\Psi(x) = \{P \in \Psi(L) : x \in P\}$$

and for each $A \in \mu$ let

$$\Psi(A) = \{\Psi(a) : a \in A\}.$$

Recall from page 9 that $(\Psi(L), \{\Psi(x) : x \in L\})$ is an Alexandroff space.

Lemma 3.4.1

Let $(L, \mu) \in \text{NoFrm}$. Then

(1) $\Psi(0) = \phi$ and $\Psi(1) = \Psi(L)$.

(2) $\Psi(a \wedge b) = \Psi(a) \cap \Psi(b)$ for any $a, b \in L$. Consequently,

$$\Psi(A \wedge B) = \{\Psi(a) \cap \Psi(b) : a \in A \text{ and } b \in B\} \text{ for each } A, B \in \mu.$$

(3) $\Psi(\bigvee A) = \bigcup_{a \in A} \Psi(a)$ for any countable $A \subseteq L$.

(4) If $Ax \leq y$ for any $A \in \mu$, then $(\Psi(A))\Psi(x) \subseteq \Psi(y)$.

Proof :

(1) Since, for any filter F on L , $0 \notin F$, particularly $\Psi(0) = \phi$. Also as $1 \in F$ for every σ -prime filter F , $\Psi(1) = \Psi(L)$.

(2) Let $F \in \Psi(a \wedge b)$. Then $a \wedge b \in F$. Since F is a filter and $a \wedge b \leq a, b$, we must have $a \in F$ and $b \in F$. Thus $\Psi(a \wedge b) \subseteq \Psi(a) \cap \Psi(b)$. If $P \in \Psi(a) \cap \Psi(b)$, then $a, b \in P$. Consequently, $a \wedge b \in P$ and thus $\Psi(a) \cap \Psi(b) \subseteq \Psi(a \wedge b)$. As a result we then have,

$$\begin{aligned} \Psi(A \wedge B) &= \{\Psi(a \wedge b) : a \in A \text{ and } b \in B\} \\ &= \{\Psi(a) \cap \Psi(b) : a \in A \text{ and } b \in B\}. \end{aligned}$$

(3) Let A be any countable subset of L and $F \in \Psi(\bigvee A)$. Then $\bigvee A \in F$. Since F is a σ -prime filter and A is countable, $F \cap A \neq \phi$. Thus $t \in F$ for some $t \in A$. Then $F \in \Psi(t) \subseteq \bigcup_{a \in A} \Psi(a)$. Also, if $P \in \Psi(a)$ for some $a \in A$. Then $a \in P$. Since $a \leq \bigvee A$ and P a filter on L , $\bigvee A \in P$. Thus $P \in \Psi(\bigvee A)$. Hence $\Psi(\bigvee A) = \bigcup_{a \in A} \Psi(a)$.

(4) Suppose that $Ax \leq y$ for some $A \in \mu$. Then

$$(\Psi(A))\Psi(x) = \bigcup \{\Psi(a) \in \Psi(A) : \Psi(a) \cap \Psi(x) \neq \phi\}$$

$$= \bigcup \{ \Psi(a) \in \Psi(A) : \Psi(a \wedge x) \neq \phi \}.$$

Let F be in some $\Psi(A)$ with $\Psi(a \wedge x) \neq \phi$. Then $a \in F$ and, since $a \wedge x \neq 0$, we have $a \leq Ax \leq y$. Thus $y \in F$, that is, $F \in \Psi(y)$. Hence $(\Psi(A))\Psi(x) \subseteq \Psi(y)$. \square

From Lemma 3.4.1, for any nearness σ -frame (L, μ) , $\Psi^*(L) = \{ \Psi(x) : x \in L \}$ is a σ -frame. Moreover, $\Psi^*(L)$ is regular. Let $\Psi(x) \in \Psi^*(L)$ for any $x \in L$. Since L is regular $x = \bigvee T$ for some countable $T \subseteq \{ a \in L : a \prec x \}$. If $t \in T$, then $t \wedge y = 0$ and $y \vee x = 1$ for some $y \in L$. Then $\Psi(y) \in \Psi^*(L)$ such that

$$\Psi(t) \cap \Psi(y) = \Psi(t \wedge y) = \Psi(0) = \phi$$

and

$$\Psi(y) \cup \Psi(x) = \Psi(y \vee x) = \Psi(1) = \Psi(L) = 1_{\Psi^*(L)}.$$

Thus $\Psi(t) \prec \Psi(x)$ in $\Psi^*(L)$. Then by regularity of L with $\{ \Psi(t) : t \in T \}$ a countable subset of $\{ \Psi(a) : \Psi(a) \prec \Psi(x) \}$, we have

$$\Psi(x) = \Psi(\bigvee T) = \bigcup_{t \in T} \Psi(t) = \bigcup \{ \Psi(t) : \Psi(t) \prec \Psi(x) \}$$

making $\Psi^*(L)$ a regular σ -frame.

Given a σ -frame L with $\nu \subseteq \text{cov}L$, we say that ν is a *base* for a nearness on L if ν satisfies the following conditions:

- (1) $A, B \in \nu \Rightarrow C \leq A \wedge B$ for some $C \in \nu$ and
- (2) for each $x \in L$ there exists a countable subset T of $\{ y \in L : y \triangleleft_{\nu} x \}$ such that $x = \bigvee T$.

If ν is a base for a nearness on the σ -frame L , let

$$\nu^* = \{ A \in \text{cov}L : B \leq A \text{ for some } B \in \nu \}.$$

Then (L, ν^*) is a nearness σ -frame.

Lemma 3.4.2

For any nearness σ -frame (L, μ) , $\nu = \{\Psi(A) : A \in \mu\}$ is a base for a nearness on the σ -frame $\Psi^*(L)$.

Proof :

For any $\Psi(A), \Psi(B) \in \nu$ with $A, B \in \mu$, $C = A \wedge B \in \mu$ and clearly $\Psi(C) \leq \Psi(A) \cap \Psi(B) = \Psi(A) \wedge \Psi(B) = \Psi(A \wedge B)$.

Let $\Psi(x) \in \Psi^*(L)$ for any $x \in L$. By the admissibility of μ , $x = \bigvee S$ for some countable subset S of $\{y \in L : y \triangleleft x\}$. Let $s \in S$. Then $As \leq x$ for some $A \in \mu$. By Lemma 3.4.1, $(\Psi(A))\Psi(s) \leq \Psi(x)$. Thus $\Psi(s) \triangleleft_\nu \Psi(x)$. Now let $T = \{\Psi(z) : z \in S\}$. Then for any $\Psi(z) \in T$, $\Psi(z) \triangleleft_\nu \Psi(x)$. Let $F \in \Psi(L)$ with $x = \bigvee S \in F$. Since F is σ -prime, $S \cap F \neq \emptyset$. Then $b \in F$ for some $b \in S$. Thus $F \in \Psi(b)$ and so $F \in \bigcup_{\Psi(z) \in T} \Psi(z)$. Also, if $F \in \Psi(z)$ for some $\Psi(z) \in T$, then $z \in F$. But $z \triangleleft_\mu x$ implies that $x \in F$. Thus $F \in \Psi(x)$. Hence T is a countable subset of $\{\Psi(y) : \Psi(y) \triangleleft_\nu \Psi(x)\}$ such that $\Psi(x) = \bigvee T$. \square

Thus for any nearness σ -frame (L, μ) , $(\Psi^*(L), \nu^*)$ is a nearness σ -frame. We next consider the space $\Psi(L)$ of all σ -prime filters on a regular σ -frame L . Let (L, μ) be a nearness σ -frame with ν a base for μ .

Proposition 3.4.1

$\mathfrak{B} = \{\Psi(A) : A \in \nu\}$ is a base for a nearness $\Psi(\mu)$ on the space $\Psi(L)$.

Proof :

Let $A \in \nu$. Then by Lemma 3.4.1,

$$\bigcup_{a \in A} \Psi(a) = \Psi(\bigvee A) = \Psi(1_L) = \Psi(L).$$

Thus $\Psi(A)$ is a cover on $\Psi(L)$ for each $A \in \mu$.

Let $\Psi(A) \in \mathfrak{B}$ for any $A \in \nu$. Since ν is a base for μ , there exists $B \in \nu$ such that $B \leq A$. Let $\Psi(a) \in \Psi(A)$ for any $a \in A$. Then there exists $b \in B$ such that $b \leq a$.

Then for any $F \in \Psi(b)$ we have $a \in F$ and thus $F \in \Psi(a)$. Hence $\Psi(B) \leq \Psi(A)$. Since $\Psi(B) \in \mathfrak{B}$, \mathfrak{B} is base for a nearness on $\Psi(L)$. \square

Thus for $(L, \mu) \in \mathbf{N}\sigma\mathbf{Frm}$, $(\Psi(L), \Psi(\mu)) \in \mathbf{Near}$ with nearness covers $\Psi(A) = \{\Psi(a) : a \in A\}$ for each $A \in \mu$.

Theorem 3.4.1

The topology $\Gamma_{\Psi(\mu)}$ induced by $\Psi(\mu)$ is exactly the topology Γ associated with $\Psi(L)$ as Alexandroff space, that is, the topology generated by the cozero sets $\Psi(x)$, $x \in L$.

Proof :

Let $X \in \Gamma$. Then $X = \bigcup\{\Psi(a_\alpha) : \alpha \in \Lambda\}$ for some arbitrary index set Λ . Let $\Psi(a_\beta) \subseteq X$ for some $\beta \in \Lambda$. Then there exists $F \in \Psi(L)$ such that $a_\beta \in F$. But $a_\beta = \bigvee T$ for some countable $T \subseteq \{y \in L : y \triangleleft_\mu a_\beta\}$. Since $\bigvee T \in F$ and F is σ -prime, $t \in F$ for some $t \triangleleft_\mu a_\beta$ with $t \in T$. Then $At \leq a_\beta$ for some $A \in \mu$. By Lemma 3.4.1, $(\Psi(A))\Psi(t) \leq \Psi(a_\beta) \subseteq X$. Thus $(\Psi(A))\Psi(t) \subseteq X$ and hence $X \in \Gamma_{\Psi(\mu)}$. Thus $\Gamma \subseteq \Gamma_{\Psi(\mu)}$.

Now let $X \subseteq \Psi(L)$ be open with respect to $\Gamma_{\Psi(\mu)}$. Then for each $F \in X$, there is $A \in \mu$ such that $(\Psi(A))\{F\} \subseteq X$ i.e.

$$\bigcup_{a \in A} \{\Psi(a) : a \in F\} \subseteq X.$$

Then $\Psi(b_F) \subseteq X$ for some $b_F \in A$ with $F \in \Psi(b_F)$. Then

$$X = \bigcup_{F \in X} \{\Psi(b_F) : F \in \Psi(b_F) \subseteq X\}.$$

Thus $X \in \Gamma$ and hence $\Gamma_{\Psi(\mu)} \subseteq \Gamma$. \square

For any uniform nearness σ -frame homomorphism $h : (L, \mu) \longrightarrow (M, \nu)$, define $\Psi h : (\Psi(M), \Psi(\nu)) \longrightarrow (\Psi(L), \Psi(\mu))$ by $\Psi h(F) = h^{-1}(F)$ for each $F \in \Psi(M)$.

Lemma 3.4.3

Ψh is uniformly continuous.

Proof :

We must show that $(\Psi h)^{-1}(\Psi(A)) \in \Psi(\nu)$ for each $A \in \mu$. To this end, let $A \in \mu$ with any $a \in A$. If $h(P) \in h(\Psi(a))$, then $a \in P$. So $h(a) \in h(P)$ and $h(P) \in \Psi(h(a))$. Also, for any $F \in \Psi(h(a))$, $h(a) \in F$. Thus $a \in h^{-1}(F)$ and then $h^{-1}(F) \in \Psi(a)$. So $F \in h(\Psi(a))$ and hence $\Psi(h(a)) = h(\Psi(a))$. Then for any $a \in A$ we have $(\Psi h)^{-1}(\Psi(a)) = \Psi(h(a))$ since

$$\begin{aligned} T \in (\Psi h)^{-1}(\Psi(a)) &\Leftrightarrow \Psi h(T) \in \Psi(a) \\ &\Leftrightarrow h^{-1}(T) \in \Psi(a) \\ &\Leftrightarrow T \in h(\Psi(a)) = \Psi(h(a)). \end{aligned}$$

Consequently, $(\Psi h)^{-1}(\Psi(A)) = \Psi(h(A))$ for each $A \in \mu$. Since h is uniform, for each $A \in \mu$, $h(A) \in \nu$. Thus $(\Psi h)^{-1}(\Psi(A)) = \Psi(h(A)) \in \Psi(\nu)$. \square

Theorem 3.4.2

$\Psi : \text{N}\sigma\text{Frm} \rightarrow \text{Near}$ is a contravariant functor.

Proof :

We have the following

$$\begin{array}{ccc} \Psi & : & \text{N}\sigma\text{Frm} \longrightarrow \text{Near} \\ \text{objects} & & (L, \mu) \longrightarrow (\Psi(L), \Psi(\mu)) \\ \text{morphisms} & & \begin{array}{ccc} L & & \Psi(L) & & h^{-1}(F) \in \Psi(L) \\ \downarrow h & & \uparrow \Psi h & & \uparrow \\ M & & \Psi(M) & & F \in \Psi(M) \end{array} \end{array}$$

By the above results Ψ is well-defined on objects and morphisms. Also, $\Psi id_L(P) = id^{-1}(P) = P = id_{\Psi(L)}(P)$ for each $P \in \Psi(L)$. Thus $\Psi id_L = id_{\Psi(L)}$. Whenever f and

g are defined, we have $\Psi(f \circ g)(P) = (f \circ g)^{-1}(P) = (g^{-1} \circ f^{-1})(P) = g^{-1}(f^{-1}(P)) = (\Psi g \circ \Psi f)(P)$. Thus $\Psi(f \circ g) = \Psi g \circ \Psi f$. Hence Ψ is a contravariant functor from the category of nearness σ -frames to the category of nearness spaces. \square

It is interesting to note that the characteristic maps of σ -prime filters on a σ -frame L are precisely σ -frame homomorphisms on L into the two point chain $\mathbf{2}$ and the spectrum functor Ψ is exactly counter to that of Σ on the category **Frm** introduced in Chapter 1. Associating a nearness σ -frame to a nearness space is not as transparent. Given a nearness space (X, \mathcal{U}) , the natural σ -frame to relate with X is the cozero-set lattice $CozX$. We may try to construct a nearness on $CozX$, say, $Coz\mathcal{U}$ by setting $Coz\mathcal{U} = \{\mathcal{A} \in \mathcal{U} : \mathcal{A} \subseteq CozX\}$. However, the definition of a nearness space involves limitations that do not provide for the admissibility criterion.

Chapter 4

Coreflective subcategories of \mathbf{NFrm}

4.1 Separable and totally bounded nearness frames

The separable coreflection

This section is devoted to the full subcategory $\mathbf{SepNFrm}$ of separable nearness frames introduced in Chapter 1. To recall, a nearness frame (L, μ) is separable if for each $A \in \mu$ there exists $B \in \mu$, B countable such that $B \leq A$. For $(L, \mu) \in \mathbf{NFrm}$ let $e_\mu = \{A \in \mu : \exists \text{ countable } B \in \mu \text{ such that } B \leq A\}$.

Theorem 4.1.1

For $(L, \mu) \in \mathbf{NFrm}$, (L, e_μ) is a separable nearness frame.

Proof :

Let $A \in e_\mu$ and suppose that $A \leq B$. Then there exists $C \in \mu$, countable such that $C \leq A$. Then as μ is a nearness $B \in \mu$ with $C \leq B$. Thus $B \in e_\mu$. If $A, B \in e_\mu$, then $A \wedge B \in \mu$. Moreover there exists countable μ -covers C and D such that $C \leq A$ and $D \leq B$. Then $C \wedge D \in \mu$, countable such that $C \wedge D \leq A \wedge B$. Thus $A \wedge B \in e_\mu$. For admissibility, let $x \in L$. By the admissibility of μ , $x = \bigvee \{y \in L : y \triangleleft_\mu x\}$. But, if $y \triangleleft_\mu x$, then $\{y^*, x\} \in e_\mu$ such that $\{y^*, x\}y \leq x$ i.e. $y \triangleleft_{e_\mu} x$. Hence

$x = \bigvee \{y \in L : y \triangleleft_{e_\mu} x\}$. Thus $(L, e_\mu) \in \mathbf{NFrm}$. That (L, e_μ) is separable is clear from the definition. \square

Lemma 4.1.1

$e : \mathbf{NFrm} \rightarrow \mathbf{SepNFrm}$ is functorial.

Proof :

We have the following

$$\begin{array}{c}
 e \quad : \quad \mathbf{NFrm} \longrightarrow \mathbf{SepNFrm} \\
 \\
 \text{objects} \quad \quad L \longrightarrow L \\
 \\
 \text{morphisms} \quad \begin{array}{ccc}
 \begin{array}{c} L \\ \downarrow h \\ M \end{array} & \begin{array}{c} L \\ \downarrow eh \\ M \end{array} & \begin{array}{c} a \\ \downarrow \\ h(a) \end{array} & \begin{array}{c} A \in e_\mu \\ \downarrow \\ h(A) \in e_\nu \end{array}
 \end{array}
 \end{array}$$

For objects, e takes any $(L, \mu) \in \mathbf{NFrm}$ to (L, e_μ) which is a separable nearness frame by the previous Theorem. For morphisms, let $h : (L, \mu) \rightarrow (M, \nu)$ be any uniform homomorphism. Define $eh : (L, e_\mu) \rightarrow (M, e_\nu)$ by $eh(a) = h(a)$. Then eh is a frame homomorphism. Moreover, if $A \in e_\mu$ then $h(A) \in \nu$ as h is uniform. Also there exists a countable $B \in \mu$ such that $B \leq A$. Then $h(B)$ is a countable ν -cover that refines $h(A)$. Thus $h(A) \in e_\nu$ and hence eh is uniform in $\mathbf{SepNFrm}$. Easily, $e(id_L) = id_L$ and if $h : (L, \mu) \rightarrow (M, \nu)$ and $g : (M, \nu) \rightarrow (N, \tau)$ are uniform homomorphisms, then $e(g \circ h) = eg \circ eh$. Hence e is functorial. \square

Lemma 4.1.2

The map $id_L : (L, e_\mu) \rightarrow (L, \mu)$ given by $a \rightsquigarrow a$ for each $a \in L$ is a uniform homomorphism.

Proof :

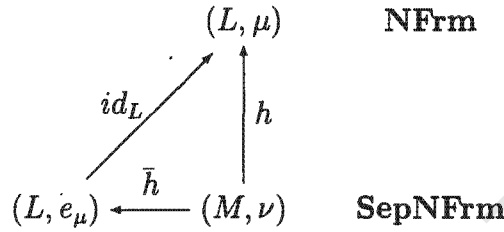
id_L is indeed a frame homomorphism. As $e_\mu \subseteq \mu$, id_L is uniform. \square

Theorem 4.1.2

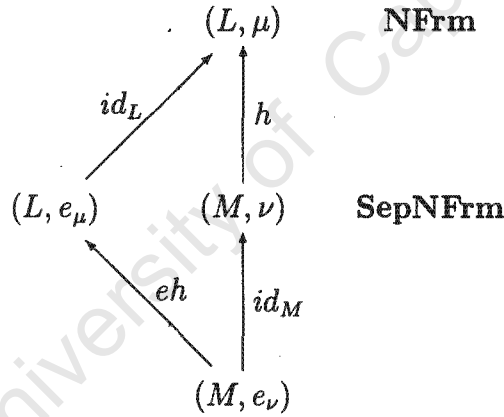
SepNFrm is coreflective in **NFrm** with coreflection functor e and coreflection map the identity.

Proof :

Let $(L, \mu) \in \mathbf{NFrm}$, $(M, \nu) \in \mathbf{SepNFrm}$ and $h : (M, \nu) \rightarrow (L, \mu)$ be any uniform homomorphism. We then need to find a uniform $\bar{h} : (M, \nu) \rightarrow (L, e_\mu)$ such that the triangle



commutes i.e. $id_L \circ \bar{h} = h$. Consider the following



Since (M, ν) is separable, $(M, \nu) = (M, e_\nu)$. Thus id_M is an isomorphism in **NFrm**.

Let $\bar{h} = eh \circ id_M^{-1}$. Then \bar{h} is a uniform homomorphism and since for each $x \in M$

$$(h \circ id_M)(x) = h(x) = eh(x) = (id_L \circ eh)(x),$$

we have that $h \circ id_M = id_L \circ eh$. So the diagram above commutes and thus $h = id_L \circ (eh \circ id_M^{-1}) = id_L \circ \bar{h}$. Hence, the original triangle above commutes and thus h factors through $id_L : (L, e_\mu) \rightarrow (L, \mu)$. This factorization is unique as id_L is monic.

□

Proposition 4.1.1

$(L, covL)$ is separable if and only if L is Lindelöf.

Proof :

Suppose that $(L, covL)$ is separable and let $A \in covL$. Then we can find a countable $B = (b_n) \in covL$ such that $B \leq A$. Then for each n there exists $a_n \in A$ such that $b_n \leq a_n$. Put $\tilde{A} = \{a_n\}$. Then $1 = \bigvee b_n \leq \bigvee a_n$. Thus \tilde{A} is a countable subcover of A making L Lindelöf. Conversely, if L is Lindelöf then for $A \in covL$, A has a countable subcover B . Clearly $B \leq A$ and hence $covL$ is separable. \square

Thus every regular frame L is Lindelöf provided that $covL$ is a separable nearness on L . Now a map $h : (L, \mu) \rightarrow (M, \nu)$ between nearness frames is a *quotient* if h is onto and $\{h(A) : A \in \mu\}$ generates ν . Clearly any nearness surjection is a quotient and any quotient map between nearness frames is a uniform homomorphism.

Theorem 4.1.3

The separable coreflection functor $e : \mathbf{NFrm} \rightarrow \mathbf{SepNFrm}$ preserves quotients.

Proof :

Let $h : (L, \mu) \rightarrow (M, \nu)$ be any quotient between nearness frames (L, μ) and (M, ν) .

Consider the following diagram

$$\begin{array}{ccc}
 (L, e_\mu) & \xrightarrow{eh} & (M, e_\nu) \\
 id_L \downarrow & & \downarrow id_M \\
 (L, \mu) & \xrightarrow{h} & (M, \nu)
 \end{array}$$

For any $y \in M$, as h is onto $y = h(x) = eh(x)$ for some $x \in L$. Thus eh is also onto. Let $A \in e_\nu$. Then $A \in \nu$ and there exists $B \in \nu$, B countable such that $B \leq A$. Since h is a quotient, $h(C) \leq B$ for some $C \in \mu$. Then $C \leq h_*(B)$ and so $h_*(B) \in \mu$. Since B is countable, $h_*(B)$ is also countable and hence $h_*(B) \in e_\mu$. Since h is onto we have that $eh((h_*(B))) = h(h_*(B)) = B \leq A$. Hence eh is a quotient. \square

Theorem 4.1.4

For any $(L, \mu) \in \mathbf{SepNFrm}$, if $h : (L, \mu) \rightarrow (M, \nu)$ is any quotient, then $(M, \nu) \in \mathbf{SepNFrm}$.

Proof :

Let $h : (L, \mu) \rightarrow (M, \nu)$ be any quotient with (L, μ) a separable nearness frame. Then $id_L : (L, e_\mu) \rightarrow (L, \mu)$ is an isomorphism. We then have

$$\begin{array}{ccc} (L, e_\mu) & \xrightarrow{eh} & (M, e_\nu) \\ id_L \downarrow \simeq & & \downarrow id_M \\ (L, \mu) & \xrightarrow{h} & (M, \nu) \end{array}$$

Let $A \in \nu$. Since h is a quotient there exists $B \in \mu$ such that $h(B) \leq A$. Then $B \in e_\mu$ and so $C \leq B$ for some countable $C \in \mu$. Clearly $h(C) \leq A$ and $h(C) \in e_\nu$. Consequently $A \in e_\nu$ and hence $\nu = e_\nu$. Thus $(M, \nu) \in \mathbf{SepNFrm}$. \square

The totally bounded coreflection

A nearness frame (L, μ) is *totally bounded* if for each $A \in \mu$ there exists $B \in \mu$, B finite such that $B \leq A$. For $(L, \mu) \in \mathbf{NFrm}$ let

$$\mu_* = \{A \in \mu : \exists \text{ finite } B \in \mu \text{ such that } B \leq A\}.$$

Theorem 4.1.5

For $(L, \mu) \in \mathbf{NFrm}$, (L, μ_*) is a totally bounded nearness frame.

The proof appears in [32] and follows Theorem 4.1.1 as "countable" may be replaced by "finite". We then have the category \mathbf{TNFrm} of totally bounded nearness frames. Similar to the case in $\mathbf{SepNFrm}$, \mathbf{TNFrm} is a coreflective subcategory of \mathbf{NFrm} with coreflection map the identity $id_L : (L, \mu_*) \rightarrow (L, \mu)$ which preserves quotients.

4.2 The Samuel compactification revisited

Baboolal and Ori [1] have shown that the Samuel compactification of a nearness frame (L, μ) can be described by its own normally regular ideals instead of the regular ideals of its uniform coreflection as follows:

$$\begin{array}{ccc}
 (L, \mu) & \xleftarrow{j} & (M, \nu) \\
 & & \uparrow \rho_M \\
 \mathfrak{NR}L & \simeq & \mathfrak{R}M
 \end{array}
 \qquad
 \begin{array}{ccc}
 \bigvee J & \longleftarrow & \bigvee (J \cap M) \\
 & & \uparrow \\
 J & \longrightarrow & J \cap M
 \end{array}$$

with (M, ν) the uniform coreflection, j the uniform coreflection map, ρ_M the compact regular coreflection map of the uniform frame (M, ν) and $\mathfrak{NR}L$ the frame of all normally regular ideals of L . As in Proposition 2.2.2 for the case in $\mathbf{N}\sigma\mathbf{Frm}$, $\rho_L : \mathfrak{NR}L \rightarrow (L, \mu)$ given by join is the compact regular coreflection map. Then it is possible to exhibit an alternative description of $\mathfrak{NR}L$ for a nearness frame (L, μ) via the corresponding compactification in $\mathbf{N}\sigma\mathbf{Frm}$ of its cozero part which coincides with Proposition 4.8 in [88].

Theorem 4.2.1

For any $(L, \mu) \in \mathbf{NFrm}$ with L compact, $\mathfrak{NR}L \simeq \mathcal{H}\mathfrak{NR}_\sigma \text{Coz } L$.

Proof:

If (L, μ) is compact then so is $\text{Coz } L$. Then $\rho_{\text{Coz } L} : \mathfrak{NR}_\sigma \text{Coz } L \rightarrow \text{Coz } L$ is an isomorphism by Lemma 2.2.2. Similarly, as L is compact, $\rho_L : \mathfrak{NR}L \rightarrow L$ is an isomorphism. Since Coz preserves isomorphisms, $\text{Coz}\rho_L : \text{Coz}\mathfrak{NR}L \rightarrow \text{Coz } L$ is an isomorphism. Then $\text{Coz}\mathfrak{NR}L \simeq \text{Coz } L \simeq \mathfrak{NR}_\sigma \text{Coz } L$. Applying \mathcal{H} gives $\mathcal{H}\text{Coz}\mathfrak{NR}L \simeq \mathcal{H}\mathfrak{NR}_\sigma \text{Coz } L$. However, $\mathfrak{NR}L \simeq \mathcal{H}\text{Coz}\mathfrak{NR}L$. Thus $\mathfrak{NR}L \simeq \mathcal{H}\mathfrak{NR}_\sigma \text{Coz } L$. \square

It should be noted that if $(L, \mu) \in \mathbf{NFrm}$ is compact, then L has a unique nearness which is a uniformity. Then $(L, \mu) \in \mathbf{UFrm}$. Moreover as L is Lindelöf, $\text{Coz } L = \text{Coz}_u L$ by Lemma 3.30 in [72] and $\text{Coz}_u \mu = \text{Coz } \mu$. Then $\mathfrak{NR}_\sigma L = \mathfrak{R}_\sigma L$.

and the result follows as in Proposition 4.8 in [88]. Next we show that the Samuel compactification, $\mathfrak{N}\mathfrak{A}L$, of a uniformly normal nearness frame (illustrated in [36]) can be described as the completion of its totally bounded coreflection (cf. the corresponding result for the Samuel compactification of *all* uniform frames in [13]). The nearness frame (L, μ) is *uniformly normal* in case both μ and μ_* are strong, with (L, μ_*) the totally bounded coreflection as in Theorem 4.1.5. Then by [36] we have the following result for any nearness frame (L, μ) .

Theorem 4.2.2

1. *If (L, μ) is totally bounded, then (L, μ) is strong if and only if (L, μ) is uniform.*
2. *(L, μ_*) is strong if and only if μ_* is a uniformity.*
3. *If (L, μ) is uniformly normal then (L, μ) has the same underlying frame as its uniform coreflection.*

Using the above result we prove the following.

Theorem 4.2.3

The Samuel compactification $\mathfrak{N}\mathfrak{A}L$ of a uniformly normal nearness frame (L, μ) can be described as the completion of its totally bounded coreflection.

Proof :

Let (L, μ) be any uniformly normal nearness frame with $(\mathcal{U}L, \mathcal{U}\mu)$ its uniform coreflection. Since (L, μ) is uniformly normal, by the above Theorem (L, μ) has the same underlying frame as its uniform coreflection. Then the coreflection map j is the identity id_L . Consider the following with $j = id_L : (\mathcal{U}L, \mathcal{U}\mu) \longrightarrow (L, \mu)$, the precompact coreflection map $id_{\mathcal{U}L} : (\mathcal{U}L, (\mathcal{U}\mu)_*) \longrightarrow (\mathcal{U}L, \mathcal{U}\mu)$ and $id_L : (L, \mu_*) \longrightarrow (L, \mu)$ the

totally bounded coreflection map.

$$\begin{array}{ccc}
 C(L, \mu_*) & \xleftarrow{Cj_*} & C(\mathcal{U}L, (\mathcal{U}\mu)_*) \\
 \downarrow \gamma_L & & \downarrow \gamma_{\mathcal{U}L} \\
 (L, \mu_*) & \xleftarrow{j_*} & (\mathcal{U}L, (\mathcal{U}\mu)_*) \\
 \downarrow id_L & & \downarrow id_{\mathcal{U}L} \\
 (L, \mu) & \xleftarrow{j} & (\mathcal{U}L, \mathcal{U}\mu)
 \end{array}$$

Since μ is strong, by the above Theorem (L, μ_*) is a uniform frame. Let $j_*(= id_L)$ be the restriction of j to the totally bounded uniformities. From [1] j is uniform as for each $A \in \mu_N$, $k(A) \in \mu_N$. Then for each $A \in (\mu_N)_*$ there exists a finite $B \in (\mu_N)_*$ such that $B \leq A$. Then $k(B)$ is finite and $k(B) \leq k(A)$ (see [13]). Thus $k(A) \in (\mu_N)_*$. So for each $A \in (\mu_N)_*$, $k(A) \in (\mu_N)_*$ rendering j_* uniform. Since j is (trivially) dense and onto, j_* is a dense surjection. Hence [13] affirms that the following is a commuting square

$$\begin{array}{ccc}
 C(L, \mu_*) & \xleftarrow{Cj_*} & C(\mathcal{U}L, (\mathcal{U}\mu)_*) \\
 \downarrow \gamma_L & \simeq & \downarrow \gamma_{\mathcal{U}L} \\
 (L, \mu_*) & \xleftarrow{j_*} & (\mathcal{U}L, (\mathcal{U}\mu)_*)
 \end{array}$$

with Cj_* an isomorphism. But the Samuel compactification of a uniform frame may be described as the completion of its precompact coreflection (see [13]). So $\mathfrak{R}(\mathcal{U}L) = \mathfrak{R}\mathcal{U}L \simeq C(\mathcal{U}L, (\mathcal{U}\mu)_*)$. Then $\mathfrak{R}\mathcal{U}L \simeq C(\mathcal{U}L, (\mathcal{U}\mu)_*) \simeq C(L, \mu_*)$. However, $\mathfrak{N}\mathfrak{R}L \simeq \mathfrak{R}\mathcal{U}L$ (see [1]). Thus $\mathfrak{N}\mathfrak{R}L \simeq C(L, \mu_*)$. \square

In [17] it is shown that, in general, for a totally bounded nearness frame (L, μ) its completion CL need not be compact. If CL is compact, then L must be uniform and hence any non-normal regular frame equipped with the nearness given by all finite covers is a totally bounded nearness frame whose completion is not compact. Also,

the Samuel compactification of a nearness frame can be described as the completion of the precompact coreflection of its uniform coreflection. We have the following as a consequence of [1] together with [13].

$$\begin{array}{ccccc}
 C(\mathcal{U}L, (\mathcal{U}\mu)_*) & \simeq & \mathfrak{R}(\mathcal{U}L) & \simeq & \mathfrak{NR}L \\
 \downarrow \gamma_{\mathcal{U}L} & & \downarrow \vee & & \downarrow \rho_L \\
 (\mathcal{U}L, (\mathcal{U}\mu)_*) & \xrightarrow{id_{\mathcal{U}L}} & ((\mathcal{U}L), (\mathcal{U}\mu)) & \xrightarrow{j} & (L, \mu)
 \end{array}$$

Since $\mathfrak{NR}L \simeq \mathfrak{R}L$ (see [1]) and $\mathfrak{R}L \simeq C(\mathcal{U}L, (\mathcal{U}\mu)_*)$ (see [13]), we have $\mathfrak{NR}L \simeq C(\mathcal{U}L, (\mathcal{U}\mu)_*)$.

Theorem 4.2.4

Consider

1. (L, μ) is a uniformly normal nearness frame.
2. $\mathfrak{NR}L \simeq C(L, \mu_*)$.
3. $C(L, \mu_*)$ is compact.

Then (1) \Rightarrow (2) \Rightarrow (3)

Proof :

(1) \Rightarrow (2) : Theorem 4.2.3 above.

(2) \Rightarrow (3) : As $\mathfrak{NR}L$ is compact the result follows. \square

In the above theorem (3) \Rightarrow (1) provided that (L, μ) is totally bounded. If (L, μ) is totally bounded, then $(L, \mu_*) = (L, \mu)$. So if $C(L, \mu_*)$ is compact, then $C(L, \mu)$ is compact and hence uniform (see [4]). Thus $\mu = \mu_*$ is strong. Hence (L, μ) is uniformly normal.

Chapter 5

Convergence

Nets and filters

In this subsection we preview some basic facts concerning nets and filters in spaces. For a more comprehensive treatment the papers [20], [21], [22], [23], [75] and [86] are suggested. Nets are described by way of *directed sets*.

To recall, given a nonempty set D with a binary relation \leq , the pair (D, \leq) is a *directed set* if \leq satisfies:

- (1) \leq is reflexive *i.e.* $d \leq d$ for each $d \in D$.
- (2) \leq is transitive *i.e.* if $m, n \in D$ with $m \leq n$, then for any $d \in D$ with $n \leq d$, $m \leq d$.
- (3) whenever $m, n \in D$ there exists $d \in D$ such that $m \leq d$ and $n \leq d$.

For any nonempty subsets R and C of D , we say that R is *residual* in D if $m, n \in D$ with $m \in R$ and $m \leq n$, then $n \in R$. C is *cofinal* in D if whenever $m \in D$ there exists $n \in C$ such that $m \leq n$.

For $X \in \mathbf{Top}$ and Λ any directed set, a function $\varphi : \Lambda \rightarrow X$ is called a *net* and is expressed as $\{\varphi_\alpha : \alpha \in \Lambda\}$ with $\varphi_\alpha = \varphi(\alpha) \forall \alpha \in \Lambda$. Let Λ be any cofinal set and for each $\alpha \in \Lambda$ let

$$\alpha^+ = \{\beta \in \Lambda : \beta \geq \alpha\} \quad \text{and} \quad \varphi(\alpha^+) = \{\varphi_\beta : \beta \in \alpha^+\}.$$

Then the collection $\mathcal{B}(\varphi) = \{\varphi(\alpha^+) : \alpha \in \Lambda\}$ is a base for a filter $\mathcal{F}(\varphi)$ called the *filter associated with* φ . Also, given any filter $\mathcal{F} \subseteq X$, the collection $\Lambda_{\mathcal{F}} = \{(x, F) : F \in \mathcal{F}, x \in F\}$ is directed by

$$(x, F) \leq (x, F') \text{ if and only if } F' \subseteq F.$$

Then the function $\varphi(\mathcal{F}) : \Lambda_{\mathcal{F}} \rightarrow X$ defined by $(x, F) \rightsquigarrow x$, for each $F \in \mathcal{F}$ is a net on X called the *net associated with* \mathcal{F} .

For $(X, \mu) \in \mathbf{Uni}$, the net $\varphi : \Lambda \rightarrow (X, \mu)$ is *cofinal* if and only if for each $\alpha \in \Lambda$ and $\mathcal{A} \in \mu$ there exists $A \in \mathcal{A}$ such that $\varphi_{\beta} \in A$ for some $\beta \in \alpha^+$. The filter $\mathcal{F} \subseteq X$ is called *Cauchy* if for each $\mathcal{A} \in \mu$ there exists $F \in \mathcal{F}$ such that $F \subseteq A$ for some $A \in \mathcal{A}$. The filter \mathcal{F} is *weakly Cauchy* or *cofinal* if for each $\mathcal{A} \in \mu$ there exists $A \in \mathcal{A}$ such that $A \cap F \neq \phi$ for each $F \in \mathcal{F}$. Using Herrlich's notation $sec\mathcal{F} = \{A \subseteq X : \forall F \in \mathcal{F}, A \cap F \neq \phi\}$ (see [46]), weakly Cauchy filters may be expressed in following sense:

$$\mathcal{F} \text{ is weakly Cauchy provided that } \mathcal{A} \cap sec\mathcal{F} \neq \phi.$$

The following result provides the translation between cofinal nets and weakly Cauchy filters.

Lemma 5.0.1

If the net $\varphi : \Lambda \rightarrow (X, \mu)$ is cofinal in X , then the associated filter $\mathcal{F}(\varphi)$ is weakly Cauchy. Also, if the filter \mathcal{F} on X is weakly Cauchy, then the associated net $\varphi(\mathcal{F})$ is cofinal.

Proof :

Suppose that $\varphi : \Lambda \rightarrow (X, \mu)$ is a cofinal net. Let $\mathcal{A} \in \mu$ and $F \in \mathcal{F}(\varphi)$. Then $\varphi(\alpha^+) \subseteq F$ for some $\alpha \in \Lambda$. Since φ is cofinal, there exists $\beta \in \alpha^+$ such that $\varphi_{\beta} \in A$ for some $A \in \mathcal{A}$. Then $\varphi_{\beta} \in \varphi(\alpha^+) \subseteq F$. Thus $A \cap F \neq \phi$. So, $\mathcal{A} \cap sec\mathcal{F} \neq \phi$ i.e. $\mathcal{F}(\varphi)$ is weakly Cauchy.

Let \mathcal{B} be a base for a weakly Cauchy filter \mathcal{F} in (X, μ) . We require that the associated net $\varphi(\mathcal{F}) = \{\varphi_{\mathcal{F}}(x, F) : x \in F \in \mathcal{F}\}$ is cofinal. Let $\mathcal{A} \in \mu$. As \mathcal{F} is cofinal, there exists $A \in \mathcal{A}$ such that $A \cap F \neq \phi, \forall F \in \mathcal{F}$. Then for each $F \in \mathcal{F}$ and $(x, F) \in \Lambda_{\mathcal{F}}$ there exists $B_F \in \mathcal{B}$ with $B_F \subseteq F$. Then for each $b \in B_F, (b, B_F) \in \Lambda_{\mathcal{F}}$ with $(b, B_F) \geq (x, F)$. As $B_F \in \mathcal{F}, A \cap B_F \neq \phi$. Thus there exists $y \in B_F$ such that $\varphi_{\mathcal{F}}(y, B_F) = y \in A$ for some $(y, B_F) \in (x, F)^+$ for $(x, F) \in \Lambda_{\mathcal{F}}$. Hence $\varphi(\mathcal{F})$ is cofinal. \square

Since $\mathcal{F} \subset \text{sec}\mathcal{F}$ for every filter \mathcal{F} , we have the following Lemma.

Lemma 5.0.2

Every Cauchy filter \mathcal{F} in (X, μ) is weakly Cauchy.

Directed nets and directed filters

Howes in [51] defines a net $\varphi : \Lambda \rightarrow X$ to be ω -directed or countably directed if for each countable $\{\lambda_i\}_{i=1}^{\infty} \subseteq \Lambda$ there is $\lambda \in \Lambda$ such that $\lambda_i \leq \lambda$ for each i . We provide a filter translation of this notion as follows.

Let $\mathcal{F} \subseteq X$ be a filter on the space X . We say that \mathcal{F} is a σ -filter if whenever \mathcal{A} is any countable subset of \mathcal{F} there is $F \in \mathcal{F}$ such that $F \subseteq A$ for each $A \in \mathcal{A}$. The following Proposition provides the translation between ω -directed nets and σ -filters in a space X .

Proposition 5.0.1

If $\varphi : \Lambda \rightarrow X$ is an ω -directed net on X , then the associated filter $\mathcal{F}(\varphi)$ is a σ -filter on X . Also, if \mathcal{F} is a σ -filter on X , then the associated net $\varphi(\mathcal{F})$ is an ω -directed net on X .

Proof :

Let $\varphi : \Lambda \rightarrow X$ be an ω -directed net. Also, let $\mathcal{A} \subseteq \mathcal{F}(\varphi)$ be any countable subset. Since $\mathcal{B}(\varphi) = \{\varphi(\lambda^+) : \lambda \in \Lambda\}$ is a base for $\mathcal{F}(\varphi)$, for each $A \in \mathcal{A}$ find $\lambda_A \in \Lambda$

such that $\varphi(\lambda_A^+) \subseteq A$. Then $\{\lambda_A : A \in \mathcal{A}\}$ is a countable subset of Λ . Since Λ is ω -directed there is $\lambda \in \Lambda$ such that $\lambda_A \leq \lambda$ for each $A \in \mathcal{A}$. Then $\lambda \in \lambda_A^+$ and thus $\varphi(\lambda) \in \varphi(\lambda_A^+)$ for each $A \in \mathcal{A}$. Consequently, $\varphi(\lambda^+) \subseteq A$ for each $A \in \mathcal{A}$ since

$$\begin{aligned} \varphi(\beta) \in \varphi(\lambda^+) &\Rightarrow \beta \in \lambda^+ \\ &\Rightarrow \beta > \lambda \geq \lambda_A \quad \forall A \in \mathcal{A} \\ &\Rightarrow \beta \in \lambda_A^+ \quad \forall A \in \mathcal{A} \\ &\Rightarrow \varphi(\beta) \in \varphi(\lambda_A^+) \subseteq A \quad \forall A \in \mathcal{A}. \end{aligned}$$

But $\varphi(\lambda^+) \in \mathcal{F}$. Thus \mathcal{F} is a σ -filter.

Now let \mathcal{F} be a σ -filter on X . The associated net is given by

$\varphi(\mathcal{F}) : \Lambda_{\mathcal{F}} \rightarrow X$ with $(x, F) \rightsquigarrow x$ where $\Lambda_{\mathcal{F}} = \{(x, F) : F \in \mathcal{F}\}$ is directed by

$$(x, F') \leq (x, F) \iff F' \subseteq F.$$

If $\Sigma \subseteq \Lambda_{\mathcal{F}}$ is any countable subset, then $\mathcal{A} = \{F : (x, F) \in \Sigma\} \subseteq \mathcal{F}$ is countable. Since \mathcal{F} is ω -directed there is $F' \in \mathcal{F}$ such that $F' \subseteq F$ for each $F \in \mathcal{A}$. Thus $(x, F) \leq (x, F')$ for each $F \in \mathcal{A}$. Since $(x, F') \in \Lambda_{\mathcal{F}}$, $\varphi(\mathcal{F})$ is an ω -directed net. \square

In the sequel the language of filters is the approach to convergence in frames.

5.1 Convergence in uniform frames

The following basic notions can be found in [50]. Let (L, μ) be a uniform frame. For a filter F in (L, μ) , let $\text{sec } F = \{y \in L : y \wedge x \neq 0 \text{ for each } x \in F\}$. We say that F is:

- *completely prime* if whenever $\bigvee_L S \in F$ for any $S \subseteq L$, then $S \cap F \neq \phi$.
- *Cauchy* if F meets every uniform cover.
- *convergent* or F *converges* if F meets every cover of L .
- *clustered* or F *clusters* in case $\text{sec } F$ meets every cover of L .
- *maximal* in case $F = G$ whenever G is a filter with $F \subseteq G$.

It is clear that every convergent filter in a uniform frame is a Cauchy filter. The following Lemma summarizes the results in [50] for a filter in a frame. We have included it here for completeness and requirements in the sequel.

Lemma 5.1.1

- (1) *Every completely prime filter clusters.*
- (2) *Every convergent filter clusters.*
- (3) *Any filter contained in a clustered filter also clusters.*
- (4) *Every maximal clustered filter converges.*
- (5) *If L is regular, then L is compact \Leftrightarrow every filter is clustered.*
- (6) *Any filter containing a completely prime filter is convergent.*
- (7) *Any filter that contains a convergent filter also converges.*
- (8) *If L is regular, then L is compact \Leftrightarrow every maximal filter converges.*
- (9) *For a maximal filter F , $\text{sec } F = F$.*

Lemma 5.1.2

If a filter F in a frame is such that $F \subseteq G$ for some convergent filter G in L , then F clusters.

Proof :

Immediate from Lemma 5.1.1 (2) and (3). \square

We call a filter F in a uniform frame (L, μ) *weakly Cauchy* if $\text{sec } F$ meets every uniform cover. Clearly every filter that clusters in a uniform frame is weakly Cauchy. The following Lemma is apparent from the fact that $F \subseteq \text{sec } F$.

Lemma 5.1.3

Every Cauchy filter in a uniform frame (L, μ) is weakly Cauchy.

The following Theorem is a pointfree filter version of the classical result in uniform spaces that if a Cauchy net clusters to a point y then it also converges to y (see for example [54]).

Theorem 5.1.1

If F is a clustered Cauchy filter in a uniform frame (L, μ) , then F converges.

Proof :

Let F be a clustered Cauchy filter in the uniform frame (L, μ) . Also, let $A \in \text{cov}L$. Then $\check{A} = \{x \in L : x \triangleleft a \text{ for some } a \in A\} \in \text{cov}L$. Since F is clustered, $\text{sec } F \cap \check{A} \neq \phi$. Thus there exists $t \in \check{A}$ such that $t \wedge y \neq 0$ for each $y \in F$. Since $t \in \check{A}$, $t \triangleleft a$ for some $a \in A$. Thus there exists $B \in \mu$ such that $Bt \leq a$. Then $B \leq \{t^*, a\}$ and hence $\{t^*, a\} \in \mu$. Since F is Cauchy, $\{t^*, a\} \cap F \neq \phi$. If $t^* \in F$, then $t \wedge t^* \neq 0$ which is a contradiction. Thus $t^* \notin F$. Hence $a \in F$. Thus $A \cap F \neq \phi$ and hence F converges. \square

A uniform frame (L, μ) is *Cauchy complete* if every Cauchy filter converges in (L, μ) . We say that the uniform frame (L, μ) is *strongly Cauchy complete* if every weakly Cauchy filter clusters in (L, μ) .

Theorem 5.1.2

Every compact regular frame is strongly Cauchy complete.

Proof :

Let L be a compact regular frame. Also let F be any weakly Cauchy filter on L . By Lemma 5.1.1 (5), F clusters. Hence L is strongly Cauchy complete. \square

Theorem 5.1.3

If the uniform frame (L, μ) is strongly Cauchy complete, then it is Cauchy complete.

Proof :

Let F be any Cauchy filter in (L, μ) . By Lemma 5.1.3, F is weakly Cauchy in L and so clusters as (L, μ) is strongly Cauchy complete. Then F is a clustered Cauchy filter in L and by Theorem 5.1.1, F converges. So, (L, μ) is Cauchy complete. \square

As an immediate consequence of the above two results we have the following.

Corollary 5.1.1

Every compact regular frame is Cauchy complete.

The above Corollary is proved differently in [13] (Lemma 10) where it is shown that any Cauchy filter in a compact regular frame contains a completely prime filter. Using Lemma 5.1.1 (6), [13] (Lemma 10) then concludes that every compact regular frame is Cauchy complete.

Lemma 5.1.4

Let (L, μ) and (M, ν) be uniform frames. If $h : (L, \mu) \rightarrow (M, \nu)$ is a uniform frame homomorphism and F is any weakly Cauchy filter in M , then $h^{-1}(F) = \{x \in L : h(x) \in F\}$ is a weakly Cauchy filter in L .

Proof :

Let $x, y \in h^{-1}(F)$. Then $h(x), h(y) \in F$ and since F is a filter in M , $h(x \wedge y) = h(x) \wedge h(y) \in F$. Thus $x \wedge y \in h^{-1}(F)$. Also, if $s \in h^{-1}(F)$ and $s \leq t$ then $h(s) \leq h(t)$ in M . As $h(s) \in F$, $h(t) \in F$. Thus $t \in h^{-1}(F)$. Hence $h^{-1}(F)$ is indeed a filter in L .

Now let $A \in \mu$. Since h is uniform, $h(A) \in \nu$. As F is weakly Cauchy in M , $\text{sec } F \cap h(A) \neq \phi$. Thus there exists $a \in A$ such that $h(a) \wedge y \neq 0$ for each $y \in F$. Then for each $x \in h^{-1}(F)$, as $h(x) \in F$, $h(a \wedge x) = h(a) \wedge h(x) \neq 0$. Since h is a frame homomorphism, $a \wedge x \neq 0$. This is valid for every $x \in h^{-1}(F)$ and so $\text{sec}(h^{-1}(F)) \cap A \neq \phi$. Hence, $h^{-1}(F)$ is weakly Cauchy in L . \square

Following the terminology of Dowker and Papert in [30], a subset T of a frame L is *conservative* if for each $S \subseteq T$ and each $x \in L$ we have

$$x \vee \bigwedge S = \bigwedge_{t \in S} (x \vee t).$$

For any $S \subseteq L$, let $S^\neg = \{s^* : s \in S\}$. Chen in [25] proves that S^\neg is conservative for any locally finite $S \subseteq L$.

Pultr and Úlehla in [72] define an element $x \in L$ to be *dense* if $a \wedge x \neq 0$ for each $0 \neq a \in L$. Equivalently, an element x in a frame L is dense if and only if $x^* = 0$. $A \subseteq L$ is a *quasicover* if $\bigvee A$ is dense in L . Easily each cover in a frame L is a quasicover. If $A \leq B$ and A is a quasicover in L we then say that A is a *quasirefinement* of B . Also recall that a regular frame L is compact if and only if every cover of L has a finite quasirefinement (see [49] Corollary 2.3(3)). Using these notions we have the following characterization for a regular frame to be Lindelöf, a generalization of the corresponding result for compact frames just mentioned above.

Theorem 5.1.4

A regular frame L is Lindelöf if and only if each cover A of L has a countable quasirefinement B such that B^\neg is conservative.

Proof :

(\Leftarrow) Let $A \in \text{cov}L$. Since L is regular, $a = \bigvee \{y \in L : y \prec a\}$ for each $a \in A$. Then $1 = \bigvee_{a \in A} \bigvee_{y \prec a} y$. By the hypothesis, we can find $T = (t_i)_{i \in I}$ (I a countable index set) with T^\neg conservative, $(\bigvee T)^* = 0$ such that for each $i \in I$, $t_i \leq y_i$ for some y_i . Thus $t_i \prec a_i$ for some $a_i \in A$. Then $t_i^* \vee a_i = 1$ for each $i \in I$. Let $B = \{a_i : i \in I\}$. Then B is a countable subset of A such that

$$\begin{aligned}
 \bigvee B &= \bigvee_{i \in I} a_i \vee 0 \\
 &= \bigvee_{i \in I} a_i \vee (\bigvee T)^* \\
 &= \bigvee_{i \in I} a_i \vee \bigwedge_{i \in I} t_i^* \\
 &= \bigwedge_{i \in I} (t_i^* \vee \bigvee_{i \in I} a_i) && \text{(since } T^\neg \text{ is conservative)} \\
 &= \bigwedge_{i \in I} (t_i^* \vee a_i \vee \bigvee_{i \neq j \in I} a_j) \\
 &= 1 && \text{(since } t_i^* \vee a_i = 1 \text{ for each } i).
 \end{aligned}$$

Thus B is a countable subcover of A rendering L Lindelöf.

(\Rightarrow) Suppose that L is Lindelöf and let $A \in \text{cov}L$. Since L is paracompact (being regular and Lindelöf, see [77]) there is a locally finite cover $B \leq A$. Then there exists a countable $C \in \text{cov}L$ such that $C \subseteq B$. Since B is locally finite and $C \subseteq B$, we have that C is locally finite. Thus by Chen [25], C^\neg is conservative and the result follows. \square

The totally bounded uniform frames are of significant interest with regard to convergence. A uniform frame is *precompact* if the uniformity is generated by its finite members. In [87] it is shown that the precompact uniform frames form a coreflective subcategory of \mathbf{UFrm} . The following results lead to alternative characterizations of precompact uniform frames in terms of filters.

Lemma 5.1.5

For any uniform frame (L, μ) ,

$$\check{A} = \{x \in L : x \triangleleft a \text{ for some } a \in A\}$$

and

$$A^\sphericalangle = \{x \in L : x \prec a \text{ for some } a \in A\}$$

are μ -covers on L for any $A \in \mu$.

Proof :

If $A \in \mu$, then $B \leq^* A$ for some $B \in \mu$ i.e. $BB \leq A$. So, $\forall b \in B, Bb \leq a$ for some $a \in A$ i.e. $b \triangleleft a$ for some $a \in A$. Thus $b \in \check{A}$. So $B \leq \check{A}$ and hence, $\check{A} \in \mu$. Now as $\check{A} \leq A^\sphericalangle, A^\sphericalangle \in \mu$. \square

A frame L is *almost compact* if for each $A \in \text{cov}L$ there is $B \in A$ such that $(\bigvee B)^* = 0$. Almost compact frames are studied in [65], [49] and [50]. We call a uniform frame (L, μ) *uniformly almost compact* provided that for each $A \in \mu$ there is $B \in A$ such that $(\bigvee B)^* = 0$. Clearly, every almost compact uniform frame is uniformly almost compact.

Lemma 5.1.6

If $h : (L, \mu) \longrightarrow (M, \nu)$ is a dense uniform homomorphism and (M, ν) is uniformly almost compact, then so is (L, μ) .

Proof:

Let $h : (L, \mu) \longrightarrow (M, \nu)$ be a dense uniform homomorphism with (M, ν) uniformly almost compact. Let $A \in \mu$. Since h is uniform, $h(A) \in \nu$. As (M, ν) is uniformly almost compact there exists $D \in h(A)$ such that $(\bigvee D)^* = 0$. Since D is finite there exists $B \in A$ such that $D = \{h(b) : b \in B\}$. Then

$$\left[h(\bigvee B) \right]^* = \left[\bigvee_{b \in B} h(b) \right]^* = (\bigvee D)^* = 0.$$

Since $h \left[(\bigvee B)^* \right] \wedge h(\bigvee B) = 0$, we have that $h \left[(\bigvee B)^* \right] \leq \left[h(\bigvee B) \right]^* = 0$. Since h is dense, $(\bigvee B)^* = 0$. Hence, (L, μ) is uniformly almost compact. \square

Theorem 5.1.5

A uniform frame (L, μ) is precompact if and only if (L, μ) is uniformly almost compact.

Proof :

Suppose that (L, μ) is precompact and let $A \in \mu$. Then there exists a finite $B \in \mu$ such that $B \leq A$. Then for each $b \in B$ there is $a_b \in A$ such that $b \leq a_b$. Then easily $\{a_b : b \in B\}$ is a finite (uniform) subcover of A and the result follows.

For the converse, under the given hypothesis let $A \in \mu$. By Lemma 5.1.5, $A^\sphericalangle \in \mu$. By the hypothesis there is a finite $Y = \{y_1, y_2, \dots, y_k\} \subseteq A^\sphericalangle$ such that $(\bigvee Y)^* = 0$. Then for each $1 \leq i \leq k$, $y_i \prec a_i$ i.e. $y_i^* \vee a_i = 1$ for some $a_i \in A$. Then $\{a_1, a_2, \dots, a_k\}$ is a finite subcover of A since

$$\begin{aligned} \bigvee_{i=1}^k a_i &= \bigvee_{i=1}^k a_i \vee 0 \\ &= \bigvee_{i=1}^k a_i \vee (\bigvee Y)^* \end{aligned}$$

$$\begin{aligned}
&= \bigvee_{i=1}^k a_i \vee \bigwedge_{i=1}^k y_i^* \\
&= \bigwedge_{i=1}^k (y_i^* \vee \bigvee_{i=1}^k a_i) \quad (\text{since } \{y^* : y \in Y\} \text{ is finite}) \\
&= \bigwedge_{i=1}^k (y_i^* \vee a_i \vee \bigvee_{i \neq j} a_j) = 1 \quad (\text{since } y_i^* \vee a_i = 1 \text{ for each } i).
\end{aligned}$$

Thus each uniform cover has a finite subcover under the hypothesis. Then for $\check{A} \in \mu$ there is a finite cover $S = \{s_1, s_2, \dots, s_k\} \subseteq \check{A}$. Then for each $1 \leq i \leq k$, $s_i \triangleleft b_i$ for some $b_i \in A$. Thus there is $W \in \mu$ such that $W s_i \leq b_i$ for each i . Now if $0 \neq w \in W$ then

$$0 \neq w = w \wedge 1 = w \wedge \bigvee_{i=1}^k s_i = \bigvee_{i=1}^k (w \wedge s_i).$$

Thus $w \wedge s_j \neq 0$ for some $1 \leq j \leq k$ and so $w \leq W s_j \leq b_j$. Thus $W \leq \{b_1, b_2, \dots, b_k\} = B$. So $B \in \mu$ and B is a finite refinement of A . Hence (L, μ) to be precompact. \square

Cauchy nets play a significant role in the characterization of precompact uniform spaces (see Howes [54] or Willard [91]). A uniform space (X, \mathcal{U}) is precompact if each uniform cover has a finite subcover or equivalently a uniform space is precompact if and only if each net has a Cauchy subnet (see [91]). We present now a pointfree version of this result using the filter approach.

Theorem 5.1.6

A uniform frame (L, μ) is precompact if and only if each filter in L is contained in a Cauchy filter.

Proof:

Suppose that (L, μ) is precompact and let F be any filter in L . Then $F \subseteq G$ for some maximal filter G in L . Let $A \in \mu$. Then there is $B \in \mu$ such that $B \leq A$ and B is finite. Then $\bigvee B \in G$. Since G is maximal and B is finite there exists $b \in B$ such that $b \in G$. Then there exists $a \in A$ such that $b \leq a$. Then $a \in G$ and so

$G \cap A \neq \phi$. Hence G is Cauchy in L .

For the converse, suppose that each filter in L is contained in a Cauchy filter but (L, μ) is not precompact. Then, by Theorem 5.1.5 there exists $A \in \mu$ such that for each $B \in A$, $(\bigvee B)^* \neq 0$. Then $\{(\bigvee B)^* : B \in A\}$ generates a (proper) filter F in L . By the hypothesis, $F \subseteq G$ for some Cauchy filter G in L . Consequently, $A \cap G \neq \phi$. Thus there exists $a \in A$ such that $a \in G$. Then $\{a\} \in A$ and hence $a^* \in G$ gives a contradiction. Thus (L, μ) must be precompact. \square

As a consequence of the above theorem we have the following characterization of precompact uniform frames.

Corollary 5.1.2

A uniform frame (L, μ) is precompact if and only if each maximal filter in L is Cauchy.

As an immediate consequence we have the following Theorem in pointfree form of the corresponding result in uniform spaces (see [91]).

Theorem 5.1.7

A uniform frame is compact if and only if it is Cauchy complete and precompact.

Proof:

Suppose that (L, μ) is compact. Then $\mu = \text{cov}L$ and easily (L, μ) is Cauchy complete. If $A \in \mu$ then by compactness A has a finite subcover which is uniform. Consequently, (L, μ) is precompact. Conversely, suppose that (L, μ) is Cauchy complete and precompact. Let F be a maximal filter in L . By the above Corollary, F is Cauchy and hence converges. By Lemma 5.1.1(8) L is compact. \square

In keeping with the terminology for uniform spaces in [54], we call a uniform frame *preparacompact* if each weakly Cauchy filter is contained in a Cauchy filter. In light of Theorem 5.1.6 preparacompactness is a filter generalization of precompactness. Preparacompactness then provides a partial converse to Theorem 5.1.3.

Theorem 5.1.8

If a uniform frame is Cauchy complete and precompact, then it is strongly Cauchy complete.

Proof :

Suppose that (L, μ) is Cauchy complete and precompact uniform frame. Let F be any weakly Cauchy filter in L . By precompactness, $F \subseteq G$ for some Cauchy filter G in L . Since (L, μ) is Cauchy complete, G converges and hence clusters. Consequently F is contained in a clustered filter and so also clusters. Hence, (L, μ) is strongly Cauchy complete. \square

Recall that a cover A in a uniform frame (L, μ) is a *normal* cover if there is a sequence of covers $(A_n) \subseteq \text{cov}(L)$ such that $A_{n+1} \leq^* A_n$ for each n and $A_0 = A$. Set

$$u_F = \{A \in \text{cov}(L) : A \text{ is normal}\},$$

$$\beta = \{A \in u_F : B \leq A \text{ for some finite } B \in \mu\} \text{ and}$$

$$e = \{A \in u_F : B \leq A \text{ for some countable } B \in \mu\}.$$

In [87] it was shown, in particular, that for any uniform frame every finite (countable) uniform cover has a finite (countable) star-refinement. Then one can show that β and e are uniformities on the frame L . Also, in [87] it is shown that u_F is a uniformity on L (called the *fine uniformity*). Further, from [14] we have that the uniform frame (L, μ) is paracompact if and only if $\mu_F = \text{cov}(L)$. We then have the following result (c.f. that for spaces in [52]).

Theorem 5.1.9

A regular frame L is compact if and only if (L, β) is strongly Cauchy complete.

Proof :

If L is compact, then L has a unique uniformity. Thus $\beta = \text{cov}L$ and directly (L, β) is strongly Cauchy complete. For the converse, suppose that (L, β) is strongly Cauchy complete. Let F be any maximal filter in L and $A \in \beta$. Then there exists a finite

Thus $0 \neq h(a) \wedge h(x) = h(a) \wedge y$. Thus $h(a) \in h(A)$ such that $h(a) \wedge y \neq 0$ showing that $h(A)$ is a quasicover in L . \square

Theorem 5.1.10

The following are equivalent for a regular frame L

1. L is paracompact.
2. (L, u_F) is strongly Cauchy complete.
3. If F is any filter in L such that for every dense uniform surjection $h : (M, d) \longrightarrow (L, u_F)$ (for any metric frame (M, d)) $h_*(F)$ clusters, then F clusters in (L, u_F) .

Proof :

(1) \Rightarrow (2) If L is paracompact then every cover of L is a normal cover. Then the fine uniformity on L consists of all covers i.e. $u_F = \text{cov}L$. Immediately, (L, u_F) is strongly Cauchy complete.

(2) \Rightarrow (3) Suppose that (L, u_F) is strongly Cauchy complete. Let F be any filter in L satisfying the hypothesis of (3). Then let $h : (M, d) \longrightarrow (L, u_F)$ be any dense uniform surjection with metric frame (M, d) . Also let $A \in u_F$. Since h is uniform there exists $\epsilon > 0$ such that $h(D_\epsilon^M) \leq A$. Since $h_*(F)$ clusters in M , $D_\epsilon^M \cap \text{sec}(h_*(F)) \neq \phi$. Find $d \in D_\epsilon^M$ such that $d \wedge h_*(x) \neq 0$ for each $x \in F$. Since h is dense and onto, for each $x \in F$,

$$0 \neq h(d \wedge h_*(x)) = h(d) \wedge h h_*(x) = h(d) \wedge x$$

Since $h(D_\epsilon^M) \leq A$ there exists $a_d \in A$ such that $h(d) \leq a_d$. Thus for each $x \in F$, $a_d \wedge x \neq 0$. Hence $A \cap \text{sec} F \neq \phi$ showing that F is weakly Cauchy in (L, u_F) . Since (L, μ) is strongly Cauchy complete, F clusters.

$B \in \beta$ such that $B \leq A$. Since B is finite and F is maximal, $B \cap F \neq \phi$. Consequently $A \cap F \neq \phi$. Hence F is β -Cauchy. Thus every maximal filter in (L, β) is Cauchy and hence by Corollary 5.1.2 (L, β) is precompact. As (L, β) is strongly Cauchy complete, (L, β) is Cauchy complete by Theorem 5.1.3. Hence (L, β) is Cauchy complete and precompact and by Theorem 5.1.7, L is compact. \square

For any frame homomorphism $h : M \rightarrow L$, consider the right adjoint $h_* : L \rightarrow M$ where $h_*(x) = \bigvee \{y \in L : h(y) \leq x\}$ and the notion of a quasicover defined on page 86. We shall make use in Theorem 5.1.10 of the following Lemma together with the result of Pultr and Úlehla in [72] that a regular frame L is paracompact if and only if every cover in L has a locally finite quasirefinement.

Lemma 5.1.7

Let $h : M \rightarrow L$ be a dense frame homomorphism. Then

1. *h preserves local finiteness and*
2. *if h is onto, h preserves quasicovers.*

Proof :

(1) Let $A \subseteq M$ be locally finite. Then there exists $S \in \text{cov}M$ such that for each $s \in S$, $A_s = \{a \in A : a \wedge s \neq 0\} \in A$. Then $h(A) \subseteq L$ and $h(S) \in \text{cov}L$. For each $s \in S$, consider $T_s = \{h(a) \in h(A) : h(a) \wedge h(s) \neq 0\} \subseteq h(A)$. Let $h(s) \in h(S)$ for any $s \in S$ and let $h(b) \in h(A)$ for any $b \in A$. If $0 \neq h(b) \wedge h(s) = h(b \wedge s)$, then $b \wedge s \neq 0$ as h is dense. Thus $b \in A_s$ and so $T_s \in h(A)$ for each $s \in S$. Consequently, each $h(s) \in h(S)$ meets at most finitely many members of $h(A)$. Thus $h(A)$ is locally finite.

(2) Suppose that h is onto. Let $A \subseteq M$ be a quasicover and let $0 \neq y \in L$. Since h is onto there exists $x \in M$ such that $h(x) = y \neq 0$. Then $0 \neq x \in M$ and since A is a quasicover there exists $a \in A$ such that $a \wedge x \neq 0$. Since h is dense $h(a \wedge x) \neq 0$.

(3) \Rightarrow (1) Suppose (3) and let $A \in \text{cov}L$. Suppose that $(\bigvee B)^* = 0$ for some $B \in A$. Then B is a finite quasirefinement of A . Since B is locally finite, B^- is conservative (see [25]). By Theorem 5.1.4, L is Lindelöf. Since L is regular and Lindelöf, L is consequently paracompact. So we may suppose that for each $B \in A$, $(\bigvee B)^* \neq 0$. Then $\{(\bigvee B)^* : B \in A\}$ generates a (proper) filter F in L . Since $A \cap \text{sec}F = \phi$, F does not cluster. Accordingly by (3) there exists a dense uniform surjection $h : (M, d) \rightarrow (L, u_F)$ with (M, d) a metric frame such that $h_*(F)$ does not cluster. Find $T \in \text{cov}M$ such that $T \cap \text{sec}(h_*(F)) = \phi$. Then for each $t \in T$, there exists $x_t \in F$ such that $t \wedge h_*(x_t) = 0$. So, for each $t \in T$, $t \leq [h_*(x_t)]^* = h_*(x_t^*)$ by Lemma 1.2.1. Then $S = \{h_*(x_t^*) : t \in T\} \in \text{cov}M$. Since (M, d) is paracompact (see [77]) there exists a locally finite quasicover $V \leq S$. Since h is dense and onto, by Lemma 5.1.7, $h(V)$ is a locally finite quasicover in L such that $h(V) \leq h(S) = \{x_t^* : t \in T\}$.

Now for each $t \in T$, $x_t \in F$ implies that $(\bigvee B_t)^* \leq x_t$ for some $B_t \in A$. Thus for each $t \in T$, $x_t^* \leq (\bigvee B_t)^{**}$. Since $h(V) \leq \{x_t^* : t \in T\}$, for each $v \in V$ there exists $t_v \in T$ such that $h(v) \leq (\bigvee B_{t_v})^{**}$. Now for each $v \in V$ let

$$h(v) \wedge B_{t_v} = \{h(v) \wedge b : b \in B_{t_v}\}$$

and put

$$J = \{h(v) \wedge B_{t_v} : v \in V\}.$$

We next show that J is a quasicover on L .

Let $0 \neq y \in L$. Then $h(v) \wedge y \neq 0$ for some $v \in V$ as $h(V)$ is a quasicover. Find $B_{t_v} \in A$ such that $h(v) \leq (\bigvee B_{t_v})^{**}$. If $h(v) \wedge y \wedge \bigvee B_{t_v} = 0$, then $h(v) \wedge y \leq (\bigvee B_{t_v})^*$. Then $h(v) \wedge y = 0$ which is a contradiction. Thus $h(v) \wedge b \wedge y \neq 0$ for some $b \in B_{t_v}$. Since $h(v) \wedge b \in J$, J is a quasicover on L . Next we show that J is locally finite.

Since $h(V)$ is locally finite we can find $W \in \text{cov}L$ such that each $w \in W$ meets at most finitely many V -members *i.e.* for each $w \in W$, $\{h(v) \in h(V) : h(v) \wedge w \neq 0\} \in h(V)$. For $W \in \text{cov}L$, we show that for each $w \in W$, $J_w = \{j \in J : j \wedge w \neq 0\} \in J$. Let $w \in W$. Then there exists $V_w \in V$ such that for each $v \in V_w$, $h(v) \wedge w \neq 0$ (and for each $v \notin V_w$, $h(v) \wedge w = 0$). Then for each $v \in V_w$, $B_{t_v} \in A$ and as before for each $v \in V_w$, $h(v) \wedge b_v \wedge w \neq 0$ for some $b_v \in B_{t_v}$. Let

$$\overline{B_{t_v}} = \{b \in B_{t_v} : h(v) \wedge b \wedge w \neq 0\} \quad \text{and} \quad K = \{h(v) \wedge \overline{B_{t_v}} : v \in V_w\}.$$

Then $K \in J$. Now, if $j \in J$ and $j \wedge w \neq 0$, then $j = h(v) \wedge b$ for some $v \in V$ and $b \in B_{t_v}$. Then $h(v) \wedge b \wedge w \neq 0$ and thus $v \in V_w$. Consequently $j = h(v) \wedge b \in K$. So, $J_w \subseteq K$. Since K is finite, $J_w \in J$. Thus J is a locally finite quasirefinement of A . Hence L is paracompact by the characterization in [72]. \square

If F is a filter in L and $h : M \rightarrow L$ is onto, then $\{h_*(x) : x \in F\}$ generates a filter, which we will denote by $h_*(F)$, in M .

Lemma 5.1.8

Let $h : (M, \nu) \rightarrow (L, \mu)$ be any uniform surjection.

1. If F is a Cauchy (respectively, weakly Cauchy) filter in L , then $h_*(F)$ is a Cauchy (respectively, weakly Cauchy) filter in M .
2. If h is dense and G is a Cauchy (respectively, weakly Cauchy) filter in M , then $h(G)$ is a Cauchy (respectively, weakly Cauchy) filter in L .

Proof :

(1) Let F be any Cauchy filter in (L, μ) . Let $A \in \nu$. Then, since h is a surjection, there exists $B \in \mu$ such that $h_*[B] \leq A$. As F is Cauchy, $B \cap F \neq \phi$. We can then find $b \in B \cap F$. Then $h_*(b) \in h_*(F)$. On the other hand, since $h_*[B] \leq A$, there exists $a \in A$ satisfying $h_*(b) \leq a$. Clearly, since $h_*(F)$ is a filter in M , $a \in h_*(F)$ and so $h_*(F) \cap A \neq \phi$. Hence $h_*(F)$ is a Cauchy filter in (M, ν) .

Now let F be a weakly Cauchy filter in (L, μ) and let $A \in \nu$. Then $h(A) \in \mu$. Since F is weakly Cauchy, $\text{sec } F \cap h(A) \neq \phi$. Thus there exists $a \in A$ such that $h(a) \wedge x \neq 0$ for each $x \in F$. Since h is onto, $0 \neq h(a) \wedge h h_*(x) = h(a \wedge h_*(x))$ for each $x \in F$. Thus $a \wedge h_*(x) \neq 0$ for each $x \in F$. Thus $\text{sec}(h_*(F)) \cap A \neq \phi$ so that $h_*(F)$ is weakly Cauchy in (M, ν) .

(2) Let F be a filter in M . If $x, y \in h(F)$, then $x = h(s)$ and $y = h(t)$ for some $s, t \in F$. Then $s \wedge t \in F$ and $x \wedge y = h(s) \wedge h(t) = h(s \wedge t) \in h(F)$. If $h(z) \in h(F)$ for any $z \in F$ and $h(z) \leq w$, then $z \leq h_*(w)$. Thus $h_*(w) \in F$. Since h is onto $w = h(h_*(w)) \in h(F)$. If $0 \in h(F)$, then $h(x) = 0$ for some $x \in F$. Since h is dense, $0 = x \in F$ gives a contradiction. Hence $0 \notin h(F)$ and thus $h(F)$ is indeed a filter in L for any filter F in M .

Now let G be a Cauchy filter in M and let $A \in \mu$. Since h is a uniform surjection $h_*(A) \in \nu$. Thus $h_*(A) \cap G \neq \phi$. Find $a \in A$ such that $h_*(a) \in G$. Then as h is onto $a \in h(G)$. Hence $h(G) \cap A \neq \phi$ showing that $h(G)$ is Cauchy in (L, μ) .

Finally, suppose that G is a weakly Cauchy filter in (M, ν) and h is dense. Let $A \in \mu$. Then $h_*(A) \in \nu$ and as G is weakly Cauchy in (L, μ) , $\text{sec } G \cap h_*(A) \neq \phi$. Find $h_*(a) \in h_*(A)$ for some $a \in A$ such that $h_*(a) \wedge x \neq 0$ for each $x \in G$. Since h is dense and onto, $0 \neq h(h_*(a) \wedge x) = a \wedge h(x)$ for each $x \in G$. Thus $\text{sec}(h(G)) \cap A \neq \phi$ showing that G is weakly Cauchy in (L, μ) . \square

Dube [32] shows that if (M, ν) is a nearness frame and $h : M \rightarrow L$ is any onto frame homomorphism, then $(L, h\nu)$ is a nearness frame where $h\nu = \{h(A) : A \in \nu\}$. Now if (M, ν) is a uniform frame then one can easily show that $(L, h\nu)$ is a uniform frame for any onto frame homomorphism $h : M \rightarrow L$. Now let (L, μ) be any uniform frame and consider its completion which we denote by $\gamma_L : (CL, c\mu) \rightarrow (L, \mu)$ (see [7] or [13] for more details). Let $(u_{CL})_F$ be the fine uniformity on CL . Since $\gamma_L : CL \rightarrow L$ is an onto frame homomorphism, $\gamma_L((u_{CL})_F)$ is a uniformity on L by the preceding remarks. Let $u^* = \gamma_L((u_{CL})_F)$. The following result shows that

for any uniform frame (L, μ) , precompactness of (L, u^*) is a sufficient condition for (L, μ) to have a paracompact completion, the pointfree analogue of the result for uniform spaces appearing in [52].

Theorem 5.1.11

A uniform frame (L, μ) has a paracompact completion whenever (L, u^) is precompact.*

Proof :

Suppose that (L, u^*) is precompact. We shall show that $(CL, (u_{CL})_F)$ is strongly Cauchy complete. Let F be any weakly Cauchy filter in $(CL, (u_{CL})_F)$. By Lemma 5.1.8 (2), $\gamma_L(F)$ is weakly Cauchy in (L, u^*) as γ_L is dense. Since (L, u^*) is precompact, $\gamma_L(F) \subseteq G$ for some Cauchy filter G in (L, u^*) . Then $F \subseteq (\gamma_L)_*(G)$. By Lemma 5.1.8 (1), $(\gamma_L)_*(G)$ is Cauchy in $(CL, (u_{CL})_F)$. Since $(CL, c\mu)$ is complete and $c\mu \subseteq (u_{CL})_F$, $(CL, (u_{CL})_F)$ is also complete (see [14]). Then $(CL, (u_{CL})_F)$ is Cauchy complete (see [13]). Thus $(\gamma_L)_*(G)$ converges and hence F clusters in $(CL, (u_{CL})_F)$. Consequently, $(CL, (u_{CL})_F)$ is strongly Cauchy complete and by the above Theorem CL is paracompact. \square

5.2 Uniform paracompactness

We now present the pointfree notion of *uniform paracompactness* in the setting of uniform frames, which is defined as the property that every cover has a uniformly locally finite refinement. A refinement B of A is *uniformly locally finite* (u.l.f.) if there exists a uniform cover C such that each member of C meets only finitely many members of B i.e. for each $c \in C$, $B_c = \{b \in B : b \wedge c \neq 0\} \in B$. In the sequel u.p. will denote *uniformly paracompact*. Let

$$(*)_0 : (L, \mu) \text{ is u.p. and}$$

$$(*)_1 : A \in \text{cov}L \Rightarrow \Omega_A = \{\bigvee B : B \in A\} \in \mu.$$

These two conditions are equivalent. For the first implication see below. The equivalence is finally shown in Corollary 5.2.1.

Theorem 5.2.1

$$(*)_0 \Rightarrow (*)_1.$$

Proof :

Let (L, μ) be u.p. and $A \in \text{cov}L$. Then there exists a u.l.f. $B \in \text{cov}L$ that refines A . Thus there exists $C \in \mu$ such that for each $c \in C$, $B_c \in B$. Let $s \in C$. Then B_s is finite and

$$\begin{aligned} s &= s \wedge 1 = s \wedge \bigvee B = \bigvee \{s \wedge b : b \in B\} \\ &= \bigvee_{b \in B_s} \{s \wedge b\} \vee \bigvee_{b \notin B_s} \{s \wedge b\} \\ &= s \wedge \bigvee B_s \vee 0 \\ &= s \wedge \bigvee B_s \\ &\leq \bigvee B_s. \end{aligned}$$

But $\bigvee B_s \in \Omega_B$. So, $C \leq \Omega_B$ and $\Omega_B \in \mu$. As $B \leq A$, $\Omega_B \leq \Omega_A$. Hence $\Omega_A \in \mu$. \square

Theorem 5.2.2

If (L, μ) is u.p. and $A \subseteq L$ is locally finite, then A is u.l.f.

Proof :

Let $A \subseteq L$ be locally finite with (L, μ) u.p. Since A is locally finite we can find $B \in \text{cov}L$ such that for each $b \in B$, $A_b \in A$. Since (L, μ) is u.p., $\Omega_B \in \mu$ by the above Theorem. Let $\bigvee C \in \Omega_B$ for any $C \in B$. Then for each $c \in C$, $A_c \in A$. If $x \in A_{\bigvee C}$, then $0 \neq x \wedge \bigvee C = \bigvee \{x \wedge c : c \in C\}$. Thus $x \wedge c_x \neq 0$ for some $c_x \in C$. Then $x \in A_{c_x}$ and so $A_{\bigvee C} \subseteq \bigcup_{c \in C} A_c$. Also, if $y \in A_c$ for some $c \in C$, then $y \wedge c \neq 0$. Consequently, $0 \neq \bigvee_{c \in C} (y \wedge c) = y \wedge \bigvee C$. So $y \in A_{\bigvee C}$. Hence $A_{\bigvee C} = \bigcup_{c \in C} A_c$. But C is finite and for each $c \in C$, $A_c \in A$. Thus $\bigcup_{c \in C} A_c = A_{\bigvee C} \in A$. Hence, $\bigvee C$ meets only finitely many members of A . Since $\Omega_B \in \mu$, A is u.l.f. \square

The following is a direct frame theoretic translation of the corresponding result by Rice (see [73]) in uniform spaces.

Theorem 5.2.3

If (L, μ) satisfies $()_1$, then (L, μ) is Cauchy complete.*

Proof :

Let F be any Cauchy filter on L . Then $F \subseteq \tilde{F}$ for some maximal filter \tilde{F} . Suppose that \tilde{F} does not converge. Then there exists $A \in \text{cov}L$ such that $A \cap \tilde{F} = \phi$. By $(*)_1$, $\Omega_A \in \mu$. As F is Cauchy, $F \cap \Omega_A \neq \phi$. Thus $\tilde{F} \cap \Omega_A \neq \phi$. Then $\exists B \in A$ such that $\bigvee B \in \tilde{F}$. Since B is finite and \tilde{F} is maximal there exists $b \in B \subseteq A$ such that $b \in \tilde{F}$. Thus $A \cap \tilde{F} \neq \phi$ which gives a contradiction. Hence, \tilde{F} must converge. As $F \subseteq \tilde{F}$, F clusters by Lemma 5.1.2. Then F is a clustered Cauchy filter on L and must converge by Theorem 5.1.1. Hence (L, μ) is Cauchy complete. \square

Let $\mu, \nu \subseteq \text{cov}L$. Dube (private communication) defines $\mu \mid \nu$ as follows :

$$A \in \mu \mid \nu \Leftrightarrow \exists B \in \nu \text{ and } \{C_b\}_{b \in B} \subseteq \mu \text{ such that } \{x \wedge y \mid x \in B, y \in C_x\} \leq A$$

Now consider the fine uniformity u_F and β as defined previously. With Dube [37] the following analogous result in [73] is realized in the setting of uniform frames.

Theorem 5.2.4 (Dube)

(L, μ) satisfies $()_1$ if and only if L is paracompact and $u_F = \beta \mid \mu$.*

Proof :

(\Rightarrow) Suppose that (L, μ) satisfies $(*)_1$ and $A \in \text{cov}L$. Then $\Omega_A \in \mu$. Since every uniform cover has a locally finite refinement (see [14]) there exists $B \in \text{cov}L$, locally finite such that $B \leq \Omega_A$. Thus for each $x \in B$ there exists $C_x \in A$ such that $x \leq \bigvee C_x$. Let $D = \{x \wedge y : x \in B, y \in C_x\}$. Then

$$\bigvee D = \bigvee_{x \in B, y \in C_x} (x \wedge y)$$

$$\begin{aligned}
&= \bigvee_{x \in B} x \wedge \bigvee_{x \in B, y \in C_x} y \\
&= \bigvee_{x \in B} x \wedge \bigvee_{x \in B} \bigvee C_x \\
&= \bigvee_{x \in B} (x \wedge \bigvee C_x) \\
&= \bigvee_{x \in B} x \\
&= \bigvee B \\
&= 1.
\end{aligned}$$

So $D \in \text{cov}L$. As B is locally finite there exists $T \in \text{cov}L$ such that for each $t \in T$, $B_t = \{b \in B : b \wedge t \neq 0\} \in B$. For each $t \in T$ let $D_t = \{x \wedge y \in D : (x \wedge y) \wedge t \neq 0\}$. Then if $(x \wedge y) \wedge t \neq 0$ for any $x \wedge y \in D$, $x \wedge t \neq 0$. So $x \in B_t$. As $B_t \in B$, $D_t \in D$. Thus D is also locally finite. Furthermore, for each $x \wedge y$ in D , $x \wedge y \leq y \in C_x \subseteq A$. Thus $D \leq A$. Hence L is paracompact.

Since L is paracompact, $u_F = \text{cov}L$. So $\beta \mid \mu \subseteq u_F$. Now let $A \in \mu$. Then $\Omega_A \in \mu$. By Lemma 5.1.5, $\check{\Omega}_A \in \mu$. Now for each $x \in \check{\Omega}_A$, let $B_x \in A$ such that $x \triangleleft \bigvee B_x$ and set $H_x = \{x^*\} \cup B_x$. Since $x \triangleleft \bigvee B_x$, $x \prec \bigvee B_x$ and thus $\bigvee H_x = x^* \vee \bigvee B_x = 1$. So H_x is a finite cover of L and hence $H_x \in \beta$. Then $D = \{x \wedge t : x \in \check{\Omega}_A \text{ and } t \in H_x\} \in \beta \mid \mu$. If $x \in \check{\Omega}_A$ and $t \in H_x$, then either $t = x^*$ or $t \in B_x$. If $t = x^*$, then $x \wedge t = 0 \leq a$ for each $a \in A$. Otherwise, $t \in B_x$ and $x \wedge t \leq t \in B_x \subseteq A$. Thus $D \leq A$ and $A \in \beta \mid \mu$. Hence, $u_F = \beta \mid \mu$.

(\Leftarrow) Suppose that L is paracompact and $u_F = \beta \mid \mu$. Let $A \in \text{cov}L$. By paracompactness, $\beta \mid \mu = u_F = \text{cov}L$. So $A \in \beta \mid \mu$. Thus there exists $B \in \mu$ and $\{C_b\}_{b \in B} \subseteq \beta$ such that $\{x \wedge y : x \in B, y \in C_x\} \leq A$. If $b \in B$, then $C_b \in \beta$. Thus there exists finite $D_b \in \mu$ such that $D_b \leq C_b$. Then

$$b = b \wedge 1 = b \wedge \bigvee_{d \in D_b} D_b = \bigvee_{d \in D_b} (b \wedge d).$$

As $\{x \wedge y : x \in B, y \in C_x\} \leq A$ and $D_b \leq C_b$, we may conclude that, for each $d \in D_b$, $b \wedge d \leq a_d$ for some $a_d \in A$. Therefore, $b = \bigvee_{d \in D_b} (b \wedge d) \leq \bigvee_{d \in D_b} a_d \in \Omega_A$ as D_b

as D_b is finite. Thus $B \leq \Omega_A$ and hence $\Omega_A \in \mu$. \square

Corollary 5.2.1

$$(*)_0 \iff (*)_1.$$

Proof :

(\Rightarrow) Theorem 5.2.1.

(\Leftarrow) Suppose $(*)_1$. Let $A \in \text{cov}L$. Since L is paracompact by Theorem 5.2.4, there exists a locally finite cover B that refines A . Since B is locally finite we can find $C \in \text{cov}L$ such that $B_c \in B$ for each $c \in C$. By $(*)_1$, $\Omega_C \in \mu$. Similar to the proof in Theorem 5.2.2, $B_{\bigvee D} \in B$ for each finite $D \subseteq C$. Thus each $\bigvee D \in \Omega_C$ meets only finitely many members of B . As $\Omega_C \in \mu$, B is u.l.f. proving $(*)_0$. \square

Thus any uniform frame (L, μ) is uniformly paracompact if and only if $\Omega_A = \{\bigvee B \in L : B \in A\} \in \mu$ for each cover A in L . As a consequence of Theorem 5.2.3 the following result is apparent for any uniform frame (L, μ) .

Corollary 5.2.2

If (L, μ) is u.p. then (L, μ) is Cauchy complete.

Since strong Cauchy completeness implies Cauchy completeness (Theorem 5.1.3), the above corollary is also immediate from the following theorem.

Theorem 5.2.5

If (L, μ) is u.p. then it is strongly Cauchy complete.

Proof :

Suppose that (L, μ) is uniformly paracompact and let F be any weakly Cauchy filter in L with $A \in \text{cov}L$. Then $\Omega_A \in \mu$ and thus $\Omega_A \cap \text{sec} F \neq \phi$. Then for some $B \in A$, $(\bigvee B) \wedge y \neq 0 \forall y \in F$. Then, $\forall y \in F$,

$$0 \neq (\bigvee B) \wedge y = \bigvee_{b \in B} (b \wedge y)$$

Thus for some $b \in B$, $b \wedge y \neq 0$ for each $y \in F$. Obviously, $b \in A$. So, $A \cap \text{sec } F \neq \phi$. Thus F clusters and so L is strongly Cauchy complete. \square

We close this chapter by showing that for Boolean uniform frames, uniform paracompactness and strong Cauchy completeness are equivalent.

Theorem 5.2.6

A Boolean uniform frame (L, μ) is u.p if and only if (L, μ) is strongly Cauchy complete.

Proof :

(\Rightarrow) Theorem 5.2.5.

(\Leftarrow) Suppose that (L, μ) is a uniform frame with L Boolean. Let $A \in \text{cov } L$. If $\bigvee B = 1$ for some $B \in A$, then $\Omega_A \in \mu$. So we may assume that for each $B \in A$, $\bigvee B \neq 1$. Let $\mathcal{B} = \{(\bigvee B)^* : B \in A\}$. If $(\bigvee B)^* = 0$ for some $B \in A$, then since L is Boolean, $\bigvee B = (\bigvee B)^{**} = 1$. Thus $0 \notin \mathcal{B}$. Then \mathcal{B} generates a proper filter F in L . Since $\text{sec } F \cap A = \phi$, F does not cluster and hence cannot be weakly Cauchy since (L, μ) is strongly Cauchy complete. Thus there exists $T \in \mu$ such that $\text{sec } F \cap T = \phi$. Let $t \in T$. Then $t \notin \text{sec } F$ and so there is $B_t \in A$ such that $t \wedge (\bigvee B_t)^* = 0$. Thus $t \leq (\bigvee B_t)^{**} = \bigvee B_t$. Consequently, $T \leq \Omega_A$ and hence $\Omega_A \in \mu$. \square

It should be noted that the notion of precompactness, uniform paracompactness, a weakly Cauchy filter and strong Cauchy completeness can also be introduced for nearness frames. All of the above results are also true for nearness frames, save those on precompactness which are true for strong nearness frames.

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