Credit Default Swaps in a Roll-Over Risk Framework

Nicholas Petersen

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

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Abstract

Spreads between swap legs referencing floating cashflows of different tenors have widened significantly since the global financial crisis of 2008. This frequency basis can be explained by the presence of "roll-over risk". Defining the roll-over risk state variables in an affine form, this dissertation prices a credit default swap using an "affine transform" methodology. This price is then compared to that obtained from a traditional Monte Carlo simulation approach. The former is shown to produce accurate results with greater computational efficiency, providing a useful way to price complex financial instruments when the state variables are defined in an appropriate form.

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Contents

List of Tables

Chapter 1

Introduction

Appropriate modelling of the term structure of interest rates is fundamentally important in quantitative finance. It is essential for the pricing and hedging of assets and options, particularly interest rate derivatives. In particular, one can model the dynamics of the short rate. Specifying these dynamics allows one to derive bond prices, and subsequently infer an arbitrage-free yield curve.

Spot-forward parity, a fundamental no-arbitrage condition linking spot and forward interest rates, has been shown not to hold in markets post the global financial crisis (GFC) of 2008. A related phenomenon also ruled out by the fundamental theory is the emergence of spreads between swap legs referencing floating cashflows of different tenors (otherwise known as the frequency basis). These have widened significantly, eclipsing transaction costs in most cases. According to conventional theory on interest rates (as shown for instance in Brigo and Mercurio (2007)), the presence of such a spread would point to the existence of an arbitrage opportunity using a simple borrowing and loan strategy. However, potential arbitrageurs are unable to take advantage of this opportunity due to the inability to roll-over the borrowing at market rates in the strategy. Alfeus *et al.* (2020) attribute this phenomenon to the presence of "roll-over risk", and seek to endogenously model this risk in a consistent way. Their paper develops a basic model framework and calibrates the model using empirical interest rates based on different interest rate instruments.

This dissertation aims to price a credit default swap under a roll-over risk framework. The underlying stochastic state variables are intentionally defined as affine processes, allowing us to use key results from Duffie *et al.* (2000) in the pricing exercise. We then compare the results to a traditional Monte Carlo simulation, showing that the Duffie *et al.* (2000) approach provides a more efficient way to price complex financial products when the underlying processes are specified appropriately.

Chapter 2 introduces the necessary theory of the term structure of interest rates – the short rate in particular. The concept of roll-over risk is then introduced and a framework to accomodate this risk is constructed. In Chapter 3, credit default swaps are introduced and pricing formulae under a roll-over risk framework are derived. Prices are then computed using both the Duffie *et al.* (2000) methodology and Monte Carlo simulations. Conclusions are then presented in Chapter 4.

Chapter 2

The Term Structure of Interest Rates

2.1 Term structure

The term structure of interest rates is defined by Filipović (2009) as a function that relates an interest rate to its maturity. This structure can be represented in practice as a yield curve. The accurate modelling of interest rates is essential in asset and derivative pricing, particularly interest rate derivatives like caps, floors and swaptions. Of particular interest in this dissertation is the concept of a short rate.

2.1.1 The short rate

The short rate, denoted $r(t)$, is an interest rate that applies over the next short period of time. A mathematical abstraction of this rate is one that applies over the next instant, i.e., as the limit of time tends to zero. The short rate is extremely important in a modelling context because many important quantities can be derived therefrom.

Many stochastic models have been posited for the dynamics of the short rate, starting with Vasicek (1977). This process is characterised by a mean reverting drift towards μ at a rate κ, and is given by

$$dr(t) = \kappa(\mu - r(t))dt + \sigma dW(t),$$

where $W(t)$ is a Brownian motion.

Over time, more short rate models were developed to take into account certain features observed in empirical interest rates. In particular, the Cox *et al.* (1985) model extended the Vasicek (1977) model to include a square root factor in the diffusion term. These dynamics are given by

$$dr(t) = \kappa(\mu - r(t))dt + \sigma\sqrt{r(t)}dW(t).$$

The square root factor ensures that $r(t)$ is positive, an assumption that has become less relevant in recent years.

Brigo and Mercurio (2007) show that we can construct the notion of the bank account $B(t)$, a locally riskless investment that compounds continuously at the short rate $r(t)$. The value of one unit invested into the bank account at time 0 grows to

$$B(t) = e^{\int_0^t r(s)ds}$$

at time t.

The stochastic discount factor $D(t,T)$ is simply the ratio of the bank accounts applying over different time periods and has the function of discounting one unit of currency from time T to time t, i.e.,

$$D(t,T) = \frac{B(t)}{B(T)} = e^{-\int_t^T r(s)ds}.$$

As stated by Filipović (2009), the absence of arbitrage in the market implies the existence of a risk-neutral measure. Suppose we denote this as $\mathbb{Q}$ – the measure under which discounted bond prices are martingales. The existence of these stochastic discount factors and the risk-neutral measure allows one to specify the zero-coupon bond (ZCB) price $P(t,T)$. This is the price at time t of a contract paying one unit at time T, with no intermediate payments. Given the information known at time t, the bond price can therefore be specified as a conditional expectation under $\mathbb{Q}$ of the stochastic discount factor. Hence, we define

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r(s)ds}|\mathcal{F}_t].$$

By definition, the terminal value of the bond is $P(T,T) = 1$. The continuously compounded yield $y(t,T)$ can then be derived directly from the bond price as

$$y(t,T) = -\frac{1}{T-t}\log(P(t,T)).$$

Thus, one way to model a term structure without arbitrage is to specify the short rate process under the risk-neutral measure and then to derive the associated bond prices. The yields can then be implied directly from the ZCB prices and represented as a yield curve.

2.1.2 Spot-forward parity

Spot-forward parity is the assumption upon which much of traditional yield curve bootstrapping is based. It can be illustrated by comparing the following example scenarios:

1. Borrowing in a single step at a spot rate applying from time 0 to time T, denoted L_{0T}. The obligation is then to repay $1 + L_{0T}T$ at maturity.

2. Borrowing at the spot rate over two steps. Suppose $\delta = \frac{T}{2}$. This would entail borrowing at spot $L_{0\delta}$ until time δ, and then rolling the borrowing over at the new spot rate $L_{\delta T}$. It is also possible to enter into a forward rate agreement (FRA) at time 0 to pay a fixed forward rate $F_{0\delta T}$ instead of the spot rate $L_{\delta T}$ at time δ. Hence, the amount owed at maturity would be $(1 + L_{0\delta}\delta)(1 + F_{0\delta T}\delta)$.

Under the assumption of no arbitrage, the value of both borrowings should be equal, i.e.,

$$1 + L_{0T}T = (1 + L_{0\delta}\delta)(1 + F_{0\delta T}\delta).$$

However, empirical evidence shows that markets do not conform to this parity after the GFC of 2008. In particular, it has been shown that

$$1 + L_{0T}T > (1 + L_{0\delta}\delta)(1 + F_{0\delta T}\delta).$$

The violation of spot-forward parity shows how longer borrowing is in fact more expensive. In other words, one has to pay a premium to avoid rolling over a shorter borrowing. When it comes to a floating-for-floating tenor swap, it therefore becomes better to receive the larger, longer interest rate. This is evidenced by a consistent frequency basis in the post-GFC interest rate market. A frequency basis refers to the spread applied to one leg of a interest rate swap to exchange some floating interest rate for another, where the rates are applied over different tenors. The basis is therefore paid by the party receiving the longer payment leg.

In classical interest rate theory posited by Brigo and Mercurio (2007), the presence of a frequency basis larger than transaction costs would imply the existence of an arbitrage opportunity. A textbook strategy to take advantage of this opportunity would be to lend at the longer tenor and borrow at the shorter tenor, where the spot rate can be exchanged for a fixed forward rate (as shown previously). In theory, this would generate a certain profit. However, it seems extremely unlikely that straightforward arbitrage opportunities are consistently present in modern markets.

2.1.3 Roll-over risk

The key limitation to the textbook strategy to take advantage of the apparent arbitrage is roll-over risk. This simply describes the risk that when attempting to roll the borrowing over (on the shorter tenor end of the strategy), the borrower is unable to access the prevailing market rates. Alfeus *et al.* (2020) state that roll-over risk can be broadly broken down into two sources: downgrade risk and funding liquidity risk.

Downgrade risk is the risk that creditors demand a greater credit spread than the prevailing market average. For instance, if a prime bank drops out of its particular interbank panel, it may no longer be able to access the prime interbank rates. Funding liquidity risk refers to the risk that market funding can only be obtained at an additional premium due to liquidity constraints (for instance, during market conditions seen in the GFC). This occurs even though the entity's credit quality has remained the same relative to the market average.

It is important to note that roll-over risk is distinct from conventional interest rate risk. The latter is simply the risk that the market rate changes stochastically. This risk can be hedged by taking out positions in certain derivative contracts, such as FRAs or swaps. On the other hand, roll-over risk cannot be hedged away with instruments like FRAs because they do not lock in future borrowing. Rather, these instruments represent a cash–settled "swaplet" paying the difference between a market interest rate and a fixed rate agreed when the FRA is put in place.

Much of the literature on this topic has been focused on trying to exogenously model the *multiple* term structures that result from the bootstrapping of forward rates that violate spot-forward parity. According to Alfeus *et al.* (2020), the limitation of this approach is that it does not link the term structure of interest rates of different tenors in any meaningful way. Backwell *et al.* (2020) also point out that relatively little attention is paid to the financial economic interpretaion of why these spreads arise in the first place.

Filipović and Trolle (2013) made a key contribution in trying to break down and identify the drivers of the basis spread seen in interest rate markets. They utilise time series data to define "interbank risk", and divide this risk into default and non-default components. Alfeus *et al.* (2020) extend the framework of Filipović and Trolle (2013), to accomodate a more explicit analysis of roll-over risk. Their model endogenously leads to basis spreads, producing multiple term structures by design. The structural links between the yield curves are also built into the model. The model dynamics are then calibrated according to Cox *et al.* (1985) processes.

2.2 Roll-over risk framework

The approach to accounting for roll-over risk in Alfeus *et al.* (2020) will form the basis of the pricing exercise in this dissertation.

2.2.1 Model variables

A number of model variables are defined in the framework.

The continuously compounded short rate, $r_c(t)$, resembles the relevant overnight rate in the market (for example, the European Overnight Index Average rate, EONIA). This is an unsecured rate subject to credit risk. It is defined as

$$r_c(t) = r(t) + \Lambda(t)q$$

for $t \in [0, T]$, where $r(t)$ denotes the risk-free (secured) continuously compounded short rate, and $\Lambda(t)$ denotes the default intensity of the market on average. The loss fraction assuming default is denoted q, so $\Lambda(t)q$ then defines the average credit spread across the panel under consideration. This is the "fractional recovery of market value at default" approach proposed by Duffie and Singleton (1999).

Roll-over risk is introduced into the model via $\pi(t)$, the total spread over $r_c(t)$ that must be paid by an arbitrary entity when borrowing overnight. This is defined as

$$\pi(t) = \phi(t) + \lambda(t)q$$

for $t \in [0, T]$, where $\phi(t)$ denotes the idiosyncratic liquidity spread of the borrower and $\lambda(t)q$ denotes the idiosyncratic credit spread of the borrower (i.e., over and above the market average $\Lambda(t)q$). Note that this roll-over risk is applied symmetrically to borrowing and lending. It follows therefore that the credit spread of an arbitrary entity in the market is $\Lambda(t)q + \lambda(t)q$, with the default intensity given by $\Lambda(t) + \lambda(t)$.

Two key points follow from this model specification. Firstly, the rate that an arbitrary entity will roll borrowing over is

$$r_c(t) + \pi(t),$$

namely the unsecured overnight market rate plus the additional roll-over risk faced by the entity. Secondly, because the credit risk of an arbitrary entity is $\Lambda(t)q+\lambda(t)q$, unsecured lending can be discounted at the default-free rate plus the credit risk, i.e.,

$$r(t) + \Lambda(t)q + \lambda(t)q = r_c(t) + \lambda(t)q.$$

As for collateralised contracts, Filipović and Trolle (2013) show that these should be discounted at the collateral rate, defined as $r_c(t)$ in this framework. Note that this is a fairly standard term in an ISDA Credit Support Annex (the document that stipulates the conditions for collateralised derivative transactions).

2.2.2 Model framework

The general model framework in Alfeus *et al.* (2020) can be illustrated by considering the LIBOR/OIS spread. The London Interbank Offered Rate (LIBOR) is a global benchmark interbank interest rate in multiple currencies. An Overnight Indexed Swap (OIS) is a swap where a fixed payment is exchanged for a floating overnight interest rate compounded over the required tenor. The fixed leg of the swap is based on the so-called OIS rate. Although they are quoted distinctly in the market, the LIBOR/OIS spread can be viewed as an extreme case of the frequency basis.

Suppose the period of borrowing is $\tau = T - t$. Then a possible strategy to take advantage of this spread in a textbook market is shown in table 2.1. Note that since the OIS contract is free to enter for either party, it has zero value at inception. The strategy yields a profit of the LIBOR/OIS spread so there appears to be a clear and simple arbitrage opportunity here.

At time t	
1. Borrow 1 unit at overnight rate and roll borrowing over continuously	1
2. Enter long OIS	0
3. Lend 1 unit at LIBOR spot $L(t,T)$	-1
	0
At time T	
1. Repay rolled-over borrowing	$-e^{\int_t^T r_c(s)ds}$
2. Long OIS payoff - receiving floating and paying fixed $\text{OIS}(t,T)$	$e^{\int_t^T r_c(s)ds} - (1+\text{OIS}(t,T)\tau)$
3. Receive LIBOR loan repayment	$1 + L(t,T)\tau$
	$(L(t,T) - \text{OIS}(t,T))\tau = \text{LIBOR/OIS spread}$

Tab. 2.1: Naive "arbitrage" strategy to take advantage of the LIBOR/OIS spread.

However, this strategy does not consider the idea of roll-over risk. In the Alfeus *et al.* (2020) framework, this LIBOR/OIS spread is attributed to compensation for roll-over risk when borrowing at the shorter tenor in the strategy (the overnight loan in table 2.1).

We now look to define roll-over risk explicitly in the strategy. For ease of notation, let $\mathbb{E}_t[\cdot]$ denote the conditional expectation under the risk-neutral measure $\mathbb{Q}$ given the filtration at time t. With the inclusion of roll-over risk, an arbitrary entity will roll borrowing over at $r_c(t) + \pi(t)$. Thus, the amount due at time T after borrowing 1 unit at t is

$$e^{\int_t^T (r_c(s)+\pi(s))ds}.$$

Since the borrowing is unsecured, it can be discounted at $r_c(t) + \lambda(t)q$. Hence, its expected present value is

$$-\mathbb{E}_t\left[e^{-\int_t^T (r_c(s)+\lambda(s)q)ds} e^{\int_t^T (r_c(s)+\pi(s))ds}\right] = -\mathbb{E}_t\left[e^{\int_t^T \phi(s)ds}\right], \tag{2.1}$$

where the value of the expectation is negative because we are repaying the loan at T.

Now we note that an OIS is a collateralised contract, so we discount at the collateral rate $r_c(t)$. Thus, taking the expectation of the discounted payoff of a long position in the OIS yields

$$\mathbb{E}_t\left[e^{-\int_t^T r_c(s)ds}\left(e^{\int_t^T r_c(s)ds} - (1+\text{OIS}(t,T)\tau)\right)\right] = \mathbb{E}_t\left[1 - e^{-\int_t^T r_c(s)ds}(1+\text{OIS}(t,T)\tau)\right]. \tag{2.2}$$

Lastly, since the LIBOR loan is defaultable, we again discount at $r_c(t) + \lambda(t)q$. The expected present value is

$$\mathbb{E}_t\left[e^{-\int_t^T (r_c(s)+\lambda(s)q)ds}(1+L(t,T)\tau)\right]. \tag{2.3}$$

As shown in table 2.1, the strategy has zero value at time t. Hence, letting equations 2.1, 2.2 and 2.3 sum to zero gives

$$0 = -\mathbb{E}_t\left[e^{\int_t^T \phi(s)ds}\right] + \mathbb{E}_t\left[1 - e^{-\int_t^T r_c(s)ds}(1+\text{OIS}(t,T)\tau)\right] + \mathbb{E}_t\left[e^{-\int_t^T (r_c(s)+\lambda(s)q)ds}(1+L(t,T)\tau)\right].$$

$$\Rightarrow \mathbb{E}_t\left[e^{\int_t^T \phi(s)ds}\right] = \mathbb{E}_t\left[1 - e^{-\int_t^T r_c(s)ds}(1+\text{OIS}(t,T)\tau) + e^{-\int_t^T (r_c(s)+\lambda(s)q)ds}(1+L(t,T)\tau)\right]. \tag{2.4}$$

In the case where the downgrade risk $\lambda(t)$ is zero, we have an explicit dependence of the LIBOR/OIS spread on funding liquidity risk, i.e.,

$$\mathbb{E}_t\left[e^{\int_t^T \phi(s)ds}\right] = \mathbb{E}_t\left[1 + e^{-\int_t^T r_c(s)ds}(L(t,T) - \text{OIS}(t,T))\tau\right].$$

In addition, observing that an OIS contract has zero value at inception, setting equation 2.2 to zero gives

$$0 = 1 - \mathbb{E}_t\left[e^{-\int_t^T r_c(s)ds}\right] - \mathbb{E}_t\left[e^{-\int_t^T r_c(s)ds}\right]\text{OIS}(t,T)\tau.$$

$$\Rightarrow \mathbb{E}_t\left[e^{-\int_t^T r_c(s)ds}\right] = \frac{1}{1+\text{OIS}(t,T)\tau} \tag{2.5}$$

The dynamics of $r_c(t)$ should therefore be consistent with equation 2.5, while $\phi(t)$ and $\lambda(t)$ should be consistent with equation 2.4.

This example illustrates why there is actually no arbitrage opportunity present in this market, and the spread can in fact be seen as compensation for taking on roll-over risk. Alfeus *et al.* (2020) extend this basic case to consider the OIS with multiple payments, rather than the single exchange of fixed and floating payments in table 2.1. They also introduce contracts with longer maturities and floating-for-fixed interest rate swaps. However, this simple case illustrates the framework sufficiently.

2.2.3 Stochastic modelling under the framework

Suppose the model is driven by affine stochastic processes $X_i(t)$ with Cox *et al.* (1985) dynamics. We have that for $i = 1, ..., d$,

$$dX_i(t) = \kappa_i(\theta_i - X_i(t))dt + \sigma_i\sqrt{X_i(t)}dW_i(t).$$

where each $W_i(t)$ is a standard Brownian motion. Note that the $W_i(t)$ are uncorrelated in this case, and the $X_i(t)$ are therefore independent.

The key model variables are then defined as

$$r_c(t) = a_0 + \sum_{i=1}^{d} a_i X_i(t), \tag{2.6}$$

$$\lambda(t) = b_0 + \sum_{i=1}^{d} b_i X_i(t), \tag{2.7}$$

$$\phi(t) = c_0 + \sum_{i=1}^{d} c_i X_i(t), \tag{2.8}$$

where a_i, b_i and c_i are constants. In general, a_0, b_0 and c_0 can be set to be time dependent but we do not consider this formulation. For the purpose of the pricing exercise in Chapter 3, we will set $d = 3$ (as is also the case in Alfeus *et al.* (2020)). In other words, each model variable will be defined as a linear combination of three independent Cox *et al.* (1985) processes.

Alfeus *et al.* (2020) then calibrate the parameters of $X_i(t)$ to market data of various instruments; the results of which will be utilised for the pricing exercise.

Chapter 3

Credit Default Swaps

A credit default swap (CDS) is defined by Hull (2012) as a credit derivative between two parties that effectively acts as an insurance contract on a company's bond. If a credit event (i.e. a default on the bond) takes place, the seller makes a protection payment to compensate for the losses experienced by the buyer. This is known as the protection leg. In return, the buyer makes periodic spread payments to the seller until the CDS terminal time or the default time, whichever is earlier. This is known as the payment leg. CDSs are typically quoted on their spread, which can be calculated as the value (as a percentage of the notional principal amount) of the periodic spread payment such that the payment leg is equal to the protection leg.

Having defined the roll-over risk framework, we now seek to price a one-year credit default swap.

3.1 Derivation of CDS pricing formulae

The protection leg of the CDS pays out the protection in the event of a default occurring before the terminal time of the contract. Suppose that τ now denotes this random default time. Recall that the default intensity in the roll-over risk framework was given earlier by $\Lambda(t) + \lambda(t)$. A well-known result, extremely useful for evaluating defaultable payments, is the following:

$$\mathbb{E}_t[X\mathbb{1}_{\{\tau>T\}}] = \mathbb{E}_t[Xe^{-\int_t^T(\Lambda(u)+\lambda(u))du}], \tag{3.1}$$

where X is an integrable random variable. Note, to be fully rigorous, a minor adjustment in the filtrations is needed when applying equation 3.1. See Backwell *et al.* (2020) for more detail.

This result is key in the construction of the pricing formulae for the CDS. Suppose the CDS contract has N spread payments, with inception time t and maturity time $T_N = T$ (since the final spread payment corresponds to the maturity date).

Since q denotes the constant loss fraction given default, it is only this fraction of the market value of the underlying that is insured as part of the protection leg of the CDS. Recall that collateralised payments can be discounted at $r_c(t)$. Hence, assuming a unit notional, the present value of the protection leg is

$$q\mathbb{E}_t[e^{-\int_t^\tau r_c(s)ds}\mathbb{1}_{\{\tau\leq T\}}].$$

The indicator reflects the fact that protection payment is only made if default occurs during the life of the contract. The stochastic discount factor accounts for the fact that the CDS is a collateralised derivative.

To derive the pricing formula for the protection leg, we utilise a disintegration technique, subdividing the indicator function into discrete time steps. Suppose the tenor of the swap is divided into an even mesh of m points, i.e., $t_i = t + i(\frac{T-t}{m})$ for $i = 0, 1, ..., m$. Applying the mesh and equation 3.1, we have

$$\begin{aligned}
&q\mathbb{E}_t\left[e^{-\int_t^\tau r_c(s)ds}\mathbb{1}_{\{\tau\leq T\}}\right] \\
&= q\lim_{m\to\infty}\sum_{i=1}^{m}\mathbb{E}_t\left[e^{-\int_t^\tau r_c(s)ds}\mathbb{1}_{\{t_{i-1}<\tau\leq t_i\}}\right] \\
&= q\lim_{m\to\infty}\sum_{i=1}^{m}\mathbb{E}_t\left[e^{-\int_t^{t_{i-1}} r_c(s)ds}\mathbb{1}_{\{t_{i-1}<\tau\leq t_i\}}\right] \\
&= q\lim_{m\to\infty}\sum_{i=1}^{m}\mathbb{E}_t\left[e^{-\int_t^{t_{i-1}} r_c(s)ds}(\mathbb{1}_{\{t_{i-1}<\tau\}} - \mathbb{1}_{\{\tau\leq t_i\}})\right] \\
&= q\lim_{m\to\infty}\sum_{i=1}^{m}\mathbb{E}_t\left[e^{-\int_t^{t_{i-1}} r_c(s)ds}(e^{-\int_t^{t_{i-1}}(\Lambda(s)+\lambda(s))ds} - e^{-\int_t^{t_i}(\Lambda(s)+\lambda(s))ds})\right].
\end{aligned} \tag{3.2}$$

Now note that if we define

$$f(u) = e^{-\int_t^u(\Lambda(s)+\lambda(s))ds},$$

then by applying the Mean Value Theorem, we get

$$f(t_{i-1})-f(t_i) = -f'(t_{i-1})(t_i-t_{i-1}) = [\Lambda(t_{i-1})+\lambda(t_{i-1})]e^{-\int_t^{t_{i-1}}(\Lambda(s)+\lambda(s))ds}(t_i-t_{i-1}). \tag{3.3}$$

Thus, equation 3.2 can be simplified to

$$\begin{aligned}
&q\lim_{m\to\infty}\sum_{i=1}^{m}\mathbb{E}_t\left[e^{-\int_t^{t_{i-1}} r_c(s)ds}(\Lambda(t_{i-1})+\lambda(t_{i-1}))e^{-\int_t^{t_{i-1}}(\Lambda(s)+\lambda(s))ds}\left(\frac{T-t}{m}\right)\right] \\
&= q\int_t^T\mathbb{E}_t\left[e^{-\int_t^u r_c(s)ds}(\Lambda(u)+\lambda(u))e^{-\int_t^u(\Lambda(s)+\lambda(s))ds}\right]du \\
&= q\int_t^T\mathbb{E}_t\left[(\Lambda(u)+\lambda(u))e^{-\int_t^u(r_c(s)+\Lambda(s)+\lambda(s))ds}\right]du.
\end{aligned} \tag{3.4}$$

Next, we look to derive an expression for the payment leg. We can divide this leg of the swap into two parts.

The first part accounts for the full spread payments assuming no default occurs. Hence, we consider the sum over over all spread payment dates of the expectation of an indicator function where the default time is greater than that particular spread payment date. Being collateralised, we again discount these cashflows at the rate $r_c(t)$. Suppose the spread is denoted C. Then the first payment leg is easily calculated as

$$\begin{aligned} & C \sum_{i=1}^{N} \mathbb{E}_t \left[e^{-\int_t^{T_i} r_c(s) ds} (T_i - T_{i-1}) \mathbb{1}_{\{\tau > T_i\}} \right] \\ &= C \sum_{i=1}^{N} \mathbb{E}_t \left[e^{-\int_t^{T_i} r_c(s) ds} (T_i - T_{i-1}) e^{-\int_t^{T_i} (\Lambda(s) + \lambda(s)) ds} \right] \\ &= C \sum_{i=1}^{N} (T_i - T_{i-1}) \mathbb{E}_t \left[e^{-\int_t^{T_i} (r_c(s) + \Lambda(s) + \lambda(s)) ds} \right], \end{aligned} \tag{3.5}$$

which follows again by application of equation 3.1.

The second payment leg accounts for all the partial spread payments that have been accrued in the event of default. Because spread payments are usually made in arrears, the proportion of protection that has been accrued when default occurs in that particular period must be valued. Thus the proportion of the spread from the previous spread payment date until the default date should be considered. We therefore sum over all spread payment dates, taking the expectation of an indicator function where the default occurs in between these dates. As before, we utilise a disintegration technique and the result from equations 3.1 and 3.3 to derive the expectation. We have

$$
\begin{aligned}
&C\sum_{i=1}^{N}\mathbb{E}_t\left[e^{-\int_t^{\tau} r_c(s)ds}(\tau-T_{i-1})\mathbb{1}_{\{T_{i-1}<\tau\leq T_i\}}\right]\\
&=C\sum_{i=1}^{N}\mathbb{E}_t\left[e^{-\int_t^{\tau} r_c(s)ds}(\tau-T_{i-1})\mathbb{1}_{\{T_{i-1}<\tau\leq T_i\}}\lim_{m\to\infty}\sum_{k=1}^{m}\mathbb{1}_{\{t_{k-1}<\tau\leq t_k\}}\right]\\
&=C\sum_{i=1}^{N}\lim_{m\to\infty}\sum_{k=1}^{m}\mathbb{E}_t\left[e^{-\int_t^{t_{k-1}} r_c(s)ds}(t_{k-1}-T_{i-1})\mathbb{1}_{\{t_{k-1}<\tau\leq t_k\}}\right]\\
&=C\sum_{i=1}^{N}\lim_{m\to\infty}\sum_{k=1}^{m}\mathbb{E}_t\left[e^{-\int_t^{t_{k-1}} r_c(s)ds}(t_{k-1}-T_{i-1})\left(\mathbb{1}_{\{\tau>t_{k-1}\}}-\mathbb{1}_{\{\tau>t_k\}}\right)\right]\\
&=C\sum_{i=1}^{N}\lim_{m\to\infty}\sum_{k=1}^{m}\mathbb{E}_t\left[e^{-\int_t^{t_{k-1}} r_c(s)ds}(t_{k-1}-T_{i-1})\left(e^{-\int_t^{t_{k-1}}(\Lambda(s)+\lambda(s))ds}-e^{-\int_t^{t_k}(\Lambda(s)+\lambda(s))ds}\right)\right]\\
&=C\sum_{i=1}^{N}\lim_{m\to\infty}\sum_{k=1}^{m}\mathbb{E}_t\left[e^{-\int_t^{t_{k-1}} r_c(s)ds}(t_{k-1}-T_{i-1})(\Lambda(t_{k-1})+\lambda(t_{k-1}))e^{-\int_t^{t_{k-1}}(\Lambda(s)+\lambda(s))ds}\left(\frac{T-t}{m}\right)\right]\\
&=C\sum_{i=1}^{N}\int_{T_{i-1}}^{T_i}(u-T_{i-1})\mathbb{E}_t\left[(\Lambda(u)+\lambda(u))e^{-\int_t^{u}(r_c(s)+\Lambda(s)+\lambda(s))ds}\right]du.
\end{aligned}
\tag{3.6}
$$

Note in this equation that the sum over k discretises the interval $[T_{i-1}, T_i]$.

To solve for the fair spread payment C, we simply find the payment that sets the value of the protection leg equal to the value of the payment leg, i.e.,

$$
\begin{aligned}
\text{Protection leg} &= C*(\text{Payment leg 1})+C*(\text{Payment leg 2})\\
\Rightarrow C &= \frac{\text{Protection leg}}{(\text{Payment leg 1}+\text{Payment leg 2})}.
\end{aligned}
\tag{3.7}
$$

3.2 Parameters

A feature of standard CDS contracts is that they have fixed roll dates, which decide the dates of the spread payments and therefore the maturity of the contract. The normal roll dates are 20 December, 20 March, 20 June and 20 September. Hence, the actual term of the CDS is almost always less than what is quoted.

In this example, we price a one-year CDS on the 31st of October 2017, since it allows us to use the model parameter values that have been calibrated to market data in Alfeus *et al.* (2020). This means the maturity date of the contract is 20 September 2018, with spread payments made on 20 December 2017, 20 March 2018, 20 June 2018 and 20 September 2018.

The time between the valuation date and the first spread payment date is $\frac{T_1 - t}{365} = 0.137 \approx 0.14$. The approximation is necessary because a discretisation interval of $dt = 0.01$ is used during the numerical integration process. The next three spread payments are assumed to be full quarters apart, i.e., $T_2 = 0.39$, $T_3 = 0.64$ and $T_3 = 0.89$.

A "time-homogenous" version of the Alfeus *et al.* (2020) model is used, where the potentially time dependent functions a_0 and b_0 (from equations 2.6 and 2.7) are set to zero. These parameters can be allowed to be time dependent (which is especially useful in a calibration exercise), but is not necessary for the purpose of pricing the product. Note also that $\phi(t)$ (equation 2.8), the idiosyncratic liquidity spread process, does not appear in any of the expectations to price the CDS.

Filipović and Trolle (2013) note that it is not possible to reliably determine the value of $\Lambda(t)$, the systemic default intensity, from market data. However, they note that any "reasonable variation" in the process value does not materially affect results in the pricing exercise, so it is safe to fix the value at five basis points.

Table 3.1 shows the parameters calibrated to market data in Alfeus *et al.* (2020) on the valuation date.

i	1	2	3
$X_i(0)$	0.773084	0.013896	0.065454
a_i	0.00334	0.00000	0.00000
b_i	0.0000539	0.113603	0.0000794
κ_i	0.260876	0.397512	0.903787
θ_i	0.798057	0.0002119	0.805810
σ_i	0.264573	0.004227	0.403512

Tab. 3.1: Calibrated model parameters from Alfeus *et al.* (2020) on 31 October 2017.

3.3 Calculation of the expectations

Using these parameters and the derived pricing formulae, we now seek to calculate the value of the CDS spread with the explicit inclusion of roll-over risk.

Our specification of the state variables in an affine form in section 2.2.3 is intentional and important. It allows us to utilise the closed form solutions to certain expectations of these state variables, detailed in Duffie *et al.* (2000). The paper provides solutions in the form of Riccati ODEs. In fact, much of the challenge in pricing this instrument lies in trying to define the form of the expectations appropriately to match the formulation in Duffie *et al.* (2000) exactly.

The simplest expectation we need to solve in the CDS pricing formulae is

$$\mathbb{E}_t\left[e^{-\int_t^{T_i}(r_c(s)+\Lambda(s)+\lambda(s))ds}\right],$$

found in equation 3.5, the first payment leg.

The "affine transform" in Duffie *et al.* (2000) solves an expectation of the form

$$\psi(u, X_t, t, T) = \mathbb{E}_t\left[e^{-\int_t^T R(X_s)ds}e^{u\cdot X_T}\right], \tag{3.8}$$

where

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dZ_t,$$

and R, u, μ and σ are specified appropriately. Note that in our calculations, we are actually using a special case of Duffie *et al.* (2000) since the jump term dZ_t is set to zero.

We then have that the solution to the expectation in equation 3.8 is

$$\psi(u, X_t, t, T) = e^{\alpha(t)+\beta(t)\cdot X_t},$$

where $\alpha(t)$ and $\beta(t)$ satisfy certain ODEs.

For our particular case, we are required to solve the following Riccati ODEs:

$$\frac{\partial\alpha}{\partial t} = (a_0 + b_0 + \Lambda(t)) - \kappa_1\theta_1\beta_1(t) + \kappa_2\theta_2\beta_2(t) + \kappa_3\theta_3\beta_3(t)$$

$$\frac{\partial\beta_1}{\partial t} = (a_1 + b_1) + \kappa_1\beta_1(t) - \frac{1}{2}\sigma_1^2\beta_1(t)^2$$

$$\frac{\partial\beta_2}{\partial t} = (a_2 + b_2) + \kappa_2\beta_2(t) - \frac{1}{2}\sigma_2^2\beta_2(t)^2$$

$$\frac{\partial\beta_3}{\partial t} = (a_3 + b_3) + \kappa_3\beta_3(t) - \frac{1}{2}\sigma_3^2\beta_3(t)^2$$

with terminal conditions $\alpha(T_i) = 0$ and $\beta(T_i) = u = [0, 0, 0]$.

The integral of the expectation is then approximated using a simple quadrature technique, where the ODEs are calculated at discrete time points and then multiplied by an interval term $dt = 0.01$.

In both equation 3.4 (the protection leg) and equation 3.6 (the second payment leg) we can split the integral into two parts since $\Lambda(t) = \Lambda$ is defined as a constant. These components with the constant coefficient Λ can be solved with the standard transform as above. We are then only left with the expectation

$$\mathbb{E}_t\left[\lambda(u)e^{-\int_t^u(r_c(s)+\Lambda(s)+\lambda(s))ds}\right]$$

to solve. This requires the use of the so-called "extended transform" in Duffie *et al.* (2000), where the solution to an expectation of the form

$$\phi(\upsilon, u, X_t, t, T) = \mathbb{E}_t\left[e^{-\int_t^T R(X_s)ds}(\upsilon\cdot X_T)e^{u\cdot X_T}\right],$$

is given by

$$\phi(v, u, X_t, t, T) = e^{\alpha(t)+\beta(t)\cdot X_t}(A(t) + B(t) \cdot X_t).$$

Here, $\alpha(t)$ and $\beta(t)$ are defined as before and $A(t)$ and $B(t)$ satisfy a new set of ODEs.

In our particular case, the new Riccati ODEs that need to be solved are

$$\frac{\partial A}{\partial t} = -\kappa_1\theta_1 B_1(t) - \kappa_2\theta_2 B_2(t) - \kappa_3\theta_3 B_3(t)$$

$$\frac{\partial B_1}{\partial t} = \kappa_1 B_1(t) - \beta_1(t)\sigma_1^2 B_1(t)$$
$$\frac{\partial B_2}{\partial t} = \kappa_2 B_2(t) - \beta_2(t)\sigma_2^2 B_2(t)$$
$$\frac{\partial B_3}{\partial t} = \kappa_3 B_3(t) - \beta_3(t)\sigma_3^2 B_3(t)$$

with terminal conditions $A(u) = 0$ and $B(u) = v = [b_1, b_2, b_3]$. Again, simple quadrature techniques are employed to calculate the required integrals.

We now seek to compute the same expectations using a Monte Carlo simulation. Unfortunately, there is no explicit solution to a Cox *et al.* (1985) stochastic differential equation, unlike under Vasicek (1977) dynamics for instance. However, we can utilise a "transition density approach" outlined by Cox *et al.* (1985) which produces exact simulations. This approach uses the fact that the conditional distribution of the Cox *et al.* (1985) process X_{i+1} given its value at the previous time point X_i, has a non-central chi-square distribution. We simulated $n = 10000$ paths for the three state processes, again with time steps of $dt = 0.01$.

Table 3.2 shows the values of each swap leg under the Duffie *et al.* (2000) methodology compared to the Monte Carlo simulation, along with the standard errors for the Monte Carlo estimates. CDS spreads were then calculated using equation 3.7.

	Duffie et al.	Monte Carlo	Standard error
Payment leg 1	0.888664354	0.887739338	$1.68255e-06$
Payment leg 2	$2.083663e-04$	$2.009702e-04$	$3.25512e-08$
Protection leg	0.001013292	0.001010743	$1.45942e-07$
CDS Spread	0.001139974	0.001138301	

Tab. 3.2: Comparison of results of Monte Carlo and Duffie et al. pricing methods.

Table 3.2 shows that the Duffie *et al.* (2000) method produces accurate prices. This method also takes a fraction of the time that it takes to produce the same prices using Monte Carlo simulations. Of course, as the sample size of the Monte Carlo simulation is increased, this disparity in efficiency will only be exacerbated. Note that the Monte Carlo standard errors are very small (several orders of magnitude smaller than the estimates themselves), so simulating 10000 paths is sufficient. The errors are small in part because we aren't actually simulating the default indicators directly. Rather, we simulate based on the result in equation 3.1 which stabilises the Monte Carlo estimates. Price differences between methodologies can also be reduced by shortening the discretisation interval.

The result of this pricing exercise has implications for the pricing of other instruments under roll-over risk, such as interest rate swaps. It shows that if the underlying state variables are defined appropriately, one can utilise the Duffie *et al.* (2000) methodology to price the product accurately and efficiently.

Chapter 4

Conclusion

Starting with the observation that a frequency basis is prevalent in post-GFC swap markets, we used the Alfeus *et al.* (2020) roll-over risk framework to explain why this is the case. We then went on to illustrate the general model through the example of a naive "arbitrage" strategy that earned the LIBOR/OIS spread. Stochastic dynamics for the roll-over risk variables were then specified, allowing us to formulate pricing equations for the various legs in a credit default swap. The affine structure of the underlying state variables allowed us to use Duffie *et al.* (2000) to find explicit solutions to the required expectations in the CDS pricing formulae. We were then able to compare these results to those obtained from a traditional Monte Carlo simulation.

The pricing exercise shown in the dissertation is significant for two main reasons. Firstly, to price a single CDS using Monte Carlo simulations is computationally intensive. For many practical applications where instruments need to be repriced for different state variables and model parameters, the Monte Carlo method is intractable. The entire calculation must be redone for any change in state variables or parameters, so this method is generally more useful for benchmarking and testing purposes. The Duffie *et al.* (2000) pricing methodology is far more powerful because the solution to the pricing calculation is given as an exponential affine function of the state variables. Hence, if the states are changed, the entire calculation does not have to be redone and is therefore far less computationally intensive for many practical purposes. For example, in time-series estimation, the state variables change over time, but the expensive calculations only have to be done once.

The other particular application of this pricing exercise is slightly more subtle. We note that the idiosyncratic liquidity spread, $\phi(t)$, does not enter into the CDS calculations. However, it does enter into calculations of the interbank rate and therefore, by extension, into the calculation for swap rates. This presents a practical problem in that if only swap data is used for pricing in the roll-over risk framework, the individual contributions of $\lambda(t)$ and $\phi(t)$ to this risk cannot be explicitly separated. Since the CDS price is not a function of $\phi(t)$, if CDS data is included in the pricing exercise, the effect of $\lambda(t)$ can be calculated explicitly. Thus, being able to accurately and efficiently price a CDS in this framework is essential to better understand the interplay between the liquidity risk and credit risk of the underlying entity being modelled.

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