An examination of kurtosis of lognormality in the Black-Scholes option pricing formula in the South African warrants market

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Hung-Hsiang Chen

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Prof. Haim Abraham

Abstract

The assumption of constant asset price volatility of classical Black-Scholes model has been challenged continuously. The symmetrical distribution emphasises a lognormalized asset. This paper aims to investigate the volatility distribution (i.e. kurtosis) of the South African warrants market at Johannesburg Stock Exchange based on a comparison of option implied distributions of the terminal price of the TOP European Call option with lognormal distribution. The result indicates that the constant volatility of Black-Scholes model does not show in the selected warrant market.

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CONTENTS

1 INTRODUCTION

This paper aims to investigate the kurtosis of the South African warrants prices at Johannesburg Stock Exchange based on a comparison of option implied distributions of the terminal price of the TOP European Call option with lognormal distribution. The methodology of this investigation has closely followed the methodology of both Shimko (1995) and Deats, Keymer, Mann, and Roffey (2000), which is based on Shimko's work.

First, though, for the sake of clarity, it is important to define some of the terms used herein. Briefly, warrants are defined in the *New Penguin Dictionary (2001)* as "a document issued by a company giving to the holder the right to purchase the capital stock of the company at a stated price, either prior to a stipulated date or at any future time." In the South African context, a warrant is an instrument issued by an independent party (the issuer), such as a member of the Johannesburg Stock Exchange, a merchant bank or investment house in respect of an underlying asset. It is important to note that, although a warrant confers rights, it creates no obligations for an investor to exercise such rights. There are currently more than 500 warrants listed on the main board of the Johannesburg Stock Exchange in South Africa. Warrants in South Africa are similar to options in the European and American contexts. In particular, the life expectancy of traded warrants options is less than a year (Taylor, 2000). These warrants include American call options, American put options, European call options, and European put options. The fundamental advantage for investors buying warrants on the JSE, as opposed to making a direct investment in the underlying asset, resides in the leverage effect offered by warrants. The term leverage means that an investor holds a position in a security by investing less than the full amount of the security's face value. The price of the underlying asset and the price of the warrant are correlated, so that any fluctuation in the price of the underlying asset will bring about a change in the price of the warrant, as measured by the delta of the warrant. This delta measures the sensitivity of the warrant to changes in the underlying asset. In other words, delta defines exactly how much the price of the warrant will move when the underlying share price changes. The delta of an option changes over time. This means that the position in the underlying asset has to be frequently rebalanced. Once an option position has been made delta neutral, it then looks at its gamma, i.e. **g**amma has frequently been used in hedging as well. The gamma of an option position is the rate of change of the delta of the position with respect to the price of the underlying asset. It is a measure of the curvature of the relationship between the option price and the asset price. Nevertheless, both delta and gamma hedging are both based on the assumption that the volatility of the underlying asset is constant. In practice, however, volatilities do change over time.

The investigation forming the subject matter of this dissertation focuses on the variance rate, σ^2. Empirical evidences from various researches **generally** show that σ^2 changes over time. That is, it appears that observed rates of return on common stock can be characterised as independent drawings from a normal population with presumably constant mean but changing variance. The key element of the resulting phenomenon is due to the stochastic nature of the volatility of the underlying assets. The volatility expectation derived from the reported option prices depends on the assumptions underlying the option valuation formula. The Black-Scholes model assumes that the asset price follows a geometric Brownian motion with constant volatility. Consequently, all options with constant volatility have the same implied volatility. In practice, however, Black-Scholes implied volatilities tend to differ across exercise prices and times to expiration (Rubinstein, 1994). S&P 500 option implied volatilities, for example, used to have so-called volatility smile pattern before the October 1987 market crash (Rubinstein, 1994). Options that are deep in-the-money or out-of-the-money have higher implied volatilities than at-the-money options. After the crash, a phenomenon of (to use the smile analogy) sneer or a slight raising of one corner of the upper lip occurs, which results from the implied volatilities. The positive relationship has been decreased monotonically as the exercise price rises relative to the index level, with the rate of decrease increasing for options with a shorter time to expiration (Dumas, Fleming, and Whaley, 1998). Furthermore, Wandmacher and Bradfield (1998) have published the methodology and parts of the results of analysing both the volatility smile index and the volatility term structure index for a close look.

The purpose of analysing the volatility smile index and the volatility term structure index was to assess the null hypothesis that the implied volatility is constant across different strike prices and over time. All the results suggest that the null hypothesis of constant volatility is not appropriate, however, because the patterns found for implied volatility differ from the constant volatility assumption. The shapes of the patterns were also found to differ systematically across the various expiration classes. Readers are thus referred to further explanations in Wandmacher and Bradfield (1998). In sum, the results have confirmed that the assumption of constant volatility required by the modified Black-Scholes model is unrealistic in the South African environment. In the pursuit of attaining greater accuracy in the pricing of options in South Africa, it is recommended that models that do not rely on the assumption of constant volatility may therefore be more suitable.

The failure of the Black-Scholes model to describe the structure of reported option is brought to an attention by rising from its constant volatility assumption. For instance, option valuation is based on the means of the Black-Scholes formula itself. Furthermore, it has been generally observed that when stock prices go up, volatility goes down, and vice versa. It is difficult to account for such non-constant volatility within an option valuation framework. For instance, in the case of stochastic volatility, it is difficult to estimate the market prices of risk parameters in valuing options.

1.1 BLACK-SCHOLES IMPLIED VOLATILITY PATTERNS

The motivation for this research arises from deficiencies in the Black-Scholes model. These deficiencies are most commonly expressed in cross section as the relation between the Black-Scholes implied volatility and the option exercise price. Dumas, Fleming, and Whaley (1998) have illustrated this relation for S&P 500 index options, associating it with its implications for option valuation.

S&P 500 index options, as emphasised by Dumas, Fleming, and Whaley (1998), provide a context where the Black-Scholes conditions seem most reasonably satisfied. To compute the implied volatilities, they have used the Black-Scholes call option formula as follows:

(1) $$C = S_0 N(d_1) - K e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

The above formula has been discussed extensively in the literature (Hull (2003), Merton (1976), and Black and Scholes (1973)). At any instance in time, C is the market value of the call option; S_0 is the current share price; K is the exercise (or strike) price; T is the time to maturity; r is risk-free interest rate (which is continuous and constant through time); σ is the standard deviation of the rate of return of the share (that is, the stock price volatility); $N(d_i)$ is the probability that a variable with a standard normal distribution will be less than d_i (e.g. $N(d_2)$ is the probability that the option will be exercised in a risk-neutral world; $Ke^{-rT}N(d_2)$ is the strike price multiplied by the probability that the strike price will be paid; $S_0\ N(d_1)$ is the expected value of a variable that equals S_T if $S_T > K$ and otherwise is zero in a risk-neutral world). In short, $N(d_i)$ is the cumulative unit normal density function with upper integral limit d_i.

For any time interval of length t, the Black-Scholes model has assumed that there are no transaction costs or taxes associated with hedging a portfolio, that the underlying asset price follows a lognormal random walk or geometric Brownian motion, that there are no restrictions on short selling of the underlying asset, and that the underlying asset pays no dividends during its lifespan. The rate on a Treasury bill of comparable maturity is used as a proxy for the risk-free rate. For each option price, the implied volatility is computed by solving for the volatility rate (σ) that equates the model price with the observed bid or ask quote. If the reported S&P 500 index level is stale, the implied volatilities of call options will be biased downward or upward, depending on whether the index is above or below its true level respectively.

The resulting S&P 500 implied volatilities do not all lie on a horizontal line, and do not generate the volatility smile, which constitutes evidence against the efficacy of the

Black-Scholes model in these circumstances. The research of Dumas, Fleming, and Whaley (1998) in this regard shows that the volatility smile of the S&P 500 implied volatilities is more like a slight raising of one corner of the upper lip (resembling, as they have defined it, a sneer rather than a smile). Dumas, Fleming, and Whaley did not find any evidence to show that the volatilities were symmetric around zero moneyness, nor any evidence of in-the-money and out-of-the-money options that have higher implied volatilities than at-the-money options. Including Dumas, Fleming, and Whaley (1998)'s work, the evidence of non-constant volatility has been recognised in many different option price models, which attempt to either model the volatility process (e.g. stochastic volatility) or to take the information necessary to determine the volatility process directly out of the market. However, at the end of Section 1.2 below, the section will show any attempt to model volatility will create additional problems relating to the ability to hedge options with the underlying asset.

1.2 THE STOCHASTIC VOLATILITY PROBLEM

The stochastic volatility problem has been examined by Geske (1979), Merton (1976), and Johnson (1979). Their works are the most important ones in the field and illustrated as follows. Geske examines the case in which the volatility of the firm value is constant, so that the volatility of the stock price changes in a systematic way as the stock price rises and falls. Merton examines the case in which the price follows a mixed jump-diffusion process. Johnson studies the general case in which the instantaneous variance of the stock price follows some form of stochastic process. However, there is a shortcoming in Johnson's work. In order to derive the differential equation that the option price must satisfy, Johnson assumes the existence of an asset with a price that is instantaneously perfectly correlated with the stochastic variance.

Hull and White (1987) provide a solution to the stochastic volatility option pricing problem in series form. The following section of the paper closely follows their . Hull and White (1987) consider a derivative asset, f, with a price that depends upon stock price, S, and its instantaneous variance, $V = \sigma^2$, which are assumed to follow the stochastic processes below:

(2) $$dS = \Phi S dt + \sigma S dw$$

(3) $$dV = \mu V dt + \xi V dz$$

The variable Φ is a parameter that may depend on S, σ, and t. The variable μ and ξ may depend on σ and t, but does not depend on S. The Wiener processes dz and dw have correlation p. The actual process that a stochastic variance follows is complex. Due to a positive-value factor, the instantaneous standard deviation must approach zero as σ^2 approaches zero. Since S and σ^2 are the only state variables affecting the price of the derivative stock, f, the risk-free rate, r, must be constant or at least deterministic. As was shown by Garman (1976), a stock f with a price that depends on state variables θ_i must satisfy the differential equation of

(4) $$\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{i,j} p_{ij}\sigma_i\sigma_j \frac{\partial^2 f}{\partial \theta_i \theta_j} - rf = \sum \theta_i \frac{\partial f}{\partial \theta_i}[-\mu_i + \beta_i(\mu^* - r)]$$

where σ_i is the instantaneous standard deviation of θ_i; p_{ij} is the instantaneous correlation between θ_i and θ_j; μ_i is the drift rate of θ_i; β_i is the vector of multiple regression betas for the regression of the state-variable returns of ($d\theta/\theta$) on the market portfolio and the portfolios most closely correlated with the state variables; μ^* is the vector of instantaneous expected returns on the market portfolio and the portfolios most closely correlated with the state variables; and r is the vector with elements that are the risk-free rate, r. When variable i is traded, it satisfies the (N+1)-factor CAPM, and the ith element of the right-hand side of Equation (4) is $-r\theta_i\partial f/\theta$i.

In considering the stochastic problem, we must look at two state variables, S and V, of which S is traded. The differential Equation (4) thus becomes

$$(5)\qquad \frac{\partial f}{\partial t}+\frac{1}{2}\left[\sigma^2S^2\frac{\partial^2 f}{\partial S^2}+2p\sigma^3\xi S\frac{\partial^2 f}{\partial S\partial V}+\xi^2V^2\frac{\partial^2 f}{\partial V^2}\right]-rf$$
$$=-rS\frac{\partial f}{\partial S}-\left[\mu-\beta v\left(\mu^*-r\right)\right]\sigma^2\frac{\partial f}{\partial V}$$

where p is the instantaneous correlation between S and V. The variable β_V is the vector of multiple regression betas for the regression of the variance "returns" (dV/V) on the market portfolio and the portfolios most closely correlated with the state variables. Since these expected returns depend on investor risk preferences, therefore, the option price will also depend on investor risk preferences. $\beta v\left(\mu^* - r\right)$ is assumed to be zero, or that the volatility is uncorrelated with aggregate consumption, i.e. that the volatility has zero systematic risk. The derivative asset must then satisfy the following:

$$(6)\qquad \frac{\partial f}{\partial t}+\frac{1}{2}\left[\sigma^2S^2\frac{\partial^2 f}{\partial S^2}+2p\sigma^3\xi S\frac{\partial^2 f}{\partial S\partial V}+\xi^2V^2\frac{\partial^2 f}{\partial V^2}\right]-rf=-rS\frac{\partial f}{\partial S}-\mu\sigma^2\frac{\partial f}{\partial V}$$

It will also be assumed that $p = 0$, that is, that the volatility is uncorrelated with the stock price. Geske (1979) shows that such assumption is equivalent to assuming no leverage and a constant volatility of firm value. An analytic solution to Equation (6) for a European call option may be derived by using the risk-neutral valuation procedure. Since neither Equation (6) nor the option boundary conditions depend upon risk preferences, Hull and White (1987) assume in calculating the option value that risk neutrality prevails. Thus, $f(S, \sigma^2, t)$ must be the present value of the expected terminal value of f discounted at the risk-free rate. The price of the option is therefore

$$f(S_t, \sigma_t^2, t) = e^{-r(T-t)} \int f(S_T, \sigma_T^2, T)\, p(S_T \mid S_t, \sigma_t^2)\, dS_T \tag{7}$$

where T is time at which the option matures; S_T is stock price at time t; σ_t is instantaneous standard deviation at time t; $p(S_T \mid S_t, \sigma_t^2)$ is the conditional distribution of S_T given the stock price and variance at time t; $E(S_T \mid S_t) = S_t e^{r(T-t)}$ and $f(S_T, \sigma_T^2, T)$ is max [0, S - X].The condition imposed on $E(S_T \mid S_t)$ is given, which clearly states that in a risk-neutral world, the expected rate of return on S is the risk-free rate. The conditional distribution of S_T depends on both the process driving S and the process driving σ^2. We can thus define $\overline{V}$ as the mean variance over the life of the derivative stock, as defined by the stochastic integral

$$\overline{V} = \frac{1}{T-t} \int \sigma_T^2\, dT$$

Thus, the conditional density functions of the distribution of S_T may be written as

$$p(S_T \mid \sigma_t^2) = \int g(S_T \mid \overline{V})\, h(\overline{V} \mid \sigma_t^2)\, d\overline{V}$$

where the dependence upon S_t is suppressed to simplify the notation. Substituting this into Equation (7) yields

$$f(S_t, \sigma_t^2, t) = \int \left[e^{-r(T-t)} \int f(S_T)\, g(S_T \mid \overline{V})\, dS_T \right] h(\overline{V} \mid \sigma_t^2)\, d\overline{V} \tag{8}$$

Please note that with respect to Itô's lemma, $e^{-r(T-t)}\int f(S_T)g(S_T|\overline{V})dS_T$ is the Black-Scholes price for a call option on a stock with a mean variance $\overline{V}$, which will be denoted C($\overline{V}$), i.e. $C(\overline{V}) = e^{-r(T-t)}\int f(S_T)g(S_T|\overline{V})dS_T$. Since log($S_T$/ S_0) conditional $\overline{V}$ is normally distributed with variance $\overline{V}T$ when S and V are instantaneously uncorrelated, it thus yields

$$C(\overline{V}) = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

where

$$d_1 = \frac{\ln(S_t / K) + (r + \overline{V}/2)(T-t)}{\sqrt{\overline{V}(T-t)}}$$

$$d_2 = d_1 - \sqrt{\overline{V}(T-t)}$$

Thus, the option value is given by

(9) $$f(S_t, \sigma_t^2) = \int C(\overline{V})h(\overline{V}|\sigma_t^2)d\overline{V}$$

Equation (9) is always true in a risk-neutral world, where the stock price and volatility are instantaneously uncorrelated. If, in addition, the volatility is uncorrelated with aggregate consumption, Hull and White (1987) have shown that the option price is independent of risk preferences and that the equation is true in a risk world as well. Equation (9) states that the option price is the Black-Scholes price integrated over the distribution of the mean volatility. It may be possible to obtain an analytic form for the distribution of $\overline{V}$ for any reasonable set of assumptions about the process driving *V*.

Let us define an at-the-money option as one for which $S = Ke^{-r(T-t)}$. When the volatility is stochastic, the Black-Scholes price tends to overprice at-the-money options and underprice deep-in-the-money and deep-out-of-the-money options. With regard to such a statement, Hull and White (1987) emphasize that Equation (9) is the expected Black-Scholes price, the expectation being taken with respect to $\overline{V}$,

$$f = E[C(\overline{V})]$$

when C is a concave function, $E[C(\overline{V})] < C(E[C(\overline{V})])$. For a convex function, the reverse is true. In other words, the Black-Scholes option price $C(\overline{V})$ is convex for low values of $\overline{V}$ and concave for higher values. Thus, at least when ξ is small, the Black-Scholes price tends to underprice for low values of $\overline{V}$ and overprice for high values of $\overline{V}$. The statement of a stochastic variance can lower the option price below the price it would have if the volatility were nonstochastic; this is, however, consistent with the results which Merton (1976) has derived for the mixed jump-diffusion process. Merton (1976) shows that if the option is priced by using the Black-Scholes results based on the expected variance (the expectation being formed over both jumps and continuous changes), then the price might be greater or less than the correct price.

In order to determine the circumstances under which the Black-Scholes price is too high or too low, it needs to examine the second derivative of $C(\overline{V})$.

$$C''(\overline{V}) = \frac{S\sqrt{T-t}}{4\overline{V}^{\frac{3}{2}}} N'(d_1)(d_1 d_2 - 1)$$

The curvature of C is determined by the sign of C'', which depends on the sign of $d_1 d_2 - 1$. The point of inflection in $C(\overline{V})$ is given when $d_1 d_2 = 1$, that is,

$$\bar{V} = \frac{2}{T-t}\left[\sqrt{1+\left[\log(S/K)+r(T-t)\right]^2} - 1\right]$$

Let us denote this value of $\bar{V}$ by *I*. When $\bar{V} < I$, then C′′ > 0, and C is a convex function of $\bar{V}$. When $\bar{V} > I$, then C′′ < 0, and C is a concave function of $\bar{V}$. If S = $Ke^{-r(T-t)}$, then *I* = 0. In other words, C is always a concave function of $\bar{V}$, and, regardless of the distribution of $\bar{V}$, the actual option price will always be lower than the Black-Scholes price. As $\log(S/K) \to \pm\infty$, *I* becomes arbitrarily large, and C is always convex so that the actual option price is always greater than the Black-Scholes price. Thus, it can be deduced that the Black-Scholes price always overprices at-the-money options but underprices options that are sufficiently deeply either in- or out-of-the-money.

We can further conclude that, in terms of Equation (9), if the stochastic volatility is independent of the stock price, the correct option price is the expected Black-Scholes price, where the expectation is taken over the distribution of mean variances.

On the other hand, if we expand C($\bar{V}$) in a Taylor series about its expected value, $\bar{V}$, it yields

$$\begin{aligned} & f(S_t, \sigma_t^2) \\ &= C\bar{\bar{V}} + \frac{1}{2}\frac{\partial^2 C}{\partial \bar{V}^2}\bigg|_{\bar{V}} \int (\bar{V} - \bar{\bar{V}})^2 h(\bar{V}) d\bar{V} + \ldots \\ &= C\bar{\bar{V}} + \frac{1}{2}\frac{\partial^2 C}{\partial \bar{V}^2}\bigg|_{\bar{V}} Var(\bar{V}) + \frac{1}{6}\frac{\partial^3 C}{\partial \bar{V}^3}\bigg|_{\bar{V}} Skew(\bar{V}) + \ldots \end{aligned}$$

where Var($\bar{V}$) and Skew($\bar{V}$) are the second and third central moments of $\bar{V}$. For sufficiently small values of ξ^2(T-t), this series converges very quickly. For any non-zero μ, options of different maturities would exhibit markedly different implied volatilities. Since this has never been observed empirically because it cannot happen, Hull and White (1987) conclude that μ is at least close to zero, $\mu \to 0$, and yields

(10)
$$\begin{aligned} f(S,\sigma^2) &= \\ &C(\sigma^2) \\ &+\frac{1}{2}\frac{S\sqrt{T-t}N'(d_1)(d_1d_2-1)}{4\sigma^3}\left[\frac{2\sigma^4(e^k-k-1)}{k^2}-\sigma^4\right] \\ &+\frac{1}{6}\frac{S\sqrt{T-t}N'(d_1)(d_1d_2-3)(d_1d_2-1)-(d_1^2d_2^2)}{8\sigma^5} \\ &\times\sigma^6\left[\frac{e^{3k}-(9+18k)e^k+(8+24k+18k^2+6k^3)}{3k^3}\right]+\ldots \end{aligned}$$

where $k = \xi^2(T\text{-}t)$. Hull and White (1987) then examined the option price given by the series solution in Equation (10) above. They found that, when the volatility is uncorrelated with the stock price, the option price is depressed relative to the Black-Scholes price for near-the-money options. Similarly, when the volatility is correlated with the stock price, this at-the-money price depression continues into the money for positive correlation and out of the money for negative correlation. These effects are expectedly exaggerated as the volatility, σ, the volatility of the volatility, ξ, or the time to maturity, T – t, increases. The unexpected result encountered by Hull and White (1987) was that longer-term options have lower implied volatilities, as calculated by the Black-Scholes equation, than do the shorter-term options whenever the Black-Scholes price overprices the option. They explain such intuition by the impact that the correlation has on the terminal distribution of stock prices. They consider the case in which the volatility is positively correlated with the stock price. High stock prices are associated with high volatilities; thus, as stock prices increase, the probability of large positive changes increases. This emphasizes that very high stock prices become more probable than when the volatility is fixed. Conversely, low stock prices are associated with low volatilities; thus, if stock prices fall, it becomes less likely that large changes will take place. Low stock prices become like absorbing states, so that it becomes more likely that the terminal stock price will be low too. The net effect is that the terminal stock price distribution is more positively skewed than the lognormal distribution arising from a fixed volatility. When volatility changes are negatively correlated with stock price changes, however, the reverse is true. Price increases reduce the volatility so that it is unlikely that very high stock prices will result. Conversely, price decreases increase volatility, increasing the change of large positive price changes; very low prices become less likely. The net effect is that the terminal stock price distribution is more peaked than the usual lognormal distribution.

The resulting phenomena above lead to time-to-maturity effects. If the time to maturity is increased, with all other variables being held constant, the effect would be the same as increasing both σ_t and ξ. Thus, longer-term near-the-money options have a price that is lower (relative to the Black-Scholes price) than that of shorter-term options. Because the Black-Scholes price is approximately linear with respect to volatility, these proportional price differences map into equivalent differences in implied volatilities. If the Black-Scholes equation is used to calculate implied volatilities, longer-term near-the-money options will exhibit lower implied volatilities than shorter-term options. Again, Hull and White (1987) has proved that this effect occurs whenever the Black-Scholes formula overprices the option.

This time-to-maturity effect is counterintuitive. We expect that uncertainty about the volatility would also increase uncertainty about the stock price, hence raising the option price, and that longer times to maturity would exacerbate this. The actual result from Hull and White (1987) shows just the opposite. Wherever the Black-Scholes formula overprices the option, it is due to the local concavity of the Black-Scholes price with respect to σ. Because of the concavity of the option price with respect to its volatility, increases in volatility do not increase the option price as much as decreases in volatility decrease the price. Thus, the average of the Black-Scholes prices for a stochastic volatility with a given mean lies below the Black-Scholes price for a fixed volatility with the same mean for all near-the-money options. As the time to maturity increases, the variance of the stochastic volatility increases, exacerbating the effect of the curvature of the option price with respect to volatility. Wherever the Black-Scholes price underprices the option, the reverse effect is observed.

In general, evidence of the striking price bias and the expiration bias (Rubinstein, 1985; Shastri and Tandon, 1986) has contradicted the assumptions of the Black-Scholes model. The assumption of a lognormal distribution of prices over time suggests that implied volatilities must be identical for options across strike prices and across expirations. As this is clearly not the case, other option price models have been developed that are able to incorporate such biases, for example, Hull and White's (1987) stochastic volatility model.

The key element in Equation (10) is the relationship between the stock price and the present price value of the exercise price. The chosen pairs matched on the basis of exercise price may also require matching in respect of the variable of interest, the present value of the exercise price. In order to support Hull and White's (1987) model, it is necessary to posit that, from one year to the next, the correlation between stock prices and the associated volatility reversed its sign. It is difficult to think of a convincing reason why this should occur. Instead, Hull and White (1987) have attempted to suggest that the observed effect may be a sampling result that can occur if some stocks have positive correlations and some have negative correlations. In this case, Hull and White point out that by changing the relative numbers of a chosen group in the sample from one period to the next, the observed result should lead to the time-to-maturity effect being strongest for out-of-the-money options and weakest for in-the-money options.

Stochastic volatility model incorporates volatility as a random process so that, for example, the pattern of volatility smiles can be accommodated. However, stochastic volatility is only one assumption about the state of volatility, and other volatility states may well be more appropriate. Nevertheless, other models do attempt to address the empirical shortcomings of the Black-Scholes price model, although most suffer new shortcomings themselves (Dupire, 1994). Hull and White (1987) derive a solution for a call option on an asset with stochastic volatility. However, this model suffers from a large disadvantage compared to the Black-Scholes pricing model because the option valuation in the model is no longer preference-free. Consequently, the replication and hedging of options in the model is not appropriate in the case of known securities and derivatives (Dupire, 1994).

2 VOLATILITY

There are two perspectives on how to obtain the volatility input of the Black-Scholes option pricing formula. The first is to use the past behaviour of asset prices to develop expectations about volatility. An alternative approach is to use reported option prices to infer volatility expectations by inverting the option valuation formula. Both the historical and the implied volatilities are estimates of the volatility of the underlying asset over the remaining life of the options. The difference is that the historical volatility gives a forecast of future expected volatility based on history of volatility, whereas the implied volatility is the market's assessment of this future expected volatility. Regarding to reinforced practice in the options market, the practitioners often quote the implied volatility of options in their statements. Thus the present study focuses on the implied volatility of the call options in warrants market at JSE.

2.1 HISTORICAL VOLATILITIES

The volatility of a stock, σ, is a measure of the market's uncertainty about the return provided by the stock. According to the lognormal assumption of the Black-Scholes formulas, below

(11)
$$\ln\frac{S_T}{S_0} \sim \phi\left[(\mu-\frac{\sigma^2}{2})T, \sigma\sqrt{T}\right]$$

Equation (11) shows that the volatility of a stock price can be defined as the standard deviation of the return provided by the stock in one year, when the return is expressed using continuous compounding. To estimate the volatility of a stock price empirically, the stock price is usually observed at fixed intervals over time such as by using daily data or weekly data. By extracting the formulas from Hull (2003), it can be defined as follows:

$n + 1$: Number of observations

S_i: Stock price at end of ith ($i = 0, 1, \ldots, n$) interval

t: Length of time interval in years

and let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

for $i = 1, 2, \ldots, n$.

More data generally leads to more accuracy, but σ does change over time, and thus data that is too old may not be relevant for predicting the future. It must also be noted that disregarding key events can have pronounced effects on the calculated historical volatility. According to Jackwerth and Rubenstein (1996), probability distributions of stock market returns have typically been estimated from historical time series. Unfortunately, common hypotheses may not capture the probability of extreme events, which are rare or may not be present in the historical record. For example, leave alone of the pronounced smile effects from lognormality in the stock market crash of October 19, 1987. The 1987 crash also sensitized historical sample statistics to sample size. For example, historical measurements of volatility are dependent on whether or not the 1987 crash is a sample point. Apart from the special problems created by the stock market crash, many other difficulties are encountered when sampling from an inherently non-stationary time series such as stock market prices. For example, even by fixing the overall sample period, historical sub-samples will exhibit systematic biases in sample statistics. These difficulties in dealing with historical time series can have a significant effect on option prices.

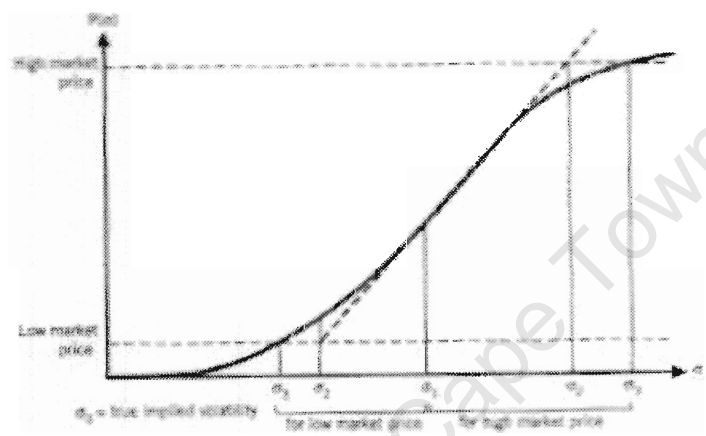
Finally, the Markov property of stock prices is consistent with the weak form of market efficiency. This states that the present price of a stock impounds all the information contained in a record of past price. This includes the historical volatility. Therefore, it can be concluded that historical volatility is by no means a perfect measure to use as a highly influential variable in the Black-Scholes option pricing model.

2.2 IMPLIED VOLATILITIES

The one parameter in the Black-Scholes pricing formulas that is difficult to observe is the volatility of the stock price. In practice, traders usually work with what are known as implied volatilities. These are the volatilities that make the Black-Scholes prices of an option equal to its market price, i.e. the volatilities implied by option prices observed in the market. To illustrate how implied volatilities are calculated, consider the value of a call option on a non-dividend-paying stock, which is 1.875 when S_0 = 21, K = 20, r = 0.1, and T = 0.25. The implied volatility is the value of σ that, when substituted into Equation (1), gives C = 1.875. Unfortunately, Equation (1) cannot be inverted and expresses the σ as a function of S_0, K, r, T, and C. However, Hull (2003) states that an iterative search procedure can be used to find the implied σ. The repeated iterations increase the approximated estimations as they converge toward the true value, provided they converge at all. There are several iterative approximation methods available, but the Newton-Raphson method is representative for all / is the most representative. The main reason for using such a method is that it guarantees a quick convergence, and provides an appropriate starting point (Hull, 2003; and Chance, 1995). Chance (1995) states that the Newton-Raphson method is particularly useful for finding implied volatilities because the derivative with respect to the volatility is often known, as in the Black-Scholes model. Moreover, the accuracy of the Newton-Raphson procedure is normally good with only a few iterations. However, Tompkins (1994) also notes that the Newton-Raphson procedure in only applicable for European options because the procedure depends on the linearity of the option price/volatility relationship, i.e. option price with respect to the volatility. This option price/volatility relationship is linear for European options, but it is non-linear for American options because of their early exercise right.

In many cases, the Newton-Raphson method will converge rapidly but one difficulty associated it are the many local optima. Nevertheless, these optima are not a serious problem in the case of the Black-Scholes model, as it is a well-behaved function over most reasonable values of volatility.

Figure 1: Newton-Raphson's Method



Source: Shimko (1995), Risk.

The Newton-Raphson method is that in Figure 1 above, the diagonal tangent line to C(σ), in other words, the Black-Scholes estimate of the option premium as a function of σ at σ_1, intersects the market price at a volatility σ_2 near the true implied volatility, and that a tangent drawn at σ_2 yields a still better estimate, σ_3. More generally, given the estimate σ_i, the improved estimate, σ_{i+1}, is given by

(12.1)
$$\sigma_{i+1} = \sigma_i - \frac{C(\sigma_i) - C_{market}}{C'(\sigma_i)}$$

The Newton-Raphson method is especially convenient for finding implied volatilities for European options because $C'(\sigma_i)$, the slope of the tangent line at volatility σ_i, can then be computed explicitly via Equation (1) as follows:

(12.2) $$C'(\sigma_i) = \frac{\partial C}{\partial \sigma_i} = Ke^{-r(T-t)} N'(d_2)\sqrt{T-t}$$

Please check Appendix III for detailed calculations in this regard. For European options, $C'(\sigma_i)$ is also the vega of the option when σ_i converges to the true implied volatility. Manaster and Koehler (1982) have exploited the Newton-Raphson method to provide a fast convergence procedure with a good initial starting value that guarantees convergence. Similar to Equation (12.1), their estimation yields a unique implied volatility for each call option with a different strike price

$$\sigma_{n+1} = \sigma_n - \frac{C_n - C}{C'_n}$$

where σ_n is the nth approximation for implied volatility, σ_{n+1} is the (n+1)th approximation for implied volatility, C is the market price (to converge); C_n is the option calculated by using σ_n (the Black-Scholes model in Equation (1)); C'_n is the derivative of C_n with respect to σ (it applies to the same Equation (12.2)). In order to start the calculation of this iteration, the present paper has closely followed the methodology of Shimko (1995) as well as Deats et al. (2000) (whose work is also based on Shimko's work). As a result, the iteration has a starting value, σ_0, via the following equation,

(12.3) $$\sigma_1^2 = \sqrt{\frac{2\left|\ln(\frac{S}{K}) + rT\right|}{T}}$$

where σ is implied volatility. Once the starting point has been calculated, this method uses the partial derivatives of the market price of the call option with respect to volatility to find the implied volatility. The resulting sigma value from Equation (12.3) will have a corresponding maximum vega value. If the estimated option price obtained by using (12.3) as the volatility is larger than the actual market price, the volatility being iterated will be a decreasing sequence bounded below by the true volatility. Such a sequence must converge, and, since (12.1) will continue to generate ever-decreasing iterates until the true implied volatility is reached, Equation (12.1) must converge to that value. A similar argument applies when the volatility estimate (12.3) is smaller than the market volatility.

Again, this paper has followed the methodology of both Deats, Keymer, Mann, and Roffey (2000) and Shimko (1995) to calculate the derivatives. In particular, Shimko (1995) states that smooth call option pricing functions are needed to calculate derivatives. An interpolation is thus set up to carry out the smoothing process. This paper will thus follow the interpolation of the Breeden and Litzenberger (1978) approach by allowing the implied volatility function to become a best-fit least-squares regression of the quadratic function. That is, it allows the implied volatility function to be a quadratic function of the exercise price (i.e. K),

(13) $$\hat{\sigma}(K,T) = A_0 + A_1 K + A_2 K^2$$

for every exercise price within the range of traded exercise prices. Outside this range, the implied volatility is assumed constant. Please note that Black-Scholes pricing holds if $A_0(T)$ for all T, and $A_1 = A_2 = 0$. The result of Equation (13) is an effective approximation for σ (the implied volatility), which incorporates the volatility skew for every strike price. These results of approximation derived for σ's are then used in the Black-Scholes model of Equation (1) to generate corresponding (smoothed) call prices.

2.3 IMPLIED DISTRIBUTIONS

The price of a European call option on an asset with strike price K and maturity is given by

(14) $$C(K,T) = e^{-rT} \int_{S_T=K}^{\infty} (S_T - K)\, g(S_T)\, dS_T$$

where r is the interest rate (assumed constant), S_T is the asset price at time T, and g is the risk neutral probability density function of S_T. Differentiating once with respect to K, it follows that

(14.1) $$\frac{\partial C}{\partial K} = -e^{-rT} \int_{S_T=K}^{\infty} g(S_T)\, dS_T$$

Differentiating again with respect to K gives

(14.2) $$\frac{\partial^2 C}{\partial K^2} = e^{-rT} g(K)$$

The only problem in implementing Equations (14.1) and (14.2) lies in calculating the cumulative normal distribution function, N. Tables for $N(x)$ are provided by Hull (2003). Nevertheless, the cumulative normal distribution functions that have been calculated in Appendix II yield the following results:

(14.3) $$\frac{\partial C}{\partial K} = -e^{-rT} N(d_2)$$

(14.4) $$\frac{\partial^2 C}{\partial K^2} = e^{-rT} N'(d_2) \frac{1}{K\sigma\sqrt{T}}$$

The calculation in this section, so far, has detailed the means of calculating the value for the cumulative distribution at each of the endpoints and of generating values for the area between the endpoints. The density function is always sum to one. Thus, the height of the calculated distribution can be scaled to match its implied area. This will allow the calculation of a marginal density at each endpoint. Deats, Keymer, Mann, and Roffey (2000) demonstrate the process of fitting appropriate lognormal tails at each endpoint by solving for the mean and standard deviation of a lognormal distribution that has the same marginal and cumulative probability. Their assumption of lognormal tails of distribution beyond the endpoints was consistent with Shimko (1995).

3 KURTOSIS

The previous section has shown how to determine the probability distribution for an asset price at a future time from the volatility smile given by options maturing at that time. This has been referred to as the implied distribution. The volatility smile for equity options corresponds to the implied probability distribution, if it has been compared to lognormal distribution (which has the same mean and standard deviation as the implied distribution).

In general, when these volatilities are graphed against strike prices of options, with equal time to maturity and using the constant volatility assumption of the Black-Scholes model, the graph should be a flat line. However, this is not happening in practice. The asset volatility is not constant over the term of the option contract. As a result, the smile patterns will become more pronounced as time to maturity shortens, and the smile becomes flatter as the time to expiration lengthens. For a call option, the implied distribution has a less heavy tail than the lognormal distribution, that is, no kurtosis exists. Let us suppose that there is a call option that is deep-out-of-the-money. A deep-out-of-the-money call with a strike price of K_t has a lower price when the implied distribution is used than when the lognormal distribution is used. This is because the option pays off only if the stock price proves to be above K_t, and the probability of this is lower for the implied probability distribution than for the lognormal distribution. Therefore, it expects the implied distribution to give a relatively low price for the option. A relative low price leads to a relatively low implied volatility.

Gujarati (1995) states that if X is any random variable with mean m and variance $s^2 > 0$, then the kurtosis of X is defined as

(15) $$Kurtosis(X) = \frac{E\left[(X-m)^4\right]}{s^4}$$

and that the excess kurtosis of X is given by Excess Kurtosis (X) = [Kurtosis(X)-3]. The measurement of kurtosis can be combined with skewness to determine whether a random variable follows a normal distribution. Furthermore, the tests serve as the tests of lognormality as well. Any normal distribution should have a kurtosis of three, both conditionally and unconditionally. To calculate the kurtosis, as well as the skewness of the implied distribution, the summary of these parameter calculations is shown below:

$$Skewness = \frac{\left[\hat{E}(S_T - S_0 e^{rT})^3\right]^2}{\left[\hat{E}(S_T - S_0 e^{rT})^2\right]^3}$$

$$Kurtosis = \frac{\hat{E}(S_T - S_0 e^{rT})^4}{\left[\hat{E}(S_T - S_0 e^{rT})^2\right]^2}$$

where the mean of the implied probability distribution, $\mu = \hat{E}(S_T) = S_0 e^{rT}$ and the variance is $\hat{E}(S_T - S_0 e^{rT})^2$. If the result of the kurtosis on the chosen dataset of particular warrant options is either left-or-right-skewed of its implied probability distribution, then we may conclude that the mean of the distribution tended to be to the right or the left mode, and that the mode tends to be more or less pronounced than the mode of the corresponding lognormal distribution. These options are not lognormal distributed under the assumptions of Black-Scholes pricing formula in Equation (1). In addition, the results of this test will also be crosschecked by performing a chi-squared goodness-of-fit test at both significant levels of 2.5% and 5%.

3.1 GOODNESS-OF-FIT TESTS

As indicated above, the present paper has also considered the chi-square goodness-of-fit test using the residuals $\hat{u}_i$ and the chi-square probability distribution. The chi-squared goodness-of-fit test is set at a significant level of 5%. The tests for lognormality were carried out using the Microsoft Excel Solver package. The statistical test was performed with the following hypotheses:

H_0: The distribution of the call options prices of chosen warrants is lognormal

H_1: The distribution of the call options prices of chosen warrants is not lognormal

If the sample size is reasonably large, the chi-square statistic approximately follows the chi-square distribution with (N-1) df, where N is the number of classes and df is the degree of freedom. Now to fit the normal distribution to the $\hat{u}_i$, associated with zero mean value of $\hat{u}_i$, it only has to estimate the variance. Lastly, if the p-value of obtaining a chi-square value is sufficiently high, the difference between the observed and expected values of the warrants options is not significant enough to reject the null hypothesis of the normality assumption.

4 THE DATA

Our data samples consist of daily closing prices of all European Call options traded on the Johannesburg Stock Exchange (JSE) namely TOP warrants index options, which are issued by Deutsche Bank (coded TOPDB), over the period of 2002/09/13 to 2003/02/14. The source of the data was downloaded from the professional data vendor service namely I-Net Bridge. In order for the analysis of data to yield a meaningful result, it must contain useful information, and the investigation in this paper has particularly focused on volatility. By referring to Deats, Keymer, Mann, and Roffey (2000), the data used in the calculation have closely followed the prerequisites below:

1. There must be a large volume of any option contract (for any given strike price) traded at any point in time.
2. There is a range of bid and ask prices at which market participants are willing to trade.
3. Many strike prices are traded for an option with a given exercise date.
4. Options should be European to simplify the pricing of the option and to define the date of exercise.
5. Market participants must be knowledgeable, having access to informative research. This will ensure that the market is operating at a weak form of market efficiency

and that the prices at which the options traded are not arbitrary, but rather close to the true value of the options.

6. Trades should take place at arm's length, implying that the deals are made due to market related factors.

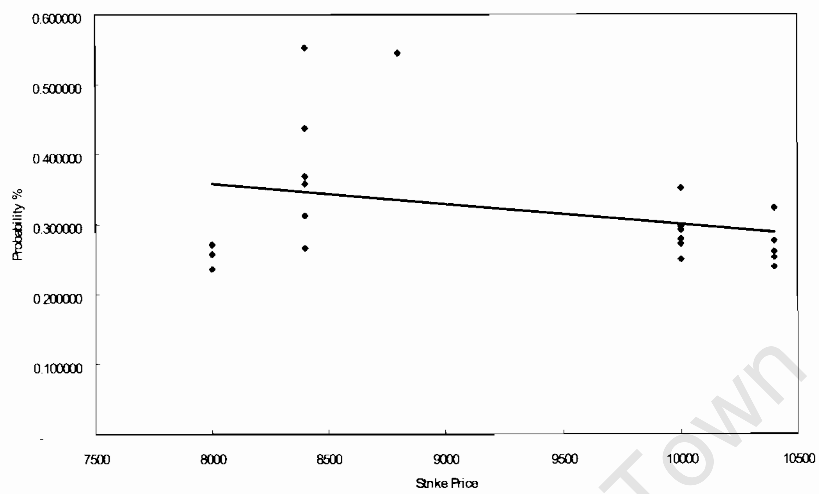
Once the above prerequisites are taken into consideration, as laid out in 3.1, the data set used is of the options traded on the TOP index warrants. In order to follow this up in the South African context, the chosen data set conforms to the requirements of liquidity, pricing volatile, and knowledge market participants.

5 TESTS FOR LOGNORMALITY

Return of daily closing prices of selected TOP European Call options are supposed to be distributed lognormally for the purpose to apply Black-Scholes model. Before examining whether or not the implied distribution has a less heavy tail than the lognormal distribution, a pre-lognormality test was carried out. Inspection of the resulting implied volatilities reveals two observations that were inconsistent throughout the test. Extensive empirical evaluation shows that the inclusion of such outliers led to meaningless results, and that they thus should be removed. Figure 2 accordingly plots the call options strike prices and implied volatilities after the outliers have been removed.

The result of the volatility skew of Figure 2 shows that when these volatilities are graphed against the strike prices of options, with equal time to maturity and the constant volatility assumption of the Black-Scholes model, the graph is not a horizontal line. It might have been reasoned that the non-constant volatility of the warrants market in JSE would lead to the downward sloping volatility skew. As a result, the smile patterns will become more pronounced as time to maturity shortens, and the smile becomes flatter as the time to expiration lengthens. The results for the tests of lognormality of prices of TOP options are presented in Table 1 below.

Figure 2: Implied Volatility of selected data plotted without outliers



From Table 1, as the p-value of obtaining a chi-square value is extremely small, the difference between the observed and expected values of the warrants options is significant enough to reject the null hypothesis of the normality assumption at the 5% significance level. In addition, the null hypothesis of the lognormality assumption is also rejected at the 2.5% significance level. For example, the p-value of 4.26386E-05 took on a value of less than $1*10^{-5}$, implying that there is a less than one 100000^{th} chance that the distribution of TOP prices is lognormal.

Table 1: Results of the test of lognormality of TOP options

Significance level	Date	p-value	chi-square	df	no.
α = 5%	Jan-03	4.08274E-21	2.7318094	3	76
	Jun-03	4.26386E-05	4.0383838	2	51
	Dec-03	6.22689E-36	2.6641089	3	156
	Feb-04	2.7262E-70	3.0307916	2	261
	Sep-04	1.13218E-69	3.0386644	2	214
α = 2.5%	Jan-03	4.08274E-21	3.3034269	3	76
	Jun-03	4.26386E-05	5.3470899	2	51
	Dec-03	6.22689E-36	3.2031835	3	156
	Feb-04	2.7262E-70	3.7421302	2	261
	Sep-04	1.13218E-69	3.7541383	2	214

Therefore, it could conclude the option prices on the TOP are not lognormal. The possible remedy for this non-lognormality is to solve sigma value, θ, from selected market option and generate new gamma. The new gamma value will produce more fitted transformation distribution (e.g. leptokurtic distribution), that lead to lognormal distribution associated with different parameter. Consequently, with regard to the derivatives from geometric Brownian motion and the same non-lognormality result already being existened in both papers of Shimko (1995) and Deats, Keymer, Mann, and Roffey (2000), the assumption of lognormality , as given this paper had followed closely with their methodology, will be maintained throughout this paper.

6 APPLICATION OF THE METHODOLOGY

There are several other concerns regarding the chosen data set:

1. The tests in this paper were conducted based on option data on the TOP over the period 1 September 2002 to 17 September 2004. This is a subset of the available TOP option data in I-Net Bridge. This may create a problem if the chosen data set is not representative of the options, and particularly if the small sample may exhibit anomalies in the TOP option data set as a whole.
2. The data required for the analysis were options traded on the TOP (European Call) options, with same time to expiry, traded on the same day. These limitations have narrowed down / have limited the availability of other data set entries. Furthermore, not all of these data are suitable.
3. After having worked on this data set (downloaded from I-Net Bridge) and attempting to derive results, the author faced a lack of historical data on options struck on TOP warrants options. As inadequate data were used, the author also used unsatisfactory assumptions (e.g. constant volatility of lognormal distribution in the market) in order to carry out the test. In general, these assumptions may affect the validity of the result.

In order to compensate for such a lack of data availability, data was then chosen from the prices of European call options traded on six trading days in year 2002, namely September 13th, September 24th, October 7th, October 18th, October 29th and November 14th 2002. The relevant information extracted from the I-Net Bridge data is listed in the Table in Appendix IV and V.

6.1 MODIFICATIONS

While Equations (1), (12.1), (12.2) and (12.3) were used to calculate the implied volatility estimation, however, the following modifications were made. Firstly, instead of writing programs to run Newton-Raphson approximations, the generalized optimizations found in the available Microsoft Excel Solver package software were used. As a result, certain changes have been made. The derivatives of the call price with respect to volatility were derived from a central numerical approximation. Thereafter, the approximations were generated by the package from quadratic approximations.

Lastly, the risk free rate was set to zero. The research firstly used the R150 government bond rate as risk-free interest rate for its constant maturity rate and less volatile characteristic in the financial market. However, the fluctuated and inconsistent daily value of this bond rate, which downloaded from I-net Bridge, had continously impacted for the calculations. Furthermore, this non-constant rate has negative impact on discounting the return of warrant prices, too. Thus, for above reasons and for simplifying reason, the risk free rate had been set to zero.

Please note that at sigma value of σ_1 in terms of Equation (12.3), the corresponding vega value and thus the slope of the curve in Figure 1 is theoretically a maximum. Since the result of the estimated option premium using Equation (12.3) as the volatility is less than the actual market premium, the volatility iterates generated by Newton-Raphson's method shows an increasing sequence bounded below by the true volatility (Please referred to Table in Appendix VI). Such a sequence may not need to converge further and since the iteration equation continuously generates increasing iterates, it may be concluded that the true implied volatility has been reached practically at σ_1.

6.2 FORMATTING THE DATA TO COMPUTE IMPLIED VOLATILITIES

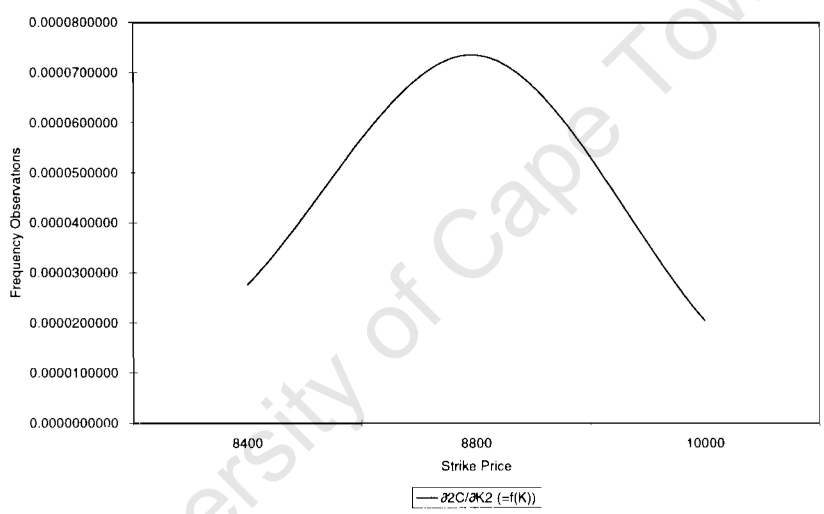
The price of the options used was based on the closing price. Due to inadequacies of the data, however, this gave relatively few points for calculating implied distributions. Thus, by using information from six trading days, I tried to compensate for such inadequacies. Extensive empirical work done by Deats, Keymer, Mann, and Roffey (2000) showed that the results became unreliable when only one or two days of trading data were used, and thus six were used in this study. Inspection of the resulting implied volatilities revealed outliers, which led to inconsistency. Eliminating these outliers, however, reduced the range of observed strike prices. The volatilities recovered after the outliers were removed are presented in the Table in Appendix V.

The values for the implied volatilities were then smoothed by using the quadratic smoothing function of the form, $\hat{\sigma}(K,t) = A_0 + A_1K + A_2K^2$, where K is the strike price. The sum-of-square differences were found by using the Microsoft Excel Solver package, as the coefficients were dependent on the initial variables used. The fitted quadratic function was then plotted in Figure 2 above.

6.3 SMOOTHING CALL PRICES

The methodology used to derive implied volatilities distribution was based on Equations (14.1) to (14.4), the theoretical basis of which has been outlined in section 2.3 above. The interval of observed prices was divided into 100 equal segments and used to plot the probability density function over the observed prices ranged from 8 800 to 10 000. The smoothed call prices were then used to derive the marginal and cumulative distributions for the probability distribution. In terms of Deats, Keymer, Mann, and Roffey (2000) calculations and approach, the area under the distribution should be the difference between the two cumulative distributions at each end of the observed strike prices. The result is portrayed in Figure 3 below, which shows the distribution (with fitted tails) with scaled probability density. The area under the graph has used the discrete approximation for the integral over the probability density function (PDF), Area = Σ (ΔK * PDF (K)). At this stage, though, it will then fit tails to the probability distribution and illustrated in Figure 3 below.

Figure 3: Implied distribution with fitted tails



6.4 DISTRIBUTION TAILS

In order to find the appropriate distribution tail, it was necessary to match both marginal and cumulative probabilities distributions to each other. The methodology used to iterate the process was found in Microsoft Excel Solver package software. Since the methodology of both Deats, Keymer, Mann, and Roffey (2000) (whom followed the methodology of Shimko (1995)) has been used extensively in this dissertation, the functions therefore have been defined as follows:

Marginal distribution function: $f(K) = \frac{1}{(\ln K)\sigma\sqrt{2\pi}} e^{-\frac{(\ln K - \mu)^2}{2\sigma^2}}$

Cumulative distribution function: $F(K) = N\left(\frac{\ln K - \mu}{\sigma}\right)$

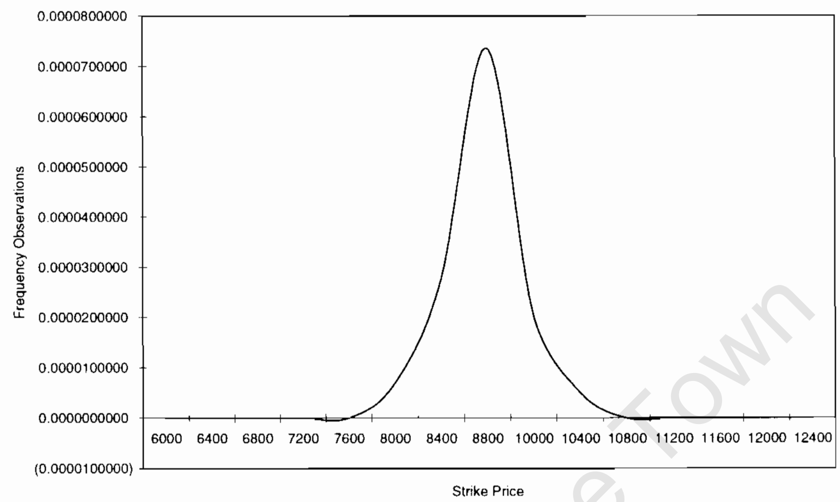
where N(K) is the cumulative normal distribution function, μ is the mean of the normal function underlying the lognormal distribution, and σ is the standard deviation of the normal function underlying the lognormal distribution. The values given in Table 2 give the figures needed to plot the tails for the implied distribution.

Table 2: Statistics for the tails of implied distribution

Strike	8400	10000
Implied volatility	0.2492777	0.250141609
$\partial^2 C / \partial K^2$ (=f(K))	0.0000276561	0.0000204362
μ	7.920328683	7.879957205

Figure 4 accordingly below shows the implied distribution complete with tails. It can be seen a narrow range of strike prices was being recovered in the distribution tails shown in Figure 3. In particular, Deats, Keymer, Mann, and Roffey (2000) have emphasized that the smoothed call prices are used to derive the marginal and cumulative distributions for the implied distribution. This has been mentioned in Section 6 above, in particular, with regard to the inadequacies of the data. This leaves relatively few points of traded data to build implied distributions on. Although the use of six trading days has compensated for the general lack of data, the methodology was found to be generally unstable and sensitive to small changes in input variables, which means that it is also extremely sensitive to small changes in mean and standard deviation.

Figure 4: Implied Distribution

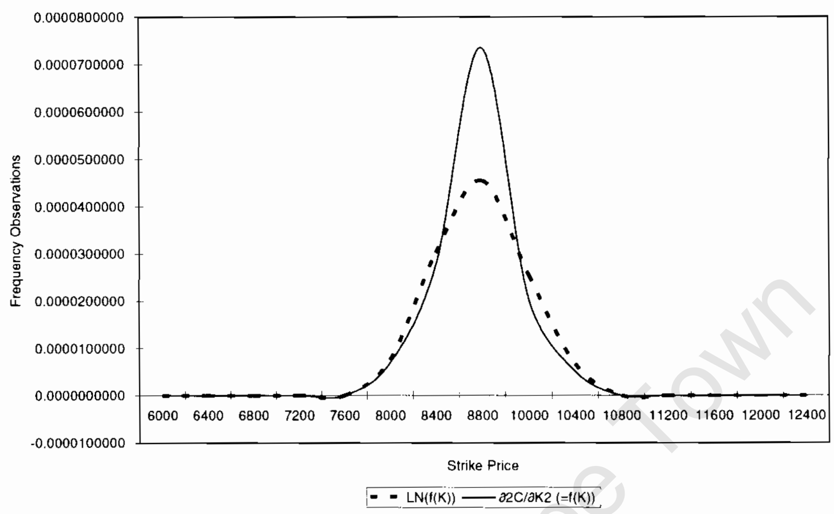


6.5 COMPARING THE IMPLIED AND LOGNORMAL DISTRIBUTIONS

This paper has used a lognormal distribution of equal mean and variance to the implied distribution to represent the market view. Figure 5 below plots the lognormal and implied distributions. Figure 2 shows that the volatility smile has a skew, whereas Figure 5 shows that the implied distribution has a less heavy tail than the lognormal distribution.

From Figure 2 it appears that the chosen data set of TOP European call options with increasing strike price has a lower price when the implied distribution is applied. This is because the option pays off only if the stock price proves to set at a higher strike price, and the probability of this is lower for the implied probability distribution than for the lognormal distribution. Therefore, it can be expected that the implied distribution will give a relatively low price for the option. A relatively low price leads to a relatively low implied volatility, which is exactly what can be observed in Figure 5. Consequently, Figures 2 and 5 are consistent with each other.

Figure 5: Comparison of Implied and Lognormal Distributions



7 APPLICATION OF KURTOSIS

According to Table 3 below, the kurtosis values are less than 3 across the chosen strike price. The probability density functions (PDFs) with kurtosis values of less than 3 lead to a fat or short-tailed distribution, which means that it is *platykurtic*. This is consistent with Figure 5 where the implied probability distribution has a less heavy tail than the lognormal distribution. It was thus found that the chosen options had a right-skewed probability distribution, that is, the mean of the distribution tended to be to the left of the mode, and the mode tended to be more pronounced than the mode of the corresponding lognormal distribution.

Table 3: Results of Kurtosis Values

Code	Expiry	Strike	Kurtosis
TOPDB	2003/01/09	8800	2.720338
TOPDB	2003/01/09	10000	1.884436
TOPDB	2003/01/09	10400	2.417728

8 CONCLUSION

The objectives of this research were to establish whether or not the implied probability distribution of the TOP warrants (European Call) option on the JSE had less of a tail than the lognormal distribution, that is, whether any kurtosis exists in the warrants market on the JSE. Given that this paper has adopted the calculations and methodologies from Deats, Keymer, Mann, and Roffey (2000), as well as Shimko (1995), the results of TOP warrants options in JSE of this paper are similar to ALSI warrants options in JSE of Deats, Keymer, Mann, and Roffey (2000)'s work. We can conclude from the results obtained in respect of the volatility skewness (Figures 2 and 5, as well as the consistency between the two), the goodness-of-fit (Table 1), as well as the test of kurtosis (both Figure 5 and Table 3), that the warrants market is not lognormal distributed nor applicable to constant volatility assumption. Another conclusion drawn from the results (associated with Figures 2 and 3) is that the further the warrants (European Call) option is from the expiry date, the more volatile the option is.

9 REFERENCES

1 Black, F. and Scholes, M., 1973, "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, Vol. 81, pp 637-659

2 Bookstaber, R.M. and McDonald, J.B., 1987, "A General Distribution for Describing Security Price Returns", *Journal of Business*, Vol. 60, pp 401-424.

3 Breeden, D.T., and Litzenberger, R.H., 1978, "Prices of State-Contingent Claims Implicit in Option Prices", *Journal of Business*, Vol. 51, pp 621-651.

4 Chance, D., "Leap into the Unknown", in: Jarrow, R. (ed.), *Over the Rainbow: Development in Exotic Options and Complex Swap*, Risk, pp 251-256.

5 Cox, J., Ross, S., and Rubinstein, M., 1979, "Option Pricing: A Simplified Approach", *Journal of Financial Economics*, Vol. 7, pp 229-263.

6 Das, S.R. and Sundaram, R.K., 1999, "Of Smiles and Smirks: A Term Structure Perspective", *Journal of Financial and Quantitative Analysis*, Vol. 34, pp 211-239.

7 Deats, W., Keymer, R., Mann, G. and Roffey, J., 2000, "Deriving Trading Strategies from Option-Implied Risk Neutral Probability Distributions", *Finance Honours Report*, School of Management Studies, University of Cape Town.

8 Dumas, B., Fleming, J. and Whaley, E.R., 1998, "Implied Volatility Functions: Empirical Tests", *Journal of Finance*, Vol. 53, pp 2059-2106.

9 Dupire, B., 1994, "Pricing with a smile", Risk, 7, pp 18-20.

10 Garman, M., 1976, "A General Theory of Asset Valuation under Diffusion State Processes", Working Paper No. 50, University of California, Berkeley.

11 Geske, R., 1979, "The Valuation of Compound Options", *Journal of Financial Economics*, Vol. 7, March, pp 63-81.

12 Gujarati, D.N., 1995, *Basic Econometrics*, 3rd Edition, McGraw-Hill Inc.

13 Hull, J.C., 2003, *Options, Futures and Other Derivatives*, 5th Edition, Prentice-Hall Inc.

14 Hull, J. C., and White, A., 1987, "The Pricing of Options on Assets with Stochastic Volatilities", *Journal of Finance* , Vol. 42, June, pp 281-300.

15 Jackwerth, J.C. and Rubinstein, M., 1996, "Recovering Probability Distributions from Option Prices", *Journal of Finance*, Vol. 51, pp 1161-1631.

16 Jarque, C.M. and Bera, A.K., 1987, "A Test for Normality of Observations and Regression Residuals", *International Statistical Review*, Vol. 55, pp 163-172.

17 Johnson, H.E., 1979, "Option Pricing When the Variance Is Changing", Graduate School of Administration Working Paper, University of California, Davis, pp 11-79.

18 Macbeth, J.D. and Merville, L.J., 1979, "An Empirical Examination of the Black-Scholes Call Option Pricing Model", *Journal of Finance*, Vol. 34, pp 1173-1186.

19 Manaster, S. and Koeler, G., 1982, "The Calculation of Implied Variance from the Black-Scholes Model: A Note", *Journal of Finance*, Vol. 37, pp 227-230.

20 Manaster, S. and Rendleman, R. Jnr., 1982, "Option Prices as Predictors of Equilibrium Stock Prices", *Journal of Finance*, Vol. 37, pp 1043-1057.

21 Merton, R.C., 1976, "Option Pricing When Underlying Stock Returns Are Discontinuous", *Journal of Financial Economics*, Vol. 3, January, pp 125-44.

22 Rubinstein, M., 1985, "Nonparametric Tests of Alternative Option Pricing Models Using All Reported Trades and Quotes on the 30 Most Active CBOE Option Classes from August 23, 1976, through August 31, 1978", *Journal of Finance*, Vol. 40, pp 455-480.

23 Rubinstein, M., 1994, "Implied Binomial Trees", *Journal of Finance*, Vol. 49, pp 771-818

24 Shastri, K. and Tandon, K., 1986, "An Empirical Test of a Valuation Model for American Options on Futures Contracts", *Journal of Financial and Quantitative Analysis*, Vol. 21, pp 377-392.

25 Shimko, K., 1995, "Bounds of Probability", in: Jarrow, R. (ed.), *Over the Rainbow: Development in Exotic Options and Complex Swap*, Risk, pp 265-270

26 Taylor, J., 2000, *Unpublished Honours Report*, Department of Economics, University of Cape Town.

27 Tompkins, R., 1994, *Options Explained* , Macmillan, Padstow, Cornwall.

28 Wandmacher, R., and Bradfield, D., 1998, "On Implied Volatilities in the South African Derivatives Market", *Investment Analysts Journal*,.

10 APPENDIX I: GLOSSARY

American option	An option that can be exercised at any time before it reaches its maturity.
At-the-money	An option in which the strike price equals the price of the underlying asset.
Call option	An option to buy an asset at a certain price by a certain date.
European option	An option that can be exercised only at maturity date.
Exercise price	The price at which the underlying asset may be brought or sold in an option contract.
Kurtosis	A measure of the tallness or fatness of the tails of a distribution.
Implied volatility	Volatility implied from an option price using the Black-Scholes formula.
In-the-money option	A call option is in-the-money where the asset price is greater than the strike price. A put option is in-the-money where the asset price is less than the strike price.
Moneyness	This is used to determine by how much profit a specific warrant was in- or out-the-money. At-the-money has a moneyness of one. In-the-money warrants have a moneyness of greater than one and out-the-money warrants have a moneyness of less than one.
Out-the-money option	A call option is out-the-money where the asset price is less than the strike price. A put option is in-the-money where the asset price is greater than the strike price.
Put option	An option to sell an asset for a certain price by a certain date.

Strike price See Exercise Price.

Volatility skew A term used to describe the volatility smile when it is non-symmetrical.

Volatility smile The variation of implied volatility with strike price.

11 APPENDIX II

Since $N(x)$ is the cumulative probability that a variable with a standardized normal distribution will be less than x, $N'(x)$ is the probability density function for a standardized normal distribution, that is,

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$N'(d_1) = N'(d_2 + \sigma\sqrt{T-t})$$

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{d_2^2}{2} - \sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]$$

$$N'(d_1) = N'(d_2)\exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]$$

Because

$$d_2 = \frac{\ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

It follows that

$$\exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] = \frac{Ke^{-r(T-t)}}{S}$$

As a result,

(1) $$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Furthermore,

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_1 = \frac{\ln S - \ln K + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Hence

(2) $$\frac{\partial d_1}{\partial K} = -\frac{1}{K\sigma\sqrt{T}}$$

Similarly,

$$d_2 = \frac{\ln S - \ln K + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T}}$$

and

(3) $$\frac{\partial d_2}{\partial K} = -\frac{1}{K\sigma\sqrt{T}}$$

Please note that $$\frac{\partial d_1}{\partial K} = \frac{\partial d_2}{\partial K}$$

From differentiating the Black-Scholes formula for a call price,

$$\frac{\partial C}{\partial K} = -e^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial K} + SN'(d_1)\frac{\partial d_1}{\partial K}$$

From Equations (1), (2) and (3) it follows that

(4) $$\frac{\partial C}{\partial K} = -e^{-r(T-t)}N(d_2)$$

For second derivative,

$$\frac{\partial^2 C}{\partial K^2} = -e^{-rT} N'(d_2) \frac{\partial d_2}{\partial K}$$

From Equation (3),

(5) $$\frac{\partial^2 C}{\partial K^2} = e^{-rT} N'(d_2) \frac{1}{K\sigma\sqrt{T}}$$

12 APPENDIX III

Since $N(x)$ is the cumulative probability that a variable with a standardized normal distribution will be less than x, $N'(x)$ is the probability density function for a standardized normal distribution, that is,

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}}$$

$$N'(d_1) = N'(d_2 + \sigma\sqrt{T-t})$$

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{d_2^2}{2} - \sigma d_2 \sqrt{T-t} - \frac{1}{2}\sigma^2 (T-t) \right]$$

$$N'(d_1) = N'(d_2) \exp\left[-\sigma d_2 \sqrt{T-t} - \frac{1}{2}\sigma^2 (T-t) \right]$$

Because

$$d_2 = \frac{\ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

It follows that

$$\exp\left[-\sigma d_2 \sqrt{T-t} - \frac{1}{2}\sigma^2 (T-t) \right] = \frac{Ke^{-r(T-t)}}{S}$$

As a result,

(1) $$SN'(d_1) = Ke^{-r(T-t)} N'(d_2)$$

Furthermore,

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_1 = \frac{\ln S - \ln K + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Hence

(2) $$\frac{\partial d_1}{\partial \sigma} = \frac{\sqrt{T-t}}{2}$$

Similarly,

$$d_2 = \frac{\ln S - \ln K + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T}}$$

and

(3) $$\frac{\partial d_2}{\partial \sigma} = -\frac{\sqrt{T-t}}{2}$$

From differentiating the Black-Scholes formula for a call price,

$$\frac{\partial C}{\partial \sigma} = SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma}$$

From Equations (1), (2) and (3), it follows that

(4) $$\frac{\partial C}{\partial \sigma} = Ke^{-r(T-t)}N'(d_2)\sqrt{T-t}$$

13 APPENDIX IV

Table: Extract of the chosen data set from I-Net Bridge

Code	Date	Expiry	Strike	Closing price
TOPDB	2002/09/13	2003/01/09	8400	1035
TOPDB	2002/09/13	2003/01/09	8800	525
TOPDB	2002/09/13	2003/01/09	10000	700
TOPDB	2002/09/13	2003/01/09	10400	913
TOPDB	2002/09/24	2003/01/09	8400	659
TOPDB	2002/09/24	2003/01/09	8800	421
TOPDB	2002/09/24	2003/01/09	10000	802
TOPDB	2002/09/24	2003/01/09	10400	1183
TOPDB	2002/10/07	2003/01/09	8400	532
TOPDB	2002/10/07	2003/01/09	8800	71
TOPDB	2002/10/07	2003/01/09	10000	1001
TOPDB	2002/10/07	2003/01/09	10400	1385
TOPDB	2002/10/18	2003/01/09	8000	830
TOPDB	2002/10/18	2003/01/09	8400	775
TOPDB	2002/10/18	2003/01/09	10000	1047
TOPDB	2002/10/18	2003/01/09	10400	703
TOPDB	2002/10/29	2003/01/09	8000	634
TOPDB	2002/10/29	2003/01/09	8400	966
TOPDB	2002/10/29	2003/01/09	10000	1334
TOPDB	2002/10/29	2003/01/09	10400	561
TOPDB	2002/11/14	2003/01/09	8000	850
TOPDB	2002/11/14	2003/01/09	8400	788
TOPDB	2002/11/14	2003/01/09	10000	1220
TOPDB	2002/11/14	2003/01/09	10400	653

14 APPENDIX V

Table: Calculated implied volatilities

Date	Strike	Implied volatility
2002/09/13	8400	0.2664156482
2002/09/13	8800	0.5447460768
2002/09/13	10000	0.2501416087
2002/09/13	10400	0.2392549811
2002/09/24	8400	0.3128813642
2002/09/24	10000	0.2724200781
2002/09/24	10400	0.2528515154
2002/10/07	8400	0.3582479001
2002/10/07	10000	0.2793203369
2002/10/07	10400	0.2614241094
2002/10/18	8000	0.2364410417
2002/10/18	8400	0.3689859940
2002/10/18	10000	0.2919747331
2002/10/18	10400	0.2766391678
2002/10/29	8000	0.2577535604
2002/10/29	8400	0.4374960766
2002/10/29	10000	0.2969770042
2002/10/29	10400	0.3233177813
2002/11/14	8000	0.2714676718
2002/11/14	8400	0.5526603098
2002/11/14	10000	0.3520561403
2002/11/14	10400	0.3233177813

15 APPENDIX VI

Table: Examples of extracted Iteration

	ATOPDB	BTOPDB	CTOPDB
σ(2,1)	0.2664156	0.5447461	0.2501416
σ(2,2)	0.373055	0.6039454	0.3850589
σ(2,3)	0.5001444	0.6382623	0.4416744
σ(2,4)	0.549068	0.6619161	0.5096987
σ(2,5)	0.5840058	0.6793032	0.5155336
σ(2,6)	0.5906009	0.6926336	0.5206106
σ(2,7)	0.5957856	0.7036053	0.5250823
σ(2,8)	0.5998659	0.7129136	0.5289546
σ(2,9)	0.6033513	0.7206046	0.5324292
σ(2,10)	0.6062569	0.7272602	0.5356847
σ(2,11)	0.6087586	0.7328984	0.5385805
σ(2,12)	0.6110618	0.7379915	0.5412683
σ(2,13)	0.6129739	0.7425332	0.5437562
σ(2,14)	0.6146685	0.7465194	0.5460579
σ(2,15)	0.6163066	0.7499544	0.5482877
σ(2,16)	0.6177253	0.7532285	0.5503381
σ(2,17)	0.6190794	0.7562954	0.5522142
σ(2,18)	0.6202105	0.7588414	0.554032
σ(2,19)	0.6212629	0.7611822	0.5557742
σ(2,20)	0.6222336	0.7636426	0.5573666
σ(2,21)	0.62312	0.7659162	0.5588927
σ(2,22)	0.6239237	0.7679727	0.5603602
σ(2,23)	0.6247685	0.7698317	0.5617663
σ(2,24)	0.6255331	0.7714872	0.5631139
σ(2,25)	0.6262082	0.7729325	0.5644007
σ(2,26)	0.626791	0.7744479	0.5656291
σ(2,27)	0.6274065	0.7757463	0.5668011
σ(2,28)	0.6279308	0.7770928	0.5679187
σ(2,29)	0.6284829	0.7782516	0.5689881
σ(2,30)	0.6289411	0.7794595	0.5700694
σ(2,31)	0.6294217	0.7804497	0.5710958
σ(2,32)	0.6298101	0.7814788	0.5720732