Fractoids

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Abstract

This dissertation will examine the properties of the algebraic structure herein named the *fractoid*. This structure will be defined and its properties closely examined. In this dissertation we will first provide context for this structure, by looking at both category theory and universal algebra. We present some first basic concepts of category theory and consider F-algebras (= algebras over an endofunctor F). We will then look at algebras in the sense of universal algebra. We will examine F-algebras and their properties in this context and compare them to the definitions used in some of the standard textbooks in universal algebra. Once *fractoids* are defined and examined, they will be compared to a similar existing algebraic structure, the *wheel*.

Declaration

I know the meaning of plagiarism and declare that all of the work in the dissertation, save for that which is properly acknowledged, is my own.

Khadija Brey
11 February 2022

I would like to dedicate this dissertation to my parents for their support and patience.

Acknowledgements

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Chapter 1

Introduction

The algebraic structures herein named *fractoids* are defined as follows: a system $(A, 0, +, -, 1, \cdot)$ is said to be a fractoid if $(A, 0, +)$ and $(A, 1, \cdot)$ are commutative monoids, $-$ is a unary operation on A, and the following identities hold:

$$0 \cdot 0 = 0,$$
$$x(y + 0z) = xy + 0z,$$
$$x(y + z)(1 + 0x) = xy + xz,$$
$$x + (-x) = 0xx$$

(here $xy = x \cdot y$, etc.). We also consider *semifractoids*, where the operation $-$ and the last of these identities are omitted. In the special case that suggested the study of such structures we have:

- $A = R \times S$, where R is a commutative ring with 1 and S is a submonoid of the multiplicative monoid of R;

- 0 and 1 of A are the pairs $(0, 1)$ and $(1, 1)$, respectively;

- $(r_1, s_1) + (r_2, s_2) = (r_1s_2 + r_2s_1, s_1s_2)$, $(r_1, s_1)(r_2, s_2) = (r_1r_2, s_1s_2)$, and $-(r, s) = (-r, s)$.

The way the operations on $R \times S$ are defined shows the connection with the fraction ring construction. Moreover, we prove our fractoid axioms are what we have on $R \times S$ to make its suitable quotient $RS^{-1} = R \times S$ a commutative ring, hence suggesting the term "fractoid". This also has the fraction semiring version with semifractoids playing the role of the fractoid. The second purpose of this dissertation is to compare the notions we introduced with the notion of *wheels* introduced by J. Carlström [2].

All this is done in the last chapter of the dissertation, while the purpose of the other chapters is to give a brief overview of general algebra that provides an environment for considering such unusual algebraic structures. It is indeed only an "environment": we neither use nor formulate any of its deep results, but only describe the applicable notions, specifically, subalgebras, products, quotient algebras, and free algebras. In fact, we begin with categories and algebras over endofunctors, and then, considering algebras in the sense of universal algebra, make comparisons with the presentations given in important textbooks of universal/general algebra, of P. M. Cohn [3], N. Jacobson [5], G. Birkhoff [1], and G. Grätzer [4], usually repeating fragments of those books and adding short comments. We hope such comparisons will make this dissertation more reader-friendly for those who studied universal algebra from one of those books.

To be precise, the dissertation has five chapters (including this one) organized as follows:

Chapter 2 is devoted to very basic, purely categorical notions: we define categories giving a couple of their examples, then define monomorphisms and epimorphisms (split or not), isomorphisms, initial objects, and products. All these notions are defined e.g. in S. Mac Lane's book [6]. Not all of them will be explicitly used later, but, for example:

- In considering subalgebras and quotient algebras in Chapters 3 and 4, the reader will see the monomorphisms and epimorphisms, respectively, behind them.
- It will then become clear, although not mentioned, that the reason for the free algebra on a set X (say, in a variety; see the description of G. Birkhoff's definition of a free algebra in Section 4.6) being generated by (the image of) X is in Theorem 2.3.5, which says that every monomorphism into an initial object is an isomorphism.
- In proving Theorem 2.3.5, although it is easy, it is useful to notice Proposition 2.2.9, which says that if a morphism is a monomorphism and a split epimorphism at the same time, then it is an isomorphism.

Chapter 3 is devoted to algebras over endofunctors (= F-*algebras*). It is useful to see that subalgebras, products of algebras, and quotient algebras can be considered already at this very general and simply described level.

Chapter 4 is devoted, as the title of its first section shows, to algebras in the sense of universal algebra (= Ω-*algebras*). It consists of the following:

1. Sections 4.1–4.4. We consider Ω-algebras, and their homomorphisms, subalgebras, products, and quotient algebras as a special case of F-algebras and the corresponding notions introduced for them in Chapter 3. We also copy the corresponding definitions from the four above-mentioned textbooks, and briefly explain that those definitions are all essentially equivalent to the definitions we use.

2. Section 4.5. We do the same with so-called word algebras, except that we have no counterpart of this notion in the theory of F-algebras, and so we simply compare its presentations in (three of) the four textbooks.

3. Section 4.6. For an arbitrary Ω, a class $\mathcal{A}$ of Ω-algebras, and a set X, we define the free Ω-algebra on X with respect to $\mathcal{A}$, and prove the existence of such algebras under the condition that $\mathcal{A}$ is (replete and) closed under subalgebras and products, which is Theorem 4.6.4. Then we return to the above-mentioned textbooks again, and, in particular, refer to results there that are the same as our Theorem 4.6.4.

4. Section 4.7. We define the notion of varieties of Ω-algebras, formulate (without proving) Birkhoff's Theorem that characterizes varieties (Theorem 4.7.2), and consider one example of a variety.

Chapter 5 collects our (definitions of and) original results on semifractoids and fractoids. It consists of the following (the numbers we use here correspond to the numbers of the sections):

1. Given a commutative (semi)ring R (with 1) and a submonoid S of its multiplicative monoid, we consider operations on $R \times S$ that induce the (semi)ring structure on the fraction semiring $RS^{-1} = R \times S/\!\sim$ (with suitable $\sim$), and we describe properties of these operations that indeed make $R \times S/\!\sim$ a (semi)ring.

2. We use the properties obtained above to define semifractoids and fractoids, and compare these structures with *wheels* in the sense of [2].

3. We discuss the independence of the semifractoid axioms, which includes a question that is still open.

4. We show that the initial semifractoid is the semiring of natural numbers, while the initial fractoid is the ring of integers.

Chapter 2

Preliminaries of Category Theory

2.1 Categories

Definition 2.1.1. A category is a system $\mathbf{C} = (\mathbf{C}_0, \mathbf{C}_1, \mathbf{d}, \mathbf{c}, \mathbf{e}, \mathbf{m})$, in which:

(a) $\mathbf{C}_0$ is a class, whose elements are called objects in $\mathbf{C}$;

(b) $\mathbf{C}_1$ is a class, whose elements are called morphisms in $\mathbf{C}$;

(c) $\mathbf{d}$ and $\mathbf{c}$ are maps from $\mathbf{C}_1$ to $\mathbf{C}_0$, called domain and codomain respectively; when $\mathbf{d}(f) = A$ and $\mathbf{c}(f) = B$, we write $f : A \to B$, or $f \in hom_{\mathbf{C}}(A, B)$ (or $f \in hom(A, B)$ if $\mathbf{C}$ is clear), and say that f is a morphism from A to B;

(d) $\mathbf{e}$ is a map from $\mathbf{C}_0$ to $\mathbf{C}_1$, called the identity and written as $\mathbf{e}(A) = 1_A$; it has $\mathbf{d}(1_A) = A = \mathbf{c}(1_A)$;

(e) $\mathbf{m}$ is a map from the set $\{(f, g) \in \mathbf{C}_1 \times \mathbf{C}_1 | \mathbf{d}(f) = \mathbf{c}(g)\}$ to $\mathbf{C}_1$, called composition, written as $\mathbf{m}(f, g) = fg$, and satisfying the equalities

  (i) $\mathbf{d}(fg) = \mathbf{d}(g)$,

  (ii) $\mathbf{c}(fg) = \mathbf{c}(f)$,

  (iii) $f 1_{\mathbf{d}(f)} = f = 1_{\mathbf{c}(f)} f$,

  (iv) $f(gh) = (fg)h$,

whenever $\mathbf{d}(f) = \mathbf{c}(g)$ and $\mathbf{d}(g) = \mathbf{c}(h)$.

The following are examples of categories:

Example 2.1.2. The category $\mathbf{C} = \mathbf{Set}$ of sets has that:

(a) $\mathbf{C}_0$ is the class of all sets;

(b) $\mathbf{C}_1$ is the class of all maps between sets;

(c) $f : A \to B$ if f is a map from set A to set B in the usual sense;

(d) for a set S, the identity morphism $1_S : S \to S$ is defined as the identity map of S in the usual sense;

(e) fg is the composite of the maps f and g in the usual sense.

Example 2.1.3. The category $\mathbf{C} = \mathbf{Mon}$ of monoids has that:

(a) $\mathbf{C}_0$ is the class of monoids;

(b) $\mathbf{C}_1$ is the class of all homomorphisms between monoids;

(c) $f : M \to N$ if f is a homomorphism from monoid M to monoid N in the usual sense;

(d) for a monoid M, the identity morphism $1_M : M \to M$ is defined as the identity map of M (which is a homomorphism) in the usual sense;

(e) fg is the composite of the homomorphisms f and g in the usual sense.

Example 2.1.4. A monoid M determines a one-object category $\mathbf{C}$ as follows:

(a) $\mathbf{C}_0$ is any one-element set, say $\{M\}$;

(b) $\mathbf{C}_1$ is M;

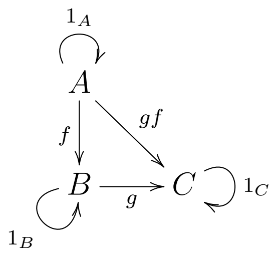
(c) since $\mathbf{C}_0$ has only one element, the maps $\mathbf{d}$ and $\mathbf{c}$ are uniquely determined;

(d) $1_M : M \to M$ is the identity element of M;

(e) fg in $\mathbf{C}$ is defined as fg in M.

Categories can be visualised by depicting them as diagrams. The objects are depicted as nodes and morphisms are depicted as arrows.

Example 2.1.5. Let the diagram



represent the category **C**.

This gives the following:

(a) The objects of **C** are A, B, and C.

(b) The morphisms of **C** are $f : A \to B$, $g : B \to C$ and $gf : A \to C$, as well as the three identity morphisms $1_A : A \to A$, $1_B : B \to B$ and $1_C : C \to C$.

2.2 Monomorphisms, Epimorphisms, and Isomorphisms

Definition 2.2.1. A morphism $f : A \to B$ is called a monomorphism, or monic, if for any C, $g_1, g_2 : C \to A$ and $fg_1 = fg_2$ implies $g_1 = g_2$.

This can be depicted by the diagram

$$C \underset{g_2}{\overset{g_1}{\rightrightarrows}} A \xrightarrow{f} B$$

where f will be a monomorphism if $fg_1 = fg_2 \Rightarrow g_1 = g_2$.

Example 2.2.2. In **Set**, the category of sets, a morphism is a monomorphism if and only if it is an injective map.

Definition 2.2.3. A morphism $f : A \to B$ is called an epimorphism, or epic, if for any C, $h_1, h_2 : B \to C$ and $h_1 f = h_2 f$ implies $h_1 = h_2$.

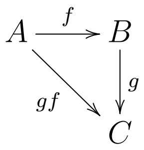
This can be depicted by the diagram

$$A \xrightarrow{f} B \underset{h_2}{\overset{h_1}{\rightrightarrows}} C$$

where f will be an epimorphism if $h_1 f = h_2 f \Rightarrow h_1 = h_2$.

Example 2.2.4. In **Set**, the category of sets, a morphism is an epimorphism if and only if it is a surjective map.

If $f : A \to B$, $g : B \to C$, as depicted in the diagram



then the following can be easily deduced from the definitions:

(1) If f and g are monic, then gf is monic

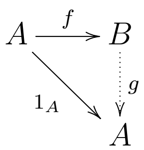
(2) If f and g are epic, then gf is epic

(3) If gf is monic, then f is monic

(4) If gf is epic, then g is epic

Definition 2.2.5. A morphism $f : A \to B$ is called a split monomorphism if there exists a $g : B \to A$ such that $gf = 1_A$.

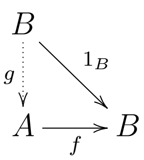
This can be depicted by the diagram



where f will be a split monomorphism if g exists and the diagram commutes.

Definition 2.2.6. A morphism $f : A \to B$ is called a split epimorphism if there exists a $g : B \to A$ such that $fg = 1_B$.

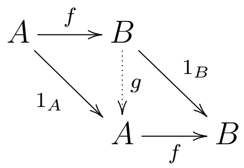
This can be depicted by the diagram



where f will be a split epimorphism if g exists and the diagram commutes.

Definition 2.2.7. A morphism $f : A \to B$ is called an isomorphism if there exists $g : B \to A$ such that $fg = 1_B$ and $gf = 1_A$.

This can be depicted by the diagram



where f will be an isomorphism if g exists and the diagram commutes.

Since g is determined by f, it can be called f^{-1}. The morphism f^{-1} is also an isomorphism and $(f^{-1})^{-1} = f$.

If f and h are isomorphisms and fh is defined, then fh is an isomorphism and $(fh)^{-1} = h^{-1}f^{-1}$.

Definition 2.2.8. Objects A and B in a given category are said to be isomorphic, and we write $A \approx B$, if there exists an isomorphism $A \to B$.

As follows from the remarks above, $\approx$ is an equivalence relation on the class of objects of any given category.

Proposition 2.2.9. If a morphism is a monomorphism and a split epimorphism at the same time, then it is an isomorphism.

Proof. Suppose $f : A \to B$ is a monomorphism and a split epimorphism at the same time. Since f is a split epimorphism, $fg = 1_B$ for some $g : B \to A$. We also have

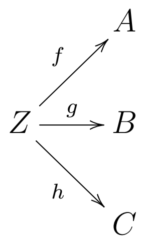
$$\begin{aligned} f(gf) &= (fg)f \\ &= 1_B f \\ &= f \\ &= f1_A \end{aligned}$$

and, since f is a monomorphism, this gives $gf = 1_A$. That is, f is an isomorphism. □

2.3 Initial Objects

Definition 2.3.1. An object Z in a category $\mathbf{C}$ is said to be initial, if for every object A in $\mathbf{C}$ there exists a unique morphism from Z to A.

This can be depicted by the diagram



where Z is an initial object if $f : Z \to A$, $g : Z \to B$ and $h : Z \to C$ are all unique morphisms.

Example 2.3.2. The empty set is an initial object in **Set**, the category of sets.

Proposition 2.3.3. Every morphism $f : A \to Z$ in a category $\mathbf{C}$, in which Z is an initial object in $\mathbf{C}$, is a split epimorphism.

Proof. The existence of $g : Z \to A$ and the equality $gf = 1_Z$ follow from the initiality of Z. □

Corollary 2.3.4. All initial objects (in the same category) are isomorphic to each other.

From Propositions 2.2.9 and 2.3.3 we obtain:

Theorem 2.3.5. *Every monomorphism $f : A \to Z$ in a category $\mathbf{C}$, in which Z is an initial object in $\mathbf{C}$, is an isomorphism.*

2.4 Products

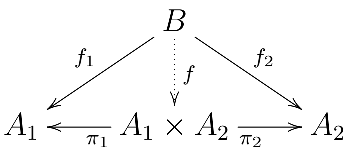
Definition 2.4.1. Let A_1 and A_2 be two objects of the category $\mathbf{C}$. The product of A_1 and A_2 in $\mathbf{C}$ consists of

(a) the object $A_1 \times A_2$

(b) the morphisms $\pi_1 : A_1 \times A_2 \to A_1$ and $\pi_2 : A_1 \times A_2 \to A_2$

and satifies the following property:

Given an object B and two morphisms $f_1 : B \to A_1$ and $f_2 : B \to A_2$, there exists a unique morphism $f : B \to A_1 \times A_2$ such that the diagram



commutes.

Definition 2.4.1 is for the product of two objects. It can be generalised for the product of a family of objects:

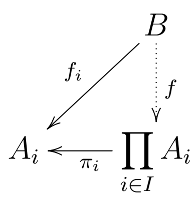
Definition 2.4.2. Let $(A_i)_{i \in I}$ be a family of objects of the category $\mathbf{C}$. The product of the A_i in $\mathbf{C}$ consists of

(a) the object $\prod\limits_{i \in I} A_i$

(b) the morphisms $\pi_i : \prod\limits_{i \in I} A_i \to A_i$ for every $i \in I$

and satisfies the following property:

Given an object B and the morphisms $f_i : B \to A_i$, there exists a unique morphism $f : B \to \prod\limits_{i \in I} A_i$ such that for each $i \in I$, the diagram



commutes.

For example, if $\mathbf{C}$ is the category of sets, then $\prod_{i \in I} A_i$ becomes (up to isomorphism) the ordinary Cartesian product of sets with π_i's defined by $\pi_i((a_j)_{j \in I}) = a_i$.

2.5 Functors and Subcategories

Definition 2.5.1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A functor F from $\mathbf{C}$ to $\mathbf{D}$ consists of maps $\mathbf{C}_0 \to \mathbf{D}_0$ and $\mathbf{C}_1 \to \mathbf{D}_1$, both written as F, and such that:

(a) $f : A \to B$ in $\mathbf{C}$ gives $F(f) : F(A) \to F(B)$ in $\mathbf{D}$;

(b) $F(1_A) = 1_{F(A)}$ for every $A \in \mathbf{C}_0$;

(c) $F(gf) = F(g)F(f)$ whenever gf is defined in $\mathbf{C}$.

Definition 2.5.2. Let $\mathbf{C}$ and $\mathbf{D}$ be categories with $\mathbf{C}_0 \subseteq \mathbf{D}_0$ and $\mathbf{C}_1 \subseteq \mathbf{D}_1$. The category $\mathbf{C}$ is said to be a subcategory of $\mathbf{D}$, if the inclusion maps $\mathbf{C}_0 \to \mathbf{D}_0$ and $\mathbf{C}_1 \to \mathbf{D}_1$ form a functor from $\mathbf{C}$ to $\mathbf{D}$. Furthermore, one then says that $\mathbf{C}$ is a full subcategory of $\mathbf{D}$, if every morphism $f : A \to B$ in $\mathbf{D}$ with A and B in $\mathbf{C}$ belongs to $\mathbf{C}$.

Chapter 3

Algebras over Endofunctors of Set

3.1 The Category of F-algebras

Definition 3.1.1. Given a functor $F : \mathbf{Set} \to \mathbf{Set}$, an F-algebra is a pair (A, α), where $A \in \mathbf{Set}_0$ and $h : F(A) \to A$.

Definition 3.1.2. A homomorphism $f : (A, \alpha) \to (B, \beta)$ between F-algebras, also called an F-algebra homomorphism, is a morphism such that the diagram

$$\begin{CD} F(A) @>{F(f)}>> F(B) \\ @V{\alpha}VV @VV{\beta}V \\ A @>>{f}> B \end{CD}$$

commutes.

F-algebras and their homomorphisms form a category, which we will call the category of F-algebras and denote by $\mathsf{Alg}(F)$.

Theorem 3.1.3. *An F-algebra homomorphism $f : (A, \alpha) \to (B, \beta)$ is an isomorphism if and only if $f : A \to B$ is a bijective map.*

Proof. $f : (A, \alpha) \to (B, \beta)$ is a homomorphism of F-algebras, which means that f is a map from A to B with $fa = bF(f)$. If f is a bijection, it has its inverse map g. We have to show that f is an isomorphism, that is, there exists a homomorphism $(B, \beta) \to (A, \alpha)$ of F-algebras, which is the inverse of f. That inverse must be g. So, we only have to prove that, if f is a homomorphism of F-algebras, then so is its inverse map g. That is, we have to prove that if g is the inverse map of f, then the equality $fa = bF(f)$ implies the equality $gb = aF(g)$.

Composing both sides of $fa = bF(f)$ with g on the left, we obtain

$$\begin{aligned} gfa &= gbF(f) \\ \implies a &= gbF(f) \end{aligned}$$

since $gf = 1$. Then, composing both sides with $F(g)$ on the right, we obtain

$$\begin{aligned} aF(g) &= gbF(f)F(g) \\ \implies aF(g) &= gb \end{aligned}$$

since $F(f)F(g) = F(fg) = F(1) = 1$. This gives the desired equality $aF(g) = gb$. □

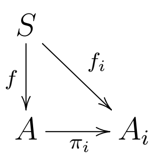
Example 3.1.4. If F is defined by $F(X) = X \times X$, then an F-algebra is a set equipped with a single binary operation, and so $\mathsf{Alg}(F)$ is the Category of Magmas.

3.2 Products of F-algebras

Theorem 3.2.1. *The product $\prod_{i\in I}(A_i, \alpha_i)$ of a family $((A_i, \alpha_i))_{i\in I}$ of F-algebras in the category $\mathsf{Alg}(F)$ can be described as the F-algebra $(\prod_{i\in I} A_i, \alpha)$, where $\prod_{i\in I} A_i$ is the usual Cartesian product of sets $A_i (i \in I)$ and α is the unique map $F(\prod_{i\in I} A_i) \to \prod_{i\in I} A_i$ with $\pi_i\alpha = \alpha_i F(\pi_i)$ for each $i \in I$; here π_i denotes the product projection $\prod_{i\in I} A_i \to A_i$.*

Proof. Given a family $(A_i)_{i\in I}$ of F-algebras (using simplified notation, we write A_i instead of (A_i, α_i), where $\alpha_i : F(A_i) \to A_i$ is the algebra structure on A_i), we indeed construct the set $A = \prod_{i\in I} A_i = \{(a_i)_{i\in I} | a_i \in A_i\}$, the usual Cartesian product of sets. This product has the usual universal property saying that for every set S and every family $(f_i : S \to A_i)_{i\in I}$ of maps, there exists a unique map $f : S \to A$ with $\pi_i f = f_i$ for all $i \in I$ (where $\pi_i : A \to A_i (i \in I)$ are the product projections).

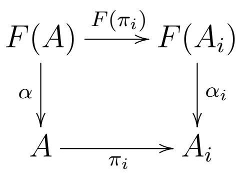
Namely, the universal property states that for all $i \in I$, there exists a unique f such that the diagram



commutes.

This universal property is easily proved. Since $\pi_i f = f_i$ means that, for every $s \in S$, the i-th component of $f(s)$ is $f_i(s)$, we can see that to say $\forall_{i\in I} \pi_i f = f_i$ is to say nothing but that f is defined by $f(s) = (f_i(s))_{i\in I}$, which determines f uniquely. We can apply this universal property to the family $(\alpha_i F(\pi_i) : F(A) \to A_i)_{i\in I}$, to obtain a map $\alpha : F(A) \to A$, which makes A an F-algebra. Explicitly, $\alpha : F(A) \to A$ is defined by $\alpha(t) = (\alpha_i F(\pi_i)(t))_{i\in I}$.

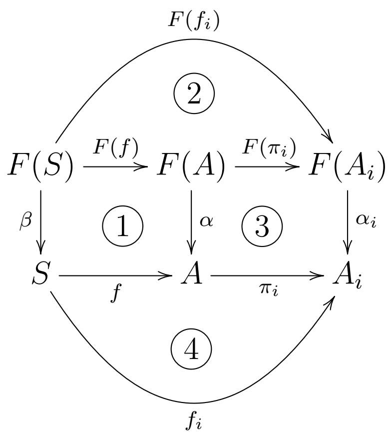
Next, we need to show that each projection $\pi_i : A \to A_i$ is an F-algebra homomorphism from (A, α) to (A_i, α_i). Namely, we need to show that for all $i \in I$, the diagram



commutes, i.e. $\pi_i \alpha = \alpha_i F(\pi_i)$.

The diagram commutes by definition of α, so each projection is an F-algebra homomorphism.

Finally, we need to show that the F-algebra $A = (A, \alpha)$ equipped with all these projections satisfies the universal property required to call it the product of the family $(A_i)_{i\in I}$ in the category of F-algebras. Namely, we need to show that the square (1) of the diagram



commutes.

To prove that ① commutes, we have to prove $\pi_i \alpha F(f) = \pi_i f \beta$.
We note that:

(a) ④ commutes by definition of f

(b) ③ commutes by definition of α

(c) ② commutes since ④ does

Therefore

$$\begin{aligned}
\pi_i \alpha F(f) &= \alpha_i F(\pi_i) F(f) && \text{(since ③ commutes)} \\
&= \alpha_i F(f_i) && \text{(since ② commutes)} \\
&= f_i \beta && \text{(since } f_i \text{ is a homomorphism)} \\
&= \pi_i f \beta && \text{(since ④ commutes)}
\end{aligned}$$

□

3.3 F-subalgebras

Definition 3.3.1. Let (A, α) be an F-algebra. Let $S \subseteq A$. Then S is an F-subalgebra if there exists $k : F(S) \to S$ such that the diagram, where $\iota : S \to A$ is an inclusion,

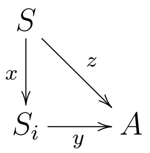
$$\begin{array}{ccc}
F(S) & \xrightarrow{F(\iota)} & F(A) \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \alpha} \\
S & \xrightarrow[\iota]{} & A
\end{array}$$

commutes.

Theorem 3.3.2. *The intersection of any family of F-subalgebras of a given F-algebra A is also an F-subalgebra of A.*

Proof. Given a family $(S_i)_{i \in I}$ of F-subalgebras of a given F-algebra A (using simplified notation, we write S_i instead of (S_i, k_i), where $k_i : F(S_i) \to S_i$ is the algebra structure on S_i), we construct $S = \bigcap_{i \in I} S_i = \{(s_i)_{i \in I} | \forall S_i \in (S_i)_{i \in I}, s_i \in S_i\}$, the intersection of sets.

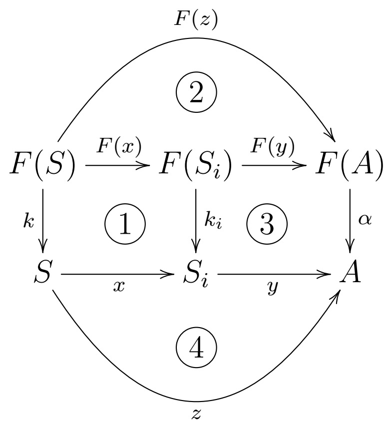
For each S_i, we have that the diagram, where x, y and z represent inclusion,



commutes.

We have that $S \subseteq A$ and now need to show that S is indeed an F-subalgebra of A. To do this, it is sufficient to show that S is an F-subalgebra of S_i, for any i, since each S_i is an F-subalgebra of A.

Let (S_i, k_i) be an F-subalgebra of A. Then S is an F-subalgebra of S_i if there exists $k : F(S) \to S$ such that the square (1) of the diagram, where x, y and z represent inclusion,



commutes.

To prove that (1) commutes, we have to prove $yk_iF(x) = yxk$.
We note that:

(a) (4) commutes as established above

(b) (3) commutes since S_i is an F-subalgebra of A

(c) (2) commutes since (4) does

Therefore

$$\begin{aligned} yk_iF(x) &= \alpha F(y)F(x) && \text{(since \textcircled{3} commutes)} \\ &= \alpha F(z) && \text{(since \textcircled{2} commutes)} \\ &= zk && \text{(since } z \text{ is a homomorphism)} \\ &= yxk && \text{(since \textcircled{4} commutes)} \end{aligned}$$

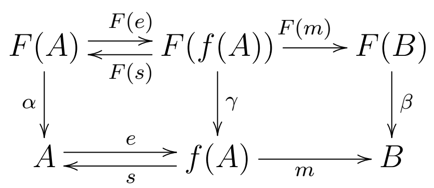
For each S_i, we have that S is an F-subalgebra of S_i. So S is an F-subalgebra of A.

□

This allows us to define, for any subset S of A, the F-subalgebra of A generated by S as the smallest F-subalgebra of A containing S, since this F-subalgebra is the same as the intersection of all F-subalgebras of A containing S.

Theorem 3.3.3. *The image of any F-algebra homomorphism $f : (A, \alpha) \to (B, \beta)$ is an F-subalgebra of B.*

Proof. Consider the diagram



in which:

- $f = me$ is the canonical image factorization of f, that is $e(a) = f(a)$ and $m(b) = b$ for all $a \in A$ and $b \in f(A)$;
- s is a map satisfying $es = 1_{f(A)}$, which does exist by the Axiom of Choice;
- $\gamma = e\alpha F(s)$.

Having in mind that $F(e)F(s) = F(es) = F(1_{f(A)}) = 1_{F(f(A))}$, we calculate $m\gamma = me\alpha F(s) = f\alpha F(s) = \beta F(f)F(s) = \beta F(m)F(e)F(s) = \beta F(m)$. That is, $m\gamma = \beta F(m)$, as desired. □

Note that in the notation above, $e : (A, \alpha) \to (f(A), \gamma)$ is a homomorphism of F-algebras. Indeed, $m\gamma F(e) = \beta F(m)F(e) = \beta F(f) = f\alpha = me\alpha$, which gives $\gamma F(e) = ea$ since m is injective.

3.4 Congruences and Quotient F-algebras

Definition 3.4.1. Let (A, α) be an F-algebra. Let $E \subseteq A \times A$ be an equivalence relation. Then E is said to be a congruence if there exists $\alpha' : F(A/E) \to A/E$ such that the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\pi)} & F(A/E) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ A & \xrightarrow[\pi]{} & A/E \end{array}$$

where π is the canonical surjection, commutes.

We have $\pi : A \to A/E$, where $A/E = (A/E, \alpha')$, with α' as in Definition 3.4.1, is what is called the quotient algebra. For each $a \in A$, we have that π sends a to its equivalence class in A/E. So $a \mapsto [a]_E$ for each $a \in A$.

Theorem 3.4.2. *Every F-algebra homomorphism $f : A \to B$ determines a congruence E_f on A and a canonical injective F-algebra homomorphism $\bar{f} : A/E_f \to B$; specifically*

$$E_f = \{(a, a') \in A \times A | f(a) = f(a')\}$$

and $\bar{f}$ is defined by $\bar{f}([a]_{E_f}) = f(a)$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \pi}\downarrow & \overset{e}{\searrow} & \uparrow{\scriptstyle m} \\ A/E_f & \xrightarrow[\bar{f}]{} & f(A) \end{array}$$

in which π is as above, e and m are as in the proof of Theorem 3.3.3, and $\bar{f}$ is the bijection defined by $\bar{f}([a]_{E_f}) = f(a)$. It immediately gives us the commutative diagram

$$\begin{array}{ccccccc} F(A) & \xrightarrow{F(\pi)} & F(A/E_f) & \xrightarrow{F(\bar{f})} & F(f(A)) & \xrightarrow{F(m)} & F(B) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \bar{f}^{-1}\gamma F(\bar{f})} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\pi]{} & A/E_f & \xrightarrow[\bar{f}]{} & f(A) & \xrightarrow[m]{} & B \end{array}$$

in which γ is as in the proof of Theorem 3.3.3 and each square represents a homomorphism of F-algebras, giving, in particular, the desired F-algebra structure $\bar{f}^{-1}\gamma F(\bar{f}) : F(A/E_f) \to A/E_f$ on A/E_f. After that all we need is to observe that $E_f = E_\pi$. □

$\bar{f}$ is an isomorphism if and only if f is surjective, since f is a monomorphism (and so is injective).

Theorem 3.4.3. *The intersection $\bigcap_{i\in I} E_i$ of a family $(E_i)_{i\in I}$ of congruences on a given F-algebra A is also a congruence on A.*

Proof. Just note that $\bigcap_{i\in I} E_i = E_f$, where $f : A \to \prod_{i\in I} A/E_i$ is induced by the family of canonical homomorphisms $A \to A/E_i$ $(i \in I)$. □

This allows us to define, for any F-algebra A and any subset R of $A \times A$, the congruence on A generated by R.

Chapter 4

Universal Algebra

4.1 Algebras in the sense of Universal Algebra

Given a functor $F : \mathbf{Set} \to \mathbf{Set}$, we have defined an F-algebra as a pair (A, α), where A is a set and $\alpha : F(A) \to A$ is an arbitrary map. Traditionally, universal algebra studies the special case where F is determined by a sequence $\Omega = (\Omega_0, \Omega_1, \Omega_2, \Omega_3, \dots)$ of disjoint sets via

$$F(X) = \Omega_0 \cup (\Omega_1 \times X) \cup (\Omega_2 \times X^2) \cup \dots;$$

given a map $u : X \to Y$, the induced map $F(u) : F(X) \to F(Y)$ is defined, of course, by

$$F(u)((\omega, x_1, \dots, x_n)) = (\omega, u(x_1), \dots, u(x_n)),$$

for all $n = 1, 2, 3, \dots$, all $\omega \in \Omega_n$, and all $x_1, \dots, x_n \in X$. In this case F-algebras are called Ω-algebras, and, for such an Ω-algebra $A = (A, \alpha)$, one simply writes

$$\alpha((\omega, a_1, \dots, a_n)) = \omega(a_1, \dots, a_n),$$

and just $\alpha(\omega) = \omega$ when $n = 0$ (unless it might create confusion).

Let us compare this with the definitions used in some standard textbooks in universal algebra:

P. M. Cohn [3, Chapter II, Section 2] introduces the following two definitions:

Definition 4.1.1. [3, Chapter II, Section 2, Definition (1)] An operator domain is a set Ω with a mapping $a : \Omega \to N$; the elements of Ω are called operators, and if $\omega \in \Omega$, then $a(\omega)$ is called the arity of ω. If $a(\omega) = n$ we also say that ω is n-ary, and we write

$$\Omega(n) = \{\omega \in \Omega | a(\omega) = n\}$$

Definition 4.1.2. [3, Chapter II, Section 2, Definition (2)] Let A be a set and Ω an operator domain; then an Ω-algebra structure on A is a family of mappings

$$\Omega(n) \to A^{A^n} (n \in N)$$

Thus with each $\omega \in \Omega(n)$ an n-ary operation on A is associated. The set A with this structure is also called an Ω-algebra and is sometimes written A_Ω to emphasize its dependence on Ω.
The underlying set A is also called the carrier of A_Ω.
Given an Ω-algebra A and $\omega \in \Omega(n)$, then ω applied to an n-tuple $(a_1, \dots, a_n)$ from A gives an element of A which we write as $a_1 a_2 \cdots a_n \omega$.
In the case $n = 0$ this merely states that ω is an element of A; thus a 0-ary operator picks out a certain distinguished element in the algebra.

We observe:

(a) To give such an operator domain is obviously the same as to give our sequence $\Omega = (\Omega_0, \Omega_1, \Omega_2, \dots)$, with Ω_n being the same as what P. M. Cohn denotes by $\Omega(n)$.

(b) To give a map from $\Omega_n = \Omega(n)$ to A^{A^n} is the same as to give a map from $\Omega_n \times A^n$ to A. Specifically, maps $\varphi : \Omega_n \to A^{A^n}$ and $\psi : \Omega_n \times A^n \to A$, corresponding to each other, determine each other via $\varphi(\omega)(a_1, \dots, a_n) = \psi((\omega, a_1, \dots, a_n))$.

(c) Hence, to give a family of maps $\Omega(n) \to A^{A^n} (n \in N)$ as [3] (see above), is the same as to give a map

$$F(A) = \Omega_0 \cup (\Omega_1 \times A) \cup (\Omega_2 \times A^2) \cup \cdots \to A,$$

and what is denoted by $a_1 a_2 \cdots a_n \omega$ in [3] corresponds to what we denoted by $\omega(a_1, \dots, a_n)$.

(d) In particular, our "$\alpha(\omega) = \omega$ when $n = 0$" fully agrees with "a 0-ary operator picks out a certain distinguished element in the algebra" said in [3].

N. Jacobson writes in [5, Chapter 2, Section 2.1]:

> We are now ready to define the concept of a "general" algebra or Ω-algebra. Roughly speaking, this is just a non-vacuous set A equipped with a set Ω of n-ary products, $n = 0, 1, 2, \ldots$. In order to compare different algebras - more precisely, to define homomorphisms - it is useful to regard Ω as having an existence apart from A and to let the elements of Ω determine products in different sets $A, B, \ldots$ in such a way that the products determined by a given $\omega \in \Omega$ in $A, B, \ldots$ all have the same arity, that is, are n-ary with a fixed n. We therefore begin with a set Ω together with a given decomposition of Ω as a disjoint union of subsets $\Omega(n), n = 0, 1, 2, \ldots$. The elements of $\Omega(n)$ are called n-ary product (or operator) symbols. For the given Ω and decomposition $\Omega = \cup\Omega(n)$ we introduce the following
>
> **Definition 4.1.3.** [5, Chapter 2, Section 2.1, Definition 2.1] An Ω-algebra is a non-vacuous set A together with a map of Ω into products on A such that if $\omega \in \Omega(n)$, then the corresponding product is n-ary on A. The underlying set A is called the carrier of the algebra, and we shall usually denote the algebra by the same symbol as its carrier.
>
> If ω is nullary, so $\omega \in \Omega(0)$, the corresponding distinguished element of A, is denoted as ω_A or, if there is no danger of confusion, as ω (e.g., the element 1 in every group G rather than 1_G). If $n \geq 1$ and $\omega \in \Omega(n)$, the corresponding product in A is
>
> $$(a_1, a_2, \ldots, a_n) \mapsto \omega(a_1, a_2, \ldots, a_n),$$
>
> $a_i \in A$. We shall now abbreviate the right-hand side as
>
> $$\omega a_1 a_2 \cdots a_n$$

This concept of Ω-algebra is again the same, except that the underlying sets of Ω-algebras are required to be non-empty (which is automatic only if Ω_0 is non-empty). Note that:

(a) Although N. Jacobson's $\Omega(n)$ is the same as P. M. Cohn's $\Omega(n)$ and our Ω_n, N. Jacobson prefers to present Ω itself neither as a set equipped with a map to the set of natural numbers (as P. M. Cohn does), nor as a sequence of sets (as we do), but as a disjoint union of all $\Omega(n)$ ($n = 0, 1, 2, \ldots,$ which is the same as $n \in N$ in P. M. Cohn's language). Of course it is still hidden that he in fact deals either with a map or, equivalently, with a sequence of disjoint sets, since, given $\omega \in \Omega$, we must know which $\Omega(n)$ it belongs to.

(b) N. Jacobson informally uses the elements of $\Omega(0)$ as elements of any given algebra, exactly as we do.

(c) N. Jacobson's $\omega(a_1, a_2, \ldots, a_n)$ is the same as ours, but he also abbreviates it as $\omega a_1 a_2 \cdots a_n$, unlike P. M. Cohn, who abbreviates the same as $a_1 a_2 \cdots a_n \omega$.

G. Birkhoff writes in [1, Chapter VI, Section 1]:

> "Universal algebra" provides general theorems about algebras with single-valued, universally defined, finitary operations. This concept may be defined as follows
>
> **Definition 4.1.4.** [1, Chapter VI, Section 1, Definition] An algebra A is a pair $[S, F]$, where S is a non-empty set of elements, and F is a specified set of operations f_α, each mapping a power $S^{n(\alpha)}$ of S into S, for some appropriate non-negative finite integer $n(\alpha)$.
>
> Otherwise stated, each operation f_α assigns to every $n(\alpha)$-ple $(x_1, \ldots, x_{n(\alpha)})$ of elements of S, a value $f_\alpha(x_1, \ldots, x_{n(\alpha)})$ in S, the result of performing the operation f_α on the sequence $x_1, \ldots, x_{n(\alpha)}$. If $n(\alpha) = 1$, the operation f_α is called unary; if $n(\alpha) = 2$, it is called binary; if $n(\alpha) = 3$, it is called ternary, etc. When $n(\alpha) = 0$, the operation f_α is called nullary; it selects a fixed element of S (e.g., a group identity).

G. Birkhoff writes in [1, Chapter VI, Section 3]:

> We shall fix the set F of operations f_α and the corresponding integers $n(\alpha)$, and shall consider algebras $A = [S, F]$ with variable S. We shall call such algebras similar.

We note the following:

(a) G. Birkhoff uses S, F and f_α, where we use A, Ω and ω. He does not use the term Ω-algebra.

(b) Algebras being similar by G. Birkhoff's definition are equivalent to algebras being Ω-algebras for the same Ω by our definition.

(c) The maps are equivalent, where f_α mapping $S^{n(\alpha)}$ into S is the same as to give a map from $\Omega_n \times A^n$ to A for $n(\alpha) = n$ (when the arity of α is n).

(d) G. Birkhoff's $f_\alpha(x_1, \ldots, x_{n(\alpha)})$ is the same as our $\omega(a_1, \ldots, a_n)$ for $n(\alpha) = n$.

(e) For a nullary operation, G. Birkhoff selects a fixed element of S, as we do.

G. Grätzer [4, Chapter 1, Section 7] writes the following:

> A type of algebras τ is a sequence $\langle n_0, n_1, \cdots, n_\gamma, \cdots \rangle$ of nonnegative integers, $\gamma < o(\tau)$, where $o(\tau)$ is an ordinal, called the order of τ. For every $\gamma < o(\tau)$ we have a symbol $\mathbf{f}_\gamma$ of an n_γ-ary operation.
> An algebra $\mathfrak{A} = \langle A; F \rangle$ of type τ is a pair, where A is a nonvoid set (the base set of $\mathfrak{A}$), and for every $\gamma < o(\tau)$, we realise $\mathbf{f}_\gamma$ as an n_γ-ary operation on $A : (\mathbf{f}_\gamma)_\mathfrak{A}$, and $F = \langle (\mathbf{f}_0)_\mathfrak{A}, (\mathbf{f}_1)_\mathfrak{A}, \cdots, (\mathbf{f}_\gamma)_\mathfrak{A}, \cdots \rangle$.
> $(\mathbf{f}_\gamma)_\mathfrak{A}$ is the realization of $\mathbf{f}_\gamma$ and if there is no danger of confusion, we will write f_γ for $(\mathbf{f}_\gamma)_\mathfrak{A}$ and $F = \langle f_0, \cdots, f_\gamma, \cdots \rangle$. Thus if $\mathfrak{A}$ and $\mathfrak{B}$ are both algebras of the same type τ, f_γ will denote the operation on A as well as on B. In general, there is no danger of confusion since, if we write $f_\gamma(a_0, \cdots, a_{n_\gamma - 1}), a_0, \cdots, a_{n_\gamma-1} \in A$, then f_γ obviously means an operation on A.
> Let us remark that this usage is generally accepted in algebra, e.g., $+$ is used to denote an operation in every abelian group.
> If $F = \langle f_0, \cdots, f_{n-1} \rangle$ we will write $\langle A; f_0, \cdots, f_{n-1} \rangle$ for $\langle A; F \rangle$.
> The class of all algebras of type τ will be denoted by $K(\tau)$; it will be called a similarity class of algebras (also called a species of algebras).

We note the following:

(a) Like G. Birkhoff, G. Grätzer does not use the term Ω-algebra.

(b) Algebras belonging to the same similarity class $K(\tau)$ (or being of the same type τ) by G. Grätzer's definition are equivalent to algebras being Ω-algebras for the same Ω by our definition.

(c) G. Grätzer's $f_\gamma(a_0, \cdots, a_{n_\gamma - 1})$ is the same as our $\omega(a_1, \ldots, a_n)$ for $n_\gamma = n$.

N. Jacobson [5, Chapter 2, Section 2.3] defines homomorphisms between Ω-algebras as follows:

> **Definition 4.1.5.** [5, Chapter 2, Section 2.3, Definition 2.2] If A and B are Ω-algebras, a homomorphism from A to B is a map f of A into B such that for any $\omega \in \Omega(n), n = 0, 1, 2, \cdots$, and every $(a_1, ..., a_n) \in A^{(n)}$ we have
>
> $$f(\omega a_1 a_2 \cdots a_n) = \omega f(a_1) f(a_2) \cdots f(a_n)$$
>
> In the case $n = 0$ it is understood that if ω_A is the element of A corresponding to ω then $f(\omega_A) = \omega_B$.

We note the following:

(a) N. Jacobson's $\omega f(a_1) f(a_2) \cdots f(a_n)$ is the same as our $\omega(f(a_1) f(a_2) \cdots f(a_n))$.

P. M. Cohn [3, Chapter II, Section 2] defines homomorphisms between Ω-algebras as follows:

> Given Ω-algebras A and B, a mapping $f : A \to B$, and $\omega \in \Omega(n)$, we say that f is compatible with ω, if for all $a_1, \cdots, a_n \in A$,
>
> $$(a_1 f) \cdots (a_n f)\omega = (a_1 \cdots a_n \omega) f$$
>
> If f is compatible with each $\omega \in \Omega$, then f is said to be a homomorphism or homomorphic mapping from A to B.

We note the following:

(a) P. M. Cohn's $(a_1 f) \cdots (a_n f)\omega$ is equivalent to our $\omega(f(a_1) f(a_2) \cdots f(a_n))$ and N. Jacobson's $\omega f(a_1) f(a_2) \cdots f(a_n)$.

(b) P. M. Cohn writes the mapping f at the end, while N. Jacobson writes it at the beginning instead, as we do.

G. Grätzer [4, Chapter 1, Section 7] defines homomorphisms between two algebras belonging to the same similarity class as follows:

> Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras belonging to the same similarity class $K(\tau)$. A mapping $\varphi : A \to B$ such that
>
> $$f_\gamma(a_0, \cdots, a_{n_\gamma - 1})\varphi = f_\gamma(a_0\varphi, \cdots, a_{n_\gamma - 1}\varphi)$$

for all $\gamma < o(\tau)$ is called a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.

Let us note that if a nullary operation f_γ picks out a from $\mathfrak{A}$ and b from $\mathfrak{B}$, then $a\varphi = b$, more precisely, $((\mathbf{f}_\gamma)_{\mathfrak{A}})\varphi = (\mathbf{f}_\gamma)_{\mathfrak{B}}$.

We note the following:

(a) G. Grätzer's $f_\gamma(a_0\varphi, \cdots, a_{n_\gamma-1}\varphi)$ for algebras belonging to the same similarity class is equivalent to our $\omega(f(a_1)f(a_2)\cdots f(a_n))$, N. Jacobson's $\omega f(a_1)f(a_2)\cdots f(a_n)$ and P. M. Cohn's $(a_1f)\cdots(a_nf)\omega$ for Ω-algebras.

(b) Like P. M. Cohn, G. Grätzer writes the mapping at the end, while N. Jacobson writes the mapping in the beginning instead, as we do.

G. Birkhoff [1, Chapter VI, Section 3] defines morphisms between similar algebras as follows:

Definition 4.1.6. [1, Chapter VI, Section 3, Definition] Let $A = [S, F]$ and $B = [T, F]$ be similar abstract algebras. A function $\phi : S \to T$ is a morphism of A into B if and only if, for all $f_\alpha \in F$ and $x_i \in S$,

$$f_\alpha(x_1\phi, \cdots, x_{n(\alpha)}\phi) = (f_\alpha(x_1, \cdots, x_{n(\alpha)}))\phi$$

We note the following:

(a) G. Birkhoff defines morphisms and not homomorphisms as N. Jacobson, P. M. Cohn and G. Grätzer do.

(b) G. Birkhoff's $f_\alpha(x_1\phi, \cdots, x_{n(\alpha)}\phi)$ for similar algebras is equivalent to our $\omega(f(a_1)f(a_2)\cdots f(a_n))$, N. Jacobson's $\omega f(a_1)f(a_2)\cdots f(a_n)$ and P. M. Cohn's $(a_1f)\cdots(a_nf)\omega$ for Ω-algebras, as well as G. Grätzer's $f_\gamma(a_0\varphi, \cdots, a_{n_\gamma-1}\varphi)$ for algebras of the same similarity class.

(c) Like P. M. Cohn and G. Grätzer, G. Birkhoff writes the function at the end, while N. Jacobson writes the mapping in the beginning instead, as we do.

4.2 Subalgebras

Let A be an Ω-algebra, and let F be determined Ω as in Section 4.1. A subalgebra (or an Ω-subalgebra) of A is the same as an F-subalgebra of A. This gives:

Definition 4.2.1. A subalgebra of an Ω-algebra A is a subset X of A with $x_1, \ldots, x_n \in X \Rightarrow \omega(x_1, \ldots, x_n) \in X$ for all $n = 0, 1, 2, \ldots$, all $\omega \in \Omega_n$, and all $x_1, \ldots, x_n \in A$.

N. Jacobson [5, Chapter 2, Section 2.2] defines a subalgebra as follows:

> Let A be an arbitrary Ω-algebra. A non-vacuous subset B of A is called a subalgebra of A if for any $\omega \in \Omega(n)$ and $(b_1, b_2, \ldots, b_n), b_i \in B$, $\omega b_1 b_2 \cdots b_n \in B$. In particular, if $n = 0$, then B contains the distinguished element ω_A associated with ω.

Note that N. Jacobson's definition agrees with Definition 4.2.1, except that he requires subalgebras to be non-empty.

P. M. Cohn [3, Chapter II, Section 2] defines a subalgebra as follows:

> Given Ω-algebras A_Ω and B_Ω, we say that B is a subalgebra of A if the carrier of B is a subset of the carrier of A; and if $\omega \in \Omega$ defines operations ω_A, ω_B in A and B respectively, then B admits ω_A and
>
> $$\omega_A | B = \omega_B$$
>
> for each $\omega \in \Omega$.
>
> Thus any subset of the carrier of A which admits each $\omega \in \Omega$ can be defined in just one way as an Ω-subalgebra of A.

Note that P. M. Cohn's definition agrees with Definition 4.2.1, except that he begins with a subset of A that already has an Ω-algebra structure. However, unlike N. Jacobson's requirement ("non-vacuous"="non-empty"), this distinction with Definition 4.2.1 is irrelevant: indeed, a subalgebra of A in the sense of Definition 4.2.1 immediately becomes an Ω-algebra itself with the structure induced from A.

G. Birkhoff [1, Chapter VI, Section 2] defines a subalgebra as follows:

> By a subalgebra of an abstract algebra $A = [S, F]$ is meant a (possibly void) subset T of S which is closed under the operations of F, or F-closed. That is, we require that if $f_\alpha \in F$ and $x_1, \cdots, x_{n(\alpha)} \in T$, then $f_\alpha(x_1, \cdots, x_{n(\alpha)}) \in T$. The couple $[T, F]$ is then also an abstract algebra.

G. Grätzer [4, Chapter 1, Section 7] defines subalgebras as follows:

> Let $\mathfrak{A}$ be an algebra of type τ and B a nonvoid subset of A. $\mathfrak{B} = \langle B; F\rangle$ is called a subalgebra of $\mathfrak{A}$ (and $\mathfrak{A}$ an extension of $\mathfrak{B}$) if and only if
>
> $$b_0, \cdots, b_{n_\gamma - 1} \in B$$
>
> implies that
>
> $$(f_\gamma)_{\mathfrak{A}}(b_0, \cdots, b_{n_\gamma-1}) = (f_\gamma)_{\mathfrak{B}}(b_0, \cdots, b_{n_\gamma-1}) \in B$$
>
> for all $\gamma < o(\tau)$, that is, if and only if B is closed under all the operations of $\mathfrak{A}$ and $(\mathbf{f}_\gamma)_{\mathfrak{B}}$ is the restriction of $(\mathbf{f}_\gamma)_{\mathfrak{A}}$ to B (or more precisely, to B^{n_γ}).

Considering again the difference between N. Jacobson's definition and Definition 4.2.1, we can add now that, out of the authors we refer to, only N. Jacobson and G. Grätzer require subalgebras to be non-empty, and otherwise all definitions are equivalent 'up to the language'.

4.3 Products

Taking the Ω-algebra case of Theorem 3.2.1 as the definition of products, easily gives us the following (well-known) definition:

Definition 4.3.1. Let $(A_i)_{i\in I}$ be a family of Ω-algebras. The (Cartesian) product

$$\prod_{i\in I} A_i$$

of the family $(A_i)_{i\in I}$ is defined as the product of the underlying sets, with the Ω-algebra structure defined component-wise, i.e. defined by $\omega((a_{1i})_{i\in I}, \dots, (a_{ni})_{i\in I}) = (\omega(a_{1i}, \dots, a_{ni}))_{i\in I}$. We will also write

$$\prod_{i=1}^{n} A_i = A_1 \times \cdots \times A_n$$

when the family is presented as a sequence $A_1, \ldots, A_n$, etc.

In particular, for the product $A \times B$ of two algebras A and B we have

$$\omega((a_1, b_1), \ldots, (a_n, b_n)) = (\omega(a_1, \ldots, a_n), \omega(b_1, \ldots, b_n)).$$

G. Birkhoff [1, Chapter VI, Section 5] defines direct products as follows:

> From any two similar abstract algebras, $A = [X, F]$ and $B = [Y, F]$ one can construct the direct product $A \times B = [X \times Y, F]$ from the elements $[x, y]$ of the Cartesian product $X \times Y$, by defining for any $f_\alpha \in F$ and for $n = n(\alpha)$:
>
> $$f_\alpha([x_1, y_1], \cdots, [x_n, y_n]) = [f_\alpha(x_1, \cdots, x_n), f_\alpha(y_1, \cdots, y_n)].$$
>
> . . . .
>
> More generally, let Γ be any set of similar algebras $A_\gamma = [S_\gamma, F]$; define the (unrestricted) direct product $\prod_\Gamma A_\gamma$ as the set of all functions $a : \gamma \to a(\gamma) \in A_\gamma$, with
>
> $$f_a(a_1, \cdots, a_{n(a)}) = b : \gamma \to f_a(a_1(\gamma), \cdots, a_{n(a)}(\gamma)) \in \prod_\Gamma A_\gamma.$$

We note the following:

(a) Up to the difference in terminology and notation, already described in Section 4.1, G. Birkhoff's definition of (direct) products matches Definition 4.3.1 of (Cartesian) products in the case of two algebras.

(b) In the general case, G. Birkhoff's definition is very brief, and, in particular, he does not explain that in fact one should consider not sets of algebras, but families of algebras. Indeed, considering only products of sets of algebras creates obvious problems, such as the impossibility of distinguishing between $A \times A$ and A (since $\{A, A\} = \{A\}$).

P. M. Cohn [3, Chapter II, Section 2] defines a direct product for any family $(A_\lambda)_{\lambda \in \Lambda}$ of Ω-algebras as follows:

> Let P be the Cartesian product of the A_λ regarded as sets, with projections $\epsilon_\lambda : P \to A_\lambda$. Then any element $a \in P$ is completely determined by its components $a\epsilon_\lambda$, and any choice of elements $a(\lambda) \in A_\lambda$ defines a unique

element a of P by $a\epsilon_\lambda = a(\lambda)(\lambda \in \Lambda)$. Therefore if $a_1, \cdots, a_n \in P$ and $\omega \in \Omega(n)$, we can define $a_1 \cdots a_n\omega$ by the equation

$$(a_1 \cdots a_n\omega)\epsilon_\lambda = (a_1\epsilon_\lambda) \cdots (a_n\epsilon_\lambda)\omega. \tag{4.1}$$

In this way an Ω-algebra structure is defined on P, and it is clear from the form of equation (4.1) that the projections are homomorphisms. The algebra so defined is called the direct product of the A_λ and is denoted by ΠA_λ.

Although P. M. Cohn's definition fully agrees with Definition 4.3.1 (up to notation), its formulation gives the impression that the author thinks of the system $(P, (\epsilon_\lambda : P \to A_\lambda)_{\lambda\in\Lambda})$ as the product defined, not set-theoretically, but by its universal property. Indeed, instead of recalling what the elements of P are, he only uses the fact that they are determined by their projections, which can be seen as an immediate consequence of the universal property.

G. Grätzer [4, Chapter 3, Section 19] defines direct products as follows:

Let $\mathfrak{A}_i = \langle A_i; F\rangle, i \in I$, be given algebras of type τ. Form the Cartesian product $\prod(A_i|i \in I)$ and define the operations f_γ on it as follows: if $p_0, \cdots, p_{n_\gamma-1} \in \prod(A_i|i \in I)$ and $\gamma < o(\tau)$, then

$$f_\gamma(p_0, \cdots, p_{n_\gamma-1})(i) = f_\gamma(p_0(i), \cdots, p_{n_\gamma-1}(i)).$$

The algebra $\langle\prod(A_i|i \in I); F\rangle = \prod(\mathfrak{A}_i|i \in I)$ is called the direct product of the algebras $\mathfrak{A}_i, i \in I$.

N. Jacobson [5, Chapter 2, Section 2.2] defines a product as follows:

Let $\{A_\alpha|\alpha \in I\}$ be a family of Ω-algebras indexed by a set I, that is, we have a map $\alpha \mapsto A_\alpha$ of I into $\{A_\alpha\}$. Moreover,we allow $A_\alpha = A_\beta$ for $\alpha \neq \beta$. In particular, we may have all of the $A_\alpha = A$. It is convenient to assume that all of the A_α are subsets of the same set A. We recall that the product set $\prod_I A_\alpha$ is the set of maps $a : \alpha \mapsto a_\alpha$ of I into A such that for every $\alpha, a_\alpha \in A_\alpha$. We now define an Ω-algebra structure on $\prod A_\alpha$ by defining the products component-wise. If $\omega \in \Omega(0)$, we define the corresponding element of $\prod A_\alpha$ to be the map $\alpha \mapsto \omega_{A_\alpha}$ where ω_{A_α} is the element of A_α singled out by ω. If $\omega \in \Omega(n)$ for $n \geq 1$ and $a^{(1)}, \cdots, a^{(n)} \in \prod A_\alpha$, then we define

$\omega a^{(1)} \cdots a^{(n)}$ to be the map

$$\alpha \mapsto \omega {a_\alpha}^{(1)} {a_\alpha}^{(2)} \cdots {a_\alpha}^{(n)},$$

which is evidently an element of $\prod A_\alpha$. If we do this for all ω, we obtain an Ω-algebra structure on $\prod A_\alpha$. $\prod A_\alpha$ endowed with this structure is called the product of the indexed family of Ω-algebras $\{A_\alpha | \alpha \in I\}$.

We note the following:

(a) Again, up to the difference in terminology and notation, all the definitions are the same. It is also interesting that all these authors use the term "Cartesian product" for sets and "direct product" for algebras. This should mean that they would not like to see sets as Ω-algebras with empty Ω, which one could also conclude from the fact that some of them require all algebras to be non-empty.

(b) As already mentioned, at some point G. Birkhoff ignores the difference between sets and families. The others don't, and N. Jacobson is especially careful: he even points out that the map $\alpha \to A_\alpha$ is not necessarily injective. Or, maybe, in saying "we allow $A_\alpha = A_\beta$ for $\alpha \neq \beta$" he does not see it as a map? Indeed, his symbol $\{A_\alpha | \alpha \in I\}$ is just a symbol of a set, not of a family.

(c) In connection to the previous remark, it is interesting that N. Jacobson wants to assume all members of a given family of algebras to be subsets of some set A. Of course, he could simply take A to be the (set-theoretic) union of those algebras, but he wants to emphasize that the union plays no role. What he does not say is that he could define an I-indexed family of algebras A_α as a map from I to the power set of A, whose values are equipped with Ω-algebra structures.

4.4 Quotient Algebras

As at the beginning of Section 4.2, let A be an Ω-algebra and let F be determined by Ω as in Section 4.1. We can use Definition 3.4.1 to define congruences on A, and we have:

Theorem 4.4.1. *An equivalence relation E on an Ω-algebra A is a congruence on A if and only if E is a subalgebra of $A \times A$.*

Proof. Each of the first three of the following four conditions is obviously equivalent to the next one:

(i) E is a congruence on A;

(ii) for all $n = 0, 1, 2, \ldots$, all $\omega \in \Omega_n$, and all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

$$(a_1, b_1), \ldots, (a_n, b_n) \in E \Rightarrow (\omega(a_1, \ldots, a_n), \omega(b_1, \ldots, b_n)) \in E;$$

(iii) for all $n = 0, 1, 2, \ldots$, all $\omega \in \Omega_n$, and all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

$$(a_1, b_1), \ldots, (a_n, b_n) \in E \Rightarrow \omega((a_1, b_1), \ldots, (a_n, b_n)) \in E;$$

(iv) E is a subalgebra of $A \times A$.

□

P. M. Cohn [3, Chapter II, Section 3] defines a congruence as follows:

> **Definition.** [3, Chapter II, Section 3, Definition] An equivalence on an Ω-algebra A which is at the same time a subalgebra of A^2 is called a congruence on A.

N. Jacobson [5, Chapter 2, Section 2.3] defines a congruence as follows:

> **Definition.** [5, Chapter 2, Section 2.3, Definition 2.3] A congruence on an Ω-algebra A is an equivalence relation on A, which is a subalgebra of $A \times A$.
>
> If Φ is an equivalence relation on the Ω-algebra A and $\omega \in \Omega(0)$, then $(\omega_A, \omega_A) \in \Phi$ and this is the element $\omega_{A \times A}$ corresponding to the nullary symbol ω in the algebra $A \times A$.

G. Birkhoff [1, Chapter VI, Section 4] defines congruences as follows, calling it the Substitution Property:

> We recall that an "equivalence relation" is a binary relation $x\theta y$, also written $x \equiv y \pmod{\theta}$, which is reflexive, symmetric and transitive; A/θ will denote the set of all equivalence classes on A.

> **Definition.** [1, Chapter VI, Section 4, Definition] A congruence relation on an algebra $A = [S, F]$ is an equivalence relation θ of A such that, for all $f_\alpha \in F, x_i \equiv y_i \pmod{\theta}$, $i = 1, \cdots, n(\alpha)$, implies $f_\alpha(x_1, \cdots, x_{n(\alpha)}) \equiv f_\alpha(y_1, \cdots, y_{n(\alpha)}) \pmod{\theta}$. (Substitution Property)

G. Grätzer [4, Chapter 1, Section 7] defines a congruence as follows:

> Let $\mathfrak{A}$ be an algebra and Θ a binary relation defined on A. Θ is called a congruence relation if it is an equivalence relation satisfying the substitution property (SP):
>
> (SP) If $\gamma < o(\tau)$, $a_i \equiv b_i(\Theta)$, $a_i, b_i \in A, 0 \leq i < n_\gamma$, then
>
> $$f_\gamma(a_0, \cdots, a_{n_\gamma - 1}) \equiv f_\gamma(b_0, \cdots, b_{n_\gamma - 1})(\Theta)$$

P. M. Cohn's and N. Jacobson's definitions are the same as what we get from Theorem 4.4.1. G. Birkhoff's and G. Grätzer's definitions are also equivalent to those, although instead of saying that E is a subalgebra of $A \times A$ (in our notation) they mention the "substitution property".

Definition 3.4.1 and Theorem 4.4.1 lead to the following theorem and definition of quotient algebras:

Theorem and Definition 4.4.2. Let $A = (A, v)$ be an Ω-algebra and E an equivalence relation on A. The following conditions are equivalent:

(a) E is a congruence on A;

(b) there exists an Ω-algebra structure w on A/E such that the natural map $A \to A/E$ is a homomorphism from (A, v) to $(A/E, w)$;

(c) there exists a unique Ω-algebra structure w on A/E such that the natural map $A \to A/E$ is a homomorphism from (A, v) to $(A/E, w)$.

If these equivalent conditions hold, we say that A/E equipped with the w above is a quotient algebra of A, and that w is the induced structure on A/E.

In other words, $A/E = (A/E, w)$ is a quotient algebra of $A = (A, v)$ if and only if the natural map $A \to A/E$ is a homomorphism from (A, v) to $(A/E, w)$.

P. M. Cohn [3, Chapter II, Section 3] defines a quotient algebra in a theorem as follows:

> **Theorem.** [3, Chapter II, Section 3, Theorem 3.5] Let A be an Ω-algebra and q a congruence on A. Then there exists a unique Ω-algebra structure on the quotient set A/q such that the natural mapping $A \to A/q$ is a homomorphism.
>
> We denote the resulting algebra again by A/q and call it the quotient algebra of A by q, with the natural homomorphism $A \to A/q$.

G. Grätzer [4, Chapter 1, Section 7] defines a quotient algebra as follows:

> If we are given an algebra $\mathfrak{A}$ and a congruence relation Θ on $\mathfrak{A}$, we can construct a new algebra called the quotient algebra as follows: the new algebra is defined on the quotient set
>
> $$A/\Theta = \{[a]\Theta | a \in A\}$$
>
> with operations defined as
>
> $$f_\gamma([a_0]\Theta, \cdots, [a_{n_\gamma - 1}]\Theta) = [f_\gamma(a_0, \cdots, a_{n_\gamma - 1})]\Theta$$
>
> The new algebra is denoted by
>
> $$\mathfrak{A}/\Theta = \langle A/\Theta; F \rangle$$
>
> Now consider the mapping
>
> $$\varphi_\Theta : a \to [a]\Theta$$
>
> which is the natural mapping of A onto the quotient set A/Θ.
>
> **Lemma.** [4, Chapter 1, Section 7, Lemma 2] φ_Θ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\Theta$.
>
> That is, every quotient algebra is a homomorphic image of the algebra.

N. Jacobson [5, Chapter 2, Section 2.3] provides no formal definition for a quotient algebra, but constructs a quotient algebra as follows:

Let A/Φ be the quotient set of A determined by the congruence relation Φ and let $\overline{a}$ be the element of A/Φ corresponding to the element a of A. Since $a_i \Phi a'_i$, $1 \leq i \leq n$, implies $\omega a_i \cdots a_n \Phi \omega a'_1 \cdots a'_n$, we see that we have a well-defined map of $(A/\Phi)^{(n)}$ into A/Φ such that

$$(\overline{a}_1, ..., \overline{a}_n) \mapsto \overline{\omega a_1 \cdots a_n}$$

We can use this to define

$$\omega \overline{a}_1 \cdots \overline{a}_n = \overline{\omega a_1 \cdots a_n} \tag{4.2}$$

and we do this for every $\omega \in \Omega(n), n \geq 1$. Also for $\omega \in \Omega(0)$ we define

$$\omega_{A/\Phi} = \overline{\omega_A} \tag{4.3}$$

In this way we endow A/Φ with an Ω-algebra structure. We call A/Φ with this structure the quotient algebra of A relative to the congruence Φ. In terms of the natural map $v : a \mapsto \overline{a}$ of A into A/Φ, equations (4.2) and (4.3) read $\omega v a_1 \cdots v a_n = v \omega a_1 \cdots a_n$, $\omega_{A/\Phi} = v \omega_A$. Evidently this means that the natural map v is a homomorphism of A into A/Φ.

G. Birkhoff [1, Chapter VI, Section 4] defines a quotient algebra in a theorem as follows:

Theorem. [1, Chapter VI, Section 4, Theorem 5] Let θ be a congruence relation on an abstract algebra $A = [S, F]$, and let $x \mapsto P_\theta(x)$ be the mapping carrying each $x \in S$ into its equivalence class in S/θ. Then the operations f_α on S/θ defined by the formula

$$f_\alpha(P_\theta(x_1), \cdots, P_\theta(x_{n(\alpha)})) = P_\theta(f_\alpha(x_1, \cdots, x_{n(\alpha)}))$$

define an algebra $B = [S/\theta, F]$ similar to A. Moreover the mapping $x \mapsto P_\theta(x)$ is an epimorphism from A onto B.

All the definitions agree again, but note the following diversity of notations: The equivalence class $[a]_E$ of an element a under the given equivalence relation E could be written as a^E, $[a]E$, $\bar{a}$, and $P_E(a)$ according to P. M. Cohn, G. Grätzer, N. Jacobson, and G. Birkhoff, respectively.

4.5 Word Algebras

P. M. Cohn [3, Chapter III, Section 2] defines free word algebras as follows:

> We shall now apply the results of the preceding section to the category (Ω). Clearly, (Ω) is subordinate to St, by the functor which associates with each algebra its carrier. Thus we have a representation of St in (Ω), and our object is to show that this representation has a universal functor. Such a functor associates with any set X an Ω-algebra which we shall call the Ω-word algebra on X and denote by $W_\Omega(X)$. It is of some interest to have a constructive existence proof of this algebra; we shall therefore begin with the construction of $W_\Omega(X)$ and verify its universal property later on.
> Let Ω be any operator domain and X any set, and define an Ω-algebra $W(\Omega; X)$, the algebra of Ω-rows in X, as follows: by an Ω-row in X we understand a finite sequence (i.e. an n-tuple for $n \geqslant 1$) of elements of the disjoint sum $\Omega \sqcup X$. On the set $W(\Omega; X)$ of all Ω-rows in X we define an Ω-algebra structure by juxtaposition; thus, if $\omega \in \Omega(n)$ and $a_i \in W(\Omega; X)$ $(i = 1, \cdots, n)$, say
>
> $$a_i = (a_{i1}, \cdots, a_{ik_1}) \qquad\qquad (a_{ij} \in \Omega \sqcup X),$$
>
> then
>
> $$a_1 \cdots a_n\omega = (a_{11}, \cdots, a_{1k_1}, a_{21}, \cdots, a_{nk_n}, \omega). \tag{4.4}$$
>
> When X is disjoint from Ω, we may replace the disjoint sum by the ordinary union, and if we identify Ω-rows consisting of a single term with the corresponding element of $\Omega \cup X$, we can regard Ω and X as subsets of $W(\Omega; X)$; then both sides of (4.4) may be denoted by
>
> $$a_{11}, \cdots, a_{1k_1}, a_{21}, \cdots, a_{nk_n}, \omega.$$
>
> For simplicity of notation we often assume that X is disjoint from Ω; this is no loss of generality, as will soon become clear.
>
> **Definition.** [3, Chapter III, Section 2, Definition] The subalgebra of $W(\Omega; X)$ generated by X is called the Ω-word algebra on X and is denoted by $W_\Omega(X)$. Its elements are called Ω-words in X, and X is called its alphabet.

The first point to notice is that $W_\Omega(X)$ is determined essentially by the cardinal of the set X.

Proposition 4.5.1. [3, Chapter III, Section 2, Proposition 2.1] If X, Y are any sets, then the Ω-word algebras on X and Y are isomorphic:

$$W_\Omega(X) \cong W_\Omega(Y),$$

if and only if X is equipotent to Y.

. . . .

We remark that the isomorphism between $W_\Omega(X)$ and $W_\Omega(Y)$ obtained in the first part of the proof is uniquely determined by θ. This follows from a quite general lemma:

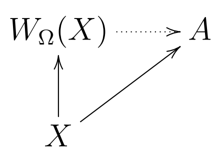
Lemma 4.5.2. [3, Chapter III, Section 2, Lemma 2.2] Let A be an Ω-algebra and X a generating set of A. Then any homomorphism of A into another Ω-algebra is completely determined by its restriction to X.

We note the following:

(a) P. M. Cohn uses St to represent **Set**, the category of sets.

(b) The underlying set of what P. M. Cohn calls the algebra of Ω-rows in X and denotes by $W(\Omega; X)$ is in fact the free monoid on $\Omega \sqcup X$ (we will not, however, consider free algebras in this section), and his $a_1 \dots a_n\omega$ is in fact the product $a_1 \dots a_n\omega$ in that monoid.

(c) P. M. Cohn's remark on the "universal functor" can be rephrased as the universal property of $W_\Omega(X)$ illustrated as



It says that every map from X to any Ω-algebra A uniquely extends to a homomorphism from $W_\Omega(X)$ to A. Note that this universal property immediately implies the "if part" of Proposition 4.5.1 and Lemma 4.5.2 above.

N. Jacobson [5, Chapter 2, Section 2.7] defines words in the alphabet Y as follows:

> Let $\Omega = \bigcup_{n=0}^{\infty} \Omega(n)$ be a given set of operator symbols where $\Omega(n)$ is the set of n-ary symbols, and let X be a non-vacuous set. Now let Y be the disjoint union of the sets Ω and X and form the disjoint union $W(\Omega, X)$ of the sets $Y^{(m)}$, $m \geq 1$, where $Y^{(m)}$ is the set of m-tuples $(y_1, y_2, \ldots, y_m)$, $y_i \in Y$. To simplify the writing, we write $y_1 y_2 \cdots y_m$ for $(y_1, y_2, \ldots, y_m)$. This suggests calling the elements of $W(\Omega, X)$ words in the alphabet Y. We define the degree of the word $w = y_1 y_2 \cdots y_m$ to be m.
>
> ....
>
> We now introduce the juxtaposition multiplication in $W(\Omega, X)$ by putting
>
> $$(y_1 \cdots y_m)(y'_1 \cdots y'_r) = y_1 \cdots y_m y'_1 \cdots y'_r.$$
>
> It is clear that this product is associative, so no parentheses are required to indicate products of more than two words.
> Next we shall use this associative product to define an Ω-algebra structure on the set $W(\Omega, X)$. If $\omega \in \Omega(0)$, we define an element of Y to be the element corresponding to ω in $W(\Omega, X)$. Now let $\omega \in \Omega(n), n \geq 1$, and let $w_1, w_2, \ldots, w_n \in W(\Omega, X)$. Then we define the action of ω on the n-tuple $(w_1, w_2, \ldots, w_n)$ as the word $\omega w_1 w_2 \cdots w_n$. Evidently, these definitions make $W(\Omega, X)$ an Ω-algebra. We now let $F(\Omega, X)$ be the subalgebra of the Ω-algebra $W(\Omega, X)$ generated by the subset X and we shall show that this is a free Ω-algebra generated by X in the sense that any map of X into an Ω-algebra A has a unique extension to a homomorphism of $F(\Omega, X)$ into A.

We note the following:

(a) We see that N. Jacobson's $W(\Omega, X)$ and $F(\Omega, X)$ are the same as P. M. Cohn's $W(\Omega; X)$ and $W_\Omega(X)$, respectively.

(b) N. Jacobson mentions the universal property of the word algebra more explicitly than P. M. Cohn does, although P. M. Cohn also does that later on in his presentation.

G. Birkhoff [1, Chapter VI, Section 6] defines free word algebras as follows:

> We now consider the "species" of all algebras having a given set F of operations. More precisely, we suppose given a fixed set of indices α and associated

nonnegative integers $n(\alpha)$. We then consider the class ("species") of all algebras $A = [S, F]$ which have for each given index α and $n(\alpha)$-ary operation $f_\alpha : S^{n(\alpha)} \to S$.
Any such "species" of algebras is closed under the formation of subalgebras and direct products, as defined earlier. Furthermore, morphisms are only defined between algebras of the same species.
For each choice of F and cardinal number r, finite or infinite, we can construct a (free) word algebra $W_r(F) = [W_r, F]$ (sometimes called "primitive algebra") having r generators or letters x_i, as follows. (The set X of x_i is often called the alphabet of $W_r(F)$.)
Call each letter x_i an F-polynomial of rank 0. For any positive integer ρ, define an F-polynomial of rank ρ recursively, as any expression ("word") of the form $f_\alpha(u_1, \cdots, u_{n(\alpha)})$, where one $u_j = p_j(x_1, \cdots, x_r)$ is an F-polynomial of rank $\rho - 1$, and every u_j is an F-polynomial of rank $\leqq \rho - 1$. Equality in $W_r(F)$ is defined to mean the formal identity: $x_i = x_j$ means $i = j$, and

$$f_\alpha(u_1, \cdots, u_{n(\alpha)}) = f_\beta(v_1, \cdots, v_{n(\beta)})$$

means that $\alpha = \beta$ and $u_k \equiv v_k$ for all $k = 1, \cdots, n(\alpha) = n(\beta)$.

G. Birkhoff's $W_r(F)$ is the same as N. Jacobson's $F(\Omega, X)$ and P. M. Cohn's $W_\Omega(X)$, except that:

(a) The letters F and r in G. Birkhoff's notation play the roles of Ω and of the cardinality of X, respectively, making P. M. Cohn's above-mentioned Proposition 4.5.1 hidden here.

(b) G. Birkhoff does not consider the counterpart of N. Jacobson's $W(\Omega, X)$ (= P. M. Cohn's $W(\Omega; X)$), which forces him to be more careful in defining his $W_r(F)$, where he explicitly 'defines' the equality of the elements.

(c) G. Grätzer [4] also introduces a counterpart of $W_r(F)$ (in Section 8 of [4]), but his approach is quite different and we will not consider it here.

4.6 Free Algebras

Definition 4.6.1. Let $\mathcal{A}$ be a class of Ω-algebras, for a fixed Ω, and let X be an arbitrary set. The free Ω-algebra on X with respect to $\mathcal{A}$ is the initial object $\mathrm{F}_{\mathcal{A}}(X) = (\mathrm{F}_{\mathcal{A}}(X), \eta_X)$ in the category of pairs (A, α), where $A \in \mathcal{A}$ and α is a map from X to the underlying set of A.

We have to explain here:

(a) One might abbreviate $\mathrm{F}_{\mathcal{A}}(X)$ as $\mathrm{F}(X)$, but this symbol should not be confused with $F(X)$, where F is as defined in Chapter 3.

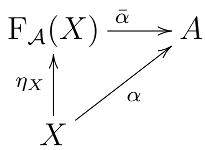
(b) Considering the category of pairs (A, α) above, we assume of course that a morphism $f : (A, \alpha) \to (B, \beta)$ is nothing but a homomorphism $f : A \to B$ with $f\alpha = \beta$.

(c) We say "the initial" rather than "an initial" having in mind that such an object is always unique up to an isomorphism, which we think of as "essentially unique".

(d) In Definition 4.6.1 we do not claim that such an object always exists; in fact it is easy to find examples of Ω and $\mathcal{A}$ for which it does not exist, even in the case $X = \emptyset$ (e.g. take $\mathcal{A} = \{A \times A\}$, where A is any Ω-algebra with more than one element, for any Ω).

(e) Writing $\mathrm{F}_{\mathcal{A}}(X) = (\mathrm{F}_{\mathcal{A}}(X), \eta_X)$ we mean that, formally speaking, $\mathrm{F}_{\mathcal{A}}(X)$ is not just an Ω-algebra, but it is equipped with map η_X from X to its underlying set. It is, however, often convenient to think of it as simply an Ω-algebra.

(f) It is convenient to rephrase the initiality replacement in Definition 4.6.1 as: for every $A \in \mathcal{A}$ and every map $\alpha : X \to A$, there exists a unique homomorphism $\bar{\alpha} : \mathrm{F}_{\mathcal{A}}(X) \to A$ making the diagram



commute.

Example 4.6.2. Let $\mathcal{A}$ be the class of all monoids. Then $\mathrm{F}(X)$ is called the free monoid on X and it can be constructed as the (disjoint) union of all $X^n (n = 0, 1, 2, \dots)$, with the monoid multiplication defined by

$$(x_1, \dots, x_m)(y_1, \dots, y_n) = (x_1, \dots, x_m, y_1, \dots, y_n).$$

As we mentioned in the previous section, P. M. Cohn's $W(\Omega; X)$, which is the same as what N. Jacobson denoted by $W(\Omega, X)$, is nothing but the free monoid on the disjoint union of Ω and X.

Example 4.6.3. (See Section 4.5 again.) Let $\mathcal{A}$ be the class of all Ω-algebras for a fixed Ω. Then $\mathrm{F}_{\mathcal{A}}(X)$ can be identified with what P. M. Cohn calls the word algebra and denotes by $W_\Omega(X)$ and what N. Jacobson denotes by $F(\Omega, X)$.

Theorem 4.6.4. *The free Ω-algebra $F_{\mathcal{A}}(X) = (F_{\mathcal{A}}(X), \eta_X)$ exists whenever $\mathcal{A}$ satisfies the following conditions:*

(a) $\mathcal{A}$ is replete and closed under subalgebras, that is, if $A \to B$ is an injective homomorphism of Ω-algebras with B in $\mathcal{A}$, then A belongs to $\mathcal{A}$;

(b) $\mathcal{A}$ is closed under products, that is, if $(A_i)_{i\in I}$ is a family in $\mathcal{A}$, then $\prod_{i\in I} A_i$ belongs to $\mathcal{A}$.

Proof. Let us write $\mathrm{F}(X)$ for the free Ω-algebra with respect to the class of all Ω-algebras (see Example 4.6.3). Let $\mathcal{E}$ be the set of all congruences on $\mathrm{F}(X)$ of the form

$$E_f = \{(u, v) \in \mathrm{F}(X) \times \mathrm{F}(X) | f(u) = f(v)\}$$

for some homomorphism $f : \mathrm{F}(X) \to A$ with $A \in \mathcal{A}$ (cf. Theorem 3.4.2). We put

$$\mathrm{F}_{\mathcal{A}}(X) = \mathrm{F}(x)/\cap\mathcal{E}$$

and define $\eta_X : X \to \mathrm{F}_{\mathcal{A}}(X)$ as the composite of the canonical maps

$$X \to \mathrm{F}(X) \to \mathrm{F}(X)/\cap\mathcal{E}.$$

To prove that $\mathrm{F}(X)/\cap\mathcal{E}$ belongs to $\mathcal{A}$, choose, for each $E \in \mathcal{E}$, a homomorphism $f : \mathrm{F}(X) \to A$ with $A \in \mathcal{A}$ and $E = E_f$, write $A = A_f$, denote the set of such homomorphisms by H, and consider the induced homomorphism

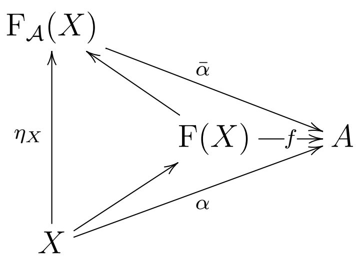
$$h : \mathrm{F}(X)/\cap\mathcal{E} \to \prod_{f\in H} A_f,$$

which is defined by $h([u]_{\cap\mathcal{E}}) = ([u]_{E_f})_{f\in H}$. We have

$$\begin{aligned} h([u]_{\cap\mathcal{E}}) = h([v]_{\cap\mathcal{E}}) &\Rightarrow \forall_{f\in H}[u]_{E_f} = [v]_{E_f} \\ &\Rightarrow \forall_{f\in H}(u,v) \in E_f \\ &\Rightarrow (u,v) \in \cap\mathcal{E} \\ &\Rightarrow [u]_{\cap\mathcal{E}} = [v]_{\cap\mathcal{E}}, \end{aligned}$$

and so h is injective. Since h is injective, $\mathrm{F}(X)/\cap\mathcal{E}$ belongs to $\mathcal{A}$ by conditions (a) and (b).

To prove the initiality of $(\mathrm{F}_{\mathcal{A}}(X), \eta_X)$, just take an arbitrary map $\alpha : X \to A$ with $A \in \mathcal{A}$ and use the commutative diagram



in which:

- the unlabelled arrows are the canonical maps whose composite is η_X;
- f is the unique homomorphism determined by α and the initiality of $\mathrm{F}(X)$ (equipped with the canonical map $X \to \mathrm{F}(X)$);
- $\bar{\alpha}$ is the composite

$$\mathrm{F}_{\mathcal{A}}(X) = \mathrm{F}(X)/\cap\mathcal{E} \to \mathrm{F}(X)/E_f \to A,$$

which is well defined since $\cap\mathcal{E} \subseteq E_f$ and by Theorem 3.4.2.

□

Let us return to the textbooks in universal algebra used in the previous sections, although now we will only briefly describe their approaches:

(a) Let $\mathcal{A}$ be as in Theorem 4.6.4, and let us consider it as a full subcategory of the category of Ω-algebras. Then sending X to $\mathrm{F}_{\mathcal{A}}(X)$ canonically determines a functor, to be called $\mathrm{F}_{\mathcal{A}}$, from $\mathcal{A}$ to the category of sets. Indeed, for any map $\varphi : X \to Y$, we can simply define $\mathrm{F}_{\mathcal{A}}(\varphi)$ as the unique homomorphism making the diagram

$$\begin{array}{ccc} \mathrm{F}_{\mathcal{A}}(X) & \xrightarrow{\mathrm{F}_{\mathcal{A}}(\varphi)} & \mathrm{F}_{\mathcal{A}}(Y) \\ {\scriptstyle \eta_X}\uparrow & \nearrow{\scriptstyle \eta_Y \varphi} & \\ X & & \end{array}$$

commute.

This is an example of a *universal functor* in the sense of P. M. Cohn (see [3, Chapter III, Section 1]), and so one can say that P. M. Cohn defines free algebras via universal functors. However, defining the free algebra on a set X, P. M. Cohn requires the canonical map from X to it to be injective (see [3, Chapter III, Section 5]). More precisely:

(i) the free $\mathcal{A}$-algebra on X in the sense of P. M. Cohn is the same as the above defined $\mathrm{F}_{\mathcal{A}}(X) = (\mathrm{F}_{\mathcal{A}}(X), \eta_X)$;

(ii) $\mathrm{F}_{\mathcal{A}}(X) = (\mathrm{F}_{\mathcal{A}}(X), \eta_X)$ is the free $\mathcal{A}$-algebra on X in the sense of P. M. Cohn if and only if η_X is injective;

(iii) if η_X is not injective, P. M. Cohn would say that the free $\mathcal{A}$-algebra on X does not exist, but still call $\mathrm{F}_{\mathcal{A}}(X) = (\mathrm{F}_{\mathcal{A}}(X), \eta_X)$ a universal $\mathcal{A}$-algebra on X.

Therefore the following very simple observation, which is a part of Proposition 5.1 in [3, Chapter III, Section 5], is important: η_X is injective if and only if there exists $A \in \mathcal{A}$ with $\mathrm{card}(X) \leq \mathrm{card}(A)$. Note also, that if condition 4.6.4(b) holds, then it suffices to have at least one $A \in \mathcal{A}$ with more than one element. Finally, note that although Theorem 4.6.4 is not formulated in [3] explicitly, it immediately follows from Theorem 5.3 (together with the sentence below it) and Proposition 5.4 of [3, Chapter III, Section 5].

(b) N. Jacobson's definition of a free algebra [5, Chapter 2, Section 2.8, the sentence above Theorem 2.10] agrees with Definition 4.6.1 (with our notation he would say "free algebra for $\mathcal{A}$ determined by X"). And his Theorem 2.10 [5, Chapter 2, Section 2.8] is a counterpart of Theorem 4.6.4, except that his assumptions are stronger.

(c) When $\eta_X : X \to \mathrm{F}_{\mathcal{A}}(X)$ is injective, having in mind that the initial object in the category is uniquely determined only up to isomorphism, we can assume that X is a subset of $\mathrm{F}_{\mathcal{A}}(X)$ and that η_X is the inclusion map. Assuming that G. Birkhoff [1, Chapter VI, Section 7] also uses, not initiality, but a property usually (but not always) equivalent to it to define $F_{\mathcal{A}}(X)$ (his notation is also very different). Specifically, he requires:

(i) $F_{\mathcal{A}}(X)$ to be generated by X (that is, $F_{\mathcal{A}}(X)$ should have no subalgebras containing X except itself);

(ii) for $A \in \mathcal{A}$, every map $X \to A$ should be extended to a homomorphism $F_{\mathcal{A}}(X) \to A$.

He does not require the uniqueness of such an extension, but it is an easy consequence of (i); indeed, if f and g are two such extensions, then

$$\{t \in F_{\mathcal{A}}(X) | f(t) = g(t)\}$$

is a subalgebra of $F_{\mathcal{A}}(X)$ containing X. G. Birkhoff's Theorems 13 and 13' in the same section can be considered as a 'more constructive counterpart' of Theorem 4.6.4.

(d) G. Grätzer [4] essentially follows G. Birkhoff in his presentation of free algebras.

Let us also specifically mention $F_{\mathcal{A}}(X)$ in the case $X = \emptyset$, which is excluded, say, in the above-mentioned Theorem 2.10 of [5, Chapter 2, Section 2.8] in order to avoid considering empty algebras. Since $\emptyset$ is the initial object in the category of sets, $F_{\mathcal{A}}(X)$ is simply the initial object $\mathcal{A}$. Therefore, it is empty if and only if the empty algebra belongs to $\mathcal{A}$. And we have:

$$(\Omega_0 = \emptyset \text{ \& } \mathcal{A} \text{ is closed under subalgebras and nonempty}) \Rightarrow \emptyset \in \mathcal{A} \Rightarrow \Omega_0 = \emptyset.$$

4.7 Varieties of Algebras

Considering Ω-algebras for a fixed Ω, we will write $F_{\Omega}(X)$ for $F_{\mathcal{A}}(X)$, where $\mathcal{A}$ is the class of all Ω-algebras, $X = \{x, y, z, \dots\}$ a fixed countable set, and by an *identity* we will mean any element of $F_{\Omega}(X) \times F_{\Omega}(X)$.

Definition 4.7.1. We say that an identity (u, v) holds in an Ω-algebra A and write $A \vDash (u, v)$ if $f(u) = f(v)$ for every homomorphism $f : F_{\Omega}(X) \to A$. For set Φ of identities, the variety $\mathrm{Var}(\Omega, \Phi)$ of Ω-algebras determined by Φ is the class of all Ω-algebras A with $A \vDash (u, v)$ for all $(u, v) \in \Phi$.

The following theorem is a classical characterisation of varieties of algebras due to G. Birkhoff:

Theorem 4.7.2. *A class of Ω-algebras is a variety, that is, it is of the form $Var(\Omega, \Phi)$ for some Φ, if and only if it satisfies conditions (a) and (b) of Theorem 4.6.4 and is closed under quotient algebras (that is, it contains all quotient algebras of algebras that belong to it).*

Let us omit the proof, but mention that this theorem is essentially the same as each of the following: Theorem 4.1 in [3, Chapter IV, Section 4], Theorem 2.14 in [5, Chapter 2, Section 2.10], Theorem 22 in [1, Chapter VI, Section 10], and Theorem 3 in [4, Chapter 4, Section 26].

It is easy to see that many familiar classes of algebras are varieties. Let us consider just one such example. A commutative semiring is a system $(R, 0, +, 1, \cdot)$ such that $(R, 0, +)$ and $(R, 1, \cdot)$ are commutative monoids with $0x = 0$ and $x(y + z) = xy + xz$ for all $x, y, z \in R$. Therefore, the class of commutative semirings is the variety $\mathrm{Var}(\Omega, \Phi)$, in which:

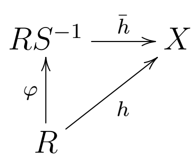
- $\Omega = \Omega_0 \cup \Omega_2$;
- $\Omega_0 = \{0, 1\}$;
- $\Omega_2 = \{+, \cdot\}$;
- Φ (in N. Jacobson's notation [5, Chapter 2, Section 2.7]) consists of $((+, x, +, y, z), (+, +, x, y, z)), ((+, x, y), (+, y, x)), ((+, 0, x), (x)),$ $((\cdot, x, \cdot, y, z)(\cdot, \cdot, x, y, z)), ((\cdot, x, y), (\cdot, y, x)), ((\cdot, 1, x), (x)), ((\cdot, 0, x), (0)),$ $((\cdot, x, +, y, z), (+, \cdot, x, y, \cdot, x, z))$.

Chapter 5

Fractoids

5.1 Commutative Fraction Semirings and Rings

Definition 5.1.1. Let R be a commutative semiring (with 1) and S a subset of R. The fraction semiring RS^{-1} equipped with the canonical homomorphism $\varphi : R \to RS^{-1}$ is defined as an initial object in the category of pairs (X, h), where X is a commutative semiring and $h : R \to X$ is a semiring homomorphism making $h(s)$ invertible for every $s \in S$. That is, for each such pair (X, h), there exists a unique semiring homomorphism $\bar{h} : RS^{-1} \to X$ making the diagram



commute.

Note that, in the notation above, the set $\{r \in R | h(r) \text{ is invertible in } X\}$ is always a submonoid of the multiplicative monoid of R, which tells us that we can always assume S to be a submonoid of the multiplicative monoid of R, and we will do so from now on.

Similarly to the classical case of commutative rings, we can construct RS^{-1} as the quotient set $R \times S/ \sim$, where $\sim$ is defined by

$$(r, s) \sim (r', s') \Leftrightarrow \exists_{t \in S} rs't = r'st,$$

with addition and multiplication defined by

$$r_1/s_1 + r_2/s_2 = (r_1s_2 + r_2s_1)/s_1s_2 \text{ and } (r_1/s_1)(r_2/s_2) = r_1s_1/r_2s_2,$$

respectively, where r_1/s_1 denotes the equivalence class of (r_1, s_1), etc. Then φ is defined by $\varphi(r) = r/1$, and, in particular, $\varphi(0) = 0/1$ and $\varphi(1) = 1/1$ are 0 and 1, respectively, of RS^{-1}.

In this construction, if R happens to be a ring, then so is RS^{-1}, with

$$-(r/s) = -r/s = r/(-s)$$

for all $r \in R$ and $s \in S$. Therefore, in this case, the fraction semiring RS^{-1} is the same as the classically defined fraction ring RS^{-1} (which justifies our notation).

Now let us observe that all the above-mentioned operations on RS^{-1} are in fact defined already on $R \times S$. Indeed, we simply put

$$\begin{aligned}(r_1, s_1) + (r_2, s_2) &= (r_1s_2 + r_2s_1, s_1s_2),\\ (r_1, s_1)(r_2, s_2) &= (r_1s_1, r_2s_2),\\ 0 &= (0, 1),\\ 1 &= (1, 1),\end{aligned}$$

and $-(r, s) = (-r, s)$ in the case of a ring. After that we can show that $\sim$ is a congruence on $R \times S = (R \times S, 0, +, 1, \cdot)$ (or on $R \times S = (R \times S, 0, +, -, 1, \cdot)$ when R is a ring) and say that RS^{-1} is constructed as the corresponding quotient algebra.

Here $(R \times S, 0, +, 1, \cdot)$ is not necessarily a semiring, but it has some properties that make $(R \times S/\sim, 0, +, 1, \cdot)$ a semiring, and we have a similar situation with rings. What are these properties? The answer consists of the following propositions:

Proposition 5.1.2. $(R \times S, 0, +)$ is a commutative monoid.

Proof. We have:

(a)

$$\begin{aligned}(r_1, s_1) + ((r_2, s_2) + (r_3, s_3)) &= (r_1, s_1) + (r_2s_3 + r_3s_2, s_2s_3)\\ &= (r_1s_2s_3 + (r_2s_3 + r_3s_2)s_1, s_1s_2s_3)\end{aligned}$$

$$= (r_1s_2s_3 + r_2s_3s_1 + r_3s_2s_1, s_1s_2s_3)$$

$$\begin{aligned}((r_1, s_1) + (r_2, s_2)) + (r_3, s_3) &= (r_1s_2 + r_2s_1, s_1s_2) + (r_3, s_3)\\ &= ((r_1s_2 + r_2s_1)s_3 + r_3s_1s_2, s_1s_2s_3)\\ &= (r_1s_2s_3 + r_2s_1s_3 + r_3s_1s_2, s_1s_2s_3)\end{aligned}$$

$$\Rightarrow (r_1, s_1) + ((r_2, s_2) + (r_3, s_3)) = ((r_1, s_1) + (r_2, s_2)) + (r_3, s_3)$$

(b)

$$(r_1, s_1) + (r_2, s_2) = (r_1s_2 + r_2s_1, s_1s_2)$$

$$(r_2, s_2) + (r_1, s_1) = (r_2s_1 + r_1s_2, s_2s_1)$$

$$\Rightarrow (r_1, s_1) + (r_2, s_2) = (r_2, s_2) + (r_1, s_1)$$

(c)

$$\begin{aligned}(0, 1) + (r, s) &= (0s + 1r, 1s)\\ &= (r, s)\end{aligned}$$

$$\begin{aligned}(r, s) + (0, 1) &= (1r + 0s, 1s)\\ &= (r, s)\end{aligned}$$

$$\Rightarrow (0, 1) + (r, s) = (r, s) + (0, 1)$$

□

Proposition 5.1.3. $(R \times S, 1, \cdot)$ is a commutative monoid.

Proof. This is simply the product of R and S considered as multiplicative monoids. □

Proposition 5.1.4. The algebra $(R \times S, 0, +, 1, \cdot)$ satisfies the following conditions:

(a) $x(y + z)(1 + 0x) = xy + xz$ (modified distributivity), for all $x, y, z \in R \times S$;

(b) $x + (-x) = 0xx$ (modified additive inverse property), for all $x \in R \times S$, whenever R is a ring.

Proof. (a) : For $x = (r_1, s_1), y = (r_2, s_2), z = (r_3, s_3)$, we have:

$$\begin{aligned} x(y+z)(1+0x) &= (r_1, s_1)((r_2, s_2) + (r_3, s_3))((1,1) + (0,1)(r_1, s_1)) \\ &= (r_1, s_1)(r_2 s_3 + r_3 s_2, s_2 s_3)((1,1) + (0, s_1)) \\ &= (r_1 r_2 s_3 + r_1 r_3 s_2, s_1 s_2 s_3)(s_1, s_1) \\ &= (r_1 r_2 s_3 s_1 + r_1 r_3 s_2 s_1, s_1 s_2 s_3 s_1) \end{aligned}$$

$$\begin{aligned} xy + xz &= (r_1, s_1)(r_2, s_2) + (r_1, s_1)(r_3, s_3) \\ &= (r_1 r_2, s_1 s_2) + (r_1 r_3, s_1 s_3) \\ &= (r_1 r_2 s_1 s_3 + r_1 r_3 s_1 s_2, s_1 s_2 s_1 s_3) \end{aligned}$$

$$\Rightarrow x(y+z)(1+0x) = xy + xz$$

(b) : For $x = (r, s)$, we have:

$$\begin{aligned} x + (-x) &= (r, s) + (-r, s) \\ &= (rs + (-r)s, ss) \\ &= (0, ss) \end{aligned}$$

$$\begin{aligned} 0xx &= (0,1)(r,s)(r,s) \\ &= (0r, 1s)(r, s) \\ &= (0, s)(r, s) \\ &= (0, ss) \end{aligned}$$

$$\Rightarrow x + (-x) = 0xx$$

□

Proposition 5.1.5. Let $(A, 0, +, 1, \cdot)$ be an algebra in which:

(a) $(A, 0, +)$ and $(A, 1, \cdot)$ are commutative monoids;

(b) the modified distributivity condition is satisfied, that is, $x(y+z)(1+0x) = xy + xz$ for all $x, y, z \in A$.

Then $(A, 0, +, 1, \cdot)$ is a semiring if and only if $0x = 0$ for all $x \in A$.

Proposition 5.1.6. Let $(A, 0, +, -, 1, \cdot)$ be an algebra in which:

(a) $(A, 0, +)$ and $(A, 1, \cdot)$ are commutative monoids;

(b) the modified distributivity condition is satisfied, that is, $x(y+z)(1+0x) = xy + xz$ for all $x, y, z \in A$;

(c) $-$ is a unary operation and the modified additive inverse property is satisfied, that is, $x + (-x) = 0xx$ for all $x \in A$.

Then $(A, 0, +, -, 1, \cdot)$ is a ring if and only if $0x = 0$ for all $x \in A$.

5.2 Semifractoids, Fractoids, and Wheels

Definition 5.2.1. An algebra $(A, 0, +, 1, \cdot)$ in which $(A, 0, +)$ and $(A, 1, \cdot)$ are commutative monoids is said to be:

(a) a commutative dimonoid, if it has $0 \cdot 0 = 0$;

(b) a commutative 0-distributive dimonoid, if it is a dimonoid satisfying the identity $x(y + 0z) = xy + 0z$;

(c) a semifractoid, if it is a 0-distributive dimonoid satisfying the modified distributivity condition, that is, the identity $x(y+z)(1+0x) = xy + xz$.

An algebra $(A, 0, +, -, 1, \cdot)$ is said to be a fractoid, if $(A, 0, +, 1, \cdot)$ is a semifractoid, and $-$ is a unary operation on A satisfying the modified additive inverse property, that is, the identity $x + (-x) = 0xx$.

Example 5.2.2. The algebra $(R \times S, 0, +, 1, \cdot)$, considered in the previous section, is a semifractoid, and, if R is a ring, then the algebra $(R \times S, 0, +, -, 1, \cdot)$ is a fractoid. Indeed, in $R \times S$ we have

$$\begin{aligned} 0 \cdot 0 &= (0,1)(0,1) \\ &= (0,1) \\ &= 0, \end{aligned}$$

for $x = (r_1, s_1)$, $y = (r_2, s_2)$, $z = (r_3, s_3)$, we have:

$$\begin{aligned}
x(y + 0z) &= (r_1, s_1)((r_2, s_2) + (0, 1)(r_3, s_3)) \\
&= (r_1, s_1)((r_2, s_2) + (0, s_3)) \\
&= (r_1, s_1)(r_2 s_3, s_2 s_3) \\
&= (r_1 r_2 s_3, s_1 s_2 s_3) \\
&= (r_1 r_2, s_1 s_2) + (0, s_3) \\
&= (r_1, s_1)(r_2, s_2) + (0, 1)(r_3, s_3) \\
&= xy + 0z,
\end{aligned}$$

and the modified distributivity condition holds here by Proposition 5.1.4 (a).

Lemma 5.2.3. Every commutative 0-distributive dimonoid satisfies the identity

$$x(y + z)(1 + 0x) = x(y + z) + 0x.$$

Proof.

$$\begin{aligned}
x(y + z)(1 + 0x) &= x(y + z)1 + 0x \\
&= x(y + z) + 0x
\end{aligned}$$

□

Remark 5.2.4. We could call our modified distributivity condition *multiplicative*, and introduce the *additive modified distributivity condition* as the identity

$$x(y + z) + 0x = xy + xz.$$

Then:

(a) replacing the multiplicative modified distributivity condition with the additive one would not change Propositions 5.1.4 and 5.1.5;

(b) Lemma 5.2.3 would say that, in the case of a 0-distributive dimonoid, these two conditions are equivalent to each other;

(c) the condition $0 \cdot 0 = 0$, the 0-distributivity condition, and the additive modified distributivity condition are used in [2], as conditions (5), (6), and (3), respectively, of the definition of a *wheel*.

Example 5.2.5. Having in mind Remark 5.2.4 (c), a wheel in the sense of [2] can be defined as an algebra $(A, 0, +, 1, \cdot, /)$ in which $(A, 0, +, 1, \cdot)$ is a semifractoid, and $/$ is a unary operation on A satisfying the following identities:

(a) $/1 = 1$,

(b) $/(xy) = (/x)(/y)$,

(c) $(x/y) + z + 0y = (x + yz)/y$,

(d) $/(x + 0y) = (/x) + 0y$,

(e) $x + (0/0) = 0/0$.

Example 5.2.6. For an arbitrary commutative monoid $(A, 0, +)$, the algebra $(A, 0, +, 1_A, 0, +)$ is a fractoid. Indeed, in this algebra we have

$$\begin{aligned} 0 \cdot 0 &= 0 + 0 \\ &= 0, \end{aligned}$$

$$\begin{aligned} x(y + 0z) &= x + y + 0 + z \\ &= xy + 0z, \end{aligned}$$

$$\begin{aligned} x(y + z)(1 + 0x) &= x + y + z + 0 + 0 + x \\ &= x + y + z + x \\ &= x + y + x + z \\ &= xy + xz, \end{aligned}$$

$$\begin{aligned} x + (-x) &= x + x \\ &= 0 + x + x \\ &= 0xx. \end{aligned}$$

5.3 On Independence of the Semifractoid Axioms

Theorem 5.3.1. *For an algebra* $(A, 0, +, 1, \cdot)$ *in which* $(A, 0, +)$ *and* $(A, 1, \cdot)$ *are commutative monoids, the three identities given in conditions (a)–(c) of Definition 5.2.1 are partly independent in the following sense:*

(a) the identity of 5.2.1(a) does not follow from the identities of 5.2.1(b) and 5.2.1(c);

(b) the identity of 5.2.1(c) does not follow from the identities of 5.2.1(a) and 5.2.1(b).

Proof. (a) Take $(A, 0, +)$ to be a commutative monoid having elements e and e' with $e + e' = 0$ and $e \neq 0$, and define 1 and $\cdot$ by $1 = e'$ and $x \cdot y = e + x + y$. Then $(A, 1, \cdot)$ is a commutative monoid, and

$$\begin{aligned} x(y + 0z) &= e + x + y + 0z \\ &= e + x + y + e + 0 + z \\ &= xy + 0z, \end{aligned}$$

$$\begin{aligned} x(y + z)(1 + 0x) &= (e + x + y + z)(e' + e + 0 + x) \\ &= (e + x + y + z)x \\ &= e + e + x + y + z + x \\ &= e + x + y + e + x + z \\ &= xy + xz, \end{aligned}$$

but $0 \cdot 0 = e + 0 + 0 = e \neq 0$. That is, in this case the identities of 5.2.1(b) and 5.2.1(c) hold, but the identity of 5.2.1(a) does not.

(b) Let $(M, 1, \circ)$ be a commutative monoid and $\{0\}$ a one-element set with $0 \notin M$. Take A to be $M \cup \{0\}$ and define $+$ and $\cdot$ as extensions of $\circ$ with

$$\begin{aligned} 0 + a &= a \\ &= a + 0 \quad \text{and} \end{aligned}$$

$$\begin{aligned} 0a &= 0 \\ &= a0 \end{aligned}$$

for all $a \in A$. Then $(A, 0, +)$ and $(A, 1, \cdot)$ are commutative monoids with

$$0 \cdot 0 = 0 \text{ and}$$

$$\begin{aligned} x(y + 0z) &= xy \\ &= xy + 0z, \end{aligned}$$

but, say for $x \in M$, the identity of 5.2.1(c) would imply that

$$\begin{aligned} x + x + x &= x(x + x) \\ &= x(x + x)(1 + 0x) \\ &= xx + xx \\ &= x + x + x + x. \end{aligned}$$

Therefore, whenever M has an element x with $x \circ x \circ x \neq x \circ x \circ x \circ x$, we obtain an algebra $(A, 0, +, 1, \cdot)$ satisfying the identities of 5.2.1(a) and 5.2.1(b), but not of 5.2.1(c).

□

Open Question 5.3.2. We also expect that the identity of 5.2.1(b) does not follow from the identities of 5.2.1(a) and 5.2.1(c), but we don't have any example confirming this.

5.4 The Initial Semifractoid and Fractoid

The results of this section are (semi)fractoid counterparts of some results of [2] on wheels.

Proposition 5.4.1. Let $(A, 0, +, 1, \cdot)$ be a semifractoid, or, more generally, a commutative dimonoid satisfying condition (c) of Definition 5.2.1. Then $R(A) = \{a \in A | 0a = 0\}$ is a subalgebra of $(A, 0, +, 1, \cdot)$, which is a semiring.

Proof. We have $0 \in R(A)$ (by Definition 5.2.1(a)), $1 \in R(A)$, and, for $a, b \in R(A)$,

$$\begin{aligned} 0(a + b) &= 0(a + b)(1 + 0) \\ &= 0(a + b)(1 + 0 \cdot 0) \\ &= 0a + 0b \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

(where the third equality follows from Definition 5.2.1(c)) and $0ab = 0b = 0$. That is, $R(A)$ is a subalgebra of $(A, 0, +, 1, \cdot)$. The fact that it is a semiring follows now from Proposition 5.1.5. □

Corollary 5.4.2. Let $(A, 0, +, 1, \cdot)$ be a fractoid. Then $R(A) = \{a \in A | 0a = 0\}$ is a subalgebra of $(A, 0, +, 1, \cdot)$, which is a ring.

Corollary 5.4.3. The smallest subalgebra of any semifractoid is a semiring, and the smallest subalgebra of any fractoid is a ring.

Corollary 5.4.4. The initial semifractoid is the semiring on natural numbers, and the initial fractoid is the ring of integers.

Corollary 5.4.5. Using category-theoretic language we say that $A \mapsto R(A)$ gives a right adjoint of the inclusion functor

$$\mathsf{Commutative\ Semirings} \to \mathsf{Semifractoids},$$

which implies that the inclusion functor preserves initial objects. The same applies to rings and fractoids.

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