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**ALGEBRAIC EXPONENTIATION AND
INTERNAL HOMOLOGY IN
GENERAL CATEGORIES**

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Abstract

We study two categorical-algebraic concepts of exponentiation:

(i) Representing objects for the so-called split extension functors in semi-abelian and more general categories, whose familiar examples are automorphism groups of groups and derivation algebras of Lie algebras. We prove that such objects exist in categories of generalized Lie algebras defined with respect to an internal commutative monoid in symmetric monoidal closed abelian category.

(ii) Right adjoints for the pullback functors between D. Bourns categories of points. We introduce and study them in the situations where the ordinary pullback functors between bundles do not admit right adjoints in particular for semi-abelian, protomodular, (weakly) Maltsev, (weakly) unital, and more general categories. We present a number of examples and counterexamples for the existence of such right adjoints.

We use the left and right adjoints of the pullback functors between categories of points to introduce internal homology and cohomology of objects in abstract categories.

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Introduction

It is well known that \mathbb{C} is a *locally cartesian closed* category when \mathbb{C} has pullbacks, and for any morphism $p : E \rightarrow B$ in \mathbb{C} the *pullback functor* $p^* : (\mathbb{C} \downarrow B) \rightarrow (\mathbb{C} \downarrow E)$ has a right adjoint. It follows that the functors p^* preserve colimits, from which many exactness properties can be derived.

For any objects B and E in \mathbb{C} and for any morphism $p : E \rightarrow B$ in \mathbb{C} , the categories $\mathbf{Pt}(B)$ and the pullback functors $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ between them were introduced and used to study exactness properties by D. Bourn in [6], see Definition 1.2.1 below. We will write p_a^* to distinguish this pullback functor from the pullback functor between comma categories, and in particular use the subscript a , since these functors play an important role in algebra, as was already observed in [6]. It can be easily seen that for any morphism $p : E \rightarrow B$, if p^* has a right adjoint, then p_a^* has a right adjoint. This provides a motivation for calling the right adjoints of p_a^* exponentiations, and leads to the question of whether it is possible for a functor p_a^* to have a right adjoint even when the functor p^* does not. Since in any category \mathbb{C} , for any isomorphism $p : E \rightarrow B$, p^* is an equivalence, it follows that in any category pullback functors along isomorphisms always have right adjoints. It was shown for any semi-abelian category (see [11]) that pullback functors p^* have a right adjoint only if p is an isomorphism. This makes the question of whether the functors p_a^* have right adjoints for semi-abelian categories more interesting. The following argument shows that at least sometimes the answer is yes. For a group B , there is a well known equivalence of categories $\mathbf{Pt}(B) \sim \mathbf{Gp}^B$, in which, on the right, we are considering the group B as a category with one object. And, for any morphism $p : E \rightarrow B$, the pullback functor $p_a^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ is equivalent to the functor $\mathbf{Gp}^p : \mathbf{Gp}^B \rightarrow \mathbf{Gp}^E$. Therefore, for any object $F \in \mathbf{Gp}^E$ the right adjoint of the functor \mathbf{Gp}^p applied to F is the right Kan extension of F along p (considered as a functor). It can be seen, Theorem 3.5.11, that $\text{Ran}_p F(1)$ is given by the well known construction $\{\theta \in \text{hom}(B, F(1)) \mid \forall (b, e) \in B \times E \theta(p(e)b) = F(e)(\theta(b))\}$, where the unique object in B and E is denoted by 1. This provides a second motivation for calling the right adjoints of pullback functors algebraic exponents. We show in Section 3.5 that this is not

the only example, in particular we show (Proposition 3.5.13) that for the category of Lie algebras over a commutative ring, all pullback functors p_a^* have right adjoints.

There is another kind of *exponential* object in the category of groups. Recall: any split epimorphism $\alpha : A \rightarrow B$, in the category of groups, is canonically isomorphic to a semidirect product projection, and so is completely determined (up to canonical isomorphism) by a morphism $B \rightarrow \text{Aut}(K)$, where K is the kernel of α and $\text{Aut}(K)$ is the group of automorphisms of K . This result can be stated categorically as: the split extension functor $\text{SplExt}(-, K) : \mathbf{Gp}^{\text{op}} \rightarrow \mathbf{Set}$ is representable, with object of representation $\text{Aut}(K)$. Since the representing object is unique (up to isomorphism) this can be considered as a categorical definition of $\text{Aut}(X)$. The functors $\text{SplExt}(-, K) : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ were studied in [4] and [5]. In Chapter 2 we give a new example.

The purpose of this thesis is to study both kinds of algebraic exponentiation mentioned above, as well internal homology/cohomology (see Chapter 4).

The thesis consists of the following chapters:

Chapter 1: We begin with some auxiliary results about weakly unital categories, which were proved in the unital case in [3]. We then give a brief overview of (weakly) Mal'tsev, Bourn protomodular and semi-abelian categories (for precise references see [3] and references there), and recall some results about preservation of certain colimits by (bi)functors, and some *lifting* theorems. We conclude the chapter by recalling some basic homology theory for an abelian category with respect to chosen class of epimorphisms.

Chapter 2: In this chapter we define categories of *generalized* Lie algebras and show that the split extension functors are still representable. The chapter is essentially the same as [12].

Chapter 3: The purpose of this chapter is to study when right adjoints of pullback functors p_a^* exist. In Section 3.1 we choose axioms on a category \mathbf{C} in such a way that:

1. all varieties are examples;
2. any Barr-exact Mal'tsev locally well-presentable category is an example;

and for which we can prove (Theorem 3.1.4): for any morphism $p : E \rightarrow B$, p_a^* has a right adjoint if and only if p_a^* preserves binary coproducts. In Section 3.2 we consider weakly

unital categories, and consider the existence of right adjoints of pullback functors along morphism into the terminal object (for the category of groups the right adjoints of these functors are known as the non-abelian zeroth cohomology functors). We show in Proposition 3.2.3 and Corollary 3.2.4, that the existence of these right adjoints is equivalent to the existence of *centralizers* (generalizing those defined in [8]). We show (Theorem 3.2.13), that when in addition the category is regular, all unions of subobjects exist and finite limits commute with filtered colimits, for any morphism $p : B \rightarrow 1$ the pullback functor p_a^* has a right adjoint if and only if p_a^* preserves binary coproducts. We conclude Section 3.2 with an example of a class of varieties generalizing categories of interest, for which these adjoints always exist. In Section 3.4 we consider pointed Bourn protomodular categories and show that if for each object B in \mathbb{C} the pullback functor $(0 \rightarrow B)_a^*$ have right adjoints then every pullback functor p_a^* is comonadic. Finally, in Sections 3.4 and 3.5 we have many semi-abelian and related examples. We show for the categories of Lie algebras over a commutative ring, internal groups in a cartesian closed category, and a subvariety of commutative rings, that all pullback functors p_a^* have right adjoints; for the categories of rings, commutative rings, and Boolean rings that in particular the kernel functors (which are not isomorphisms) do not have right adjoints, and that all pullback functors p_a^* along regular epimorphisms have right adjoints; we conclude with an example for which only pullback functors p_a^* along isomorphisms have right adjoints.

Chapter 4: In this chapter we introduce *internal homology* and *cohomology* as well as *induced* and *coinduced* functors. In Section 4.1 under suitable conditions we show (Theorem 4.1.2) that the internal cohomology and coinduced functors exist. In Section 4.2 we give for weakly unital categories a sufficient condition for the existence of the zeroth cohomology functors. In Section 4.3 we show in particular for regular Mal'tsev categories that the homology and induced functors exist. Finally in Section 4.4 we show for semi-abelian varieties that the internal cohomology and internal homology functors can be expressed via suitable functors Ext and Tor respectively.

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Chapter 1

Preliminaries

1.1 Weakly unital and unital categories

Weakly unital categories were introduced by N. Martins-Ferreira in [19].

Definition 1.1.1. *A category \mathbb{C} is weakly unital when:*

1. \mathbb{C} is pointed;
2. \mathbb{C} is finitely complete;
3. for every pair of objects $X, Y \in \mathbb{C}$ the morphisms $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ in the diagram

$$X \xrightarrow{\langle 1, 0 \rangle} X \times Y \xleftarrow{\langle 0, 1 \rangle} Y$$

are jointly epimorphic.

Proposition 1.1.2. *Any pointed quasivariety \mathcal{V} with binary $+$ satisfying the following axioms:*

1. $x + 0 = 0 + x$;
2. $x + 0 = y + 0 \Rightarrow x = y$;

is weakly unital.

Proof. Conditions 1.1.1(1) and 1.1.1(2) are trivially satisfied. For Condition 1.1.1(3), we

need to show that for any diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{\langle 1,0 \rangle} & X \times Y & \xleftarrow{\langle 0,1 \rangle} & Y \\
 & \searrow f & \downarrow \theta & \downarrow \phi & \swarrow g \\
 & & Z & &
 \end{array}$$

in \mathbb{C} , in which:

- $\theta\langle 1,0 \rangle = f = \phi\langle 1,0 \rangle$;
- $\theta\langle 0,1 \rangle = g = \phi\langle 0,1 \rangle$;

we have $\theta = \phi$. For any $x \in X$ and $y \in Y$, we have:

$$\begin{aligned}
 \theta(x,y) + 0 &= \theta(x,y) + \theta(0,0) \\
 &= \theta(x+0,y+0) \\
 &= \theta(x+0,0+y) \\
 &= \theta(x,0) + \phi(0,y) \\
 &= f(x) + g(y).
 \end{aligned}$$

Similarly $\phi(x,y) + 0 = f(x) + g(y)$ and so by the second axiom we have that $\theta(x,y) = \phi(x,y)$. \square

The following definition was introduced for unital categories in [3] except we replace *cooperate* with *commute*.

Definition 1.1.3. Any pair of morphisms $f : A \rightarrow C$ and $g : B \rightarrow C$, with common codomain, commute when there exists a morphism $\phi : A \times B \rightarrow C$ such that the diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{\langle 1,0 \rangle} & A \times B & \xleftarrow{\langle 0,1 \rangle} & B \\
 & \searrow f & \downarrow \phi & \downarrow \phi & \swarrow g \\
 & & C & &
 \end{array}$$

commutes.

The following proposition essentially comes from [3].

Proposition 1.1.4. *Let \mathbb{C} be a weakly unital category. For a fixed object $C \in \mathbb{C}$. Let (A, f) and (B, g) be any objects in $(\mathbb{C} \downarrow C)$. For any morphism $h : W \rightarrow A$, if f commutes with g then fh commutes with g .*

Proof. The following diagram

$$\begin{array}{ccccc}
 W & \xrightarrow{h} & A & & \\
 \langle 1,0 \rangle \downarrow & & \langle 1,0 \rangle \downarrow & \searrow f & \\
 W \times B & \xrightarrow{h \times 1} & A \times B & \xrightarrow{\phi} & C \\
 \langle 0,1 \rangle \uparrow & & & \nearrow g & \\
 B & & & &
 \end{array}$$

commutes, where ϕ is the unique morphism with $\phi\langle 1,0 \rangle = f$ and $\phi\langle 0,1 \rangle = g$. □

Definition 1.1.5. *A category \mathbb{C} is unital (see [7]) when:*

1. \mathbb{C} is pointed;
2. \mathbb{C} is finitely complete;
3. for all objects $X, Y \in \mathbb{C}$ the morphisms $\langle 1,0 \rangle : X \rightarrow X \times Y$ and $\langle 0,1 \rangle : Y \rightarrow X \times Y$ are jointly extremal epimorphic, that is, in any commutative diagram

$$\begin{array}{ccccc}
 & & T & & \\
 & \nearrow x & \downarrow m & \nwarrow y & \\
 X & \xrightarrow{\langle 1,0 \rangle} & X \times Y & \xleftarrow{\langle 0,1 \rangle} & Y
 \end{array}$$

in which m is a monomorphism, m is an isomorphism.

The following Proposition simply shows that jointly extremal epimorphic and jointly strongly epimorphic are the same.

Proposition 1.1.6. *For a pointed finitely complete category \mathbb{C} the following conditions are equivalent:*

1. \mathbb{C} is unital;

2. in any commutative diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{\langle 1,0 \rangle} & X \times Y & \xleftarrow{\langle 0,1 \rangle} & Y \\
 & \searrow f & \downarrow \phi & & \swarrow g \\
 & & W & & \\
 & & \uparrow m & & \\
 & & Z & &
 \end{array} \tag{1}$$

in \mathbb{C} , in which m is a monomorphism, there exists $\theta : X \times Y \rightarrow Z$ such that $m\theta = \phi$.

Proof. $1 \Rightarrow 2$: In the situation as in diagram (1), let (P, p, q) be the pullback of m and ϕ , since m is a monomorphism q is a monomorphism. The induced morphisms $\langle f, \langle 1, 0 \rangle \rangle$ and $\langle g, \langle 0, 1 \rangle \rangle$ make the diagram

$$\begin{array}{ccccc}
 & & P & & \\
 \langle f, \langle 1,0 \rangle \rangle & \nearrow & \downarrow q & \nwarrow & \langle g, \langle 0,1 \rangle \rangle \\
 X & \xrightarrow{\langle 1,0 \rangle} & X \times Y & \xleftarrow{\langle 0,1 \rangle} & Y
 \end{array}$$

commute and therefore q is an isomorphism. Take $\theta = pq^{-1}$, then $m\theta = \phi$, as desired.

$2 \Rightarrow 1$: For any commutative diagram

$$\begin{array}{ccccc}
 & & T & & \\
 x & \nearrow & \downarrow m & \nwarrow & y \\
 X & \xrightarrow{\langle 1,0 \rangle} & X \times Y & \xleftarrow{\langle 0,1 \rangle} & Y
 \end{array}$$

in which m is a monomorphism, the diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{\langle 1,0 \rangle} & X \times Y & \xleftarrow{\langle 0,1 \rangle} & Y \\
 & \searrow x & \downarrow 1_{X \times Y} & & \swarrow y \\
 & & X \times Y & & \\
 & & \uparrow m & & \\
 & & T & &
 \end{array}$$

commutes and so there exists a morphism $\theta : X \times Y \rightarrow T$ such that $m\theta = 1_{X \times Y}$. That is,

m is a split epimorphism as well as a monomorphism, therefore it is an isomorphism. \square

Proposition 1.1.7. *Every unital category is weakly unital.*

Proof. The proof follows trivially from the fact that in the presence of equalizers, jointly strongly epimorphic implies jointly epimorphic. \square

The following observation was made for regular unital categories (see [3], Chapter 1, Proposition 1.6.4) and the proof lifts to regular weakly unital categories.

Proposition 1.1.8. *Let \mathbb{C} be a regular weakly unital category. For any diagram*

$$W \xrightarrow{h} A \xrightarrow{f} C \xleftarrow{g} B$$

in which h is a regular epimorphism, f commutes with g if and only if fh commutes with g .

Proof. If f commutes with g then by Proposition 1.1.4 fh commutes with g . Conversely, suppose that fh commutes with g , that is, there exists $\phi : W \times B \rightarrow C$ such that $\phi\langle 1, 0 \rangle = hf$ and $\phi\langle 0, 1 \rangle = g$. Let (K, k_1, k_2) be the kernel pair of h , it easily follows that $\phi(k_1 \times 1)\langle 1, 0 \rangle = \phi(k_2 \times 1)\langle 1, 0 \rangle$ and trivially $\phi(k_1 \times 1)\langle 0, 1 \rangle = \phi(k_2 \times 1)\langle 0, 1 \rangle$, consequently $\phi(k_1 \times 1) = \phi(k_2 \times 1)$. Since \mathbb{C} is regular, $h \times 1$ is a regular epimorphism and therefore the coequalizer of its kernel pair $(K \times 1, k_1 \times 1, k_2 \times 1)$, and so the morphism ϕ factors through $h \times 1$. Therefore, there exists a morphism $\theta : A \times B \rightarrow C$ with the property that $\theta(h \times 1) = \phi$. Since $\theta\langle 1, 0 \rangle h = \theta(h \times 1)\langle 1, 0 \rangle = \phi\langle 1, 0 \rangle = fh$ and h is an epimorphism, $\theta\langle 1, 0 \rangle = f$ and $\theta\langle 0, 1 \rangle = \theta(h \times 1)\langle 0, 1 \rangle = \phi\langle 0, 1 \rangle = g$. \square

For a pointed finitely complete category \mathbb{C} , consider the following condition:

Condition 1.1.9. *For every object X and for every commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{\langle 1, 0 \rangle} & X \times X & \xleftarrow{\langle 0, 1 \rangle} & X \\ & & \downarrow \phi & & \\ & & Y & & \\ & \searrow f & \uparrow m & \swarrow f & \\ & & W & & \end{array}$$

in \mathbb{C} , in which m is a monomorphism, there exists a morphism $\psi : X \times X \rightarrow W$ such that $m\psi = \phi$.

Proposition 1.1.10. *Every unital category \mathbb{C} satisfies Condition 1.1.9.*

Proof. This trivially follows from the fact that Condition 1.1.9 is a special case of the second condition in Proposition 1.1.6 . \square

Remark 1.1.11. *We do not have any examples of a weakly unital category \mathbb{C} , that satisfies Condition 1.1.9, that is not unital.*

Proposition 1.1.12. *Let \mathbb{C} be a weakly unital category with coequalizers satisfying Condition 1.1.9. For any object $X \in \mathbb{C}$, if (Y, c) is the coequalizer of $\langle 1, 0 \rangle, \langle 0, 1 \rangle : X \rightarrow X \times X$, then the morphism $c\langle 1, 0 \rangle$ is an extremal epimorphism.*

Proof. Since c is an epimorphism (regular epimorphism), $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are jointly epimorphic, and $c\langle 1, 0 \rangle = c\langle 0, 1 \rangle$, the morphism $c\langle 1, 0 \rangle$ is an epimorphism. Suppose $c\langle 1, 0 \rangle : X \rightarrow Y$ factorizes as in the diagram

$$\begin{array}{ccc} X & \xrightarrow{c\langle 1, 0 \rangle} & Y \\ & \searrow g & \nearrow m \\ & S & \end{array}$$

in which m is a monomorphism. Since \mathbb{C} satisfies Condition 1.1.9 and the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\langle 1, 0 \rangle} & X \times X & \xleftarrow{\langle 0, 1 \rangle} & X \\ & \searrow g & \downarrow c & & \nearrow g \\ & & Y & & \\ & & \uparrow m & & \\ & & S & & \end{array}$$

is commutative, there exists a morphism $\phi : X \times X \rightarrow S$ such that $m\phi = c$. Consequently $\phi\langle 1, 0 \rangle = g = \phi\langle 0, 1 \rangle$, and since (Y, c) is the coequalizer of $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$, there exists a morphism $s : Y \rightarrow S$ with $sc = \phi$. It follows that $msc = m\phi = c$, and since c is an epimorphism we conclude that m is a split epimorphism, therefore m is an isomorphism. \square

Definition 1.1.13. *Let \mathbb{C} be a weakly unital category. An object X in \mathbb{C} is commutative when the morphism 1_X commutes with itself.*

Proposition 1.1.14. *Let \mathbb{C} be a weakly unital category. The forgetful functor $\mathbf{CMon}(\mathbb{C}) \rightarrow \mathbb{C}$ induces an isomorphism between $\mathbf{CMon}(\mathbb{C})$ and the category of commutative objects in \mathbb{C} .*

Proof. It is obvious that if $(Y, m, 0)$ is a commutative monoid in \mathbb{C} then Y is a commutative object. For the converse suppose that X is a commutative object in \mathbb{C} , it follows that there exists a morphism $\phi : X \times X \rightarrow X$ such that $\phi\langle 1, 0 \rangle = 1_X = \phi\langle 0, 1 \rangle$. We will show that the $(X, \phi, 0)$ is a commutative monoid. It only remains to show that ϕ is associative and commutative. Let α and σ be the associativity and symmetry isomorphisms for the cartesian product monoidal structure on \mathbb{C} . The commutative diagrams

$$\begin{array}{ccccc}
 X & \xrightarrow{\langle 1, 0 \rangle} & X \times (X \times X) & \xleftarrow{\langle 0, 1 \rangle} & X \times X \\
 & \searrow \langle 1, 0 \rangle & \downarrow 1 \times \phi & & \downarrow \phi \\
 & & X \times X & \xleftarrow{\langle 0, 1 \rangle} & X \\
 & \searrow 1_X & \downarrow \phi & & \downarrow 1_X \\
 & & X & &
 \end{array}$$

$$\begin{array}{ccccc}
 X & \xrightarrow{\langle 1, 0 \rangle} & X \times (X \times X) & \xleftarrow{\langle 0, 1 \rangle} & X \times X \\
 & \searrow \langle \langle 1, 0 \rangle, 0 \rangle & \downarrow \alpha & & \searrow \langle 0, 1 \rangle \times 1 \\
 & & (X \times X) \times X & & \\
 & \searrow \langle 1, 0 \rangle & \downarrow \phi \times 1 & & \searrow 1_{X \times X} \\
 & & X \times X & & \\
 & \searrow 1_X & \downarrow \phi & & \\
 & & X & &
 \end{array}$$

show that the morphisms $\phi(1 \times \phi)$ and $\phi(\phi \times 1)\alpha$ are equal after composing them with $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$, and so, as $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are jointly epimorphic, they are equal. This proves associativity. Commutativity: note that $\phi\sigma\langle 1, 0 \rangle = \phi\langle 0, 1 \rangle = 1_X$, and similarly $\phi\sigma\langle 0, 1 \rangle = 1_X$, and therefore $\phi = \phi\sigma$. \square

Proposition 1.1.15. *Let \mathbb{C} be a regular weakly unital category with coequalizers satisfying Condition 1.1.9. If (Y, c) is the coequalizer of the morphisms $\langle 1, 0 \rangle, \langle 0, 1 \rangle : X \rightarrow X \times X$, then Y is commutative. Moreover, the functor with object function assigning to each X in \mathbb{C} the coequalizer of $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$, is the left adjoint of the forgetful functor $U : \mathbf{CMon}(\mathbb{C}) \rightarrow \mathbb{C}$.*

Proof. By Proposition 1.1.14 the category of commutative objects and the category of

commutative monoids are isomorphic, and so we can replace the category of commutative monoids with the category of commutative objects. By Proposition 1.1.12 the composite $c\langle 1, 0 \rangle$ is an extremal epimorphism, and since \mathbb{C} is regular, $c\langle 1, 0 \rangle$ is a regular epimorphism. Note that $1_Y c\langle 1, 0 \rangle$ commutes with $1_Y c\langle 1, 0 \rangle$, therefore by Proposition 1.1.8 we have that 1_Y commutes with 1_Y . The morphism $c\langle 1, 0 \rangle$ is the X component of the unit of the adjunction. To see this, consider any morphism $f : X \rightarrow Z$ with Z a commutative object. Clearly the diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{\langle 1, 0 \rangle} & X \times X & \xleftarrow{\langle 0, 1 \rangle} & X \\
 \downarrow f & & \downarrow f \times f & & \downarrow f \\
 Z & \xrightarrow{\langle 1, 0 \rangle} & Z \times Z & \xleftarrow{\langle 0, 1 \rangle} & Z \\
 & \searrow 1_Z & \downarrow m & \swarrow 1_Z & \\
 & & Z & &
 \end{array}$$

in which m is the morphism exhibiting that 1_Z commutes with 1_Z , commutes and consequently, there exists a unique morphism $\bar{f} : Y \rightarrow Z$ with $\bar{f}c = m(f \times f)$. Note that $\bar{f}c\langle 1, 0 \rangle = m(f \times f)\langle 1, 0 \rangle = f$ and such \bar{f} is unique since $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are jointly epimorphic and $c\langle 1, 0 \rangle = c\langle 0, 1 \rangle$. \square

Proposition 1.1.16. *Let \mathbb{C} be a finitely complete category. The category $\mathbf{CMon}(\mathbb{C})$ is unital and every object in $\mathbf{CMon}(\mathbb{C})$ is commutative.*

Proof. It is obvious that $\mathbf{CMon}(\mathbb{C})$ is a pointed finitely complete category. It is easy to see that if C is an object in $\mathbf{CMon}(\mathbb{C})$, the morphism $+ : C \times C \rightarrow C$ in \mathbb{C} is actually a morphism in $\mathbf{CMon}(\mathbb{C})$. This follows from the fact that the diagram

$$\begin{array}{ccc}
 (C \times C) \times (C \times C) & \xrightarrow{+\times+} & C \times C \\
 \downarrow m & & \downarrow + \\
 (C \times C) \times (C \times C) & & C \\
 \downarrow +\times+ & & \downarrow + \\
 C \times C & \xrightarrow{+} & C
 \end{array}$$

in which m is the *middle interchange morphism*, commutes by a coherence result (for the case of monoids, see [18], Chapter VII, Section 3, Proposition 1). To prove that $\mathbf{CMon}(\mathbb{C})$

is unital, consider any diagram

$$\begin{array}{ccccc}
 & & T & & \\
 & x \nearrow & \downarrow n & \nwarrow y & \\
 X & \xrightarrow{\langle 1,0 \rangle} & X \times Y & \xleftarrow{\langle 0,1 \rangle} & Y
 \end{array}$$

in $\mathbf{CMon}(\mathbb{C})$ with n a monomorphism. It is easy to see that the diagram

$$\begin{array}{ccccc}
 X \times Y & \xrightarrow{x \times y} & T \times T & \xrightarrow{+} & T \\
 \searrow \langle 1,0 \rangle \times \langle 0,1 \rangle & & \downarrow n \times n & & \downarrow n \\
 & & (X \times Y) \times (X \times Y) & & \\
 \searrow \langle 1,0 \rangle \times \langle 0,1 \rangle & & \downarrow m & & \\
 & & (X \times X) \times (Y \times Y) & \xrightarrow{+\times+} & X \times Y
 \end{array}$$

is commutative. Since $n(+ (x \times y)) = (+ \times +) \langle 1,0 \rangle \times \langle 0,1 \rangle = 1_{X \times Y}$, n is a split epimorphism and therefore an isomorphism. Let $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ be any morphisms in $\mathbf{CMon}(\mathbb{C})$, then since $+(f \times g) \langle 1,0 \rangle = f$ and $+(f \times g) \langle 0,1 \rangle = g$, f commutes with g . \square

Remark 1.1.17. Note that the category $\mathbf{CMon}(\mathbb{C})$ is half-additive, which makes its finite products biproducts (that is, products and coproducts at the same time and $\langle 1,0 \rangle$ and $\langle 0,1 \rangle$ there are jointly isomorphic).

Proposition 1.1.18. Let \mathbb{C} be a unital category with cokernels, in which every object is commutative. The forgetful functor $W : \mathbf{Ab}(\mathbb{C}) \rightarrow \mathbb{C}$ has a left adjoint.

Proof. For any object $X \in \mathbb{C}$ we will write $+ : X \times X \rightarrow X$ for the morphism that exhibits that 1_X commutes with 1_X , it follows from Proposition 1.1.14 that $(X, +, 0)$ is commutative monoid. Let C be an object in \mathbb{C} and let (D, c) be the cokernel of the morphism $\langle 1,1 \rangle : C \rightarrow C \times C$. Since $c\sigma \langle 1,1 \rangle = 0$, there exists a unique morphism $s : D \rightarrow D$ making the diagram

$$\begin{array}{ccccccc}
 & & \langle 1,1 \rangle & & & & \\
 & \curvearrowright & & \searrow & & & \\
 C & \xrightarrow{\langle 1,1 \rangle} & C \times C & \xrightarrow{\sigma} & C \times C & \xrightarrow{c} & D \\
 & & & \searrow c & & & \uparrow s \\
 & & & & & & D
 \end{array}$$

commute. Since c is an epimorphism and the diagram

$$\begin{array}{ccccc}
 & & \langle 1, s \rangle c & & \\
 & \swarrow & & \searrow & \\
 C \times C & \xrightarrow{\langle 1, \sigma \rangle} & (C \times C) \times (C \times C) & \xrightarrow{c \times c} & D \times D \\
 \downarrow + & \searrow \langle 1, \sigma \rangle & \downarrow m & & \downarrow + \\
 & & (C \times C) \times (C \times C) & & \\
 & & \downarrow + \times + & & \\
 C & \xrightarrow{\langle 1, 1 \rangle} & C \times C & \xrightarrow{c} & D \\
 & \searrow & & \swarrow & \\
 & & 0 & &
 \end{array}$$

commutes, $+\langle 1, s \rangle = 0$ and $(D, 0, +, s)$ is an abelian group. The component of the unit of the adjunction at C is the morphism $c\langle 1, 0 \rangle$. To see this, consider any morphism $f : C \rightarrow A$ where A is an abelian group in \mathbb{C} . The diagram

$$\begin{array}{ccccccc}
 C & \xrightarrow{\langle 1, 1 \rangle} & C \times C & \xrightarrow{f \times f} & A \times A & \xrightarrow{1 \times -} & A \times A \\
 & \searrow f & & \searrow \langle 1, 1 \rangle & & & \downarrow + \\
 & & A & \xrightarrow{\langle 1, - \rangle} & A \times A & \xrightarrow{+} & A
 \end{array}$$

commutes, and consequently the morphism $+(1 \times -)(f \times f)$ factors through c ; that is, there exists a morphism $\bar{f} : D \rightarrow A$ such that $\bar{f}c = +(1 \times -)(f \times f)$. Since c is an epimorphism and

$$\begin{aligned}
 \bar{f}sc &= \bar{f}c\sigma = +(1 \times -)(f \times f)\sigma = +\sigma(- \times 1)(f \times f) \\
 &= +(- \times 1)(f \times f) = +(- \times -)(1 \times -)(f \times f) = -+(1 \times -)(f \times f) = -\bar{f}c,
 \end{aligned}$$

it follows that $\bar{f}s = -\bar{f}$. Since

$$\begin{aligned}
 \bar{f}c\langle 1, 0 \rangle &= +(1 \times -)(f \times f)\langle 1, 0 \rangle = f, \\
 \bar{f}c\langle 0, 1 \rangle &= \bar{f}c\sigma\langle 1, 0 \rangle = \bar{f}sc\langle 1, 0 \rangle = -\bar{f}c\langle 1, 0 \rangle, \\
 -+(1 \times -)(f \times f)\langle 1, 0 \rangle &= -f = +(1 \times -)(f \times f)\langle 0, 1 \rangle,
 \end{aligned}$$

and $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are jointly epimorphic, the original uniqueness property of \bar{f} implies

that \bar{f} is the unique morphism with $\bar{f}c\langle 1, 0 \rangle = f$. \square

Corollary 1.1.19. *Let \mathbb{C} be a regular weakly unital category with finite colimits satisfying Condition 1.1.9. The forgetful functor from $\mathbf{Ab}(\mathbb{C})$ to \mathbb{C} has a left adjoint.*

Proof. The forgetful functor can be decomposed as the forgetful functor from $\mathbf{Ab}(\mathbb{C})$ to $\mathbf{CMon}(\mathbb{C})$ followed by the forgetful functor from $\mathbf{CMon}(\mathbb{C})$ to \mathbb{C} . By Proposition 1.1.15 the forgetful functor from $\mathbf{CMon}(\mathbb{C})$ to \mathbb{C} has a left adjoint. It is obvious that $\mathbf{Ab}(\mathbf{CMon}(\mathbb{C})) \cong \mathbf{Ab}(\mathbb{C})$, and by Proposition 1.1.16 $\mathbf{CMon}(\mathbb{C})$ satisfies the requirements of Proposition 1.1.18, therefore the forgetful functor from $\mathbf{Ab}(\mathbb{C})$ to $\mathbf{CMon}(\mathbb{C})$ has a left adjoint. \square

1.2 Weakly Mal'tsev and Mal'tsev categories

Definition 1.2.1. *Let \mathbb{C} be a category and let B be a fixed object in \mathbb{C} . The category $\mathbf{Pt}(B)$ is defined to be $((B \downarrow \mathbb{C}) \downarrow (B, 1_B))$, or equivalently $((B, 1_B) \downarrow (\mathbb{C} \downarrow B))$. Explicitly, the objects are triples (A, α, β) , where $A \in \mathbb{C}$ and $\alpha : A \rightarrow B$ and $\beta : B \rightarrow A$ are morphisms in \mathbb{C} with $\alpha\beta = 1_B$. A morphism $f : (A, \alpha, \beta) \rightarrow (A', \alpha', \beta')$ in $\mathbf{Pt}(B)$ is a morphism $f : A \rightarrow A'$ in \mathbb{C} with $\alpha'f = \alpha$ and $f\beta = \beta'$. When \mathbb{C} is finitely complete, any morphism $p : E \rightarrow B$ induces a pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ defined as follows: let (A, α, β) be an object in $\mathbf{Pt}(B)$ and let $(A \times_B E, \pi_1, \pi_2)$ be the pullback of α and p ; then $p^*(A, \alpha, \beta) = (A \times_B E, \pi_2, \langle \beta p, 1 \rangle)$.*

For a category \mathbb{C} with pullbacks of split epimorphisms along arbitrary morphisms, consider the following condition:

Condition 1.2.2. *For each $p : E \rightarrow B$ in \mathbb{C} the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ preserves reflexive coequalizers.*

Remark 1.2.3. *Recall the following well known facts:*

1. *For a diagram $D : G \rightarrow (\mathbb{C} \downarrow B)$, the limit $\lim D$ can be calculated as the wide pullback over B of limits of the restrictions of D on the connected components of G - and in particular, for a connected G , $\lim D$ is calculated as in \mathbb{C} ;*
2. *All limits in $(B \downarrow \mathbb{C})$ are calculated as in \mathbb{C} .*

Putting these two facts (and/or their duals) together we get complete descriptions of limits and colimits in $\mathbf{Pt}(B)$.

Proposition 1.2.4. *Let \mathbb{C} be a category. The projection functor $P : \mathbf{Pt}(B) \rightarrow \mathbb{C}$ creates equalizers (coequalizers) and pullbacks (pushouts).*

Proof. The proof follows trivially from Remark 1.2.3. \square

Weakly Mal'tsev categories were introduced and studied in [19] Chapter 3, Definition 25, it was proved:

Theorem 1.2.5. \mathbb{C} is a weakly Mal'tsev category when for each $B \in \mathbb{C}$ the category $\mathbf{Pt}(B)$ is weakly unital.

Definition 1.2.6. \mathbb{C} is a Mal'tsev category when:

1. \mathbb{C} is finitely complete;
2. every reflexive relation in \mathbb{C} is an equivalence relation in \mathbb{C} .

Proposition 1.2.7. For a finitely complete category \mathbb{C} the following conditions are equivalent:

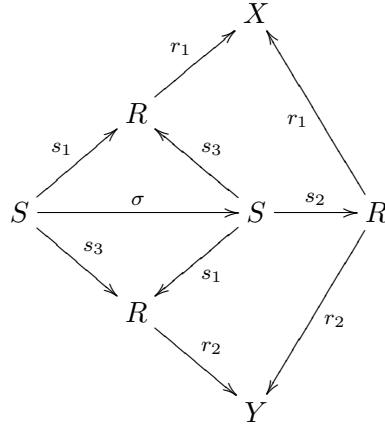
1. \mathbb{C} is Mal'tsev;
2. every relation is difunctional;
3. for each B the category $\mathbf{Pt}(B)$ is unital;

Proof. 1 \Rightarrow 2: Suppose $(R, r_1 : R \rightarrow X, r_2 : R \rightarrow Y)$ is a relation in \mathbb{C} . Consider the diagram

$$\begin{array}{ccccc}
 R & \xleftarrow{s_1} & S & \xrightarrow{s_3} & R \\
 \downarrow r_2 & & \downarrow s_2 & & \downarrow r_1 \\
 Y & \xleftarrow{r_2} & R & \xrightarrow{r_1} & X
 \end{array}$$

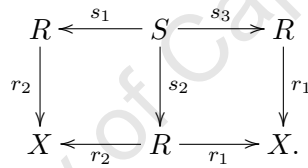
in which (S, s_1, s_2, s_3) is the limiting cone. For any morphisms $f, g : A \rightarrow S$ with $s_1 f = s_1 g$ and $s_3 f = s_3 g$, it is easy to see that $r_2 s_2 f = r_2 s_1 f = r_2 s_1 g = r_2 s_2 g$ and similarly $r_1 s_2 f = r_1 s_2 g$, therefore $s_2 f = s_2 g$ since r_1 and r_2 are jointly monic. It follows that $f = g$ and (S, s_1, s_3) is a relation. The morphism $\langle 1, 1, 1 \rangle$ clearly makes (S, s_1, s_3) into a reflexive relation, and since \mathbb{C} is a Mal'tsev, it follows that (S, s_1, s_3) is an equivalence relation. Let

$\sigma : S \rightarrow S$ be the symmetry isomorphism, since the diagram

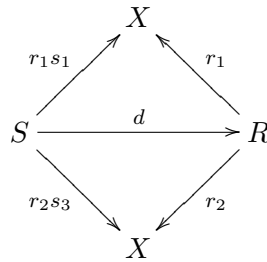


commutes, R is difunctional.

$2 \Rightarrow 1$: Let $(R, r_1 : R \rightarrow X, r_2 : R \rightarrow X)$ be a reflexive relation with splitting $s : X \rightarrow R$ and let (S, s_1, s_2, s_3) be the limiting cone of the diagram

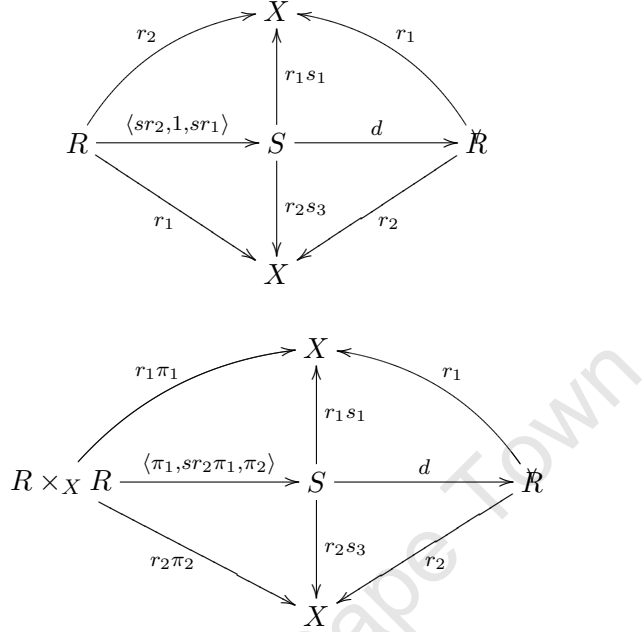


Since every relation is difunctional, there exists a morphism $d : S \rightarrow R$ making the diagram



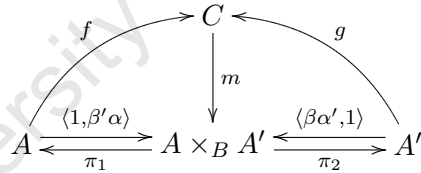
commute. Let $(R \times_X R, \pi_1, \pi_2)$ be the pullback of r_2 and r_1 with $r_2\pi_1 = r_1\pi_2$. The

morphisms $d\langle sr_2, 1, sr_1 \rangle : R \rightarrow R$ and $d\langle \pi_1, sr_2\pi_1, \pi_2 \rangle : R \times_X R \rightarrow R$ make the diagrams



commute and exhibit that (R, r_1, r_2) is an equivalence relation.

2 \Rightarrow 3: For a fixed object B in \mathbb{C} , consider any diagram



in which:

- $(A \times_B A', \alpha\pi_1, \langle \beta, \beta' \rangle)$ is the product of (A, α, β) and (A', α', β') in $\mathbf{Pt}(B)$;
- $mf = \langle 1, \beta' \alpha \rangle$;
- $mg = \langle \beta \alpha', 1 \rangle$.

It is easy to see that $(C, \pi_1 m, \pi_2 m)$ is a relation in \mathbb{C} and therefore, by assumption, difunc-

tional. Consider the diagram

$$\begin{array}{ccccc}
 C & \xleftarrow{s_1} & S & \xrightarrow{s_3} & C \\
 \pi_2 m \downarrow & & \downarrow s_2 & & \downarrow \pi_1 m \\
 A' & \xleftarrow{\pi_2 m} & C & \xrightarrow{\pi_1 m} & A
 \end{array}$$

in which (S, s_1, s_2, s_3) is the limiting cone. Since $(C, \pi_1 m, \pi_2 m)$ is difunctional, there exists a morphism $d : S \rightarrow C$ making the diagram

$$\begin{array}{ccc}
 & A & \\
 \pi_1 m s_1 \nearrow & & \nwarrow \pi_1 m \\
 S & \xrightarrow{d} & C \\
 \pi_2 m s_3 \searrow & & \swarrow \pi_2 m \\
 & A' &
 \end{array}$$

commute. And, since the diagram

$$\begin{array}{ccccc}
 C & \xleftarrow{f\pi_1} & A \times_B A' & \xrightarrow{g\pi_2} & C \\
 \pi_2 m \downarrow & & \downarrow g\beta' \alpha \pi_1 & & \downarrow \pi_1 m \\
 A' & \xleftarrow{\pi_2 m} & C & \xrightarrow{\pi_1 m} & A
 \end{array}$$

is commutative, there exists a morphism $\langle f\pi_1, g\beta' \alpha \pi_1, g\pi_2 \rangle : A \times_B A' \rightarrow S$. It easily follows that the composite $md\langle f\pi_1, g\beta' \alpha \pi_1, g\pi_2 \rangle$ makes the diagram

$$\begin{array}{ccccc}
 & A \times_B A' & & & \\
 & \pi_1 \swarrow & \downarrow \langle f\pi_1, g\beta' \alpha \pi_1, g\pi_2 \rangle & \searrow \pi_2 & \\
 A & \xleftarrow{\pi_1 m s_1} & S & \xrightarrow{\pi_2 m s_3} & A' \\
 & \swarrow \pi_1 & \downarrow d & \searrow \pi_2 & \\
 & & C & & \\
 & & \downarrow m & & \\
 & & A \times_B A' & &
 \end{array}$$

commute. Since π_1 and π_2 are jointly monomorphic, m is a split epimorphism and therefore an isomorphism.

3 \Rightarrow 2: For any relation $(R, r_1 : R \rightarrow X, r_2 : R \rightarrow Y)$, consider the diagrams

$$\begin{array}{ccccc} R & \xleftarrow{s_1} & S & \xrightarrow{s_3} & R \\ r_2 \downarrow & & \downarrow s_2 & & \downarrow r_1 \\ Y & \xleftarrow{r_2} & R & \xrightarrow{r_1} & X \end{array}$$

$$\begin{array}{ccccc} X & \xleftarrow{r_1} & R & \xrightarrow{r_2} & Y \\ r_1 \uparrow & & \uparrow t_4 & & \uparrow r_2 \\ R & \xleftarrow{t_1} & T & \xrightarrow{t_3} & R \\ r_2 \downarrow & & \downarrow t_2 & & \downarrow r_1 \\ Y & \xleftarrow{r_2} & R & \xrightarrow{r_1} & X, \end{array}$$

in which (S, s_1, s_2, s_3) and (T, t_1, t_2, t_3, t_4) are the limiting cones. Let (P, p_1, p_2) be the kernel pair of r_2 , and let (Q, q_1, q_2) be the kernel pair of r_1 . It easy to see that the diagram

$$\begin{array}{ccc} S & \begin{array}{c} \xleftarrow{\langle q_1, q_1, q_2 \rangle} \\ \xrightarrow{\langle s_2, s_3 \rangle} \end{array} & Q \\ \langle p_1, p_2, p_2 \rangle \updownarrow & \begin{array}{c} \langle s_1, s_2 \rangle \\ \langle 1, 1 \rangle \end{array} & \updownarrow q_1 \\ P & \begin{array}{c} \xleftarrow{\langle 1, 1 \rangle} \\ \xrightarrow{p_2} \end{array} & R \end{array}$$

is a pullback of split epimorphisms, therefore, $(S, s_2, \langle 1, 1, 1 \rangle)$ is the product of $(P, p_2, \langle 1, 1 \rangle)$ and $(Q, q_1, \langle 1, 1 \rangle)$ in $\mathbf{Pt}(R)$. Since the diagram

$$\begin{array}{ccc} T & \xrightarrow{\langle t_1, t_2, t_3 \rangle} & S \\ t_4 \downarrow & & \downarrow \langle r_1 s_1, r_2 s_3 \rangle \\ R & \xrightarrow{\langle r_1, r_2 \rangle} & X \times Y \end{array}$$

is a pullback, $\langle t_1, t_2, t_3 \rangle$ is a monomorphism. Since the diagram

$$\begin{array}{ccccc}
 & & T & & \\
 & \nearrow & \downarrow & \nwarrow & \\
 P & & \langle t_1, t_2, t_3 \rangle & & Q \\
 & \searrow & \downarrow & \swarrow & \\
 & & S & & \\
 \langle p_1, p_2, p_2, p_1 \rangle & & & & \langle q_1, q_1, q_2, q_2 \rangle \\
 P & \xrightarrow{\langle p_1, p_2, p_2 \rangle} & S & \xleftarrow{\langle q_1, q_1, q_2 \rangle} & Q
 \end{array}$$

commutes, it follows that $\langle t_1, t_2, t_3 \rangle$ is an isomorphism. The composite $t_4 \langle t_1, t_2, t_3 \rangle^{-1}$ makes the diagram

$$\begin{array}{ccc}
 & X & \\
 r_1 s_1 \nearrow & & \nwarrow r_1 \\
 S & \xrightarrow{t_4 \langle t_1, t_2, t_3 \rangle^{-1}} & R \\
 r_2 s_3 \searrow & & \swarrow r_2 \\
 & Y &
 \end{array}$$

commute, and shows that the relation (R, r_1, r_2) is difunctional. \square

Corollary 1.2.8. *Every Mal'tsev category is weakly Mal'tsev.*

Proof. Let \mathbb{C} be a Mal'tsev category. Since, by Proposition 1.2.7, for each B in \mathbb{C} the category $\mathbf{Pt}(B)$ is unital, and therefore by Proposition 1.1.7 weakly unital, it follows that \mathbb{C} is weakly Mal'tsev. \square

Definition 1.2.9. *A category \mathbb{C} is Barr-exact when:*

1. \mathbb{C} is a regular category;
2. equivalence relations in \mathbb{C} are effective, that is, equivalence relations are kernel pairs.

Proposition 1.2.10. *Condition 1.2.2 holds in a Barr-exact Mal'tsev category.*

Proof. Let \mathbb{C} be a Barr-exact Mal'tsev category. Consider any reflexive coequalizer diagram

$$(X, \chi, \psi) \begin{array}{c} \xrightarrow{f_1} \\ \xleftarrow{s} \\ \xrightarrow{f_2} \end{array} (A, \alpha, \beta) \xrightarrow{c} (C, \gamma, \delta)$$

in $\mathbf{Pt}(B)$; since \mathbb{C} is Barr-exact and consequently regular, there exists a regular epimorphism $e : X \rightarrow R$ and a monomorphism $\langle r_1, r_2 \rangle : R \rightarrow A \times A$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\langle f_1, f_2 \rangle} & A \times A \\ & \searrow e & \nearrow \langle r_1, r_2 \rangle \\ & R & \end{array}$$

commutes. Since \mathbb{C} is Mal'tsev and (R, r_1, r_2) is a reflexive relation ($r_1 e s = f_1 s = 1_A = f_2 s = r_2 e s$), it is an equivalence relation. Moreover, since \mathbb{C} is Barr-exact, (R, r_1, r_2) is a kernel pair. Since e is a regular epimorphism it easily follows that (R, r_1, r_2) is the kernel pair of c . Let $p : E \rightarrow B$ be a morphism in \mathbb{C} . Since p^* preserves regular epimorphisms (since \mathbb{C} is a regular category) and pullbacks, it preserves diagrams of the form

$$(X', \chi', \psi') \xrightarrow{e'} (R', \rho', \sigma') \begin{array}{c} \xrightarrow{r'_1} \\ \xrightarrow{r'_2} \end{array} (A', \alpha', \beta') \xrightarrow{c'} (C', \gamma', \delta'),$$

in which e' and c' are regular epimorphisms and $((R', \rho', \sigma'), r'_1, r'_2)$ is the kernel pair of c' . Since in any such diagram c' is the coequalizer of $r'_1 e'$ and $r'_2 e'$ it follows that $p^*(c)$ is the coequalizer of $p^*(f_1)$ and $p^*(f_2)$ as required. \square

1.3 Protomodular categories

Definition 1.3.1. A category \mathbb{C} is protomodular when:

1. pullbacks of split epimorphisms along arbitrary morphisms exist in \mathbb{C} ;
2. for each morphism $p : E \rightarrow B$ in \mathbb{C} the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ reflects isomorphisms.

Proposition 1.3.2. Let \mathbb{C} be a pointed finitely complete category. The following are equivalent:

1. \mathbb{C} is protomodular;

2. the split short five lemma holds in \mathbb{C} ; that is, for any diagram

$$\begin{array}{ccccc} K & \xrightarrow{\kappa} & A & \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} & B \\ \downarrow u & & \downarrow v & & \downarrow w \\ K' & \xrightarrow{\kappa'} & A' & \begin{array}{c} \xleftarrow{\alpha'} \\ \xrightarrow{\beta'} \end{array} & B' \end{array}$$

in \mathbb{C} , in which:

- (A, α, β) and (A', α', β') are objects in $\mathbf{Pt}(B)$;
- (K, κ) is the kernel α ;
- (K', κ') is the kernel α' ;
- $v\kappa = \kappa'u$, $w\alpha = \alpha'v$ and $v\beta = \beta'w$;

v is an isomorphism if both u and w are isomorphisms.

Proof. Since \mathbb{C} is pointed we can identify the category $\mathbf{Pt}(1)$ with \mathbb{C} . For any morphism $p : E \rightarrow B$, let $0_B : 1 \rightarrow B$ and $0_E : 1 \rightarrow E$ be the unique morphisms from the initial object. Since $p0_E = 0_B$, we see that $0_E^*p^* = 0_B^*$; therefore, reflection of isomorphisms by 0_B^* and preservation of isomorphisms by 0_E^* implies reflection of isomorphisms by p^* . The functor 0_B^* is defined by pulling back along the morphism $0 \rightarrow B$; consequently, it is equivalent to the kernel functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbb{C}$ which assigns to each triple (A, α, β) the kernel of α . The split short five lemma is simply a reformulation of reflection of isomorphisms by the functor kernel functor. \square

Proposition 1.3.3. *Every pointed finitely complete protomodular category is unital.*

Proof. In any commutative diagram

$$\begin{array}{ccccc} & & W & & \\ & \nearrow f & \downarrow m & \nwarrow g & \\ X & \xrightarrow{\langle 1,0 \rangle} & X \times Y & \xleftarrow{\langle 0,1 \rangle} & Y \end{array}$$

in which m is a monomorphism, we must show that m is an isomorphism. Consider the

diagram

$$\begin{array}{ccccc}
 X & & & & \\
 \searrow^{(1,f)} & & f & & \\
 & P & \xrightarrow{q} & W & \\
 \downarrow p & & & & \downarrow m \\
 X & \xrightarrow{\langle 1,0 \rangle} & X \times Y & & \\
 \downarrow & & & & \downarrow \pi_2 \\
 0 & \longrightarrow & Y & &
 \end{array}$$

in which (P, p, q) is the pullback of $\langle 1, 0 \rangle$ and m , and $\langle 1, f \rangle$ is canonical morphism into the pullback. It follows that p is a split epimorphism, and therefore an isomorphism; since both the top and the bottom squares are pullbacks their composite is also a pullback, and therefore (X, f) is the kernel of $\pi_2 m$. In the diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & W & \xrightleftharpoons[\pi_2 m]{} & Y \\
 \parallel & & \downarrow m & \xrightarrow{g} & \parallel \\
 X & \xrightarrow{\langle 1,0 \rangle} & X \times Y & \xrightleftharpoons[\langle 0,1 \rangle]{\pi_2} & Y,
 \end{array}$$

since $\pi_2 m g = \pi_2 \langle 0, 1 \rangle = 1_Y$ and since the split short five lemma holds in \mathbb{C} (by Proposition 1.3.2), m is an isomorphism. \square

Remark 1.3.4. *It well know that a category \mathbb{C} with finite limits is protomodular if and only if for each B in \mathbb{C} and for every (A, α, β) in $\mathbf{Pt}(B)$, κ and β are jointly extremal epimorphic. It is easy to see that Proposition 1.3.3 follows trivially from this, however, we do not need this characterization in what follows.*

Proposition 1.3.5. *Let \mathbb{C} be a category in which pullbacks of split epimorphisms along arbitrary morphisms exist. For any object $(B, \gamma, \delta) \in \mathbf{Pt}(C)$, the category $\mathbf{Pt}(B, \gamma, \delta)$ is isomorphic to the category $\mathbf{Pt}(B)$. Moreover, for any morphism $p : (E, \theta, \phi) \rightarrow (B, \gamma, \delta)$ these isomorphisms commute with pullback functors, that is, the diagram*

$$\begin{array}{ccc}
 \mathbf{Pt}(B, \gamma, \delta) & \xrightarrow{\cong} & \mathbf{Pt}(B) \\
 \downarrow p^* & & \downarrow p^* \\
 \mathbf{Pt}(E, \theta, \phi) & \xrightarrow{\cong} & \mathbf{Pt}(E)
 \end{array}$$

is commutative.

Proof. For any object $((A, \lambda, \rho), \alpha, \beta) \in \mathbf{Pt}(B, \gamma, \delta)$, since $\lambda = \gamma\alpha$ and $\rho = \beta\rho$, there exists an isomorphism between $\mathbf{Pt}(B, \gamma, \delta)$ and $\mathbf{Pt}(B)$. Since, by Proposition 1.2.4, the forgetful functor $P : \mathbf{Pt}(B) \rightarrow \mathbb{C}$ creates pullbacks, it is easy to see that pullback functors commute with these isomorphisms of categories. \square

Proposition 1.3.6. *If \mathbb{C} is a protomodular category, then for any B in \mathbb{C} the category $\mathbf{Pt}(B)$ is pointed protomodular.*

Proof. The category $\mathbf{Pt}(B)$ is trivially pointed. Since, by Proposition 1.2.4, the forgetful functor $P : \mathbf{Pt}(B) \rightarrow \mathbb{C}$ creates pullbacks, $\mathbf{Pt}(B)$ has pullbacks of split epimorphisms along arbitrary morphisms when \mathbb{C} does. It easily follows from the second part of Proposition 1.3.5, that since pullback functors between categories of points in \mathbb{C} reflect isomorphisms, the same is true for pullback functors between categories of points in $\mathbf{Pt}(B)$. \square

Proposition 1.3.7. *Every finitely complete protomodular category is Mal'tsev.*

Proof. For each $B \in \mathbb{C}$ by Proposition 1.3.6 the category $\mathbf{Pt}(B)$ is pointed protomodular, and so by Proposition 1.3.3, $\mathbf{Pt}(B)$ is unital. Consequently, by Proposition 1.2.7, the category \mathbb{C} is Mal'tsev. \square

Proposition 1.3.8. *Let \mathbb{C} be a pointed protomodular category with reflexive coequalizers and pushouts of split monomorphisms along arbitrary morphisms, satisfying Condition 1.2.2. For any morphism $p : E \rightarrow B$ the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ is monadic.*

Proof. For an object (D, δ, ϵ) in $\mathbf{Pt}(E)$, consider the diagram

$$\begin{array}{ccccc}
 & & B & \xleftarrow{p\delta} & D \\
 & & \uparrow [1, p\delta] & & \uparrow \epsilon \\
 & & B +_E D & \xleftarrow{\iota_2} & D \\
 & & \uparrow \iota_1 & & \uparrow \epsilon \\
 & & B & \xleftarrow{p} & E
 \end{array}$$

in which:

- $(B +_E D, \iota_1, \iota_2)$ is the pushout of p and ϵ ;
- $[1, p\delta]$ is the unique morphism with $[1, p\delta]\iota_1 = 1_B$ and $[1, p\delta]\iota_2 = p\delta$.

It is easy to see that the functor assigning to each triple (D, δ, ϵ) the triple $(B+{}_E D, [1, p\delta], \iota_1)$, is the right adjoint to p^* . Since by Proposition 1.2.4 the forgetful functor $P : \mathbf{Pt}(B) \rightarrow \mathbb{C}$ creates coequalizers, the category $\mathbf{Pt}(B)$ has reflexive coequalizers. It follows by Condition 1.2.2 that the functor p^* preserves coequalizers of reflexive pairs, and since \mathbb{C} is protomodular, p^* reflects isomorphisms. It follows by the reflexive form of Beck's monadicity theorem that the functor p^* is monadic. \square

1.4 Semi-abelian categories

Definition 1.4.1. *A category \mathbb{C} is semi-abelian when:*

1. \mathbb{C} is pointed;
2. \mathbb{C} is finitely cocomplete;
3. \mathbb{C} is Barr-exact;
4. \mathbb{C} is protomodular.

Proposition 1.4.2. *Let \mathbb{C} be a semi-abelian category. For any morphism $p : E \rightarrow B$ the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ is monadic.*

Proof. Since \mathbb{C} is a finitely complete protomodular category, it follows from Proposition 1.3.7 that \mathbb{C} is Mal'tsev, and by Proposition 1.2.10 Condition 1.2.2 holds in \mathbb{C} . It then follows from Proposition 1.3.8 that for any $p : E \rightarrow B$ the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ is monadic. \square

1.5 Properties of functors

We begin with some observations about colimits. The proposition and the lemma that follow can be found in [17], Proposition 4 and Lemma 3.

Proposition 1.5.1. *Let \mathbb{C} be a category with finite colimits. In any diagram*

$$\begin{array}{ccc}
 A & \xrightarrow{f_1} & B \\
 \downarrow f'_2 & \begin{array}{c} \parallel \\ f_2 \end{array} & \downarrow g'_1 \\
 A' & \xrightarrow[g_2]{g_1} & B'
 \end{array}$$

in which:

- f_1 and f_2 are a reflexive pair with section s ;
- f'_1 and f'_2 are a reflexive pair with section s' ;
- $g'_i f_j = g_j f'_i$ $i, j \in \{1, 2\}$.

The coequalizer of $g'_1 f_1$ and $g'_2 f_2$ is equal to the diagonal of the pushout of the coequalizers of g_1 and of g_2 , and g'_1 and g'_2 .

Proof. It is easy to see that to prove the proposition is equivalent to showing that for each morphism $x : B' \rightarrow X$ that the following conditions are equivalent:

1. $xg_1 = xg_2$ and $xg'_1 = xg'_2$;
2. $xg'_1 f_1 = xg'_2 f_2$.

If $xg_1 = xg_2$ and $xg'_1 = xg'_2$, then pre-composing the first identity with f'_1 and the second with f_2 and noting that $g'_2 f_1 = g'_1 f_2$ we observe that: $xg'_1 f_1 = xg_1 f'_1 = xg_2 f'_1 = xg'_1 f_2 = xg'_2 f_2$. Conversely if $xg'_1 f_1 = xg'_2 f_2$, pre-composing with s gives $xg'_1 = xg'_2$. Noting that $g'_1 f_1 = g_1 f'_1$ and that $g'_2 f_2 = g_2 f'_2$ it easily follows that $xg_1 f'_1 = xg'_1 f_1 = xg'_2 f_2 = xg_2 f'_2$, pre-composing this identity with s' gives $xg_1 = xg_2$. \square

Lemma 1.5.2. *Let $T : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ be any functor. If*

$$A \begin{array}{c} \xrightarrow{f_1} \\ \xleftarrow{s} \\ \xrightarrow{f_2} \end{array} B \xrightarrow{c} C$$

and

$$A' \begin{array}{c} \xrightarrow{f'_1} \\ \xleftarrow{s'} \\ \xrightarrow{f'_2} \end{array} B' \xrightarrow{c'} C'$$

are reflexive coequalizers, preserved, for any $D \in \mathbb{C}$, by the functors $T(D, -)$ and $T(-, D)$, then

$$T(A, A') \begin{array}{c} \xrightarrow{T(f_1, f'_1)} \\ \xleftarrow{T(s, s')} \\ \xrightarrow{T(f_2, f'_2)} \end{array} T(B, B') \xrightarrow{T(c, c')} T(C, C')$$

is a reflexive coequalizer.

Proof. The diagram

$$\begin{array}{ccc}
T(A, A') & \xrightarrow[T(1, f'_2)]{T(1, f'_1)} & T(A, B') \\
\downarrow T(f_2, 1) & & \downarrow T(f_2, 1) \\
T(B, A') & \xrightarrow[T(1, f'_2)]{T(1, f'_1)} & T(B, B')
\end{array}$$

$T(f_1, 1) \quad T(f_1, 1)$

satisfies the conditions of Proposition 1.5.1 and so it follows that the diagonal of the pushout of the regular epimorphisms $T(c, 1) : T(B, B') \rightarrow T(C, B')$ and $T(1, c') : T(B, B') \rightarrow T(B, C')$ is the coequalizer of $T(f_1, f'_1)$ and $T(f_2, f'_2)$. Consider the diagram

$$\begin{array}{ccccc}
T(B, A') & \xrightarrow[T(1, f'_2)]{T(1, f'_1)} & T(B, B') & \xrightarrow{T(1, c')} & T(B, C') \\
\downarrow T(c, 1) & & \downarrow T(c, 1) & & \downarrow T(c, 1) \\
T(C, A') & \xrightarrow[T(1, f'_2)]{T(1, f'_1)} & T(C, B') & \xrightarrow{T(1, c')} & T(C, C') \\
& & & & \searrow \bar{h} \\
& & & & Z
\end{array}$$

h (curved arrow from $T(C, B')$ to Z)
 h' (curved arrow from $T(B, C')$ to Z)

in which h and h' are any morphisms with $hT(c, 1) = h'T(1, c')$. Since

$$hT(1, f'_1)T(c, 1) = hT(c, 1)T(1, f'_1) = hT(c, 1)T(1, f'_2) = hT(1, f'_2)T(c, 1)$$

and since $T(c, 1)$ is an epimorphism (regular epimorphism) $hT(1, f'_1) = hT(1, f'_2)$. Since $T(1, c') : T(C, B') \rightarrow T(C, C')$ is the coequalizer of $T(1, f'_1)$ and $T(1, f'_2)$ there exists a unique morphism $\bar{h} : T(C, C') \rightarrow Z$ with $\bar{h}T(1, c') = h$. Since $h'T(1, c') = hT(c, 1) = \bar{h}T(1, c')T(c, 1) = \bar{h}T(c, 1)T(1, c')$ and $T(1, c')$ is an epimorphism (regular epimorphism) $\bar{h}T(c, 1) = h'$. \square

Proposition 1.5.3. *Let $D : \mathbb{G} \rightarrow \mathbb{C}$ and $D' : \mathbb{G}' \rightarrow \mathbb{C}$ be functors, and let $(\lambda_x : D(x) \rightarrow L)_{x \in \mathbb{G}}$ and $(\rho_y : D'(y) \rightarrow R)_{y \in \mathbb{G}'}$ be the colimiting cocones over D and D' respectively. If for each $C \in \mathbb{C}$*

$$(T(\lambda_x, 1_C) : T(D(x), C) \rightarrow T(L, C))_{x \in \mathbb{G}}$$

is the colimiting cocone for the diagram $T(D(-), C) : \mathbb{G} \rightarrow \mathbb{C}$, and

$$(T(1_C, \rho_y) : T(C, D'(y)) \rightarrow T(C, R))_{y \in \mathbb{G}'}$$

is the colimiting cocone for the diagram $T(C, D'(-)) : \mathbb{G}' \rightarrow \mathbb{C}$, then

$$(T(\lambda_x, \rho_y) : T(D(x), D'(y)) \rightarrow T(L, R))_{(x,y) \in \mathbb{G} \times \mathbb{G}'}$$

is the colimiting cocone of the diagram $T(D \times D') : \mathbb{G} \times \mathbb{G}' \rightarrow \mathbb{C}$. If in addition \mathbb{G} is filtered and $\mathbb{G}' = \mathbb{G}$ then

$$(T(\lambda_x, \lambda_x) : T(D(x), D'(x)) \rightarrow T(L, R))_{x \in \mathbb{G}}$$

is the colimiting cocone of the diagram $T(D \times D')(1, 1) : \mathbb{G} \rightarrow \mathbb{C}$.

Proof. Let $(\gamma_{x,y} : T(D(x), D'(y)) \rightarrow C)_{(x,y) \in \mathbb{G} \times \mathbb{G}'}$ be a cocone over the diagram $T(D \times D') : \mathbb{G} \times \mathbb{G}' \rightarrow \mathbb{C}$. Since for each $y \in \mathbb{G}'$ and for any morphism $f : x \rightarrow x'$ in \mathbb{G} the diagram

$$\begin{array}{ccc} & C & \\ \gamma_{x,y} \nearrow & & \nwarrow \gamma_{x',y} \\ T(D(x), D(y)) & \xrightarrow{T(D(f), 1)} & T(D(x'), D'(y)) \end{array}$$

commutes, there exists a unique morphism $\gamma_y : T(L, D(y)) \rightarrow C$ making the diagram

$$\begin{array}{ccc} T(L, D(y)) & \xrightarrow{\gamma_y} & C \\ & \nwarrow T(\lambda_x, 1) & \nearrow \gamma_{x,y} \\ & T(D(x), D'(y)) & \end{array}$$

commute. Since for any $g : y \rightarrow y'$ in \mathbb{G} and since T is a functor, the square in the diagram

$$\begin{array}{ccccc} T(D(x), D'(y)) & \xrightarrow{T(\lambda_x, 1)} & T(L, D'(y)) & & \\ \downarrow T(1, D'(g)) & & \downarrow T(1, D'(g)) & \searrow \gamma_y & \\ & & & & C \\ T(D(x), D'(y')) & \xrightarrow{T(\lambda_x, 1)} & T(L, D'(y')) & \nearrow \gamma'_{y'} & \end{array}$$

commutes, the outer arrows commute since $\gamma_y T(\lambda_x, 1) = \gamma_{x,y}$, $\gamma'_{y'} T(\lambda_x, 1) = \gamma_{x,y'}$ and $\gamma_{x,y} =$

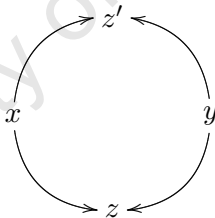
$\gamma_{x,y}T(1, G(y))$. Consequently, since the morphisms $T(\lambda_x, 1)$ are jointly epimorphic, the triangle commutes and there exists a unique morphism $\gamma : T(L, R) \rightarrow C$ making the diagram

$$\begin{array}{ccc}
 T(L, R) & \xrightarrow{\gamma} & C \\
 & \swarrow T(1, \rho_y) & \nearrow \gamma_y \\
 & T(L, D'(y)) &
 \end{array}$$

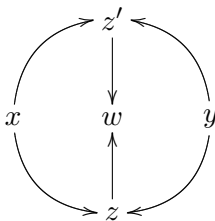
commute. Note that the diagram

$$\begin{array}{ccc}
 T(L, R) & \xrightarrow{\gamma} & C \\
 & \swarrow T(1, \rho_y) & \nearrow \gamma_y \\
 & T(L, D'(y)) & \\
 & \uparrow T(\lambda_x, 1) & \\
 & T(D(x), D'(y)) & \\
 & \searrow T(\lambda_x, \rho_y) & \swarrow \gamma_{x,y} \\
 & & C
 \end{array}$$

commutes. Let $(\tau_x : T(D(x), D'(x)) \rightarrow T)_{x \in \mathbb{G}}$ be a cocone over the diagram $T(D \times D')\langle 1, 1 \rangle : \mathbb{G} \rightarrow \mathbb{C}$. Since \mathbb{G} is filtered for any x and y in \mathbb{G} there exist morphisms $x \rightarrow z$ and $y \rightarrow z$ in \mathbb{G} , and for any diagram



in \mathbb{G} there exist morphisms $z \rightarrow w$ and $z' \rightarrow w$ such that the diagram



commutes. Since the diagram

$$\begin{array}{ccccc}
 & & T(D(z), D'(z)) & & \\
 & \nearrow & \downarrow & \searrow & \\
 T(D(x), D'(y)) & \longrightarrow & T(D(w), D'(w)) & \xrightarrow{\tau_w} & T \\
 & \searrow & \uparrow & \nearrow & \\
 & & T(D(z'), D'(z')) & &
 \end{array}$$

commutes, we can define $\tau_{x,y}$ to be either of the outer composites in the diagram above. Again since \mathbb{G} is filtered for any morphisms $f : x \rightarrow x'$ and $g : y \rightarrow y'$, there exist morphisms $h : z \rightarrow z'$, $x \rightarrow z$, $x' \rightarrow z'$, $y \rightarrow z$ and $y' \rightarrow z'$ such that the diagram

$$\begin{array}{ccccc}
 x & \longrightarrow & z & \longleftarrow & y \\
 \downarrow f & & \downarrow h & & \downarrow g \\
 x & \longrightarrow & z & \longleftarrow & y
 \end{array}$$

commutes. Consequently, since $(\tau_x : T(D(x), D'(x)) \rightarrow T)_{x \in \mathbb{G}}$ is a cocone over the diagram $T(D \times D')\langle 1, 1 \rangle : \mathbb{G} \rightarrow \mathbb{C}$, the diagram

$$\begin{array}{ccccc}
 T(D(x), D'(y)) & \longrightarrow & T(D(z), D'(z)) & & \\
 \downarrow T(D(f), D'(g)) & & \downarrow T(D(h), D'(h')) & \searrow \tau_z & \\
 T(D(x'), D'(y')) & \longrightarrow & T(D(z'), D'(z')) & \nearrow \tau_{z'} & T
 \end{array}$$

commutes and $(\tau_{x,y} : T(D(x), D'(y)) \rightarrow T)_{(x,y) \in \mathbb{G} \times \mathbb{G}}$ is a cocone over the diagram $T(D \times D') : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{C}$. \square

Theorem 1.5.4. *Let \mathbb{C} be finitely cocomplete. A functor $U : \mathbb{C} \rightarrow \mathbb{D}$ preserves all finite colimits when:*

1. U preserves the initial object;
2. U preserves binary coproducts;
3. U preserves reflexive coequalizers.

When \mathbb{C} is also small cocomplete, U preserves all small colimits if in addition to the other conditions:

4. U preserves filtered colimits.

Proof. It is obvious that Conditions 1 and 2 imply that U preserves all finite coproducts. It is well known that in any category with finite coproducts, every finite colimit of a diagram can be presented as a coequalizer of a reflexive pair with domain and codomain constructed as finite coproducts of objects from the diagram (see [18], Chapter V, Section 2, Exercise 1(a)). Consequently, if U preserves reflexive coequalizers, it will preserve all finite colimits. Now let \mathbb{C} be small cocomplete. It is well known that any small cocomplete category, any small colimit can be presented as a small filtered colimit of its finite subcolimits. Consequently, if U preserves finite colimits as well as small filtered colimits, it preserves all small colimits. \square

Proposition 1.5.5. *For any adjunction $(F, U, \eta, \epsilon) : \mathbb{X} \rightarrow \mathbb{A}$, if U is faithful and if \mathbb{X} has a small generating set, then \mathbb{A} has a small generating set.*

Proof. Let \mathbb{Q} be a small generating set of \mathbb{X} , let $\mathbb{R} = \{FQ \mid Q \in \mathbb{Q}\}$, and let $f, g : A \rightarrow A'$ be any morphisms in \mathbb{A} with $f \neq g$. Since U is faithful and \mathbb{Q} is a generating set, $Uf \neq Ug$ and there exists an object $Q \in \mathbb{Q}$ and a morphism $h : Q \rightarrow UA$ such that $Ufh \neq Ugh$. Since (F, U, η, ϵ) is an adjunction and ϵ is a natural transformation, we have $f\epsilon_A Fh = \epsilon_{A'} F(Ufh) \neq \epsilon_{A'} F(Ugh) = g\epsilon_A Fh$. \square

Theorem 1.5.6. *Consider a diagram of functors*

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{V} & \mathbb{A}' \\ \swarrow U & & \searrow F' \\ & \mathbb{X} & \\ \nwarrow F & & \nearrow U' \end{array}$$

where (F, U, η, ϵ) and $(F', U', \eta', \epsilon')$ are adjunctions and $U'V = U$. If \mathbb{A} has coequalizers, and if for each $A' \in \mathbb{A}'$ the diagram

$$F'U'F'U'A' \begin{array}{c} \xrightarrow{F'U'\epsilon'_{A'}} \\ \xrightarrow{\epsilon'_{F'U'A'}} \end{array} \rightrightarrows F'U'A' \xrightarrow{\epsilon'_{A'}} A'$$

is a coequalizer diagram, then the functor V has a left adjoint. Moreover, for an object

$A' \in \mathbb{A}'$ this left adjoint L appears as a coequalizer diagram

$$\begin{array}{ccccc}
FU'F'U'A' & \xrightarrow{FU'\epsilon'_{A'}} & FU'A' & \dashrightarrow & L(A') \\
\downarrow FU'F'\eta_{U'A'} & & \uparrow \epsilon_{FU'A'} & & \\
FU'F'UFU'A' & \xrightarrow{FU'\epsilon'_{VFU'A'}} & FUFU'A' & &
\end{array}$$

If in addition for each $A \in \mathbb{A}$, ϵ_A is the coequalizer of the morphisms $FU\epsilon_A, \epsilon_{FUA} : FUFUA \rightarrow FUA$, then the counit $\bar{\epsilon} : LV \rightarrow 1_{\mathbb{A}}$ has the property that for each $A \in \mathbb{A}$, $\bar{\epsilon}_A$ is the coequalizer of the morphisms $LV\bar{\epsilon}_A, \bar{\epsilon}_{LVA} : LVLVA \rightarrow LVA$.

Proof. Since (F, U, η, ϵ) and $(F', U', \eta', \epsilon')$ are adjunctions, there exist isomorphisms $\phi : \text{hom}(F^{\text{op}} \times 1_{\mathbb{A}}) \rightarrow \text{hom}(1_{\mathbb{X}}^{\text{op}} \times U)$ and $\phi' : \text{hom}(F'^{\text{op}} \times 1_{\mathbb{A}'}) \rightarrow \text{hom}(1_{\mathbb{X}'}^{\text{op}} \times U')$, and from these natural transformations we can form $\lambda = \phi'^{-1}((U')^{\text{op}} \times V) \cdot \phi(U'^{\text{op}} \times 1_{\mathbb{A}}) : \text{hom}(F'U'^{\text{op}} \times V) \rightarrow \text{hom}(FU'^{\text{op}} \times 1_{\mathbb{A}})$. Let $\alpha = (\epsilon'VF) \cdot (F'\eta)$. Since, for each $A \in \mathbb{A}$ the diagram

$$\begin{array}{ccc}
F'UA & & \\
\downarrow F'\eta_{UA} & \searrow 1_{F'UA} & \\
F'UFUA & \xrightarrow{F'U\epsilon_A} & F'UA \\
\downarrow \epsilon'_{VFUA} & & \downarrow \epsilon'_{VA} \\
VFUA & \xrightarrow{V\epsilon_A} & VA
\end{array}$$

commutes, it easily follows that the diagram

$$\begin{array}{ccccc}
F'U'F'X & \xrightarrow{F'U'\alpha_X} & F'UF'X & \xrightarrow{\alpha_{UF'X}} & VFUF'X \\
\downarrow \epsilon'_{F'X} & & \downarrow \epsilon'_{VF'X} & \swarrow V\epsilon_{F'X} & \\
F'X & \xrightarrow{\alpha_X} & VF'X & &
\end{array}$$

also commutes. Therefore, the identities $\epsilon'V = (V\epsilon) \cdot (\alpha U)$ and $\alpha \cdot (\epsilon F') = (V\epsilon F') \cdot (\alpha UF) \cdot (F'U'\alpha)$ hold. From the definition of λ it is easy to see that for any $g : FU'A' \rightarrow A$ the

diagram

$$\begin{array}{ccc}
 F'U'A' & \xrightarrow{\lambda_{(A',A)}(g)} & VA \\
 \downarrow F'\eta_{U'A'} & \searrow \alpha_{U'A'} & \nearrow Vg \\
 & VFU'A' & \\
 & \nearrow \epsilon'_{VFU'A} & \uparrow \epsilon'_{VA} \\
 F'UFU'A' & \xrightarrow{F'Ug} & F'UA
 \end{array}$$

commutes. Let $H, I : \mathbb{A}'^{\text{op}} \times \mathbb{A} \rightarrow \mathbf{Set}$ be the functors defined on objects $(A', A) \in \mathbb{A}'^{\text{op}} \times \mathbb{A}$ by:

$$H(A', A) = \left\{ f : F'U'A' \rightarrow VA \mid F'U'F'U'A' \xrightarrow[\epsilon_{F'U'A'}]{F'U'\epsilon'_{A'}} F'U'A' \xrightarrow{f} VA \text{ is a fork} \right\},$$

$$I(A', A) = \left\{ g : FU'A' \rightarrow A \mid FU'F'U'A' \xrightarrow[\epsilon_{FU'A'}]{FU'\epsilon'_{A'}} FU'A' \xrightarrow{g} A \text{ is a fork} \right\}.$$

For any $g \in I(A', A)$, considering the diagrams

$$\begin{array}{ccc}
 F'U'F'U'A' & \xrightarrow{\alpha_{U'F'U'A'}} & VFU'F'U'A' \\
 \downarrow F'U'\epsilon'_{A'} & & \downarrow VFU'\epsilon'_{A'} \\
 F'U'A' & \xrightarrow{\alpha_{U'A'}} & VFU'A' \\
 & \searrow \lambda_{(A',A)}(g) & \downarrow Vg \\
 & & VA
 \end{array}$$

$$\begin{array}{ccc}
F'U'F'U'A' & \xrightarrow{\alpha_{U'F'U'A'}} & VFU'F'U'A' \\
\downarrow F'U'\alpha_{U'A'} & & \downarrow VFU'\alpha_{U'A'} \\
F'UFU'A' & \xrightarrow{\alpha_{UFU'A'}} & VFUFU'A' \\
\downarrow V\epsilon_{FU'A'} & & \downarrow V\epsilon_{FU'A'} \\
F'U'A' & \xrightarrow{\alpha_{U'A'}} & VFU'A' \\
\downarrow \lambda_{(A',A)}(g) & & \downarrow Vg \\
& & VA
\end{array}$$

$\epsilon'_{F'U'A'}$ (curved arrow from $F'U'F'U'A'$ to $F'U'A'$)

we see that $\lambda_{(F'U'A',A)}(gFU'\epsilon'_{A'}) = VgVFU'\epsilon'_{A'}\alpha_{U'F'U'A'} = \lambda_{(A',A)}(g)F'U'\epsilon'_{A'}$ and

$$\lambda_{(F'U'A',A)}(g\epsilon_{FU'A'}FU'\alpha_{U'A'}) = VgV\epsilon_{FU'A'}VFU'\alpha_{U'A'}\alpha_{U'F'U'A'} = \lambda_{(A',A)}(g)\epsilon'_{F'U'A'}.$$

Therefore $\lambda_{(A',A)}(g) \in H(A', A)$, that is, λ restricts to a isomorphism $\tau : I \rightarrow H$. Note that under the assumptions of the theorem $I \cong \text{hom}(L^{\text{op}} \times 1_{\mathbb{A}})$ and $H \cong \text{hom}(1_{\mathbb{A}}^{\text{op}} \times V)$, proving that L and V are adjoint functors.

Since (LVA, c_A) is the coequalizer of the morphisms $FU'\epsilon'_{VA}, \epsilon_{FUA}FU'\alpha_{UA} : FU'F'UA \rightarrow FUA$, it easy to see that $\bar{\epsilon}_A$ is the unique morphism making the diagram

$$\begin{array}{ccccc}
FU'F'UA & \xrightarrow{FU'\epsilon'_{VA}} & FUA & \xrightarrow{c_A} & LVA \\
\downarrow FU'\alpha_{UA} & \xrightarrow{\epsilon_{FUA}FU'\alpha_{UA}} & \nearrow FU\epsilon_A & \searrow \epsilon_A & \downarrow \bar{\epsilon}_A \\
FUFUA & & FUA & & A \\
& & \nearrow \epsilon_{FUA} & & \\
& & & &
\end{array}$$

commute. Since the diagram

$$\begin{array}{ccc}
FULVA & \xrightarrow{c_{LVA}} & LVLVA \\
\downarrow \epsilon_{LVA} & & \downarrow \bar{\epsilon}_{LVA} \\
& & LVA \\
\downarrow c_A FU \bar{\epsilon}_A & & \downarrow LV \bar{\epsilon}_A
\end{array}$$

commutes and c_{LVA} is an epimorphism, to prove that $\bar{\epsilon}_A$ is the coequalizer of $\bar{\epsilon}_{LVA}$ and

$LV\bar{\epsilon}_A$, it suffices to show that $\bar{\epsilon}_A$ is the coequalizer of ϵ_{LVA} and $c_A FU\bar{\epsilon}_A$. It is easy to see that $\bar{\epsilon}_A c_A FU\bar{\epsilon}_A = \epsilon_A FU\bar{\epsilon}_A = \bar{\epsilon}_A \epsilon_{LVA}$. Let $g : LVA \rightarrow C$ be any morphism with $g\epsilon_{LVA} = gcFU\bar{\epsilon}_A$, consider the diagram:

$$\begin{array}{ccccc}
 FULVA & \begin{array}{c} \xrightarrow{\epsilon_{LVA}} \\ \xrightarrow{c_A FU\bar{\epsilon}_A} \end{array} & LVA & \xrightarrow{g} & C \\
 \uparrow FUc_A & & \uparrow c_A & & \uparrow \bar{g} \\
 FUFUA & \begin{array}{c} \xrightarrow{\epsilon_{FUA}} \\ \xrightarrow{FU\epsilon_A} \end{array} & FUA & \xrightarrow{\epsilon_A} & A
 \end{array}$$

Since c_A is an epimorphism and $gc_A = \bar{g}\epsilon_A = \bar{g}\bar{\epsilon}_A c_A$, we have $g = \bar{g}\bar{\epsilon}_A$. Since $\bar{\epsilon}_A c_A = \epsilon_A$ and ϵ_A is an epimorphism, $\bar{\epsilon}_A$ is also an epimorphism, therefore such \bar{g} is unique. \square

1.6 Relative homology

Throughout this section we consider an abelian category \mathbb{A} and a class of epimorphisms \mathbb{E} in \mathbb{A} containing all split epimorphisms and satisfying the properties:

1. for any pair (f, g) of composable morphisms from \mathbb{E} , the composite gf is in \mathbb{E} ;
2. for any morphisms e in \mathbb{E} and f in \mathbb{A} with common codomain, if (P, q, r) is the pullback of e and f then r is in \mathbb{E} .

Definition 1.6.1. An object $P \in \mathbb{A}$ is called \mathbb{E} -projective, if for any $h : P \rightarrow B'$ in \mathbb{A} and any $e : B \rightarrow B'$ in \mathbb{E} there exists a morphism $h' : P \rightarrow B$ in \mathbb{A} such that the diagram

$$\begin{array}{ccc}
 & P & \\
 h' \swarrow & & \downarrow h \\
 B & \xrightarrow{e} & B'
 \end{array}$$

commutes.

Definition 1.6.2. A family $(A_n, \delta_n)_{n \in \mathbb{Z}}$ of objects $A_n \in \mathbb{A}$ and morphisms $\delta_n : A_n \rightarrow A_{n-1}$ is called a chain complex when $\delta_n \delta_{n+1} = 0$. We will write A for the chain complex $(A_n, \delta_n)_{n \in \mathbb{Z}}$ when convenient. Suppose $A' = (A'_n, \delta'_n)_{n \in \mathbb{Z}}$ is another chain complex, a chain

morphism $f : A \rightarrow A'$ is a family of morphisms $f_n : A_n \rightarrow A'_n$ such that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A_{n+1} & \xrightarrow{\delta_{n+1}} & A_n & \xrightarrow{\delta_n} & A_{n-1} & \longrightarrow & \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ \cdots & \longrightarrow & A'_{n+1} & \xrightarrow{\delta'_{n+1}} & A'_n & \xrightarrow{\delta'_n} & A'_{n-1} & \longrightarrow & \cdots \end{array}$$

commutes. If $g : A \rightarrow A'$ is another chain morphism, then a chain homotopy $s : f \rightarrow g$ is a family of morphisms $s_n : A_n \rightarrow A'_{n+1}$ such that $f_n - g_n = s_{n-1}\delta_n + \delta'_{n+1}s_n$. The homology of the complex A is defined to be $H_n(A) = \text{Ker}(\delta_n)/\text{Im}(\delta_{n+1})$. Consider the diagram

$$\begin{array}{ccccc} A_{n+1} & \xrightarrow{\delta_{n+1}} & A_n & \xrightarrow{\delta_n} & A_{n-1} \\ \downarrow f_{n+1} & \searrow \bar{\delta}_{n+1} & \nearrow \kappa_n & & \downarrow f_{n-1} \\ & K_n & \xrightarrow{\gamma_n} & H_n(A) & \\ & \downarrow \bar{f}_n & & \downarrow H_n(f) & \\ & K'_n & \xrightarrow{\gamma'_n} & H_n(A') & \\ \downarrow f_{n+1} & \nearrow \bar{\delta}'_{n+1} & \searrow \kappa'_n & & \downarrow f_{n-1} \\ A'_{n+1} & \xrightarrow{\delta'_{n+1}} & A'_n & \xrightarrow{\delta'_n} & A'_{n-1} \end{array}$$

in which:

- (K_n, κ_n) is the kernel of δ_n ;
- (K'_n, κ'_n) is the kernel of δ'_n ;
- $\bar{\delta}_{n+1}$ is the unique morphism with $\kappa_n \bar{\delta}_{n+1} = \delta_{n+1}$;
- $\bar{\delta}'_{n+1}$ is the unique morphism with $\kappa'_n \bar{\delta}'_{n+1} = \delta'_{n+1}$;
- γ_n is the cokernel of $\bar{\delta}_{n+1}$;
- γ'_n is the cokernel of $\bar{\delta}'_{n+1}$.

$H_n(f) : H_n(A) \rightarrow H_n(A')$ is the unique morphism such that $H_n(f)\gamma_n = \gamma'_n \bar{f}_n$. A complex A is exact when $H_n(A) = 0$, that is, when $\text{Ker}(\delta_n) = \text{Im}(\delta_{n+1})$.

Proposition 1.6.3. If there exists a chain homotopy $s : f \rightarrow g$ between chain morphisms $f, g : A \rightarrow A'$, then $H_n(f) = H_n(g)$.

Proof. Let $t : K_n \rightarrow K'_n$ be the unique morphism with $\kappa'_n t = (f_n - g_n)\kappa_n = (\delta'_{n+1}s_n + s_{n-1}\delta_n)\kappa_n$. Since $(\delta'_{n+1}s_n + s_{n-1}\delta_n)\kappa_n = \delta'_{n+1}s_n\kappa_n = \kappa'_n\bar{\delta}'_{n+1}s_n\kappa_n$ and κ'_n is a monomorphism, it follows that $t = \bar{\delta}'_{n+1}s_n\kappa_n$ and consequently $\gamma't = 0$. It is easy to see that $H_n(f - g) = H_n(f) - H_n(g)$, therefore $H_n(f) = H_n(g)$. \square

Definition 1.6.4. Two chain complexes A and A' are said to be homotopy equivalent, if there exist chain morphisms $f : A \rightarrow A'$ and $g : A' \rightarrow A$ and chain homotopies $s : gf \rightarrow 1_A$ and $t : fg \rightarrow 1_{A'}$.

It is easy to see that homotopy equivalent chain complexes have isomorphic homology.

Definition 1.6.5. A chain complex P in \mathbb{A} is called an \mathbb{E} -projective resolution of A when P is exact, $P_n = 0$ for $n < -1$, $P_{-1} = A$, P_n is \mathbb{E} -projective for $n > -1$, and δ_n factors as a morphism in \mathbb{E} followed by a monomorphism.

Proposition 1.6.6. If P is an \mathbb{E} -projective resolution of A in \mathbb{A} , P' is an \mathbb{E} -projective resolution of A' in \mathbb{A} , and if $f : A \rightarrow A'$ is a morphism in \mathbb{A} , then f lifts to a morphism of \mathbb{E} -projective resolutions.

Proof. It is obvious that δ'_0 is in \mathbb{E} , and consequently there exists a morphism $f_0 : P_0 \rightarrow P'_0$ such that the diagram

$$\begin{array}{ccc} P_0 & \xrightarrow{\delta_0} & A \\ \downarrow f_0 & & \downarrow f \\ P'_0 & \xrightarrow{\delta'_0} & A' \end{array}$$

commutes. The morphisms f_n are constructed by induction as follows. Consider the diagram

$$\begin{array}{ccccc} P_n & \xrightarrow{\delta_n} & P_{n-1} & \xrightarrow{\delta_{n-1}} & P_{n-2} \\ & \searrow h & \downarrow f_{n-1} & & \downarrow f_{n-2} \\ & & S & & \\ \downarrow f_n & \nearrow e & & \searrow m & \\ P'_n & \xrightarrow{\delta'_n} & P'_{n-1} & \xrightarrow{\delta'_{n-1}} & P'_{n-2} \end{array}$$

in which:

- (e, m) is the factorization of δ'_n ;
- m is the kernel of δ'_{n-1} (since P' is exact);

- h is the unique morphism with $mh = f_{n-1}\delta_n$ (since $\delta'_{n-1}f_{n-1}\delta_n = f_{n-2}\delta_{n-1}\delta_n = 0$);
- f_n is any morphism with $ef_n = h$ (f_n exists since P_n is \mathbb{E} -projective and e is in \mathbb{E}).

□

Proposition 1.6.7. *For any morphisms $f, g : P \rightarrow P'$ between \mathbb{E} -projective resolutions with $f_{-1} = g_{-1} = f$ and with P' exact, there exists a chain homotopy $s : f \rightarrow g$.*

Proof. We construct s inductively. To construct s_0 , consider the diagram

$$\begin{array}{ccccc}
 P_1 & \xrightarrow{\delta_1} & P_0 & \xrightarrow{\delta_0} & A \\
 \downarrow f_1 & & \downarrow f_0 & & \downarrow f \\
 \downarrow g_1 & & \downarrow g_0 & & \\
 P'_1 & \xrightarrow{\delta'_1} & P'_0 & \xrightarrow{\delta'_0} & A \\
 & & \downarrow m & & \\
 & & S & &
 \end{array}$$

$\begin{array}{c} \nearrow s_0 \\ \searrow h \\ \nearrow e \end{array}$

in which:

- (e, m) is the factorization of δ'_1 ;
- m is the kernel of δ'_0 ;
- h is the unique morphism with $mh = f_0 - g_0$ (h exists since $\delta'_0(f_0 - g_0) = (f - f)\delta_0 = 0$);
- s_0 is any morphism such that $es_0 = h$ (s_0 exists since P_0 is \mathbb{E} -projective and e is in \mathbb{E}).

To construct s_n , consider the diagram

$$\begin{array}{ccccccc}
 P_{n+1} & \xrightarrow{\delta_{n+1}} & P_n & \xrightarrow{\delta_n} & P_{n-1} & \xrightarrow{\delta_{n-1}} & P_{n-2} \\
 \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \downarrow f_{n-2} \\
 \downarrow g_{n+1} & & \downarrow g_n & & \downarrow g_{n-1} & & \downarrow g_{n-2} \\
 P'_{n+1} & \xrightarrow{\delta'_{n+1}} & P'_n & \xrightarrow{\delta'_n} & P'_{n-1} & \xrightarrow{\delta'_{n-1}} & P'_{n-2} \\
 & & \downarrow m & & & & \\
 & & S & & & &
 \end{array}$$

$\begin{array}{c} \nearrow s_n \\ \searrow h \\ \nearrow e \end{array}$

in which:

- (e, m) is the factorization of δ'_{n+1} ;

- m is the kernel of δ'_n ;
- $\delta'_n(f_n - g_n - s_{n-1}\delta_n) = ((f_{n-1} - g_{n-1})\delta_n - (f_{n-1} - g_{n-1} - s_{n-2}\delta_{n-1})\delta_n) = 0$;
- h is the unique morphism with $mh = f_n - g_n - s_{n-1}\delta_n$;
- s_n is any morphism with $es_n = h$ (s_n exists since P_n is \mathbb{E} -projective).

□

Proposition 1.6.8. *If P and P' are both \mathbb{E} -projective resolutions of A in \mathbb{A} , then $P \sim P'$.*

Proof. By Proposition 1.6.6, the morphism 1_A lifts to chain morphisms $f : P \rightarrow P'$ and $g : P' \rightarrow P$. The composite gf and 1_P are both chain morphisms of P with $(gf)_{-1} = (1_P)_{-1} = 1_A$; therefore, by Proposition 1.6.7, there exists a chain homotopy $s : gf \rightarrow 1_P$. Similarly, there exists a chain homotopy $t : fg \rightarrow 1_{P'}$, and so P and P' are homotopy equivalent. □

Lemma 1.6.9. *Let $K \xrightarrow{\kappa} A \xrightarrow{\alpha} B$ be a short exact sequence in \mathbb{A} with α is in \mathbb{E} . Let P and P' be \mathbb{E} -projective resolutions of K and B respectively. The short exact sequence lifts to a short exact sequence of \mathbb{E} -projective resolutions and moreover, this short exact sequence splits at each $n \geq 0$.*

Proof. Since P'_0 is \mathbb{E} -projective and α is in \mathbb{E} , there exists a morphism $h : P'_0 \rightarrow A$ such that $\alpha h = \delta'_0$. Consequently, since \mathbb{A} is additive, the diagram

$$\begin{array}{ccc} K \oplus P'_0 & \xrightarrow{\pi_2} & P'_0 \\ [\kappa, h] \downarrow & & \downarrow \delta'_0 \\ A & \xrightarrow{\alpha} & B \end{array}$$

in which π_2 is the second biproduct projection, is a pullback and $[\kappa, h]$ is in \mathbb{E} . It is easy to see that the diagram

$$\begin{array}{ccccc} & P_0 \oplus P'_0 & \xrightarrow{p_2} & P'_0 & \\ & \downarrow \delta_0 \oplus 1 & \nearrow \pi_2 & \downarrow \delta'_0 & \\ [\kappa \delta_0, h] \curvearrowright & K \oplus P'_0 & & & \\ & \downarrow [\kappa, h] & & & \\ & A & \xrightarrow{\alpha} & B & \end{array}$$

in which p_2 is second biproduct projection, is commutative. Since the diagram

$$\begin{array}{ccc} P_0 \oplus P'_0 & \xrightarrow{p_1} & P_0 \\ \delta_0 \oplus 1 \downarrow & & \downarrow \delta_0 \\ K \oplus P'_0 & \xrightarrow{\pi_1} & K \end{array}$$

is a pullback, it follows that $\delta_0 \oplus 1$ is in \mathbb{E} and therefore the composite $[\kappa\delta_0, h] = [\kappa, h](\delta_0 \oplus 1)$ is in \mathbb{E} . Since the diagram

$$\begin{array}{ccccc} P_0 & \xrightarrow{i_1} & P_0 \oplus P'_0 & \xrightarrow{p_2} & P'_0 \\ \delta_0 \downarrow & & \downarrow [\kappa\delta_0, h] & & \downarrow \delta'_0 \\ X & \xrightarrow{\kappa} & A & \xrightarrow{\alpha} & B \end{array}$$

commutes and $P_0 \oplus P'_0$ is trivially \mathbb{E} -projective, we have proved the lemma at level 0. Since for each $n > 0$, δ_n factorizes as $e_n : P_n \rightarrow K_n$ in \mathbb{E} followed by a monomorphism $\kappa_n : K_n \rightarrow P_{n-1}$, and δ'_n factorizes as $e'_n : P'_n \rightarrow K'_n$ in \mathbb{E} followed by a monomorphism $\kappa'_n : K'_n \rightarrow P'_{n-1}$, the diagram

$$\begin{array}{ccccc} P_n & \xrightarrow{i_1} & P_n \oplus P'_n & \xrightarrow{p_2} & P'_n \\ \downarrow e_n & & \downarrow e_n \oplus e'_n & & \downarrow e'_n \\ \delta_n \downarrow & & \downarrow \delta_n & & \downarrow \delta'_n \\ K_n & \xrightarrow{i_1} & K_n \oplus K'_n & \xrightarrow{p_2} & K'_n \\ \downarrow \kappa_n & & \downarrow \kappa_n \oplus \kappa'_n & & \downarrow \kappa'_n \\ P_{n-1} & \xrightarrow{i_1} & P_{n-1} \oplus P'_{n-1} & \xrightarrow{p_2} & P'_{n-1} \end{array}$$

commutes. It is easy to see that the morphisms $e_n \oplus e'_n$ are in \mathbb{E} and that the composites $(\kappa_n \oplus \kappa'_n)(e_n \oplus e'_n)$ make the collection of objects $P_n \oplus P'_n$ an \mathbb{E} -projective resolution. \square

Proposition 1.6.10. *Let \mathbb{B} be a abelian category and let $F : \mathbb{A} \rightarrow \mathbb{B}$ be a left exact functor. Let $\mathbb{E}\text{-SES}(\mathbb{A})$ be the category of short exact sequences in \mathbb{A} with the cokernel in \mathbb{E} , and let $P, Q : \mathbb{E}\text{-SES}(\mathbb{A}) \rightarrow \mathbb{A}$ be the functors taking a short exact sequence to the kernel and the cokernel respectively. If there are enough \mathbb{E} -projectives, then there exist unique functors $H_n : \mathbb{A} \rightarrow \mathbb{B}$ and natural transformations $\lambda_n : H_n Q \rightarrow H_{n-1} P$ with the following properties:*

1. $H_0 = F$;
2. $H_n(P) = 0$ if P is \mathbb{E} -projective and $n > 0$;
3. for each \mathbb{E} -short exact sequence $E = K \xrightarrow{\kappa} A \xrightarrow{\alpha} B$, the diagram

$$\begin{array}{ccccccc}
& \dots & & \dots & & \dots & \\
& & \searrow & & \searrow & & \searrow \\
& & & \lambda_{3E} & & & \\
H_2(K) & \xrightarrow{H_2(\kappa)} & H_2(A) & \xrightarrow{H_2(\alpha)} & H_2(B) & & \\
& & \searrow & & \searrow & & \searrow \\
& & & \lambda_{2E} & & & \\
H_1(K) & \xrightarrow{H_1(\kappa)} & H_1(A) & \xrightarrow{H_1(\alpha)} & H_1(B) & & \\
& & \searrow & & \searrow & & \searrow \\
& & & \lambda_{1E} & & & \\
H_0(K) & \xrightarrow{H_0(\kappa)} & H_0(A) & \xrightarrow{H_0(\alpha)} & H_0(B) & \xrightarrow{0} & 0
\end{array}$$

is a long exact sequence.

Proof. For an object A , we choose any \mathbb{E} -projective resolution P , and form the chain complex $F(P)$ with $F(P)_n = F(P_n)$ for $n \geq 0$ and $F(P)_{-1} = 0$; we define $H_n(A)$ to be $H_n(F(P))$ for $n \geq 0$. By Proposition 1.6.8, if P' is another \mathbb{E} -projective resolution, then P and P' are homotopy equivalent. Since F is left exact, it is additive, and consequently $F(P)$ and $F(P')$ are homotopy equivalent and have the same homology. It is easy to see that this passage is functorial. Suppose δ_1 factors as $e_1 \in \mathbb{E}$ followed by κ_1 with κ_1 the kernel of δ_0 . Note that δ_0 is the cokernel of κ_1 and the kernel of $0 : F(P_0) \rightarrow 0$ is $(F(P_0), 1_{F(P_0)})$, therefore, $F\delta_1$ factors through the kernel as $F\delta_0$. Since F is right-exact, $F(e_1)$ is an epimorphism, and the cokernel of $F(\delta_1)$ is equal to the cokernel of $F(\kappa_1)$. Moreover, $F(\kappa_1)$ has cokernel $(F(A), F(\delta_0))$, proving $H_0 = F$.

Now let P be an \mathbb{E} -projective object in \mathbb{A} . Since F preserves 0 and the diagram

$$\longrightarrow 0 \longrightarrow P \xrightarrow{1_P} P$$

is an \mathbb{E} -projective resolution, it follows that $H_n(P) = 0$ for $n \geq 1$. Finally, to show that the third property holds true, let $E = K \xrightarrow{\kappa} A \xrightarrow{\alpha} B$ be an \mathbb{E} -short exact sequence. By Lemma 1.6.9, E lifts to a short exact sequence of complexes split for $n \geq 0$. Therefore, since F is additive, applying F to this short exact sequence we obtain a short exact sequence of

complexes. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(P_n) & \longrightarrow & F(P''_n) & \longrightarrow & F(P'_n) & \longrightarrow & 0 \\
& & \downarrow c_n & & \downarrow c''_n & & \downarrow c'_n & & \\
& & C_n & \longrightarrow & C''_n & \longrightarrow & C'_n & \longrightarrow & 0 \\
& & \downarrow \underline{\delta}_n & & \downarrow \underline{\delta}''_n & & \downarrow \underline{\delta}'_n & & \\
0 & \longrightarrow & K_{n-1} & \longrightarrow & K''_{n-1} & \longrightarrow & K'_{n-1} & & \\
& & \downarrow k_{n-1} & & \downarrow k''_{n-1} & & \downarrow k'_{n-1} & & \\
0 & \longrightarrow & F(P_{n-1}) & \longrightarrow & F(P''_{n-1}) & \longrightarrow & F(P'_{n-1}) & \longrightarrow & 0
\end{array}$$

in which c_n , c''_n and c'_n are the cokernels of δ_{n+1} , δ''_{n+1} and δ'_{n+1} respectively, and k_{n-1} , k''_{n-1} and k'_{n-1} are the kernels of δ_{n-1} , δ''_{n-1} and δ'_{n-1} respectively. It is easily seen that $H_n(K) = \text{Ker}(\underline{\delta}_n)$, $H_n(A) = \text{Ker}(\underline{\delta}''_n)$, $H_n(B) = \text{Ker}(\underline{\delta}'_n)$, $H_{n-1}(K) = \text{Coker}(\underline{\delta}_n)$, $H_{n-1}(A) = \text{Coker}(\underline{\delta}''_n)$ and $H_{n-1}(B) = \text{Coker}(\underline{\delta}'_n)$. Consequently, by the snake lemma (see e.g. [18]) there exists a morphism $\gamma : H_n(B) \rightarrow H_{n-1}(K)$ such that the diagram

$$H_n(K) \longrightarrow H_n(A) \longrightarrow H_n(B) \xrightarrow{\gamma} H_{n-1}(K) \longrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(B)$$

is an exact sequence. The following observation shows that functors above are unique up to isomorphism. For any B in \mathbb{A} there exists an epimorphism $\epsilon : P \rightarrow B$ in \mathbb{E} ; let (K, κ) be the kernel of ϵ , then the properties above imply that $H_1(B) \cong \text{Ker}(F(\epsilon))$ and $H_n(B) \cong H_{n-1}(K)$ for $n > 1$. \square

Chapter 2

Representability of the split extension functor for categories of generalized Lie algebras

Let \mathbb{C} be a pointed protomodular category. A diagram

$$K \xrightarrow{\kappa} A \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} B$$

in which $\alpha\beta = 1_B$, and (K, κ) is the kernel of α , is called a split extension of B with kernel K , and will be denoted by the quadruple $(A, \alpha, \beta, \kappa)$. A diagram

$$\begin{array}{ccccc} K & \xrightarrow{\kappa} & A & \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} & B \\ \parallel & & \downarrow f & & \parallel \\ K & \xrightarrow{\kappa'} & A' & \begin{array}{c} \xleftarrow{\alpha'} \\ \xrightarrow{\beta'} \end{array} & B, \end{array}$$

in which $(A, \alpha, \beta, \kappa)$ and $(A', \alpha', \beta', \kappa')$ are both split extensions of B with kernel K , is called a morphism of split extensions and will be denoted by $f : (A, \alpha, \beta, \kappa) \rightarrow (A', \alpha', \beta', \kappa')$. Since \mathbb{C} is pointed protomodular, the split short five lemma holds, and every morphism of split extensions is necessarily invertible. Consequently, there is an equivalence relation on the set of all split extensions of B with kernel K , a pair of extensions are equivalent as soon as there exists a morphism between them. The functor $\text{SplExt}(-, K) : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$, is defined on an object B as the set of equivalence classes of split extensions of B with kernel K . Let

$p : E \rightarrow B$ be a morphism in \mathbb{C} , since \mathbb{C} is protomodular, the pullback $(A \times_B E, \pi_1, \pi_2)$ of α and p exists. It is easy to see that the diagram

$$\begin{array}{ccccc}
 K & \xrightarrow{\langle \kappa, 0 \rangle} & A \times_B E & \xrightleftharpoons[\langle \beta p, 1 \rangle]{\pi_2} & E \\
 \parallel & & \downarrow \pi_1 & & \downarrow p \\
 K & \xrightarrow{\kappa} & A & \xrightleftharpoons[\beta]{\alpha} & B
 \end{array}$$

commutes and that $(A \times_B E, \pi_2, \langle \beta p, 1 \rangle, \langle \kappa, 0 \rangle)$ is a split extension of E with kernel K . Therefore, there is an induced morphism from the set of split extensions of B with kernel K to the set of split extensions of E with kernel K . Since \mathbb{C} is protomodular, this morphism induces a morphism $\text{SplExt}(p, K) : \text{SplExt}(B, K) \rightarrow \text{SplExt}(E, X)$, making $\text{SplExt}(-, K)$ into a functor.

Recall: for a Lie algebra X over a commutative ring R , a map $f : X \rightarrow X$ is called a derivation of X if f is linear and, for all x and y in X , $f(xy) = f(x)y + xf(y)$. The set $\text{Der}(X)$ of all derivations on X can be made into a Lie algebra with Lie multiplication $fg = f \circ g - g \circ f$ and all other operations defined pointwise. A well-known classical result can be stated as: the functor $\text{SplExt}(-, X)$ is representable with $\text{Der}(X)$ the object of the representation, that is, there is a natural isomorphism $\text{SplExt}(-, X) \cong \mathbf{Lie}_R(-, \text{Der}(X))$. This result can be extended to any category of internal Lie algebras defined in a cartesian closed category (see Theorem 5.2 in [4]). We will generalize this result in a different direction, namely to suitably define Lie algebras over a monoid M in an additive symmetric monoidal closed category. Introducing this concept requires some auxiliary observations:

Recall that a commutative monoid in a symmetric monoidal category $(\mathbb{C}, \otimes, I, \alpha, \rho, \lambda, \sigma)$ is an object M together with two morphisms

$$\mu : M \otimes M \rightarrow M, \quad \eta : I \rightarrow M$$

such that the diagrams

$$\begin{array}{ccc}
 M \otimes (M \otimes M) & \xrightarrow{\alpha} & (M \otimes M) \otimes M \\
 \downarrow 1 \otimes \mu & & \downarrow \mu \otimes 1 \\
 M \otimes M & \xrightarrow{\mu} & M \xleftarrow{\mu} M \otimes M \\
 \downarrow \sigma & \nearrow \mu & \\
 M \otimes M & &
 \end{array}$$

$$\begin{array}{ccccc}
 I \otimes M & \xrightarrow{\eta \otimes 1} & M \otimes M & \xleftarrow{1 \otimes \mu} & M \otimes I \\
 & \searrow \lambda & \downarrow \mu & \nearrow \rho & \\
 & & M & &
 \end{array}$$

are commutative. Let us recall that when $(\mathbb{C}, \otimes, I, \alpha, \rho, \lambda, \sigma) = (\mathbf{Ab}, \otimes, \mathbb{Z}, \alpha, \rho, \lambda, \sigma)$ is the usual symmetric monoidal category of abelian groups, a commutative monoid in it is the same as a commutative ring. In this case the morphism $\mu : M \otimes M \rightarrow M$ corresponds, via the universal property of the tensor product, to a map $M \times M \rightarrow M$, call it multiplication, which is bilinear (distributive with respect to the addition of the abelian group M). The morphism $\eta : \mathbb{Z} \rightarrow M$ is determined by picking an element u in M , the image of 1. Furthermore, the commutativity of the first diagram means that multiplication is associative and commutative, while the commutativity of the second means that η makes u the identity element of M .

For an ordinary Lie algebra X over a commutative ring M , the scalar multiplication $M \times X \rightarrow X$ and the Lie multiplication $X \times X \rightarrow X$ are bilinear maps, and so by the universal property of the tensor product in \mathbf{Ab} they can be described as morphisms $a : M \otimes X \rightarrow X$ and $b : X \otimes X \rightarrow X$ respectively. The commutativity of the diagrams

$$\begin{array}{ccc}
 M \otimes (M \otimes X) & \xrightarrow{\alpha} & (M \otimes M) \otimes X \\
 \downarrow 1 \otimes a & & \downarrow \mu \otimes 1 \\
 M \otimes X & & M \otimes X \\
 & \searrow a & \downarrow a \\
 & & X
 \end{array}$$

$$\begin{array}{ccccc}
(M \otimes X) \otimes X & \xleftarrow{\alpha} & M \otimes (X \otimes X) & \xrightarrow{\sigma\alpha(1 \otimes \sigma)} & X \otimes (M \otimes X) \\
\downarrow a \otimes 1 & & \downarrow 1 \otimes b & & \downarrow 1 \otimes a \\
X \otimes X & & M \otimes X & & X \otimes X \\
& \searrow b & \downarrow a & \swarrow b & \\
& & X & & \\
\end{array}$$

$$\begin{array}{ccc}
X \otimes X & \xrightarrow{\sigma} & X \otimes X \\
\downarrow b & \swarrow -b & \\
X & & \\
\end{array}
\qquad
\begin{array}{ccc}
X \otimes (X \otimes X) & \xrightarrow{1 + \sigma\alpha + \sigma\alpha\sigma\alpha} & X \otimes (X \otimes X) \\
\downarrow 0 & & \downarrow 1 \otimes b \\
X & \xleftarrow{b} & X \otimes X
\end{array}$$

state that

$$(mn)x = m(nx), \quad (mx)y = m(xy) = x(my), \quad xy = -yx,$$

$$x(yz) + z(xy) + y(zx) = 0$$

for all $m, n \in M$ and for all $x, y, z \in X$. These identities correspond to the axioms of a Lie algebra except that we have replaced the axiom $xx = 0$ ($x \in X$), with the axiom $xy = -yx$ ($x, y \in X$). Assuming the axiom $xx = 0$, the well known argument

$$xy = xx + xy + yx + yy - yx = (x + y)(x + y) - yx = -yx$$

shows that we have actually replaced an axiom with a formally weaker one. Assuming the axiom $xy = -yx$, the argument

$$2xx = xx + xx = xx - xx = 0$$

shows that when 2 has a multiplicative inverse in M , the two axioms are equivalent. When M is a field this corresponds to saying that M is not of characteristic 2. Since the axiom $xx = 0$ has a repeated variable in it, it is not possible to express it as the commutativity of a diagram involving tensor products. Therefore, in order to define a Lie algebra in an abstract symmetric monoidal category $(\mathbb{C}, \otimes, I, \alpha, \rho, \lambda, \sigma)$ we introduce an additional structure on \mathbb{C} . The structure we choose in this paper consists of a category \mathbb{D} , functors $U, V : \mathbb{C} \rightarrow \mathbb{D}$, and a natural transformation $\delta : U \rightarrow V(- \otimes -)$, satisfying suitable conditions (see Section 1). In Section 1 we define a generalized Lie algebra following the above as motivation. In Section 2 we define in this new setting the generalized Lie algebra of derivations and show, in Section 3, that the functor of split extensions from the category of these generalized Lie algebras to

the category of sets is representable. We conclude Section 3 by remarking that the functor of split extensions of crossed modules of these generalized Lie algebras is representable.

2.1 Algebraic structures in monoidal categories

In this section we introduce the needed algebraic structures to define a generalized Lie algebra and construct in this context the functor which in the classical case takes associative algebras to Lie algebras. Throughout this chapter we will assume that:

1. $\mathbb{C} = (\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is an additive symmetric monoidal category with all finite limits; in addition we assume it to be monoidal closed, although in this section we only use the fact that the tensor is distributive with respect to finite products;
2. $(M, \mu : M \otimes M \rightarrow M, \eta : I \rightarrow M)$ is a commutative monoid in \mathbb{C} ;
3. \mathbb{D} is a category in which hom-sets are abelian groups;
4. Composition of morphisms in \mathbb{D} is distributive on the right with respect to addition of morphisms, that is, for any morphisms $f, g : B \rightarrow C$ and $h : A \rightarrow B$ we have $(f + g)h = fh + gh$;
5. U and V are functors from \mathbb{C} to \mathbb{D} and V restricted to hom-sets is an abelian group homomorphism;
6. δ is a natural transformation from U to $V(- \otimes -)$ such that:

Condition 2.1.1. For any $C \in \mathbb{C}$ the diagram

$$\begin{array}{ccc}
 UC & \xrightarrow{\delta_C} & V(C \otimes C) \\
 & \searrow \delta_C & \downarrow V\sigma \\
 & & V(C \otimes C)
 \end{array}$$

commutes.

Example 2.1.2. $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \sigma) = (\mathbf{Ab}, \otimes, \mathbb{Z}, \alpha, \rho, \lambda, \sigma)$ is the usual symmetric monoidal category of abelian groups, $\mathbb{D} = \mathbf{ab}$ is the category with objects all abelian groups and morphisms all maps between their underlying sets, $U = V : \mathbf{Ab} \rightarrow \mathbf{ab}$ is the inclusion functor, and δ is defined by $\delta_C(c) = c \otimes c$ for all C in \mathbf{Ab} and c in C . This example explains the

main purpose of introducing \mathbb{D} , U , V , and δ : the axiom $xx = 0$, mentioned at the beginning of this chapter, can now be expressed categorically as $V(b)\delta_X = 0$, where $b : X \otimes X \rightarrow X$ is a multiplication morphism on an object X .

We recall: (i) an M -action is a pair (X, a) , where X is an object in \mathbb{C} and $a : M \otimes X \rightarrow X$ is a morphism in \mathbb{C} , such that the diagrams

$$\begin{array}{ccc}
 M \otimes (M \otimes X) & \xrightarrow{\alpha} & (M \otimes M) \otimes X \\
 \downarrow 1 \otimes a & & \downarrow a \otimes 1 \\
 M \otimes X & \xrightarrow{a} & X \xleftarrow{a} M \otimes X
 \end{array}
 \qquad
 \begin{array}{ccc}
 I \otimes X & \xrightarrow{\eta \otimes 1} & M \otimes X \\
 \searrow \lambda & & \downarrow a \\
 & & X
 \end{array}$$

commute; (ii) a magma defined with respect to the monoidal structure in \mathbb{C} is a pair (X, b) , where X is an object in \mathbb{C} and $b : X \otimes X \rightarrow X$ is a morphism in \mathbb{C} .

Definition 2.1.3. A triple $(X, a : M \otimes X \rightarrow X, b : X \otimes X \rightarrow X)$ is said to be an M -magma if (X, a) is an M -action for the monoid M . For M -magmas (X, a, b) and (X', a', b') , a morphism $f : X \rightarrow X'$ in \mathbb{C} is an M -magma morphism if the diagrams

$$\begin{array}{ccc}
 M \otimes X & \xrightarrow{1 \otimes f} & M \otimes X' \\
 \downarrow a & & \downarrow a' \\
 X & \xrightarrow{f} & X'
 \end{array}
 \qquad
 \begin{array}{ccc}
 X \otimes X & \xrightarrow{f \otimes f} & X' \otimes X' \\
 \downarrow b & & \downarrow b' \\
 X & \xrightarrow{f} & X'
 \end{array}$$

commute; that is, f must be a morphism of magmas and a morphism of M -actions at the same time. The category of M -magmas will be denoted $M\text{-Mag}_0$.

For an M -magma (X, a, b) consider the following condition:

Condition 2.1.4. (a) The diagram

$$\begin{array}{ccc}
 M \otimes (X \otimes X) & \xrightarrow{\alpha} & (M \otimes X) \otimes X \\
 \downarrow 1 \otimes b & & \downarrow a \otimes 1 \\
 M \otimes X & \xrightarrow{a} & X \xleftarrow{b} X \otimes X
 \end{array}$$

commutes;

(b) The diagram

$$\begin{array}{ccc}
M \otimes (X \otimes X) & \xrightarrow{\sigma\alpha(1 \otimes \sigma)} & X \otimes (M \otimes X) \\
\downarrow 1 \otimes b & & \downarrow 1 \otimes a \\
M \otimes X & \xrightarrow{a} X \xleftarrow{b} & X \otimes X
\end{array}$$

commutes.

Let $M\text{-Mag}_1$ be the full subcategory of $M\text{-Mag}_0$ with objects all M -magmas satisfying Conditions 2.1.4(a) and 2.1.4(b). Let $M\text{-Mag}_2$ be the full subcategory of $M\text{-Mag}_1$ with objects all (X, a, b) , in which the pair (X, b) is a semigroup, that is, the diagram

$$\begin{array}{ccc}
X \otimes (X \otimes X) & \xrightarrow{\alpha} & (X \otimes X) \otimes X \\
\downarrow 1 \otimes b & & \downarrow b \otimes 1 \\
X \otimes X & \xrightarrow{b} X \xleftarrow{b} & X \otimes X
\end{array}$$

commutes. In the situation of Example 2.1.2 the categories $M\text{-Mag}_1$ and $M\text{-Mag}_2$ are the categories of non-associative and associative M -algebras respectively.

For a magma (X, b) we are also going to use the following conditions:

Condition 2.1.5. $V(b)\delta_X = 0$.

Condition 2.1.6. (a) $b(1 + \sigma) = 0$ (anticommutativity);

(b) $b(1 \otimes b)(1 + \sigma\alpha + \sigma\alpha\sigma\alpha) = 0$ (Jacobi identity).

Remark 2.1.7. When $\mathbb{C} = \mathbb{D}$, $V = 1$, $U = (- \otimes -)$ and $\delta = 1 + \sigma$, Condition 2.1.6(a) becomes an instance of Condition 2.1.5.

Let $\mathbf{Lie}(M, \delta)$ be the full subcategory of $M\text{-Mag}_1$ with objects all (X, a, b) , in which the magma (X, b) satisfies Conditions 2.1.5, 2.1.6(a) and 2.1.6(b). In the situation of Example 2.1.2, Conditions 2.1.5, 2.1.6(a) and 2.1.6(b) correspond to the identities

$$xx = 0, \quad xy + yx = 0, \quad x(yz) + z(xy) + y(zx) = 0$$

respectively, and recalling that the category $M\text{-Mag}_1$ is the category of non-associative algebras we see that the category $\mathbf{Lie}(M, \delta)$ is the category of Lie algebras over the commutative ring M .

Remark 2.1.8. If $\mathbb{D} = \mathbb{C}$, $U = V = 1_{\mathbb{C}}$ and $\delta_{\mathbb{C}}$ is the zero morphism, then Condition 2.1.5 is trivially satisfied by any magma (X, b) . If in addition, as in Example 2.1.2, $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \sigma) = (\mathbf{Ab}, \otimes, \mathbb{Z}, \alpha, \lambda, \rho, \sigma)$ is the usual symmetric monoidal category of abelian groups, the category $\mathbf{Lie}(M, \delta)$ has as objects Lie algebras, except that the axiom $xx = 0$ has been replaced by the axiom $xy = -yx$.

If $(X, a : R \times X \rightarrow X, b : X \times X \rightarrow X)$ is an associative algebra over a ring R and if we define $\tilde{b} : X \times X \rightarrow X$ as

$$\tilde{b}(x, y) = b(x, y) - b(y, x)$$

for all $x, y \in X$, then the triple (X, a, \tilde{b}) is a Lie algebra defined with respect to the ring R . This correspondence of associative algebras and Lie algebras is functorial and can be extended to our setting.

Theorem 2.1.9. If $(X, a, b) \in M\text{-Mag}_2$, then $(X, a, b(1 - \sigma)) \in \mathbf{Lie}(M, \delta)$ and the assignment $(X, a, b) \mapsto (X, a, b(1 - \sigma))$ defines a functor $L : M\text{-Mag}_2 \rightarrow \mathbf{Lie}(M, \delta)$ which is identity on morphisms.

Proof. Let $\tilde{b} = b(1 - \sigma)$. It is clear that (X, a, \tilde{b}) is an M -magma. Condition 2.1.4(a) holds for (X, a, \tilde{b}) since

$$\begin{aligned} a(1 \otimes \tilde{b}) &= a(1 \otimes (b(1 - \sigma))) = a(1 \otimes b) - a(1 \otimes b)(1 \otimes \sigma) \\ &= b(a \otimes 1)\alpha - b(1 \otimes a)\sigma\alpha(1 \otimes \sigma)(1 \otimes \sigma) \\ &= b(1 - \sigma)(a \otimes 1)\alpha = \tilde{b}(a \otimes 1)\alpha, \end{aligned}$$

where the third equality follows by Conditions 2.1.4(a) and 2.1.4(b) for (X, a, b) . Similarly, it can easily be seen that Condition 2.1.4(b) holds for (X, a, \tilde{b}) . To show that the Jacobi identity, Condition 2.1.6(b), holds for (X, a, \tilde{b}) , consider the equation:

$$\begin{aligned} &\tilde{b}(1 \otimes \tilde{b})(1 + \sigma\alpha + \sigma\alpha\sigma\alpha) \\ &= b(1 - \sigma)(1 \otimes b)(1 - 1 \otimes \sigma)(1 + \sigma\alpha + \sigma\alpha\sigma\alpha) \\ &= b((1 \otimes b)(1 - 1 \otimes \sigma) - \sigma(1 \otimes b)(1 - 1 \otimes \sigma))(1 + \sigma\alpha + \sigma\alpha\sigma\alpha) \\ &= b(1 \otimes b)^{(1)} + b(1 \otimes b)\sigma\alpha^{(2)} + b(1 \otimes b)\sigma\alpha\sigma\alpha^{(3)} \\ &\quad - b(1 \otimes b)(1 \otimes \sigma)^{(4)} - b(1 \otimes b)(1 \otimes \sigma)\sigma\alpha^{(5)} - b(1 \otimes b)(1 \otimes \sigma)\sigma\alpha\sigma\alpha^{(6)} \\ &\quad - b(b \otimes 1)\sigma^{(3)} - b(b \otimes 1)\alpha^{(1)} - b(b \otimes 1)\alpha\sigma\alpha^{(2)} \\ &\quad + b(b \otimes 1)(\sigma \otimes 1)\sigma^{(5)} + b(b \otimes 1)(\sigma \otimes 1)\alpha^{(6)} + b(b \otimes 1)(\sigma \otimes 1)\alpha\sigma\alpha^{(4)} \\ &= 0, \end{aligned}$$

where composites labelled with the same superscript are equal. For, we only need to observe that $b(1 \otimes b) = b(b \otimes 1)\alpha$ since (X, b) is a semigroup, and use that directly for (1) and (2), or together with $\alpha\sigma\alpha\sigma\alpha = \sigma$ for (3), or together with $\alpha(1 \otimes \sigma) = (\sigma \otimes 1)\alpha\sigma\alpha$ for (4), or together with $\alpha(1 \otimes \sigma)\sigma\alpha = (\sigma \otimes 1)\sigma$ for (5), or together with $\alpha(1 \otimes \sigma)\sigma\alpha\sigma\alpha = (\sigma \otimes 1)\sigma$ for (6). From Condition 2.1.1 and the definition of \tilde{b} it follows that Conditions 2.1.5 and 2.1.6(a) hold for (X, \tilde{b}) . For a morphism

$$(X, a, b) \xrightarrow{f} (X', a', b')$$

let $\tilde{b}' = b'(1 - \sigma)$. By calculating

$$\begin{aligned} \tilde{b}'(f \otimes f) &= b'(1 - \sigma)(f \otimes f) \\ &= b'(f \otimes f - \sigma(f \otimes f)) \\ &= b'(f \otimes f - (f \otimes f)\sigma) \\ &= b'(f \otimes f)(1 - \sigma) \\ &= fb(1 - \sigma) \\ &= f\tilde{b}, \end{aligned}$$

we see that f is a morphism in $\mathbf{Lie}(M, \delta)$. □

2.2 Construction of derivations

In this section we construct, for an object (X, a, b) in $\mathbf{Lie}(M, \delta)$, the object $\text{Der}(X)$, which will be shown in Section 2.3 to be the representing object for the functor $\text{SplExt}(-, X) : \mathbf{Lie}(M, \delta) \rightarrow \mathbf{Set}$.

Recall that, for a Lie algebra X over a commutative ring M , the Lie algebra of derivations, $\text{Der}(X)$, can be constructed as follows. For abelian groups A and B , let $\text{Hom}(A, B)$ be the abelian group of homomorphisms from A to B . Defining multiplication by composition and scalar multiplication pointwise, it is easily seen that $\text{Hom}(X, X)$ satisfies the axioms of a ring as well as those of an M -module and, moreover has scalar multiplication with the property

$$m(h_1 \circ h_2) = (mh_1) \circ h_2$$

for all $m \in M$ and $h_1, h_2 \in \text{Hom}(X, X)$. The abelian group $E(X)$ of M -module morphisms from X to X can be constructed as the equalizer of the diagram

$$\text{Hom}(X, X) \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{array} \text{Hom}(M \times X, X)$$

where f_1 and f_2 are defined by

$$f_1(h)(m, x) = mh(x), \quad f_2(h)(m, x) = h(mx)$$

for all $h \in \text{Hom}(X, X)$, $m \in M$ and $x \in X$. It is easily seen that $E(X)$ is closed under the operations defined for $\text{Hom}(X, X)$ and has the property

$$m(h_1 \circ h_2) = h_1 \circ (mh_2)$$

for all $m \in M$ and $h_1, h_2 \in E(X)$, i.e. $E(X)$ is an associative M -algebra. As described before, any associative M -algebra $E(X)$ becomes a Lie algebra with Lie multiplication defined by

$$h_1 h_2 = h_1 \circ h_2 - h_2 \circ h_1$$

for all $h_1, h_2 \in E(L)$. Finally, the Lie algebra of derivations $\text{Der}(X)$, can be constructed as the equalizer of the diagram

$$E(X) \begin{array}{c} \xrightarrow{g_1 e} \\ \xrightarrow{g_2 e} \end{array} \text{Hom}(X \times X, X)$$

where $e : E(X) \rightarrow \text{Hom}(X, X)$ is the equalizer of f_1 and f_2 , and g_1 and g_2 are defined by

$$g_1(h)(x_1, x_2) = h(x_1 x_2), \quad g_2(h)(x_1, x_2) = h(x_1) x_2 + x_1 h(x_2)$$

for all $h \in \text{Hom}(X, X)$ and $x_1, x_2 \in L$. $\text{Der}(X)$ can be seen to be closed under the operations defined for $E(X)$ and hence is a Lie algebra.

We show that this construction extends to our general context. We begin by showing that for $(X, a, b) \in \mathbf{Lie}(M, \delta)$ the internal hom-object X^X can be given a semigroup structure as well as an M -magma structure that satisfies Condition 2.1.4(a). We then construct the semigroup $E(X)$ as a regular sub- M -magma of the internal hom-object X^X and show that it satisfies Condition 2.1.4(b). We then apply the functor $L : M\text{-Mag}_2 \rightarrow \mathbf{Lie}(M, \delta)$ to $E(X)$ and construct $\text{Der}(X)$ as a regular subobject of $L(E(X))$.

For each object B in \mathbb{C} , we will denote the chosen right adjoint to the functor $- \otimes B$ by $-^B$ and denote the chosen counit of the associated adjunction by ϵ^B . For functors $F : \mathbb{X} \rightarrow \mathbb{A}$ and $G : \mathbb{A} \rightarrow \mathbb{X}$, where G is the right adjoint of F , given a morphism $h : FX \rightarrow A$, the corresponding morphism $X \rightarrow GA$ will be called the right adjunct of h (as in [18]). Similarly, given a morphism $g : X \rightarrow GA$, the corresponding morphism $FX \rightarrow A$ will be called the left adjunct of g . That is, for $g : A \rightarrow C^B$, the left adjunct of g is $\epsilon_C^B(g \otimes 1) : A \otimes B \rightarrow C$.

For a pair $(X, a_X : M \otimes X \rightarrow X)$ where $M = (M, \mu, \eta)$ is a monoid in \mathbb{C} as above, consider the following condition, which is part of the definition of an action for a monoid:

Condition 2.2.1. *The diagram*

$$\begin{array}{ccc}
 I \otimes X & \xrightarrow{\eta \otimes 1} & M \otimes X \\
 \searrow \lambda & & \swarrow a_X \\
 & X &
 \end{array}$$

commutes.

Proposition 2.2.2. *If (X, a_X) satisfies Condition 2.2.1 and if $a_{X^X} : M \otimes X^X \rightarrow X^X$ is the right adjunct of $a_X(1 \otimes \epsilon_X^X)\alpha^{-1} : (M \otimes X^X) \otimes X \rightarrow X$ then (X^X, a_{X^X}) satisfies Condition 2.2.1.*

Proof. In the diagram

$$\begin{array}{ccc}
 (I \otimes X^X) \otimes X & \xrightarrow{(\eta \otimes 1) \otimes 1} & (M \otimes X^X) \otimes X \\
 \downarrow \alpha^{-1} & & \downarrow \alpha^{-1} \\
 I \otimes (X^X \otimes X) & \xrightarrow{\eta \otimes 1} & M \otimes (X^X \otimes X) \\
 \downarrow 1 \otimes \epsilon_X^X & & \downarrow 1 \otimes \epsilon_X^X \\
 I \otimes X & \xrightarrow{\eta \otimes 1} & M \otimes X \\
 \downarrow \lambda & & \downarrow a_X \\
 X & & X \\
 \uparrow \epsilon_X^X & & \uparrow \epsilon_X^X \\
 X^X \otimes X & & X^X \otimes X
 \end{array}$$

The diagram is a large commutative diagram with several sub-diagrams. The top row is $(I \otimes X^X) \otimes X \xrightarrow{(\eta \otimes 1) \otimes 1} (M \otimes X^X) \otimes X$. Below it, a triangle ① has vertices $(I \otimes X^X) \otimes X$, $I \otimes (X^X \otimes X)$, and $M \otimes (X^X \otimes X)$. A triangle ② has vertices $(I \otimes X^X) \otimes X$, $I \otimes X$, and $X^X \otimes X$. A triangle ③ has vertices $M \otimes (X^X \otimes X)$, $M \otimes X$, and $X^X \otimes X$. A triangle ④ has vertices $I \otimes X$, $I \otimes (X^X \otimes X)$, and $X^X \otimes X$. A triangle ⑤ has vertices $M \otimes X$, $M \otimes (X^X \otimes X)$, and $X^X \otimes X$. A triangle ⑥ has vertices $I \otimes X$, $M \otimes X$, and $X^X \otimes X$. The morphisms are: α^{-1} (downward from top row to middle row), $1 \otimes \epsilon_X^X$ (downward from middle row to bottom row), λ (downward from $I \otimes X$ to $X^X \otimes X$), ϵ_X^X (upward from $X^X \otimes X$ to X), $(\eta \otimes 1) \otimes 1$ (top horizontal), $\eta \otimes 1$ (middle horizontal), $\eta \otimes 1$ (bottom horizontal), λ (left diagonal), a_X (right diagonal), $a_{X^X} \otimes 1$ (right diagonal), and $\lambda \otimes 1$ (left diagonal).

① commutes since α is a natural transformation; ② commutes as an immediate consequence of the axioms of a monoidal category; ③ commutes since \otimes is a bifunctor; ④ commutes since λ is a natural transformation; ⑤ commutes by definition of $a_{X^X} : M \otimes X^X \rightarrow X^X$; ⑥ commutes by assumption on (X, a_X) . That is, $\lambda \otimes 1 = (a_{X^X} \otimes 1)((\eta \otimes 1) \otimes 1) = (a_{X^X}(\eta \otimes 1)) \otimes 1$, which tells us that the left adjoints of the morphisms $\lambda, a_{X^X}(\eta \otimes 1) : I \otimes X^X \rightarrow X^X$ are equal to each other. Therefore these two morphisms are equal to each other themselves, as desired. \square

For a sextuple $(P, Q, X, u : P \otimes Q \rightarrow Q, p : P \otimes X \rightarrow X, q : Q \otimes X \rightarrow X)$ we consider the following condition:

Condition 2.2.3. *The diagram*

$$\begin{array}{ccc}
 P \otimes (Q \otimes X) & \xrightarrow{\alpha} & (P \otimes Q) \otimes X \\
 \downarrow 1 \otimes q & & \downarrow u \otimes 1 \\
 P \otimes X & \xrightarrow{p} X \xleftarrow{q} & Q \otimes X
 \end{array}$$

commutes.

Lemma 2.2.4. *Suppose $(P, Q, X, u : P \otimes Q \rightarrow Q, p : P \otimes X \rightarrow X, q : Q \otimes X \rightarrow X)$ satisfies Condition 2.2.3, $p' : P \otimes X^X \rightarrow X^X$ is the right adjunct of $p(1 \otimes \epsilon_X^X)\alpha^{-1} : (P \otimes X^X) \otimes X \rightarrow X$ and $q' : Q \otimes X^X \rightarrow X^X$ is the right adjunct of $q(1 \otimes \epsilon_X^X)\alpha^{-1} : (Q \otimes X^X) \otimes X \rightarrow X$ then (P, Q, X^X, u, p', q') satisfies Condition 2.2.3.*

Proof. In the diagram

$$\begin{array}{ccccc}
(P \otimes (Q \otimes X^X)) \otimes X & \xrightarrow{\alpha \otimes 1} & ((P \otimes Q) \otimes X^X) \otimes X & & \\
\downarrow \alpha^{-1} & & \downarrow \alpha^{-1} & & \\
P \otimes ((Q \otimes X^X) \otimes X) & \xrightarrow{1 \otimes \alpha^{-1}} & P \otimes (Q \otimes (X^X \otimes X)) & \xrightarrow{\alpha} & (P \otimes Q) \otimes (X^X \otimes X) \\
\downarrow 1 \otimes (q' \otimes 1) & & \downarrow 1 \otimes (1 \otimes \epsilon_X^X) & & \downarrow 1 \otimes \epsilon_X^X \\
P \otimes (X^X \otimes X) & \xrightarrow{1 \otimes \epsilon_X^X} & P \otimes (Q \otimes X) & \xrightarrow{\alpha} & (P \otimes Q) \otimes X \\
\downarrow (1 \otimes q') \otimes 1 & & \downarrow 1 \otimes q & & \downarrow u \otimes 1 \\
P \otimes (X^X \otimes X) & \xrightarrow{1 \otimes \epsilon_X^X} & P \otimes X & & P \otimes X \\
\downarrow \alpha^{-1} & & \downarrow p & & \downarrow u \otimes 1 \\
(P \otimes X^X) \otimes X & \xrightarrow{p' \otimes 1} & X^X \otimes X & \xleftarrow{q} & Q \otimes X \\
\downarrow \alpha^{-1} & & \downarrow \epsilon_X^X & & \downarrow 1 \otimes \epsilon_X^X \\
(P \otimes X^X) \otimes X & \xrightarrow{p' \otimes 1} & X^X \otimes X & \xleftarrow{q'} & (Q \otimes X^X) \otimes X \\
\downarrow \alpha^{-1} & & \downarrow \epsilon_X^X & & \downarrow \alpha^{-1} \\
(P \otimes X^X) \otimes X & \xrightarrow{p' \otimes 1} & X^X \otimes X & \xleftarrow{q'} & (Q \otimes X^X) \otimes X
\end{array}$$

① commutes by the axioms of a monoidal category; ②, ③ and ⑤ commute since α is natural transformation; ④ and ⑨ commute from the definition of q' ; ⑧ commutes by the definition of p' ; ⑦ commutes since \otimes is a bifunctor; ⑥ commutes by assumption on u , p and q (Condition 2.2.3). That is, $(q' \otimes 1)((u \otimes 1) \otimes 1)(\alpha \otimes 1) = (p' \otimes 1)((1 \otimes q') \otimes 1)$, or, equivalently, $(q'(u \otimes 1)\alpha) \otimes 1 = (p'(1 \otimes q')) \otimes 1$ – which means that the left adjuncts of the morphisms $p'(1 \otimes q')$, $q'(u \otimes 1)\alpha : P \otimes (Q \otimes X^X) \rightarrow X^X$ are equal to each other. Therefore these two morphisms are equal to each other themselves, as desired. \square

Proposition 2.2.5. *Let (X, a_X) be an M -action and, let $a_{X^X} : M \otimes X^X \rightarrow X^X$ and $b_{X^X} : X^X \otimes X^X \rightarrow X^X$ be the right adjuncts of $a(1 \otimes \epsilon_X^X)\alpha^{-1} : (M \otimes X^X) \otimes X \rightarrow X$ and $\epsilon_X^X(1 \otimes \epsilon_X^X)\alpha^{-1} : (X^X \otimes X^X) \otimes X \rightarrow X$ respectively. Then (X^X, a_{X^X}) is an M -action, (X^X, b_{X^X}) is a semigroup, and Condition 2.1.4(a) is satisfied.*

Proof. It is clear that since (X, a_X) is an M -action, the sextuple (M, M, X, μ, a_X, a_X) satisfies Condition 2.2.3. From Lemma 2.2.4 it follows that $(M, M, X^X, \mu, a_{X^X}, a_{X^X})$ satisfies Condition 2.2.3. This together with Proposition 2.2.2 applied to (X, a_X) shows that (X^X, a_{X^X}) is an M -action. From the definition of b_{X^X} we see that $(X^X, X^X, X, b_{X^X}, \epsilon_X^X, \epsilon_X^X)$ satisfies Condition 2.2.3 and by Lemma 2.2.4 $(X^X, X^X, X^X, b_{X^X}, b_{X^X}, b_{X^X})$ satisfies Con-

dition 2.2.3 and therefore (X^X, b_{X^X}) is a semigroup. From the definition of a_{X^X} the sextuple $(M, X^X, X, a_{X^X}, a_X, \epsilon_X^X)$ satisfies Condition 2.2.3 and by Lemma 2.2.4 the sextuple $(M, X^X, X^X, a_{X^X}, a_{X^X}, b_{X^X})$ satisfies Condition 2.2.3 and therefore (X^X, a_{X^X}, b_{X^X}) satisfies Condition 2.1.4(a). \square

Let $f_1 : X^X \rightarrow X^{M \otimes X}$ and $f_2 : X^X \rightarrow X^{M \otimes X}$ be the right adjoints of $\epsilon_X^X(1 \otimes a_X) : X^X \otimes (M \otimes X) \rightarrow X$ and $a_X(1 \otimes \epsilon_X^X)\sigma\alpha(1 \otimes \sigma) : X^X \otimes (M \otimes X) \rightarrow X$ respectively, and let $e : E(X) \rightarrow X^X$ be the equalizer of f_1 and f_2 .

Proposition 2.2.6. *For the object $E(X)$ there exist unique morphisms $b_{E(X)} : E(X) \otimes E(X) \rightarrow E(X)$ and $a_{E(X)} : M \otimes E(X) \rightarrow E(X)$ for which e becomes an M -magma morphism and $(E(X), a_{E(X)}, b_{E(X)})$ is in $M\text{-Mag}_2$.*

Proof. In the diagram

$$\begin{array}{ccccc}
 E(X) \otimes E(X) & \xrightarrow{e \otimes e} & X^X \otimes X^X & & \\
 \downarrow b_{E(X)} & & \downarrow b_{X^X} & & \\
 E(X) & \xrightarrow{e} & X^X & \xrightarrow[f_2]{f_1} & X^{M \otimes X} \\
 \uparrow a_{E(X)} & & \uparrow a_{X^X} & & \\
 M \otimes E(X) & \xrightarrow{1 \otimes e} & M \otimes X^X & &
 \end{array}$$

it can be seen, by considering the left adjoints of $f_1 b_{X^X}(e \otimes e)$ and $f_2 b_{X^X}(e \otimes e)$ and the left adjoints of $f_1 a_{X^X}(1 \otimes e)$ and $f_2 a_{X^X}(1 \otimes e)$, that the arrows $b_{X^X}(e \otimes e)$ and $a_{X^X}(1 \otimes e)$ equalize f_1 and f_2 and so, by the universal property of the equalizer e , there exist unique arrows $b_{E(X)}$ and $a_{E(X)}$ making the diagram commute. The left adjoints of the morphisms $ea_{E(X)}(1 \otimes b_{E(X)})$ and $eb_{E(X)}(1 \otimes a_{E(X)})\sigma\alpha(1 \otimes \sigma)$ can be seen to be equal and since e is a monomorphism this shows that $(E(X), a_{E(X)}, b_{E(X)})$ satisfies Condition 2.1.4(b). On the other hand, according to our construction of a_{X^X} and b_{X^X} , the monomorphism e becomes an M -magma morphism from $(E(X), a_{E(X)}, b_{E(X)})$ to (X^X, a_{X^X}, b_{X^X}) , which implies that $(E(X), a_{E(X)}, b_{E(X)})$ satisfies Condition 2.1.4(a) and that $(E(X), b_{E(X)})$ is a semigroup. This completes the proof. \square

By Theorem 2.1.9 we have that $L(E(X), b_{E(X)}, a_{E(X)}) = (E(X), \tilde{b}_{E(X)} = b_{E(X)}(1 - \sigma), a_{E(X)})$ is in $\mathbf{Lie}(M, \delta)$. For $(X, a_X, b_X) \in \mathbf{Lie}(M, \delta)$ let $g_1 : X^X \rightarrow X^{X \otimes X}$ be the right adjoint of $\epsilon_X^X(1 \otimes b_X) : X^X \otimes (X \otimes X) \rightarrow X$, let $g_2 : X^X \rightarrow X^{X \otimes X}$ be the right adjoint of

the sum of the morphisms $b_X(\epsilon_X^X \otimes 1)\alpha : X^X \otimes (X \otimes X) \rightarrow X$ and $b_X(1 \otimes \epsilon_X^X)\sigma\alpha(1 \otimes \sigma) : X^X \otimes (X \otimes X) \rightarrow X$, and let $d : D(X) \rightarrow E(X)$ be the equalizer of g_1e and g_2e .

Proposition 2.2.7. *For the object $D(X)$ there exist unique morphisms $b_{D(X)} : D(X) \otimes D(X) \rightarrow D(X)$ and $a_{D(X)} : M \otimes D(X) \rightarrow D(X)$ for which d is an M -magma morphism from $(D(X), a_{D(X)}, b_{D(X)})$ to $L(E(X), a_E(X), b_E(X))$ and $(D(X), a_{D(X)}, b_{D(X)})$ is in $\mathbf{Lie}(M, \delta)$.*

Proof. In the diagram

$$\begin{array}{ccccccc}
D(X) \otimes D(X) & \xrightarrow{d \otimes d} & E(X) \otimes E(X) & & & & \\
\vdots \downarrow b_{D(X)} & & \downarrow (1-\sigma) & & & & \\
& & E(X) \otimes E(X) & \xrightarrow{e \otimes e} & X^X \otimes X^X & & \\
& & \downarrow b_{E(X)} & & \downarrow b_{X^X} & & \\
D(X) & \xrightarrow{d} & E(X) & \xrightarrow{e} & X^X & \xrightarrow[g_2]{g_1} & X^X \otimes X^X \\
\uparrow a_{D(X)} & & \uparrow a_{E(X)} & & \uparrow a_{X^X} & & \\
M \otimes D(X) & \xrightarrow{1 \otimes d} & M \otimes E(X) & \xrightarrow{1 \otimes e} & M \otimes X^X & &
\end{array}$$

it can be seen, by considering the left adjoints of $g_1b_{X^X}(e \otimes e)(1 - \sigma)(d \otimes d)$ and $g_2b_{X^X}(e \otimes e)(1 - \sigma)(d \otimes d)$ and the left adjoints of $g_1a_{X^X}(1 \otimes e)(1 \otimes d)$ and $g_2a_{X^X}(1 \otimes e)(1 \otimes d)$, that the morphisms $b_{X^X}(e \otimes e)(1 - \sigma)(d \otimes d)$ and $a_{X^X}(1 \otimes e)(1 \otimes d)$ equalize g_1 and g_2 and so, by the universal property of the equalizer d , there exist unique arrows $b_{D(X)}$ and $a_{D(X)}$ making the diagram commute. Since d is a monomorphism we see that $(D(X), a_{D(X)}, b_{D(X)})$ is in $\mathbf{Lie}(M, \delta)$. \square

We now define the object $\text{Der}(X)$ of a derivation of $X = (X, a_X, b_X)$ as $\text{Der}(X) = D(X) = (D(X), a_{D(X)}, b_{D(X)})$.

2.3 Representability of split extension functor for the category $\mathbf{Lie}(M, \delta)$

In this section we show that the functor $\text{SplExt}(-, X)$ can be defined for the category $\mathbf{Lie}(M, \delta)$ and prove that it is representable by showing that $\text{Der}(X) = D(X)$ is the representing object.

and j making the diagram commute.

Proposition 2.3.1. *The morphism $j : G \rightarrow D(X)$ is a morphism in $\mathbf{Lie}(M, \delta)$.*

Proof. Consider the diagrams

$$\begin{array}{ccccc}
 & & 1 \otimes h & & \\
 & \curvearrowright & & \curvearrowleft & \\
 M \otimes G & \xrightarrow{1 \otimes j} & M \otimes D(X) & \xrightarrow{1 \otimes ed} & M \otimes X^X \\
 \downarrow a_G & & \downarrow a_{D(X)} & & \downarrow a_{X^X} \\
 G & \xrightarrow{j} & D(X) & \xrightarrow{ed} & X^X \\
 & \curvearrowleft & & \curvearrowright & \\
 & & h & &
 \end{array}$$

$$\begin{array}{ccccc}
 & & h \otimes h & & \\
 & \curvearrowright & & \curvearrowleft & \\
 G \otimes G & \xrightarrow{j \otimes j} & D(X) \otimes D(X) & \xrightarrow{ed \otimes ed} & X^X \otimes X^X \\
 \downarrow b_G & & \downarrow b_{D(X)} & & \downarrow 1 - \sigma \\
 G & \xrightarrow{j} & D(X) & \xrightarrow{ed} & X^X \\
 & \curvearrowleft & & \curvearrowright & \\
 & & h & &
 \end{array}$$

where $G = (G, a_G, b_G)$. Considering the left adjoints of $a_{X^X}(1 \otimes h)$ and ha_G (in the first diagram), and considering the left adjoints of $b_{X^X}(1 - \sigma)(h \otimes h)$ and hb_G (in the second diagram), the diagram formed by the outer arrows can be seen to commute. Therefore, since e and d are monomorphisms and the right hand square in each diagram commutes, the left hand squares also commute. \square

For each G in $\mathbf{Lie}(M, \delta)$, using the above construction we define the map $\tau_G : \text{SplExt}(G, X) \rightarrow \mathbf{Lie}(M, \delta)(G, \text{Der}(X))$ as follows:

$$\tau_G([X \xrightarrow{k} A \xrightleftharpoons[s]{p} G]) = j.$$

Proposition 2.3.2. *The maps τ_G form a natural transformation.*

Proof. Let $A = (A, a, b)$ and $A' = (A', a', b')$ be objects in $\mathbf{Lie}(M, \delta)$ and let $f : G' \rightarrow G$ be

any morphism in $\mathbf{Lie}(M, \delta)$, such that in the diagram

$$\begin{array}{ccccc} X & \xrightleftharpoons[k']{l'} & A' & \xrightleftharpoons[s']{p'} & G' \\ \downarrow 1_X & & \downarrow f' & & \downarrow f \\ X & \xrightleftharpoons[k]{l} & A & \xrightleftharpoons[s]{p} & G \end{array}$$

(A', f', p') is the pullback of f and p in $\mathbf{Lie}(M, \delta)$, l and l' are the unique M -action morphisms with $kl = 1_A - sp$ and $k'l' = 1_{A'} - s'p'$, and the top and bottom rows excluding l and l' are split extensions. Let h' be the right adjunct of $l'b'(s' \otimes k')$ and j' be the unique morphism with $edj' = h'$, that is,

$$\tau'_G([X \xrightarrow{k'} A' \xrightleftharpoons[s']{p'} G']) = j'.$$

Since $lb(s \otimes k)(f \otimes 1) = lb(sf \otimes k) = lb(f's' \otimes f'k') = lf'b'(s' \otimes k') = l'b'(s' \otimes k')$ and h and h' are the right adjuncts of $lb(s \otimes k)$ and $l'b'(s' \otimes k')$ respectively, it follows that $hf = h'$. Therefore we have $edjf = hf = h' = edj'$ and since ed is monomorphism we conclude that $jf = j'$ and that the diagram

$$\begin{array}{ccc} \text{SplExt}(G, X) & \xrightarrow{\tau_G} & \mathbf{Lie}(M, \delta)(G, \text{Der}(X)) \\ \downarrow \text{SplExt}(f, X) & & \downarrow \mathbf{Lie}(M, \delta)(f, \text{Der}(X)) \\ \text{SplExt}(G', X) & \xrightarrow{\tau_{G'}} & \mathbf{Lie}(M, \delta)(G', \text{Der}(X)) \end{array}$$

commutes. □

Theorem 2.3.3. *The functor $\text{SplExt}(-, X) : \mathbf{Lie}(M, \delta) \rightarrow \mathbf{Set}$ is representable with representation $(\tau, \text{Der}(X))$.*

Proof. We show that the natural transformation $\tau : \text{SplExt}(-, X) \rightarrow \mathbf{Lie}(M, \delta)(-, \text{Der}(X))$ is a natural isomorphism. For an arrow $z : G \rightarrow \text{Der}(X)$ in $\mathbf{Lie}(M, \delta)$ let $r : G \otimes X \rightarrow X$ be the left adjunct of edz , and let $X \rtimes_z G = (X \oplus G, a, b)$, where

$$a = \iota_1 a_X(1 \otimes \pi_1) + \iota_2 a_G(1 \otimes \pi_2)$$

and

$$b = \iota_1(b_X(\pi_1 \otimes \pi_1) + r(\pi_2 \otimes \pi_1)(1 - \sigma)) + \iota_2 b_G(\pi_2 \otimes \pi_2),$$

in obvious notation. It can be seen that $X \rtimes_z G$ is in $\mathbf{Lie}(M, \delta)$ and that the diagram

$$X \xrightarrow{\iota_1} X \rtimes_r G \begin{array}{c} \xleftarrow{\pi_2} \\ \xrightarrow{\iota_2} \end{array} G$$

is a split extension in $\mathbf{Lie}(M, \delta)$. Let $\hat{\tau}_G : \mathbf{Lie}(M, \delta)(G, \text{Der}(X)) \rightarrow \text{SplExt}(G, X)$ be the map defined as follows:

$$\hat{\tau}_G(z) = [X \xrightarrow{\iota_1} X \rtimes_r G \begin{array}{c} \xleftarrow{\pi_2} \\ \xrightarrow{\iota_2} \end{array} G]$$

It can be seen that $\hat{\tau}_G = \tau_G^{-1}$ and hence $(\tau, \text{Der}(X))$ is a representation of $\text{SplExt}(-, X)$. □

Remark 2.3.4. *Since the category $\mathbf{Cat}(\mathbf{Lie}(M, \delta))$ of internal categories in $\mathbf{Lie}(M, \delta)$ can be presented as $\mathbf{Lie}(M', \delta')$ for suitable M' and δ' (it essentially follows from the results of [16]), by Theorem 2.3.3 the functor $\text{SplExt} : \mathbf{Cat}(\mathbf{Lie}(M, \delta)) \rightarrow \mathbf{Set}$ is representable.*

Chapter 3

Right adjoints to pullback functors

In this chapter we consider the question of existence of right adjoints of pullback functors between categories of points. In Section 3.1 we consider arbitrary pullback functors for categories which generalize locally well-presentable categories. In Section 3.2 we consider pullback functors along morphisms into the terminal object for weakly unital categories. In Section 3.3 we consider pullback functors along split epimorphisms for weakly mal'tsev categories. In Sections 3.4 and 3.5 we again consider arbitrary pullback functors for pro-modular and semi-abelian categories respectively, and conclude Section 3.5 with many examples.

3.1 Locally well-presentable and related categories

In this section we consider locally small finitely complete well-cocomplete categories satisfying Condition 1.2.2, with a small generating set, and in which finite limits commute with filtered colimits. We show (Theorem 3.1.4) that in such categories, pullback functors have right adjoints when they preserve binary coproducts.

Definition 3.1.1. \mathbb{C} is a well-cocomplete category when \mathbb{C} has small colimits and wide pushouts of epimorphisms.

Proposition 3.1.2. Let \mathbb{C} be a locally small finitely cocomplete category. If \mathbb{C} has a small generating set, then for any object $B \in \mathbb{C}$ the category $\mathbf{Pt}(B)$ has a small generating set.

Proof. Let \mathbb{Q} be a small generating set for \mathbb{C} , let B be a fixed object in \mathbb{C} , and let $\mathbb{Q}' = \{(Q, \rho) \in (\mathbb{C} \downarrow B) \mid Q \in \mathbb{Q}\}$. We will show that \mathbb{Q}' is a generating set for $(\mathbb{C} \downarrow B)$. Since for any pair of morphisms $f, g : (A, \alpha) \rightarrow (A, \alpha')$ in $(\mathbb{C} \downarrow B)$ with $f \neq g$, since $f \neq g$ in \mathbb{C} , there

exists $Q \in \mathbb{Q}$ and a morphism $h : Q \rightarrow A$ with $fh \neq gh$. It is easy to see that if $\rho = \alpha h$, then (Q, ρ) is in \mathbb{Q}' and $h : (Q, \rho) \rightarrow (A, \alpha)$ is a morphism in $(\mathbb{C} \downarrow B)$.

Let $U : \mathbf{Pt}(B) \rightarrow (\mathbb{C} \downarrow B)$ be the functor defined on objects as $U(A, \alpha, \beta) = (A, \alpha)$, and let $F : (\mathbb{C} \downarrow B) \rightarrow \mathbf{Pt}(B)$ be the functor defined on objects as $F(A, \alpha) = (B + A, [1_B, \alpha], \iota_1)$. It is easy to see that F is the left adjoint of U and that U is faithful, consequently by Proposition 1.5.5, $\mathbf{Pt}(B)$ has a small generating set. \square

Proposition 3.1.3. *Let \mathbb{C} be a locally small finitely complete well-cocomplete category with a small generating set, in which finite limits commute with filtered colimits. For any object $B \in \mathbb{C}$, the category $\mathbf{Pt}(B)$ is also locally small finitely complete well-cocomplete category with a small generating set, in which finite limits commute with filtered colimits.*

Proof. It follows from Remark 1.2.3, if \mathbb{C} is small cocomplete then $\mathbf{Pt}(B)$ is also small cocomplete. Note that for any objects $(A, \alpha, \beta), (A', \alpha', \beta') \in \mathbf{Pt}(B)$, since $\text{hom}(A, A')$ is small, so is $\text{hom}((A, \alpha, \beta), (A', \alpha', \beta'))$. Since \mathbb{C} is locally small, small cocomplete, and has a small generating set, it follows by Proposition 3.1.2 that $\mathbf{Pt}(B)$ has a small generating set. It follows from Proposition 1.2.4 that the forgetful functor $P : \mathbf{Pt}(B) \rightarrow \mathbb{C}$ preserves pushouts. Therefore, P preserves epimorphisms, and since \mathbb{C} is well-cocomplete and the projection functor creates pushouts, it follows that $\mathbf{Pt}(B)$ is well-cocomplete. \square

Theorem 3.1.4. *Let \mathbb{C} be a finitely complete finitely cocomplete category satisfying Condition 1.2.2. For a fixed morphism $p : E \rightarrow B$ in \mathbb{C} the following are equivalent:*

1. *the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ preserves binary coproducts;*
2. *the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ preserves finite colimits.*

When \mathbb{C} is also locally small well-cocomplete category with a small generating set, in which finite limits commute with filtered colimits, those conditions are further equivalent to:

3. *the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ has a right adjoint.*

Proof. 2 \Rightarrow 1: Obvious.

1 \Rightarrow 2: By Theorem 1.5.4 it is sufficient to prove that p^* preserves the initial object as well as all reflexive coequalizers. However, since Condition 1.2.2 holds, we need only prove that p^* preserves the initial object. Since categories of points are always pointed and p^* preserves all limits (limits commute with limits), it trivially follows that p^* preserves the initial object.

Suppose next that \mathbb{C} is also a locally small finitely complete well-cocomplete category with a small generating set, in which finite limits commute with filtered colimits.

2 \Rightarrow 3: Since finite limits commute with filtered colimits, it follows from Proposition 1.2.4 that p^* preserves filtered colimits; by Theorem 1.5.4, p^* preserves all small colimits. By Proposition 3.1.3, both $\mathbf{Pt}(B)$ and $\mathbf{Pt}(E)$ are well-cocomplete, locally small, and have small generating sets. Since p^* preserves all small colimits, The Special Adjoint Functor Theorem (see [18], Chapter V, Section 8, Theorem 2) gives a right adjoint.

3 \Rightarrow 1: Obvious. □

Definition 3.1.5. *A category \mathbb{C} is locally well-presentable when:*

1. \mathbb{C} is locally presentable;
2. finite limits commute with filtered colimits in \mathbb{C} .

Proposition 3.1.6. *Every locally presentable category is locally small, finitely complete, well-cocomplete, and has a small generating set.*

Proof. By definition, locally presentable categories are locally small, cocomplete, and have a small generating set. In [1] the authors show that every locally presentable category is complete (Corollary 1.28) and co-well-powered (Theorem 1.58). The proof is completed by noting that any category which is cocomplete and co-well-powered is well-cocomplete. □

3.2 Weakly unital categories

In this section we consider the question of existence of right adjoints of pullback functors $(B \rightarrow 1)^* : \mathbf{Pt}(1) \rightarrow \mathbf{Pt}(B)$ in weakly unital categories. We begin by showing that the existence of a right adjoint is equivalent to the existence of a *centralizer* object (as defined bellow). We then find sufficient conditions for the existence of these pullback functors.

Throughout this section we assume that \mathbb{C} is a weakly unital category. Since the category \mathbb{C} is pointed there is a canonical equivalence of categories $\mathbf{Pt}(1) \sim \mathbb{C}$. Let $T : \mathbb{C} \rightarrow \mathbf{Pt}(B)$ be the functor defined as composite of the pullback functor $(B \rightarrow 1)^* : \mathbf{Pt}(1) \rightarrow \mathbf{Pt}(B)$ and this canonical equivalence.

The original categorical concept of a centre comes from [13] and was also studied in [14]. The concept of the centralizer of a morphism for a unital category was introduced in [8], here we introduce the related concept:

Definition 3.2.1. *For an object $C \in \mathbb{C}$ and for any objects $(A, f), (B, g) \in (\mathbb{C} \downarrow C)$, we will denote by $\mathbb{Z}_{(A,f)}(B, g)$ the full subcategory of the comma category $(\mathbb{C} \downarrow A)$, with objects*

(E, λ) such that $f\lambda$ and g commute. This category will be called the centralizer category of (B, g) with respect to (A, f) (in $(\mathbb{C} \downarrow C)$), and if this category has a terminal object it will be called the centralizer of (B, g) with respect to (A, f) and denoted by $Z_{(A,f)}(B, g)$.

Definition 3.2.2. The centralizer of a morphism $g : B \rightarrow C$, as defined in [8], is (when it exists) the terminal object in the category $\mathbb{Z}_{(C,1_C)}(B, g)$, we will also denote the object $Z_{(C,1_C)}(B, g)$ by Z_g .

Proposition 3.2.3. For each $(A, \alpha, \beta) \in \mathbf{Pt}(B)$, the comma category $(T \downarrow (A, \alpha, \beta))$ is isomorphic to $\mathbb{Z}_{(K,\kappa)}(B, \beta)$, where (K, κ) is the kernel of α .

Proof. Since the functor T assigns to each $X \in \mathbb{C}$ the triple $(X \times B, \pi_2, \langle 0, 1 \rangle) \in \mathbf{Pt}(B)$, it follows that for any object $(E, \psi) \in (T \downarrow (A, \alpha, \beta))$ the diagram

$$\begin{array}{ccc}
 E & & \\
 \langle 1, 0 \rangle \downarrow & \searrow^{\psi \langle 1, 0 \rangle} & \\
 E \times B & \xrightarrow{\psi} & A \\
 \langle 0, 1 \rangle \uparrow & \nearrow_{\beta} & \\
 B & &
 \end{array}$$

commutes, and consequently the morphisms $\psi \langle 1, 0 \rangle$ and β commute. Since ψ is a morphism in $\mathbf{Pt}(B)$, $\alpha\psi = \pi_2$ and therefore $\alpha\psi \langle 1, 0 \rangle = \pi_2 \langle 1, 0 \rangle = 0$. Since (K, κ) is the kernel of α there exists a unique morphism $\lambda : E \rightarrow K$ in \mathbb{C} with $\kappa\lambda = \psi \langle 1, 0 \rangle$, therefore (E, λ) is an object in $\mathbb{Z}_{(K,\kappa)}(B, \beta)$. Conversely, for a pair $(E', \lambda') \in \mathbb{Z}_{(K,\kappa)}(B, \beta)$, let $\psi' : E' \times B \rightarrow A$ be the unique morphism making the diagram

$$\begin{array}{ccc}
 E' & & \\
 \langle 1, 0 \rangle \downarrow & \searrow^{\kappa\lambda'} & \\
 E' \times B & \xrightarrow{\psi'} & A \\
 \langle 0, 1 \rangle \uparrow & \nearrow_{\beta} & \\
 B & &
 \end{array}$$

commute. Since the morphisms $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are jointly epimorphic, $\alpha\psi' \langle 1, 0 \rangle = \alpha\kappa\lambda' = 0 = \pi_2 \langle 1, 0 \rangle$ and $\alpha\psi' \langle 0, 1 \rangle = \alpha\beta = 1_B = \pi_2 \langle 0, 1 \rangle$, it follows that $\alpha\psi' = \pi_2$ and (E', ψ') is an object in $(T \downarrow (A, \alpha, \beta))$. It is clear that these assignments are inverse to each other. \square

Corollary 3.2.4. *The following conditions are equivalent:*

1. T has a right adjoint;
2. for every $(A, \alpha, \beta) \in \mathbf{Pt}(B)$ with (K, κ) the kernel of α , the category $\mathbb{Z}_{(K, \kappa)}(B, \beta)$ has a terminal object.

Proof. Immediately follows from Proposition 3.2.3. □

Proposition 3.2.5. *For $C \in \mathbb{C}$ and $(A, f), (B, g) \in (\mathbb{C} \downarrow C)$ the following are equivalent:*

1. the category $\mathbb{Z}_{(A, f)}(B, g)$ has a terminal object;
2. the forgetful functor from $\mathbb{Z}_{(A, f)}(B, g)$ to $(\mathbb{C} \downarrow C)$ has a right adjoint.

Proof. $2 \Rightarrow 1$: The proof follows trivially from the fact that $(C, 1_C)$ is the terminal object in $(\mathbb{C} \downarrow C)$, and that right adjoints preserve limits.

$1 \Rightarrow 2$: Suppose that $\mathbb{Z}_{(A, f)}(B, g)$ has a terminal object (T, τ) , for an object $(E, \lambda) \in (\mathbb{C} \downarrow C)$ let

$$\begin{array}{ccc} D & \xrightarrow{\rho_2} & T \\ \downarrow \rho_1 & & \downarrow f\tau \\ E & \xrightarrow{\lambda} & C \end{array}$$

be the pullback of λ and $f\tau$. Since, by Proposition 1.1.4, $f\tau\rho_2$ commutes with g , it is clear that $(D, \tau\rho_2)$ is in $\mathbb{Z}_{(A, f)}(B, g)$ and that the functor $R: (\mathbb{C} \downarrow C) \rightarrow \mathbb{Z}_{(A, f)}(B, g)$, defined on objects as $R(E, \lambda) = (D, \tau\rho_2)$, is the right adjoint of the forgetful functor. □

Proposition 3.2.6. *Let $h: (A', f') \rightarrow (A, f)$ be any morphism in $(\mathbb{C} \downarrow C)$. The functor $h_*: \mathbb{Z}_{(A', f')}(B, g) \rightarrow \mathbb{Z}_{(A, f)}(B, g)$ defined on objects as $h_*(E', \lambda') = (E', h\lambda')$, has a right adjoint.*

Proof. For $(E, \lambda) \in \mathbb{Z}_{(A, f)}(B, g)$, let (E', λ', h') be the pullback of h and λ . It easily follows that the functor assigning (E, λ) to (E', λ') is the right adjoint of h_* . □

Corollary 3.2.7. *If the centralizer of $g: B \rightarrow C$ exists then the centralizer of (B, g) with respect to (A, f) exists.*

Proof. Immediately follows from Proposition 3.2.6. □

It follows that if the centralizer Z_g exists then $\mathbb{Z}_{(A, f)}(B, g)$ can be calculated as the pullback of the inclusion of Z_g in C along g . Moreover, if f is a monomorphism $\mathbb{Z}_{(A, f)}(B, g)$

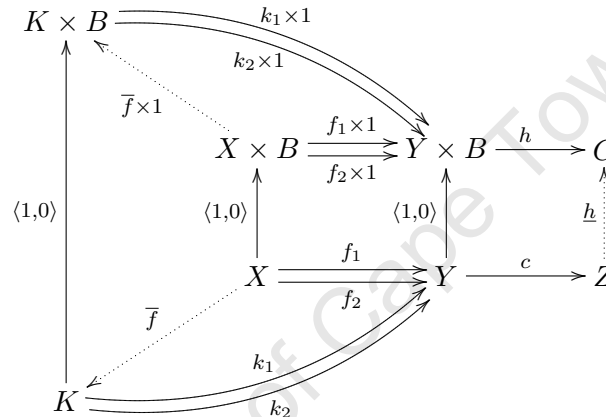
is simply the intersection of Z_g and A (in C).

Throughout the rest of the section we assume, in addition, that \mathbb{C} is regular.

It can be seen that the following proposition follows from Proposition 1.1.8, we give a direct proof here.

Proposition 3.2.8. *For any object $B \in \mathbb{C}$, the functor $- \times B : \mathbb{C} \rightarrow \mathbb{C}$ preserves coequalizers.*

Proof. Consider any diagram



in \mathbb{C} , in which:

- c is the coequalizer of f_1 and f_2 ;
- k_1 and k_2 are the kernel pair of c ;
- h is any morphism with $h(f_1 \times 1) = h(f_2 \times 1)$;
- \underline{h} is the unique morphism $\underline{h}c = h\langle 1, 0 \rangle$;
- \bar{f} is the unique morphism with $k_1\bar{f} = f_1$ and $k_2\bar{f} = f_2$.

Note that $h(k_1 \times 1)\langle 1, 0 \rangle = h\langle 1, 0 \rangle k_1 = \underline{h}ck_1$ and similarly $h(k_2 \times 1)\langle 1, 0 \rangle = \underline{h}ck_2$. Since $ck_1 = ck_2$, it follows that $h(k_1 \times 1)\langle 1, 0 \rangle = h(k_2 \times 1)\langle 1, 0 \rangle$. Trivially $h(k_1 \times 1)\langle 0, 1 \rangle = h(k_2 \times 1)\langle 0, 1 \rangle$, and since $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are jointly epimorphic, $h(k_1 \times 1) = h(k_2 \times 1)$. From this it easily follows that the coequalizer of $f_1 \times 1$ and $f_2 \times 1$ and the coequalizer of $k_1 \times 1$ and $k_2 \times 1$ are the same. Since \mathbb{C} is regular, $c \times 1$ is a regular epimorphism and therefore the coequalizer of its kernel pair $k_1 \times 1$ and $k_2 \times 1$. \square

Corollary 3.2.9. For any object $B \in \mathbb{C}$, the functor $T : \mathbb{C} \rightarrow \mathbf{Pt}(B)$ preserves coequalizers.

Proof. Immediately follows from Proposition 3.2.8. \square

Proposition 3.2.10. Let C be an object in \mathbb{C} , and let (A, f) and (B, g) be objects in the comma category $(\mathbb{C} \downarrow C)$. For any $(E, \lambda) \in \mathbb{Z}_{(A, f)}(B, g)$, there exists $(S, \eta) \in \mathbb{Z}_{(A, f)}(B, g)$ with η a monomorphism in \mathbb{C} , and there exists a regular epimorphism $e : (E, \lambda) \rightarrow (S, \eta)$ in $\mathbb{Z}_{(A, f)}(B, g)$.

Proof. Since \mathbb{C} is a regular category, there exists a regular epimorphism e and a monomorphism η such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\lambda} & A \\ & \searrow e & \nearrow \eta \\ & S & \end{array}$$

commutes. Since e is a regular epimorphism, by Proposition 1.1.8 $f\eta$ commutes with g . It is easy to see that $e : (E, \lambda) \rightarrow (S, \eta)$ is a regular epimorphism in $\mathbb{Z}_{(A, f)}(B, g)$. \square

Proposition 3.2.11. For $C \in \mathbb{C}$ and $(A, f), (B, g) \in (\mathbb{C} \downarrow C)$ the following are equivalent:

1. $\mathbb{Z}_{(A, f)}(B, g)$ has a terminal object;
2. $\mathbb{Z}_{(A, f)}(B, g)$ has a weak terminal object.

Proof. 1 \Rightarrow 2: The proof is trivial since every terminal object is a weak terminal object.

2 \Rightarrow 1: Suppose (E, λ) is a weak terminal object. By Proposition 3.2.10 there exists a morphism (regular epimorphism) $e : (E, \lambda) \rightarrow (S, \eta)$ where η is mono, it follows that (S, η) is a terminal object. \square

Corollary 3.2.12. Suppose in addition that \mathbb{C} is a small cocomplete well-powered category. Let C be an object in \mathbb{C} , and let (A, f) and (B, g) be objects in the comma category $(\mathbb{C} \downarrow C)$. The category $\mathbb{Z}_{(A, f)}(B, g)$ has a terminal object if it is small cocomplete.

Proof. Immediately follows from Proposition 3.2.11. \square

Theorem 3.2.13. The following are equivalent:

1. the functor $T : \mathbb{C} \rightarrow \mathbf{Pt}(B)$ preserves binary coproducts;
2. the functor $T : \mathbb{C} \rightarrow \mathbf{Pt}(B)$ preserves finite colimits.

When \mathbb{C} is, in addition, a cocomplete well-powered category in which filtered colimits in \mathbb{C} commute with finite limits, those conditions are further equivalent to:

3. the functor $T : \mathbb{C} \rightarrow \mathbf{Pt}(B)$ has a right adjoint.

Proof. $1 \Rightarrow 2$: Since both \mathbb{C} and $\mathbf{Pt}(B)$ are pointed and T clearly preserves limits (limits commute with limits), the functor T preserves the initial object. The functor T preserves binary coproducts by assumption and (reflexive) coequalizers by Corollary 3.2.9, and so by Theorem 1.5.4, T preserves all finite colimits.

Suppose \mathbb{C} is also well-powered and cocomplete with filtered colimits commuting with finite limits.

$2 \Rightarrow 3$: It follows from Theorem 1.5.4 that T preserves all small colimits. Therefore, for any object $(A, \alpha, \beta) \in \mathbf{Pt}(B)$, the comma category $(T \downarrow (A, \alpha, \beta))$ is small cocomplete; equivalently, by Proposition 3.2.3, the category $\mathbb{Z}_{(K, \kappa)}(B, \beta)$ where (K, κ) is the kernel of α , is small cocomplete. By Corollary 3.2.12, the category $\mathbb{Z}_{(K, \kappa)}(B, \beta)$ has a terminal object and consequently, by Corollary 3.2.4, T has a right adjoint. \square

Remark 3.2.14. *The implication $2 \Rightarrow 3$ of Theorem 3.2.13 holds in a more general setting, namely, when \mathbb{C} is a regular unital category with large directed unions of subobjects preserved for each $B \in \mathbb{C}$ by the functor $B \times -$.*

For a variety \mathcal{V} with signature $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$, where Ω_n is the set of n -ary terms, consider the following condition:

Condition 3.2.15. 1. $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

2. $\Omega_0 = \{0\}$;

3. $\Omega_2 = \{+\} \cup \Omega'_2$;

4. $x + 0 = x = 0 + x$;

5. $x + (y + z) = (x + y) + z$;

6. for each $u \in \Omega_1$, $u(x + y) = u(x) + u(y)$ or $u(x + y) = u(y) + u(x)$;

7. for each $t \in \Omega'_2$ and for each $u \in \Omega_1$:

(a) t^{op} is in Ω'_2 ;

(b) $t(x + x', y) = t(x, y) + t(x', y)$;

(c) $t(x, 0) = 0$;

$$(d) \ t(x, y) + z = z + t(x, y);$$

$$(e) \ u(t(x, y)) = t(ux, y);$$

8. for each ordered pair $(s, t) \in \Omega'_2 \times \Omega'_2$, there exist ordered pairs $(s_1, t_1), \dots, (s_n, t_n) \in \Omega'_2 \times \Omega'_2$ and an n -ary word ω such that

$$s(x, t(y, z)) = \omega(s_1(z, t_1(x, y)), \dots, s_m(z, t_m(x, y)), s_{m+1}(y, t_{m+1}(x, z)), \dots, s_n(y, t_n(x, z)))$$

Definition 3.2.16. \mathcal{V} is a category of interest when \mathcal{V} is a variety satisfying:

1. Conditon 3.2.15;
2. there exists $- \in \Omega_1$ such that every object has the structure of a group;
3. for any $u \in \Omega_1 \setminus \{-\}$, $u(x + y) = u(x) + u(y)$.

Categories of interest were introduced and studied by G. Orzech (see [20]), and include the categories of groups, rings, Lie algebras over a ring and many others.

Proposition 3.2.17. Let \mathcal{V} be a variety satisfying Condition 3.2.15. A pair of morphisms $f : A \rightarrow C$ and $g : B \rightarrow C$ commute if and only if for each $a \in A$ and $b \in B$:

1. $f(a) + g(b) = g(b) + f(a)$;
2. $t(f(a), f(b)) = 0$ for each $t \in \Omega'_2$.

Proof. Suppose $f : A \rightarrow C$ and $g : B \rightarrow C$ commute, then there exists a morphism $\phi : A \times B \rightarrow C$ such that the diagram

$$\begin{array}{ccc}
 A & & C \\
 \langle 1, 0 \rangle \downarrow & \searrow f & \\
 A \times B & \xrightarrow{\phi} & C \\
 \langle 0, 1 \rangle \uparrow & \nearrow g & \\
 B & &
 \end{array}$$

commutes. For any $(a, b) \in A \times B$, applying ϕ to the equation $(a, 0) + (0, b) = (a, b) = (0, b) + (a, 0)$ gives $f(a) + g(b) = g(b) + f(a)$. We have:

$$\phi(t((a, 0), (0, b))) = \phi(t(a, 0), t(0, b)) = \phi(0, 0) = 0,$$

$$\begin{aligned}\phi(t((a, 0), (0, b))) &= t(\phi(a, 0), \phi(0, b)) = t(f(a), g(b)) \\ &= t(f(a), g(b)),\end{aligned}$$

and consequently $t(f(a), g(b)) = 0$. Now suppose $f : A \rightarrow C$ and $g : B \rightarrow C$ satisfy the conditions above and let $\phi : A \times B \rightarrow C$ be defined as $\phi(a, b) = f(a) + g(b)$. We must show that ϕ is a morphism, which in fact follows from the identities:

$$u\phi(a, b) = u(f(a) + g(b)) = uf(a) + ug(b) = fua + gub = \phi(ua, ub) = \phi(u(a, b)),$$

$$\begin{aligned}\phi((a, b) + (a', b')) &= \phi(a, b) + \phi(a', b') = f(a) + g(b) + f(a') + g(b') \\ &= f(a + a') + g(b + b') = \phi(a + a', b + b'),\end{aligned}$$

$$\begin{aligned}t(\phi(a, b), \phi(a', b')) &= t(f(a) + g(b), f(a') + g(b)) \\ &= t(f(a), f(a')) + t(f(a), g(b')) + t^{\text{op}}(f(a'), g(b)) + t(g(b), g(b')) \\ &= f(t(a, a')) + g(t(b, b')) = \phi(t(a, a'), t(b, b')).\end{aligned}$$

□

The following result is known when \mathcal{V} is a category of interest (see [20]).

Proposition 3.2.18. *Let \mathcal{V} be a variety satisfying Condition 3.2.15. Every morphism $g : B \rightarrow C$ has a centralizer.*

Proof. Let

$$\begin{aligned}A &= \{c \in C \mid \forall b \in B \forall u_1, u_2, \dots, u_n \in \Omega_1 \ u_1 u_2 \dots u_n c + g(b) \\ &= g(b) + u_1 u_2 \dots u_n c \text{ and } \forall t \in \Omega_2' \ t(c, g(b)) = 0\}.\end{aligned}$$

By Proposition 3.2.17, every morphism that commutes with g has image in A , and the inclusion of A into C clearly satisfies the conditions of Proposition 3.2.17. Therefore, all that remains is to show that A is a subalgebra of C . It is easy to see that A is closed under unary operations, and for each $u \in \Omega_1$ and for each $x, y \in A$ we have $u(x + y) = ux + uy$ or $u(x + y) = uy + ux$ and $u(t(a, y)) = t(u(x), y)$. Since for each $a, a' \in A$ and for each $b \in B$ the equalities

$$-(a + a') + g(b) = a + a' + g(b) = a + g(b) + a' = g(b) + a + a';$$

- for all $t \in \Omega'_2$, $t(a + a', g(b)) = t(a, g(b)) + t(a', g(b)) = 0$;
- for all $t \in \Omega'_2$, $t(a, a') + g(b) = g(b) + t(a, a')$;
- for all $t, s \in \Omega'_2$,

$$\begin{aligned}
s(t(a, a'), g(b)) &= s^{\text{op}}(g(b), t(a, a')) \\
&= \omega(s_1(a', t_1(g(b), a)), \dots, s_m(a', t_m(g(b), a)), \\
&\quad s_{m+1}(a, t_{m+1}(g(b), a')), \dots, s_n(a, t_n(g(b), a')),
\end{aligned}$$

since $t_i(g(b), a) = t_i^{\text{op}}(a, g(b)) = 0$, $t_i(g(b), a') = t_i^{\text{op}}(a', g(b)) = 0$, and since for any $v \in \Omega'_2$ $v(x, 0) = 0$, it follows that $s(t(a, a'), g(b)) = \omega(0, \dots, 0) = 0$;

hold, it follows that A is closed under all operations. \square

3.3 Weakly Mal'tsev categories

In this section we consider regular weakly Mal'tsev categories. We give a sufficient condition for the existence of a right adjoint of a pullback functor along a split epimorphism.

Proposition 3.3.1. *Let \mathbb{C} be a regular weakly Mal'tsev category. For any object B in \mathbb{C} and for any (C, γ, δ) in $\mathbf{Pt}(B)$, the functor $(C, \gamma, \delta) \times - : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(B)$ preserves reflexive coequalizers.*

Proof. This is simply a reformulation of Proposition 3.2.8 in terms of the category $\mathbf{Pt}(B)$. \square

Let $p : E \rightarrow B$ be any split epimorphism with splitting $s : B \rightarrow E$. Since by Proposition 1.3.5 the functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ can be constructed from the functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E, p, s)$ by composing with the canonical isomorphism $\mathbf{Pt}(E, p, s) \cong \mathbf{Pt}(E)$, and since $\mathbf{Pt}(B)$ is weakly unital, the following results follow directly from Corollary 3.2.9 and Theorem 3.2.13 respectively:

Corollary 3.3.2. *Let \mathbb{C} be a regular weakly Mal'tsev category. For any split epimorphism $p : E \rightarrow B$, the functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ preserves reflexive coequalizers.*

Theorem 3.3.3. *Let \mathbb{C} be a regular weakly Mal'tsev category. For any split epimorphism $p : E \rightarrow B$, the following are equivalent:*

1. the functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ preserves binary coproducts;

2. the functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ preserves finite colimits.

When \mathbb{C} , in addition, is well complete and filtered colimits commute with finite limits, those conditions are further equivalent to:

3. the functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ has a right adjoint.

3.4 Protomodular categories

In this section we show that for a pointed protomodular category \mathbb{C} , if every kernel functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbb{C}$ has a right adjoint, then every pullback functor is comonadic.

Theorem 3.4.1. *For a pointed protomodular category \mathbb{C} with finite limits, the following conditions are equivalent:*

1. for each object $B \in \mathbb{C}$ the functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbb{C}$ has a right adjoint;
2. for each object $B \in \mathbb{C}$ the functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbb{C}$ is comonadic;
3. for each morphism $p : E \rightarrow B$ in \mathbb{C} the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ has a right adjoint;
4. for each morphism $p : E \rightarrow B$ in \mathbb{C} the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ is comonadic.

Proof. The implication $4 \Rightarrow 1$ is trivial. Recall that for any morphism $p : E \rightarrow B$ in \mathbb{C} , p^* preserves all limits and reflects isomorphisms. Therefore if p^* has a right adjoint, then by the dual of the Weak Tripleability Theorem ([18], Chapter VI, Section 7, Exercise 2-3), p^* is comonadic; this proves the implications $1 \Rightarrow 2$ and $3 \Rightarrow 4$. The implication $2 \Rightarrow 3$ follows from the dual of Theorem 1.5.6 applied to the diagram of functors

$$\begin{array}{ccc}
 \mathbf{Pt}(B) & \xrightarrow{p^*} & \mathbf{Pt}(E) \\
 \swarrow \text{Ker}_B & & \searrow R_E \\
 & & \mathbb{C} \\
 \swarrow R_B & & \searrow \text{Ker}_E
 \end{array}$$

in which R_B and R_E are the right adjoints to the functors Ker_B and Ker_E respectively. \square

Remark 3.4.2. *Theorem 3.4.1 could be made more general, but at the same time weaker by omitting the protomodularity requirement and considering a general pointed category \mathbb{C} with finite limits, and by adding to conditions 1 and 3 of Theorem 3.4.1, that the unit*

of the adjunction satisfies the requirements of the dual of Theorem 1.5.6, and by omitting conditions 2 and 4 of Theorem 3.4.1.

Proposition 3.4.3. *Let \mathbb{C} be a cartesian closed category with equalizers. The pullback functor along any morphism $p : E \rightarrow B$ in the category $\mathbf{Gp}(\mathbb{C})$ of internal groups has a right adjoint.*

Proof. Let $B\text{-Act}(\mathbb{C})$ be the category with objects X having the structure of an internal group in \mathbb{C} as well as being equipped with a morphism $a : B \times X \rightarrow X$ in \mathbb{C} making the diagrams

$$\begin{array}{ccc}
 B \times (X \times X) & \xrightarrow{m} & (B \times X) \times (B \times X) \\
 \downarrow 1 \times + & & \downarrow a \times a \\
 B \times X & \xrightarrow{a} & X \xleftarrow{+} X \times X, \\
 \\
 B \times (B \times X) & \xrightarrow{\alpha} & (B \times B) \times X \\
 \downarrow 1 \times a & & \downarrow + \times 1 \\
 B \times X & \xrightarrow{a} & X \xleftarrow{a} B \times X, \\
 \\
 & & \begin{array}{ccc}
 X & \xrightarrow{\langle 0, 1 \rangle} & B \times X \\
 & \searrow 1_X & \downarrow a \\
 & & X,
 \end{array}
 \end{array}$$

commute. For an object $B \in \mathbf{Gp}(\mathbb{C})$ the category $\mathbf{Pt}(B)$ is equivalent to the category $B\text{-Act}(\mathbb{C})$. Let $V : \mathbf{Gp}(\mathbb{C}) \rightarrow \mathbf{Pt}_{\mathbb{C}}(1)$ (where $\mathbf{Pt}_{\mathbb{C}}(1)$ is the category of points over the terminal object in \mathbb{C}) be the functor defined for an internal group $(X, +, -, 0)$ as $V(X, +, -, 0) = (X, u, 0)$, where $u : X \rightarrow 1$ is the unique morphism to the terminal object. It is clear that the functor V preserves all limits and that the morphisms $+$ and $-$ in \mathbb{C} lift to morphisms in $\mathbf{Pt}_{\mathbb{C}}(1)$. Let (A, α, β) be an object in $\mathbf{Pt}(B)$, let (K, κ) be the kernel of α , let $\lambda : A \rightarrow K$ be the unique morphism in $\mathbf{Pt}_{\mathbb{C}}(1)$ with $V\kappa\lambda = +(1 \times -)\langle 1_A, V(\beta\alpha) \rangle$, and let a be the composite of the morphisms

$$B \times K \xrightarrow{\beta \times \kappa} A \times A \xrightarrow{\langle \pi_1, \langle \pi_2, \pi_1 \rangle \rangle} A \times (A \times A) \xrightarrow{1 \times (1 \times -)} A \times (A \times A) \xrightarrow{+(1 \times +)} A \xrightarrow{\lambda} K.$$

It can be seen that (K, a) is an object in $B\text{-Act}(\mathbb{C})$ and that this passage is an equivalence of categories. It follows that the functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbf{Gp}(\mathbb{C})$ is equivalent to the functor

$U : B\text{-Act}(\mathbb{C}) \rightarrow \mathbf{Gp}(\mathbb{C})$ which forgets the action. For an object X in $\mathbf{Gp}(\mathbb{C})$, we can equip X^B with a group structure as well as an action making the appropriate diagrams commute. Let the morphism $a' : B \times X^B \rightarrow X^B$ be the right adjunct of the morphism

$$(B \times X^B) \times B \xrightarrow{\sigma} B \times (B \times X^B) \xrightarrow{\alpha} (B \times B) \times X^B \xrightarrow{+\times 1} B \times X^B \xrightarrow{\sigma} X^B \times B \xrightarrow{\text{ev}} X,$$

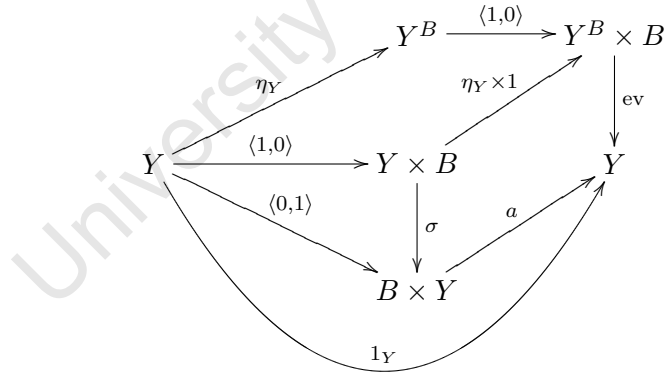
and let $+$: $X^B \times X^B \rightarrow X^B$ and $-$: $X^B \rightarrow X^B$ be defined *pointwise*, that is, the right adjuncts of

$$(X^B \times X^B) \times B \xrightarrow{1 \times \langle 1, 1 \rangle} (X^B \times X^B) \times (B \times B) \xrightarrow{m} (X^B \times B) \times (X^B \times B) \xrightarrow{+(\text{ev} \times \text{ev})} X$$

and

$$(X^B \times B) \xrightarrow{\text{ev}} X \xrightarrow{-} X$$

respectively; σ is the symmetry isomorphism, α is the associativity isomorphism, ev is the evaluation morphism, and m the middle interchange morphism. Let $R(X) = (X^B, a')$, it easily follows that R determines a functor from $\mathbf{Gp}(\mathbb{C})$ to $B\text{-Act}(\mathbb{C})$. We will show that R is the right adjoint of U . The component of the unit of the adjunction η is defined for any object $Y \in B\text{-Act}(\mathbb{C})$ as the right adjunct of the morphism $a'\sigma : X \times B \rightarrow X$. The counit ϵ has as its X component $\epsilon_X : UR(X) \rightarrow X$ the morphism $\text{ev}\langle 1, 0 \rangle : X^B \rightarrow X$. Since the diagram



commutes, it follows that $\epsilon_{UY}U\eta_Y = 1_{UY}$. Since the diagram

$$\begin{array}{ccccc}
X^B \times B & \xrightarrow{\eta_{R(X)} \times 1} & (X^B)^B \times B & & \\
\downarrow \sigma & & \searrow \text{ev} & \xrightarrow{\langle 1,0 \rangle^{B \times 1}} & (X^B \times B)^B \times B \\
B \times X^B & \xrightarrow{a'} & X^B & & \\
\downarrow \langle 1,0 \rangle & & \downarrow \langle 1,0 \rangle & & \downarrow \text{ev}^{B \times 1} \\
(B \times X^B) \times B & \xrightarrow{a \times 1} & X^B \times B & & X^B \times B \\
\downarrow \alpha\sigma & & \downarrow \text{ev} & & \downarrow \text{ev} \\
(B \times B) \times X^B & & & & X \\
\downarrow + \times 1 & & \downarrow \sigma & \xrightarrow{\text{ev}} & \\
B \times X^B & \xrightarrow{\sigma} & X^B \times B & &
\end{array}$$

$\langle 0,1 \rangle \times 1$ (curved arrow from $B \times X^B$ to $(B \times B) \times X^B$)
 $1_{B \times X^B}$ (curved arrow from $B \times X^B$ to $B \times X^B$)

commutes, it follows that $\text{ev}((R\epsilon_X\eta_{RX}) \times 1) = \text{ev}\sigma 1_{B \times X^B}\sigma = \text{ev} = \text{ev}(1_{X^B} \times 1)$, and therefore $R\epsilon_X\eta_{RX} = 1_{RX}$. Since $\mathbf{Gp}(\mathbb{C})$ is protomodular, it follows from Theorem 3.4.1 that all pullback functors have right adjoints. \square

3.5 Semi-abelian categories

In this section we consider many semi-abelian examples. Examples include: the category of Lie algebras, for which right adjoints to all pullback functors exist; the categories of rings, boolean rings, and commutative rings, for which right adjoints to pullback functors along regular epimorphisms exist; a semi-abelian category, for which only right adjoints of pullback functors along isomorphisms exist.

Proposition 3.5.1. *Let \mathbb{C} be a semi-abelian category. The functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbb{C}$ preserves binary coproducts if and only if the functor $Bb-$ preserves binary coproducts.*

Proof. It is obvious that if the functor Ker preserves coproducts, then $Bb-$, being the composite of Ker with its left adjoint, will also do. Conversely, suppose $Bb-$ preserves binary coproducts. It follows from Proposition 1.4.2 that $\mathbf{Pt}(B) \sim \mathbb{C}^{Bb-}$, and since under these conditions binary coproducts in \mathbb{C}^{Bb-} are computed as $(X, \zeta) + (X', \zeta') = (X + X', [\zeta, \zeta'])$, they are therefore preserved by the forgetful functor $\mathbb{C}^{Bb-} \rightarrow \mathbb{C}$. \square

Proposition 3.5.2. *Let \mathbf{CRng} be the category of (non-unital) commutative rings. For any $B \in \mathbf{CRng}$ the functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbf{CRng}$ has a right adjoint if and only if $B = 0$.*

Proof. It is obvious that if $B = 0$ then the functor Ker has a right adjoint (it is an isomorphism of categories). Now suppose that $B \neq 0$. Let $B\text{-}\mathbf{CAlg}$ be the category with objects having the structure of a commutative ring as well as that of a module over the ring B , and in addition satisfying the identity $b(xy) = (bx)y$. The category $\mathbf{Pt}(B)$ is equivalent to the category $B\text{-}\mathbf{CAlg}$. Indeed, for any object (A, α, β) let (K, κ) be the kernel of α . Clearly, K is a commutative ring. The abelian group morphism $1 - \beta\alpha$ has composite $\alpha(1 - \beta\alpha) = \alpha - \alpha = 0$, and consequently factors through the kernel κ . Let λ be the unique abelian group morphism with $\kappa\lambda = 1 - \beta\alpha$. For $k \in K$ and $b \in B$ we can define $bk = \lambda(\beta(b)\kappa(k))$; since λ is an abelian group morphism, this action of B on K is distributive on the left and right. Since κ is a monomorphism, the equation

$$\begin{aligned} \kappa(\lambda(\beta b \kappa(k_1 k_2))) &= \beta b \kappa(k_1 k_2) - \beta \alpha(\beta b \kappa(k_1 k_2)) = \beta b \kappa(k_1 k_2) = \beta b \kappa k_1 \kappa k_2 \\ &= (\beta b \kappa k_1 - \beta \alpha(\beta b \kappa k_1)) \kappa k_2 = \kappa(\lambda(\beta b \kappa k_1) k_2) \end{aligned}$$

shows that $b(k_1 k_2) = (b k_1) k_2$. Similarly, the equation

$$\lambda(\beta b \kappa \lambda(\beta b' \kappa k)) = \lambda(\beta b (\beta b' \kappa k - \beta \alpha(\beta b' \kappa k))) = \lambda(\beta b \beta b' \kappa k)$$

shows that $b(b'k) = (bb')k$. For any object $X \in B\text{-}\mathbf{CAlg}$ we can construct the object $B \times X$ as follows, it has as underlying structure $(B \oplus X, \pi_1, \langle 1, 0 \rangle)$. Elements (b, x) and (b', x') in $B \times X$ have product defined by $(bb', xx' + bx' + b'x)$. Since the multiplication in X and B is commutative, it easily follows that the multiplication defined is commutative. Since the multiplication is distributive, commutative, and the monomorphisms $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ preserve multiplication, associativity follows from the equalities:

$$\begin{aligned} (b, 0)((b', 0), (0, x)) &= (0, b(b'x)) = (0, (bb')x) = ((0, b)(0, b'))(0, x), \\ (b, 0)((x, 0), (x', 0)) &= (0, b(xx')) = (0, (bx)x') = ((0, b)(x, 0))(x', 0). \end{aligned}$$

The functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbf{CRng}$ is equivalent to the functor $U : B\text{-}\mathbf{CAlg} \rightarrow \mathbf{CRng}$ which simply forgets the module structure. Let \tilde{B} be the commutative unital ring $\mathbb{Z} \times B$; that is, the commutative ring with underlying abelian group $\mathbb{Z} \oplus B$, and with multiplication defined for all (n, b) and (n', b') in \tilde{B} by $(n, b)(n', b') = (nn', nb' + n'b + bb')$. Let B_0 and B_1 be the objects in $B\text{-}\mathbf{CAlg}$ with underlying commutative ring \tilde{B} , and with actions defined for each $b' \in B$ and $(n, b) \in \tilde{B}$ as $b'(n, b) = 0$ and $b'(n, b) = (0, nb' + b'b)$ respectively. We will

show that the following construction defines the coproduct of B_0 and B_1 . $B_0 + B_1$ has as underlying abelian group $\tilde{B} \oplus \tilde{B} \oplus \tilde{B}$, with coproduct injections $\iota_1 = \langle 1, 0, 0 \rangle : B_0 \rightarrow B_0 + B_1$ and $\iota_2 = \langle 0, 1, 0 \rangle : B_1 \rightarrow B_0 + B_1$. These morphisms and the identities

$$\begin{aligned} ((n, b), (0, 0), (0, 0))((0, 0), (n', b'), (0, 0)) &= ((0, 0), (0, 0), (nn', n'b)), \\ ((n, b), (0, 0), (0, 0))((0, 0), (0, 0), (n', b')) &= ((0, 0), (0, 0), (nn', nb' + n'b + bb')), \\ ((0, 0), (n, b), (0, 0))((0, 0), (0, 0), (n', b')) &= ((0, 0), (0, 0), (nn', n'b)), \\ ((0, 0), (0, 0), (n, b))((0, 0), (0, 0), (n', b')) &= ((0, 0), (0, 0), (nn', nb' + n'b + bb')), \end{aligned}$$

define multiplication (by distributivity). The action is defined by $b'((n_1, b_1), (n_2, b_2), (n_3, b_3)) = ((0, 0), (0, n_2b' + b'b_2), (0, 0))$. Suppose that the diagram

$$B_0 \xrightarrow{f} A \xleftarrow{g} B_1$$

is a co-span in $B\text{-CAlg}$. It follows that the abelian group morphism $[f, g] : B_0 + B_1 \rightarrow A$, defined by

$$[f, g]((n_1, b_1), (n_2, b_2), (n_3, b_3)) = f(n_1, b_1) + g(n_2, b_2) + f(n_3, b_3)g(1, 0),$$

is a morphism since $f(n, b)g(n', b') = f(n, b)g(n', 0) + f(n, b)g(b'(1, 0)) = f(nn', n'b)g(1, 0) + (b'(f(n, b))g(1, 0)) = f(nn', n'b)g(1, 0) + f(0, 0)g(1, 0) = f(nn', n'b)g(1, 0)$. It is obvious that $[f, g]\iota_1 = f$ and that $[f, g]\iota_2 = g$. The underlying abelian group of the coproduct $U(B_0) + U(B_1) = \tilde{B} + \tilde{B}$ is $\tilde{B} \oplus \tilde{B} \oplus (\tilde{B} \otimes \tilde{B})$, and the canonical morphism $[U\iota_1, U\iota_2] : UB_0 + UB_1 \rightarrow U(B_0 + B_1)$ is clearly not a monomorphism when $B \neq 0$ since

$$\begin{aligned} [U\iota_1, U\iota_2]((0, 0), (0, 0), (0, b) \otimes (1 \otimes b')) &= ((0, 0), (0, 0), (0, b)) \\ &= [U\iota_1, U\iota_2]((0, 0), (0, 0), (0, b) \otimes (1 \otimes 0)). \end{aligned}$$

□

Proposition 3.5.3. *For any regular epimorphism $p : E \rightarrow B$ in \mathbf{CRng} the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ has a right adjoint*

Proof. Since in Proposition 3.5.2 we showed that the category $\mathbf{Pt}(B)$ is equivalent to the category $B\text{-CAlg}$, it is easy to see that the functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ is equivalent to the functor $B\text{-CAlg} \rightarrow E\text{-CAlg}$ which assigns to each object X the object X with action defined by $ex = p(e)x$ for each $e \in E$ and $x \in X$. For some object $Y \in E\text{-CAlg}$ let

$R(Y) = \{y \in Y \mid \forall (e_1, e_2) \in E \times_B E \ e_1 y = e_2 y\}$. Since by forgetting multiplication on Y , R becomes the right adjoint of the functor $B\text{-Mod} \rightarrow E\text{-Mod}$, and since for any $e_1, e_2 \in E$ and $y_1 \in Y$ if $e_1 y_1 = e_2 y_1$ then for any $y_2 \in Y$ $e_1(y_1 y_2) = (e_1 y_2) y_2 = (e_2 y_1) y_2 = e_2(y_1 y_2)$, it follows that $R(Y)$ is closed under the multiplication defined in Y . \square

Proposition 3.5.4. *Let \mathbf{BRng} be the category of (non-unital) boolean rings. For any $B \in \mathbf{BRng}$ the functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbf{BRng}$ has a right adjoint if and only if $B = 0$.*

Proof. It is obvious that if $B = 0$ then the functor Ker has a right adjoint (it is an isomorphism of categories). Now suppose that $B \neq 0$. Let $B\text{-BAlg}$ be the category with objects having the structure of a boolean ring as well as that of a module over the ring B , and in addition satisfying the identity $b(xy) = (bx)y$. The category $\mathbf{Pt}(B)$ is equivalent to the category $B\text{-BAlg}$. The equivalence constructed for commutative rings lifts to an equivalence for boolean rings. Since the kernel of a split epimorphism between boolean rings clearly is boolean, to show that the equivalence lifts is to show that for some X in $B\text{-BAlg}$, $B \times X$ is boolean, which follows from $(b, x)(b, x) = (bb, xx + bx + bx) = (b, x + 0) = (b, x)$. The functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbf{BRng}$ is equivalent to the functor $U : B\text{-BAlg} \rightarrow \mathbf{BRng}$. Let \tilde{B} be the boolean unital ring $\mathbb{Z}_2 \times B$; that is, the commutative ring with underlying abelian group $\mathbb{Z}_2 \oplus B$, and with multiplication defined for all (n, b) and (n', b') in \tilde{B} by $(n, b)(n', b') = (nn', nb' + n'b + bb')$. Let B_0 and B_1 be the objects in $B\text{-BAlg}$ with underlying boolean ring \tilde{B} , and with actions defined for each $b' \in B$ and $(n, b) \in \tilde{B}$ as $b'(n, b) = 0$ and $b'(n, b) = (0, nb' + b'b)$ respectively. We will show that the following construction defines the coproduct of B_0 and B_1 . $B_0 + B_1$ has as underlying abelian group $\tilde{B} \oplus \tilde{B} \oplus \tilde{B}$, with coproduct injections $\iota_1 = \langle 1, 0, 0 \rangle : B_0 \rightarrow B_0 + B_1$ and $\iota_2 = \langle 0, 1, 0 \rangle : B_1 \rightarrow B_0 + B_1$. These morphisms and the identities

$$\begin{aligned} ((n, b), (0, 0), (0, 0))((0, 0), (n', b'), (0, 0)) &= ((0, 0), (0, 0), (nn', n'b)), \\ ((n, b), (0, 0), (0, 0))((0, 0), (0, 0), (n', b')) &= ((0, 0), (0, 0), (nn', nb' + n'b + bb')), \\ ((0, 0), (n, b), (0, 0))((0, 0), (0, 0), (n', b')) &= ((0, 0), (0, 0), (nn', nb')), \\ ((0, 0), (0, 0), (n, b))((0, 0), (0, 0), (n', b')) &= ((0, 0), (0, 0), (nn', nb' + n'b + bb')), \end{aligned}$$

define multiplication (by distributivity). The action is defined by $b'((n_1, b_1), (n_2, b_2), (n_3, b_3)) = ((0, 0), (0, n_2 b' + b' b_2), (0, 0))$. Suppose that the diagram

$$B_0 \xrightarrow{f} A \xleftarrow{g} B_1$$

is a co-span in $B\text{-BAlg}$. It follows that the abelian group morphism $[f, g] : B_0 + B_1 \rightarrow A$,

defined by

$$[f, g]((n_1, b_1), (n_2, b_2), (n_3, b_3)) = f(n_1, b_1) + g(n_2, b_2) + f(n_3, b_3)g(1, 0),$$

is a morphism since $f(n, b)g(n', b') = f(n, b)g(n', 0) + f(n, b)g(b'(1, 0)) = f(nn', n'b)g(1, 0) + (b'(f(n, b))g(1, 0)) = f(nn', n'b)g(1, 0) + f(0, 0)g(1, 0) = f(nn', n'b)g(1, 0)$. It is obvious that $[f, g]_{\iota_1} = f$ and that $[f, g]_{\iota_2} = g$. The underlying abelian group of the coproduct $U(B_0) + U(B_1) = \tilde{B} + \tilde{B}$ is $\tilde{B} \oplus \tilde{B} \oplus (\tilde{B} \otimes \tilde{B})$ and the canonical morphism $[U_{\iota_1}, U_{\iota_2}] : UB_0 + UB_1 \rightarrow U(B_0 + B_1)$ is clearly not a monomorphism when $B \neq 0$ since

$$\begin{aligned} [U_{\iota_1}, U_{\iota_2}]((0, 0), (0, 0), (0, b) \otimes (1 \otimes b')) &= ((0, 0), (0, 0), (0, b)) \\ &= [U_{\iota_1}, U_{\iota_2}]((0, 0), (0, 0), (0, b) \otimes (1 \otimes 0)). \end{aligned}$$

□

Remark 3.5.5. *It was shown in [5] that the category \mathbf{BRng} is representable.*

Proposition 3.5.6. *For any regular epimorphism $p : E \rightarrow B$ in \mathbf{BRng} the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ has a right adjoint*

Proof. The proof is analogous to the proof of Proposition 3.5.3. □

Proposition 3.5.7. *Let \mathbf{Rng} be the category of (non-unital) rings. For any $B \in \mathbf{Rng}$ the functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbf{Rng}$ has a right adjoint if and only if $B = 0$.*

Proof. It is obvious that if $B = 0$ then the functor Ker has a right adjoint (it is an isomorphism of categories). Suppose $B \neq 0$. Let $B\text{-biAlg}$ be the category with objects having the structure of a ring as well as the structure of a bimodule, and in addition satisfying the identities:

1. $b(xy) = (bx)y$;
2. $x(by) = (xb)y$;
3. $x(yb) = (xy)b$;
4. $b(b'x) = (bb')x$;
5. $b(xb') = (bx)b'$.

It can be seen that the category $\mathbf{Pt}(B)$ is equivalent to the category $B\text{-biAlg}$, and that the functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbf{Rng}$ is equivalent to the functor $U : B\text{-biAlg} \rightarrow \mathbf{Rng}$ which forgets the bimodule structure. Let \tilde{B} be the unital ring $(\mathbb{Z} \times B) \otimes (\mathbb{Z} \times B)$, in which $\mathbb{Z} \times B$ is the ring with underlying abelian group $(\mathbb{Z} \oplus B)$ and with multiplication defined for all (n, b) and (n', b') in $\mathbb{Z} \times B$ by $(n, b)(n', b') = (nn', nb' + n'b + bb')$. Let B_0 and B_1 be the objects in $B\text{-biAlg}$ with underlying commutative ring \tilde{B} , and with left and right actions defined for each $b'' \in B$ and $(n, b) \otimes (n', b') \in \tilde{B}$ as $b''(n, b) \otimes (n', b') = 0 = (n, b) \otimes (n', b')b''$, $b''(n, b) \otimes (n', b') = (0, nb'' + b''b) \otimes (n', b')$ and $(n, b) \otimes (n', b')b'' = (n, b) \otimes (0, nb'' + b'b'')$ respectively. Suppose that the diagram

$$B_0 \xrightarrow{f} A \xleftarrow{g} B_1$$

is a co-span in $B\text{-BiAlg}$, it follows that $f((1, 0) \otimes (0, 1))g((0, b) \otimes (1, 0)) = f((1, 0) \otimes (1, 0))g(b((1, 0) \otimes (1, 0))) = (f((1, 0) \otimes (0, 1))b)g((1, 0) \otimes (1, 0)) = 0$. But in the co-span

$$U(B_0) \xrightarrow{\langle 1, 1 \rangle} \tilde{B} \times \tilde{B} \xleftarrow{\langle 1, 1 \rangle} U(B_1),$$

we have $\langle 1, 1 \rangle((1, 0) \otimes (1, 0))\langle (0, b) \otimes (1, 0) \rangle = ((0, b) \otimes (1, 0), (0, b) \otimes (1, 0)) \neq 0$. \square

Proposition 3.5.8. *For any regular epimorphism $p : E \rightarrow B$ in \mathbf{Rng} the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ has a right adjoint*

Proof. The proof is analogous to the proof of Proposition 3.5.3. \square

Proposition 3.5.9. *Let \mathbf{DRng} be the full subcategory of the category of commutative rings \mathbf{CRng} in which for every object X the identity $xyz = 0$ holds. For any $B \in \mathbf{DRng}$ the functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbf{DRng}$ has a right adjoint.*

Proof. Let $B\text{-DAIlg}$ be the category with objects having the structure of a commutative ring as well as that of a module over the ring B , and in addition satisfying the identities $b(xy) = (bx)y = 0$ and $b(b'x) = (bb')x = 0$. The category $\mathbf{Pt}(B)$ is equivalent to the category $B\text{-DAIlg}$. Indeed, for any object (A, α, β) , let (K, κ) be the kernel of α . Clearly, K is a commutative ring. The abelian group morphism $1 - \beta\alpha$ has composite $\alpha(1 - \beta\alpha) = \alpha - \alpha = 0$, and consequently factors through the kernel κ . Let λ be the unique abelian group morphism with $\kappa\lambda = 1 - \beta\alpha$. For $k \in K$ and $b \in B$ we can define $bk = \lambda(\beta b \kappa k)$; since λ is an abelian group morphism, this action of B on K is clearly distributive on the left and right. Since

κ is a monomorphism, the equation

$$\begin{aligned}\kappa(\lambda(\beta b \kappa(k_1 k_2))) &= \beta b \kappa(k_1 k_2) - \beta \alpha(\beta b \kappa(k_1 k_2)) = \beta b \kappa(k_1 k_2) = \beta b \kappa k_1 \kappa k_2 = 0 \\ &= (\beta b \kappa k_1 - \beta \alpha(\beta b \kappa k_1)) \kappa k_2 = \kappa(\lambda(\beta b \kappa k_1) k_2)\end{aligned}$$

shows that $b(k_1 k_2) = (b k_1) k_2 = 0$. Similarly, the equation

$$\lambda(\beta b \kappa \lambda(\beta b' \kappa k)) = \lambda(\beta b (\beta b' \kappa k - \beta \alpha(\beta b' \kappa k))) = \lambda(\beta b \beta b' \kappa k) = 0$$

shows that $b(b' k) = (b b') k = 0$. For any object $X \in B\text{-DAlg}$ we can construct the object $B \times X$ as follows, it has as underlying structure $(B \oplus X, \pi_1, \langle 1, 0 \rangle)$. Elements (b, x) and (b', x') in $B \times X$ have product defined by $(b b', x x' + b x' + b' x)$. Since the multiplication in X and B is commutative, it easily follows that the multiplication defined is commutative. Since the multiplication is distributive, commutative and the monomorphisms $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ preserve multiplication, to show all triple products are zero, it is sufficient to consider the following equalities:

$$\begin{aligned}(b, 0)((b', 0), (0, x)) &= (0, b(b' x)) = 0 = (0, (b b') x) = ((0, b)(0, b'))(0, x), \\ (b, 0)((x, 0), (x', 0)) &= (0, b(x x')) = 0 = (0, (b x) x') = ((0, b)(x, 0))(x', 0).\end{aligned}$$

The functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbf{DRng}$ is equivalent to the functor $U : B\text{-DAlg} \rightarrow \mathbf{DRng}$ which simply forgets the module structure. Let $\tilde{B} = \mathbb{Z} \times B$, and let $R(X) = \{f \in \text{Hom}(\tilde{B}, X) \mid \forall b, b' \in B \forall x \in X f(0, b)x = 0 \text{ and } f(0, b b') = 0\}$ with multiplication defined by $(fg)(n, b) = nf(1, 0)g(1, 0)$ and with scalar multiplication defined by $(b' f)(n, b) = f(0, n b')$ for all $f, g \in R(X)$, $b' \in B$ and $(n, b) \in \tilde{B}$. Clearly $fgh = 0$. It easily follows that $R(X) \in B\text{-DAlg}$ and that $R : \mathbf{DRng} \rightarrow B\text{-DAlg}$ defines a functor. Let $\eta : 1_{B\text{-DAlg}} \rightarrow RU$ be the natural transformation defined for an object X in $B\text{-DAlg}$ as $\eta_X(x)(n, b) = nx + bx$ for any $x \in X$ and $(n, b) \in \tilde{B}$. Let $\epsilon : UR \rightarrow 1_{\mathbf{DRng}}$ be the natural transformation defined for an object Y in \mathbf{DRng} as $\epsilon_Y(f) = f(1, 0)$ for any $f \in UR(Y)$. Now $\epsilon_{UX} U \eta_X(x) = U \eta_X(x)(1, 0) = x$ and $(R(\epsilon_Y) \eta_{RY}(f))(n, b) = \eta_{RY}(f)(n, b)(1, 0) = nf(1, 0) + (bf)(1, 0) = f(n, 0) + f(0, b) = f(n, b)$. \square

Proposition 3.5.10. *Let \mathcal{V} be the variety with objects having the structure of an abelian group as well as a unary operation which preserves the nullary operation 0. For any morphism $p : E \rightarrow B$, the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ has a right adjoint if and only if p is an isomorphism.*

Proof. Let \mathcal{V}^B be the variety with objects having the structure of an abelian group as well as unary operations, one for each $b \in B$, each of which preserves the nullary operation 0. The category $\mathbf{Pt}(B)$ is equivalent to the \mathcal{V}^B ; the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ is equivalent to the functor $V : \mathcal{V}^B \rightarrow \mathcal{V}^E$ which takes $X \in \mathcal{V}^B$ to X where $e(x) = p(e)(x)$ for all $e \in E$ and $x \in X$. We show that the coproducts are only preserved when p is an isomorphism. Let X have underlying abelian group \mathbb{Z} and $b(x) = 0$, and for any $b' \in B$ let $Z_{b'}$ have underlying abelian group $\mathbb{Z} \times \mathbb{Z}$ and unary operation defined for each $b \in B$ by

$$b(x, y) = \begin{cases} (0, 0) & \text{if } x = 0 \text{ or } y = 0 \text{ or } b \neq b' \\ (x, y) & \text{if } x \neq 0 \text{ and } y \neq 0 \text{ and } b = b'. \end{cases}$$

Let $f, g : X \rightarrow Z_{b'}$ be the morphisms defined by $f(x) = (x, 0)$ and $g(x) = (0, x)$. Clearly, if $b \neq b'$ then $b(f(1) + g(1)) = (0, 0) \neq (1, 1) = b'(f(1) + g(1))$. Consequently, in the coproduct $X + X$ $b(\iota_1(1) + \iota_2(1)) = b'(\iota_1(1) + \iota_2(1))$ if and only if $b = b'$. In the diagram

$$\begin{array}{ccccc} VX & \xrightarrow{i_1} & VX + VX & \xleftarrow{i_2} & VX \\ & \searrow V\iota_1 & \downarrow [V\iota_1, V\iota_2] & \swarrow V\iota_2 & \\ & & V(X + Y) & & \end{array}$$

$[V\iota_1, V\iota_2](e(i_1(1) + i_2(1))) = p(e)(\iota_1(1) + \iota_2(1))$, and if $[V\iota_1, V\iota_2]$ is an isomorphism then $e(i_1(1) + i_2(1)) = e'(i_1(1) + i_2(1))$ if and only if $p(e)(\iota_1(1) + \iota_2(1)) = p(e')(\iota_1(1) + \iota_2(1))$. Therefore $e = e'$ if and only if $p(e) = p(e')$, and so we see that p must be injective. Now suppose p is not surjective, then there exists $b' \in B$ such that $b' \notin p(E)$. Let W have underlying abelian group $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ with unary operations defined for each $b \in B$ by

$$b(x, y, z) = \begin{cases} (0, 0, 1) & \text{if } x = y = 1, z = 0 \text{ and } b = b' \\ (0, 0, 0) & \text{otherwise,} \end{cases}$$

and let $m, n : X \rightarrow W$ be the morphisms defined by $m(x) = (x, 0, 0)$ and $n(x) = (0, x, 0)$. Consider the diagram:

$$\begin{array}{ccccc} VX & \xrightarrow{i_1} & VX + VX & \xleftarrow{i_2} & VX \\ & \searrow Vm & \downarrow [Vm, Vn] & \swarrow Vn & \\ & & VW & & \end{array} .$$

Since $VX + VX$ is always generated by $\{\iota_1(x)|x \in V(X)\} \cup \{\iota_2(x)|x \in V(X)\}$, the image $[Vm, Vn](VX + VX)$ is generated by $\{m(x)|x \in V(X)\} \cup \{n(x)|x \in V(X)\}$. Since $p(e)(x, y, z) = 0$, $[Vm, Vn](VX + VX) = \{(x, y, 0)|x, y \in \mathbb{Z}\}$ and $b'(n(1) + m(1)) \notin [Vm, Vn](VX + VX)$. But it is obvious that $b'(n(1) + m(1)) \in V[n, m](V(X + Y))$, and since the diagram

$$\begin{array}{ccc} V(X) + V(X) & \xrightarrow{[V\iota_1, V\iota_2]} & V(X + Y) \\ & \searrow [Vm, Vn] & \swarrow V[n, m] \\ & & W \end{array}$$

commutes, it follows that $[V\iota_1, V\iota_2]$ is not surjective. \square

Theorem 3.5.11. *For any morphism $p : E \rightarrow B$ in \mathbf{Gp} the functor $\mathbf{Gp}^p : \mathbf{Gp}^B \rightarrow \mathbf{Gp}^E$ has right adjoint $R_p : \mathbf{Gp}^E \rightarrow \mathbf{Gp}^B$, defined for any object $F \in \mathbf{Gp}^E$ by $\text{Ran}_p F(1) = \{\theta \in \text{hom}(B, F(1)) \mid \forall (b, e) \in B \times E \theta(p(e)b) = F(e)(\theta(b))\}$ where 1 denotes the unique object in B .*

Proof. It is well known that right adjoints of such functors \mathbf{Gp}^p defined on some object $F \in \mathbf{Gp}^E$ is the right Kan extension of F along p . This Right Kan extension can be expressed as a point-wise limit (see [18], Chapter X, Section 3, Theorem 1). It follows that

$$\text{Ran}_p(F)(1) = \lim((1 \downarrow p) \xrightarrow{Q} E \xrightarrow{F} \mathbf{Gp})$$

where 1 denotes the unique object in B . An object in the category $(1 \downarrow p)$ is an element $b \in B$ a morphism $f : b \rightarrow 1$ is a pair (b, e) such that $p(e)b = 1$. As with every limit, this limit can be expressed as an equalizer of a pair of morphisms between products as in the following diagram:

$$\begin{array}{ccccc} & & F(1) & \xrightarrow{F(e)} & F(1) \\ & & \uparrow \pi_b & & \uparrow \pi_{(e,b)} \\ \text{Ran}_p(F)(1) & \xrightarrow{eq} & \prod_{b \in B} F(1) & \xrightleftharpoons[g]{f} & \prod_{(b,e) \in B \times E} F(1) \\ & & \searrow \pi_{bp(e)} & & \downarrow \pi_{(b,e)} \\ & & & & F(1) \end{array}$$

Identifying $\prod_{b \in B} F(1)$ with $\text{hom}(B, F(1))$ and $\prod_{(b,e) \in B \times E} F(1)$ with $\text{hom}(B \times E, F(1))$, it follows that $\text{Ran}_p F(1) = \{\theta \in \text{hom}(B, F(1)) \mid \forall (b, e) \in B \times E \theta(p(e)b) = F(e)(\theta(b))\}$. \square

Proposition 3.5.12. *Let \mathbf{Gp} be the category of groups. For any morphism $p : E \rightarrow B$ the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ has a right adjoint.*

Proof. There is a well know equivalence of categories $\mathbf{Pt}(B) \sim \mathbf{Gp}^B$, in which on the right we are considering the group B as a one point category. It is also easy to show that for any morphism $p : E \rightarrow B$ the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ is equivalent to the functor $\mathbf{Gp}^p : \mathbf{Gp}^B \rightarrow \mathbf{Gp}^E$. It follows from Theorem 3.5.11 that the functor $\mathbf{Gp}^p : \mathbf{Gp}^B \rightarrow \mathbf{Gp}^E$ has a right adjoint. \square

Proposition 3.5.13. *Let \mathbf{Lie}_D be the category of Lie algebras over a commutative unital ring D . For any $B \in \mathbf{Lie}_D$ the functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbf{Lie}_D$ has a right adjoint.*

Proof. Let $B\text{-Act}_D$ be the category with objects having the structure of a Lie algebra as well as the structure of a module over the Lie algebra B , satisfying the following identities:

1. $(b_1 b_2)x = b_1(b_2 x) - b_2(b_1 x)$;
2. $b(x_1 x_2) = (b x_1)x_2 + x_1(b x_2)$.

Let $B\text{-Mod}_D$ be the category of modules over the Lie algebra B . The category $\mathbf{Pt}(B)$ is equivalent to the category $B\text{-Act}_D$. Indeed, for any object (A, α, β) let (K, κ) be the kernel of α . Clearly, K is a Lie ring. The $B\text{-Mod}_D$ morphism $1 - \beta\alpha$ has composite $\alpha(1 - \beta\alpha) = \alpha - \alpha = 0$, and consequently factors through the kernel κ . Let λ be the unique $B\text{-Mod}_D$ morphism with $\kappa\lambda = 1 - \beta\alpha$. For $k \in K$ and $b \in B$ we can define $bk = \lambda(\beta b \kappa k)$; since λ is an abelian group morphism this action of B on K is distributive on the left and right. Since κ is a monomorphism, the equation

$$\begin{aligned} \kappa(\lambda(\beta b \kappa(k_1 k_2))) &= \beta b \kappa(k_1 k_2) - \beta \alpha(\beta b \kappa(k_1 k_2)) = \beta b \kappa(k_1 k_2) = -\kappa k_2(\beta b \kappa k_1) - \kappa k_1(\kappa k_2 \beta b) \\ &= (\beta b \kappa k_1) \kappa k_2 + \kappa k_2(\beta b \kappa k_2) \\ &= (\beta b \kappa k_1 - \beta \alpha(\beta b \kappa k_1)) \kappa k_2 + \kappa k_1(\beta b \kappa k_1 - \beta \alpha(\beta b \kappa k_2)) \\ &= \kappa(\lambda(\beta b \kappa k_1) k_2) + \kappa(k_1 \lambda(\beta b \kappa k_2)) \end{aligned}$$

shows that $b(k_1 k_2) = (b k_1) k_2 + k_1 (b k_2)$. Similarly, the equation

$$\begin{aligned} \lambda(\beta(b b') \kappa k) &= -\lambda(\kappa k(\beta(b) \beta(b'))) = \lambda(\beta b'(\kappa k \beta b)) + \lambda(\beta b(\beta b' \kappa k)) \\ &= \lambda(\beta b(\beta b' \kappa k)) - \lambda(\beta b'(\beta b \kappa k)) \\ &= \lambda(\beta b((\beta b' \kappa k) - \beta \alpha(\beta b' \kappa k))) - \lambda(\beta b'((\beta b \kappa k) - \beta \alpha(\beta b \kappa k))) \\ &= \lambda(\beta b \kappa \lambda(\beta b' \kappa k)) - \lambda(\beta b' \kappa \lambda(\beta b \kappa k)) \end{aligned}$$

shows that $(bb')k = b(b'k) - b'(bk)$. For any object $X \in B\text{-Act}_D$ we can construct the object $B \times X$ as follows, it has as underlying structure $(B \oplus X, \pi_1, \langle 1, 0 \rangle)$. Elements (b, x) and (b', x') in $B \times X$ have product defined by $(bb', xx' + bx' - b'x)$. It easily follows that for the multiplication defined, $(b, x)(b, x) = (bb, xx + bx - bx) = 0$. Since the multiplication is distributive, anti-commutative, and the monomorphisms $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ preserve multiplication, to show that the Jacobi identity holds it is sufficient to consider the following equalities:

$$\begin{aligned} & (b, 0)((b', 0), (0, x)) + (0, x)((b, 0)(b', 0)) + (b', 0)((0, x)(b, 0)) \\ &= (0, b(b'x) - (bb')x - b'(bx)) = 0, \\ & (b, 0)((x, 0), (x', 0)) + (x', 0)((b, 0), (x, 0)) + (x, 0)((x', 0)(b, 0)) \\ &= (0, b(xx') + x'(bx) - x(bx')) = 0. \end{aligned}$$

The functor $\text{Ker} : \mathbf{Pt}(B) \rightarrow \mathbf{Lie}_D$ can be identified with the functor $U : B\text{-Act}_D \rightarrow \mathbf{Lie}_D$ which simply forgets the module structure. Let

$$T(B) = \bigoplus_{n \in \mathbb{N}} \underbrace{B \otimes B \otimes \dots \otimes B}_n,$$

in which \otimes is the tensor product as D -modules, and let $\tilde{B} = D \times T(B)$. We will identify $(m, 0) \in \tilde{B}$ with m and $(0, b_1 \otimes \dots \otimes b_n) \in \tilde{B}$ with $b_1 \otimes \dots \otimes b_n$, and simply write $m + b_1 \otimes \dots \otimes b_n$ for a pair $(m, b_1 \otimes \dots \otimes b_n)$. For an object $X \in \mathbf{Lie}_D$ let

$$\begin{aligned} R(X) &= \{ f \in \text{Hom}_D(\tilde{B}, X) \mid \forall b_1 \otimes \dots \otimes b_n \in \tilde{B} \\ & f(b_1 \otimes \dots \otimes b_{i-1} \otimes b_i b_{i+1} \otimes b_{i+2} \otimes \dots \otimes b_n) \\ &= f(b_1 \otimes \dots \otimes b_{i-1} \otimes b_i \otimes b_{i+1} \otimes b_{i+2} \otimes \dots \otimes b_n) \\ & - f(b_1 \otimes \dots \otimes b_{i-1} \otimes b_{i+1} \otimes b_i \otimes b_{i+2} \otimes \dots \otimes b_n) \}. \end{aligned}$$

Since any abelian group morphism $f : \tilde{B} \rightarrow X$ is completely determined if we know f on elements of the form $m + b_1 \otimes \dots \otimes b_n$, in what follows we will work exclusively with elements of this form. For any $f \in R(X)$, any $b \in B$ and any $m + b_1 \otimes \dots \otimes b_n \in \tilde{B}$, let

$(bf)(m + b_1 \otimes \dots \otimes b_n) = f(mb + b_1 \otimes \dots \otimes b_n \otimes b)$. The following calculation

$$\begin{aligned}
(bf)(b_1 \otimes \dots \otimes b_i b_{i+1} \otimes \dots \otimes b_n) &= f(b_1 \otimes \dots \otimes b_i b_{i+1} \otimes \dots \otimes b_n \otimes b) \\
&= f(b_1 \otimes \dots \otimes b_i \otimes b_{i+1} \otimes \dots \otimes b_n \otimes b) \\
&\quad - f(b_1 \otimes \dots \otimes b_{i+1} \otimes b_i \otimes \dots \otimes b_n \otimes b) \\
&= (bf)(b_1 \otimes \dots \otimes b_i \otimes b_{i+1} \otimes \dots \otimes b_n) \\
&\quad - (bf)(b_1 \otimes \dots \otimes b_{i+1} \otimes b_i \otimes \dots \otimes b_n)
\end{aligned}$$

shows that bf is in $R(X)$. For any $w = b_1 \otimes \dots \otimes b_n \in \tilde{B}$ and for any subset S of $\{b_1, \dots, b_n\}$, when S is non empty let $w_S = b_{s_1} \otimes \dots \otimes b_{s_k}$ in which $b_{s_1}, b_{s_2}, \dots, b_{s_k}$ are all the elements of S appearing in the same order as in w , and when S is empty let $w_S = 1$. For f and g in $R(X)$, the product fg is defined on an element $n + w$ by

$$(fg)(n + w) = nf(1)g(1) + \sum_{S \subseteq \{b_1, \dots, b_n\}} f(w_S)g(w_{\neg S})$$

in which $\neg S = \{b_1, \dots, b_n\} \setminus S$. Let $w' = b_1 \otimes \dots \otimes b_{i-1} \otimes b_{i+1} \otimes b_i \otimes b_{i+2} \dots \otimes b_n$, and $w'' = (b_1 \otimes \dots \otimes b_{i-1} \otimes b_i b_{i+1} \otimes b_{i+2} \otimes \dots \otimes b_n)$. It is easy to see that if $b_i \in S$ and $b_{i+1} \notin S$ then $w_S = w'_S$. Similarly, if $b_i \notin S$ and $b_{i+1} \in S$, or if $b_i \notin S$ and $b_{i+1} \notin S$, then $w_S = w''_S$.

Consequently, one shows

$$\begin{aligned}
fg(w) - fg(w') &= \sum_{\substack{S \subseteq \{b_1, \dots, b_n\}, \\ b_i \in S, b_{i+1} \in S}} f(w_S)g(w_{\neg S}) - f(w'_S)g(w'_{\neg S}) \\
&+ \sum_{\substack{S \subseteq \{b_1, \dots, b_n\}, \\ b_i \notin S, b_{i+1} \notin S}} f(w_S)g(w_{\neg S}) - f(w'_S)g(w'_{\neg S}) \\
&= \sum_{\substack{S \subseteq \{b_1, \dots, b_n\}, \\ b_i \in S, b_{i+1} \in S}} (f(w_S) - f(w'_S))g(w_{\neg S}) \\
&+ \sum_{\substack{S \subseteq \{b_1, \dots, b_n\}, \\ b_i \notin S, b_{i+1} \notin S}} f(w_S)(g(w_{\neg S}) - g(w'_{\neg S})) \\
&= \sum_{\substack{S \subseteq \{b_1, \dots, b_n\}, \\ b_i \in S, b_{i+1} \in S}} f(w''_{S \setminus \{b_i, b_{i+1}\} \cup \{b_i b_{i+1}\}})g(w''_{\neg S}) \\
&+ \sum_{\substack{S \subseteq \{b_1, \dots, b_n\}, \\ b_i \notin S, b_{i+1} \notin S}} f(w''_S)g(w''_{\neg S \setminus \{b_i, b_{i+1}\} \cup \{b_i b_{i+1}\}}) \\
&= \sum_{T \subseteq \{b_1, \dots, b_i b_{i+1}, \dots, b_n\}} f(w''_T)g(w''_{\neg T}) \\
&= (fg)(w'')
\end{aligned}$$

and therefore fg is in $R(X)$. We have:

$$\begin{aligned}
(ff)(m + w) &= mf(1)f(1) + \sum_{S \subseteq \{b_1, \dots, b_n\}} f(w_S)f(w_{\neg S}) \\
&= \sum_{\substack{S \subseteq \{b_1, \dots, b_n\} \\ b_1 \in S}} f(w_S)f(w_{\neg S}) + \sum_{\substack{S \subseteq \{b_1, \dots, b_n\} \\ b_1 \notin S}} f(w_S)f(w_{\neg S}) \\
&= \sum_{\substack{S \subseteq \{b_1, \dots, b_n\} \\ b_1 \in S}} f(w_S)f(w_{\neg S}) + f(w_{\neg S})f(w_S) \\
&= \sum_{\substack{S \subseteq \{b_1, \dots, b_n\} \\ b_1 \in S}} f(w_S)f(w_{\neg S}) - f(w_S)f(w_{\neg S}) \\
&= 0.
\end{aligned}$$

From the calculation

$$\begin{aligned}
(f(gh))(n+w) &= mf(1)(gh)(1) + \sum_{S \subseteq \{b_1, \dots, b_n\}} f(w_S)(gh)(w_{\gamma_S}) \\
&= mf(1)(g(1)h(1)) + \sum_{S \subseteq \{b_1, \dots, b_n\}} f(w_S) \sum_{T \subseteq \gamma_S} f(w_{\gamma_{S_T}})g(w_{\gamma_{S-T}}) \\
&= mf(1)(g(1)h(1)) + \sum_{\substack{S, T \subseteq \{b_1, \dots, b_n\} \\ S \cap T = \emptyset}} f(w_S)(g(w_T)h(w_{\gamma_{S \cup T}})),
\end{aligned}$$

it easily follows that the Jacobi identity holds for $R(X)$. It is easy to check that the identities $b(fg) = (bf)g - (bg)f$ and $(b_1b_2)f = b_1(b_2f) - b_2(b_1f)$ hold for all $b, b_1, b_2 \in B$ and for all $f, g \in R(X)$. We see that $R(X) \in B\text{-Act}_D$ and that R determines a functor from \mathbf{Lie}_D to $B\text{-Act}_D$. For any object $Y \in B\text{-Act}_D$, for any $y \in Y$, and for any $m + b_1 \otimes \dots \otimes b_n \in \tilde{B}$, let $\eta_Y(y)(m + b_1 \otimes \dots \otimes b_n) = my + b_1(b_2(\dots(b_{n-1}(b_n y))\dots))$; it is easy to see that η_Y preserves addition and scalar multiplication. The calculation

$$\begin{aligned}
\eta_Y(y)(b_1 \otimes \dots \otimes b_i b_{i+1} \otimes \dots \otimes b_n) &= b_1(\dots [b_i b_{i+1}(\dots (b_n y) \dots)] \dots) \\
&= b_1(\dots [b_i(b_{i+1}(\dots (b_n y) \dots)) - b_{i+1}(b_i(\dots (b_n y) \dots))] \dots) \\
&= \eta_Y(y)(b_1 \otimes \dots \otimes b_n) - \eta_Y(y)(b_1, \dots, b_{i+1}, b_i, \dots, b_n)
\end{aligned}$$

shows that $\eta_Y(y) \in R(X)$. We have $\eta_Y(y_1 y_2)(m) = m(y_1 y_2) = (\eta_Y(y_1) \eta_Y(y_2))(m)$, we show by induction that η_Y preserves multiplication. It is obvious that $\eta_Y(y y')(b) = b(y y') = (by)y' + y(by') = (\eta_Y(y) \eta_Y(y'))(b)$. Now suppose that for any y and y' and for any $b_1 \otimes \dots \otimes b_{n-1}$ that $\eta_Y(y y')(b_1 \otimes \dots \otimes b_{n-1}) = (\eta_Y(y) \eta_Y(y'))(b_1 \otimes \dots \otimes b_{n-1})$. Let $u = b_1 \otimes \dots \otimes b_n$ and $v = b_1 \otimes \dots \otimes b_{n-1}$, we have:

$$\begin{aligned}
\eta_Y(y_1 y_2)(0, u) &= \eta_Y(b_n(y_1 y_2))(v) \\
&= \eta_Y((b_n y_1) y_2 + y_1 (b_n y_2))(v) \\
&= \eta_Y((b_n y_1) y_2)(v) + \eta_Y(y_1 (b_n y_2))(v) \\
&= (\eta_Y(b_n y_1) \eta_Y(y_2))(v) + (\eta_Y(y_1) \eta_Y(b_n y_2))(v) \\
&= (\eta_Y(y_1) \eta_Y(y_2))(u),
\end{aligned}$$

where the last equality follows from the following observation:

$$\begin{aligned}
(fg)(u) &= \sum_{S \subseteq \{b_1, \dots, b_n\}} f(u_S)g(u_{\neg S}) \\
&= \sum_{\substack{S \subseteq \{b_1, \dots, b_n\} \\ s_n \in S}} f(u_S)g(u_{\neg S}) + \sum_{\substack{S \subseteq \{b_1, \dots, b_n\} \\ s_n \notin S}} f(u_S)g(u_{\neg S}) \\
&= \sum_{S \subseteq \{b_1, \dots, b_{n-1}\}} (b_n f)(v_S)g(v_{\neg S}) + \sum_{S \subseteq \{b_1, \dots, b_{n-1}\}} f(v_S)(b_n g)(v_{\neg S}) \\
&= ((b_n f)g)(v) + (f(b_n g))(v).
\end{aligned}$$

Therefore, η_Y preserves multiplication, and the components η_Y form part of a natural transformation $\eta : 1_{B\text{-Act}} \rightarrow RU$. Let $\epsilon : UR \rightarrow 1_{\mathbf{Lie}}$ be the natural transformation defined for an object X as $\epsilon_X(f) = f(1)$ for any $f \in R(X)$. Since $\epsilon_{UY}U\eta_Y(y) = U\eta_Y(y)(1) = y$ and $(R\epsilon_X\eta_{RX}(f))(m + b_1 \otimes \dots \otimes b_n) = (mf + (b_1(\dots(b_n(f)\dots))))(1) = mf(1) + f(b_1 \otimes \dots \otimes b_n) = f(m + b_1 \otimes \dots \otimes b_n)$, it follows that (R, U, η, ϵ) is an adjunction. \square

Chapter 4

Internal homology and cohomology

In this chapter we introduce *internal homology* and *internal cohomology*, and find relationships between these homological theories and existing homological theories.

Let \mathbb{C} be an arbitrary category. In [2], the author calls the category $\mathbf{Ab}(\mathbb{C} \downarrow B)$ of internal abelian groups ($\mathbb{C} \downarrow B$), the category of modules over B , we will call $\mathbf{Ab}(\mathbb{C} \downarrow B)$ the category of Beck modules. Since for the unit $\beta : (B, 1_B) \rightarrow (A, \alpha)$ of any abelian group in $\mathbf{Ab}(\mathbb{C} \downarrow B)$ we have $\alpha\beta = 1_B$, it easily follows that the category $\mathbf{Ab}(\mathbb{C} \downarrow B)$ is isomorphic to the category $\mathbf{Ab}(\mathbf{Pt}(B))$.

Definition 4.0.1. *Let \mathbb{C} be a category with finite limits and let $T : \mathbf{Ab}(\mathbb{C}) \rightarrow \mathbf{Ab}(\mathbf{Pt}(B))$ be the functor determined by the pullback functor along $B \rightarrow 1$, where we have identified $\mathbf{Ab}(\mathbf{Pt}(1))$ with $\mathbf{Ab}(\mathbb{C})$. The zeroth homology and zeroth cohomology functors are defined to be (when they exist) the left and right adjoints of T . The induced and coinduced point functors are defined to be (when they exist) the left and right adjoints of the functor $\text{Ker} : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Ab}(\mathbb{C})$.*

Let \mathbb{C} be Barr-exact. It easily follows that for each $B \in \mathbb{C}$ the category $\mathbf{Pt}(B)$ is Barr-exact, and therefore the category $\mathbf{Ab}(\mathbf{Pt}(B))$ is abelian. If the induced and the zeroth homology functors exist and if \mathbb{E} is the class of all epimorphisms whose images under the functor Ker are split epimorphisms, then since the image of any object under the induced functor is an \mathbb{E} -projective, it follows by Proposition 1.6.10 that the left \mathbb{E} -derived functors exist. We will call the n^{th} left \mathbb{E} -derived functor the n^{th} homology functor. Dually, if the coinduced and the zeroth cohomology functors exist and if \mathbb{M} is the class of all monomorphisms with image being a split monomorphism under the functor Ker , then by the dual of Proposition 1.6.10 the right \mathbb{M} -derived functors exist, and we will call n^{th} right \mathbb{M} -derived functor the n^{th} cohomology functor.

4.1 Existence

In this section we consider the question of existence of left and right adjoints to pullback functors between categories of Beck modules. We begin by looking at the existence of right adjoints.

Proposition 4.1.1. *Let \mathbb{C} be a locally small finitely complete well cocomplete category with the following properties:*

1. *for each D in \mathbb{C} the functor $D \times -$ preserves reflexive coequalizers;*
2. *finite limits commute with filtered colimits.*

The forgetful functor $U : \mathbf{Ab}(\mathbb{C}) \rightarrow \mathbb{C}$ creates reflexive coequalizers, creates filtered colimits and reflects isomorphisms.

Proof. It is clear that the functor sending a pair of objects in $\mathbb{C} \times \mathbb{C}$ to their product in \mathbb{C} , satisfies for every pair of coequalizer diagrams the requirements for Lemma 1.5.2. Consequently, for any reflexive pair $f_1, f_2 : A \rightarrow B$ in $\mathbf{Ab}(\mathbb{C})$ with coequalizer (C, c) in \mathbb{C} , the diagrams

$$\begin{array}{ccccc}
 A & \xrightarrow{f_1} & B & \xrightarrow{c} & C \\
 \downarrow - & & \downarrow - & & \downarrow - \\
 A & \xrightarrow{f_2} & B & \xrightarrow{c} & C \\
 \\
 A \times A & \xrightarrow{f_1 \times f_1} & B \times B & \xrightarrow{c \times c} & C \times C \\
 \downarrow + & & \downarrow + & & \downarrow + \\
 A & \xrightarrow{f_1} & B & \xrightarrow{c} & C \\
 & \xrightarrow{f_2} & & &
 \end{array}$$

indicate how to construct the operations for C . Since addition is associative in B and $c \times (c \times c)$ is an epimorphism, addition is associative in C . Similarly, all the axioms that hold for B hold for C , therefore C is an abelian group. It easily follows that (C, c) is the coequalizer of f_1 and f_2 in $\mathbf{Ab}(\mathbb{C})$ and that U creates reflexive coequalizers. Let $(\gamma_x : D(x) \rightarrow A)_{x \in \mathbb{G}}$ be a colimiting cocone in \mathbb{C} of some functor $D : \mathbb{G} \rightarrow \mathbf{Ab}(\mathbb{C})$ for which \mathbb{G} is filtered. Since filtered colimits commute with finite limits, Proposition 1.5.3 shows that $(\gamma_x \times \gamma_x : D(x) \times D(x) \rightarrow A \times A)$ is the colimiting cocone over the diagram $D(1, 1) : \mathbb{G} \rightarrow$

$\mathbf{Pt}(1)$. Since the morphisms

$$D(x) \times D(x) \xrightarrow{+} D(x) \xrightarrow{\gamma_z} A$$

form a cocone over $D\langle 1, 1 \rangle : \mathbb{G} \rightarrow \mathbf{Pt}(1)$, there exists a unique morphism $+$: $A \times A \rightarrow A$ such that the diagram

$$\begin{array}{ccc} A \times A & \xrightarrow{+} & A \\ \uparrow \gamma_x \times \gamma_x & & \uparrow \gamma_x \\ D(x) \times D(x) & \xrightarrow{+} & D(x) \end{array}$$

commutes. It is obvious that the forgetful functor reflects isomorphisms. \square

Theorem 4.1.2. *Let \mathbb{C} be a locally small finitely complete well cocomplete category with the following properties:*

1. *Condition 1.2.2 holds in \mathbb{C} ;*
2. *finite limits commute with filtered colimits;*
3. *for each $B \in \mathbb{C}$, the category $\mathbf{Ab}(\mathbf{Pt}(\mathbb{C}))$ has a small generating set.*

Then for any morphism $p : E \rightarrow B$ in \mathbb{C} , the pullback functor $p^ : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Ab}(\mathbf{Pt}(E))$ has a right adjoint.*

Proof. It is easy to see that when \mathbb{C} is a finitely complete well cocomplete category satisfying Condition 1.2.2, the category $\mathbf{Pt}(B)$ is also well cocomplete. Therefore, it follows from Proposition 4.1.1 that the category $\mathbf{Ab}(\mathbf{Pt}(B))$ is a well cocomplete and the forgetful functor $U : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Pt}(B)$ creates reflexive coequalizers and filtered colimits. Note that binary coproducts coincide with binary products in any additive category; since $\mathbf{Ab}(\mathbf{Pt}(B))$ is additive, it follows that the forgetful functor creates all colimits. Since the pullback functor $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ preserves all limits, reflexive coequalizers, and filtered colimits (see Theorem 3.1.4), it follows that $p^* : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Ab}(\mathbf{Pt}(E))$ preserves binary coproducts (binary products), reflexive coequalizers, and filtered colimits. Moreover, it follows from Theorem 1.5.4 that p^* preserves all colimits, and therefore by The Special Adjoint Functor Theorem (see [18], Chapter V, Section 8, Theorem 2) p^* has a right adjoint. \square

Proposition 4.1.3. *Let \mathbb{C} be a category with finite limits. If $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$ has a right adjoint then the functor $p^* : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Ab}(\mathbf{Pt}(E))$ has a right adjoint.*

Proof. This follows trivially from the fact that right adjoints preserve limits. \square

4.2 Weakly unital categories

In this section we show, in much the same way we showed for the functor $T : \mathbb{C} \rightarrow \mathbf{Pt}(B)$ in Section 3.2, that the existence of zeroth cohomology functor is equivalent to existence of *centralizers*. We conclude the section with a theorem in which we show that the zeroth cohomology functor exists in any regular weakly unital category which is cocomplete and well-powered.

Throughout this section we assume that \mathbb{C} is a weakly unital category. There is an equivalence of categories $\mathbf{Ab}(\mathbf{Pt}(1)) \sim \mathbf{Ab}(\mathbb{C})$. Let $U : \mathbf{Ab}(\mathbb{C}) \rightarrow \mathbb{C}$ be the forgetful functor and let $T : \mathbf{Ab}(\mathbb{C}) \rightarrow \mathbf{Ab}(\mathbf{Pt}(B))$ be the functor equivalent to the pullback functor $(B \rightarrow 1)^* : \mathbf{Ab}(\mathbf{Pt}(1)) \rightarrow \mathbf{Ab}(\mathbf{Pt}(B))$.

Definition 4.2.1. For an object $C \in \mathbb{C}$ and for any objects $(A, f) \in (U \downarrow C)$, $(B, g) \in (\mathbb{C} \downarrow C)$, we will denote by $\overline{\mathbb{Z}}_{(A,f)}(B, g)$ the full subcategory of the comma category $(\mathbf{Ab}(\mathbb{C}) \downarrow A)$, with objects (E, λ) such that $fU\lambda$ and g commute. This category will be called the *abelian centralizer category* of (B, g) with respect to (A, f) (in $(\mathbb{C} \downarrow C)$) and if this category has a terminal object it will be called the *abelian centralizer* of (B, g) with respect to (A, f) .

Remark 4.2.2. Abelian centralizers are usually (e.g. when \mathbb{C} is a regular Mal'tsev category) the same as centralizers defined in Chapter 3.

Proposition 4.2.3. For each $(A, \alpha, \beta) \in \mathbf{Pt}(B)$, the comma category $(T \downarrow (A, \alpha, \beta))$ is isomorphic to $\overline{\mathbb{Z}}_{(K, \kappa)}(B, \beta)$ where (K, κ) is the kernel of α .

Proof. Since the pullback functor T assigns to each $X \in \mathbb{C}$ the triple $(X \times B, \pi_2, \langle 0, 1 \rangle) \in \mathbf{Ab}(\mathbf{Pt}(B))$, it follows that for any object $(E, \psi) \in (T \downarrow (A, \alpha, \beta))$ the diagram

$$\begin{array}{ccc}
 E & & \\
 \langle 1, 0 \rangle \downarrow & \searrow^{\psi \langle 1, 0 \rangle} & \\
 E \times B & \xrightarrow{\psi} & A \\
 \langle 0, 1 \rangle \uparrow & \nearrow_{\beta} & \\
 B & &
 \end{array}$$

commutes, and consequently the morphisms $\psi \langle 1, 0 \rangle$ and β commute. Since ψ is a morphism in $\mathbf{Ab}(\mathbf{Pt}_B(\mathbb{C}))$, $\alpha\psi = \pi_2$ and therefore $\alpha\psi \langle 1, 0 \rangle = \pi_2 \langle 1, 0 \rangle = 0$. Since (K, κ) is the kernel

of α there exists a unique morphism $\lambda: E \rightarrow K$ in $\mathbf{Ab}(\mathbb{C})$ with $\kappa\lambda = \psi\langle 1, 0 \rangle$. Conversely, for a pair $(E', \lambda') \in \overline{\mathbb{Z}}_{(K, \kappa)}(B, \beta)$, let $\psi': E' \times B \rightarrow A$ be the unique morphism making the diagram

$$\begin{array}{ccc}
 E' & & \\
 \langle 1, 0 \rangle \downarrow & \searrow^{\kappa\lambda'} & \\
 E' \times B & \xrightarrow{\psi'} & A \\
 \langle 0, 1 \rangle \uparrow & \nearrow_{\beta} & \\
 B & &
 \end{array}$$

commute. Since the morphisms $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are jointly epimorphic, $\alpha\psi'\langle 1, 0 \rangle = \alpha\kappa\lambda' = 0 = \pi_2\langle 1, 0 \rangle$ and $\alpha\psi'\langle 0, 1 \rangle = \alpha\beta = 1_B = \pi_2\langle 0, 1 \rangle$, it follows that $\alpha\psi' = \pi_2$ and (E', ψ') is an object in $(T \downarrow (A, \alpha, \beta))$. It is clear that these assignments are inverse to each other. \square

Corollary 4.2.4. *The following conditions are equivalent:*

1. T has a right adjoint.
2. for every $(A, \alpha, \beta) \in \mathbf{Pt}(B)$ with (K, κ) the kernel of α , the category $\overline{\mathbb{Z}}_{(K, \kappa)}(B, \beta)$ has a terminal object.

Proof. Immediately follows from Proposition 4.2.3. \square

Proposition 4.2.5. *Let $h: (A', f') \rightarrow (A, f)$ be any morphism in $(U \downarrow C)$. The functor $h_*: \overline{\mathbb{Z}}_{(A', f')}(B, g) \rightarrow \overline{\mathbb{Z}}_{(A, f)}(B, g)$ defined on objects as $h_*(E', \lambda') = (E', h\lambda')$, has a right adjoint.*

Proof. For $(E, \lambda) \in \overline{\mathbb{Z}}_{(A, f)}(B, g)$, let (E', λ', h') be the pullback of h and λ . It easily follows that the functor assigning (E, λ) to (E', λ') is the right adjoint of h_* . \square

Proposition 4.2.6. *The functor $T: \mathbf{Ab}(\mathbb{C}) \rightarrow \mathbf{Ab}(\mathbf{Pt}(B))$ preserves coequalizers.*

Proof. Since binary products and coproducts coincide in $\mathbf{Ab}(\mathbb{C})$ and since T preserves limits, it only remains to show that T preserves reflexive coequalizers. Let $(X, +, -, 0)$ and $(Y, +, -, 0)$ be objects in $\mathbf{Ab}(\mathbb{C})$, and let

$$X \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{array} \rightrightarrows Y \xrightarrow{c} Z$$

be a reflexive coequalizer in \mathbb{C} . By Proposition 3.2.8, the functor assigning to each pair of objects in $\mathbb{C} \times \mathbb{C}$ their product in \mathbb{C} , satisfies the requirements for Lemma 1.5.2 for every pair of reflexive coequalizer diagrams. Therefore the diagram

$$X \times X \begin{array}{c} \xrightarrow{f_1 \times f_1} \\ \xrightarrow{f_2 \times f_2} \end{array} \rightrightarrows Y \times Y \xrightarrow{c \times c} Z \times Z$$

is a reflexive coequalizer. By Corollary 3.2.9 we know that $T : \mathbb{C} \rightarrow \mathbf{Pt}(B)$ preserves coequalizers, and since it preserves all products the diagram

$$T(X) \times T(X) \begin{array}{c} \xrightarrow{T(f_1) \times T(f_1)} \\ \xrightarrow{T(f_2) \times T(f_2)} \end{array} \rightrightarrows T(Y) \times T(Y) \xrightarrow{T(c) \times T(c)} T(Z) \times T(Z)$$

a reflexive coequalizer. Therefore, $T(Z)$ can be equipped with abelian group structure as follows:

$$\begin{array}{ccccc} T(X) \times T(X) & \begin{array}{c} \xrightarrow{T(f_1) \times T(f_1)} \\ \xrightarrow{T(f_2) \times T(f_2)} \end{array} \rightrightarrows & T(Y) \times T(Y) & \xrightarrow{T(c) \times T(c)} & T(Z) \times T(Z) \\ \downarrow + & & \downarrow + & & \downarrow + \\ T(X) & \begin{array}{c} \xrightarrow{T(f_1)} \\ \xrightarrow{T(f_2)} \end{array} \rightrightarrows & T(Y) & \xrightarrow{T(c)} & T(Z), \\ & & & & \downarrow - \\ & & T(X) & \begin{array}{c} \xrightarrow{T(f_1)} \\ \xrightarrow{T(f_2)} \end{array} \rightrightarrows & T(Y) & \xrightarrow{T(c)} & T(Z). \\ & & \downarrow - & & \downarrow - & & \downarrow - \end{array}$$

□

Proposition 4.2.7. *Let C be an object in \mathbb{C} , let (A, f) be an object in $(U \downarrow C)$, and let (B, g) be an object in the comma category $(\mathbb{C} \downarrow C)$. For any $(E, \lambda) \in \overline{\mathbb{Z}}_{(A, f)}(B, g)$, there exists $(S, \eta) \in \overline{\mathbb{Z}}_{(A, f)}(B, g)$ with η a monomorphism in \mathbb{C} , and there exists a regular epimorphism $e : (E, \lambda) \rightarrow (S, \eta)$ in $\overline{\mathbb{Z}}_{(A, f)}(B, g)$.*

Proof. Since \mathbb{C} is a regular category, there exists a regular epimorphism e and a monomor-

phism η such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\lambda} & A \\ & \searrow e & \nearrow \eta \\ & S & \end{array}$$

commutes. Since $U(e)$ is a regular epimorphism, by Proposition 1.1.8 $fU\eta$ commutes with g . It is easy to see that $e : (E, \lambda) \rightarrow (S, \eta)$ is a regular epimorphism in $\overline{\mathbb{Z}}_{(A,f)}(B, g)$. \square

Proposition 4.2.8. *For $C \in \mathbb{C}$, $(A, f) \in (U \downarrow C)$ and $(B, g) \in (\mathbb{C} \downarrow C)$ the following are equivalent:*

1. $\overline{\mathbb{Z}}_{(A,f)}(B, g)$ has a terminal object.
2. $\overline{\mathbb{Z}}_{(A,f)}(B, g)$ has a weak terminal object.

Proof. 1 \Rightarrow 2: The proof is trivial since every terminal object is a weak terminal object.

2 \Rightarrow 1: Suppose that (E, λ) is a weak terminal object. By the Proposition 4.2.7 there exists a morphism (regular epimorphism) $e : (E, \lambda) \rightarrow (S, \eta)$ where η is a monomorphism, it follows that (S, η) is a terminal object. \square

Proposition 4.2.9. *Suppose in addition that \mathbb{C} is a small cocomplete well-powered category. Let C be an object in \mathbb{C} , let (A, f) be an object in $\mathbf{Ab}(\mathbb{C})$, and let (B, g) be an object in the comma category $(\mathbb{C} \downarrow C)$. The category $\overline{\mathbb{Z}}_{(A,f)}(B, g)$ has a terminal object if it is small cocomplete.*

Proof. Let (T, τ) be the colimit of the inclusion of the full subcategory of $(E, \lambda) \in \overline{\mathbb{Z}}_{(A,f)}(B, g)$ for which λ is a monomorphism, into the category $\overline{\mathbb{Z}}_{(A,f)}(B, g)$. It is easy to see that (T, τ) is a weak terminal object and we conclude by Proposition 4.2.8 that the category $\overline{\mathbb{Z}}_{(A,f)}(B, g)$ has a terminal object. \square

Theorem 4.2.10. *Suppose in addition that \mathbb{C} is a cocomplete well-powered category in which filtered colimits commute with finite limits. The Zeroth cohomology functor exists.*

Proof. Since both $\mathbf{Ab}(\mathbb{C})$ and $\mathbf{Ab}(\mathbf{Pt}(B))$ are pointed and T clearly preserves limits (limits commute with limits), the functor T preserves the initial object and binary coproducts. T preserves coequalizers by Proposition 4.2.6, and so by Theorem 1.5.4 T preserves all finite colimits. Since \mathbb{C} is also cocomplete and filtered colimits commute with finite limits, again by Theorem 1.5.4 we see that T preserves all small colimits. Therefore, for any object

$(A, \alpha, \beta) \in \mathbf{Pt}(B)$, the comma category $(T \downarrow (A, \alpha, \beta))$ is small cocomplete; equivalently, by Proposition 4.2.3, the category $\overline{\mathbb{Z}}_{(K, \kappa)}(B, \beta)$ where (K, κ) is the kernel of α , is small cocomplete. By Proposition 4.2.9, the category $\overline{\mathbb{Z}}_{(K, \kappa)}(B, \beta)$ has a terminal object and consequently, by Corollary 4.2.4, T has a right adjoint. \square

4.3 Weakly Mal'tsev categories

Proposition 4.3.1. *Let \mathbb{C} be a regular weakly Mal'tsev category. For any morphism $p : E \rightarrow B$ in \mathbb{C} , the pullback functor $p^* : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Ab}(\mathbf{Pt}(E))$ has a left adjoint if the categories $\mathbf{Pt}(B)$ and $\mathbf{Pt}(E)$ satisfy Condition 1.1.9.*

Proof. Since $\mathbf{Pt}(B)$ and $\mathbf{Pt}(E)$ are regular weakly unital categories and satisfy Condition 1.1.9, it follows from Proposition 1.1.19 that the forgetful functors $U_B : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Pt}(B)$ and $U_E : \mathbf{Ab}(\mathbf{Pt}(E)) \rightarrow \mathbf{Pt}(E)$ have left adjoints. Since by Lemma 1.5.2 the category $\mathbf{Ab}(\mathbf{Pt}(E))$ has reflexive coequalizers, and since binary products are binary coproducts in $\mathbf{Ab}(\mathbf{Pt}(E))$, the category $\mathbf{Ab}(\mathbf{Pt}(E))$ has all coequalizers. Examining the proof of Proposition 1.1.19, it is easy to see that the counit of the adjunction is an isomorphism. Consider the diagram of functors

$$\begin{array}{ccc}
 \mathbf{Ab}(\mathbf{Pt}(B)) & \xrightarrow{p^*} & \mathbf{Ab}(\mathbf{Pt}(E)) \\
 \swarrow U_B & & \nearrow F_E L \\
 \mathbf{Pt}(B) & & \mathbf{Pt}(E) \\
 \nwarrow F_B & & \nearrow p^* U_E
 \end{array}$$

in which L is the left adjoint of $p^* : \mathbf{Pt}(B) \rightarrow \mathbf{Pt}(E)$, it follows from Theorem 1.5.6 that the left adjoint of $p^* : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Ab}(\mathbf{Pt}(E))$ exists. \square

Remark 4.3.2. *When \mathbb{C} is Mal'tsev, the categories $\mathbf{Pt}(B)$ and $\mathbf{Pt}(E)$ trivially satisfy Condition 1.1.9 (since they are unital).*

4.4 Semi-abelian categories

Recall: for a ring homomorphism $f : R \rightarrow S$ there is a functor assigning to each S -module X the R -module X with action defined by $rx = f(r)x$ for each $r \in R$ and $x \in X$. We will denote this functor by $f^* : S\text{-Mod} \rightarrow R\text{-Mod}$.

Proposition 4.4.1. *Let \mathcal{V} be a semi-abelian variety and let $U : \mathcal{V} \rightarrow \mathbf{Set}$ be the forgetful functor. For each $B \in \mathcal{V}$, the category $\mathbf{Ab}(\mathbf{Pt}(B))$ is equivalent to the category $R_B\text{-Mod}$ for some ring R_B ; moreover, for each morphism $p : E \rightarrow B$ induces a ring homomorphism $R_p : R_E \rightarrow R_B$. such that the functor $p^* : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Ab}(\mathbf{Pt}(E))$ is equivalent to the functor $R_p^* : R_B\text{-Mod} \rightarrow R_E\text{-Mod}$.*

Proof. Let $V : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Pt}(B)$ be the forgetful functor. Since \mathcal{V} is semi-abelian it easily follows that the category $\mathbf{Pt}(B)$ is weakly unital and satisfies Condition 1.1.9. Corollary 1.1.19 and Proposition 1.4.2 show, respectively, that the functors V and Ker have left adjoints, and Proposition 4.1.1 shows that the functor V preserves reflexive coequalizers, preserves filtered colimits, and reflects isomorphisms. Proposition 1.2.10 shows that the functor Ker preserves reflexive coequalizers, and trivially Ker reflects isomorphisms. Since \mathcal{V} is a variety, finite limits commute with filtered colimits and therefore the functor Ker preserves filtered colimits. It is well known that the functor U preserves reflexive coequalizers, preserves filtered colimits, and reflects isomorphisms, therefore the composite $U \circ \text{Ker} \circ V$ preserves reflexive coequalizers, preserves filtered colimits, reflects isomorphisms, and has a left adjoint; consequently, Becks monadicity theorem shows that the composite is monadic. It is well known that the category of algebras for a monad on the category of sets is equivalent to a variety when the monad preserves filtered colimits, therefore the category $\mathbf{Ab}(\mathbf{Pt}(B))$ is equivalent to a variety. Since this variety is additive, it is equivalent to a category of modules over the ring with underlying set $R_B = T_B(\{1\})$. The functor p^* is equivalent to a functor $\mathbf{Set}^{T_B} \rightarrow \mathbf{Set}^{T_E}$ which makes the diagram

$$\begin{array}{ccc} \mathbf{Set}^{T_B} & \longrightarrow & \mathbf{Set}^{T_E} \\ & \searrow & \swarrow \\ & \mathbf{Set} & \end{array}$$

commute, and therefore induces a morphism between the monads $T_E \rightarrow T_B$. The map $T_E(\{1\}) \rightarrow T_B(\{1\})$ turns out to be a ring homomorphism. \square

Remark 4.4.2. *It is easy to see that the monad $Bb-$ preserves filtered colimits, from which it follows that $\mathbf{Pt}(B)$ is a variety. From this observation it is easy to construct an alternate proof of Proposition 4.4.1.*

Corollary 4.4.3. *Let \mathcal{V} be a semi-abelian variety and let $U : \mathcal{V} \rightarrow \mathbf{Set}$ be the forgetful functor. For any morphism $p : E \rightarrow B$ in \mathcal{V} the left and right adjoints of the pullback functor $p^* : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Ab}(\mathbf{Pt}(E))$ exist. Moreover, the left adjoint can be described as*

$R_B \otimes_{R_E} - : R_B\text{-Mod} \rightarrow R_E\text{-Mod}$ and the right adjoint as $\text{Hom}_{R_E}(R_B, -) : R_B\text{-Mod} \rightarrow R_E\text{-Mod}$.

Proof. The left and right adjoints (and their derived functors) of functors between modules, as described above, are well known classically. \square

Remark 4.4.4. *It follows from Corollary 4.4.3 that $H_n(B, -) \cong \text{Tor}_n(R_B, -)$ and $H^n(B, -) \cong \text{Ext}^n(R_B, -)$.*

Proposition 4.4.5. *The functors $H^2 : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Ab}(\mathbf{Gp})$ and $H^2 : \mathbf{Ab}(\mathbf{Pt}(B)) \rightarrow \mathbf{Ab}(\mathbf{Lie})$ classify extensions with abelian kernel in \mathbf{Gp} and \mathbf{Lie} respectively.*

Proof. This follows from the fact that in both cases H^0 coincides with classical zeroth cohomology. \square

Example 4.4.6. *Let \mathcal{V} be the variety with objects having the structure of abelian groups as well as an additional unary operation which preserves the nullary operation 0. As mentioned in the proof of Proposition 3.5.10, $\mathbf{Pt}(B)$ is equivalent to the variety \mathcal{V}^B with objects having the structure of an abelian group as well as unary operations one for each $b \in B$ each of which preserve the nullary operation 0. Therefore, it easily follows that $\mathbf{Ab}(\mathbf{Pt}(B))$ is equivalent to the subvariety of \mathcal{V}^B in which each unary operation is an abelian group homomorphism. Consequently, the category $\mathbf{Ab}(\mathbf{Pt}(B))$ is equivalent to the category $R_B\text{-Mod}$ where R_B is the free unital ring on the set B .*

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