ANALYTICAL APPROXIMATIONS OF SURFACE FIELDS INDUCED ON CONVEX SCATTERERS BY EXTERIORLY INCIDENT SCALAR FIELDS

BY

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CONTENTS

ABSTRACT

The boundary value problems for the Helmholtz equation give rise to boundary integral equations for the unknown surface field or its normal derivative. These integral equations involve the Helmholtz surface potentials in the form of weakly singular surface integrals. This thesis is based on a method of parameterisation of the surface integrals which removes the weak singularities provided that the surface satisfies certain convexity conditions. Firstly this method of parameterisation is applied to investigate the properties of the Helmholtz surface potentials on convex surface elements, and some new proofs are given. The theory is then applied to the boundary integral equations which arise when a scalar field is incident on a bounded scatterer. The surface integrals in these integral equations are Helmholtz potentials and can be regularised by suitable parameterisation. It is assumed that the unknown density function is an analytical function on the boundary of the scatterer, and can therefore be expanded as a Taylor series at any point of the surface. If this expansion is substituted into the regularised integral equation and if the operations of integration and summation are formally interchanged, then the end result is a partial differential equation of infinite order involving only the field coordinates and having analytical coefficients. However, if the Taylor expansions are truncated then partial differential equations of finite orders result. The view is taken that analytical solutions of such differential equations of finite orders can serve as approximations for the surface field or its normal derivative provided that suitable initial conditions are imposed to ensure uniqueness. On the other hand the general solution of such a differential equation can serve as a local approximation at any point on the surface. Some basic properties of the differential equations and their solutions, called analytical approximations, are discussed and the theory is then applied to the problem of acoustic scattering from a sound hard sphere.

CHAPTER I

INTRODUCTION

1.1 Prospectus.

This thesis is a report of work done on obtaining approximations to surface fields induced on convex bodies D by externally incident scalar fields. Such fields arise from boundary conditions imposed on the scalar Helmholtz equation

$$\nabla^2 \phi + k^2 \phi = 0 \tag{1.1.1}$$

in the region D^e exterior to D. The Helmholtz equation, also known as the reduced wave equation, is obtained when we seek steady-state solutions of the scalar wave equation for mono-frequency sources, and $k = 2\pi/\lambda$ is the wave number. If an external field ϕ_i is incident on a bounded scatterer then reflected and diffracted waves are produced which travel outward from the scatterer and the resulting field is called the total field, denoted by ϕ_t. The scattered field ϕ_s is then defined by

$$\phi_s = \phi_t - \phi_i, \tag{1.1.2}$$

and is subject to a radiation condition which ensures that it is a diverging wave.

On the boundary ∂D of the scatterer D conditions are imposed on the total wave ϕ_t (Dirichlet condition), or its normal derivative $\frac{\partial \phi_t}{\partial n}$ (Neumann condition), and the mathematical problem is to find the exterior wave function ϕ_t which satisfies these conditions and is such that the scattered wave obeys the radiation condition. Empirical evidence indicates that the exterior wave field is unique. Mathematically the existence of a solution is given at points $r \in D^e$ by the Helmholtz exterior formula

$$\phi_t(r) = \iint_{\partial D} \left[\phi_t(r) \frac{\partial G_k(r,r')}{\partial n'} - G_k(r,r') \frac{\partial \phi_t(r')}{\partial n'}\right] d\sigma', \tag{1.1.3}$$

provided that the radiation condition is satisfied. Here r is a field point in D^e and r' is a source point in ∂D, and G_k is the free-space Green's function defined by

$$G_k(\mathbf{r},\mathbf{r}') = \frac{e^{ikR}}{4\pi R} \qquad (1.1.4)$$

where $R = \|\mathbf{r}' - \mathbf{r}\|$ is the distance between the field point and the source point. The uniqueness of the solution of the exterior Dirichlet or Neumann boundary value problem subject to the radiation condition was proved by Atkinson (5) and is also based on the Helmholtz exterior formula given above.

If the field point $\mathbf{r} \in D^e$ approaches a point in ∂D along the normal direction then the Helmholtz exterior formula assumes the limiting form

$$\frac{1}{2}\phi_t(\mathbf{r}) = \iint_{\partial D} \left\{\phi_t(\mathbf{r}) \frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}') \frac{\partial \phi_t(\mathbf{r}')}{\partial n'}\right\} d\sigma', \qquad (1.1.5)$$

when $\mathbf{r} \in \partial D$. This equation can be used to obtain the solution for the exterior Neumann boundary value problem for the Helmholtz equation. If the known value of the normal derivative $\frac{\partial \phi_t}{\partial n}$ is inserted in the Helmholtz boundary formula (1.1.5) an integral equation for ϕ_t on ∂D is obtained. Using the solution of this equation and the value of $\frac{\partial \phi_t}{\partial n}$ in the Helmholtz exterior formula yields the solution of the boundary value problem.

If the boundary condition is homogeneous Neumann, that is if $\frac{\partial \phi_t}{\partial n} = 0$, and if the incident field does not satisfy the radiation condition, (1.1.5) becomes

$$\frac{1}{2}\phi(\mathbf{r}) = \phi_i(\mathbf{r}) + \iint_{\partial D} \frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} \phi(\mathbf{r}')d\sigma' , \qquad (1.1.6)$$

where ϕ denotes the total surface field and

$$\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} = \frac{(ikR - 1)e^{ikR}\, \mathbf{R}.\mathbf{n}'}{4\pi R^3} . \qquad (1.1.7)$$

In view of the weak singularity of the integrand in the neighbourhood of the field point $\mathbf{r} \in \partial D$ it was suggested by du Plessis (17) that the main contribution to the integral in (1.1.6) could be attributed to a surface element S of which the points $\mathbf{r}'$ are situated in the immediate proximity of the point $\mathbf{r}$. Consequently

$$\phi(\mathbf{r}) \approx \psi(\mathbf{r})$$

where ψ satisfies the integral equation

$$\frac{1}{2}\,\psi(r) = \phi_i(r) + \iint_S \frac{\partial G_k(r,r')}{\partial n'}\,\psi(r')d\sigma' \ . \qquad (1.1.8)$$

If the radii of curvature of the surface are large compared with the wave length of the incident radiation, the surface element S can be assumed to be plane in which case the inner product $\mathbf{R}.\mathbf{n}' = 0$. Thus the integral in (1.1.8) is zero and we find that

$$\phi(r) \approx \psi(r) = 2\phi_i(r). \qquad (1.1.9)$$

This result corresponds to the geometic-optics surface field which is twice the incident field on the whole surface. In physical optics the approximation (1.1.9) is used only on the illuminated side of the convex body; the field on the shadow side of the body is taken to be zero.

In view of the assumption that the main contribution of the surface integral to the surface field results from the portion of ∂D in the immediate neighbourhood of the singularity, one may expect that an expansion of the integrand in an arbitrary neighbourhood of the singular point would lead to equations progressively giving the effect of non-local contributions to the surface field as the region of integration is expanded and more terms are taken into account. Thus we propose that a tangent-normal coordinate system be introduced at the field point $\mathbf{r} \in \partial D$ and that the integrand be expanded in terms of spherical coordinates with respect to the chosen tangent-normal coordinate axes. The effect of this transformation on the surface integral in equation (1.1.8) is discussed in Chapter II, section (2.3). There we use tangent-normal axes at a point $\mathbf{r}_0 \in \partial D$, and they are defined so that the 1-axis and 2-axis are mutually orthogonal tangents to the surface and the 3-axis is in the direction of the outward normal. The position of a point $P \in \mathbb{R}^3$ can be expressed in terms of spherical coordinates (R,θ,φ) where $R = P_0P$, φ is the azimuth of P with respect to the 1-axis and θ is the colatitude of P. Source points and coordinates of source points lying on the surface ∂D are denoted by primed letters. If the region D is convex then the colatitude θ' of a point P′ on the surface is not less than 90^0, and we define the altitude or declination χ' of P′ by $\theta' = 90^0 + \chi'$. Thus the altitude of a point is positive if it lies below the tangent plane.

If the surface integral in equation (1.1.8) is parameterised in terms of the azimuth-altitude coordinates it turns out that the transformed integrand is free of singularities. For example, it is shown in section (2.3) that for a function ϕ on ∂D

$$\iint_S \frac{G_k(\mathbf{r},\mathbf{r}')}{\partial n'}\phi(\mathbf{r}')\, d\sigma' = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\chi'(\varphi')}(ikR - 1)e^{ikR}\cos\chi'\,\phi(\mathbf{r}+\mathbf{R})\,d\chi'\,d\varphi' \tag{1.1.10}$$

where $R = \|\mathbf{r}' - \mathbf{r}\|$, $\mathbf{r}' - \mathbf{r} = \mathbf{P}_0\mathbf{P}'$ and

$$\chi' = \chi'(\varphi') \ , \quad 0 \le \varphi' \le 2\pi,$$

is the equation of the boundary of the surface element S.

This result shows that if $\mathbf{r}$ is any point of the surface and if S is a surface element containing $\mathbf{r}$ then the surface integral

$$\Delta\Phi(\mathbf{r}) = \iint_S \frac{G_k(\mathbf{r},\mathbf{r}')}{\partial n'}\phi(\mathbf{r}')\, d\sigma'$$

can be made arbitrarily small by choosing the region S sufficiently small. If it is granted that in the high frequency limit the surface field on the illuminated side of the surface is approximately twice the incident field, then equation (1.1.8) implies that the surface integral

$$\Phi(\mathbf{r}) = \iint_{\partial D} \frac{G_k(\mathbf{r},\mathbf{r}')}{\partial n'}\phi(\mathbf{r}')\, d\sigma'$$

is small when the field point $\mathbf{r}$ is in the illuminated portion of the surface. Thus it may be argued that if $\mathbf{r}$ is a point on the illuminated side of the surface then the non-local contribution of the surface integral, i.e.

$$\Phi(\mathbf{r}) - \Delta\Phi(\mathbf{r}) = \iint_{\partial D-S} \frac{G_k(\mathbf{r},\mathbf{r}')}{\partial n'}\phi(\mathbf{r}')\, d\sigma' \ ,$$

is vanishingly small, and that the local contribution of the surface integral, i.e. $\Delta\Phi(\mathbf{r})$, although small, still accounts for the small difference existing between the surface field and $2\phi_i$.

However, a different situation pertains in the shadow region, for in this case the surface field is small when the incident radiation is in the high frequency range. Equation (1.1.8) therefore implies that if $\mathbf{r}$ is in the shadow region then

$$\Phi(\mathbf{r}) \approx -\phi_i(\mathbf{r}) \ ,$$

and since $\Delta\Phi(\mathbf{r})$ is always small, we conclude that the main contribution of the surface integral is due to non-local effects.

In the light of the foregoing discussion we therefore have the following problem : to what extent do local contributions to the surface field at a point $\mathbf{r}_0 \in \partial D$ make up the surface field at this point ? This problem may be formulated in the following way. Let $\mathbf{r}_0 \in S$ where $S \subset \partial D$

and let ϕ_S be a solution of the integral equation

$$\tfrac{1}{2}\,\phi_S(\mathbf{r}) = \phi_i(\mathbf{r}) + \Phi_S(\mathbf{r})\,,\quad \mathbf{r}\in S, \tag{1.1.11}$$

where

$$\Phi_S(\mathbf{r}) = \iint_S \frac{G_k(\mathbf{r},\mathbf{r}')}{\partial n'}\phi_S(\mathbf{r}')\,d\sigma'. \tag{1.1.12}$$

If S = ∂D then of course $\phi_S = \phi$, where ϕ is given by (1.1.6). However, if S is a subset of ∂D, the question is how large S must be in order that $\phi_S(\mathbf{r}_0) \approx \phi(\mathbf{r}_0)$. Tentatively we can say that if if $\mathbf{r}_0$ is a point on the illuminated side of the scatterer then S can be a small surface element containing $\mathbf{r}_0$, while if $\mathbf{r}_0$ is in the shadow region of the scatterer it is likely that we must take S = ∂D. The function ϕ_S defined above may be referred to as a ***local integral approximation*** of the surface field. As the choice of S on the illuminated side of the scatterer imposes a limit on $R = |\mathbf{r}' - \mathbf{r}|$, it is appropriate to replace $\phi(\mathbf{r}') = \phi(\mathbf{r} + \mathbf{R})$ by a suitably truncated power series expansion.

By using azimuth-altitude coordinates it is also possible to define ***global approximations*** of the surface field. In terms of these coordinates it is found that for a closed convex surface

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{\frac{\pi}{2}} (ikR - 1)e^{ikR}\cos\chi'\,\phi(\mathbf{r}+\mathbf{R})\,d\chi'\,d\varphi'. \tag{1.1.13}$$

If it is assumed that the surface field ϕ is an analytical function on the boundary, we can expand $\phi(\mathbf{r}+\mathbf{R})$ in powers of $R = \|\mathbf{R}\|$, obtaining

$$\phi(\mathbf{r}+\mathbf{R}) = \phi(\mathbf{r}) + (\mathbf{R}.\nabla)\phi(\mathbf{r}) + \tfrac{1}{2}(\mathbf{R}.\nabla)^2\phi(\mathbf{r}) + \ldots\,. \tag{1.1.14}$$

In equation (1.1.14) partial differentiation is with respect to the field coordinate $\mathbf{r}$. It follows that the surface integral can be expanded in the form

$$\Phi(\mathbf{r}) = T_0\phi(\mathbf{r}) + (T_1\phi)(\mathbf{r}) + (T_2\phi)(\mathbf{r}) + \ldots \tag{1.1.15}$$

where T_0 is given by

$$T_0 = -\frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{\frac{\pi}{2}} (ikR - 1)e^{ikR}\cos\chi'\,d\chi'\,d\varphi', \tag{1.1.16}$$

and $T_1, T_2, \ldots$ are linear partial differential operators defined by

$$T_n = -\frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{\frac{\pi}{2}} (ikR - 1)e^{ikR}\cos\chi'\,(\mathbf{R}.\nabla)^n\,d\chi'\,d\varphi',\quad n = 1,2,\ldots\,. \tag{1.1.17}$$

This and similar expansions are dealt with in Chapter IV, where a different notation is used. The integral equation (1.1.6) can now be expressed in the form

$$\frac{1}{2}\,\phi(\mathbf{r}) = \phi_i(\mathbf{r}) + T_0\phi(\mathbf{r}) + (T_1\phi)(\mathbf{r}) + (T_2\phi)(\mathbf{r}) + \ldots\,. \tag{1.1.18}$$

Thus in place of the singular integral equation (1.1.6) for the surface field we now have a partial differential equation of infinite order, and we also note that the coefficients are analytical functions of $\mathbf{r} \in \partial D$. Bearing in mind that for large values of k the contribution of the integral $\Phi(\mathbf{r})$ is small if $\mathbf{r}$ is a point in the illuminated side of the surface, we assume that equation (1.1.15) can be replaced by the truncated expression

$$\Phi(\mathbf{r}) = T_0\phi(\mathbf{r}) + (T_1\phi)(\mathbf{r}), \tag{1.1.19}$$

which results in a first order partial differential equation

$$\frac{1}{2}\,\phi(\mathbf{r}) = \phi_i(\mathbf{r}) + T_0\phi(\mathbf{r}) + (T_1\phi)(\mathbf{r}) \tag{1.1.20}$$

in place of equation (1.1.8). The general solution of equation (1.1.20), and similar equations of higher orders, will of course involve arbitrary constants or functions. To obtain a definite solution relevant to the problem, we will assume that for large values of the wave number k the surface field and its derivatives at the specular point are respectively twice the incident field and its corresponding derivatives at this point. This condition can then serve as initial condition for equation (1.1.20) and similar equations of higher orders.

The methods outlined above for obtaining global approximations to the surface field can of course also be applied to the local integral approximations defined by equation (1.1.11). The analysis outlined above remains the same and we only have to replace equations (1.1.16) and (1.1.17) respectively by

$$T_0 = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\chi'(\varphi')}(ikR-1)e^{ikR}\cos\chi'\,d\chi'\,d\varphi',$$

and

$$T_n = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\chi'(\varphi')}(ikR-1)e^{ikR}\cos\chi'\,(\mathbf{R}.\nabla)^n d\chi'\,d\varphi',\quad n=1,2,\ldots\,.$$

where $\chi' = \chi'(\varphi')\ ,\quad 0 \le \varphi' \le 2\pi,$

is the equation of the boundary of the surface element S.

Henceforth we refer to the differential equations resulting from a local integral approximation as a ***local approximation***. These local approximations suffer from the disadvantage that no obvious initial or boundary conditions other than $2\phi_i$ at the source point are available in order to obtain unique solutions. Another method of solution is to cover the boundary surface with a finite set of overlapping surface elements S and to assign suitable initial conditions at a specific point on the surface, such as the specular point. Solving the differential equation for the surface element containing this point enables us to continue the solution over the remainder of the surface.

When formulated in this way the distinction between *local* and *global* approximations is not very sharp. For instance, the general solution of the *global* approximation (1.1.20) can be used as a local approximation at any point of the surface. The main advantage of the *local* approximations is that their analytical coefficients can also be replaced by local approximations, a consideration of some significance when considering arbitrary convex bodies.

In summary we can say that the main theme of this work is to determine if solutions of partial differential equations such as (1.1.20) can serve as approximations to the surface field on the illuminated side of the surface. In addition, successful approximation of the surface field in the shadow region may require that equations of higher orders have to be considered. Such local and global approximations of successively higher orders may also have progressively wider ranges of applicability below the high frequency limit, thus making it possible to penetrate the resonance region from above.

1.2 Summary of contents and review of literature.

Chapter II is primarily concerned with establishing the properties of the Helmholtz surface potentials. These potentials can be included in the class of functions having the form

$$H(r) = \iint_S \frac{g(R)}{R^\alpha} h(r') \, d\sigma'$$

where g and h are continuous functions defined on a surface element S. The regularisation of such integrals using tangent-normal axes at $r \in S$ succeeds because the displacement vector $\mathbf{R} = \mathbf{r}' - \mathbf{r}$ can then be expressed in the form $\mathbf{R} = R\boldsymbol{\eta}$, where $R = \|\mathbf{R}\|$ and $\boldsymbol{\eta}$ is a unit vector depending only on the azimuth φ' and altitude χ' of the source point $\mathbf{r}' \in S$. If the surface S satisfies certain convexity conditions then the distance R can be expressed as a function of φ' and χ'. It is then found that the element of surface area is a function homogeneous of degree 2 in R. Consequently regularisation is ensured if $\alpha \leq 2$.

The behaviour of the Helmholtz surface potentials and their derivatives regarding continuity is fundamental for the derivation of boundary integral equations describing the surface field. These continuity properties are derived in section (2.4) in some detail, and new proofs using azimuth-altitude coordinates are given in some cases. In other cases use is made of local Cartesian coordinates, as was done by Kellogg (24) and Günter (20). An attempt was made to formulate some of the work of this section in such a way that the required continuity properties are special cases of a few general theorems.

In section (2.5) the interior and exterior Helmholtz integral formulae are derived with a discussion of the Wilcox radiation condition (Wilcox (53)). These results are used in section (2.6) to prove the uniqueness of the exterior Dirichlet and Neumann boundary value problems for the Helmholtz equation (Atkinson (5)).

Some essential results of functional analysis are summarised in section (3.1) of Chapter III, and the Helmholtz surface potentials are expressed in terms of boundary integral operators L_k and M_k defined by

$$\left(L_k\phi\right)(\mathbf{r}) = \iint_{\partial D} G_k(\mathbf{r},\mathbf{r}')\,\phi(\mathbf{r}')\,d\sigma'$$

and

$$\left(M_k\phi\right)(\mathbf{r}) = \iint_{\partial D} \frac{G_k(\mathbf{r},\mathbf{r}')}{\partial n'}\,\phi(\mathbf{r}')\,d\sigma'$$

and called, respectively, the single and double layer Helmholtz potentials. We also define an operator N_k by

$$\left(N_k\phi\right)(\mathbf{r}) = \frac{\partial}{\partial n}\iint_{\partial D} \frac{G_k(\mathbf{r},\mathbf{r}')}{\partial n'}\,\phi(\mathbf{r}')\,d\sigma',$$

the notation being the same as that used by Burton (9) and Burton and Miller (10). These linear operators are defined on $L^2(\partial D)$ and into $L^2(\partial D)$. It is convenient to extend the range of these operators so that for each $\phi \in L^2(\partial D)$ the domain of the functions $L_k\phi$, $M_k\phi$ and $N_k\phi$ is all of $\mathbb{R}^3$. The continuity properties of section (2.4) are now formulated in operator notation, and likewise for the Helmholtz integral formulae. The limiting forms of these integral formulae when the field point approaches the boundary along the normal direction are determined using the continuity properties of the surface potentials. Various boundary integral equations result when these Helmholtz formulae are applied to the interior and exterior boundary value problems for the Helmholtz equation.

Alternatively one may assume a representation of the interior or exterior field in terms of either of the Helmholtz potentials or a linear combination of these potentials. The limiting form of these fields when the boundary is approached from the interior or exterior yield integral equations for the surface field. However, in this work we give attention only to integral equations obtained from the Helmholtz formulation, and refer to Burton (9) for a discussion of the equations obtained from the potential formulation.

In sections (3.4) and (3.5) the interior Dirichlet and Neumann problems for the Laplace equation are dealt with, and it is shown that in these problems the Laplace operator has a discrete spectrum. It is then shown that there are certain homogeneous boundary integral equations whose adjoints are related to integral equations for the exterior problems for the Helmholtz equation, which have non-trivial solutions for certain values of the wave number k. Thus the integral equation approach to exterior boundary value problems suffer from the disadvantage that these integral equations do not always have unique solutions for certain values of the wave number k. When equations obtained from the potential formulation are used it is found that these equations may even fail to possess a solution at all for certain critical values of the wave number (Burton (9)). As the exterior field exists and is unique methods have to be devised for obtaining the surface field which gives the correct exterior field.

The failure of the integral equation formulation to produce solutions at certain frequencies has been known for some time (see eg. Lamb (32)). Attempts to overcome these difficulties appear in the work of Weyl (52), Leis (35) and others. Burton (9) gives a careful analysis of the work of Brundrit (8), Copley (15,16), Schenk (44), Brakhage and Werner (7), Panich (39) and Leis (35.). The methods devised by Brundrit, Copely and Schenk were not completely free of defects at wave numbers corresponding to eigenvalues for the adjoint interior problem. One of the first methods for the exterior Dirichlet problem to be completely free of this defect was discovered independently by Brakhage and Werner, Panich and Leis. This method is based on a potential formulation whose kernel is a linear combination of the single and double layer kernels, and is extended by Kussmaul (31) to the Neumann case. Greenspan and Werner (19) used this formulation to develop a successful numerical method for the solution of the two-dimensional exterior Dirichlet problem. This was followed by a paper presented by Burton and Miller (10) in which they apply the Helmholtz formulation to the exterior Neumann problem. They show that there exist two boundary integral equations which always have only one solution in common. They go on to show that if a linear combination of these two equations is formed, the resulting equation has a unique solution. Later Burton (9) gave a comprehensive review of both the potential and Helmholtz formulation as it applies to the Dirichlet and Robin boundary value problems. In the latter problem the boundary condition imposed is a linear combination of the Dirichlet and Neumann conditions. Another review of the theory was presented by Kleinman and Roach (26) in which the properties of the relevant integral operators and their eigenfunctions are derived in greater generality. These authors give a complete classification of all boundary integral equations applicable to the classical boundary value problems for the Helmholtz equation. Thus their work consolidates previous results based on the potential and Helmholtz formulations, but they exclude equations resulting from linear combinations of single and double layer distributions. Likewise they exclude linear combinations of equations resulting from the Helmholtz formulation.

The work described in the preceding paragraph is general in that it applies to any boundary value problem for the Helmholtz equation, and is not specifically aimed at the scattering problem. In section (3.6) we derive Helmholtz representation formulae for the total field of a scatterer in terms of the incident field as was first done by Noble (38). The corresponding integral equations for the surface field in terms of the incident field are then obtained. Following Kleinman and Roach (*loc. cit.*) it is shown that in both the Dirichlet and Neumann cases there exist two integral equations for the surface field and that these equations always have only one solution in common for all wave numbers. Chapter III is concluded by showing that a linear combination of the two integral equations found in the Neumann case always have a unique solution provided that the coupling constants satisfy certain conditions (Burton *(loc.cit)*. Similar results can be formulated for the Dirichlet problem.

Chapter IV deals only with the surface field resulting when a scalar wave is incident on a convex scatterer. When the surface integrals in these integral equations are parameterised in terms of azimuth-altitude coordinates the singularities are removed and the integrand is then in a form which can be expanded in a Taylor series. If these Taylor series are truncated then various partial differential equations of arbitrary orders result. A number of partial differential operators corresponding to the different integral equations of scattering theory are introduced and some of their basic properties are deduced. These results are then applied to the case of a sphere, and the coefficients of the different kinds of equations are worked out in detail. It is observed that the differential equations for the sphere share some of the properties of the corresponding integral equations. For instance, the solutions of the differential equations depend only on the product ka where k is the wave number and a is the radius of the sphere. We have previously noted that certain linear combinations of the integral equations of scattering theory have a common unique solution. It appears that to some extent this property is inherited by the corresponding approximating differential equations of a given order, as is shown in section (4.6) in the case of first order global approximations for a sphere. The expectation is that this property will also apply to the differential equations for higher order approximations.

CHAPTER II

THE SCALAR HELMHOLTZ SURFACE POTENTIALS AND THEIR PROPERTIES

In the solution of boundary value problems involving the Helmholtz or time independent scalar wave equation, the Helmholtz surface potentials play a fundamental role. In this chapter we define and investigate these surface potentials and derive the interior and exterior Helmholtz formulae for them. The literature on the subject is extensive, but here we will only mention the work of Kellogg (24), Günter (20) and Colton and Kress (14). We will proceed along the lines of Kellogg, who uses a tangent-normal Cartesian system of axes at a point on a surface element to derive the limit relations for Laplace surface potentials. However, instead of consistently using rectangular coordinates, we will also use spherical coordinates in some proofs. These coordinates can be introduced in such a way that the weak singularities disappear in the singular integrals representing the Helmholtz surface potentials. Thus the existence of these singular surface integrals at points on the surface is immediately apparent. In some cases this method of parameterisation can also be utilised in determining the limiting behaviour of the surface potentials across the surface, and it forms the basis of the method of analytical approximations for the determination of surface and scattered fields which is developed in Chapter IV.

2.1 The scalar boundary value problem.

The Helmholtz surface potentials and the Helmholtz integral formulae enable one to obtain solutions of the time independent scalar wave equations with suitable boundary conditions. Let $D \subset \mathbb{R}^3$ be a bounded region; then the homogeneous time dependent wave equation is given by

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad (2.1.1)$$

where u is a complex valued function of a point P in the interior (D^i) or exterior (D^e) of D and t is the time, and the real scalar c is the velocity of propagation of the wave. Assume that the point P has Cartesian coordinates (x_1, x_2, x_3) with respect to a suitable origin O; then in (2.1.1) ∇^2 is the Laplace operator given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = \partial_i \partial_j u \qquad (2.1.2)$$

where $\partial_i = \frac{\partial}{\partial x_i}$ and the summation convention applies.

The time independent form of (2.1.1) is obtained by assuming that u has harmonic time dependence, i.e.

$$u(\mathbf{r},t) = \phi(\mathbf{r})e^{-i\omega t} \qquad (2.1.3)$$

where the function ϕ is explicitly independent of the time t. In (2.1.3) $\mathbf{r} = \mathbf{OP}$ denotes the position vector of the point P with respect to the origin O and here and in the sequel we identify the position vector $\mathbf{r}$ of a point P with the coordinates (x_1, x_2, x_3) of the point. If (2.1.3) is substituted into (2.1.1) we obtain

$$\nabla^2 \phi + k^2 \phi = 0 \qquad (2.1.4)$$

which is known as Helmholtz's equation. In (2.1.4) $k = \frac{\omega}{c}$ is the wave number.

A unique solution of (2.1.4) in D^i or D^e can be obtained only if ϕ satisfies suitable boundary conditions on ∂D, the boundary of D. In this work we will consider only two types of boundary value problems, namely the homogeneous Dirichlet boundary value problem

$$\left.\begin{aligned} \nabla^2 \phi + k^2 \phi &= 0 \quad \text{on } D^i \text{ (or } D^e\text{)} \\ \phi &= 0 \quad \text{on } \partial D \end{aligned}\right\} \qquad (2.1.5)$$

and the homogeneous Neumann boundary value problem

$$\left.\begin{aligned} \nabla^2 \phi + k^2 \phi &= 0 \quad \text{on } D^i \text{ (or } D^e\text{)} \\ \frac{\partial \phi}{\partial n} &= 0 \quad \text{on } \partial D \end{aligned}\right\} \qquad (2.1.6)$$

Here $\mathbf{n}$ denotes the unit positive normal to ∂D. In both these boundary value problems we will make use of the free space Green's function G_k for the Helmholtz equation (2.1.4). If $\mathbf{r}$ and $\mathbf{r}'$ are any two points of $\mathbb{R}^3$ then G_k is a symmetric function of $\mathbf{r}$ and $\mathbf{r}'$ and it satisfies the equations

$$\left.\begin{aligned} \nabla^2 G_k(\mathbf{r},\mathbf{r}') + k^2 G_k(\mathbf{r},\mathbf{r}') &= 0 \\ \nabla'^2 G_k(\mathbf{r},\mathbf{r}') + k^2 G_k(\mathbf{r},\mathbf{r}') &= 0 \end{aligned}\right\} \quad \mathbf{r} \neq \mathbf{r}' \qquad (2.1.7)$$

where $\nabla'^2 = \partial_i' \partial_j'$ and $\partial_i' = \frac{\partial}{\partial x_i'}$. A suitable form of this Green's function is

given by
$$G_k(\mathbf{r},\mathbf{r}') = \frac{e^{ikR}}{4\pi R} \quad , \ \mathbf{r} \neq \mathbf{r}' \tag{2.1.8}$$

where $R = \|\mathbf{r} - \mathbf{r}'\|$.

The solution of these boundary value problems in the form of the well-known Helmholtz integral formulae consists of surface integrals over ∂D which represent the surface of a scattering body. These surface integrals are of two basic types, namely the Helmholtz single and double layer potentials. Prior to the derivations of the Helmholtz integral formulae in sections (2.5) and (2.6) we briefly discuss the types of surfaces on which the surface potentials are defined. The properties of the surface potentials are derived in sections (2.3) to (2.4).

2.2 Helmholtz surface potentials; types of surfaces.

Let D be a closed bounded region of $\mathbb{R}^3$ (this means that D is the closure of a bounded open connected subset of $\mathbb{R}^3$), then the boundary of D is a closed surface $S = \partial D$. Let ϕ and ψ be functions defined on S. Then the ***Helmholtz single layer potential*** Φ is defined on $\mathbb{R}^3$ by

$$\Phi(\mathbf{r}) = \iint_S G_k(\mathbf{r},\mathbf{r}')\phi(\mathbf{r}')d\sigma' \tag{2.2.1}$$

and the ***Helmholtz double layer potential*** Ψ is defined on $\mathbb{R}^3$ by

$$\Psi(\mathbf{r}) = \iint_S \frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} \psi(\mathbf{r}')d\sigma' \ . \tag{2.2.2}$$

In (2.2.1) and (2.2.2) $\mathbf{r} = (x_1, x_2, x_3)$ denotes a field point, $\mathbf{r}' = (x_1', x_2', x_3')$ a source point on the boundary surface S and $d\sigma'$ an element of surface area, the prime indicating that integration is with respect to the primed variables. Also $\mathbf{n}' = \mathbf{n}(\mathbf{r}')$ denotes the unit outward normal to S at $\mathbf{r}' \in S$. Moreover, the functions ϕ and ψ are referred to as the single and double layer densities respectively. According to (2.1.8)

$$\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} = (ikR - 1)e^{ikR} \frac{\mathbf{R}.\mathbf{n}(\mathbf{r}')}{4\pi \mathbf{R}^3} \quad , \tag{2.2.3}$$

and we see that the surface integrals (2.2.1) and (2.2.2) are singular when $\mathbf{r} \in \partial D$.

For later applications we need information on the existence and continuity properties of (2.2.1) and (2.2.2), as well as the existence and continuity properties of their normal derivatives. These questions will be investigated for the case where S is not necessarily a closed surface. First of all the nature of the surface S and its boundary curve must be specified.

A point set $C \subset \mathbb{R}^3$ is a ***regular arc*** iff there is a continuously differentiable parametric representation

$$\mathbf{r} = \mathbf{f}(t) \quad , \quad a \le t \le b,$$

of C such that the derivative of $\mathbf{f}(t)$ is not zero at all points $t \in [a,b]$. Here $\mathbf{f} = (f_1, f_2, f_3)$ is a triplet of continuously differentiable real valued functions f_i on $[a,b]$, and the condition $\mathbf{f}'(t) \neq 0$ means that at least one of $f_1(t), f_2(t)$ or $f_3(t)$ is not zero at each point $t \in [a,b]$.

A ***regular curve*** C in $\mathbb{R}^3$ is a finite chain of regular arcs $C_1, \ldots, C_n$ in $\mathbb{R}^3$ such that the terminal point of each arc (other than the last) is the initial point of the contiguous arc. The regular arcs have no other points in common, i.e. the regular curve does not intersect itself.

A ***closed regular curve*** is a regular curve of which the terminal point of the last arc is the initial point of the first arc.

Analogous definitions are made for regular arcs and curves in $\mathbb{R}^2$. Thus we define a ***regular region of*** $\mathbb{R}^2$ as the closure of a bounded region of $\mathbb{R}^2$ whose boundary is a closed regular curve of $\mathbb{R}^2$. Here a region in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is an open connected subset.

A point set $S \subset \mathbb{R}^3$ is a ***regular surface element*** if there is a continuously differentiable parametric representation

$$\mathbf{r} = \mathbf{f}(u_1, u_2) \quad , \quad (u_1, u_2) \in D \tag{2.2.4}$$

of S, where D is a regular region of $\mathbb{R}^2$, and if the parametric representation is such that it admits of a unique normal vector at each point of S, i.e. the vector $\frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2}$ is never zero. The unique ***unit normal vector*** is then defined by

$$\mathbf{n} = \mathbf{n}(\mathbf{r}) = \frac{\frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2}}{\left\| \frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2} \right\|} \tag{2.2.5}$$

A ***standard representation*** of a regular surface element S is given by

$$\left.\begin{aligned} x_1 &= u_1 \\ x_2 &= u_2 \\ x_3 &= f(u_1, u_2) \end{aligned}\right\} \quad (u_1, u_2) \in D \tag{2.2.6}$$

where f is a continuously differentiable function on the regular region $D \subset \mathbb{R}^2$.

A standard representation is usually constructed in such a way that the x_1,x_2-plane is tangential to an interior point of the regular surface element. Then (2.2.6) can be written as

$$x_3 = f(x_1,x_2) \ , \ (x_1,x_2) \in D$$

where we have identified the regular region $D \subset \mathbb{R}^2$ with the Cartesian product $D \times \{0\} \subset \mathbb{R}^3$.

The ***boundary*** of a regular surface element S is defined as the image of the boundary of the regular parameter region $D \subset \mathbb{R}^2$. It is shown by Kellogg (24) that the boundary of a regular surface element S is a regular curve C, and that a regular surface element S can be covered by a finite set of regular surface elements Σ each of which admits of a standard representation.

The regular closed curve bounding a regular surface element may consist of a finite number of regular arcs; each of these arcs is called an ***edge*** of the regular surface element. A ***vertex*** is a point at which two or more edges meet, and there can only be a finite number of vertices.

A ***regular surface*** S consists of a finite number of regular surface elements S_α , $\alpha = 1,\ldots,n$, related as follows:

(a) two of the regular surface elements may have in common either a single point, which is a vertex of both, or a single regular arc, which is an edge of both, but no other points;

(b) three or more of the regular surface elements may have, at most, vertices in common;

(c) any two of the regular surface elements are the first and last in a chain, such that each has an edge in common with the next, and

(d) all the regular surface elements having a vertex in common form a chain such that each has an edge, terminating in that vertex, in common with the next; the last may, or may not, have an edge in common with the first.

A regular surface is ***closed*** iff all the edges of the regular surface belong to each of two of the finite number of regular surface elements compounding the regular surface. A closed regular surface is ***smooth*** iff it has a unique normal vector at each of its points.

A ***regular region*** D in $\mathbb{R}^3$ is a region bounded by a regular surface $S = \partial D$. For such regions the divergence theorem holds (Kellogg (24)):

Theorem(2.2.1) Suppose that **A** is a vector valued function defined on a regular region $D \subset \mathbb{R}^3$, and that **A** is continuous and continuously differentiable on D. If **n** is the unit outward normal to $S = \partial D$, then

$$\iiint_D \operatorname{div}(\mathbf{A})\, d\tau = \iint_{\partial D} \mathbf{A}.\mathbf{n}\, d\sigma \tag{2.2.8}$$

where $\operatorname{div}(\mathbf{A}) = \nabla\cdot\mathbf{A} = \partial_i A_i$, and $\partial_i A_i = \dfrac{\partial A_i}{\partial x_i}$, the x_i's being the coordinates of a point in D.

From the divergence theorem one obtains Green's identities.

Theorem(2.2.2) Let D be a regular region contained in $\mathbb{R}^3$ and let ϕ and ψ be a continuously differentiable functions defined on D. Then Green's first identity states that

$$\iiint_D \phi\, \nabla^2\psi\, d\tau = - \iiint_D \nabla\phi.\nabla\psi\, d\tau + \iint_{\partial D} \phi\, \frac{\partial\psi}{\partial n}\, d\sigma \tag{2.2.9}$$

and Green's second identity states that

$$\iiint_D \left(\phi\, \nabla^2\psi - \psi\, \nabla^2\phi \right) d\tau = \iint_{\partial D} \left(\phi\, \frac{\partial\psi}{\partial n} - \psi\, \frac{\partial\phi}{\partial n} \right) d\sigma \tag{2.2.10}$$

Here $\frac{\partial\phi}{\partial n} = \mathbf{n}.\nabla\phi$ is the derivative of ϕ in the direction of the outward normal **n**.

We conclude this section by briefly discussing the paremeterisation of surface integrals in terms of a tangent-normal system of axes. Let an arbitrary point O be the origin of an orthogonal coordinate system with axes x_1, x_2 and x_3, and let $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ be the corresponding orthonormal basis vectors. All coordinate systems are assumed to be right-handed. An arbitrary point P with coordinates (x_1, x_2, x_3) has a position vector OP denoted by **r** , so that

$$\mathbf{r} = x_i \mathbf{e}_i \; .$$

Points on S will in general be denoted by P' with coordinates (x_1', x_2', x_3') and position vector

$$\mathbf{r}' = x_i'\, \mathbf{e}_i$$

relative to the origin O.

We assume that S is a regular surface element having a standard representation in terms of a tangent-normal system of axes with origin at some convenient point $P_0 \in S$. Let $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ be the orthonormal basis vectors for the tangent-normal axes ξ_1, ξ_2, ξ_3. Here $\mathbf{f}_3 = \mathbf{n}_0 = \mathbf{n}(\mathbf{r}_0)$, the unit positive normal to S at P_0. If the surface S is closed, the unit positive normal is directed outward of S. In other cases we adopt the following standard convention. An arbitrary positive orientation is assigned to the boundary curve $C = \partial S$ of S. The positive direction on the normal at P_0 is then determined by the direction in which a right-handed screw advances when rotated in the positive direction of the boundary of S.

Let D be the projection of S onto the tangent plane and let the coordinates of a point $P' \in S$ be (ξ_1', ξ_2', ξ_3') relative to the tangent-normal axes at P_0. Then S has a standard representation of the form

$$\xi_3' = f(\xi_1', \xi_2') \ , \ (\xi_1', \xi_2') \in D,$$

the function f being continuously differentiable on D. If the coordinates of P relative to the tangent-normal axes are denoted by (ξ_1, ξ_2, ξ_3), then for the vector $\mathbf{R} = \mathbf{PP'}$ we have

$$\mathbf{R} = \mathbf{r}' - \mathbf{r} = (x_i' - x_i)\mathbf{e}_i = (\xi_i' - \xi_i)\mathbf{f}_i \ .$$

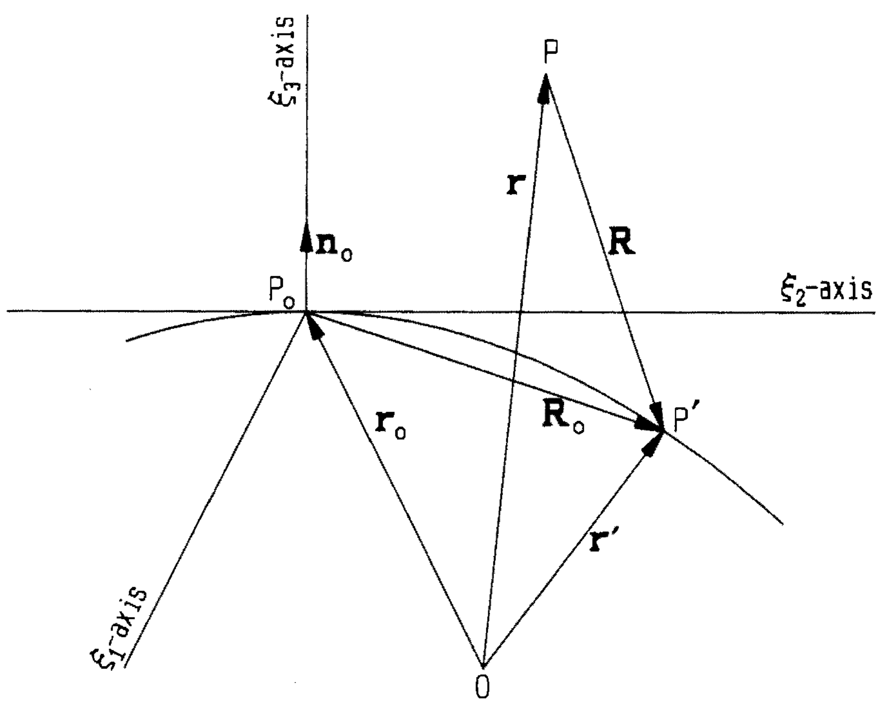


Fig. (2.2.1).

If the direction cosines of the $P_0\xi_1\xi_2\xi_3$ axes with respect to the $Ox_1x_2x_3$ axes are a_{ij} ,i.e., $a_{ij} = \mathbf{e}_i.\mathbf{f}_j$, then

$$x_i = x_{0i} + a_{ij}\xi_j ,$$

$$x_i' = x_{0i} + a_{ij}\xi_j'$$

and

$$x_i' - x_i = a_{ij}(\xi_j' - \xi_j) .$$

Here (x_{01}, x_{02}, x_{03}) are the coordinates of P_0 with respect to $Ox_1x_2x_3$.

The Helmholtz potenials defined by (2.2.1) and (2.2.2) and their normal derivatives are of the basic form

$$H(\mathbf{r}) = \iint_S \frac{h(\mathbf{r}')}{R^\alpha}\, d\sigma' \qquad (2.2.11)$$

where $R = \|\mathbf{r}' - \mathbf{r}\|$, h is a suitably well behaved function defined on S and $\alpha > 0$. If this integral is parameterised in terms of the plane region D we obtain

$$H(\mathbf{r}) = \iint_D \frac{h(\mathbf{r}')}{R^\alpha} \left\| \mathbf{r}'_{\xi_1'} \times \mathbf{r}'_{\xi_2'} \right\| d\xi_1'\, d\xi_2' \qquad (2.2.12)$$

where

$$\mathbf{r}' = (x_{0i} + a_{ij}\,\xi_j')\mathbf{f}_i ,$$

$$R = \left((\xi_i' - \xi_i)(\xi_i' - \xi_i)\right)^{1/2} ,$$

and

$$\left\| \mathbf{r}'_{\xi_1'} \times \mathbf{r}_{\xi_2'} \right\| = \left\| \frac{\partial \mathbf{r}'}{\partial \xi_1'} \times \frac{\partial \mathbf{r}'}{\partial \xi_2'} \right\| .$$

If $\mathbf{R}_0 = \mathbf{P}_0\mathbf{P}'$ then $\mathbf{r}' = \mathbf{r}_0 + \mathbf{R}_0$ where $\mathbf{r}_0 = x_{0i}\mathbf{e}_i$ is independent of ξ_1', ξ_2' and ξ_3', and it follows that

$$\frac{\partial \mathbf{r}'}{\partial \xi_1'} \times \frac{\partial \mathbf{r}'}{\partial \xi_2'} = \frac{\partial \mathbf{R}_0}{\partial \xi_1'} \times \frac{\partial \mathbf{R}_0}{\partial \xi_2'} ,$$

$$= \varepsilon_{ijk}\mathbf{f}_i \frac{\partial \xi_j'}{\partial \xi_1'} \frac{\partial \xi_k'}{\partial \xi_2'}$$

$$= -\frac{\partial \xi_3'}{\partial \xi_1'}\mathbf{f}_1 - \frac{\partial \xi_3'}{\partial \xi_2'}\mathbf{f}_2 + \mathbf{f}_3 .$$

Hence

$$\left\| \mathbf{r}'_{\xi_1'} \times \mathbf{r}'_{\xi_2'} \right\| = \left[\left(1 + \left(\frac{\partial \xi_3'}{\partial \xi_1'}\right)^2 + \left(\frac{\partial \xi_3'}{\partial \xi_2'}\right)^2\right]^{1/2} . \qquad (2.2.13)$$

If $\gamma' = \gamma(\mathbf{r}')$ is the angle between the unit normal vectors $\mathbf{n}_0 = \mathbf{n}(\mathbf{r}_0)$ and $\mathbf{n}' = \mathbf{n}(\mathbf{r}')$, then

$$\cos\gamma' = \mathbf{n}_0 . \mathbf{n}' = \frac{\mathbf{f}_3 . \left(\mathbf{r}'_{\xi'_1} \times \mathbf{r}'_{\xi'_2} \right)}{\| \mathbf{r}'_{\xi'_1} \times \mathbf{r}'_{\xi'_2} \|} = \left[\left(1 + \left(\frac{\partial \xi'_3}{\partial \xi'_1} \right)^2 + \left(\frac{\partial \xi'_3}{\partial \xi'_2} \right)^2 \right] \right]^{-1/2} ,$$

and we therefore have

$$\left[\left(1 + \left(\frac{\partial \xi'_3}{\partial \xi'_1} \right)^2 + \left(\frac{\partial \xi'_3}{\partial \xi'_2} \right)^2 \right] \right]^{1/2} = \sec\gamma' . \qquad (2.2.14)$$

From (2.2.11),(2.2.12) and (2.2.13) it follows that

$$H(r) = \iint_D \frac{h(r)}{R^{\alpha}} \sec\gamma' \, d\xi'_1 \, d\xi'_2 \quad . \qquad (2.2.15)$$

In section (2.4) where the existence and continuity properties of the Helmholtz surface potentials are determined , frequent use is made of this general form whenever tangent-normal local coordinate systems are required.

2.3 Paremeterisation of surface integrals in terms of surface spherical coordinates.

Let S be a regular surface element having a natural representation

$$\xi'_3 = f(\xi'_1, \xi'_2), \quad (\xi'_1, \xi'_2) \in D \qquad (2.3.1)$$

with respect to tangent-normal axes at a point $P_0 \in S$. It is assumed that the two-dimensional region D lies in the tangent plane, and that it is so small that any straight line itersects the surface in at most two distinct points. At the point P_0 we introduce azimuth and altitude coordinates φ and χ respectively. Here φ is measured from an arbitrary tangent to the surface at P_0 , and χ is measured positive below the tangent plane and negative above the tangent plane. Thus χ is positive (negative) when ξ_3 is negative (positive).

As in the previous section , $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ denotes an orthonormal triad determined by the surface at P_0 , with $\mathbf{f}_3 = \mathbf{n}_0$, the unit positive normal to S at P_0 . For each pair (φ, χ) there is a unique unit vector

$$\boldsymbol{\eta} = \eta_i \mathbf{f}_i \qquad (2.3.2)$$

determined by

$$\left. \begin{aligned} \eta_1 &= \cos\varphi \ \cos\chi \\ \eta_2 &= \sin\varphi \ \cos\chi \\ \eta_3 &= - \sin\chi \end{aligned} \right\} \qquad (2.3.3)$$

where $0 \le \varphi \le 2\pi$ and $-\frac{\pi}{2} \le \chi \le \frac{\pi}{2}$.

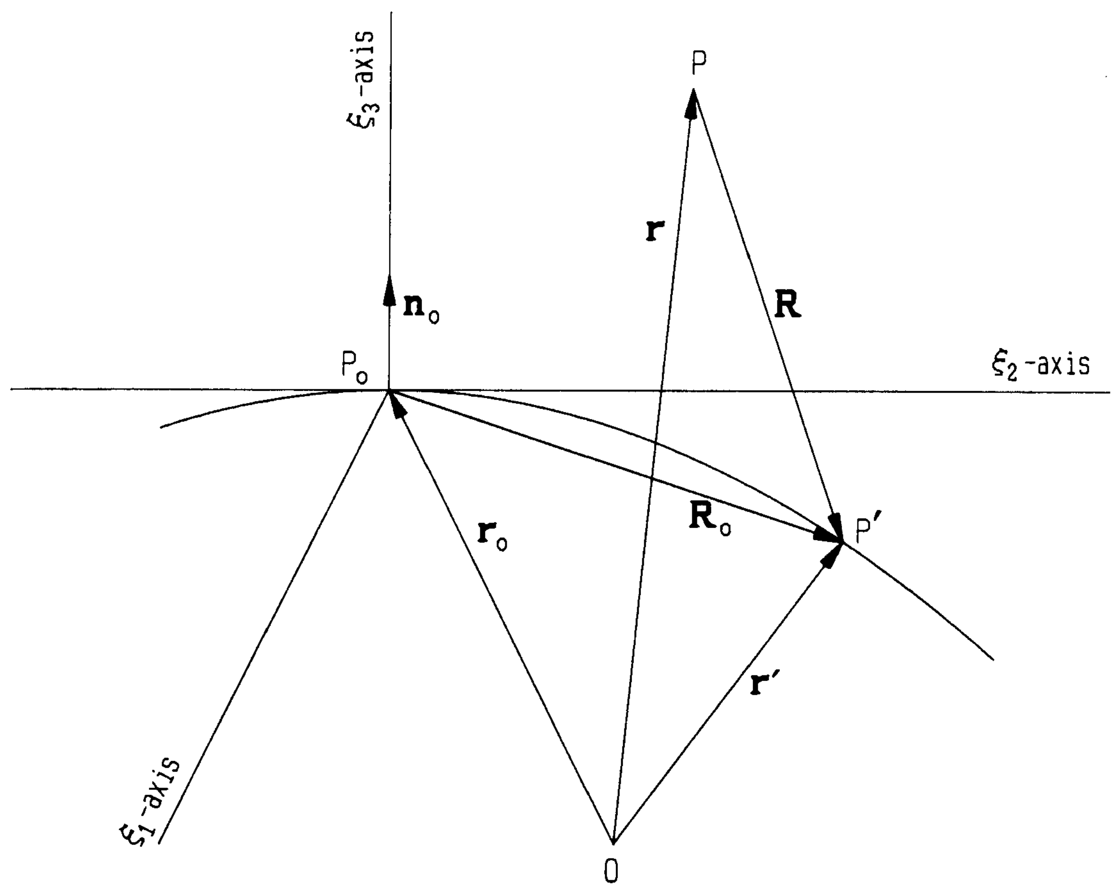


Fig. (2.3.1).

Let P′ be a point of S with rectangular coordinates (ξ_1', ξ_2', ξ_3') with respect to the tangent-normal axes at P_0 and azimuth-altitude coordinates (φ', χ') with respect to these axes. If we let

$$\mathbf{R}_0 = \mathbf{P}_0\mathbf{P}' \tag{2.3.4}$$

then

$$\mathbf{R}_0 = R_0 \boldsymbol{\eta}'$$

or

$$\xi_i' = R_0 \eta_i \;, \quad i = 1,2,3 \tag{2.3.5}$$

where

$$R_0 = \| \mathbf{R}_0 \| \tag{2.3.6}$$

Using (2.3.1) and (2.3.5) when $i = 3$ yields

$$R_0 \sin\chi' = - f(R_0 \eta_1' \;, R_0 \eta_2') \tag{2.3.8}$$

Clearly $R_0 = 0$ is a solution of this equation, since $f(0,0) = 0$. We now require that every straight line through P_0 intersects S in at most one other point. This means that we can solve for a unique R_0 from (2.3.8) as a function of φ' and χ' in some neighbourhood of P_0 , i.e.

$$R_0 = R_0(\varphi',\chi') \tag{2.3.9}$$

where (φ',χ') lies in a subset of $[0,2\pi]\times[0,\pi/2]$. If S is a convex surface element it will have a unique representation of the form (2.3.9). For non-convex surface elements it is necessary to partition S into non-overlapping regular surface elements S_α ,α = 1,2,...,n, on each of which (2.3.8) will have a unique solution of the form (2.3.9). It is to be noted that such a partitioning will depend on the point $\mathbf{r}_0 \in S$. This could lead to considerable complications, and here we will deal only with convex surface elements and convex closed surfaces.

Using (2.3.8) and (2.3.9) we can determine the partial derivatives of R_0 with respect to φ' and χ'. Since $R_0 \sin\chi' = -\xi_3$,

$$\frac{\partial R_0}{\partial\varphi'}\sin\chi' = -\frac{\partial\xi_3'}{\partial\xi_1'}\frac{\partial\xi_1'}{\partial\varphi'} - \frac{\partial\xi_3'}{\partial\xi_2'}\frac{\partial\xi_2'}{\partial\varphi'}$$

and
$$\frac{\partial R_0}{\partial\chi'}\sin\chi' - R_0\cos\chi' = -\frac{\partial\xi_3'}{\partial\xi_1'}\frac{\partial\xi_1'}{\partial\chi'} - \frac{\partial\xi_3'}{\partial\xi_2'}\frac{\partial\xi_2'}{\partial\chi'} .$$

If we write $\partial_1'\xi_3' = \dfrac{\partial\xi_3'}{\partial\xi_1'}$ and $\partial_2'\xi_3' = \dfrac{\partial\xi_3'}{\partial\xi_2'}$, we obtain from (2.3.5) and (2.3.3),

$$\frac{\partial R_0}{\partial\varphi'} = \frac{R_0\left(\partial_1'\xi_3'\sin\varphi'\cos\chi' - \partial_2'\xi_3'\cos\varphi'\cos\chi'\right)}{\sin\chi' + \partial_1'\xi_3'\cos\varphi'\cos\chi' + \partial_2'\xi_3'\sin\varphi'\cos\chi'} \tag{2.3.10}$$

and
$$\frac{\partial R_0}{\partial\chi'} = \frac{R_0\left(\partial_1'\xi_3'\cos\varphi'\sin\chi' + \partial_2'\xi_3'\sin\varphi'\sin\chi' - \cos\chi'\right)}{\sin\chi' + \partial_1'\xi_3'\cos\varphi'\cos\chi' + \partial_2'\xi_3'\sin\varphi'\cos\chi'} \tag{2.3.11}$$

We define ω_1 and ω_2 by

$$\omega_1(\varphi',\chi') = \partial_1'\xi_3'\cos\varphi' + \partial_2'\xi_3'\sin\varphi' \tag{2.3.12}$$

and

$$\omega_2(\varphi',\chi') = \partial_1'\xi_3'\sin\varphi' - \partial_2'\xi_3'\cos\varphi'. \tag{2.3.13}$$

Then (2.3.10) and (2.3.11) can be written as

$$\frac{\partial R_0}{\partial\varphi'} = \frac{R_0\omega_2\cos\chi'}{\omega_1\cos\chi' + \sin\chi'} \tag{2.3.14}$$

and
$$\frac{\partial R_0}{\partial \chi'} = \frac{R_0(\omega_1 \sin\chi' - \cos\chi')}{\omega_1 \cos\chi' + \sin\chi'} \qquad (2.3.15)$$

These derivatives will be finite unless
$$\omega_1 \cos\chi' + \sin\chi' = 0,$$
or
$$\omega_1 = -\tan\chi' .$$
From (2.3.1) we see that ω_1 is the derivative of ξ_3' in the direction φ' with respect to the positive ξ_1-axis. Let $\ell(\varphi')$ be the straight line through P_0 and lying in the tangent plane and making an angle φ' with the postive ξ_1-axis. Distances measured from P_0 along $\ell(\varphi')$ are denoted by s, and we call $\ell(\varphi')$ the s-axis. The plane formed by this line and the ξ_3-axis intersects the surface S along a plane normal curve. The equation of this curve in the s,ξ_3-plane is
$$\xi_3' = f(s \cos\varphi', s \sin\varphi') ,$$
φ' being fixed; then
$$\frac{d\xi_3'}{ds} = \frac{\partial \xi_3'}{\partial \xi_1'} \cos\varphi' + \frac{\partial \xi_3'}{\partial \xi_2'} \sin\varphi' = \omega_1 .$$

Let τ be the angle the tangent to this curve makes with the positive direction of the s-axis. Then
$$\frac{d\xi_3'}{ds} = -\tan\tau \qquad (\xi_3' < 0) .$$
We therefore have $\omega_1 = -\tan\tau$. If follows that $\tan\tau = \tan\chi'$, and hence $\tau = \chi'$ or $\tau = \chi' + \pi$. Thus the condition that $\omega_1 \cos\varphi' + \sin\chi'$ is never zero is equivalent to the condition that no tangent (other than those at P_0) passes through P_0. Clearly a convex surface always satisfies this condition.

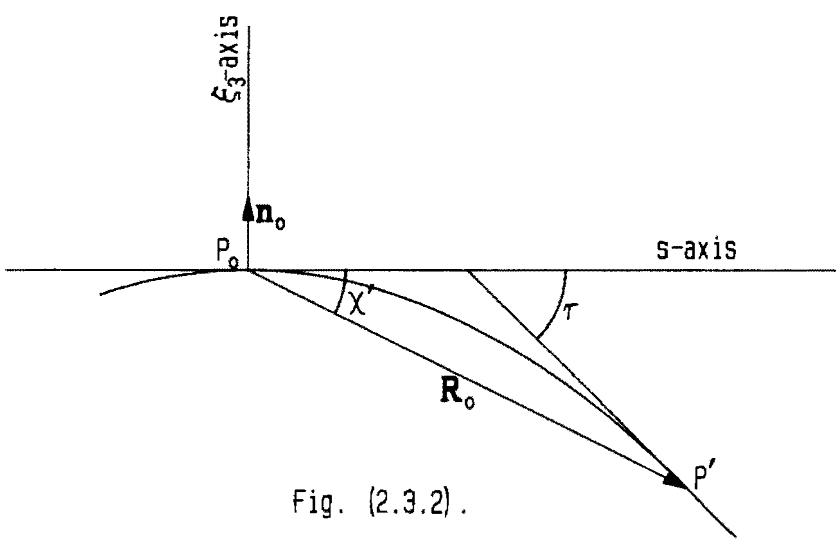


Fig. (2.3.2).

We can now consider the mapping $T:[0,2\pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to D$ defined by

$$\left.\begin{aligned} \xi_1' &= R_0 \cos\varphi' \cos\chi' \\ \xi_2' &= R_0 \sin\varphi' \cos\chi' \end{aligned}\right\} \qquad (2.3.16)$$

where R_0 is given by (2.3.9). Using (2.3.14) and (2.3.15) we find that the Jacobian J of the mapping T is given by

$$J = R_0^2 \cos\chi' \sin\chi' - \frac{\partial R_0}{\partial \chi'} \cos^2\chi'$$

or

$$J = \frac{R_0^2 \cos\chi'}{\omega_1 \cos\chi' + \sin\chi'} \qquad (2.3.17)$$

For a convex surface J will always be finite. We recall that $R_0 = 0$ if and only if $\chi' = 0$. It follows that $J = 0$ when $\chi' = 0$ or $\chi' = \mp \pi/2$. For a convex surface element ξ_3' is always negative and $\chi' \in [0,\pi/2]$. Thus if $\chi' = 0$ or $\chi' = \pi/2$, the mapping T is restricted to the sets $[0,2\pi]\times\{0\}$ and $[0,2\pi]\times\{\pi/2\}$. As these two sets have two dimensional measure zero, we may parameterise surface integrals over S with respect to an arbitrary point $P_0 \in S$ and the azimuth-altitude coordinate system associated with the point P_0. Suppose now S is a convex surface element bounded by a curve $C = \partial S$. We will asume that the boundary C is such that it can be represented by an equation of the form

$$\chi' = \chi'(\varphi') \ , \quad 0 \le \varphi' \le 2\pi \qquad (2.3.18)$$

The various Helmholtz potentials and their normal derivatives differ significantly in their continuity properties, a fact which is concealed by the general form (2.2.11). In order to display these differences we introduce the following three types of the form (2.2.11):

$$H_1(\mathbf{r}) = \iint_S \frac{h(\mathbf{r},\mathbf{r}')}{R} \, d\sigma' \qquad (2.3.19)$$

$$H_2(\mathbf{r}) = \iint_S \frac{\mathbf{R}.\mathbf{n}'}{R^3} \, h(\mathbf{r},\mathbf{r}') \, d\sigma' \qquad (2.3.20)$$

$$H_3(\mathbf{r}) = \iint_S \frac{\mathbf{R}.\mathbf{n}_0}{R^3} \, h(\mathbf{r},\mathbf{r}') \, d\sigma' \qquad (2.3.21)$$

The functions H_1 and H_2 obviously derive from respectively the single and double layer potentials, whereas H_3 is a surface integral of the form of the normal derivative of the single layer potential.

In terms of azimuth-altitude coordinates (φ',χ') we have

$$H_1(\mathbf{r}) = \int_0^{2\pi}\int_0^{\chi'(\varphi')} \frac{h(\mathbf{r},\mathbf{r}_0+\mathbf{R}_0)}{R} \left\| \mathbf{r}'_{\varphi'}\times \mathbf{r}'_{\chi'}\right\| d\chi'\ d\varphi' \qquad (2.3.22)$$

$$H_2(\mathbf{r}) = \int_0^{2\pi}\int_0^{\chi'(\varphi')} \frac{\mathbf{R}.\left(\mathbf{r}'_{\varphi'}\times \mathbf{r}'_{\chi'}\right)}{R^3} h(\mathbf{r},\mathbf{r}_0+\mathbf{R}_0)\ d\chi'\ d\varphi' \qquad (2.3.23)$$

$$H_3(\mathbf{r}) = \int_0^{2\pi}\int_0^{\chi'(\varphi')} \frac{\mathbf{R}.\mathbf{n}_0}{R^3} h(\mathbf{r},\mathbf{r}_0+\mathbf{R}_0)\left\| \mathbf{r}'_{\varphi'}\times \mathbf{r}'_{\chi'}\right\| d\chi'\ d\varphi' \qquad (2.3.24)$$

Here we have used $\mathbf{r}' = \mathbf{r}_0 + \mathbf{R}_0$. Since P_0 is a fixed point of S, it follows that

$$\mathbf{r}'_{\varphi'}\times \mathbf{r}'_{\chi'} = \mathbf{R}_{0\varphi'} \times \mathbf{R}_{0\chi'} \ .$$

But
$$\mathbf{R}_0 = \xi'_i\ \mathbf{f}_i = R_0\ \eta'_i\ \mathbf{f}_i \ ,$$

and so
$$\begin{aligned}\mathbf{r}'_{\varphi'}\times \mathbf{r}'_{\chi'} &= \varepsilon_{ijk}\mathbf{f}_i\left(\frac{\partial R_0}{\partial\varphi'}\eta'_j + R_0\frac{\partial\eta'_j}{\partial\varphi'}\right)\left(\frac{\partial R_0}{\partial\chi'}\eta'_k + R_0\frac{\partial\eta_k}{\partial\chi'}\right)\\
&= \frac{\partial R_0}{\partial\varphi'}\frac{\partial R_0}{\partial\chi'}\ \varepsilon_{ijk}\ \mathbf{f}_i\ \eta'_j\ \eta'_k\\
&+ R_0\frac{\partial R_0}{\partial\varphi'}\ \varepsilon_{ijk}\ \mathbf{f}_i\ \eta_j\ \frac{\partial\eta_k}{\partial\chi'}\\
&+ R_0\frac{\partial R_0}{\partial\chi'}\ \varepsilon_{ijk}\ \mathbf{f}_i\ \frac{\partial\eta_j}{\partial\varphi'}\ \eta_k\\
&+ R_0^2\ \varepsilon_{ijk}\ \mathbf{f}_i\ \frac{\partial\eta_j}{\partial\varphi'}\ \frac{\partial\eta_k}{\partial\chi'}\end{aligned}$$

The first term on the right is zero, and using (2.3.3) for $P' \in S$ yields

$$\begin{aligned}\mathbf{r}'_{\varphi'}\times \mathbf{r}'_{\chi'} &= \left(-R_0\frac{\partial R_0}{\partial\varphi'}\sin\varphi' - R_0\frac{\partial R_0}{\partial\chi'}\cos\varphi'\cos\chi'\sin\chi' - R_0^2\cos\varphi'\sin\chi'\right)\mathbf{f}_1\\
&+ \left(R_0\frac{\partial R_0}{\partial\varphi'}\cos\varphi'\cos\chi' - R_0\frac{\partial R_0}{\partial\chi'}\sin\varphi'\cos\chi'\sin\chi' - R_0^2\sin\varphi'\cos\chi'\right)\mathbf{f}_2\\
&+\left(-R_0\frac{\partial R_0}{\partial\chi'}\cos\chi' + R_0^2\cos\chi'\sin\chi'\right)\mathbf{f}_3 \ . \qquad (2.3.25)\end{aligned}$$

We now find that

$$\left\| \mathbf{r}'_{\varphi'} \times \mathbf{r}'_{\chi'} \right\| = R_0 \left[\left(\frac{\partial R_0}{\partial \varphi'}\right)^2 + \left(\frac{\partial R_0}{\partial \chi'}\right)^2 \cos^2\chi' + R_0^2 \cos^2\chi' \right]^{1/2}, \tag{2.3.26}$$

and on substituting for $\frac{\partial R_0}{\partial \varphi'}$ and $\frac{\partial R_0}{\partial \chi'}$ from (2.3.14) and (2.3.15) into (2.3.26) gives

$$\left\| \mathbf{r}'_{\varphi'} \times \mathbf{r}'_{\chi'} \right\| = \frac{\omega\, R_0^2 \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|}, \tag{2.3.27}$$

where ω_1 and ω_2 are given by (2.3.12) and (2.3.14) respectively, and

$$\omega = \left[1 + \omega_1^2 + \omega_2^2 \right]^{1/2}. \tag{2.3.28}$$

Thus for a convex surface, $\left\| \mathbf{r}'_{\varphi'} \times \mathbf{r}'_{\chi'} \right\|$ is always finite, and it is zero only at the singular points of the mapping T defined in (2.3.16). We can now substitute (2.3.27) into (2.3.22) and (2.3.24), and (2.3.25) can be substituted into (2.3.23). However, with later appliations in mind, we specialize for the particular case when the field point lies on the normal to S at P_0. The field point is now denoted by P_λ, where λ is a real parameter such that the position vector

$$\mathbf{OP}_\lambda = \mathbf{r}_\lambda = \mathbf{r}_0 + \lambda \mathbf{n}_0 .$$

If $\mathbf{R}_\lambda = \mathbf{P}_\lambda \mathbf{P}'$ then $\mathbf{R}_\lambda = \mathbf{r}' - \mathbf{r}_\lambda = \mathbf{R}_0 - \lambda \mathbf{n}_0$ and

$$\mathbf{R}_\lambda . \left(\mathbf{r}'_{\varphi'} \times \mathbf{r}'_{\chi'}\right) = \mathbf{R}_0 . \left(\mathbf{r}'_{\varphi'} \times \mathbf{r}'_{\chi'}\right) - \lambda\, \mathbf{n}_0 . \left(\mathbf{r}'_{\varphi'} \times \mathbf{r}'_{\chi'}\right)$$

$$= - R_0^3 \cos\chi' - \lambda R_0 \left(\frac{\partial R_0}{\partial \chi'} \cos\chi' - R_0 \sin\chi'\right)\cos\chi' \tag{2.3.29}$$

Using (2.3.25) in (2.3.29) gives

$$\mathbf{R}_\lambda . \left(\mathbf{r}'_{\varphi'} \times \mathbf{r}'_{\chi'}\right) = - R_0^3 \cos\chi' - \frac{\lambda\, R_0^2 \cos\chi'}{\omega_1 \cos\chi' + \sin\chi'} \tag{2.3.30}$$

Thus when $\mathbf{r} = \mathbf{r}_\lambda$ in (2.3.22)-2.3.24)

$$H_1(\mathbf{r}_\lambda) = \int_0^{2\pi} \int_0^{\chi'(\varphi')} \frac{h(\mathbf{r}_\lambda, \mathbf{r}_0 + \mathbf{R}_0)}{R_\lambda} \frac{\omega\, R_0^2 \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|}\, d\chi' d\varphi' \tag{2.3.31}$$

$$H_2(\mathbf{r}_\lambda) = - \int_0^{2\pi} \int_0^{\chi'(\varphi')} \frac{1}{R_\lambda^3} \left(R_0^3 + \frac{\lambda R_0^2}{\omega_1 \cos\chi' + \sin\chi'}\right) h(\mathbf{r}_\lambda, \mathbf{r}_0 + \mathbf{R}_0) \cos\chi'\, d\chi' d\varphi' \tag{2.3.32}$$

$$H_3(\mathbf{r}_\lambda) = -\int_0^{2\pi}\int_0^{\chi'(\varphi')} \frac{(R_0 \sin\chi' + \lambda)}{R_\lambda^3} \left[\frac{\omega\, R_0^2 \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|}\right] h(\mathbf{r}_0, \mathbf{r}_0 + \mathbf{R}_0)\, d\chi' d\varphi' \qquad (2.3.33)$$

In particular, if $\lambda = 0$ in (2.3.32)-(2.3.34), or, in other words, if (2.3.22)-(2.3.24) are parameterised when the field point P_λ is the point $P_0 \in S$, we obtain

$$H_1(\mathbf{r}_0) = \int_0^{2\pi}\int_0^{\chi'(\varphi')} h(\mathbf{r}_0, \mathbf{r}_0 + \mathbf{R}_0) \left[\frac{\omega\, R_0 \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|}\right] d\chi' d\varphi' \qquad (2.3.34)$$

$$H_2(\mathbf{r}_0) = -\int_0^{2\pi}\int_0^{\chi'(\varphi')} h(\mathbf{r}_0, \mathbf{r}_0 + \mathbf{R}_0) \cos\chi'\, d\chi' d\varphi' \qquad (2.3.35)$$

$$H_3(\mathbf{r}_0) = -\int_0^{2\pi}\int_0^{\chi'(\varphi')} h(\mathbf{r}_0, \mathbf{r}_0 + \mathbf{R}_0) \left[\frac{\omega \sin\chi' \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|}\right] d\chi' d\varphi' \qquad (2.3.36)$$

Thus if h is integrable over S then H_1, H_2 and H_3 exist on S. For later reference we note that if S in (2.3.19)-(2.3.21) is a closed convex surface then (2.3.31)-(2.3.36) become:

$$H_1(\mathbf{r}_\lambda) = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{h(\mathbf{r}_\lambda, \mathbf{r}_0 + \mathbf{R}_0)}{R_\lambda} \left[\frac{\omega\, R_0^2 \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|}\right] d\chi' d\varphi' \qquad (2.3.37)$$

$$H_2(\mathbf{r}_\lambda) = -\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{R_\lambda^3} \left[R_0^3 + \frac{\lambda R_0^2}{\omega_1 \cos\chi' + \sin\chi'}\right] h(\mathbf{r}_\lambda, \mathbf{r}_0 + \mathbf{R}_0) \cos\chi'\, d\chi'\, d\varphi' \qquad (2.3.38)$$

and

$$H_3(\mathbf{r}_\lambda) = -\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{(R_0 \sin\chi' + \lambda)}{R_\lambda^3} \left[\frac{\omega\, R_0^2 \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|}\right] h(\mathbf{r}_\lambda, \mathbf{r}_0 + \mathbf{R}_0)\, d\chi' d\varphi' \qquad (2.3.39)$$

$$H_1(\mathbf{r}_0) = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} h(\mathbf{r}_0, \mathbf{r}_0 + \mathbf{R}_0) \left[\frac{\omega\, R_0 \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|}\right] d\chi' d\varphi' \qquad (2.3.40)$$

$$H_2(\mathbf{r}_0) = -\int_0^{2\pi}\int_0^{\frac{\pi}{2}} h(\mathbf{r}_0, \mathbf{r}_0+\mathbf{R}_0)\cos\chi'\, d\chi'\, d\varphi' \qquad (2.3.41)$$

$$H_3(\mathbf{r}_0) = -\int_0^{2\pi}\int_0^{\frac{\pi}{2}} h(\mathbf{r}_0, \mathbf{r}_0+\mathbf{R}_0)\left(\frac{\omega\sin\chi'\cos\chi'}{|\omega_1\cos\chi'+\sin\chi'|}\right) d\chi'\, d\varphi' \qquad (2.3.42)$$

We conclude this section with a few useful inequalities. It will be assumed that the surface element S is of class C^2. Then the second order partial derivatives of f in (2.3.1) are bounded on D. If $(\xi_1', \xi_2') \in D$, and

$$\xi_3' = f(\xi_1', \xi_2')$$

then by Taylors theorem

$$\xi_3' = \frac{1}{2}\left(\frac{\partial^2 f(\xi_1^*,\xi_2^*)}{\partial\xi_1'^2}\xi_1'^2 + 2\frac{\partial^2 f(\xi_1^*,\xi_2^*)}{\partial\xi_1'\,\partial\xi_2'}\xi_1'\xi_2' + \frac{\partial^2 f(\xi_1^*,\xi_2^*)}{\partial\xi_2'^2}\xi_2'^2\right)$$

where $(\xi_1^*, \xi_2^*) \in D$. It follows that there is a real number $M > 0$ such that

$$|\xi_3'| \le M\,(\xi_1'^2 + \xi_2'^2) \qquad (2.3.43)$$

We denote by Q′ the projection of $P' \in S$ onto the tangent plane to S at P_0. Thus the coordinates of Q′ are $(\xi_1', \xi_2', 0)$ relative to the tangent-normal axes. The polar coordinates of Q′ are then (ρ_0, φ'), where

$$\rho_0^2 = \xi_1'^2 + \xi_2'^2 = R_0\cos^2\varphi'.$$

Thus we have

$$|\xi_3'| \le M\,\rho_0^2 = MR_0^2\cos^2\chi' \le MR_0^2 \qquad (2.3.43)$$

We now prove that the unit normal $\mathbf{n}'$ satisfies a Hölder condition at P_0. Since $\mathbf{n}_0 = (0,0,1)$ and

$$\mathbf{n}' = \frac{\left(-\partial_1'\xi_3', -\partial_2'\xi_3', -1\right)}{\left[1 + \left(\partial_1'\xi_3'\right)^2 + \left(\partial_2'\xi_3'\right)^2\right]^{1/2}},$$

$$\|\mathbf{n}_0 - \mathbf{n}'\|^2 = \frac{\left(\partial_1'\xi_3'\right)^2 + \left(\partial_2'\xi_3'\right)^2 + \left[\left(1+\left(\partial_1'\xi_3'\right)^2+\left(\partial_2'\xi_3'\right)^2\right)^{1/2} - 1\right]^2}{1 + \left(\partial_1'\xi_3'\right)^2 + \left(\partial_2'\xi_3'\right)^2}.$$

Hence

$$\|\mathbf{n}_0 - \mathbf{n}'\|^2 \le \left(\partial_1'\xi_3'\right)^2 + \left(\partial_2'\xi_3'\right)^2 + \left[\left(1 + \left(\partial_1'\xi_3'\right)^2 + \left(\partial_2'\xi_3'\right)^2\right)^{1/2} - 1\right]^2,$$

since $1 + \left(\partial_1'\xi_3'\right)^2 + \left(\partial_2'\xi_3'\right)^2 \ge 1$.

We apply Taylor's theorem to the partial derivatives $\frac{\partial\xi_3'}{\partial\xi_1'}$ and $\frac{\partial\xi_3'}{\partial\xi_2'}$. Thus

$$\frac{\partial \xi_3'}{\partial \xi_1'} = \frac{\partial f(0,0)}{\partial \xi_1'} + \xi_1' \frac{\partial^2 f(\xi_1^*,\xi_2^*)}{\partial \xi_1'^2} + \xi_2' \frac{\partial^2 f(\xi_1^*,\xi_2^*)}{\partial \xi_2' \partial \xi_1'}$$

where $(\xi_1^*,\xi_2^*) \in D$. Since $\frac{\partial f(0,0)}{\partial \xi_1'} = 0$, we have

$$\left| \frac{\partial \xi_3'}{\partial \xi_1'} \right| \le M(|\xi_1'| + |\xi_2'|) \le 2M\rho_0 \;.$$

Similarly
$$\left| \frac{\partial \xi_3'}{\partial \xi_2'} \right| \le 2M\rho_0 \;.$$

Hence $\left[1 + (\partial_1' \xi_3')^2 + (\partial_2' \xi_3')^2 \right]^{1/2} - 1 \le \left(1 + 8M\rho_0^2 \right)^{1/2} - 1 \le 2\sqrt{2}M\rho_0$

and therefore
$$\| \mathbf{n}_0 - \mathbf{n}' \|^2 \le 16M^2 \rho_0^2 \;,$$

or
$$\| \mathbf{n}_0 - \mathbf{n}' \| \le 4M\rho_0 \le 4MR_0 \;, \qquad (2.3.44)$$

which is the required Hölder condition.

Finally we prove that

$$|\mathbf{R}_0 . \mathbf{n}'| \le O(R_0^2).$$

Firstly we observe that

$$|\mathbf{R}_0 . \mathbf{n}_0| = |\xi_3'| \le MR_0^2 \;,$$

and secondly that

$$|\mathbf{R}_0 . (\mathbf{n}_0 - \mathbf{n}')| \le R_0 \; \| \mathbf{n}_0 - \mathbf{n}' \| \le 4MR_0^2 \;.$$

But as
$$|\mathbf{R}_0 . \mathbf{n}'| \le |\mathbf{R}_0 . (\mathbf{n}_0 - \mathbf{n}')| + |\mathbf{R}_0 . \mathbf{n}_0| \;,$$

it follows that

$$|\mathbf{R}_0 . \mathbf{n}'| \le 5MR_0^2 \;. \qquad (2.3.45)$$

2.4 Helmholtz single and double layer potentials.

In this section we investigate the existence, continuity and differentiabilty of the Helmholtz single and double layer potentials (2.2.1) and (2.2.2). First of all we define functions γ_k and θ_k on $\mathbb{R}^3 \times S$ by

$$\gamma_k(\mathbf{r},\mathbf{r}') = e^{ikR} \qquad (2.4.1)$$

and

$$\theta_k(\mathbf{r},\mathbf{r}') = (ikR - 1)e^{ikR} \qquad (2.4.2)$$

where $\mathbf{r} \in \mathbb{R}^3$, $\mathbf{r}' \in S$, $R = \|\mathbf{r}' - \mathbf{r}\|$, k is an arbitrary real or complex number and S is a regular surface element. Using (2.2.3), (2.4.1) and (2.4.2) we can now write the Helmholtz single and double layer potentials respectively as

$$\Phi(\mathbf{r}) = \frac{1}{4\pi} \iint_S \frac{\gamma_k(\mathbf{r},\mathbf{r}')}{R} \phi(\mathbf{r}')d\sigma' \tag{2.4.3}$$

and

$$\Psi(\mathbf{r}) = \frac{1}{4\pi} \iint_S \frac{\mathbf{R}.\mathbf{n}'}{R^3} \theta_k(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\sigma' \tag{2.4.4}$$

Here ϕ and ψ are respectively the single and double layer densities on S. We note that the Helmholtz potentials are analytic in any region of $\mathbb{R}^3$ which contains no point of S.

Before deriving the properties of Φ and Ψ we prove the following lemma.

Lemma (2.4.1) Let G be a closed and bounded region of $\mathbb{R}^3$, and let d_G be the diameter of G. Then there is a positive number H, depending only on k and d_G, such that if $\mathbf{r}_1$ and $\mathbf{r}_2 \in G$ and $\mathbf{r}' \in S$ then

$$|\gamma_k(\mathbf{r}_2,\mathbf{r}') - \gamma_k(\mathbf{r}_1,\mathbf{r}')| \le H|k|\,\|\mathbf{r}_2 - \mathbf{r}_1\| \tag{2.4.5}$$

and

$$|\theta_k(\mathbf{r}_2,\mathbf{r}') - \theta_k(\mathbf{r}_1,\mathbf{r}')| \le \left(1 + H\,(1+k^2d^2)^{1/2}\right)|k|\,\|\mathbf{r}_2 - \mathbf{r}_1\| \tag{2.4.6}$$

where

$$d = \sup\left\{\|\mathbf{r}' - \mathbf{r}\| : \mathbf{r}' \in S,\ \mathbf{r} \in G\right\}.$$

Proof. If $R_1 = \|\mathbf{r}' - \mathbf{r}_1\|$ and $R_2 = \|\mathbf{r}' - \mathbf{r}_2\|$ then

$$\gamma_k(\mathbf{r}_2,\mathbf{r}') - \gamma_k(\mathbf{r}_1,\mathbf{r}') = e^{ikR_1}(R_2 - R_1) \sum_{n=0}^{\infty} \frac{(ik(R_2 - R_1))^n}{(n+1)!} \tag{2.4.7}$$

The series $\displaystyle\sum_{n=0}^{\infty} \frac{|k|^n d_G^n}{(n+1)!}$ is convergent and we denote its sum by H.

We note that $|R_2 - R_1| \le \|\mathbf{r}_2 - \mathbf{r}_1\|$. Thus if $\mathbf{r}_1$ and $\mathbf{r}_2 \in G$ then the series

$\displaystyle\sum_{n=0}^{\infty} \frac{(ik(R_2 - R_1))^n}{(n+1)!}$ is uniformly absoluteley convergent and is bounded by H.

Hence (2.4.5) follows from (2.4.7). We can write

$$\theta_k(\mathbf{r}_2,\mathbf{r}') - \theta_k(\mathbf{r}_1,\mathbf{r}') = ik(R_2 - R_1)\gamma_k(\mathbf{r}_2,\mathbf{r}') + (ikR_1 - 1)(\gamma_k(\mathbf{r}_2,\mathbf{r}') - \gamma_k(\mathbf{r}_1,\mathbf{r}')),$$

from which (2.4.6) follows immediately.

For the Helmholtz single layer potential Φ we can now prove the following theorem.

Theorem (2.4.1). Let S be a regular surface element of class C^2 and suppose that the single layer density ϕ is bounded and integrable over S; then the Helmholtz single layer potential Φ is uniformly Hölder continuous on any bounded region $G \subset \mathbb{R}^3$.

Proof. Let $\mathbf{r}_1, \mathbf{r}_2 \in G$. If $\|\phi\|_S = \sup\{|\phi(\mathbf{r}')| : \mathbf{r}' \in S\}$, then

$$|\Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_1)| \le \frac{\|\phi\|_S}{4\pi} \iint_S \left| \frac{\gamma_k(\mathbf{r}_2,\mathbf{r}')}{R_2} - \frac{\gamma_k(\mathbf{r}_1,\mathbf{r}')}{R_1} \right| d\sigma'$$

where $R_1 = \|\mathbf{r}' - \mathbf{r}_1\|$ and $R_2 = \|\mathbf{r}' - \mathbf{r}_2\|$. Now

$$\left| \frac{\gamma_k(\mathbf{r}_2,\mathbf{r}')}{R_2} - \frac{\gamma_k(\mathbf{r}_1,\mathbf{r}')}{R_1} \right| \le |\gamma_k(\mathbf{r}_2,\mathbf{r}')| \left| \frac{1}{R_2} - \frac{1}{R_1} \right| + \frac{1}{R_1} |\gamma_k(\mathbf{r}_2,\mathbf{r}') - \gamma_k(\mathbf{r}_1,\mathbf{r}')|$$

$$\le \frac{\|\mathbf{r}_2 - \mathbf{r}_1\|}{R_1 R_2} + \frac{1}{R_1} H|k| \|\mathbf{r}_2 - \mathbf{r}_1\|$$

where we have used Lemma (2.4.1) and $|\gamma_k(\mathbf{r}_2,\mathbf{r}_1)| = 1$. Hence

$$|\Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_1)| \le \frac{\|\phi\|_S \|\mathbf{r}_2 - \mathbf{r}_1\|}{4\pi} \iint_S \frac{d\sigma'}{R_1 R_2} + \frac{\|\phi\|_S H|k|}{4\pi} \|\mathbf{r}_2 - \mathbf{r}_1\| \iint_S \frac{d\sigma'}{R_1}.$$

We now choose α so that $0 \le \alpha < 1$. Then

$$|\Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_1)| \le \frac{\|\phi\|_S \|\mathbf{r}_2 - \mathbf{r}_1\|^\alpha}{4\pi} \iint_S \frac{\|\mathbf{r}_2 - \mathbf{r}_1\|^{1-\alpha}}{R_1 R_2} d\sigma' + \frac{\|\phi\|_S H|k|}{4\pi} \|\mathbf{r}_2 - \mathbf{r}_1\| \iint_S \frac{d\sigma'}{R_1}. \qquad (2.4.8)$$

By the triangle inequality, $\|\mathbf{r}_2 - \mathbf{r}_1\| \le R_1 + R_2$. If we use the inequality

$$(R_1 + R_2)^\alpha \le 2^\alpha (R_1^\alpha + R_2^\alpha),$$

then

$$\|\mathbf{r}_1 - \mathbf{r}_2\|^{1-\alpha} \le 2^{1-\alpha} (R_1^{1-\alpha} + R_2^{1-\alpha}).$$

It follows that

$$\iint_S \frac{\|\mathbf{r}_2 - \mathbf{r}_1\|^{1-\alpha}}{R_1 R_2}\, d\sigma' \le 2^{1-\alpha} \iint_S \frac{d\sigma'}{R_1^{\alpha} R_2}\, d\sigma' \;+\; 2^{1-\alpha} \iint_S \frac{d\sigma'}{R_1 R_2^{\alpha}}$$

Let $S_1 = \left\{\mathbf{r}' \in S : R_2 \ge R_1\right\}$ and $S_2 = \left\{\mathbf{r}' \in S : R_1 \ge R_2\right\}$
Then

$$\iint_S \frac{d\sigma'}{R_1^{\alpha} R_2}\, d\sigma' \le \iint_{S_1} \frac{d\sigma'}{R_1^{1+\alpha}} + \iint_{S_2} \frac{d\sigma'}{R_2^{1+\alpha}}$$

$$\le \iint_S \frac{d\sigma'}{R_1^{1+\alpha}} + \iint_S \frac{d\sigma'}{R_2^{1+\alpha}}$$

since $S_1 \subset S$ and $S \subset S_2$. Similarly

$$\iint_S \frac{d\sigma'}{R_1 R_2^{\alpha}}\, d\sigma' \le \iint_S \frac{d\sigma'}{R_1^{1+\alpha}} + \iint_S \frac{d\sigma'}{R_2^{1+\alpha}}$$

We therefore have

$$\iint_S \frac{\|R_2 - \mathbf{r}_1\|^{1-\alpha}}{R_1 R_2}\, d\sigma' \le 2^{2-\alpha} \left\{ \iint_S \frac{d\sigma'}{R_1^{1+\alpha}} + \iint_S \frac{d\sigma'}{R_2^{1+\alpha}} \right\} \qquad (2.4.9)$$

Suppose now that the regular surface element S has a standard representation

$$\xi_3' = f(\xi_1', \xi_2') \;,\; (\xi_1', \xi_2') \in D \;,$$

with respect to tangent-normal axes at a point $P_0 \in S$. Let the coordinates of P_1 , P_2 and $P' \in S$ relative to the tangent-normal axes be $(\xi_{11}, \xi_{12}, \xi_{13})$, $(\xi_{21}, \xi_{22}, \xi_{23})$ and (ξ_1', ξ_2', ξ_3') respectively. Let Q_1 , Q_2 and Q' be the projections of P_1, P_2 and P' respectively onto the tangent plane, and let $\rho_1 = Q_1Q'$, $\rho_2 = Q_2Q'$, and $\rho' = P_0Q'$. Then

$$\begin{aligned} R_1^2 &= (\xi_{11} - \xi_1')^2 + (\xi_{12} - \xi_2')^2 + (\xi_{13} - \xi_3')^2 \\ &\ge (\xi_{11} - \xi_1')^2 + (\xi_{12} - \xi_2')^2 \\ &= \rho_1^2 \;, \end{aligned}$$

i.e. $\quad R_1 \ge \rho_1$.

Similarly $\quad R_2 \ge \rho_2$.

If we let $A = \|\sec\gamma\|_S = \sup\{|\sec\gamma(r')| : r' \in S\}$, then

$$\iint_S \frac{d\sigma'}{R_1^{1+\alpha}} \le A \iint_D \frac{d\xi_1' d\xi_2'}{\rho_1^{1+\alpha}} .$$

Let $D(Q_1,\alpha)$ be the closed disk in the tangent plane with centre Q_1 and radius $a = \surd(\text{area}(D)/\pi)$. It is shown by Kellogg (24),Chapter VI, that

$$\iint_D \frac{d\xi_1' d\xi_2'}{\rho_1^{1+\alpha}} \le \iint_{D(Q_1,a)} \frac{d\xi_1' d\xi_2'}{\rho_1^{1+\alpha}} .$$

It follows that

$$\iint_D \frac{d\xi_1' d\xi_2'}{\rho_1^{1+\alpha}} \le \iint_{D(Q_1,a)} \frac{d\xi_1' d\xi_2'}{\rho_1^{1+\alpha}} = \iint_{D(P_0,a)} \frac{d\xi_1' d\xi_2'}{\rho_1^{1+\alpha}} = \frac{2\pi a^{1-\alpha}}{1-\alpha} ,$$

and from (2.4.10) we therefore have

$$\iint_S \frac{d\sigma'}{R_1^{1+\alpha}} = \frac{2\pi A a^{1-\alpha}}{1-\alpha} . \tag{2.4.11}$$

In the same way we obtain

$$\iint_S \frac{d\sigma'}{R_1^{1+\alpha}} = \frac{2\pi A a^{1-\alpha}}{1-\alpha} . \tag{2.4.12}$$

Using (2.4.11) and (2.4.12) in (2.4.9) yields

$$\iint_S \frac{\| r_2 - r_1\|^{1-\alpha}}{R_1 R_2} d\sigma' \le \frac{2^{4-\alpha}\pi A a^{1-\alpha}}{1-\alpha} . \tag{2.4.13}$$

If $\alpha = 0$ in (2.4.11) then

$$\iint_S \frac{d\sigma'}{R_1} \le 2\pi A a . \tag{2.4.14}$$

Now using (2.4.13) and (2.4.14) in (2.4.8) yields

$$|\Phi(r_2 - \Phi(r_1)| \le C \|r_2 - r_1\|^{\alpha}$$

where

$$C = \left(\frac{2^{2-\alpha} A \|\phi\|_S a^{1-\alpha}}{1-\alpha} + \frac{A \|\phi\|_S H|k|a}{2} \|r_2 - r_1\|^{1-\alpha} \right)$$

We may suppose that $\|r_2 - r_1\| \le c$, where c is the maximum chord of S.

Then $$|\Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_1)| \le \|\phi\|_S K \|\mathbf{r}_2 - \mathbf{r}_1\|^\alpha$$
whenever $\|\mathbf{r}_2 - \mathbf{r}_1\| \le c$, and $K \ge 0$ depends only on S, k and α.

In order to establish the behaviour of the Helmholtz double layer potential across S, it is convenient to decompose it as the sum of two terms. Referring to (2.4.2) we see that when $k = 0$,

$$\theta_0(\mathbf{r},\mathbf{r}') = -1 \tag{2.4.15}$$

and (2.4.4) then has the form of a Laplace double layer potential. We now write

$$\Psi(\mathbf{r}) = \Psi_1(\mathbf{r}) - \Psi_2(\mathbf{r}) \tag{2.4.16}$$

where

$$\Psi_1(\mathbf{r}) = \frac{1}{4\pi}\iint_S \frac{\mathbf{R}.\mathbf{n}'}{R^3} (\theta_k(\mathbf{r},\mathbf{r}') - \theta_0(\mathbf{r},\mathbf{r}'))\psi(\mathbf{r}')d\sigma' \tag{2.4.17}$$

and

$$\Psi\ (\mathbf{r}) = \frac{1}{4\pi}\iint_S \frac{\mathbf{R}.\mathbf{n}'}{R^3} \psi(\mathbf{r}')d\sigma' \tag{2.4.18}$$

Lemma(2.4.2) For all $\mathbf{r} \in \mathbb{R}^3$ and $\mathbf{r}' \in S$ we have

$$\theta_k(\mathbf{r},\mathbf{r}') - \theta_0(\mathbf{r},\mathbf{r}') = -k^2R^2\beta_k(\mathbf{r},\mathbf{r}') \tag{2.4.19}$$

where β_k is given by the power series

$$\beta_k(\mathbf{r},\mathbf{r}') = \sum_{n=0}^{\infty} \frac{(n+1)}{(n+2)!}(ikR)^n , \quad R = \|\mathbf{r}' - \mathbf{r}\| , \tag{2.4.20}$$

which has an infinite radius of convergence. Moreover, if G is an arbitrary closed and bounded region of $\mathbb{R}^3$ then there is a positive number B such that

$$|\beta_k(\mathbf{r}_2,\mathbf{r}') - \beta_k(\mathbf{r}_1,\mathbf{r}')| \le B\ k\ \|\mathbf{r}_2 - \mathbf{r}_1\| \tag{2.4.21}$$

where $\mathbf{r}' \in S$ and $\mathbf{r}_1, \mathbf{r}_2 \in G$.

Proof. Equation (2.4.19) follows directly from (2.4.2) and (2.4.15). Since

$$\beta_k(\mathbf{r},\mathbf{r}') = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(n+1)}{(n+2)!}(ikR)^n ,$$

$$\beta_k(\mathbf{r}_2,\mathbf{r}') - \beta_k(\mathbf{r}_1,\mathbf{r}') = \sum_{n=1}^{\infty} \frac{(n+1)}{(n+2)!}(ik)^n(R_2^n - R_1^n) .$$

$$= \frac{2}{3!}ik(R_2 - R_1) + \sum_{n=2}^{\infty} \frac{(n+1)}{(n+2)!}(ik)^n(R_2^n - R_1^n).$$

If $n \ge 2$ then

$$R_2^n - R_1^n = (R_2 - R_1)(R_2^{n-1} + R_2^{n-2}R_1 + \dots + R_2 R_1^{n-2} + R_1^{n-1})$$

and $$|R_2^n - R_1^n| \le (n-1)|R_2 - R_1| \max\left\{ R_1^{n-1}, R_2^{n-1} \right\}$$

$$\le (n-1)|R_2 - R_1|(R_1^{n-1} + R_2^{n-1}) .$$

Hence

$$|\beta_k(\mathbf{r}_2,\mathbf{r}') - \beta_k(\mathbf{r}_1,\mathbf{r}')| \le B|k|\|R_2 - R_1\|$$

where

$$B = \frac{2}{3!} + \sum_{n=1}^{\infty} \frac{(n-1)(n+1)}{(n+1)!} |k|^{n-1}\left(R_1^{n-1} + R_2^{n-1}\right) .$$

The power series has an infinite radius of convergence; thus if we let

$$d = \sup\left\{R = \|\mathbf{r}' - \mathbf{r}\| : \mathbf{r}' \in S, \mathbf{r} \in G \right\}$$

and

$$B = \frac{2}{3!} + 2 \sum_{n=1}^{\infty} \frac{(n-1)(n+1)}{(n+1)!} (|k|d)^{n-1} ,$$

then (2.4.21) follows, since $|R_2 - R_1| \le \|\mathbf{r}_2 - \mathbf{r}_1\|$.

Using (2.4.19) we can write (2.4.17) as

$$\Psi_1(\mathbf{r}) = -\frac{k^2}{4\pi} \iint_S \frac{\mathbf{R}.\mathbf{n}'}{R} \beta_k(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\sigma' . \qquad (2.4.22)$$

In view of theorem (2.4.1) the next result is almost obvious.

Lemma (2.4.3) Let S be a regular surfce element of class C^2 and let the double layer density ψ be bounded and integrable on S. Then Ψ_1, as defined by (2.4.22), is uniformly Hölder continuous on any bounded region $G \subset \mathbb{R}^3$.

Proof. If $\mathbf{r}_1, \mathbf{r}_2 \in G$ and if $\|\psi\|_S = \sup\left\{|\psi(\mathbf{r}')| : \mathbf{r}' \in S \right\}$ then

$$|\Psi_1(\mathbf{r}_2) - \Psi_1(\mathbf{r}_1)| \le \frac{|k|^2\|\psi\|_S}{4\pi} \iint_S \left| \frac{\mathbf{R}_2.\mathbf{n}'}{R_2} \beta_k(\mathbf{r}_2,\mathbf{r}') - \frac{\mathbf{R}_1.\mathbf{n}'}{R_1} \beta_k(\mathbf{r}_1,\mathbf{r}') \right| d\sigma'$$

and we find that

$$\left| \frac{\mathbf{R}_2.\mathbf{n}'}{R_2} \beta_k(\mathbf{r}_2,\mathbf{r}') - \frac{\mathbf{R}_1.\mathbf{n}'}{R_1} \beta_k(\mathbf{r}_1,\mathbf{r}') \right|$$

$$\le \left(\frac{|R_1 - R_2|}{R_1} + \frac{\|\mathbf{R}_2 - \mathbf{R}_1\|}{R_1}\right) |\beta_k(\mathbf{r}_2,\mathbf{r}')| + |\beta_k(\mathbf{r}_2,\mathbf{r}') - \beta_k(\mathbf{r}_1,\mathbf{r}')|$$

$$\le \left(\frac{2b_k}{R_1} + Bk\right) \|\mathbf{r}_2 - \mathbf{r}_1\| ,$$

where $b_k = \sup\left\{|\beta_k(\mathbf{r},\mathbf{r}')| \;:\; \mathbf{r} \in G,\ \mathbf{r}' \in S\right\}$ and we have used (2.4.21). We assume a tangent-normal ccordinate system at a suitable point $P_0 \in S$ and we obtain

$$|\Psi_1(\mathbf{r}_2) - \Psi_1(\mathbf{r}_2)| \le \frac{|k|^2 \|\psi\|_S}{4\pi}\left[2b_k \iint_S \frac{d\sigma'}{R_1} + Bk \iint_S d\sigma'\right]\|\mathbf{r}_2 - \mathbf{r}_1\| .$$

If we now use (2.4.14) with $a = \surd(\text{Area}(D)/\pi)$, D being the projection of S onto the tangent plane at P_0 , and if $A = \sup\left\{\sec\gamma(\mathbf{r}') \;:\; \mathbf{r}' \in S\right\}$, we find that

$$|\Psi_1(\mathbf{r}_2) - \Psi_1(\mathbf{r}_1)| \le C\|\psi\|_S \|\mathbf{r}_2 - \mathbf{r}_1\| ,$$

where

$$C = \frac{|k|^2}{4\pi}\left(4\pi a A b_k + Bk\ \text{Area}(S)\right)$$

depends only on k, S and G.

We will now show that the Laplace double layer potential is discontinuous across S. Towards this end we will prove two lemmas in which we will make use of the following construction. Let P_0 be an interior point of a regular surface element S of class C^2, and let P_λ be a point on the normal to S at P_0. Let $\bar{S}$ be a sphere of radius $c > 0$ tangential to the surface at P_0 . We assume that this sphere lies on the negative side of the tangent plane to S at P_0 . Any small circle on $\bar{S}$ and parallel to the tangent plane is characterised by a constant altitude $\alpha > 0$. We denote this circle by $\bar{C}_\alpha$ and it is represented by $\chi = \alpha$. On $\bar{C}_\alpha$ we construct a cone with vertex at P_λ . If c is sufficiently small this cone intersects the surface S in a closed curve C_α , which is represented by

$$\chi' = \chi_\alpha(\varphi') \ , \ 0 \le \varphi' \le 2\pi, \tag{2.4.23}$$

and is such that

$$\lim_{\alpha\to 0} \chi_\alpha(\varphi') = 0, \ 0 \le \varphi' \le 2\pi . \tag{2.4.24}$$

The portion of S bounded by C_α and containing P_0 is denoted by S_α , and $\bar{S}_\alpha$ is the portion of $\bar{S}$ bounded by $\bar{C}_\alpha$ and containing P_0. The projection of C_α onto the tangent plane to S at P_0 is a closed plane curve, denoted by Γ_α, whose equation in plane polar coordinates has the form

$$\rho_0 = a_\alpha(\varphi') \ , \ 0 \le \varphi' \le 2\pi . \tag{2.4.25}$$

According to section (2.3) we have

$$a_\alpha(\varphi') = R_0\left(\chi_\alpha'(\varphi'),\varphi'\right).\cos\left(\chi_\alpha'(\varphi')\right). \tag{2.4.26}$$

From (2.4.24) and (2.4.26) it is clear that

$$\lim_{\alpha\to 0} a_\alpha(\varphi') = 0 \ , \ 0 \le \varphi' \le 2\pi , \tag{2.4.27}$$

and we write

$$A_\alpha = \sup\left\{a_\alpha(\varphi') \; : \; 0 \le \varphi' \le 2\pi\right\} . \tag{2.4.28}$$

Now (2.4.27) implies that

$$\lim_{\alpha\to 0} A_\alpha = 0. \tag{2.4.29}$$

As before we let $Q'(\xi_1,\xi_2',0)$ be the projection of $P'(\xi_1',\xi_2',\xi_3')$ in S onto the tangent plane at P_0, and we write $\rho_\lambda = P_\lambda Q'$. Then

$$\rho_\lambda^2 = \xi_1'^2 + \xi_2'^2 + \lambda^2 = \rho_0{}^2 + \lambda^2$$

and
$$R_\lambda^2 = \rho_0^2 + (\lambda - \xi')^2 .$$

Hence

$$R_\lambda^2 - \rho_\lambda^2 = \xi_3' \; (\xi_3' - 2\lambda)$$

and we have, using (2.3.43), that

$$|R_\lambda^2 - \rho_\lambda^2| \le |\xi_3'| \; (|\xi_3' - \lambda| + \lambda) \le M \; \rho_0^2 \; (R_\lambda + \rho_\lambda).$$

Consequently

$$|R_\lambda - \rho_\lambda| \le M \; \rho_0^2 \; . \tag{2.4.30}$$

and
$$\left| \frac{R_\lambda}{\rho_\lambda} - 1 \right| \le \frac{M \; \rho_0^2}{\rho_\lambda} \le M \; \rho_0 \tag{2.4.31}$$

Now if $r' \in S_\alpha$ then $\rho_0 \le a_\alpha$. According to (2.4.2) we can choose $\alpha_0 > 0$ so that $A_\alpha < \frac{1}{2M}$ if $0 \le \alpha \le \alpha_0$, and (2.4.23) then yields

$$\left| \frac{R_\lambda}{\rho_\lambda} - 1 \right| \le \frac{1}{2}$$

or

$$\frac{1}{2}\,\rho_\lambda \le \; R_\lambda \le \; \frac{3}{2}\,\rho_\lambda \; , \quad 0 \le \alpha \le \alpha_0 \; . \tag{2.4.32}$$

We also have

$$R_0^2 = \rho_0^2 + \xi_3'^2 \le \rho_0^2 \; (\; 1 + M^2\rho_0^2 \;) \le (\; 1 + M^2A_\alpha^2 \;)\rho_0^2 \; .$$

where we have again used (2.3.43). Hence

$$R_0 = \rho_0\sqrt{(1 + M^2A_\alpha^2 \;)} \tag{2.4.33}$$

If $0 \le \alpha \le \alpha_0$, then $A_\alpha \le 1/2\ M$ and (2.4.33) becomes

$$R_0 \le \frac{\sqrt{5}}{2}\,\rho_0 \; . \tag{2.4.34}$$

Lemma (2.4.4) Let S be a regular surface element of class C^2, let f be a continuous function on S and let

$$K(\alpha,\lambda) = \iint_{S_\alpha} \frac{\mathbf{R}_\lambda \cdot \mathbf{n}'}{R_\lambda^3} f(\mathbf{r}')\, d\sigma'; \qquad (2.4.35)$$

then there is a positive number N_α, depending on α but not on λ, such that

$$|K(\alpha,\lambda)| \le N_\alpha \sup\{|f(\mathbf{r}')| : \mathbf{r}' \in S_\alpha\} \qquad (2.4.36)$$

Proof. Using the methods of section (2.3) we have

$$K(\alpha,\lambda) = \lambda \int_0^{2\pi}\int_0^{\chi'_\alpha(\varphi')} \frac{R_0 \cos\chi' \left\{ \frac{\partial R_0}{\partial \chi'} \cos\chi' - R_0 \sin\chi' \right\}}{R_\lambda^3} f(\mathbf{r}')\, d\chi' d\varphi'$$

$$- \int_0^{2\pi}\int_0^{\chi'_\alpha(\varphi')} \frac{R_0^3}{R_\lambda^3} f(\mathbf{r}') \cos\chi'\, d\chi' d\varphi' .$$

Noting that $\rho_0 = R_0 \cos\chi'$, we can write

$$K(\alpha,\lambda) = \lambda \int_0^{2\pi}\int_0^{a_\alpha(\varphi')} \frac{\rho_0}{R_\lambda^3} f(\mathbf{r}')\, d\rho_0\, d\varphi' - \int_0^{2\pi}\int_0^{\chi'_\alpha(\varphi')} \frac{R_0^2 \rho_0}{R_\lambda^3} f(\mathbf{r}')\, d\chi' d\varphi'$$

and so

$$|K(\alpha,\lambda)| \le \left(|\lambda|\,|K_1(\alpha,\lambda)| + |K_2(\alpha,\lambda)|\right) \sup\{|f(\mathbf{r}')| : \mathbf{r}' \in S_\alpha\} \qquad (2.4.37)$$

where

$$K_1(\alpha,\lambda) = \int_0^{2\pi}\int_0^{a_\alpha(\varphi')} \frac{\rho_0}{R_\lambda^3}\, d\rho_0\, d\varphi' \qquad (2.4.38)$$

and

$$K_2(\alpha,\lambda) = \int_0^{2\pi}\int_0^{\chi'_\alpha(\varphi')} \frac{R_0^2 \rho_0}{R_\lambda^3}\, d\chi' d\varphi' \qquad (2.4.39)$$

If $0 \le \alpha \le \alpha_0$, then from (2.4.32) we have that

$$R_\lambda \ge \tfrac{1}{2}\rho_\lambda = \tfrac{1}{2}\sqrt{(\rho_0^2 + \lambda^2)}$$

and so

$$|\lambda|\,|K_1(\alpha,\lambda)| \le 8 \int_0^{2\pi}\int_0^{a_\alpha(\varphi')} \frac{\rho_0}{(\rho_0^2 + \lambda^2)}\, d\rho_0\, d\varphi' .$$

Using the substitution $\rho_0 = |\lambda| \tan u$ we obtain for $\lambda \neq 0$

$$|\lambda| |K_1(\alpha,\lambda)| \leq 8 \int_0^{2\pi}\int_0^{v} \cos u \, du \leq 16\pi , \tag{2.4.40}$$

where $v = \arctan \left(a_\alpha(\varphi')/|\lambda|\right)$. Next, using (2.4.33) and $R_\lambda \geq \rho_0$ we obtain

$$|K_2(\alpha,\lambda)| \leq (1 + M^2 A_\alpha^2) \int_0^{2\pi}\int_0^{\chi_\alpha'(\varphi')} d\chi' d\varphi'$$

$$\leq 2\pi(1 + M^2 A_\alpha^2) \sup\left\{|\chi'(\varphi')| \; : \; 0 \leq \varphi' \leq 2\pi\right\} \tag{2.4.41}$$

Thus if $0 \leq \alpha \leq \alpha_0$ then

$$|\lambda| |K_1(\alpha,\lambda)| + |K_2(\alpha,\lambda)| \leq N_\alpha$$

where

$$N_\alpha = 16\pi + \frac{5\pi}{2} \sup\left\{|\chi'(\varphi')| \; : \; 0 \leq \varphi' \leq 2\pi\right\} ,$$

and the proof of the lemma is complete.

Since $\chi'(\varphi') \leq \frac{\pi}{2}$ we see that

$$N = 16\pi + \frac{5\pi^2}{4} \tag{2.4.42}$$

is an upper bound for N_α .

Lemma (2.4.5) If $\bar{P}$ is a point of the spherical surface element $\bar{S}_\alpha$ of radius c and if $\bar{R}_\lambda = P_\lambda \bar{P}$ then

$$\left.\begin{aligned} &\lim_{\lambda\to 0+} \iint_{\bar{S}_\alpha} \frac{\bar{R}_\lambda . \bar{n}}{\bar{R}_\lambda^3} d\bar{\sigma} = 2\pi(1 - \sin \alpha) \\ \text{and} \quad &\lim_{\lambda\to 0-} \iint_{\bar{S}_\alpha} \frac{\bar{R}_\lambda . \bar{n}}{\bar{R}_\lambda^3} d\bar{\sigma} = - 2\pi(1 + \sin \alpha) \end{aligned}\right\} \tag{2.4.43}$$

Proof. If $R_0 = P_0 \bar{P}$ then $R_0 = 2c \sin \bar{\chi}$ where $(\bar{\chi},\bar{\varphi})$ are the altitude and azimuth coordinates of $\bar{P} \in \bar{S}_\alpha$. Then from (2.3.20) and (2.3.32) we find that

$$\iint_{\bar{S}_\alpha} \frac{\bar{R}_\lambda . \bar{n}}{\bar{R}_\lambda^3} d\bar{\sigma} = I_1(\alpha,\lambda) - I_2(\alpha,\lambda)$$

where

$$I_1(\alpha,\lambda) = \lambda \int_0^{2\pi}\int_0^{\alpha} \frac{\bar{R}_0\left[\dfrac{\partial \bar{R}_0}{\partial \bar{\chi}}\cos\bar{\chi} - \bar{R}_0\sin\bar{\chi}\right]}{\bar{R}_\lambda^3}\cos\bar{\chi}\, d\bar{\chi}\, d\bar{\varphi}$$

$$= 4c^2\lambda \int_0^{2\pi}\int_0^{\alpha} \frac{\sin\bar{\chi}\cos\bar{\chi}\,(\cos^2\bar{\chi} - \sin^2\bar{\chi})}{(4c(c+\lambda)\sin^2\bar{\chi} + \lambda^2)^{3/2}}\, d\bar{\chi}\, d\bar{\varphi}$$

$$= 8\pi c^2\lambda \int_0^{\sin\alpha} \frac{u(1-2u^2)}{(4c(c+\lambda)u^2+\lambda^2)^{3/2}}\, du$$

and

$$I_2(\alpha,\lambda) = \int_0^{2\pi}\int_0^{\alpha} \frac{\bar{R}_0^3}{\bar{R}_\lambda^3}\cos\bar{\chi}\, d\bar{\chi}\, d\bar{\varphi}$$

$$= 16\pi c^3 \int_0^{\alpha} \frac{\sin^3\bar{\chi}\cos\bar{\chi}}{(4c(c+\lambda)\sin^2\bar{\chi}+\lambda^2)^{3/2}}\, d\bar{\chi}$$

$$= 16\pi c^3 \int_0^{\sin\alpha} \frac{u^3}{(4c(c+\lambda)u^2+\lambda^2)^{3/2}}\, du \quad .$$

We may clearly suppose that $|\lambda| \le c$, so that $c + \lambda$ is always positive . If $\mu^2 = \lambda^2/(4c(c+\lambda))$ then

$$I_1(\alpha,\lambda) = \frac{8\pi\lambda c^2}{(4c(c+\lambda))^{3/2}}(1+2\mu^2)\left[\frac{1}{|\mu|} - \frac{1}{\left(\sin^2\alpha+\mu^2\right)^{3/2}}\right]$$

$$- \frac{16\pi\lambda c^2}{(4c(c+\lambda))^{3/2}}\left[\left(\sin^2\alpha+\mu^2\right)^{1/2} - |\mu|\right]$$

and

$$I_2(\alpha,\lambda) = \frac{16\pi c^2}{(4c(c+\lambda))^{3/2}}\left\{\left(\sin^2\alpha+\mu^2\right)^{1/2} - |\mu| + \frac{1}{\left(\sin^2\alpha+\mu^2\right)^{3/2}} - \frac{\mu^2}{|\mu|}\right\} .$$

If $\lambda \to 0$ then also $\mu \to 0$, and so

$$\lim_{\lambda\to 0\pm} \frac{\lambda}{|\mu|} = \pm 2c ,$$

which leads to

$$\lim_{\lambda\to 0\pm} I_1(\alpha,\lambda) = \pm 2\pi .$$

Also

$$\lim_{\lambda\to 0} I_2(\alpha,\lambda) = 2\pi \sin\alpha ,$$

and this completes the proof of the lemma.

We shall distinguish between the limits obtained when a point $\mathbf{r}_0 \in S$ is approached respectively along the positive and negative normal direction by means of the symbols + an - . Thus, for example

$$\Psi(\mathbf{r}_0-) = \lim_{\lambda\uparrow 0} \Psi(\mathbf{r}_\lambda) \qquad (2.4.44)$$

and

$$\Psi(\mathbf{r}_0+) = \lim_{\lambda\downarrow 0} \Psi(\mathbf{r}_\lambda) \qquad (2.4.45)$$

where $\mathbf{r}_\lambda = \mathbf{r}_0 + \lambda\mathbf{r}_0$ and λ is a real number.

Lemma (2.4.6) Let S be a regular surface element of class C^2 and let ψ be a continuous funtion on S. Then the Laplace double layer potential Ψ_2 is discontinuous across S , and

$$-\tfrac{1}{2}\psi(\mathbf{r}_0) + \Psi_2(\mathbf{r}_0-) = \Psi_2(\mathbf{r}_0) = \Psi_2(\mathbf{r}_0+) + \tfrac{1}{2}\psi(\mathbf{r}_0) \qquad (2.4.46)$$

Proof. The existence of Ψ_2 has been shown in section (2.3) – see (2.3.20) and (2.3.35). We now write

$$\Psi_2(\mathbf{r}_\lambda) = I(\alpha,\lambda) + J(\alpha,\lambda) \qquad (2.4.47)$$

where

$$I(\alpha,\lambda) = \frac{1}{4\pi}\iint_{S_\alpha} \frac{\mathbf{R}_\lambda.\mathbf{n}'}{R_\lambda^3}\,\psi(\mathbf{r}')\,d\sigma' \qquad (2.4.48)$$

and

$$J(\alpha,\lambda) = \frac{1}{4\pi}\iint_{S-S_\alpha} \frac{\mathbf{R}_\lambda.\mathbf{n}'}{R_\lambda^3}\,\psi(\mathbf{r}')\,d\sigma' . \qquad (2.4.49)$$

Then

$$I(\alpha,\lambda) = \frac{\psi(\mathbf{r}_0)}{4\pi}\iint_{S_\alpha} \frac{\mathbf{R}_\lambda.\mathbf{n}'}{R_\lambda^3}\,d\sigma' + \frac{1}{4\pi}\iint_{S_\alpha} \frac{\mathbf{R}_\lambda.\mathbf{n}'}{R_\lambda^3}\,\Delta(\mathbf{r}_0,\mathbf{r}')\,d\sigma' \qquad (2.4.50)$$

where

$$\Delta(\mathbf{r}_0,\mathbf{r}') = \psi(\mathbf{r}') - \psi(\mathbf{r}_0) \qquad (2.4.51)$$

$$\left| I(\alpha,\lambda) - \frac{\psi(\mathbf{r}_0)}{4\pi}\iint_{S_\alpha} \frac{\mathbf{R}_\lambda.\mathbf{n}'}{R_\lambda^3}\,d\sigma' \right| \le \frac{1}{4\pi}\left| \iint_{S_\alpha} \frac{\mathbf{R}_\lambda.\mathbf{n}'}{R_\lambda^3}\,\Delta(\mathbf{r}_0,\mathbf{r}')\,d\sigma' \right| .$$

According to lemma (2.4.4)

$$\left| \iint_{S_\alpha} \frac{R_\lambda . n'}{R_\lambda^3} \Delta(r_0, r')d\sigma' \right| \le N_\alpha \sup \left\{ |\Delta(r_0, r')| \; : \; r' \in S_\alpha \right\}$$

and N_α is independent of λ . Since ψ is continuous on the closed set S , for any $\varepsilon > 0$ we can choose $\alpha_0 > 0$ so that

$$\sup \left\{ |\Delta(r_0, r')| \; : \; r' \in S_\alpha \right\} \le \frac{\varepsilon}{N}$$

whenever $0 \le \alpha \le \alpha_0$. Here N is given by (2.4.42) . We therefore have

$$\left| I(\alpha,\lambda) - \frac{\psi(r_0)}{4\pi} \iint_{S_\alpha} \frac{R_\lambda . n'}{R_\lambda^3} d\sigma' \right| \le \varepsilon$$

whenever $0 \le \alpha \le \alpha_0$. We now observe that the solid angles subtended by S_α and $\bar{S}_\alpha$ at P_λ are the same . Thus

$$\iint_{S_\alpha} \frac{R_\lambda . n'}{R_\lambda^3} d\sigma' = \iint_{\bar{S}_\alpha} \frac{\bar{R}_\lambda . \bar{n}}{\bar{R}_\lambda^3} d\bar{\sigma} .$$

Accordingly,

$$\left| I(\alpha,\lambda) - \frac{\psi(r_0)}{4\pi} \iint_{\bar{S}_\alpha} \frac{\bar{R}_\lambda . \bar{n}}{\bar{R}_\lambda^3} d\bar{\sigma} \right| \le \varepsilon$$

whenever $0 \le \alpha \le \alpha_0$ and for arbitrary $\lambda \ne 0$. From lemma (2.4.5) it follows that $\lim_{\lambda \to 0} I(\alpha,\lambda) = \frac{1}{2} \psi(r_0)(1 - \sin\alpha)$

and $\lim_{\lambda \to 0 -} I(\alpha,\lambda) = - \frac{1}{2} \psi(r_0) (1 + \sin\alpha)$.

Consequently:

$$\left. \begin{aligned} \lim_{\alpha \to 0} \lim_{\lambda \to 0+} I(\alpha,\lambda) &= 2\pi \\ \lim_{\alpha \to 0} \lim_{\lambda \to 0-} I(\alpha,\lambda) &= - 2\pi \end{aligned} \right\} \qquad (2.4.52)$$

The existence of these limits implies the existence of $\lim_{\alpha \to 0} \lim_{\lambda \to 0} J(\alpha,\lambda)$, for if $\alpha > 0$ and $\beta > 0$ then (2.4.47) implies that

$$| J(\alpha,\lambda) - J(\beta,\lambda) | = | I(\alpha,\lambda) - I(\beta,\lambda) | .$$

Using (2.4.52) and the Cauchy convergence criterion it follows that $\lim\limits_{\alpha\to 0} \lim\limits_{\lambda\to 0\pm} J(\alpha,\lambda)$ exists . But

$$\lim_{\lambda\to 0} J(\alpha,\lambda) = J(\alpha,0) ,$$

since $\mathbf{r}_\lambda \in S-S_\alpha$ if $\alpha \neq 0$; hence from the definition of an improper integral

$$\lim_{\alpha\to 0} J(\alpha,0) = \Psi_2(\mathbf{r}_0) .$$

Thus

$$\lim_{\alpha\to 0} \lim_{\lambda\to 0} J(\alpha,\lambda) = \Psi_2(\mathbf{r}_0) . \tag{2.4.53}$$

Using (2.4.52) and (2.4.53) in (2.4.47) we obtain

$$\Psi_2(\mathbf{r}_0-) = -\tfrac{1}{2}\,\psi(\mathbf{r}_0) + \Psi_2(\mathbf{r}_0) \tag{2.4.54}$$

and

$$\Psi_2(\mathbf{r}_0+) = \tfrac{1}{2}\,\psi(\mathbf{r}_0) + \Psi_2(\mathbf{r}_0), \tag{2.4.55}$$

from which (2.4.46) follows .

Combining lemmas (2.4.3) and (2.4.6) in equation (2.4.16) we arrive at:

Theorem (2.4.2) Let S be a regular surface element of class C^2 and let ψ be a continuous function on S . Then the Helmholtz double layer potential Ψ is discontinuous across S , and

$$-\tfrac{1}{2}\,\psi(\mathbf{r}_0) + \Psi(\mathbf{r}_0-) = \Psi(\mathbf{r}_0) = \tfrac{1}{2}\,\psi(\mathbf{r}_0) + \Psi(\mathbf{r}_0+) . \tag{2.4.56}$$

We now turn to the behaviour of the normal derivative of the Helmholtz single layer potential Φ given by (2.4.3) . For the remainder of this section we let

$$\Theta(\mathbf{r}) = \frac{\partial\Phi(\mathbf{r})}{\partial n_0} , \tag{2.4.57}$$

where $\mathbf{n}_0$ is the unit normal to a regular surface element at a point P_0 in S. From (2.4.3) we find that

$$\Theta(\mathbf{r}) = -\frac{1}{4\pi} \iint_S \frac{\mathbf{R}.\mathbf{n}_0}{R^3}\, \theta_k(\mathbf{r},\mathbf{r}')\phi(\mathbf{r}')d\sigma' , \tag{2.4.58}$$

which we write in the form

$$\Theta(\mathbf{r}) = \Theta_1(\mathbf{r}) + \Theta_2(\mathbf{r}). \qquad (2.4.59)$$

Here

$$\Theta_1(\mathbf{r}) = -\frac{1}{4\pi}\iint_S \frac{\mathbf{R}.\mathbf{n}_0}{R^3}\left(\theta_k(\mathbf{r},\mathbf{r}') - \theta_0(\mathbf{r},\mathbf{r}')\right)\phi(\mathbf{r}')d\sigma' \qquad (2.4.60)$$

and

$$\Theta_2(\mathbf{r}) = \frac{1}{4\pi}\iint_S \frac{\mathbf{R}.\mathbf{n}_0}{R^3}\phi(\mathbf{r}')d\sigma' . \qquad (2.4.61)$$

Referring to (2.4.19) we see that (2.4.60) can be written as

$$\Theta_1(\mathbf{r}) = -\frac{k^2}{4\pi}\iint_S \frac{\mathbf{R}.\mathbf{n}_0}{R^3}\beta_k(\mathbf{r},\mathbf{r}')\phi(\mathbf{r}')d\sigma'. \qquad (2.4.62)$$

and we can formulate a lemma for Θ_1 analogous to lemma (2.4.3) for Ψ_1 .

Lemma (2.4.7) Let S be a regular surface element of class C^2 and let the single layer potential be bounded and integrable on S. Then Θ_1 defined by (2.4.60) is uniformly Hölder continuous on any bounded region $G \subset \mathbb{R}^3$.

This lemma may be proved by replacing $\mathbf{n}'$ in the proof of lemma (2.4.3) by $\mathbf{n}_0$. For Θ_2 we can state a lemma which is analogous to lemma (2.4.6) for Ψ_2 .

Lemma (2.4.8) Let S be a regular surface element of class C^2 and let ϕ be a continuous function on S . Then the normal derivative of the Laplace single layer potential Θ_2 is discontinuous across S , and we have

$$\tfrac{1}{2}\psi(\mathbf{r}_0) + \Theta_2(\mathbf{r}_0-) = \Theta_2(\mathbf{r}_0) = -\tfrac{1}{2}\psi(\mathbf{r}_0) + \Theta_2(\mathbf{r}_0+) . \qquad (2.4.63)$$

Proof. The existence of $\Theta_2(\mathbf{r}_0)$ has been shown in section (2.3) . We now write

$$\Theta_2(\mathbf{r}_\lambda) = \Theta_{21}(\alpha,\lambda) + \Theta_{22}(\alpha,\lambda) \qquad (2.4.64)$$

where

$$\Theta_{21}(\alpha,\lambda) = \frac{1}{4\pi}\iint_S \frac{\mathbf{R}_\lambda.\mathbf{n}_0}{R_\lambda^3}\phi(\mathbf{r}')d\sigma' \qquad (2.4.65)$$

and

$$\Theta_{22}(\alpha,\lambda) = \frac{1}{4\pi}\iint_{S-S_\alpha} \frac{\mathbf{R}_\lambda.\mathbf{n}_0}{R_\lambda^3}\phi(\mathbf{r}')d\sigma' \qquad (2.4.66)$$

We futhermore write

$$\Theta_{21}(\alpha,\lambda) = \Theta_{211}(\alpha,\lambda) + \Theta_{212}(\alpha,\lambda) \tag{2.4.67}$$

where

$$\Theta_{211}(\alpha,\lambda) = \frac{1}{4\pi} \iint_{S_\alpha} \frac{\mathbf{R}_\lambda.(\mathbf{n}_0 - \mathbf{n}')}{R_\lambda^3} \, d\sigma' \tag{2.4.68}$$

and

$$\Theta_{212}(\alpha,\lambda) = \frac{1}{4\pi} \iint_{S_\alpha} \frac{\mathbf{R}_\lambda.\mathbf{n}'}{R_\lambda^3} \phi(\mathbf{r}')d\sigma' \tag{2.4.69}$$

We apply the inequality (2.3.44) to (2.4.68) , obtaining

$$|\Theta_{211}(\alpha,\lambda)| \le \frac{M}{\pi} \| \phi \|_S \iint_{S_\alpha} \frac{\rho_0}{R_\lambda^2} \, d\sigma'$$

$$= \frac{M}{\pi} \| \phi \|_S \int_0^{2\pi} \int_0^{a_\alpha(\varphi')} \frac{\rho_0^2}{R_\lambda^2} \, d\rho_0 \, d\varphi'$$

$$\le \frac{M}{\pi} \|\phi\|_S \int_0^{2\pi} \int_0^{a_\alpha(\varphi')} \frac{\rho_0^2}{\rho_0^2 + \lambda^2} \, d\rho_0 \, d\varphi'$$

where we used (2.4.32) when $0 \le \alpha \le \alpha_0$:

$$R_\lambda \ge \tfrac{1}{2} \rho_\lambda = \tfrac{1}{2} \surd(\rho_0^2 + \lambda^2) .$$

Then

$$|\Theta_{211}(\alpha,\lambda)| \le \frac{M}{\pi} \|\phi\|_S \int_0^{2\pi} \int_0^{a_\alpha(\varphi')} \left(1 - \frac{\lambda^2}{\rho_0^2 + \lambda^2} \right) d\rho_0 \, d\varphi'$$

$$\le M \|\phi\|_S (2A_\alpha + |\lambda|) ,$$

and according to (2.4.29)

$$\lim_{\alpha\to 0} \lim_{\lambda\to 0} \Theta_{211}(\alpha,\lambda) = 0 . \tag{2.4.70}$$

Now , since $\Theta_{212}(\alpha,\lambda) = \Psi_{21}(\alpha,\lambda)$ as given by (2.4.48) , we obtain from (2.4.52)

$$\lim_{\alpha\to 0}\ \lim_{\lambda\to 0\pm}\ \Theta_{212}(\alpha,\lambda) = \begin{cases} \frac{1}{2}\,\phi(\mathbf{r}_0) \text{ if } \lambda\downarrow 0 \\ -\frac{1}{2}\,\phi(\mathbf{r}_0) \text{ if } \lambda\uparrow 0\ . \end{cases} \qquad (2.4.71)$$

From (2.4.70) and (2.4.71) it follows that

$$\lim_{\alpha\to 0}\ \lim_{\lambda\to 0\pm}\ \Theta_{21}(\alpha,\lambda) = \begin{cases} \frac{1}{2}\,\phi(\mathbf{r}_0) \text{ if } \lambda\downarrow 0 \\ -\frac{1}{2}\,\phi(\mathbf{r}_0) \text{ if } \lambda\uparrow 0\ . \end{cases} \qquad (2.4.72)$$

The existence of this limit therefore implies the existence of the limit $\lim_{\alpha\to 0}\lim_{\lambda\to 0}\Theta_{22}(\alpha,\lambda)$ and, in fact,

$$\lim_{\alpha\to 0}\lim_{\lambda\to 0}\Theta_{22}(\alpha,\lambda) = \Theta_2(\mathbf{r}_0) \qquad (2.4.73)$$

Accordingly from (2.4.63) it follows that

$$\Theta_2(\mathbf{r}_0-) = -\tfrac{1}{2}\,\phi(\mathbf{r}_0) + \Theta_2(\mathbf{r}_0) \qquad (2.4.74)$$

and

$$\Theta_2(\mathbf{r}_0+) = \tfrac{1}{2}\,\phi(\mathbf{r}_0) + \Theta_2(\mathbf{r}_0) \qquad (2.4.75)$$

and (2.4.63) now follows.

If lemmas (2.4.7) and (2.4.8) are applied to equation (2.4.59) we obtain:

Theorem (2.4.3) Let S be a regular surface element of class C^2 and let ϕ be a continuous function on S ; then the normal derivative of the Helmholtz single layer potential Φ is discontinuous across S and

$$\tfrac{1}{2}\,\phi(\mathbf{r}_0) + \frac{\partial\Phi(\mathbf{r}_0-)}{\partial n_0} = \frac{\partial\Phi(\mathbf{r}_0)}{\partial n_0} = -\tfrac{1}{2}\,\phi(\mathbf{r}_0) + \frac{\partial\Phi(\mathbf{r}_0+)}{\partial n_0}\ . \qquad (2.4.76)$$

We note that these limits are approached uniformly with respect to $\mathbf{r}_0 \in S$, provided $\mathbf{r}_0$ is not a boundary point of S.

If one attempts to derive the behaviour of the tangential derivatives of the single layer potential across S , some difficulty is encountered when azimuth-altitude coordinates are used. Essentially this is due to the lack of a common parameter region when integrals such as

$$\iint_{S_\alpha} \frac{\mathbf{R}_\lambda \cdot \mathbf{t}_0}{R_\lambda^3}\, d\sigma' \quad \text{and} \quad \iint_{\bar{S}_\alpha} \frac{\bar{\mathbf{R}}_\lambda \cdot \mathbf{t}_0}{\bar{R}_\lambda^3}\, d\bar{\sigma}$$

have to be compared, where $\mathbf{t}_0$ is an arbitrary tangent vector . Accordinly we will now use rectangular and polar coordinates (see for example Kellogg (24) or Günter (20)) to deduce the behaviour of the tangential derivatives of the single and double layer potentials across S .

We introduce a general density θ defined on the Cartesian product $\mathbb{R}^3 \times S$ and assume that θ is continuous on the Cartesian product $G \times S$, where G is a closed and bounded region containing S and contained in $\mathbb{R}^3$. We then define Θ by

$$\Theta(\mathbf{r}) = \frac{1}{4\pi} \iint_S \frac{\mathbf{R}.\mathbf{e}(\mathbf{r}')}{R^3}\, \theta(\mathbf{r},\mathbf{r}')d\sigma' \tag{2.4.77}$$

where the unit vector $\mathbf{e}$ is either $\mathbf{n}'$ or $\mathbf{n}_0$ or $\mathbf{t}_0$.

We again let P_0 with position vector $\mathbf{r}_0 = \mathbf{OP}_0$ be an interior point of a regular surface element S of class C^2 and we assume that S has the natural representation

$$\xi_3' = f(\xi_1',\xi_2') \ , \ (\xi_1',\xi_2') \in D$$

in terms of tangent-normal axes at P_0 . For any $a > 0$ let D_a be a closed disk of radius a in the tangent plane to S at P_0 . We assume that a is so small that $D_a \subset D$. The projection of D_a onto S is denoted by S_a. At a point P_λ on the normal to S at P_0, we have

$$\Theta(\mathbf{r}_\lambda) = I(a,\lambda) + J(a,\lambda) \tag{2.4.78}$$

where

$$I(a,\lambda) = \frac{1}{4\pi} \iint_{S_a} \frac{\mathbf{R}_\lambda.\mathbf{e}(\mathbf{r}')}{R_\lambda^3}\, \theta(\mathbf{r}_\lambda,\mathbf{r}')d\sigma' \tag{2.4.79}$$

and

$$J(a,\lambda) = \frac{1}{4\pi} \iint_{S-S_\alpha} \frac{\mathbf{R}_\lambda.\mathbf{e}(\mathbf{r}')}{R_\lambda^3}\, \theta(\mathbf{r}_\lambda,\mathbf{r}')d\sigma' . \tag{2.4.80}$$

We compare the integral $I(a,\lambda)$ with

$$I_0(a,\lambda) = \frac{\theta(\mathbf{r}_\lambda,\mathbf{r}_0)}{4\pi} \iint_{D_a} \frac{\boldsymbol{\rho}_\lambda.\mathbf{e}(\mathbf{r}_0)}{\rho_\lambda^3}\, d\xi_1' d\xi_2' \tag{2.4.81}$$

where $\boldsymbol{\rho}_\lambda = \mathbf{P}_\lambda\mathbf{Q}'$ and Q' is the projection of $P' \in S$ onto the tangent plane at P_0. We can now write

$$I(a,\lambda) - I_0(a,\lambda) = I_1(a,\lambda) + I_2(a,\lambda) + I_3(a,\lambda), \tag{2.4.82}$$

where

$$I_1(a,\lambda) = \frac{1}{4\pi}\iint_{D_a} \frac{\mathbf{R}_\lambda.(\mathbf{e}(\mathbf{r}') - \mathbf{e}(\mathbf{r}_0))}{R_\lambda^3}\,\theta(\mathbf{r}_\lambda,\mathbf{r}')\sec\gamma(\mathbf{r}')d\xi_1'd\xi_2' \tag{2.4.83}$$

$$I_2(a,\lambda) = \frac{1}{4\pi}\iint_{D_a}\left[\frac{\mathbf{R}_\lambda.\mathbf{e}(\mathbf{r}_0)}{R_\lambda^3} - \frac{\rho_\lambda.\mathbf{e}(\mathbf{r}_0)}{\rho_\lambda^3}\right]\theta(\mathbf{r}_\lambda,\mathbf{r}')\sec\gamma(\mathbf{r}')d\xi_1'd\xi_2' \tag{2.4.84}$$

and

$$I_3(a,\lambda) = \frac{1}{4\pi}\iint_{D_a}\frac{\rho_\lambda.\mathbf{e}(\mathbf{r}_0)}{\rho_\lambda^3}\Big(\theta(\mathbf{r}_\lambda,\mathbf{r}')\sec\gamma(\mathbf{r}') - \theta(\mathbf{r}_\lambda,\mathbf{r}_0)\Big)d\xi_1'd\xi_2'. \tag{2.4.85}$$

In the above integrals $\gamma(\mathbf{r}')$ is the angle between the unit vectors $\mathbf{n}_0$ and $\mathbf{n}'$.

If $\mathbf{e}(\mathbf{r}')$ is a constant unit vector then $I_1(\alpha,\lambda) = 0$ for all a and λ .

If $\mathbf{e}(\mathbf{r}') = \mathbf{n}(\mathbf{r}')$ then $\mathbf{e}(\mathbf{r}_0) = \mathbf{n}_0$ and by (2.3.44)

$$\|\mathbf{e}(\mathbf{r}') - \mathbf{e}(\mathbf{r}_0)\| \le 4MR_0 \ .$$

Using (2.3.43) we have

$$R_0^2 = \rho_0^2 + \xi_3'^2 \le \rho_0^2 + M^2\rho_0^4 \ .$$

Hence, if $0 \le \rho_0 \le a$ then

$$R_0 \le \rho_0\surd(1 + M^2a^2) \le a\surd(1 + M^2a^2) \tag{2.4.86}$$

and there is a number $C \ge 0$ such that

$$\|\mathbf{e}(\mathbf{r}') - \mathbf{e}(\mathbf{r}_0)\| \le C\rho_0 \ . \tag{2.4.87}$$

From (2.4.83) we then obtain , for all λ ,

$$\begin{aligned}|I_1(\alpha,\lambda)| &\le \frac{C\|\,\theta\,\|_{G,S}\ \|\sec\gamma\|_S}{4\pi}\iint_{D_a}\frac{\rho_0}{R_\lambda^2}\,d\xi_1'd\xi_2' \\ &\le \tfrac{1}{2}\,C\|\,\theta\,\|_{G,S}\ \|\sec\gamma\|_S\ ,\end{aligned} \tag{2.4.88}$$

where we have used the relation $R_\lambda \le \rho_0$.

Secondly we consider $I_2(a,\lambda)$ as defined by (2.4.48) . Since

$$\mathbf{R}_\lambda = \rho_\lambda + \xi_3'\mathbf{n}_0$$

we have

$$\frac{\mathbf{R}_\lambda.\mathbf{e}(\mathbf{r}_0)}{R_\lambda^3} - \frac{\rho_\lambda.\mathbf{e}(\mathbf{r}_0)}{\rho_\lambda^3} = \rho_\lambda.\mathbf{e}(\mathbf{r}_0)\left(\frac{1}{R_\lambda^3} - \frac{1}{\rho_\lambda^3}\right) + \frac{\xi_3'\mathbf{e}(\mathbf{r}_0).\mathbf{n}_0}{R_\lambda^3}$$

and

$$\left| \frac{\mathbf{R}_\lambda \cdot \mathbf{e}(\mathbf{r}_0)}{R_\lambda^3} - \frac{\boldsymbol{\rho}_\lambda \cdot \mathbf{e}(\mathbf{r}_0)}{\rho_\lambda^3} \right| \le \frac{|R_\lambda - \rho_\lambda|}{R_\lambda} \left(\frac{1}{R_\lambda^2} + \frac{1}{R_\lambda \rho_\lambda} + \frac{1}{\rho_\lambda^2} \right) + \frac{|\xi_3'|}{R_\lambda^3} .$$

Because $R_\lambda^2 - \rho_\lambda^2 = (\xi_3' - \lambda)^2 - \lambda^2$ we obtain

$$|R_\lambda^2 - \rho_\lambda^2| \le |\xi'| \left(|\xi' - \lambda| + |\lambda| \right) \le M\rho_0^2 (R_\lambda + \rho_\lambda) ,$$

and

$$|R_\lambda - \rho_\lambda| \le M\rho_0^2 . \tag{2.4.89}$$

Using the relations (2.3.44) , (2.4.89) and $R_\lambda \ge \rho_0$, $\rho_\lambda \ge \rho_0$, we find that

$$\left| \frac{\mathbf{R}_\lambda \cdot \mathbf{e}(\mathbf{r}_0)}{R_\lambda^3} - \frac{\boldsymbol{\rho}_\lambda \cdot \mathbf{e}(\mathbf{r}_0)}{\rho_\lambda^3} \right| \le \frac{4M}{\rho_0} .$$

and

$$|I_2(a,\lambda)| \le \frac{M \, \| \phi \|_{G,S} \, \| \sec\gamma \|_S}{\pi} \iint_{D_\alpha} \frac{d\xi_1' d\xi_2'}{\rho_0}$$

or

$$|I_2(a,\lambda)| \le 2Ma \, \| \phi \|_{G,S} \, \| \sec\gamma \|_S \tag{2.4.90}$$

Choosing an $\varepsilon > 0$, we now determine $a_1 > 0$ so that

$$|I_1(a,\lambda)| \le \tfrac{1}{3} \varepsilon \tag{2.4.91}$$

and

$$|I_2(a,\lambda)| \le \tfrac{1}{3} \varepsilon \tag{2.4.92}$$

whenever $0 \le a \le a_1$, and for all λ such that $\mathbf{r}_\lambda \in G$.

The two relations (2.4.91) and (2.4.92) have been proved when $\mathbf{e}$ is $\mathbf{n}_0$, $\mathbf{t}_0$ or $\mathbf{n}(\mathbf{r}')$.

We now consider $I_3(\alpha,\lambda)$ for these three cases.

• **CASE 1:** $\mathbf{e}(\mathbf{r}') = \mathbf{n}(\mathbf{r}') = \mathbf{n}'$, and θ continous on $G \times S$.

As $\sec\gamma$ is continuous on S and $\sec\gamma(\mathbf{r}_0) = 1$, for any $\varepsilon > 0$ we can find $a_2 > 0$ such that

$$|\theta(\mathbf{r}_\lambda,\mathbf{r}')\sec\gamma(\mathbf{r}') - \theta(\mathbf{r}_\lambda,\mathbf{r}_0)\sec\gamma(\mathbf{r}_0)| < \frac{\varepsilon}{6\pi}$$

whenever $|\lambda| \le \delta$ and $a \le a_2$. Then

$$|I_3(a,\lambda)| \le \frac{\varepsilon}{6\pi} \iint_{D_a} \frac{|\lambda|}{\rho_\lambda^3} \, d\xi_1' d\xi_2' = \frac{\varepsilon|\lambda|}{3} \iint_{D_a} \frac{\rho_0}{(\rho_0^2 + \lambda^2)^{3/2}} \, d\rho_0$$

$$= \frac{\varepsilon|\lambda|}{3} \left(\frac{1}{|\lambda|} - \frac{1}{\sqrt{(a^2 + \lambda^2)}} \right) .$$

Hence

$$|I_3(a,\lambda)| \le \tfrac{1}{3} \varepsilon \tag{2.4.93}$$

independent of λ. Thus from (2.4.91), (2.4.92) and (2.4.93) we obtain

$$|I(a,\lambda) - I_0(a,\lambda)| > \varepsilon \qquad (2.4.94)$$

whenever $0 \le a \le \min\{a_1, a_2\}$ and $|\lambda| < \delta$.

• **CASE 2:** $e(r') = n_0$ and θ continuous on $G \times S$.

As in Case 1 , $e(r_0) = n_0$ and so (2.4.93) and (2.4.94) are again valid .

• **CASE 3:** $e(r') = t_0$, a unit tangent vector to S at P_0, θ a uniformly Hölder continuous function on S for each $r \in G$, and θ continuous on $G \times S$. As $\sec\gamma$ is continuously differentiable on S , there exists $A > 0$ such that

$$|\sec\gamma(\mathbf{r}') - \sec\gamma(\mathbf{r}_0)| < AR_0$$

whenever $\mathbf{r}' \in S_a$. The Hölder condition on θ means that there are positive numbers B , β and c such that

$$|\theta(\mathbf{r},\mathbf{r}') - \theta(\mathbf{r},\mathbf{r}_0)| < B\, R_0^{\beta}$$

whenever $\mathbf{r} \in G$ and $R_0 = \|\mathbf{r}' - \mathbf{r}_0\| \le c$. We therefore see that

$$\begin{aligned} |\theta(\mathbf{r}_\lambda,\mathbf{r}')\sec\gamma(\mathbf{r}') - \theta(\mathbf{r}_\lambda,\mathbf{r}_0)| &\le |\sec\gamma(\mathbf{r}')|\;|\theta(\mathbf{r}_\lambda,\mathbf{r}') - \theta(\mathbf{r}_\lambda,\mathbf{r}_0)| + \\ &\quad |\theta(\mathbf{r}_\lambda,\mathbf{r}_0)|\;|\sec\gamma(\mathbf{r}_\lambda) - 1| \\ &\le B\|\sec\gamma\|_S\, R_0^{\beta} + A\|\theta\|_{G,S}\, R_0 \,. \end{aligned}$$

From (2.4.86) ,

$$R_0 = \rho_0\sqrt{(1 + M^2a^2)} \,.$$

If $\beta < 1$ then $R_0 = R_0^{\beta}\, R_0^{1-\beta}$, and if $\beta > 1$ then $R_0 < R_0^{\beta}$. Hence in either case there exists $C > 0$ such that

$$|\theta(\mathbf{r}_\lambda,\mathbf{r}')\sec\gamma(\mathbf{r}') - \theta(\mathbf{r}_\lambda,\mathbf{r}_0)| \le C\,\rho_0^{\beta} \,.$$

Then

$$|I_3(a,\lambda)| \le \frac{C}{4\pi}\iint_{D_a}\frac{\rho_0^{\beta}}{\rho_\lambda^{2}}\,d\xi_1'd\xi_2' \le \frac{C}{2}\int_0^a\frac{d\rho_0}{\rho_0^{1-\beta}} = \frac{C}{2\beta}\,a^{\beta} \,.$$

We can therefore determine $a_3 > 0$ so that

$$|I_3(a,\lambda)| \le \tfrac{1}{3}\,\varepsilon$$

whenever $a \le a_3$.

Thus we can assert that in the three cases considered there is $a_0 > 0$ and $\delta > 0$ such that

$$|I(a,\lambda) - I_0(a,\lambda)| < \varepsilon \qquad (2.4.95)$$

whenever $a \le a_0$ and $|\lambda| < \delta$, uniformly with respect to λ .

We now observe that if $a \ne 0$ and $\lambda \ne 0$ then

$$\iint_{D_a}\frac{\rho_\lambda\cdot e(\mathbf{r}_0)}{\rho_\lambda^{3}}\,d\xi_1'd\xi_2' = 2\pi e_3(\mathbf{r}_0)\,\lambda\left[\frac{1}{|\lambda|} - \frac{1}{\sqrt{(a^2+\lambda^2)}}\right] \qquad (2.4.96)$$

where $e_3(\mathbf{r}_0) = n_0 . e(\mathbf{r}_0)$. Using this formulae we can evaluate the limits of the integral on the left for the three cases considered above.

• **CASES 1 and 2.** In both cases $\mathbf{e}(\mathbf{r}_0) = \mathbf{n}_0$, and so, since D_a is a plane region $e_3(\mathbf{r}_0) = 1$. Hence

$$\lim_{\lambda\to 0+} \iint_{D_a} \frac{\rho_\lambda \cdot \mathbf{e}(\mathbf{r}_0)}{\rho_\lambda^3} d\xi_1' d\xi_2' = 2\pi e_3(\mathbf{r}_0)$$

and

$$\lim_{\lambda\to 0-} \iint_{D_a} \frac{\rho_\lambda \cdot \mathbf{e}(\mathbf{r}_0)}{\rho_\lambda^3} d\xi_1' d\xi_2' = - 2\pi e_3(\mathbf{r}_0).$$

As these limits are independent of a, we deduce from (2.4.95) that

$$\lim_{a\to 0} \lim_{\lambda\to 0+} I(a,\lambda) = \tfrac{1}{2}\, \theta(\mathbf{r}_0,\mathbf{r}_0) e_3(\mathbf{r}_0)$$

and

$$\lim_{a\to 0} \lim_{\lambda\to 0-} I(a,\lambda) = - \tfrac{1}{2}\, \theta(\mathbf{r}_0,\mathbf{r}_0) e_3(\mathbf{r}_0) .$$

On the other hand we know that $\Theta(\mathbf{r}_0)$ exists , and if $a \neq 0$ then

$$\lim_{\lambda\to 0} J(a,\lambda) = J(a,0) ,$$

since $\mathbf{r}_\lambda \notin S - S_a$.

Hence

$$\Theta(\mathbf{r}_0) = \lim_{a\to 0} J(a,\lambda) = \lim_{a\to 0} \lim_{\lambda\to 0} J(a,\lambda) .$$

Hence from (2.4.78) we deduce the limit relations

$$\left.\begin{aligned} \Theta(\mathbf{r}_0+) &= \tfrac{1}{2}\, \theta(\mathbf{r}_0,\mathbf{r}_0) e_3(\mathbf{r}_0) + \Theta(\mathbf{r}_0) \\ \text{and} \quad & \\ \Theta(\mathbf{r}_0-) &= - \tfrac{1}{2}\, \theta(\mathbf{r}_0,\mathbf{r}_0) e_3(\mathbf{r}_0) + \Theta(\mathbf{r}_0) \end{aligned}\right\} \qquad (2.4.97)$$

• **CASE 3.** If $\mathbf{e}(\mathbf{r}') = \mathbf{t}_0$ then $\mathbf{e}(\mathbf{r}_0) = \mathbf{t}_0$. Hence the right-hand side of (2.4.96) is zero , and the limit on the left-hand side exists and is zero when $\lambda \to 0$. From (2.4.95) it then follows that

$$\lim_{a\to 0} \lim_{\lambda\to 0} I(a,\lambda) = 0 .$$

Since

$$\begin{aligned} \Theta(\mathbf{r}_\lambda) &= I(a,\lambda) + J(a,\lambda) \\ &= I(b,\lambda) + J(b,\lambda) \end{aligned}$$

we have

$$|J(a,\lambda) - J(b,\lambda)| = |I(a,\lambda) - I(b,\lambda)| ,$$

and the Cauchy convergence criterion implies that $\lim_{a\to 0} \lim_{\lambda\to 0} J(a,\lambda)$ exists.

But if $a \neq 0$ then $J(a,\lambda)$ is a continuous function of λ; hence

$$J(a,0) = \lim_{\lambda \to 0} J(a,\lambda) \quad , \quad a \neq 0.$$

It follows that $\lim_{a \to 0} J(a,0) = \lim_{a \to 0} \lim_{\lambda \to 0} J(a,\lambda)$ exists. Thus, by the definition of an improper integral,

$$\Theta(\mathbf{r}_0) = \lim_{a \to 0} J(a,0) = \lim_{a \to 0} \lim_{\lambda \to 0} J(a,\lambda)$$

exists . Since $\Theta(\mathbf{r}_0)$ is independent of a ,

$$\lim_{\lambda \to 0} \Theta(\mathbf{r}_\lambda) = \lim_{a \to 0} \lim_{\lambda \to 0} J(a,\lambda) = \Theta(\mathbf{r}_0) \ . \tag{2.4.98}$$

We summarise the above results as

Theorem (2.4.4) Let S be a regular surface element of class C^2 and let $G \subset \mathbb{R}^3$ be a closed bounded region of $\mathbb{R}^3$ containing S . Let θ be a continuous function defined on $G \times S$ and let $\mathbf{e}$ be a unit vector function on S. Then at an interior point $\mathbf{r}_0 \in S$ we have

$$\Theta(\mathbf{r}_0-) + \tfrac{1}{2}\,\theta(\mathbf{r}_0,\mathbf{r}_0)e_3(\mathbf{r}_0) = \Theta(\mathbf{r}_0) \tag{2.4.99}$$
$$= \Theta(\mathbf{r}_0+) - \tfrac{1}{2}\,\theta(\mathbf{r}_0,\mathbf{r}_0)e_3(\mathbf{r}_0)$$

in the folowing cases:
- **CASE 1.** $\mathbf{e}(\mathbf{r}') = \mathbf{n}(\mathbf{r}') = \mathbf{n}'$;
- **CASE 2.** $\mathbf{e}(\mathbf{r}') = \mathbf{n}(\mathbf{r}_0) = \mathbf{n}_0$;
- **CASE 3.** $\mathbf{e}(\mathbf{r}') = \mathbf{t}_0$ and θ uniformly Hölder continuous on S for each $\mathbf{r} \in G$.

The normal derivative of the single layer potential is given at a point $\mathbf{r} \notin S$ by

$$\frac{\partial \phi(\mathbf{r})}{\partial n_0} = -\frac{1}{4\pi} \iint_S \frac{\mathbf{R}.\mathbf{n}_0}{R}\, \theta_k(\mathbf{r},\mathbf{r}')\, \phi(\mathbf{r}')d\sigma' \ ,$$

where θ_k is defined by (2.4.2) . If we define θ by

$$\theta(\mathbf{r},\mathbf{r}') = -\,\theta_k(\mathbf{r},\mathbf{r}')\,\phi(\mathbf{r}')$$

and $\mathbf{e}(\mathbf{r}') = \mathbf{n}_0$, then from Theorem (2.4.4) we obtain

Theorem (2.4.5) If S is a regular surface element of class C^2 and if ϕ is a continuous function on S, then the normal derivative of the Helmholtz single layer potential Φ is discontinuous across S , and at an interior point $\mathbf{r}_0$ of S we have

$$\frac{\partial\Phi(\mathbf{r}_0-)}{\partial n_0} + \frac{1}{2}\,\phi(\mathbf{r}_0) = \frac{\partial\Phi(\mathbf{r}_0)}{\partial n_0} = \frac{\partial\Phi(\mathbf{r}_0+)}{\partial n_0} - \frac{1}{2}\,\phi(\mathbf{r}_0) \cdot$$

We now consider the tangential derivative of the single layer potential. Let $\mathbf{t}_0$ be a unit tangent vector to S at an interior point P_0 of S, then

$$\frac{\partial\Phi(\mathbf{r})}{\partial t_0} = -\frac{1}{4\pi}\iint_S \frac{\mathbf{R}.\mathbf{t}_0}{R^3}\,\theta_k(\mathbf{r},\mathbf{r}')\,\phi(\mathbf{r}')d\sigma' \ .$$

Let $$\theta(\mathbf{r},\mathbf{r}') = -\,\theta_k(\mathbf{r},\mathbf{r}')\phi(\mathbf{r}') \ ,$$

and assume that ϕ is Hölder continuous on S . Equation (2.4.2) shows that $\theta_k(\mathbf{r}',\mathbf{r}) = \theta_k(\mathbf{r},\mathbf{r}')$ and it follows from lemma (2.4.1) that θ is uniformly Hölder continuous on S for each $\mathbf{r} \in G$, where $S \subset G$. Thus with $\mathbf{e}(\mathbf{r}') = \mathbf{t}_0$, **CASE 3** of theorem (2.4.4) yields :

Theorem (2.4.6) If S is a regular surface element of class C^2 and ϕ is a uniformly Hölder continuous function on S then the tangential derivatives of the Helmholtz single layer potential are continuous across S at an interior point $\mathbf{r}_0$ of S ; i.e.

$$\frac{\partial\Phi(\mathbf{r}_0-)}{\partial t_0} = \frac{\partial\Phi(\mathbf{r}_0)}{\partial t_0} = \frac{\partial\Phi(\mathbf{r}_0+)}{\partial t_0} \ .$$

For the double layer potential

$$\Psi(\mathbf{r}) = \frac{1}{4\pi}\iint_S \frac{\mathbf{R}_\lambda.\mathbf{n}'}{R_\lambda^3}\,\theta_k(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\sigma'$$

we let $$\theta(\mathbf{r},\mathbf{r}') = \theta_k(\mathbf{r},\mathbf{r}')\,\psi(\mathbf{r}')$$

and $\mathbf{e}(\mathbf{r}') = \mathbf{n}(\mathbf{r}') = \mathbf{n}'$. Referring to **CASE 1** of theorem (2.4.4) we obtain

Theorem (2.4.7). If S is a regular surface element of class C^2 and if ψ is a continuous function on S, then the Helmholtz double layer potential Ψ is discontinuous across S, and at an interior point $\mathbf{r}_0$ of S,

$$\Psi(\mathbf{r}_0-) - \frac{1}{2}\,\psi(\mathbf{r}_0) = \Psi(\mathbf{r}_0) = \Psi(\mathbf{r}_0+) + \frac{1}{2}\,\psi(\mathbf{r}_0) \ .$$

Finally the normal derivative of the Helmholtz double layer potential Ψ defined by (2.4.4) is considered. It is assumed that the double layer density ψ has uniformly Hölder continouous derivatives of the first order on a regular surface element S of class C^2. From (2.4.16)

$$\Psi(\mathbf{r}) = \Psi_1(\mathbf{r}) - \Psi_2(\mathbf{r})$$

where Ψ_1 and Ψ_2 are given respectively by (2.4.17) and (2.4.18). At a point $\mathbf{r} \notin S$

$$\frac{\partial \Psi_1(\mathbf{r})}{\partial \mathbf{e}_0} = \frac{1}{4\pi} \iint_S \left[\left(-\frac{\mathbf{e}_0 . \mathbf{n}'}{R^3} + \frac{3\ \mathbf{R}.\mathbf{e}_0\ \mathbf{R}.\mathbf{n}'}{R^3} \right) \left(\theta_k(\mathbf{r},\mathbf{r}') - \theta_0(\mathbf{r},\mathbf{r}') \right) + \frac{\mathbf{R}.\mathbf{n}'}{R^3} \frac{\partial \theta_k(\mathbf{r},\mathbf{r}')}{\partial \mathbf{e}_0} \right] \psi(\mathbf{r}')d\sigma'$$

We use (2.4.19) , viz.

$$\theta_k(\mathbf{r},\mathbf{r}') - \theta_0(\mathbf{r},\mathbf{r}') = - k^2R^2\beta_k(\mathbf{r},\mathbf{r}'),$$

where $R = \|\mathbf{r}' - \mathbf{r}\|$ and $\beta_k(\mathbf{r},\mathbf{r}')$ is given by (2.4.20). Then

$$\frac{\partial \Psi_1(\mathbf{r})}{\partial \mathbf{e}_0} = \frac{k^2}{4\pi} \iint_S \left[\frac{\mathbf{e}_0 . \mathbf{n}'}{R} \beta_k(\mathbf{r},\mathbf{r}') - \frac{\mathbf{R}.\mathbf{e}_0\ \mathbf{R}.\mathbf{n}'}{R^3} (3\beta_k(\mathbf{r},\mathbf{r}') - 1) \right] \psi(\mathbf{r}')d\sigma' \tag{2.4.100}$$

where we have used

$$\frac{\partial \theta_k}{\partial \mathbf{e}_0} = \frac{d\theta_k}{dR} \frac{\partial R}{\partial \mathbf{e}_0} = k^2\mathbf{R}.\mathbf{e}_0\ e^{ikR} .$$

Now write

$$\frac{\partial \Psi_1(\mathbf{r})}{\partial \mathbf{e}_0} = \Psi_{11}(\mathbf{r}) + \Psi_{12}(\mathbf{r}) \tag{2.4.101}$$

where

$$\Psi_{11}(\mathbf{r}) = \frac{k^2}{4\pi} \iint_S \frac{\mathbf{e}_0 . \mathbf{n}'}{R} \beta_k(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\sigma' \tag{2.4.102}$$

and

$$\Psi_{12}(\mathbf{r}) = - \frac{k^2}{4\pi} \iint_S \frac{\mathbf{R}.\mathbf{n}'}{R^3} \mathbf{R}.\mathbf{e}_0 (3\beta_k(\mathbf{r},\mathbf{r}') - 1)\psi(\mathbf{r}')d\sigma' . \tag{2.4.103}$$

Then

$$\Psi_{11}(\mathbf{r}_2) - \Psi_{12}(\mathbf{r}_1) = \frac{k^2}{4\pi} \iint_S \left[\frac{\beta_k(\mathbf{r}_2,\mathbf{r}')}{R_2} - \frac{\beta_k(\mathbf{r}_1,\mathbf{r}')}{R_1} \right] (\mathbf{e}_0 . \mathbf{n}')\psi(\mathbf{r}')d\sigma'$$

and

$$|\Psi_{11}(\mathbf{r}_2) - \Psi_{12}(\mathbf{r}_1)| \le \frac{k^2}{4\pi} \|\psi\|_S \iint_S \left| \frac{\beta_k(\mathbf{r}_2,\mathbf{r}')}{R_2} - \frac{\beta_k(\mathbf{r}_1,\mathbf{r}')}{R_1} \right| d\sigma' ,$$

where $R_1 = \|\mathbf{r}' - \mathbf{r}_1\|$ and $R_2 = \|\mathbf{r}' - \mathbf{r}_2\|$.

Since β_k satisfies the Hölder condition (2.4.21), we can proceed as in the proof of Theorem (2.4.1) to deduce that Ψ_{11} is continuous across S at interior points of S .

As regards Ψ_{12} , we let

$$\theta(\mathbf{r},\mathbf{r}') = - k^2(\mathbf{R}.\mathbf{e}_0)(3\beta_k(\mathbf{r},\mathbf{r}') - 1) .$$

Then θ is continuous on $\mathbb{R}^3 \times S$. Let G be a closed and bounded region containing S . Now β_k , as defined by (2.4.20) , is continuously differentiable on S for each $\mathbf{r} \in G$, and we can apply the mean value theorem to β_k on S to obtain

$$\theta(\mathbf{r},\mathbf{r}_2') - \theta(\mathbf{r},\mathbf{r}_1') = (\mathbf{r}_2' - \mathbf{r}_1').\ \nabla'\ \theta(\mathbf{r},\mathbf{r}_1' + \alpha(\mathbf{r}_2' - \mathbf{r}_1')),$$

where $0 < \alpha < 1$ and $\mathbf{r}_1'$,$\mathbf{r}_2' \in S$. Hence, if C is an upper bound of $\|\nabla'\ \theta(\mathbf{r},\mathbf{r}')\|$ for $\mathbf{r} \in G$ and $\mathbf{r}' \in S$, then

$$|\theta(\mathbf{r},\mathbf{r}_2') - \theta(\mathbf{r},\mathbf{r}_1')| \le C\|\mathbf{r}_1' - \mathbf{r}_2'\|$$

whenever $\mathbf{r} \in G$ and $\mathbf{r}_1'$, $\mathbf{r}_2' \in S$.

Note that if $\mathbf{r}_0$ is an interior point of S then

$$\theta(\mathbf{r}_0,\mathbf{r}_0) = 0 ;$$

thus by Theorem (2.4.4), Cases 1 and 3, it follows that Ψ_{12} is continuous across S at $\mathbf{r}_0 \in S$. Thus we can assert that $\dfrac{\partial\Psi_1(\mathbf{r})}{\partial e_0}$ is continuous across S at an interior point $\mathbf{r}_0 \in S$.

We now turn to Ψ_2 and consider its derivatives at an interior point $\mathbf{r}_0$ of S. By hypothesis, ψ is Hölder continuously differentiable on S, and according to the mean value theorem

$$\psi(\mathbf{r}') - \psi(\mathbf{r}_0) = (\mathbf{r}' - \mathbf{r}_0).\nabla'\psi(\mathbf{r}_0 + \beta(\mathbf{r}' - \mathbf{r}_0)) \ , \ 0 < \beta < 1,$$

or

$$\psi(\mathbf{r}') - \psi(\mathbf{r}_0) = \mathbf{R}_0.\mathbf{A} + \mathbf{R}_0.\mathbf{B}(\mathbf{r}_0,\mathbf{r}')$$

where $$\mathbf{A} = \nabla'\ \psi(\mathbf{r}_0)$$

and $$\mathbf{B}(\mathbf{r}_0,\mathbf{r}') = \nabla'\ \psi(\mathbf{r}_0 + \beta(\mathbf{r}' - \mathbf{r}_0)) - \nabla'\ \psi(\mathbf{r}_0).$$

A is a constant vector, and as ψ is uniformly Hölder continuously differentiable , there are positive numbers C, c and α such that

$$\|\mathbf{B}(\mathbf{r}_0,\mathbf{r}')\| \le C\ R_0^{\alpha} \tag{2.4.104}$$

whenever $R_0 = \|\mathbf{r}' - \mathbf{r}_0\| < c$. We assume $\alpha < 1$; if $\alpha \ge 1$ a shorter proof is possible .

Let $S_c = \{\mathbf{r}' \in S : \|\mathbf{r}' - \mathbf{r}_0\| \le c\ \}$ and write

$$\Psi_2(\mathbf{r}) = \Psi_{21}(\mathbf{r}) + \Psi_{22}(\mathbf{r}) + \Psi_{23}(\mathbf{r}) + \Psi_{24}(\mathbf{r}) \tag{2.4.105}$$

where

$$\Psi_{21}(\mathbf{r}) = \frac{1}{4\pi} \iint_{S-S_c} \frac{\mathbf{R}.\mathbf{n}'}{R^3} \psi(\mathbf{r}')d\sigma' \tag{2.4.106}$$

$$\Psi_{22}(\mathbf{r}) = \frac{1}{4\pi}.\iint_{S_c} \frac{\mathbf{R}.\mathbf{n}'}{R^3} \mathbf{R}_0.\mathbf{A}\ d\sigma' \tag{2.4.107}$$

$$\Psi_{23}(\mathbf{r}) = \frac{1}{4\pi} \iint_{S_c} \frac{\mathbf{R}.\mathbf{n}'}{R^3} \mathbf{R}_0.\mathbf{B}(\mathbf{r}_0,\mathbf{r}'))d\sigma' \tag{2.4.108}$$

$$\Psi_{24}(\mathbf{r}) = \frac{\psi(\mathbf{r}_0)}{4\pi} \iint_{S_c} \frac{\mathbf{R}.\mathbf{n}'}{R^3} d\sigma'. \tag{2.4.109}$$

As $\mathbf{r}_0 \notin S-S_c$, the derivatives of Ψ_{21} are continuous across S at $\mathbf{r}_0$. We show that the normal limits of the derivatives of the remaining terms in the decomposition (2.4.105) are also continuous across S at $\mathbf{r}_0$.

Let G be a regular region such that $S_c \subset \partial G$, and assume that a point of G can approach S_c only from the positive side of S_c. Let $\mathbf{r}_\lambda$ be a point on the normal to S at $\mathbf{r}_0$.

If $\mathbf{r}_\lambda$ is an interior point of G , then there is an a > 0 such that the neighbourhood $B(\mathbf{r}_\lambda,a) \subset G$. To the regular region $G - B(\mathbf{r}_\lambda,a)$ we apply Green's second identity (2.2.10) with $\phi(\mathbf{r}') = \mathbf{R}_0.\mathbf{A}$ and $\psi(\mathbf{r}') = 1/R_\lambda$. Thus

$$\iint_{\partial(G-B(\mathbf{r}_\lambda,a))} \left[\mathbf{R}_0.\mathbf{A} \frac{\partial}{\partial \tilde{n}}\left(\frac{1}{R_\lambda}\right) - \frac{1}{R_\lambda}\frac{\partial}{\partial \tilde{n}}\left(\mathbf{R}_0.\mathbf{A}\right)\right] d\sigma'$$

$$= \iiint_{G-B(\mathbf{r}_\lambda,a)} \left[\mathbf{R}_0.\mathbf{A}\ \nabla'^2 \left(\frac{1}{R_\lambda}\right) - \frac{1}{R_\lambda} \nabla'^2\left(\mathbf{R}_0.\mathbf{A}\right)\right] d\tau'. \tag{2.4.110}$$

Here we denote the outward normal to $\partial(G - B(R_\lambda,a))$ by $\tilde{\mathbf{n}}$.

Since $\nabla'^2(\mathbf{R}_0.\mathbf{A}) = 0$ and $\nabla'^2\left(1/R_\lambda\right) = 0$ when $\mathbf{r}' \in G - B(\mathbf{r}_\lambda,a)$ we have

$$\iint_{\partial G} \left[\mathbf{R}_0 . \mathbf{A} \frac{\partial}{\partial \tilde{n}}\left(\frac{1}{R_\lambda}\right) - \frac{1}{R_\lambda} \frac{\partial}{\partial \tilde{n}}\left(\mathbf{R}_0 . \mathbf{A}\right) \right] d\sigma'$$

$$= - \iint_{\partial B(\mathbf{r}_\lambda, a)} \left[\mathbf{R}_0 . \mathbf{A} \frac{\partial}{\partial \tilde{n}}\left(\frac{1}{R_\lambda}\right) - \frac{1}{R_\lambda} \frac{\partial}{\partial \tilde{n}}\left(\mathbf{R}_0 . \mathbf{A}\right) \right] d\sigma' \qquad (2.4.111)$$

On $\partial B(\mathbf{r}_\lambda, a)$ we have $\tilde{n} = R_\lambda / R_\lambda$ and $R_\lambda = a$; hence

$$\iint_{\partial B(\mathbf{r}_\lambda, a)} \mathbf{R}_0 . \mathbf{A} \frac{\partial}{\partial \tilde{n}}\left(\frac{1}{R_\lambda}\right) d\sigma' = - \iint_{\partial B(\mathbf{r}_\lambda, a)} \frac{\mathbf{R}_0 . \mathbf{A}}{R_\lambda^2} d\sigma' = - \int_0^{2\pi} \int_0^{\pi} \mathbf{R}_0 . \mathbf{A} \sin\theta \, d\theta d\varphi ,$$

whence it follows that

$$\lim_{a \to 0} \iint_{\partial B(\mathbf{r}_\lambda, a)} \mathbf{R}_0 . \mathbf{A} \frac{\partial}{\partial \tilde{n}}\left(\frac{1}{R_\lambda}\right) d\sigma' = - 4\pi(\mathbf{r}_\lambda - \mathbf{r}_0) . \mathbf{A} .$$

Similarly

$$\iint_{\partial B(\mathbf{r}_\lambda, a)} \frac{1}{R_\lambda} \frac{\partial}{\partial \tilde{n}}\left(\mathbf{R}_0 . \mathbf{A}\right) d\sigma' = - \iint_{\partial B(\mathbf{r}_\lambda, a)} \frac{\mathbf{A} . \mathbf{R}_\lambda}{R_\lambda^2} d\sigma'$$

$$= - \int_0^{2\pi} \int_0^{\pi} \mathbf{A} . \mathbf{R}_\lambda \sin\theta \, d\theta d\varphi ,$$

and

$$\lim_{a \to 0} \iint_{\partial B(\mathbf{r}_\lambda, a)} \frac{1}{R_\lambda} \frac{\partial}{\partial \tilde{n}}\left(\mathbf{R}_0 . \mathbf{A}\right) d\sigma' = 0 ,$$

since $\mathbf{R}_\lambda = \mathbf{r}' - \mathbf{r}_\lambda \to \mathbf{r}_\lambda - \mathbf{r}_\lambda = 0$ when $a \to 0$. Now on S_c we have $\tilde{n} = - \mathbf{n}'$, and from (2.4.111), when $a \to 0$,

$$- \iint_{S_c} \mathbf{R}_0 . \mathbf{A} \frac{\mathbf{R}_\lambda . \mathbf{n}'}{R_\lambda^3} d\sigma' + \iint_{S_c} \frac{\mathbf{A} . \mathbf{n}'}{R_\lambda} d\sigma'$$

$$= 4\pi(\mathbf{r}_\lambda - \mathbf{r}_0) . \mathbf{A} - \iint_{\partial G - S_c} \left[\mathbf{R}_0 . \mathbf{A} \frac{\partial}{\partial \tilde{n}}\left(\frac{1}{R_\lambda}\right) - \frac{\mathbf{A} . \tilde{n}}{R_\lambda} \right] d\sigma'$$

or

$$\Psi_{22}(\mathbf{r}_\lambda) = \frac{1}{4\pi}\iint_{S_c} \frac{\mathbf{A}.\mathbf{n}'}{R_\lambda}\, d\sigma' - (\mathbf{r}_\lambda - \mathbf{r}_0).\mathbf{A} +$$

$$\frac{1}{4\pi}\iint_{\partial G - S_c} \left[\mathbf{R}_0.\mathbf{A}\,\frac{\mathbf{R}_\lambda.\tilde{\mathrm{n}}}{R_\lambda^3} - \frac{\mathrm{A}.\tilde{\mathrm{n}}}{R_\lambda} \right] d\sigma' \qquad (2.4.112)$$

Thus if $\lambda \neq 0$ and $\mathbf{r}_\lambda \in G$ then

$$\frac{\partial\Psi_{22}(\mathbf{r}_\lambda)}{\partial \mathrm{n}_0} = -\frac{1}{4\pi}\iint_{S_c} \frac{\mathbf{R}_\lambda.\mathbf{n}_0}{R_\lambda^3}\,\mathbf{A}.\mathbf{n}'d\sigma' - \mathbf{A}.\mathbf{n}_0 +$$

$$\frac{1}{4\pi}\frac{\partial}{\partial \mathrm{n}_0}\iint_{\partial G - S_c} \left[\mathbf{R}_0.\mathbf{A}\,\frac{\mathbf{R}_\lambda.\tilde{\mathrm{n}}}{R_\lambda^3} - \frac{\mathbf{A}.\tilde{\mathrm{n}}}{R_\lambda} \right] d\sigma' \qquad (2.4.113)$$

As $\mathbf{r}_0 \notin \partial G - S_c$, the third term on the right-hand side is continuous across S at $\mathbf{r}_0$. If we now apply Theorem (2.4.4) , Case 1 , we arrive at

$$\frac{\partial\Psi_{22}(\mathbf{r}_0+)}{\partial \mathrm{n}_0} = -\tfrac{1}{2}\,\mathbf{A}.\mathbf{n}_0 - \frac{1}{4\pi}\iint_{S_c} \frac{\mathbf{R}_0.\mathbf{n}_0}{R_0^3}\,\mathbf{A}.\mathbf{n}'d\sigma' +$$

$$\frac{1}{4\pi}\frac{\partial}{\partial \mathrm{n}_0}\iint_{\partial G - S_c} \left[\mathbf{R}_0.\mathbf{A}\,\frac{\mathbf{R}_0.\tilde{\mathrm{n}}}{R_0^3} - \frac{\mathrm{A}.\tilde{\mathrm{n}}}{R_0} \right] d\sigma' . \qquad (2.4.114)$$

If $\mathbf{r}_\lambda \notin G$ then Green's second identity holds:

$$\iint_{\partial G} \left[\mathbf{R}_0.\mathbf{A}\,\frac{\partial}{\partial\tilde{\mathrm{n}}}\left(\frac{1}{R_\lambda}\right) - \frac{1}{R_\lambda}\frac{\partial}{\partial\tilde{\mathrm{n}}}\left(\mathbf{R}_0.\mathbf{A}\right) \right] d\sigma' = 0 .$$

From this we obtain , for $\lambda < 0$,

$$\Psi_{22}(\mathbf{r}_\lambda) = \frac{1}{4\pi}\iint_{S_c} \frac{\mathbf{A}.\mathbf{n}'}{R_\lambda}\, d\sigma + \frac{1}{4\pi}\iint_{\partial G - S_c} \left[\mathbf{R}_0.\mathbf{A}\,\frac{\mathbf{R}_\lambda.\tilde{\mathrm{n}}}{R_\lambda^3} - \frac{\mathrm{A}.\tilde{\mathrm{n}}}{R_\lambda} \right] d\sigma' \qquad (2.4.115)$$

and

$$\frac{\partial\Psi_{22}(\mathbf{r}_\lambda)}{\partial \mathrm{n}_0} = -\frac{1}{4\pi}\iint_{S_c} \frac{\mathbf{R}_\lambda.\mathbf{n}_0}{R_\lambda^3}\,\mathbf{A}.\mathbf{n}'d\sigma' +$$

$$\frac{1}{4\pi}\frac{\partial}{\partial \mathrm{n}_0}\iint_{\partial G - S_c} \left[\mathbf{R}_0.\mathbf{A}\,\frac{\mathbf{R}_\lambda.\tilde{\mathrm{n}}}{R_\lambda^3} - \frac{\mathrm{A}.\tilde{\mathrm{n}}}{R_\lambda} \right] d\sigma' \qquad (2.4.116)$$

Hence , by Theorem (2.4.4) ,

$$\frac{\partial \Psi_{22}(\mathbf{r}_0-)}{\partial n_0} = -\frac{1}{2}\mathbf{A}.\mathbf{n}_0 - \frac{1}{4\pi}\iint_{S_c} \frac{\mathbf{R}_0.\mathbf{n}_0}{R_0^3}\mathbf{A}.\mathbf{n}'\, d\sigma' +$$

$$\frac{1}{4\pi}\frac{\partial}{\partial n_0}\iint_{\partial G - S_c}\left[\mathbf{R}_0.\mathbf{A}\frac{\mathbf{R}_0.\tilde{\mathbf{n}}}{R_0^3} - \frac{\mathbf{A}.\tilde{\mathbf{n}}}{R_0}\right] d\sigma' \qquad (2.4.117)$$

Thus from (2.4.114) and (2.4.117) we see that

$$\frac{\partial \Psi_{22}(\mathbf{r}_0-)}{\partial n_0} = \frac{\partial \Psi_{22}(\mathbf{r}_0+)}{\partial n_0} \qquad (2.4.118)$$

and we define $\frac{\partial \Psi_{22}(\mathbf{r}_0)}{\partial n_0}$ to be this common limit. Thus the normal derivative of Ψ_{22} is continuous across S at $\mathbf{r}_0$.

As the regular surface element S is of class C^2, the unit normal $\mathbf{n}'$ is continuously differentiable on S. Hence by the mean value theorem it follows that there is a positive number C > 0 such that

$$\|\mathbf{n}_2' - \mathbf{n}_1'\| \le C\|\mathbf{r}_2' - \mathbf{r}_1'\| \ ,$$

where $\mathbf{r}_1'$,$\mathbf{r}_2'$ $\in$ S. Thus we can assert that $\mathbf{A}.\mathbf{n}'$ is uniformly Hölder continuous on S . Noting that

$$(\mathbf{r}_\lambda - \mathbf{r}_0).\mathbf{A} = \lambda\, \mathbf{n}_0.\mathbf{A} \ ,$$

we see that the derivative of $(\mathbf{r}_\lambda - \mathbf{r}_0).\mathbf{A}$ in a tangential direction t_0 is zero . From (2.4.112) and (2.4.115) it therefore follows that

$$\frac{\partial \Psi_{22}(\mathbf{r}_\lambda)}{\partial t_0} = -\frac{1}{4\pi}\iint_{S_c} \frac{\mathbf{R}_\lambda.\mathbf{t}_0}{R_\lambda^3}\mathbf{A}.\mathbf{n}'\, d\sigma' +$$

$$\frac{1}{4\pi}\frac{\partial}{\partial t_0}\iint_{\partial G - S_c}\left[\mathbf{R}_0.\mathbf{A}\frac{\mathbf{R}_\lambda.\tilde{\mathbf{n}}}{R_\lambda^3} - \frac{\mathbf{A}.\tilde{\mathbf{n}}}{R_\lambda}\right] d\sigma' \qquad (2.4.119)$$

whenever $\lambda \neq 0$. Thus by Theorem (2.4.4) ,Case 3 , we conclude that the tangential derivatives of Ψ_{22} are continuous across S , and

$$\frac{\partial \Psi_{22}(\mathbf{r}_0-)}{\partial t_0} = \frac{\partial \Psi_{22}(\mathbf{r}_0)}{\partial t_0} = \frac{\partial \Psi_{22}(\mathbf{r}_0+)}{\partial t_0} \ . \qquad (2.4.120)$$

Next we consider $\Psi_{23}(\mathbf{r})$ as defined by (2.4.107) , and show that here we may differentiate under the integral sign . For if $\lambda \neq 0$ then

$$\frac{\partial \Psi_{23}(\mathbf{r}_\lambda)}{\partial t_0} = \frac{1}{4\pi} \iint_{S_c} \left[-\frac{\mathbf{t}_0 . \mathbf{n}'}{R_\lambda^3} + \frac{3\mathbf{R}_\lambda . \mathbf{e}_0 \ \mathbf{R}_\lambda . \mathbf{n}'}{R_\lambda^3} \right] \mathbf{R}_0 . \mathbf{B}(\mathbf{r}_0, \mathbf{r}') \, d\sigma' ,$$

and the integrals are proved convergent by comparison with the corresponding integrals on the tangent plane . Let $0 < a \le C$ and let D_a be the closed circular disk , of radius a , lying in the tangent plane to S at P_0 . From (2.4.32) , if $a \le \min\{C, 2/M\}$ then

$$\tfrac{1}{2} \rho_\lambda \le R_\lambda \le \tfrac{3}{2} \rho_\lambda ,$$

and from (2.4.86) ,

$$R_0 \le \rho_0 \sqrt{(1 + M^2 a^2)} \le \frac{\sqrt{5}}{2} \rho_0 .$$

Using these relations and (2.4.104) gives

$$\left| \iint_{S_a} \frac{\mathbf{R}_\lambda . \mathbf{t}_0 \ \mathbf{R}_\lambda . \mathbf{n}'}{R_\lambda^3} \mathbf{R}_0 . \mathbf{B}(\mathbf{r}_0, \mathbf{r}') d\sigma' \right|$$

$$\le C \|\sec\gamma\|_S \iint_{D_a} \frac{R_0^{1+\alpha}}{R_\lambda^3} d\xi_1' d\xi_2'$$

$$\le 8C \left(\frac{\sqrt{5}}{2}\right)^{1+\alpha} \|\sec\gamma\|_S \int_0^{2\pi} \int_0^a \frac{\rho_0^{2+\alpha}}{(\rho_0^2 + \lambda^2)^{3/2}} d\rho_0 \, d\varphi$$

$$\le 16C \left(\frac{\sqrt{5}}{2}\right)^{1+\alpha} \|\sec\gamma\|_S \int_0^a \frac{\rho_0^3}{(\rho_0^2 + \lambda^2)^{3/2}} d\rho_0 ,$$

since $0 < \alpha < 1$ implies that $\rho_0^\alpha \le \rho_0$. But

$$\int_0^a \frac{\rho_0^3}{(\rho_0^2 + \lambda^2)^{3/2}} d\rho_0 = (a^2 + \lambda^2)^{1/2} + \frac{\lambda^2}{(a^2 + \lambda^2)^{1/2}} - |\lambda| - \frac{\lambda^2}{|\lambda|} ,$$

and therefore

$$\lim_{\lambda \to 0} \int_0^a \frac{\rho_0^3}{(\rho_0^2 + \lambda^2)^{3/2}} d\rho_0 = a .$$

Hence we can assert that

$$\lim_{\lambda\to 0} \iint_{S_a} \frac{\mathbf{R}_\lambda \cdot \mathbf{t}_0 \ \mathbf{R}_\lambda \cdot \mathbf{n}'}{R_\lambda^3} \mathbf{R}_0 \cdot \mathbf{B}(\mathbf{r}_0, \mathbf{r}') d\sigma'$$

exists , and that

$$\lim_{a\to 0} \lim_{\lambda\to 0} \iint_{S_a} \frac{\mathbf{R}_\lambda \cdot \mathbf{t}_0 \ \mathbf{R}_\lambda \cdot \mathbf{n}'}{R_\lambda^3} \mathbf{R}_0 \cdot \mathbf{B}(\mathbf{r}_0, \mathbf{r}') d\sigma' = 0 .$$

Secondly,

$$\left| \iint_{S_a} \frac{\mathbf{t}_0 \cdot \mathbf{n}'}{R_\lambda^3} \mathbf{R}_0 \cdot \mathbf{B}(\mathbf{r}_0, \mathbf{r}') d\sigma' \right| \leq C \ \|\sec\gamma\|_s \iint_{D_a} \frac{R_0^{1+\alpha}}{R_\lambda^3} d\xi_1' d\xi_2' ,$$

and it follows as above that

$$\lim_{a\to 0} \lim_{\lambda\to 0} \iint_{S_a} \frac{\mathbf{t}_0 \cdot \mathbf{n}'}{R_\lambda^3} \mathbf{R}_0 \cdot \mathbf{B}(\mathbf{r}_0, \mathbf{r}') d\sigma' = 0 . .$$

Thus we have completed the proof that $\dfrac{\partial \Psi_{23}(\mathbf{r})}{\partial t_0}$ is continuous across S at $\mathbf{r}_0 \in S$.

It remains only to consider $\dfrac{\partial \Psi_{24}(\mathbf{r})}{\partial t_0}$. Let G be a regular region such that $S_c \subset \partial G$ and assume that a point of G can approach S_c only from the positive side of S_c . Let $\mathbf{r}_\lambda \in G^i$ and choose $a > 0$ so that $B(\mathbf{r}_\lambda, a) \subset G^i$. We apply the divergence theorem to the function $\nabla'(1/R_\lambda)$ on the regular region $G^* = G - B(r_\lambda, a)$, and denote by $\tilde{\mathbf{n}}$ the unit outward normal to ∂G^* . Then

$$0 = \iiint_{G^*} \nabla'^2 \left(\frac{1}{R_\lambda}\right) d\mathbf{r}' = - \iint_{\partial G^*} \frac{\partial}{\partial \tilde{n}} \left(\frac{1}{R_\lambda}\right) d\sigma' .$$

Hence

$$\iint_{\partial G} \frac{\partial}{\partial \tilde{n}} \left(\frac{1}{R_\lambda}\right) d\sigma' = - \iint_{\partial B(\mathbf{r}_\lambda, a)} \frac{\partial}{\partial \tilde{n}} \left(\frac{1}{R_\lambda}\right) d\sigma'$$

$$= - 4\pi$$

since $\dfrac{\partial}{\partial \tilde{n}} \left(\dfrac{1}{R_\lambda}\right) = \dfrac{1}{a^2}$ on $\partial B(\mathbf{r}_\lambda, a)$.

Since $\tilde{n} = -n'$ on S_C ,

$$-\iint_{S_C} \frac{\partial}{\partial n'}\left(\frac{1}{R_\lambda}\right) d\sigma' + \iint_{\partial G - S_C} \frac{\partial}{\partial \tilde{n}}\left(\frac{1}{R_\lambda}\right) d\sigma' = -4\pi,$$

and so

$$\Psi_{24}(\mathbf{r}_\lambda) = \psi(\mathbf{r}_0) - \frac{\psi(\mathbf{r}_0)}{4\pi} \iint_{\partial G - S_C} \frac{\mathbf{R}_\lambda \cdot \tilde{n}}{R_\lambda^3} d\sigma' .$$

Thus as $\mathbf{r}_0 \notin G - S_C$, the right-hand side has continuous derivatives across S at $\mathbf{r}_0$. We have therefore established the theorem

Theorem (2.4.5) Let S be a regular surface element of class C^2 and let the double layer density ψ have uniformly Hölder continuous derivatives of the first order; then the first order derivatives of the Helmholtz double layer potential Ψ are continuous across S at interior points of S.

2.5 Helmholtz formulae for interior regions.

Let D be a regular region of $\mathbb{R}^3$ bounded by a smooth regular surface $S = \partial D$. Suppose that ϕ satisfies the inhomogeneous Helmholtz equation

$$\nabla^2 \phi + k^2 \phi = f \qquad (2.5.1)$$

on the interior of D. It is assumed that ϕ is twice continuously differentiable on D^i and that f is continuous on D. In this section we shall derive an integral representation of ϕ in terms of f and the boundary values of ϕ and its normal derivative $\frac{\partial \phi}{\partial n}$. In this derivation use is made of the free-space Green's function G_k defined by (2.1.8).

Let P be an interior point of D with position vector $\mathbf{r}$ relative to an origin O ; then there is $a_0 > 0$ such that the open ball $U(\mathbf{r},a)$, with centre P and radius a, is contained in D^i whenever $0 < a < a_0$. Let $D_a = D - U(\mathbf{r},a)$; the boundary of D_a is $\partial D_a = S \cap S_a$, where $S_a = \partial U(P,a)$. Since $G_k(\mathbf{r},\mathbf{r}')$ is not singular when $\mathbf{r} \in D_a$, we can apply Green's second identity to ϕ and G_k on D_a , obtaining

$$\iiint_{D_a} \left[\phi(\mathbf{r}')\, \nabla'^2\, G_k(\mathbf{r},\mathbf{r}') - G_k(\mathbf{r},\mathbf{r}')\, \nabla'^2\, \phi(\mathbf{r}')\right] d\tau'$$

$$= \iint_{S \cup S_a} \left[\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\, \frac{\partial \phi(\mathbf{r}')}{\partial n'}\right] d\sigma' \, . \tag{2.5.2}$$

Here $\mathbf{n}' = \mathbf{n}(\mathbf{r}')$ is the unit outward normal to the boundary $S \cup S_a$ at a point $\mathbf{r}' \in S \cup S_a$. Using (2.1.7) and (2.5.1) in (2.5.2) it follows that

$$- \iiint_{D_a} G_k(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d\tau' = \iint_S \left[\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\, \frac{\partial \phi(\mathbf{r}')}{\partial n'}\right] d\sigma'$$

$$+ \iint_{S_a} \left[\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\, \frac{\partial \phi(\mathbf{r}')}{\partial n'}\right] d\sigma' \tag{2.5.3}$$

If we put $$I(a) = \iiint_{D_a} G_k(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d\tau' \, ,$$

and if $0 < a < b < a_0$, then

$$I(b) - I(a) = \frac{1}{4\pi} \int_0^{2\pi}\int_0^{\pi}\int_a^b e^{ikR}\, f(\mathbf{r} + \mathbf{R})R^2 \sin\theta \; dR d\theta d\varphi$$

where $\mathbf{r}' = \mathbf{r} + \mathbf{R}$. Hence $|I(b) - I(a)|$ can be made arbitarily small by choosing b sufficiently small , and this implies that

$$\iiint_D G_k(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d\tau' = \lim_{a\to 0} I(a) \tag{2.5.4}$$

exists. It remains to determine the limit of the surface integral over S_a when $a \to 0$. We introduce Cartesian reference frames with origins at O and at P . Let the coordinates of P relative to O be (x_1, x_2, x_3) , and let the coordinates of an arbitary point P′ be (x_1', x_2', x_3') relative to O and (ξ_1', ξ_2', ξ_3') relative to P. Let $\mathbf{r}' = \mathbf{OP}'$, $\mathbf{R} = \mathbf{r}' - \mathbf{r} = \mathbf{PP}'$, and denote the azimuth and zenith distances of P′ relative to P by θ and φ respectively. Thus if P′ lies on the sphere S_a then

$$\left.\begin{aligned} \xi_1' &= a \cos\varphi \sin\theta \\ \xi_2' &= a \sin\varphi \sin\theta \\ \xi_3' &= a \cos\theta \end{aligned}\right\}$$

On the sphere S_a the unit normal $\mathbf{n}'$ is directed towards the centre P , so that $\mathbf{n}' = -\mathbf{R}/R$, and

$$\frac{\partial G_k(\mathbf{r}')}{\partial n'} = -\frac{(ika-1)e^{ika}}{4\pi a^2} \quad , \text{ if } \mathbf{r}' \in S_a \ .$$

Therefore

$$\iint_{S_a} \phi(\mathbf{r}') \frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} \, d\sigma' = -\frac{(ika-1)e^{ika}}{4\pi} \int_0^{2\pi}\int_0^{\pi} \phi(\mathbf{r}')\sin\theta \, d\theta d\phi$$

$$= -(ika-1)e^{ika}\,\phi(\mathbf{r}^*) \ ,$$

where $\mathbf{r}^*$ is some point in S_a , which implies that

$$\lim_{a\to 0} \iint_{S_a} \phi(\mathbf{r}') \frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} \, d\sigma' = \phi(\mathbf{r}) \ . \tag{2.5.5}$$

We also have

$$\iint_{S_a} G_k(\mathbf{r},\mathbf{r}')\frac{\partial \phi(\mathbf{r}')}{\partial n'} \, d\sigma' = \frac{ae^{ika}}{4\pi} \int_0^{2\pi}\int_0^{\pi} \frac{\partial \phi(\mathbf{r}')}{\partial n'} \sin\theta \, d\theta d\varphi$$

from which follows

$$\lim_{a\to 0} \iint_{S_a} G_k(\mathbf{r},\mathbf{r}')\frac{\partial \phi(\mathbf{r}')}{\partial n'} \, d\sigma' = 0 \ . \tag{2.5.6}$$

Letting $a \to 0$ in (2.5.3) and using (2.5.4),(2.5.5) and (2.5.6) we find that

$$\phi(\mathbf{r}) = -\iiint_D G_k(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d\tau' - \iint_S \left(\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\frac{\partial \phi(\mathbf{r}')}{\partial n'} \right) d\sigma' \tag{2.5.6}$$

whenever $\mathbf{r} \in D^i$.

When $\mathbf{r} \in D^e$,

$$\phi(\mathbf{r}')\nabla'^2 G_k(\mathbf{r},\mathbf{r}') - G_k(\mathbf{r},\mathbf{r}')\nabla'^2\phi(\mathbf{r}') = -G_k(\mathbf{r},\mathbf{r}')f(\mathbf{r}')$$

for all $\mathbf{r}' \in D^i$, and so

$$\iiint_D G_k(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d\tau' + \iint_S \left(\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\frac{\partial \phi(\mathbf{r}')}{\partial n'}\right) d\sigma' = 0 \ . \tag{2.5.7}$$

These results yield the following two theorems:

Theorem (2.5.1) (Inhomogeneous Helmholtz interior formulae) .
Let D be a regular region of $\mathbb{R}^3$ with a smooth regular boundary ∂D , and suppose that $\phi \in C^2(D^i)\cap C^1(\partial D)$ and satisfies the inhomogeneous Helmholtz equation

$$\nabla^2\phi + k^2\phi = f$$

in the interior of D where f is continuous on D; then

$$\iiint_D G_k(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d\tau' + \iint_S \left(\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r}')}{\partial n'}\right) d\sigma'$$

$$= \begin{cases} -\phi(\mathbf{r}) & \text{if } \mathbf{r} \in D^i \\ 0 & \text{if } \mathbf{r} \in D^e \end{cases}$$

In the special case when $f = 0$, we have:

Theorem (2.5.2) (Helmholtz interior formulae)
Let D be a regular region of $\mathbb{R}^3$ with a smooth regular boundary ∂D , and suppose that $\phi \in C^2(D^i)\cap C^1(\partial D)$ and satisfies the homogeneous Helmholtz equation

$$\nabla^2\phi + k^2\phi = 0$$

in the interior of D; then

$$\iint_S \left(\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r}')}{\partial n'}\right) d\sigma' = \begin{cases} -\phi(\mathbf{r}) & \text{if } \mathbf{r} \in D^i \\ 0 & \text{if } \mathbf{r} \in D^e \end{cases}$$

2.6 Helmholtz formulae for exterior regions.

Let D denote a closed and bounded region of $\mathbb{R}^3$ and $S = \partial D$, and let ϕ be a solution of the inhomogeneous Helmholtz equation

$$\nabla^2\phi + k^2\phi = f$$

in the exterior D^e of D .

Let P be any point of $\mathbb{R}^3$ with position vector $\mathbf{r} = x_i\mathbf{e}_i$ with respect to a Cartesian reference frame with origin at O. Suppose first that $\mathbf{r} \in D^e$ and let $B(\mathbf{r},a)$ be a closed ball with centre at $\mathbf{r}$ and radius a so large that $D \subset B(\mathbf{r},a)$. For the remainder of this section let $B_a = B(\mathbf{r},a)$, $S_a = \partial B_a$, and let D_a denote the closure of $B_a - D$, i.e. $D_a = (B_a - D)^c$. Helmholtz's formulae for interior regions applies to D_a , and since $\mathbf{r} \in D_a$,

$$\iint_{\partial D_a} \left[\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\frac{\partial \phi(\mathbf{r}')}{\partial n'}\right] d\sigma' = \phi(\mathbf{r}) \qquad (2.6.2)$$

where $\mathbf{n}'$ is the unit inward normal to $\partial D_a = S \cup S_a$. Since $S \cap S_a = \emptyset$, the surface integral in (2.6.2) can be written as the sum of two surface integrals :

$$\phi(\mathbf{r}) = \iint_{S} \left[\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\frac{\partial \phi(\mathbf{r}')}{\partial n'}\right] d\sigma'$$

$$+ \iint_{S_a} \left[\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\frac{\partial \phi(\mathbf{r}')}{\partial n'}\right] d\sigma' \qquad (2.6.3)$$

In the first integral $\mathbf{n}' = \mathbf{n}(\mathbf{r}')$ is the unit outward normal to S and in the second integral $\mathbf{n}'$ is the unit inward normal to S_a . Hence on S_a ,

$$\mathbf{n}' = -\mathbf{R}/R$$

where $\mathbf{R} = \mathbf{r}' - \mathbf{r}$, $\mathbf{r}' \in S_a$, and $R = a$. Hence

$$\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} = -\frac{1}{4\pi R^2}(ikR - 1)e^{ikR} \quad , \quad \text{for } \mathbf{r}' \in S_a .$$

On the sphere S_a , $\phi(\mathbf{r}')$ can be expressed as a function of $\mathbf{R}$, since $\mathbf{r}' = \mathbf{r} + \mathbf{R}$, and $\mathbf{R}$ can be expressed in terms of spherical ccordinates (R,θ,ϕ) with $\mathbf{r}$ as origin. Hence

$$\frac{\partial \phi(\mathbf{r}')}{\partial n'} = \frac{\partial \phi(\mathbf{r}')}{\partial x_i'} n_i(\mathbf{r}') = \frac{\partial \phi(\mathbf{R})}{\partial R}\frac{\partial R}{\partial x_i'} n_i' = \frac{\partial \phi(\mathbf{R})}{\partial R}\frac{\mathbf{R}.\mathbf{n}'}{R} = -\frac{\partial \phi(\mathbf{R})}{\partial R} ,$$

and so

$$\iint_{S_a} \left[\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\frac{\partial \phi(\mathbf{r}')}{\partial n'}\right] d\sigma'$$

$$= \frac{1}{4\pi}\iint_{S_a} \frac{e^{ikR}}{R}\left[\frac{\partial \phi(\mathbf{R})}{\partial R} - \left(ik - \frac{1}{R}\right)\phi(\mathbf{R})\right] d\sigma$$

$$= \frac{1}{4\pi}\iint_{S_a} \frac{e^{ikR}}{R}\left[\frac{\partial \phi(\mathbf{R})}{\partial R} - ik\phi(\mathbf{R})\right] d\sigma + \frac{1}{4\pi}\iint_{S_a} \frac{e^{ikR}}{R^2}\phi(\mathbf{R})\, d\sigma ,$$

the integrations now being with respect to **R** . Accordingly (2.6.2) can now be written as

$$\phi(\mathbf{r}) = \iint_S \left\{\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\frac{\partial \phi(\mathbf{r}')}{\partial n'}\right\} d\sigma'$$

$$+ \frac{1}{4\pi}\iint_{S_a} \frac{e^{ikR}}{R}\left[\frac{\partial\phi(\mathbf{R})}{\partial R} - ik\phi(\mathbf{R})\right]d\sigma + \frac{1}{4\pi}\iint_{S_a} \frac{e^{ikR}}{R^2}\phi(\mathbf{R})\, d\sigma. \qquad (2.6.4)$$

Applying Schwartz's inequality to the second integral on the right-hand side yields

$$\left|\iint_{S_a} \frac{e^{ikR}}{R}\left[\frac{\partial\phi(\mathbf{R})}{\partial R} - ik\phi(\mathbf{R})\right] d\sigma\right|^2$$

$$\leq \iint_{S_a} \frac{d\sigma}{R^2} \iint_{S_a} \left|ik\phi(\mathbf{R}) - \frac{\partial\phi(\mathbf{R})}{\partial R}\right|^2 d\sigma$$

$$= 4\pi \iint_{S_a} \left|ik\phi(\mathbf{R}) - \frac{\partial\phi(\mathbf{R})}{\partial R}\right|^2 d\sigma \, . \qquad (2.5.6)$$

Hence if ϕ satisfies the ***Wilcox radiation condition*** (Wilcox (53))

$$\lim_{a\to\infty} \iint_{S_a} \left|ik\phi(\mathbf{R}) - \frac{\partial\phi(\mathbf{R})}{\partial R}\right|^2 d\sigma = 0 \qquad (2.6.6)$$

then the second integral in (2.6.4) vanishes when $a \to \infty$.

This condition also implies that

$$\lim_{a\to 0} \iint_{S_a} \frac{e^{ikR}}{R^2}\phi(\mathbf{R})\, d\sigma = 0 \, . \qquad (2.6.7)$$

This may be proved as follows . The radiation condition (2.6.6) can be written as

$$\lim_{a\to 0} \iint_{S_a} \left(|k|^2 |\phi(\mathbf{R})|^2 - ik\phi(\mathbf{R}) \frac{\partial \overline{\phi(\mathbf{R})}}{\partial R} + i\bar{k}\, \overline{\phi(R)} \frac{\partial \phi(\mathbf{R})}{\partial R} + \left| \frac{\partial \phi(\mathbf{R})}{\partial R} \right|^2 \right) d\sigma = 0. \tag{2.6.8}$$

The second and third terms of (2.6.8) can be transformed by applying Green's first identity (2.2.9) to the functions ϕ and $\bar{\phi}$ on the closed region D_a. If $\mathbf{n}'$ is the unit inward normal to ∂D_a then

$$\iiint_{D_a} \phi(\mathbf{r}')\nabla'^2\overline{\phi(\mathbf{r}')}d\tau'$$

$$= - \iint_{\partial D_a} \phi(\mathbf{r}')\frac{\partial\overline{\phi(\mathbf{r}')}}{\partial n'}\, d\sigma' - \iiint_{D_a} \|\nabla'\phi(\mathbf{r}')\|^2 d\tau'$$

$$= - \iint_{S_a} \phi(\mathbf{r}')\frac{\partial\overline{\phi(\mathbf{r}')}}{\partial n'}\, d\sigma' - \iint_{\partial D} \phi(\mathbf{r}')\frac{\partial\overline{\phi(\mathbf{r}')}}{\partial n'}\, d\sigma' - \iiint_{D_a} \|\nabla'\phi(\mathbf{r}')\|^2 d\tau'$$

$$= \iint_{S_a} \phi(\mathbf{R})\frac{\partial\overline{\phi(\mathbf{R})}}{\partial R}\, d\sigma - \iint_{\partial D} \phi(\mathbf{r}')\frac{\partial\overline{\phi(\mathbf{r}')}}{\partial n'}\, d\sigma' - \iiint_{D_a} \|\nabla'\phi(\mathbf{r}')\|^2 d\tau'$$

from which we obtain

$$\iint_{S_a}\phi(\mathbf{R})\frac{\partial\overline{\phi(\mathbf{R})}}{\partial R}\, d\sigma = \iiint_{D_a}\left(\phi(\mathbf{r}')\nabla'^2\overline{\phi(\mathbf{r}')} + \|\nabla'\phi(\mathbf{r}')\|^2\right) d\tau' + \iint_{\partial D}\phi(\mathbf{r}')\frac{\partial\overline{\phi(\mathbf{r}')}}{\partial n'}\, d\sigma'$$

$$= \iiint_{D_a}\left(- \bar{k}^2 \|\phi(\mathbf{r}')\|^2 + \|\nabla'\phi(\mathbf{r}')\|^2\right) d\tau' + \iint_{\partial D}\phi(\mathbf{r}')\frac{\partial\overline{\phi(\mathbf{r}')}}{\partial n'}\, d\sigma'$$

since $\quad \nabla'^2\overline{\phi(\mathbf{r}')} + \bar{k}^2\, \overline{\phi(\mathbf{r}')} = 0$

if $\mathbf{r}' \in D_a$. Taking the conjugate of the above integral expression we obtain

$$\iint_{S_a}\overline{\phi(\mathbf{R})}\, \frac{\partial\phi(\mathbf{R})}{\partial R}\, d\sigma = \iiint_{D_a}\left(- k^2\|\phi(\mathbf{r}')\|^2 + \|\nabla'\phi(\mathbf{r}')\|^2\right)d\tau' + \iint_{\partial D}\overline{\phi(\mathbf{r}')}\, \frac{\partial\phi(\mathbf{r}')}{\partial n'}\, d\sigma'$$

Using these integral expressions in (2.6.8) we find that

$$\lim_{a\to\infty}\left[\iint_{S_a}\left(|k|^2|\phi(\mathbf{R})|^2+\left|\frac{\partial\phi(\mathbf{R})}{\partial R}\right|^2\right)d\sigma+2\,\mathrm{Im}(k)\;|k|^2\iiint_{D_a}|\phi(\mathbf{r}')|^2\,d\tau'\right.$$

$$\left.+\,2\,\mathrm{Im}(k)\iiint_{D_a}\|\nabla'\phi(\mathbf{r}')\|^2\,d\tau'\right]=-\,2\,\mathrm{Im}\left(k\iint_{\partial D}\phi(\mathbf{r}')\overline{\frac{\partial\phi(\mathbf{r}')}{\partial n'}}\,d\sigma'\right).$$

(2.6.9)

Following Wilcox we note that the right-hand side of (2.6.9) is a fixed number independent of the radius a of S_a and hence each of the non-negative terms on the left is bounded provided $\mathrm{Im}(k)\geq 0$. Hence for all $a>0$

$$\lim_{a\to\infty}\iint_{S_a}|\phi(\mathbf{R})|^2\,d\sigma<\infty \qquad (2.6.10)$$

and

$$\lim_{a\to\infty}\iint_{S_a}\left|\frac{\partial\phi(\mathbf{R})}{\partial R}\right|^2 d\sigma<\infty\,. \qquad (2.6.11)$$

If $\mathrm{Im}(k)>0$ then also

$$\lim_{a\to\infty}\iiint_{D_a}|\phi(\mathbf{r}')|^2\,d\tau'\;<\infty \qquad (2.6.12)$$

and

$$\lim_{a\to\infty}\iiint_{D_a}\|\nabla\,\phi(\mathbf{r}')\|^2\,d\tau'<\infty\,. \qquad (2.6.13)$$

We can now prove (2.6.7) . By Schwartz's inequality

$$\left|\iint_{S_a}\frac{e^{ikR}}{R^2}\,\phi(\mathbf{R})\,d\sigma\right|^2\leq\iint_{S_a}\left|\frac{e^{ikR}}{R^2}\right|^2 d\sigma.\iint_{S_a}\left|\frac{\phi(\mathbf{R})}{R}\right|^2 d\sigma$$

$$=16\pi^2\int_0^{2\pi}\int_0^{\pi}|\phi(a\hat{r})|\;\sin\theta\,d\theta\,d\varphi$$

where $\hat{r}=(\cos\varphi\sin\theta\;,\sin\varphi\sin\theta\;,\cos\theta\;)$. Now if M is an upper bound for (2.6.10) then we find that

$$\int_0^{2\pi}\int_0^{\pi}|\phi(a\hat{r})|\;\sin\theta\,d\theta\,d\varphi\;<\frac{M}{a^2}\;,$$

and hence (2.6.7) follows.

Thus when $r \in D$ we have proved that

$$\phi(r) = \iint_{\partial D} \left[\phi(r')\frac{\partial G_k(r,r')}{\partial n'} - G_k(r,r')\frac{\partial \phi(r')}{\partial n'}\right] d\sigma' \tag{2.6.14}$$

provided that ϕ satisfies the radiation condition (2.6.6).

If $r \in D^i$, then

$$\nabla'^2 G_k(r,r') + k^2 G_k(r,r') = 0$$

for all $r' \in D^e$. Hence

$$\iiint_{D_a} \left[\phi(r') \nabla'^2 G_k(r,r') - G_k(r,r') \nabla'^2 \phi(r')\right] d\tau' = 0$$

and so

$$\iint_{\partial D_a} \left[\phi(r')\frac{\partial G_k(r,r')}{\partial n'} - G_k(r,r')\frac{\partial \phi(r')}{\partial n'}\right] d\sigma' = 0 .$$

But $\partial D_a = S \cup S_a$, and the surface integral over S_a converges to zero when $a \to 0$; as $S = \partial D$, it follows that

$$\iint_{\partial D} \left[\phi(r')\frac{\partial G_k(r,r')}{\partial n'} - G_k(r,r')\frac{\partial \phi(r')}{\partial n'}\right] d\sigma' = 0 . \tag{2.6.15}$$

Theorem (2.6.1) (Helmholtz exterior formulae).
Let D be a region of $\mathbb{R}^3$ such that the boundary ∂D of D is a smooth regular surface , and let $\phi \in C^2(D^e) \cap C^1(\partial D)$ and satisfy the Helmholtz equation

$$\nabla^2 \phi + k^2 \phi = 0$$

in the exterior D^e of D . Then

$$\iint_{\partial D} \left[\phi(r')\frac{\partial G_k(r,r')}{\partial n'} - G_k(r,r')\frac{\partial \phi(r')}{\partial n'}\right] d\sigma' = \begin{cases} 0 & \text{if } r \in D^i \\ \phi(r) & \text{if } r \in D^e \end{cases}$$

provided that $\mathrm{Im}(k) \geq 0$ and that ϕ satisfies the radiation condition

$$\lim_{a\to\infty} \iint_{S_a} \left| ik\phi(R) - \frac{\partial \phi(R)}{\partial R}\right|^2 d\sigma = 0$$

when $r \in D^e$ and $S_a = \partial B(r,a) \supset D$.

Finally we note that the radiation condition (2.6.6) can be replaced by

$$\lim_{a\to\infty} \iint_{S(0,a)} \left| ik\phi(\mathbf{r}) - \frac{\partial\phi(\mathbf{r})}{\partial r} \right|^2 d\sigma = 0 . \qquad (2.6.16)$$

To prove this we note that (2.6.6) implies (2.6.16) since we may take the point $P \in D^e$ as the origin O. Conversely, (2.6.16) implies (2.6.6) since all the formulae used above are independent of the choice of origin.

If the integral (2.6.16) is expanded and transformed by means of Green's first identity we obtain the expression

$$\lim_{a\to\infty} \left[\iint_{S_a} \left(|k|^2 |\phi(\mathbf{r})|^2 + \left| \frac{\partial\phi(\mathbf{r})}{\partial r} \right|^2 \right) d\sigma + 2\,\mathrm{Im}(k)\, |k|^2 \iiint_{D_a} |\phi(\mathbf{r})|^2 \, d\tau \right.$$

$$\left. + 2\mathrm{Im}(k) \iiint_{D_a} \|\nabla\phi(\mathbf{r})\|^2 \, d\tau \right] = -\,2\,\mathrm{Im} \left\{ k \iint_{\partial D} \phi(\mathbf{r}) \overline{\frac{\partial\phi(\mathbf{r})}{\partial n}} \, d\sigma \right\} . \qquad (2.6.17)$$

2.7 Uniqueness of solutions of the exterior boundary value problem for the Helmholtz equation .

In this section we shall prove that the Helmholtz equation

$$\nabla^2\phi + k^2\phi = 0 \qquad (2.1.7)$$

in the exterior region D^e and with the Dirichlet or Neumann boundary conditions has a unique solution, provided that the solution satisfies a radiation condition at infinity. This problem has been dealt with by Sommerfeld , Atkinson (5) , Rellich (41) and Wilcox (53). The need for a uniqueness theorem for the exterior problem arises when the Helmholtz exterior formula (2.6.14) is used to derive integral equations for the field on the surface of the scatterer. If, for example, the Dirichlet boundary condition is incorporated into the integral equations, one obtains integral equations for the normal derivative $\frac{\partial\phi}{\partial n}$ of the surface field. However, these integral equations do not have unique solutions, and would then lead to non-unique exterior fields when use is made of the Helmholtz exterior formula. From physical considerations it is known that the scattered field is unique for a given incident field. The mathematical problem is solved by showing that the surface field on the scatterer satisfies two different integral equations which always have one and only one solution in common. The uniqueness theorem ensures that this is the physically correct solution. Other methods exist for obtaining the unique solution from one integral equation.

The uniqueness theorem is proved in the usual way by showing that the difference between any two solutions of the exterior boundary value problem is identically zero. Towards this end we require an expansion theorem due to Atkinson (5).

For any vector $\mathbf{r} \neq \mathbf{0}$ we denote by $\hat{\mathbf{r}}$ the unit vector corresponding to $\mathbf{r}$; i.e

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{\|\mathbf{r}\|} \quad , \text{ if } \mathbf{r} \neq \mathbf{0} \; .$$

Theorem (2.7.1) If ϕ satisfies the Helmholtz equation on D^e and the radiation condition (2.6.6), then ϕ can be expanded in a series

$$\phi(\mathbf{r}) = \frac{e^{ikr}}{4\pi r} \sum_{n=0}^{\infty} \frac{a_n(\hat{\mathbf{r}})}{r^n} \; , \tag{2.7.2}$$

the series being uniformly and absolutely convergent when $r > c$, where

$$c = (\sqrt{2} + 1) \sup \{ r : \mathbf{r} \in S \} \; ,$$

and $r = \| \mathbf{r} \|$. The positive number c depends only on the choice of origin.

Proof. By the Helmholtz exterior formula (2.6.14) we have

$$\phi(\mathbf{r}) = \iint_S \left(\phi(\mathbf{r}')\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} - G_k(\mathbf{r},\mathbf{r}')\frac{\partial \phi(\mathbf{r}')}{\partial n'} \right) d\sigma'$$

where $S = \partial D$ and G_k and $\frac{\partial G_k}{\partial n'}$ are given respectively by (2.1.8) and (2.2.3) . Then

$$G_k(\mathbf{r},\mathbf{r}') = \frac{e^{ikr} \, e^{ik(R- r)}}{4\pi R}$$

where

$$R = \sqrt{r^2 + r'^2 - 2rr' \cos\gamma} \; ,$$

γ being the angle between $\mathbf{r}$ and $\mathbf{r}'$, and $r = \|\mathbf{r}\|$, $r' = \|\mathbf{r}'\|$. If we let

$$x = \left(\frac{r'}{r}\right)^2 - 2 \left(\frac{r'}{r}\right) \cos\gamma,$$

then

$$R = r \sqrt{1 + x}$$

and $$G_k(r,r') = \frac{e^{ikr}}{4\pi r} \frac{e^{ikr(\sqrt{1+x}-1)}}{\sqrt{1+x}} .$$

If $|x| < 1$ then
$$\sqrt{1+x} - 1 = \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} x^n$$

the series being uniformly and absolutely convergent .
If $n \geq 1$ then
$$x^n = \left[\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\gamma\right]^n$$
$$= \left(\frac{r'}{r}\right)^n \sum_{m=0}^{n} \binom{n}{m} (-2\cos\gamma)^{n-m} \left(\frac{r'}{r}\right)^m$$

and
$$\sqrt{1+x} - 1 = \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} \left(\frac{r'}{r}\right)^n \sum_{m=0}^{n} \binom{n}{m} (-2\cos\gamma)^{n-m} \left(\frac{r'}{r}\right)^m$$
$$= \sum_{n=1}^{\infty} A_n \left(\frac{r'}{r}\right)^n ,$$

where
$$A_n = \sum_{m=0}^{N} \binom{\frac{1}{2}}{n-m} \binom{n-m}{n} (-2\cos\gamma)^{n-2m} ,$$

where $N = \left[\frac{n}{2}\right]$.
The above rearrangement in powers of $\frac{r'}{r}$ is permissible since the series is absolutely convergent when $|x| < 1$, that is when

$$\left|\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\gamma\right| < 1 .$$

It follows that the series is absolutely convergent when

$$0 < \frac{r'}{r} < \cos\gamma + \left(1 + \cos^2\gamma\right)^{1/2} .$$

The minimum value of the right hand side is $\sqrt{2} - 1$. The series is therefore uniformly and absolutely covergent when $r > (\sqrt{2} + 1)r'$. We have now obtained

$$G_k(r,r') = \frac{e^{ikr}}{4\pi r} (1 + x)^{-1/2} \exp\left[ikr \sum_{n=1}^{\infty} A_n \left(\frac{r'}{r}\right)^n\right] .$$

Again, $(1 + x)^{-1/2}$, and therefore the product

$$(1 + x)^{-1/2} \exp \left(ikr \sum_{n=1}^{\infty} A_n \left(\frac{r'}{r} \right)^n \right)$$

can be expanded in powers of $\frac{r'}{r}$, the series being uniformly and absolutely convergent when $r > (\sqrt{2} + 1)r'$. Thus

$$G_k(r,r') = \frac{e^{ikr}}{4\pi r} \sum_{n=0}^{\infty} B_n \left(\frac{r'}{r} \right)^n , \quad \text{if } r' > (\sqrt{2} + 1) ,$$

where the coefficients B_n depend only $\mathbf{r}'$ and γ . Consequently

$$\iint_S G_k(\mathbf{r},\mathbf{r}') \frac{\partial \phi(\mathbf{r}')}{\partial n'} d\sigma' = \frac{e^{ikr}}{4\pi r} \sum_{n=1}^{\infty} C_n(\hat{\mathbf{r}}) r^{-n} \tag{2.7.3}$$

and this series is uniformly and absolutely convergent when $r > c$. The coefficients C_n can only have the form

$$\iint_S f(\mathbf{r}',\gamma) \, d\sigma' ,$$

and can therefore depend only on the direction of $\mathbf{r}$, i.e. on the unit vector $\hat{\mathbf{r}}$.

In the same way

$$\frac{dG_k}{dR} = \left(ik - \frac{1}{R} \right) \frac{e^{ikR}}{4\pi R}$$

can be expanded in an uniformly and absolutely convergent series of powers of $\frac{r'}{r}$ when $r > (\sqrt{2} + 1)r'$. Consider now

$$\frac{\partial G_k}{\partial n'} = \frac{dG_k}{dR} \frac{\partial R}{\partial n'} ,$$

where $$R = \sqrt{r^2 + r'^2 - 2rr' \cos\gamma} ,$$

and

$$\frac{\partial R}{\partial n'} = \frac{\partial R}{\partial x_i'} n_i' = \frac{1}{R} \left(x_i' - \frac{2rx_i'}{r} \cos\gamma + 2rr' \sin\gamma \frac{\partial \gamma}{\partial x_i'} \right) n_i'$$

$$= \frac{1}{R} \left(\mathbf{r}'.\mathbf{n}' - \frac{2r}{r'} (\mathbf{r}'.\mathbf{n}') \cos\gamma + 2rr' \sin\gamma \frac{\partial \gamma}{\partial n_i'} \right) .$$

Since $$\cos\gamma = \frac{\mathbf{r}.\mathbf{r}'}{rr'} = \frac{x_j \, x_j'}{rr'} ,$$

differentiation in the direction $\mathbf{n}'$ yields

$$- \sin\gamma \frac{\partial\gamma}{\partial n'} = \frac{x_j}{r}\frac{\partial}{\partial x'}\left(\frac{x_j'}{r'}\right) n_i'$$

$$= \frac{x_j}{r}\left(\frac{\delta_{ij}}{r'} - \frac{x_j' x_i'}{r'^3}\right) n_i'$$

$$= \frac{x_j n_j'}{rr'} - \frac{x_j x_j' x_i' n_i'}{rr'^3} ,$$

and so

$$2rr' \sin\gamma \frac{\partial\gamma}{\partial n'} = 2rr' \left(\frac{x_j x_j' x_i' n_i'}{rr'^3} - \frac{x_j n_j'}{rr'}\right)$$

$$= \frac{2r}{r'} (\mathbf{r}'.\mathbf{n}') \cos\gamma - 2 \mathbf{r}.\mathbf{n}' .$$

Hence

$$\frac{\partial R}{\partial n'} = \frac{1}{R} \left(\mathbf{r}'.\mathbf{n}' - 2 \mathbf{r}.\mathbf{n}'\right) ,$$

i.e.
$$\frac{\partial R}{\partial n'} = (1 + x)^{-1/2} \left(\left(\frac{r'}{r}\right)(\hat{\mathbf{r}}'.\mathbf{n}') - 2 \hat{\mathbf{r}}.\mathbf{n}'\right) .$$

This expression can also be expanded in powers of $\frac{r'}{r}$ and the resulting series is uniformly and absolutely convergent when $r > (\sqrt{2} + 1)r'$. Consequently $\frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'}$ is an analytical function of $\frac{r'}{r}$ when $r > (\sqrt{2} + 1)r'$. Accordingly we can write

$$\iint_S \phi(\mathbf{r}') \frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'} d\sigma' = \frac{e^{ikr}}{4\pi r} \sum_{n=0}^{\infty} D(\hat{\mathbf{r}}) r^{-n} \qquad (2.7.4)$$

the series being uniformly and absolutely convergent when $r > c$. Thus, finally, (2.7.2) follows from (2.7.3) and (2.7.4) .

In the special case where S is a sphere, the coefficients a_n in (2.7.2) may be determined from the recurrence relation

$$2ika_n = n(n-1)a_n + Da_n \quad , \text{ if } n \geq 1 , \qquad (2.7.5)$$

where

$$Da_n = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial a_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 a_n}{\partial\varphi^2} \qquad (2.7.6)$$

in terms of spherical coordinates.

This recurrence relation follows when the expansion (2.4.2) is substituted into the Helmholtz equation (2.4.1) . If the Laplace operator is expressed in terms of spherical coordinates , one obtains

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} + k^2\phi = 0 .$$

Using ϕ as defined by (2.7.2) , we may write

$$\phi(r) \doteq \frac{e^{ikr}}{r}\psi(r)$$

where

$$\psi(r) = \sum_{n=0}^{\infty} a_n(\hat{r})\, r^{-n} .$$

Then

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + k^2 r\,\phi = re^{ikr}(\,2ik\,\psi'(r) + \psi''(r))$$

where

$$\psi'(r) = -\sum_{n=1}^{\infty}\frac{n\,a_n}{r^{n+1}} ,$$

and

$$\psi''(r) = \sum_{n=1}^{\infty}\frac{n(n+1)a_n}{r^{n+1}} .$$

It follows that

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + k^2r^2\phi = re^{ikr}\left(\sum_{n=2}^{\infty}\frac{n(n+1)a_n}{r^{n+1}} - 2ik\sum_{n=1}^{\infty}\frac{n\,a_n}{r^{n+1}}\right) .$$

Helmholtz's equation now assumes the form

$$re^{ikr}\left(\sum_{n=1}^{\infty}\frac{n(n+1)a_n}{r^{n+1}} - 2ik\sum_{n=1}^{\infty}\frac{n\,a_n}{r^{n+1}}\right) +$$

$$\frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{1}{r^n}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial a_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 a_n}{\partial\phi^2}\right] = 0 ,$$

or

$$r^2\left(\sum_{n=1}^{\infty}\frac{n(n+1)a_n}{r^{n+1}} - 2ik\sum_{n=1}^{\infty}\frac{n\,a_n}{r^{n+1}}\right) + \sum_{n=0}^{\infty}\frac{Da_{n-1}}{r^n} = 0 .$$

It follows that

$$2ika_1 = Da_0 ,$$

and for $n > 1$,

$$2ika_n = n(n-1)\,a_{n-1} + Da_{n-1} .$$

The uniqueness theorem can now be proved. Suppose ϕ and ψ are solutions of the exterior Dirichlet (or Neumann) problem for the Helmholtz equation:

$$\left.\begin{array}{ll} \nabla^2 \phi + k^2 \phi = 0 & \text{on } D^e \\ \phi = f \ (\text{ or } \dfrac{\partial \phi}{\partial n} = f \ ) & \text{on } \partial D \end{array}\right\} \tag{2.7.8}$$

and ϕ satisfies the radiation condition .

Then $\theta = \phi - \psi$ is a solution of the corresponding homogeneous problem

$$\left.\begin{array}{ll} \nabla^2 \theta + k^2 \theta = 0 & \text{on } D^e \\ \theta = 0 \ (\text{ or } \dfrac{\partial \theta}{\partial n} = 0 \ ) & \text{on } \partial D \end{array}\right\} \tag{2.7.9}$$

and θ satisfies the radiation condition .

That θ does satisfy the radiation condition is not quite obvious , since the radiation condition is not linear . To prove this we may use (2.7.2) as suggested by Wilcox (52), or use Schwartz's inequality for integrals . Thus if $S_a = \partial B(0,a)$, then

$$\iint_{S_a} \left| ik\theta(r) - \frac{\partial \theta(r)}{\partial n} \right|^2 d\sigma = \iint_{S_a} \left| A(r) - B(r) \right|^2 d\sigma$$

where

$$A(r) = ik\phi(r) - \frac{\partial \phi(r)}{\partial n}$$

and

$$B(r) = ik\psi(r) - \frac{\partial \psi(r)}{\partial n} .$$

Hence

$$\iint_{S_a} \left| ik\theta(r) - \frac{\partial \theta(r)}{\partial n} \right|^2 d\sigma = \iint_{S_a} \left(|A|^2 + |B|^2 - A\bar{B} - \bar{A}B \right) d\sigma. \tag{2.7.10}$$

Applying Schwartz's inequlity to the last two terms in the above integral, we obtain

$$\left| \iint_{S_a} A\bar{B} \, d\sigma \right|^2 \le \left(\iint_{S_a} |A|^2 \, d\sigma \right) \left(\iint_{S_a} |B|^2 \, d\sigma \right)$$

and

$$\left| \iint_{S_a} \bar{A}B \, d\sigma \right|^2 \le \left(\iint_{S_a} |A|^2 \, d\sigma \right) \left(\iint_{S_a} |B|^2 \, d\sigma \right) .$$

As ϕ and ψ satisfy the radiation condition ,

$$\lim_{a \to \infty} \iint_{S_a} |A|^2 \, d\sigma = 0 \quad \text{and} \quad \lim_{a \to \infty} \iint_{S_a} |B|^2 \, d\sigma = 0 ,$$

and from the above two inequalities we have that

$$\lim_{a\to\infty} \iint_{S_a} A\bar{B}\, d\sigma = 0 \qquad \text{and} \qquad \lim_{a\to\infty} \iint_{S_a} \bar{A}B\, d\sigma = 0 \;;$$

consequently

$$\lim_{a\to\infty} \iint_{S_a} \left| ik\theta(\mathbf{r}) - \frac{\partial\theta(\mathbf{r})}{\partial n} \right|^2 d\sigma = 0 \;.$$

It only remains to prove that θ is identically zero. If $\theta = 0$ on ∂D or if $\frac{\partial\theta}{\partial n} = 0$ on ∂D, then according to (2.6.17) we have

$$\lim_{a\to\infty} \left[\iint_{S_a} \left(|k|^2 |\theta(\mathbf{r})|^2 + \left|\frac{\partial\theta(\mathbf{r})}{\partial r}\right|^2 \right) d\sigma + 2\,\mathrm{Im}(k)\,|k|^2 \iiint_{D_a} |\theta(\mathbf{r})|^2\, d\tau \right.$$

$$\left. + 2\,\mathrm{Im}(k) \iiint_{D_a} \|\nabla\theta(\mathbf{r})\|^2\, d\tau \right] = 0 \qquad (2.7.11)$$

If $\mathrm{Im}(k) > 0$, then (2.7.11) implies that

$$\lim_{a\to\infty} \iiint_{D_a} |\theta(\mathbf{r})|^2\, d\tau = 0$$

and it follows that $\theta = 0$ on D_a .
If $\mathrm{Im}(k) = 0$ and $k \neq 0$ then (2.7.11) implies that

$$\lim_{a\to\infty} \iint_{S_a} |\theta(\mathbf{r})|^2\, d\sigma = 0 \;. \qquad (2.7.12)$$

But by the Atkinson expansion theorem , we can write

$$\theta(\mathbf{r}) = \frac{e^{ikr}}{4\pi r} \sum_{n=0}^{\infty} \frac{a_n(\hat{\mathbf{r}})}{r^n} \;, \quad r > c \;,$$

and

$$|\theta(\mathbf{r})|^2 = \theta(\mathbf{r})\overline{\theta(\mathbf{r})}$$

$$= \frac{1}{16\pi r^2} \sum_{n=0}^{\infty} \left(\sum_{p+q=n} a_p(\hat{\mathbf{r}})\, \overline{a_q(\hat{\mathbf{r}})} \right) r^{-n} \;.$$

Substituting into (2.7.12) yields

$$\lim_{a\to\infty} \sum_{n=0}^{\infty} a^{-n} \int_0^{2\pi}\int_0^{\pi} \left(\sum_{p+q=n} a_p(\hat{r})\, \overline{a_q(\hat{r})} \right) \sin\theta \; d\theta d\phi = 0 \; .$$

As $a^{-n} \to 0$ when $a \to \infty$ and $n \geq 1$, it follows that

$$\int_0^{2\pi}\int_0^{\pi} |a_0(\hat{r})|^2 \sin\theta \; d\theta d\phi = 0 \; ,$$

which implies that $a_0(\hat{r}) = 0$ for all directions $\hat{r}$.

It follows from the recurrence relation (2.7.5) that

$$a_n(\hat{r}) = 0 \; , \quad n = 0,1,2,3,\ldots \; , \quad \text{and for all } \hat{r} \; .$$

Hence we have shown that $\theta(r) = 0$ provided $r \in D^e$ and $r > c$. The proof is completed by noting that according to Theorem (2.6.1)

$$\theta(r) = \iint_S \left[\theta(r') \frac{\partial G_k(r,r')}{\partial n'} - G_k(r,r') \frac{\partial \theta(r')}{\partial n'} \right] d\sigma' \; , \; r \in D^e \; ,$$

which implies that θ is an analytical function on D^e. Thus the analycity of θ together with the fact that $\theta(r) = 0$ whenever $r > c$ implies that $\theta = 0$ on D^e .

Theorem (2.7.2) (Uniqueness theorem)

If D is a bounded regular region of $\mathbb{R}^3$ then the exterior Dirichlet or Neumann boundary value problem for the Helmholtz equation

$$\nabla^2 \phi + k^2 \phi = 0 \quad \text{on } D^e$$

has a unique solution provided that the radiation condition is satisfied and $Im(k) \geq 0$.

CHAPTER III

INTEGRAL EQUATION FORMULATION OF BOUNDARY VALUE PROBLEMS FOR THE SCALAR HELMHOLTZ EQUATION

3.1 Operators on the Hilbert Space $L^2(\partial D)$.

In this section we briefly recall those parts of the theory of operators on the Hilbert space $L^2(\partial D)$ which are required for a complete solution of the boundary value problems of the scalar Helmholtz equation. We again let D denote a bounded regular region of $\mathbb{R}^3$. Then ∂D is compact and so $C^n(\partial D) \subset L^2(\partial D)$ for $n = 0,1,2,\ldots$.

If ϕ and ψ are members of $L^2(\partial D)$, then their ***inner product*** is denoted by $\langle\phi,\psi\rangle$ and is defined by

$$\langle\phi,\psi\rangle = \iint_{\partial D} \phi(r)\ \overline{\psi(r)}\ d\sigma \tag{3.1.1}$$

where $\overline{\psi(r)}$ is the complex conjugate of $\psi(r)$.

Corresponding to a complex valued function K defined on the Cartesian product $\partial D \times \partial D$ there is an integral operator A on $L^2(\partial D)$ given by

$$(A\phi)(r) = \iint_{\partial D} K(r,r')\ \phi(r')\ d\sigma' \ . \tag{3.1.2}$$

We say that A is the integral operator generated by the ***kernel*** K . We assume, temporarily, that the operator A is properly defined, that is $\phi \in L^2(\partial D)$ implies $A\phi \in L^2(\partial D)$, and make the following definitions:

The ***transpose kernel*** K^T of K is defined by

$$K^T(r,r') = K(r',r) \tag{3.1.3}$$

and the ***transpose operator*** A^T generated by K^T is given by

$$(A^T\phi)(r) = \iint_{\partial D} K(r',r)\phi(r')d\sigma' \tag{3.1.4}$$

We say that the kernel K is ***symmetrical*** iff

$$K^T = K$$

and then the operator A is also ***symmetrical:***

$$A^T = A \ .$$

The *adjoint operator* A^* of A is defined by the inner product relation

$$\langle A\phi,\psi\rangle = \langle \phi, A^*\psi\rangle . \tag{3.1.5}$$

From (3.1.1) it follows that

$$\langle A\phi,\psi\rangle = \iint_{\partial D} (A\phi)(\mathbf{r}')\ \overline{\psi(\mathbf{r}')}\ d\sigma'$$

$$= \iint_{\partial D} \left[\iint_{\partial D} K(\mathbf{r},\mathbf{r}')\ \phi(\mathbf{r})\ d\sigma \right] \overline{\psi(\mathbf{r}')}\ d\sigma'$$

$$= \iint_{\partial D} \phi(\mathbf{r}) \left[\iint_{\partial D} K(\mathbf{r}',\mathbf{r})\ \psi(\mathbf{r}')\ d\sigma' \right] d\sigma ,$$

and hence

$$(A^*\phi)(r) = \iint_{\partial D} \overline{K(r',r)}\ \psi(r')\ d\sigma' .$$

Thus the adjoint operator A^* is generated by the *adjoint kernel* K^* defined by

$$K^*(r,r') = \overline{K(r',r)} = \overline{K^T(r,r')} ;$$

or

$$K^* = \overline{K^T} \tag{3.1.6}$$

Hence

$$A^* = \overline{A^T} \tag{3.1.7}$$

We say that the operator A is *self-adjoint* iff

$$A^* = A ,$$

or, equivalently , iff

$$K^* = K .$$

We now return to (3.1.2) and state conditions ensuring the existence of the integral operator A .

If $K \in L^2(\partial D \times \partial D)$, i.e. if

$$\iint_{\partial D} \iint_{\partial D} |K(r,r')|^2\ d\sigma\ d\sigma' < \infty , \tag{3.1.8}$$

we say that K is a ***Fredholm kernel*** , and the operator A generated by K is called a ***Fredholm operator***. It is shown by Mikhlin(37) that a Fredholm operator is defined on all of $L^2(\partial D)$, i.e. if $\phi \in L^2(\partial D)$ then $A\phi \in L^2(\partial D)$, or $A(L^2(\partial D)) \subset L^2(\partial D)$. Moreover, a Fredholm operator A on $L^2(\partial D)$ is ***completely continuous***; i.e. A maps bounded subsets of $L^2(\partial D)$ into compact subsets of $L^2(\partial D)$.

The kernels of the Helmholtz potentials are, however, not Fredholm kernels in general. We say that a kernel $K : \partial D \times \partial D \to \mathbb{C}$ is ***weakly singular*** iff

$$K(r,r') = \frac{f(r,r')}{\|r - r'\|^{\alpha}} \tag{3.1.9}$$

where $0 \le \alpha < 2$ and the function $f:\partial D \times \partial D \to \mathbb{C}$ is continuous. The weakly singular operator generated by a weakly singular kernel is defined on all of $L^2(\partial D)$ and is completely continuous on $L^2(\partial D)$. If $0 \le \alpha < 1$ then a weakly singular kernel (operator) is also a Fredholm kernel (operator). These results are proved by Mikhlin(37). He also shows that a Fredholm or a weakly singular operator on $C^2(\partial D)$ has its range in $C^2(\partial D)$, and the operator $A:C^2(\partial D) \to C^2(\partial D)$ is completely continuous. In the literature Fredholm kernels (operators) are also called Hilbert-Schmidt kernels (operators). See e.g. Akhierzer and Glazman (5).

The ***resolvent*** set $\rho(A)$ of an operator A on $L^2(\partial D)$ is the set of all $\lambda \in \mathbb{C}$ such that the range of the operator $(A - \lambda I)$ is dense in $L^2(\partial D)$ and $(A - \lambda I)$ has a continuous (i.e. bounded) inverse. Here I is the identity operator on $L^2(\partial D)$. The ***spectrum*** of A is the set $\sigma(A) = C - \rho(A)$. The ***eigenvalues*** of A is the set of all complex numbers λ for which the equation

$$(A - \lambda I)\phi = 0 \tag{3.1.10}$$

has non-trivial solutions ϕ, which are called ***eigenfunctions*** of A. The ***eigenspace*** of A correponding to the eigenvalue λ is the class of all non-trivial solutions of (3.1.10), and is denoted by $E_\lambda(A)$. The eigenspace $E_\lambda(A)$ of A is the same as the null space $N(A - \lambda I)$ of the operator $A - \lambda I$, i.e.

$$E_\lambda(A) = N(A-\lambda I) = \{ \phi \in L^2(\partial D) : A\phi = \lambda\phi \} \tag{3.1.11}$$

For a completely continuous operator A on $L^2(\partial D)$ we can state the following theorems. (See e.g. Taylor and Lay (47) , Akhierzer and Glazman (5)).

(1) $\lambda \neq 0$ is an eigenvalue of the completely continuous operator A if and only if its complex conjugate $\bar{\lambda}$ is an eigenvalue of the adjoint operator A^*.

(2) For any $\lambda \neq 0$ the null space $N(A - \lambda I)$ is finite dimensional.

(3) If λ is an eigenvalue of the completely continuous operator A then

$$\dim(N(A^* - \lambda I)) = \dim(N(A - \lambda I)) < \infty .$$

(4) The spectrum of a completely continuous operator contains at most a countable number of points, and these have no accumulation points, except possibly the point $\lambda = 0$. Each non-zero point of the spectrum is an eigenvalue.

(5) If A is completely continuous then the equation

$$(A - \lambda I)\phi = \psi$$

has a solution if and only if ψ is orthogonal to the null space $N(A^* - \lambda I)$.

In the theory of integral equations one encounters equations of the form

$$(I - \lambda A)\phi = \psi \qquad (3.1.12)$$

where $\psi \in L^2(\partial D)$. The corresponding homogeneous equation is

$$(I - \lambda A)\phi = 0 \qquad (3.1.13)$$

A complex number λ for which this equation has a non-trivial solution ϕ is called a ***characteristic value*** of A, and ϕ is then the ***characteristic function*** corresponding to λ. The dimension of the null space $N(I - \lambda A)$ is also called the ***rank*** or ***multiplicity*** of λ. If A is an integral operator on $L^2(\partial D)$ then (3.1.12) is called a ***Fredholm integral equation of the second kind***. If the operator A is completely continuous, then the above theorems on completely continuous operators are usually stated in the following form: (See e.g. Mikhlin (37), Smirnov (45), Smithies (46))

(1) Every characteristic value of (3.3.13) has finite rank.

(2) Equation (3.1.13) has either a finite or countable set of characteristic values; if the set of characteristic values is infinite, they have a unique accumulation point.

(3) If λ is a characteristic value of (3.1.13), then its complex conjugate $\bar{\lambda}$ is a characteristic value of the homogeneous adjoint equation ; i.e.

$$(I - \bar{\lambda}A^*)\chi = 0 \ , \ \chi \neq 0 \ , \qquad (3.1.14)$$

and they have the same rank.

(4) The inhomogeneous equation (3.1.12) has a solution if and only if ψ is orthogonal to the characteristic functions of the homogeneous adjoint equation (3.1.14).

From (3) and (4) above there follows the theorem known as Fredholm's Alternative:

If $\lambda \neq 0$, either the homogeneous equation (3.1.13) has only the trivial solution, in which case the inhomogeneous equation (3.1.12) has a unique solution for all $\psi \in L^2(\partial D)$, or the homogeneous equation (3.1.13) has ν linearly independent solutions, in which case the inhomogeneous equation has ν solutions if and only if ψ is orthogonal to the null space $N(I - \bar{\lambda}A^*)$.

In the sequel we will also have occasion to use ***Fredholm equations of the first kind***. These are equations of the form

$$A\phi = \psi \tag{3.1.15}$$

where A is an integral operator. All that need be said about (3.1.15) is that a solution exists if and only if ψ is an element of the range of A, that is, $\psi \in A(L^2(\partial D))$. Since the closure of the range of A is equal to the orthogonal complement of the null space of A^*, it follows that a necessary condition for (3.1.15) to have a solution is that ψ is orthogonal to the null space of A^*.

3.2 The Helmholtz potential operators.

With reference to section (2.4) we see that the theorems proved there hold if we assume that the single and double layer densities on a regular surface element S are of class C^2 on S, the surface element S also being of class C^2. Henceforth we consider only single and double layer densities of class C^2 on the boundary ∂D of a regular region $D \subset \mathbb{R}^3$, and in addition we assume that ∂D is of class C^2. The results of section (2.4) then show that the Helmholtz potentials are sectionally continuous across ∂D. Accordingly we introduce the subspace $SC(\mathbb{R}^3, \partial D)$ of $L(\mathbb{R}^3)$, consisting of bounded functions continuous on D^i and D^e but possibly discontinuous across ∂D. We then define linear operators $L_k : C^2(\partial D) \to SC(\mathbb{R}^3, \partial D)$ and $M_k : C^2(\partial D) \to SC(\mathbb{R}^3, \partial D)$ by

$$(L_k\phi)(r) = \iint_{\partial D} G_k(r,r')\, \phi(r')\, d\sigma' \tag{3.2.1}$$

and

$$(M_k\phi)(r) = \iint_{\partial D} \frac{\partial G_k(r,r')}{\partial n'}\, \phi(r')\, d\sigma' \tag{3.2.2}$$

for all $r \in \mathbb{R}^3$, where G_k is the free space Green's function defined by (2.1.8). Strictly speaking L_k and M_k are not integral operators in the sence of section (3.1). We will nevertheless call G_k the ***kernel*** of L_k and $K_k = \frac{\partial G_k}{\partial n'}$ the kernel of M_k. We call L_k the ***Helmholtz single layer potential operator*** and M_k the ***Helmholtz double layer potential operator***.

If the field point r in (3.2.1) and (3.2.2) is restricted to ∂D, we obtain ***boundary integral operators*** in the sence of section (3.1), and we denote these integral operators by ∂L_k and ∂M_k . Thus if $\phi \in C^2(\partial D)$, Then

$$(\partial L_k \phi)(r) = (L_k \phi)(r) \ , \ \text{if } r \in \partial D \ ,$$

and

$$(\partial M_k \phi)(r) = (M_k \phi)(r) \ , \ \text{if } r \in \partial D \ .$$

It is immediately apparent that the kernel G_k of the boundary operator ∂L_k is weakly singular for any regular region D. Thus ∂L_k is a weakly singular operator in the sence of section (3.1). It is also true that M_k is a weakly singular integral operator provided that ∂D is a regular surface of class C^2. To prove this statement, we note that

$$\frac{\partial G_k(r,r')}{\partial n'} = (ikR - 1)\ e^{ikR}\ \frac{R.n(r')}{4\pi R^3} \ , \ \text{if } r' \neq r \ ,$$

and

$$\left| \frac{\partial G_k(r,r')}{\partial n'} \right| \le \frac{1}{4\pi} \left(\frac{k}{R} + \frac{|R.n'|}{R^3} \right) .$$

Let r be an interior point of a regular suface element S(r) of class C^2 and contained in ∂D, and assume that S(r) has a standard representation

$$\xi_3' = f(\xi_1', \xi_2')$$

with respect to tangent-normal axes with origin at r. Using the relation (2.3.45) we obtain

$$\frac{|R.n'|}{R^3} = O(R^{-1}) \ , \ \text{if } r' \in S(r) \ ;$$

hence $$\left| \frac{\partial G_k(r,r')}{\partial n'} \right| = O(R^{-1})$$

whenever $r' \in S(r)$.

The kernel G_k of ∂L_k is symmetrical, i.e. $G_k^T = G_k$ and hence ∂L_k is symmetrical:

$$(\partial L_k)^T = \partial L_k \ .$$

The adjoint kernel of G_k is given by

$$G_k^*(\mathbf{r},\mathbf{r}') = \overline{G_k(\mathbf{r}',\mathbf{r})} = \frac{e^{-ikR}}{4\pi R} = \overline{G_k(\mathbf{r},\mathbf{r}')} ,$$

that is

$$G_k^* = \overline{G_k} ,$$

and hence

$$(\partial L_k)^* = \overline{\partial L_k} .$$

The transpose of the kernel K_k of ∂M_k is given by

$$K_k^T(\mathbf{r},\mathbf{r}') = K_k(\mathbf{r}',\mathbf{r}) = \frac{\partial G_k(\mathbf{r},\mathbf{r}')}{\partial n'}$$

$$= - (ikR - 1)\ e^{ikR}\ \frac{\mathbf{R}.\mathbf{n}(\mathbf{r})}{R^3} ,$$

which is again weakly singular, ∂D being of class C^2. For the transpose of ∂M_k we therefore have

$$((\partial M_k)^T\phi)(\mathbf{r}) = \iint_{\partial D} \frac{\partial G_k(\mathbf{r}',\mathbf{r})}{\partial n}\ \phi(\mathbf{r}')\ d\sigma'$$

$$= \frac{\partial}{\partial n} \iint_{\partial D} G_k(\mathbf{r},\mathbf{r}')\ \phi(\mathbf{r}')\ d\sigma'$$

$$= \frac{\partial}{\partial n}(\partial L_k\phi)(\mathbf{r}) ;$$

thus

$$(\partial M_k)^T = \frac{\partial}{\partial n}(\partial L_k) .$$

So far we have defined the transpose and adjoint operators only for boundary integral operators. However, G_k is always symmetrical; in particular

$$G_k(\mathbf{r}',\mathbf{r}) = G_k(\mathbf{r},\mathbf{r}')$$

whenever $\mathbf{r} \in \mathbb{R}^3$ and $\mathbf{r}' \in \partial D$. Thus we define the transpose of L_k by

$$(L_k^T\phi)(\mathbf{r}) = \iint_{\partial D} G_k(\mathbf{r}',\mathbf{r})\ \phi(\mathbf{r}')d\sigma' , \quad \mathbf{r} \in \mathbb{R}^3 ,$$

and the adjoint of L_k by

$$(L_k^*\phi)(\mathbf{r}) = \iint_{\partial D} \overline{G_k(\mathbf{r}',\mathbf{r})}\ \phi(\mathbf{r}')\ d\sigma' .$$

Therefore

$$L_k^T = L_k$$

and

$$L_k^* = \overline{L_k} .$$

It is clear that $\partial(L_k^T) = (\partial L_k)^T$

and

$$\partial(L_k^*) = (\partial L_k)^* .$$

Similarly the transpose of M_k is defined by

$$(M_k^T \phi)(r) = \iint_{\partial D} \frac{\partial G_k(r',r)}{\partial n} \phi(r') \, d\sigma'$$

and its adjoint by

$$(M_k^* \phi)(r) = \iint_{\partial D} \overline{\frac{\partial G_k(r',r)}{\partial n}} \phi(r') \, d\sigma'$$

We then find that

$$M_k^T = \frac{\partial}{\partial n}(L_k) \tag{3.2.3}$$

and

$$M_k^* = \frac{\partial}{\partial n}(\overline{L_k}) .$$

Moreover it is clear that

$$\partial(M_k^T) = (\partial M_k)^T$$

and

$$\partial(M_k^*) = (\partial M_k)^* .$$

We also define an operator $N_k : C^2(\partial D) \to SC(\mathbb{R}^3, \partial D)$ by

$$(N_k\phi)(r) = \frac{\partial}{\partial n} \iint_{\partial D} \frac{\partial G_k(r,r')}{\partial n'} \phi(r') \, d\sigma' \tag{3.2.4}$$

or

$$N_k = \frac{\partial}{\partial n}(M_k) \tag{3.2.5}$$

The corresponding boundary operator ∂N_k cannot be represented by a weakly singular kernel. For the transpose and adjoint of N_k we find that

$$N_k^T = \frac{\partial^2}{\partial n^2}(L_k)$$

and

$$N_k^* = \frac{\partial^2}{\partial n^2}(\overline{L_k}) .$$

Because of the similar properties of the operator L_k and the corresponding boundary operator ∂L_k , we will not distinguish between them in the sequel. The same convention holds for the other operators, viz. M_k and N_k .

We now restate the continuity and discontinuity relations of section (2.4) in terms of the operators L_k , M_k and N_k . We assume that ∂D is of class C^2 and that $\phi \in C^2(\partial D)$.

(1) $L_k\phi$ is continuous across ∂D, i.e. if $\mathbf{r} \in \partial D$ then

$$(L_k\phi)(\mathbf{r}-) = (L_k\phi)(\mathbf{r}) = (L_k\phi)(\mathbf{r}+). \qquad (3.2.6)$$

(2) The normal derivative $M_k^T\phi$ of $L_k\phi$ is discontinuous across ∂D; i.e if $\mathbf{r} \in \partial D$ then

$$(M_k^T\phi)(\mathbf{r}-) - \tfrac{1}{2}\,\phi(\mathbf{r}) = (M_k^T\phi)(\mathbf{r}) = (M_k^T\phi)(\mathbf{r}+) + \tfrac{1}{2}\,\phi(\mathbf{r}). \qquad (3.2.7)$$

(3) $M_k\phi$ is discontinuous across ∂D; i.e. if $\mathbf{r} \in \partial D$ then

$$(M_k\phi)(\mathbf{r}-) + \tfrac{1}{2}\,\phi(\mathbf{r}) = (M_k\phi)(\mathbf{r}) = (M_k\phi)(\mathbf{r}+) - \tfrac{1}{2}\,\phi(\mathbf{r}). \qquad (3.2.8)$$

(4) The normal derivative $N_k\phi$ of $M_k\phi$ is continuous across ∂D; i.e. if $\mathbf{r} \in \partial D$ then

$$(N_k\phi)(\mathbf{r}-) = (N_k\phi)(\mathbf{r}) = (N_k\phi)(\mathbf{r}+). \qquad (3.2.9)$$

We now express the Helmholtz formulae of sections (2.5) and (2.6) in terms of the linear boundary operators L_k and M_k . If these formulae are differentiated in a normal direction, the results can be expressed in terms of the operators M_k and N_k .

- (a) ***Helmholtz interior formulae:***

  If $\nabla^2\phi + k^2\phi = 0$ on D^i then

$$(M_k\phi)(\mathbf{r}) - \left[L_k \frac{\partial\phi}{\partial n}\right](\mathbf{r}) = \begin{cases} -\,\phi(\mathbf{r}) & \text{if } \mathbf{r} \in D^i \\ 0 & \text{if } \mathbf{r} \in D^e \end{cases} \qquad (3.2.10)$$

- (b) ***Differentiated Helmholtz interior formulae:***

  If $\nabla^2\phi + k^2\phi = 0$ on D^i then

$$(N_k\phi)(\mathbf{r}) - \left[M_k^T \frac{\partial\phi}{\partial n}\right](\mathbf{r}) = \begin{cases} -\,\dfrac{\partial\phi(\mathbf{r})}{\partial n} & \text{if } \mathbf{r} \in D^i \\ 0 & \text{if } \mathbf{r} \in D^e \end{cases} \qquad (3.2.11)$$

$$(N_k\phi)(r) - \left[M_k^T \frac{\partial\phi}{\partial n}\right](r) = \begin{cases} -\dfrac{\partial\phi(r)}{\partial n} & \text{if } r \in D^i \\ -\dfrac{1}{2}\dfrac{\partial\phi(r)}{\partial n} & \text{if } r \in \partial D \\ 0 & \text{if } r \in D^e \end{cases} \tag{3.2.15}$$

- (b) ***Extended Helmholtz exterior formulae:***

  If $\nabla^2 \phi + k^2 \phi = 0$ on D^e and if ϕ satisfies the radiation condition (2.6.6) then

$$(M_k\phi)(r) - \left[L_k \frac{\partial\phi}{\partial n}\right](r) = \begin{cases} 0 & \text{if } r \in D^i \\ \frac{1}{2}\phi(r) & \text{if } r \in \partial D \\ \phi(r) & \text{if } r \in D^e \end{cases} \tag{3.2.16}$$

and

$$(N_k\phi)(r) - \left[M_k^T \frac{\partial\phi}{\partial n}\right](r) = \begin{cases} 0 & \text{if } r \in D^i \\ \dfrac{1}{2}\dfrac{\partial\phi(r)}{\partial n} & \text{if } r \in \partial D \\ \dfrac{\partial\phi(r)}{\partial n} & \text{if } r \in D^e \end{cases} \tag{3.2.17}$$

These extended Helmholtz formulae will be used in the following sections to derive integral equations for boundary value problems involving the Helmholtz equation.

3.3 Boundary integral equations for the Helmholtz equation.

Here we briefly review various boundary integral equations which are of use in solving interior and exterior Dirichlet or Neumann boundary value problems for the Helmholtz equation. A complete classification of all such boundary integral equations together with the solution of the boundary value problem is given by Kleinman and Roach (26). The boundary integral equations are obtained using either the Helmholtz boundary formulae, or by assuming that the required solution can be represented by a single or double layer potential, and then applying the relations (3.2.6)-(3.2.9) to obtain the boundary integral equation.

- ***The interior Dirichlet problem.***

$$\left.\begin{aligned} \phi &\in C^2(D^i) \cap C^0(D) \\ \nabla^2\phi + k^2\phi &= 0 \quad \text{on } D^i \\ \phi &= f \quad \text{on } \partial D \end{aligned}\right\} \tag{3.3.1}$$

(a) Helmholtz formulation.

From the Helmholtz boundary formulae we obtain

$$L_k \frac{\partial\phi}{\partial n} = (\tfrac{1}{2}\phi + M_k)f \quad \text{on } \partial D , \tag{3.3.2}$$

or $$(\tfrac{1}{2} I - M_k^T) \frac{\partial \phi}{\partial n} = - N_k f \quad \text{on } \partial D . \tag{3.3.3}$$

Hence if ψ is a solution of (3.3.2) or (3.3.3) then

$$\phi = L_k\psi - M_k f \tag{3.3.4}$$

is a solution of (3.3.1).

(b) Single layer formulation.

We assume that there is a function ζ on ∂D such that

$$\phi = L_k\zeta \quad \text{on } D^i . \tag{3.3.5}$$

If $r \in \partial D$ we have from (3.2.6) that

$$f(r) = \phi(r-) = (L_k\zeta)(r-) = (L_k\zeta)(r) ;$$

hence the boundary integral equation is

$$L_k\zeta = f \quad \text{on } \partial D. \tag{3.3.6}$$

(c) Double layer formulation:

Assume that there is a function μ on ∂D such that

$$\phi = M_k\mu \quad \text{on } D^i . \tag{3.3.7}$$

If $r \in \partial D$ then

$$f(r) = \phi(r-) = (M_k\mu)(r-) = (M_k\mu)(r) - \tfrac{1}{2}\mu(r).$$

Hence the boundary integral equation is

$$(\tfrac{1}{2} I - M_k)\mu = f \quad \text{on } \partial D. \tag{3.3.8}$$

• ***Interior Neumann problem.***

$$\left.\begin{aligned} &\phi \in C^2(D^i) \cap C^0(D) \\ &\nabla^2\phi + k^2\phi = 0 \quad \text{on } D^i \\ &\frac{\partial \phi}{\partial n} = g \quad \text{on } \partial D \end{aligned}\right\} \tag{3.3.9}$$

(a) Helmholtz formulation.

From the Helmholtz boundary formulae we obtain

$$(\tfrac{1}{2} I + M_k)\phi = L_k g , \quad \text{on } \partial D , \tag{3.3.10}$$

$$N_k \phi = M_k^T g' , \quad \text{on } \partial D . \tag{3.3.11}$$

Hence if ψ is any solution of (3.3.10) or (3.3.11) then

$$\phi = L_k g - M_k\psi. \tag{3.3.12}$$

(b) Single layer formulation.

Assume that there is a function ζ on ∂D such that

$$\phi = L_k \zeta \quad \text{on } \partial D. \tag{3.3.13}$$

Then the boundary integral equation is

$$(\tfrac{1}{2} I + M_k^{T'}) \zeta = g \quad \text{on } \partial D . \tag{3.3.14}$$

(c) Double layer formulation.

Assume that there is a function μ on ∂D such that

$$\phi = M_k \mu \quad \text{on } D^i. \tag{3.3.15}$$

Then

$$N_k \mu = g \quad \text{on } \partial D. \tag{3.3.16}$$

• ***Exterior Dirichlet problem.***

$$\left.\begin{aligned} \phi \in C^2(D^e) \cap C^0(D^e \cup \partial D) \\ \nabla^2 \phi + k^2 \phi = 0 \quad \text{on } D^e \\ \phi = f \quad \text{on } \partial D \end{aligned}\right\} \tag{3.3.17}$$

and ϕ satisfies the radiation condition (2.6.6).

(a) Helmholtz formulation.

From the exterior Helmholtz boundary formulae we have

$$L_k \frac{\partial \phi}{\partial n} = (M_k - \tfrac{1}{2} I) f \quad \text{on } \partial D \tag{3.3.18}$$

and

$$(\tfrac{1}{2} I + M_k^T) \frac{\partial \phi}{\partial n} = N_k f \quad \text{on } \partial D. \tag{3.3.19}$$

If ψ is any solution of (3.3.18) or (3.3.19) then (3.3.17) must have a solution of the form

$$\phi = M_k f - L_k \psi. \tag{3.3.20}$$

However, by the uniqueness theorem (Theorem (2.7.2)), (3.3.17) has a unique solution. We note that (3.3.19) is a Fredholm integral equation of the second kind, and if $-\frac{1}{2}$ is an eigenvalue of M_k^T, then the homogeneous equation corresponding to (3.3.19) has a finite number of linearly independent solutions. In this case (3.3.20) is not unique, and a method for determining the correct solution must be found. This problem is dealt with in section (3.6).

(b) Single layer formulation.

Assume that there is a function ζ on ∂D such that

$$\phi = L_k \zeta \quad \text{on } D^e, \tag{3.3.21}$$

then

$$L_k \zeta = f \quad \text{on } \partial D. \tag{3.3.22}$$

(c) Double layer formulation.

Assume that there is a function μ on ∂D such that

$$\phi = M_k \mu \quad \text{on } D^e \tag{3.3.23}$$

then

$$(\tfrac{1}{2} I + M_k) \mu = f \quad \text{on } \partial D. \tag{3.3.24}$$

• ***Exterior Neumann problem.***

$$\left.\begin{array}{ll} \phi \in C^2(D\ )\cap C^0(D^e\cup\partial D) & \\ \nabla^2\ \phi + k^2\phi = 0 & \text{on } D^e \\ \dfrac{\partial\phi}{\partial n} = g & \text{on } \partial D \end{array}\right\} \qquad (3.3.25)$$

and ϕ satisfies the radiation condition (2.6.6).

(a) Helmholtz formulation.

From the exterior Helmholtz boundary formulae we obtain

$$(\tfrac{1}{2}I - M_k)\,\phi = L_k\,g \quad \text{on } \partial D \qquad (3.3.26)$$

and

$$N_k\,\phi = (\tfrac{1}{2}I + M_k^T)\,g \quad \text{on } \partial D. \qquad (3.3.27)$$

In section (3.7) we prove that the two equations above always have a unique solution ψ. Then the unique exterior field ϕ is given by

$$\phi = M_k\,\psi - L_k\,g\ . \qquad (3.3.28)$$

(b) Single layer formulation.

Assume that there is a function ς on ∂D such that

$$\phi = L_k\,\varsigma \quad \text{on } D^e\ . \qquad (3.3.29)$$

We then find that

$$(-\tfrac{1}{2}I + M_k^T)\,\varsigma = g \quad \text{on } \partial D\ . \qquad (3.3.30)$$

(c) Double layer formulation.

Asume that there is a function μ on ∂D such that

$$\phi = M_k\,\mu \quad \text{on } D^e\ . \qquad (3.3.31)$$

In this case

$$N_k\,\mu = g \quad \text{on } \partial D\ . \qquad (3.3.32)$$

We conclude this section by proving the following theorem which has frequent application in the following sections of this chapter.

Theorem (3.3.1). If $\varsigma,\mu \in C^2(\partial D)$ and if

$$\Phi = L_k\,\varsigma$$

or

$$\Phi = M_k\,\mu$$

on D^i and/or D^e, then Φ is a solution of the Helmholtz equation in D^i and/or D^e, and in the latter case Φ also satisfies the Wilcox radiation condition.

Proof. If $r \notin \partial D$ then the order of integration over ∂D and differentiation w.r.t. r can be interchanged and so $\nabla^2\Phi + k^2\Phi = 0$ follows trivially. Suppose now $\Phi = L_k\varsigma$ on D^e , and let $S_a = \partial B(0,a)$ be a sphere containing D in its interior; then

$$\iint_{S_a} \left| ik\Phi(r) - \frac{\partial\Phi(r)}{\partial r} \right|^2 d\sigma$$

$$= \iint_{S_a} \left| ik \iint_{\partial D} G_k(r,r')\, \varsigma(r')\, d\sigma' - \frac{\partial}{\partial r} \iint_{\partial D} G_k(r,r')\, \varsigma(r')\, d\sigma' \right|^2 d\sigma$$

$$= \iint_{S_a} \left| \iint_{\partial D} \left(ikG_k(r,r') - \frac{\partial G_k(r,r')}{\partial r} \right) \varsigma(r')\, d\sigma' \right|^2 d\sigma$$

$$\le \iint_{S_a} \left[\iint_{\partial D} \left| ikG_k(r,r') - \frac{\partial G_k(r,r')}{\partial r} \right|^2 d\sigma' \iint_{\partial D} |\varsigma(r')|^2\, d\sigma' \right] d\sigma$$

where we have used Schwartz's inequality. But $\varsigma \in C^2(\partial D)$ implies that there is $M > 0$ such that

$$\iint_{\partial D} |\varsigma(r')|^2\, d\sigma' < M .$$

Hence

$$\iint_{S_a} \left| ik\Phi(r) - \frac{\partial\Phi(r)}{\partial r} \right|^2 d\sigma \le M \iint_{\partial D} \iint_{S_a} \left| ikG_k(r,r') - \frac{\partial G_k(r,r')}{\partial r} \right|^2 d\sigma\, d\sigma' .$$

Since G_k satisfies the radiation condition, it follows that ϕ also satisfies the radiation condition. The proof for $\Phi = M_k\mu$ is similar.

(3.4) The interior Dirichlet problem for the Laplace operator.

Here we review the theory of the Laplace operator on $C^2(D^i)\cap C^0(D)$ with the Dirichlet boundary condition. It is proved that this problem has a unique solution if D is a regular region of $\mathbb{R}^3$, and that this operator has a discrete spectrum K_D . The purpose of this section is to show that certain homogeneous boundary integral equations which also occur in the exterior Dirichlet problem for the Helmholtz equation have non-trivial solutions.

Let $F \in C^0(D)$ and $f \in C^0(\partial D)$. The operator A on $C^2(D^i) \cap C^0(D)$ is defined by

$$A = -\nabla^2 , \tag{3.4.1}$$

and we consider the inhomogeneous Dirichlet interior problem

$$\left. \begin{aligned} A\phi &= F \quad \text{on } D^i \\ \phi &= f \quad \text{on } \partial D \end{aligned} \right\} , \tag{3.4.2}$$

and the homogeneous Dirichlet eigenvalue problem

$$\left. \begin{aligned} A\phi &= \lambda\phi \quad \text{on } D^i \\ \phi &= 0 \quad \text{on } \partial D \end{aligned} \right\} . \tag{3.4.3}$$

If we let

$$\mathcal{D}_1 = \{\phi : \phi \in C^2(D^i) \cap C^0(D),\ \phi = f \text{ on } \partial D\}$$

and

$$\mathcal{D}_2 = \{\phi : \phi \in C^2(D^i) \cap C^0(D),\ \phi = 0 \text{ on } \partial D\}$$

then problem (3.4.2) is characterized by A on $\mathcal{D}_1$, and problem (3.4.3) is characterized by A on $\mathcal{D}_2$. We consider the latter problem first. Forming the inner product of $A\phi$ and ϕ on $\mathcal{D}_2$ yields, by Green's first identity,

$$\langle A\phi,\phi \rangle = \iiint_D \| \nabla \phi \|^2 \, d\tau - \iint_{\partial D} \overline{\phi(\mathbf{r})} \, \frac{\partial \phi(\mathbf{r})}{\partial n} \, d\sigma .$$

If ϕ is an eigenfunction of A on $\mathcal{D}_2$ corresponding to an eigenvalue $\lambda = k^2$, then $\phi = 0$ on ∂D and we obtain

$$k^2 \|\phi\|^2_{\mathcal{H}} = \iiint_D \| \nabla \phi \|^2 \, d\tau ,$$

where $\|.\|_{\mathcal{H}}$ denotes the Hilbert space norm of $L^2(D)$. It follows that k^2 is zero or positive, and hence k is always real.

If $k = 0$, then

$$\iiint_D \| \nabla \phi \|^2 \, d\tau = 0 .$$

This implies that $\nabla\phi = 0$ on D and hence $\phi = c$, a constant on D. But $\phi = 0$ on ∂D and the assumed continuity of ϕ on D implies that $c = 0$. Hence $\phi = 0$ on D, which means that $k = 0$ is not an eigenvalue . We conclude that A on $\mathcal{D}_2$ has a unique inverse. It follows that A on $\mathcal{D}_1$ also has a unique inverse; for suppose that ϕ_1 and ϕ_2 are solutions of the problem (3.4.2), and let $\phi = \phi_1 - \phi_2$. Then $\phi \in \mathcal{D}_2$ and

$$A\phi = 0 \quad \text{on } D^i .$$

Thus $\phi = 0$ on D , or $\phi_1 = \phi_2$ on D.

We now determine the unique solution of (3.4.2). Setting k = 0 in the inhomogeneous Helmholtz interior formulae (Theorem (2.5.2)), the unique solution of (3.4.2) when $\mathbf{r} \in D^i$ is given by

$$-\phi(\mathbf{r}) = -\iiint_D G_0(\mathbf{r},\mathbf{r}')\, F(\mathbf{r}')\, d\tau' + (M_0\phi)(\mathbf{r}) - \left(L_0\frac{\partial\phi}{\partial n}\right)(\mathbf{r}) ,$$

or, since $\phi = f$ on ∂D,

$$\phi(\mathbf{r}) = \iiint_D G_0(\mathbf{r},\mathbf{r}')\, F(\mathbf{r}')\, d\tau' - (M_0 f)(\mathbf{r}) + \left(L_0\frac{\partial\phi}{\partial n}\right)(\mathbf{r}) .$$

If $U : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$U(\mathbf{r}) = \iiint_D G_0(\mathbf{r},\mathbf{r}')\, F(\mathbf{r}')\, d\tau', \qquad (3.4.4)$$

then U is continuously differentiable throughout $\mathbb{R}^3$ (see eg. Kellogg (23)). Thus we write

$$\phi(\mathbf{r}) = U(\mathbf{r}) - (M_0 f)(\mathbf{r}) + \left(L_0\frac{\partial\phi}{\partial n}\right)(\mathbf{r}) \qquad (3.4.5)$$

and

$$\frac{\partial\phi(\mathbf{r})}{\partial n} = \frac{\partial U(\mathbf{r})}{\partial n} - (N_0 f)(\mathbf{r}) + \left(M_0^T\frac{\partial\phi}{\partial n}\right)(\mathbf{r})$$

whenever $\mathbf{r} \in D^i$.

Suppose now $\mathbf{r} \in \partial D$. Then the limit relation

$$\frac{\partial\phi(\mathbf{r}-)}{\partial n} = \frac{\partial U(\mathbf{r}-)}{\partial n} - (N_0 f)(\mathbf{r}-) + \left(M_0^T\frac{\partial\phi}{\partial n}\right)(\mathbf{r}-)$$

holds. Using (3.2.7) and (3.2.9) gives

$$\frac{\partial\phi(\mathbf{r})}{\partial n} = \frac{\partial U(\mathbf{r})}{\partial n} - (N_0 f)(\mathbf{r}) + \left(M_0^T\frac{\partial\phi}{\partial n}\right)(\mathbf{r}) + \frac{1}{2}\frac{\partial\phi(\mathbf{r})}{\partial n}$$

or

$$\left((M_0^T - \tfrac{1}{2} I)\frac{\partial\phi}{\partial n}\right)(\mathbf{r}) = (N_0 f)(\mathbf{r}) - \frac{\partial U(\mathbf{r})}{\partial n} .$$

Thus the boundary values of $\frac{\partial\phi}{\partial n}$ satisfy the boundary integral equation

$$(M_0^T - \tfrac{1}{2} I)\psi = N_0 f - \frac{\partial U}{\partial n} . \qquad (3.4.6)$$

We now prove that this integral equation has a unique solution. First consider the homogeneous equation

$$(M_0^T - \tfrac{1}{2} I)\,\psi = 0 \quad \text{on } \partial D . \qquad (3.4.7)$$

Suppose that ψ is a non-trivial solution of this equation, and define Ψ on D^i and D^e by

$$\Psi = L_0\,\psi .$$

According to Theorem (3.3.1), Ψ satisfies the Laplace equation in D^i and D^e and $\quad \Psi(r) = O(\, r^{-1}) \quad$ on D^e .

If $r \in \partial D$ is approached from the exterior then, using (3.4.7),

$$\frac{\partial \Psi(r+)}{\partial n} = (M_0^T\psi)(r+) = (M_0^T\psi)(r) - \tfrac{1}{2}\,\psi(r) = 0$$

Hence from the uniqueness theorem it follows that $\Psi = 0$ on D^e and hence $\Psi = 0$ on ∂D, since $\Psi = L_0\psi$ is continuous across ∂D.

Secondly, if $r \in \partial D$ is approached from the interior then, (3.4.7) gives

$$\frac{\partial \Psi(r-)}{\partial n} = (M_0^T\psi)(r-)$$

$$= (M_0^T\psi)(r) + \tfrac{1}{2}\,\psi(r)$$

$$= \psi(r) \ ,$$

According to Green's first identity

$$\iiint_D \Psi\, \nabla^2\, \Psi\, d\tau + \iiint_D \|\nabla\,\Psi\|^2\, d\tau = \iint_{\partial D} \Psi\, \frac{\partial \Psi}{\partial n}\, d\sigma \ .$$

Now $\nabla^2\, \Psi = 0$ in D^i and $\Psi = 0$ on ∂D ; hence

$$\iiint_D \|\nabla\,\Psi\|^2\, d\tau = 0 \ ,$$

which implies that $\Psi = c$, a constant on D^i. However Ψ is continuous across ∂D and zero on ∂D. Therefore $\Psi = 0$ on D and it follows that

$$\Psi = 0 \quad \text{on } \mathbb{R}^3;$$

hence $$\frac{\partial \Psi}{\partial n} = 0 \quad \text{on } \mathbb{R}^3$$

and consequently

$$\psi = \frac{\partial \Psi}{\partial n} = 0 \quad \text{on } \partial D \ .$$

We have proved that the homogeneous equation (3.4.7) has only the trivial solution, and according to Fredholm's theorem the inhomogeneous equation (3.4.6) has a unique solution on ∂D . If this unique solution is denoted by g, then the Dirichlet problem (3.4.2) has the unique solution

$$\phi = U - M_0 f + L_0 g . \tag{3.4.8}$$

We state this result as:

Theorem (3.4.1) The inhomogeneous interior Dirichlet problem for the Laplace equation, namely

$$\begin{aligned} \nabla^2 \phi &= -F \quad \text{on } D^i \\ \phi &= f \quad \text{on } \partial D \end{aligned}$$

has the unique solution (3.4.8).

We can now establish the existence of the Green's function of the first kind for the inhomogeneous interior Dirichlet problem (3.4.2). This function is denoted by $G_0^D : D \times D \to \mathbb{R}$, and is defined by

(i) $G_0^D(\mathbf{r},\mathbf{r}') = G_0(\mathbf{r},\mathbf{r}') + H(\mathbf{r},\mathbf{r}')$ for all $\mathbf{r},\mathbf{r}' \in D$, where $H(\mathbf{r}, \cdot) \in C^1(D) \cap C^2(D^i)$ and $\nabla'^2 H(\mathbf{r},\mathbf{r}') = 0$;

(ii) $G_0^D(\mathbf{r},\mathbf{r}') = 0$ whenever $\mathbf{r} \in D$ and $\mathbf{r}' \in \partial D$, and

(iii) $\nabla'^2 G_0^D(\mathbf{r},\mathbf{r}') = 0$ whenever $\mathbf{r} \in D$ and $\mathbf{r}' \in D^i$.

The existence of G_0^D is proved if the existence of an H such that

$$\left.\begin{aligned} \nabla'^2 H(\mathbf{r},\mathbf{r}') &= 0 && \text{if } \mathbf{r}, \mathbf{r}' \in D^i \\ H(\mathbf{r},\mathbf{r}') &= -G_0(\mathbf{r},\mathbf{r}') && \text{if } \mathbf{r} \in D^i \text{ and } \mathbf{r}' \in \partial D \end{aligned}\right\}$$

is known. Since G_0^D is continuous on $D^i \times \partial D$, the existence of H follows from Theorem (3.4.1).

We can now use G_0^D in place of G_0 in the Helmholtz interior formulae with $k = 0$. Thus

$$\phi(\mathbf{r}) = -\iiint_D G_0^D(\mathbf{r},\mathbf{r}')\, F(\mathbf{r}')\, d\tau' - \iint_{\partial D} \frac{\partial G_0^D(\mathbf{r},\mathbf{r}')}{\partial n'}\, \phi(\mathbf{r}')\, d\sigma' + \iint_{\partial D} G_0^D(\mathbf{r},\mathbf{r}')\, \phi(\mathbf{r}')\, d\sigma' .$$

But $\phi = 0$ on ∂D and $G_0^D(\mathbf{r},\mathbf{r}') = 0$ if $\mathbf{r}' \in \partial D$, and so

$$\phi(r) = - \iiint_D G_0^D(r,r')\, F(r')\, d\tau' .$$

We have now shown that the inverse of A in $\mathcal{D}_1$ is given by

$$(A^{-1}F)(r) = - \iiint_D G_0^D(r,r')\, F(r')\, d\tau' .$$

It is proved by eg. Hellwig (21) and Mikhlin (37) that A^{-1} is completely continuous on $C^0(D)$, which is a dense subspace of $L^2(D)$. Moreover, A^{-1} is self-adjoint on $C^0(D)$. Thus the spectrum of A^{-1} is an enumerable set with at most a limit point at infinity.

Suppose that k^2 is an eigenvalue of A , and let ϕ be a corresponding eigenfunction; then $k \neq 0$ and the equation

$$A\phi = k^2\phi$$

is equivalent to the equation

$$A^{-1}\phi = \frac{1}{k^2}\phi .$$

Thus k^2 is an eigenvalue of A if and only if $\frac{1}{k^2}$ is an eigenvalue of A^{-1} , and we conclude that the set K_D of eigenvalues of $A = -\nabla^2$ is enumerable.

Now, given that $k^2 \in K_D$. Then the system

$$\left.\begin{aligned} \nabla^2\phi + k^2\phi &= 0 \quad \text{on } D^i \\ \phi &= 0 \quad \text{on } \partial D \end{aligned}\right\}$$

has a non-trivial solution ϕ, and $\frac{\partial\phi}{\partial n} \neq 0$ on ∂D. For, otherwise, from (3.2.14), $\phi = - L_k \frac{\partial\phi}{\partial n} = 0$ on D^i, and we have a contradiction. Hence it follows from (3.2.14) and (3.2.15) that $\frac{\partial\phi}{\partial n}$ is a non-trivial solution of the equations

$$L_k\psi = 0 .$$

and

$$(M_k^T - \tfrac{1}{2} I)\psi = 0 .$$

Thus we have established the following theorem :

Theorem (3.4.2) If $k^2 \in K_D$ the boundary integral equations

$$L_k\psi = 0$$

and

$$(M_k^T - \tfrac{1}{2} I)\psi = 0$$

on ∂D have non-trivial solutions.

3.5 The interior Neumann problem for the Laplace operator.

Let $F \in C^0(D)$ and $g \in C^0(\partial D)$. We consider the inhomogeneous Neumann interior problem

$$\left.\begin{aligned} A\phi &= F \quad \text{on } D^i \\ \frac{\partial \phi}{\partial n} &= g \quad \text{on } \partial D \end{aligned}\right\} , \tag{3.5.1}$$

and the homogeneous Neumann eigenvalue problem

$$\left.\begin{aligned} A\phi &= \lambda\phi \quad \text{on } D^i \\ \frac{\partial \phi}{\partial n} &= 0 \quad \text{on } \partial D \end{aligned}\right\} \tag{3.5.2}$$

where the operator $A = -\nabla^2$ on $C^2(D^i) \cap C^0(D)$.

Let $\mathfrak{D}_3 = \{\phi : \phi \in C^2(D^i) \cap C^0(D) \ , \ \frac{\partial \phi}{\partial n} = g \text{ on } \partial D\}$,

and $\mathfrak{D}_4 = \{\phi : \phi \in C^2(D^i) \cap C^0(D) \ , \ \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial D\}$.

If ϕ is an eigenfunction of A on $\mathfrak{D}_4$ corresponding to the eigenvalue $\lambda = k^2$, then

$$\langle A\phi, \phi\rangle = k^2 \|\phi\|^2_{\mathcal{H}} = \iiint_D \|\nabla\phi\|^2 \, d\tau$$

where we have used Green's first identity. Hence k is always real and $k = 0$ if and only if $\phi = c$, a constant. This is compatible with $\frac{\partial \phi}{\partial n} = 0$ on ∂D , and we cannot conclude that $c = 0$. Then $k^2 = 0$ is an eigenvalue of (3.5.2) and hence A on $\mathfrak{D}_4$ does not have an inverse.

We observe that if $\lambda = k^2 \neq 0$ and if ϕ is a solution of (3.5.2) then, by the divergence theorem,

$$\begin{aligned} k^2 \iiint_D \phi \, d\tau &= \iiint_D A\phi \, d\tau \\ &= -\iiint_D \nabla^2 \phi \, d\tau \\ &= -\iint_{\partial D} \frac{\partial \phi}{\partial n} \, d\sigma \\ &= 0 . \end{aligned}$$

In other words ϕ is orthogonal to 1 , i.e. the function whose value equals 1 at all points of D. Now if $\phi \perp 1$ then ϕ is not a non-zero constant on D. For otherwise, if $\phi = c \neq 0$ on D , then

$$0 = \langle \phi, 1 \rangle = \iiint_D \phi \, d\tau = c \iiint_D d\tau ,$$

which implies that $c = 0$.

We now define

$$\mathfrak{D}_5 = \{\phi \in \mathfrak{D}_3 \ : \ \phi \perp 1\}$$

and

$$\mathfrak{D}_6 = \{\phi \in \mathfrak{D}_4 \ : \ \phi \perp 1\} .$$

Then the Neumann problems

$$\left.\begin{array}{lll} A\phi = F & \text{on } D^i \\ \dfrac{\partial \phi}{\partial n} = g & \text{on } \partial D \\ \phi \perp 1 & \text{on } D \end{array}\right\} \qquad (3.5.3)$$

and

$$\left.\begin{array}{lll} A\phi = \lambda\phi & \text{on } D^i \\ \dfrac{\partial \phi}{\partial n} = 0 & \text{on } \partial D \\ \phi \perp 1 & \text{on } D \end{array}\right\} \qquad (3.5.4)$$

are characterised by $A\phi = F$, $\phi \in \mathfrak{D}_5$ and $(A - \lambda I)\phi = 0$, $\phi \in \mathfrak{D}_6$, respectively.

We can now assert that $\lambda = 0$ is not an eigenvalue of A on $\mathfrak{D}_6$. Consequently, A on $\mathfrak{D}_6$ and hence A on $\mathfrak{D}_5$ have unique inverses, and according to Theorem (2.5.1) the unique solution of (3.5.3) is given by

$$\phi = U - M_0\phi + L_0 g, \qquad (3.5.5)$$

where U on D is given by

$$U(\mathbf{r}) = \iiint_D G_0(\mathbf{r},\mathbf{r}') \, F(\mathbf{r}') \, d\tau' \qquad (3.5.6)$$

In (3.5.5) the boundary values of ϕ are unknown. To determine the values of ϕ on ∂D we form a pair of integral equations on ∂D and prove that under certain conditions the solution is unique except for an arbitrary additive constant.

If $\mathbf{r} \in \partial D$ is approached from the interior of D along the normal direction, we obtain

$$\phi(\mathbf{r}) = U(\mathbf{r}) - (M_0\phi)(\mathbf{r}) + \tfrac{1}{2}\,\phi(\mathbf{r}) + (L_0 g)(\mathbf{r}),$$

where we have used (3.2.8). Hence the boundary value of ϕ satisfy the integral equation

$$(M_0 + \tfrac{1}{2} I)\phi = U + L_0 g \quad \text{on } \partial D . \tag{3.5.7}$$

From (3.5.5) we also obtain

$$\frac{\partial\phi(\mathbf{r})}{\partial n} = \frac{\partial U(\mathbf{r})}{\partial n} - (N_0\phi)(\mathbf{r}) + (M_0^T g)(\mathbf{r}), \quad \text{if } \mathbf{r} \in D^i , \tag{3.5.8}$$

by differentiating in the normal direction. Now, if $\mathbf{r} \in \partial D$ is approached from the interior along the normal direction, we find that

$$(N_0\phi)(\mathbf{r}) = \frac{\partial U(\mathbf{r})}{\partial n} + (M_0^T - \tfrac{1}{2} I) g(\mathbf{r}),$$

or

$$N_0\phi = \frac{\partial U}{\partial n} + (M_0^T - \tfrac{1}{2} I) g \quad \text{on } \partial D. \tag{3.5.9}$$

If we put

$$\hat{\phi} = U + L_0 g \quad \text{on } D^i, \tag{3.5.10}$$

we find, using (3.2.6) and (3.2.7), that

$$\hat{\phi} = U + L_0 g \quad \text{on } \partial D \tag{3.5.11}$$

and

$$\frac{\partial\hat{\phi}^+}{\partial n} = \frac{\partial U}{\partial n} + (M_0^T - \tfrac{1}{2} I) g \quad \text{on } \partial D , \tag{3.5.12}$$

where

$$\frac{\partial\hat{\phi}^+(\mathbf{r})}{\partial n} = \frac{\partial\hat{\phi}(\mathbf{r}+)}{\partial n} .$$

Hence we can write (3.5.7) and (3.5.9), respectively, in the forms

$$(M_0 + \tfrac{1}{2} I)\phi = \hat{\phi} \quad \text{on } \partial D \tag{3.5.13}$$

and

$$N_0\phi = \frac{\partial\hat{\phi}^+}{\partial n} \quad \text{on } \partial D . \tag{3.5.14}$$

We now prove that solutions of the simultaneous pair (3.5.13) and (3.5.14) differ only by an arbitary constant. First we consider the homogeneous equations corresponding to (3.5.13) and (3.5.14).

Theorem (3.5.1) If θ on ∂D is a solution of the homogeneous equations

$$(M_0 + \tfrac{1}{2} I)\phi = 0 \quad \text{on } \partial D \tag{3.5.15}$$

and

$$N_0 = 0 \quad \text{on } \partial D \tag{3.5.16}$$

then $\theta = c$, an arbitary constant, on ∂D.

Proof. Define Θ on D^i and D^e by

$$\Theta = M_0\,\theta ;$$

then $$\frac{\partial \theta}{\partial n} = N_0\,\theta \quad \text{on } D^i \text{ and } D^e .$$

If $\mathbf{r} \in \partial D$ we obtain from (3.2.7)

$$\theta(\mathbf{r}-) = M_0\theta(\mathbf{r}-) - \tfrac{1}{2}\,\theta(\mathbf{r}) = -\,\theta(\mathbf{r}) \tag{3.5.17}$$

and

$$\theta(\mathbf{r}+) = M_0\theta(\mathbf{r}) + \tfrac{1}{2}\,\theta(\mathbf{r}) = 0 .$$

Using (3.2.6) we see that

$$\frac{\partial\theta(\mathbf{r}-)}{\partial n} = (N_0\theta)(\mathbf{r}) = \frac{\partial\theta(\mathbf{r}+)}{\partial n} .$$

Since $\theta(\mathbf{r}+) = 0$ when $\mathbf{r} \in \partial D$, it follows from Theorem (3.3.1) that θ is a solution of the homogeneous exterior Dirichlet problem for the Laplace equation on D^e. Thus from the uniqueness theorem (Theorem (2.7.2)) it follows that $\theta = 0$ on D^e. Hence $\frac{\partial\theta}{\partial n} = 0$ on D^e and so

$$\frac{\partial\theta(\mathbf{r}-)}{\partial n} = \frac{\partial\theta(\mathbf{r}+)}{\partial n} = 0 \quad \text{on } \partial D .$$

Therefore θ on D^i is a solution of the homogeneous interior Neumann problem for the Laplace equation on D^i. Then Green's first identity in the form

$$\iiint_D \theta\,\nabla^2\theta\,d\tau + \iiint_D \|\nabla\theta\|^2\,d\tau = \iint_{\partial D} \theta\,\frac{\partial\theta}{\partial n}\,d\sigma$$

reduces to $$\iiint_D \|\nabla\theta\|^2\,d\tau = 0 ,$$

which implies that $\theta = -\,c$, a constant, on D^i. Thus for $\mathbf{r} \in \partial D$ it follows from (3.5.17) that

$$\theta(\mathbf{r}) = -\,\theta(\mathbf{r}-) = c, \tag{3.5.18}$$

which completes the proof of the theorem.

Theorem (3.5.2) The inhomogeneous equation

$$(M_0 + \tfrac{1}{2} I)\,\phi = \hat{\phi}$$

always has a solution, provided that the compatibility condition

$$\iiint_D F\,d\tau + \iint_{\partial D} g\,d\sigma = 0 \tag{3.5.19}$$

is satisfied.

Proof. The homogeneous adjoint equation is

$$(M_0^* + \tfrac{1}{2} I)\,\theta = 0 ,$$

and so

$$(M_0^T + \tfrac{1}{2} I)\,\bar{\theta} = 0 .$$

We assume that θ is a non-trivial solution of this equation and define Θ on $\mathbb{R}^3$ by

$$\Theta = L_0 \bar{\theta} .$$

If $\mathbf{r} \in \partial D$ then

$$\frac{\partial \Theta(\mathbf{r}-)}{\partial n} = (M_0^T\,\bar{\theta})(\mathbf{r}-) = (M_0^T\,\bar{\theta})(\mathbf{r}) + \tfrac{1}{2}\,\theta(\mathbf{r}) = 0 .$$

Since $\nabla^2\Theta = 0$ on D^i (Theorem (3.3.1)) it follows from Green's first identity that

$$\iiint_D \|\nabla \Theta\|^2 \, d\tau = 0$$

and hence

$$\Theta = c \text{ ,a constant on } D^i .$$

Since Θ is continuous across ∂D , $\Theta(\mathbf{r}) = c$ if $\mathbf{r} \in \partial D$, and so

$$\Theta = L_0\,\bar{\theta} = c \quad \text{on } D.$$

Now using (3.5.11) (3.5.6) and (3.2.1) we obtain

$$\langle \hat{\phi}, \theta \rangle = \iint_{\partial D} \bar{\theta}\,\hat{\phi}\, d\sigma$$

$$= \iint_{\partial D} \overline{\theta(\mathbf{r})} \left\{ \iiint_D G_0(\mathbf{r},\mathbf{r}')\, F(\mathbf{r}')\, d\tau' + \iint_{\partial D} G_0(\mathbf{r},\mathbf{r}')\, g(\mathbf{r}')\, d\sigma' \right\} d\sigma$$

$$= \iiint_D F(\mathbf{r}') \iint_{\partial D} G_0(\mathbf{r},\mathbf{r}')\, \overline{\theta(\mathbf{r})}\, d\sigma\, d\tau' + \iint_{\partial D} g(\mathbf{r}') \iint_{\partial D} G_0(\mathbf{r},\mathbf{r}')\, \overline{\theta(\mathbf{r})}\, d\sigma d\sigma'$$

$$= \iiint_D F(\mathbf{r}')\, \Theta(\mathbf{r}')\, d\tau' + \iint_{\partial D} g(\mathbf{r}')\, \Theta(\mathbf{r}')\, d\sigma'$$

$$= c \left\{ \iiint_D F(\mathbf{r}')\, d\tau' + \iint_{\partial D} g(\mathbf{r}')\, d\sigma' \right\} .$$

As c is an arbitary constant, it follows that $\hat{\phi} \perp \theta$ if and only if

$$\iiint_D F(\mathbf{r}')\, d\tau' + \iint_{\partial D} g(\mathbf{r}')\, d\sigma' = 0 \ .$$

We observe that (3.5.19) is also a necessary condition for (3.5.1) to have a solution. This follows directly from (3.5.1) by applying the divergence theorem to $A\phi = -\nabla^2\phi$.

Theorem (3.5.3) If

$$\iiint_D F(\mathbf{r}')\, d\tau' + \iint_{\partial D} g(\mathbf{r}')\, d\sigma' = 0 \ ,$$

then the inhomogeneous equations

$$(M_0 + \tfrac{1}{2} I)\, \phi = \hat{\phi} \quad \text{on } \partial D$$

and

$$N_0\, \phi = \frac{\partial \hat{\phi}^+}{\partial n} \quad \text{on } \partial D$$

have common solutions and any pair of simultaneous solutions differ by an arbitary constant.

Proof. Suppose ϕ_0 is a solution of (3.5.13), i.e.

$$(M_0 + \tfrac{1}{2} I)\, \phi_0 = \hat{\phi} \ ,$$

and define Φ on D^i and D^e by

$$\Phi = \hat{\phi} - M_0\, \phi_0 \ .$$

Then for $\mathbf{r} \in \partial D$

$$\begin{aligned} \Phi(\mathbf{r}-) &= \hat{\phi}(\mathbf{r}-) - (M_0\phi_0)(\mathbf{r}-) \\ &= \hat{\phi}(\mathbf{r}) - (M_0\phi_0)(\mathbf{r}) + \tfrac{1}{2}\, \phi_0(\mathbf{r}) \\ &= \phi_0(\mathbf{r}) \end{aligned}$$

and

$$\begin{aligned} \Phi(\mathbf{r}+) &= \hat{\phi}(\mathbf{r}+) - (M_0\phi_0)(\mathbf{r}+) \\ &= \hat{\phi}(\mathbf{r}) - (M_0\phi_0)(\mathbf{r}) - \tfrac{1}{2}\, \phi_0(\mathbf{r}) \\ &= 0 \ . \end{aligned}$$

By Theorems (3.3.1) and (2.7.2) we deduce that $\Phi = 0$ on D^e; it follows that

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on } D^e \ ,$$

and hence

$$\frac{\partial \Phi(\mathbf{r}+)}{\partial n} = 0 \ , \quad \text{for } \mathbf{r} \in \partial D.$$

But on D^e

$$\frac{\partial \Phi}{\partial n} = \frac{\partial \hat{\phi}}{\partial n} - N_0 \phi_0 \ ,$$

and so, if $\mathbf{r} \in \partial D$ is approached from the exterior D^e ,

$$0 = \frac{\partial\Phi(r+)}{\partial n} = \frac{\partial\hat{\phi}(r+)}{\partial n} - (N_0\phi_0)(r+)$$

$$= \frac{\partial\Phi(r)}{\partial n} - (N_0\phi_0)(r) \ .$$

It follows that

$$N_0\phi_0 = \frac{\partial\hat{\phi}^+}{\partial n} \quad \text{on } \partial D \ .$$

Thus, if ϕ_0 is a solution of (3.5.13) then it is also a solution of (3.5.14). If ϕ_1 is any other common solution of these two equations, then

$$\phi = \phi_1 - \phi_0$$

is a solution of the homogeneous equations (3.5.15) and (3.5.16), and hence $\phi = c$ according to Theorem (3.5.1).

We can therefore state that the simultaneous equations (3.5.13) and (3.5.14) have a solution f which is unique exceptig for an arbitary additive constant.

It follows that

$$\phi = U - M_0 f + L_0 g$$

is a solution of (3.5.1). If c is an arbitary constant and if we put

$$f_c = f + c \ ,$$

then

$$\psi = U - M_0 f_c + L_0 g = \phi + c$$

is also a solution, since $M_0 c = c$.

If we choose

$$c = - \frac{\iiint_D \phi \, d\tau}{\iiint_D d\tau}$$

then $\psi \perp 1$ is the unique solution of (3.5.3).

We have therefore proved the following theorem.

Theorem (3.5.4) The inhomogeneous interior Neumann problem for the Laplace equation, namely

$$\nabla^2 \phi = F \quad \text{on } D^i$$

$$\frac{\partial\phi}{\partial n} = g \quad \text{on } \partial D \ ,$$

is solvable provided

$$\iiint_D F(r')\, d\tau' + \iint_{\partial D} g(r')\, d\sigma' = 0 \ .$$

The solution is unique except for an arbitary constant and, if $\phi \perp 1$ then the boundary value problem has a unique solution.

We can now prove the existence of the Green's function of the second kind, denoted by G_0^N, for the inhomogeneous interior Neumann problem (3.5.1) with $g = 0$ on ∂D. The function $G_0^N : D \times D \to \mathbb{R}$ is defined by

(i) $G_0^N(\mathbf{r},\mathbf{r}') = G_0(\mathbf{r},\mathbf{r}') - K(\mathbf{r},\mathbf{r}')$, for all $\mathbf{r}$, $\mathbf{r}' \in D$, where $K(\mathbf{r},\cdot) \in C^1(D) \cap C^2(D^i)$ and $\nabla'^2 K(\mathbf{r},\mathbf{r}') = 0$;

(ii) $\dfrac{\partial G_0^N(\mathbf{r},\mathbf{r}')}{\partial n} = C$ whenever $\mathbf{r} \in D$ $\mathbf{r}' \in \partial D$, where $C = -\left[\iint_{\partial D} d\sigma\right]^{-1}$;

(iii) $\nabla'^2 G_0^N(\mathbf{r},\mathbf{r}') = 0$ whenever $\mathbf{r} \in D$, $\mathbf{r}' \in D^i$ and $\mathbf{r} \neq \mathbf{r}'$.

This definition of the Green's function of the second kind differs from that given by Pogorzelski (40).

We see therefore that for each $\mathbf{r} \in D^i$ the function $K(\mathbf{r},\cdot)$ is the solution of the boundary value problem

$$\left.\begin{aligned} \nabla'^2 K(\mathbf{r},\mathbf{r}') &= 0 && \text{if } \mathbf{r} \in D^i \\ \frac{\partial K(\mathbf{r},\mathbf{r}')}{\partial n} &= \frac{\partial G_0(\mathbf{r},\mathbf{r}')}{\partial n} - C && \text{if } \mathbf{r}' \in \partial D \end{aligned}\right\} \tag{3.5.20}$$

This system is solvable since

$$\iint_{\partial D} \left[\frac{\partial G_0(\mathbf{r},\mathbf{r}')}{\partial n'} - C \right] d\sigma = 0 .$$

As G_0 is continuous on $D^i \times \partial D$, the function K exists and is unique except for an arbitary constant. Now using G_0^N in place of G_0 in Theorem (2.5.1) with $k = 0$, we obtain

$$\phi(\mathbf{r}) = -\iiint_D G_0^N(\mathbf{r},\mathbf{r}')F(\mathbf{r}')d\tau' - \iint_{\partial D} \frac{\partial G_0^N(\mathbf{r},\mathbf{r}')}{\partial n'}\phi(\mathbf{r}')d\sigma' + \iint_{\partial D} G_0^N(\mathbf{r},\mathbf{r}')\frac{\partial \phi(\mathbf{r}')}{\partial n'}\, d\sigma' .$$

Since $\dfrac{\partial \phi(\mathbf{r}')}{\partial n} = g(\mathbf{r}') = 0$ when $\mathbf{r}' \in \partial D$ and $\dfrac{\partial G_0^N(\mathbf{r},\mathbf{r}')}{\partial n'} = C$ when $\mathbf{r} \in D$

and $\mathbf{r}' \in \partial D$, we have

$$\phi(\mathbf{r}') = -\iiint_D G_0^N(\mathbf{r}.\mathbf{r}')\, F(\mathbf{r}')\, d\tau' + c ,$$

where
$$c = -C \iint_{\partial D} \phi(\mathbf{r}')\, d\sigma' ,$$

and $c = 0$ if and only if $\phi \perp 1$ on ∂D.

Theorem (3.5.5) If

$$\iiint_D F \, d\tau = 0$$

then the boundary value problem (3.5.3) with $g = 0$ has the unique solution

$$\phi(r) = (A^{-1}F)(r) = -\iiint_D G_0^N(r,r') \, F(r') \, d\tau'.$$

As the operator A^{-1} is completely continuous (Hellwig (21), Mikhlin (37)) we deduce, as in section (3.4), that the spectrum of A on $\mathcal{D}_8$ is enumerable. It follows immediately that the spectrum K_N of A on $\mathcal{D}_4$ is also enumerable. Thus we finally obtain the following result :

Theorem (3.5.6) If $k^2 \in K_N$ then the integral equations

$$(M_k + \tfrac{1}{2} I)\,\phi = 0$$

and

$$N_k \phi = 0$$

have non-trivial solutions.

3.6 Integral formulation of scalar scattering problems.

When considering the scattering of a time independent scalar wave by a scatterer D, it is usual to express the total wave field ϕ_t as the sum of a know incident field ϕ_i and a scattered field ϕ_s ; thus

$$\phi_t = \phi_i + \phi_s \ . \tag{3.6.1}$$

Both ϕ_s and ϕ_i are assumed to satisfy the Helmholtz equation in the exterior region D^e, and we also assume that the scattered field satifies the radiation condition (2.6.6). However, we do not make this latter assumption for the incident field. For example, if ϕ_i is a plane wave, then (2.6.6) is not satisfied.

It follows that the scattered wave satisfies the extended Helmholtz exterior formulae (3.2.16):

$$(M_k\phi_s)(r) - \left[L_k \frac{\partial \phi_s}{\partial n}\right](r) = \begin{cases} 0 & \text{if } r \in D^i \\ \tfrac{1}{2}\,\phi_s(r) & \text{if } r \in \partial D \\ \phi_s(r) & \text{if } r \in D^e \end{cases} \tag{3.6.2}$$

We cannot write similar equations for the total field, since the incident field, and therefore also the total field, need not satisfy the radiation condition. However the incident field will satisfy the extended Helmholtz interior formulae. Thus from (3.2.14) we have

$$(M_k\phi_i)(\mathbf{r}) - \left(L_k \frac{\partial\phi_i}{\partial n}\right)(\mathbf{r}) = \begin{cases} -\phi_i(\mathbf{r}) & \text{if } \mathbf{r} \in D^i \\ -\frac{1}{2}\phi_i(\mathbf{r}) & \text{if } \mathbf{r} \in \partial D \\ 0 & \text{if } \mathbf{r} \in D^e \end{cases} \qquad (3.6.3)$$

Adding (3.6.2) and (3.6.3) we obtain equations for the total field:

$$(M_k\phi_t)(\mathbf{r}) - \left(L_k \frac{\partial\phi_t}{\partial n}\right)(\mathbf{r}) = \begin{cases} -\phi_i(\mathbf{r}) & \text{if } \mathbf{r} \in D^i \\ \frac{1}{2}(\phi_s(\mathbf{r}) - \phi_i(\mathbf{r})) & \text{if } \mathbf{r} \in \partial D \\ \phi_s(\mathbf{r}) - \phi_i(\mathbf{r}) & \text{if } \mathbf{r} \in D^e \end{cases}$$

Substituting $\phi_s = \phi_t - \phi_i$ in the right-hand side we obtain

$$(M_k\phi_t)(\mathbf{r}) - \left(L_k \frac{\partial\phi_t}{\partial n}\right)(\mathbf{r}) = \begin{cases} -\phi_i(\mathbf{r}) & \text{if } \mathbf{r} \in D^i \\ \frac{1}{2}\phi_t(\mathbf{r}) - \phi_i(\mathbf{r}) & \text{if } \mathbf{r} \in \partial D \\ \phi_t(\mathbf{r}) - \phi_i(\mathbf{r}) & \text{if } \mathbf{r} \in D^e \end{cases} \qquad (3.6.4)$$

This is the form used by Noble (37) in his account of the integral equations of diffraction theory, and also used by Burton (9) in his work on the exterior problems for the Helmholtz equation.

The last equation of (3.6.4) represents the total field at exterior points of the scatterer D as the sum of the incident field and boundary values of the total field:

$$\phi_t = \phi_i + M_k\phi_t - L_k \frac{\partial\phi_t}{\partial n} \quad \text{on } D^e . \qquad (3.6.5)$$

The second equation of (3.6.4) shows that the total surface field satisfies the boundary integral equation

$$(\tfrac{1}{2} I - M_k)\phi_t + L_k \frac{\partial\phi_t}{\partial n} = \phi_i . \qquad (3.6.6)$$

If we repeat the above derivation with equations (3.2.17) and (3.2.15) in place of equations (3.2.16) and (3.2.14) respectively, we arrive at the differentiated form of (3.6.4):

$$(N_k\phi_t)(\mathbf{r}) - \left[M_k^T \frac{\partial \phi_t}{\partial n}\right](\mathbf{r}) = \begin{cases} -\dfrac{\partial \phi_i(\mathbf{r})}{\partial n} & \text{if } \mathbf{r} \in D^i \\ \dfrac{1}{2}\dfrac{\partial \phi_t(\mathbf{r})}{\partial n} - \dfrac{\partial \phi_i(\mathbf{r})}{\partial n} & \text{if } \mathbf{r} \in \partial D \\ \dfrac{\partial \phi_t(\mathbf{r})}{\partial n} - \dfrac{\partial \phi_i(\mathbf{r})}{\partial n} & \text{if } \mathbf{r} \in D^e \end{cases} \quad (3.6.7)$$

Thus the normal derivative of the total field is given by

$$\frac{\partial \phi_t}{\partial n} = \frac{\partial \phi_i}{\partial n} + N_k\phi_t - M_k^T \frac{\partial \phi_t}{\partial n} \quad \text{on } D^e \quad (3.6.8)$$

and the boundary values of ϕ_t and $\frac{\partial \phi_t}{\partial n}$ satisfy the boundary integral equation

$$(\tfrac{1}{2} I - M_k^T)\frac{\partial \phi_t}{\partial n} - N_k\phi_t = \frac{\partial \phi_i}{\partial n} \quad \text{on } \partial D . \quad (3.6.9)$$

If it is assumed that both equations (3.6.6) and (3.6.9) are solvable, then there will always be at least one common solution to these two integral equations. For example, if we are dealing with the exterior Dirichlet problem, equation (3.6.9) can be used to determine $\frac{\partial \phi_t}{\partial n}$ on ∂D and then equation (3.6.5) can be used to determine ϕ_t on D^e. The limiting value of ϕ_t when the boundary is approached from the exterior must agree with the prescribed boundary value of the Dirichlet problem. The limiting form of (3.6.5) when ∂D is approached from D^e is in fact equation (3.6.6). Thus it is seen that (3.6.6) and (3.6.9) must have common solutions. The following two sections of this chapter is devoted to proving that for the Dirichlet and Neumann boundary value problems these two equations always have a unique solution.

3.7 The exterior Dirichlet problem for the Helmholtz equation.

If the boundary condition for the Dirichlet problem is

$$\phi = f \quad \text{on } \partial D , \quad (3.7.1)$$

then from equations (3.6.6) and (3.6.9) we have respectively

$$L_k \frac{\partial \phi}{\partial n} = \phi_i - (\tfrac{1}{2} I - M_k) f \quad \text{on } \partial D \quad (3.7.2)$$

and

$$(\tfrac{1}{2} I + M_k^T)\frac{\partial \phi}{\partial n} = N_k f + \frac{\partial \phi_i}{\partial n} \quad \text{on } \partial D . \quad (3.7.3)$$

If we define
$$\tilde{\phi}_i = \phi_i + M_k f \quad \text{on } D^i \text{ and } D^e \tag{3.7.4}$$

then the continuity relation (3.2.9) shows that $\frac{\partial \phi_i}{\partial n}$ is continuous across ∂D and
$$\frac{\partial \tilde{\phi}_i}{\partial n} = \frac{\partial \phi}{\partial n} + N_k f \tag{3.7.5}$$

But (3.2.8) shows that $\tilde{\phi}_i$ is discontinuous across ∂D, and if $\mathbf{r} \in \partial D$, then
$$\tilde{\phi}_i(\mathbf{r}-) = \phi_i(\mathbf{r}) + (M_k f)(\mathbf{r}) - \tfrac{1}{2} f(\mathbf{r}) .$$

Hence we define $\tilde{\phi}_i$ on ∂D by
$$\tilde{\phi}_i = \phi_i - (\tfrac{1}{2}I - M_k) f \tag{3.7.6}$$

Equations (3.7.2) and (3.7.3) can now be written in the form
$$L_k \frac{\partial \phi}{\partial n} = \tilde{\phi}_i \quad \text{on } \partial D \tag{3.7.7}$$

and
$$(\tfrac{1}{2} I + M_k^T) \frac{\partial \phi}{\partial n} = \frac{\partial \tilde{\phi}_i}{\partial n} \quad \text{on } \partial D. \tag{3.7.8}$$

Equations (3.7.7) and (3.7.8) are respectively Fredholm integral equations of the first and second kind. The correspponding homogeneous equations are
$$L_k \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial D \tag{3.7.9}$$

and
$$(\tfrac{1}{2} I + M_k^T) \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial D \tag{3.7.10}$$

In section (3.6) it was shown that these equations have non-trival solutions when $k^2 \in K_N$. According to Fredholm theory, equation (3.7.8) can have a solution only if $\frac{\partial \tilde{\phi}_i}{\partial n}$ is orthogonal to the null-space $N(\frac{1}{2} I + \overline{M_k})$. Also, equation (3.7.7) can have a solution only if $\tilde{\phi}_i$ is orthogonal to $N(L_k)$, since L_k is self-adjoint. Assuming that these conditions are satisfied, it follows that equations (3.7.7) and (3.7.8) have non-unique solutions when $k^2 \in K_N$. However, it was shown by Kleinman and Roach (26) that the two equations (3.7.7) and (3.7.8) together have a unique solution. The account we give here follows along the lines laid down by these authors.

We first prove that the homogeneous equations (3.7.9) and (3.7.10) have only the trivial solution in common.

Theorem(3.7.1) Suppose that

$$L_k \psi = 0 \quad \text{on } \partial D$$

and

$$(\tfrac{1}{2} I + M_k^T) \psi = 0 \quad \text{on } \partial D .$$

Then $\psi = 0$ on ∂D.

Proof. We define Ψ on D^i and D^e by

$$\Psi(\mathbf{r}) = (L_k\psi)(\mathbf{r}) .$$

If $\mathbf{r} \in \partial D$ then

$$\begin{aligned} \frac{\partial \Psi(\mathbf{r}-)}{\partial n} &= (M_k^T\psi)(\mathbf{r}-) \\ &= (M_k^T\psi)(\mathbf{r}) + \tfrac{1}{2}\psi(\mathbf{r}) \\ &= 0 , \end{aligned}$$

and

$$\Psi(\mathbf{r}-) = (L_k\psi)(\mathbf{r}-) = (L_k\psi)(\mathbf{r}) = 0 .$$

Hence by the Helmholtz interior formulae we have

$$\Psi = 0 \quad \text{on } D = \partial D \cup \partial D .$$

Moreover, Ψ satisfies the Helmholtz equation in the exterior region D^e and the radiation condition. If $\mathbf{r} \in \partial D$, then

$$\Psi(\mathbf{r}+) = (L_k\psi)(\mathbf{r}+) = (L_k\psi)(\mathbf{r}) = 0 .$$

Hence

$$\Psi = 0 \quad \text{on } \partial D ,$$

and we have from the uniqueness theorem for the Dirichlet problem that

$$\Psi = 0 \quad \text{on } D^e .$$

Futhermore it follows that $\Psi = 0$ $\mathbb{R}^3$ on and hence $\frac{\partial \Psi}{\partial n} = 0$ on $\mathbb{R}^3$. But if $\mathbf{r} \in \partial D$ then

$$\begin{aligned} \frac{\partial \Psi(\mathbf{r}+)}{\partial n} &= (M_k^T\psi)(\mathbf{r}+) \\ &= (M_k^T\psi)(\mathbf{r}) - \tfrac{1}{2}\psi(\mathbf{r}) \\ &= -\psi(\mathbf{r}) , \end{aligned}$$

and consequently $\psi = 0$ on ∂D.

Theorem (3.7.2) The inhomogeneous boundary integral equation

$$(\tfrac{1}{2} I + M_k^T) \psi = \frac{\partial \Phi_i}{\partial n} \quad \text{on } \partial D$$

always has a solution.

Proof. Assume θ to be a non-trivial solution of the homogeneous adjoint equation

$$(\tfrac{1}{2} I + \overline{M_k})\theta = 0 \quad \text{on } \partial D ,$$

which implies that

$$(\tfrac{1}{2} I + M_k)\bar{\theta} = 0 \quad \text{on } \partial D .$$

Define Θ by

$$\Theta = - M_k\bar{\theta} \qquad \text{on } \mathbb{R}^3$$

If $\mathbf{r} \in \partial D$ then

$$\Theta(\mathbf{r}-) = - (M_k\bar{\theta})(\mathbf{r}-)$$

$$= - (M_k\bar{\theta})(\mathbf{r}) + \tfrac{1}{2}\,\bar{\theta}(\mathbf{r})$$

$$= \bar{\theta}(\mathbf{r})$$

and

$$\Theta(\mathbf{r}+) = - (M_k\bar{\theta})(\mathbf{r}+)$$

$$= - (M_k\bar{\theta})(\mathbf{r}) - \tfrac{1}{2}\,\bar{\theta}(\mathbf{r})$$

$$= 0$$

Hence by Theorem (3.3.1) and the uniqueness theorem for the exterior Dirichlet problem it follows that $\Theta = 0$ on D^e, and therefore

$$\frac{\partial\Theta}{\partial n} = 0 \quad \text{on } D^e .$$

However,

$$\frac{\partial\Theta}{\partial n} = - \frac{\partial}{\partial n} M_k\,\bar{\theta} = - N_k\,\bar{\theta} ,$$

which is continuous across ∂D by (3.2.9) . Hence

$$\frac{\partial\Theta}{\partial n} = 0 \qquad \text{on } \partial D .$$

We now apply Green's second identity to $\bar{\phi}_i$ and Θ on D, obtaining

$$0 = \iiint_D (\Theta\nabla^2\bar{\phi}_i - \bar{\phi}_i\nabla^2\Theta)\, d\tau = \iint_{\partial D} \left(\Theta\frac{\partial\bar{\phi}_i}{\partial n} - \bar{\phi}_i\frac{\partial\Theta}{\partial n}\right) d\sigma .$$

Since $\Theta(\mathbf{r}-) = \bar{\theta}(\mathbf{r})$ and $\frac{\partial\Theta(\mathbf{r})}{\partial n} = 0$ when $\mathbf{r} \in \partial D$, we have

$$\iint \Theta\,\frac{\partial\bar{\phi}_i}{\partial n}\, d\sigma = 0;$$

i.e

$$\frac{\partial\bar{\phi}_i}{\partial n} \perp \theta .$$

Thus the compatibilty condition of Fredholm's theorem is satisfied.

Theorem (3.7.3) The boundary integral equations

$$(\tfrac{1}{2} I + M_k^T)\,\psi = 0$$

and

$$(\tfrac{1}{2} I + M_k)\,\psi = 0$$

have non-trivial solutions if and only if k^2 is an eigenvalue of the homogeneous interior Neumann problem; i.e. if and only if $k^2 \in K_N$.

Proof. Suppose θ is a non-trivial solution of the equation $(\tfrac{1}{2} I + M_k^T)\psi = 0$, i.e.

$$(\tfrac{1}{2} I + M_k^T)\,\theta = 0 \quad \text{on } \partial D.$$

Define Θ by

$$\Theta = L_k\,\theta \quad \text{on } \mathbb{R}^3;$$

then for $\mathbf{r} \in \partial D$

$$\frac{\partial \Theta(\mathbf{r}-)}{\partial n} = (M_k^T\theta)(\mathbf{r}-)$$

$$= (M_k^T\theta)(\mathbf{r}) + \tfrac{1}{2}\,\theta(\mathbf{r})$$

$$= 0\ .$$

Hence, according to Theorem (3.3.1) Θ is a solution of the homogeneous interior Neumann problem for the Helmholtz equation. We assert that $\Theta \neq 0$ on D^i. For if $\Theta = 0$ on D^i, then also $\Theta = 0$ on ∂D as $\Theta = L_k\,\theta$ is continuous across ∂D. Hence, it follows from Theorem (3.3.1) and the uniqueness theorem for the exterior Dirichlet problem that $\Theta = 0$ on D^e. Consequently $\dfrac{\partial \Theta}{\partial n} = 0$ on D^e and hence

$$\frac{\partial \Theta(\mathbf{r}+)}{\partial n} = 0 \text{ if } \mathbf{r} \in \partial D.$$

But

$$\frac{\partial \Theta(\mathbf{r}+)}{\partial n} = (M_k^T\theta)(\mathbf{r}+)$$

$$= (M_k^T\theta)(\mathbf{r}) - \tfrac{1}{2}\,\theta(\mathbf{r})$$

$$= -\,\theta(\mathbf{r})\ ,$$

and we have that $\theta = 0$ on ∂D . But this contradicts the assumption that θ is a non-trivial solution of $(\tfrac{1}{2} I + M_k^T)\psi = 0$. Hence Θ is an eigenfunction of the homogeneous interior Neumann problem, or in other words, $k^2 \in K_N$.

Conversely, suppose that $k^2 \in K_N$ and let Θ be a corresponding eigenfunction for the homogeneous interior Neumann problem for the Helmholtz equation. Then $\Theta \neq 0$ on ∂D, for otherwise the Helmholtz interior formula

Conversely, assume that

$$\phi = - M_k\theta \quad \text{on } D^i ,$$

where $\theta \neq 0$ on ∂D and

$$(\tfrac{1}{2} I + M_k)\theta = 0 \text{ on } \partial D .$$

If $r \in \partial D$,

$$\begin{aligned}\phi(r-) &= (M_k\theta)(r-) \\ &= (M_k\theta)(r) - \tfrac{1}{2}\,\theta(r) \\ &= -\,\theta(r) ,\end{aligned}$$

and hence $\phi \neq 0$ on ∂D. We also define ϕ on D^e by $\phi = M_k\,\theta$.
Then, if $r \in \partial D$,

$$\begin{aligned}\phi(r+) &= (M_k\theta)(r+) \\ &= (M_k\theta)(r) + \tfrac{1}{2}\,\theta(r) = 0 .\end{aligned}$$

From Theorem (3.3.1) we have that ϕ is a solution of the exterior Dirichlet problem which satisfies the radiation condition, and so the uniqueness theorem implies that

$$\phi = 0 \quad \text{on } D^e .$$

Hence

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } D^e ,$$

and

$$\frac{\partial\phi(r+)}{\partial n} = 0 \quad \text{when } r \in \partial D .$$

However, $\frac{\partial\phi}{\partial n} = N_k\theta$ is continuous across ∂D; thus

$$\frac{\partial\phi(r-)}{\partial n} = 0 \quad \text{if } r \in \partial D .$$

Again, using Theorem (3.3.1), it follows that ϕ is a non-trivial solution of the homogeneous interior Neumann problem.

Corollary. If k^2 is an eigenvalue of the homogeneous interior Neumann problem then the corresponding eigenspace is finite dimensional and equal to the dimension of the null space of the boundary operator $(\tfrac{1}{2} I + M_k)$.

Proof. Let ϕ be an eigenfunction corresponding to the eigenvalue k^2. Then it follows from the theorem there is a $\theta \neq 0$ such that

$$\phi = - M_k\,\theta \quad \text{on } D^i ,$$

where

$$(\tfrac{1}{2} I + M_k)\theta = 0 \quad \text{on } \partial D .$$

According to Fredholm's theorem the eigenspace of the boundary operator $(\tfrac{1}{2} I + M_k)$ is finite dimensional. Assume that $n = \dim N(\tfrac{1}{2} I + M_k)$, and let the functions $\theta_1,\dots,\theta_n$ on ∂D be a basis for the null space $N(\tfrac{1}{2} I + M_k)$.
Then

$$(\tfrac{1}{2} I + M_k)\theta_\alpha = 0$$

for each $\alpha = 1,\ldots,n$, and there are scalars a_α, $\alpha = 1,\ldots,n$, such that

$$\theta = \sum_{\alpha=1}^{n} a_\alpha \theta_\alpha .$$

For each $\alpha = 1,\ldots,n$ define ϕ_α on D^i by $\phi_\alpha = -M_k\theta_\alpha$. Since M_k is a linear operator,

$$\phi = \sum_{\alpha=1}^{n} a_\alpha \phi_\alpha ,$$

and $\phi_1,\ldots,\phi_n$ is a generating set for the eigenspace of the homogeneous interior Neumann problem. We now show that $\phi_1,\ldots,\phi_n$ are linearly independent. Suppose that ϕ_n is linearly dependent on $\phi_1,\ldots,\phi_{n-1}$, then there are scalars $c_1,\ldots,c_{n-1}$ such that

$$\phi_n = \sum_{\alpha=1}^{n-1} c_\alpha \phi_\alpha .$$

Now let

$$\phi_0 = \phi_n - \sum_{\alpha=1}^{n-1} c_\alpha \phi_\alpha$$

and

$$\theta_0 = \theta_n - \sum_{\alpha=1}^{n-1} c_\alpha \theta_\alpha ;$$

then $\phi_0 = 0$ on ∂D and, since M_k and $\frac{1}{2}I + M_k$ are linear operators,

$$\phi_0 = -M_k\theta_0 \quad \text{on } D^i ,$$

and

$$(\tfrac{1}{2}I + M_k)\theta_0 = 0 \quad \text{on } \partial D.$$

But $\theta_0 \neq 0$ and hence ϕ_0 is an eigenfunction of the homogeneous interior Neumann problem. This implies that $\phi_0 \neq 0$, which contradicts our assumption that ϕ_n is a linear combination of $\phi_1,\ldots,\phi_{n-1}$.

Theorem (3.7.5) The function ϕ is an eigenfunction of the homogeneous interior Neumann problem if and only if there is a function $\chi \neq 0$ on ∂D such that

$$\phi = L_k\chi \quad \text{on } D^i$$

and

$$(\tfrac{1}{2}I + M_k^T)\chi = 0 \quad \text{on } \partial D .$$

Proof. Suppose that $\phi = L_k\chi$ on D^i, where $\chi \neq 0$ on ∂D and

$$(\tfrac{1}{2}I + M_k^T)\chi = 0 \quad \text{on } \partial D ;$$

then ϕ is continuous across ∂D and if $\mathbf{r} \in \partial D$,

$$\frac{\partial\phi(r-)}{\partial n} = (M_k^T\chi)(r-)$$

$$= (\tfrac{1}{2}I + M_k^T)\chi(r) = 0.$$

Moreover, $\phi \neq 0$ on D^i. For if $\phi = 0$ on D^i then by the continuity of $\phi = L_k\chi$ it follows that $\phi = 0$ on ∂D . We now prove that $\phi \neq 0$ on D^i. First we define ϕ on all of $\mathbb{R}^3$ by $\phi = L_k\chi$. Assuming that $\phi = 0$ on D^i, the continuity of $\phi = L_k\chi$ across ∂D implies that $\phi = 0$ on ∂D . Hence, from Theorem (3.3.1), ϕ is a solution of the homogeneous exterior Dirichlet problem , and from the uniqueness theorem $\phi = 0$ on D^e. Hence $\frac{\partial\phi}{\partial n} = 0$ on D^e, and thus

$$\frac{\partial\phi(r+)}{\partial n} = 0 \quad \text{if } r \in \partial D .$$

However
$$\frac{\partial\phi(r+)}{\partial n} = (M_k^T\chi)(r+)$$

$$= (M_k^T\chi)(r) - \tfrac{1}{2}\chi(r)$$
$$= -\chi(r) ,$$

and so $\chi = 0$ on ∂D, which contradicts our assumption that $\chi \neq 0$ on ∂D . It follows that $\phi \neq 0$ on D^i; in other words ϕ is an eigenfunction of the homogeneous interior Neumann problem.

Conversely assume that ϕ is an eigenfunction of the homogeneous interior Neumann problem corresponding to the eigenvalue k^2. Let the corresponding eigenspace have dimension n and let $\phi_1,\ldots,\phi_n$ be a basis for this eigenspace. According to the corollary to Theorem (3.7.4), the eigenspace of the operator $(\tfrac{1}{2}I + M_k)$ has a basis $\theta_1,\ldots\theta_n$ such that for each α

$$\phi_\alpha = -M_k\theta_\alpha \quad \text{on } D^i ,$$

where $\theta_\alpha \neq 0$ on ∂D and

$$(\tfrac{1}{2}I + M_k)\theta_\alpha = 0 \quad \text{on } \partial D .$$

However by Fredholm's theorem

$$\dim N(\tfrac{1}{2}I + M_k^T) = \dim N(\tfrac{1}{2}I + M_k) = n .$$

Let $\chi_1,\ldots,\chi_n$ be a basis for $N(\tfrac{1}{2}I + M_k^T)$ and define ψ_α , $\alpha = 1,\ldots,n$, by

$$\psi_\alpha = L_k\,\chi_\alpha \quad \text{on } D^i ;$$

then $(\tfrac{1}{2}I + M_k^T)\chi_\alpha = 0$ for each α. Thus from the first part of the theorem it follows that $\psi_1,\ldots,\psi_n$ are eigenfunctions of the homogeneous interior Neumann problem corresponding to the eigenvalue k^2 , and that they are linearly independent. For, suppose that ψ_n is linearly dependent on

$\psi_1, \ldots, \psi_{n-1}$; then there are scalars a_α , $\alpha = 1, \ldots, n$ such that

$$\psi_n = \sum_{\alpha=1}^{n-1} a_\alpha \psi_\alpha .$$

Now let

$$\psi_0 = \psi_n - \sum_{\alpha=1}^{n-1} a_\alpha \psi_\alpha$$

and

$$\chi_0 = \chi_n - \sum_{\alpha=1}^{n-1} a_\alpha \chi_\alpha ;$$

then $\psi_0 = 0$, and since L_k and $(\frac{1}{2} I + M_k^T)$ are linear operators,

$$L_k \chi_0 = \psi_0 = 0 \quad \text{on } D^i ,$$

and

$$(\tfrac{1}{2} I + M_k^T)\chi_0 = 0 \quad \text{on } \partial D .$$

But $\chi_0 \neq 0$, and so from the first part of the theorem ψ_0 is an eigenfunction of the homogeneous interior Neumann problem and, of course, $\psi_0 \neq 0$. Thus we have a contradiction and it follows that $\{\psi_1, \ldots, \psi_n\}$ is a basis for the eigenspace of the homogeneous interior Neumann problem corresponding to the eigenvalue k^2. Hence there are scalars $c_1, \ldots, c_n$, not all zero, such that

$$\phi = \sum_{\alpha=1}^{n} c_\alpha \psi_\alpha .$$

Define χ on ∂D by

$$\chi = \sum_{\alpha=1}^{n} c_\alpha \chi_\alpha ,$$

Then $\chi \neq 0$ on ∂D, and

$$\phi = L_k \chi \quad \text{on } D^i ,$$

and

$$(\tfrac{1}{2} I + M_k^T)\chi = 0 \quad \text{on } \partial D.$$

Theorem (3.7.6) If $k^2 \notin K_N$ then the inhomogeneous boundary integral equation

$$(\tfrac{1}{2} I + M_k^T)\psi = \frac{\partial \tilde{\phi}_i}{\partial n} \quad \text{on } \partial D \qquad (3.7.11)$$

has a unique solution for all functions $\tilde{\phi}_i \in C^1(\partial D)$, and this solution also satisfies the equation

$$L_k \psi = \tilde{\phi}_i \quad \text{on } \partial D . \qquad (3.7.12)$$

Proof. As $k^2 \notin K_N$ the homogeneous equation

$$(\tfrac{1}{2} I + M_k^T)\psi = 0$$

has only the trivial solution according to Theorem (3.7.3). Hence, from

Theorem (3.7.2), the equation (3.7.8) has a unique solution ψ_0. Define Φ on D^i by

$$\Phi = \tilde{\phi}_i - L_k\psi_0 \ ;$$

then Φ is a solution of the Helmholtz equation on D^i and, recalling that $\frac{\partial\tilde{\phi}_i}{\partial n}$ is continuous across ∂D we have by (3.7.8) that

$$\frac{\partial\phi(r-)}{\partial n} = \frac{\partial\tilde{\phi}_i(r-)}{\partial n} - (M_k^T\psi_0)(r-)$$

$$= \frac{\partial\tilde{\phi}_i(r)}{\partial n} - (M_k^T\psi_0)(r) - \tfrac{1}{2}\psi_0(r) = 0.$$

Hence Φ is a solution of the homogeneous interior Neumann problem. However Φ cannot be an eigenfunction of this problem, for otherwise the homogeneous boundary integral equation $(\frac{1}{2}I + M_k^T)\psi = 0$ would have a non-trivial solution by Theorem (3.7.6). Hence $\Phi = 0$ on D^i and by continuity $\Phi = 0$ on ∂D. Consequently

$$L_k\psi_0 = \tilde{\phi}_i \ .$$

Theorem (3.7.8) If $k^2 \in K_N$ then the two equations

$$L_k\psi = \tilde{\phi}_i \quad \text{on } \partial D \tag{3.7.13}$$

and

$$(\tfrac{1}{2}I + M_k^T)\psi = \frac{\partial\tilde{\phi}_i}{\partial n} \quad \text{on } \partial D \tag{3.7.14}$$

have one and only one solution in common.

Proof. If $k^2 \in K_N$ then the equation (3.7.8) has a finite set of solutions. Let ψ_0 be any such solution, and define Φ by

$$\frac{\partial\Phi(r-)}{\partial n} = \frac{\partial\tilde{\phi}_i(r)}{\partial n} - (M_k^T\psi_0)(r) - \tfrac{1}{2}\psi_0(r) = 0 \ .$$

Since $k^2 \in K_N$, $\Phi \neq 0$ on D^i, and it follows that Φ is an eigenfunction of the homogeneous interior Neumann problem. Hence according to Theorem (3.7.6) we can represent Φ on D^i by

$$\Phi = L_k\,\chi_0 \ ,$$

where $\chi_0 \neq 0$ on ∂D and

$$(\tfrac{1}{2}I + M_k^T)\,\chi_0 = 0 \quad \text{on } \partial D \ .$$

We now have

$$\tilde{\phi}_i - L_k\psi_0 = L_k\chi_0 \quad \text{on } D^i \ ,$$

or $$L_k(\psi_0 + \chi_0) = \tilde{\phi}_i \quad \text{on } D^i .$$

Thus, if we let $\theta_0 = \psi_0 + \chi_0$ then

$$L_k\theta_0 = \tilde{\phi}_i \quad \text{on } D^i ,$$

and due to the continuity of $L_k\theta_0$,

$$L_k\theta_0 = \tilde{\phi}_i \quad \text{on } \partial D ,$$

where $\tilde{\phi}_i$ on ∂D is given by (3.7.6). Since

$$(\tfrac{1}{2} I + M_k^T)\chi_0 = 0 \quad \text{on } \partial D$$

and

$$(\tfrac{1}{2} I + M_k^T)\psi_0 = \frac{\partial \tilde{\phi}_i}{\partial n} \quad \text{on } \partial D ,$$

we see that $$(\tfrac{1}{2} I + M_k^T)\theta_0 = \frac{\partial \tilde{\phi}_i}{\partial n} \quad \text{on } \partial D .$$

Thus the two equations (3.7.7) and (3.7.8) have at least one solution in common. If θ_1 is any other solution of both these equations, then

$$\theta_2 = \theta_1 - \theta_0$$

is a solution of the homogeneous equations corresponding to (3.7.7) and (3.7.8). From Theorem (3.7.1) it then follows that $\theta_2 = 0$ on ∂D, and therefore the two equations (3.7.7) and (3.7.8) have one and only one solution in common.

3.8 The exterior Neumann problem for the Helmholtz equation.

For the exterior Neumann problem with boundary condition

$$\frac{\partial \phi}{\partial n} = f \quad \text{on } \partial D \tag{3.8.1}$$

we obtain, from (3.6.6) and (3.6.9), the boundary integral equations

$$(M_k - \tfrac{1}{2} I)\phi = - \phi_i + L_k f \quad \text{on } \partial D \tag{3.8.2}$$

and

$$N_k\phi = - \frac{\partial \phi_i}{\partial n} + (\tfrac{1}{2} I + M_k^T) f \quad \text{on } \partial D \tag{3.8.3}$$

If we define $$\hat{\phi}_i = L_k f - \phi_i \quad \text{on } \mathbb{R}^3 , \tag{3.8.4}$$

then

$$\frac{\partial \hat{\phi}_i(\mathbf{r})}{\partial n} = (M_k^T f)(\mathbf{r}) - \frac{\partial \phi_i(\mathbf{r})}{\partial n} \quad \text{on } D^i \text{ and } D^e .$$

We note that $\hat{\phi}_i$ is continuous across ∂D, but $\dfrac{\partial \hat{\phi}_i}{\partial n}$ is discontinuous across ∂D. If $\mathbf{r} \in \partial D$ then

$$\frac{\partial \hat{\phi}_i(\mathbf{r}-)}{\partial n} = (M_k^T f)(\mathbf{r}-) - \frac{\partial \phi_i(\mathbf{r}-)}{\partial n}$$

i.e. $$\frac{\partial \hat{\phi}_i(r-)}{\partial n} = (M_k^T f)(r) + \frac{1}{2} f(r) - \frac{\partial \phi_i(r)}{\partial n}$$

and we define $\frac{\partial \hat{\phi}_i}{\partial n}$ on ∂D by

$$\frac{\partial \hat{\phi}_i}{\partial n} = (\tfrac{1}{2} I + M_k^T) f - \frac{\partial \hat{\phi}_i}{\partial n} . \tag{3.8.5}$$

Hence we can write (3.8.2) and (3.8.3) in the form

$$(M_k - \tfrac{1}{2} I)\phi = \hat{\phi}_i \qquad \text{on } \partial D \tag{3.8.6}$$

and

$$N_k \phi = \frac{\partial \hat{\phi}_i}{\partial n} \qquad \text{on } \partial D. \tag{3.8.7}$$

Equation (3.8.5) is a Fredholm integral equation of the second kind, and equation (3.8.6) is a Fredholm integral equation of the first kind. The corresponding homogeneous equations are

$$(M_k - \tfrac{1}{2} I)\phi = 0 \quad \text{on } \partial D \tag{3.8.7}$$

and

$$N_k \phi = 0 \quad \text{on } \partial D . \tag{3.8.8}$$

We saw in section (3.5) that (3.8.7) has non-trivial solutions when $k^2 \in K_D$. According to Fredholm's theory, equation (3.8.5) has a solution if and only if $\hat{\phi}_i$ is orthogonal to $N(M_k^* - \frac{1}{2} I)$. In this section we show that the system (3.8.5) and (3.8.6) always has a unique solution.

Theorem (3.8.1) If ϕ is a solution of the homogeneous equations

$$(M_k - \tfrac{1}{2} I)\phi = 0 \quad \text{on } \partial D$$

and

$$N_k \phi = 0 \quad \text{on } \partial D ,$$

then $\phi = 0$ on ∂D .

Proof. Define Φ on D^i and D^e by $\Phi = M_k \phi$. Then

$$\nabla^2 \Phi + k^2 \Phi = 0 \quad \text{on } D^i \text{ and } D^e .$$

If $r \in \partial D$ is approached from the interior of D then

$$\begin{aligned} \Phi(r-) &= (M_k \phi)(r-) \\ &= (M_k \phi)(r) - \tfrac{1}{2} \phi(r) \\ &= 0 . \end{aligned}$$

Since $\frac{\partial \Phi(r)}{\partial n} = N_k \phi$, we obtain for $r \in \partial D$

$$\frac{\partial \Phi(r)}{\partial n} = (N_k \phi)(r) = 0 .$$

Hence by the Helmholtz interior formulae we find that $\Phi = 0$ on D^i, and so $\Phi = 0$ on ∂D.

If $r \in \partial D$ is approached from the exterior of D, then

$$\begin{aligned}\Phi(r+) &= (M_k\phi)(r+)\\ &= (M_k\phi)(r) + \tfrac{1}{2}\phi(r)\\ &= \phi(r),\end{aligned}$$

and

$$\frac{\partial\Phi(r+)}{\partial n} = (N_k\phi)(r+) = (N_k\phi)(r) = 0 .$$

Hence according to Theorem (3.3.1) Φ is a solution of the homogeneous exterior Neumann problem, and from the uniqueness theorem it follows that $\Phi = 0$ on D^e. Hence if $\mathbf{r} \in \partial D$ then $\phi(r) = \Phi(r+) = 0$; i.e. $\phi = 0$ on ∂D.

Theorem (3.8.2) The boundary integral equation

$$(M_k - \tfrac{1}{2} I)\phi = \hat{\phi}_i \qquad \text{on } \partial D$$

always has a solution.

Proof. Suppose θ is a non-trivial solution of the homogeneous adjoint of equation (3.8.5); i.e.

$$(M_k^* - \tfrac{1}{2} I)\theta = 0 \qquad \text{on } \partial D .$$

Then

$$(M_k^T - \tfrac{1}{2} I)\overline{\theta} = 0 \qquad \text{on } \partial D .$$

Define Θ by

$$\Theta = L_k\overline{\theta} .$$

If $\mathbf{r} \in \partial D$ is approached from the exterior then

$$\begin{aligned}\frac{\partial\Theta(r+)}{\partial n} &= (M_k^T\,\overline{\theta})(r+)\\ &= (M_k^T\,\overline{\theta})(r) - \tfrac{1}{2}\overline{\theta}(r)\\ &= 0 .\end{aligned}$$

According to Theorem (3.3.1) Θ is a solution of the homogeneous exterior Neumann problem, and the uniqueness theorem implies that $\Theta = 0$ on D^e. But $\Theta = L_k\overline{\theta}$ is continuous across ∂D and hence $\Theta = 0$ on ∂D. We now apply Green's second identity to $\hat{\phi}_i$ and Θ on the interior of D, obtaining

$$0 = \iiint_D (\Theta\, \nabla^2\hat{\phi}_i - \hat{\phi}_i\, \nabla^2\Theta)\, d\tau = \iint_{\partial D} \left(\Theta\, \frac{\partial\hat{\phi}_i}{\partial n} - \hat{\phi}_i\, \frac{\partial\Theta}{\partial n}\right) d\sigma$$

and it follows that $\displaystyle\iint_{\partial D} \hat{\phi}_i\, \frac{\partial\Theta}{\partial n}\, d\sigma = 0$.

However, if $\mathbf{r} \in \partial D$ then

$$\begin{aligned}\frac{\partial \Theta(\mathbf{r}-)}{\partial n} &= (M_k^T \bar{\theta})(\mathbf{r}-) \\ &= (M_k^T \bar{\theta})(\mathbf{r}) + \tfrac{1}{2}\bar{\theta}(\mathbf{r}) \\ &= \bar{\theta}(\mathbf{r}).\end{aligned}$$

Hence we have

$$\iint_{\partial D} \hat{\phi}_i \, \bar{\theta} \, d\sigma = 0 \ ;$$

and so $\hat{\phi}_i$ is orthogonal to θ.

Theorem (3.8.3) The homogeneous boundary integral equations

$$(M_k - \tfrac{1}{2} I)\phi = 0$$

and

$$(M_k^T - \tfrac{1}{2} I)\phi = 0$$

have non-trivial solutions if and only if k^2 is an eigenvalue of the homogeneous interior Dirichlet problem, i.e. if and only if $k^2 \in K_D$.

Proof. Suppose that θ is a non-trivial solution of the equation

$$(M_k^T - \tfrac{1}{2} I)\phi = 0 \quad \text{on } \partial D \ ,$$

and define Θ on D^i and D^e by

$$\Theta = L_k \, \theta \ .$$

Theorem (3.3.1) implies that Θ satisfies the Wilcox radiation condition and $\quad \nabla^2 \Theta + k^2 \Theta = 0 \quad$ on D^e .

If $\mathbf{r} \in \partial D$ then

$$\begin{aligned}\frac{\partial \Theta(\mathbf{r}+)}{\partial n} &= (M_k^T\theta)(\mathbf{r}+) \\ &= (M_k^T\theta)(\mathbf{r}) - \tfrac{1}{2}\theta(\mathbf{r}) \ ,\end{aligned}$$

and so Θ is a solution of the homogeneous exterior Neumann problem. Hence the uniqueness theorem implies that $\Theta = 0$ on D^e. However $\Theta = L_k \, \theta$ is continuous across ∂D and we conclude that $\Theta = 0$ on ∂D. We now show that $\Theta \neq 0$ on D^i; for if $\Theta = 0$ on D^i then also $\frac{\partial \Theta}{\partial n} = 0$ on D^i, and so if $\mathbf{r} \in \partial D$ then

$$\begin{aligned}0 = \frac{\partial \Theta(\mathbf{r}-)}{\partial n} &= (M_k^T \theta)(\mathbf{r}-) \\ &= (M_k^T \theta)(\mathbf{r}) + \tfrac{1}{2}\theta(\mathbf{r}) \\ &= \theta(\mathbf{r}) \ .\end{aligned}$$

But this contradicts our assumption that θ is a non-trivial solution of $(M_k - \frac{1}{2} I)\phi = 0$ on ∂D and so $\Theta \neq 0$ on D^i. Accordingly Θ is an eigenfunction of the homogeneous interior Dirichlet problem; i.e. $k^2 \in K_D$.

Conversely let $k^2 \in K_D$ and let θ be an eigenfunction of the homogeneous interior Dirichlet problem. Then $\theta = 0$ on ∂D and the Helmholtz interior formulae yield

$$\theta = L_k \frac{\partial \theta}{\partial n} \quad \text{on } D^i .$$

Therefore $\frac{\partial \theta}{\partial n} \neq 0$ on ∂D, for otherwise $\theta = 0$ on D^i, and by the Helmholtz boundary formulae (3.2.1)

$$(M_k^T - \tfrac{1}{2} I)\frac{\partial \theta}{\partial n} = 0 \quad \text{on } \partial D.$$

In the following two theorems we prove that the homogeneous interior Dirichlet problem for the Helmholtz equation can be solved by assuming either a single layer or double layer representation.

Theorem (3.8.4) The function ϕ on D is an eigenfunction of the homogeneous interior Dirichlet problem for the Helmholtz equation if and only if there exists a function $\theta \neq 0$ on ∂D such that

$$\phi = L_k \theta \quad \text{on } D^i$$

and

$$(M_k^T - \tfrac{1}{2} I)\theta = 0 \quad \text{on } \partial D .$$

Proof. If ϕ is an eigenfunction of the homogeneous interior Dirichlet problem then $\phi = 0$ on ∂D and the Helmholtz interior formulae (3.2.14) and (3.2.15) respectively yield

$$L_k \frac{\partial \phi}{\partial n} = \phi \quad \text{on } D^i$$

and

$$(M_k^T - \tfrac{1}{2} I)\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial D .$$

If we define θ on ∂D by $\theta(\mathbf{r}) = \frac{\partial \phi(\mathbf{r}-)}{\partial n}$ when $\mathbf{r} \in \partial D$ then $\phi = L_k\theta$ on D^i, and $(M_k^T - \frac{1}{2} I)\theta = 0$ on ∂D. Moreover, $\theta \neq 0$ on ∂D, for otherwise $\phi = 0$ on D^i.

Conversely suppose that

$$\phi = L_k\theta \quad \text{on } D^i ,$$

where θ is a non-trivial solution of

$$(M_k^T - \tfrac{1}{2} I)\theta = 0 \quad \text{on } \partial D.$$

According to Theorwm (3.3.1)

$$\nabla^2\phi + k^2\phi = 0 \quad \text{on } D^i ;$$

hence if $r \in \partial D$ then

$$\begin{aligned} \frac{\partial \hat{\phi}_i(\mathbf{r}-)}{\partial n} &= (M_k^T \theta)(\mathbf{r}-) \\ &= (M_k^T \theta)(\mathbf{r}) + \tfrac{1}{2}\theta(\mathbf{r}) = \theta(\mathbf{r}) , \end{aligned}$$

and it follows that $\phi \neq 0$ on D^i. For if $\phi = 0$ on D^i, then also $\frac{\partial \phi}{\partial n} = 0$ on D^i and so $\frac{\partial \phi(r-)}{\partial n} = 0$ for $r \in \partial D$. But this implies that $\theta = 0$ on ∂D , and we have a contradiction.

By the Helmholtz interior boundary formula (3.2.14) we obtain

$$M_k\phi - L_k\theta = -\frac{1}{2}\phi \quad \text{on } \partial D ,$$

where we have used $\frac{\partial \phi(r-)}{\partial n} = \theta(r)$ for $r \in \partial D$. But $\phi = L_k\theta$ is continuous across ∂D ,and so

$$(M_k - \frac{1}{2} I)\phi = 0 \quad \text{on } \partial D .$$

Accordin to the boundary formula in (3.2.15)

$$N_k\phi - M_k^T \theta = -\frac{1}{2}\theta \quad \text{on } \partial D .$$

But $(M_k^T - \frac{1}{2} I)\theta = 0$ on ∂D, and so

$$L_k\phi = 0 \quad \text{on } \partial D.$$

We see therefore that ϕ satisfies the two equations

$$(M_k - \frac{1}{2} I)\phi = 0 \quad \text{on } \partial D$$

and

$$N_k\phi = 0 \quad \text{on } \partial D .$$

Hence by Theorem (3.8.1) $\phi = 0$ on ∂D . But $\phi \neq 0$ on D^i and hence ϕ is an eigenfunction of the homogeneous interior Dirichlet problem for the Helmholtz equation.

Corollary. If $k^2 \in K_D$ then the correspnding eigenspace is finite dimensional, and its dimension equals the dimension of the null space of the boundary operator $(M_k^T - \frac{1}{2} I)$ on ∂D .

Proof. Let ϕ be an eigenfunction corresponding to the eigenvalue k^2 . Then it follows from the theorem that there is $\theta \neq 0$ on ∂D such that

$$\phi = L_k\theta \quad \text{on } D^i ,$$

where $\quad (M_k^T - \frac{1}{2} I)\theta = 0 \quad \text{on } \partial D .$

Fredholm's theorem implies that the null space of the operator $(M_k^T - \frac{1}{2} I)$ on ∂D is finite dimensional. If $\theta_1,\ldots,\theta_n$ be a basis for this null space, then there are scalars a_α , α ¨ 1,...,n, such that

$$\theta = \sum_{\alpha=1}^{n} a_\alpha \theta_\alpha .$$

Let $\phi_\alpha = L_k\theta_\alpha$ for $\alpha = 1,\ldots,n$; then

$$\phi = L_k\theta = \sum_{\alpha=1}^{n} a_\alpha \phi_\alpha .$$

Thus $\{\phi_1,\ldots,\phi_n\}$ is a generating set for the eigenspace of the homogeneous interior Dirichlet problem, and we now prove that $\{\phi_1,\ldots,\phi_n\}$ is a basis for this eigenspace. Suppose that ϕ_n is linearly dependent on $\phi_1,\ldots,\phi_{n-1}$; then there are scalars c_α, $\alpha = 1,\ldots,n-1$, not all zero such that

$$\phi_n = \sum_{\alpha=1}^{n-1} c_\alpha\phi_\alpha \quad \text{on } D^i .$$

Let

$$\phi_0 = \phi_n - \sum_{\alpha=1}^{n-1} c_\alpha \phi_\alpha$$

and

$$\theta_0 = \theta_n - \sum_{\alpha=1}^{n-1} c_\alpha \theta_\alpha .$$

Then $\phi_0 = 0$ on D^i, and so

$$L_k\theta_0 = \phi_0 = 0 \quad \text{on } D^i ,$$

and

$$(M_k^T - \tfrac{1}{2} I)\theta_0 = 0 \quad \text{on } \partial D ,$$

since L_k and $(M_k^T - \frac{1}{2} I)$ are linear operators. But $\theta_0 \neq 0$, and hence ϕ_0 is an eigenfunction of the homogeneous interior Dirichlet problem. Therefore $\phi_0 \neq 0$, which contradicts our assumption that ϕ_n is a linear combination of $\phi_1,\ldots,\phi_{n-1}$.

Theorem (3.8.5) The function ϕ is an eigenfunction of the homogeneous interior Dirichlet problem for the Helmholtz equation if and only if there exists a function $\theta \neq 0$ on ∂D such that

$$\phi = M_k\theta \quad \text{on } D^i$$

where

$$(M_k - \tfrac{1}{2} I)\theta = 0 \quad \text{on } \partial D .$$

Proof. Assume that $\phi = M_k\theta$ on D^i (and on D^e) where $\theta \neq 0$ on ∂D and

$$(M_k - \tfrac{1}{2} I)\theta = 0 \quad \text{on } \partial D.$$

Then ϕ satifies the Helmholtz equation on D^i, and if $\mathbf{r} \in \partial D$

$$\phi(\mathbf{r}-) = (M_k\theta)(\mathbf{r}) - \tfrac{1}{2}\,\theta(\mathbf{r}) = 0 .$$

As $\theta \neq 0$ on ∂D, $\phi \neq 0$ on D^i. For if $\phi = 0$ on D^i, then $\frac{\partial\phi}{\partial n} = 0$ on D^i, and

$$0 = \frac{\partial\phi(r-)}{\partial n} = \frac{\partial\phi(r)}{\partial n} = \frac{\partial\phi(r+)}{\partial n} ,$$

since $\frac{\partial\phi}{\partial n} = N_k\theta$ is continuous across ∂D. But ϕ is a solution of the Helmholtz equation on D^e and according to the uniqueness theorem $\phi = 0$ on D^e. Hence if $r \in \partial D$

$$\begin{aligned} 0 &= \phi(r+) = (M_k\theta)(r+) \\ &= (M_k\theta)(r) + \tfrac{1}{2}\,\theta(r) \\ &= \theta(r) ; \end{aligned}$$

i.e. $\theta = 0$ on ∂D and θ is therefore not an eigenfunction of M_k . Thus the assumption that $\phi = 0$ on D^i is false. Accordingly $\phi \neq 0$ on D^i implies that ϕ is an eigenfunction of the homogeneous interior Dirichlet problem. Conversely, suppose that ϕ is an eigenfunction corresponding to the eigenvalue k^2 of the Dirichlet problem. Let $\theta_1,\ldots,\theta_n$ be a basis for the null space of the operator $(M_k - \frac{1}{2}I)$ on ∂D; thus each θ_α is an eigenfunction of M_k , i.e.

$$(M_k - \tfrac{1}{2}I)\theta_\alpha = 0 \quad \text{on } \partial D .$$

Hence if we define

$$\phi_\alpha = M_k\theta_\alpha \quad \text{on } D^i,$$

then by the first part of the proof each ϕ_α is an eigenfunction corresponding to the eigenvalue k^2. Moreover $\{\phi_1,\ldots,\phi_n\}$ is a basis for this eigenspace. To prove this we assume the contrary. Thus suppose that ϕ_n is linearly dependent on $\phi_1,\ldots,\phi_{n-1}$; then there are scalars a_α , $\alpha = 1,\ldots,n-1$, not all zero such that

$$\phi_n = \sum_{\alpha=1}^{n-1} a_\alpha\phi_\alpha \qquad \text{on } D^i .$$

Define
$$\phi_0 = \phi_n - \sum_{\alpha=1}^{n-1} a_\alpha\,\phi_\alpha \quad \text{on } D^i ,$$

and
$$\theta_0 = \theta_n - \sum_{\alpha=1}^{n-1} a_\alpha\,\theta_\alpha \quad \text{on } \partial D .$$

Then $\phi_0 = 0$ on D^i , and because of the linearity of the operators M_k and $(M_k - \frac{1}{2}I)$ we find

$$0 = \phi_0 = M_k\,\theta_0 \quad \text{on } D^i$$

and
$$(M_k - \tfrac{1}{2}I)\,\theta_0 = 0 \quad \text{on } \partial D .$$

But $\theta_0 \neq 0$ on ∂D, and by the first part of the proof it follows that $\phi_0 \neq 0$ on D^i. Thus we have a contradiction, and hence the assumption that $\{\phi_1, \ldots, \phi_n\}$ is not a basis is false. Thus there are scalars c_α, α ¨ 1,...,n-1, such that

$$\phi = \sum_{\alpha=1}^{n-1} c_\alpha \phi_\alpha$$

If we now define θ on ∂D by

$$\theta = \sum_{\alpha=1}^{n-1} c_\alpha \theta_\alpha ,$$

then $$\phi = M_k \theta \quad \text{on } D^i$$

and

$$(M_k - \tfrac{1}{2} I)\theta = 0 \quad \text{on } \partial D .$$

Hence $\theta \neq 0$ on ∂D, for otherwise each $c_\alpha = 0$ and $\phi = 0$.

Theorem (3.8.6) If $k^2 \notin K_D$ then the equation

$$(M_k - \tfrac{1}{2} I)\phi = \hat{\phi}_i \quad \text{on } \partial D$$

has a unique solution for all functions $\hat{\phi}_i \in L^2(\partial D)$, and this solution also satisfies the equation

$$N_k \phi = \frac{\partial \hat{\phi}_i}{\partial n} \quad \text{on } \partial D.$$

Proof. If $k^2 \notin K_D$ then, by Theorem (3.8.3), the homogeneous equation

$$(M_k - \tfrac{1}{2} I)\phi = 0$$

has only the trivial solution and hence the equation

$$(M_k - \tfrac{1}{2} I)\phi = \hat{\phi}_i$$

has a unique solution.

We now show that this solution is also a solution of the equation

$$N_k \phi = \frac{\partial \phi_i}{\partial n} \quad \text{on } \partial D.$$

On D^i define Φ by

$$\Phi = \hat{\phi}_i - M_k \phi ,$$

where $\hat{\phi}_i$ is given by (3.8.4). If $\mathbf{r} \in \partial D$ then

$$\begin{aligned} \Phi(\mathbf{r}-) &= \hat{\phi}_i(\mathbf{r}-) - (M_k\phi)(\mathbf{r}-) \\ &= \hat{\phi}_i(\mathbf{r}) - (M_k\phi)(\mathbf{r}) + \tfrac{1}{2}\phi(\mathbf{r}) \\ &= 0 . \end{aligned}$$

According to Theorem (3.3.1) Φ is a solution of the Helmholtz equation on D^i and $\Phi = 0$ on ∂D . But $k^2 \notin K_D$ so that $\Phi = 0$ on D. Hence $\frac{\partial \Phi}{\partial n} = 0$ on D^i and therefore $\frac{\partial \Phi(\mathbf{r}-)}{\partial n} = 0$ on ∂D. However,

$$\frac{\partial\Phi(r-)}{\partial n} = \frac{\partial\hat{\phi}_i(r-)}{\partial n} - (N_k\phi)(r-)$$

$$= \frac{\partial\hat{\phi}_i(r)}{\partial n} - (N_k\phi)(r) .$$

It follows that

$$N_k\phi = \frac{\partial\hat{\phi}_i}{\partial n} \quad \text{on } \partial D.$$

Theorem (3.8.7) If $k^2 \in K_D$ then the system of equations

$$(M_k - \tfrac{1}{2} I)\phi = \hat{\phi}_i$$

$$N_k\phi = \frac{\partial\hat{\phi}_i}{\partial n}$$

on ∂D has a unique solution.

Proof. If $k^2 \in K_D$ then the equation

$$(M_k - \tfrac{1}{2} I)\phi = \hat{\phi}_i$$

has a finite number of linearly independent solutions. Let ϕ_0 be any such solution and define Φ on D^i by

$$\Phi = \hat{\phi}_i - M_k\phi_0 .$$

If $r \in \partial D$ then, as in Theorem (3.8.6), $\Phi(r-) = 0$, and Φ satisfies the Helmholtz equation on D^i. Therefore Φ is an eigenfunction of the homogeneous interior Dirichlet problem for the Helmholtz equation on D^i. According to Theorem (3.8.5) there is a function $\theta_0 \neq 0$ on ∂D such that

$$\Phi = M_k\,\theta_0 \quad \text{on } D^i ,$$

where

$$(M_k - \tfrac{1}{2} I)\theta_0 = 0 \quad \text{on } \partial D .$$

Hence we now have

$$\hat{\phi}_i - M_k\phi_0 = M_k\,\theta_0 \quad \text{on } D^i$$

or

$$M_k(\theta_0 + \phi_0) = \hat{\phi}_i \quad \text{on } D^i .$$

If we define

$$\chi_0 = \theta_0 + \phi_0 \quad \text{on } \partial D ,$$

then

$$M_k\,\chi_0 = \hat{\phi}_i \quad \text{on } D^i .$$

If $r \in \partial D$ then

$$(M_k\chi_0)(r-) = \hat{\phi}_i(r-) .$$

Using (3.2.8) and recalling that $\hat{\phi}_i$ is continuous across ∂D we obtain

$$(M_k - \tfrac{1}{2} I)\chi_0(r) = \hat{\phi}_i(r) .$$

Thus

$$(M_k - \tfrac{1}{2} I)\ \chi_0 = \hat{\phi}_i \qquad \text{on } \partial D \ .$$

From the equation $M_k\chi_0 = \hat{\phi}_i$ on D^i we also obtain

$$\frac{\partial}{\partial n}(M_k\chi_0) = \frac{\partial \hat{\phi}_i}{\partial n} \qquad \text{on } D^i \ ,$$

i.e.

$$N_k\chi_0 = \frac{\partial \hat{\phi}_i}{\partial n} \qquad \text{on } D^i \ .$$

But by (3.2.9) we find that

$$N_k\chi_0 = \frac{\partial \hat{\phi}_i}{\partial n} \qquad \text{on } \partial D \ ,$$

where $\frac{\partial \hat{\phi}_i}{\partial n}$ on ∂D is given by equation (3.8.5). Thus χ_0 on ∂D is a solution of the system

$$(M_k - \tfrac{1}{2}I)\phi = \hat{\phi}_i$$

and

$$N_k\phi = \frac{\partial \hat{\phi}_i}{\partial n}$$

on ∂D . If χ_1 is any other solution of this system then the function ψ on ∂D defined by

$$\psi = \chi_1 - \chi_0$$

is a solution of the system

$$(M_k - \tfrac{1}{2} I)\psi = 0$$

and

$$N_k\psi = 0 \ .$$

Hence from Theorem (3.8.1) it follows that $\psi = 0$ on ∂D and therefore the solution is unique.

3.9 The metod of Burton and Miller.

In the preceding two sections we have described the method of Kleinman and Roach for securing unique solutions for the exterior boundary value problems at all frequencies. As described in the introduction attempts were made by various authors to overcome the difficulties inherent in formulations using a single integral equation for solving the exterior problems. Here we describe the method put forward by Burton and Miller (10) whereby composite integral equations are formed by taking a linear combination of the two Helmholtz boundary formulae. In Chapter IV this method is usd to determine analytical approximations of surface fields on convex bodies. Here we consider only the case of the Neumann boundary value problem with homogeneous boundary condition $\frac{\partial \phi}{\partial n} = 0$. Then f = 0 in (3.8.2) and (3.8.3), and from these two equations we form the composite equation

$$\alpha(M_k - \tfrac{1}{2} I)\phi + \beta N_k\phi = -\left(\alpha\phi_i + \beta \frac{\partial \phi_i}{\partial n}\right) \quad \text{on } \partial D \tag{3.9.1}$$

where α and β are complex numbers. The corresponding homogeneous equation is

$$\alpha(M_k - \tfrac{1}{2} I)\phi + \beta N_k\phi = 0 \quad \text{on } \partial D \tag{3.9.2}$$

Assume that ϕ_0 is a non-trivial solution of this equation, and define Φ on D^i and D^e by

$$\Phi = M_k \phi_0 . \tag{3.9.3}$$

If $r \in \partial D$ is approached from the interior then

$$\Phi(r-) = M_k\phi_0(r) - \tfrac{1}{2}\phi_0(r)$$

and from (3.9.2) it follows that

$$\alpha\Phi(r-) = -\beta N_k\phi_0(r). \tag{3.9.4}$$

From (3.2.5)

$$\frac{\partial \Phi(r)}{\partial n} = N_k\phi_0(r) \tag{3.9.5}$$

for all $r \in D^i \cup D^e$. Since $N_k\phi_0$ is continuous across ∂D equation (3.9.4) yields

$$\alpha\Phi(r-) + \beta \frac{\partial \Phi(r)}{\partial n} = 0 \tag{3.9.6}$$

if $r \in \partial D$.

Since both Φ and $\bar{\Phi}$ satisfy the Helmholtz equation in the interior region D^i,

$$\Phi \nabla^2 \bar{\Phi} - \bar{\Phi} \nabla^2 \Phi = (k^2 - \bar{k}^2)|\Phi|^2 = 4ik_1k_2|\Phi|^2 ,$$

where $k = k_1 + ik_2$. Hence if we apply Green's second identity to Φ and $\bar{\Phi}$ on D^i then

$$4ik_1k_2 \iiint_D |\Phi(r)|^2 d\tau = \iint_{\partial D} \left(\Phi(r-)\frac{\partial \overline{\Phi(r)}}{\partial n} - \overline{\Phi(r-)}\,\frac{\partial \Phi(r)}{\partial n}\right) d\sigma$$

Using (3.9.6) gives

$$4ik_1k_2 \iiint_D |\Phi(r)|^2 d\tau = \frac{2\,\mathrm{Im}\,(\alpha\bar{\beta})}{|\alpha|^2} \iint_{\partial D} \left|\frac{\partial\Phi(r)}{\partial n}\right|^2 d\sigma , \qquad (3.9.7)$$

provided $\alpha \neq 0$, and we consider two cases.

If k is real or imaginary and $\mathrm{Im}(\alpha\bar{\beta}) \neq 0$ then

$$\iint_{\partial D} \left|\frac{\partial\Phi(r)}{\partial n}\right|^2 d\sigma = 0 .$$

Hence
$$\frac{\partial\Phi(r-)}{\partial n} = 0 \quad \text{if } r \in \partial D ,$$

and according to (3.9.5)

$$\frac{\partial\Phi(r+)}{\partial n} = \frac{\partial\Phi(r-)}{\partial n} = 0 \quad \text{if } r \in \partial D.$$

If k is complex we choose α and β so that $\mathrm{Im}(\alpha\bar{\beta}) = 0$. Then equation (3.9.7) implies that $\Phi = 0$ on D^i and so

$$\Phi(r-) = 0 \quad \text{if } r \in \partial D.$$

From (3.9.6) it follows that

$$\frac{\partial\Phi(r+)}{\partial n} = \frac{\partial\Phi(r)}{\partial n} = 0 , \quad \text{if } r \in \partial D.$$

Since Φ satisfies the Helmholtz equation in D^e it follows by the uniqueness theorem for the homogeneous exterior Neumann problem that $\Phi = 0$ on D^e. From (3.9.3) we now have

$$\begin{aligned} 0 = \Phi(r+) &= M_k\phi_0(r+) \\ &= (M_k\phi_0)(r) + \tfrac{1}{2}\,\phi_0(r) \\ &= \phi_0(r), \end{aligned}$$

and this contradicts the assumption that ϕ_0 is a non-trivial solution of (3.9.2).

Theorem (3.9.1) The boundary integral equation

$$\alpha(M_k - \tfrac{1}{2} I)\phi + \beta N_k\phi = -\left(\alpha\phi_i + \beta \frac{\partial\phi_i}{\partial n}\right) \quad \text{on } \partial D$$

always has a unique solution provided

(i) k is real or imaginary and $\mathrm{Im}(\alpha\bar{\beta}) \neq 0$, or

(ii) k is complex and $\mathrm{Im}(\alpha\bar{\beta}) = 0$.

Finally we note that the unique solution of the two equations (3.8.2) and (3.8.3) (with $f = 0$) is also a solution of (3.9.1), and hence the two problems are equivalent.

for each $\alpha = 1,\ldots,n$, and there are scalars a_α, $\alpha = 1,\ldots,n$, such that

$$\theta = \sum_{\alpha=1}^{n} a_\alpha \theta_\alpha .$$

For each $\alpha = 1,\ldots,n$ define ϕ_α on D^i by $\phi_\alpha = -M_k\theta_\alpha$. Since M_k is a linear operator,

$$\phi = \sum_{\alpha=1}^{n} a_\alpha \phi_\alpha ,$$

and $\phi_1,\ldots,\phi_n$ is a generating set for the eigenspace of the homogeneous interior Neumann problem. We now show that $\phi_1,\ldots,\phi_n$ are linearly independent. Suppose that ϕ_n is linearly dependent on $\phi_1,\ldots,\phi_{n-1}$, then there are scalars $c_1,\ldots,c_{n-1}$ such that

$$\phi_n = \sum_{\alpha=1}^{n-1} c_\alpha \phi_\alpha .$$

Now let
$$\phi_0 = \phi_n - \sum_{\alpha=1}^{n-1} c_\alpha \phi_\alpha$$

and
$$\theta_0 = \theta_n - \sum_{\alpha=1}^{n-1} c_\alpha \theta_\alpha ;$$

then $\phi_0 = 0$ on ∂D and, since M_k and $\frac{1}{2}I + M_k$ are linear operators,

$$\phi_0 = -M_k\theta_0 \quad \text{on } D^i ,$$

and
$$(\tfrac{1}{2}I + M_k)\theta_0 = 0 \quad \text{on } \partial D.$$

But $\theta_0 \neq 0$ and hence ϕ_0 is an eigenfunction of the homogeneous interior Neumann problem. This implies that $\phi_0 \neq 0$, which contradicts our assumption that ϕ_n is a linear combination of $\phi_1,\ldots,\phi_{n-1}$.

Theorem (3.7.5) The function ϕ is an eigenfunction of the homogeneous interior Neumann problem if and only if there is a function $\chi \neq 0$ on ∂D such that

$$\phi = L_k\chi \quad \text{on } D^i$$

and
$$(\tfrac{1}{2}I + M_k^T)\chi = 0 \quad \text{on } \partial D .$$

Proof. Suppose that $\phi = L_k\chi$ on D^i, where $\chi \neq 0$ on ∂D and

$$(\tfrac{1}{2}I + M_k^T)\chi = 0 \quad \text{on } \partial D ;$$

then ϕ is continuous across ∂D and if $\mathbf{r} \in \partial D$,

$$\frac{\partial\phi(r-)}{\partial n} = (M_k^T\chi)(r-)$$

$$= (\tfrac{1}{2}I + M_k^T)\chi(r) = 0.$$

Moreover, $\phi \neq 0$ on D^i. For if $\phi = 0$ on D^i then by the continuity of $\phi = L_k\chi$ it follows that $\phi = 0$ on ∂D. We now prove that $\phi \neq 0$ on D^i. First we define ϕ on all of $\mathbb{R}^3$ by $\phi = L_k\chi$. Assuming that $\phi = 0$ on D^i, the continuity of $\phi = L_k\chi$ across ∂D implies that $\phi = 0$ on ∂D. Hence, from Theorem (3.3.1), ϕ is a solution of the homogeneous exterior Dirichlet problem, and from the uniqueness theorem $\phi = 0$ on D^e. Hence $\frac{\partial\phi}{\partial n} = 0$ on D^e, and thus

$$\frac{\partial\phi(r+)}{\partial n} = 0 \quad \text{if } r \in \partial D .$$

However
$$\frac{\partial\phi(r+)}{\partial n} = (M_k^T\chi)(r+)$$

$$= (M_k^T\chi)(r) - \tfrac{1}{2}\chi(r)$$
$$= -\chi(r) ,$$

and so $\chi = 0$ on ∂D, which contradicts our assumption that $\chi \neq 0$ on ∂D. It follows that $\phi \neq 0$ on D^i; in other words ϕ is an eigenfunction of the homogeneous interior Neumann problem.

Conversely assume that ϕ is an eigenfunction of the homogeneous interior Neumann problem corresponding to the eigenvalue k^2. Let the corresponding eigenspace have dimension n and let $\phi_1,\ldots,\phi_n$ be a basis for this eigenspace. According to the corollary to Theorem (3.7.4), the eigenspace of the operator $(\tfrac{1}{2}I + M_k)$ has a basis $\theta_1,\ldots\theta_n$ such that for each α

$$\phi_\alpha = -M_k\theta_\alpha \quad \text{on } D^i ,$$

where $\theta_\alpha \neq 0$ on ∂D and

$$(\tfrac{1}{2}I + M_k)\theta_\alpha = 0 \quad \text{on } \partial D .$$

However by Fredholm's theorem

$$\dim N(\tfrac{1}{2}I + M_k^T) = \dim N(\tfrac{1}{2}I + M_k) = n .$$

Let $\chi_1,\ldots,\chi_n$ be a basis for $N(\tfrac{1}{2}I + M_k^T)$ and define ψ_α, $\alpha = 1,\ldots,n$, by

$$\psi_\alpha = L_k\,\chi_\alpha \quad \text{on } D^i ;$$

then $(\tfrac{1}{2}I + M_k^T)\chi_\alpha = 0$ for each α. Thus from the first part of the theorem it follows that $\psi_1,\ldots,\psi_n$ are eigenfunctions of the homogeneous interior Neumann problem corresponding to the eigenvalue k^2, and that they are linearly independent. For, suppose that ψ_n is linearly dependent on

$\psi_1,\ldots,\psi_{n-1}$; then there are scalars a_α , $\alpha = 1,\ldots,n$ such that

$$\psi_n = \sum_{\alpha=1}^{n-1} a_\alpha \psi_\alpha .$$

Now let

$$\psi_0 = \psi_n - \sum_{\alpha=1}^{n-1} a_\alpha \psi_\alpha$$

and

$$\chi_0 = \chi_n - \sum_{\alpha=1}^{n-1} a_\alpha \chi_\alpha ;$$

then $\psi_0 = 0$, and since L_k and $(\frac{1}{2} I + M_k^T)$ are linear operators,

$$L_k\chi_0 = \psi_0 = 0 \quad \text{on } D^i ,$$

and

$$(\tfrac{1}{2} I + M_k^T)\chi_0 = 0 \quad \text{on } \partial D .$$

But $\chi_0 \neq 0$, and so from the first part of the theorem ψ_0 is an eigenfunction of the homogeneous interior Neumann problem and, of course, $\psi_0 \neq 0$. Thus we have a contradiction and it follows that $\{\psi_1,\ldots,\psi_n\}$ is a basis for the eigenspace of the homogeneous interior Neumann problem corresponding to the eigenvalue k^2. Hence there are scalars $c_1,\ldots,c_n$, not all zero, such that

$$\phi = \sum_{\alpha=1}^{n} c_\alpha \psi_\alpha .$$

Define χ on ∂D by

$$\chi = \sum_{\alpha=1}^{n} c_\alpha \chi_\alpha ,$$

Then $\chi \neq 0$ on ∂D, and

$$\phi = L_k \chi \quad \text{on } D^i ,$$

and

$$(\tfrac{1}{2} I + M_k^T)\chi = 0 \quad \text{on } \partial D.$$

Theorem **(3.7.6)** If $k^2 \notin K_N$ then the inhomogeneous boundary integral equation

$$(\tfrac{1}{2} I + M_k^T)\psi = \frac{\partial\tilde{\phi}_i}{\partial n} \quad \text{on } \partial D \tag{3.7.11}$$

has a unique solution for all functions $\tilde{\phi}_i \in C^1(\partial D)$, and this solution also satisfies the equation

$$L_k \psi = \tilde{\phi}_i \quad \text{on } \partial D . \tag{3.7.12}$$

Proof. As $k^2 \notin K_N$ the homogeneous equation

$$(\tfrac{1}{2} I + M_k^T)\psi = 0$$

has only the trivial solution according to Theorem (3.7.3). Hence, from

Theorem (3.7.2), the equation (3.7.8) has a unique solution ψ_0 . Define Φ on D^i by

$$\Phi = \tilde{\phi}_i - L_k\psi_0 \ ;$$

then Φ is a solution of the Helmholtz equation on D^i and, recalling that $\frac{\partial\tilde{\phi}_i}{\partial n}$ is continuous across ∂D we have by (3.7.8) that

$$\frac{\partial\phi(r-)}{\partial n} = \frac{\partial\tilde{\phi}_i(r-)}{\partial n} - (M_k^T\psi_0)(r-)$$

$$= \frac{\partial\tilde{\phi}_i(r)}{\partial n} - (M_k^T\psi_0)(r) - \frac{1}{2}\,\psi_0(r) = 0.$$

Hence Φ is a solution of the homogeneous interior Neumann problem. However Φ cannot be an eigenfunction of this problem, for otherwise the homogeneous boundary integral equation $(\frac{1}{2}I + M_k^T)\psi = 0$ would have a non-trivial solution by Theorem (3.7.6). Hence $\Phi = 0$ on D^i and by continuity $\Phi = 0$ on ∂D. Consequently

$$L_k\psi_0 = \tilde{\phi}_i \ .$$

Theorem (3.7.8) If $k^2 \in K_N$ then the two equations

$$L_k\psi = \tilde{\phi}_i \quad \text{on } \partial D \tag{3.7.13}$$

and

$$(\tfrac{1}{2}I + M_k^T)\psi = \frac{\partial\tilde{\phi}_i}{\partial n} \quad \text{on } \partial D \tag{3.7.14}$$

have one and only one solution in common.

Proof. If $k^2 \in K_N$ then the equation (3.7.8) has a finite set of solutions. Let ψ_0 be any such solution, and define Φ by

$$\frac{\partial\Phi(r-)}{\partial n} = \frac{\partial\tilde{\phi}_i(r)}{\partial n} - (M_k^T\psi_0)(r) - \frac{1}{2}\,\psi_0(r) = 0 \ .$$

Since $k^2 \in K_N$, $\Phi \neq 0$ on D^i, and it follows that Φ is an eigenfunction of the homogeneous interior Neumann problem. Hence according to Theorem (3.7.6) we can represent Φ on D^i by

$$\Phi = L_k\,\chi_0 \ ,$$

where $\chi_0 \neq 0$ on ∂D and

$$(\tfrac{1}{2}I + M_k^T)\,\chi_0 = 0 \quad \text{on } \partial D \ .$$

We now have

$$\tilde{\phi}_i - L_k\psi_0 = L_k\chi_0 \quad \text{on } D^i \ ,$$

or $$L_k(\psi_0 + \chi_0) = \tilde{\phi}_i \quad \text{on } D^i \ .$$

Thus, if we let $\theta_0 = \psi_0 + \chi_0$ then

$$L_k\theta_0 = \tilde{\phi}_i \quad \text{on } D^i \ ,$$

and due to the continuity of $L_k\theta_0$,

$$L_k\theta_0 = \tilde{\phi}_i \quad \text{on } \partial D \ ,$$

where $\tilde{\phi}_i$ on ∂D is given by (3.7.6). Since

$$(\tfrac{1}{2} I + M_k^T)\chi_0 = 0 \quad \text{on } \partial D$$

and

$$(\tfrac{1}{2} I + M_k^T)\psi_0 = \frac{\partial \tilde{\phi}_i}{\partial n} \quad \text{on } \partial D \ ,$$

we see that $$(\tfrac{1}{2} I + M_k^T)\theta_0 = \frac{\partial \tilde{\phi}_i}{\partial n} \quad \text{on } \partial D \ .$$

Thus the two equations (3.7.7) and (3.7.8) have at least one solution in common. If θ_1 is any other solution of both these equations, then

$$\theta_2 = \theta_1 - \theta_0$$

is a solution of the homogeneous equations corresponding to (3.7.7) and (3.7.8). From Theorem (3.7.1) it then follows that $\theta_2 = 0$ on ∂D, and therefore the two equations (3.7.7) and (3.7.8) have one and only one solution in common.

3.8 The exterior Neumann problem for the Helmholtz equation.

For the exterior Neumann problem with boundary condition

$$\frac{\partial \phi}{\partial n} = f \quad \text{on } \partial D \tag{3.8.1}$$

we obtain, from (3.6.6) and (3.6.9), the boundary integral equations

$$(M_k - \tfrac{1}{2} I)\phi = - \phi_i + L_k f \quad \text{on } \partial D \tag{3.8.2}$$

and

$$N_k\phi = - \frac{\partial \phi_i}{\partial n} + (\tfrac{1}{2} I + M_k^T) f \quad \text{on } \partial D \tag{3.8.3}$$

If we define $$\hat{\phi}_i = L_k f - \phi_i \quad \text{on } \mathbb{R}^3 \ , \tag{3.8.4}$$

then

$$\frac{\partial \hat{\phi}_i(r)}{\partial n} = (M_k^T f)(r) - \frac{\partial \phi_i(r)}{\partial n} \quad \text{on } D^i \text{ and } D^e .$$

We note that $\hat{\phi}_i$ is continuous across ∂D, but $\dfrac{\partial \hat{\phi}_i}{\partial n}$ is discontinuous across ∂D. If $r \in \partial D$ then

$$\frac{\partial \hat{\phi}_i(r-)}{\partial n} = (M_k^T f)(r-) - \frac{\partial \phi_i(r-)}{\partial n}$$

i.e. $$\frac{\partial \hat{\phi}_i(r-)}{\partial n} = (M_k^T f)(r) + \frac{1}{2} f(r) - \frac{\partial \phi_i(r)}{\partial n}$$

and we define $\frac{\partial \hat{\phi}_i}{\partial n}$ on ∂D by

$$\frac{\partial \hat{\phi}_i}{\partial n} = (\tfrac{1}{2} I + M_k^T) f - \frac{\partial \hat{\phi}_i}{\partial n} . \qquad (3.8.5)$$

Hence we can write (3.8.2) and (3.8.3) in the form

$$(M_k - \tfrac{1}{2} I)\phi = \hat{\phi}_i \qquad \text{on } \partial D \qquad (3.8.6)$$

and

$$N_k \phi = \frac{\partial \hat{\phi}_i}{\partial n} \qquad \text{on } \partial D. \qquad (3.8.7)$$

Equation (3.8.5) is a Fredholm integral equation of the second kind, and equation (3.8.6) is a Fredholm integral equation of the first kind. The corresponding homogeneous equations are

$$(M_k - \tfrac{1}{2} I)\phi = 0 \qquad \text{on } \partial D \qquad (3.8.7)$$

and

$$N_k \phi = 0 \qquad \text{on } \partial D . \qquad (3.8.8)$$

We saw in section (3.5) that (3.8.7) has non-trivial solutions when $k^2 \in K_D$. According to Fredholm's theory, equation (3.8.5) has a solution if and only if $\hat{\phi}_i$ is orthogonal to $N(M_k^* - \frac{1}{2} I)$. In this section we show that the system (3.8.5) and (3.8.6) always has a unique solution.

Theorem (3.8.1) If ϕ is a solution of the homogeneous equations

$$(M_k - \tfrac{1}{2} I)\phi = 0 \qquad \text{on } \partial D$$

and

$$N_k \phi = 0 \qquad \text{on } \partial D ,$$

then $\phi = 0$ on ∂D .

Proof. Define Φ on D^i and D^e by $\Phi = M_k \phi$. Then

$$\nabla^2 \Phi + k^2 \Phi = 0 \qquad \text{on } D^i \text{ and } D^e .$$

If $r \in \partial D$ is approached from the interior of D then

$$\begin{aligned} \Phi(r-) &= (M_k\phi)(r-) \\ &= (M_k\phi)(r) - \tfrac{1}{2} \phi(r) \\ &= 0 . \end{aligned}$$

Since $\frac{\partial \Phi(r)}{\partial n} = N_k \phi$, we obtain for $r \in \partial D$

$$\frac{\partial \Phi(r)}{\partial n} = (N_k \phi)(r) = 0 .$$

Hence by the Helmholtz interior formulae we find that $\Phi = 0$ on D^i, and so $\Phi = 0$ on ∂D.

If $r \in \partial D$ is approached from the exterior of D, then

$$\begin{aligned}\Phi(r+) &= (M_k\phi)(r+)\\ &= (M_k\phi)(r) + \tfrac{1}{2}\,\phi(r)\\ &= \phi(r),\end{aligned}$$

and

$$\frac{\partial\Phi(r+)}{\partial n} = (N_k\phi)(r+) = (N_k\phi)(r) = 0 .$$

Hence according to Theorem (3.3.1) Φ is a solution of the homogeneous exterior Neumann problem, and from the uniqueness theorem it follows that $\Phi = 0$ on D^e. Hence if $r \in \partial D$ then $\phi(r) = \Phi(r+) = 0$; i.e. $\phi = 0$ on ∂D.

Theorem (3.8.2) The boundary integral equation

$$(M_k - \tfrac{1}{2}I)\phi = \hat{\phi}_i \quad \text{on } \partial D$$

always has a solution.

Proof. Suppose θ is a non-trivial solution of the homogeneous adjoint of equation (3.8.5); i.e.

$$(M_k^* - \tfrac{1}{2}I)\theta = 0 \quad \text{on } \partial D .$$

Then

$$(M_k^T - \tfrac{1}{2}I)\bar{\theta} = 0 \quad \text{on } \partial D .$$

Define Θ by

$$\Theta = L_k\bar{\theta} .$$

If $r \in \partial D$ is approached from the exterior then

$$\begin{aligned}\frac{\partial\Theta(r+)}{\partial n} &= (M_k^T\,\bar{\theta})(r+)\\ &= (M_k^T\,\bar{\theta})(r) - \tfrac{1}{2}\,\bar{\theta}(r)\\ &= 0 .\end{aligned}$$

According to Theorem (3.3.1) Θ is a solution of the homogeneous exterior Neumann problem, and the uniqueness theorem implies that $\Theta = 0$ on D^e. But $\Theta = L_k\bar{\theta}$ is continuous across ∂D and hence $\Theta = 0$ on ∂D. We now apply Green's second identity to $\hat{\phi}_i$ and Θ on the interior of D, obtaining

$$0 = \iiint_D (\Theta\,\nabla^2\hat{\phi}_i - \hat{\phi}_i\,\nabla^2\Theta)\,d\tau = \iint_{\partial D}\left(\Theta\,\frac{\partial\hat{\phi}_i}{\partial n} - \hat{\phi}_i\,\frac{\partial\Theta}{\partial n}\right)d\sigma$$

and it follows that $\displaystyle\iint_{\partial D}\hat{\phi}_i\,\frac{\partial\Theta}{\partial n}\,d\sigma = 0$.

However, if $r \in \partial D$ then

$$\frac{\partial \Theta(r-)}{\partial n} = (M_k^T \bar{\theta})(r-) = (M_k^T \bar{\theta})(r) + \tfrac{1}{2}\bar{\theta}(r) = \bar{\theta}(r).$$

Hence we have

$$\iint_{\partial D} \hat{\phi}_i \, \bar{\theta} \, d\sigma = 0 \;;$$

and so $\hat{\phi}_i$ is orthogonal to θ.

Theorem (3.8.3) The homogeneous boundary integral equations

$$(M_k - \tfrac{1}{2} I)\phi = 0$$

and

$$(M_k^T - \tfrac{1}{2} I)\phi = 0$$

have non-trivial solutions if and only if k^2 is an eigenvalue of the homogeneous interior Dirichlet problem, i.e. if and only if $k^2 \in K_D$.

Proof. Suppose that θ is a non-trivial solution of the equation

$$(M_k^T - \tfrac{1}{2} I)\phi = 0 \quad \text{on } \partial D ,$$

and define Θ on D^i and D^e by

$$\Theta = L_k\, \theta \ .$$

Theorem (3.3.1) implies that Θ satisfies the Wilcox radiation condition and $\qquad \nabla^2 \Theta + k^2 \Theta = 0 \quad \text{on } D^e$.

If $r \in \partial D$ then

$$\frac{\partial \Theta(r+)}{\partial n} = (M_k^T\theta)(r+) = (M_k^T\theta)(r) - \tfrac{1}{2}\theta(r) ,$$

and so Θ is a solution of the homogeneous exterior Neumann problem. Hence the uniqueness theorem implies that $\Theta = 0$ on D^e. However $\Theta = L_k\, \theta$ is continuous across ∂D and we conclude that $\Theta = 0$ on ∂D. We now show that $\Theta \neq 0$ on D^i; for if $\Theta = 0$ on D^i then also $\frac{\partial \Theta}{\partial n} = 0$ on D^i, and so if $r \in \partial D$ then

$$0 = \frac{\partial \Theta(r-)}{\partial n} = (M_k^T \theta)(r-) = (M_k^T \theta)(r) + \tfrac{1}{2}\theta(r) = \theta(r) \ .$$

But this contradicts our assumption that θ is a non-trivial solution of $(M_k - \frac{1}{2} I)\phi = 0$ on ∂D and so $\Theta \neq 0$ on D^i. Accordingly Θ is an eigenfunction of the homogeneous interior Dirichlet problem; i.e. $k^2 \in K_D$.

Conversely let $k^2 \in K_D$ and let θ be an eigenfunction of the homogeneous interior Dirichlet problem. Then $\theta = 0$ on ∂D and the Helmholtz interior formulae yield

$$\theta = L_k \frac{\partial \theta}{\partial n} \quad \text{on } D^i .$$

Therefore $\frac{\partial \theta}{\partial n} \neq 0$ on ∂D, for otherwise $\theta = 0$ on D^i, and by the Helmholtz boundary formulae (3.2.1)

$$(M_k^T - \tfrac{1}{2} I)\frac{\partial \theta}{\partial n} = 0 \quad \text{on } \partial D.$$

In the following two theorems we prove that the homogeneous interior Dirichlet problem for the Helmholtz equation can be solved by assuming either a single layer or double layer representation.

Theorem (3.8.4) The function ϕ on D is an eigenfunction of the homogeneous interior Dirichlet problem for the Helmholtz equation if and only if there exists a function $\theta \neq 0$ on ∂D such that

$$\phi = L_k \theta \quad \text{on } D^i$$

and

$$(M_k^T - \tfrac{1}{2} I)\theta = 0 \quad \text{on } \partial D .$$

Proof. If ϕ is an eigenfunction of the homogeneous interior Dirichlet problem then $\phi = 0$ on ∂D and the Helmholtz interior formulae (3.2.14) and (3.2.15) respectively yield

$$L_k \frac{\partial \phi}{\partial n} = \phi \quad \text{on } D^i$$

and

$$(M_k^T - \tfrac{1}{2} I)\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial D .$$

If we define θ on ∂D by $\theta(\mathbf{r}) = \frac{\partial \phi(\mathbf{r}-)}{\partial n}$ when $\mathbf{r} \in \partial D$ then $\phi = L_k\theta$ on D^i, and $(M_k^T - \frac{1}{2} I)\theta = 0$ on ∂D. Moreover, $\theta \neq 0$ on ∂D, for otherwise $\phi = 0$ on D^i.

Conversely suppose that

$$\phi = L_k\theta \quad \text{on } D^i ,$$

where θ is a non-trivial solution of

$$(M_k^T - \tfrac{1}{2} I)\theta = 0 \quad \text{on } \partial D.$$

According to Theorwm (3.3.1)

$$\nabla^2\phi + k^2\phi = 0 \quad \text{on } D^i ;$$

hence if $\mathbf{r} \in \partial D$ then

$$\begin{aligned} \frac{\partial \hat{\phi}_i(\mathbf{r}-)}{\partial n} &= (M_k^T \theta)(\mathbf{r}-) \\ &= (M_k^T \theta)(\mathbf{r}) + \tfrac{1}{2}\theta(\mathbf{r}) = \theta(\mathbf{r}) , \end{aligned}$$

and it follows that $\phi \neq 0$ on D^i. For if $\phi = 0$ on D^i, then also $\frac{\partial \phi}{\partial n} = 0$ on D^i and so $\frac{\partial \phi(\mathbf{r}-)}{\partial n} = 0$ for $\mathbf{r} \in \partial D$. But this implies that $\theta = 0$ on ∂D , and we have a contradiction.

By the Helmholtz interior boundary formula (3.2.14) we obtain

$$M_k\phi - L_k\theta = -\frac{1}{2}\phi \quad \text{on } \partial D ,$$

where we have used $\frac{\partial \phi(\mathbf{r}-)}{\partial n} = \theta(\mathbf{r})$ for $\mathbf{r} \in \partial D$. But $\phi = L_k\theta$ is continuous across ∂D ,and so

$$(M_k - \frac{1}{2} I)\phi = 0 \quad \text{on } \partial D .$$

Accordin to the boundary formula in (3.2.15)

$$N_k\phi - M_k^T \theta = -\frac{1}{2}\theta \quad \text{on } \partial D .$$

But $(M_k^T - \frac{1}{2} I)\theta = 0$ on ∂D, and so

$$L_k\phi = 0 \quad \text{on } \partial D.$$

We see therefore that ϕ satisfies the two equations

$$(M_k - \frac{1}{2} I)\phi = 0 \quad \text{on } \partial D$$

and

$$N_k\phi = 0 \quad \text{on } \partial D .$$

Hence by Theorem (3.8.1) $\phi = 0$ on ∂D . But $\phi \neq 0$ on D^i and hence ϕ is an eigenfunction of the homogeneous interior Dirichlet problem for the Helmholtz equation.

Corollary. If $k^2 \in K_D$ then the correspnding eigenspace is finite dimensional, and its dimension equals the dimension of the null space of the boundary operator $(M_k^T - \frac{1}{2} I)$ on ∂D .

Proof. Let ϕ be an eigenfunction corresponding to the eigenvalue k^2 . Then it follows from the theorem that there is $\theta \neq 0$ on ∂D such that

$$\phi = L_k\theta \quad \text{on } D^i ,$$

where $$(M_k^T - \frac{1}{2} I)\theta = 0 \quad \text{on } \partial D .$$

Fredholm's theorem implies that the null space of the operator $(M_k^T - \frac{1}{2} I)$ on ∂D is finite dimensional. If $\theta_1, \ldots, \theta_n$ be a basis for this null space, then there are scalars a_α , α ¨ 1,...,n, such that

$$\theta = \sum_{\alpha=1}^{n} a_\alpha \theta_\alpha .$$

Let $\phi_\alpha = L_k\theta_\alpha$ for $\alpha = 1,\ldots,n$; then

$$\phi = L_k\theta = \sum_{\alpha=1}^{n} a_\alpha \phi_\alpha .$$

Thus $\{\phi_1,\ldots,\phi_n\}$ is a generating set for the eigenspace of the homogeneous interior Dirichlet problem, and we now prove that $\{\phi_1,\ldots,\phi_n\}$ is a basis for this eigenspace. Suppose that ϕ_n is linearly dependent on $\phi_1,\ldots,\phi_{n-1}$; then there are scalars c_α , $\alpha = 1,\ldots,n-1$, not all zero such that

$$\phi_n = \sum_{\alpha=1}^{n-1} c_\alpha\phi_\alpha \quad \text{on } D^i .$$

Let

$$\phi_0 = \phi_n - \sum_{\alpha=1}^{n-1} c_\alpha \phi_\alpha$$

and

$$\theta_0 = \theta_n - \sum_{\alpha=1}^{n-1} c_\alpha \theta_\alpha .$$

Then $\phi_0 = 0$ on D^i, and so

$$L_k\theta_0 = \phi_0 = 0 \quad \text{on } D^i ,$$

and

$$(M_k^T - \tfrac{1}{2} I)\theta_0 = 0 \quad \text{on } \partial D ,$$

since L_k and $(M_k^T - \frac{1}{2} I)$ are linear operators. But $\theta_0 \neq 0$, and hence ϕ_0 is an eigenfunction of the homogeneous interior Dirichlet problem. Therefore $\phi_0 \neq 0$, which contradicts our assumption that ϕ_n is a linear combination of $\phi_1,\ldots,\phi_{n-1}$.

Theorem (3.8.5) The function ϕ is an eigenfunction of the homogeneous interior Dirichlet problem for the Helmholtz equation if and only if there exists a function $\theta \neq 0$ on ∂D such that

$$\phi = M_k\theta \qquad \text{on } D^i$$

where

$$(M_k - \tfrac{1}{2} I)\theta = 0 \quad \text{on } \partial D .$$

Proof. Assume that $\phi = M_k\theta$ on D^i (and on D^e) where $\theta \neq 0$ on ∂D and

$$(M_k - \tfrac{1}{2} I)\theta = 0 \quad \text{on } \partial D.$$

Then ϕ satifies the Helmholtz equation on D^i, and if $\mathbf{r} \in \partial D$

$$\phi(\mathbf{r}-) = (M_k\theta)(\mathbf{r}) - \tfrac{1}{2}\, \theta(\mathbf{r}) = 0 .$$

As $\theta \neq 0$ on ∂D, $\phi \neq 0$ on D^i. For if $\phi = 0$ on D^i, then $\frac{\partial\phi}{\partial n} = 0$ on D^i , and

$$0 = \frac{\partial\phi(r-)}{\partial n} = \frac{\partial\phi(r)}{\partial n} = \frac{\partial\phi(r+)}{\partial n} ,$$

since $\frac{\partial\phi}{\partial n} = N_k\theta$ is continuous across ∂D. But ϕ is a solution of the Helmholtz equation on D^e and according to the uniqueness theorem $\phi = 0$ on D^e. Hence if $r \in \partial D$

$$\begin{aligned} 0 &= \phi(r+) = (M_k\theta)(r+) \\ &= (M_k\theta)(r) + \tfrac{1}{2}\theta(r) \\ &= \theta(r) ; \end{aligned}$$

i.e. $\theta = 0$ on ∂D and θ is therefore not an eigenfunction of M_k . Thus the assumption that $\phi = 0$ on D^i is false. Accordingly $\phi \neq 0$ on D^i implies that ϕ is an eigenfunction of the homogeneous interior Dirichlet problem. Conversely, suppose that ϕ is an eigenfunction corresponding to the eigenvalue k^2 of the Dirichlet problem. Let $\theta_1,\ldots,\theta_n$ be a basis for the null space of the operator $(M_k - \frac{1}{2}I)$ on ∂D; thus each θ_α is an eigenfunction of M_k , i.e.

$$(M_k - \tfrac{1}{2}I)\theta_\alpha = 0 \quad \text{on } \partial D .$$

Hence if we define

$$\phi_\alpha = M_k\theta_\alpha \quad \text{on } D^i ,$$

then by the first part of the proof each ϕ_α is an eigenfunction corresponding to the eigenvalue k^2. Moreover $\{\phi_1,\ldots,\phi_n\}$ is a basis for this eigenspace. To prove this we assume the contrary. Thus suppose that ϕ_n is linearly dependent on $\phi_1,\ldots,\phi_{n-1}$; then there are scalars a_α , $\alpha = 1,\ldots,n-1$, not all zero such that

$$\phi_n = \sum_{\alpha=1}^{n-1} a_\alpha\phi_\alpha \quad \text{on } D^i .$$

Define

$$\phi_0 = \phi_n - \sum_{\alpha=1}^{n-1} a_\alpha \phi_\alpha \quad \text{on } D^i ,$$

and

$$\theta_0 = \theta_n - \sum_{\alpha=1}^{n-1} a_\alpha \theta_\alpha \quad \text{on } \partial D .$$

Then $\phi_0 = 0$ on D^i , and because of the linearity of the operators M_k and $(M_k - \frac{1}{2}I)$ we find

$$0 = \phi_0 = M_k\,\theta_0 \quad \text{on } D^i$$

and

$$(M_k - \tfrac{1}{2}I)\,\theta_0 = 0 \quad \text{on } \partial D .$$

But $\theta_0 \neq 0$ on ∂D, and by the first part of the proof it follows that $\phi_0 \neq 0$ on D^i. Thus we have a contradiction, and hence the assumption that $\{\phi_1, \ldots, \phi_n\}$ is not a basis is false. Thus there are scalars c_α, $\alpha = 1, \ldots, n-1$, such that

$$\phi = \sum_{\alpha=1}^{n-1} c_\alpha \phi_\alpha$$

If we now define θ on ∂D by

$$\theta = \sum_{\alpha=1}^{n-1} c_\alpha \theta_\alpha ,$$

then

$$\phi = M_k \theta \quad \text{on } D^i$$

and

$$(M_k - \tfrac{1}{2} I)\theta = 0 \quad \text{on } \partial D .$$

Hence $\theta \neq 0$ on ∂D, for otherwise each $c_\alpha = 0$ and $\phi = 0$.

Theorem (3.8.6) If $k^2 \notin K_D$ then the equation

$$(M_k - \tfrac{1}{2} I)\phi = \hat{\phi}_i \quad \text{on } \partial D$$

has a unique solution for all functions $\hat{\phi}_i \in L^2(\partial D)$, and this solution also satisfies the equation

$$N_k \phi = \frac{\partial \hat{\phi}_i}{\partial n} \quad \text{on } \partial D.$$

Proof. If $k^2 \notin K_D$ then, by Theorem (3.8.3), the homogeneous equation

$$(M_k - \tfrac{1}{2} I)\phi = 0$$

has only the trivial solution and hence the equation

$$(M_k - \tfrac{1}{2} I)\phi = \hat{\phi}_i$$

has a unique solution.

We now show that this solution is also a solution of the equation

$$N_k \phi = \frac{\partial \hat{\phi}_i}{\partial n} \quad \text{on } \partial D.$$

On D^i define Φ by

$$\Phi = \hat{\phi}_i - M_k \phi ,$$

where $\hat{\phi}_i$ is given by (3.8.4). If $r \in \partial D$ then

$$\begin{aligned} \Phi(r-) &= \hat{\phi}_i(r-) - (M_k\phi)(r-) \\ &= \hat{\phi}_i(r) - (M_k\phi)(r) + \tfrac{1}{2}\phi(r) \\ &= 0 . \end{aligned}$$

According to Theorem (3.3.1) Φ is a solution of the Helmholtz equation on D^i and $\Phi = 0$ on ∂D . But $k^2 \notin K_D$ so that $\Phi = 0$ on D. Hence $\frac{\partial \Phi}{\partial n} = 0$ on D^i and therefore $\frac{\partial \Phi(r-)}{\partial n} = 0$ on ∂D. However,

$$\frac{\partial\Phi(r-)}{\partial n} = \frac{\partial\hat{\phi}_i(r-)}{\partial n} - (N_k\phi)(r-)$$

$$= \frac{\partial\hat{\phi}_i(r)}{\partial n} - (N_k\phi)(r) \ .$$

It follows that

$$N_k\phi = \frac{\partial\hat{\phi}_i}{\partial n} \quad \text{on } \partial D.$$

Theorem (3.8.7) If $k^2 \in K_D$ then the system of equations

$$(M_k - \tfrac{1}{2} I)\phi = \hat{\phi}_i$$

$$N_k\phi = \frac{\partial\hat{\phi}_i}{\partial n}$$

on ∂D has a unique solution.

Proof. If $k^2 \in K_D$ then the equation

$$(M_k - \tfrac{1}{2} I)\phi = \hat{\phi}_i$$

has a finite number of linearly independent solutions. Let ϕ_0 be any such solution and define Φ on D^i by

$$\Phi = \hat{\phi}_i - M_k\phi_0 \ .$$

If $r \in \partial D$ then, as in Theorem (3.8.6), $\Phi(r-) = 0$, and Φ satisfies the Helmholtz equation on D^i. Therefore Φ is an eigenfunction of the homogeneous interior Dirichlet problem for the Helmholtz equation on D^i. According to Theorem (3.8.5) there is a function $\theta_0 \neq 0$ on ∂D such that

$$\Phi = M_k\,\theta_0 \quad \text{on } D^i \ ,$$

where

$$(M_k - \tfrac{1}{2} I)\theta_0 = 0 \quad \text{on } \partial D \ .$$

Hence we now have

$$\hat{\phi}_i - M_k\phi_0 = M_k\,\theta_0 \quad \text{on } D^i$$

or

$$M_k(\theta_0 + \phi_0) = \hat{\phi}_i \quad \text{on } D^i \ .$$

If we define

$$\chi_0 = \theta_0 + \phi_0 \quad \text{on } \partial D \ ,$$

then

$$M_k\,\chi_0 = \hat{\phi}_i \quad \text{on } D^i \ .$$

If $r \in \partial D$ then

$$(M_k\chi_0)(r-) = \hat{\phi}_i(r-) \ .$$

Using (3.2.8) and recalling that $\hat{\phi}_i$ is continuous across ∂D we obtain

$$(M_k - \tfrac{1}{2} I)\chi_0(r) = \hat{\phi}_i(r) \ .$$

Thus

$$(M_k - \tfrac{1}{2} I) \chi_0 = \hat{\phi}_i \qquad \text{on } \partial D .$$

From the equation $M_k\chi_0 = \hat{\phi}_i$ on D^i we also obtain

$$\frac{\partial}{\partial n}(M_k\chi_0) = \frac{\partial\hat{\phi}_i}{\partial n} \qquad \text{on } D^i ,$$

i.e.

$$N_k\chi_0 = \frac{\partial\hat{\phi}_i}{\partial n} \qquad \text{on } D^i .$$

But by (3.2.9) we find that

$$N_k\chi_0 = \frac{\partial\hat{\phi}_i}{\partial n} \qquad \text{on } \partial D ,$$

where $\frac{\partial\hat{\phi}_i}{\partial n}$ on ∂D is given by equation (3.8.5). Thus χ_0 on ∂D is a solution of the system

$$(M_k - \tfrac{1}{2}I)\phi = \hat{\phi}_i$$

and

$$N_k\phi = \frac{\partial\hat{\phi}_i}{\partial n}$$

on ∂D . If χ_1 is any other solution of this system then the function ψ on ∂D defined by

$$\psi = \chi_1 - \chi_0$$

is a solution of the system

$$(M_k - \tfrac{1}{2} I)\psi = 0$$

and

$$N_k\psi = 0 .$$

Hence from Theorem (3.8.1) it follows that $\psi = 0$ on ∂D and therefore the solution is unique.

3.9 The metod of Burton and Miller.

In the preceding two sections we have described the method of Kleinman and Roach for securing unique solutions for the exterior boundary value problems at all frequencies. As described in the introduction attempts were made by various authors to overcome the difficulties inherent in formulations using a single integral equation for solving the exterior problems. Here we describe the method put forward by Burton and Miller (10) whereby composite integral equations are formed by taking a linear combination of the two Helmholtz boundary formulae. In Chapter IV this method is usd to determine analytical approximations of surface fields on convex bodies. Here we consider only the case of the Neumann boundary value problem with homogeneous boundary condition $\frac{\partial \phi}{\partial n} = 0$. Then f = 0 in (3.8.2) and (3.8.3), and from these two equations we form the composite equation

$$\alpha(M_k - \tfrac{1}{2} I)\phi + \beta N_k\phi = -\left(\alpha\phi_i + \beta \frac{\partial \phi_i}{\partial n}\right) \quad \text{on } \partial D \qquad (3.9.1)$$

where α and β are complex numbers. The corresponding homogeneous equation is

$$\alpha(M_k - \tfrac{1}{2} I)\phi + \beta N_k\phi = 0 \quad \text{on } \partial D \qquad (3.9.2)$$

Assume that ϕ_0 is a non-trivial solution of this equation, and define Φ on D^i and D^e by

$$\Phi = M_k \phi_0 \ . \qquad (3.9.3)$$

If $r \in \partial D$ is approached from the interior then

$$\Phi(r-) = M_k\phi_0(r) - \tfrac{1}{2}\phi_0(r)$$

and from (3.9.2) it follows that

$$\alpha\Phi(r-) = -\beta N_k\phi_0(r). \qquad (3.9.4)$$

From (3.2.5)

$$\frac{\partial\Phi(r)}{\partial n} = N_k\phi_0(r) \qquad (3.9.5)$$

for all $r \in D^i \cup D^e$. Since $N_k\phi_0$ is continuous across ∂D equation (3.9.4) yields

$$\alpha\Phi(r-) + \beta \frac{\partial\Phi(r)}{\partial n} = 0 \qquad (3.9.6)$$

if $r \in \partial D$.

Since both Φ and $\bar{\Phi}$ satisfy the Helmholtz equation in the interior region D^i,

$$\Phi \nabla^2\bar{\Phi} - \bar{\Phi}\nabla^2\Phi = (k^2 - \bar{k}^2)|\Phi|^2 = 4ik_1k_2|\Phi|^2 \ ,$$

where $k = k_1 + ik_2$. Hence if we apply Green's second identity to Φ and $\bar{\Phi}$ on D^i then

$$4ik_1k_2 \iiint_D |\Phi(r)|^2 d\tau = \iint_{\partial D} \left[\Phi(r-)\frac{\partial\overline{\Phi(r)}}{\partial n} - \overline{\Phi(r-)}\,\frac{\partial\Phi(r)}{\partial n}\right] d\sigma$$

Using (3.9.6) gives

$$4ik_1k_2 \iiint_D |\Phi(r)|^2 dr = \frac{2\,\mathrm{Im}\,(\alpha\bar{\beta})}{|\alpha|^2} \iint_{\partial D} \left|\frac{\partial\Phi(r)}{\partial n}\right|^2 d\sigma\,, \qquad (3.9.7)$$

provided $\alpha \neq 0$, and we consider two cases.

If k is real or imaginary and $\mathrm{Im}(\alpha\bar{\beta}) \neq 0$ then

$$\iint_{\partial D} \left|\frac{\partial\Phi(r)}{\partial n}\right|^2 d\sigma = 0\,.$$

Hence $$\frac{\partial\Phi(r-)}{\partial n} = 0 \quad \text{if } r \in \partial D\,,$$

and according to (3.9.5)

$$\frac{\partial\Phi(r+)}{\partial n} = \frac{\partial\Phi(r-)}{\partial n} = 0 \quad \text{if } r \in \partial D.$$

If k is complex we choose α and β so that $\mathrm{Im}(\alpha\bar{\beta}) = 0$. Then equation (3.9.7) implies that $\Phi = 0$ on D^i and so

$$\Phi(r-) = 0 \quad \text{if } r \in \partial D.$$

From (3.9.6) it follows that

$$\frac{\partial\Phi(r+)}{\partial n} = \frac{\partial\Phi(r)}{\partial n} = 0\,, \quad \text{if } r \in \partial D.$$

Since Φ satisfies the Helmholtz equation in D^e it follows by the uniqueness theorem for the homogeneous exterior Neumann problem that $\Phi = 0$ on D^e. From (3.9.3) we now have

$$\begin{aligned} 0 = \Phi(r+) &= M_k\phi_0(r+) \\ &= (M_k\phi_0)(r). + \tfrac{1}{2}\,\phi_0(r) \\ &= \phi_0(r), \end{aligned}$$

and this contradicts the assumption that ϕ_0 is a non-trivial solution of (3.9.2).

Theorem (3.9.1) The boundary integral equation

$$\alpha(M_k - \tfrac{1}{2} I)\phi + \beta N_k\phi = -\left(\alpha\phi_i + \beta \frac{\partial\phi_i}{\partial n}\right) \quad \text{on } \partial D$$

always has a unique solution provided

(i) k is real or imaginary and $\mathrm{Im}(\alpha\bar{\beta}) \neq 0$, or

(ii) k is complex and $\mathrm{Im}(\alpha\bar{\beta}) = 0$.

Finally we note that the unique solution of the two equations (3.8.2) and (3.8.3) (with $f = 0$) is also a solution of (3.9.1), and hence the two problems are equivalent.

CHAPTER IV

ANALYTICAL APPROXIMATIONS OF SURFACE FIELDS FOR A CONVEX SCATTERER

4.1 Azimuth-altitude parameterisation of boundary integral operators.

We assume that D is a regular convex region of $\mathbb{R}^3$ and that the boundary ∂D of D is a smooth surface of class C^2. Let O be the origin of an arbitrary Cartesian reference frame. For any field point P in $\mathbb{R}^3$ we again denote the position vector **OP** by **r** and the coordinates of P by (x_1, x_2, x_3), and source points $P' \in \partial D$ are indicated by primed variables. For the boundary integral operators introduced in Chapter II the field point P is also in ∂D. At such a field point P in ∂D we select tangent-normal axes, and the coordinates of a point P' in ∂D relative to these axes is denoted by (ξ_1', ξ_2', ξ_3'). It is assumed that the coordinate systems introduced are right-handed. Then there is a proper orthogonal matrix $[a_{ij}]$ such that

$$x_i' = x_i + a_{ij}\ \xi_j' \tag{4.1.1}$$

and the coefficients a_{ij} are functions of $\mathbf{r} = (x_1, x_2, x_3)$. Let $R = \|\mathbf{R}\|$ where $\mathbf{R} = \mathbf{PP'} = \mathbf{r}' - \mathbf{r}$. If (φ', χ') are the azimuth-altitude coordinates of $P' \in \partial D$ then

$$R^2 = \xi_i' \xi_i' \tag{4.1.2}$$

and

$$\xi_i' = R\ \eta_i' \tag{4.1.3}$$

where

$$\left.\begin{aligned} \eta_1' &= \cos\varphi' \sin\chi' \\ \eta_2' &= \cos\varphi' \sin\chi' \\ \eta_3' &= -\sin\chi' \end{aligned}\right\} \tag{4.1.4}$$

Consider first the Helmholtz single layer potential $L_k\phi$ with density $\phi \in C^2(\partial D)$. Then

$$(L_k\phi)(\mathbf{r}) = \frac{1}{4\pi} \iint_{\partial D} \frac{e^{ikR}}{R}\ \phi(\mathbf{r}')\ d\sigma'$$

With $r' = r + R$ and $h(r,r') = \frac{1}{4\pi} e^{ikR} \phi(r')$, it follows from (2.3.40) that

$$(L_k\phi)(r) = \frac{1}{4\pi} \int_0^{2\pi}\int_0^{\frac{\pi}{2}} Re^{ikR} \frac{\omega \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|} \phi(r+R) d\chi' d\varphi' , \qquad (4.1.5)$$

where ω is defined by

$$\omega^2 = 1 + \omega_1^2 + \omega_2{}^2$$

and ω_1 and ω_2 are respectively given by equations (2.3.12) and (2.3.13).

The Helmholtz double layer potential is given by

$$(M_k\phi)(r) = \frac{1}{4\pi} \iint_{\partial D} (ikR - 1)\ e^{ikR} \frac{R.n'}{R^3} \phi(r + R)\ d\sigma' ,$$

and we obtain from (2.3.41), with $r' = r + R$ and

$$h(r,r') = \frac{1}{4\pi} (ikR - 1)e^{ikR} \phi(r') ,$$

$$(M_k\phi)(r) = - \frac{1}{4\pi} \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (ikR - 1)e^{ikR} \phi(r + R)\cos\chi' d\chi' d\varphi' . \qquad (4.1.6)$$

For the operator M_k^T it follows from section (3.2) that

$$(M_k^T \phi)(r) = - \frac{1}{4\pi} \iint_{\partial D} (ikR - 1\ )\ e^{ikR} \frac{R.n}{R^3} \phi(r')\ d\sigma' .$$

From equation (2.3.42) with $r' = r + R$ and

$$h(r,r') = \frac{1}{4\pi} (ikR - 1)e^{ikR} \phi(r')$$

it follows that

$$(M_k^T \phi)(r) = \frac{1}{4\pi} \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (ikR - 1)\ e^{ikR} \frac{\omega \sin\chi' \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|} \phi(r+R)\ d\chi' d\varphi' .$$

(4.1.7)

The operator N_k was defined in (3.2.5) by

$$N_k\phi = \frac{\partial}{\partial n} (M_k\phi) ,$$

and so

$$(N_k\phi)(\mathbf{r}) = -\frac{1}{4\pi}\frac{\partial}{\partial n}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}(ikR - 1)e^{ikR}\,\phi(\mathbf{r} + \mathbf{R})\cos\chi' d\chi' d\varphi' . \qquad (4.1.8)$$

In section (2.3) it was proved that for a fixed point $P \in \partial D$ the distance $R = \|\mathbf{r}' - \mathbf{r}\|$ can be expressed as a function of the azimuth-altitude coordinates (φ', χ'). The distance R also depends on the reference point $P \in \partial D$; thus in general R is a function of $\mathbf{r}$, φ' and χ':

$$R = R(\mathbf{r}, \varphi', \chi') \qquad (4.1.9)$$

As the region D is convex, R is a well defined function of φ' and χ' for each $\mathbf{r} \in \partial D$.

If we assume that ϕ is an analytical function on ∂D then $\phi(\mathbf{r}')$ can be expanded as a uniformly convergent Taylor series, where $\mathbf{r}' = \mathbf{r} + \mathbf{R}$. Thus

$$\begin{aligned}\phi(\mathbf{r}') &= \phi(\mathbf{r} + \mathbf{R}) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{R}.\nabla)^n \phi(\mathbf{r}) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (x'_{i_1} - x_{i_1})\ldots(x'_{i_n} - x_{i_n})\, \partial_{i_1 \ldots i_n}\phi(\mathbf{r}) \qquad (4.1.10)\end{aligned}$$

Unless the contrary is explicitly indicated, the summation convention with range 1,2,3 applies to small repeated latin suffixes , and

$$\partial_{i_1 \ldots i_n}\phi(\mathbf{r}) = \frac{\partial^n \phi(\mathbf{r})}{\partial x_{i_1} \ldots \partial x_{i_n}} ,$$

the partial differentiation being with respect to the field coordinates.

Now using (4.1.1) and (4.1.3) in (4.1.10) yields

$$\phi(\mathbf{r} + \mathbf{R}) = \sum_{n=0}^{\infty} \frac{R^n}{n!} a_{i_1 j_1} \ldots a_{i_n j_n}\, \eta'_{j_1} \ldots \eta'_{j_n}\, \partial_{i_1 \ldots i_n}\phi(\mathbf{r}) , \qquad (4.1.11)$$

where the direction cosines a_{ij} are now functions of $\mathbf{r} \in \partial D$. If (4.1.11) is introduced into (4.1.5) then the assumed analycity of ϕ on ∂D allows interchanging the order of integration and summation and we obtain

$$(L_k\phi)(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{1}{n!} a_{i_1 j_1} \ldots a_{i_n j_n}\, A_{j_1 \ldots j_n}(\mathbf{r})\, \partial_{i_1 \ldots i_n}\phi(\mathbf{r}) \qquad (4.1.12)$$

where

$$A_{j_1 \ldots j_n}(\mathbf{r}) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} R^{n+1}\ e^{ikR}\ \eta'_{j_1} \ldots \eta'_{j_n}\ \frac{\omega \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|}\ d\chi'\ d\varphi' \tag{4.1.13}$$

In (4.1.13) R is an analytical function of $\mathbf{r} \in \partial D$ and so $A_{j_1 \ldots j_n}$ is also an analytical function of $\mathbf{r}$.

Similarly we find that

$$(M_k\phi)(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{1}{n!}\ a_{i_1 j_1} \ldots a_{i_n j_n}\ B_{j_1 \ldots j_n}(\mathbf{r})\ \partial_{i_1 \ldots i_n}\phi(\mathbf{r}) \tag{4.1.14}$$

and

$$(M_k^T\phi)(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{1}{n!}\ a_{i_1 j_1} \ldots a_{i_n j_n}\ C_{j_1 \ldots j_n}(\mathbf{r})\ \partial_{i_1 \ldots i_n}\phi(\mathbf{r}) \tag{4.1.15}$$

where

$$B_{j_1 \ldots j_n}(\mathbf{r}) = -\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} R^n (ikR - 1)\ e^{ikR}\ \eta'_{j_1} \ldots \eta'_{j_n}\ \cos\chi'\ d\chi'\ d\varphi' \tag{4.1.16}$$

and

$$C_{j_1 \ldots j_n}(\mathbf{r}) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} R^n (ikR - 1) e^{ikR}\ \eta'_{j_1} \ldots \eta'_{j_n}\ \frac{\omega \cos\chi' \sin\chi'}{|\omega_1 \cos\chi' + \sin\chi'|}\ d\chi' d\varphi' \tag{4.1.17}$$

The functions $B_{j_1 \ldots j_n}$ and $C_{j_1 \ldots j_n}$ are also analytical on ∂D. A similar expansion formula for $N_k\phi$ is derived following equation (4.1.33).

Now consider for example the exterior Dirichlet problem formulated in section (3.6) and (3.7). There we found that the surface field ϕ induced on a scatterer by an incident field ϕ_i satisfies the equations

$$L_k\phi = \phi_i \tag{4.1.18}$$

and

$$(M_k^T + \tfrac{1}{2}\ I)\phi = \frac{\partial \phi_i}{\partial n} \tag{4.1.19}$$

Using (4.1.14) and (4.1.16) in these equations yields

$$\sum_{n=0}^{\infty} \frac{1}{n!}\ a_{i_1 j_1} \ldots a_{i_n j_n}\ A_{j_1 \ldots j_n}(\mathbf{r})\ \partial_{i_1 \ldots i_n}\phi(\mathbf{r}) = \phi_i(\mathbf{r}) \tag{4.1.20}$$

$$(C + \tfrac{1}{2})\phi + \sum_{n=1}^{\infty} \frac{1}{n!}\ a_{i_1 j_1} \ldots a_{i_n j_n}\ C_{j_1 \ldots j_n}(\mathbf{r})\ \partial_{i_1 \ldots i_n}\phi(\mathbf{r}) = \frac{\partial \phi_i(\mathbf{r})}{\partial n} \tag{4.1.21}$$

As these expressions involve derivatives of arbitrary orders they cannot be used to determine the unknown surface field ϕ. But bearing in mind that an analytical function can be approximated arbitrarily closely by truncated Taylor expansions, we replace the infinite sum on the left-hand side of (4.1.20) and (4.1.21) by a finite sum and investigate the extent to which solutions of the resulting partial differential equations approximate the surface field ϕ.

Motivated by these considerations we associate with each of the operators L_k, M_k and M_k^T, respectively, a corresponding nth-order partial differential operator $L_k^{(n)}$, $M_k^{(n)}$ and $M_k^{T(n)}$ defined on $C^n(\partial D)$ by

$$(L_k^{(n)}\phi)(r) = \sum_{p=0}^{n} \frac{1}{p!} a_{i_1 j_1} \cdots a_{i_p j_p} A_{j_1 \ldots j_p} \partial_{i_1 \ldots i_p}\phi(r) \tag{4.1.22}$$

$$(M_k^{(n)}\phi)(r) = \sum_{p=0}^{n} \frac{1}{p!} a_{i_1 j_1} \cdots a_{i_p j_p} B_{j_1 \ldots j_p} \partial_{i_1 \ldots i_p}\phi(r) \tag{4.1.23}$$

and

$$(M_k^{T(n)}\phi)(r) = \sum_{p=0}^{n} \frac{1}{p!} a_{i_1 j_1} \cdots a_{i_p j_p} C_{j_1 \ldots j_p} \partial_{i_1 \ldots i_p}\phi(r) \tag{4.1.24}$$

We now prove that if ϕ is analytical on ∂D, i.e. if $\phi \in C^\infty(\partial D)$, then the integral operators L_k, M_k and M_k^T can respectively be approximated arbitrarily closely by the partial differential operators $L_k^{(n)}$, $M_k^{(n)}$ and $M_k^{T(n)}$, provided n is sufficiently large. For if $\phi \in C^\infty(\partial D)$ then by Taylor's theorem for any $r, r' \in \partial D$ we have

$$\phi(r') = \sum_{p=0}^{n} \frac{1}{p!} (x'_{i_1} - x_{i_1}) \ldots (x'_{i_p} - x_{i_p}) \partial_{i_1 \ldots i_p}\phi(r)$$

$$+ \frac{1}{(n+1)!} (x'_{i_1} - x_{i_1}) \ldots (x'_{i_{n+1}} - x_{i_{n+1}}) \partial_{i_1 \ldots i_{n+1}}\phi(r + \theta \mathbf{R})$$

where $r' = r + \mathbf{R}$ and $0 < \theta < 1$ – in general θ will be a function of r and r'. Using (4.1.1)-(4.1.3) we obtain

$$\phi(r') = \sum_{p=0}^{n} \frac{R^p}{p!} a_{i_1 j_1} \cdots a_{i_p j_p} \eta'_{j_1} \cdots \eta'_{j_p} \partial_{i_1 \ldots i_p}\phi(r) + \phi^{(n)}(r + \theta \mathbf{R}) \tag{4.1.25}$$

where

$$\phi^{(n)}(\mathbf{r} + \theta\mathbf{R}) = \frac{R^{n+1}}{(n+1)!} a_{i_1 j_1} \cdots a_{i_{n+1} j_{n+1}} \eta'_{j_1} \cdots \eta'_{j_{n+1}} \partial_{i_1 \ldots i_{n+1}} \phi(\mathbf{r} + \theta\mathbf{R}) \tag{4.1.26}$$

is the remainder after n terms.

If we define a vector valued function $\boldsymbol{\theta}$ by

$$\boldsymbol{\theta}(\mathbf{r}') = \mathbf{r} + \theta\mathbf{R} \tag{4.1.27}$$

where $\mathbf{r}' = \mathbf{r} + \mathbf{R}$ and θ depends on $\mathbf{r}$ and $\mathbf{r}'$, then we can write

$$\phi^{(n)}(\mathbf{r} + \theta\mathbf{R}) = (\phi^{(n)} \circ \boldsymbol{\theta})(\mathbf{r}')$$

where $\phi^{(n)} \circ \boldsymbol{\theta}$ denotes the composite function.

Substituting for $\phi(\mathbf{r}') = \phi(\mathbf{r} + \mathbf{R})$ from (4.1.25) into the right-hand side of (4.1.5) and using (4.1.13) we obtain

$$(L_k\phi)(\mathbf{r}) = \frac{1}{4\pi} \sum_{p=0}^{n} \frac{1}{p!} a_{i_1 j_1} \cdots a_{i_p j_p} A_{j_1 \ldots j_p}(\mathbf{r}) \, \partial_{i_1 \ldots i_p} \phi(\mathbf{r})$$

$$+ \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} R \, e^{ikR} \phi^{(n)}(\mathbf{r}+\theta\mathbf{R}) \frac{\omega \cos\chi' \, d\chi' d\varphi'}{|\omega_1 \cos\chi' + \sin\chi'|} \, .$$

Using (4.1.22) and (4.1.5) yields

$$(L_k\phi)(\mathbf{r}) = (L_k^{(n)}\phi)(\mathbf{r}) + (L_k\phi^{(n)} \circ \boldsymbol{\theta})(\mathbf{r})$$

or

$$L_k\phi = L_k^{(n)}\phi + L_k \, \phi^{(n)} \circ \boldsymbol{\theta} \tag{4.1.28}$$

Similarly we can prove that

$$M_k\phi = M_k^{(n)}\phi + M_k \, \phi^{(n)} \circ \boldsymbol{\theta} \tag{4.1.29}$$

and

$$M_k^T\phi = M_k^{T(n)}\phi + M_k \, \phi^{(n)} \circ \boldsymbol{\theta} \, , \tag{4.1.30}$$

it being understood that $\boldsymbol{\theta}$ is not necessarily the same in the three cases.

Since $\phi \in C^{\infty}(\partial D)$, its partial derivatives of all orders are bounded. Let U_p be an upper bound for all the derivatives of order p. From (4.1.26) we obtain

$$(L_k\phi^{(n)} \circ \boldsymbol{\theta})(\mathbf{r}) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} R \, e^{ikR} \phi^{(n)}(\mathbf{r}+\theta\mathbf{R}) \frac{\omega \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|} \, d\chi' \, d\varphi'$$

$$= \frac{a_{i_1 j_1} \cdots a_{i_{n+1} j_{n+1}}}{4\pi (n+1)!} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} R^{n+2} e^{ikR} \eta'_{j_1} \cdots \eta'_{j_{n+1}} \partial_{i_1 \ldots i_{n+1}} \phi(\mathbf{r}+\theta\mathbf{R}) \, .$$

$$\frac{\omega \cos\chi'}{|\omega_1 \cos\chi' + \sin\chi'|} \, d\chi' \, d\varphi' \, ,$$

and it follows that

$$|(L_k\phi^{(n)}{}_\circ\theta)(\mathbf{r})| \le \frac{d^n\ U_{n+1}}{4\pi(n+1)!}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{R^2\ \omega\ \cos\chi'}{|\omega_1\cos\chi'+\sin\chi'|}\, d\chi'\ d\varphi' ,$$

where d is the diameter of D. Denoting the area of ∂D by $A(\partial D)$ and using equation (2.3.27), we obtain

$$|(L_k\phi^{(n)}{}_\circ\theta)(\mathbf{r})| \le \frac{A(\partial D)\ d^n\ U_{n+1}}{4\pi(n+1)!} \quad , \ n \ge 0 \ . \tag{4.1.31}$$

The right-hand side of (4.1.33) can be made arbitrarily small by choosing n sufficiently large, provided $U = \sup\{\ U_n\ :\ n \ge 0\ \} < \infty$. In the same way we can prove that

$$|(M_k\phi^{(n)}{}_\circ\theta)(\mathbf{r})| \le \frac{U_{n+1}\ \ d^{n+1}\sqrt{(k^2d^2\ +\ 1\ )}}{2(n+1)!} \ , \quad n \ge 0 \ , \tag{4.1.32}$$

and

$$|(M_k^T\ \phi^{(n)}{}_\circ\theta)(\mathbf{r})| \le \frac{A(\partial D)\ U_{n+1}\ \ d^{n-1}\ \sqrt{(k^2d^2\ +\ 1)}}{4\pi(n+1)!} \ , \ n \ge 1 \ . \tag{4.1.33}$$

It remains to consider the operator N_k defined by (4.1.8). Here we can not differentiate under the integral sign as $(N_k\phi)(\mathbf{r}) = \frac{\partial}{\partial n}(M_k\phi)(\mathbf{r})$ is the normal limit of the normal derivative of $(M_k\phi)(\mathbf{r} + \lambda\mathbf{n})$ for $\lambda \neq 0$. We recall that in section (3.2) the function $M_k\phi$ was defined at points not in ∂D.

Now let $\mathbf{r}$ be a point of ∂D and let

$$\mathbf{r}_\lambda = \mathbf{r} + \lambda\mathbf{n}$$

where $\mathbf{n}$ is the unit outward normal to ∂D at $\mathbf{r} \in \partial D$. An arbitrary point of ∂D is denoted by $\mathbf{r}'$,

$$\mathbf{R}_\lambda = \mathbf{r}' - \mathbf{r}_\lambda \ ,$$

and

$$\mathbf{R} = \mathbf{r}' - \mathbf{r} \ .$$

Then

$$(M_k\phi)(\mathbf{r}_\lambda) = -\frac{1}{4\pi}\iint_{\partial D} (ikR_\lambda - 1)\ e^{ikR_\lambda}\ \frac{\mathbf{R}_\lambda.\mathbf{n}'}{R_\lambda^3}\ \phi(\mathbf{r}')\ d\sigma' \tag{4.1.34}$$

where $\mathbf{n}'$ is the unit normal to ∂D at $\mathbf{r}' \in \partial D$. From (2.3.38) we obtain

$$(M_k\phi)(\mathbf{r}_\lambda) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}(ikR_\lambda - 1)\frac{e^{ikR_\lambda}}{R_\lambda^3}\ \phi(\mathbf{r}+\mathbf{R})\ \Omega\ \cos\chi' d\chi' d\varphi' \tag{4.1.35}$$

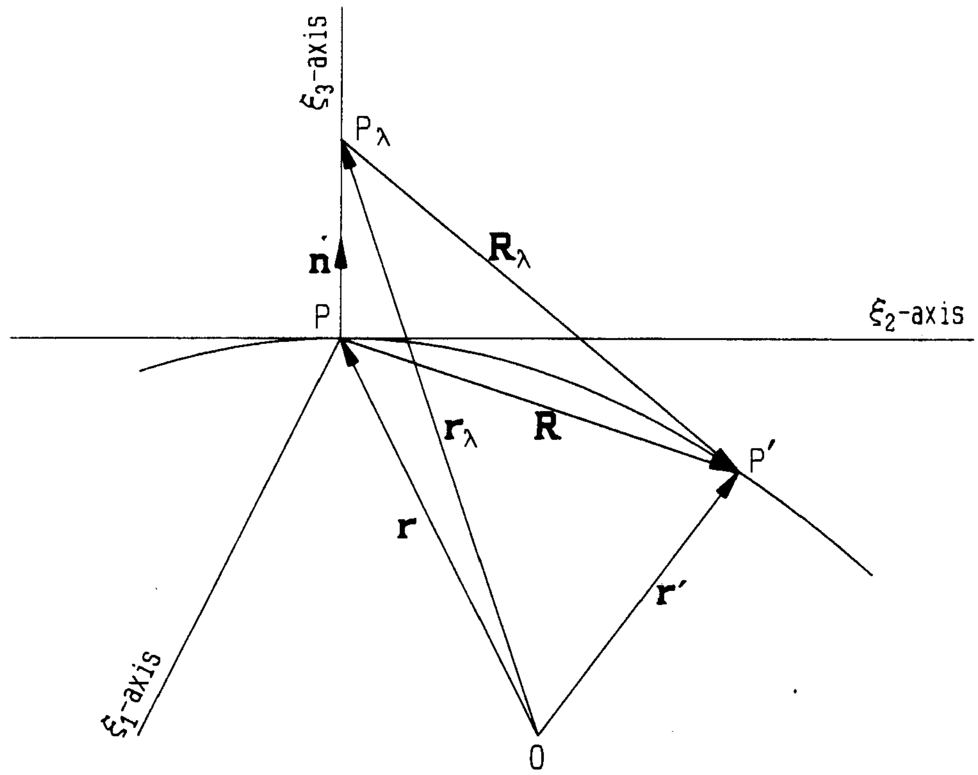


Fig. (4.1.1).

where
$$\Omega = R^3 + \frac{\lambda\ R^2}{\omega_1 \cos\chi' + \sin\chi'} \ . \tag{4.1.36}$$

Here $\mathbf{r}$ and $\mathbf{r}' = \mathbf{r} + \mathbf{R}$ are points on ∂D and we may therefore use (4.1.11), remembering that the direction cosines a_{ij} are functions of $\mathbf{r}$ only. Then

$$(M_k\phi)(\mathbf{r}_\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} a_{i_1 j_1} \cdots a_{i_n j_n} B_{j_1 \ldots j_n}(\mathbf{r}_\lambda)\ \partial_{i_1 \ldots i_n}\phi(\mathbf{r}) \tag{4.1.37}$$

where
$$B_{j_1 \ldots j_n} = -\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} R^n (ikR_\lambda - 1) \frac{e^{ikR_\lambda}}{R_\lambda^3}\ \Omega\ \eta'_{j_1} \cdots \eta'_{j_n} \cos\chi'\ d\chi'\ d\varphi' . \tag{4.1.38}$$

Formally it follows that

$$(N_k\phi)(\mathbf{r}_\lambda) = \frac{\partial}{\partial n} (M_k\phi)(\mathbf{r}_\lambda) = \sum_{n=0}^{\infty} \frac{a_{i_1 j_1} \cdots a_{i_n j_n}}{n!} D_{j_1 \ldots j_n}(\mathbf{r}_\lambda)\ \partial_{i_1 \ldots i_n}\phi(\mathbf{r}) \tag{4.1.39}$$

where
$$D_{j_1 \ldots j_n}(\mathbf{r}_\lambda) = \frac{\partial}{\partial n} B_{j_1 \ldots j_n}(\mathbf{r}_\lambda). \tag{4.1.40}$$

From (4.1.38) and (4.1.40) we now have

$$D_{j_1 \ldots j_n}(\mathbf{r}_\lambda) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} R^n\Bigg[- ik\,\frac{\mathbf{R}_\lambda.\mathbf{n}}{R_\lambda^4}\,e^{ikR_\lambda}\,\Omega$$

$$- (ikR_\lambda - 1)ik\frac{\mathbf{R}_\lambda.\mathbf{n}}{R_\lambda^4}\,e^{ikR_\lambda}\,\Omega + 3(ikR_\lambda - 1)\frac{\mathbf{R}_\lambda.\mathbf{n}}{R_\lambda^5}\,e^{ikR_\lambda}\,\Omega$$

$$+ (ikR_\lambda - 1)\,\frac{e^{ikR_\lambda}}{R_\lambda^3}\,\frac{R^2}{\omega_1\cos\chi'+\sin\chi'}\Bigg]\,\eta'_{j_1}\ldots\eta'_{j_n}\,\cos\chi' d\chi' d\varphi' . \qquad (4.1.41)$$

If $\lambda \to 0$ in (4.1.41) so that $\mathbf{r}_\lambda \to \mathbf{r} \in \partial D$, then

$$D_{j_1 \ldots j_n}(\mathbf{r}) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} R^n\Bigg[- ik\,\frac{\mathbf{R}.\mathbf{n}}{R}\,e^{ikR}$$

$$- (ikR - 1)ik\,\frac{\mathbf{R}.\mathbf{n}}{R}\,e^{ikR} + 3(ikR - 1)\,\frac{\mathbf{R}.\mathbf{n}}{R^2}\,e^{ikR}$$

$$+ (ikR - 1)\,\frac{e^{ikR}}{R}\,\frac{1}{\omega_1\cos\chi'+\sin\chi'}\Bigg]\,\eta'_{j_1}\ldots\eta'_{j_n}\,\cos\chi' d\chi' d\varphi' . \qquad (4.1.42)$$

Now $\mathbf{R}.\mathbf{n} = -R\sin\chi'$, so that

$$D_{j_1 \ldots j_n}(\mathbf{r}) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} W_k^{(n)} e^{ikR}\,\eta'_{j_1}\ldots\eta'_{j_n}\cos\chi' d\chi' d\varphi' \qquad (4.1.43)$$

where

$$W_k^{(n)} = R^{n-2}\left[ikR^2\sin\chi' + (ikR - 1)\left\{ikR^2\sin\chi' - 3\,R\sin\chi' + \frac{R}{\omega_1\cos\chi'+\sin\chi'}\right\}\right]. \qquad (4.1.44)$$

From equation (2.3.15),

$$\omega_1 = \frac{R\cos\chi' + \frac{\partial R}{\partial\chi'}\sin\chi'}{R\sin\chi' - \frac{\partial R}{\partial\chi'}\cos\chi'} \;;$$

hence

$$\frac{R}{\omega_1\cos\chi' + \sin\chi'} = R\sin\chi' - \frac{\partial R}{\partial\chi'}\cos\chi'$$

is always finite, and therefore the integral in (4.1.43) is convergent provided that $n \geq 2$.

When n = 0,

$$D(r) = \lim_{\lambda\to 0} \frac{\partial}{\partial n} B(r_\lambda)$$

$$= -\frac{1}{4\pi} \lim_{\lambda\to 0} \frac{\partial}{\partial n} \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (ikR_\lambda - 1)\frac{e^{ikR_\lambda}}{R_\lambda^3} \;\Omega \cos\chi' d\chi' d\varphi'$$

$$= \frac{1}{4\pi} \lim_{\lambda\to 0} \frac{\partial}{\partial n} \iint_{\partial D} (ikR_\lambda - 1) \frac{R_\lambda . n'}{R_\lambda^3} e^{ikR_\lambda} d\sigma'. \tag{4.1.45}$$

For n = 1,

$$D_j(r) = \lim_{\lambda\to 0} \frac{\partial}{\partial n} B_j(r_\lambda)$$

$$= -\frac{1}{4\pi} \lim_{\lambda\to 0} \frac{\partial}{\partial n} \int_0^{2\pi}\int_0^{\frac{\pi}{2}} R(ikR_\lambda - 1)\frac{e^{ikR_\lambda}}{R_\lambda^3} \;\Omega\; \eta_j' \cos\chi' d\chi' d\varphi'$$

$$= \frac{1}{4\pi} \lim_{\lambda\to 0} \frac{\partial}{\partial n} \iint_{\partial D} (ikR_\lambda - 1) \frac{R_\lambda . n'}{R_\lambda^3} e^{ikR_\lambda} \xi_j' \; d\sigma' ,$$

$$= \frac{1}{4\pi} a_{ij} \lim_{\lambda\to 0} \frac{\partial}{\partial n} \iint_{\partial D} (ikR_\lambda - 1) \frac{R_\lambda . n'}{R_\lambda^3} e^{ikR_\lambda} (x_i' - x_i) d\sigma' , \tag{4.1.46}$$

where we have used the inverse of (4.1.1), namely

$$R\eta_j' = \xi_j' = a_{ij}(x_i' - x_i).$$

According to Theorem (2.4.8) the normal derivatives of the integrals

$$\iint_{\partial D} (ikR_\lambda - 1) \frac{R_\lambda . n'}{R_\lambda^3} e^{ikR_\lambda} d\sigma'$$

and

$$\iint_{\partial D} (ikR_\lambda - 1) \frac{R_\lambda . n'}{R_\lambda^3} e^{ikR_\lambda} (x_i' - x_i) d\sigma'$$

are continuous across ∂D, and hence the limits in (4.1.45) and (4.1.46) exist.

Returning to (4.1.40) and formally letting $\lambda \to 0$, we obtain

$$(N_k\phi)(r) = \sum_{n=0}^{\infty} \frac{a_{i_1 j_1} \cdots a_{i_n j_n}}{n!} D_{j_1 \ldots j_n}(r)\; \partial_{i_1 \ldots i_n} \phi(r) \tag{4.1.47}$$

where

$$D_{j_1\ldots j_n}(r) = \lim_{\lambda\to 0} D_{j_1\ldots j_n}(r_\lambda). \qquad (4.1.48)$$

We can now define an nth order linear partial differential operator $N_k^{(n)}$ by

$$(N_k^{(n)}\phi)(r) = \sum_{p=0}^{n} \frac{a_{i_1 j_1}\cdots a_{i_p j_p}}{n!} D_{j_1\ldots j_p}(r)\, \partial_{i_1\ldots i_p}\phi(r) \qquad (4.1.49)$$

Following the same procedure used in deriving equations (4.1.28) - (4.1.30), it is found that

$$N_k\phi = N_k^{(n)}\phi + N_k\phi^{(n)}\circ\theta, \qquad (4.1.50)$$

where

$$(N_k\phi^{(n)}\circ\theta)(r) = \lim_{\lambda\to 0}\frac{\partial}{\partial n}(M_k\phi^{(n)}\circ\theta)(r_\lambda)$$

$$= -\frac{1}{4\pi}\lim_{\lambda\to 0}\frac{\partial}{\partial n}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}(ikR_\lambda - 1)\frac{e^{ikR_\lambda}}{R_\lambda^3}\,\phi^{(n)}(r+\theta R)\,\Omega\cos\chi'\,d\chi'\,d\varphi'.$$

Substituting for $\phi^{(n)}(r + \theta R)$ from (4.1.26) yields

$$(N_k\phi^{(n)}\circ\theta)(r) = -\frac{a_{i_1 j_1}\cdots a_{i_{n+1} j_{n+1}}}{4\pi(n+1)!}\lim_{\lambda\to 0}\frac{\partial}{\partial n}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} R^{n+1}(ikR_\lambda - 1)\frac{e^{ikR_\lambda}}{R_\lambda^3}\,\Omega.$$

$$\partial_{i_1\ldots i_{n+1}}\phi(r+\theta R)\,\eta'_{j_1}\cdots\eta'_{j_{n+1}}\cos\chi'\,d\chi'\,d\varphi',$$

where Ω is given by (4.1.36). Proceeding now as in (4.1.41) and letting $\lambda \to 0$ gives

$$(N_k\phi^{(n)}\circ\theta)(r) = -\frac{a_{i_1 j_1}\cdots a_{i_{n+1} j_{n+1}}}{4\pi(n+1)!}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} W_k^{(n+1)}\,e^{ikR}\,\eta'_{j_1}\cdots\eta'_{j_{n+1}}.$$

$$\partial_{i_1\ldots i_{n+1}}\phi(r+\theta R)\quad\cos\chi'\quad d\chi'\quad d\varphi'$$

where, according to (1.4.44),

$$W_k^{(n+1)} = R^{n-1}\left[ikR^2\sin\chi' + (ikR - 1)\left(ikR^2\sin\chi' - 3R\sin\chi' + \frac{R}{\omega_1\cos\chi'+\sin\chi'}\right)\right],$$

and so

$$|(N_k\phi^{(n)}\circ\theta)(r)| \le \frac{H\,U_{n+1}\,d^{n+2}}{4\pi(n+1)!},\quad n \ge 1, \qquad (4.1.51)$$

where H is a positive number.

The formulae derived above are applicable to the global approximations as defined in section (1.1). In the case of the local approximations the analysis as outlined above is identical in all respects. Here we briefly summarise the formulae applicable to local approximations. Let S be a surface element contained in ∂D, and suppose that the boundary of S is described by

$$\chi' = \chi'(\varphi'), \qquad 0 \le \varphi' \le 2\pi. \tag{4.1.52}$$

Corresponding to the boundary or global integral operators L_k, M_k, M_k^T and N_k we can define local integral operators $L_{k,s}$,$M_{k,s}$, $M_{k,s}^T$ and $N_{k,s}$ by

$$(L_{k,s}\phi)(\mathbf{r}) = \frac{1}{4\pi}\iint_S \frac{e^{ikR}}{R}\ \phi(\mathbf{r}')\ d\sigma', \tag{4.1.53}$$

$$(M_{k,s}\phi)(\mathbf{r}) = \frac{1}{4\pi}\iint_S (ikR - 1)\ e^{ikR}\ \frac{\mathbf{R}.\mathbf{n}'}{R^3}\ \phi(\mathbf{r} + \mathbf{R})\ d\sigma', \tag{4.1.54}$$

$$(M_{k,s}^T\phi)(\mathbf{r}) = -\frac{1}{4\pi}\iint_S (ikR - 1\ )\ e^{ikR}\ \frac{\mathbf{R}.\mathbf{n}}{R^3}\ \phi(\mathbf{r}')\ d\sigma' \tag{4.1.55}$$

and

$$(N_{k,s}\phi)(\mathbf{r}) = \frac{\partial}{\partial n}\ (M_{k,s}\phi). \tag{4.1.56}$$

If these integral operators are parameterised in terms of azimuth-altitude coordinates we obtain

$$(L_{k,s}\phi)(\mathbf{r}) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\chi'(\varphi')} Re^{ikR}\frac{\omega\ \cos\chi'}{|\omega_1\cos\chi'+\sin\chi'|}\ \phi(\mathbf{r}+\mathbf{R})d\chi'd\varphi', \tag{4.1.57}$$

$$(M_{k,s}\phi)(\mathbf{r}) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\chi'(\varphi')} (ikR - 1)e^{ikR}\ \phi(\mathbf{r} + \mathbf{R})\cos\chi' d\chi' d\varphi' \tag{4.1.58}$$

$$(M_{k,s}^T\phi)(\mathbf{r}) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\chi'(\varphi')} (ikR - 1)\ e^{ikR}\ \frac{\omega\ \sin\chi'\ \cos\chi'}{|\omega_1\cos\chi'+\sin\chi'|}\ \phi(\mathbf{r}+\mathbf{R})\ d\chi' d\varphi' \tag{4.1.58}$$

and

$$(N_{k,s}\phi)(\mathbf{r}) = -\frac{1}{4\pi}\frac{\partial}{\partial n}\int_0^{2\pi}\int_0^{\chi'(\varphi')} (ikR - 1)e^{ikR}\ \phi(\mathbf{r} + \mathbf{R})\cos\chi' d\chi' d\varphi'. \tag{4.1.59}$$

According to (4.1.11)

$$\phi(\mathbf{r} + \mathbf{R}\ ) = \sum_{n=0}^{\infty} \frac{R^n}{n!}\ a_{i_1 j_1}\cdots a_{i_n j_n}\ \eta'_{j_1}\cdots\eta'_{j_n}\ \partial_{i_1\ldots i_n}\phi(\mathbf{r}),$$

and substituting into (4.1.57)-(4.1.59) gives the formal expansions

$$(L_{k,s}\phi)(r) = \sum_{n=0}^{\infty} \frac{1}{n!} a_{i_1j_1}\cdots a_{i_nj_n} A_{s,j_1\ldots j_n} \partial_{i_1\ldots i_n}\phi(r) \tag{4.1.60}$$

$$(M_{k,s}\phi)(r) = \sum_{n=0}^{\infty} \frac{1}{n!} a_{i_1j_1}\cdots a_{i_nj_n} B_{s,j_1\ldots j_n} \partial_{i_1\ldots i_n}\phi(r) \tag{4.1.61}$$

$$(M^T_{k,s}\phi)(r) = \sum_{n=0}^{\infty} \frac{1}{n!} a_{i_1j_1}\cdots a_{i_nj_n} C_{s,j_1\ldots j_n} \partial_{i_1\ldots i_n}\phi(r) \tag{4.1.62}$$

and

$$(N_{k,s}\phi)(r) = \sum_{n=0}^{\infty} \frac{1}{n!} a_{i_1j_1}\cdots a_{i_nj_n} D_{s,j_1\ldots j_n} \partial_{i_1\ldots i_n}\phi(r) \tag{4.1.63}$$

where

$$A_{s,j_1\ldots j_n}(r) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\chi'(\varphi')} R^{n+1}\, e^{ikR}\, \eta'_{j_1}\cdots\eta'_{j_n} \frac{\omega\cos\chi'}{|\omega_1\cos\chi'+\sin\chi'|}\, d\chi'\, d\varphi' \tag{4.1.64}$$

$$B_{s,j_1\ldots j_n}(r) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\chi'(\varphi')} R^{n}(ikR-1)\, e^{ikR}\, \eta'_{j_1}\cdots\eta'_{j_n} \cos\chi'\, d\chi'\, d\varphi' \tag{4.1.65}$$

$$C_{s,j_1\ldots j_n}(r) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\chi'(\varphi')} R^{n}(ikR-1)e^{ikR}\, \eta'_{j_1}\cdots\eta'_{j_n} \frac{\omega\cos\chi'\sin\chi'}{|\omega_1\cos\chi'+\sin\chi'|}\, d\chi' d\varphi' \tag{4.1.66}$$

and if $n \geq 2$

$$D_{s,j_1\ldots j_n}(r) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\chi'(\varphi')} W_k^{(n)} e^{ikR}\, \eta'_{j_1}\cdots\eta'_{j_n}\cos\chi' d\chi' d\varphi', \tag{4.1.67}$$

where $W_k^{(n)}$ is given by (4.1.44). The formulae for D_s and $D_{s,j}$ can be obtained from respectively (4.1.45) and (4.1.46) with S in place of ∂D.

Corresponding to each of the local integral operators $L_{k,s}$, $M_{k,s}$, $M^T_{k,s}$ and $N_{k,s}$ we can associate a partial differential operator $L^{(n)}_{k,s}$, $M^{(n)}_{k,s}$, $M^{T(n)}_{k,s}$ and $N^{(n)}_{k,s}$ defined by

$$(L^{(n)}_{k,s}\phi)(r) = \sum_{p=0}^{n} \frac{1}{p!} a_{i_1j_1}\cdots a_{i_pj_p} A_{s,j_1\ldots j_p} \partial_{i_1\ldots i_p}\phi(r) \tag{4.1.68}$$

$$(M^{(n)}_{k,s}\phi)(r) = \sum_{p=0}^{n} \frac{1}{p!} a_{i_1j_1}\cdots a_{i_pj_p} B_{s,j_1\ldots j_p} \partial_{i_1\ldots i_p}\phi(r) \tag{4.1.69}$$

$$(M^{T(n)}_{k,s}\,\phi)(\mathbf{r}) = \sum_{p=0}^{n} \frac{1}{p!}\, a_{i_1 j_1}\cdots a_{i_p j_p}\, C_{s,j_1\ldots j_p}\, \partial_{i_1\ldots i_p}\phi(\mathbf{r}) \qquad (4.1.70)$$

and

$$(N^{(n)}_{k,s}\,\phi)(\mathbf{r}) = \sum_{p=0}^{n} \frac{1}{p!}\, a_{i_1 j_1}\cdots a_{i_p j_p}\, D_{s,j_1\ldots j_p}\, \partial_{i_1\ldots i_p}\phi(\mathbf{r}). \qquad (4.1.68)$$

Finally we note that each of the analytical coefficients $A_{s,j_1\ldots j_p}$, $B_{s,j_1\ldots j_p}$, $C_{s,j_1\ldots j_p}$ and $D_{s,j_1\ldots j_p}$ can be replaced by suitable local approximations on S, which are obtained by expanding the respective integrands to a sufficient degree of accuracy.

4.2 Analytical approximations for the surface field.

In sections (3.7) and (3.8) we found that when a scalar field ϕ_i is incident on a scatterer D the surface field ψ generated on the scatterer satisfies the boundary integral equations

$$L_k \frac{\partial\psi}{\partial n} = \phi_i \qquad (4.2.1)$$

$$(M^T_k - \tfrac{1}{2} I)\frac{\partial\psi}{\partial n} = \frac{\partial\phi_i}{\partial n} \qquad (4.2.2)$$

in the Dirichlet case, and

$$N_k\psi = -\frac{\partial\phi_i}{\partial n} \qquad (4.2.3)$$

$$(M_k - \tfrac{1}{2} I)\psi = -\phi_i \qquad (4.2.4)$$

in the Neumann case. With each of these two sets of boundary integral equations we now associate, for all positive integral values of n, two sets of nth order linear partial differential equations, namely

$$L^{(n)}_k \phi = \phi_i \qquad (4.2.5)$$

$$(M^{T(n)}_k + \tfrac{1}{2} I)\phi = \frac{\partial\phi_i}{\partial n} \qquad (4.2.6)$$

for the Dirichlet case, and

$$N^{(n)}_k\phi = -\frac{\partial\phi_i}{\partial n} \qquad (4.2.7)$$

$$(M^{(n)}_k - \tfrac{1}{2} I)\phi = -\phi_i \qquad (4.2.8)$$

for the Neumann case. In the sequel we will only consider the latter problem. Now for the Neumann case the coefficients of the partial differential equations (4.2.7) and (4.2.8) are $D_{j_1\ldots j_n}$ and $B_{j_1\ldots j_n}$

respectively, as defined in section (4.1). There it was shown that these coefficients are analytical functions on ∂D. Thus at any point $r \in \partial D$ which is not a singular point of the differential equation a formal power series solution can be constructed. The radius of convergence of this power series will depend on the location of the singular points of the differential equation. These singular points are usually some or all of the zeros of the leading coefficients of the equation. As the coefficients are complex valued functions their zeros will in general be complex vectors which do not lie on the surface of the scatterer. In this case it is possible to construct a general analytical solution by analytical continuation over the bounding surface ∂D of the scatterer.

In section (3.8) we saw that the boundary integral equations (4.2.3) and (4.2.4) always have a unique solution for all wave numbers k. We cannot expect a similar result to hold for the corresponding differential system (4.2.7) and (4.2.8). As neither of these equations are subject to boundary or initial conditions, either of these equations can have many analytical solutions and none of these may be common to both equations.

Suppose that $\{\phi_n\}$ is a sequence of analytical solutions of (4.2.8) for $n = 1,2,3,\ldots$. Then

$$M_k^{(n)} \phi_n - \frac{1}{2} \phi_n = - \phi_i \ , \ n = 1,2,3,\ldots \qquad (4.2.9)$$

and the Taylor expansion

$$\phi_n(r+R) = \phi_n(r) + \sum_{m=1}^{\infty} \frac{R^m}{m!} a_{i_1 j_1} \cdots a_{i_m j_m} \eta'_{j_1} \cdots \eta'_{j_m} \partial_{j_1 \ldots j_m} \phi_n(r) \qquad (4.2.10)$$

is valid at all points r and $r' = r + R$ in ∂D. The convergence of (4.2.10) implies that the mth term converges to zero as $m \to \infty$. Since each $|a_{ij}| \leq 1$ and $|\eta'_j| \leq 1$, it follows that

$$\frac{d^m \, U_m^{(n)}}{m!} \to 0 \qquad (4.2.11)$$

as $m \to \infty$, where d is the diameter of D and $U_m^{(n)}$ is an upper bound for the partial derivatives of order m of ϕ_n.

Using (4.1.29) in (4.2.9) yields

$$- \phi_i = M_k^{(n)} \phi_n - \frac{1}{2} \phi_n$$

$$= M_k \phi_n - M_k(\phi_n^{(n)} \circ \theta_n) - \frac{1}{2} \phi_n$$

or $$M_k\phi_n - \tfrac{1}{2}\phi_n + \phi_i = M_k(\phi_n^{(n)} \circ \theta_n).$$

Hence $$|M_k\phi_n - \tfrac{1}{2}\phi_n + \phi_i| \le \frac{U_{n+1}^{(n)} d^{n+1} \sqrt{(k^2 d^2 + 1)}}{2(n+1)!} \qquad (4.2.12)$$

where we have used (4.1.32). From (4.2.11) it is now clear that
$$\lim_{n\to\infty} (M_k\phi_n - \tfrac{1}{2}\phi_n) = -\phi_i . \qquad (4.2.13)$$

If we assume that there exists a sequence of functions ϕ_n on ∂D satisfying (4.2.9) and that there is a function ψ_0 on ∂D such that
$$\lim_{n\to\infty} \phi_n = \psi_0 \qquad (4.2.14)$$

uniformly on ∂D, then (4.2.13) implies that
$$M_k\psi_0 - \tfrac{1}{2}\psi_0 = -\phi_i . \qquad (4.2.15)$$

These considerations show that any convergent sequence $\{\phi_n\}$ of solutions of (4.2.9) converge to the unique solution ψ_0 of (4.2.3) and (4.2.4) provided that $k^2 \notin K_D$,and we have reason to consider the ϕ_n's as approximations to the surface field. According to (4.2.12) the "best" approximation of order n would be that function ϕ_n for which $U_{n+1}^{(n)}$ is a minimum with respect to all other solutions of order n. No attempt has been made to carry out this line of reasoning, for according to the theory of Chapter 3 additional conditions on the functions ϕ_n are clearly required when $k^2 \in K_D$.

We again recall that the system of equations (4.2.3) and (4.2.4) always have a unique solution for all wave numbers k. We also recall that the nth order differential equations (4.2.7) and (4.2.8) each have non-unique solutions, of which none may be common to both equations. If the view is taken that the nth order differential equations (4.2.7) and (4.2.8) can serve as an approximate representation of the system (4.2.3) and (4.2.4), we remain with the problem of determining a unique "solution" of this system which can then be considered as an nth order approximation to the exact solution of the system (4.2.3) and (4.2.4). In this connection it is appropriate to mention the work of Burton and Miller (10). They observe that as these equations are always consistent, a possible numerical method of solution would be to approximate each by a system of linear algebraic equations and solve the combined system by a least-squares procedure. An analogous procedure can be formulated for the two differential equations (4.2.7) and (4.2.8). One may for example derive the general solution of each of these two equations and then determine the arbitrary constants or functions in these general solutions in such a way that the norm of their difference is a minimum. Alternatively we may use a linear combination of these equations as was done in section (3.9). There it was shown that the boundary integral equation

$$\alpha(M_k - \tfrac{1}{2} I)\phi + \beta N_k\phi = -\alpha \phi_i - \beta \frac{\partial\phi_i}{\partial n} \qquad (4.2.16)$$

has a unique solution provided $\mathrm{Im}(\alpha\bar{\beta}) \neq 0$. If the procedure of section (4.1) is applied to this equation, we obtain a sequence of partial differential equations

$$\alpha(M_k^{(n)} - \tfrac{1}{2} I)\phi_n + \beta N_k^{(n)}\phi_n = -\alpha \phi_i - \beta \frac{\partial\phi_i}{\partial n} \qquad (4.2.17)$$

for $n = 0,1,2,\ldots$. In (4.2.17) ϕ_n represents the general solution of the nth order equation, and we assume that α and β satisfy $\mathrm{Im}(\alpha\bar{\beta}) \neq 0$ in order to retain connection with equation (4.2.16). We point out that the use of two parameters is unnecessary, as equations (4.2.16) and (4.2.17) depend only on the ratio β/α. However, the use of two parameters has certain advantages of symmetry and also simplifies the design of computer programs. The cases $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$ correspond respectively to equations (4.2.8) and (4.2.7). On the other hand equation (4.2.17) has the serious disadvantage that the solutions of this equation are dependent on the ratio β/α, whereas the solution of (4.2.16) is unique and therefore independent of the ratio β/α. Nevertheless, the following result is easily proved: for a given α and β such that $\mathrm{Im}(\alpha\bar{\beta}) \neq 0$ assume that $\{\phi_n\}$ is a sequence of solutions of the sequence of equations (4.2.17) such that ϕ_n converges uniformly to a function ψ_0. Then ψ_0 is the unique solution of equation (4.2.16). It is not known if such a sequence of functions exists, and here we will only speculate on how to construct such a sequence of solutions.

With reference to (4.1.24) and (4.1.49) we see that when $n = 0$ equation (4.2.17) reduces to an ordinary equation for the function ϕ_0 :

$$\alpha(B - \tfrac{1}{2}) \phi_0 + \beta D \phi_0 = -\alpha \phi_i - \beta \frac{\partial\phi_i}{\partial n}$$

or

$$\phi_0 = - \frac{\alpha\phi_i + \beta \dfrac{\partial\phi_i}{\partial n}}{\alpha(B - \frac{1}{2}) + \beta D} , \qquad (4.2.18)$$

and ϕ_0 is referred to as a zero order global approximation.

As outlined in the introduction our main purpose is to obtain low order approximations for the surface field on the illuminated side of the surface when the incident field has a high frequency. We consider one specific instance, namely when a monochromatic plane wave is incident at the point $\mathbf{r}_0$ in the boundary of a convex body D. We can now formulate a sequence of initial value problems for the differential equations (4.2.17) by using the values of ϕ_i and its derivatives at the specular point r_0 as initial conditions. Thus for example we consider the first order initial value problem

$$\left.\begin{aligned} \alpha(M_k^{(1)} - \tfrac{1}{2} I)\phi_1 + \beta N_k^{(1)}\phi_1 &= -\alpha\phi_i - \beta\frac{\partial\phi_i}{\partial n} \\ \phi_1(\mathbf{r}_0) &= \phi_i(\mathbf{r}_0) \end{aligned}\right\} \qquad (4.2.19)$$

and the second order initial value problem

$$\left.\begin{aligned} \alpha(M_k^{(2)} - \tfrac{1}{2} I)\phi_2 + \beta N_k^{(2)}\phi_2 &= -\alpha\phi_i - \beta\frac{\partial\phi_i}{\partial n} \\ \phi_2(\mathbf{r}_0) &= \phi_i(\mathbf{r}_0) \\ \partial_{ij}\phi_2(\mathbf{r}_0) &= \partial_{ij}\phi_i(\mathbf{r}_0) \end{aligned}\right\} \qquad (4.2.20)$$

and so on. This procedure is applied in the following sections to the sphere.

The theoretical considerations of this section does not apply to local integral approximations as defined in section (1.1) for a surface element S contained in the boundary of a convex region D. For instance, the results of Chapter III rely heavily on the use of the Helmholtz integral formulae for bounded regions and their closed boundaries. Nevertheless we can define local approximations, in the sense of section (1.1), simply by replacing the global differential operators $M_k^{(n)}$ and $N_k^{(n)}$ respectively, by the corresponding local differential operator $M_{k,S}^{(n)}$ and $N_{k,S}^{(n)}$ defined in section (4.1).

4.3 The differential operators $L_{k,s}^{(n)}$, $M_{k,s}^{(n)}$, $M_{k,s}^{T(n)}$ and $N_{k,s}^{(n)}$ for a sphere.

The centre O of a sphere D of radius a is taken as the origin of a Cartesian reference frame $Ox_1x_2x_3$, and P ($\mathbf{r} = x_i\mathbf{e}_i$) is an arbitrary point on the sphere. Here $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a right-handed orthonormal triad localized at O. At P we choose a tangent-normal reference frame $P\xi_1\xi_2\xi_3$ so that the ξ_1- axis is tangential to the meridian through P and positive in the direction of increasing colatitude θ, and the ξ_2- axis is tangential to the parallel of latitude and in the direction of increasing azimuth φ. The ξ_3- axis is in the direction of the outward normal to the sphere at P. As in chapter II we denote the orthonormal triad formed by these axes by $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$, and the azimuth-altitude coordinates of a point P′ with respect to these axes is denoted by φ', χ'. Let S be an element of surface containing P and contained in the boundary ∂D of the sphere D. We assume that the boundary curve C of S is a circle with centre on the normal at P. Then this circular boundary of S has a constant altitude with respect to the tangent- normal axes at $\mathbf{r}$. Denoting this altitude by ε, the equation of C is

$$\chi' = \varepsilon \ , \ 0 \le \varphi' \le 2\pi,$$

where $0 \le \varepsilon \le \frac{\pi}{2}$. We note that if $\varepsilon = \frac{\pi}{2}$, then $S = \partial D$.

Now let P′ ($\mathbf{r}' = x_i'\mathbf{e}_i$) be an arbitrary point of S with coordinates (ξ_1', ξ_2', ξ_3') with respect to the $P\xi_1\xi_2\xi_3$-axes. Then

$$x_i' = x_i + a_{ij}\xi_j' \qquad (4.3.1)$$

where a_{ij} is the direction cosine of the x_i-axis with respect to the ξ_j- axis:

$$[\, a_{ij} \,] = \begin{bmatrix} \cos\varphi\,\cos\theta & -\sin\varphi & \cos\varphi\,\sin\theta \\ \sin\varphi\,\cos\theta & \cos\varphi & \sin\varphi\,\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad (4.3.2)$$

If R = PP′ then

$$\xi_i' = R\,\eta_i' \qquad (4.3.3)$$

where

$$\left.\begin{aligned} \eta_1' &= \cos\varphi'\,\cos\chi' \\ \eta_2' &= \sin\varphi'\,\cos\chi' \\ \eta_3' &= -\sin\chi' \end{aligned}\right\} . \qquad (4.3.4)$$

If γ is the angle subtended at the centre of the sphere by the chord PP′, then from the isoceles triangle OPP′, $\gamma = 2\chi'$, and

$$R = PP' = 2a\sin(\gamma/2) = 2a\sin\chi' \qquad (4.3.5)$$

where $0 \le \chi' \le \varepsilon$. Equation (4.3.5) is the equation of the spherical surface element S in terms of azimuth-altitude coordinates. It now follows that

$$\left.\begin{aligned} \xi_1' &= R\eta_1' = 2a\cos\varphi'\cos\chi'\sin\chi' \\ \xi_2' &= R\eta_2' = 2a\sin\varphi'\cos\chi'\sin\chi' \\ \xi_3' &= R\eta_3' = -2a\sin^2\chi' \end{aligned}\right\}. \qquad (4.3.6)$$

The Cartesian equation of the sphere relative to the $P\xi_1\xi_2\xi_3$-frame is

$$\xi_1'^2 + \xi_2'^2 + (\xi_3' + a)^2 = a^2,$$

and so

$$\frac{\partial\xi_3'}{\partial\xi_1'} = -\frac{\xi_1'}{\xi_3' + a} = -\cos\varphi'\tan2\chi', \qquad (4.3.7)$$

$$\frac{\partial\xi_3'}{\partial\xi_2'} = -\frac{\xi_2'}{\xi_3' + a} = -\sin\varphi'\tan2\chi'. \qquad (4.3.8)$$

If equations (4.3.7) and (4.3.8) are applied to equations (2.3.12) and (2.3.13), then

$$\omega_1 = \frac{\partial\xi_3'}{\partial\xi_1'}\cos\varphi' + \frac{\partial\xi_3'}{\partial\xi_2'}\sin\varphi' = -\tan2\chi',$$

$$\omega_2 = \frac{\partial\xi_3'}{\partial\xi_1'}\sin\varphi' - \frac{\partial\xi_3'}{\partial\xi_2'}\cos\varphi' = 0.$$

Hence we find that

$$\omega_1\cos\chi' + \sin\chi' = -\sin\chi'\sec2\chi', \qquad (4.3.9)$$

$$\omega = \surd(1 + \omega_1^2 + \omega_2^2) = \surd(1 + \tan^2 2\chi') = |\sec2\chi'|, \qquad (4.3.10)$$

$$\frac{\omega\sin\chi'\cos\chi'}{|\omega_1\cos\chi' + \sin\chi'|} = \cos\chi',$$

and

$$\frac{\omega R\cos\chi'}{|\omega_1\cos\chi' + \sin\chi'|} = 2a\cos\chi',$$

where $0 \le \chi' \le \varepsilon$.

We can now evaluate the coefficients $A_{S,j_1\ldots j_n}(\mathbf{r})$ etc. as given by equations (4.1.64)-(4.1.67). However, in the sequel we will not explicitly indicate the dependence of these coefficients on the surface element S, and in general the value of ε will be clear from the context. This convention will also apply to the differential operators $L_{k,s}^{(n)}$ etc. Thus:

$$A_{j_1\ldots j_n}(r) = \frac{a}{2\pi}\int_0^{2\pi}\int_0^{\varepsilon} R^n e^{ikR}\, \eta'_{j_1}\ldots\eta'_{j_n}\, \cos\chi'\, d\chi' d\varphi' \,, \qquad (4.3.11)$$

$$B_{j_1\ldots j_n}(r) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\varepsilon} R^n (ikR - 1)\, e^{ikR}\, \eta'_{j_1}\ldots\eta'_{j_n}\, \cos\chi'\, d\chi' d\varphi' , \qquad (4.3.12)$$

$$C_{j_1\ldots j_n}(r) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\varepsilon} R^n (ikR - 1)\, e^{ikR}\, \eta'_{j_1}\ldots\eta'_{j_n}\, \cos\chi'\, d\chi' d\varphi' , \qquad (4.3.13)$$

and, for $n \geq 2$,

$$D_{j_1\ldots j_n}(r) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\varepsilon} W_k^{(n)} e^{ikR}\, \eta'_{j_1}\ldots\eta'_{j_n} \cos\chi' d\chi' d\varphi' \qquad (4.3.14)$$

where

$$W_k^{(n)} = R^{n-2}\left[ikR^2\sin\chi' + (ikR - 1)\left(ikR^2\sin\chi' - 3\, R\sin\chi' + \frac{R}{\omega_1\cos\chi' + \sin\chi'}\right)\right].$$

Since $R = 2a\sin\chi'$ is independent of r, so are the coefficients $A_{j_1\ldots j_n}$, $B_{j_1\ldots j_n}$, $C_{j_1\ldots j_n}$ and $D_{j_1\ldots j_n}$. Thus in the case of the sphere the analytical functions (4.3.11)-(4.3.14) are constants depending only on the radius a and the altitude ε of the boundary C of S. As mentioned previously (section (4.1)) we notice that if the surface element S is small, i.e. if ε is small, then the integrals (4.3.11)- (4.3.14) can be evaluated by expanding their respective integrands to a sufficient degree of accuracy followed by integration. However, in the case of the sphere, the integrals are easily evaluated, thus we first evaluate these integrals and then expand in powers of ε. We also note that the values of the above integrals is not affected by permutation of the suffixes $j_1\ldots j_n$, and that

$$C_{j_1\ldots j_n} = -\, B_{j_1\ldots j_n} \,. \qquad (4.3.15)$$

In Appendix A formulae are derived for computing the coefficients $A_{j_1\ldots j_n}$ and $B_{j_1\ldots j_n}$ for all $n \geq 0$. These formulae are also applicable to the evaluation of $D_{j_1\ldots j_n}$ when $n \geq 2$; the two remaining terms D and D_j are dealt with in Appendix B. The method by means of which these coefficients are found in Appendix A is based on the observation that the integrals of (4.3.11), (4.3.12) and (4.3.13) are linear combinations of integrals of the form

$$\int_0^{2\pi}\int_0^{\varepsilon} R^n e^{ikR} \eta_{j_1}\cdots\eta_{j_n}\cos\chi' d\chi' d\varphi' \tag{4.3.16}$$

which can be obtained by repeated partial differentiation with respect to k of the integral

$$\int_0^{2\pi}\int_0^{\varepsilon} e^{ikR} \eta_{j_1}\cdots\eta_{j_n}\cos\chi' d\chi' d\varphi' \tag{4.3.17}$$

for all $n \geq 1$. The same procedure applies to (4.3.14) when $n \geq 2$, since, according to (4.3.5) and (4.3.9), $W_k^{(n)}$ can be written in the form

$$W_k^{(n)} = R^{n-2}\left(-\frac{k^2R^4}{2a} - \frac{5ikR^3}{2a} + \frac{5R^2}{2a} + 2ikaR - 2a\right). \tag{4.3.18}$$

Now consider a time harmonic plane wave

$$u(r,t) = \phi_i(\mathbf{r})e^{-i\omega t} \tag{4.3.19}$$

incident on a sphere of radius a. The spatial dependence of the wave has the form

$$\phi_i(\mathbf{r}) = e^{ik\mathbf{r}.\mathbf{e}} \tag{4.3.20}$$

where $\mathbf{e}$ is a unit vector in the direction of propagation of the wave. If the medium surrounding the sphere is non-dissipative then the wave number k is real. Choose the Cartesian reference frame $Ox_1x_2x_3$ with origin O at the centre of the sphere and oriented so that the negative x_3-axis is in the direction of propagation; then

$$\phi_i(\mathbf{r}) = e^{-ikx_3} . \tag{4.3.21}$$

Due to symmetry the time-independent part of the surface field ϕ generated on the sphere is independent of x_1 and x_2. Replacing x_3 by z, this means that

$$\phi(\mathbf{r}) = \phi(z).$$

It follows that all partial derivative of the form

$$\frac{\partial^n\phi(\mathbf{r})}{\partial x_1^p \partial x_2^q \partial x_3^r} ,$$

where $p+q+r = n$, is zero if p or q is not zero. Equations (4.1.22), (4.1.23), (4.1.24) and (4.1.49) now respectively assume the forms

$$(L_k^{(n)}\phi)(z) = \sum_{p=0}^{n} \frac{1}{p!} a_{3j_1}\cdots a_{3j_p} A_{j_1\cdots j_p} \frac{d^p\phi(z)}{dz^p} , \tag{4.3.22}$$

$$(M_k^{(n)}\phi)(z) = \sum_{p=0}^{n} \frac{1}{p!} a_{3j_1}\cdots a_{3j_p} B_{j_1\cdots j_p} \frac{d^p\phi(z)}{dz^p} , \tag{4.3.23}$$

$$(M_k^{T(n)}\phi)(z) = \sum_{p=0}^{n} \frac{1}{p!} \; a_{3j_1}\cdots a_{3j_p} \; C_{j_1\cdots j_p} \; \frac{d^p\phi(z)}{dz^p} \,, \qquad (4.3.24)$$

$$(N_k^{(n)}\phi)(z) = \sum_{p=0}^{n} \frac{1}{p!} \; a_{3j_1}\cdots a_{3j_p} \; D_{j_1\cdots j_p} \; \frac{d^p\phi(z)}{dz^p} \,. \qquad (4.3.25)$$

From (4.3.2):

$$\left.\begin{aligned} a_{31} &= -\sin\theta = -\left[1-\left(\frac{z}{a}\right)^2\right]^{1/2} \\ a_{32} &= 0 \\ a_{33} &= \cos\theta = \frac{z}{a} \end{aligned}\right\} \qquad (4.3.26)$$

With reference to Appendix A the successive terms of the above differential operators can now be determined. Consider for instance the mth term of $L_k^{(n)}\phi$ as defined by (4.3.22). As $a_{32} = 0$ and as permutation of the suffixes $j_1\ldots j_m$ does not affect the value of $A_{j_1\ldots j_m}$, it follows that

$$a_{3j_1}\cdots a_{3j_m} \, A_{j_1\cdots j_m} = \sum_{p+q=m} \frac{m!}{p!q!} (a_{31})^p (a_{33})^q \, A_{1_p 3_q} \qquad (4.3.27)$$

where $\qquad A_{1_p3_q} = A_{1\ldots13\ldots3}$, $\quad (p \geq 0, \; q \geq 0)$,
the notation meaning that the suffix 1 is repeated p times and the suffix 3 is repeated q times, where $p+q = m$. According to (4.3.11) and (4.3.4) we therefore have

$$A_{1_p3_q}(r) = \frac{a}{2\pi}\int_0^{2\pi}\int_0^{\varepsilon} R^m \, e^{ikR} \, (\eta_1')^p (\eta_3')^q \, \cos\chi' \, d\chi' d\varphi'$$

$$= \frac{(-1)^q a}{2\pi}\int_0^{2\pi}\int_0^{\varepsilon} R^m \, e^{ikR} \cos^p\varphi' \, \cos^{p+1}\chi' \, \sin^q\chi' \, d\chi' d\varphi'$$

$$= \frac{(-1)^q a}{2\pi}\int_0^{2\pi}\cos^p\varphi' \, d\varphi' \int_0^{\varepsilon} R^m \, e^{ikR} \cos^{p+1}\chi' \, \sin^q\chi' \, d\chi' ,$$

since $R = 2a\sin\chi'$ is independent of φ'. Using the formula

$$\int_0^{2\pi} \cos^p\varphi' \, d\varphi' = \begin{cases} 0 & \text{if p is odd} \\ \dfrac{(p-1)(p-3)\ldots5.3.1}{p(p-2)\ldots6.4.2}\, 2\pi & \text{if p is even} \end{cases}$$

and the relation

$$\int_0^{\varepsilon} R^m \, e^{ikR} \cos^{p+1}\chi' \, \sin^q\chi' \, d\chi' = (2a)^m \, J(m,p,q)$$

according to the notation of Appendix A, equation (A.5), we find that

$$A_{1_p 3_q} = \begin{cases} 0 & \text{if } p \text{ is odd} \\ (-1)^q \dfrac{(2a)^{m+1}}{2} \dfrac{(p-1)(p-3)\ldots 3.1}{p(p-2)\ldots 4.2} J(m,p,q) & \text{if } p \text{ is even} \end{cases} \quad (4.3.28)$$

where $p+q = m$.

Thus the coefficient of $\frac{d^m\phi(z)}{dz^m}$ in $(L_k^{(n)}\phi)(z)$ is obtained by using (4.3.28) in (4.3.27). Note that terms containing odd powers of a_{31} in (4.3.26) are always zero.

The mth term of the operator $M_k^{(n)}$ defined by (4.3.23) is given by

$$a_{3j_1}\ldots a_{3j_m} B_{j_1\ldots j_m} = \sum_{p+q=m} \frac{m!}{p!q!} (a_{31})^p (a_{33})^q B_{1_p 3_q} \quad (4.3.29)$$

From (4.3.12) it now follows that

$$B_{1_p 3_q}(\mathbf{r}) = -\frac{1}{4\pi} \int_0^{2\pi}\int_0^{\varepsilon} R^m(ikR - 1)\, e^{ikR} (\eta_1')^p (\eta_3')^q \cos\chi'\, d\chi' d\varphi'$$

$$= \frac{(-1)^{q+1}}{4\pi} \int_0^{2\pi}\int_0^{\varepsilon} R^m(ikR - 1)\, e^{ikR} \cos^p\varphi' \cos^{p+1}\chi' \sin^q\chi'\, d\chi' d\varphi'$$

$$= \frac{(-1)^{q+1}}{4\pi} \int_0^{2\pi} \cos^p\varphi'\, d\varphi' \int_0^{\varepsilon} R^m(ikR - 1)\, e^{ikR} \cos^{p+1}\chi' \sin^q\chi'\, d\chi'$$

and

$$\int_0^{\varepsilon} R^m(ikR - 1)\, e^{ikR} \cos^{p+1}\chi' \sin^q\chi'\, d\chi' = (2a)^m K(m,p,q)$$

according to Appendix A (eq. (A.7)). Hence

$$B_{1_p 3_q} = \begin{cases} 0 & \text{if } p \text{ is odd} \\ (-1)^{q+1} \dfrac{(2a)^m}{2} \dfrac{(p-1)(p-3)\ldots 3.1}{p(p-2)\ldots 4.2} K(m,p,q) & \text{if } p \text{ is even} \end{cases} \quad (4.3.30)$$

where $p+q = m$.

Finally we consider the mth term of the operator $N_k^{(n)}$ as defined by (4.3.25). As in the previous two cases it follows that

$$a_{3j_1}\ldots a_{3j_m} D_{j_1\ldots j_m} = \sum_{p+q=m} \frac{m!}{p!q!} (a_{31})^p (a_{33})^q D_{1_p 3_q} \quad (4.3.31)$$

if $m \geq 2$.

From (4.3.14) and (4.3.18) with p even we obtain the relation

$$D_{1_p 3_q} = \frac{(-1)^{q+1}(p-1)(p-3)\ldots 3.1}{2p(p-2)\ldots 4.2}\Big[-k^2(2a)^{m+1}J(m+2,p,q) - 5ik(2a)^m J(m+1,p,q)$$

$$+ 5(2a)^{m-1}J(m,p,q) + ik(2a)^m J(m-1,p,q) - (2a)^{m-1}J(m-2,p,q)\Big].$$

If, as in Appendix A, we put

$$2ika = h\ ,$$

then the $D_{1_p 3_q}$ can be written in the form

$$D_{1_p 3_q} = (-1)^{q+1}\frac{(2a)^{m-1}}{2}\frac{(p-1)(p-3)\ldots 3.1}{p(p-2)\ldots 4.2}\Big\{h^2 J(m+2,p,q) - 5hJ(m+1,p,q) +$$

$$5J(m,p,q) + hJ(m-1,p,q) - J(m-2,p,q)\Big].$$

If $H(m,p,q)$ is defined by

$$H(m,p,q) = h^2 J(m+2,p,q) - 5hJ(m+1,p,q) + 5J(m,p,q) + hJ(m-1,p,q) - J(m-2,p,q)$$

then

$$D_{1_p 3_q} = \left\{\begin{array}{ll} 0 & \text{if } p \text{ is odd} \\ (-1)^{q+1}\dfrac{(2a)^{m-1}}{2}\dfrac{(p-1)(p-3)\ldots 3.1}{p(p-2)\ldots 4.2}H(m,p,q) & \text{if } p \text{ is even} \end{array}\right\} \quad (4.3.32)$$

Finally the dependence of the coefficients $A_{1_p 3_q}$, $B_{1_p 3_q}$ and $D_{1_p 3_q}$ on the radius a of the sphere is made explicit by writing, for all $m \geq 0$,

$$A_{1_p 3_q} = a^{m+1}\mathfrak{A}_{1_p 3_q}\ , \quad (4.3.33)$$

$$B_{1_p 3_q} = a^m \mathfrak{B}_{1_p 3_q}\ , \quad (4.3.34)$$

and

$$D_{1_p 3_q} = a^{m-1}\mathfrak{D}_{1_p 3_q}\ . \quad (4.3.35)$$

From these equations and respectively (4.3.28), (4.3.30) and (4.3.32) it follows that

$$\mathfrak{A}_{1_p 3_q} = \left\{\begin{array}{ll} 0 & \text{if } p \text{ is odd} \\ 2^m(-1)^q\dfrac{(p-1)(p-3)\ldots 3.1}{p(p-2)\ldots 4.2}J(m,p,q) & \text{if } p \text{ is even}\end{array}\right\}, \quad (4.3.36)$$

$$\mathfrak{B}_{1_p 3_q} = \begin{cases} 0 & \text{if p is odd} \\ 2^{m-1}(-1)^{q+1} \dfrac{(p-1)(p-3)\ldots 3.1}{p(p-2)\ldots 4.2} K(m,p,q) & \text{if p is even} \end{cases} \qquad (4.3.37)$$

and if $m \geq 2$ then

$$\mathfrak{D}_{1_p 3_q} = \begin{cases} 0 & \text{if p is odd} \\ (-1)^{q+1} 2^{m-2} \dfrac{(p-1)(p-3)\ldots 3.1}{p(p-2)\ldots 4.2} H(m,p,q) & \text{if p is even} \end{cases}. \qquad (4.3.38)$$

In the equations above $p + q = m$.
If $m = 0$ or $m = 1$, then $\mathfrak{D}$ and $\mathfrak{D}_j$ are also defined by (4.3.35), but the values of D and D_j are those given in Appendix B.

4.4 Application to the Hard Sphere.

The results of the preceding sections are now applied to the case of the scattering of scalar waves from an acoustically hard sphere, i.e. the normal derivative of the total field is zero on the surface of the sphere. We therefore have a Neumann problem and equations (4.2.7) and (4.2.8) apply. In particular we derive formulae for the coefficients of linear combinations of the equations

$$(M_k^{(n)} - \tfrac{1}{2} I)\phi = - \phi_i$$

and

$$N_k^{(n)}\phi = - \frac{\partial \phi_i}{\partial n}$$

for the cases $n = 0,1,2,3,4$, and solutions are derived for $n = 0,1$ and 2.

When $n = 4$ it follows from equations (4.2.23) and (4.3.34) that

$$(M_k^{(4)} - \tfrac{1}{2}I)\phi = (\mathfrak{B} - \tfrac{1}{2})\phi + a\, a_{3p}\mathfrak{B}_p \frac{d\phi}{dz} + \frac{a^2}{2!} a_{3p}a_{3q}\mathfrak{B}_{pq} \frac{d^2\phi}{dz^2} +$$
$$\frac{a^3}{3!} a_{3p}a_{3q}a_{3r}\mathfrak{B}_{pqr} \frac{d^3\phi}{dz^3} + \frac{a^4}{4!} a_{3p}a_{3q}a_{3r}a_{3s}\mathfrak{B}_{pqrs} \frac{d^4\phi}{dz^4} \qquad (4.4.1)$$

and equations (4.3.25) and (4.3.38) give

$$N_k^{(4)}\phi = a^{-1}\mathfrak{D}\phi + a_{3p}\mathfrak{D}_p \frac{d\phi}{dz} + \frac{a}{2!} a_{3p}a_{3q}\mathfrak{D}_{pq} \frac{d^2\phi}{dz^2} + \frac{a^2}{3!} a_{3p}a_{3q}a_{3r}\mathfrak{D}_{pqr} \frac{d^3\phi}{dz^3} +$$
$$\frac{a^3}{4!} a_{3p}a_{3q}a_{3r}a_{3s}\mathfrak{D}_{pqrs} \frac{d^4\phi}{dz^4}. \qquad (4.4.2)$$

Using equations (4.3.26) and (4.3.37) in equation (4.4.1) gives

$$a_{3p}\mathcal{B}_p = a_{31}\mathcal{B}_1 + a_{33}\mathcal{B}_3 = a_{33}\mathcal{B}_3 = \mathcal{B}_3\left(\frac{z}{a}\right)$$

and

$$\frac{1}{2!}\, a_{3p}a_{3q}\mathcal{B}_{pq} = \frac{1}{2}\left(a_{31}^2\mathcal{B}_{11} + a_{31}a_{33}(\mathcal{B}_{13} + \mathcal{B}_{31}) + a_{33}^2\mathcal{B}_{33}\right)$$

$$= \frac{1}{2}\left(a_{31}^2\mathcal{B}_{11} + a_{33}^2\mathcal{B}_{33}\right)$$

$$= \frac{1}{2}\left[\left(1 - \left(\frac{z}{a}\right)^2\right)\mathcal{B}_{11} + \left(\frac{z}{a}\right)^2\mathcal{B}_{33}\right]$$

$$= \frac{1}{2}\,\mathcal{B}_{11} - \frac{1}{2}\left(\mathcal{B}_{11} - \mathcal{B}_{33}\right)\left(\frac{z}{a}\right)^2 .$$

For the sum in the coefficient of $\frac{d^3\phi}{dz^3}$ in (4.4.1) we get

$$\frac{1}{3!}\, a_{3p}a_{3q}a_{3r}\mathcal{B}_{pqr} = \frac{1}{6}\left[3a_{31}^2a_{33}\,\mathcal{B}_{113} + a_{33}^3\,\mathcal{B}_{333}\right]$$

$$= \frac{1}{2}\,\mathcal{B}_{113}\left(\frac{z}{a}\right) - \frac{1}{6}\left(3\mathcal{B}_{113} - \mathcal{B}_{333}\right)\left(\frac{z}{a}\right)^3 .$$

Similarly

$$\frac{1}{4!}\, a_{3p}a_{3q}a_{3r}a_{3s}\mathcal{B}_{pqrs} = \frac{1}{24}\left(a_{31}^4\mathcal{B}_{1111} + 6a_{31}^2a_{33}^2\mathcal{B}_{1133} + a_{33}^4\mathcal{B}_{3333}\right)$$

$$= \frac{1}{24}\,\mathcal{B}_{1111} - \frac{1}{12}\left(\mathcal{B}_{1111} - 3\mathcal{B}_{1133}\right)\left(\frac{z}{a}\right)^2 +$$

$$\frac{1}{24}\left(\mathcal{B}_{1111} - 6\mathcal{B}_{1133} + \mathcal{B}_{3333}\right)\left(\frac{z}{a}\right)^4 .$$

It is convenient to introduce the following notation:

$$\begin{aligned}
\mathcal{P}_1 &= \mathcal{B} - \frac{1}{2} \\
\mathcal{P}_2 &= \mathcal{B}_3 \\
\mathcal{P}_3 &= \frac{1}{2}\,\mathcal{B}_{11} \\
\mathcal{P}_4 &= \frac{1}{2}\left(\mathcal{B}_{11} - \mathcal{B}_{33}\right) \\
\mathcal{P}_5 &= \frac{1}{2}\,\mathcal{B}_{113} \\
\mathcal{P}_6 &= \frac{1}{6}\left(3\mathcal{B}_{113} - \mathcal{B}_{333}\right) \\
\mathcal{P}_7 &= \frac{1}{24}\,\mathcal{B}_{1111} \\
\mathcal{P}_8 &= \frac{1}{12}\left(\mathcal{B}_{1111} - 3\mathcal{B}_{1133}\right) \\
\mathcal{P}_9 &= \frac{1}{24}\left(\mathcal{B}_{1111} - 6\mathcal{B}_{1133} + \mathcal{B}_{3333}\right)
\end{aligned} \qquad (4.4.3)$$

Using the results obtained above, (4.4.1) now has the form

$$\mathfrak{L}_1\phi + a\mathfrak{L}_2\left(\frac{z}{a}\right)\frac{d\phi}{dz} + a^2\left(\mathfrak{L}_3 - \mathfrak{L}_4\left(\frac{z}{a}\right)^2\right)\frac{d^2\phi}{dz^2} + a^3\left(\mathfrak{L}_5 - \mathfrak{L}_6\left(\frac{z}{a}\right)^2\right)\left(\frac{z}{a}\right)\frac{d^3\phi}{dz^3} +$$

$$a^4\left(\mathfrak{L}_7 - \mathfrak{L}_8\left(\frac{z}{a}\right)^2 + \mathfrak{L}_9\left(\frac{z}{a}\right)^4\right)\frac{d^4\phi}{dz^4} = -\phi_i,$$

from which the equations with n = 0,1,2 and 3 can be obtained. In these equations $\phi_i(z) = e^{-ikz}$, and using the transformation

$$\zeta = \frac{z}{a},$$

the equations above can be written in the forms

n = 0:
$$\mathfrak{L}_1\phi = -e^{-ika\zeta} \tag{4.4.4)(a}$$

n = 1:
$$\mathfrak{L}_1\phi + \mathfrak{L}_2\zeta\frac{d\phi}{d\zeta} = -e^{-ika\zeta} \tag{4.4.4)(b}$$

n = 2:
$$\mathfrak{L}_1\phi + \mathfrak{L}_2\zeta\frac{d\phi}{d\zeta} + \left(\mathfrak{L}_3 - \mathfrak{L}_4\zeta^2\right)\frac{d^2\phi}{d\zeta^2} = -e^{-ika\zeta} \tag{4.4.4)(c}$$

n = 3:
$$\mathfrak{L}_1\phi + \mathfrak{L}_2\zeta\frac{d\phi}{d\zeta} + \left(\mathfrak{L}_3 - \mathfrak{L}_4\zeta^2\right)\frac{d^2\phi}{d\zeta^2} + \left(\mathfrak{L}_5 - \mathfrak{L}_6\zeta^2\right)\zeta\frac{d^3\phi}{d\zeta^3} = -e^{-ika\zeta} \tag{4.4.4)(d}$$

n = 4:
$$\mathfrak{L}_1\phi + \mathfrak{L}_2\zeta\frac{d\phi}{d\zeta} + \left(\mathfrak{L}_3 - \mathfrak{L}_4\zeta^2\right)\frac{d^2\phi}{d\zeta^2} + \left(\mathfrak{L}_5 - \mathfrak{L}_6\zeta^2\right)\zeta\frac{d^3\phi}{d\zeta^3} +$$
$$\left(\mathfrak{L}_7 - \mathfrak{L}_8\zeta^2 + \mathfrak{L}_9\zeta^4\right)\frac{d^4\phi}{d\zeta^4} = -e^{-ika\zeta} \tag{4.4.4)(e}$$

The formulae derived in Appendix A show that the coefficients $\mathfrak{L}_1,\ldots\mathfrak{L}_9$ are only dependent on the product ka. As the inhomogeneity $e^{-ika\zeta}$ is also dependent on the product ka, it follows that the solutions of equations (4.4.4) are likewise dependent on ka.

Now consider the operator $N_k^{(n)}$ as defined by (4.4.2) and apply equations (4.3.26) and (4.3.38). We find that

$$a_{3p}\mathfrak{D}_p = \mathfrak{D}_3\left(\frac{z}{a}\right),$$

$$\frac{1}{2!}a_{3p}a_{3q}\mathfrak{D}_{pq} = \frac{1}{2}\mathfrak{D}_{11} - \frac{1}{2}\left(\mathfrak{D}_{11} - \mathfrak{D}_{33}\right)\left(\frac{z}{a}\right)^2,$$

$$\frac{1}{3!}a_{3p}a_{3q}a_{3r}\mathfrak{D}_{pqr} = \frac{1}{6}\mathfrak{D}_{113}\left(\frac{z}{a}\right) - \frac{1}{6}\left(3\mathfrak{D}_{113} - \mathfrak{D}_{333}\right)\left(\frac{z}{a}\right)^3,$$

and
$$\frac{1}{4!}a_{3p}a_{3q}a_{3r}a_{3s}\mathfrak{D}_{pqrs} = \frac{1}{24}\mathfrak{D}_{1111} - \frac{1}{12}\left(\mathfrak{D}_{1111} - 3\mathfrak{D}_{1133}\right)\left(\frac{z}{a}\right)^2 +$$
$$\frac{1}{24}\left(\mathfrak{D}_{1111} - 6\mathfrak{D}_{1133} + \mathfrak{D}_{3333}\right)\left(\frac{z}{a}\right)^4.$$

The coefficients $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_9$ are defined by

$$
\begin{aligned}
\mathfrak{Q}_1 &= \mathfrak{D} \\
\mathfrak{Q}_2 &= \mathfrak{D}_3 \\
\mathfrak{Q}_3 &= \tfrac{1}{2}\,\mathfrak{D}_{11} \\
\mathfrak{Q}_4 &= \tfrac{1}{2}\left(\mathfrak{D}_{11} - \mathfrak{D}_{33}\right) \\
\mathfrak{Q}_5 &= \tfrac{1}{2}\,\mathfrak{D}_{113} \\
\mathfrak{Q}_6 &= \tfrac{1}{6}\left(3\mathfrak{D}_{113} - \mathfrak{D}_{333}\right) \\
\mathfrak{Q}_7 &= \tfrac{1}{24}\,\mathfrak{D}_{1111} \\
\mathfrak{Q}_8 &= \tfrac{1}{12}\left(\mathfrak{D}_{1111} - 3\mathfrak{D}_{1133}\right) \\
\mathfrak{Q}_9 &= \tfrac{1}{24}\left(\mathfrak{D}_{1111} - 6\mathfrak{D}_{1133} + \mathfrak{D}_{3333}\right).
\end{aligned}
\qquad (4.4.5)
$$

Thus for the equation

$$N_k^{(4)}\phi = -\frac{\partial \phi_i}{\partial n}$$

we obtain

$$a^{-1}\mathfrak{Q}_1\phi + \mathfrak{Q}_2\left(\frac{z}{a}\right)\frac{d\phi}{dz} + a\left[\mathfrak{Q}_3 - \mathfrak{Q}_4\left(\frac{z}{a}\right)^2\right]\frac{d^2\phi}{dz^2} + a^2\left[\mathfrak{Q}_5 - \mathfrak{Q}_6\left(\frac{z}{a}\right)^2\right]\left(\frac{z}{a}\right)\frac{d^3\phi}{dz^3} +$$

$$a^3\left[\mathfrak{Q}_7 - \mathfrak{Q}_8\left(\frac{z}{a}\right)^2 + \mathfrak{Q}_9\left(\frac{z}{a}\right)^4\right]\frac{d^4\phi}{dz^4} = -\frac{\partial \phi_i}{\partial n},$$

from which the equations for $n = 0,1,2$ and 3 can be found. In these equations

$$\frac{\partial \phi_i(z)}{\partial n} = \frac{d\phi_i(z)}{dz}\,n_3 = -\frac{ikz}{a}\,e^{-ikz}.$$

Using the transformation $\zeta = z/a$ we obtain, after multiplication of each equation by the radius a, the set of equations

$n=0$: $$\mathfrak{Q}_1\phi = ika\zeta\, e^{-ika\zeta} \qquad (4.4.6)(a)$$

$n=1$: $$\mathfrak{Q}_1\phi + \mathfrak{Q}_2\zeta\frac{d\phi}{d\zeta} = ika\zeta\, e^{-ika\zeta} \qquad (4.4.6)(b)$$

$n=2$: $$\mathfrak{Q}_1\phi + \mathfrak{Q}_2\zeta\frac{d\phi}{d\zeta} + \left[\mathfrak{Q}_3 - \mathfrak{Q}_4\zeta^2\right]\frac{d^2\phi}{d\zeta^2} = ika\zeta\, e^{-ika\zeta} \qquad (4.4.6)(c)$$

$n=3$: $$\mathfrak{Q}_1\phi + \mathfrak{Q}_2\zeta\frac{d\phi}{d\zeta} + \left[\mathfrak{Q}_3 - \mathfrak{Q}_4\zeta^2\right]\frac{d^2\phi}{d\zeta^2} + \left[\mathfrak{Q}_5 - \mathfrak{Q}_6\zeta^2\right]\zeta\frac{d^3\phi}{d\zeta^3} = ika\zeta\, e^{-ika\zeta} \qquad (4.4.6)(d)$$

$n=4$: $$\mathfrak{Q}_1\phi + \mathfrak{Q}_2\zeta\frac{d\phi}{d\zeta} + \left[\mathfrak{Q}_3 - \mathfrak{Q}_4\zeta^2\right]\frac{d^2\phi}{d\zeta^2} + \left[\mathfrak{Q}_5 - \mathfrak{Q}_6\zeta^2\right]\zeta\frac{d^3\phi}{d\zeta^3} +$$

$$\left[\mathfrak{Q}_7 - \mathfrak{Q}_8\zeta^2 + \mathfrak{Q}_9\zeta^4\right]\frac{d^4\phi}{d\zeta^4} = ika\zeta e^{-ika\zeta} \qquad (4.4.6)(e)$$

Finally coefficients $\mathfrak{R}_1,\ldots,\mathfrak{R}_9$ are defined by

$$\mathfrak{R}_n = \alpha\mathfrak{P}_n + \beta\mathfrak{Q}_n , \qquad n = 1,\ldots,9 \tag{4.4.7}$$

where α and β are constants. Thus forming the linear combinations

$$\alpha(M_k^{(n)} - \tfrac{1}{2} I)\phi + \beta N_k^{(n)}\phi = -\alpha\phi_i - \beta \frac{\partial\phi_i}{\partial n}$$

for n = 0,...,4 yields the equations

$$\mathfrak{R}_1\phi = -(\alpha-\beta ika\zeta)e^{-ika\zeta} \tag{4.4.8}$$

$$\mathfrak{R}_1\phi + \mathfrak{R}_2\zeta \frac{d\phi}{d\zeta} = -(\alpha-\beta ika\zeta)e^{-ika\zeta} \tag{4.4.9}$$

$$\mathfrak{R}_1\phi + \mathfrak{R}_2\zeta \frac{d\phi}{d\zeta} + \left(\mathfrak{R}_3 - \mathfrak{R}_4\zeta^2\right) \frac{d^2\phi}{d\zeta^2} = -(\alpha-\beta ika\zeta)e^{-ika\zeta} \tag{4.4.10}$$

$$\mathfrak{R}_1\phi + \mathfrak{R}_2\zeta \frac{d\phi}{d\zeta} + \left(\mathfrak{R}_3 - \mathfrak{R}_4\zeta^2\right) \frac{d^2\phi}{d\zeta^2} + \left(\mathfrak{R}_5 - \mathfrak{R}_6\zeta^2\right)\zeta \frac{d^3\phi}{d\zeta^3} = -(\alpha-\beta ika\zeta)e^{-ika\zeta} \tag{4.4.11}$$

$$\mathfrak{R}_1\phi + \mathfrak{R}_2\zeta \frac{d\phi}{d\zeta} + \left(\mathfrak{R}_3 - \mathfrak{R}_4\zeta^2\right) \frac{d^2\phi}{d\zeta^2} + \left(\mathfrak{R}_5 - \mathfrak{R}_6\zeta^2\right)\zeta \frac{d^3\phi}{d\zeta^3} + \left(\mathfrak{R}_7 - \mathfrak{R}_8\zeta^2 + \mathfrak{R}_9\zeta^4\right) \frac{d^4\phi}{d\zeta^4} = -(\alpha-\beta ika\zeta)e^{-ika\zeta} \tag{4.4.12}$$

In the following sections the behaviour of the solutions of some of these equations is discussed for a selected set of wave numbers k; in all cases the radius a = 1. As regards the coupling constants, particular attention is given to the following cases:

$$(\alpha,\beta) = (1,0),(0,1) \text{ and } (1,i),$$

corresponding respectively to the equations

$$(M_k^{(n)} - \tfrac{1}{2} I)\phi = -\phi_i ,$$

$$N_k^{(n)}\phi = -\frac{\partial\phi_i}{\partial n}$$

and

$$(M_k^{(n)} - \tfrac{1}{2} I)\phi + iN_k^{(n)}\phi = -\phi_i - i\frac{\partial\phi_i}{\partial n} .$$

for n = 0,1,2.

The eigh coefficients $\mathfrak{P}_1,\ldots,\mathfrak{P}_4,\mathfrak{Q}_1,\ldots,\mathfrak{Q}_4$ required for the solutions of these equations for the cases n = 0,1 and 2 are programed in SUBROUTINE HCOEF2, and is based on the formulae derived in Appendices A and B and equations (4.4.3) and (4.4.5).

In all the cases dealt with in the following sections the solutions of the equations (4.4.8)-(4.4.12) are compared with the exact solution of the surface field for the acoustically hard sphere. This solution is derived from the boundary integral equation

$$(M_k - \tfrac{1}{2} I)\psi = - \phi_i$$

and is given (Hönl,Maue and Westpfahl (22)) in terms of Legendre polynomials $P_n(\zeta)$,where $\zeta = \cos\theta$ and $0 \le \theta \le \pi$, by

$$\psi(\zeta) = \frac{i}{(ka)^2} \sum_{n=0}^{\infty} (-i)^n (2n+1) \left[h_n^{(1)\prime}(ka)\right]^{-1} P_n(\zeta) \;,\; |\zeta| \le 1, \qquad (4.4.13)$$

where $h_n^{(1)}$ is the spherical Bessel function of the third kind and

$$h_n^{(1)\prime}(ka) = \left[\frac{dh_n^{(1)}(t)}{dt}\right]_{t=ka} .$$

The Legendre polynomials are computed by means of the recurrence formula

$$nP_n(\zeta) - (2n-1)\zeta P_{n-1}(\zeta) + (n-1)P_{n-2}(\zeta) = 0$$

where $$P_0(\zeta) = 1$$

and $$P_1(\zeta) = \zeta .$$

The program is listed in SUBROUTINE LP.

The spherical Bessel functions of the first and second kind of integer order n are respectively denoted by j_n and y_n, and are linearly independent solutions of the differential equation (Abramowitz and Stegun (1))

$$t^2\frac{d^2w}{dt^2} + 2t\frac{dw}{dt} + (t^2 - n(n+1))w = 0.$$

The Wronskian W of this differential equation is given by

$$W(t) = \frac{1}{t^2} .$$

The functions j_n and y_n are related to the Bessel functions of the first and second kind J_n and Y_n by the formulae

$$j_n(t) = \left(\frac{\pi}{2t}\right)^{1/2} J_{n+\frac{1}{2}}(t)$$

and

$$y_n(t) = \left(\frac{\pi}{2t}\right)^{1/2} Y_{n+\frac{1}{2}}(t)$$

4-28

The formulae $$h_n^{(1)}(t) = j_n(t) + iy_n(t)$$

and $$h_n^{(1)\prime}(t) = j_n'(t) + iy_n'(t)$$

are then used to calculate the Bessel function of the third kind and its derivative. The calculation of j_n and y_n and their first order derivatives for all orders n and all values of the argument t is based on the recurrence formulae (Abramowitz and Stegun (1))

$$f_{n-1}(t) + f_{n+1}(t) = (2n+1)f_n(t)/t \qquad (4.4.14)$$

and

$$nf_{n-1}(t) - (n+1)f_{n+1}(t) = (2n+1)\frac{df_n(t)}{dt}, \qquad (4.4.15)$$

where f_n denotes either j_n or y_n. Then there are constants C_j and C_y such that

$$j_n(t) = C_j\, f_n(t)$$

and

$$y_n(t) = C_y\, f_n(t)$$

for all $n \geq 0$.

It is also necessary to take note of the asymptotic relations

$$J_\nu(t) \sim \frac{1}{\surd(2\pi\nu)}\left(\frac{et}{2\nu}\right)^\nu$$

and

$$Y_\nu(t) \sim -\left(\frac{2}{\pi\nu}\right)^{1/2}\left(\frac{et}{2\nu}\right)^{-\nu}$$

where $\nu \to \infty$ through real positive values. Setting $u = et/2$ and $N = n+\frac{1}{2}$, we obtain

$$j_n(t) \sim \frac{1}{2}\frac{1}{\surd(Nt)}\left(\frac{u}{N}\right)^N \qquad (4.4.16)$$

and

$$y_n(t) \sim -\frac{1}{\surd(Nt)}\left(\frac{u}{N}\right)^{-N} \qquad (4.4.17)$$

Relation (4.4.17) shows that for a given argument t the value of $y_n(t)$ increases extremely rapidly with increasing order n. If therefore $y_n(t)$ and $y_{n+1}(t)$ are given then the recurrence relation (4.4.15) can be used to calculate $y_{n+m}(t)$ with good accuracy (Goldstein and Thaler (18)) up to a certain maximum order n depending on the hardware and software used for for computations. Using Fortran software the upper bound on numerical values is of the order of 10^{300}, assuming that scaling is not used. This means that an estimate of the smallest positive integer $n_y(t)$ for which $|y_n(t)|$ exceeds 10^{300} whenever $n \geq n_y(t)$ is obtained by considering when

$$\frac{1}{\surd(Nt)}\left(\frac{u}{N}\right)^{-N} \geq 10^{300}$$

or

$$N \log(u/N) + 0.5 \log(Nt) + 300 \leq 0.$$

This relation is used in SUBROUTINE SBF to determine $n_y(t)$ and $y_n(t)$ is then computed for all $n = 0,\ldots,n_y(t)$.

For the spherical Bessel function j_n we observe from (4.4.16) that for a given value of t the value of $j_n(t)$ decreases rapidly as the order n increases beyond $u = et/2$. Thus if one is given $j_n(t)$ and $j_{n+1}(t)$ then (4.4.14) gives poor accuracy for $j_{n+m}(t)$ as m increases (Goldstein and Thaler, *loc.cit.*). We can however still use (4.4.14) provided we recur in the direction of decreasing order. Relation (4.4.16) implies that $j_n(t)$ decreases to zero as $n \to \infty$, and yields an estimate of the smallest positive integer $n_j(t)$ such that $|j_n(t)| \leq 10^{-300}$ whenever $n \geq n_j(t)$:

$$\frac{1}{2}\,\frac{1}{\sqrt{(Nt)}}\left(\frac{u}{N}\right)^N \leq 10^{-300}$$

or

$$N\log(u/N) + 0.5\log(Nt) + 300 \leq 0.$$

Assuming now that

$$f_{n_j+2}(t) = 0$$

and

$$f_{n_j+1}(t) = 10^{-300},$$

the functions $f_n(t)$ are computed in SUBROUTINE SBF for $n = 0,\ldots,n_j(t)$. Since $j_0(t)$ is known, viz.

$$j_0(t) = \frac{\sin t}{t},$$

the coupling constant C_j is given by

$$C_j = \frac{j_0(t)}{f_0(t)}.$$

A printed list of the spherical Bessel functions and their first derivatives is given by PROGRAM SBF for orders $n = 0,\ldots n_0(t)$, where $n_0(t)$ is the minimum of $n_j(t)$ and $n_y(t)$. This program also calculates the theoretical Wronskian W given by

$$W(t) = \frac{1}{t^2}$$

and the computed Wronskian W_n given by

$$W_n(t) = \begin{vmatrix} j_n(t) & y_n(t) \\ j_n'(t) & y_n'(t) \end{vmatrix}.$$

In all cases considered ($0 \leq t \leq 200$ and $n = 0,\ldots,n_0(t)$) the absolute difference between $W(t)$ and $W_n(t)$ was found to be less than 10^{-16} on a computer using 18-digit double precision arithmetic.

The output of SUBROUTINE LP and SUBROUTINE SBF is used by PROGRAM

SFHSE to compute the surface field on a sphere as given by equation (4.4.13). In this program t = ka and θ = x, where θ is the colatitude expressed in radians. As the absolute values of the spherical Bessel functions j_n and y_n and their first derivatives vary greatly in magnitude, the function subprogram FCABS is used to calculate the reciprocal of $h_n^{(1)\prime}(ka)$. The output of this program is a printed list of the surface field at selected points of the surface, and a list of the amplitude of the field at these points which is used for plotting purposes.

4.5 Zero order approximations for a sphere.

According to equation (4.4.8) the zero order approximation ϕ_0 is given by

$$\phi_0(\zeta) = -\frac{(\alpha-\beta ika\zeta)e^{-ika\zeta}}{\mathfrak{R}_1}, \tag{4.5.1}$$

and its amplitude is

$$|\phi_0(\zeta)| = \frac{|\alpha-i\beta ka\zeta|}{|\mathfrak{R}_1|}. \tag{4.5.2}$$

• **Case 1.** If $\alpha = 1$ and $\beta = 0$ then, according to (4.4.3) and (4.4.7),

$$\mathfrak{R}_1 = \mathfrak{P}_1$$

where

$$\mathfrak{P}_1 = \mathfrak{B} - \tfrac{1}{2}$$

and

$$\mathfrak{B} = -\tfrac{1}{2}(h\partial_h - 1)I .$$

According to Appendix A,

$$I = \frac{e^{h\tau} - 1}{h}$$

where

$$h = 2ika.$$

and

$$\tau = \sin\epsilon,\ 0 \le \epsilon \le \tfrac{\pi}{2} .$$

Hence

$$\mathfrak{P}_1 = -\tfrac{1}{2}\left[1 + e^{h\tau} + 2\left(1 - e^{h\tau}\right)h^{-1}\right], \tag{4.5.3}$$

$$\phi_{0,1}(\zeta) = \frac{2e^{ika\zeta}}{\left[1 + \tau e^{h\tau} - 2\left(1 - e^{h\tau}\right)h^{-1}\right]} \tag{4.5.4}$$

and

$$|\phi_{0,1}(\zeta)| = \frac{2}{|1 + e^{h\tau} - 2(1 - e^{h\tau})h^{-1}|}. \tag{4.5.5}$$

This should be compared with the geometric-optics field ϕ_g valid for large values of k:

$$\phi_g(\zeta) = 2\phi_i(\zeta) = 2e^{-ika\zeta} \tag{4.5.6}$$

and

$$|\phi_g(\zeta)| = 2.$$

If ε is so small that powers of ε higher than ε^3 are negligible then, as in Appendix A,

$$\mathcal{P}_1 = -\frac{1}{2}\,(1 + 3\varepsilon + 3h\varepsilon^2).$$

Hence

$$\phi_{0,1}(\zeta) = \frac{2e^{-ika\zeta}}{1 + 3\varepsilon + 3h\varepsilon^2}$$

and

$$|\phi_{0,1}(\zeta)| = \frac{2}{|1 + 3\varepsilon + 3h\varepsilon^2|} \le 2.$$

It follows that

$$\lim_{\varepsilon\to 0} \phi_{0,1}(\zeta) = 2e^{ika\zeta} = \phi_s(\zeta)$$

for all $\zeta \in [-1, 1]$.

If $\varepsilon = \frac{\pi}{2}$, i.e. $\tau = 1$, then

$$\phi_{0,1}(\zeta) = \frac{2e^{ika\zeta}}{\left[1 + e^h - 2\left(1 - e^h\right)h^{-1}\right]} \tag{4.5.7}$$

and

$$|\phi_{0,1}(\zeta)| = \frac{2}{|1 + e^h - 2(1 - e^h)h^{-1}|}\ . \tag{4.5.8}$$

Equations (4.5.7) and (4.5.8) are infinite when

$$1 + e^h - 2\left[\frac{e^h - 1}{h}\right] = 0 \tag{4.5.9}$$

or

$$h = 2\left[\frac{e^h - 1}{e^h + 1}\right]$$

from which it follows that

$$ka = \tan ka\ . \tag{4.5.10}$$

Equation (4.5.10) shows that the zeros of equation (4.5.9) are real if ka is real, and the approximate location of these zeros can be read off from Fig.(4.5.1), where $x = ka$. We observe that these zeros are the same as those of the spherical Bessel function

$$j_1(x) = 0\ .$$

The points at which $\phi_{0,1}(\zeta)$ is infinite therefore correspond to some of the eigenvalues of the homogeneous interior Dirichlet problem for the sphere. Note also that for large values of $x = ka$ the zeros of equation (4.5.9) are approximately equal to an odd multiple of $\frac{\pi}{2}$.

• Case 2. If $\alpha = 0$ and $\beta = 1$ then

$$\mathfrak{R}_1 = \mathfrak{Q}_1$$

and

$$\mathfrak{Q}_1 = \mathfrak{D} = \frac{1}{4}\left((4 + (2 - h)\tau - 4h^{-1} + \tau^{-1})e^{h\tau} - h + 4h^{-1}\right). \qquad (4.5.11)$$

Hence

$$\phi_{0,2}(\zeta) = \frac{4ika\zeta e^{ika\zeta}}{\left(4 + (2-h)\tau - 4h^{-1} + \tau^{-1}\right)e^{h\tau} - h + 4h^{-1}} .$$

First we note that

$$\lim_{\varepsilon \to 0} \phi_{0,2}(\zeta) = 0 , \quad -1 \le \zeta \le 1 ,$$

and this corresponds to the surface field in the shadow region in the case of physical optics.

If $\tau = 1$, i.e. $\varepsilon = \frac{\pi}{2}$, then

$$\phi_{0,2}(\zeta) = - \frac{2h\zeta e^{-ika\zeta}}{(4 - h - 4h^{-1})e^{h} + 4h^{-1} - h} \qquad (4.5.12)$$

and

$$|\phi_{0,2}(\zeta)| = - \frac{2|h\zeta|}{|(4 - h - 4h^{-1})e^{h} + 4h^{-1} - h|} . \qquad (4.5.13)$$

This expression is infinite when

$$(4h - h^2 - 4)e^h + 4 - h^2 = 0. \qquad (4.5.14)$$

Since $h = 2ika$, we obtain

$$(8ika + 4k^2a^2 - 4)(\cos 2ka + i \sin 2ka) + 4 + 4k^2a^2 = 0.$$

Separating the real and imaginary parts yields

$$(4k^2a^2 - 4)\cos 2ka - 8ka \sin 2ka + 4 + 4k^2a^2 = 0$$

and $$(4k^2a^2 - 4)\sin 2ka + 8ka \cos 2ka = 0.$$

Eliminating terms containing $\sin 2ka$ we find that

$$\cos 2ka = \frac{1 - k^2a^2}{1 + k^2a^2} . \qquad (4.5.15)$$

If $ka \to \infty$ then the right-hand side converges to -1. Thus for large values of ka the zeros of equation (4.5.14) converge to the odd multiples of $\frac{\pi}{2}$. The distribution of the zeros of equation (4.5.14) for $x = ka \ge 0$ is shown in Fig.(4.5.2).

The considerations above show that when $\varepsilon = \frac{\pi}{2}$ the two zero order forms (4.5.7) and (4.5.12) are not suitable as approximations for the surface field when ka is near an odd multiple of $\frac{\pi}{2}$, and it may be expected that higher order approximations with $(\alpha,\beta) = (1,0)$ or $(0,1)$ will exhibit similar undesirable properties.

• **Case 3**. If $\alpha = 1$ and $\beta = i$, then

$$\mathfrak{R}_1 = \mathfrak{P}_1 + i\mathfrak{Q}_1$$

where $\mathfrak{P}_1$ and $\mathfrak{Q}_1$ are given respectively by (4.5.3) and (4.5.11), and

$$\phi_{0,3}(\zeta) = -\frac{(1 + ka\zeta)e^{-ika\zeta}}{\mathfrak{R}_1} \tag{4.5.16}$$

and

$$|\phi_{0,3}(\zeta)| = \frac{|1 + ka\zeta|}{|\mathfrak{R}_1|} . \tag{4.5.17}$$

Here we also find that $\phi_{0,3}(\zeta) \to 0$ when $\tau \to 0$.

If $\varepsilon = \frac{\pi}{2}$, then

$$\mathfrak{R}_1 = -\tfrac{1}{2}\left[1 + e^h + 2\left(1 - e^h\right)h^{-1}\right] + \tfrac{1}{4}i\left[\left(4 - h - 4h^{-1}\right)e^h - h + 4h^{-1}\right].$$

Separating the real and the imaginary parts in the equation

$$\mathfrak{R}_1 = 0, \tag{4.5.18}$$

we obtain the two equations

$$-(x^2 - x + 1)\sin 2x + (1 - 2x)\cos 2x = 1 + x^2 \tag{4.5.19}$$

and

$$(1 - 2x)\sin 2x + (x^2 - x - 1)\cos 2x = -1 + x - x^2 \tag{4.5.20}$$

where $x = ka$. Multiplying equation (4.5.19) by $(1 - 2x)$ and equation (4.5.20) by $(x^2 - x + 1)$ and adding we get

$$\cos 2x = -\frac{x(x^2 + 2x + 2)}{(x - 1)(x^2 - x + 4)} . \tag{4.5.21}$$

Fig.(4.5.3) shows that when $x = ka \geq 0$, then the two curves

$$y = \cos 2x$$

and

$$y = -\frac{x(x^2 + 2x + 2)}{(x - 1)(x^2 - x + 4)}$$

have only one point of intersection at approximately $x = 0.4387$. The denominator $\mathfrak{R}_1$ in (4.5.16) and (4.5.17) will be small but non-zero when ka is a large odd multiple of $\frac{\pi}{2}$.

It appears that $\alpha = 1$ and $\beta = i$ is a more suitable choice for constructing approximations than the preceding two cases when $\varepsilon = \frac{\pi}{2}$. However, if ka is near to an odd multiple of $\frac{\pi}{2}$ then $\mathfrak{R}_1$ will be approximately zero and hence $|\phi_{0,3}(\zeta)|$ will be very large. In Fig.(4.5.4) the values of the reciprocals of $|\mathfrak{P}_1|$, $|\mathfrak{Q}_1|$ and $|\mathfrak{R}_1|$ are plotted against ka for the range $1 \le ka \le 19$. Thus we see that the graphs of the zero order approximations differ widely in appearance, a few typical examples being given in Figs.(4.5.5)-(4.5.10).

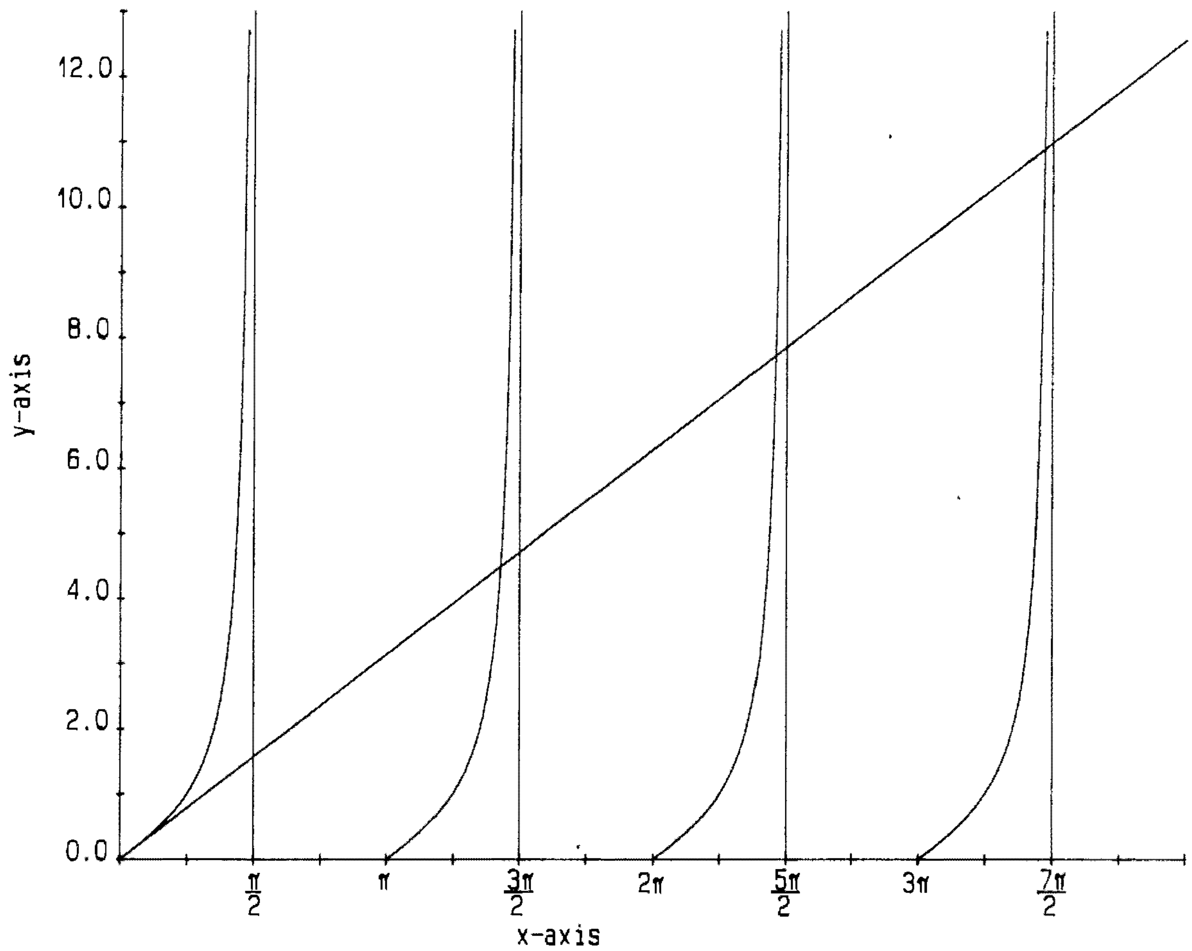


Fig.(4.5.1). Grahps of $y \doteq x$ and $y = \tan x$ for locating the zeros of equation (4.5.9) when $x = ka > 0$.

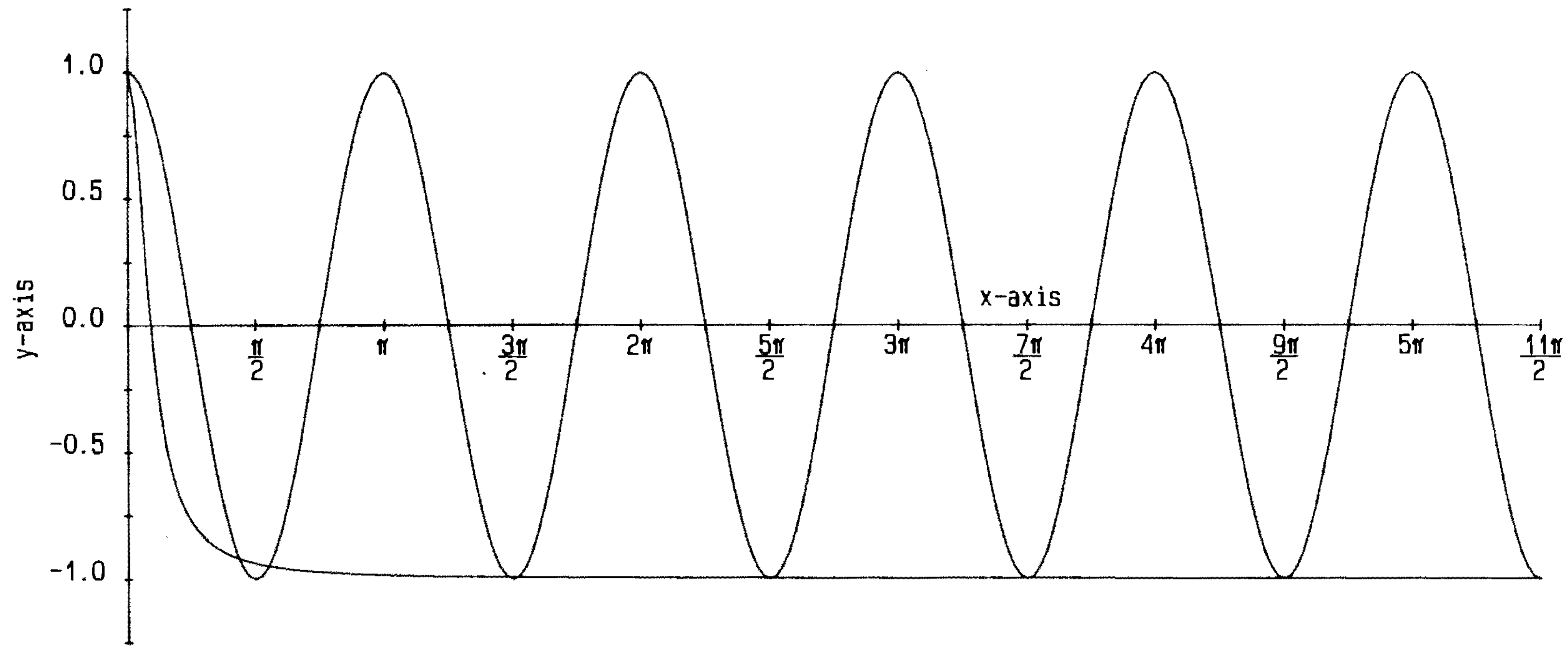


Fig. (4.5.2). Graphs of $y = \cos 2x$ and $y = \dfrac{1 - x^2}{1 + x^2}$ for locating the zeros of equation (4.5.15) when $x = ka > 0$.

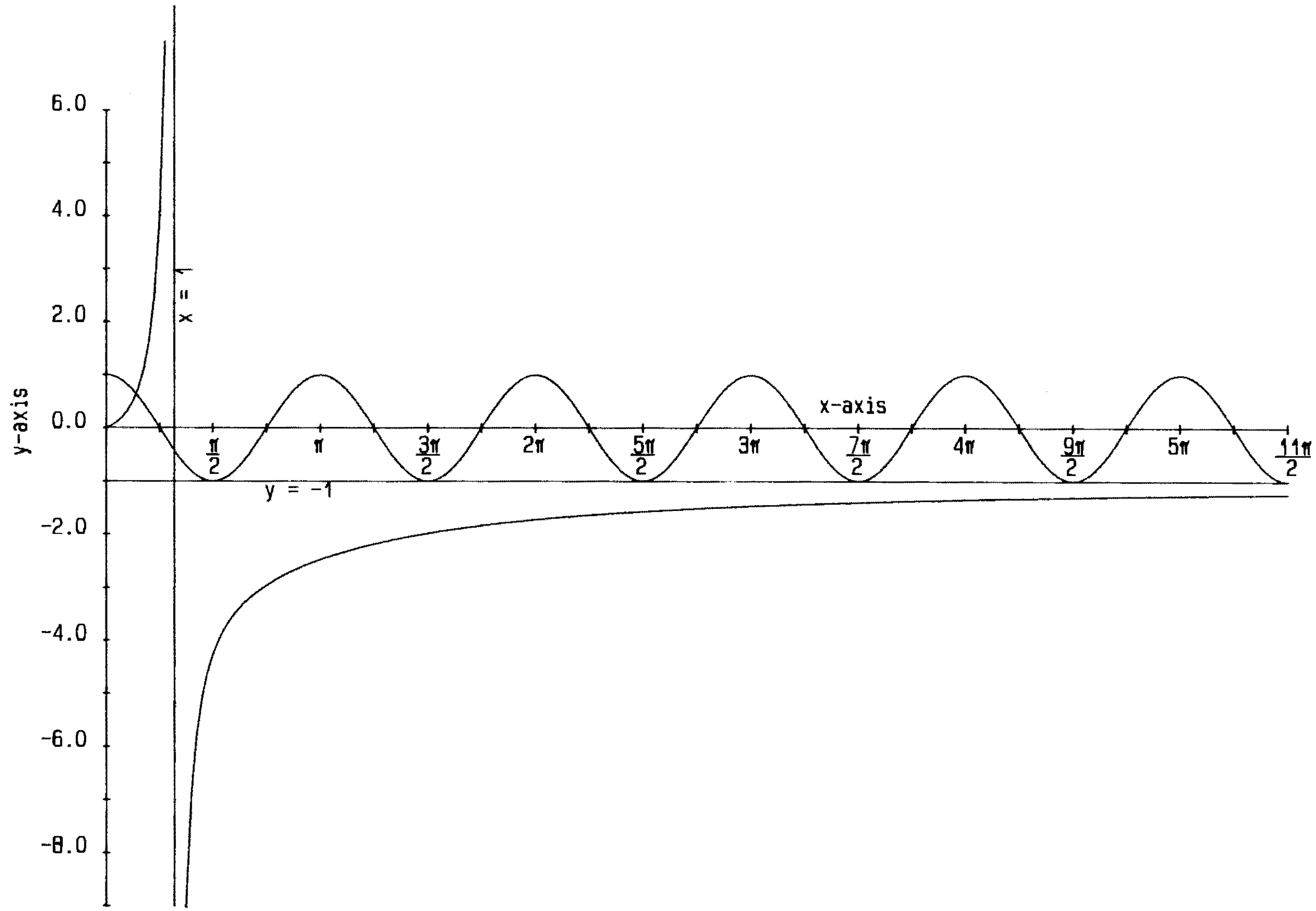


Fig. (4.5.3). Graphs of $y = \cos 2x$ and $y = -\dfrac{x(x^2 + 2x + 2)}{(x - 1)(x^2 - x + 4)}$ for locating the real zeros of equation (4.5.18) when $x = ka > 0$.

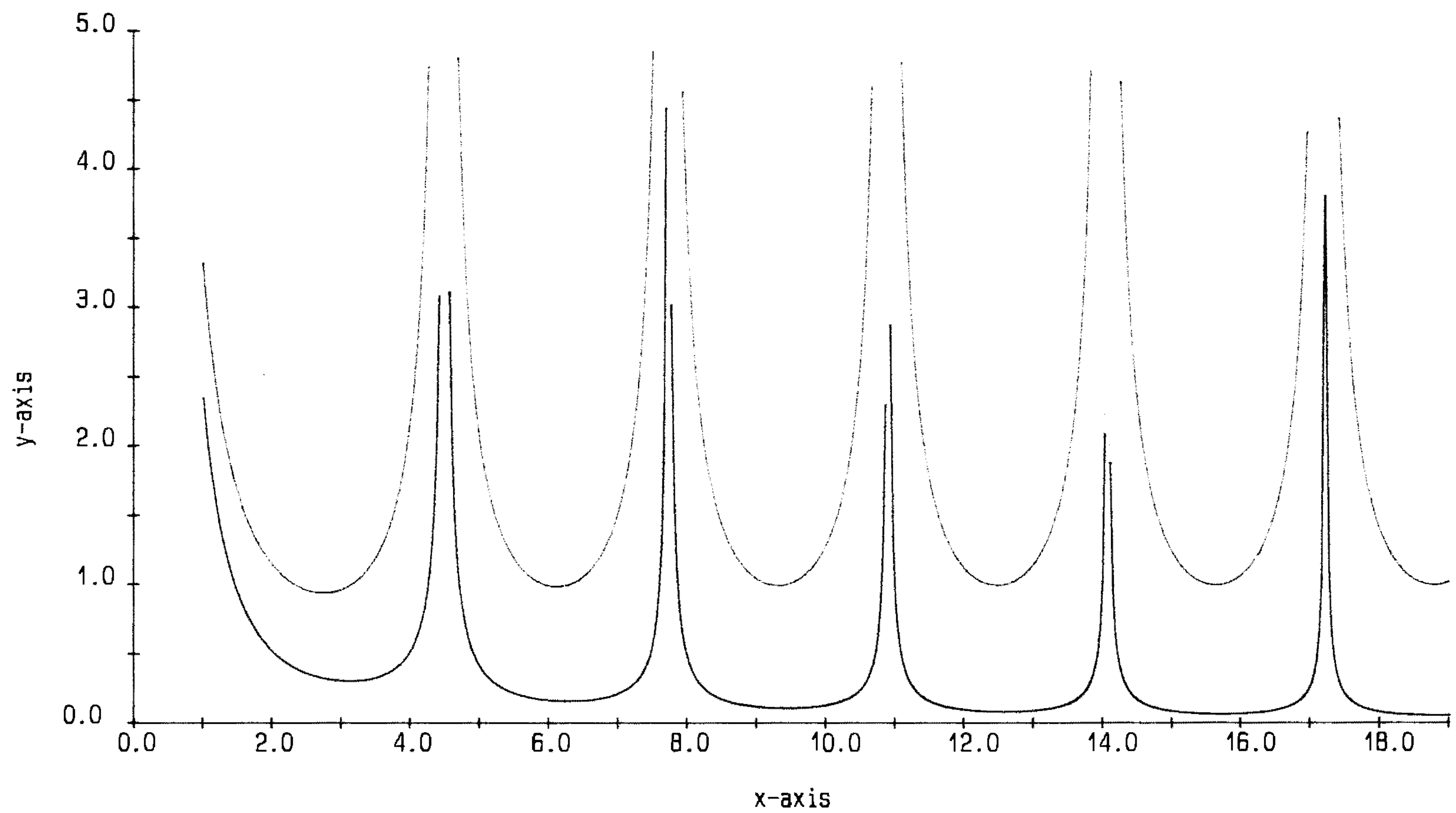


Fig. (4.5.4). Graphs of $y = |\mathfrak{R}_1|^{-1}$ plotted against $x = ka$ for ka in the range 1 to 19.

(i) Blue: $\alpha = 1$, $\beta = 0$, (ii) Red: $\alpha = 1$, $\beta = i$ and (iii) Green: $\alpha = 0$, $\beta = 1$.

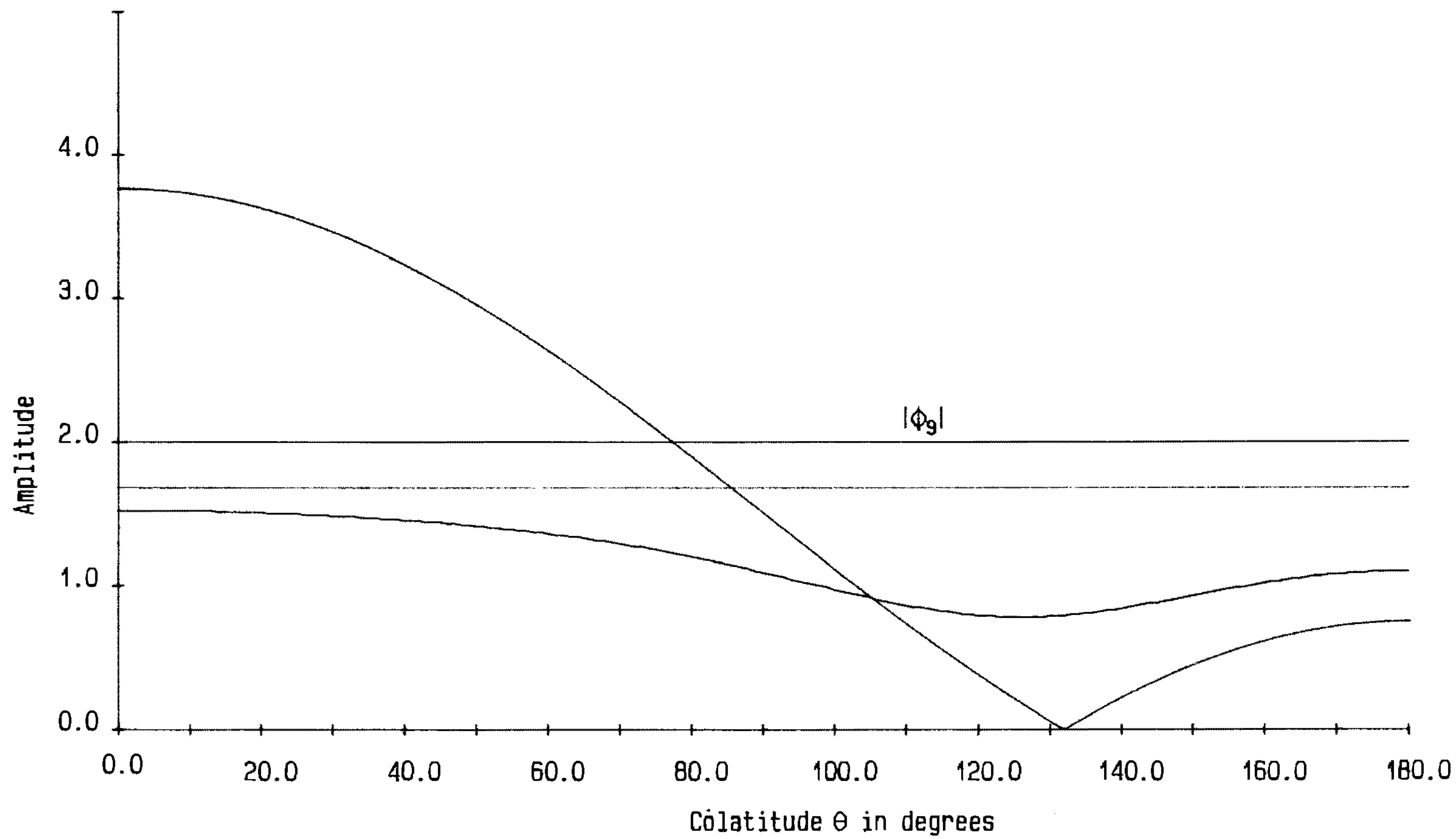


Fig. (4.5.5). Surface field amplitude (Black) for a sphere, ka = 1.5, and amplitudes of zero order approximations: (i) Blue: $\alpha = 1$, $\beta = 0$, (ii) Red: $\alpha = 1$, $\beta = i$ and (iii) Green: $\alpha = 0$, $\beta = 1$.

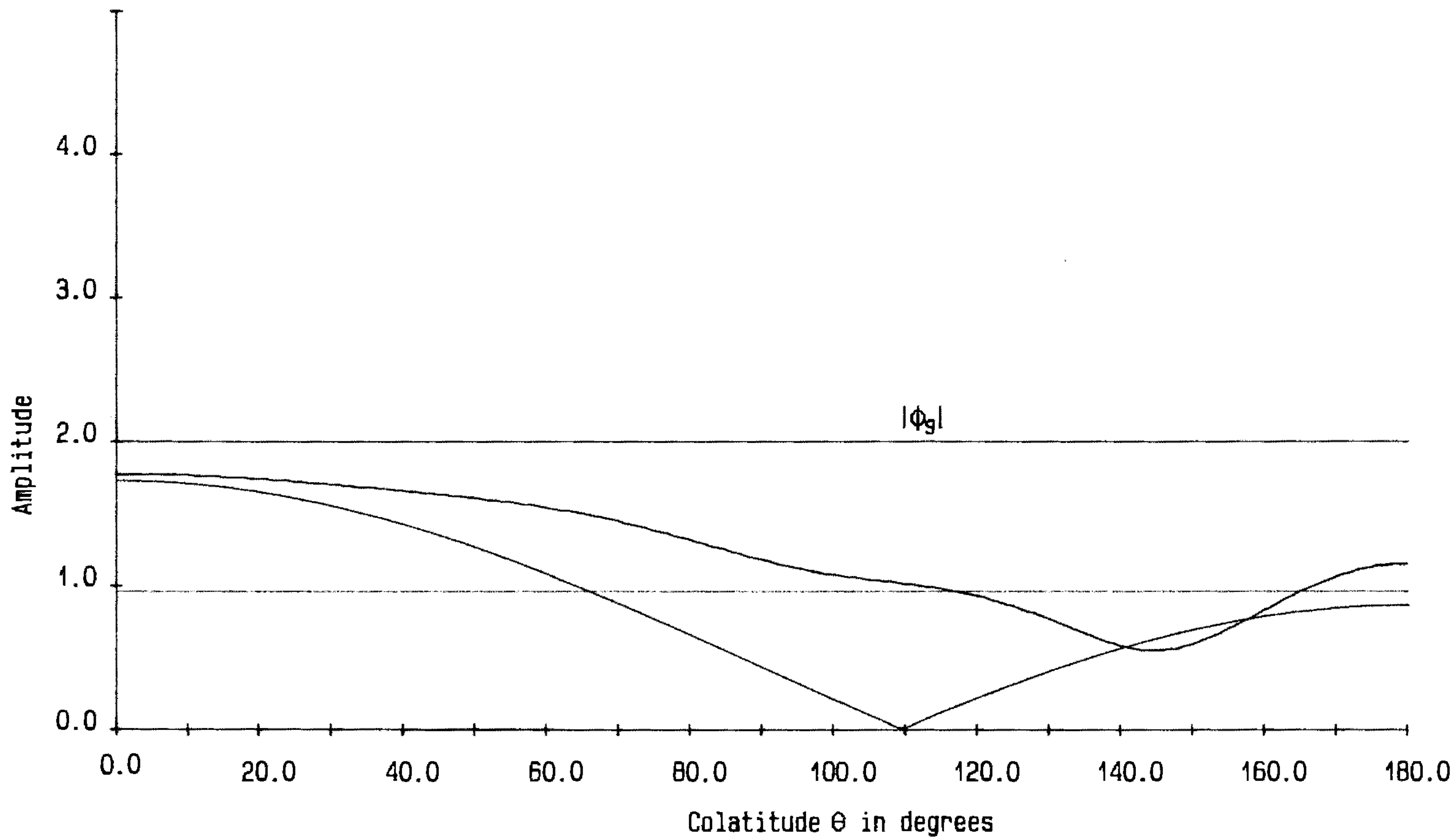


Fig. (4.5.6). Surface field amplitude (Black) for a sphere, ka = 3.0, and amplitudes of zero order approximations: (i) Blue: $\alpha = 1$, $\beta = 0$, (ii) Red: $\alpha = 1$, $\beta = i$ and (iii) Green: $\alpha = 0$, $\beta = 1$.

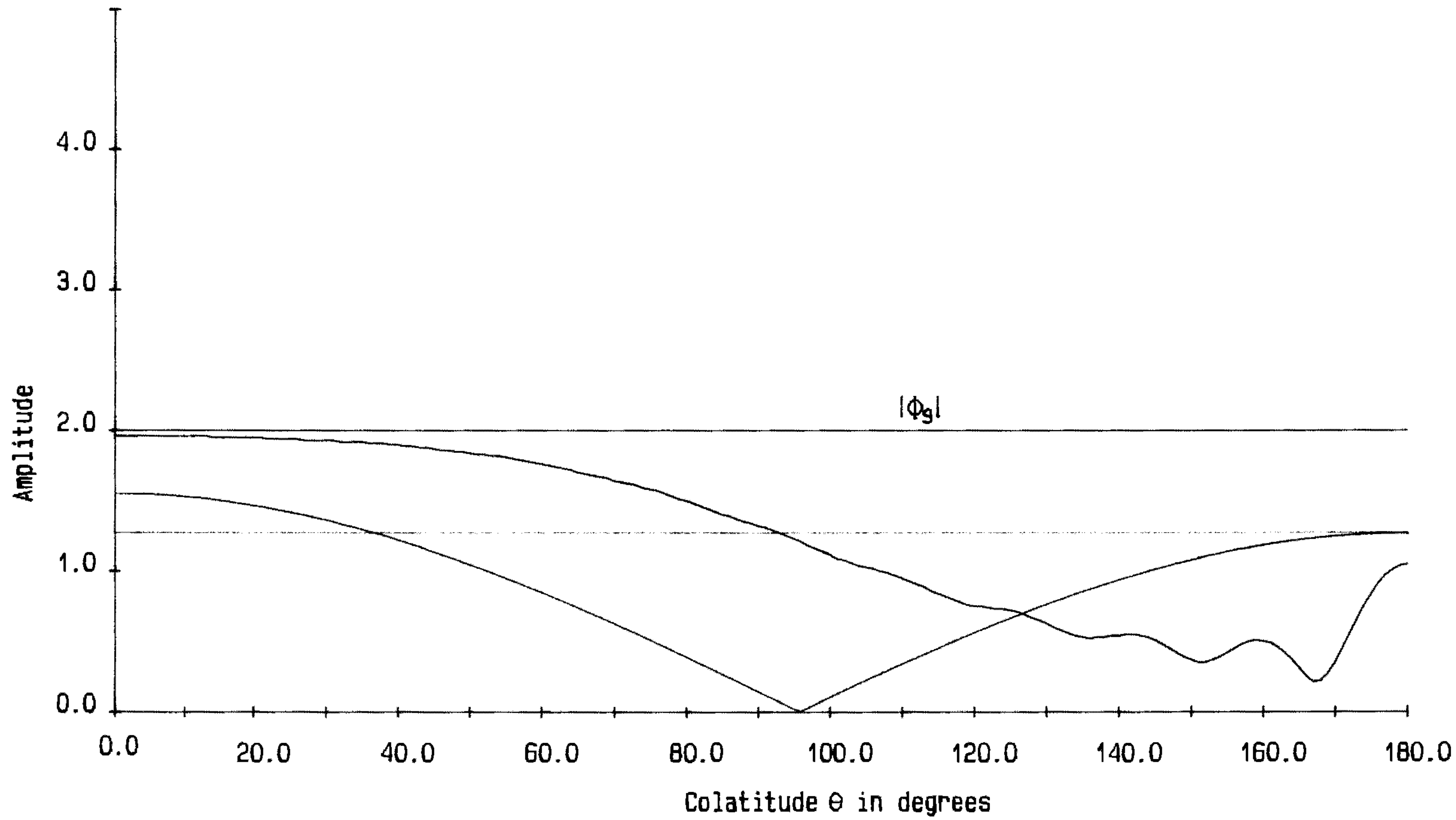


Fig. (4.5.7). Surface field amplitude (Black) for a sphere, ka = 10.0, and amplitudes of zero order approximations: (i) Blue: $\alpha = 1$, $\beta = 0$, (ii) Red: $\alpha = 1$, $\beta = i$ and (iii) Green: $\alpha = 0$, $\beta = 1$.

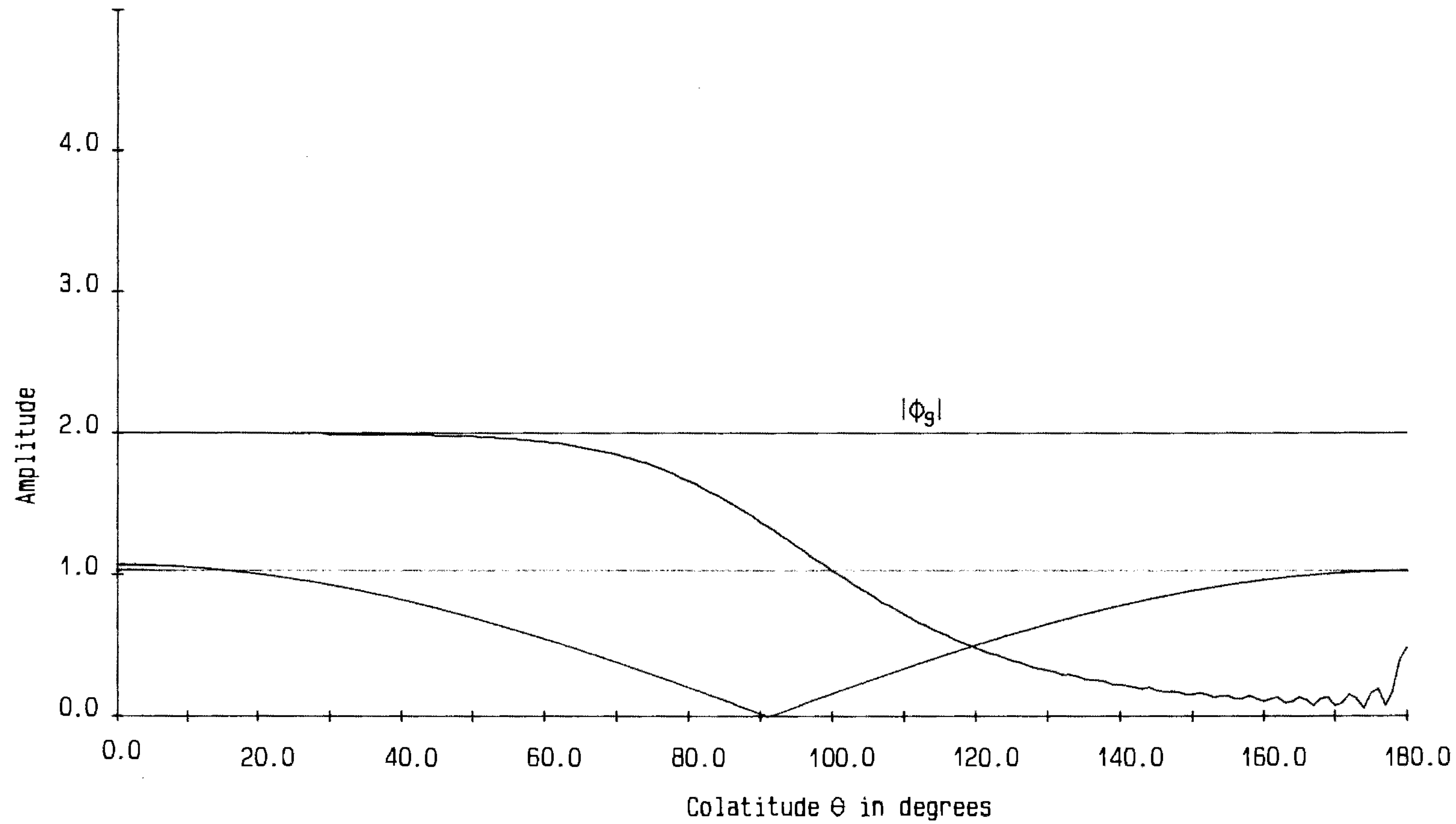


Fig. (4.5.9). Surface field amplitude (Black) for a sphere, ka = 50.0, and amplitudes of zero order approximations: (i) Blue: $\alpha = 1$, $\beta = 0$, (ii) Red: $\alpha = 1$, $\beta = i$ and (iii) Green: $\alpha = 0$, $\beta = 1$.

4.6 First order global approximations for a sphere.

For the global approximation $\varepsilon = \frac{\pi}{2}$ as in section (4.3), and from equations (4.4.9) and (4.4.7) we have

$$\mathfrak{R}_1\phi + \mathfrak{R}_2\zeta\frac{d\phi}{d\zeta} = f(\zeta), \; -1 \le \zeta \le 1 , \tag{4.6.1}$$

where
$$f(\zeta) = -(\alpha-\beta ika\zeta)e^{-ika\zeta} , \tag{4.6.2}$$
$$\mathfrak{R}_1 = \alpha\mathfrak{P}_1 + \beta\mathfrak{Q}_1$$
and
$$\mathfrak{R}_2 = \alpha\mathfrak{P}_2 + \beta\mathfrak{Q}_2 .$$

The equation has only one singular point at the origin and analytical solutions can be obtained in any region not containing the origin. Thus if C is a continuous path in the complex plane $\mathbb{C}$ with initial point 1 and end point -1, and if C does not contain the origin, then a solution of equation (4.6.1) can be constructed by analytical continuation along C. Any such solutions will be the same as any other such solution provided that their respective paths are homotopic with respect to the origin. We also note that equation (4.6.1) has a unique singular solution which is analytical at all points of the complex plane. This solution is obtained by writing (4.6.2) in the form

$$f(\zeta) = \sum_{n=0}^{\infty} f_n\zeta^n \tag{4.6.3}$$

where
$$f_n = -(\alpha + n\beta)\frac{(-ika)^n}{n!} , \; n = 0,1,2.... , \tag{4.6.4}$$
and assuming that equation (4.6.1) has an analytical solution

$$\phi_{1,s}(\zeta) = \sum_{n=0}^{\infty} a_n\zeta^n . \tag{4.6.5}$$

We then obtain
$$a_n = \frac{f_n}{(\mathfrak{R}_1 + n\mathfrak{R}_2)} , \; n = 0,1,2.... , \tag{4.6.6}$$
and
$$\phi_{1,s}(\zeta) = -\sum_{n=0}^{\infty} \frac{(\alpha + n\beta)}{(\mathfrak{R}_1 + n\mathfrak{R}_2)} \frac{(-ika)^n}{n!} \zeta^n . \tag{4.6.7}$$

This series is absolutely and uniformly convergent in any bounded region of the complex plane. However for values of ka > 20 the series is not suitable for numerical computations - the initial terms of the series are then very large and the accumulated errors are consequently also large. In general the functions represented by equation (4.6.7) are not suitable as global approximations of the surface field as is apparent in Fig.(4.6.1) for the case ka = 3. For larger values of ka the values predicted by (4.6.7) at the specular point may become very large.

The homogeneous equation corresponding to (4.6.1) is

$$\mathcal{R}_1\phi + \mathcal{R}_2\zeta \frac{d\phi}{d\zeta} = 0, \tag{4.6.8}$$

and the general solution of this equation has a branch point at the origin $\zeta = 0$. If

$$s = \frac{\mathcal{R}_1}{\mathcal{R}_2} \tag{4.6.9}$$

we can write equation (4.6.8) in the form

$$\frac{d\phi}{d\zeta} + \frac{s}{\zeta} = 0 \tag{4.6.10}$$

which has the general solution

$$\phi_C(\zeta) = C\zeta^{-s}, \quad \zeta \neq 0, \tag{4.6.11}$$

where C is an arbirary constant. In (4.6.11) $-1 \leq \zeta \leq 1$ and we can select any one of the branches of the many-valued function ζ^{-s}. In general

$$\zeta^{-s} = e^{-s\log\zeta}$$

where

$$\log\zeta = \log|\zeta| + i\arg(\zeta),$$

$$\arg(\zeta) = \mathrm{Arg}(\zeta) + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots,$$

and $\mathrm{Arg}(\zeta)$ denotes the principal argument of ζ, i.e.

$$-\pi < \mathrm{Arg}(\zeta) \leq \pi.$$

Any one of the branches of ζ^{-s} is therefore discontinuous across the negative real axis, but continuity can be maintained if we imagine the complex plane to be cut along the negative real axis from 0 to $-\infty$. Then each of the infinite number of branches

$$g_n(\zeta) = e^{-s(\log|\zeta| + i(\theta + 2n\pi))}$$

is an analytical function of ζ in the cut complex plane, where $\theta = \mathrm{Arg}(\zeta)$. The behaviour of $g_n(\zeta)$ near the origin depends on the value of s. Computations show that the value of s is essentially independent of the values of α and β if the wave number k is sufficiently large. Let

$$s_P = \frac{\mathcal{P}_1}{\mathcal{P}_2}$$

and

$$s_Q = \frac{\mathcal{Q}_1}{\mathcal{Q}_2}.$$

With reference to the asymptotic relations given in Appendix A, we find that

$$\lim_{ka\to\infty} s_P = \lim_{ka\to\infty} s_Q = s_0 ,$$

where
$$s_0 = -\frac{1+e^h}{2e^h}$$

and h = 2ika. It follows that

$$\lim_{ka\to\infty} s = \lim_{ka\to\infty} \frac{\mathfrak{R}_1}{\mathfrak{R}_2} = \lim_{ka\to\infty} \frac{\alpha\mathfrak{P}_1 + \beta\mathfrak{Q}_1}{\alpha\mathfrak{P}_2 + \beta\mathfrak{Q}_2} = s_0$$

for all values of α and β. Since

$$s_0 = -\frac{1+\cos 2ka - i\sin 2ka}{2} ,$$

we see that $-1 \le Rl(s_0) \le 0$.

Fig.(4.6.2) exhibits the graphs of $Rl(s_0)$, $Rl(s_P)$ and $Rl(s_Q)$ plotted against ka for values of ka within the range 1 to 20, and the graphs of $Im(s_0)$, $Im(s_P)$ and $Im(s_Q)$ are shown in Fig.(4.6.3) for the same range of values of ka. In both instances we observe that

$$Rl(s_P) \approx Rl(s_Q) \approx Rl(s_0)$$

and

$$Im(s_P) \approx Im(s_Q) \approx Im(s_0)$$

if ka ≥ 10. It follows that

$$\frac{\mathfrak{P}_1}{\mathfrak{P}_2} \approx \frac{\mathfrak{Q}_1}{\mathfrak{Q}_2} \approx \frac{\mathfrak{R}_1}{\mathfrak{R}_2} \quad \text{if } ka \ge 10.$$

We conclude that the behaviour of the homogeneous equation (4.6.8) is essentially independent of the values of α and β. Moreover we notice that $-1 \le Rl(s) \le 0$ for any choice of α and β provided ka is sufficiently large, where s is given by (4.6.9).

A particular solution for the inhomogeneous equation (4.6.1) can be found by assuming a solution of the form

$$\phi_P(\zeta) = \zeta^{-s}\,\psi(\zeta) \tag{4.6.12}$$

If (4.6.12) is substituted into (4.6.1) we obtain

$$\psi'(\zeta) = \frac{\zeta^{s-1}}{\mathfrak{R}_2}\, f(\zeta).$$

According to the asymptotic relations in Appendix A, $|\mathfrak{R}_2| \sim ka$ and hence $\mathfrak{R}_2$ is never zero if ka is sufficiently large.

A particular solution for the inhomogeneous equation (4.6.1) is therefore

$$\phi_p(\zeta) = \frac{\zeta^{-s}}{\mathfrak{R}_2} \int_1^{\zeta} t^{s-1} f(t)dt . \qquad (4.6.13)$$

Since $Rl(s) \leq 0$ the integral diverges when $\zeta = 0$. However the function ϕ_p defined by this integral is an analytical function of ζ if the complex-plane is cut along the negative real axis from 0 to $-\infty$. We denote this cut complex plane by $\mathbb{C}' = \mathbb{C} - [-\infty, 0]$. Thus if ζ is a point on the negative real axis then we can replace ζ by either $\zeta + i0$ or $\zeta - i0$ and there are two distinct paths from 1 to ζ in $\mathbb{C}'$. If ζ is real and $-1 \leq \zeta \leq 1$ we use as path of integration a straight line from 1 to ζ which is indented at the origin if $\zeta < 0$, and which lies either in the upper-half or lower-half of the cut plane $\mathbb{C}'$. These two paths are denoted respectively by C_ζ^+ and C_ζ^-. Thus if $0 < \delta < 1$ and if $-1 \leq \zeta \leq -\delta$ then in the first case

$$\zeta^{-s} = e^{-s(\log|\zeta| + i\pi)}$$

$$= e^{(-u \log|\zeta| + v\pi) + i(v \log|\zeta| - u\pi)}$$

and in the second case

$$\zeta^{-s} = e^{-s(\log|\zeta| - i\pi)}$$

$$= e^{(-u \log|\zeta| - v\pi) + i(v \log|\zeta| + u\pi)} ,$$

where $u = Rl(s)$ and $v = Im(s)$. That portion of the two paths C_ζ^+ and C_ζ^- which is common to both corresponds to the illuminated side of the sphere. The choice of path is motivated by the fact that for large values of ka the surface field is small in the shadow region. Now if $v > 0$ then the factor $e^{v\pi}$ is large along C_ζ^+ but $e^{-v\pi}$ is small along C_ζ^- , and if $v < 0$ then the factor $e^{-v\pi}$ is large along C_ζ^- but $e^{v\pi}$ is small along C_ζ^+. Accordingly the choice of path is determined by the sign of Im(s): if $Im(s) < 0$ we choose C_ζ^+ as the path of integration, while if $Im(s) > 0$ we choose C_ζ^- as the path of integration.

The general solution of equation (4.6.1) can now be given as

$$\phi(\zeta) = C\zeta^{-s} + \frac{\zeta^{-s}}{\mathfrak{R}_2} \int_1^{\zeta} t^{s-1} f(t)dt . \qquad (4.6.14)$$

If we assume that in the high frequency case the surface field at the specular point $\zeta = 1$ is twice the incident field at this point, we find that $$C = 2e^{-ika} .$$

Thus the first order high frequency approximation to the surface field is given by

$$\phi_1(\zeta) = \zeta^{-s}\left[2e^{-ika} + \frac{1}{\mathfrak{R}_2} \int_1^{\zeta} t^{s-1} f(t)dt\right] . \qquad (4.6.15)$$

Although the integral $\int_1^{\zeta} t^{s-1} f(t)dt$ is divergent if $\zeta \to 0$ in $\mathbb{C}'$, it is nevertheless true that the product $\zeta^{-s} \int_1^{\zeta} t^{s-1} f(t)dt$ converges to a finite limit when $\zeta \to 0$. To prove this note that if $\zeta = \delta > 0$ then

$$\left|\delta^{-s} \int_1^{\delta} t^{s-1} e^{-ikat}\, dt - \delta^{-s} \int_1^{\delta} t^{s-1}\, dt\right| \le \delta^{-u} \int_1^{\delta} t^{u-1}\, |e^{-ikat} - 1|\, dt$$

where $u = Rl(s)$, and

$$e^{-ikat} - 1 = t\, g(t) ,$$

where

$$g(t) = -\, ika \sum_{n=1}^{\infty} \frac{(-ikat)^{n-1}}{n!} .$$

The series representing $g(t)$ is uniformly and absolutely convergent for all values of t, and if $0 \le t \le 1$ then

$$|g(t)| \le |ka|\surd 2 .$$

This result is found by applying the mean value theorem to the real and imaginary parts of $e^{-ikat} - 1$. Hence

$$\left|\delta^{-s} \int_1^{\delta} t^{s-1} e^{-ikat}\, dt - \delta^{-s} \int_1^{\delta} t^{s-1}\, dt\right| \le \surd 2\, |ka|\, \delta^{-u} \int_1^{\delta} t^{u}\, dt .$$

and it is easily shown that if $u < 0$ then

$$\lim_{\delta \to 0} \delta^{-u} \int_1^{\delta} t^{u}\, du = 0$$

and

$$\lim \delta^{-s} \int_1^{\delta} t^{s-1}\, dt = \lim_{\delta \to 0} \frac{\delta^{-s}(\delta^{s} - 1)}{s} = \frac{1}{s} .$$

Hence if $Rl(s) < 0$ then

$$\lim_{\delta \to 0} \delta^{-s} \int_1^{\delta} t^{s-1} e^{-ikat}\, dt = \frac{1}{s} . \qquad (4.6.16)$$

Now using the inequality

$$|e^{-ika\zeta} - 1| \le 2,$$

it follows that

$$\left| \delta^{-s} \int_1^{\delta} t^s e^{-ikat}\, dt - \delta^{-s} \int_1^{\delta} t^s\, dt \right| \le 2\, \delta^{-u} \int_1^{\delta} t^u\, dt ,$$

and if Rl(s) < 0 then

$$\lim_{\delta\to 0} \delta^{-s} \int_1^{\delta} t^s e^{-ikat}\, dt = \lim_{\delta\to 0} \delta^{-s} \int_1^{\delta} t^s\, dt$$

$$= \lim_{\delta\to 0} \frac{\delta^{-s}(\delta^{s+1} - 1)}{s+1} = 0 . \qquad (4.6.17)$$

Thus

$$\lim_{\delta\to 0} \frac{\delta^{-s}}{\mathfrak{R}_2} \int_1^{\delta} t^{s-1} f(t)\, dt = - \frac{\alpha}{\mathfrak{R}_2\, s} = - \frac{\alpha}{\mathfrak{R}_1} \qquad (4.6.18)$$

if Rl(s) < 0, where $f(t) = -(\alpha - \beta ika\zeta)e^{-ika\zeta}$ and $\zeta \in \mathbb{C}'$. Hence according to equation (4.6.15), if Rl(s) < 0 then

$$\lim_{\delta\to 0+} \phi_1(\delta) = - \frac{\alpha}{\mathfrak{R}_1} . \qquad (4.6.19)$$

Next we determine the limit of the expression

$$I = \zeta^{-s} \int_{C_\zeta^+} t^{s-1}\, e^{-ikat}\, dt \qquad (4.6.20)$$

where $\zeta = -\delta + i0$ and $0 < \delta \to 0$. The path of integration in (4.6.20) is

$$C_\zeta^+ = \Gamma_1 \cup \Gamma_\delta^+ ,$$

where Γ_1 is the line segment from 1 to δ and Γ_δ^+ is the semi-circular arc

$$t = \delta\, e^{i\varphi} , \; 0 \le \varphi \le \pi .$$

We write (4.6.20) in the form

$$I = I_1 + I_2 \qquad (4.6.21)$$

where

$$I_1 = (-\delta)^{-s} \int_1^{\delta} t^{s-1}\, e^{-ikat}\, dt \qquad (4.6.22)$$

and

$$I_2 = (-\delta)^{-s} \int_{\Gamma_\delta^+} t^s\, e^{-ikat}\, dt . \qquad (4.6.23)$$

Since

$$(-\delta)^{-s} = e^{-s(\log\delta + i\pi)} = e^{-i\pi s}\, \delta^{-s} ,$$

$$I_1 = e^{-i\pi s}\, \delta^{-s} \int_1^{\delta} t^{s-1}\, e^{-ikat}\, dt$$

and from equation (4.6.16) it follows that

$$\lim_{\delta \to 0} I_1 = \frac{e^{-i\pi s}}{s} . \tag{4.6.24}$$

If $t \in C_\delta^+$, then $t = \delta e^{i\varphi}$, where $0 \le \varphi \le \pi$. Hence

$$I_2 = e^{-s(\log\delta + i\pi)} \int_0^\pi e^{(s-1)(\log\varphi + i\varphi)} e^{ka\delta(\sin\varphi + i\cos\varphi)} i\delta e^{i\varphi} \, d\varphi$$

i.e. $$I_2 = ie^{-i\pi s} \int_0^\pi e^{is\varphi} e^{ka\delta(\sin\varphi + i\cos\varphi)} \, d\varphi ,$$

and hence

$$\lim_{\delta \to 0} I_2 = ie^{-i\pi s} \int_0^\pi e^{i\pi s} \, d\varphi = \frac{1 - e^{-i\pi s}}{s} . \tag{4.6.25}$$

From (4.6.24), (4.6.25) and (4.6.21) we find that

$$\lim_{\delta \to 0} I = \frac{1 - e^{-i\pi s}}{s} + \frac{e^{-i\pi s}}{s} = \frac{1}{s} , \tag{4.6.26}$$

provided that Rl(s) < 0. Similarly, if

$$J = \zeta^{-s} \int_{C_\zeta^+} t^s e^{-ikat} \, dt$$

where $\zeta = -\delta + i0$, then we find that

$$\lim_{\delta \to 0} J = 0 \tag{4.6.27}$$

provided that Rl(s) < 0.

Using equations (4.6.26) and (4.6.27) in (4.6.15) and comparing the result with equation (4.6.19) gives

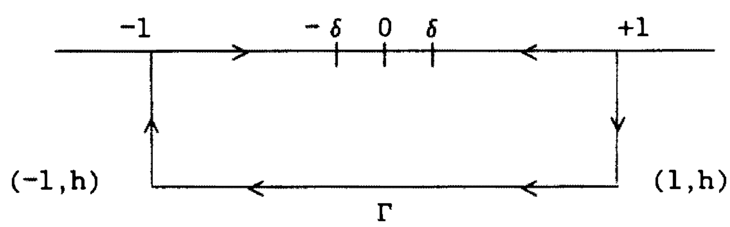
$$\lim_{\delta \to 0+} \phi_1(\delta) = \lim_{\delta \to 0-} \phi_1(\delta) = -\frac{\alpha}{\Re_1} \tag{4.6.28}$$

This means that $\phi_1(0)$ is uniquely defined by

$$\phi_1(0) = -\frac{\alpha}{\Re_1} \tag{4.6.29}$$

if Rl(s) < 0.

The solution (4.6.15) of equation (4.6.1) is computed by PROGRAM SFHS1A at a discrete set of points in the interval [-1 ,1], the path of integration being either C_ζ^+ or C_ζ^- with $\delta \approx 0.1$, and the integrals are evaluated by using Simpson's one-third rule. No convergence criteria or tests of accuracy are built into this program, as the results can be tested by running the program with successively finer partions of the intervals involved. We do note, however, that for large values of ka the function f(t) given by (4.6.2) oscillates very rapidly and there may be some dought as to the accuracy of the computed product $\zeta^{-s}\int_1^{\zeta} t^{s-1}\ f(t)dt$ if $\zeta \in [-\delta,\delta]$ and δ is very small. This is particularly relevant when $u = Rl(s) \approx -1$, for then $|t^{s-1}| \approx |t|^{-2}$ inceases very rapidly as $|t|$ decreases. Another disadvantage is that errors resulting from computations along the semi-circular portions also affect subsequent results. In an attempt to avoid thses difficulties we can compute the integral along some other path which is homotopic to either C_ζ^+ or C_ζ^-. PROGRAM SFHS1B is desiged to compute (4.6.15) along the path shown in the figure below for the case where $Im(s) > 0$:



Thus, choosing a suitable $h < 0$, we first determine $\phi_1(\zeta)$ along the interval $[\delta,1]$ starting from (1,0), and then proceed to the point (-1,0) along the path Γ as shown, again starting from the point (1,0). Finally the values of the function can be computed along the interval $[-1,-\delta]$, and $\phi_1(0)$ is given by (4.6.29). With reference to Fig.(4.5.4) equation (4.6.29) shows that $\phi_1(0)$ oscillates widely with different values of ka, and accordingly $\phi_1(0)$ is in general not a good representation of the surface field at this point. The effect of the singularity at $\zeta = 0$ is therefore that equation (4.6.15) does not give a good representation of the surface field in the penumbral and shadow region.

In PROGRAM SFHS1C the solution of equation (4.6.1) is computed by analytical continuation along a path as shown in the figure above. This method of solution is very much slower in execution than the two former programs, but it does provide a convenient means of determining the accuracy of the calculations, and for cases where ka is large and $u \approx -1$ the results are generally better than those obtained by SFHS1B. The path along which the solution is constructed is subdivided into a sufficiently large number of non-overlapping segments by a sequence of points ζ_ν, $\nu = 0,1,2,\ldots,N$, and such that

$$|\zeta_\nu - \zeta_{\nu-1}| < |\zeta_\nu|.$$

We assume that $\phi_1(\zeta_\nu)$ is known and apply the transformation

$$\zeta = \zeta_\nu + \zeta' \tag{4.6.30}$$

to equation (4.6.1), which assumes the form

$$\mathfrak{R}_1\psi_\nu + \mathfrak{R}_2(\zeta_\nu + \zeta')\frac{d\psi_\nu}{d\zeta'} = F_\nu(\zeta') \tag{4.6.31}$$

where

$$\psi_\nu(\zeta') = \phi_1(\zeta) = \phi_1(\zeta_\nu + \zeta')$$

and

$$F_\nu(\zeta') = -(\alpha - \beta ika\zeta_\nu - \beta ika\zeta')e^{-ika(\zeta_\nu+\zeta')}.$$

We can write $$F_\nu(\zeta') = \sum_{n=0}^{\infty} F_{\nu,n}\zeta'^n$$

where

$$F_{\nu,n} = -e^{-ika\zeta_\nu}(\alpha - \beta ika\zeta_\nu + n\beta)\frac{(-ika)^n}{n!} \tag{4.6.32}$$

for all $n = 0,1,2,3,\ldots$. Assuming that (4.6.31) has a solution of the form

$$\psi_\nu(\zeta') = \sum_{n=0}^{\infty} a_{\nu,n}\zeta'^n, \tag{4.6.33}$$

substitution into (4.6.31) gives the recurrence relations

$$(\mathfrak{R}_1 + n\mathfrak{R}_2)a_{\nu,n} + (n+1)\mathfrak{R}_2\zeta_\nu a_{\nu,n+1} = F_{\nu,n},$$

true for all $n = 0,1,2,3,\ldots$. Since $\zeta_\nu \neq 0$ for $\nu = 1,\ldots,N-1$,

$$a_{\nu,n+1} = \frac{F_{\nu,n} - (\mathfrak{R}_1 + n\mathfrak{R}_2)a_{\nu,n}}{(n+1)\mathfrak{R}_2\zeta_\nu} \tag{4.6.34}$$

and if $n = 0$, we find that

$$a_{\nu,1} = \frac{F_{\nu,0} - \mathfrak{R}_1 a_{\nu,0}}{\mathfrak{R}_2\zeta_\nu}.$$

Since $a_{\nu,0} = \psi_\nu(0) = \phi_1(\zeta_\nu)$ is known the coefficients $a_{\nu,n}$, $n \geq 1$, are uniquely determined by (4.6.34). According to the theory of linear ordinary differential equations the solution (4.6.33) of (4.6.31) is uniformly absolutely convergent, the radius of convergence being $|\zeta_\nu|$, i.e. the distance of ζ_ν to the nearest singular point, in this case the point $\zeta = 0$. Since $|\zeta_{\nu+1} - \zeta_\nu| < |\zeta_\nu|$, we can use (4.6.33) to determine $\phi_1(\zeta_{\nu+1}) = \psi(\zeta_{\nu+1} - \zeta_\nu)$. Thus starting for instance at the specular point with suitable initial conditions the solution ϕ_1 is obtained in a finite number of steps. If the distance between successive points $\zeta_{\nu-1}$ and ζ_ν is small, say less than .01, then the error at every step is found to be of the order of 10^{-14} or less, and the resultant error is therefore of the order of $N \cdot 10^{-14}$.

Comparison of results obtained over a wide range of values of ka by each of the three programs mentioned show that their respective results are almost identical on the illuminated side, and good agreement exists on the shadow side. This is illustrated by amplitude curves corresponding to ka = 20 as computed by each of these programs and given in Figs. (4.6.4)-(4.6.6) for the three cases $(\alpha,\beta) = (1,0)$, $(1,i)$ and $(0,1)$.

The amplitude curves plotted in Figs. (4.6.7)-(4.6.11) were computed by program SFHS1A. In all cases considered the curves corresponding to $(\alpha,\beta) = (1,i)$ and $(0,1)$ are found to be almost identical in the illuminated region and in many cases good agreement exists in the shadow region. In cases where $(\alpha,\beta) = (1,0)$ and $v = \text{Im}(s) < 0$ the computations in the penumbral and shadow region were found to be unreliable. Finally we note that the oscillatory behaviour of s as a function of ka influences the form of the amplitude curves. Thus the first order amplitude curves for different values of ka will be nearly the same if the corresponding values of s are approximately equal.

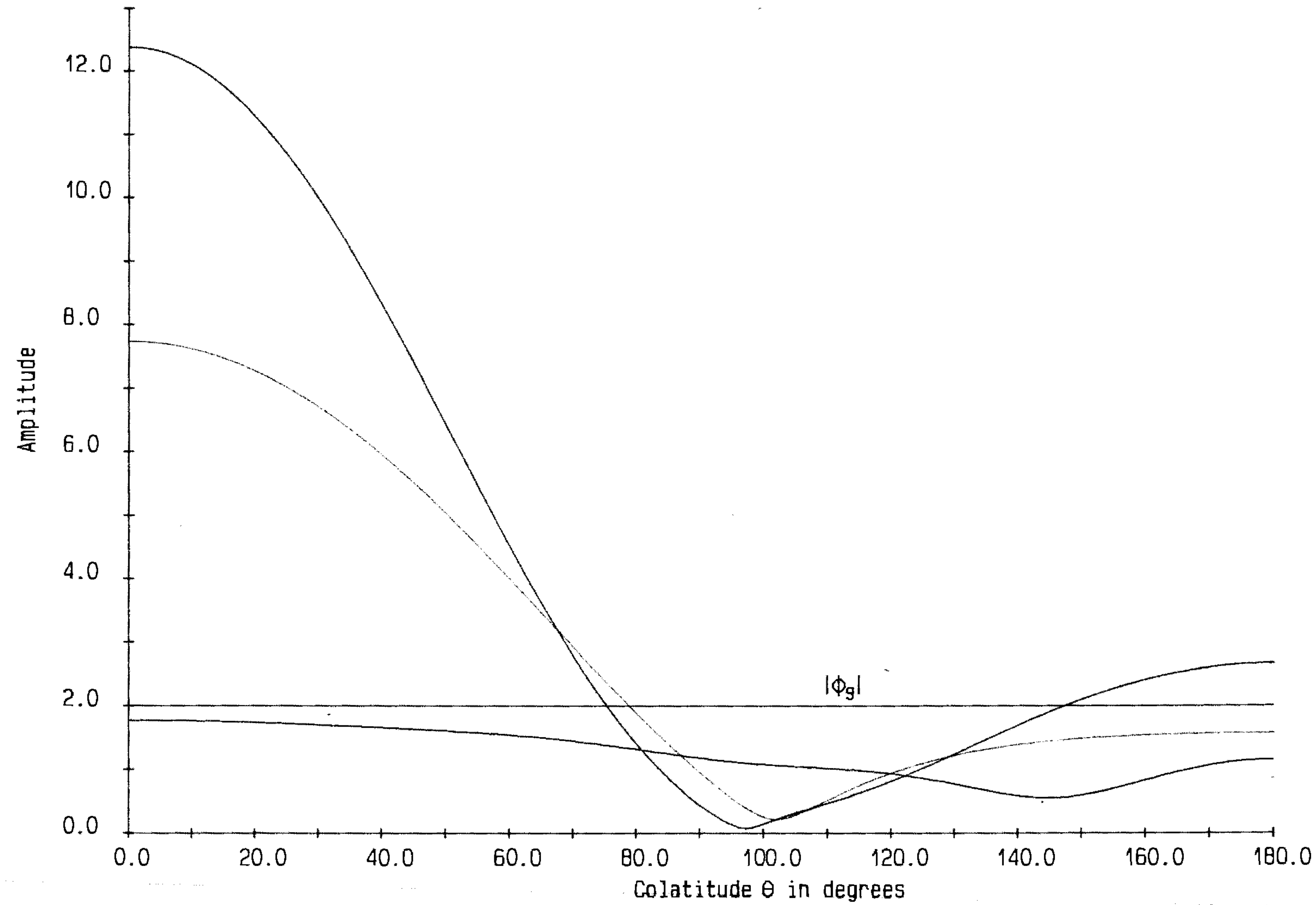


Fig. (4.6.1). Surface field amplitude (Black) for a sphere, ka = 3.0, and amplitudes of first order approximations as given by equation (4.6.7): (i) Blue: $\alpha = 1$, $\beta = 0$, (ii) Red: $\alpha = 1$, $\beta = i$ and (iii) Green: $\alpha = 0$, $\beta = 1$.

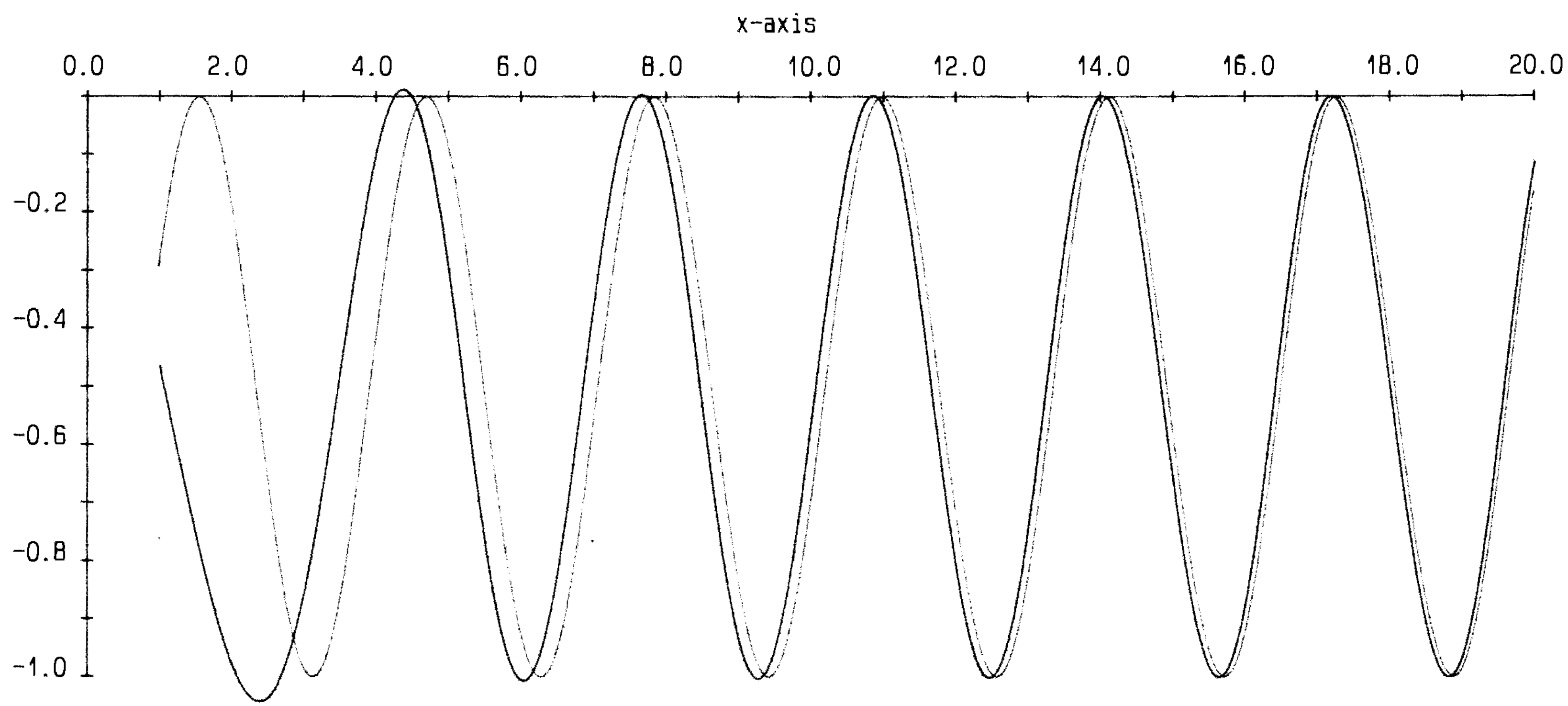


Fig. (4.6.2). Graphs of (i) $Rl(s_O)$ in Blue, (ii) $Rl(s_P)$ in Red and (iii) $Rl(s_Q)$ in Green, for $1 < x < 20$ where $x = ka$. The graphs of (ii) and (iii) are almost indentical if $x > 10$

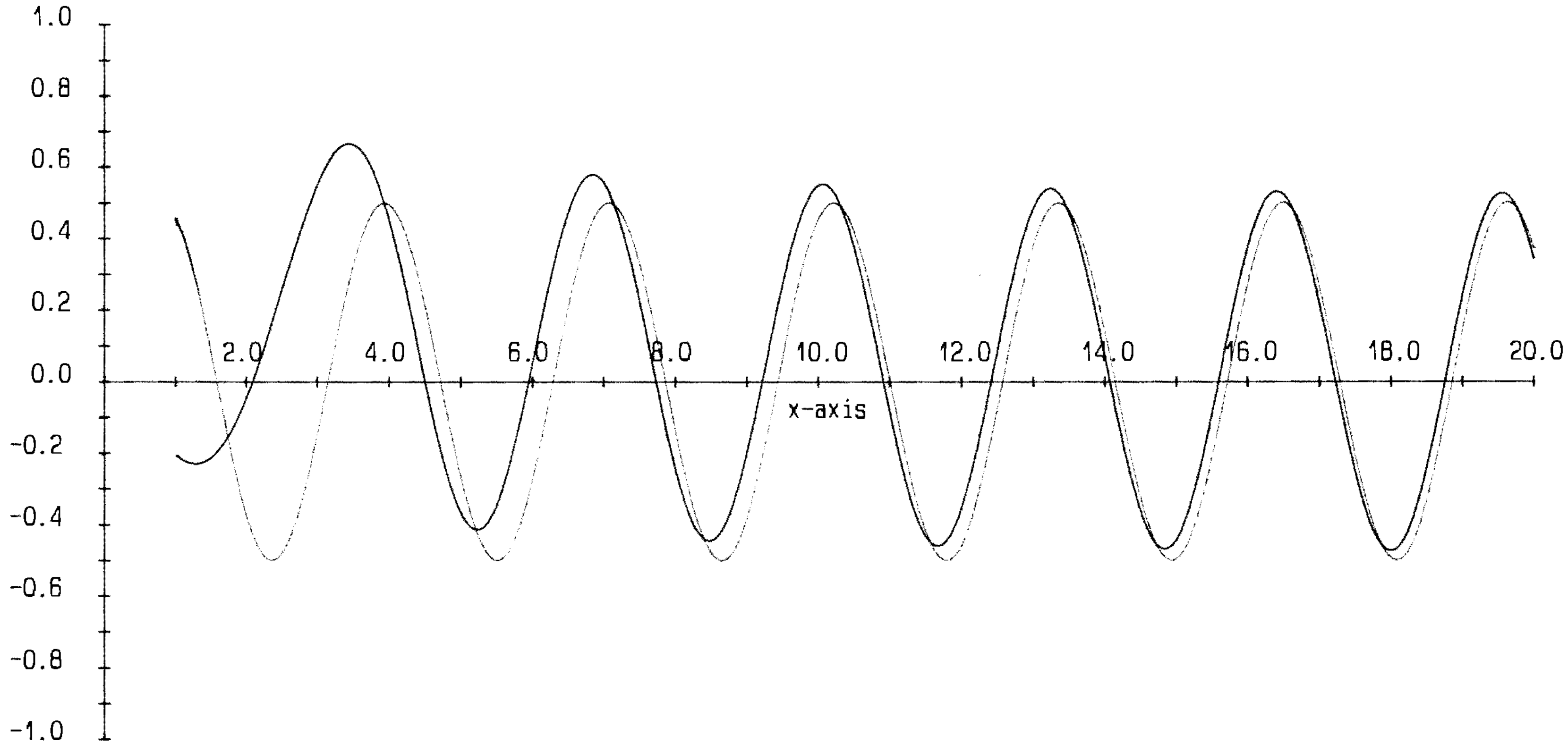


Fig. (4.6.3). Graphs of (i) $Im(s_0)$ in Blue, (ii) $Im(s_P)$ in Red and (iii) $Im(s_Q)$ in Green, for $1 < x < 20$ where $x = ka$. The graphs of (ii) and (iii) are almost indentical if $x > 10$.

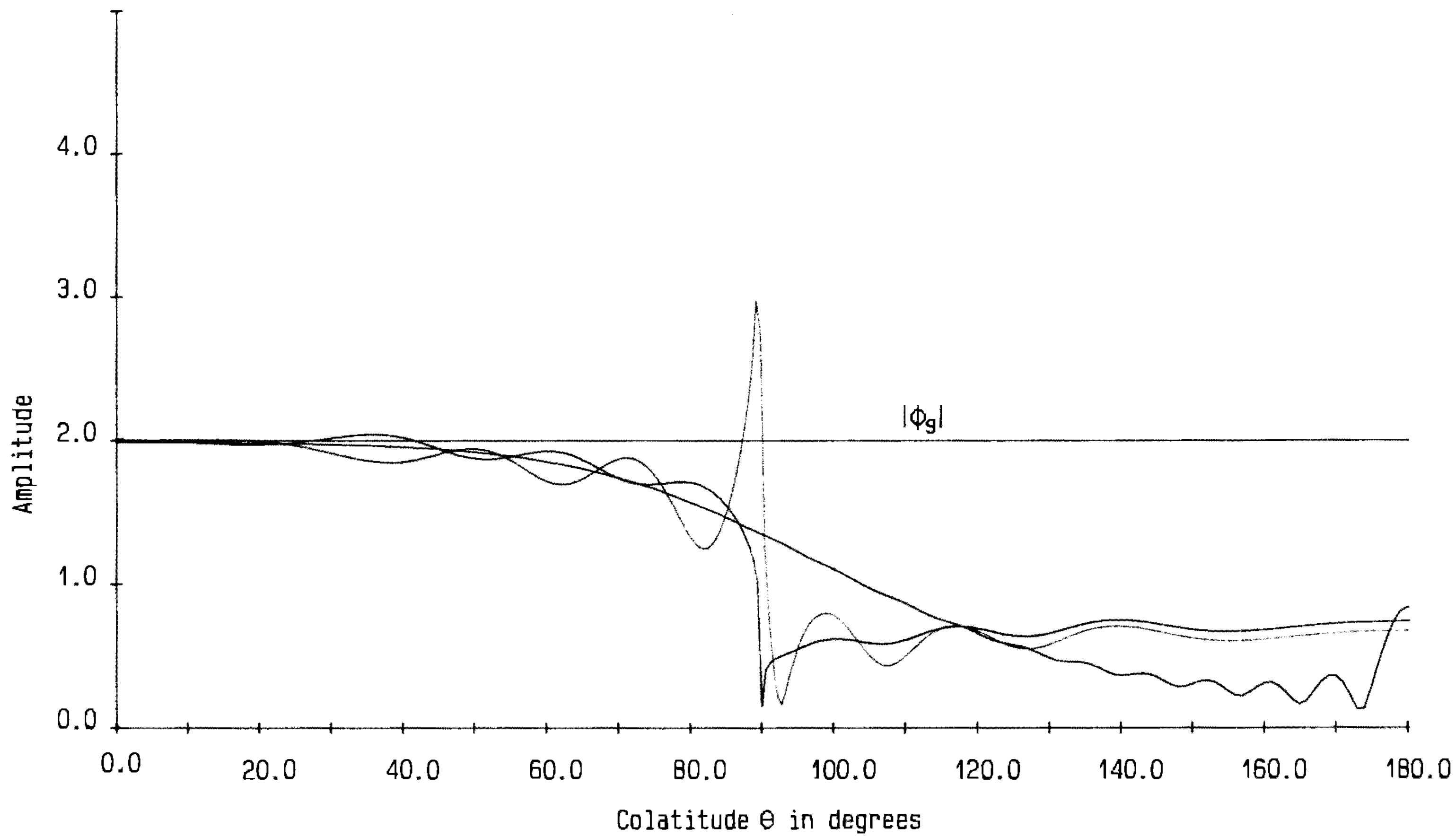


Fig. (4.6.4). Surface field amplitude (Black) for a sphere, ka = 20.0, and amplitudes of first order approximations computed by PROGRAM SFHS1A with δ = .01. (i) Blue: $\alpha = 1$, $\beta = 0$, u = −.1141, v = .344, (ii) Red: $\alpha = 1$, $\beta = i$, u = −.1235, v = .3421 and (iii) Green: $\alpha = 0$, $\beta = 1$, u = −.1231, v = .3422.

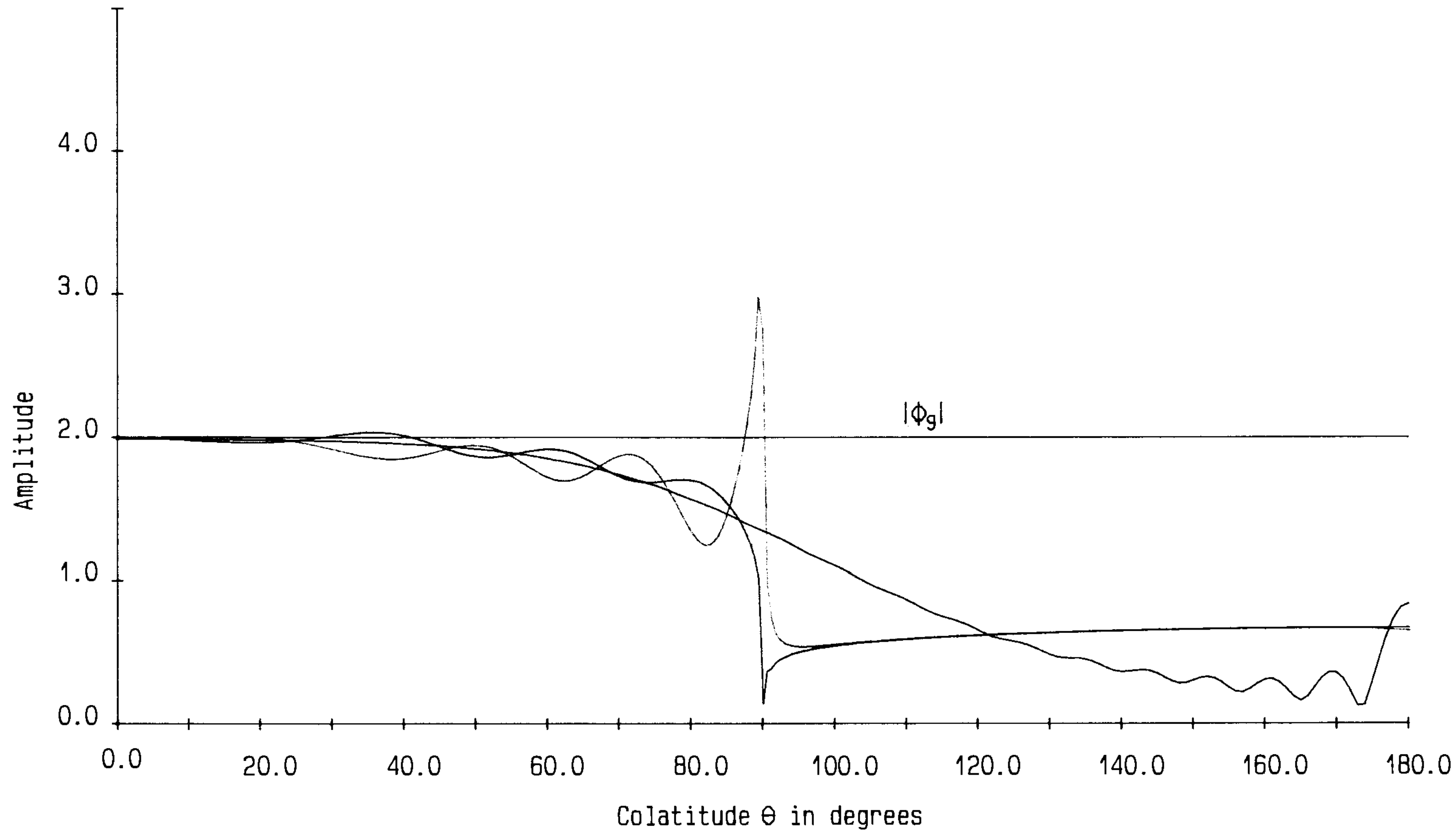


Fig. (4.6.5). Surface field amplitude (Black) for a sphere, ka = 20.0, and amplitudes of first order approximations computed by PROGRAM SFHS1B with δ=.01 and h=−1. (i) Blue: α = 1, β = 0, u=−.1141, v= .344, (ii) Red: α = 1, β = i, u = −.1235, v = .3421 and (iii) Green: α = 0, β = 1, u = −.1231, v = .3422.

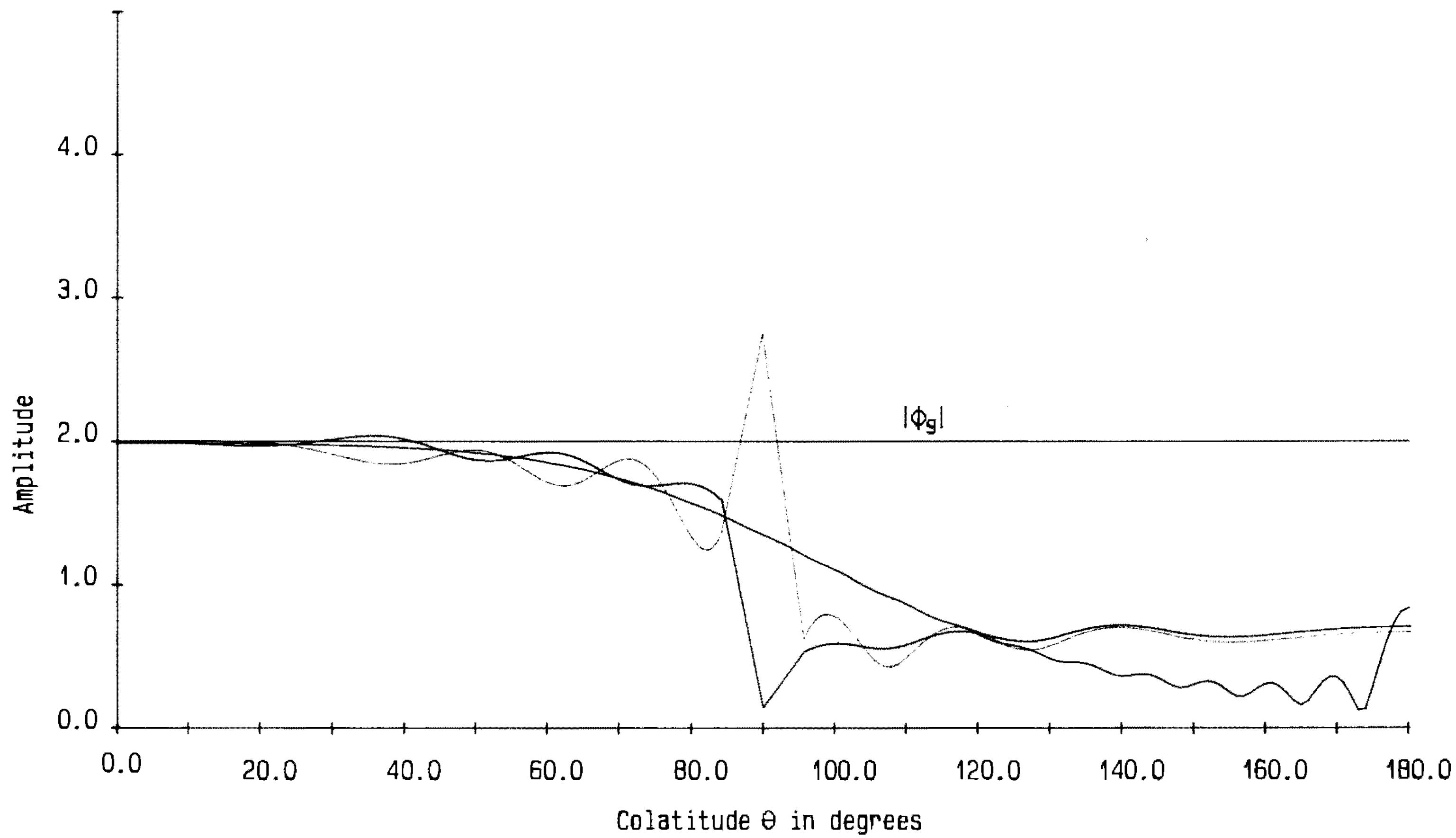


Fig. (4.6.6). Surface field amplitude (Black) for a sphere, ka = 20.0, and amplitudes of first order approximations computed by PROGRAM SFHS1C with δ=.1 and h=−1. (i) Blue: $\alpha = 1$, $\beta = 0$, u=−.1141, v= .344, (ii) Red: $\alpha = 1$, $\beta = i$, u = −.1235, v = .3421 and (iii) Green: $\alpha = 0$, $\beta = 1$, u = −.1231, v = .3422.

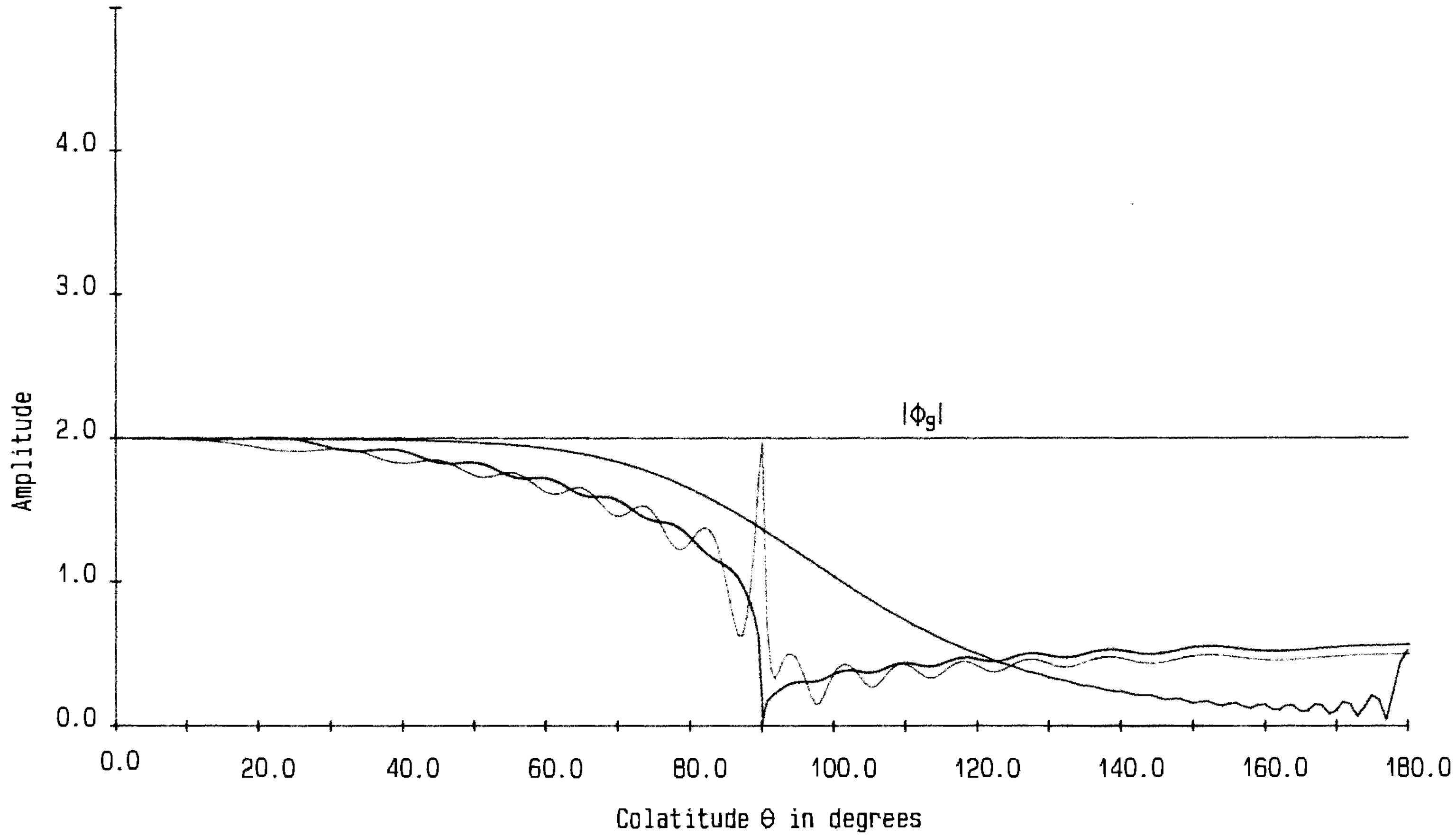


Fig. (4.6.7). Surface field amplitude (Black) for a sphere, ka = 45.0, and amplitudes of first order approximations computed by PROGRAM SFHS1A with δ = .01. (i) Blue: $\alpha = 1$, $\beta = 0$, u = −.2466, v = .4423, (ii) Red: $\alpha = 1$, $\beta = i$, u = −.2518, v = .4398 and (iii) Green: $\alpha = 0$, $\beta = 1$, u = −.2517, v = .4398.

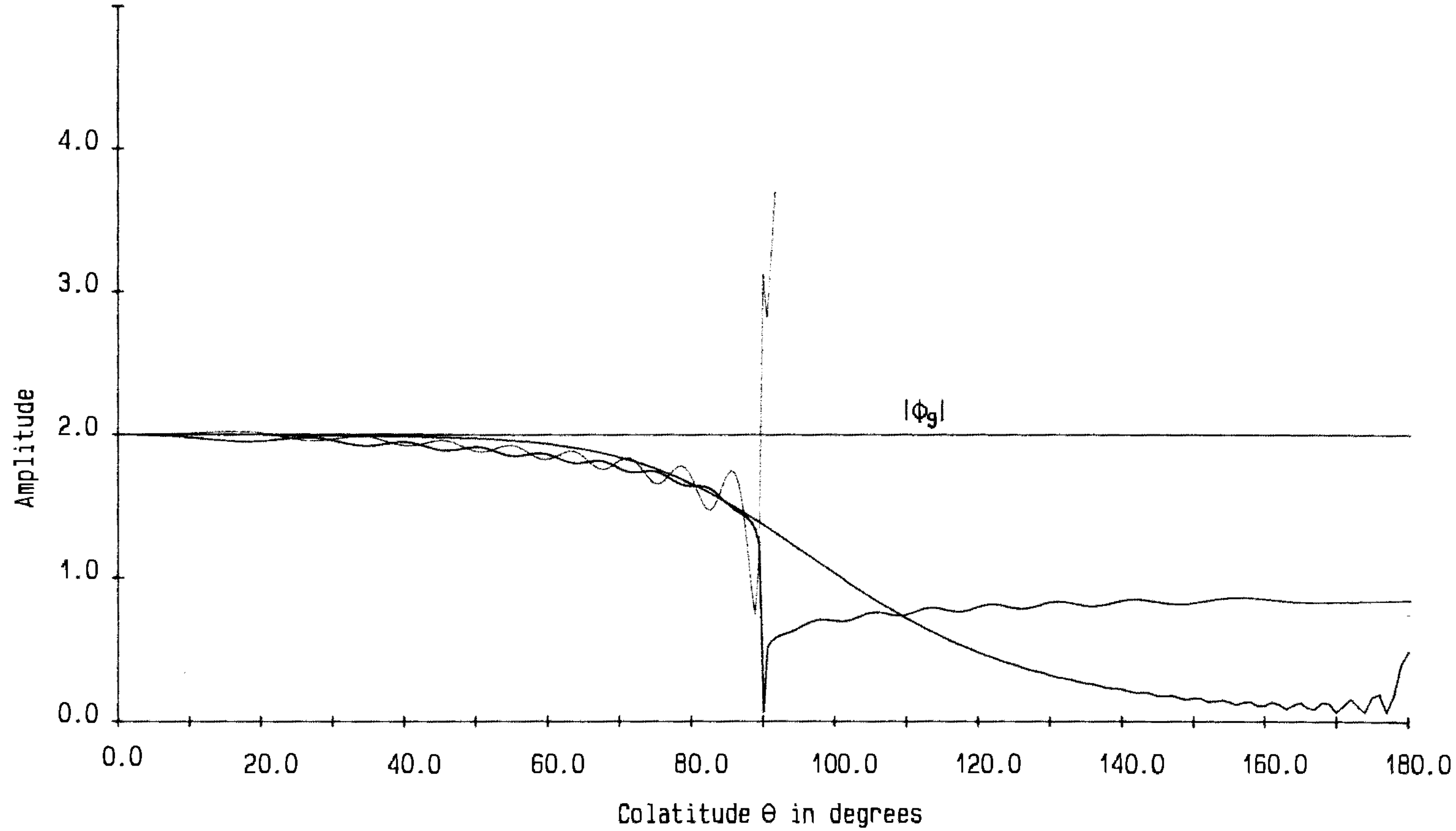


Fig. (4.6.8). Surface field amplitude (Black) for a sphere, ka = 49.0, and amplitudes of first order approximations computed by PROGRAM SFHS1A with δ = .01. (i) Blue: $\alpha = 1$, $\beta = 0$, u = −.1086, v = −.3011, (ii) Red: $\alpha = 1$, $\beta = i$, u = −.1055, v = −.3024 and (iii) Green: $\alpha = 0$, $\beta = 1$, u = −.1056, v = −.3023.

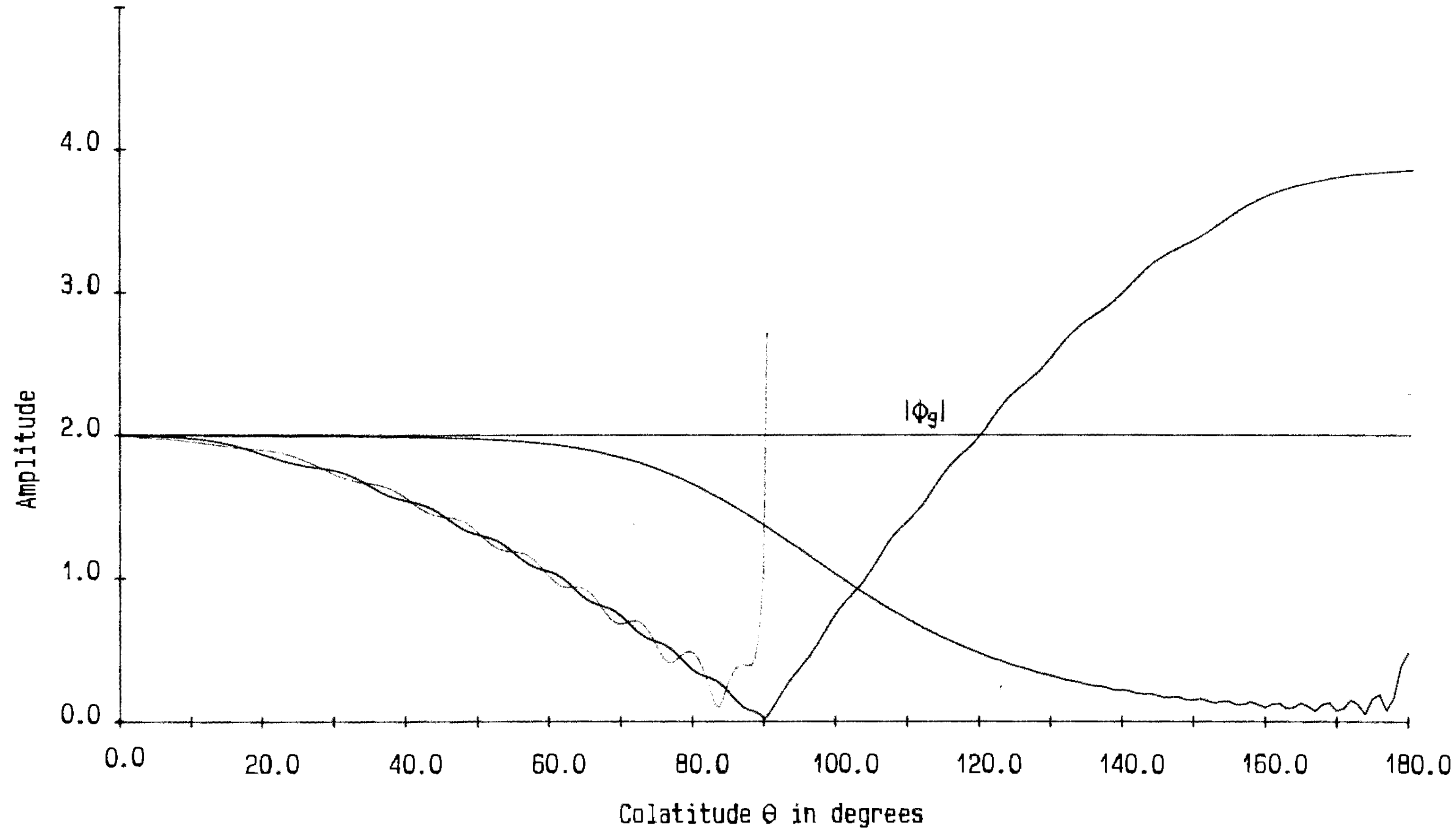


Fig. (4.6.9). Surface field amplitude (Black) for a sphere, ka = 50.0, and amplitudes of first order approximations computed by PROGRAM SFHS1A with δ = .01. (i) Blue: α = 1, β = 0, u = −.9457, v = −.2169, (ii) Red: α = 1, β = i, u = −.9439, v = −.2267 and (iii) Green: α = 0, β = 1, u = −.944, v = −.2265.

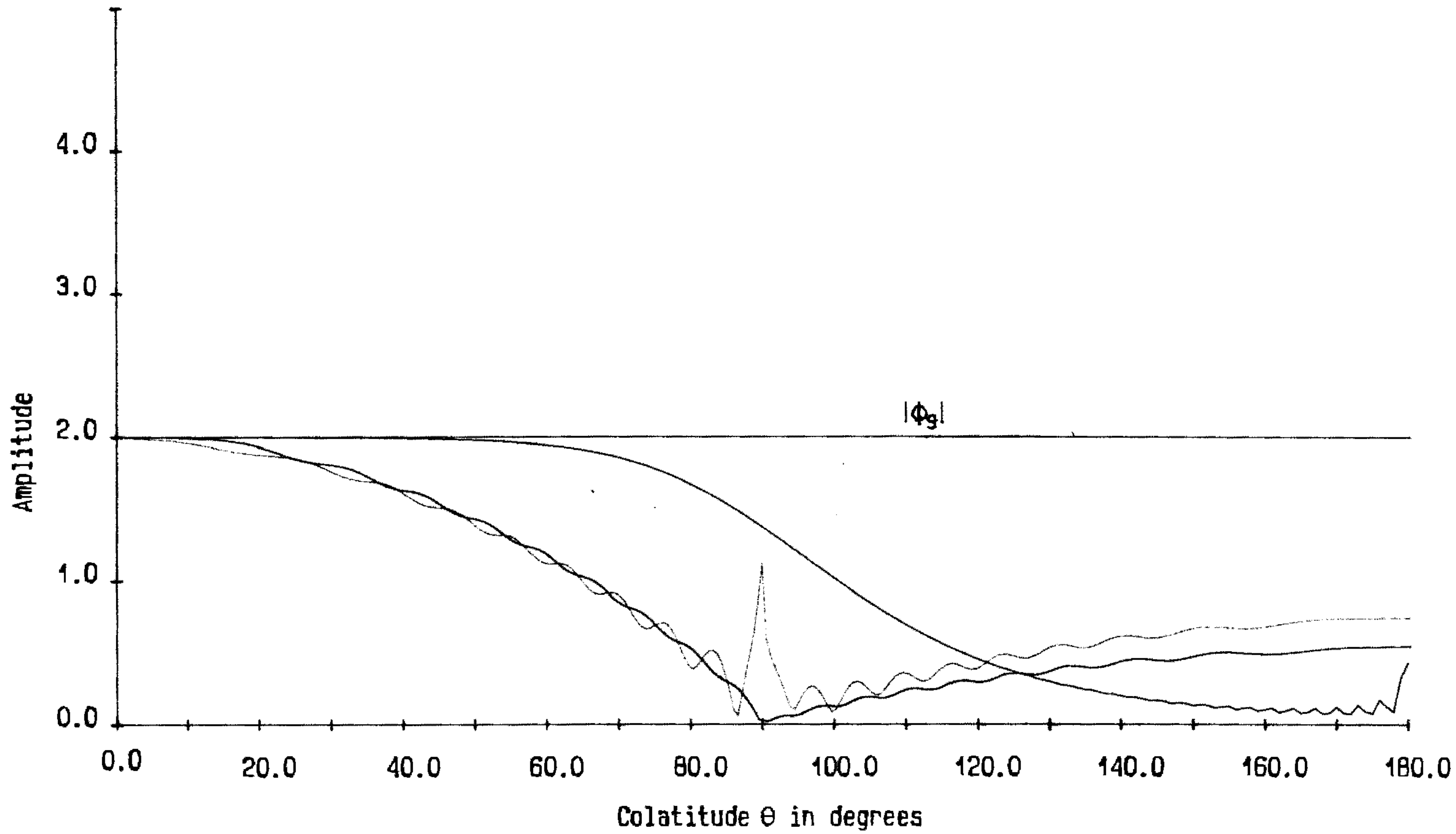


Fig. (4.6.10). Surface field amplitude (Black) for a sphere, ka = 57.0, and amplitudes of first order approximations computed by PROGRAM SFHS1A with δ = .01. (i) Blue: α = 1, β = 0, u = −.7887, v = .4171, (ii) Red: α = 1, β = i, u = −.7928, v = .4102 and (iii) Green: α = 0, β = 1, u = −.7927, v = .4103.

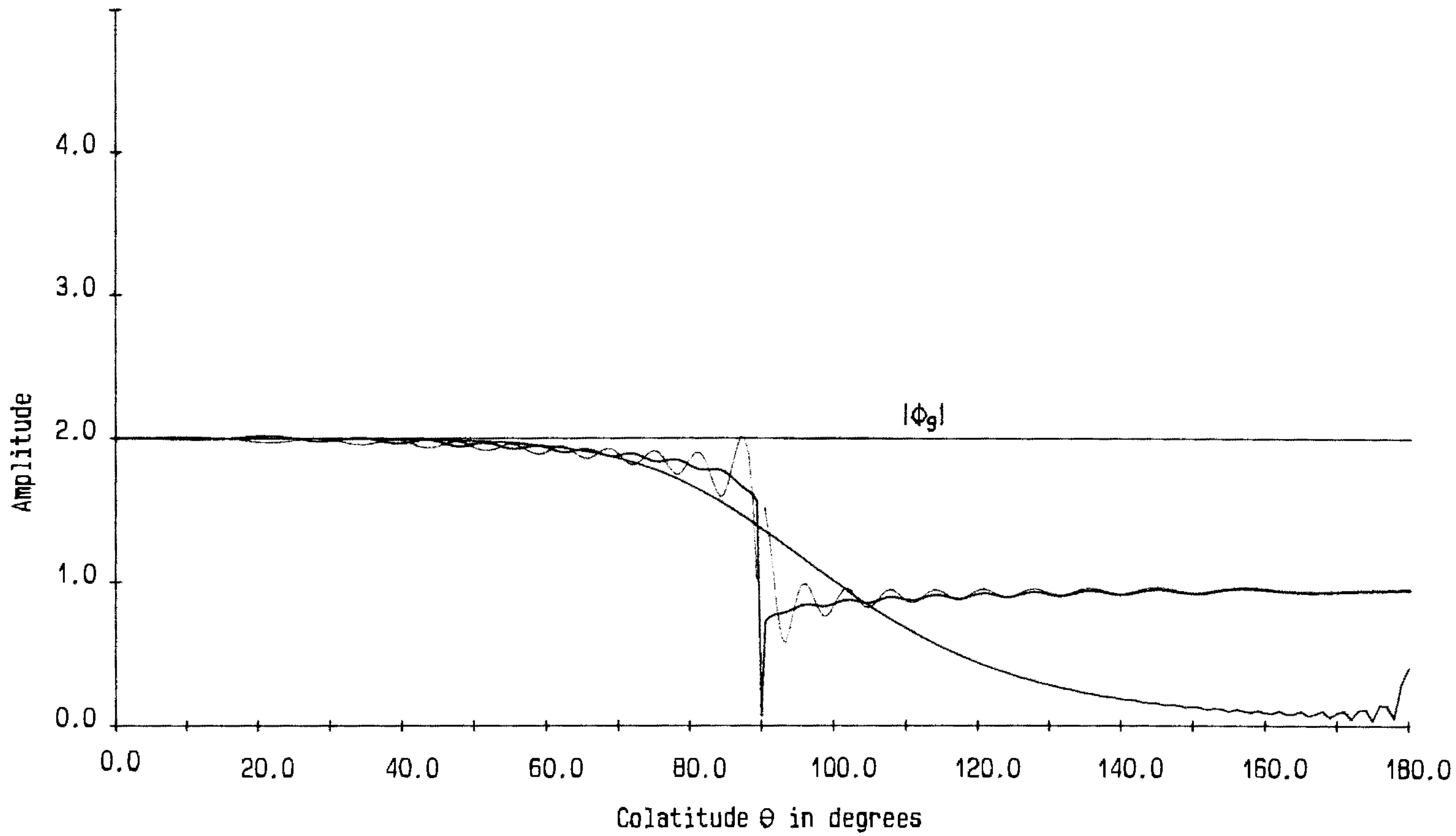


Fig. (4.6.11). Surface field amplitude (Black) for a sphere, ka = 61.0, and amplitudes of first order approximations computed by PROGRAM SFHS1A with δ = .01. (i) Blue: α = 1, β = 0, u = −.0548, v = .236, (ii) Red: α = 1, β = i, u = −.0568, v = .2356 and (iii) Green: α = 0, β = 1, u = −.0568, v = .2356.

4.7 First order local approximations for a sphere.

Let P_0, with position vector $r_0 = OP_0$, be a point on the surface of a sphere D of radius a and centre O, and let S be a surface element containing P_0 and contained in ∂D. We assume that the boundary curve C of S is a circle with centre on the normal at $P_0 \in \partial D$, and ε is the altitude of C with respect to tangent-normal axes at P_0. Let r be an arbitrary point in S with colatitude θ with respect to Cartesian axes $Ox_1x_2x_3$ at the centre of the sphere, and let $\zeta = \cos\theta$. With reference to sections (4.3) and (4.4) we find that the differential equation for the local approximation of the surface field on S is given by

$$\mathfrak{R}_1\phi + \mathfrak{R}_2\zeta\,\frac{d\phi}{d\zeta} = f(\zeta), \tag{4.7.1}$$

where $\qquad f(\zeta) = -\,(\alpha-\beta ika\zeta)e^{-ika\zeta}$.

We recall from sections (4.3) and (4.4) that the coeffcients $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are constants depending only on the radius a and ε, and we assume that ε is so small that powers of ε^3 and higher can be neglected. Thus from Appendix A we find that

$$\mathfrak{P}_1 = -\tfrac{1}{2}\,(1 + 3\varepsilon + 3h\varepsilon^2),$$

$$\mathfrak{P}_2 = 0$$

$$\mathfrak{Q}_1 = \tfrac{1}{2}\left(\varepsilon^{-1} + 1 - (2 - \tfrac{1}{2}h^2)\,\varepsilon + (-\tfrac{1}{2}\,h^2 + \tfrac{1}{6}\,h^3)\varepsilon^2\right)$$

and $\qquad \mathfrak{Q}_2 = -\tfrac{1}{2}\,\varepsilon^2$,

from which we can obtain

$$\mathfrak{R}_1 = \alpha\mathfrak{P}_1 + \beta\mathfrak{Q}_1$$

and

$$\mathfrak{R}_2 = \alpha\mathfrak{P}_2 + \beta\mathfrak{Q}_2\,.$$

Here we briefly investigate the nature of the singular solution given by equation (4.6.7).

- **Case 1.** If $\alpha = 1$ and $\beta = 0$ then

$$\mathfrak{R}_1 = \mathfrak{P}_1 = -\tfrac{1}{2}\,(1 + 3\varepsilon + 3h\varepsilon^2),$$

$$\mathfrak{R}_2 = \mathfrak{P}_2 = 0,$$

and hence the singular solution of equation (4.7.1) is given by

$$\phi_{1,1}(\zeta) = -\frac{1}{\mathfrak{P}_1}\sum_{n=0}^{\infty}\frac{(-ika)^n}{n!}\,\zeta^n$$

or
$$\phi_{1,1}(\zeta) = -\frac{e^{-ika\zeta}}{\mathcal{P}_1} = -\frac{\phi_i(\zeta)}{\mathcal{P}_1},$$

and $\mathcal{P}_1$ is not zero provided ε is real. For the amplitude we have

$$|\phi_{1,1}(\zeta)| = \frac{2}{(1 + 3\varepsilon + 3h\varepsilon^2)} \leq 2,$$

since the expression $1 + 3\varepsilon + 3h\varepsilon^2$ is always positive. In the limit when $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon \to 0} \phi_{1,1}(\zeta) = 2\phi_i(\zeta) = \phi_g(\zeta).$$

• **Case 2.** If $\alpha = 0$ and $\beta = 1$ then

$$\mathcal{R}_1 = \mathcal{Q}_1,$$
$$\mathcal{R}_2 = \mathcal{Q}_2,$$

and according to equation (4.6.7) the corresponding first order local approximation is given by

$$\phi_{1,2}(\zeta) = -\sum_{n=0}^{\infty} \frac{n}{(\mathcal{Q}_1 + n\mathcal{Q}_2)} \frac{(-ika)^n}{n!} \zeta^n$$

$$= ika\zeta \sum_{n=1}^{\infty} \frac{1}{(\mathcal{Q}_1 + n\mathcal{Q}_2)} \frac{(-ika)^n}{n!} \zeta^n.$$

Since
$$\lim_{\varepsilon \to 0} \frac{1}{(\mathcal{Q}_1 + n\mathcal{Q}_2)} = 0$$

for all $n = 0, 1, 2, \ldots$, it follows that
$$\lim_{\varepsilon \to 0} \phi_{1,2}(\zeta) = 0.$$

• **Case 3.** If $\alpha = 1$ and $\beta = i$ then the corresponding first order local approximation as given by equation (4.6.7) is

$$\phi_{1,3}(\zeta) = -\sum_{n=0}^{\infty} \frac{(1 + in)}{(\mathcal{R}_1 + n\mathcal{R}_2)} \frac{(-ika)^n}{n!} \zeta^n$$

where
$$\mathcal{R}_1 = \mathcal{P}_1 + i\mathcal{Q}_1$$
and
$$\mathcal{R}_2 = \mathcal{P}_2 + i\mathcal{Q}_2.$$

Here too we find that

$$\lim_{\varepsilon \to 0} \phi_{1,3}(\zeta) = 0.$$

Amplitude curves of first order local approximations, as computed by PROGRAM SFHS1L, are given in Figs. (4.7.1) and (4.7.2) for the cases where ka is respectively equal to 20 and 30 and $\varepsilon = 1/ka$. In general these curves are similar in form to the corresponding zero order approximations and reduce to the same limiting forms when $\varepsilon \to 0$. For values of ka larger than 35 the computations are not reliable on the hardware used.

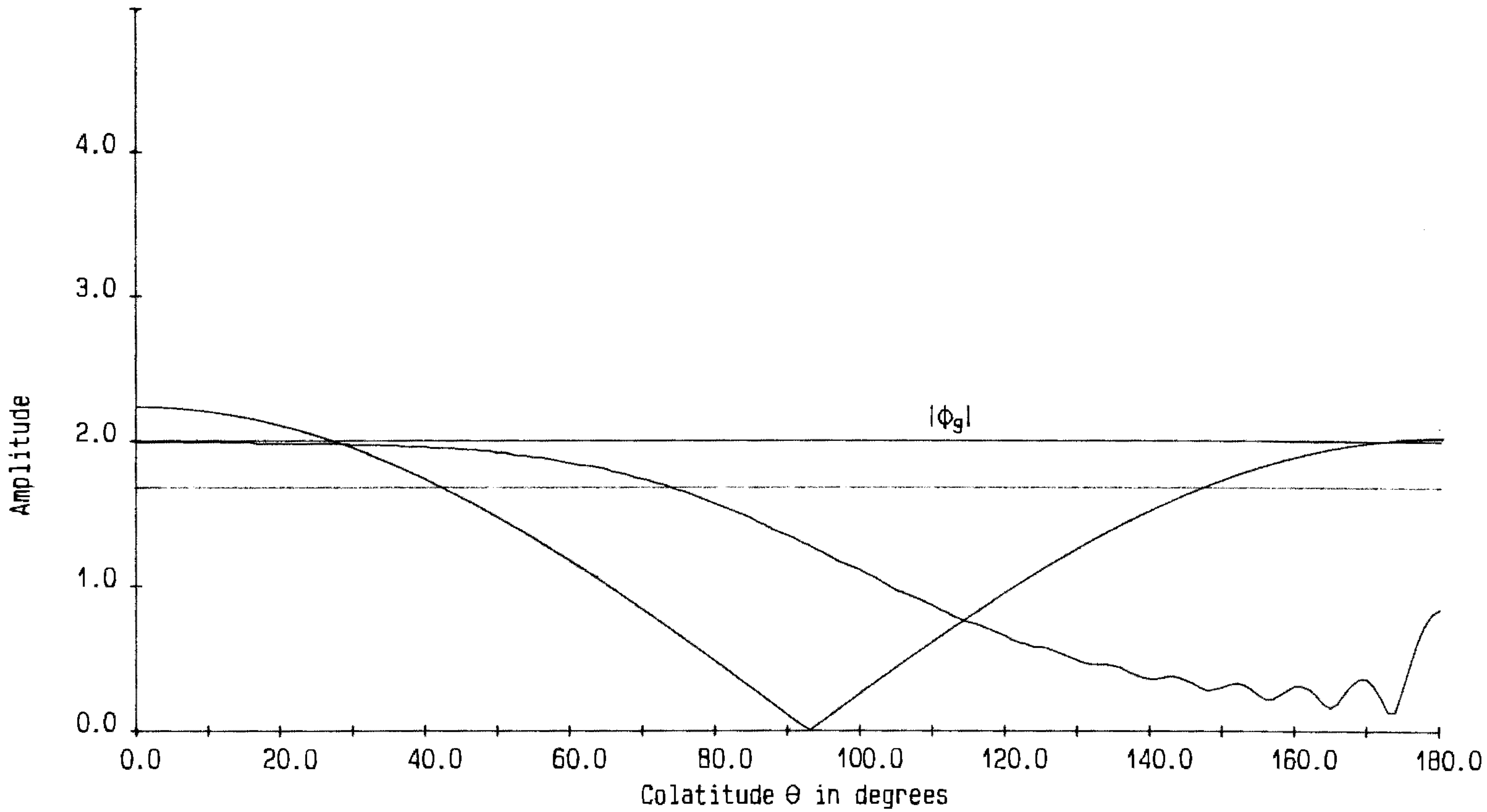


Fig. (4.7.1). Surface field amplitude (Black) for a sphere, ka = 20.0, and amplitudes of first order local approximations as given by equation (4.6.7): (i) Blue: $\alpha = 1$, $\beta = 0$, (ii) Red: $\alpha = 1$, $\beta = i$ and (iii) Green: $\alpha = 0$, $\beta = 1$.

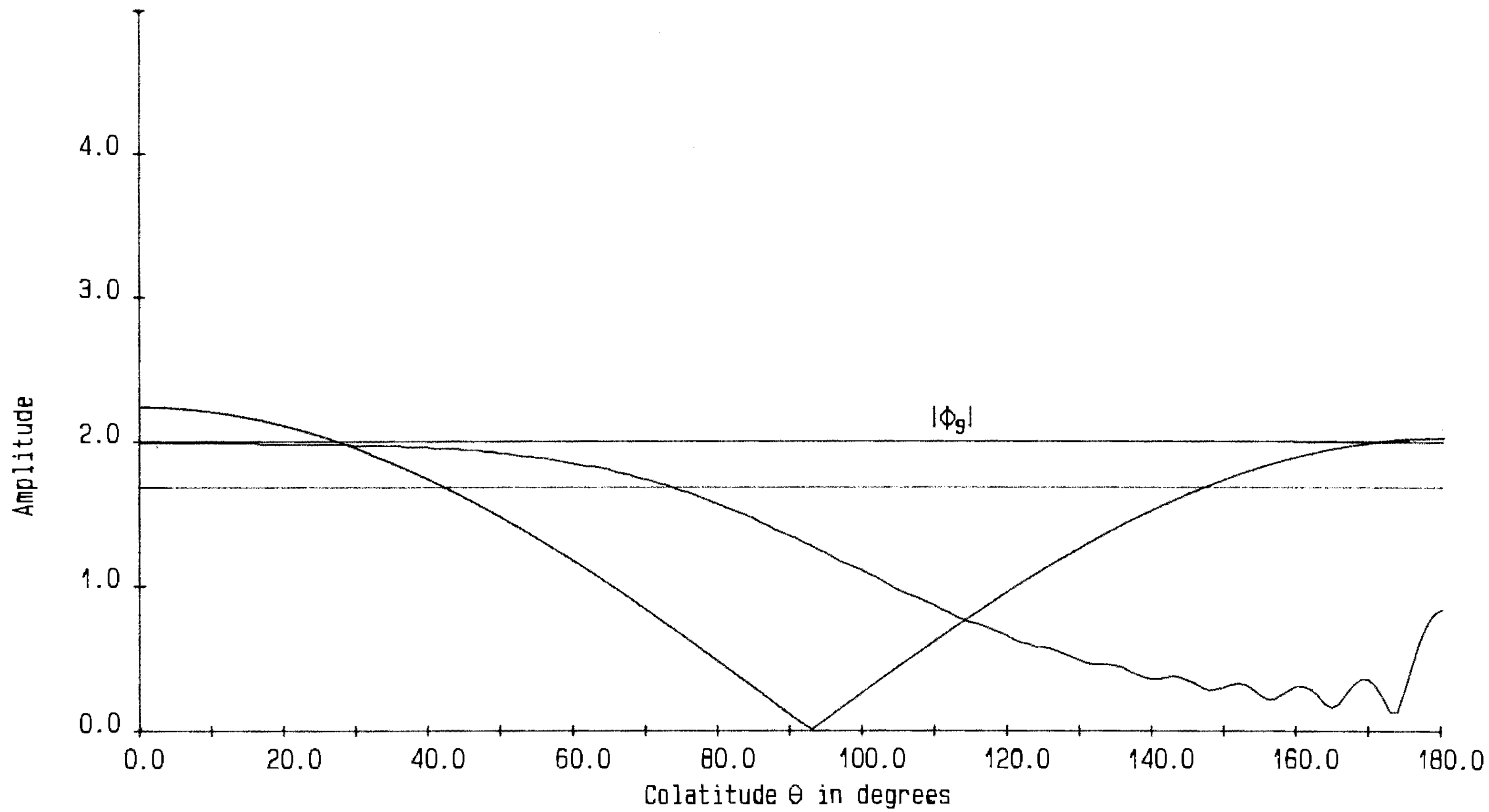


Fig. (4.7.2). Surface field amplitude (Black) for a sphere, ka = 30.0, and amplitudes of first order local approximations as given by equation (4.6.7): (i) Blue: $\alpha = 1$, $\beta = 0$, (ii) Red: $\alpha = 1$, $\beta = i$ and (iii) Green: $\alpha = 0$, $\beta = 1$.

CHAPTER V

SUMMARY AND CONCLUSION

In Chapter II we introduced spherical or azimuth-altitude coordinates at an arbitrary point of a convex surface and discussed the basic properties of such coordinate systems. The main advantage of these coordinates is that weakly singular surface integrals are regularised when such integrals are parameterised in terms of these coordinates. However, the use of such coordinates is resticted to convex surfaces not containing any lines of zero curvature, and thus any type of convex ruled surface is excluded. The use of azimuth-altitude coordinates is specifically aimed at the regularisation of the Helmholtz surface potentials, which of course include the Laplace surface potentials as special cases. The interior and exterior boundary value problems for the Helmholtz equation were discussed in some detail in Chapter III, and boundary integral equations for the surface fields of such problems were obtained, the main emphasis being on the exterior scattering problem. The fundamental result concerning this problem is that there exists a pair of boundary integral equations which always have a unique solution for the surface field, and that this pair of equations is equivalent to a linear combinations of these two equations, provided that the coupling constants α and β satisfy the condition $\mathrm{Im}(\alpha\bar{\beta}) \neq 0$. The surface integrals in these integral equations are Helmholtz surface potentials and can therefore be regularised by means of azimuth-altitude coordinates.

Chapter IV is concerned with the construction of analytical approximations of the boundary integral equations mention above. The essential assumption underlying the construction of such analytical approximations is that the integral equation defining the surface field has a unique analytical solution. This assumption permits the unknown function representing the surface field to be expanded in a Taylor series, and results in a partial differential equation of infinite order and with analytical coefficients. On the other hand, if the Taylor expansion is truncated then partial differential equations of finite orders are obtained. The solutions of these truncated differential equations are referred to as analytical approximations of the surface field.

The work of Chapter IV rests on the assumption that a sequence of analytical approximations exists which converge to the surface field. In section (4.2) it was shown that if a convergent sequence of analytical approximations exists, then their limit is necessarily the surface field. A disadvantage of using a differential equation as an approximation to the surface field is that suitable initial conditions or boundary conditions are required to obtain unique solutions. In the case where the incident radiation has a high frequency we have assumed that the surface field at the specular point is twice the incident field at this point. However, if the incident radiation is in the low frequency range, then such an assumption is unwarrented, and a different method for obtaining unique approximations is required. Another method for obtaining unique solutions for the differential equations defining the analytical approximations is to use the value predicted by the zero order approximation at the specular point as initial condition for the first order approximation, and so on for higher order approximations. But as was shown in the case of the sphere in section (4.5) the oscillatory behaviour of the zero order coefficient precludes the usefulness of such an approach.

As mentioned in section (4.2) an alternative method for obtaining approximations of a given order is first to obtain the general solution of the differential equations corresponding to the two integral equations for the problem. The arbitrary constants or functions in these general solutions can then be determined by requiring that the norm of their difference is a minimum. Howevere, apart from the possibly difficult existence problem involved here, the problem lacks physical content when formulated in this way. Nevertheless, in the absence of better means of obtaining unique solutions, this method may deserve some attention.

It was shown in section (4.5) that in the case of the sphere the solutions of the differential equation for the first order approximation possess a reasonable measure of stabilty as regards the choice of coupling constants α and β provided ka is sufficiently large, where a is the radius of the sphere and k is the wave number. We note, however, that this partial stability of the first order approximations is not dependent on the condition that $\mathrm{Im}(\alpha\bar{\beta}) \neq 0$, as is required in the case of the related integral equation. Preliminary investigation shows that a similar behaviour may be expected for higher order approximations. Thus for the second order approximation for the sphere we have the equation

$$\mathfrak{R}_1\phi + \mathfrak{R}_2\zeta \frac{d\phi}{d\zeta} + \left(\mathfrak{R}_3 - \mathfrak{R}_4\zeta^2\right)\frac{d^2\phi}{d\zeta^2} = f(\zeta)$$

where $$f(\zeta) = -(\alpha-\beta ika\zeta)e^{-ika\zeta}\ .$$

The corresponding homogeneous equation is

$$\frac{d^2\phi}{d\zeta^2} + \frac{\mathfrak{R}_2\zeta}{(\mathfrak{R}_3 - \mathfrak{R}_4\zeta^2)}\frac{d\phi}{d\zeta} + \frac{\mathfrak{R}_1}{(\mathfrak{R}_3 - \mathfrak{R}_4\zeta^2)}\phi = 0.$$

This equation has two regular singular points when

$$\mathfrak{R}_3 - \mathfrak{R}_4\zeta^2 = 0,$$

and an irregular singular point at infinity. Writing

$$s_1 = \frac{\mathfrak{R}_1}{\mathfrak{R}_4}\ ,\ s_2 = \frac{\mathfrak{R}_2}{\mathfrak{R}_4}\ \text{ and } s^2 = \frac{\mathfrak{R}_3}{\mathfrak{R}_4}$$

the homogeneous equation has the form

$$\frac{d^2\phi}{d\zeta^2} + \frac{s_2\zeta}{(s^2-\zeta^2)}\frac{d\phi}{d\zeta} + \frac{s_1}{(s^2-\zeta^2)}\phi = 0.$$

The singular points are $\zeta = \pm s$ and it is found that s is small but never zero and that $s \to 0$ when $ka \to \infty$. The values of s_1 and s_2 are found to be nearly independent of the choice of coupling constants α and β. In contrast to the first order equations, the second order equations have an infinite number of solutions which are analytical in a region containing the real axis between -1 and 1 and excluding the singular points. As there are two singular points on either side of the real axes, there are three homotopy classes of curves with initial point 1 and end point -1 . Thus given a set of initial conditions at the point $\zeta = 1$ correspnding to the specular point, essentially three disticnt analytical solutions can be obtained by analytical continuation along representative members of the three homotopy classes. Because of the close proximity of the singular points to the origin, the process of continuation along the real axis from 1 to -1 must proceed in small incements, and a marked deviation of the solution from the surface field predicted in the shadow region is observed. We may therefore expect that such solutions will agree reasonably well with the predicted surface field only on the illuminated side of the surface. However, solutions obtained along a member of one of the other two remaining homotopy classes appear to be improvements on the first order approximations. The method of analytical continuation can also be used to obtain a general solution of the second order equation, and such solutions can be used as local approximations at any point of the surface. We also note that when $ka \to \infty$ a limiting form of the second order equation is obtained which has a unique solution analytical at all points of the complex plane.

For the second order equation we can also construct solutions of the form ζ'^{λ} $F(\zeta')$ for the homogeneous equation, where $\zeta' = \zeta - s$ and F is analytical in a disc not containing the other singular point. Using variation of parameters the general solution of the inhomogeneous equation can be found in a neighbourhood of $\zeta = s$, and this solution can then be continued so as to include the real interval [-1,1] and matched to suitable initial conditions at the specular point. Such a solution may have certain advantages as it partly involves an integral representation and is constructed from fewer elements.

On the other hand, for arbitrary convex bodies the differential equations involve partial derivatives, and for such equations solutions obtained by analytical continuation in small increments is probably the most feasible line of approach. For a general convex body D the approximations will of necessity be local. For example the distance R between two points P_0 and P on the closed surface ∂D can be expanded in terms of some suitable surface parameter, such as the eccentricity in the case of a spheroid. Thus relative to the point P_0 the surface ∂D is replaced by some other closed surface or surface element peculiar to P_0.

APPENDIX A

The integrals (4.3.11)-(4.3.14) for the coefficients $A_{j_1\ldots j_n}$, $B_{j_1\ldots j_n}$, $C_{j_1\ldots j_n}$ and $D_{j_1\ldots j_n}$ are linear combinations of integrals of the form (see. p. 4-21)

$$\int_0^{2\pi}\int_0^{\varepsilon} R^m e^{ikR} \cos^p\varphi' \cos^{p+1}\chi' \sin^q\chi' \, d\varphi' d\chi' ,$$

where p and q are non-negative integers, and $0 \le \varepsilon \le \frac{\pi}{2}$.

For a surface element S on a sphere of radius a (see section (4.3))

$$R = 2a\sin\chi', \quad 0 \le \chi' \le \varepsilon , \qquad (A.1)$$

and as R is independent of φ' the integral can be written as a repeated integral

$$\int_0^{2\pi} \cos^p\varphi' \, d\varphi' \int_0^{\varepsilon} R^m e^{ikR} \cos^{p+1}\chi' \sin^q\chi' \, d\chi' ,$$

where

$$\int_0^{2\pi} \cos^p\varphi' \, d\varphi' = \left\{ \begin{array}{ll} 0 & \text{if p is odd} \\ \dfrac{(p-1)(p-3)\ldots5.3.1}{p(p-2)\ldots6.4.2}\, 2\pi & \text{if p is even} \end{array} \right\} \qquad (A.2)$$

Now put

$$h = 2ika; \qquad (A.3)$$

then

$$\int_0^{\varepsilon} R^m e^{ikR} \cos^{p+1}\chi' \sin^q\chi' \, d\chi' = (2a)^m \, J(m,p,q) \qquad (A.4)$$

where

$$J(m,p,q) = \int_0^{\varepsilon} e^{h\sin\chi'} \cos^{p+1}\chi' \sin^{m+q}\chi' \, d\chi' , \qquad (A.5)$$

and

$$\int_0^{\varepsilon} R^m (ikR - 1) e^{ikR} \cos^{p+1}\chi' \sin^q\chi' \, d\chi' = (2a)^m \, K(m,p,q) \qquad (A.6)$$

where

$$K(m,p,q) = \int_0^{\varepsilon} (h\sin\chi' - 1) e^{h\sin\chi'} \cos^{p+1}\chi' \sin^{m+q}\chi' \, d\chi' . \qquad (A.7)$$

Now consider

$$I(p,q) = \int_0^{\varepsilon} e^{ikR} \cos^{p+1}\chi' \sin^q\chi' \, d\chi'$$

$$= \int_0^{\varepsilon} e^{h\sin\chi'} \cos^{p+1}\chi' \sin^q\chi' \, d\chi' , \qquad (A.8)$$

then
$$I(p,0) = \int_0^{\varepsilon} e^{h \sin \chi'} \cos^{p+1} \chi' \, d\chi',$$

and it follows that
$$I(p,q) = \frac{\partial^q I(p,0)}{\partial h^q} = \partial_h^q \, I(p,0)$$

where the operator ∂_h denotes partial differentiation with respect to h. From (A.5) and (A.7) we also have that
$$J(m,p,q) = \partial_h^{m+q} \, I(p,0) \tag{A.9}$$

and
$$K(m,p,q) = hJ(m+1,p,q) - J(m,p,q)$$

$$= h \, \partial_h^{m+q+1} \, I(p,0) - \partial_h^{m+q} \, I(p,0)$$

$$= (h \, \partial_h^{m+q+1} - \partial_h^{m+q}) \, I(p,0). \tag{A.10}$$

According to (A.2) we need only evaluate the integral I(p,0) when p is even. Replacing p by 2p we obtain
$$I(2p,0) = \int_0^{\varepsilon} e^{h \sin \chi'} \cos^{2p+1} \chi' \, d\chi' \; .$$

$$= \int_0^{\varepsilon} e^{h \sin \chi'} (1 - \sin^2 \chi')^p \cos\chi' \, d\chi' \; .$$

$$= \sum_{r=0}^{p} \binom{p}{r} (-1)^r \int_0^{\varepsilon} e^{h \sin \chi'} \sin^{2r} \chi' \cos\chi' \, d\chi' \; .$$

$$= \sum_{r=0}^{p} \binom{p}{r} (-1)^r I(0,2r)$$

$$= \sum_{r=0}^{p} \binom{p}{r} (-1)^r \, \partial_h^{2r} I(0,0)$$

$$= (1 - \partial_h^2)^p \, I(0,0),$$

and
$$I(2p,q) = \partial_h^q \, (1 - \partial_h^2)^p \, I(0,0), \tag{A.11}$$

where $$I(0,0) = \int_0^{\varepsilon} e^{h\sin\chi'} \cos\chi' \, d\chi',$$

i.e., $$I(0,0) = \frac{e^{h\tau} - 1}{h}, \tag{A.12}$$

and $\tau = \sin\varepsilon$.

Substituting (A.11) in (A.9) and (A.10) yields

$$J(m,2p,q) = \partial_h^{m+q} (1 - \partial_h^2)^p \, I(0,0) \tag{A.13}$$

and

$$K(m,2p,q) = (h\,\partial_h^{m+q+1} - \partial_h^{m+q})(1 - \partial_h^2)^p \, I(0,0). \tag{A.14}$$

Using these relations and equation (4.3.37) and writing I for I(0,0) we obtain the following results:

$$\begin{aligned}
\mathfrak{B} &= -\tfrac{1}{2}(h\partial_h - 1)I \\
\mathfrak{B}_3 &= (h\partial_h^3 - \partial_h^2)I \\
\mathfrak{B}_{11} &= (h\partial_h^5 - \partial_h^4 - h\partial_h^3 + \partial_h^2)I \\
\mathfrak{B}_{33} &= -2(h\partial_h^5 - \partial_h^4)I \\
\mathfrak{B}_{113} &= 2(-h\partial_h^7 + \partial_h^6 + h\partial_h^5 - \partial_h^4)I \\
\mathfrak{B}_{333} &= 4(h\partial_h^7 - \partial_h^6)I \\
\mathfrak{B}_{1111} &= -3(h\partial_h^9 - \partial_h^8 - 2h\partial_h^7 + 2\partial_h^6 + h\partial_h^5 - \partial_h^4)I \\
\mathfrak{B}_{1133} &= 4(h\partial_h^9 - \partial_h^8 - h\partial_h^7 + \partial_h^6)I \\
\mathfrak{B}_{3333} &= -8(h\partial_h^9 - \partial_h^8)I .
\end{aligned} \tag{A.15}$$

From equations (B.17) and (B.25), Appendix B, we obtain

$$\mathfrak{D} = \frac{1}{4}\left[\left(1 - \frac{4}{h} + (2-h)\tau + \frac{1}{\tau}\right)e^{h\tau} - h + \frac{4}{h}\right] \tag{A.16}$$

and

$$\mathfrak{D}_3 = -2\left[\left(1 - \frac{h}{4}\right)\tau^3 + \left(\frac{1}{4} - \frac{4}{h}\right)\tau^2 + \frac{8\tau}{h^2} - \frac{8}{h^3}\right]e^{h\tau} - \frac{16}{h^3} . \tag{A.17}$$

Using equation (4.3.38) and (A.13) we find that:

$$\mathfrak{D}_{11} = \frac{1}{2}(-h^2T_1 + 5hT_2 - 5T_3 + hT_4 - T_5)I \tag{A.18}$$

where
$$\begin{aligned}
T_1 &= \partial_h^5 - \partial_h^7 \\
T_2 &= \partial_h^4 - \partial_h^6 \\
T_3 &= \partial_h^3 - \partial_h^5 \\
T_4 &= \partial_h^2 - \partial_h^4 \\
T_5 &= \partial_h^1 - \partial_h^3 ,
\end{aligned}$$

$$\mathfrak{D}_{33} = (-h^2T_1 + 5hT_2 - 5T_3 + hT_4 - T_5)I \tag{A.19}$$

where
$T_1 = \partial_h^7$
$T_2 = \partial_h^6$
$T_3 = \partial_h^5$
$T_4 = \partial_h^4$
$T_5 = \partial_h^3$,

$$\mathfrak{D}_{113} = (-h^2T_1 + 5hT_2 - 5T_3 + hT_4 - T_5)I \tag{A.20}$$

where
$T_1 = \partial_h^7 - \partial_h^9$
$T_2 = \partial_h^6 - \partial_h^8$
$T_3 = \partial_h^5 - \partial_h^7$
$T_4 = \partial_h^4 - \partial_h^6$
$T_5 = \partial_h^3 - \partial_h^5$

and

$$\mathfrak{D}_{333} = 2(-h^2T_1 + 5hT_2 - 5T_3 + hT_4 - T_5)I \tag{A.21}$$

where
$T_1 = \partial_h^9$
$T_2 = \partial_h^8$
$T_3 = \partial_h^7$
$T_4 = \partial_h^6$
$T_5 = \partial_h^5$.

Setting $y = h^{-1}$ and using (A.12), viz.

$$I = \frac{e^{h\tau} - 1}{h} = ye^{h\tau} - y ,$$

we find that

$\partial_h I = (\tau y - y^2)e^{h\tau} + y^2$

$\partial_h^2 I = (\tau^2 y - 2\tau y^2 + 2y^3)e^{h\tau} - 2y^3$

$\partial_h^3 I = (\tau^3 y - 3\tau^2 y^2 + 6\tau y^3 - 6y^4)e^{h\tau} + 6y^4$

$\partial_h^4 I = (\tau^4 y - 4\tau^3 y^2 + 12\tau^2 y^3 - 24\tau y^4 + 24y^5)e^{h\tau} - 24y^5$

$\partial_h^5 I = (\tau^5 y - 5\tau^4 y^2 + 20\tau^3 y^3 - 60\tau^2 y^4 + 120\tau y^5 - 120y^6)e^{h\tau} + 120y^6$

$\partial_h^6 I = (\tau^6 y - 6\tau^5 y^2 + 30\tau^4 y^3 - 120\tau^3 y^4 + 360\tau^2 y^5 - 720\tau y^6 + 720y^7)e^{h\tau}$
$- 720y^7$

$\partial_h^7 I = (\tau^7 y - 7\tau^6 y^2 + 42\tau^5 y^3 - 210\tau^4 y^4 + 840\tau^3 y^5 - 2520\tau^2 y^6 + 5040\tau y^7$
$- 5040y^8)e^{h\tau} + 5040y^8$

$\partial_h^8 I = (\tau^8 y - 8\tau^7 y^2 + 56\tau^6 y^3 - 336\tau^5 y^4 + 1680\tau^4 y^5 - 6720\tau^3 y^6 + 20160\tau^2 y^7$
$- 40320\tau y^8 + 40320y^9)e^{h\tau} - 40320y^9$

$\partial_h^9 I = (\tau^9 y - 9\tau^8 y^2 + 72\tau^7 y^3 - 504\tau^6 y^4 + 3024\tau^5 y^5 - 15120\tau^4 y^6 + 60480\tau^3 y^7$
$- 181440\tau^2 y^8 + 362880\tau y^9 - 362880y^{10})e^{h\tau} + 362880y^{10}$.

If these relations are inserted into equations (A.15) we obtain expansions for the $\mathfrak{B}$-coefficients in powers of τ, and the same procedure applies to the $\mathfrak{D}$-coefficients. Since $\tau = \sin\varepsilon$, these expansions can be rearranged in powers of ε. Here we take ε so small that powers of ε higher than ε^3 are negligible. Then $\tau = \sin\varepsilon \approx \varepsilon$, and it follows that

$$\mathfrak{B} = -\tfrac{3}{2}\,\varepsilon(1 + h\varepsilon)$$

$$\mathfrak{B}_3 = 0$$

$$\mathfrak{D} = \tfrac{1}{4}\left(\varepsilon^{-1} + 1 - (2 - \tfrac{1}{2}h^2)\varepsilon + (-\tfrac{1}{2}h^2 + \tfrac{1}{6}h^3)\varepsilon^2\right)$$

and

$$\mathfrak{D}_3 = -\tfrac{1}{2}\,\varepsilon^2.$$

Hence we find that

$$\mathfrak{P}_1 = \mathfrak{B} - \tfrac{1}{2} = -\tfrac{1}{2}\,(1 + 3\varepsilon + 3h\varepsilon^2)$$

$$\mathfrak{P}_2 = \mathfrak{B}_3 = 0$$

$$\mathfrak{Q}_1 = \mathfrak{D} = \tfrac{1}{4}\left(\varepsilon^{-1} + 1 - (2 - \tfrac{1}{2}h^2)\varepsilon + (-\tfrac{1}{2}h^2 + \tfrac{1}{6}h^3)\varepsilon^2\right)$$

and

$$\mathfrak{Q}_2 = \mathfrak{D}_3 = -\tfrac{1}{2}\,\varepsilon^2\,.$$

When the relations for the partial derivatives $\partial_h^n I$ are inserted into equations (A.15)-(A.21) and we allow $k \to \infty$ then certain limit and asymptotic relations are obtained, some of which are given here for the case $\varepsilon = \frac{\pi}{2}$:

$$\begin{aligned} \mathfrak{B} &\sim -\tfrac{1}{2}\,e^h \\ \mathfrak{B}_3 &\sim e^h \\ \mathfrak{B}_{11} &\to 0 \\ \mathfrak{B}_{33} &\sim -2e^h \end{aligned} \qquad \text{(A.22)}$$

and

$$\begin{aligned} \mathfrak{D} &\sim (4 - h)e^h - h \\ \mathfrak{D}_3 &\sim \tfrac{1}{2}\,(h - 5)e^h \\ \mathfrak{D}_{11} &\sim -e^h \\ \mathfrak{D}_{33} &\sim (13 - h)e^h\,. \end{aligned} \qquad \text{(A.23)}$$

Refering to equation (4.4.3) and (4.4.5) we find that

$$\begin{aligned} \mathfrak{P}_1 &\sim -\tfrac{1}{2}\,(e^h + 1) \\ \mathfrak{P}_2 &\sim e^h \\ \mathfrak{P}_3 &\to 0 \\ \mathfrak{P}_4 &\sim e^h \end{aligned} \qquad \text{(A.24)}$$

and

$$\begin{aligned}
\mathfrak{Q}_1 &\sim \tfrac{1}{4}\left[(4 - h)e^{h} - h\right] \\
\mathfrak{Q}_2 &\sim \tfrac{1}{2}(h - 5)e^{h} \\
\mathfrak{Q}_3 &\sim -\tfrac{1}{2}e^{h} \\
\mathfrak{Q}_4 &\sim -\tfrac{1}{2}(14 - h)e^{h}
\end{aligned} \tag{A.25}$$

Inserting these relations into equations (4.4.7) with $\alpha = 1$ and $\beta = i$ we find that

$$\begin{aligned}
\mathfrak{R}_1 &\sim -\tfrac{1}{2}(e^{h} + 1) + \tfrac{1}{4}i\left[(4 - h)e^{h} - h\right] \\
\mathfrak{R}_2 &\sim e^{h} + \tfrac{1}{2}i(h - 5)e^{h} \\
\mathfrak{R}_3 &\sim -\tfrac{1}{2}e^{h} \\
\mathfrak{R}_4 &\sim e^{h} - \tfrac{1}{2}i(14 - h)e^{h} .
\end{aligned} \tag{A.25}$$

APPENDIX B

With reference to equations (4.1.45) and (4.1.46) the coefficients D and D_3 are evaluated here for the case of a surface element S on a sphere of radius a, as used in section (4.3). From equation (4.1.40) we have

$$D(\mathbf{r}_\lambda) = \frac{\partial B(\mathbf{r}_\lambda)}{\partial n} . \tag{B.1}$$

Replacing ∂D by S in (4.1.38) gives

$$B(\mathbf{r}_\lambda) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\varepsilon} (ikR_\lambda - 1)\frac{e^{ikR_\lambda}}{R_\lambda^3}\,\Omega\,\cos\chi' d\chi' d\varphi' , \tag{B.2}$$

where $0 \le \varepsilon \le \frac{\pi}{2}$.

In this integral $\quad \mathbf{R}_\lambda = \mathbf{R} - \lambda\mathbf{n},$

$$R_\lambda^2 = R^2 - 2\lambda\mathbf{R}.\mathbf{n} + \lambda^2, \tag{B.3}$$

$$R = 2a\sin\chi' \ , \ 0 \le \chi' \le \varepsilon \ , \tag{B.4}$$

and

$$\Omega = R^3 + \frac{\lambda R^2}{\omega_1\cos\chi' + \sin\chi'} . \tag{B.5}$$

As the integrand is independent of φ'

$$B(\mathbf{r}_\lambda) = -\frac{1}{2}\int_0^{\varepsilon} (ikR_\lambda - 1)\frac{e^{ikR_\lambda}}{R_\lambda^3}\,\Omega\,\cos\chi' d\chi' . \tag{B.6}$$

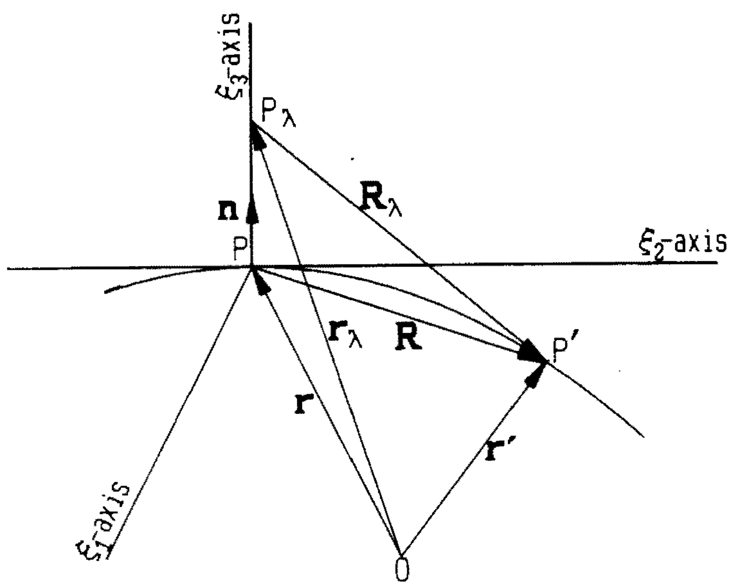


Fig. (B.1).

Since $\xi_3' = \mathbf{R}.\mathbf{n} = - R\sin\chi' = - 2a\sin^2\chi'$, we find

$$R_\lambda^2 = 4a(a + \lambda)\sin^2\chi' + \lambda^2 . \tag{B.7}$$

In the integral the independent variable χ' is replaced by

$$X = R_\lambda .$$

Substituting for $\sin\chi'$ from (B.4) into (B.7) yields

$$X^2 = \left(\frac{a + \lambda}{a}\right) R^2 + \lambda^2$$

$$= 4a(a + \lambda)\sin^2\chi' + \lambda^2 . \tag{B.8}$$

From (B.8) it follows that

$$\cos\chi'\, d\chi' = \frac{XdX}{4a(a + \lambda)\sin\chi'} = \frac{XdX}{2(a + \lambda)R} .$$

Using (4.3.9) and (B.8) we obtain Ω as a function of X:

$$\Omega = R\left(X^2 - \lambda^2 - 2a\lambda\right) .$$

Equation (B.6) now becomes

$$B(r_\lambda) = - \frac{1}{4(a+\lambda)} \int_\lambda^\gamma (ikX - 1)(X^2 - \lambda^2 - 2a\lambda)\frac{e^{ikX}}{X^2}\, dX , \tag{B.9}$$

where $\gamma = \left(4a(a + \lambda)\tau^2 + \lambda^2\right)^{1/2}$

and $\tau = \sin\varepsilon$.

Hence

$$B(\mathbf{r}_\lambda) = - \frac{1}{4(a+\lambda)} \int_\lambda^\gamma \left\{ikX - 1 - \frac{ik\lambda(2a+\lambda)}{X} + \frac{\lambda(2a+\lambda)}{X^2}\right\} e^{ikX}\, dX$$

$$= - \frac{1}{4(a+\lambda)} F_1(\lambda) - \frac{\lambda(2a+\lambda)}{4(a+\lambda)} F_2(\lambda) , \tag{B.10}$$

where

$$F_1(\lambda) = \int_\lambda^\gamma (ikX - 1)e^{ikX}\, dX \tag{B.11}$$

and

$$F_2(\lambda) = \int_\lambda^\gamma \left\{ \frac{1}{X^2} - \frac{ik}{X} \right\} e^{ikX}\, dX . \tag{B.12}$$

Hence

$$\frac{\partial B(\mathbf{r}_\lambda)}{\partial\lambda} = \frac{1}{4(a+\lambda)^2} F_1(\lambda) - \frac{1}{4(a+\lambda)} \frac{\partial F_1(\lambda)}{\partial\lambda} - \frac{\partial}{\partial\lambda}\left(\frac{\lambda(2a+\lambda)}{4(a+\lambda)} F_2(\lambda)\right)$$

and

$$\lim_{\lambda\to 0} \frac{\partial B(r_\lambda)}{\partial\lambda} = \frac{1}{4a^2} F_1(0) - \frac{1}{4a}\frac{\partial F_1(0)}{\partial\lambda} - \lim_{\lambda\to 0} \frac{\partial}{\partial\lambda}\left(\frac{\lambda(2a+\lambda)}{4(a+\lambda)} F_2(\lambda)\right). \quad (B.13)$$

Differentiating (B.11) with respect to λ gives

$$\frac{\partial F_1(\lambda)}{\partial\lambda} = \left[(ikX - 1)e^{ikX} \right]_\lambda^\gamma ,$$

and integration by parts yields

$$F_1(\lambda) = \left[Xe^{ikX} - \frac{2\,e^{ikX}}{ik} \right]_\lambda^\gamma .$$

Since $\gamma \to 2a\tau$ when $\lambda \to 0$, we find that

$$F_1(0) = 2a\tau \left[\left(1 - \frac{2}{h\tau}\right)e^{h\tau} + \frac{2}{h\tau}\right] \quad (B.14)$$

and

$$\frac{\partial F_1(0)}{\partial\lambda} = (h\tau - 1)e^{h\tau} + 1 , \quad (B.15)$$

where $h = 2ika.$

Since $$\frac{d}{dX}\left(\frac{e^{ikX}}{X}\right) = \left(\frac{ik}{X} - \frac{1}{X^2}\right)e^{ikX} ,$$

(B.12) gives

$$F_2(\lambda) = \frac{e^{ik\lambda}}{\lambda} - \frac{e^{ik\gamma}}{\gamma} .$$

Hence $$\frac{\lambda(2a+\lambda)}{4(a+\lambda)} F_2(\lambda) = \frac{(2a+\lambda)e^{ik\lambda}}{4(a+\lambda)} - \frac{\lambda(2a+\lambda)e^{ik\gamma}}{4(a+\lambda)\gamma}$$

and

$$\frac{\partial}{\partial\lambda}\left(\frac{\lambda(2a+\lambda)}{4(a+\lambda)} F_2(\lambda)\right) = - \frac{ae^{ik\lambda}}{4(a+\lambda)^2} + \frac{ik(2a+\lambda)e^{ik\lambda}}{4(a+\lambda)} - \left(\frac{1}{2} - \frac{\lambda(2a+\lambda)}{4(a+\lambda)^2}\right)\frac{e^{ik\gamma}}{\gamma}$$
$$- \frac{\lambda(2a+\lambda)}{4(a+\lambda)}\,\frac{(ik\gamma-1)}{\gamma^2}\,\frac{(2a\tau^2+\lambda)}{\gamma}\,e^{ik\gamma},$$

from which we obtain

$$\lim_{\lambda\to 0} \frac{\partial}{\partial\lambda}\left(\frac{\lambda(2a+\lambda)}{4(a+\lambda)} F_2(\lambda)\right) = - \frac{1}{4a}\left(1 - h + \frac{e^{h\tau}}{\tau}\right) \quad (B.16)$$

provided $\tau \neq 0$. Inserting (B.14)-(B.16) into (B.13) gives

$$D = \frac{1}{4a}\left[\left(1 - \frac{4}{h} + (2-h)\tau + \frac{1}{\tau}\right)e^{h\tau} - h + \frac{4}{h}\right], \text{ if } \tau \neq 0. \tag{B.17}$$

Next we consider

$$D_j(\mathbf{r}_\lambda) = \lim_{\lambda\to 0} \frac{\partial B_j(\mathbf{r}_\lambda)}{\partial \lambda}$$

where

$$B_j(\mathbf{r}_\lambda) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\varepsilon} R(ikR_\lambda - 1)\frac{e^{ikR_\lambda}}{R_\lambda^3}\,\Omega\,\eta_j' \cos\chi' d\chi' d\varphi'. \tag{B.18}$$

Here $\eta_1' = \cos\varphi' \cos\chi'$ and $\eta_2' = \sin\varphi' \cos\chi'$; hence $B_1(\mathbf{r}_\lambda) = 0$ and $B_2(\mathbf{r}_\lambda) = 0$. When $j = 3$, $\eta_3' = -\sin\chi'$, and

$$B_3(\mathbf{r}_\lambda) = \frac{1}{2}\int_0^{\varepsilon} R(ikR_\lambda - 1)\frac{e^{ikR_\lambda}}{R_\lambda^3}\,\Omega \sin\chi' \cos\chi' d\chi'. \tag{B.19}$$

Using the substitution (B.8) gives

$$B_3(\mathbf{r}_\lambda) = \frac{1}{8(a+\lambda)^2}\int_\lambda^\gamma (ikX - 1)(X^2 - \lambda^2)(X^2 - \lambda^2 - 2a\lambda)\,\frac{e^{ikX}}{X^2}\,dX ,$$

i.e., $$B_3(\mathbf{r}_\lambda) = \frac{1}{8(a+\lambda)^2}\int_\lambda^\gamma (ikX - 1)(X^2 - \lambda^2 - 2a\lambda)e^{ikX}\,dX$$

$$- \frac{\lambda^2}{8(a+\lambda)^2}\int_\lambda^\gamma (ikX - 1)(X^2 - \lambda^2 - 2a\lambda)\,\frac{e^{ikX}}{X^2}\,dX ,$$

or $$B_3(\mathbf{r}_\lambda) = \frac{1}{8(a+\lambda)^2}F_3(\lambda) + \frac{\lambda^2}{2(a+\lambda)}B(\mathbf{r}_\lambda) \tag{B.20}$$

where $$F_3(\lambda) = \int_\lambda^\gamma (ikX - 1)(X^2 - \lambda^2 - 2a\lambda)e^{ikX}\,dX$$

$$= \int_\lambda^\gamma \left[ikX^3 - X^2 - ik\lambda(2a+\lambda)X + \lambda(2a+\lambda)\right]e^{ikX}\,dX . \tag{B.21}$$

Differentiating (B.20) and allowing $\lambda \to 0$ gives

$$\lim_{\lambda\to 0}\frac{\partial B_3(\mathbf{r}_\lambda)}{\partial\lambda} = -\frac{1}{4a^3}F_3(0) + \frac{1}{8a^2}\frac{\partial F_3(0)}{\partial\lambda} . \tag{B.22}$$

From (B.21)

$$F_3(0) = \int_0^{2a\tau} (ikX^3 - X^2)e^{ikX}\,dX$$

$$= \left[\left(X^3 + \frac{4i}{k}X^2 - \frac{8}{k^2}X - \frac{8i}{k^3}\right)e^{ikX}\right]_0^{2a\tau}$$

$$= 8a^3\tau^3\left[\left(1 - \frac{4}{h\tau} + \frac{8}{h^2\tau^2} - \frac{8}{h^3\tau^3}\right)e^{h\tau} + \frac{8}{h^3\tau^3}\right]. \tag{B.23}$$

Again from (B.21)

$$\frac{\partial F_3(\lambda)}{\partial\lambda} = \left[\left(ikX^3 - X^2 - ik\lambda(2a+\lambda)X + \lambda(2a+\lambda)\right)e^{ikX}\right]_\lambda^\gamma$$

and

$$\frac{\partial F_3(0)}{\partial\lambda} = \left[\left(ikX^3 - X^2\right) e^{ikX}\right]_0^{2a\tau}$$

$$= 4a^2\tau^2(h\tau - 1)e^{h\tau} . \tag{B.24}$$

Inserting (B.23) and (B.24) into (B.22) we finally obtain

$$D_3 = -2\left[\left(1-\frac{h}{4}\right)\tau^3 + \left(\frac{1}{4} - \frac{4}{h}\right)\tau^2 + \frac{8\tau}{h^2} - \frac{8}{h^3}\right]e^{h\tau} - \frac{16}{h^3} . \tag{B.25}$$

APPENDIX C

PROGRAMS FOR COMPUTING THE EXACT AND ANALYTICAL APPROXIMATIONS FOR THE SURFACE FIELD ON A SOUND HARD SPHERE

CONTENTS

```
      PROGRAM SFHSE
      DOUBLE PRECISION A,AK,AK2,AMP,DELTAX,DM,DNX,DSJ,DSY,ETA,EPLN,
     1FCABS,P,PIE,RAD,THETA,WNK,X,Z
      COMPLEX*16 C,EYE,FE,RDH1,S,T,V,W
      CHARACTER PATH*2,FNAME*13,PFNAME*15
      DIMENSION FE(0:180),DSJ(0:550),DSY(0:550),RDH1(0:550),P(0:550),
     1S(0:180),T(0:180)
      OPEN(1,FILE='SFHSE.INP')
      READ(1,100)NX,NT,NO,NC,EPLN,RAD,WNK,PATH
      WRITE(*,100)NX,NT,NO,NC,EPLN,RAD,WNK,PATH
      CLOSE(1,STATUS='KEEP')
      PIE=4.D0*DATAN(1.D0)
      DELTAX=PIE/DNX
      EYE=DCMPLX(0.D0,1.D0)
      AK=RAD*WNK
      CALL SBF(AK,NT,DSJ,DSY,NSJ,NSY)
      NSB=MIN0(NT,NSJ,NSY)
      DO 10 N=0,NSB
      W=DCMPLX(DSJ(N),-DSY(N))
      A=FCABS(W)
10    RDH1(N)=W/(A*A)
      AK2=AK*AK
      C=EYE/AK2
      DO 20 M=0,NX
      X=DM*DELTAX
      Z=DCOS(X)
      CALL LP(Z,NT,P)
      T(0)=P(0)*RDH1(0)
      S(0)=T(0)
      V=-EYE
      DO 30 N=1,NT
      T(N)=V*DBLE(2*N+1)*P(N)*RDH1(N)
      V=-V*EYE
      S(N)=S(N-1)+T(N)
      IF(N.LT.NO) GOTO 30
      ETA=CDABS(S(N)-S(N-NC))
      IF(ETA.LT.EPLN) THEN
```

```
      FE(M)=C*S(N)
      GOTO 20
      ELSEIF(ETA.GT.EPLN.AND.N.LT.NT)THEN
      GOTO 30
      ELSEIF(ETA.GT.EPLN.AND.N.EQ.NT)THEN
      FE(M)=C*S(N)
      WRITE(*,110)M,ETA
      GOTO 20
      ENDIF
30    CONTINUE
20    CONTINUE
      FNAME='"SFHSE.AMP'
      PFNAME=PATH//FNAME
      OPEN(2,FILE=PFNAME)
      WRITE(2,120)NX
      DO 40 M=0,NX
      DM=DBLE(M)
      X=DM*DELTAX
      THETA=(180.D0*X)/PIE
      AMP=CDABS(FE(M))
40    WRITE(2,130)THETA,AMP
100   FORMAT(4(2X,I4),/,2X,D8.1,/,2(2X,F7.3),/,2X,A2)
110   FORMAT(2X,'FE(',I4,') DOES NOT CONVERGE, ETA =',D22.15)
120   FORMAT(2X,I4)
130   FORMAT(2(2X,F8.4))
      STOP
      END


      SUBROUTINE SBF(Z,NT,DSJ,DSY,NSJ,NSY)
      DOUBLE PRECISION Z,SJ,SY,DSJ,DSY,A,B,C,E,F,U,V,DNT,X,Y
      DIMENSION F(0:1000),SJ(0:550),SY(0:550),DSJ(0:550),DSY(0:550)
      E=DEXP(1.D0)
      U=(E*Z)/2.D0
      WRITE(*,*)' U=',U
      DO 40 N=0,550
      SJ(N)=0.D0
```

```
      SY(N)=0.D0
      DSJ(N)=0.D0
40    DSY(N)=0.D0
      DO 50 N=0,1000
50    F(N)=0.D0
      DNT=DBLE(NT)
      IF(U.LT.DNT)GOTO 1000
      DO 60 N=1,1000
      V=DBLE(N)+0.5D0
      X=U/V
      Y=Z*V
      A=V*DLOG(X)-0.5D0*DLOG(Y)-DLOG(2.D0)+300.D0*DLOG(10.D0)
      IF(A.GT.0.D0) GOTO 60
      IF(A.LT.0.D0) NSJ=N
      WRITE(*,*)' NSJ=',NSJ
      GOTO 61
60    CONTINUE
61    NSJ2=MIN0(NT,NSJ)
      SJ(0)=DSIN(Z)/Z
      SJ(1)=(SJ(0)-DCOS(Z))/Z
      DO 70 N=1,NSJ2
70    SJ(N+1)=(DBLE(2*N+1)*SJ(N))/Z-SJ(N-1)
      DSJ(0)=(DCOS(Z)-SJ(0))/Z
      DO 80 N=1,NSJ2
80    DSJ(N)=SJ(N-1)-(DBLE(N+1)*SJ(N))/Z
      GOTO 2000
1000  DO 90 N=1,1000
      V=DBLE(N)+0.5D0
      X=U/V
      Y=Z*V
      A=V*DLOG(X)-0.5D0*DLOG(Y)-DLOG(2.D0)+300.D0*DLOG(10.D0)
      IF(A.GT.0.D0) GOTO 90
      IF(A.LT.0.D0) NSJ=N
      WRITE(*,*)' NSJ=',NSJ
      GOTO 3000
90    CONTINUE
3000  IF(NSJ.GE.550)GOTO 5000
```

```
      F(NSJ+2)=0.D0
      F(NSJ+1)=1.D-300
      DO 100 N=NSJ,0,-1
100   F(N)=(DBLE(2*N+3)*F(N+1))/Z-F(N+2)
      SJ(0)=DSIN(Z)/Z
      C=SJ(0)/F(0)
      NSJ2=MINO(NT,NSJ)
      DO 110 N=1,NSJ2
110   SJ(N)=C*F(N)
      DSJ(0)=(DCOS(Z)-SJ(0))/Z
      DO 120 N=1,NSJ2
120   DSJ(N)=SJ(N-1)-(DBLE(N+1)*SJ(N))/Z
2000  DO 130 N=1,1000
      V=DBLE(N)+0.5D0
      X=U/V
      Y=Z*V
      B=V*DLOG(X)+0.5*DLOG(Y)+300*DLOG(10.D0)
      IF(B.GT.0.D0) GOTO 130
      IF(B.LT.0.D0) NSY=N
      WRITE(*,*)' NSY=',NSY
      GO TO 4000
130   CONTINUE
4000  SY(0)=-DCOS(Z)/Z
      SY(1)=(SY(0)-DSIN(Z))/Z
      NSY2=MINO(NT,NSY)
      DO 140 N=2,NSY2
140   SY(N)=(DBLE(2*N-1)*SY(N-1))/Z-SY(N-2)
      DSY(0)=(DSIN(Z)-SY(0))/Z
      DO 150 N=1,NSY2
150   DSY(N)=SY(N-1)-(DBLE(N+1)*SY(N))/Z
      GOTO 6000
5000  WRITE(*,*)' NSJ GREATER THAN 550; INCREASE DIMENSIONS OF
     1SEQUENCES.'
      STOP
6000  RETURN
      END
```

```
      SUBROUTINE LP(Z,NT,P)
      DOUBLE PRECISION Z,P,A,B,C
      DIMENSION P(0:550)
      P(0)=1.D0
      P(1)=Z
      P(2)=(3.D0*Z*Z-1.D0)/2.D0
      DO 10 M=3,NT
      A=DBLE(2*M-1)
      B=DBLE(M-1)
      C=DBLE(M)
10    P(M)=(A*Z*P(M-1)-B*P(M-2))/C
      RETURN
      END

      DOUBLE PRECISION FUNCTION FCABS(Z)
      DOUBLE PRECISION X,Y,AX,AY,U,V,W,W1
      COMPLEX*16 Z
      X=DREAL(Z)
      Y=DIMAG(Z)
      AX=DABS(X)
      AY=DABS(Y)
      U=DMAX1(AX,AY)
      V=DMIN1(AX,AY)
      IF(U.EQ.V)THEN
      FCABS=U*DSQRT(2.D0)
      RETURN
      ENDIF
      W=V/U
      IF(W.LT.1.D-150)THEN
      FCABS=U
      RETURN
      ELSEIF(W.GT.1.D-150.AND.W.LT.1.D0)THEN
      W1=1.D0+W*W
      FCABS=U*DSQRT(W1)
      RETURN
      ENDIF
      END
```

```
      PROGRAM SBF
      DOUBLE PRECISION X,CW,TW,SJ,SY,DSJ,DSY,A,B,C,E,F,U,V,DNT,ARDH1,
     1FCABS
      COMPLEX*16 W,RDH1
      CHARACTER PATH*13,FNAME*13,PFNAME*26
      DIMENSION F(0:1000),CW(0:1000),SJ(0:1000),SY(0:1000),DSJ(0:1000),
     1DSY(0:1000)
      OPEN(1,FILE='SBF.INP')
      READ(1,10)NT,X,PATH
10    FORMAT(2X,I4,/,2X,F7.3,/,2X,A13)
      FNAME='"SBF.OUT'
      PFNAME=PATH//FNAME
      OPEN(2,FILE=PFNAME)
      WRITE(2,20)
20    FORMAT(53X,'SPHERICAL BESSEL FUNCTIONS',/,/)
      TW=1.D0/(X*X)
      WRITE(2,30)X,TW
30    FORMAT(44X,'X=',F9.4,5X,'TW=',D22.15,/,/)
      E=DEXP(1.D0)
      U=(E*X)/2.D0
      WRITE(*,*)' U=',U
      DO 40 N=0,1000
      SJ(N)=0.D0
      SY(N)=0.D0
      DSJ(N)=0.D0
40    DSY(N)=0.D0
      DO 50 N=0,1000
50    F(N)=0.D0
      DNT=DBLE(NT)
      IF(U.LT.DNT)GOTO 1000
      DO 60 N=1,1000
      V=DBLE(N)+0.5D0
      A=V*DLOG10(U/V)-0.5D0*DLOG10(X*V)-DLOG10(2.D0)+300.D0
      IF(A.GT.0.D0) GOTO 60
      IF(A.LT.0.D0) NSJ=N
      WRITE(*,*)' NSJ=',NSJ
      GOTO 61
```

```
60      CONTINUE
61      NSJ2=MINO(NT,NSJ)
        SJ(0)=DSIN(X)/X
        SJ(1)=(SJ(0)-DCOS(X))/X
        DO 70 N=1,NSJ2
70      SJ(N+1)=(DBLE(2*N+1)*SJ(N))/X-SJ(N-1)
        DSJ(0)=(DCOS(X)-SJ(0))/X
        DO 80 N=1,NSJ2
80      DSJ(N)=SJ(N-1)-(DBLE(N+1)*SJ(N))/X
        GOTO 2000
1000    DO 90 N=1,1000
        V=DBLE(N)+0.5D0
        A=V*DLOG10(U/V)-0.5D0*DLOG10(X*V)-DLOG10(2.D0)+300.D0
        IF(A.GT.0.D0) GOTO 90
        IF(A.LT.0.D0) NSJ=N
        WRITE(*,*)NSJ
        GOTO 3000
90      CONTINUE
3000    IF(NSJ.GE.1000)GOTO 5000
        F(NSJ+2)=0.D0
        F(NSJ+1)=1.D-300
        DO 100 N=NSJ,0,-1
100     F(N)=(DBLE(2*N+3)*F(N+1))/X-F(N+2)
        SJ(0)=DSIN(X)/X
        C=SJ(0)/F(0)
        NSJ2=MINO(NT,NSJ)
        DO 110 N=1,NSJ2
110     SJ(N)=C*F(N)
        DSJ(0)=(DCOS(X)-SJ(0))/X
        DO 120 N=1,NSJ2
120     DSJ(N)=SJ(N-1)-(DBLE(N+1)*SJ(N))/X
2000    DO 130 N=1,1000
        V=DBLE(N)+0.5D0
        B=V*DLOG10(U/V)+0.5D0*DLOG10(X*V)+300.D0
        IF(B.GT.0.D0) GOTO 130
        IF(B.LT.0.D0) NSY=N
        WRITE(*,*)' NSY=',NSY
```

```
      GO TO 4000
130   CONTINUE
4000  SY(0)=-DCOS(X)/X
      SY(1)=(SY(0)-DSIN(X))/X
      NSY2=MIN0(NT,NSY)
      DO 140 N=2,NSY2
140   SY(N)=(DBLE(2*N-1)*SY(N-1))/X-SY(N-2)
      DSY(0)=(DSIN(X)-SY(0))/X
      DO 150 N=1,NSY2
150   DSY(N)=SY(N-1)-(DBLE(N+1)*SY(N))/X
      NT=MIN0(NSJ2,NSY2)
      WRITE(*,*)' NT=',NT
      DO 160 N=0,NT
160   CW(N)=SJ(N)*DSY(N)-SY(N)*DSJ(N)
      WRITE(2,170)
170   FORMAT(3X,'N',13X,'SJ',23X,'SJD',22X,'SY',23X,'SYD',22X,'CW',/)
      DO 180 N=0,NT
      WRITE(2,190) N,SJ(N),DSJ(N),SY(N),DSY(N),CW(N)
190   FORMAT(I5,5D24.15)
180   CONTINUE
      CLOSE(1,STATUS='KEEP')
      CLOSE(2,STATUS='KEEP')
      FNAME='"RDH1.OUT'
      PFNAME=PATH//FNAME
      OPEN(2,FILE=PFNAME)
      NSB=MIN0(NT,NSJ,NSY)
      DO 200 N=0,NSB
      W=DCMPLX(DSJ(N),-DSY(N))
      A=FCABS(W)
      RDH1(N)=W/(A*A)
200   WRITE(2,210)N,RDH1(N)
210   FORMAT(2X,I4,4X,D16.10,2X,D16.10)
      CLOSE(2,STATUS='KEEP')
      GOTO 6000
5000  WRITE(*,*)' NSJ GT. 1000; INCREASE DIM. OF SEQ.'
6000  STOP
      END
```

```
      PROGRAM SFHSOA
      DOUBLE PRECISION RAD,WNK,AK,PIE,DELTAX,X,Z,DM,AMP,AKZ,AG,AR,DNX,
     1RAR
      COMPLEX*16 EYE,P,Q,R,A,B,ALPHA,BETA,E,G,CAKZI
      CHARACTER PATH*2,FNAME(10)*13,XNAME*13,PFNAME*15
      DIMENSION P(4),Q(4),R(4),A(10),B(10)
      OPEN(1,FILE='SFHSOA.INP')
      READ(1,100)NX,NF,RAD,WNK,PATH
      DO 10 I=1,NF
10    READ(1,110)A(I),B(I)
      DO 20 I=1,NF
20    READ(1,120)FNAME(I)
      DNX=DBLE(NX)
      EYE=DCMPLX(0.D0,1.D0)
      PIE=4.D0*DATAN(1.D0)
      DELTAX=PIE/DNX
      DO 30 I=1,NF
      XNAME=FNAME(I)
      PFNAME=PATH//XNAME
      OPEN(2,FILE=PFNAME)
      CALL HCOEF2(RAD,WNK,P,Q)
      ALPHA=A(I)
      BETA=B(I)
      R(1)=ALPHA*P(1)+BETA*Q(1)
      AR=CDABS(R(1))
      RAR=1.D0/AR
      WRITE(*,*)' R(1)=',R(1),' AR=',AR,' RAR=',RAR
      AK=RAD*WNK
      DO 40 M=0,NX
      DM=DBLE(M)
      X=DM*DELTAX
      Z=DCOS(X)
      AKZ=AK*Z
      CAKZI=DCMPLX(0.D0,AKZ)
      E=BETA*CAKZI
      G=ALPHA-E
      AG=CDABS(G)
```

```
      AMP=AG*RAR
      WRITE(*,130)M,AMP
40    WRITE(2,130)M,AMP
      CLOSE(2,STATUS='KEEP')
30    CONTINUE
100   FORMAT(2(2X,I4),/,2(2X,F7.3),/,2X,A2)
110   FORMAT(4(2X,F7.3))
120   FORMAT(2X,A13)
130   FORMAT(2X,I4,2X,F9.4)
      STOP
      END


      PROGRAM SFHS1A
      DOUBLE PRECISION AK,AKD,AZ,D,DELTA,DELTAT,DLNAZ,DLND,DN,
     1DNPHI,DNT,PHI,PIE,RAD,SGN,U,V,T,ZETA,WNK
      COMPLEX*16 A,B,ALPHA,BETA,AR,ARG,C0,C1,C2,C4,CAKI,CD,CDELTA,
     1CLNZ,EYE,F,G,P,Q,R,S,S1,SEQA,SEQB,SET,SOT,SUMC,W,W0,W1,W2,W3,W4,
     2W5,X,X0,X1,X2,Y,Z
      CHARACTER FNAME(3)*13,PATH*2,PFNAME*15,XNAME*13
      DIMENSION P(4),Q(4),R(4),F(0:202),G(0:202),ZETA(0:202),W(0:400),
     1SEQA(3),SEQB(3)
      OPEN(1,FILE='SFHS1A.INP')
      READ(1,200)NF,NT,NPHI,RAD,WNK,D,PATH
      DO 1 I=1,NF
1     READ(1,210)SEQA(I),SEQB(I)
      DO 2 I=1,NF
2     READ(1,220)FNAME(I)
      DNT=DBLE(NT)
      DNPHI=DBLE(NPHI)
      PIE=4.D0*DATAN(1.D0)
      AK=RAD*WNK
      AKD=AK*D
      C0=DCMPLX(0.D0,0.D0)
      C1=DCMPLX(1.D0,0.D0)
      C2=DCMPLX(2.D0,0.D0)
      C4=DCMPLX(4.D0,0.D0)
```

```
      EYE=DCMPLX(0.D0,1.D0)
      CD=DCMPLX(D,0.D0)
      CAKD=DCMPLX(AKD,0.D0)
      CAKI=DCMPLX(0.D0,AK)
      DO 3 I=1,NF
      XNAME=FNAME(I)
      PFNAME=PATH//XNAME
      OPEN(2,FILE=PFNAME)
      ALPHA=SEQA(I)
      BETA=SEQB(I)
      A=ALPHA
      B=-BETA*CAKI
      CALL HCOEF2(RAD,WNK,P,Q)
      DO 10 J=1,4
10    R(J)=ALPHA*P(J)+BETA*Q(J)
      S=R(1)/R(2)
      S1=S-C1
      WRITE(*,*)' S=',S
      V=DIMAG(S)
      IF(V.GT.0.D0)THEN
      SGN=-1.D0
      ELSE
      SGN=1.D0
      ENDIF
      WRITE(*,*)' V=',V,' SGN=',SGN
C     *********(1)**************
      DELTAT=1.D0/DNT
      DO 20 N=0,NT
      DN=DBLE(N)
      T=DN*DELTAT
      U=1.D0+(D-1.D0)*T
      Z=DCMPLX(U,0.D0)
      AZ=CDABS(Z)
      DLNAZ=DLOG(AZ)
      CLNZ=DCMPLX(DLNAZ,0.D0)
      X1=S1*CLNZ
      W1=CDEXP(X1)
```

```
      X2=S*CLNZ
      W2=CDEXP(X2)
      Y=-CAKI*Z
      W3=CDEXP(Y)
      W4=A*W1*W3+B*W2*W3
20    W(N)=(CD-C1)*W4
      DELTA=DELTAT/3.D0
      CDELTA=DCMPLX(DELTA,0.D0)
      G(0)=C0
      ZETA(0)=1.D0
      DO 30 N=2,NT,2
      M=N/2
      DN=DBLE(N)
      AR=W(N)+C4*W(N-1)+W(N-2)
      G(M)=G(M-1)+CDELTA*AR
      T=DN*DELTAT
      U=1.D0+(D-1.D0)*T
30    ZETA(M)=U
C     ********(2)*************
      DELPHI=(SGN*PIE)/DNPHI
      DLND=DLOG(D)
      AKD=AK*D
      CD=DCMPLX(D,0.D0)
      CAKI=DCMPLX(0.D0,AK)
      DO 40 N=0,NPHI
      DN=DBLE(N)
      PHI=DN*DELPHI
      X=DCMPLX(DCOS(PHI),DSIN(PHI))
      Z=CD*X
      CLNZ=DCMPLX(DLND,PHI)
      X1=S1*CLNZ
      W1=CDEXP(X1)
      X2=S*CLNT
      W2=CDEXP(X2)
      Y=-CAKI*Z
      W3=CDEXP(Y)
      W4=EYE*Z
```

```
      W5=A*W1*W3+B*W2*W3
40    W(N)=W4*W5
      SOT=C0
      DO 50 N=1,NPHI-1,2
50    SOT=SOT+W(N)
      SET=C0
      DO 60 N=2,NPHI-2,2
60    SET=SET+W(N)
      DELTA=DELPHI/3.D0
      CDELTA=DCMPLX(DELTA,0.D0)
      SUMC=CDELTA*(W(0)+W(NPHI)+C4*SOT+C2*SET)
C     ********** (3) *************
      MT=NT/2
      DELTAT=1.D0/DNT
      PHI=SGN*PIE
      DO 70 N=0,NT
      DN=DBLE(N)
      T=DN*DELTAT
      U=-D+(D-1.D0)*T
      Z=DCMPLX(U,0.D0)
      AZ=CDABS(Z)
      DLNAZ=DLOG(AZ)
      CLNZ=DCMPLX(DLNAZ,PHI)
      X1=S1*CLNZ
      W1=CDEXP(X1)
      X2=S*CLNZ
      W2=CDEXP(X2)
      Y=-CAKI*Z
      W3=CDEXP(Y)
      W4=A*W1*W3+B*W2*W3
70    W(N)=(CD-C1)*W4
      DELTA=DELTAT/3.D0
      CDELTA=DCMPLX(DELTA,0.D0)
      M=MT+2
      G(M)=G(MT)+SUMC
      ZETA(M)=-D
      U=-D
```

```
      DO 80 N=2,NT,2
      M=MT+2+N/2
      DN=DBLE(N)
      AR=W(N)+C4*W(N-1)+W(N-2)
      G(M)=G(M-1)+CDELTA*AR
      T=DN*DELTAT
      U=-D+(D-1.D0)*T
80    ZETA(M)=U
C     ********************
      X0=-CAKI
      W0=C2*CDEXP(X0)
      DO 90 M=0,MT
      DM=DBLE(M)
      U=ZETA(M)
      Z=DCMPLX(U,0.D0)
      AZ=CDABS(Z)
      DLNAZ=DLOG(AZ)
      CLNZ=DCMPLX(DLNAZ,0.D0)
      X1=-S*CLNZ
      W1=CDEXP(X1)
      W2=G(M)/R(2)
      W3=W0+W2
90    F(M)=W1*W3
      M=MT+1
      ZETA(M)=0.D0
      F(M)=ALPHA/R(1)
      U=0.D0
      DO 100 M=MT+2,NT+2
      DM=DBLE(M)
      U=ZETA(M)
      Z=DCMPLX(U,0.D0)
      AZ=CDABS(Z)
      DLNAZ=DLOG(AZ)
      CLNZ=DCMPLX(DLNAZ,PHI)
      X1=-S*CLNZ
      W1=CDEXP(X1)
      W2=G(M)/R(2)_
```

```
      W3=W0+W2
100   F(M)=W1*W3
C     ****************************
      U=DREAL(S)
      V=DIMAG(S)
      WRITE(2,230)U,V
      NP=NT+2
      WRITE(2,240)NP
      DO 170 M=0,NT+2
      U=ZETA(M)
      PHI=DACOS(U)
      PHI=(PHI*180.D0)/PIE
      AMP=CDABS(F(M))
      WRITE(*,250)PHI,AMP
170   WRITE(2,250)PHI,AMP
      CLOSE(2,STATUS='KEEP')
3     CONTINUE
C     ********************************
200   FORMAT(3(2X,I4),/,3(2X,F7.3),/,2X,A2)
210   FORMAT(4(2X,F7.3))
220   FORMAT(2X,A13)
230   FORMAT(2(3X,F8.4))
240   FORMAT(2X,I4)
250   FORMAT(2(2X,F8.4))
      STOP
      END


      PROGRAM SFHS1B
      DOUBLE PRECISION AH,AK,AZ,D,DELTA,DELTAT,DLNAZ,DMT,DN,
     1DNT,DNT1,DNT2,DK,H,PHI,PIE,RAD,SEQZ,SGN,U,V,T,WNK,ZETA
      COMPLEX*16 A,B,ALPHA,BETA,AR,ARG,C0,C1,C2,C4,CAKI,CD,CDELTA,CHI,
     1CLNZ,EYE,F,G,P,Q,R,S,S1,SEQA,SEQB,SET,SOT,SUM,SUM1,SUM2,SUM3,
     2W,W0,W1,W2,W3,W4,X0,X1,X2,Y,Z
      CHARACTER FNAME(3)*13,PATH*2,PFNAME*15,XNAME*13
      DIMENSION P(4),Q(4),R(4),F(0:202),G(0:202),ZETA(0:202),W(0:400),
     1SEQA(3),SEQB(3)
```

```
      OPEN(1,FILE-'SFHS1B.INP')
      READ(1,200)NF,NT,NT1,NT2,D,AH,RAD,WNK,PATH
      WRITE(*,200)NF,NT,NT1,NT2,D,AH,RAD,WNK,PATH
      DO 1 I-1,NF
1     READ(1,210)SEQA(I),SEQB(I)
      DO 2 I-1,NF
2     READ(1,220)FNAME(I)
      DNT-DBLE(NT)
      DNT1-DBLE(NT1)
      DNT2-DBLE(NT2)
      PIE-4.D0*DATAN(1.D0)
      C0-DCMPLX(0.D0,0.D0)
      C1-DCMPLX(1.D0,0.D0)
      C2-DCMPLX(2.D0,0.D0)
      C4-DCMPLX(4.D0,0.D0)
      CD-DCMPLX(D,0.D0)
      EYE-DCMPLX(0.D0,1.D0)
      AK-RAD*WNK
      CAKI-DCMPLX(0.D0,AK)
      DO 3 I-1,NF
      XNAME-FNAME(I)
      PFNAME-PATH//XNAME
      OPEN(2,FILE-PFNAME)
      ALPHA-SEQA(I)
      BETA-SEQB(I)
      A-ALPHA
      B--BETA*CAKI
      CALL HCOEF2(RAD,WNK,P,Q)
      DO 10 J-1,4
10    R(J)-ALPHA*P(J)+BETA*Q(J)
      S-R(1)/R(2)
      S1-S-C1
      WRITE(*,*)' S-',S
      V-DIMAG(S)
      IF(V.GE.0.D0)THEN
      SGN--1.D0
      ELSE
```

```
      SGN=1.D0
      ENDIF
      H=SGN*AH
C     *********(1)*************
      DELTAT=1.D0/DNT
      DO 20 N=0,NT
      DN=DBLE(N)
      T=DN*DELTAT
      U=1.D0+(D-1.D0)*T
      V=0.D0
      Z=DCMPLX(U,V)
      AZ=CDABS(Z)
      DLNAZ=DLOG(AZ)
      CLNZ=DCMPLX(DLNAZ,0.D0)
      X1=S1*CLNZ
      W1=CDEXP(X1)
      X2=S*CLNZ
      W2=CDEXP(X2)
      Y=-CAKI*Z
      W3=CDEXP(Y)
      W4=A*W1*W3+B*W2*W3
20    W(N)=(CD-C1)*W4
      DELTA=DELTAT/3.D0
      CDELTA=DCMPLX(DELTA,0.D0)
      G(0)=0.D0
      ZETA(0)=1.D0
      DO 30 N=2,NT,2
      M=N/2
      DN=DBLE(N)
      AR=W(N)+C4*W(N-1)+W(N-2)
      G(M)=G(M-1)+CDELTA*AR
      T=DN*DELTAT
      U=1.D0+(D-1.D0)*T
30    ZETA(M)=U
C     ********(2)*************
      DELTAT=1.D0/DNT1
      CHI=DCMPLX(0.D0,H)
```

```
      DO 40 N=0,NT1
      DN=DBLE(N)
      T=DN*DELTAT
      U=1.D0
      V=H*T
      Z=DCMPLX(U,V)
      PHI=ARG(Z)
      AZ=CDABS(Z)
      DLNAZ=DLOG(AZ)
      CLNZ=DCMPLX(DLNAZ,PHI)
      X1=S1*CLNZ
      X2=S*CLNZ
      W1=CDEXP(X1)
      W2=CDEXP(X2)
      X3=-CAKI*Z
      W3=CDEXP(X3)
      W4=A*W1*W3+B*W2*W3
40    W(N)=CHI*W4
      SOT=0.D0
      DO 50 N=1,NT1-1,2
50    SOT=SOT+W(N)
      SET=0.D0
      DO 60 N=2,NT1-2,2
60    SET=SET+W(N)
      DELTA=DELTAT/3.D0
      CDELTA=DCMPLX(DELTA,0.D0)
      SUM1=CDELTA*(W(0)+W(NT1)+C4*SOT+C2*SET)
C     **********(3)****************
      DELTAT=1.D0/DNT2
      DO 70 N=0,NT2
      DN=DBLE(N)
      T=DN*DELTAT
      U=1.D0-2.D0*T
      V=H
      Z=DCMPLX(U,V)
      PHI=ARG(Z)
      AZ=CDABS(Z)
```

```
      DLNAZ=DLOG(AZ)
      CLNZ=DCMPLX(DLNAZ,PHI)
      X1=S1*CLNZ
      X2=S*CLNZ
      W1=CDEXP(X1)
      W2=CDEXP(X2)
      X3=-CAKI*Z
      W3=CDEXP(X3)
      W4=A*W1*W3+B*W2*W3
70    W(N)=-C2*W4
      SOT=0.D0
      DO 80 N=1,NT2-1,2
80    SOT=SOT+W(N)
      SET=0.D0
      DO 90 N=2,NT2-2,2
90    SET=SET+W(N)
      DELTA=DELTAT/3.D0
      CDELTA=DCMPLX(DELTA,0.D0)
      SUM2=CDELTA*(W(0)+W(NT2)+C4*SOT+C2*SET)
C     **********(4)*******************
      DELTAT=1.D0/DNT1
      DO 100 N=0,NT1
      DN=DBLE(N)
      T=DN*DELTAT
      U=-1.D0
      V=H*(1.D0-T)
      Z=DCMPLX(U,V)
      PHI=ARG(Z)
      AZ=CDABS(Z)
      DLNAZ=DLOG(AZ)
      CLNZ=DCMPLX(DLNAZ,PHI)
      X1=S1*CLNZ
      X2=S*CLNZ
      W1=CDEXP(X1)
      W2=CDEXP(X2)
      X3=-CAKI*Z
      W3=CDEXP(X3)
```

```
      W4=A*W1*W3+B*W2*W3
100   W(N)=-CHI*W4
      SOT=0.D0
      DO 110 N=1,NT1-1,2
110   SOT=SOT+W(N)
      SET=0.D0
      DO 120 N=2,NT1-2,2
120   SET=SET+W(N)
      DELTA=DELTAT/3.D0
      CDELTA=DCMPLX(DELTA,0.D0)
      SUM3=CDELTA*(W(0)+W(NT1)+C4*SOT+C2*SET)
C     ***************************************
      SUM=SUM1+SUM2+SUM3
C     ********** (5) ************************
      DELTAT=1.D0/DNT
      DO 130 N=0,NT
      DN=DBLE(N)
      T=DN*DELTAT
      U=-1.D0+(1.D0-D)*T
      Z=DCMPLX(U,0.D0)
      PHI=SGN*PIE
      AZ=CDABS(Z)
      DLNAZ=DLOG(AZ)
      CLNZ=DCMPLX(DLNAZ,PHI)
      X1=S1*CLNZ
      X2=S*CLNZ
      W1=CDEXP(X1)
      W2=CDEXP(X2)
      X3=-CAKI*Z
      W3=CDEXP(X3)
      W4=A*W1*W3+B*W2*W3
130   W(N)=(C1-CD)*W4
      DELTA=DELTAT/3.D0
      CDELTA=DCMPLX(DELTA,0.D0)
      M=NT+2
      ZETA(M)=-1.D0
      G(M)=SUM
```

```
      U=-1.D0
      DO 140 N=2,NT,2
      L=N/2
      M=NT+2-L
      DN=DBLE(N)
      AR=W(N)+C4*W(N-1)+W(N-2)
      G(M)=G(M-1)+CDELTA*AR
      T=DN*DELTAT
      U=-1.D0+(1.D0-D)*T
140   ZETA(M)=U
C     ********************************
      MT=NT/2
      X0=-CAKI
      W0=C2*CDEXP(X0)
      DO 150 M=0,MT
      DM=DBLE(M)
      U=ZETA(M)
      Z=DCMPLX(U,0.D0)
      AZ=CDABS(Z)
      DLNAZ=DLOG(AZ)
      CLNZ=DCMPLX(DLNAZ,0.D0)
      X1=-S*CLNZ
      W1=CDEXP(X1)
      W2=G(M)/R(2)
      W3=W0+W2
150   F(M)=W1*W3
      M=MT+1
      ZETA(M)=0.D0
      F(M)=ALPHA/R(1)
      Z=C0
      W1=C0
      W2=C0
      DO 160 M=MT+2,NT+2
      DM=DBLE(M)
      U=ZETA(M)
      Z=DCMPLX(U,0.D0)
      PHI=SGN*PIE
```

```
      AZ=CDABS(Z)
      DLNAZ=DLOG(AZ)
      CLNZ=DCMPLX(DLNAZ,PHI)
      X1=-S*CLNZ
      W1=CDEXP(X1)
      W2=G(M)/R(2)
      W3=W0+W2
160   F(M)=W1*W3
C     *****************************
      U=DREAL(S)
      V=DIMAG(S)
      WRITE(2,230)U,V
      NP=NT+2
      WRITE(2,240)NP
      DO 170 M=0,NT+2
      U=ZETA(M)
      PHI=DACOS(U)
      PHI=(PHI*180.D0)/PIE
      AMP=CDABS(F(M))
      WRITE(*,250)PHI,AMP
170   WRITE(2,250)PHI,AMP
      CLOSE(2,STATUS='KEEP')
3     CONTINUE
C     ********************************
200   FORMAT(2X,I4,/,3(2X,I4),2(/,2(2X,F7.3)),/,2X,A2)
210   FORMAT(4(2X,F7.3))
220   FORMAT(2X,A13)
230   FORMAT(2(2X,F8.4))
240   FORMAT(2X,I4)
250   FORMAT(2(2X,F8.4))
      STOP
      END
```

```
      PROGRAM SFHS1C
      DOUBLE PRECISION ABERR,AH,AK,AMP,AMP1,AMP2,AMP3,D,DELTAT,DNT,DNT1,
     1DNT2,EPLN,ESLN,H,PHI,PIE,RAD,U,U0,V,V0,T,T0,ZETA,WNK
      COMPLEX*16 A0,A,ALPHA,BETA,C1,C2,CAK,EYE,ERR,FIN,G,P,Q,R,S,SEQA,
     1SEQB,Y,Y0,D1Y,W,W0,W1,W2,X,Z,Z0
      CHARACTER FNAME(3)*13,PATH*2,PFNAME*15,XNAME*13
      DIMENSION A(0:300),AMP(0:202),P(4),Q(4),R(4),SEQA(3),SEQB(3),
     1ZETA(0:202)
      OPEN(1,FILE='SFHS1C.INP')
      READ(1,100)NF,NT,NT1,NT2,NTRM,N0,NC,EPLN,ESLN,D,AH,RAD,WNK,PATH
      DO 1 I=1,NF
1     READ(1,210)SEQA(I),SEQB(I)
      DO 2 I=1,NF
2     READ(1,220)FNAME(I)
      CLOSE(1,STATUS='KEEP')
      WRITE(*,100)NF,NT,NT1,NT2,NTRM,N0,NC,EPLN,ESLN,D,AH,RAD,WNK,PATH
      PIE=4.D0*DATAN(1.D0)
      DNT=DBLE(NT)
      DNT1=DBLE(NT1)
      DNT2=DBLE(NT2)
      C1=DCMPLX(1.D0,0.D0)
      C2=DCMPLX(2.D0,0.D0)
      EYE=DCMPLX(0.D0,1.D0)
      AK=RAD*WNK
      CAK=DCMPLX(AK,0.D0)
      W0=-EYE*CAK
      Y0=-C2*CDEXP(W0)
      CALL HCOEF2(RAD,WNK,P,Q)
      DO 3 I=1,NF
      XNAME=FNAME(I)
      PFNAME=PATH//XNAME
      OPEN(2,FILE=PFNAME)
      ALPHA=SEQA(I)
      BETA=SEQB(I)
      DO 20 J=1,4
20    R(J)=ALPHA*P(J)+BETA*Q(J)
      S=R(1)/R(2)
```

```
      V=DIMAG(S)
      IF(V.GT.0.D0)THEN
      SGN=-1.D0
      ELSE
      SGN=1.D0
      ENDIF
      H=SGN*AH
C     *********** (1) ****************
      DELTAT=1.D0/DNT
      Y=Y0
      A0=Y0
      M=0
      Z=C1
      ZETA(M)=1.D0
      X=-EYE*CAK*Z
      W1=CDEXP(X)
      W2=ALPHA+BETA*X
      FIN=-W1*W2
      D1Y=(FIN-R(1)*Y)/(Z*R(2))
      AMP(M)=CDABS(Y)
      ERR=R(1)*Y+R(2)*Z*D1Y-FIN
      ABERR=CDABS(ERR)
      WRITE(*,110)M,Z,ABERR,AMP(M)
      DO 30 M=1,NT
      T0=DBLE(M-1)*DELTAT
      U0=1.D0+(D-1.D0)*T0
      V0=0.D0
      Z0=DCMPLX(U0,V0)
      CALL ACOEF1(NTRM,AK,ALPHA,BETA,Z0,A0,R,A)
      T=DBLE(M)*DELTAT
      U=1.D0+(D-1.D0)*T
      V=0.D0
      ZETA(M)=U
      Z=DCMPLX(U,V)
      CALL PHI1(NTRM,N0,NC,EPLN,ESLN,Z0,Z,A,Y,D1Y)
      AMP(M)=CDABS(Y)
      X=-EYE*CAK*Z
```

```
      W1=CDEXP(X)
      W2=ALPHA+BETA*X
      FIN=-W1*W2
      ERR=R(1)*Y+R(2)*Z*D1Y-FIN
      ABERR=CDABS(ERR)
      A0=Y
      WRITE(*,110)M,Z,ABERR,AMP(M)
30    CONTINUE
C     *************
      M=NT+1
      ZETA(M)=0.D0
      Y=-ALPHA/R(1)
      AMP(M)=CDABS(Y)
C     *********** (2) ***************
      DELTAT=1.D0/DNT1
      Y=Y0
      A0=Y0
      M=0
      Z=C1
      X=-EYE*CAK*Z
      W1=CDEXP(X)
      W2=ALPHA+BETA*X
      FIN=-W1*W2
      D1Y=(FIN-R(1)*Y)/(Z*R(2))
      AMP1=CDABS(Y)
      ERR=R(1)*Y+R(2)*Z*D1Y-FIN
      ABERR=CDABS(ERR)
      WRITE(*,110)M,Z,ABERR,AMP1
      DO 40 M=1,NT1
      T0=DBLE(M-1)*DELTAT
      U0=1.D0
      V0=H*T0
      Z0=DCMPLX(U0,V0)
      CALL ACOEF1(NTRM,AK,ALPHA,BETA,Z0,A0,R,A)
      T=DBLE(M)*DELTAT
      U=1.D0
      V=H*T
```

```
      Z=DCMPLX(U,V)
      CALL PHI1(NTRM,NO,NC,EPLN,ESLN,Z0,Z,A,Y,D1Y)
      AMP1=CDABS(Y)
      X=-EYE*CAK*Z
      W1=CDEXP(X)
      W2=ALPHA+BETA*X
      FIN=-W1*W2
      ERR=R(1)*Y+R(2)*Z*D1Y-FIN
      ABERR=CDABS(ERR)
      A0=Y
      WRITE(*,110)M,Z,ABERR,AMP1
40    CONTINUE
C     *********** (3) ***************
      DELTAT=1.D0/DNT2
      DO 50 M=1,NT2
      T0=DBLE(M-1)*DELTAT
      U0=1.D0-2.D0*T0
      V0=H
      Z0=DCMPLX(U0,V0)
      CALL ACOEF1(NTRM,AK,ALPHA,BETA,Z0,A0,R,A)
      T=DBLE(M)*DELTAT
      U=1.D0-2.D0*T
      V=H
      Z=DCMPLX(U,V)
      CALL PHI1(NTRM,NO,NC,EPLN,ESLN,Z0,Z,A,Y,D1Y)
      AMP2=CDABS(Y)
      X=-EYE*CAK*Z
      W1=CDEXP(X)
      W2=ALPHA+BETA*X
      FIN=-W1*W2
      ERR=R(1)*Y+R(2)*Z*D1Y-FIN
      ABERR=CDABS(ERR)
      A0=Y
      WRITE(*,110)M,Z,ABERR,AMP2
50    CONTINUE
C     *********** (4) ***************
      DELTAT=1.D0/DNT1
```

```
      DO 60 M=1,NT1
      T0=DBLE(M-1)*DELTAT
      U0=-1.D0
      V0=H*(1.D0-T0)
      Z0=DCMPLX(U0,V0)
      CALL ACOEF1(NTRM,AK,ALPHA,BETA,Z0,A0,R,A)
      T=DBLE(M)*DELTAT
      U=-1.D0
      V=H*(1.D0-T)
      Z=DCMPLX(U,V)
      CALL PHI1(NTRM,N0,NC,EPLN,ESLN,Z0,Z,A,Y,D1Y)
      AMP3=CDABS(Y)
      X=-EYE*CAK*Z
      W1=CDEXP(X)
      W2=ALPHA+BETA*X
      FIN=-W1*W2
      ERR=R(1)*Y+R(2)*Z*D1Y-FIN
      ABERR=CDABS(ERR)
      A0=Y
      WRITE(*,110)M,Z,ABERR,AMP3
60    CONTINUE
C     *********** (5) **************
      NP=2*NT+2
      ZETA(NP)=-1.D0
      AMP(NP)=AMP3
      DELTAT=1.D0/DNT
      DO 70 M=1,NT
      N=2*NT-M+2
      T0=DBLE(M-1)*DELTAT
      U0=-1.D0+(1.D0-D)*T0
      V0=0.D0
      Z0=DCMPLX(U0,V0)
      CALL ACOEF1(NTRM,AK,ALPHA,BETA,Z0,A0,R,A)
      T=DBLE(M)*DELTAT
      U=-1.D0+(1.D0-D)*T
      ZETA(N)=U
      V=0.D0
```

```
      Z=DCMPLX(U,V)
      CALL PHI1(NTRM,NO,NC,EPLN,ESLN,ZO,Z,A,Y,D1Y)
      AMP(N)=CDABS(Y)
      X=-EYE*CAK*Z
      W1=CDEXP(X)
      W2=ALPHA+BETA*X
      FIN=-W1*W2
      ERR=R(1)*Y+R(2)*Z*D1Y-FIN
      ABERR=CDABS(ERR)
      AO=Y
      WRITE(*,110)M,Z,ABERR,AMP(N)
70    CONTINUE
C     **********************************
      U=DREAL(S)
      V=DIMAG(S)
      WRITE(2,240)U,V
      WRITE(2,230)NP
      DO 80 N=0,NP
      U=ZETA(N)
      PHI=DACOS(U)
      PHI=(PHI*180.D0)/PIE
      WRITE(*,120)PHI,AMP(N)
80    WRITE(2,120)PHI,AMP(N)
      CLOSE(2,STATUS='KEEP')
3     CONTINUE
C     ***********************************
100   FORMAT(2X,I4,/,3(2X,I4),/,3(2X,I4),/,2(2X,D8.1),2(/,2(2X,F7.3)),
     1/,2X,A2)
110   FORMAT(2X,I4,2X,'(',F7.3,',',F7.3,')',2X,D22.15,4X,F10.5)
120   FORMAT(2(2X,F9.4))
210   FORMAT(4(2X,F7.3))
220   FORMAT(2X,A13)
230   FORMAT(2X,I4)
240   FORMAT(2(2X,F7.4))
310   FORMAT(2X,I4,2X,F9.4)
      STOP
      END
```

```
      PROGRAM SFHS1S
      DOUBLE PRECISION ABERR,AK,AMP,DELTAT,DN,DNT,EPLN,ESLN,PIE,RAD,
     1T,TO,WNK,ZETA
      COMPLEX*16 A,ALPHA,BETA,CA,CB,CO,C1,CAK,CN,E,ERR,EYE,F,FIN,
     1P,Q,R,U,V,W1,W2,X,Y,D1Y,Z,Z0
      CHARACTER PATH*2,FNAME(10)*13,XNAME*13,PFNAME*15
      DIMENSION CA(3),CB(3),P(4),Q(4),R(4),A(0:300)
      OPEN(1,FILE-'SFHS1S.INP')
      READ(1,100)NF,NT,NTRM,NO,NC,EPLN,ESLN,RAD,WNK,PATH
      WRITE(*,100)NF,NT,NTRM,NO,NC,EPLN,ESLN,RAD,WNK,PATH
      DO 10 I-1,NF
10    READ(1,110)CA(I),CB(I)
      DO 20 I-1,NF
20    READ(1,120)FNAME(I)
      CLOSE(1,STATUS-'KEEP')
      DNT-DBLE(NT)
      EYE-DCMPLX(0.D0,1.D0)
      PIE-4.D0*DATAN(1.D0)
      DELTAT-PIE/DNT
      C0-DCMPLX(0.D0,0.D0)
      C1-DCMPLX(1.D0,0.D0)
      AK-RAD*WNK
      CAK-DCMPLX(AK,0.D0)
      DO 30 I-1,NF
      XNAME-FNAME(I)
      PFNAME-PATH//XNAME
      OPEN(2,FILE-PFNAME)
      CALL HCOEF2(RAD,WNK,P,Q)
      ALPHA-CA(I)
      BETA-CB(I)
      DO 40 J-1,4
40    R(J)-ALPHA*P(J)+BETA*Q(J)
      WRITE(*,*)R
      A(0)--ALPHA/R(1)
      U-C1
      V--EYE*CAK
      DO 50 N-1,NT
```

```
      DN=DBLE(N)
      CN=DCMPLX(DN,0.D0)
      U=U*V/CN
      E=ALPHA+CN*BETA
      F=R(1)+CN*R(2)
50    A(N)=-E*U/F
      T=0.D0
      Z0=C0
      DO 60 N=0,NT
      DN=DBLE(N)
      T=DN*DELTAT
      ZETA=DCOS(T)
      Z=DCMPLX(ZETA,0.D0)
      Z0=C0
      CALL PHI1(NTRM,N0,NC,EPLN,ESLN,Z0,Z,A,Y,D1Y)
      X=-CAK*Z*EYE
      W1=ALPHA+BETA*X
      W2=CDEXP(X)
      FIN=-W1*W2
      ERR=R(1)*Y+R(2)*Z*D1Y-FIN
      ABERR=CDABS(ERR)
      AMP=CDABS(Y)
      WRITE(*,130)N,ABERR,AMP
60    WRITE(2,140)N,AMP
      CLOSE(2,STATUS='KEEP')
30    CONTINUE
100   FORMAT(5(2X,I4),/,2(2X,D8.1),/,2(2X,F7.3),/,2X,A2)
110   FORMAT(4(2X,F7.3))
120   FORMAT(2X,A13)
130   FORMAT(2X,I4,2X,D16.10,2X,F9.4)
140   FORMAT(2X,I4,2X,F9.4)
      STOP
      END
```

```
      PROGRAM SFHS1L
      DOUBLE PRECISION ABERR,AK,AMP,DELTAT,DN,DNT,E,EPLN,ESLN,PIE,RAD,
     1T,T0,WNK,ZETA
      COMPLEX*16 A,ALPHA,BETA,CA,CB,C0,C1,C2,C3,C6,CE,CE2,CE3,CAK,CN,
     1EYE,F,FIN,G,H,H2,H3,P,Q,R,U,V,W1,W2,W3,X,Y,D1Y,Z,Z0
      CHARACTER PATH*2,FNAME(10)*13,XNAME*13,PFNAME*15
      DIMENSION CA(3),CB(3),P(4),Q(4),R(4),A(0:300)
      OPEN(1,FILE='SFHS1L.INP')
      READ(1,100)NF,NT,NTRM,NO,NC,EPLN,ESLN,RAD,WNK,E,PATH
      WRITE(*,100)NF,NT,NTRM,NO,NC,EPLN,ESLN,RAD,WNK,E,PATH
      DO 10 I=1,NF
10    READ(1,110)CA(I),CB(I)
      DO 20 I=1,NF
20    READ(1,120)FNAME(I)
      CLOSE(1,STATUS='KEEP')
      DNT=DBLE(NT)
      EYE=DCMPLX(0.D0,1.D0)
      PIE=4.D0*DATAN(1.D0)
      DELTAT=PIE/DNT
      C0=DCMPLX(0.D0,0.D0)
      C1=DCMPLX(1.D0,0.D0)
      C2=DCMPLX(2.D0,0.D0)
      C3=DCMPLX(3.D0,0.D0)
      C6=DCMPLX(6.D0,0.D0)
      AK=RAD*WNK
      CAK=DCMPLX(AK,0.D0)
      H=C2*EYE*CAK
      H2=H*H
      H3=H2*H
      CE=DCMPLX(E,0.D0)
      CE2=CE*CE
      CE3=CE2*CE
      P(1)=-(C1+C3*CE+C3*H*CE*CE)/C2
      P(2)=C0
      W1=C1/CE
      W2=C2-(H2/C2)
      W3=-(H2/C2)+(H3/C6)
```

```
      Q(1)=(W1+C1-W2*CE2+W3*CE3)/C2
      Q(2)=-CE2/C2
      DO 30 I=1,NF
      XNAME=FNAME(I)
      PFNAME=PATH//XNAME
      OPEN(2,FILE=PFNAME)
      ALPHA=CA(I)
      BETA=CB(I)
      DO 40 J=1,2
40    R(J)=ALPHA*P(J)+BETA*Q(J)
      WRITE(*,*)R
      A(0)=-ALPHA/R(1)
      U=C1
      V=-EYE*CAK
      DO 50 N=1,NT
      DN=DBLE(N)
      CN=DCMPLX(DN,0.D0)
      U=U*V/CN
      F=ALPHA+CN*BETA
      G=R(1)+CN*R(2)
50    A(N)=-F*U/G
      T=0.D0
      Z0=C0
      DO 60 N=0,NT
      DN=DBLE(N)
      T=DN*DELTAT
      ZETA=DCOS(T)
      Z=DCMPLX(ZETA,0.D0)
      Z0=C0
      CALL PHI1(NTRM,NO,NC,EPLN,ESLN,Z0,Z,A,Y,D1Y)
      X=-CAK*Z*EYE
      W1=ALPHA+BETA*X
      W2=CDEXP(X)
      FIN=-W1*W2
      ERR=R(1)*Y+R(2)*Z*D1Y-FIN
      ABERR=CDABS(ERR)
      AMP=CDABS(Y)
```

```
      WRITE(*,130)N,ABERR,AMP
60    WRITE(2,140)N,AMP
      CLOSE(2,STATUS='KEEP')
30    CONTINUE
100   FORMAT(5(2X,I4),/,2(2X,D8.1),/,3(2X,F7.3),/,2X,A2)
110   FORMAT(4(2X,F7.3))
120   FORMAT(2X,A13)
130   FORMAT(2X,I4,2X,D16.10,2X,F9.4)
140   FORMAT(2X,I4,2X,F9.4)
      STOP
      END
```

```
      SUBROUTINE HCOEF2(RAD,WNK,P,Q)
      DOUBLE PRECISION RAD,WNK,BC,AK,AH,A,DN
      COMPLEX*16 W,W1,W2,W3,W4,S,H,X,Y,Z,DX,DY,DZ,B,B3,B11,B33,
     1D,D3,D11,D33,T,T1,T2,T3,T4,T5,P,Q,CBC,C,C102
      DIMENSION DX(0:15),DY(0:15),DZ(0:15),BC(0:15,0:15),S(0:15,0:15),
     1P(4),Q(4),C(32),CBC(0:15,0:15)
      DO 30 N=0,15
      BC(N,0)=1.D0
      A=BC(N,0)
      CBC(N,0)=DCMPLX(A,0.D0)
      DO 40 M=1,N
      BC(N,M)=BC(N,M-1)*DBLE(N-M+1)/DBLE(M)
      A=BC(N,M)
40    CBC(N,M)=DCMPLX(A,0.D0)
30    CONTINUE
      DO 70 N=1,32
      DN=DBLE(N)
70    C(N)=DCMPLX(DN,0.D0)
      AH=2.D0*WNK*RAD
      H=DCMPLX(0.D0,AH)
      W=DCMPLX(DCOS(AH),DSIN(AH))
      X=W-C(1)
      Y=C(1)/H
      Z=X*Y
      DY(0)=Y
      DO 10 N=1,15
10    DY(N)=-C(N)*Y*DY(N-1)
      DX(0)=X
      DO 20 N=1,15
20    DX(N)=W
      DZ(0)=Z
      DO 50 N=0,15
      S(N,0)=CBC(N,0)*DX(0)*DY(N)
      DO 60 M=1,N
60    S(N,M)=S(N,M-1)+CBC(N,M)*DX(M)*DY(N-M)
50    DZ(N)=S(N,N)
      B=-(H*DZ(1)-DZ(0))/C(2)
```

```
B3=H*DZ(3)-DZ(2)
B11=H*DZ(5)-DZ(4)-H*DZ(3)+DZ(2)
B33=-C(2)*(H*DZ(5)-DZ(4))
W1=C(4)-H-C(4)*Y
W2=W1*W
W3=C(4)*Y-H
W4=W2+W3
D=W4/C(4)
C102=DCMPLX(0.5D0,0.D0)
W1=H-C(5)+C(16)*Y-C(32)*Y*Y+C(32)*Y*Y*Y
W2=C(16)*Y*Y*Y
W3=C102*W1*W
D3=W3-W2
T1=DZ(4)-DZ(6)
T2=DZ(3)-DZ(5)
T3=DZ(2)-DZ(4)
T4=DZ(1)-DZ(3)
T5=DZ(0)-DZ(2)
T=H*H*T1-C(5)*H*T2+C(5)*T3+H*T4-T5
D11=-T/C(2)
T1=DZ(6)
T2=DZ(5)
T3=DZ(4)
T4=DZ(3)
T5=DZ(2)
T=H*H*T1-C(5)*H*T2+C(5)*T3+H*T4-T5
D33=-T
P(1)=B-C102
P(2)=B3
P(3)=C102*B11
P(4)=C102*(B11-B33)
Q(1)=D
Q(2)=D3
Q(3)=C102*D11
Q(4)=(D11-D33)/C(2)
RETURN
END
```

```
      SUBROUTINE ACOEF1(NT,AK,ALPHA,BETA,Z0,A0,R,A)
      DOUBLE PRECISION AK,DN
      COMPLEX*16 A0,A,ALPHA,BETA,B1,B2,C1,CN,CN1,CAK,EYE,F0,F,
     1R,U,V,X0,W,W0,W1,Z0
      DIMENSION A(0:300),F(0:300),R(4)
      C1=DCMPLX(1.D0,0.D0)
      EYE=DCMPLX(0.D0,1.D0)
      CAK=DCMPLX(AK,0.D0)
      X0=-EYE*CAK*Z0
      W0=CDEXP(X0)
      W1=ALPHA+BETA*X0
      F(0)=-W0*W1
      U=C1
      V=-EYE*CAK
      DO 2 N=1,NT
      DN=DBLE(N)
      CN=DCMPLX(DN,0.D0)
      W=W1+CN*BETA
      U=(U*V)/CN
2     F(N)=-W0*W*U
      A(0)=A0
      A(1)=(F(0)-R(1)*A(0))/(R(2)*Z0)
      DO 10 N=2,NT
      DN=DBLE(N)
      CN=DCMPLX(DN,0.D0)
      CN1=CN-C1
      B1=R(1)+CN1*R(2)
      B2=Z0*CN*R(2)
      A(N)=(F(N-1)-B1*A(N-1))/B2
10    CONTINUE
      RETURN
      END
```

```
      SUBROUTINE PHI1(NT,NO,NC,EPLN,ESLN,Z0,Z,A,Y,D1Y)
      DOUBLE PRECISION AS,AT,AW,CN,DN,EPLN,ESLN,ETA
      COMPLEX*16 A,C0,C1,S,T,W,Y,D1Y,Z0,Z
      DIMENSION A(0:300),AS(0:300),S(0:300),W(0:300)
      C0=DCMPLX(0.D0,0.D0)
      C1=DCMPLX(1.D0,0.D0)
      W(0)=C1
      W(1)=Z-Z0
      DO 1 N=2,NT
      AW=CDABS(W(N-1))
      IF(AW.LT.ESLN)THEN
      W(N)=C0
      ELSE
      W(N)=W(N-1)*W(1)
      ENDIF
1     CONTINUE
      S(0)=A(0)
      AS(0)=CDABS(A(0))
      DO 10 N=1,NT
      T=A(N)*W(N)
      AT=CDABS(T)
      S(N)=S(N-1)+T
      AS(N)=AS(N-1)+AT
      IF(N.LT.NO) GOTO 10
      ETA=AS(N)-AS(N-NC)
      IF(ETA.LT.EPLN) THEN
      Y=S(N)
      GOTO 12
      ELSEIF(ETA.GT.EPLN.AND.N.LT.NT) THEN
      GOTO 10
      ELSEIF(ETA.GT.EPLN.AND.N.EQ.NT) THEN
      Y=S(NT)
      WRITE(*,11)Z,ETA
11    FORMAT(2X,'F(',F8.3,') DOES NOT CONVERGE,ETA=',D22.15)
      ENDIF
10    CONTINUE
12    S(0)=A(1)
```

```
      AS(0)=CDABS(A(1))
      NT1=NT-1
      DO 20 N=1,NT1
      DN=DBLE(N)
      CN=DCMPLX(DN,0.D0)
      T=(CN+C1)*A(N+1)*W(N)
      AT=CDABS(T)
      S(N)=S(N-1)+T
      AS(N)=AS(N-1)+AT
      IF(N.LT.NO) GOTO 20
      ETA=AS(N)-AS(N-NC)
      IF(ETA.LT.EPLN) THEN
      D1Y=S(N)
      GOTO 22
      ELSEIF(ETA.GT.EPLN.AND.N.LT.NT1) THEN
      GOTO 20
      ELSEIF(ETA.GT.EPLN.AND.N.EQ.NT1) THEN
      D1Y=S(NT1)
      WRITE(*,21)Z,ETA
21    FORMAT(2X,'D1F(',F8.3,') DOES NOT CONVERGE,ETA=',D22.15)
      ENDIF
20    CONTINUE
22    RETURN
      END
```

```
      COMPLEX*16 FUNCTION ARG(Z)
      DOUBLE PRECISION HPIE,PHI,PIE,U,V,W
      COMPLEX*16 Z
      PIE=4.D0*DATAN(1.D0)
      HPIE=PIE/2.D0
      U=DREAL(Z)
      V=DIMAG(Z)
      IF(U.EQ.0.D0)GOTO 10
      W=V/U
      PHI=DATAN(W)
      IF(U.GT.0.D0)THEN
      ARG=DATAN(W)
      ELSEIF(U.LT.0.D0.AND.V.GT.0.D0)THEN
      ARG=HPIE+PHI
      ELSEIF(U.LT.0.D0.AND.V.LT.0.D0)THEN
      ARG=-HPIE+PHI
      ENDIF
      RETURN
10    IF(V.GT.0.D0)THEN
      ARG=HPIE
      ELSEIF(V.LT.0.D0)THEN
      ARG=-HPIE
      ENDIF
      RETURN
      END
```

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