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STATE DIAGRAMS FOR BOUNDED AND UNBOUNDED LINEAR OPERATORS

by

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SYNOPSIS

The theme of this thesis is the construction of state diagrams and their implications. The author generalises most of the theorems in Chapter II of Goldberg [G1] by dropping the assumption that the domain of the operator is dense in X. The author also presents the standard Taylor–Halberg–Goldberg state diagrams [G1, 61, 66]. Chapters II and III deal with F_+– and F_-–operators, which are generalisations of the ϕ_+– and ϕ_-–operators in Banach spaces of Gokhberg–Krein [GK]. Examples are given of F_+– and F_-–operators.

Also, in Chapter III, the main theorems needed to construct the state diagrams of Chapter IV are discussed. The state diagrams of Chapter IV are based on states corresponding to F_+– and F_-–operators; in addition state diagrams relating T and T'' under the assumptions $\gamma(T) > 0$ and $\gamma(T') > 0$ are derived. Second adjoints are important in Tauberian Theory (see Cross [C1]).

Chapters I and IV are the main chapters. In Chapter I of this thesis the author modifies many of the proofs appearing in Goldberg [G1], to take account of the new definition of the adjoint.

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CONTENTS

NOTATION

INTRODUCTION

In this introduction we present the definitions of vector spaces, normed linear spaces and Banach spaces. We also provide examples of these spaces. Quotient spaces, adjoint spaces, and reflexive spaces are introduced and the main theorems are stated.

0.1 Normed Linear Spaces

0.1.1 Definition [T] Let X be a set of elements, sometimes called *points*, and denoted by small italic letters: x, y, ... We assume that each pair of elements x, y can be combined by a process called addition to yield another element z denoted by $z = x + y$. We also assume that each complex number α and each element x can be combined by a process called multiplication to yield another element y denoted by $y = \alpha x$. The set X with these two processes is called a *vector space* if the following axioms are satisfied :

1. $x + y = y + x$.
2. $x + (y + z) = (x + y) + z$.
3. There is in X a unique element, denoted by 0 and called the zero element, such that $x + 0 = x$ for each x .
4. To each x in X corresponds a unique element, denoted by $-x$, such that $x + (-x) = 0$.
5. $\alpha(x + y) = \alpha x + \alpha y$.
6. $(\alpha + \beta)x = \alpha x + \beta x$.
7. $\alpha(\beta x) = (\alpha\beta)x$.

8. $1.x = x$.

9. $0.x = 0$.

We now introduce the concept of normed linear spaces.

0.1.2 Definition [G1] Let X be a vector space over the field of real or complex numbers. A norm on X , denoted by $\| \ \|$, is a real–valued function on X with the following properties.

i. $\|x\| \geq 0$ for all $x \in X$.

ii. $x \neq 0$ implies $\|x\| \neq 0$.

iii. $\|\alpha x\| = |\alpha| \, \|x\|$

iv. $\|x + y\| \leq \|x\| + \|y\|$ (triangular inequality).

The vector space X , together with a norm on X , is called a *normed linear space.* When the scalars over X are the reals, X is called a *real normed linear space.*

The following are examples of normed linear spaces.

0.1.3 Example [G1] Euclidean n–space, denoted by E^n , is the normed linear space of n–tuples of real numbers over the reals with norm

$$\|(a_1, a_2, \ldots, a_n)\| = (\Sigma \, |a_j|^2)^{\frac{1}{2}} .$$

Unitary n–space, denoted by U^n , is the normed linear space of n–tuples of complex numbers over the complex numbers with the above norm.

0.1.4 Example [G1] For S an arbitrary set, B(S) is the normed linear space of bounded complex–valued functions over the complex numbers with norm

$$\|f\| = \sup_{s \in S} |f(s)| .$$

0.1.5 Example [G1] For S a compact topological space, C(S) is the subspace of B(S) consisting of the continuous functions.

0.1.6 Example [G1] Let $1 \leq p < \infty$ and let μ be a measure defined on a σ–ring Σ of subsets of a set S. Define $L_p^0(S,\Sigma,\mu)$ to be the class of all those μ–measurable complex–valued functions f for which $|f|^p$ is integrable. Functions f and g in $L_p^0(S,\Sigma,\mu)$ are said to be equivalent if $f = g$ almost everywhere. $L_p(S,\Sigma,\mu)$ will denote the normed linear space of equivalence classes [f] of $f \in L_p^0(S,\Sigma,\mu)$ with norm given by

$$\|[f]\|_p = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}} .$$

Addition and scalar multiplication are defined by

$$\alpha[f] + \beta[g] = [\alpha f + \beta g] .$$

As is customary, we shall use f instead of [f] as an element in $L_p(S,\Sigma,\mu)$. The proof of the triangular inequality, called Minkowski's inequality, appears in Dunford and Schwartz [DS], Lemma II.3.3.

When μ is Lebesgue measure and Σ is the class of Lebesgue–measurable sets, we write $L_p(S)$ instead of $L_p(S,\Sigma,\mu)$.

0.2 Complete Normed Linear Spaces

0.2.1 Definition [G1] Let X be a normed linear space. The metric d induced by the norm is defined by $d(x,y) = \|x-y\|$. For $x \in X$ and M a subset of X , d(x,M) will denote the distance from x to M ; that is ,

$$d(x,M) = \inf_{m \in M} \|x-m\| .$$

If X is a complete metric space with respect to d , X is called a *complete normed linear space* or a *Banach space.* The metric topology on X determined by d is the topology used throughout this thesis.

0.2.2 Example [G1] B(S) is complete. To see this, suppose $\{f_n\}$ is a Cauchy sequence in B(S) . Given $\epsilon > 0$, there exists an integer N such that for all $s \in S$,

(1) $$|f_n(s) - f_m(s)| \leq \|f_n - f_m\| \leq \epsilon \quad m,n \geq N .$$

Hence for each $s \in S$, $\{f_n(s)\}$ is a Cauchy sequence of scalars and therefore converges. Define f on S by $f(s) = \lim_{n \to \infty} f_n(s)$ It is now shown that f is in B(S) and that $f_n \to f$ in B(S) . Since $\{f_n\}$ converges pointwise to f , it follows from (1), after fixing $n \geq N$ and letting $m \to \infty$, that for all $s \in S$

(2) $$|f_n(s) - f(s)| \leq \epsilon \quad n \geq N .$$

Since f_N is bounded, (2) implies that f is in B(S) and that

$$\|f_n - f\| = \sup |f_n(s) - f(s)| \leq \epsilon \quad n \geq N .$$

Thus $f_n \to f$ in B(S) . □

0.2.3 Definition [G1] Let X and Y be normed linear spaces. Define the norm on the vector space of bounded linear operators which map X into Y by

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} .$$

Then this is a norm and [X,Y] will denote the normed linear space of bounded linear operators with the above norm.

0.2.4 Theorem [G1] If X is a normed linear space and Y is a Banach space, then [X,Y] is a Banach space.

Proof [G1] The proof is essentially the same as the proof in 0.2.2, which shows that B(S) is complete.

The converse to the theorem is proved in Corollary 0.3.9. □

0.2.5 Definition [G1] Let T be a 1–1 linear operator with domain $D(T) \subset X$ and range $R(T) \subset Y$, X and Y normed linear spaces. The inverse of T , written T^{-1} , is the map from subspace R(T) into X given by $T^{-1}(Tx) = x$. It is clear that T^{-1} is linear. N(T) denotes the subspace $\{x \mid Tx = 0\}$. N(T) is called the null–space or kernel of T . T is called *1–1* if distinct elements in D(T) are mapped into distinct elements in R(T) . Since a linear operator T has the property that $T0 = 0$, T is 1–1 if and only if $N(T) = (0)$.

For a function f which is not necessarily 1–1 , $f^{-1}(B)$ will be used to denote the set $\{x \mid f(x) \in B\}$.

0.2.6 Theorem [G1] Let T be a linear map from normed linear space X into normed linear space Y. T^{-1} exists and is continuous if and only if there exists an $m > 0$ such that

$$\|Tx\| \geq m\|x\| \quad x \in X .$$

Proof [G1] Suppose $\|Tx\| \geq m\|x\|$ for all $x \in X$. Then $x \neq 0$ implies $Tx \neq 0$. Hence T is 1–1. Since

$$\|T^{-1}Tx\| = \|x\| \leq m^{-1}\|Tx\| .$$

T^{-1} is bounded and therefore continuous. On the other hand, if T^{-1} is continuous, then

$$\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\| \, \|Tx\| \quad , \ x \in X .$$

The theorem follows upon taking $m = 1/\|T^{-1}\|$. □

0.3 Hahn–Banach Extension Theorem

0.3.1 Definition [G1] A *functional* on a vector space V is a map from V to the scalars. The *adjoint* X' of a normed linear space X is the Banach space of bounded linear functionals on X; that is, $X' = [X,Y]$, where Y is the Banach space of scalars with absolute value taken as norm. Note that X' is complete by Theorem 0.2.4.

The following theorem is one of the most fundamental theorems in functional analysis. For a proof the reader is referred to Goldberg [G1].

0.3.2 Hahn–Banach Extension Theorem [G1] Suppose X is a vector space over the real or complex numbers. Let M be a subspace of X and let p be a real–valued function on X with the following properties.

(i) $p(x+y) \leq p(x) + p(y)$.

(ii) $p(\alpha x) = |\alpha| p(x)$.

If f is a linear functional on M such that

$$|f(m)| \leq p(m) \quad m \in M$$

then there exists a linear functional F which is an extension of f to all of X such that

$$F(x) \leq p(x) \quad x \in X.$$

0.3.3 Remark [G1] In the proof of the above theorem, the following result has been shown.

Suppose X is a vector space over the reals. Let M be a subspace of X and let p be a real–valued function on X with the following properties :

$$p(x+y) \leq p(x) + p(y)$$

$$p(\alpha x) = \alpha p(x) \ , \ \alpha \geq 0 .$$

If f is a linear functional on M such that

$$f(m) \leq p(m) \quad m \in M$$

then there exists a linear functional F which is an extension of f to all of X such that

$$F(x) \leq p(x) \quad x \in X.$$

0.3.4 Corollary [G1] Let m' be a continuous linear functional on a subspace M of normed linear space X. There exists an $x' \in X'$ such that $x' = m'$ on M and $\|x'\| = \|m'\|$.

Proof. [G1] Define p on X by $p(x) = \|m'\|\,\|x\|$. Since p satisfies (i) and (ii) of Theorem 0.3.2 and $|m'm| \leq \|m'\|\,\|m\| = p(m)$, $m \in M$, there exists a linear functional x' on X such that $x' = m'$ on M and

$$|x'x| \leq p(x) = \|m'\|\,\|x\| \ , \ x \in X .$$

Thus x' is in X' and $\|x'\| \leq \|m'\|$. On the other hand ,

$$\|x'\| = \sup_{\substack{\|x\|=1 \\ x \in X}} |x'x| \geq \sup_{\substack{\|x\|=1 \\ x \in M}} |x'x| = \sup_{\substack{\|x\|=1 \\ x \in M}} |m'x| = \|m'\| .$$

Hence $\|x'\| = \|m'\|$. □

0.3.5 Corollary [G1] Let M be a subspace of normed linear space X . Given $x \in X$ with $d = d(x,M) > 0$, there exists an $x' \in X'$ such that

$$\|x'\| = 1 \ , \ x'M = 0 \text{ and } x'x = d(x,M) .$$

Proof [G1] Let M_1 be the subspace spanned by x and the elements of M . Define linear functional v' on M_1 by

$$v'(\alpha x + m) = \alpha d \quad m \in M .$$

Then $v'M = 0$ and $v'x = d$. We assert that v' is in M_1' with $\|v'\| = 1$. For $\alpha \neq 0$ and $m \in M$,

$$\|\alpha x + m\| = |\alpha|\ \|x + \frac{m}{\alpha}\| \geq |\alpha| d .$$

Thus for all α ,

$$|v'(\alpha x + m)| = |\alpha| d \leq \|\alpha x + m\| \quad m \in M .$$

Hence $v' \in M_1'$ and $\|v'\| \leq 1$. There exists a sequence $\{m_k\}$ in M such that $\|x - m_k\| \to d$. Since

$$d = v'(x - m_k) \leq \|v'\|\,\|x - m_k\| \to \|v'\| d$$

it follows that $\|v'\| \geq 1$. Thus $\|v'\| = 1$. The corollary follows upon taking $x' \in X'$ to be an extension of v' so that $\|x'\| = \|v'\| = 1$. □

0.3.6 Corollary [G1] Given $x \in X$, there exists an $x' \in X'$ such that $\|x'\| = 1$ and $x'x = \|x\|$. In particular, if $x \neq y$, there exists an $x' \in X'$ such that $0 \neq \|x-y\| = x'x - x'y$.

Proof [G1] Take $M = (0)$ in Corollary 0.3.5. □

0.3.7 Definition [G1] By the *1–sphere* of X' we mean the $\{x' \mid \|x'\| = 1\}$.

0.3.8 Corollary [G1] For any x in normed linear space X,

$$\|x\| = \sup_{\substack{\|x'\|=1 \\ x' \in X'}} |x'x| .$$

Proof [G1] For x' in the 1–sphere of X'

(1) $$|x'x| \leq \|x'\| \, \|x\| \leq \|x\| .$$

By Corollary 0.3.5 there exists a z' in the 1–sphere of X' such that

(2) $$z'x = \|x\| .$$

The corollary follows from (1) and (2). □

As a simple application of Corollary 0.3.5, we prove the converse to Theorem 0.2.4.

0.3.9 Corollary [G1] Let X and Y be normed linear spaces. If $[X,Y]$ is complete, then Y is complete.

Proof [G1] Let $\{y_n\}$ be a Cauchy sequence in Y. Choose $x_0 \in X$ such that $\|x_0\| = 1$. There exists an $x' \in X'$ such that $x'x_0 = \|x_0\| = 1$. Define $T_n \in [X,Y]$ by

$$T_n x = x'(x) y_n .$$

Now

$$\|(T_n - T_m)x\| = |x'x| \, \|y_n - y_m\| \le \|x'\| \, \|y_n - y_m\| \, \|x\| \, , \ x \in X \, .$$

Hence $\|T_n - T_m\| \le \|x'\| \, \|y_n - y_m\|$ which implies that $\{T_n\}$ is a Cauchy sequence in $[X,Y]$. Thus, by hypothesis, $\{T_n\}$ converges in $[X,Y]$ to some T. Since

$$\|y_n - Tx_0\| = \|T_n x_0 - Tx_0\| \le \|T_n - T\| \, \|x_0\|$$

$\{y_n\}$ converges to Tx_0 and therefore Y is complete. □

The following theorem is very useful and is needed in the proof of Theorem I.4.3. The theorem is a consequence of the Hahn Banach Theorem and is sometimes referred to as the "geometric form of the Hahn–Banach Theorem".

0.3.10 Definition [G1] A subset K of a vector space over the real or complex numbers is called *convex* if for every x and y in K, the set

$$\{ax + (1 - a)y \mid 0 \le a \le 1\}$$

is contained in K.

0.3.11 Definition By *Re z* the author denotes the real part of a complex number z.

0.3.12 Theorem [G1] Let K be a closed convex subset of a normed linear space X. Given $x \in X$ but not in K, there exists an $f \neq 0 \in X'$ such that

$$\text{Re}\, f(x) \ge \text{Re}\, f(k) \quad k \in K \, .$$

The proof of the theorem depends on the following lemma.

0.3.13 Definition [G1] Let 0 be an interior point of a convex subset K of the normed linear space X. For each $x \in X$ let

$$A(x) = \{a \mid a > 0 , \; x \in aK\}$$

where $aK = \{ak \mid k \in K\}$. Define the functional p on X by $p(x) = \inf A(x)$.

We shall call p the Minkowski functional of K. Since 0 is an interior point of K, $A(x) \neq \phi$ and $0 \leq p(x) < \infty$.

The proof of the following lemma is to be found in [G1] and [Sc1].

0.3.14 Lemma [G1] Let K and p be as in the above definition. Then for x and y in X

(i) $p(\alpha x) = \alpha p(x)$, $\alpha \geq 0$

(ii) $p(x + y) \leq p(x) + p(y)$

(iii) $p(z) \geq 1$ for all $z \notin K$.

The proof of Theorem 0.3.12 is to be found in [G1] and [Sc1].

0.4 Quotient Spaces

In linear algebra one encounters the concept of a quotient space X/M, where M is a subspace of vector space X. When X is a normed linear space and M is a closed subspace of X, a norm is put on X/M so that certain topological properties of operators defined on X are shared by corresponding operators on X/M.

0.4.1 Definition [G1] Let M be a closed subspace of normed linear space X. Define an equivalence relation R on X by xRy if $x-y$ is in M. Let X/M denote the corresponding set of equivalence classes and let $[x]$, called a coset, denote the set of elements equivalent to x. Thus

$$[x] = \{x + m \mid m \in M\} = x + M$$

Vector addition and scalar multiplication on X/M are defined by

$$[x] + [y] = [x + y]$$
$$\alpha[x] = [\alpha x] .$$

Define a norm on X/M by

$$\|[x]\| = d(x, M)$$

where d(x, M) is the distance from x to M. It is left to the reader to verify that X/M is a normed linear space. M is required to be closed in order that $\|[x]\| = 0$ implies $[x] = [0]$.

Geometrically, if X is the plane and M is a line through the origin then X/M is the space whose elements are M together with the lines parallel to M. The norm of [x] is the distance between the line containing x and the line M.

0.4.2 Remarks [G1]

(i) Since any $y \in [x]$ is of the form $x - m$, $m \in M$, it follows that $\|[x]\| = \inf_{y \in [x]} \|y\|$.

(ii) If [x] and [z] are such that $\|[x] - [z]\| < \epsilon$, there exists a $v \in [z]$ such that $\|x - v\| < \epsilon$.

0.4.3 Theorem [G1] If X is a Banach space and M is a closed subspace of X, then X/M is a Banach space.

Thinking of coset as being lines, the proof proceeds by considering a given Cauchy sequence of parallel lines (lines "crowded" together) and choosing points, one on each line, which are crowded together in the sense of being a Cauchy sequence. The Cauchy sequence of points converges to a point x, and the sequence of lines containing the points converges to a line containing x.

Proof [G1] Let $\{[x_n]\}$ be a Cauchy sequence in X/M. There exists a subsequence $\{[y_n]\}$ of $\{[x_n]\}$ such that

$$\|[y_{n+1}] - [y_n]\| < 2^{-n} \quad 1 \leq n$$

By Remark (ii) of 0.4.2, we may choose $v_n \in [y_n]$ so that

$$\|v_{n+1} - v_n\| < 2^{-n}$$

The sequence $\{v_n\}$ is a Cauchy sequence, since

$$\|v_{n+i} - v_n\| \leq \|v_{n+1} - v_n\| + \|v_{n+2} - v_{n+1}\| + \dots + \|v_{n+i} - v_{n+i-1}\|$$
$$\leq \sum_{k=0}^{\infty} 2^{-n-k} = 2^{-n+1} \to 0 \text{ as } n \to \infty .$$

The completeness of X assures the existence of a $v \in X$ such that $v_n \to v$. By Remark (i) of 0.4.2

$$\|[v_n] - [v]\| = \|[v_n - v]\| \leq \|v_n - v\| .$$

Thus $[v_n]$ converges to $[v]$ in X/M. Since $\{[v_n]\} = \{[y_n]\}$ is a subsequence of the Cauchy sequence $\{[x_n]\}$, $\{[x_n]\}$ also converges to $[v]$. Hence X/M is complete. □

0.5 Adjoint Spaces

0.5.1 Definition [G1] Normed linear spaces X and Y are called *equivalent* if there exists a linear isometry from X onto Y.

0.5.2 Definition [G1] A set K in normed linear space X is called *orthogonal* to a set $F \subset X'$ if $x'k = 0$ for all $k \in K$ and $x' \in F$.

The *orthogonal complement* in X' of K, denoted by $K^{\perp}$, is the set of elements in X' which is orthogonal to K.

Even if K is not a subspace, $K^{\perp}$ is a closed subspace of X'.

0.5.3 Theorem [G1] Let M be a subspace of normed linear space X. Then

(i) $X'/M^{\perp}$ is equivalent to M' under the map U defined by $U[x'] = x'_M$ where $[x']$ is in $X'/M^{\perp}$ and x'_M is the restriction of x' to M.

(ii) If M is closed (so that X/M is a normed linear space), then $(X/M)'$ is equivalent to $M^{\perp}$ under the map V defined by

$$(Vz')x = z'[x] \ , \ z' \in \left(\frac{X}{M}\right)' .$$

Proof of (i) [G1]. Note that U is unambiguously defined, since $[y'] = [x']$ implies $0 = y'm - x'm$, $m \in M$. Clearly, U is linear with range in M'. Given $m' \in M'$, there exists, by Corollary 0.3.4, an $x' \in X'$ which is an extension of m'. Hence $U[x'] = x'_M = m'$ which shows that $R(U) = M'$. For any $y' \in [x']$,

$$\|U[x']\| = \|y'_M\| \leq \|y'\| .$$

Thus

(1) $$\|U[x']\| \leq \inf_{y' \in [x']} \|y'\| = \|[x']\| .$$

On the other hand, there exists a $v' \in X'$ which is an extension of x'_M such that $\|v'\| = \|x'_M\|$. Therefore v' is in $[x']$ and

(2) $$\|U[x']\| = \|x'_M\| = \|v'\| \geq \|[x']\| .$$

By (1) and (2), $\|U[x']\| = \|[x']\|$.

Proof of (ii). [G1] For $z' \in (X/M)'$,

$$|(Vz')x| = |z'[x]| \le \|z'\| \, \|[x]\| \le \|z'\| \, \|x\| \qquad x \in X$$

and

$$(Vz')m = z'[m] = z'[0] = 0 \qquad m \in M .$$

Thus Vz' is in $M^{\perp}$ with

(3) $$\|Vz'\| \le \|z'\| .$$

Since

$$|z'[x]| = |(Vz')y| \le \|Vz'\| \, \|y\| \qquad y \in [x]$$

it follows that

$$|z'[x]| \le \|Vz'\| \, \|[x]\| .$$

Thus

$$\|z'\| \le \|Vz'\|$$

which, together with (3), proves that V is an isometry. Given $x' \in M^{\perp}$, let z' be the linear functional on X/M defined by $z'[x] = x'x$. Since

$$|z'[x]| = |x'y| \le \|x'\| \, \|y\| \qquad y \in [x]$$

it follows that $|z'[x]| \le \|x'\| \, \|[x]\|$. Hence z' is in $(X/M)'$. Furthermore $Vz' = x'$, proving that $R(V) = M^{\perp}$. □

0.5.4 Definition [G1] The *natural map*, denoted by J_X , of normed linear space X into its second conjugate space X'' (the Banach space of bounded linear functionals on X') is defined by

$$(J_X x)x' = x'x \qquad x' \in X' .$$

If the range of J_X is all of X'' , then X is called *reflexive.*

0.5.5 Remarks [G1]

(i) The natural map J from X into X'' is a linear isometry. The linearity of J is clear, while from Corollary 0.3.8 we have

$$\|Jx\| = \sup_{\|x'\|=1} |(Jx)x'| = \sup_{\|x'\|=1} |x'x| = \|x\| .$$

(ii) Every reflexive space is complete, since an adjoint space is complete and isomorphisms preserve completeness.

(iii) $L_p(S,\Sigma,\mu)$, $1 < p < \infty$, is reflexive.

The following cautionary comment is made in [J]: Sometimes the argument given to show, for example, that $L_p = L_p(S,\Sigma,\mu)$, $1 < p < \infty$, is reflexive is the following.

L_p'' is equivalent to $L_{p'}'$, which in turn is equivalent to $L_{p''} = L_p$. Hence L_p is reflexive.

The flaw in the argument is that the equivalence of a normed linear space with its second conjugate does not guarantee the reflexivity of the space. James [J] gave an ingenious construction of a Banach space X which is equivalent to X'', yet the dimension of $X''/J_X X$ is 1 .

0.5.6 Theorem [G1] A closed subspace of a reflexive space is reflexive.

Proof. [G1] Let M be a closed subspace of reflexive space X . Given $m'' \in M''$, define $x'' \in X''$ by

$$x''x' = m''x_M'$$

where x_M' is the restriction of $x' \in X'$ to M . Let $m = J_X^{-1}x''$. It will be shown that m is in M and $J_M m = m''$, proving that M reflexive. Suppose $m \notin M$. Then there exists an $x' \in X'$ such that

$x'm \neq 0$ while $x'M = 0$. Thus $x'_M = 0$ and

$$0 \neq x'm = x'J_X^{-1}x'' = x''x' = m''x'_M$$
$$= m''0 = 0$$

which is impossible. Hence m is in M. For each $m' \in M'$, let m'_e be an element in X', which is an extension of m'. Then

$$m''m' = x''m'_e = m'_e J_X^{-1} x'' = m'_e m = m'm .$$

Thus $J_M m = m''$, completing the proof of the theorem. □

0.6 Weak Convergence

0.6.1 Definition [G1] A sequence $\{x_n\}$ in normed linear space X is said to *converge weakly* to $x \in X$ if for every $x' \in X'$, $x'x_n \to x'x$. This is written $x_n \xrightarrow{w} x$.

The author quotes the next theorem without proof. For the proof the reader is referred to ([G1], 30).

0.6.2 Theorem [G1] Every bounded sequence in a reflexive space contains a weakly convergent subsequence.

CHAPTER I

THE ADJOINT OF A LINEAR OPERATOR

In this chapter we study the adjoint of a linear operator in depth and present two state diagrams, both due to Goldberg [G2], one for linear operators in general and the other for closed linear operators.

Throughout this chapter, X and Y are normed linear spaces over the same scalars, and T is a linear operator having domain a subspace of X and range a subspace of Y. X and Y are assumed complete only when specifically stated.

For any set $M \subset X$, TM denotes the set $\{Tm \mid m \in M \cap D(T)\}$.

I.1 Closed Linear Operators

I.1.1 Definition [G1]. $X \times Y$ is defined as the normed linear space of all ordered pairs (x,y), $x \in X$, $y \in Y$, with the usual definitions of addition and scalar multiplication and with norm given by $\|(x,y)\| = \max\{\|x\|, \|y\|\}$.

I.1.2 Definition [G1]. The graph $G(T)$ of T is the set $\{(x,Tx) \mid x \in D(T)\}$. Since T is linear, G(T) is a subspace of $X \times Y$.

If the graph of T is closed in $X \times Y$, then T is said to be closed in X. When there is no ambiguity concerning the space X, we say that T is closed.

I.1.3 Remarks [G1]

(i) T is closed if and only if $\{x_n\}$ in $D(T)$, $x_n \to x$, $Tx_n \to y$, imply $x \in D(T)$ and $Tx = y$.

(ii) If T is 1–1 and closed, then T^{-1} is closed.

(iii) The null manifold of a closed operator is closed.

(iv) If D(T) is closed and T is continuous, then T is closed.

(v) The continuity of T does not necessarily imply that T is closed. Conversely, T closed does not necessarily imply that T is continuous. This statement can be verified by the following examples.

Let D(T) be any proper dense subspace of $X = Y$ and let T be the identity map. T is obviously continuous but not closed.

I.1.4 Example [G1]

Let $X = Y = C([0,1])$ and let $C'([0,1])$ be the subspace of X consisting of the functions with continuous first derivatives. Define the linear differential operator T mapping $C'([0,1])$ into Y by $(Tx)(t) = x'(t)$, $t \in [0,1]$. T is closed; for if $x_n \to x$ and $Tx_n \to y$, then $\{x_n\}$ converges uniformly to x and $\{x_n'\}$ converges uniformly to y on [0,1]. It follows from taking antiderivatives of x_n' and y that x is in $C'([0,1])$ and that $Tx = x' = y$ on [0,1]. Thus T is closed. However, T is unbounded, since the sequence $\{x_n(t)\} = \{t^n\}$ has the properties $\|Tx_n\| = n$ and $\|x_n\| = 1$.

I.1.5 Remark [G1] We now prove the closed–graph theorem, which is another fundamental theorem in functional analysis. It shows when a closed linear operator is continuous. Goldberg [G1] uses basic Lemma I.1.8 to prove the theorem for Banach spaces.

The lemma is also used later to prove the important Theorem I.4.3.

I.1.6 Definition Let E be a subspace of X. Let the *closed unit ball, open unit ball* and *unit sphere* of E be denoted by B_E, U_E and S_E respectively, where $B_E = \{x \in E: \|x\| \le 1\}$, $U_E = \{x \in E: \|x\| < 1\}$ and $S_E = \{x \in E: \|x\| = 1\}$.

The following lemma is a simple consequence of the properties of a norm.

I.1.7 Lemma [G1]

(i) Given $x \in X$, the translation map f_x from X onto X defined by $f_x(z) = x + z$ is a homeomorphism.

(ii) For any nonzero scalar a, the map g_a from X onto X, defined by $g_a(x) = ax$, is a homeomorphism.

(iii) For any open set $V \subset X$ and any $x \in X$, $f_x(V) = x + V$ is open. Therefore $A + V = \bigcup_{x \in A} x + V$ is open, where A is an arbitrary set in X.

(iv) Given a set $K \subset X$, $a\overline{K} = g_a(\overline{K}) = \overline{g_a(K)} = \overline{aK}$ for any scalar a.

I.1.8 Basic Lemma [G1] Let X be complete and let T be closed. If $rU_Y \subset \overline{TB_X}$, then $rU_Y \subset TB_X$.

Proof: [G1] To prove the lemma, it suffices to prove that $rU_Y \subset T(1/1-\epsilon)B_X$ whenever $0 < \epsilon < 1$; for if this is the case, then given $y \in rU_Y$, there exists an ϵ, $0 < \epsilon < 1$, such that $y/1-\epsilon$ is also in rU_Y. Hence there exists an $x \in (1/1-\epsilon)B_X$ such that $Tx = y/1-\epsilon$ or $T((1-\epsilon)x) = y$. Since $\|(1-\epsilon)x\| \le 1$, y is in TB_X. Let $0 < \epsilon < 1$ and let $y \in rU_Y$ be given. By hypothesis and (iv) of Lemma I.1.7 it follows that for each non–negative integer n, $r\epsilon^n U_Y \subset \overline{T(\epsilon^n B_X)}$. Taking $n = 0$, there exists an

$x_0 \in B_X$ such that $\|y - Tx_0\| < r\epsilon$; that is, $y - Tx_0 \in (r\epsilon)U_Y$. Hence, taking $n = 1$, there exists an $x_1 \in \epsilon B_X$ such that $\|y - Tx_0 - Tx_1\| < r\epsilon^2$; that is, $y - Tx_0 - Tx_1 \in (r\epsilon^2)U_Y$. Proceeding in this manner, a sequence $\{x_n\}$ is obtained with the properties that (1) $x_n \in \epsilon^n B_X$ and $\|y - \sum_{i=0}^{n} Tx_i\| < r\epsilon^{n+1}$.

Consequently, $\sum_{n=0}^{\infty} \|x_n\| \leq 1/1 - \epsilon$ and therefore the sequence $\{z_n\}$ defined by $z_n = \sum_{i=0}^{n} x_i$ is a Cauchy sequence in Banach space X . Thus there exists an $x \in X$ such that $\|x\| \leq 1/1 - \epsilon$ and $z_n \to x$. From (1) it is clear that $Tz_n \to y$. Since T is closed, x must be in $D(T) \cap (1/1 - \epsilon)B_X$ and $Tx = y$, whence $y \in T(1/1 - \epsilon)B_X$. □

I.1.9 Open–Mapping Theorem [G1] Let X be complete and let Y be of the second category. If T is closed and $R(T) = Y$, then T is an open map ; i.e., the map takes open sets onto open sets.

Proof: [G1] To show that T is an open map, one need only prove that given $x \in X$ and $\epsilon > 0$, the set $Tx + T(\epsilon B_X)$ contains an open set $Tx + rU_Y$ for some $r > 0$. By the basic Lemma I.1.8, it suffices to show the existence of some $r > 0$ such that $rU_Y \subset \overline{T(\epsilon B_X)}$. Since $Y = \bigcup_{n=1}^{\infty} n\, TB_X$ and Y is of the second category, there exists a positive integer p such that $p\,\overline{TB_X} = \overline{p\ TB_X}$ contains a non–empty open set. It follows from (ii) of Lemma I.1.7 that $\overline{T(\epsilon/2)B_X}$ must also contain a non–empty open set V . Thus

$$0 \in V - V \subset \overline{T(\epsilon/2)B_X} - \overline{T(\epsilon/2)B_X} \subset \overline{T(\epsilon B_X)} .$$

Since $V - V$ is an open set about 0 , there exists an $r > 0$ such that $rU_Y \subset V - V \subset \overline{T(\epsilon B_X)}$. Thus the proof of the theorem is complete. □

I.1.10 Remark [G1] For an example of an incomplete normed linear space of the second category, the reader is referred to Bourbaki [Bo], Exercise 6, page 3.

I.1.11 Definition The author will refer to a continuous everywhere defined operator as a *bounded* operator.

I.1.12 Closed–Graph Theorem [G1] A closed linear operator mapping a Banach space into a Banach space is bounded.

Proof: [G1] Suppose the domain of T is all of X with X and Y complete. The graph G(T) may be considered as a Banach space, since it is a closed subspace of Banach space $X \times Y$. Define linear maps P_1 and P_2 from G(T) into X and Y, respectively, by $P_1(x,Tx) = x$ and $P_2(x, Tx) = Tx$. Clearly, P_1 is 1–1 and satisfies the hypotheses of the open–mapping theorem. Consequently, P_1 has a bounded inverse. Since P_2 is continuous, $T = P_2P_1^{-1}$ is also bounded. □

I.1.13 Remarks [G1] In the closed–graph theorem it is essential that both X and Y be complete, as may be seen in the following two examples.

The first–order differential operator is Example I.1.4 is closed but not continuous. The domain of the operator is $C'([0,1])$, which is not complete, while the range of the operator is $Y = C([0,1])$, which is complete.

I.1.14 Example [G1] Let X be any infinite–dimensional Banach space and let H be a set of basis elements for X. H is usually referred to as a Hamel basis. It may be assumed that the elements in H are of norm 1. Define X_1 as the vector space X with norm $\| \|_1$ given by

$$\|\sum_{i=1}^{n} a_i h_i\|_1 = \sum_{i=1}^{n} |a_i| \ , \ h_i \in H .$$

Clearly, $\|x\|_1 \geq \|x\|$. X_1 is not complete. Let $y_1, y_2, \ldots$ be any infinite countable subset of H and let $s_n = \sum_{k=1}^{n} k^{-2} y_k$. Then $\{s_n\}$ is a Cauchy sequence in X_1 which does not converge in X_1 . Let T be the identity map from X onto X_1 . Then T is closed and has a continuous inverse. However, T is not continuous; otherwise T would be an isomorphism, which would imply that X_1 is complete.

I.1.15 Remark [G1] As an application of the closed–graph theorem, the following fundamental theorem is obtained. It is also referred to in the literature as the Banach–Steinhaus theorem.

I.1.16 Uniform–Boundedness Principle [G1] Let X be a Banach space and let F be a set of continuous linear operators mapping X into Y such that for each $x \in X$, $\sup_{T \in F} \|Tx\| < \infty$. Then $\sup_{T \in F} \|T\| < \infty$; that is, the operators in F , when restricted to a bounded set in X , are uniformly bounded.

I.1.17 Remark [G1]. Goldberg [G1] motivates the proof of the above theorem as follows: It is required to find some constant M so that for all $x \in X$, $\sup \|Tx\| \leq M\|x\|$. A technique which is often used is to fix x and thereby induce a vector–valued function $Ax : F \to Y$ defined by $(Ax)T = Tx$. In this context, we seek a constant M so that

(1) $$\sup \|(Ax)T\| \leq M\|x\| \ , \ x \in X .$$

By hypothesis, $\sup_{T \in F} \|(Ax)T\| < \infty$, for each $x \in X$. Thus, Ax is a member of B(F,Y) , the normed linear space of bounded functions from F into Y with norm defined by $\|h\| = \sup_{T \in F} \|h(T)\|$. Hence, by (1), an M is required so that $\|Ax\| \leq M\|x\|$ for all

$x \in X$. This, in turn, suggests defining the operator $A: X \to B(F,Y)$, with the hope that A is bounded. The proof of the theorem consists of showing that A is continuous by applying the closed–graph theorem. Since $B(F,Y)$ is not necessarily complete, $B(F,\tilde{Y})$ is considered instead, where $\tilde{Y}$ is the completion of Y.

Proof: [G1] Let A be the linear map from X into $B(F,\tilde{Y})$ defined by $(Ax)T = Tx$, $T \in F$. It is easy to verify that A is closed, whence, by the closed–graph theorem A is continuous. Therefore

$$\sup_{T \in F} \|Tx\| = \|Ax\| \leq \|A\| \, \|x\| \quad , \ x \in X .$$

Consequently

$$\sup_{T \in F} \|T\| \leq \|A\| .$$

□

I.1.18 Remark [G1] In the above theorem it is essential that X be complete. Let X be the subspace of ℓ_2 consisting of all elements of the form $\sum_{i=1}^{k} a_i u_i$, where $u_1 = (1,0,0,\ldots)$, $u_2 = (0,1,0,\ldots)$, etc. For each positive integer n, let $T_n: X \to \ell_2$ be the linear operator defined by setting

$$T_n u_i = \begin{cases} 0 & \text{if } i \neq n \\ n u_n & \text{if } i = n \end{cases} .$$

Then $\|T_n\| = n$ while for each $x \in X$, $T_n x \to 0$.

I.1.19 Remark [G1] By taking Y to be the space of scalars in Theorem I.1.16, we obtain the result that if X is complete and F is a subset of X' such that $\sup_{x' \in F} |x'x| < \infty$, $x \in X$, then F is bounded in X'. The following theorem is what one might call the dual result.

I.1.20 Theorem [G1] Suppose K is a subset of X such that

$$\sup_{k \in K} |x'k| < \infty \quad x' \in X' .$$

Then K is bounded.

Proof: [G1] Let J be the natural map from X into X″. By hypothesis,

$$\sup_{k \in K} |(Jk)x'| = \sup_{k \in K} |x'k| < \infty \quad x' \in X'$$

Since X′ is complete and J is a linear isometry, the uniform–boundedness principle assures us that $\sup_{k \in K} \|k\| = \sup_{k \in K} \|Jk\| < \infty$. □

I.1.21 Definition Let X and Y be normed linear spaces. The class of linear operators T defined on a linear subspace D(T) of X with range contained in Y is denoted by $L(X,Y)$. Throughout this thesis T ∈ L(X,Y) unless stated otherwise.

I.2 Adjoint of a Linear Operator

The concept of the adjoint of a linear operator is very useful in obtaining information about the range, inverse and null space of T. In this section we show how certain properties of an operator and its adjoint are related. Another reason for considering the adjoint of a linear operator is that it provides a necessary and sufficient condition for a solution to the equation Tx = y to exist. A necessary condition that y ∈ R(T) is that y′(y) = 0 for all y′ ∈ N(T′). If R(T) is closed in Y then this condition is also sufficient. As many problems in mathematics and its applications can be put in the form of an equation Tx = y, as above, this means that the concept of the adjoint of a linear operator is very important.

If X, Y are normed linear spaces and $T \in B(X,Y)$, then we define the adjoint T' of T as follows: T' takes each $y' \in Y'$ to $y'T \in X'$, i.e., $T'y'(x) = y'T(x)$ and T' is a bounded linear operator with $\|T'\| = \|T\|$. The author emphasizes that if T is bounded on all of X , then T' is also bounded and $\|T'\| = \|T\|$.

However, in many applications, the operators we need to consider are unbounded instead of bounded e.g., differential operators. If we consider the operator $d|dt$ on $C([0,1])$, it is a closed operator with domain consisting of continuously differentiable functions. It is clearly unbounded as the sequence $x_n(t) = t^n$ satisfies $\|x_n\| = 1$, $\|dx_n|dt\| = n$ which tends to infinity as $n \to \infty$.

To define the adjoint of an unbounded linear operator we follow the definition for bounded operators, exercising a bit of care along the way. We want $T'y'(x) = y'(Tx)$, $x \in D(T)$. Thus, we say that $y' \in D(T')$ if there is an $x' \in X'$ such that $x'(x) = y'(Tx)$, $x \in D(T)$. Then we define $T'y'$ to be x' . We need x' to be unique, i.e., that $x'(x) = 0$ for all $x \in D(T)$ should imply that $x' = 0$. This is true if and only if $D(T)$ is dense in X .

The fact that we do not need $D(T) = X$ means that we can consider cases in which T is a differential operator whose domain, considered as a subspace of $L_2(\Omega)$, consists of certain smooth functions. In most applications, $D(T) \neq X$ but $\overline{D(T)} = X$, where X is complete.

However, we can relax the restriction that $\overline{D(T)} = X$. The author indicates how one might define the adjoint of an unbounded linear operator T whose domain $D(T)$ is not necessarily dense in X as follows :

I.2.1 Definition (adjoint of T) Let $T \in L(X,Y)$. The *adjoint* T' of T is defined by $D(T') = \{y' \in Y' : y'T \text{ is continuous on } D(T)\}$, $T' \in L(Y', D(T)')$ and $T'y'x = y'Tx$ for $x \in D(T)$.

1.2.2 Notation [Pi] For a given subspace E of X the operator $J_E^X \in L(E,X)$ is defined to be the natural injection of E into X. When X is understood we abbreviate $J_{D(T)}^X$ to $J_{D(T)}$.

I.2.3 Remark: The author remarks that T' is just the adjoint of $TJ_{D(T)}$ in the sense of [G1, II.2.2].

The author shows in this section that most of the theorems still hold when T is not necessarily densely defined.

The following simple example is presented in detail in order to give some "feeling" for the definition of an adjoint operator.

I.2.4 Example [G1] Let $X = Y = \ell_p$, $1 \leq p < \infty$, and let $u_1 = (1,0,0,...)$, $u_2 = (0,1,0,...)$, etc. be the unit vectors in ℓ_p. Define T by

$$D(T) = \text{sp}\,\{u_k\}$$

$$T(x_1, x_2, ..., x_n, 0, 0, ...) = (\sum_{j=1}^{n} j\,x_j, x_2, x_3, ..., x_n, 0, 0, ...)$$

Suppose $y' = (a_1, a_2, ...) \in D(T')$. Then for $k \geq 1$,

$$|y'\,Tu_k| = |a_1k + a_k| \geq |a_1|k - |a_k| \geq |a_1|k - \|y'\|.$$

Since $\|u_k\| = 1$ and $y'T$ is bounded on $D(T)$, $a_1 = 0$. Also, any element $(0, b_1, b_2, ...) \in \ell_{p'} = \ell_p'$ is in $D(T')$. Hence the domain of T' consists of all the elements in ℓ_p' which have zero as their first term. Suppose $T'y' = (c_1, c_2, \ldots)$, where $y' = (0, a_2, a_3, \ldots) \in D(T')$. Then

$$c_k = T'y'u_k = y'Tu_k = a_k \quad , k \geq 2$$

and $c_1 = 0$. Thus $T'y' = y'$.

I.2.5 Theorem [G1] T' is a closed linear operator.

Proof: Suppose $y_n' \to y'$ and $(TJ_{D(T)})'y_n' \to x'$. Then for each $x \in D(T)$, $y_n'(TJ_{D(T)})x \to y'Tx$ and $y_n'Tx = (TJ_{D(T)})'y_n'x \to x'x$.

Thus $y'T = x'$ on $D(T)$. Hence, by the definition of $(TJ_{D(T)})'$, $y' \in D((TJ_{D(T)})')$ and $(TJ_{D(T)})'y' = x'$. Therefore $(TJ_{D(T)})'$ is closed. □

Even though $D(T)$ may be all of X, it is still possible for $D(T')$ to consist solely of the zero vector. The following example which demonstrates this is due to Berberian.

I.2.6 Example [G1] Take $X = Y = \ell_2$ and take $D(T)$ to be the span of the unit vectors $u_k = (0, 0, \ldots, 1, 0, 0, \ldots) \in \ell_2$. Let

$$\{u_{kj} \mid k,j = 1, 2, \ldots\}$$

be any double indexing of $\{u_k\}$. For each k, define

$$Tu_{kj} = u_k \, , \; j = 1, 2, \ldots$$

and extend T linearly to X. Suppose $y' = (a_1, a_2, \ldots) \in D(T')$. Then for each k, $T'y'u_{kj} = y'u_k = a_k$, $j = 1, 2, \ldots$. Now

$$\sum_{j=1}^{\infty} |T'y'u_{kj}|^2 \leq \|T'y'\|^2 .$$

Hence

$$0 = \lim_{j\to\infty} T'y'u_{kj} = a_k \ , \ 1 \le k \ .$$

Therefore $y' = 0$.

For an example of a differential operator T with $D(T') = (0)$, the reader is referred to Example III.1.7, p.66. A special case of the latter is Stone [St], Theorem 10.10, p.447.

I.2.7 Theorem [G1] $D(T') = Y'$ if and only if T is continuous. If that is the case, then T' is also continuous and $\|T'\| = \|T\|$.

Proof: Clearly, if T is continuous, then $y'T$ is continuous for each $y' \in Y'$. Thus $D((TJ_{D(T)})') = Y'$. Suppose $D((TJ_{D(T)})') = Y'$. Let S be the 1–sphere of $D(T)$. For each $y' \in Y'$, $\sup_{z\in S} |y'Tx| \le \|(TJ_{D(T)})'y'\|$. Hence, by Theorem I.1.19 , $\|T\| = \sup_{x\in S} \|Tx\| < \infty$. Now, for each $x \in S$, $|(TJ_{D(T)})'y'x| \le \|y'\| \, \|T\|$. Thus $\|(TJ_{D(T)})'y'\| \le \|T\| \, \|y'\|$, and therefore $\|(TJ_{D(T)})'\| \le \|T\|$. By Corollary 0.3.7

$$\|Tx\| = \sup_{\|y'\|=1} |y'Tx| = \sup_{\|y'\|=1} |(TJ_{D(T)})'y'x|$$

$$\le \|(TJ_{D(T)})'\| \, \|x\| \qquad x \in D(T) \ .$$

Hence $\|T\| \le \|(TJ_{D(T)})'\|$ and the theorem follows. □

The linear operators which one usually encounters have the property that they are restrictions of closed operators. Theorem I.2.10 characterizes such operators.

I.2.8 Definition [G1] A set F of linear functionals on a vector space V is said to be *total* if given any $v \neq 0$ in V, there exists an $f \in F$ such that $f(v) \neq 0$.

In Examples I.2.4 and I.2.6 the domain of T' is not total. Corollary 0.3.6 shows that an adjoint space is total.

I.2.9 Definition [G1] T is called *closable* if there exists a linear extension of T which is closed in X.

I.2.10 Theorem [G1] Statements (i), (ii) and (iii) are equivalent. (D(T) is not required to be dense in X.)

(i) T is closable.

(ii) T has a minimal closed linear extension; i.e., there exists a closed linear extension $\overline{T}$ of T such that any closed linear extension of T is a closed linear extension of $\overline{T}$,

(iii) For any $y \neq 0$ in Y, (0,y) is not in the closure of the graph of T.

(iv) $D(T')$ is total.

Furthermore if T is closable then, $T' = (\overline{T})'$, where $\overline{T}$ is the minimal closed extension of T.

Proof [G1] (i) implies (iii). Let $\overline{T}$ be a closed linear extension of T. If $y \in Y$ and $y \neq 0$, then $(0,y) \notin G(\overline{T}) \supset G(T)$. Hence $(0,y) \notin \overline{G(T)}$, since $G(\overline{T})$ is closed in $X \times Y$.

(iii) implies (ii). Suppose $(0,y) \notin \overline{G(T)}$ for any $y \neq 0$ in Y. Define $\overline{T}$ as the operator whose graph is $\overline{G(T)}$; that is,

$$D(\overline{T}) = \{x \mid (x,z) \in \overline{G(T)} \text{ for some } z \in Y\}$$

$\overline{T}x = z$.

I.2.8 Definition [G1] A set F of linear functionals on a vector space V is said to be *total* if given any $v \neq 0$ in V, there exists an $f \in F$ such that $f(v) \neq 0$.

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(i) T is closable.

(ii) T has a minimal closed linear extension; i.e., there exists a closed linear extension $\bar{T}$ of T such that any closed linear extension of T is a closed linear extension of $\bar{T}$,

(iii) For any $y \neq 0$ in Y, $(0,y)$ is not in the closure of the graph of T.

(iv) $D(T')$ is total.

Furthermore if T is closable then, $T' = (\bar{T})'$, where $\bar{T}$ is the minimal closed extension of T.

Proof [G1]. (i) implies (iii). Let $\bar{T}$ be a closed linear extension of T. If $y \in Y$ and $y \neq 0$, then $(0,y) \notin G(\bar{T}) \supset G(T)$. Hence $(0,y) \notin \overline{G(T)}$, since $G(\bar{T})$ is closed in $X \times Y$.

(iii) implies (ii). Suppose $(0,y) \notin \overline{G(T)}$ for any $y \neq 0$ in Y. Define $\bar{T}$ as the operator whose graph is $\overline{G(T)}$; that is,

$$D(\bar{T}) = \{x \mid (x,z) \in \overline{G(T)} \text{ for some } z \in Y\}$$

$\bar{T}x = z$.

Then $\overline{T}$ is unambiguously defined and is a closed linear extension of T. Furthermore, $\overline{T}$ is the minimal closed linear extension of T; for if T_1 is a closed linear extension of T, then $G(T_1) \supset \overline{G(T)} = G(\overline{T})$.

(ii) implies (i) trivially.

For the next part of the proof, Goldberg assumes that $\overline{D(T)} = X$. However, we are able to drop this assumption, as the author has discussed above, and proceed to show that $D((TJ_{D(T)})')$ is total if and only if statement (iii) is valid. Let $D((TJ_{D(T)})')$ be total and let $(0,y)$ be in $\overline{G(T)}$. Then there exists a sequence $\{x_n\}$ in $D(T)$ such that $x_n \to 0$ and $Tx_n \to y$. Thus, for each $y' \in D((TJ_{D(T)})')$, $y'Tx_n \to 0$ and $y'Tx_n \to y'y$. Since $D((TJ_{D(T)})')$ is total, it follows that $y = 0$.

Assume (iii). Let $y \neq 0$ be an element in Y. Then $(0,y) \notin \overline{G(T)}$ and therefore there exists a $z' \in (X \times Y)'$ such that $z'(0,y) \neq 0$ and $z'\overline{G(T)} = 0$. Defining $x' \in X'$ and $y' \in Y'$ by $x'x = z'(x,0)$ and $y'y = z'(0,y)$, we obtain

$$0 = z'(x,Tx) = x'x + y'Tx \qquad x \in D(T)$$

$$0 \neq z'(0,y) = y'y .$$

From these two equations, we have $y' \in D((TJ_{D(T)})')$ and $y'y \neq 0$. Thus $D((TJ_{D(T)})')$ is total.

Suppose $y' \in D((\overline{T})')$. Then $y'\overline{T}$ is continous on $D(\overline{T})$ and, in particular, continuous on $D(T)$. Thus $y' \in D((TJ_{D(T)})')$. Suppose $y' \in D((TJ_{D(T)})')$ and $x \in D(\overline{T})$. Since $(x,\overline{T}x) \in G(\overline{T}) = \overline{G(T)}$, there exists a sequence $\{x_n\}$ in $D(T)$ such that $x_n \to x$ and $Tx_n \to \overline{T}x$. Hence

$$|y'\overline{T}x| = \lim_{n\to\infty} |y'Tx_n| \leq \|(TJ_{D(T)})'y'\| \lim_{n\to\infty} \|x_n\| = \|(TJ_{D(T)})'y'\| \, \|x\|$$

Therefore $y'\overline{T}$ is bounded, which means $y' \in D((\overline{T}J_{D(\overline{T})})')$. Hence

$$D((TJ_{D(T)})') = D((\overline{T}J_{D(\overline{T})})') .$$

Now $(TJ_{D(T)})'y' = (\overline{T}J_{D(\overline{T})})'y'$ on $D(T)$. Hence $T' = \overline{T}'$. □

The theorem just proved shows that the operator in Example I.2.2 is not closable.

I.2.11 Corollary [G1] A linear operator which maps a Banach space into a Banach space is bounded if and only if the domain of its conjugate operator is total.

Proof: [G1] Let X and Y be complete and let $D(T) = X$. Suppose $D(T')$ is total. Then by Theorem I.2.8 T is closable. Since $D(T) = X$, T must be its own closed extension. Hence T is bounded by the closed–graph theorem. If T is bounded, then $D(T') = Y'$, which is total.

I.2.12 Lemma [G1] If X is reflexive and F is a subspace of X' , then F is total if and only if F is dense in X' .

Proof [G1] If F is total but not dense in X' , there exists an $x'' \neq 0$ in X'' such that $x''F = 0$. By the reflexivity of X , there exists an $x \neq 0$ such that $0 = x''x' = x'x$ for all $x' \in F$. But this is impossible since F is total. If F is dense in X' , then F is total, since any set dense in a total set is also total.

I.2.13 Theorem Suppose Y is reflexive. Then T is closable if and only if $D(T')$ is dense in Y' . In that case, the minimal closed extension of T is $J_Y^{-1}T''J_{D(T)}$, where $J_{D(T)}$ and J_Y are the natural maps from $D(T)$ into $D(T)''$ and Y onto Y'' , respectively.

The author refers the reader to [G1, II.2.14] for the proof (with X replaced by $D(T)$).

I.2.14 Theorem [G1] T' is continuous if and only if $D(T')$ is closed in Y' .

Proof: Suppose T' is continuous and $y'_n \to y'$, $y'_n \in D(T')$. Since $(TJ_{D(T)})'$ is continuous, and $\{y'_n\}$ are bounded, there exists some constant M such that $\|(TJ_{D(T)})'y'_n\| \leq M$.

Therefore

$$|y'Tx| = \lim_{n\to\infty} |y'_n Tx| = \lim_{n\to\infty} |(TJ_{D(T)})'y'_n x)| \leq M\|x\| .$$

As a result, $y' \in D((TJ_{D(T)})')$ whence $D((TJ_{D(T)})')$ is closed.

Conversely, if $D((TJ_{D(T)})')$ is closed, then $(TJ_{D(T)})'$ is a closed linear map from the Banach space $D((TJ_{D(T)})')$ into the Banach space X'. By the closed–graph theorem, $(TJ_{D(T)})'$ is continuous. □

Under the assumptions that X and Y are complete and that T is closed and densely defined, it is shown in Corollary I.4.8 that the statements "T bounded," "T' continuous," and "$D(T')$ closed" are all equivalent.

I.2.15 Theorem [G1] Every operator in $[Y',X']$ is a densely defined adjoint of an operator in $[X,Y]$ if and only if Y is reflexive.

Proof [G1] Suppose Y is reflexive and $A \in [Y',X']$. Define $T \in [X,Y]$ by $T = J_Y^{-1}A'J_X$. Then $T' = A$, since for all $y' \in Y'$ and $x \in X$,

$$T'y'x = y'(J_Y^{-1}A'J_X x) = (A'J_X x)y' = (J_X x)Ay' = Ay'x .$$

Suppose every operator in $[Y',X']$ is an adjoint operator. Let $x'_0 \in X'$ and $x_0 \in X$ be chosen so that $x'_0\, x_0 = 1$. Given $y'' \in Y''$, define $A \in [Y',X']$ by $Ay' = y''(y')x'_0$. By hypothesis, there exists a $T \in [X,Y]$ such that $A = T'$. Hence,

$$y'Tx_0 = T'y'x_0 = Ay'x_0 = y''(y')x'_0x_0 = y''y' \quad y' \in Y' .$$

Therefore Y is reflexive. □

I.3 States of Linear Operators

The author shall first explain the notion of a state diagram. The state diagram is a "checkerboard" diagram which enables one to keep track of and to summarize some theorems concerning the range and inverse of T as well as T'.

Guided by the known theorems relating T and T', we classify various possibilities of $R(T)$ and T^{-1} by

I: $R(T) = Y$

II: $R(T) \neq Y$ but $\overline{R(T)} = Y$

III: $\overline{R(T)} \neq Y$

1: T^{-1} exists and is continuous.

2: T^{-1} exists but is not continuous.

3: T has no inverse.

If $R(T) = Y$, we say that T is in state I or that T is surjective, written $T \in I$. Similarly, we say that T is in state 3, written $T \in 3$, if T has no inverse. If $T \in II$ and $T \in 1$, we write $T \in II_1$. The same notation is used for T'. If $T \in III_1$ and $T' \in I_3$, we write $(T,T') \in (III_1,I_3)$. Thus (T,T') has 81 possibilities, which are described in the "checkerboard" I.3.14, which is referred to as a state diagram.

In this section we show the impossibility of certain states for (T,T'). These eliminated states are shown by shading the corresponding squares. If we assume that X is reflexive, or Y is complete, for example, then certain additional squares can be eliminated. We indicate these squares with the letters $X - R$ or Y. The state diagrams I.3.14 and I.4.11 were obtained by Goldberg [G2].

I.3.1 Theorem [G1] If T' has a continuous inverse, then $R(T')$ is closed.

Proof: Suppose $(TJ_{D(T)})'y_n' \to x' \in D(T)'$. Since $(TJ_{D(T)})'$ has a bounded inverse, there exists an $m > 0$ such that

$$\|(TJ_{D(T)})'y_n' - (TJ_{D(T)})'y_k'\| \geq m\|y_n' - y_k'\| .$$

Thus $\{y_n'\}$ is a Cauchy sequence which converges to some y' in the Banach space Y'. Since $(TJ_{D(T)})'$ is closed, y' is in $D((TJ_{D(T)})')$ and $(TJ_{D(T)})'y' = x'$. Hence $R((TJ_{D(T)})')$ is closed. □

The theorem shows that state II_1 is impossible for T'. For brevity, we write $T' \notin II_1$.

I.3.2 Definition [G1] If C is a subset of X', then the *orthogonal complement* of C in X is the set

$${}^{\perp}C = \{x \mid x \in X, x'x = 0 \text{ for all } x' \in C\} .$$

I.3.3 Remarks [G1] $K^{\perp}$ and ${}^{\perp}C$ are closed subspaces of X' and X, respectively. Also $K^{\perp} = \overline{K}^{\perp}$ and ${}^{\perp}C = {}^{\perp}\overline{C}$.

I.3.4 Theorem [G1] If M is a subspace of X, then ${}^{\perp}(M^{\perp}) = \overline{M}$.

Proof: $M \subset {}^{\perp}(M^{\perp})$ and ${}^{\perp}(M^{\perp})$ is a closed subspace of X. But $\overline{M}$ is the intersection of all closed subspaces containing M. Therefore $\overline{M} \subset {}^{\perp}(M^{\perp})$.

If $\overline{M} = X$, if follows that $\overline{M} = {}^{\perp}(M^{\perp})$.

If $\overline{M} \neq X$, $\exists x_0 \in X$, $x_0 \notin \overline{M} \Rightarrow d(x_0,\overline{M}) > 0$.

By Corollary 0.3.5 $\exists x' \in X'$ such that $x'(x_0) \neq 0$ and $x'(x) = 0 \ \forall x \in \overline{M}$. Since $\overline{M} \supset M$, $x' \in M^{\perp}$. Also $x'(x_0) \neq 0$ implies that $x_0 \notin {}^{\perp}(M^{\perp})$.

We have shown that $x_0 \notin \overline{M}$ implies $x_0 \notin {}^{\perp}(M^{\perp})$ i.e., ${}^{\perp}(M^{\perp}) \subset \overline{M}$ and the proof is complete. □

I.3.5 Theorem [G1] If N is a subspace of X', then $({}^{\perp}N)^{\perp} \supset \overline{N}$. If X is reflexive, then $({}^{\perp}N)^{\perp} = \overline{N}$.

Proof: ${}^{\perp}N = \{x \mid x \in X,\ n'x = 0 \text{ for all } n' \in N\}$. Hence $({}^{\perp}N)^{\perp} \supset N$ but $({}^{\perp}N)^{\perp}$ is a closed subspace of X'. Hence $\overline{N} \subset ({}^{\perp}N)^{\perp}$.

If X is reflexive, then $({}^{\perp}N)^{\perp} = N''$. Hence $({}^{\perp}N)^{\perp} = \overline{N}$.

I.3.6 Remarks [G1] Dieudonne [D] has shown that if N is a reflexive subspace of X', then $({}^{\perp}N)^{\perp} = N$. Since we later use the fact that $({}^{\perp}N)^{\perp} = N$, for N finite dimensional, we shall prove this special case of Dieudonne's theorem.

Proof: [G1] Since $N \subset ({}^{\perp}N)^{\perp}$, it follows from Theorem 0.5.3 that

(1) $$\dim X/{}^{\perp}N = \dim (X/{}^{\perp}N)' = \dim ({}^{\perp}N)^{\perp} \geq \dim N .$$

If $x_1', x_2', \ldots, x_n'$ is a basis for N, then the map A from $X/{}^{\perp}N$ into U^n defined by $A[x] = (x_1'x, x_2'x, \ldots, x_n'x)$ is 1–1 and linear. Thus

(2) $$\dim X/{}^{\perp}N \leq n = \dim N .$$

It follows from (1) and (2) that $N = ({}^{\perp}N)^{\perp}$. □

I.3.7 Theorem [G1]

(i) $\overline{R(T)}^{\perp} = R(T)^{\perp} = N(T')$

(ii) $\overline{R(T)} = {}^{\perp}N(T')$.

In particular, T has a dense range if and only if T' is 1–1.

Proof: (i) We have $\overline{R(T)}^{\perp} = R(T)^{\perp}$ by Remark I.3.3. Let $y' \in N(T')$ and $y \in R(T)$. Then $T'y' = 0$ and there exists $x \in D(T)$ such that $Tx = y$. Hence

$$\begin{aligned} y'y &= y'Tx = T'y'x \\ &= 0 . \end{aligned}$$

Hence $N(T') \subset R(T)^{\perp}$.

For the opposite inclusion, let $y' \in R(T)^{\perp}$. Then $y'Tx = 0$ for every $x \in D(T)$. Hence

$$\begin{aligned} T'y'x &= y'Tx \\ &= 0 \quad \forall x \in D(T) . \end{aligned}$$

Hence $T'y' = 0$, i.e. $y' \in N(T')$.
Therefore $R(T)^{\perp} \subset N(T')$.

(ii) This follows from (i) and Theorem I.3.4. □

By interchanging the roles of T and T' in the above theorem, we do not quite obtain the dual type theorems.

I.3.8 Theorem [G1]

(i) $^{\perp}R(T') \supset N(T)$, where the "$\perp$" is taken in $D(T)'$.

(ii) If $D(T')$ is total, then $N(T) = {}^{\perp}R(T')$.

(iii) $\overline{R(T')} \subset N(T)^{\perp}$.

In particular, if $R(T')$ is total, then T is 1–1.

Proof:

(i) Let $x \in N(TJ_{D(T)})$ and $x' \in R((TJ_{D(T)})')$. Then $TJ_{D(T)}x = 0$ and $\exists y' \in D((TJ_{D(T)})')$ such that $x' = (TJ_{D(T)})'y'$. Hence

$$x'x = (TJ_{D(T)})'y'x$$

$= y'TJ_{D(T)}x$

$= 0$ since $x \in N(TJ_{D(T)})$.

Hence $N(TJ_{D(T)}) \subset {}^{\perp}R((TJ_{D(T)})')$.

(ii) Let $x \in {}^{\perp}R((TJ_{D(T)})') \cap D(TJ_{D(T)})$. Then

$(TJ_{D(T)})'y'x = 0 \quad (\forall y' \in D((TJ_{D(T)})'))$

$y'TJ_{D(T)}x = 0 \quad (\forall y' \in D((TJ_{D(T)})'))$.

Since $D((TJ_{D(T)})')$ is total, $TJ_{D(T)}x = 0$, i.e. $x \in N(TJ_{D(T)})$. The other inclusion follows from (i).

(iii) This follows from (i) and Theorem I.3.5 as

$$N(TJ_{D(T)})^{\perp} \supset {}^{\perp}R((TJ_{D(T)})')^{\perp} \supseteq \overline{R((TJ_{D(T)})')}$$

I.3.9 Theorem [G1] If T and T' each has an inverse, then $((TJ_{D(T)})^{-1})' = (T')^{-1}$.

Before presenting the proof of this theorem, the author checks whether it is possible to compare $((TJ_{D(T)})^{-1})'$ and $(T')^{-1}$. The following information is presented.

Write $T_1 = TJ_{D(T)}$. Then $T_1' = T'$ by definition, and

$T_1 \in L(D(T), Y)$

$T_1' \in L(Y', D(T)')$

$(T_1')^{-1} \in L(D(T'), Y')$

$T_1^{-1} \in L(Y, D(T))$

$(T_1^{-1})' \in L(D(T)', D(T_1^{-1})')$

But $D((T_1^{-1})') = R(T)' = Y'$ since $R(T)$ is dense in Y. So $(T_1^{-1})' \in L(D(T)', Y')$.

From the above the author observes that it is possible to compare $(T_1^{-1})'$ and $(T')^{-1}$.

Proof: By Theorem I.3.7, $D(T^{-1}) = R(T)$ is dense in Y. Suppose $x' \in D(((TJ_{D(T)})')^{-1}) = R((TJ_{D(T)})')$. Then there exists a $y' \in D((TJ_{D(T)})')$ such that $(TJ_{D(T)})'y' = x'$. To show $x' \in D((T^{-1})')$, it suffices to prove that $x'T^{-1}$ is continuous on $R(T)$. This is certainly the case since

$$x'T^{-1}(Tx) = (TJ_{D(T)})'y'x = y'Tx \quad x \in D(T).$$

Thus $(T^{-1})'x' = y'$ on $R(T)$, whence $y' = (T^{-1})'x' = (T^{-1})'(TJ_{D(T)})'y'$ since $R(T)$ is dense in Y. This shows that $(T^{-1})' = ((TJ_{D(T)})')^{-1}$ on $D(((TJ_{D(T)})')^{-1})$. It remains to prove $D((T^{-1})') \subset D(((TJ_{D(T)})')^{-1})$.

Suppose $z' \in D((T^{-1})')$. To show $z' \in D(((TJ_{D(T)})')^{-1}) = R((TJ_{D(T)})')$, an element $v' \in D((TJ_{D(T)})')$ will be exhibited so that $(TJ_{D(T)})'v' = z'$ or, equivalently, $v'T = z'$ on $D(T)$. It is clear that we should define v' as the continuous linear extension of $z'T^{-1}$ to all of Y, thereby obtaining $(TJ_{D(T)})'v' = z'$. Thus

$$D((T^{-1})') \subset D(((TJ_{D(T)})')^{-1}).$$

□

I.3.10 Lemma [G1] If T does not have a continuous inverse, there exists a sequence $\{x_n\}$ in $D(T)$ such that $\|x_n\| \to \infty$ and $Tx_n \to 0$.

Proof: [G1] There exists a sequence $\{z_n\}$ in $D(T)$ such that $\|z_n\| = 1$ and $Tz_n \to 0$. Define

$$x_n = \begin{cases} \dfrac{z_n}{\|Tz_n\|^{1/2}} & \text{if } Tz_n \neq 0 \\ nz_n & \text{if } Tz_n = 0 . \end{cases}$$

□

I.3.11 Theorem [G1] $R(T') = D(T)'$ if and only if T has a continuous inverse.

Proof: Suppose $R((TJ_{D(T)})') = D(T)'$. If T does not have a continuous inverse, there exists a sequence $\{x_n\}$ in $D(T)$ such that $\|x_n\| \to \infty$ while $Tx_n \to 0$. Thus, for each $y' \in D((TJ_{D(T)})')$, $(TJ_{D(T)})'y'x_n \to 0$ whence $x'x_n \to 0$ for each $x' \in D(T)'$. But then, by Theorem I.1.19, $\{x_n\}$ is bounded, which is a contradiction.

Conversely, suppose T has a continuous inverse. For $x' \in D(T)'$, $x'T^{-1}$ is continuous on $R(T)$. Let y' be any continuous linear extension of $x'T^{-1}$ to all of Y . Then $y'T = x'$ on $D(T)$, whence $y' \in D((TJ_{D(T)})')$ and $(TJ_{D(T)})'y' = x'$. Since $x' \in D(T)'$ was arbitrary, $R((TJ_{D(T)})') = D(T)'$. □

I.3.12 Theorem [G1] $\overline{R(T)} = Y$ and T has a continuous inverse if and only if $R(T') = D(T)'$ and T' has a continuous inverse.

Proof: By Theorems I.3.7 and I.3.11, $\overline{R(T)} = Y$ and T has a continuous inverse if and only if $(TJ_{D(T)})'$ has an inverse defined on all of $D(T)'$. Since $(TJ_{D(T)})'$ is closed, so is $((TJ_{D(T)})')^{-1}$. Hence, by the closed–graph theorem, $D((TJ_{D(T)})')^{-1} = D(T)'$ if and only if $((TJ_{D(T)})')^{-1}$ is continuous on $D(T)'$. □

The next theorem shows that additional states for (T,T') fail to exist when Y is complete.

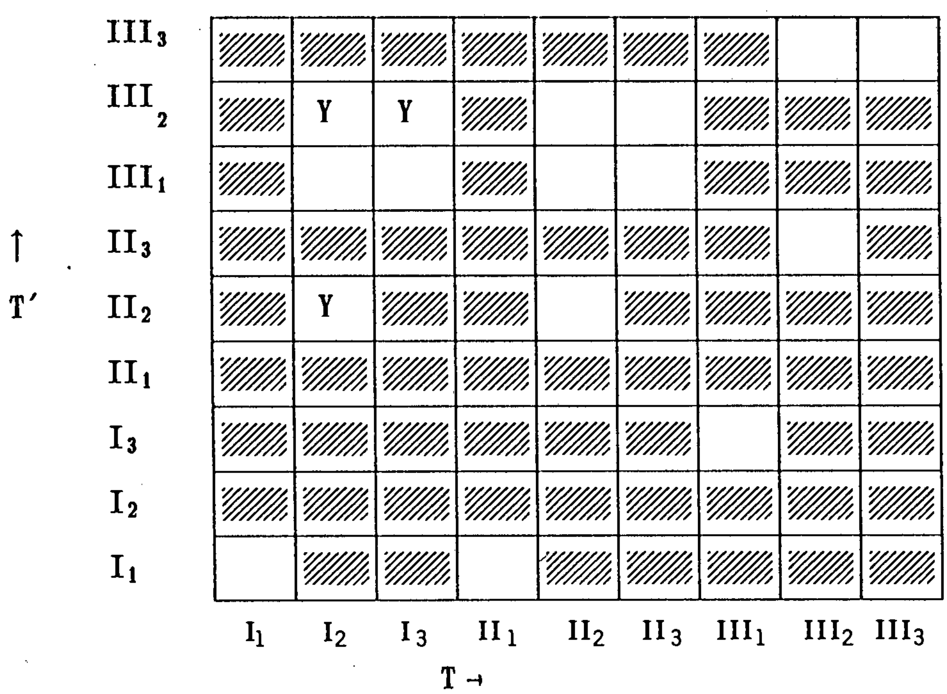
I.3.13 Theorem [G1] Let Y be complete. If $R(T) = Y$, then T' has a continuous inverse.

Proof: Suppose $R(T) = Y$. If $(TJ_{D(T)})'$ does not have a continuous inverse, there exists a sequence $\{y_n'\}$ in $D((TJ_{D(T)})')$ such that $\|y_n'\| \to \infty$ while $(TJ_{D(T)})'y_n' \to 0$. Thus, for each $x \in D(T)$, $y_n'Tx \to 0$. Since $R(T) = Y$, $y_n'y \to 0$ for each $y \in Y$. By

Theorem I.1.11 $\|y_n'\| \leq M$ for some constant $M > 0$. But this contradicts the fact that $\|y_n'\| \to \infty$. Hence $(TJ_{D(T)})'$ has a continuous inverse. □

I.3.14 Remark [G1] The following state diagram summarizes the theorems we have presented so far. For example, we see from an inspection of the diagram that T has a continuous inverse if and only if the range of T' is all of $D(T)'$.

I.3.14 State Diagram for Linear Operators [G1]



Y: Cannot occur if Y is complete.

I.3.15 Remark The author notices that the assumption that Y is reflexive has played no part in the theorems we have presented so far. Many implications can be read from the state diagram, e.g., if $T \in I_1$ or II_1 then $T' \in I_1$, i.e., Theorem I.3.12 is implied by the diagram.

Consequently, $y' \in D((TJ_{D(T)})')$ and

$$\|y'\| \, \|y\| \geq |y'y| \geq \sup_{x \in S_1} |y'Tx| = \|(TJ_{D(T)})'y'\| \geq r\|y'\| \, .$$

Thus $\|y\| \geq r$, which contradicts the supposition that $y \in rU_Y$. □

I.4.2 Lemma [G1] Suppose $TB_X \supset rU_Y$. If T^{-1} exists, then it is continuous with $\|T^{-1}\| \leq 1/r$.

Proof: If T^{-1} exists, then for $x \neq 0$ and $0 < \epsilon < 1$, $(1-\epsilon)r \, Tx \, / \, \|Tx\|$ is in rU_Y. By hypothesis, there exists some $z \in B_X \cap D(T)$ such that $Tz = (1-\epsilon)r \, Tx \, / \, \|Tx\|$. Since T is 1–1 , $z = (1-\epsilon)rx \, / \, \|Tx\|$. Hence $\|Tx\| \geq (1-\epsilon)r\|x\|$, which implies that T^{-1} is continuous with $\|T^{-1}\| \leq 1/r$. □

I.4.3 Theorem [G1] Suppose X is complete. If T is closed and T' has a continuous inverse, then $TB_X \supset rU_Y$, where $r = 1/\|(T')^{-1}\|$. Thus $R(T) = Y$ and T is an open map. If T^{-1} exists, it is continuous and $(1/\|T^{-1}\|)U_Y$ is the largest open a–ball which is contained in TB_X . □

Proof: All but the last statement of the theorem are immediate consequences of Lemmas I.4.1, I.1.8 and I.4.2. Suppose T^{-1} exists and $aU_Y \subset TB_X$. Then from Lemma I.4.2 and Theorems I.3.9 and I.2.6, it follows that

$$a \leq \frac{1}{\|T^{-1}\|} = \frac{1}{\|((TJ_{D(T)})')^{-1}\|} = r \, .$$

□

As a summary of Theorems I.3.11, I.3.13, and I.4.3, we have the following dual results. Note that neither X nor Y need be complete, nor do we require that T be closed in (i).

I.4.7 Lemma [G1] Suppose $N(T)$ is closed and $\hat{T}$ is the 1–1 operator induced by T. Then

(i) T is closed if and only if $\hat{T}$ is closed.

(ii) T is continuous if and only if $\hat{T}$ is continuous, in which case $\|T\| = \|\hat{T}\|$.

(iii) $D((\hat{T})') = D(T')$ and $\|(\hat{T})'y'\| = \|T'y'\|$.

Proof:

(i) [G1] Let T be closed. Suppose $[x_n] \to [x]$, $[x_n] \in D(T)/N(T)$, and $\hat{T}[x_n] \to y$. Then there exists a sequence $v_n \in N(T)$ such that $x_n - v_n \to x$. Since $T(x_n - v_n) = \hat{T}[x_n] \to y$ and T is closed, x is in $D(T)$ and $Tx = y$. Thus $[x] \in D(\hat{T})$ and $\hat{T}[x] = y$, which proves that $\hat{T}$ is closed. Conversely, let $\hat{T}$ be closed. Suppose $x_n \to x$ and $Tx_n \to y$. Then $[x_n] \to [x]$ and $\hat{T}[x_n] = Tx_n \to y$. Hence $[x] \in D(\hat{T})$ and $\hat{T}[x] = y$, or, equivalently, $x \in D(T)$ and $Tx = y$. Therefore T is closed.

(ii) [G1] Suppose T is continuous. Then

$$\|\hat{T}[x]\| = \|Tz\| \le \|T\| \, \|z\| \quad , \; z \in [x] .$$

Hence

$$\|\hat{T}[x]\| \le \|T\| \inf_{z \in [x]} \|z\| = \|T\| \, \|[x]\| .$$

Thus $\|\hat{T}\| \le \|T\|$. On the other hand, if $\hat{T}$ is continuous, then

$$\|Tx\| = \|\hat{T}[x]\| \le \|\hat{T}\| \, \|[x]\| \le \|\hat{T}\| \, \|x\| .$$

Therefore $\|\hat{T}\| \ge \|T\|$. Combining these result, we obtain $\|T\| = \|\hat{T}\|$.

(iii) Let $y' \in D((\hat{T})')$. Then $y'\hat{T}$ is continuous on $D(\hat{T})$ and

$$\|y'Tx\| = \|y'\hat{T}[x]\| \le \|y'\hat{T}\| \, \|[x]\| \le \|y'\hat{T}\| \, \|x\|$$

I.4.4 Theorem [G1]

(i) $R(T') = D(T)'$ if and only if T has a continuous inverse.

(ii) Suppose that X and Y are complete and that T is closed. Then $R(T) = Y$ if and only if T' has a continuous inverse.

I.4.5 Corollary (Banach–Mazur) [G1] If Y is a separable Banach space, then there exists a continuous linear operator mapping ℓ_1 onto Y. Hence Y' is equivalent to a subspace of ℓ_∞.

Proof: [G1] Let $\{y_k\}$ be a sequence of elements of norm 1 which is dense in the unit sphere of Y. Define $T \in [\ell_1, Y]$ by $T(\{\alpha_k\}) = \sum_{k=1}^{\infty} \alpha_k y_k$. Let $u_1 = (1,0,0,\ldots)$, $u_2 = (0,1,0,0,\ldots)$, etc. Then for $y' \in Y'$

$$\|T'y'\| = \sup_k |T'y'(u_k)| = \sup_k |y'y_k| = \sup_{\|y\|=1} |y'y| = \|y'\| .$$

Hence T' is an isometry and the range of T is Y by Theorem I.4.4. □

I.4.6 Definition [G1] The 1–1 operator $\hat{T}$ induced by T is the mapping from $D(T)/N(T)$ into Y defined by

$$\hat{T}[x] = Tx .$$

Note that $\hat{T}$ is 1–1 and linear with the same range as T.

The importance of considering $\hat{T}$ is that certain results which hold for 1–1 linear operators may be applied to $\hat{T}$ in order to obtain information about T. The proof of Corollary I.4.8 is a case in point.

The next lemma shows that $\hat{T}$ has some of the essential properties of T.

I.3.16 Remark [G1] In regard to the squares which remain open, we shall present in Section I.5 examples which exhibit the existence of the corresponding states when both X and Y are reflexive.

I.4 States of Closed Linear Operators

In this section we construct a state diagram under the assumption that T is a closed linear operator. The most important theorem in this section is Theorem I.4.3. The theorem was first proved for bounded, instead of closed, operators by Banach [Ba], Theorem 1, page 146. If we consider reflexivity and completeness assumptions about the normed space X under the hypothesis that T is closed we find that still further states are eliminated. However, in section I.5, we show that these states can occur when the corresponding hypotheses on X and Y are removed.

I.4.1 Lemma [G1] If T' has a continuous inverse (T not necessarily closed), then $\overline{TB_X} \supset rU_Y$, where $r = 1/\|(T')^{-1}\|$.

Proof: Suppose $y \in rU_Y$ but $y \notin \overline{TB_X}$. Since $\overline{TB_X}$ is closed and convex, there exists, by Theorem 0.3.12, a nonzero $y' \in Y'$ such that $\operatorname{Re} y'(y) \geq \operatorname{Re} y'(\overline{TB_X})$. Assert that $\operatorname{Re} y'(y) \geq |y'Tx|$ for all $x \in B_X \cap D(T) = S_1$. Indeed, if $x \in S_1$ and $y'Tx$ is written in polar form $|y'Tx|e^{i\theta}$, then $e^{-i\theta}x \in S_1$. Hence

$$\operatorname{Re} y'(y) \geq \operatorname{Re} y'T(e^{-i\theta}x) = |y'Tx| .$$

Consequently, $y' \in D((TJ_{D(T)})')$ and

$$\|y'\|\,\|y\| \geq |y'y| \geq \sup_{x \in S_1} |y'Tx| = \|(TJ_{D(T)})'y'\| \geq r\|y'\| .$$

Thus $\|y\| \geq r$, which contradicts the supposition that $y \in rU_Y$. □

I.4.2 Lemma [G1] Suppose $TB_X \supset rU_Y$. If T^{-1} exists, then it is continuous with $\|T^{-1}\| \leq 1/r$.

Proof: If T^{-1} exists, then for $x \neq 0$ and $0 < \epsilon < 1$, $(1-\epsilon)r\,Tx / \|Tx\|$ is in rU_Y. By hypothesis, there exists some $z \in B_X \cap D(T)$ such that $Tz = (1-\epsilon)r\,Tx / \|Tx\|$. Since T is 1–1, $z = (1-\epsilon)rx / \|Tx\|$. Hence $\|Tx\| \geq (1-\epsilon)r\|x\|$, which implies that T^{-1} is continuous with $\|T^{-1}\| \leq 1/r$. □

I.4.3 Theorem [G1] Suppose X is complete. If T is closed and T' has a continuous inverse, then $TB_X \supset rU_Y$, where $r = 1/\|(T')^{-1}\|$. Thus $R(T) = Y$ and T is an open map. If T^{-1} exists, it is continuous and $(1/\|T^{-1}\|)U_Y$ is the largest open a–ball which is contained in TB_X. □

Proof: All but the last statement of the theorem are immediate consequences of Lemmas I.4.1, I.1.8 and I.4.2. Suppose T^{-1} exists and $aU_Y \subset TB_X$. Then from Lemma I.4.2 and Theorems I.3.9 and I.2.6, it follows that

$$a \leq \frac{1}{\|T^{-1}\|} = \frac{1}{\|((TJ_{D(T)})')^{-1}\|} = r .$$ □

As a summary of Theorems I.3.11, I.3.13, and I.4.3, we have the following dual results. Note that neither X nor Y need be complete, nor do we require that T be closed in (i).

I.4.4 Theorem [G1]

(i) $R(T') = D(T)'$ if and only if T has a continuous inverse.

(ii) Suppose that X and Y are complete and that T is closed. Then $R(T) = Y$ if and only if T' has a continuous inverse.

I.4.5 Corollary (Banach–Mazur) [G1] If Y is a separable Banach space, then there exists a continuous linear operator mapping ℓ_1 onto Y. Hence Y' is equivalent to a subspace of ℓ_∞.

Proof: [G1] Let $\{y_k\}$ be a sequence of elements of norm 1 which is dense in the unit sphere of Y. Define $T \in [\ell_1, Y]$ by $T(\{\alpha_k\}) = \sum_{k=1}^{\infty} \alpha_k y_k$. Let $u_1 = (1,0,0,\ldots)$, $u_2 = (0,1,0,0,\ldots)$, etc. Then for $y' \in Y'$

$$\|T'y'\| = \sup_k |T'y'(u_k)| = \sup_k |y'y_k| = \sup_{\|y\|=1} |y'y| = \|y'\| .$$

Hence T' is an isometry and the range of T is Y by Theorem I.4.4. □

I.4.6 Definition [G1] The 1–1 operator $\hat{T}$ induced by T is the mapping from $D(T)/N(T)$ into Y defined by

$$\hat{T}[x] = Tx .$$

Note that $\hat{T}$ is 1–1 and linear with the same range as T.

The importance of considering $\hat{T}$ is that certain results which hold for 1–1 linear operators may be applied to $\hat{T}$ in order to obtain information about T. The proof of Corollary I.4.8 is a case in point.

The next lemma shows that $\hat{T}$ has some of the essential properties of T.

I.4.7 Lemma [G1] Suppose $N(T)$ is closed and $\hat{T}$ is the 1–1 operator induced by T. Then

(i) T is closed if and only if $\hat{T}$ is closed.

(ii) T is continuous if and only if $\hat{T}$ is continuous, in which case $\|T\| = \|\hat{T}\|$.

(iii) $D((\hat{T})') = D(T')$ and $\|(\hat{T})'y'\| = \|T'y'\|$.

Proof:

(i) [G1] Let T be closed. Suppose $[x_n] \to [x]$, $[x_n] \in D(T)/N(T)$, and $\hat{T}[x_n] \to y$. Then there exists a sequence $v_n \in N(T)$ such that $x_n - v_n \to x$. Since $T(x_n - v_n) = \hat{T}[x_n] \to y$ and T is closed, x is in $D(T)$ and $Tx = y$. Thus $[x] \in D(\hat{T})$ and $\hat{T}[x] = y$, which proves that $\hat{T}$ is closed. Conversely, let $\hat{T}$ be closed. Suppose $x_n \to x$ and $Tx_n \to y$. Then $[x_n] \to [x]$ and $\hat{T}[x_n] = Tx_n \to y$. Hence $[x] \in D(\hat{T})$ and $\hat{T}[x] = y$, or, equivalently, $x \in D(T)$ and $Tx = y$. Therefore T is closed.

(ii) [G1] Suppose T is continuous. Then

$$\|\hat{T}[x]\| = \|Tz\| \leq \|T\| \, \|z\| \quad , \quad z \in [x] .$$

Hence

$$\|\hat{T}[x]\| \leq \|T\| \inf_{z \in [x]} \|z\| = \|T\| \, \|[x]\| .$$

Thus $\|\hat{T}\| \leq \|T\|$. On the other hand, if $\hat{T}$ is continuous, then

$$\|Tx\| = \|\hat{T}[x]\| \leq \|\hat{T}\| \, \|[x]\| \leq \|\hat{T}\| \, \|x\| .$$

Therefore $\|\hat{T}\| \geq \|T\|$. Combining these result, we obtain $\|T\| = \|\hat{T}\|$.

(iii) Let $y' \in D((\hat{T})')$. Then $y'\hat{T}$ is continuous on $D(\hat{T})$ and

$$\|y'Tx\| = \|y'\hat{T}[x]\| \leq \|y'\hat{T}\| \, \|[x]\| \leq \|y'\hat{T}\| \, \|x\|$$

therefore $y' \in D((TJ_{D(T)})')$ and $\|y'T\| \le \|y'\hat{T}\|$. For the opposite inclusion let $y' \in D((TJ_{D(T)})')$. Then $y'T$ is continuous and

$$\|y'\hat{T}[x]\| \le \|y'T\| \, \|[x]\| .$$

Hence $y' \in D((\hat{T})')$ and $\|y'\hat{T}\| \le \|y'T\|$. Hence $D((\hat{T})') = D((TJ_{D(T)})')$ and $\|(\hat{T})'y'\| = \|(TJ_{D(T)})'y'\|$. □

I.4.8 Corollary [G1] Let Y be complete and let T be closed and densely defined. If T' is continuous, then $D(T) = X$.

If both X and Y are complete and T is closed, then the following three statements are equivalent.

(i) T' is continuous.

(ii) T is bounded.

(iii) $D(T')$ is closed.

The author refers the reader to [G1, II.4.8] for the proof. The author also remarks that the corollary holds only if domain of T is dense in X .

I.4.9 Corollary [G1] Suppose X is reflexive. If there exists a bounded linear operator which maps X onto Banach space Y , then Y is reflexive.

If M is a closed subspace of X , then X/M is reflexive.

Proof: [G1] Assume that T is an isomorphism from X onto Y . It follows from Theorem I.4.4 that T'' is also an isomorphism. (One can also easily prove directly that T'' is an isomorphism.) Since $T''J_X = J_Y T$ and both J_X and T are surjective, it follows that

$$Y = R(T''J_X) = R(J_Y T) \subset R(J_Y) .$$

Thus Y is reflexive. Let us only assume that $R(T) = Y$. Then by Theorem I.3.13, $T'Y'$ is isomoprhic to Y'. Since Y' is complete, $T'Y'$ is also complete and therefore a closed subspace of reflexive space X'. Hence $T'Y'$ is reflexive by Theorem 0.5.6. Thus, by what was just proved, Y' is reflexive and therefore so is Y. The last statement of the theorem now follows, since the map from X onto X/M defined by $x \to [x]$ is bounded and linear. □

We come to the final theorem which is needed to complete the state diagram for closed operators.

I.4.10 Theorem [G1] Suppose X is reflexive. If T is closed and 1–1, then $R(T')$ is dense in $D(T)'$.

Proof: Let T_1 be the operator T considered as a map onto

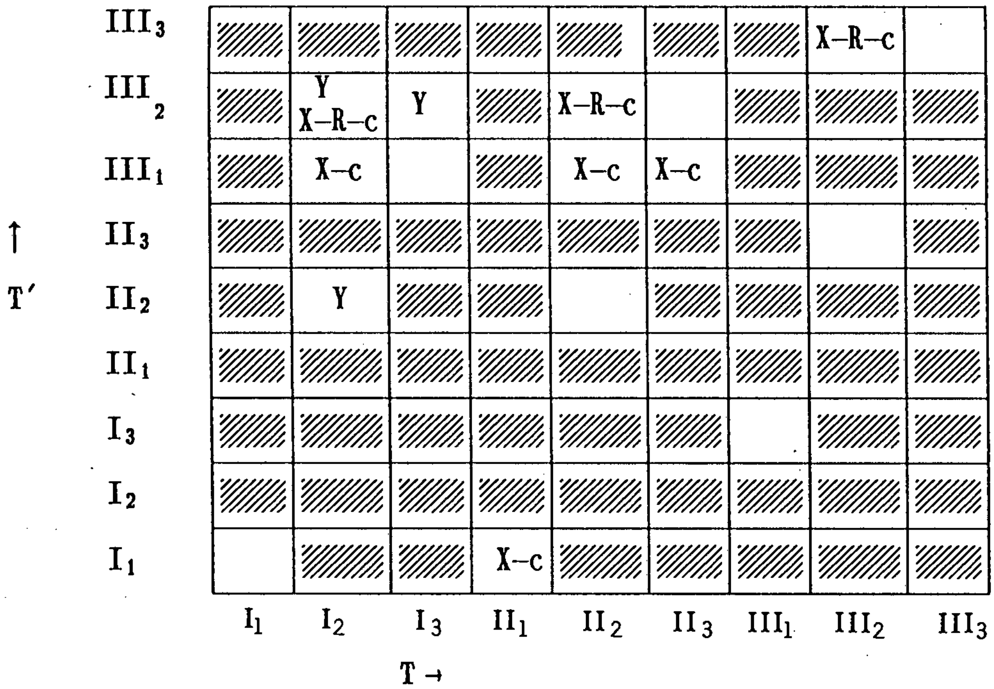
$$Y_1 = R(T).$$

It is easy to see that $R((TJ_{D(T)})') = R((T_1J_{D(T)})')$. By Theorems I.3.7 and I.3.9, $(T_1J_{D(T)})'$ has an inverse and $((T_1J_{D(T)})')^{-1} = (T_1^{-1})'$. Since T is closed, it is clear that T_1 is closed and therefore so is T_1^{-1}. Applying Theorem I.2.15 to T_1^{-1}, we have that $D((T_1^{-1})')$ is dense in $D(T)'$. Since

$$D((T_1^{-1})') = D(((T_1')^{-1}) = R((TJ_{D(T)})') = R((TJ_{D(T)})')$$

the theorem follows. □

I.4.11 State Diagram for Closed Linear Operators [G1]



Y: Cannot occur if Y is complete

X – c : Cannot occur if X is complete and T is closed.

X – R – c : Cannot occur if X is reflexive and T is closed.

I.5 Examples of States [G1]

In this section examples are given which exhibit those states of (T,T′) which can occur even when both X and Y are reflexive. Examples are also given which show that the states of (T,T′) corresponding to the squares with entries can occur when the corresponding hypotheses on X or Y are removed.

Since some of the arguments depend on properties of compact operators, we give a brief introduction to them.

I.5.1 Definition [G1] Let T be a linear operator with domain in X and range in Y. If TB_X is totally bounded in Y, where B_X is the 1–ball in X, then T is called *precompact.* If $\overline{TB_X}$ is compact in Y, then T is called *compact.*

I.5.2 Remarks [G1]

(i) Every continuous linear operator with finite–dimensional range is compact.

(ii) An operator is compact if and only if it takes every bounded sequence into a sequence which has a convergent subsequence. An operator is precompact if and only if it takes every bounded sequence into a sequence which has a Cauchy subsequence.

(iii) If T is precompact as a mapping from X into Y and $\tilde{Y}$ is the completion of Y, then T, when considered as a map into $\tilde{Y}$, is compact. Thus $T \in [X,Y]$ is compact if and only if it is precompact, provided Y is complete.

I.5.3 Theorem A continuous linear operator is precompact if and only if its adjoint is compact.

Proof: If T is precompact then $TJ_{D(T)}$ is precompact. Hence $(TJ_{D(T)})' = T'$ is compact.

Conversely, suppose T' is compact. Then $(TJ_{D(T)})'$ is compact. But T is continuous by hypothesis and so $TJ_{D(T)}$ is bounded. Hence, by [G1, III.1.11], $TJ_{D(T)}$ is precompact. Hence T is precompact. □

I.5.4 Theorem If T is precompact then $R(T)$ is separable.

Proof: If Y is complete then T is precompact if and only if T is compact. Hence $R(T)$ is separable by [T, 5.5–A]. If Y is not complete then T precompact implies $J_Y^{\tilde{Y}}T$ is compact. Hence $R(T)$ is a separable subspace of $\tilde{Y}$ and thus a separable subspace of Y. □

I.5.5 [G1] In our examples of states, we shall, where possible, construct compact linear operators.

$$X = Y = \ell_2 \quad T \text{ continuous on } X$$

(I_1, I_1): Let T be the identity operator on X.

(I_3, III_1): Let T be the left–shift operator defined by $T(\{x_k\}) = \{x_{k+1}\}$.

Obviously, $T \in I_3$ and by I.3.14 $T' \in III_1$.

(III_1, I_3): Let T be the right–shift operator defined by

$$T(\{x_k\}) = \{x_{k-1}\} \ , \ 1 < k$$

where $x_0 = 0$. $T \in III_1$ and by I.4.11, $T' \in I_3$.

It is easy to see that the left–shift and right–shift operators are conjugates of each other.

I.5.6 [G1] $X = Y = \ell_2$ T compact on X.

(II_2, II_2): Let T be defined by $T(\{x_k\}) = \{x_k/k\}$. T is compact since $T_n \to T$, $n = 1, 2, \ldots$, where T_n is the compact operator from X into Y defined by $T_n(\{x_k\}) = \{y_k\}$, $y_k = x_k/k$, $1 \leq k \leq n$, and $y_k = 0$, $k > n$. T_n is compact since it is bounded and its range is finite–dimensional. Clearly, $R(T)$ is dense in Y, and T has an unbounded inverse. Since a compact operator cannot have an infinite–dimensional range which is complete, $T \in II_2$. From I.4.11, $T' \in II_2$.

(II_3, III_2): Let L be the left–shift operator on ℓ_2, and let A be the compact operator in example (II_2, II_2). Define $T = AL$. Then T is compact and in II_3. From I.4.11, $T' \in III_2$.

(III_2, II_3): Let T be the adjoint of the operator in example (II_3, III_2). Since the adjoint of a compact operator is compact, T is compact and from I.4.11, $T \in III_2$ and $T' \in II_3$.

(III_3, III_3): Let T be the zero operator. Less trivially, let T_1 be the compact operator in example (II_2, II_2) and let L and R be the left–shift and right–shift operators, respectively. Then $T = T_1RL$ is a compact operator in III_3. By I.4.11, $T' \in III_3$.

I.5.7 [G1] $X = \ell_2$ Y not complete T compact on X.

The next example depends on the following observation.

Suppose T is a compact linear operator from reflexive space X into Y. Let T_1 be the operator T considered as a map from X onto $Y_1 = R(T)$. Then T_1 is also compact. To see this, suppose $\{x_n\}$ is a bounded sequence in X. Since T is compact, there exists a subsequence $\{v_n\}$ of $\{x_n\}$ such that $Tv_n \to y \in Y$. By Theorem 0.6.2, there exists a subsequence $\{z_n\}$ of $\{v_n\}$ which converges weakly to some $x \in X$. The continuity of T implies that $Tz_n \xrightarrow{w} Tx$. Since $Tz_n \to y$, it is clear that $Tz_n \xrightarrow{w} y$. Hence $y = Tx$, since $y'y = y'Tx$ for all $y' \in Y'$. Thus $T_1v_n \to Tx \in Y_1$, showing that T_1 is compact.

(I_2, II_2): Let T_0 be the compact operator in example (II_2,II_2) and let T be the operator T_0 considered as a map from X onto $R(T_0)$. Then T is a compact operator in I_2, and by I.4.11, $T' \in II_2$.

(I_3, III_2): Let L be the left–shift operator on ℓ_2. Choose K as the operator in the preceeding example (I_2, II_2). Defining $T = KL$, T is a compact operator in I_3. Hence T' is also compact. Since a compact operator on an infinite–dimensional normed linear space cannot have a bounded inverse, $T' \notin 1$. Thus $T \in III_2$, by I.4.11.

I.5.8 [G1] X not complete $Y = \ell_2$ T continuous on X.

(I_2, III_1): Let X be the normed linear space obtained by renorming ℓ_2 as in Example I.1.14. Define T as the identity map from X onto ℓ_2. Then $T \notin 1$; otherwise the incomplete space X would be isomorphic to Banach space Y. Thus $T \in I_2$ and $T' \in III_1$ by I.4.11.

(II_1, I_1): Choose X as any proper subspace dense in $Y = \ell_2$ with T as the identity map on X.

(II_2, III_1): Let A be the operator in example (I_2, III_1). Setting X_1 as the domain of A and Y_1 as any proper subspace dense in ℓ_2, define $T: X_1 \times Y_1 \to \ell_2 \times \ell_2$ by $(x,y) = (Ax,y)$. Take the norm on $\ell_2 \times \ell_2$ to be given by $\|(x,y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$. With respect to the inner product

$$\langle (x,y), (u,v)\rangle = \langle x,u\rangle + \langle y,v\rangle$$

$\ell_2 \times \ell_2$ is a separable Hilbert space which, by Theorem I.7.24 of Goldberg [G1], is equivalent to ℓ_2. T is easily seen to be in II_2. Given $z' \in (\ell_2 \times \ell_2)'$, define x' and $y' \in \ell_2'$ by

$x'(x) = z'(x,0)$ and $y'(y) = z'(0,y)$. Then

$$|z'(x,y)| \le \|x'\|\,\|x\| + \|y'\|\,\|y\| \le (\|x'\| + \|y'\|)\,\|(x,y)\|.$$

Hence $\|z'\| \le \|x'\| + \|y'\|$. For $\|x\| = 1$,

$$\|T'z'\| \ge |T'z'(x,0)| = |(A'x')x|.$$

Thus

(1) $$\|T'z'\| \geq \|A'x'\| \geq \frac{\|x'\|}{\|(A')^{-1}\|}.$$

Similarly,

$$\|T'z'\| \geq |T'z'(0,y)| = |y'y| \quad \|y\| = 1.$$

Thus

(2) $$\|T'z'\| \geq \|y'\|.$$

Hence, by (1) and (2), there exists some $m > 0$ such that

$$\|T'z'\| \geq m(\|x'\| + \|y'\|) \geq m\|z'\|.$$

This shows $T' \in 1$. It follows from I.4.11 that $T' \in III_1$.

(II_3, III_1): Let A, X_1, and Y_1 be as above. Define T: $X_1 \times Y_1 \to \ell_2 \times \ell_2$ by $T(x,y) = (Ax, Ly)$, where L is the left–shift operator on ℓ_2. T is easily seen to be in II_3. Arguing as in the above example, we obtain

$$\|T'z'\| \geq \|L'y'\| = \|y'\|.$$

Recall that L' is the right–shift operator and, in particular, is an isometry. Thus $T' \in III_1$, by I.4.11.

I.5.9 [G1] X complete but not reflexive $Y = \ell_2$ T compact on X.

(II_2, III_2): If $0 < p \leq q \leq \infty$, then $x = (x_1, x_2, \ldots) \in \ell_p$ implies $x \in \ell_q$ and $\|x\|_p \geq \|x\|_q$. Indeed, if $q < \infty$, then $\|x\|_q^q = \sum_{k=1}^{\infty} |x_k|^p |x_k|^{q-p} < \infty$, since $\{x_k\}$ is bounded. Now $\|x\|_q \geq |x_k|$, $1 \leq k$. Thus $\|x\|_q^q \leq \|x\|_q^{q-p}\|x\|_p^p$ whence $\|x\|_q \leq \|x\|_p$. If $q = \infty$, then clearly $\|x\|_\infty \leq \|x\|_p$. Let A be the identity map from ℓ_1 into ℓ_2. Then by what was just proved, A is continuous. Define T: $\ell_1 \to \ell_2$ by $T = KA$, where K is the compact operator in example (II_2, II_2). Since T' is compact, it is not in 1 and its range is separable. However, ℓ_∞ is not separable. Thus $T' \in III_2$ by I.4.11.

(III_2, III_3): Let T_1 be the compact operator T in the preceding example and let R be the right–shift operator on ℓ_1. Define $T: \ell_1 \to \ell_2$ as the operator T_1R. Then T is clearly in III_2. Since compact operator T' has a separable range, T' is in III. Thus $T' \in III_3$, by I.4.11.

The next example is for the only square in I.4.11 which has not been accounted for.

I.5.10 [G1] X complete but not reflexive Y not complete T compact on X.

(I_2, III_2): Let T_1 be the operator from $\ell_1 \to \ell_2$ defined by

$$T_1(\{x_k\}) = \{x_k/k\}.$$

Then by the argument given in example (II_2, II_2), T_1 is a compact operator with an unbounded inverse. Define T to be the operator T_1 considered as a map from ℓ_1 onto $Y = R(T_1)$. Then $T \in I_2$. We show that T is compact. Suppose $\{x_n\}$ is a sequence of elements in the 1–sphere of ℓ_1. Since T_1 is compact, there exists a subsequence $\{v_n\}$ of $\{x_n\}$ such that $Tv_n \to y = (y_1, y_2, \ldots) \in R(T_1)$. Writing $v_n = (v_1^n, v_2^n, \ldots)$, it follows that $\lim_{n\to\infty} v_k^n = ky_k$, $1 \leq k$. Hence, for any positive integer M,

$$\sum_{k=1}^{M} |ky_k| \leq \sum_{k=1}^{M} |ky_k - v_k^n| + \sum_{k=1}^{M} |v_k^n| \leq \sum_{k=1}^{M} |ky_k - v_k^n| + 1 \to 1 \text{ as } n \to \infty.$$

Thus $\sum_{k=1}^{M} |ky_k| \leq 1$ for all M, and therefore $x = \{ky_k\}$ is an element in ℓ_1. Hence $Tv_n \to y = Tx$, which shows that T is compact. Since T' is compact, its range is a separable subspace of the inseparable space ℓ_∞. Thus $T' \in III$. It follows from I.4.11 that T' is in III_2.

To summarize the chapter, the author has shown that Goldberg's [G1] assumption of denseness of the domain of the operator T can be dispensed with. However, in Theorem I.2.15 it is necessary to assume that the domain of T is dense in X.

CHAPTER II

F_+– AND ϕ_+–OPERATORS

II.1 Notation If M and N are linear subspaces of X such that $M \cap N = 0$ then we write $M \oplus N$ for $M + N$.

II.2 Lemma If E, M are subspaces of X with $\dim (X/E) < \infty$ then $M = E \cap M \oplus F$ where $\dim F < \infty$ (see e.g. [C6, 1]).

II.3 Definition [C6] A closed finite codimensional subspace of X is called a *principal subspace* and the restriction of T to such a subspace is called a *principal restriction.*

II.4 Lemma Let Z be a subspace of X. For each principal subspace M of Z there exists a principal subspace M_0 of X such that $M = M_0 \cap Z$.

Proof: [C6] Let M be a principal subspace of Z. There exists a finite dimensional subspace F of Z such that $M \oplus F = Z$. Let $x_1, ..., x_n$ be a basis for F. Choose $f_1, ..., f_n \in X'$ such that $f_i(x_j) = \delta_{ij}$ and $f_i(m) = 0$ for $m \in M$ $(i,j \leq n)$; this is possible since M is closed. Then $M_0 = \bigcap_{i \leq n} f_i^{-1}(0)$ is a principal subspace of X with $M = M_0 \cap Z$. □

II.5 Definition [C6] A restriction T/M is said to be *nontrivial* if $M \cap D(T)$ is infinite dimensional. The family of infinite dimensional linear subspaces of a given linear subspace M of X is denoted by $\mathcal{I}(M)$.

II.6 Lemma [C6] The operator T has no principal restriction having a continuous inverse if and only if corresponding to each $\epsilon > 0$ there exists $M \in \mathcal{J}(D(T))$ such that T/M is precompact and $\|T/M\| \leq \epsilon$.

Proof: [C6] If T has no principal restriction having a continuous inverse then the conclusion follows from the theorem of Kato [Ka] and Goldberg (see [G1], 80). Conversely, let T/M be precompact where $M \in \mathcal{J}(D(T))$. Suppose that E is principal in X. Then $M \cap E \in \mathcal{J}(X)$ by Lemma II.2, and therefore $T/M \cap E$ has no continuous inverse. Consequently T/E has no continuous inverse. □

II.7 Definition [C6] The operator $T \in L(X,Y)$ is called an *F_+–operator* if there exists a subspace M of finite codimension in X for which T/M has a continuous inverse.

II.8 Corollary [C6] The following statements are equivalent :

(i) $T \in F_+$.

(ii) T has a principal restriction having a continuous inverse.

(iii) T has no nontrivial precompact restriction.

II.9 Corollary [C6] The operator $T \in F_+$ if and only if there exists a principal subspace M of D(T) for which T/M has a continuous inverse.

Proof: [C6] Combine Lemma II.4 and Corollary II.8. □

II.10 Definition [C6] We call a closed operator T a *ϕ_+–operator* if $\dim N(T) < \infty$ and $R(T)$ is closed.

II.11 Remark [C6] The next proposition relates the two classes F_+ and ϕ_+ when the operators are closed. A normed space is called an *operator range* [C2] if it is the range of a bounded linear operator whose domain is a Banach space. An example of an operator range is $D(T)$, when $T \in L(X,Y)$ is closed and X and Y are complete.

II.12 Proposition [C6] Let $T \in L(X,Y)$ be a closed operator. Then

(a) if X is complete, we have $T \in F_+ \Rightarrow T \in \phi_+$

(b) if X is an operator range and Y is complete, we have $T \in \phi_+ \Rightarrow T \in F_+$.

Proof: [C6] (a) Assume X complete and let $T \in F_+$. By Corollary II.9, $D(T) = M \oplus F \oplus N(T)$ where $\dim N(T) < \infty$, $\dim F < \infty$, M is closed in $D(T)$ and $(T/M)^{-1}$ is continuous. Let $Tm_n \to y$ $(m_n \in M)$. Then (m_n) is Cauchy, so $m_n \to x$ $(\exists x \in X)$ since X is complete. Thus $(m_n, Tm_n) \to (x,y)$ in $X \times Y$. But T is closed; hence $x \in D(T)$ and $y = Tx$. Since M is closed in $D(T)$, we have $x \in M$. Therefore TM is closed, and so $R(T) = TF + TM$ is closed (see e.g. [G1], 16). Hence $T \in \phi_+$.

(b) [C6] Assume $T \in \phi_+$, X an operator range and Y complete. Then $D(T) = M \oplus N(T)$ where $\dim N(T) < \infty$, M is principal in $D(T)$ and TM is closed. It is an easy consequence of the Closed Graph Theorem that any closed operator mapping a Banach space into an operator range is continuous. Therefore $(T/M)^{-1}$ is continuous. Hence $T \in F_+$ by Corollary II.9. □

For the purpose of obtaining a computable characterization of F_+–operators we introduce for a given $L(X,Y)$ the following function (cf. [C7]):

$$\Gamma(T) = \begin{cases} \inf \{\|T/M\| : M \in \mathcal{J}(D(T))\} & \text{if } \dim D(T) = \infty \\ 0 & \text{if } \dim D(T) < \infty \end{cases}$$

($\Gamma(T) = \infty$ can occur; see Proposition II.31.)

II.13 Proposition [C6] If $D(T)$ is infinite dimensional then $T \in F_+$ if and only if $\Gamma(T) > 0$.

Proof: [C3] If $T \notin F_+$ then $\Gamma(T) = 0$ by Lemma II.6 and Corollary II.8. Conversely, let $T \in F_+(X,Y)$. Then T/M has a continuous inverse for some principal subspace M of $D(T)$ (Corollary II.9), and we have $\|Tm\| \geq c\|m\|$ for some $c > 0$ and all $m \in M$. Suppose $\Gamma(T) = 0$. Then $\|T/N\| < c$ for some $N \in \mathcal{J}(D(T))$, while $M \cap N \in \mathcal{J}(D(T))$ by Lemma II.1. Hence $\|Tx\| < c$ for $x \in M \cap N$, which is a contradiction. Therefore $\Gamma(T) > 0$. □

II.14 Definition [C7] We shall call the operator $T \in L(X,Y)$ *strictly singular* if there is no infinite dimensional subspace M of $D(T)$ for which T/M has a continuous inverse. The class of all such operators will be denoted by $SS(X,Y)$ or simply SS.

We state the next few results without proof. For proofs the reader is is referred to R. W. Cross ([C3], page 8).

II.15 Definition [C3] We define an extended nonnegative real valued function on the class of all operators as follows: If $\dim D(T) < \infty$ then $\Delta(T) = 0$. If $\dim D(T) = \infty$ then $\Delta(T) = \sup_{M \in \mathcal{J}(D(T))} \Gamma(T/M)$.

II.16 Proposition [C3] $T \in SS \Leftrightarrow \Delta(T) = 0$.

II.17 Corollary [C3] The sum of two strictly singular operators is strictly singular.

II.18 Definition [C3] Let $\mathscr{S}$ be a subset of $L(X,Y)$. We denote by $P(\mathscr{S})$ the class of operators in $L(X,Y)$ such that if $T \in \mathscr{S}$ and $A \in P(\mathscr{S})$ then $T + A \in \mathscr{S}$. $P(\mathscr{S})$ is called the *perturbation class* of $\mathscr{S}$.

II.19 Theorem [C3] $P(F_+) = SS$.

II.20 Corollary [C6] $T \in F_+(X,Y)$ if and only if $\dim N(T + K) < \infty$ for every precompact operator $K \in L(X,Y)$.

Proof: [C6] We may clearly suppose $\dim D(T) = \infty$. Assume $T \notin F_+$. By Corollary II.8 there exists $E \in \mathscr{I}(D(T))$ such that T/E is precompact. Define $K \in L(X,Y)$ by $D(K) = E$ and $Kx = Tx$ $(x \in E)$. Then K is precompact and $N(T - K) \supset E$. The converse is immediate from Theorem II.19. □

II.21 Definition [G1] Let M be a subspace of X. An operator P is called a *projection* from X onto M if P is an everywhere defined linear map from X onto M such that $P^2 = P$.

II.22 Lemma [C6] If $X = M \oplus N$ where M is a principal subspace then the projection of X onto M with null space N is bounded.

Proof: [C6] Let $\{x_1, ..., x_n\}$ be a basis for N and let $N_i = \text{sp}\{x_1, ..., x_{i-1}, x_{i+1}, ..., x_n\}$. Since each $M + N_i$ is closed (see e.g. [G1], 16) there exists by the Hahn Banach Theorem an $f_i \in X'$ such that $f_i(x_i) = 1$ and $f_i(x) = 0$ for $x \in M + N_i$. Define $Q = \sum_{i \le n} f_i \otimes x_i$. Then Q is a bounded projection with range N and null space M, and $P = I - Q$ is the required projection. □

II.23 Corollary [C6] If M is a principal subspace of X and if T/M is continuous, then T is continuous.

II.24 Definition [C6] We say that T has *finite rank* if $\dim R(T) < \infty$.

II.25 Lemma [C6] Let $T \in F_+$. Then any bounded sequence (x_n) in $D(T)$ such that Tx_n is Cauchy has a Cauchy subsequence.

Proof: [C6] There exists by Corollary II.8 and Lemma II.22 a bounded projection P defined on $D(T)$ with $\dim R(I-P) < \infty$ and $(T/PX)^{-1}$ continuous. Let (x_n) be a bounded sequence in $D(T)$ satisfying $\|T(x_n - x_m)\| = f(n,m)$ where $f(n,m) \to 0 (n,m \to \infty)$. Since $T/R(I-P)$ is continuous, the sequence $T(I-P)x_n$ is bounded in the finite dimensional space $R(T(I-P))$. Select a subsequence (x_{k_n}) of (x_n) for which $T(I-P)x_{k_n}$ is Cauchy. Then $\|TP(x_{k_n} - x_{k_m})\| \leq \|T(x_{k_n} - x_{k_m})\| + \|T(I-P)(x_{k_n} - x_{k_m})\| \leq f(k_n, k_m) + \|T(I-P)(x_{k_n} - x_{k_m})\| \to 0$ $(n,m \to \infty)$. Hence (TPx_{k_n}) is Cauchy. Therefore (Px_{k_n}) is Cauchy by the continuouty of $(T/PX)^{-1}$. Choose a subsequence (w_n) of (x_{k_n}) such that $(I-P)w_n$ is Cauchy.
Then $w_n = Pw_n + (I-P)w_n$ is Cauchy. □

II.26 Theorem [C6] The following statements are equivalent :

(i) $T \notin F_+$

(ii) there exists a non precompact bounded subset W of $D(T)$ such that TW is precompact

(iii) T has a singular sequence, i.e. a sequence (x_n) of norm one elements of $D(T)$ such that (x_n) has no Cauchy subsequence and $\lim Tx_n = 0$.

Proof: [C6] The implication (iii) ⇒ (ii) is trivial, while (ii) ⇒ (i) is clear from Lemma II.25. Assume (i). Then by Corollary II.8 there exists $M \in \mathcal{J}(D(T))$ for which T/M is precompact. Select a sequence (z_n) in M of norm one elements for which $\inf_{j \neq k} \|z_j - z_k\| > 0$. Since $T(\{z_n\} \cup 0)$ is totally bounded, given k there exists j such that $\|Tz_j\| \leq 1/k$. Therefore it is possible to select a subsequence (x_n) of (z_n) such that $\lim_n Tx_n = 0$. Hence (i) ⇒ (iii). □

II.27 Remark [C6] Theorem II.26 extends the result for ϕ_+–operators of Wolf [Wo] (also E. Balslev and C.F. Schubert, c.f. [LS]).

II.28 Definition [C6] Given $T \in L(X,Y)$ let X_T denote the linear space $D(T)$ with the norm $\|x\|_T = \|x\| + \|Tx\|$ $(x \in D(T))$. The graph operator $G \in L(X_T,X)$ is defined by $Gx = x (x \in X_T)$. The operator TG is a bounded operator from X_T into Y.

II.29 Definition [C4] An operator T is called *partially continuous* if there exists a finite codimensional subspace E such that T/E is continuous. If there is no infinite dimensional subspace of $D(T)$ upon which T is continuous then T is called *nowhere continuous*; otherwise it is said to be *somewhere continuous.*

II.30 Theorem [C3] The following statements are equivalent:

(i) $T \in F_+$

(ii) there exists a partially continuous operator $S \in L(Y,X)$ with domain $R(T)$ and a bounded finite rank operator $F \in L(X,X)$ such that $ST = I_{D(T)} + F$

(iii) the same as (ii) but with "precompact" in place of "bounded finite rank"

(iv) the same as (iii) but with "strictly singular" in place of "precompact"

Proof: See [C3], Theorem 5.9.

II.31 Proposition [C6] Let T be injective. The following statements are equivalent:

(i) T^{-1} is strictly singular.

(ii) $\Gamma(T) = \infty$.

(iii) T is a nowhere continuous F_+–operator.

Proof: [C6] Assume (i). Then $\|T/M\| = \infty$ for $M \in \mathscr{I}(D(T))$. Hence (i) $\Rightarrow$ (ii). Next assume (ii). Then $T \in F_+$ by Proposition II.13, and $\|T/M\| = \infty$ for $M \in \mathscr{I}$ $(D(T))$. Thus (ii) $\Rightarrow$ (iii) . Finally, if T is nowhere continuous then $T^{-1} \in SS$ by definition. Therefore (iii) $\Rightarrow$ (i). □

II.32 Example [C6] A bounded injective F_+–operator which is not an isomorphism. Let f be a discontinuous linear functional with domain X , and let $G: X_f \to X$ be the graph operator associated with f . Then G^{-1} is unbounded. However $G^{-1}/N(f)$ is an isometry and hence $G \in F_+$. Thus G is a bounded injective F_+–operator which is not an isomorphism.

Notation: Let E be a subspace of X . The author denotes the finite codimensional subspaces of E by $\mathscr{C}(E)$ and the principal subspaces of E by $\mathscr{P}(E)$.

II.33 Definition [C3] If $\dim D(T) < \infty$ then we put $\tau_0(T) = \bar{\tau}_0(T) = 0$. If $\dim D(T) = \infty$ then $\tau_0(T) = \sup_{M \in \mathscr{C}(X)} \inf_{x \in S_M} \|Tx\|$ and $\bar{\tau}_0(T) = \sup_{M \in \mathscr{P}(X)} \inf_{x \in S_M} \|Tx\|$.

The proof of the following theorem is to be found in ([C3], p.289).

II.34 Theorem [C3] Let $\dim D(T) = \infty$. Then the following statements are equivalent:

(i) $T \in F_+$

(ii) T has a principal restriction having a continuous inverse

(iii) T has no nontrivial precompact restriction

(iv) $\Gamma(T) > 0$

(v) $\tau_0(T) > 0$

(vi) $\overline{\tau}_0(T) > 0$.

The author follows Pietsch's notation and denotes the natural quotient map of X onto X/E by Q_E^X . We have by Theorem 0.5.3 that $(J_E^X)' = Q_{E^\perp}^{X'}$, and if E is closed then $(Q_E^X)' = J_{E^\perp}^{X'}$.

II.35 Definition The closed operator T is called ϕ_-–operator if $R(T)$ is a principal subspace of Y .

II.36 Lemma [C6] Let M be a principal subspace of $D(T)$. Then $(J_M^X)'T' = (TJ_M^X)'$.

For the proof the author refers the reader to [C6, Lemma 36].

II.37 Lemma [C2] If R_1 and R_2 are disjoint complementary operator ranges in a Banach space X then R_1 and R_2 are closed.

II.38 Theorem [C6] $T \in F_+ \Leftrightarrow T' \in \phi_-$.

Proof: [C6] Let $T \in F_+$. By Corollary II.8 there exists a principal subspace E of $D(T)$ such that $TJ_E^X \in 1$. Write $J = J_E^X$. Then $(TJ)' \in I$ by ([G1], 61). Hence $J'T' \in I$ by Lemma II.36. Now let $x' \in X'$ and choose $y' \in Y'$ such that $x' + E^\perp = T'y' + w$. This shows that $X' = R(T') + E^\perp$. But $R(T')$ is an operator range (see e.g. [C2], 228). Hence by Lemma II.37, $R(T')$ is closed. Therefore $T' \in \phi_-$.

Conversely, let $T' \in \phi_-$. Then there exists a finite dimensional subspace F of X' such that $Q_F^{X'} T \in I$. Since $Q_F^{X'} = (J_{T^\perp}^X)'$, we have $(TJ_{F^\perp}^X)' \in I$ by Lemma II.36 where $F^\perp$ is a principal subspace. Hence by the state diagram (I.3.14), $TJ_{F^\perp}^X \in 1$. Thus $T \in F_+$. □

CHAPTER III

F_- AND ϕ_--OPERATORS

In this chapter the author discusses the results on F_- and ϕ_--operators which are needed to construct the state diagrams of Chapter IV. The minimum modulus function $\gamma(T)$ and the function $\Gamma'(T)$ will play an important role.

III.1 The General Case

III.1.1 Definition An *F_--operator* $T : D(T) \subset X \to Y$ is defined as one whose adjoint is a ϕ_+-operator, where X and Y are normed sapces.

III.1.2 Remark The author notes that if X and Y are complete and T is closed then $T \in F_+ \Leftrightarrow T \in \phi_+$. (Proposition II.12.) Similarly, $T \in F_- \Leftrightarrow T \in \phi_-$. Hence in the general case we have the dual relations $T \in F_+ \Leftrightarrow T' \in F_-$ and $T \in F_- \Leftrightarrow T' \in F_+$.

III.1.3 Notation [CL] We denote by $\mathscr{E}(X)$ the collection of closed infinite codimensional subspaces of X.

III.1.4 Definition [Pi] T is called *nuclear* if $T = \sum_1^\infty x_i' \otimes y_i$ where $\{x_i'\} \subset X'$, $\{y_i\} \subset Y$ and $\sum_1^\infty \|x_i'\| \, \|y_i\| < \infty$, and where the convergence of $\Sigma\, x_i'(x)y_i$ $(x \in D(T))$ is in the norm topology.

III.1.5 Definition [Pi] T is called *approximable* if there exists a sequence of continuous finite rank operators F_n such that $D(F_n) \subset D(T)$ for each n and $\lim_n \|F_n - T\| = 0$.

To simplify the notation the author will write J_X for $J_X^{\tilde{X}}$ and Q_E for Q_E^X when X is understood.

III.1.6 Theorem [CL] The following statements are equivalent :

(i) $T' \notin \phi_+$

(ii) there is no subspace $F \in \mathscr{I}(Y)$ for which $T'/F^{\perp}$ has a continuous inverse

(iii) for each $\epsilon > 0$ there exists $M \in \mathscr{E}(\tilde{Y})$ such that $Q_M J_Y T$ is a nuclear (compact approximable) operator with norm not exeeding $\mathscr{E}$.

The author refers the reader to [CL, 4.2] for the proof.

III.1.7 Example [CL] Let $\tau = \sum_0^n a_k D^k$ where $D = \frac{d}{dt}$, $D^0 f = f$, $a_k \in C^k(I)$ for $0 \leq k \leq n$ and $a_0 \in L_\infty(I)$ where I is an interval containing one of its endpoints. Define $T: L_p(I) \to L_q(I)$ as follows:

$$D(T) = \{f: f \in L_p(I) , \ \tau f \in L_q(I)\} ;$$
$$Tf = \tau f .$$

(a) The operator T is F_- with $D(T') = (0)$ whenever $1 \leq p,q < \infty$ and either $a_0 = 0$ or $p = q$.

Proof: Let $1 \leq p,q < \infty$ and let E denote the set of all functions in $L_p(I)$ for which $Df = 0 - ae$. It is a simple matter to verify that this is indeed a subspace of $D(T)$. Furthermore for any step–function g, $Dg = 0 - ae$ and hence E contains the step functions and is therefore dense in $L_p(I)$ [CO; 3.4.3]. Consequently $E' \equiv D(T)' \equiv L_{p'}(I)$ $(\frac{1}{p}+\frac{1}{p'}=1)$ and so T' and $(TJ_E)'$ belong to the same system $L(L_{q'}(I), L_{p'}(I))$ $(\frac{1}{q}+\frac{1}{q'}=1)$. Suppose $a_0=0$. Then $TJ_E=0 \Rightarrow (TJ_E)'=0$ with T' a restriction of $(TJ_E)'$. Therefore $T'=0$. Clearly $D(T)$ contains the domain of the maximal operator $T_{\tau,p,q}$ defined as in [G1; VI.1.6]. Hence $R(T)$ contains $R(T_{\tau,p,q})$ and is moreover dense in $L_q(I)$ by [G1; VI.2.10]. It follows that T' is injective and thus $\{0\}=D(T')$. Therefore T is F_- in this case. Now suppose $a_0 \neq 0$ and $p=q$. Let J denote the identity map on $L_p(I)$. Observe that $a_0 J$ is a bounded map as $a_0 \in L_\infty(I)$. The fact that $D(T')=\{0\}$ follows from the observation that $T'=(T-a_0J)'+(a_0J)'$ since from the first part it is clear that $D((T-a_0J)')=\{0\}$.

(b) Suppose the interval I has finite Lebesgue measure with $1 \leq p,q < \infty$ and $\frac{1}{a_0} \in L_\infty(I)$. Then $T \in F_-$. Furthermore $D(T')=\{0\}$ whenever $p \geq q$.

Proof: Let E be as in (a). We consider two cases.

Case 1 $(p<q)$:

Observe that $K=E \cap D(T)$ contains the step functions and is therefore dense in both $L_p(I)$ and $L_q(I)$. Furthermore $TJ_K=a_0 I_K$ where I_K denotes the identity map from $K \subset L_p(I)$ into $L_q(I)$. Let $f \in L_q(I)$ be arbitrary. Then $\frac{1}{a_0} f \in L_q(I)$ as $\frac{1}{a_0} \in L_\infty(I)$. Hence there exists a sequence of step functions $\{g_n\} \subset L_q(I)$ such that $g_n \to \frac{1}{a_0} f$. Consequently $\{a_0 g_n\} \subset R(a_0 I_K)$ with $a_0 g_n \to f$ and therefore $R(a_0 I_K)$ is dense in $L_q(I)$. In addition $a_0 I_K$ has a continuous inverse by [CO; Exercise 3.4.9] and the fact that $\frac{1}{a_0} \in L_\infty(I)$. Hence $(TJ_K)'=(a_0 I_K)'$ has a continuous inverse [G1; II.3.14]. The result now follows from the observation that as in (a), T' is a restriction of $(TJ_K)'$.

Case 2 $(p \geq q)$:

Observe that in this case $L_p(I)$ may be continuously injected into $L_q(I)$ [CO; Exercise 3.4.9]. Denote this injection by U. By (a) we then have $(T - a_0 U) \in F_-$ with $D((T - a_0 U)') = \{0\}$. As $a_0 \in L_\infty(I)$, $a_0 U$ is bounded and hence $T' = (T - a_0 U)' + (a_0 U)'$. Consequently $D(T') = \{0\}$.

III.1.8 Corollary [CL] The following statements are equivalent:

(i) $T \notin F_-$

(ii) there exists $M \in \tilde{\mathscr{E}}(Y)$ for which $Q_M J_Y T$ is compact (nuclear, approximable)

(iii) for each $\epsilon > 0$ there exists $M \in \tilde{\mathscr{E}}(Y)$ for which $\|Q_M J_Y T\| < \epsilon$.

The author refers the reader to [CL, 4.3] for the proof.

III.1.9 Definition [We 1] If $\dim Y < \infty$ then $\Gamma'(T) = 0$, while if $\dim Y = \infty$ then $\Gamma'(T) = \inf_{M \in \tilde{\mathscr{E}}(Y)} \|Q_M J_Y T\|$.

III.1.10 Corollary [CL] Let $\dim Y = \infty$. Then $T \in F_- \Leftrightarrow \Gamma'(T) > 0$.

III.1.11 Proposition The following statements are equivalent:

(i) $T \in F_-$

(ii) $Q_F T \in F_-$ for every $F \in \mathscr{F}(Y)$

(iii) $Q_F T \in F_-$ for some $F \in \mathscr{F}(Y)$.

The author refers the reader to [CL, 4.5] for the proof.

III.1.12 Definition [C2] We note that a subspace R of a Banach space is an operator range if and only if there exists a stronger norm $\|\cdot\|_1$ on R under which R is complete. We will refer to $(R_1, \|\cdot\|_1)$ as the *pre–image space* of R and it will be denoted by R_1. The bounded bijection from R_1 onto R will be denoted by α_R with β_R denoting the open bijection α_R^{-1}.

III.1.13 Theorem [L2] (Generalised Open Mapping Theorem)
Let X be an operator range and let Y be of second category. If T is closed, everywhere defined and surjective then T is open.

Proof: [L2] Note that $T\alpha_X$ is still closed and hence by the classical open mapping theorem $T\alpha_X$ and so $T\alpha_X\beta_X = T$ is an open map if $R(T\alpha_X) = R(T) = Y$. □

III.1.14 Definition [CL] T is said to be *range open* if $(J^Y_{R(T)})^{-1}T$ (i.e. T regarded as a member of L(X,R(T)) is an open map. We denote the range open maps T by $T \in RO$.

III.1.15 Definition [N] We define the *minimum modulus* function $\gamma(T)$ of T by $\gamma(T) = \sup\{\gamma\colon \|Tx\| \geq \gamma\, d(x,N(T))\ \ x \in D(T)\}$ where $d(x,N(T))$ is the distance of x from $N(T)$. Note that $\gamma(T) = \infty$ if and only if $N(T)$ is dense in $D(T)$.

III.1.16 Lemma [CL] T is range open $\Leftrightarrow \gamma(T) > 0$.

Proof: [CL] We have $\gamma(T) > 0 \Leftrightarrow (\exists\gamma > 0)(\|Tx\| \geq \gamma\|x + N(T)\|) \Leftrightarrow (\exists\gamma > 0)(\|Tx\| < \gamma \Rightarrow \|x + N(T)\| < 1) \Leftrightarrow$ T is range open. □

III.1.17 Lemma [CL] $\quad$ T is range open $\Leftrightarrow$ TG is range open.

Proof: [CL] $\quad$ Suppose TG is range open. Then clearly $T = TG.G^{-1}$ is range open since G^{-1} is an open map. Conversely, suppose that T is range open. Then there exists $\lambda > 0$ such that $\lambda U_{R(T)} \subset TU_{D(T)}$, i.e. $\|Tx\| < \lambda \Rightarrow \|x + z\| < 1$ for some $z \in N(T)$. Hence $\|TGx\| = \|Tx\| = \|T(x + z)\| < \lambda \Rightarrow \|x + z\| + \|T(x + z)\| < 1 + \lambda$ for some $z \in N(T)$, whence $(\frac{\lambda}{1 + \lambda}) U_{R(TG)} \subset TGU_{X_T}$ showing that TG is range open. $\square$

III.1.18 Proposition [CL]. $\quad$ Consider the following statements :

(i) $\quad \gamma(T) > 0$

(ii) $\quad \gamma(T') > 0$

(iii) $\quad R(T') = N(T)^{\perp D(T)'}$

(iv) $\quad R(T)$ is closed (or equivalently, $R(T) = {}^{\perp}N(T')$).

We have

(a) $\quad$ in general (i) $\Leftrightarrow$ (iii) and (iii) $\Rightarrow$ (ii)

(b) $\quad$ if X is complete and T is closed then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and (i) $\Rightarrow$ (iv)

(c) $\quad$ if X is an operator range, Y is complete and T closed then (iv) $\Rightarrow$ (i)

(d) $\quad$ if X and Y are complete and T is closed then all the statements (i)–(iv) are equivalent.

Proof: [CL] $\quad$ The author gives a brief outline of the proof, which is due to L.E. Labuschagne and R.W. Cross. We shall assume without loss of generality that $\overline{D(T)} = X$.

(a) We divide the proof of (a) into two parts. In the first part we prove (a) for the case when $N(T)$ is closed in $D(T)$ and in the second part we prove (a) for the case $N(T)$ not closed in $D(T)$.

If $N(T)$ is closed in $D(T)$ then $\hat{T}$ has a continuous inverse. By Theorem I.3.11 we have $R(\hat{T}') = N(T)^{\perp}$. From Lemma I.4.7(iii) we conclude that $R(T') = N(T)^{\perp}$.

If $N(T)$ is not closed then by Lemma III.1.16 and III.1.17 $\gamma(TG) > 0$. Hence $R((TG)') = N(TG)^{\perp}$ by what we have just shown. Noting that $T' = (G^{-1})'(TG)'$, we have

(1) $$R(T') = R((G^{-1})' \, J_{N(TG)^{\perp}}).$$

Now $\gamma(Q_N G^{-1}) > 0$ by Lemma III.1.16 and hence by what has been proved above we have

(2) $$R((Q_N G^{-1})') = R((G^{-1})' J_{N^{\perp}}) = N(T)^{\perp}.$$

Now $(G^{-1})' J_{N^{\perp}}$ is a restriction of $(G^{-1})' J_{N(TG)^{\perp}}$ and so comparing (1) and (2), we conclude that $R(T') \supset N(T)^{\perp}$. Since in general $R(T') = N(T)^{\perp}$ equality follows. Therefore (i) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (ii) is immediate from (c) proved below.

(b) Let X be complete and T closed. (i) $\Rightarrow$ (iv): Let $\gamma(T) > 0$. Then since $N(T)$ is closed, $\hat{T}$ exists and has a continuous inverse and is moreover closed. Let $y \in \overline{R(T)} = \overline{R(\hat{T})}$ and choose a sequence (x_n) in X such that $Tx_n \to y$. Then $(Q_{N(T)}x_n)$ is a Cauchy sequence and hence convergent in the Banach space $X/N(T)$. Let $Q_{N(T)}x_n \to Q_{N(T)}x$. Since $\hat{T}$ is closed, $y = \hat{T}Q_{N(T)}x \in R(T)$. Therefore $R(T)$ is closed.

Thus (i) ⇒ (iv). (ii) ⇒ (i): Suppose $\gamma(T') > 0$. Then $R(T')$ is closed by what has just been proved. Let T_1 be T considered as an element of $L(X,\overline{R(T)})$, i.e. $T_1 = (J^{Y}_{\overline{R(T)}})^{-1}T$. Then T_1 is closed and T'_1 is injective. Moreover it is easily seen that $R(T'_1) = R(T')$. Thus $R(T'_1)$ is closed and by the Open Mapping Theorem, T'_1 has a continuous inverse. Consequently T_1 is an open map (Theorem I.4.3) i.e. T is range open. Hence $\gamma(T) > 0$ by lemma III.1.16. Thus (ii) ⇒ (i).

(c) Let $R(T)$ be closed. Then by the Generalised Open Mapping Theorem (Theorem III.1.13) T is range open. Hence $\gamma(T) > 0$ by Lemma III.1.16.

(d) Immediate from (b) and (c). □

III.1.19 Definition The dimension of $N(T)$, written $a(T)$, will be called the *kernel index* of T and the deficiency of $R(T)$ in Y, written $b(T)$, will be called the *deficiency index* of T. Thus $a(T)$ and $b(T)$ will be either a nonnegative integer or ∞. We denote $\dim \frac{Y}{\overline{R(T)}}$ by $\overline{b}(T)$.

III.1.20 Theorem [L1]

(I) $a(T') = \overline{b}(T)$ (cf. [G1; IV.2.3]) :

(II) $\overline{b}(T') \geq a(T)$ with $b(T') = \overline{b}(T') = a(T)$ if $\gamma(T) > 0$.

Proof: (I)

$$\begin{aligned}\dim \frac{Y}{\overline{R(T)}} &= \dim \left(\frac{Y}{\overline{R(T)}}\right)' \\ &= \dim R(T)^{\perp} \\ &= \dim N(T') \\ &= a(T').\end{aligned}$$

(II) By Theorem III.1.18 $R(T') = N(TJ_{D(T)})^{\perp}$ if $\gamma(T) > 0$. Hence $\overline{R(T')} = R(T') = N(TJ_{D(T)})^{\perp}$.

$$\begin{aligned}\text{codim } \overline{R(T')} &= \text{codim } R(T') \\ &= \text{codim } N(TJ_{D(T)})^{\perp} \\ &= \dim N(TJ_{D(T)}) \\ &= a(T).\end{aligned}$$

By (iii) of Theorem I.3.8

$$\overline{R(T')} \subset N(T)^{\perp}.$$

Hence codim $\overline{R(T')} \geq$ codim $N(T)^{\perp} = \dim N(T)$.
i.e. $\overline{b}(T') \geq a(T)$. □

The following theorem generalises [G1; IV.1.8] where only the case where $N(T)$ is closed is considered. The proof is an alternative to that for [G1; IV.1.8].

III.1.21 Theorem [L1] $\gamma(T') = \gamma(T) > 0$ whenever $\gamma(T) > 0$.

Proof: [L1] Without loss of generality assume $D(T) = X$. Suppose $\gamma(T) = \infty$. From the definition of $\gamma(T)$ we note that this can only be the case if $d(x, N(T)) = 0$ for each $x \in D(T)$, that is if $N(T)$ is dense in $D(T) = X$. Moreover for any $y' \in D(T')$ we have $N(y'T) \supset N(T)$. Now since $y'T$ is continuous, $N(T'y')$ is closed and so $N(T'y') \supset \overline{N(T)} = X$. Hence $T' = 0$ and therefore $\gamma(T') = \infty$.

Now let $0 < \gamma(T) < \infty$. As $\gamma(T) < \infty$ we conclude that $\overline{N(T)} \neq X$. Noting that $d(x, N(T)) = d(x, \overline{N(T)}) = \|Q_{\overline{N(T)}}x\|$ for each $x \in X$ we deduce that since $\overline{N(T)} \subsetneq X$, we have

$$\gamma(T) = \sup\{\lambda \in \mathbb{R} : \|Tx\| \geq \lambda \,.\, \|Q_{\overline{N(T)}}x\| \text{ for each } x \in X \setminus \overline{N(T)}\}$$

and hence we can write

(2) $$\gamma(T) = \inf_{x \in X\setminus \overline{N(T)}} \frac{\|Tx\|}{\|Q_{\overline{N(T)}}x\|} .$$

Now let $\epsilon > 0$ be arbitrary and select $x \in X \setminus \overline{N(T)}$ such that

(3) $$\gamma(T) + \epsilon \geq \frac{\|Tx\|}{\|Q_{\overline{N(T)}}x\|} .$$

By Theorem III.1.18 and Theorem 0.5.3 we have that $R(T') = N(T)^{\perp} \equiv (X / \overline{N(T)})'$ and so by Corollary 0.3.6 there exists $T'y' \in R(T')$ such that

(4) $$\|T'y'\| = 1 \text{ and } T'y'x = \|Q_{\overline{N(T)}}x\| .$$

From Theorem 0.5.3 and Theorem I.3.7 we see that for any $z' \in Y'$

(5) $$\begin{aligned} d(z', N(T')) &= d(z', R(T)^{\perp}) \\ &= \|Q_{R(T)^{\perp}} z'\| \\ &= \|z' J_{R(T)}\| . \end{aligned}$$

We now conclude from (3), (4) and (5) that

$$d(y', N(T')) = \|y'J_{R(T)}\| \geq \frac{|y'Tx|}{\|Tx\|} = \|Q_{\overline{N(T)}}x\| \,/\, \|Tx\| \geq \frac{1}{\gamma(T) + \epsilon} .$$

Hence $\gamma(T) + \epsilon \geq \frac{\|T'y'\|}{d(y', N(T'))}$ since $\|T'y'\| = 1$. (Note that $y' \in D(T') \setminus N(T')$). As with $\gamma(T)$ we now have that

$$\gamma(T') = \inf_{z' \in D(T')\setminus N(T')} \frac{\|T'z'\|}{d(z', N(T'))}$$

since $N(T')$ is closed and $T' \neq 0$. Consequently $\gamma(T) + \epsilon \geq \gamma(T')$ and since $\epsilon > 0$ was chosen arbitrarily,

(6) $$\gamma(T) \geq \gamma(T') .$$

Conversely, let $\delta > 0$ be arbitrary and select $z' \in D(T') \setminus N(T')$ such that

$$\gamma(T') + \delta > \frac{\|T'z'\|}{d(z', N(T'))} = \frac{\|T'z'\|}{\|z'J_{R(T)}\|}.$$

Now select $\mu > 0$ such that

$$\gamma(T') + \delta \geq \frac{\|T'z'\|}{\|z'J_{R(T)}\| - \mu} > 0. \tag{7}$$

Let $Tx \in R(T)$ be such that

$$\|Tx\| = 1 \text{ and } |z'Tx| \geq \|z'J_{R(T)}\| - \mu > 0. \tag{8}$$

Note that $x \notin N(T'z') \supset \overline{N(T)}$ and so

$$d(x, \overline{N(T)}) = \|Q_{\overline{N(T)}}x\| > 0. \tag{9}$$

But since $R(T') = N(T)^{\perp} \equiv (X/\overline{N(T)})'$, we have by (2), (7), (8), (9) and 0.3.7 that

$$\begin{aligned}
\frac{1}{\gamma(T)} &\geq \|Q_{N(T)}x\| \,/\, \|Tx\| = \|Q_{\overline{N(T)}}x\| \\
&= \sup_{\substack{r' \in R(T') \\ \|r'\| = 1}} |r'x| \\
&\geq \frac{|z'Tx|}{\|T'z'\|} \\
&\geq \frac{\|z'J_{R(T)}\| - \mu}{\|T'z'\|} \\
&\geq \frac{1}{\gamma(T') + \delta}.
\end{aligned}$$

Hence $\gamma(T') + \delta \geq \gamma(T)$. Considering (6) and the fact that $\delta > 0$ was chosen arbitrarily, we conclude that $\gamma(T') = \gamma(T)$. □

III.1.22 Notation The author denotes the class of closed linear operators T in $L(X,Y)$ by $T \in C(X,Y)$.

III.1.23 Corollary [L1] Let $T \in C(X,Y)$ with X complete. Then $\gamma(T) = \gamma(T')$.

III.1.24 Proposition [L1] Let Y be complete. Then $\gamma(T') > 0$ whenever $R(T)$ is closed.

Proof: [L1] Defining T_1 as in [L1, 2.9] we note that $R(T) = R(T_1)$ is closed by Theorem I.3.1 and Theorem I.3.13. Hence $\gamma(T') > 0$ by [G1, IV.1.6]. □

III.1.25 Proposition [CL] $J_Y T \in F_- \Rightarrow T \in F_-$. (In particular, if Y is complete, then $T \in \phi_- \Rightarrow T \in F_-$.)

Proof: [CL] Since $T \in F_- \Leftrightarrow J_Y T \in F_-$ we assume without loss of generality that Y is complete. Let $T \in \phi_-$. Then there exists $F \in \mathcal{F}(Y)$ such that $R(T) \oplus F = Y$ and then $Q_F T$ is surjective. Hence, by state diagram I.3.14, $(Q_F T)'$ has a continuous inverse. Hence $T \in F_-$ by Proposition III.1.11. □

III.2 Closed F_--operators

III.2.1 Lemma [CL] Let M and N be subspaces of X with $\dim M = \infty$ and $\dim N < \infty$. Then there exists $m \neq 0$ in M such that $\|m\| = d(m,N)$ ([KKM]; see [G1], VI.1.1).

III.2.2 Lemma [CL]

(a) If $a(T) < \infty$ and $\dim D(T) = \infty$ then $\Gamma(T) \geq \gamma(T)$.

(b) If $\bar{b}(T) < \infty$ and $\dim Y = \infty$ then $\Gamma'(T) \geq \gamma(T')$.

The proof uses Lemma III.2.1 and is to be found in [CL, 5.4(1), (c)].

III.2.3 Corollary [CL] Let $\gamma(T') > 0$. Then $T \in F_- \Leftrightarrow \bar{b}(T) < \infty$.

Proof: [CL] The case when $\dim Y < \infty$ is trivial and so is the implication $\Rightarrow$. Assume that $\dim Y = \infty$ and that $\bar{b}(T) < \infty$. Then $\Gamma'(T) \geq \gamma(T')$ by Lemma III.2.2(b). Hence $T \in F_-$ by Corollary III.1.10. □

III.2.4 Corollary [C3] Let $\gamma(T) > 0$. Then $T \in F_+ \Leftrightarrow \Gamma(T) > 0$ unless $\dim D(T) < \infty$.

III.2.5 Proposition [CL; 5.7] Let T be closed and let X be complete. Then $T \in F_- \Rightarrow T \in \phi_-$.

III.2.6 Corollary [CL] Let T be closed and let X and Y be complete. Then

$$T \in F_- \Leftrightarrow T \in \phi_- .$$

Proof: Combine Propositions III.1.25 and III.2.5. □

CHAPTER IV

STATE DIAGRAMS

IV.1 First State Diagram for Linear Operators

The first classification of $T \in L(X,Y)$ is as follows :

I : $R(T)$ is closed and $b(T) < \infty$.

II : $R(T)$ is not closed but $\bar{b}(T) < \infty$.

III : $\bar{b}(T) = \infty$.

1 : $\gamma(T) > 0$ and $a(T) < \infty$

2 : $\gamma(T) = 0$ but $a(T) < \infty$

3 : $a(T) = \infty$.

The author now obtains the state diagram corresponding to this classification of $T \in L(X,Y)$.

IV.1.1 Proposition $T \in I_1$ implies $T' \in I_1$.

Proof: We have $a(T') = \bar{b}(T) < \infty$ by (I) of Theorem III.1.20. Also, since $\gamma(T) > 0$, we have $\bar{b}(T') = b(T') = a(T) < \infty$ by (II) of Theorem III.1.20. Since $\gamma(T) > 0$ we have $\gamma(T') = \gamma(T) > 0$ by Theorem III.1.21. □

IV.1.2 Proposition. Let Y be complete. Then $T \in I_2$ implies $T' \in 1$ and $T' \notin II$.

Proof: We have $a(T') = \bar{b}(T) < \infty$ by (I) of Theorem III.1.20. By Proposition III.1.24 $\gamma(T') > 0$. Hence $R(T')$ is closed by [G1, IV.1.6]. □

IV.1.3 Proposition Let Y be complete. Then $T \in I_3$ implies $T' \in III_1$.

Proof: We have $a(T') = \bar{b}(T) < \infty$ by (I) of Theorem III.1.20. Also, by (II) of Theorem III.1.20 $\bar{b}(T') \geq a(T) = \infty$. Hence $\bar{b}(T') = \infty$. By Proposition III.1.24 $\gamma(T') > 0$. □

IV.1.4 Proposition $T \in II_1$ implies $T' \in I_1$.

Proof: We have $a(T') = \bar{b}(T) < \infty$ by (I) of Theorem III.1.20 and $\bar{b}(T') = b(T') = a(T) < \infty$ by (II) of Theorem III.1.20. By Theorem III.1.21 $\gamma(T') = \gamma(T) > 0$. □

IV.1.5 Proposition $T \in II_2$ implies $T' \notin 3$.

Proof: We have $a(T') = \bar{b}(T) < \infty$ by (I) of Theorem III.1.20. □

IV.1.6 Proposition $T \in II_3$ implies $T' \in III$ and $T' \notin 3$.

Proof: We have $a(T') = \bar{b}(T) < \infty$ by (I) of Theorem III.1.20. By (II) of Theorem III.1.20 $\bar{b}(T') \geq a(T) = \infty$. Hence $\bar{b}(T') = \infty$. □

IV.1.7 Proposition $T \in III_1$ implies $T' \in I_3$.

Proof: We have $a(T') = \bar{b}(T) = \infty$ by (I) of Theorem III.1.20. By (II) of Theorem III.1.20 $\bar{b}(T') = b(T') = a(T) < \infty$, i.e., $R(T')$ is closed. □

IV.1.8 Proposition $T \in III_2$ implies $T' \in 3$.

Proof: We have $a(T') = \bar{b}(T) = \infty$ by (I) of Theorem III.1.20.

IV.1.9 Proposition $T \in III_3$ implies $T' \in III_3$.

Proof: By (I) of Theorem III.1.20 $a(T') = \bar{b}(T) = \infty$ and by (II) of Theorem III.1.20 $\bar{b}(T') \geq a(T) = \infty$. Hence $\bar{b}(T') = \infty$. □

IV.1.10 Proposition $T' \in I_1$ implies $T \notin III$ and $T \notin 3$.

Proof: We have $a(T') = \bar{b}(T) < \infty$ by (I) of Theorem III.1.20. By (II) of Theorem III.1.20 $a(T) < \infty$. □

IV.1.11 Proposition $T' \notin I_2$.

Proof: This follows from [G1, IV.1.6]. □

IV.1.12 Proposition $T' \in I_3$ implies $T \in III_1 \cup III_2$.

Proof: By (I) of Theorem III.1.20 $a(T') = \bar{b}(T) = \infty$. By (II) of Theorem III.1.20 $\bar{b}(T') \geq a(T)$. Hence $a(T) < \infty$. □

IV.1.13 Proposition $T' \notin II_1$.

Proof: See [G1, IV.1.6].

IV.1.14 Proposition Let Y be complete. Then $T' \in II_2$ implies $T \in II_2$.

Proof: By (I) of Theorem III.1.20 $a(T') = \bar{b}(T) < \infty$ and by (II) of Theorem III.1.20 $\bar{b}(T') \geq a(T)$. Hence $a(T) < \infty$. By Proposition III.1.24 $R(T)$ is not closed. □

IV.1.15 Proposition $T' \in II_3$ implies $T \in III_2$.

Proof: By (I) of Theorem III.1.20 $a(T') = \bar{b}(T) = \infty$ and by (II) of Theorem III.1.20 $\bar{b}(T') \geq a(T)$. Hence $a(T) < \infty$. □

IV.1.16 Proposition $T' \in III_1$ implies $T \notin III$.

Proof: By (I) of Theorem III.1.20 $\bar{b}(T) = a(T') < \infty$. □

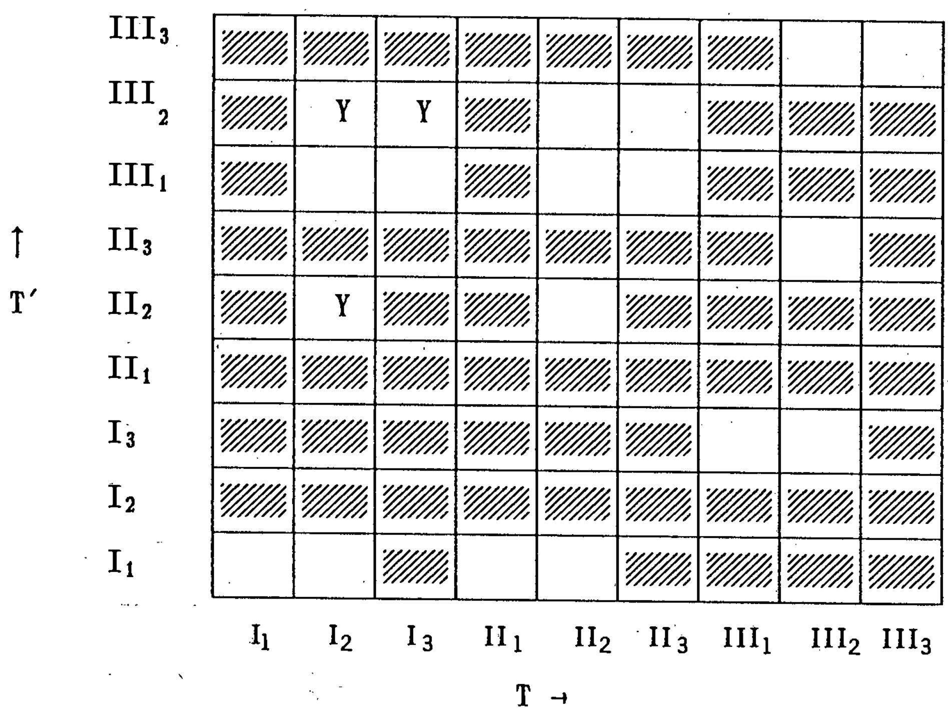
IV.1.17 Proposition Let Y be complete. Then $T' \in III_2$ implies $T \in II$.

Proof: By (I) of Theorem III.1.20 $a(T') = \bar{b}(T) < \infty$. By Proposition III.1.24 $R(T)$ is not closed. □

IV.1.18 Proposition $T' \in III_3$ implies $T \in III$.

Proof: By (I) of Theorem III.1.20 we have $a(T') = \bar{b}(T) = \infty$. □

IV.1.19 State Diagram for Linear Operators (First classification IV.1)



Y: Cannot occur if Y is complete.

IV.2 First State Diagram for Closed Linear Operators

IV.2.1 Proposition Let X be complete with T closed. Then $T \in I_2$ implies $T' \in 2$ and $T' \notin I$.

Proof: By (I) of Theorem III.1.20 $a(T') = \bar{b}(T) < \infty$. Also by Theorem III.1.23 $\gamma(T') = 0$. Hence, by [G1, IV.1.6], $R(T')$ is not closed □

IV.2.2 Proposition Let $T \in C(X,Y)$ with X reflexive. Then $T \in I_2$ implies $T' \in II_2$.

Proof: If $T \in F_+$ then $\gamma(T') > 0$ and $T' \notin 2$. If $T \notin F_+$ then $T \in (I_2)e$, where $(I_2)e$ is the essential state (see IV.12.2). Hence $T' \notin III_2$ for X reflexive. □

IV.2.3 Proposition Let $T \in C(X,Y)$ with X complete. Then $T \in II_1$ implies $T' \notin I_1$.

Proof: By Proposition III.1.18 we have that if X is complete and T is a closed operator then $\gamma(T) > 0$ if and only if $\gamma(T') > 0$ in which case $R(T)$ is closed. But this contradicts the fact that $T \in II$. □

IV.2.4 Proposition Let $T \in C(X,Y)$ with X complete. Then $T \in II_2$ implies $T \notin I$ and $T' \in 2$.

Proof: By Theorem III.1.23 $\gamma(T') = \gamma(T) = 0$. Hence $R(T')$ is not closed. □

IV.2.5 Proposition Let $T \in C(X,Y)$ with X reflexive. Then $T \in II_2$ implies $T' \notin III_2$.

Proof: If $T \in F_+$ then $\gamma(T') > 0$ and $T' \notin 2$. On the other hand, $T \notin F_+$ implies $T \in (II_2)e$ and $T' \notin III_2$ for X reflexive. □

IV.2.6 Proposition Let X be complete and T closed. Then $T \in II_3$ implies $T' \notin III_1$.

Proof: This follows from the essential state diagram. See [IV.12.2] □

IV.2.7 Proposition Let X be complete and T closed. Then $T \in III_2$ implies $T' \notin I_3$.

Proof: If $\gamma(T) = 0$ then $T' \notin \phi_-$. □

IV.2.8 Proposition If X is reflexive and T is closed then $T \in III_2$ implies $T' \notin III_3$.

Proof: The proof is similar to that of Proposition IV.2.5. □

IV.2.9 State Diagram for Closed Linear Operators (First classification IV.1)

T′ ↑ / T →	I_1	I_2	I_3	II_1	II_2	II_3	III_1	III_2	III_3
III_3	///	///	///	///	///	///	///	X–R–c	
III_2	///	Y X–R–c	Y	///	X–R–c		///	///	///
III_1	///	X–c		///	X–c	X–c	///	///	///
II_3	///	///	///	///	///	///	///		///
II_2	///	Y	///	///		///	///	///	///
II_1	///	///	///	///	///	///	///	///	///
I_3	///	///	///	///	///	///		X–c	///
I_2	///	///	///	///	///	///	///	///	///
I_1		X–c	///	X–c	X–c	///	///	///	///

Y: Cannot occur if Y is complete.

X–c: Cannot occur if X is complete and T closed.

X–R–c: Cannot occur if X is reflexive and T closed.

IV.3 Examples of States for First Two State Diagrams

In this section we give examples of states which can occur and thus show that the blank squares appearing in the two diagrams V.1.19 and V.2.9 all eventuate.

IV.3.1 $X = Y = \ell_2$ T bounded.

(I_1, I_1): Let T be the identity operator on X .

(I_3, III_1): Let T be defined by $T((x_k)) = (x_{2k})$. Then $T \in I_3$ and $T' \in III_1$ by IV.1.19.

(II_2, II_2): The same example as for (II_2, II_2) in I.5.5, i.e. $T((x_k)) = (x_k/k)$, using 3.1 of ([C5],13).

(II_3, III_2): The operator $T: (x_k) \to (x_{2k}/k)$ is compact and has dense range. Therefore $T \in II_3$ and $T' \in III_2$ by 2.7 of ([C5],12) .

(III_1, I_3): Define T by $T((x_k)) = (0, x_1, 0, x_2, \ldots)$. Then $T \in III_1$ and $T' \in I_3$ by IV.1.19.

(III_2, II_3): Let T be the adjoint of the operator in example (II_3, III_2). Then $T' \in II_3$ by 2.7 of [C5, 12].

(III_3, III_3): Let T be the zero operator.

IV.3.2 X not complete $Y = \ell_2$ T bounded.

(I_2, III_1): Let $\{x_\alpha\}$ be a Hamel base for ℓ_2 and let X be the linear space ℓ_2 renormed by $\|\Sigma\lambda_\alpha x_\alpha\| = \Sigma|\lambda_\alpha|$. Define T as the identity map from X onto ℓ_2 . Then $T \notin 1$; otherwise the incomplete space X would be isomorphic to Banach space Y. Thus $T \in I_2$ and $T' \in III_1$.

(II_1, I_1): See I.5.7.

(II_2, III_1): Similar to the example for (II_2, III_1) in I.5.7.

(II_3, III_1): Similar to the example for (II_3, III_1) in I.5.7, using the operator $(x_k) \to (x_{2k})$ in place of the left shift operator.

IV.3.3 X complete but not reflexive $Y = \ell_2$ T compact on X.

(II_2, III_2): The same example in I.5.8 for (II_2, III_2) serves.

(III_2, III_3): Similar to example for (III_2, III_3) in I.5.8 but using the operator $(x_n) \to (0, x_1, 0, x_2, \ldots)$ (on ℓ_1) in place of the right–shift operator.

IV.3.4 X complete but not reflexive Y not complete T compact on X.

(I_2, III_2): The same example as for (I_2, III_2) of I.5.9.

IV.3.5 $X = \ell_2$ Y not complete T compact.

(I_2, II_2): The same example as that given for (I_2, II_2) in Chapter I serves.

IV.3.6 $X = Y = c_0$.

(I_2, I_1): Let $X = Y = c_0$, $T = A + f \otimes e_1$ where f is a discontinuous linear functional and A the right shift operator, $e_1 = (1,0,0,\ldots)$. Then A^{-1} is continuous (= left shift) and hence $A \in F_+$, while $N(T) = 0$. Now $\gamma(T) = 0$ since otherwise T would be a surjective isomorphism by the Closed Graph Theorem. Also $T' \in \phi_-$, so $T' \in I$, and $T' = A' + F'$ is injective. Since $\gamma(T') > 0$ ($R(T')$ is closed), we have $T' \in I_1$.

The author was not able to find examples for states (II_2, I_1), (III_2, I_3), (I_3, III_2), (II_2, I_1), and (III_3, I_3).

IV.4 Second State Diagram for Linear Operators

The second classification of $T \in L(X,Y)$ is as follows :

I: $R(T)$ is closed and $b(T) < \infty$.

II: $R(T)$ is not closed but $\bar{b}(T) < \infty$.

III: $\bar{b}(T) = \infty$.

1: $a(T) < \infty$ and $R(T')$ is closed (i.e. $\gamma(T') > 0$)

2: $a(T) < \infty$ but $R(T')$ is not closed ($\gamma(T') = 0$)

3: $a(T) = \infty$.

IV.4.1 Proposition $T \in I_1$ implies $T' \in I_1 \cup III_1$.

Proof: By (I) of Theorem III.1.20 $a(T') = \bar{b}(T) < \infty$. Also $R(T'')$ is closed. □

IV.4.2 Proposition $T \in I_2$ implies $T' \in II_2 \cup III_2$.

Proof: By (I) of Theorem III.1.20 $a(T') = \bar{b}(T) < \infty$, and by [G1, IV.1.9] $R(T'')$ is not closed. □

IV.4.3 Proposition Let Y be complete. Then $T \notin I_2$.

Proof: By Proposition III.1.24 we have that $\gamma(T') > 0$ whenever $R(T)$ is closed. [G1, IV.1.6] implies that $R(T')$ is closed. But $T \in 2$. Hence there is a contradiction and the author concludes that $T \notin I_2$. □

IV.4.4 Proposition $T \in I_3$ implies that $T' \in III_1 \cup III_2$.

Proof: From (I) of Theorem III.1.20 $a(T') = \delta(T) < \infty$. Also, from (II) of Theorem III.1.20 it follows that $\delta(T') \geq a(T) = \infty$. Hence $\delta(T') = \infty$. □

IV.4.5 Proposition Let Y be complete. Then $T \in I_3$ implies $T' \in III_1$.

Proof: By Proposition III.1.24 $\gamma(T') > 0$. This means by [G1, IV.1.6] that $R(T')$ is closed. Hence $R(T'')$ is closed. □

IV.4.6 Proposition $T \in II_1$ implies $T' \in I_1 \cup III_1$.

Proof: By (I) of Theorem III.1.20 $a(T') = \delta(T) < \infty$. By [G1, IV.1.6] $R(T'')$ is closed. □

IV.4.7 Proposition $T \in II_2$ implies $T' \in II_2 \cup III_2$.

Proof: By (I) of Theorem III.1.20 $a(T') = \delta(T) < \infty$. Also $R(T'')$ is not closed by [G1, IV.1.9]. □

IV.4.8 Proposition $T \in II_3$ implies $T' \in III_1 \cup III_2$.

Proof: By (I) of Theorem III.1.20 $a(T') = \delta(T) < \infty$. By (II) of Theorem III.1.20 $\delta(T') \geq a(T) = \infty$. Hence $\delta(T') = \infty$. □

IV.4.9 Proposition $T \in III_1$ implies $T' \in I_3 \cup III_3$.

Proof: By (I) of Theorem III.1.20 $a(T') = \overline{b}(T) = \infty$. □

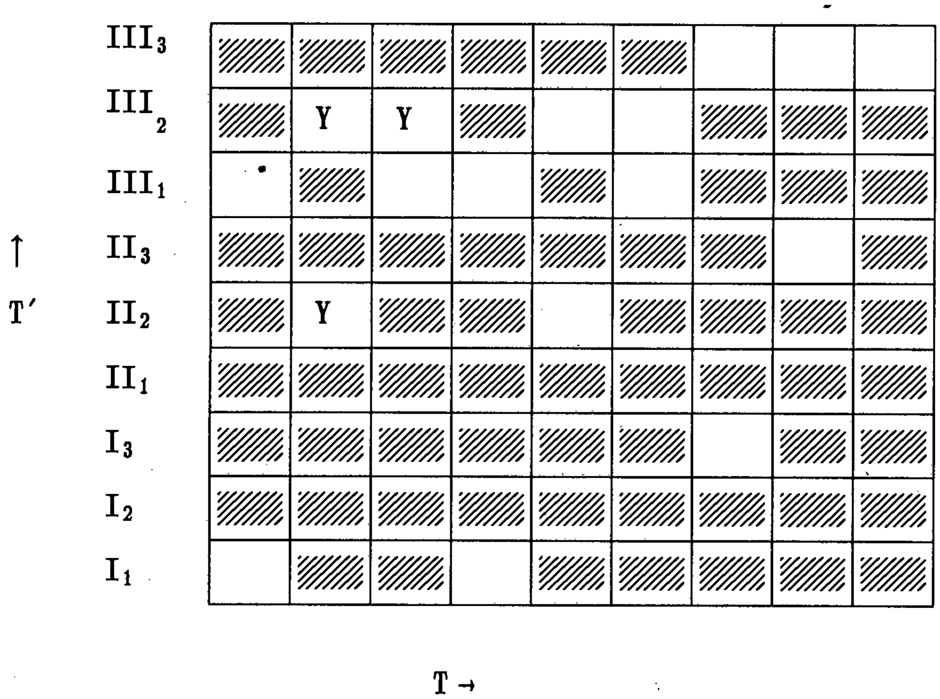
IV.4.10 Proposition $T \in III_2$ implies $T' \in II_3 \cup III_3$.

Proof: By (I) of Theorem III.1.20 $a(T') = \overline{b}(T) = \infty$. Also $R(T')$ is not closed. □

IV.4.11 Proposition $T \in III_3$ implies $T' \in III_3$.

Proof: By Theorem III.1.20 $a(T') = \overline{b}(T) = \infty$ and $\overline{b}(T') \geq a(T) = \infty$. Hence $\overline{b}(T') = \infty$. □

IV.4.12 State Diagram for Linear Operators (Second classification IV.4)



Y: Cannot occur if Y is complete.

IV.5 Second State Diagram for Closed Linear Operators

IV.5.1 Proposition Let X be complete and $T \in C(X,Y)$. Then $T' \in I_1$ implies $T \in I_1$.

Proof: Since $R(T')$ is closed, we have $\gamma(T') > 0$ by [G1, IV.1.6]. Hence by Theorem III.1.18 $\gamma(T) > 0$ and $R(T)$ is closed. □

IV.5.2 Proposition Let $T \in C(X,Y)$ with X complete. Then $T' \in III_1$ implies $T \in I_3$.

Proof: Since $\gamma(T') > 0$ we have that $\gamma(T) > 0$ and $R(T)$ is closed by Theorem III.1.18. Also $R(T')$ is closed, and by Theorem III.1.20 $a(T) = \infty$. □

IV.5.3 Proposition Let X be reflexive and $T \in C(X,Y)$. Then $T \in II_2$ implies $T' \notin III_2$.

Proof: If $T \in F_+$ then $\gamma(T') > 0$ and $T' \notin 2$. On the other hand, $T \notin F_+$ implies $T \in (II_2)e$ and $T' \notin III_2$ for X reflexive. □

IV.5.4 Proposition Let X be reflexive and $T \in C(X,Y)$. Then $T \in III_2$ implies $T' \notin III_3$.

Proof: If $T \in F_+$, then $\gamma(T') > 0$, so $\gamma(T) > 0$ and then $T' \notin III_3$ by IV.2.9. If $T \notin F_+$ then it follows as in the previous proof that $T' \notin III_3$. □

IV.5.5 Proposition Let $T \in C(X,Y)$ with X complete. Then $T \in III_1$ implies $T' \notin III$.

Proof: Since $\gamma(T) > 0$ we have $\bar{b}(T') = b(T') = a(T) < \infty$ so $T' \notin III$. □

V.5.6 State Diagram for Closed Linear Operators (Second classification IV.4)

↑ T′									
III_3	////	////	////	////	////	////	X–c	X–R–c	
III_2	////	Y	Y	////	X–R–c		////	////	////
III_1	X–c	////		X–c	////	X–c	////	////	////
II_3	////	////	////	////	////	////	////		////
II_2	////	Y	////	////		////	////	////	////
II_1	////	////	////	////	////	////	////	////	////
I_3	////	////	////	////	////	////		////	////
I_2	////	////	////	////	////	////	////	////	////
I_1		////	////	X–c	////	////	////	////	////
T →	I_1	I_2	I_3	II_1	II_2	II_3	III_1	III_2	III_3

Y: Cannot occur if Y is complete.

X–c: Cannot occur if X is complete and T is closed.

X–R–c: Cannot occur if X is reflexive and T is closed.

IV.6 Examples of States for Second Two State Diagrams

IV.6.1 $X = Y = \ell_2$ T bounded.

(I_1, I_1): Let T be the identity operator on X.

(I_3, III_1): Let T be defined by $T((x_k)) = (x_{2k})$. Then $T \in I_3$ and $T' \in III_1$ by 1.17 of ([C5], 8).

(II_2, II_2): The same example as for (II_2, II_2) in I.5.5, i.e., $T((x_k)) = (x_k/k)$, using 2.7 of ([C5], 12).

(II_3, III_2): The operator $T: (x_k) \to (x_{2k}/k)$ is compact and has dense range. Therefore $T \in II_3$, and then $T' \in III_2$ by 2.7 of ([C5], 12).

(III_1, I_3): Define T by $T((x_k)) = (0, x_1, 0, x_2, \dots)$. Then $T' \in III_1$ and $T' \in I_3$ by 1.17 of ([C5], 8).

(III_2, II_3): Let T be the adjoint of the operator in example (II_3, III_2). Then $T' \in II_3$ by 2.7 of ([C5], 12).

(III_3, III_3): Let T be the zero operator.

IV.6.2 $X = \ell_2$ Y not complete T compact.

(I_2, II_2): The same example as that given for (I_2, II_2) in I.5.7 serves.

(I_3, III_2): Let T be defined by $T((x_k)) = (x_{2k}/k)$ and let $Y = R(T) \subset \ell_2$. Then $T \in I_3$, and since T' is compact it is clear that $T' \notin 1$. Hence $T' \in III_2$ by 1.7 of ([C5], 8).

IV.6.3 X not complete $Y = \ell_2$ T bounded.

(II_1, I_1): See example in I.5.8.

(II_3, III_1): Similar to the example for (II_3, III_1) in I.5.8 using the operator $(x_k) \to (x_{2k})$ in place of the left–shift operator.

IV.6.4 X complete but not reflexive $Y = \ell_2$ T compact on X.

(II_2, III_2): The same example in I.5.9 for (II_2, III_2) serves.

(III_2, III_3): Similar to example for (III_2, III_3) in I.5.9 but using the operator $(x_n) \to (0, x_1, 0, x_2, \ldots)$ (on ℓ_1) in place of the right–shift operator.

The author notes that this classification results in a different configuration to that of the Taylor–Halberg–Goldberg state diagram.

The author was not able to find examples for states (I_1, III_1), (II_1, III_1) and (III_1, III_3).

IV.7 Classification based on F_+ and F_-–Operators

The classification for F_-–operators is as follows :

I: $T \in F_-$ and $b(T) < \infty$.

II: $T \notin F_-$ but $\bar{b}(T) < \infty$.

III: $\bar{b}(T) = \infty$.

1: $T \in F_+$.

2: $T \notin F_+$ but $a(T) < \infty$.

3: $a(T) = \infty$.

The state diagram is based upon the following Propositions.

IV.7.1 Proposition $T \in I_1$ implies $T' \in I_1$.

Proof: We have from Theorem II.38 that $T' \in \phi_-$. Hence, by definition of the ϕ_--operators, it follows that $b(T') = \bar{b}(T') < \infty$. Thus we have $T' \in F_-$ and $b(T') < \infty$ and $T' \in F_+$. □

IV.7.2 Proposition $T \in I_2$ implies $T' \in III_1$.

Proof: We have that $T' \in F_+$ but $T' \notin F_-$. By Proposition II.12 $T' \in \phi_+$ and by Corollary III.2.9 $T' \notin \phi_-$. Hence $\bar{b}(T') = \infty$. □

IV.7.3 Proposition $T \in I_3$ implies $T' \in III_1$.

Proof: We have that $T' \in F_+$ and by (II) of Theorem III.1.20 $\bar{b}(T') \geq a(T) = \infty$. Hence $\bar{b}(T') = \infty$. □

IV.7.4 Proposition $T \notin II_1$.

Proof: We have that $T' \in F_-$ but $T' \notin F_+$. Hence $T' \in \phi_-$ but $T' \notin \phi_+$. This implies that $a(T') = \infty$ but by Theorem III.1.20 $a(T') = \bar{b}(T) < \infty$. Hence we have a contradiction. Thus $T \notin II_1$. □

IV.7.5 Proposition $T \in II_2$ implies $T' \in II_2 \cup III_2$.

Proof: Since $T \notin F_-$ and $T \notin F_+$ it follows by definition that $T' \notin F_+$ and $T' \notin F_-$. By (I) of Theorem III.1.20 $a(T') = \bar{b}(T) < \infty$. □

IV.7.6 Proposition $T \in II_3$ implies $T' \in III_2$.

Proof: Since $T \notin F_-$, $T' \notin F_+$, but $a(T') = \bar{b}(T) < \infty$ and $\bar{b}(T') \geq a(T) = \infty$. Hence $\bar{b}(T') = \infty$. □

IV.7.7 Proposition $T \in III_1$ implies $T' \in I_3$.

Proof: Since $T \in F_+$, we have $T' \in F_-$. By (I) of Theorem III.1.20 $a(T') = \bar{b}(T) = \infty$. From Corollary III.2.9 we conclude that $R(T')$ is closed and $b(T') < \infty$. □

IV.7.8 Proposition $T \in III_2$ implies $T' \in II_3 \cup III_3$.

Proof: Since $T \notin F_+$ we have that $T' \notin F_-$. By (I) of Theorem III.1.20 $a(T') = \bar{b}(T) = \infty$. □

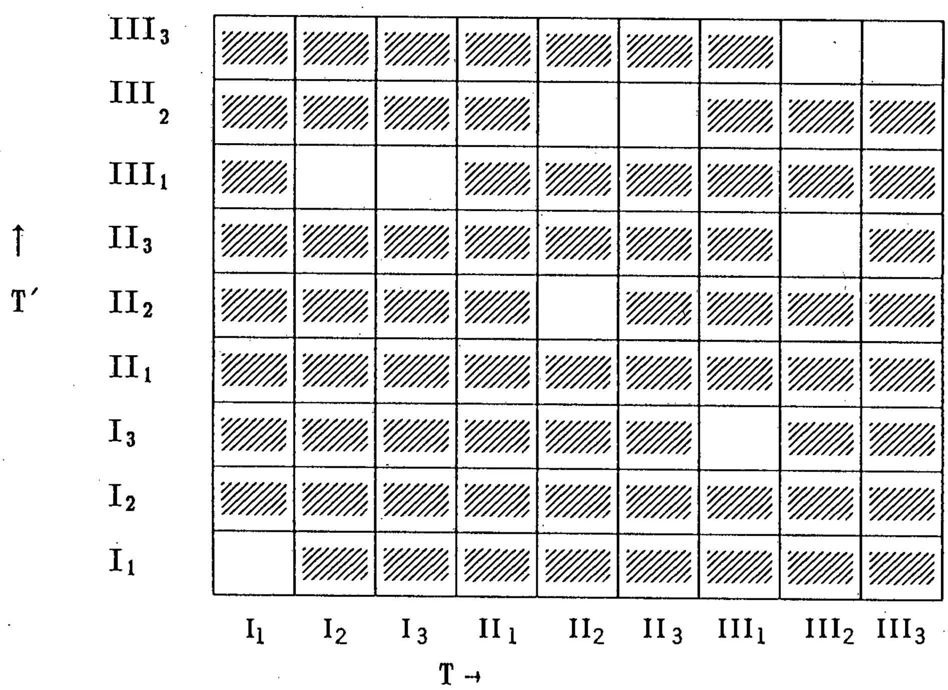
IV.7.9 Proposition $T \in III_3$ implies $T' \in III_3$.

Proof: By (I) of Corollary IV.5.5 $a(T') = \bar{b}(T) = \infty$ and by (II) of Theorem III.1.20 $\bar{b}(T') \geq a(T) = \infty$. Hence $\bar{b}(T') = \infty$. □

The author does not consider closed F_-–operators as this does not yield any new information.

IV.8 State Diagram based on F_+ and F_-–Operators

(F_- – classification IV.7)



IV.8.1 The author remarks that this diagram is stable under perturbation by everywhere defined finite rank operators, since the defining classes are so stable, c.f. ([C8], 478). This remark can also be justified by theorems in the thesis e.g. Theorem II.19.

IV.9 Completeness of the Diagram for F_-–Operators

$X = Y = \ell_2$ T bounded.

(I_1, I_1): Let T be the identity operator on X.

(I_3, III_1): Let T be defined by $T((x_k)) = (x_{2k})$. Then $T \in I_3$ and $T' \in III_1$ by 1.17 of ([C5], 8).

(III_1, I_3): Define T by $T((x_k)) = (0, x_1, 0, x_2, \ldots)$. Then $T \in III_1$ and $T' \in I_3$ by 1.17 of ([C5], 8).

(II_2, II_2): The same example as for (II_2, II_2) in I.5.5 serves.

(II_3, III_2): The operator $T: (x_k) \to (x_{2k}/k)$ is compact and has dense range. Therefore $T \in II_3$, and then $T' \in III_2$ by 2.7 of ([C5], 12).

(III_2, II_3): Let T be the adjoint of the operator in the previous example. Then $T' \in II_3$ by 2.7 of ([C5], 12).

(III_3, III_3): Let T be the zero operator.

X not complete $Y = \ell_2$ T bounded.

(I_2, III_1): The same example as for (I_2, III_1) in I.5.8 serves.

X complete but not reflexive $Y = \ell_2$ T compact on X

(II_2, III_2): The same example as for (II_2, III_2) in I.5.9 serves.

(III_2, III_3): Similar to example for (III_2, III_3) in I.5.9 but using the operator $(x_n) \to (0, x_1, 0, x_2, \ldots)$ (on ℓ_1) in place of the right shift operator.

IV.10 State Diagrams for T and T''

In these sections the author investigate the states that are available for a linear operator T and its second adjoint T'' under the following assumptions : first, $\gamma(T) > 0$ and secondly, $\gamma(T') > 0$.

These diagrams are of interest in Tauberian operator theory, where an operator $T \in L(X,Y)$ is Tauberian if

$$(T'')^{-1}(Q\hat{Y}) \subset \hat{\tilde{D}(T)}$$

where $Q = Q^{Y''}_{D(T')^{\perp}}$, $\hat{Y}$ denotes the canonical embedding of Y in Y'' and $\tilde{D}(T)$ denotes the completion of $D(T)$.

This definition is a generalisation of N. Kalton and A. Wilansky's definition [KW] for the classical case : $(T'')^{-1}(Y) \subset X$.

The following lemma, which is to be found in Dunford and Schwartz ([DS], 479), is of importance to the ensuing state diagrams :

IV.10.1 Lemma [DS] If T is in $B(X,Y)$, the second adjoint $T'': X'' \to Y''$ is an extension of T . If X is reflexive, then $T'' = T$.

Proof: [DS] Let $x \in X$, $y' \in Y'$. Then

$$(T''\hat{x})y' = \hat{x}T'y' = (T'y')x = y'Tx = (\hat{Tx})y' .$$

The author now constructs a state diagram for closed operators T and T'' under the assumption that $\gamma(T) > 0$. This implies by Theorem III.1.21 that $\gamma(T') > 0$ and in turn $\gamma(T'') > 0$. Hence T' and T'' have closed ranges by [G1, IV.1.6]. Also $T \notin 2$, $T' \notin 2$ and $T'' \notin 2$ since we are assuming $\gamma(T) > 0$. State 2 belongs to Goldberg's classification as stated in I.3.

IV.11 State Diagrams based on Goldberg's states $(\gamma(T) > 0)$:

I: $R(T) = Y$

II: $R(T) \neq Y$ but $\overline{R(T)} = Y$

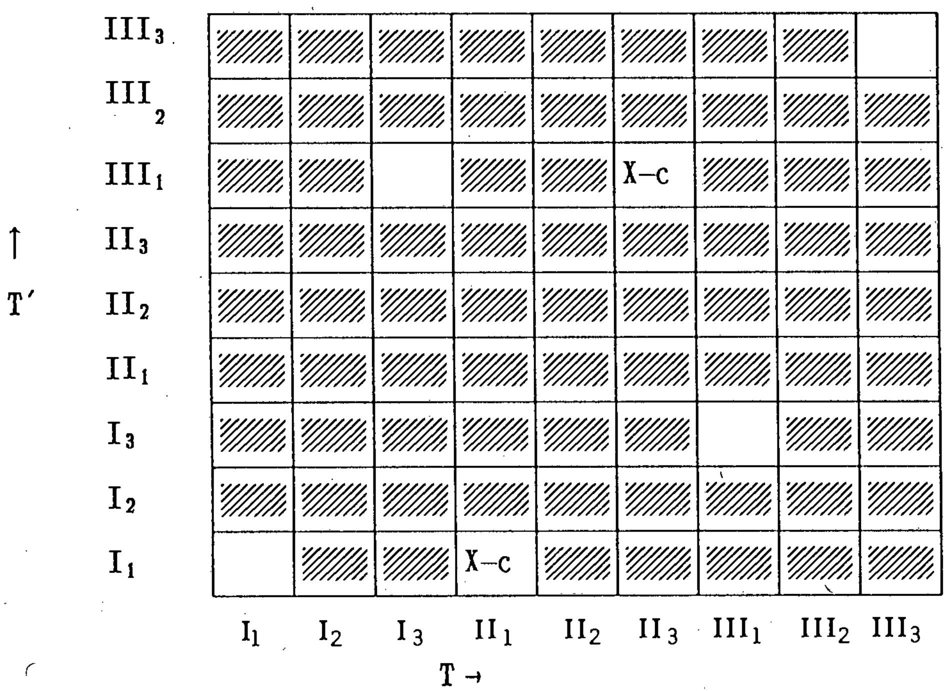
III: $\overline{R(T)} \neq Y$

1: T^{-1} exists and is continuous.

2: T^{-1} exists but is not continuous.

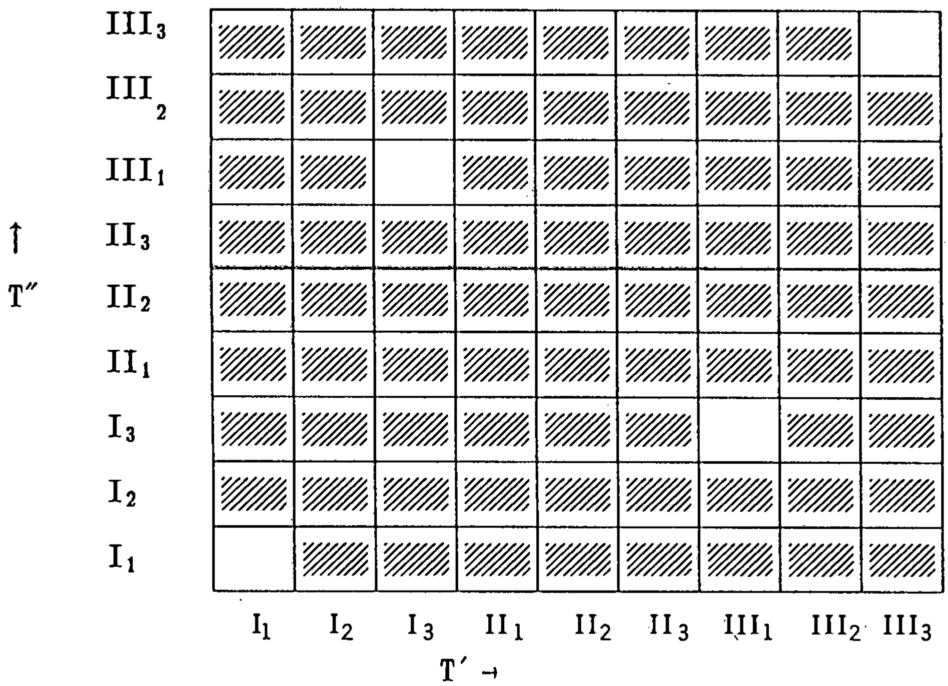
3: T has no inverse.

IV.11.1 State diagram for T and T' under the assumption $\gamma(T) > 0$. The states considered are as above. See I.3.14 and I.4.11 for Taylor–Halberg–Goldberg state diagrams.



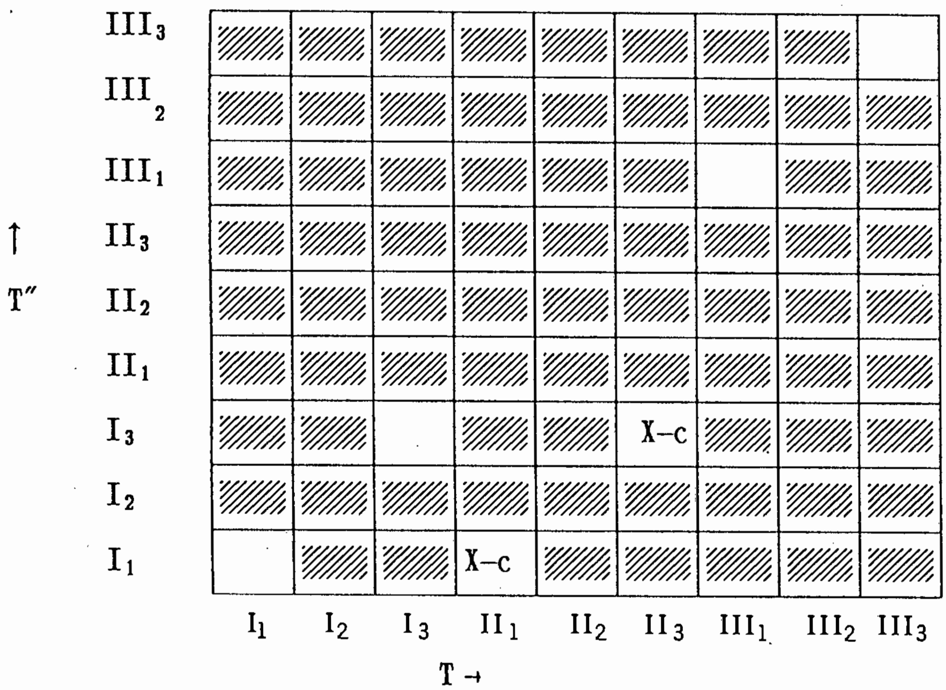
Proof: Since $\gamma(T) > 0$ we have that $\gamma(T') > 0$ by Theorem III.1.21. Now if T is 1–1 and $\gamma(T) > 0$ then T has a bounded inverse since then $d(x, N(T)) = \|x\|$, and $\|Tx\| \geq c\|x\|$, i.e. $T \notin 2$. Similarly, $T' \notin 2$ if $\gamma(T') > 0$. Also T' has closed range by [G1; IV.1.6], since T' is closed and Y' and $D(T)'$ are complete. □

IV.11.2 State diagram for T' and T'' $(\gamma(T) > 0)$. The states are as above.



Proof: Since $\gamma(T'') > 0$, $T'' \not\subseteq \text{II}$, as T'' is closed and $D(T)''$ and $D(T')'$ are complete. Also, T', $T'' \not\subseteq 2$. □

IV.11.3 State diagram for T and T″ based on the above classification (also see [G1, 58]. The author assumes $\gamma(T) > 0$.



Proof: This diagram follows from IV.11.1 and IV.11.2. □

IV.12 Essential State Diagram for T and T″ based on the assumption $\gamma(T') > 0$.

The author constructs the state diagram in three steps. In each of these three state diagrams the essential states are considered. The essential states are as follows:

I: $R(T)$ is closed and $b(T) < \infty$

II: $R(T)$ is not closed but $\bar{b}(T) < \infty$

III: $\bar{b}(T) = \infty$.

1: $T \in F_+$

2: $T \notin F_+$ but $a(T) < \infty$

3: $a(T) = \infty$.

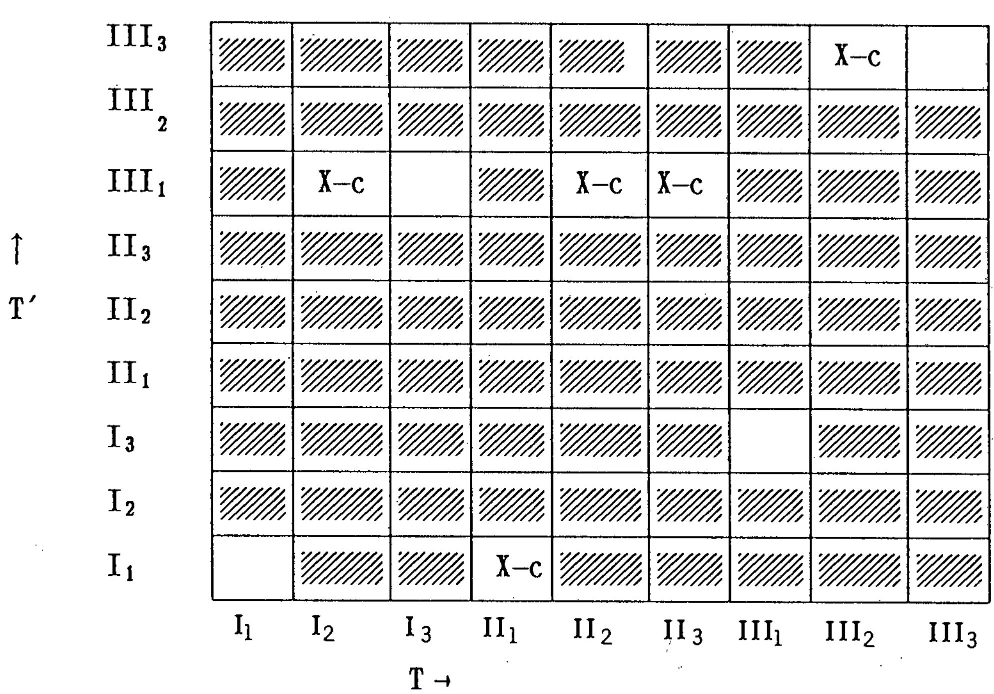
IV.12.1 The Essential State Diagram for linear operators [C5; 1.17].

This diagram has the same configuration as the Taylor–Halberg–Goldberg diagram I.3.14.

IV.12.2 The Essential State Diagram for closed linear operators [C5; 2.7].

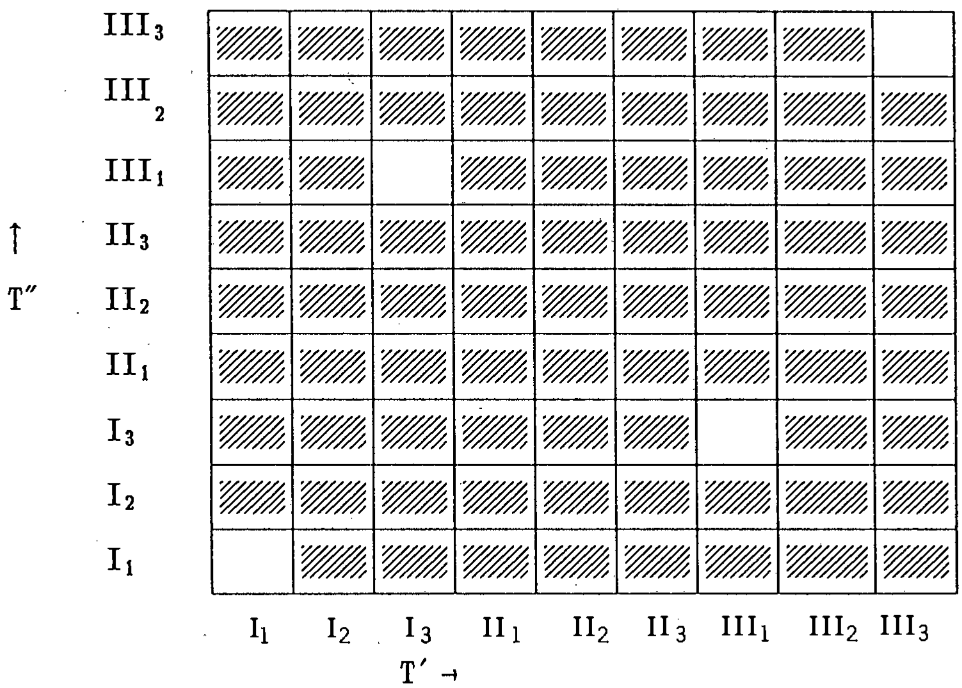
This diagram has the same configuration as the diagram for closed linear operators I.4.11.

IV.12.3 Essential State Diagram for T and T′ under the assumption $\gamma(T') > 0$.



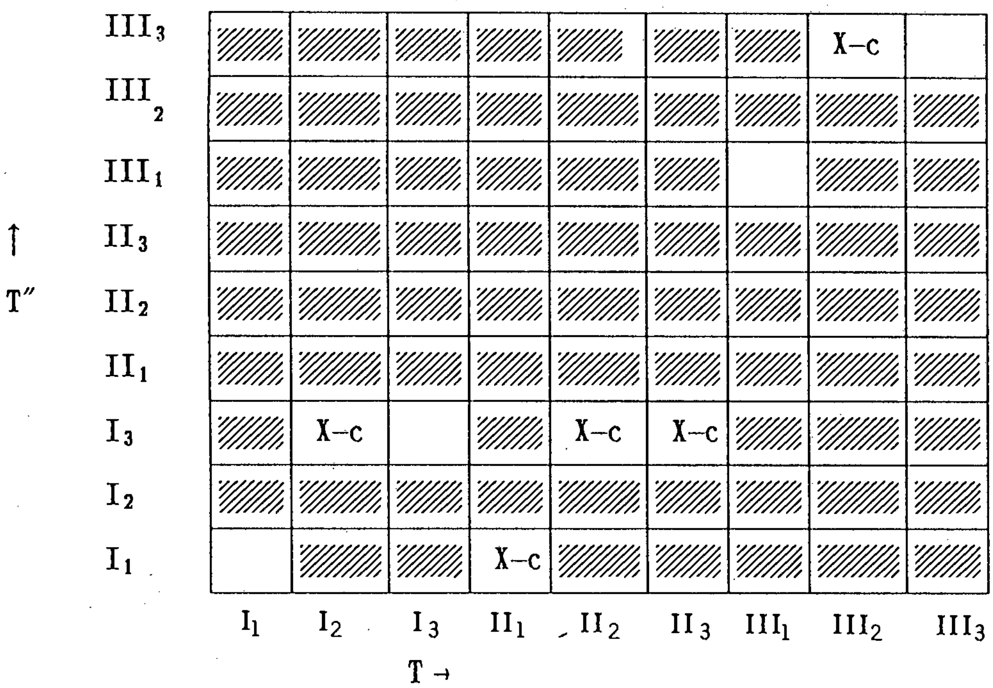
Proof: Since $\gamma(T') > 0$, $R(T')$ is closed. Hence $T' \notin II$ and $T' \notin 2$. Now suppose T is closed and X is complete. Then $\gamma(T) > 0$ by Corollary III.1.23. Hence $T \in F_+ \Leftrightarrow a(T) < \infty$ [CL; 5.6] whence $T \notin 2$. In particular $T \notin III_2$. □

IV.12.4 Essential State Diagram for T' and T'' under the assumption $\gamma(T') > 0$.



Proof: Immediate from IV.12.3. □

IV.12.5 Essential State Diagram for T and T″ $(\gamma(T') > 0)$.



Proof: See diagrams IV.12.3 and IV.12.4. □

IV.13 Completeness of the Diagrams for T and T″

IV.13.1 Completeness of State Diagram IV.11.3

$$X = Y = \ell_2 \qquad T \text{ continuous on } X$$

(I_1, I_1): Let T be the identity operator on X .

(I_3, I_3): See (I_3, III_1) in I.5.5.

(III_1, III_1): See (III_1, I_3) in I.5.5.

(III_3, III_3): Let T be the zero operator.

X not complete $Y = \ell_2$ T continuous on X

(II_1, I_1): See example in I.5.8.

(II_3, I_3): See example in I.5.8.

IV.13.2 Completeness of Essential State Diagram IV.12.5

$X = Y = \ell_2$ T bounded

(I_1, I_1): See corresponding example of [C5, 13].

(I_3, I_3): See corresponding example of [C5, 13].

(III_1, III_1): See corresponding example of [C5, 13].

(III_3, III_3): See corresponding example of [C5, 13].

X not complete $Y = \ell_2$ T bounded

(iii_2, III_3): See corresponding example in [C5, 14].

All the state diagrams in this chapter were constructed by the author and are based on classifications suggested to the author by R.W. Cross.

For further information regarding state diagrams the reader is referred to the survey monograph of V.M. Onieva [O].

REFERENCES

[Ba] Banach, S., "Théorie des opérations linéaires," Monografje Matematyczne, Warsaw, 1932.

[Bo] Bourbaki, N., "Espaces vectoriels topologiques", "Elements de mathematique," livre V, Actualités Sci. Ind., 1229, Hermann & Cie, Paris, 1955.

[C1] Cross, R.W., Linear Transformations of Tauberian Type in Normed Spaces, Preprint, November 1989.

[C2] Cross, R.W., On the continuous linear image of a Banach space, J. Austral. Math. Soc., Series A, 29 (1980), 219–234.

[C3] Cross, R.W., Properties of Some Norm Related Functions of Unbounded Linear Operators, Mathematische Zeitschrift, 199, (1988), 285–302.

[C4] Cross, R.W., Some continuity properties of linear transformations in normed spaces. Glasgow Math. J. 30 (1988), 243–247.

[C5] Cross, R.W., The Essential State Diagram of a Linear Operator. Preprint.

[C6] Cross, R.W., Unbounded linear transformations of upper semi–Fredholm type in normed spaces. Port. Math. 47, No.1 (1990), 1–19.

[C7] Cross, R.W., Unbounded strictly singular operators, Proc. Netherlands Acad. Sc. Series A = Indagationes Math. 91, No.3 (1988), 245–248.

[C8] Cross, R.W., Unilateral quasi–regularizers of closed operators, Mathematical Proceedings of the Cambridge Philosophical Society (1980), 87, 471–480.

[CL] Cross, R.W. and L.E. Labuschagne, Characterizations of Operators of Lower Semi–Fredholm Type in Normed Linear Spaces. Preprint.

[CO] Cohn, D.L., Measure Theory, Birkhäuser, Boston, 1980.

[D] Dieudonné, J., La dualité dans les espaces vectoriels topologiques, Ann. Sci. Ecole Norm. Super. Ser. 3, Vol. 59, (1942), 107–139.[11]

[DS] Dunford, N., and J.T. Schwartz, "Linear Operators", Parts I and II, Interscience Publishers, Inc., New York, 1958, 1963.

[G1] Goldberg, S., Unbounded linear operators, McGraw–Hill, New York, 1966.

[G2] Goldberg, S., Linear operators and their conjugates, Pacific J. Math., 9 (1959), 69–79.

[GK] Gokhberg, I.C., and M.G. Krein, Fundamental Theorems on Deficiency Numbers, Root Numbers, and Indices of Linear Operators (Russian), Usp. Mat. Nauk, 12, 1957, pp.43–118. Translated in Am. Math. Soc. Transls., ser.2, Vol. 13.

[J] James, R.C., A Non–reflexive Banach Space Isometric with Its Second Conjugate Space, Proc. Natl. Acad. Sci. U.S., Vol.37, (1951), 174–177.

[KW] Kalton, N.J. and Wilansky, A. "Tauberian operators in Banach spaces", Proc. Amer. Math. Soc. 57 (1976), 251–255.

[Ka] Kato, T., Perturbation Theory for Nullity, Deficiency and Other Quantities of Linear Operators, J. d'Analyse Math. **6** (1958), 273–322.

[KKM] Krein, M.G., M.A. Krasnoselskii and D.P. Milman, On the defect numbers of linear operators in a Banach space and on some geometrical questions (Russian), Sb. Tr. Inst. Mat. Akad. Nauk Ukr. SSR, No.**11** (1948), 97–112.

[L1] Labuschagne, L.E., On the Minimum Modulus of an Arbitrary Linear Operator. Preprint.

[L2] Labuschagne, L.E., Unbounded Linear Operators in Operator Ranges, Thesis Reprints, No.TR002, Univ. Cape Town, 1987.

[LS] Lebow, A. and M. Schechter, Semigroups of Operators and Measures of Noncompactness, J. Functional Anal. **7** (1971), 1–26.

[N] Nikaido, G., Product of linear operators with closed range, Proc. Japn Acad. Ser. A, **62** (1986), 338–340.

[O] Onieva, V.M., Adjoint operators and state diagrams in Functional Analysis, (Spanish), University of Santander, 1980.

[Pi] Pietsch, A., Operator ideals, North Holland, Amsterdam (1980).

[Sc1] Schechter, M., Principles of Functional Analysis, Academic Press, New York and London, 1971.

[Sc2] Schechter, M., Quantities related to strictly singular operators, Indiana Math. J. **21** (1972), 1061–1071.

[St] Stone, M.M., "Linear Transformations in Hilbert Spaces and Their Applications to Analysis", American Mathematical Society (American Mathematical Society Colloquium Publications, Vol.15), New York, 1932.

[T] Taylor, A.E., "Introduction to Functional Analysis", John Wiley & Sons, Inc., New York, 1958.

[TH] Taylor, A.E. and C.J.A. Halberg, General theorems about a linear operator and its conjugate, J. Reine Angew. Math. 198 (1957), 93–11.

[We1] Weis, L., On the computation of some quantitites in the theory of Fredholm operators, Supplemento Rendi Circ. Math. Palermo, Series 2, 5 (1984), 109–133.

[We2] Weis, L., Über strikt singuläre und strikt consiguläre operatoren in Banachräumen, Dissertation, Bonn, 1974.

[Wo] Wolf, F., On the essential spectrum of partial differential boundary problems, Commun. Pure Appl. Math. 12 (1959), 211–228.