Beyond Concordance Cosmology

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I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other university for a degree and that it represents my own work.

Amare Abebe Gidelew
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Dedication

To those of old, thru all the pains,
Who made science so logic reigns;
To those of now, who came before,
And made it known a little more;
To those to come to make it shine,
This work is yours as much as mine.

A Parable

The cheese-mites asked how the cheese got there,
And warmly debated the matter;
The Orthodox said that it came from the air,
And the Heretics said from the platter.
They argued it long and they argued it strong,
And I hear they are arguing now;
But of all the choice spirits who lived in the cheese,
Not one of them thought of a cow.

—Arthur Conan Doyle

Beyond Concordance Cosmology
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Abstract

For almost a hundred years now, Einstein's General Relativity theory has played a pivotal role in our understanding of the Universe. The recent realization of the accelerated cosmic expansion, however, has put modern cosmology in crisis, mainly because this new realization implies most of the matter and energy content of the Universe exists in a not-yet-understood dark energy form. The existence of dark energy and dark matter (another little-understood, non-luminous form of matter) has led to the conclusion that the general relativistic formulation of gravity does not provide a complete description of the Universe on large scales, and has therefore motivated the cosmology community in the search for new underlying physics.

One possible modification of General Relativity comes in the form of $f(R)$ theories of gravity. In this thesis we look at the possible implications to cosmology of this class of modified gravitational physics. In particular, we study the formation of large scale structures using the 1+3 covariant and gauge-invariant formulation of cosmological perturbations and the application of dynamical systems in the background analysis.

Given that we are in an era of precision cosmology, there are multiple observational surveys (such as the Sloan Digital Sky Survey) against which a comparison of predicted matter power spectra of these theories can be made, thus constraining the viability of some $f(R)$ models. In this regard, we study the predicted power spectra using both a dynamical systems approach for the background and solving for the matter perturbations, comparing the theoretical results with several SDSS Data. The importance of studying the first order perturbed equations by assuming the correct background evolution and the relevance of the initial conditions are stressed. Moreover, we analyze constraints of the cosmic evolution history using the geometric information that can be extracted from baryon acoustic oscillations (BAO) data.

We also look at shear-free perturbations in these theories and show that for some models, some classical results arising from General Relativity can be avoided. General Relativity does not have a proper Newtonian limit on cosmological scales. However, a quasi-Newtonian rendering of the theory in the linearized regime of the Friedmann-Lemaître- Robertson-Walker (FLRW) cosmological model gives integrability conditions for a consistent propagation of the field equations. In this thesis, we prove that these integrability conditions are also satisfied in the broader class of gravitational theories.

Keywords: general relativity, cosmic acceleration, dark energy, modified gravity, $f(R)$, cosmological perturbations, multi-component fluids, shear-free, quasi-Newtonian

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PART I

Chapter 1

Introduction

> Research is what I'm doing when I don't know what I'm doing.
>
> Werner von Braun

Humanity's eternal quest for a better understanding of the cosmos has always brought more surprises and more questions than answers. Nevertheless, the evolution of our perception of the Universe over the last few millennia has been dramatic. From an earth supported by an infinite chain of "turtles all-the-way-down" to a fixed firmament of heavens and hells surrounding us to Ptolemy's geocentric universe, from Copernicus' heliocentric universe to Newton's "falling-apple inspired" gravitation to Kant's "island universes" to Einstein's fabric of static spacetime to the accelerating, albeit still little-known, universe, humankind has traced all sorts of mysticism and reason to satisfy its curiousity and at times to determine its place and fate in the grandest scheme of things.

The development by Einstein of the General Relativity Theory (henceforth shortened by GR) [113] is arguably the greatest scientific contribution to the understanding of the universe we live in. Standard cosmology is based on GR. Through GR it has been established that the Universe expands. This fact has led to the discovery of the Big Bang, the start of everything from a point. Through GR we understand how the first elements in the Universe formed and how large scale structures in the Universe form and grow. The age, size, geometry and future fate of the Universe have been deduced from GR. Moreover, GR has been able to perfectly explain solar system and other astrophysical phenomena.

Since its inception, however, there have always been challenges to the unique status of GR as the ultimate theory of gravitation.

The greatest challenge to General Relativity came when, in 1998, the High-z Supernova Search Team observations of Type Ia supernovae showed [249, 260] that the present universe (i.e., since a redshift of $z \sim 0.5$)[1] is in a state of accelerated expansion. This phenomenon came as a complete surprise since neither GR nor any other theoretical framework could explain it.

If GR is the correct theory of gravitation that controls the expansion of the Universe, then observational analyses [29, 85, 139, 182, 280] show that most of its energy content exists neither in the luminous (baryonic) nor in the dark, invisible forms of matter known as *dark matter*, but instead in the form of an exotic, unclustered, invisible *dark energy*. In light of this, there have been several recent attempts to explain the observed discrepancy between theory and observation. Many candidates have been put forward as an explanation for DE, but most of them fall under one of these three forms: the cosmological constant, exotic scalar fields (such as Quintessence) and geometrical dark energy in which the gravitational Lagrangian is modified with respect to the usual Einstein-Hilbert one (see [52, 78, 98, 108, 174, 277] for an extensive review).

An important class of modified gravity are the scalar-tensor and $f(R)$ theories [23, 49, 51, 60, 67, 71, 77, 135, 221, 222, 278, 279, 282, 306]. Both these candidates have their own serious shortcomings [135, 221, 306], and have to pass rigorous theoretical and observational scrutiny before they can be accepted as viable theories [52, 252]. In this work we will only concentrate on the interpretation of DE as geometrical manifestation of a more fundamental theory, focusing on $f(R)$-gravity.

It is well known that the dynamical evolution of small density perturbations, seeded in the early universe, led to the large-scale structure we see today [22, 31, 33, 41, 89, 99, 179, 205, 215, 240, 244, 246]. An excellent framework for studying cosmological perturbations is the 1+3 covariant approach, which has been developed, among other things, to analyze the evolution of linear perturbations of Friedmann-Lemaître-Robertson-Walker (FLRW) models in GR [12, 28, 44, 45, 68, 101, 104, 106, 128, 169, 171, 172, 252, 283, 298].

In recent years, higher order theories of gravity have attracted a great deal of attention. A detailed analysis of the background FLRW models using dynamical system techniques has shown that there exist classes of fourth order theories which admit a transient decelerated expansion phase during which structure formation can take place, followed by a DE-like era which drives the present cosmological acceleration (see [92, 132, 210, 223, 267, 276] among many others). However, it has been proved [93] that when dust matter scalar cosmological perturbations are studied in the metric formalism, $f(R)$ theories, even mimicking the standard cosmological expansion, provide a different matter power spectrum from that predicted by the ΛCDM model [95]. In [58] the evolution of scalar perturbations of FLRW models in fourth order gravity was developed for single barotropic fluids using the 1+3 covariant approach. The solutions of the perturbation equations on large-scales showed that a decelerated phase is not necessarily required to form large scale structures. This divergence from the standard GR provides us with a distinguishable signature of fourth order theories, which can be tested against observations.

In this dissertation, we will discuss some of the different alternatives to GR put

[1]The *late-time* universe, on the other hand, is usually taken to be the Universe after $z \lesssim 1000$.

forward to solve the cosmological conundrum and will also give a more detailed exposition of natural generalization of GR known as $f(R)$ theories of gravity.

For convenience, the thesis has been divided in three parts.

Part I gives a general review of basic cosmological concepts and the theoretical tools needed for developing them, giving particular emphasis to the work included in this dissertation. In Chapter 2, we will give an overview of modern cosmology and shortcomings of the concordance model. Some of the suggested solutions to these shortcomings will then be outlined.

We then give a review of $f(R)$ theories of gravitation as generalizations of GR and introduce the corresponding field equations in Chapter 3.

In Chapter 4 we give an outline of the $1+3$ covariant cosmology applied to $f(R)$ gravity. This is where we provide the necessary background material on linearized field equations and where gauge-invariant cosmological perturbations will be treated.

Chapter 5 will discuss dynamical systems in cosmology, with particular emphasis on the application of dynamical systems to $f(R)$ gravity theories.

Chapters 6 and 7 comprise **Part II** of this thesis, where we provide a detailed discussion of original research contributions. Since the Universe consists of a mixture of *fluids*, a complete treatment of perturbations in fourth order theories requires taking this into account. This part of the thesis presents a general framework for studying multi-fluid cosmological perturbations with a completely general equation of state in any $f(R)$ theory of gravity, using the 1+3 covariant approach. In Chapter 6 we develop a general framework for analysing covariant, gauge-invariant cosmological perturbations for a multi-fluid cosmic medium. After deriving the complete set of equations describing the evolution of matter and curvature fluctuations for a multi-fluid cosmological medium, we specialize to a radiation-dust fluid described by barotropic equations of state and solve the perturbation equations around a background solution of R^n gravity. In particular we study exact solutions for scales much smaller and much larger than the Hubble radius and show that $n > \frac{2}{3}$ in order to have a growth rate compatible with the *Mészáros effect*.

Even though much attention has been given to the study of modified gravity theories in order to find a more natural explanation for the late time acceleration of the Universe, a comparison of the matter power spectrum predictions made by these theories with available data has not yet been subjected to a detailed analysis. In the context of $f(R)$ theories of gravity, Chapter 7 combines background dynamics with perturbations to do power-spectrum analysis and constrain the viability of some $f(R)$ models using the Sloan Digital Sky Survey (SDSS) data. We study the predicted power spectra using both a dynamical systems approach for the background and solving for the matter perturbations, comparing the theoretical results with the SDSS data. The importance of studying the first order perturbed equations by assuming the correct background evolution and the relevance of the initial conditions are also stressed. We determine the statistical significance in relation to the observational data and demonstrate their conflict with existing observations.

The remaining chapters comprise **Part III**. As further cosmological applications of $f(R)$ gravitation, we discuss shear-free perturbations in Chapter 8. Recently it was shown that if the matter congruence of a general relativistic perfect fluid flow in an almost FLRW universe is shear-free, then it must be either expansion or rotation-free. Here we generalize this result for a general $f(R)$ theory of gravity and show that scenarios exist where this result can be avoided. This suggests that there are situations where linearized fourth-order gravity shares properties with Newtonian theory not valid in General Relativity.

Chapter 9 will give a rendering of quasi-Newtonian $f(R)$ cosmologies. We will derive the integrability conditions necessary for consistent linearized shear-free $f(R)$ models in a purely covariant way. We also present the velocity and density perturbation equations that arise as a result of the imposed integrability conditions.

Finally, the general conclusions and future outlook of the work done in this thesis will be presented in Chapter 10.

The traditional sign and natural unit conventions are used. Thus the natural units ($\hbar = c = k_B = 8\pi G = 1$), i.e., the normalized reduced Planck's constant, speed of light, Boltzmann constant and Newton's gravitational constant (also denoted by κ), in that order, are assumed unless otherwise purposefully retained. The Planck mass is defined by $M_{Pl} = \sqrt{\frac{\hbar c}{8\pi G}}$. Latin indices of tensors run from 0 to 3. The symbols ∇ and ; represent the usual covariant derivative, ∂ and , correspond to partial differentiation and an over dot shows differentiation with respect to proper time.

We use the $(-,+,+,+)$ spacetime signature and the Riemann tensor is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^f{}_{bc}\Gamma^a{}_{df}, \tag{1.1}$$

where the $\Gamma^a{}_{bd}$, etc., are the usual affine connection symbols defined by

$$\Gamma^a{}_{bd} = \frac{1}{2}g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}). \tag{1.2}$$

The Ricci tensor is obtained by contracting the first and the third indices:

$$R_{ab} = g^{cd}R_{acbd}, \tag{1.3}$$

and the scalar curvature (Ricci scalar) is defined to be the trace of the Ricci tensor:

$$R = g^{ab}R_{ab} = R^a{}_a. \tag{1.4}$$

Moreover the following are standard notations used in the thesis:

g :	$\det(g_{ab})$, the determinant of the metric g_{ab}	(1.5)
(ab) :	symmetrization over the indices a and b	(1.6)
$[ab]$:	anti-symmetrization over the indices a and b	(1.7)

The completely anti-symmetric pseudotensor η^{abcd} is defined such that

$$\eta_{0123} = \sqrt{-g}\ . \tag{1.8}$$

Chapter 2

Review of Modern Cosmology

> No amount of experimentation can ever prove me right; a single experiment can prove me wrong.
>
> Albert Einstein

2.1 General Relativity : a tool for Cosmology

Cosmology as the scientific study of the large scale properties of the Universe as a whole endeavors to use the scientific method to understand the origin, evolution and ultimate fate (i.e., past, present and future) of the entire Universe. Although merely a century old as a modern science, cosmology as a curiosity about the origin and nature of the world is certainly as old as mankind itself.

Einstein initiated the development of modern cosmology with the publication of his *Allgemeine Relativitätstheorie* in 1915 [113]. By then he had realized the effect of spacetime and matter on each other, and that to derive the field equations of motion, he needed to start with an action of the form

$$\mathcal{A}_{EH} = \frac{1}{2\kappa}\int_{all\ \ specetime} d^4x\sqrt{-g}R\ . \tag{2.1}$$

This has now come to be known as the *Einstein-Hilbert action* because it was David Hilbert who proposed the action first [34, 86]. Here $\kappa \equiv \dfrac{8\pi G}{c^4}$ where G is Newton's universal gravitational constant. The action $\mathcal{A}_{EH}$ is defined if the whole space-time converges, but if the integral diverges, it can always be redefined over arbitrarily large compact domains.

If we now include a matter term into the Einstein-Hilbert action

$$\mathcal{A} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[R + 2\mathcal{L}_m\right]\ , \tag{2.2}$$

where $\mathcal{L}_m$ is the Lagrangian of any kind of matter field, and use variational principle of least action with respect to the metric g_{ab},

$$\delta\mathcal{A} = 0 \ , \tag{2.3}$$

we obtain the Einstein field equations (EFEs)

$$R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab} \ , \tag{2.4}$$

where

$$T_{ab} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} = -2\frac{\delta\mathcal{L}_m}{\delta g^{ab}} + g_{ab}\mathcal{L}_m \tag{2.5}$$

is the energy-momentum tensor (EMT) (also often called the stress-energy or stress-energy-momentum tensor).

Einstein had a static universe in mind when he derived his field equations, but it was soon discovered by Friedmann [140] and Lemaître [190] that the equations do actually describe an expanding universe. Einstein included a cosmological constant (Λ) in the Lagrangian

$$\mathcal{A} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[R + 2(\mathcal{L}_m - \Lambda)\right] \ , \tag{2.6}$$

so the new field equations

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = \kappa T_{ab} \tag{2.7}$$

could balance the cosmic expansion.

The most common, and most compact, way of writing (2.7) is

$$G_{ab} + \Lambda g_{ab} = T_{ab} \ , \tag{2.8}$$

where

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} \tag{2.9}$$

and $\kappa = 1$ in the geometrized natural units, the units that we will be adopting throughout the rest of the thesis.

When fully written out, these field equations are a system of 10 coupled, nonlinear, hyperbolic-elliptic partial differential equations with at most second-order derivatives in the metric (in four dimensions [78, 196]). Because of the highest order of the derivatives in the field equations, GR is a *second-order theory of gravity.*

The left-hand-side (LHS) of (2.8) encodes the curvature of spacetime determined by the spacetime metric, whereas the right-hand-side (RHS) represents the matter-energy content of spacetime. Thus the EFEs tell us that "matter and energy dictate spacetime how to curve" and "spacetime dictates matter how to move."

If the metric of a spacetime manifold (M, g) is given by

$$ds^2 = g_{ab}x^a x^b \ , \tag{2.10}$$

then a freely-falling matter moving through spacetime will be dictated by the geodesic equation

$$\frac{d^2x^a}{d\lambda^2} + \Gamma^a{}_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0 \ , \tag{2.11}$$

where λ is an affine parameter representing proper time (for a time-like curve) or distance (for a space-like curve) such that

$$\frac{d\mathbf{t}}{d\lambda} = 0 \ , \tag{2.12}$$

where $\Gamma^a{}_{bc}$ is the Levi-Civita connection defined as

$$\Gamma^a{}_{bc} \equiv \frac{1}{2}g^{ad}\left[\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right] \ , \tag{2.13}$$

and $\mathbf{t}$ is a tangent vector to the curve x^a with components given by

$$t^a = \frac{dx^a}{d\lambda} \ . \tag{2.14}$$

If T_p is a vector space for each point on the manifold (M, g), then T_p will not be entirely spanned if the geodesics run into a singularity, which can be thought of as the "edge of the manifold" [66]. Manifolds endowed with such singularities are said to be *geodesically incomplete*. On the other hand, if any geodesic can be extended indefinitely, the manifold (M, g) is said to be *geodesically complete* [175].

While Eqns (2.8) and (2.11) form the core of the mathematical formulation of GR, strictly speaking, modern cosmology started with the study of exact solutions of the EFEs. The solutions are metrics of spacetime that describe the structure (such as the geometry of, and the inertial motion of matter in) spacetime. But because of the complicated nature of the field equations[1], there are only a few exact solutions and much of the study of applications of GR uses perturbative techniques.

2.2 The Concordance Model

One of the earliest exact solutions to be found of the EFEs is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which describes a globally homogenous, isotropically expanding (or contracting) spacetime geometry.

Comoving coordinates in this geometry are usually chosen such that the metric in Eqn (2.15) takes the form:

[1]Because of the nonlinearity of the field equations, Einstein assumed that they were unsolvable. However, in 1916 Schwarzschild discovered [268] the first exact solution for the case of a spherically symmetric spacetime surrounding a massive object in spherical coordinates.

$$ds^2 = -dt^2 + a^2(t)d\sigma^2, \quad u^a = \delta^a{}_0(a = 0, 1, 2, 3), \tag{2.15}$$

where $a(t)$ is the cosmological scale factor, a time-dependent parameter that tells us the relative expansion of the Universe. The worldlines with tangent vector

$$u^a = \frac{dx^a}{dt} \tag{2.16}$$

represent the histories of fundamental observers. The (t = constant) space sections are surfaces of homogeneity and have maximal symmetry: they are 3-spaces of constant curvature

$$K = \frac{k}{a^2(t)} , \tag{2.17}$$

where k is the sign of K and hence takes the values $-1, 0$ or $+1$ depending on whether the Universe is open, flat or closed, respectively. The normalized metric $d\sigma^2$ characterizes a 3-space of normalized constant curvature k whose spatial coordinates (r, θ, ϕ) can be chosen such that

$$d\sigma^2 = d^2r + f^2(r)(d^2\theta + \sin^2\theta d^2\phi), \tag{2.18}$$

where

$$f(r) = \begin{cases} \sin(r) & \text{for } k = +1, \\ r & \text{for } k = 0, \\ \sinh(r) & \text{for } k = -1. \end{cases}$$

The rate of expansion at any time t is characterized by the Hubble parameter

$$H(t) = \frac{\dot{a}}{a} , \tag{2.19}$$

the parameter that appeared in the original Hubble's law

$$v = HD \tag{2.20}$$

as the 'constant' of proportionality between the proper distance D to a galaxy and its velocity v of recession. In fact, this relation is a first-order term of a Taylor expansion whose higher-order terms are not significant.

2.2.1 Cosmological Equations

As mentioned earlier, the EFEs determine the evolution of the spacetime metric and show the effect of matter on space-time curvature. If an assumption of local isotropy is made, the EMT T_{ab} necessarily takes a perfect fluid form relative to the preferred worldlines with tangent vector u^a:

$$T_{ab} = \mu u_a u_b + p h_{ab} , \tag{2.21}$$

where

$$h_{ab} = g_{ab} + u_a u_b \tag{2.22}$$

is the projection tensor into the tangent 3-spaces orthogonal to u^a.

The energy density and the pressure terms $\mu(t)$ and $p(t)$ are the time-like and space-like eigenvalues of T_{ab}, respectively. The evolution of the energy density gives the conservation equation

$$T^{ab}{}_{;b} = 0 \Leftrightarrow \dot{\mu} + (\mu + p)\Theta = 0 \,, \tag{2.23}$$

where

$$\Theta \equiv 3H = 3\frac{\dot{a}}{a} \,, \tag{2.24}$$

and determines the integrability conditions for the EFEs.

With a suitable prescription of the equation of state function such that $w = \frac{p}{\mu}$ where w is a function of energy density ($\mu(t)$) and temperature (T), the integrability conditions become determinate. For example, Cold Dark Matter (*dust*) is pressureless and hence $w_d = 0$ whereas radiation has

$$p_r = \mu_r/3 \Leftrightarrow w = 1/3, \mu_r = aT_r^4 \,. \tag{2.25}$$

Thus according to the conservation equation (2.23) one obtains

$$\mu_d \propto a^{-3}, \quad \mu_r \propto a^{-4}, \quad T_r \propto a^{-1}. \tag{2.26}$$

We can also think of the cosmological constant Λ as a fluid with pressure p related to the energy density μ by $p_\Lambda = -\mu_\Lambda$ and hence with a corresponding equation of state$w_\Lambda = -1$.

The *Raychaudhuri equation* [258] gives the basic evolution equation for the scale factor $a(t)$

$$3\frac{\ddot{a}}{a} = -\tfrac{1}{2}(\mu + 3p) + \Lambda \,, \tag{2.27}$$

and hence the basic equation of gravitational interactions and the basis of singularity theorems in GR [270]. This equation shows that the active gravitational mass density of the matter and fields present is

$$\mu_{grav} \equiv \mu + 3p \,. \tag{2.28}$$

For ordinary matter the *Strong Energy Condition* (SEC) imposes a positive gravitational mass density according to

$$\mu + 3p > 0 \Leftrightarrow w > -1/3 \,. \tag{2.29}$$

This means that ordinary matter will tend to cause the Universe to decelerate ($\ddot{a} < 0$) whereas a positive cosmological constant, according to Eqn (2.27), causes an accelerated expansion ($\ddot{a} > 0$).

When matter and a cosmological constant are both present, the resulting expansion dynamics (i.e., accelerated, decelerated or constant expansion) may occur depending on which effect is dominant or if the two offset each other.

The first integral of Eqns (2.23) and (2.27) when $\dot{a} \neq 0$ yields the Gauß equation relating the 3-space curvature to the 4-space curvature, showing how matter directly causes a curvature of 3-spaces. [2]

$$\frac{\dot{a}^2}{a^2} = \frac{\mu}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}. \tag{2.30}$$

This equation is known as the *Friedmann equation* [140], although, strictly speaking, both Eqns (2.27) and (2.30) are are often referred to as *Friedmann equations*. From now on in this thesis we will identify the Friedmann equation by (2.30) and the Raychaudhuri equation by (2.27).

Because of the spacetime symmetries, the ten EFEs (2.8) are equivalent to the two Friedmann equations. Cosmological models based on a Robertson-Walker (RW) geometry with metric defined by Eqns (2.15) and(2.18) and dynamics governed by Eqns (2.23), (2.27) and (2.30) are known as the Friedmann-Lemaître-Robertson-Walker (FLRW) universes. The Friedmann equation (2.30) controls the expansion of the Universe, and the conservation equation (2.23) controls the density of matter as the Universe expands; when $\dot{a} \neq 0$, Eqn (2.27) will necessarily hold if Eqns (2.23)and (2.30) are both satisfied [121].

2.2.2 Initial Conditions

To ensure the existence of unique cosmological solutions both for a single matter component and for a combination of different kinds of matter, an integrable matter description (specifying the equation of state $w = w(\mu, T)$ explicitly or implicitly) for each matter component is required. If we include baryons (b), radiation (r), Cold Dark Matter (CDM) and neutrinos (ν), for example,

$$\mu = \mu_b + \mu_r + \mu_{CDM} + \mu_\nu \ , \tag{2.31}$$

this requires the specification of the w_i for each component i of matter. At an arbitrary time t_0 (say, today) [3], initial data for such solutions consists of [121]:

- The Hubble constant

$$H_0 = \left[\frac{\dot{a}}{a}\right]|_{t=0} = 100\tilde{h} \ \ \text{km/sec/Mpc} \ , \tag{2.32}$$

where $\tilde{h}$ reprsents a normalized dimensionless parameter quantifying our uncertainty in H_0;

[2] Note that the Raychaudhuri equation can be obtained from the Friedmann equation, the energy conservation equation and the definition of the Ricci scalar. Hence, any solution of the Friedmann equation automatically solves the Raychaudhuri equation.

[3] In forthcoming chapters, however, a subscript 0, unless otherwise indicated, will refer to the present epoch.

- A dimensionless normalized density parameter

$$\Omega_{i0} = \frac{\mu_{i0}}{3H^2} \tag{2.33}$$

for each type of matter present;

- For a non-vanishing cosmological constant, i.e., $\Lambda \neq 0$, either the fractional energy density

$$\Omega_{\Lambda 0} = \frac{\Lambda}{3H_0^2} \ , \tag{2.34}$$

or the dimensionless *deceleration parameter*

$$q_0 = -\left[\frac{\ddot{a}}{a}\right]|_{t=0} H_0^{-2} \ . \tag{2.35}$$

Thus the equations of state for the matter are known, a unique solution for $(a(t), \mu(t))$, and hence a unique corresponding cosmic history is determined.

The total energy density is given by the sum of the terms Ω_{i0} for each type of matter present, i.e., the equations

$$\Omega_{m0} = \Omega_{d0} + \Omega_{r0} + \Omega_{\nu 0}, \tag{2.36}$$
$$\Omega_0 = \Omega_{m0} + \Omega_{\Lambda 0}, \tag{2.37}$$

give us the total matter energy density and the total energy density (the cosmological constant included). Here the energy densities of baryons and Cold Dark Matter have been combined such that

$$\mu_d \equiv \mu_b + \mu_{CDM} \Rightarrow \Omega_{d0} \equiv \Omega_{b0} + \Omega_{CDM0} \ . \tag{2.38}$$

If the pressure term p is negligible[4] relative to the matter term μ in (2.27), then we get

$$q_0 = \frac{1}{2}\Omega_{m0} - \Omega_{\Lambda 0} \ . \tag{2.39}$$

This equation clearly shows that a dominant cosmological constant ($\Omega_{\Lambda 0} > \Omega_{m0}$) causes an accelerated cosmic expansion ($q_0 < 0$). On the other hand, a vanishing Λ ($q_0 = \frac{1}{2}\Omega_{m0}$) shows that matter can cause deceleration ($q > 0$) of the expansion.

The spatial curvature

$$K_0 = \frac{k}{a_0^2} = H_0^2(\Omega_0 - 1) \tag{2.40}$$

can be obtained by evaluating the Friedmann equation (2.30) at the present time t_0. The Universe is said to be *open*, *flat* or *closed* depending on whether $K_0 < 0$ ($\Omega_0 < 1$), $K_0 = 0$ ($\Omega_0 = 1$), or $K_0 > 0$ ($\Omega_0 > 1$). To completely define the geometry of the

[4] It is important to note here that, since CDM is by far the dominant component in a baryon-CDM mixture, it is customary to approximate the mixture as dust and hence pressureless.

homogeneous and isotropic Universe, we define the normalized, dimensionless density parameter Ω_k to measure the curvature of space, in such a way that

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1. \tag{2.41}$$

The FLRW models are the most widely explored cosmological models, mainly because of their extreme geometrical simplicity and the ever-increasing accuracy of observational data that seem to support these models. They have been used extensively in the analysis of the gravitational effect of matter (*dust* and radiation) on the global evolutionary dynamics of the Universe and the local background physics of the evolution of matter itself [121].

Defining $X_D \equiv \mu_{d0} a_0^3$ and $X_R \equiv \mu_{r0} a_0^4$ such that $\dot{X}_D = 0$ and $\dot{X}_R = 0$, one can rewrite Eqn (2.30) for dust and non-interacting radiation as

$$3\frac{\dot{a}^2}{a^2} = \frac{X_D}{a^3} + \frac{X_R}{a^4} + \frac{\Lambda}{3} - 3\frac{k}{a^2}\,. \tag{2.42}$$

Here the cosmological constant plays a prominent role in determining the behaviour of the equation [121]. For a vanishing cosmological constant ($\Lambda = 0$) the Universe starts off at a very dense initial state, where its energy density and curvature tend to infinity. Its future fate depends on the value of the spatial curvature, or equivalently the density parameter Ω_0. The Universe expands forever if ($k = 0 \Leftrightarrow \Omega_0 = 1$) or ($k < 0 \Leftrightarrow \Omega_0 < 1$), but collapses to a future singularity if ($k > 0 \Leftrightarrow \Omega_0 > 1$). Thus $\Omega_0 = 1$ corresponds to the critical density μ_{crit} separating $\Lambda = 0$ FLRW models that recollapse in the future from those that expand forever, and Ω_0 is just the ratio of the matter density to this critical density:

$$\Omega_{crit} = 1 \Leftrightarrow \mu_{crit} = 3H_0^2 \Rightarrow \Omega_0 = \frac{\mu_0}{3H_0^2} = \frac{\mu_0}{\mu_{crit}}. \tag{2.43}$$

When $\Lambda < 0$, all solutions start at a singularity and recollapse.

When Λ is positive, there are some interesting possible scenarios [121]:

- If $k = 0$ or $k = -1$, all solutions start at a singularity and expand forever.
- If $k = +1$, there can again be models with a singular start, either expanding forever or collapsing to a future singularity. In this case, however, a static solution (the Einstein static universe) is also possible, as well as models asymptotic to this static state in either the future or the past.
- Models with $k = +1$ can bounce (collapsing from infinity to a minimum radius and re-expanding).

The Universe as we know it in the standard cosmological models contains a realistic mixture of matter components (baryons, radiation, neutrinos, cold dark matter, a scalar field, and perhaps a cosmological constant). Here we give a list of some very specific models with simple expanding solutions [121]:

- The Einstein-de Sitter model. This is the simplest $(p = 0, \Lambda = 0, k = 0)$ $(\Rightarrow \Omega_0 = 1)$ expanding non-empty solution

$$a(t) = Ct^{2/3} \,, \tag{2.44}$$

  where C is an integration constant. This solution starts from a singular state at time $t = 0$. The age of the Universe in this model (the proper time since the start of the Universe) when the Hubble constant takes the value H_0 is $\tau_0 = \frac{2}{3H_0}$. This is a good model of the expansion of the Universe because it represents a dust-dominated, Minkowski universe until the recent times when a cosmological constant started to dominate the expansion. If k and Λ remain vanishing, this model is also a good model of the far future universe.

- The Milne model. This model is characterized by $(\mu = p = 0, \Lambda = 0, k = -1) \Rightarrow \Omega_0 = 0$ and represents a linearly expanding empty solution

$$a(t) = Ct \tag{2.45}$$

  in a flat spacetime as seen by a uniformly expanding set of observers, singular at $t = 0$. The age of the Universe in this model is given by $\tau_0 = \frac{1}{H_0}$. It is a good model of the far future universe if k remains negative and Λ remains vanishing.

- The de Sitter universe. For this model $(\mu = p = 0, \Lambda \neq 0, k = 0) \Rightarrow \Omega_0 = 0$ and the Universe is in a steady state of expansion without matter, the empty solution given by

$$a(t) = Ce^{Ht}, \tag{2.46}$$

  where C and H are constants. As can be seen, since H is a constant, the Universe expands at a constant rate in this model, and hence there is no start and its age is infinite. It is a good model of the far future universe for those cases which expand forever with a positive definite Λ. It can alternatively be understood as a solution with a vanishing Λ and containing matter with the exceptional equation of state $\mu + p = 0$.

Other FLRW forms of the de Sitter universe include: a geodesically complete form with $k = +1$ and a regular bounce solution

$$a(t) = a_0 \cosh(Ht) \,, \tag{2.47}$$

and another geodesically incomplete form with $k = -1$ and a singular start solution

$$a(t) = a_0 \sinh(Ht) \,. \tag{2.48}$$

The fact that there are no preferred time-like directions or space sections in this spacetime of constant curvature has led to there being no uniqueness in the solutions of such a universe.

In general, the Raychaudhuri equation (2.27) and the SEC (9.45) lead to the following theorem [112, 116, 121]:

Friedmann-Lemaître Universe Singularity Theorem. *In a FL universe with $\Lambda \leq 0$ and $\mu + 3p > 0$ at all times, at any instant t_0 when $H_0 \equiv \left[\frac{\dot{a}}{a}\right]_0 > 0$ there is a finite time $t_* : t_0 - (\frac{1}{H_0}) < t_* < t_0$, such that $a(t) \to 0$ as $t \to t_*$; the Universe starts at a spacetime singularity there, with $\mu \to \infty$ and $T \to \infty$ if $\mu + p > 0$.*

This state of singularity, famously known as the *Big Bang*, is generally taken to be not only the start of matter and spacetime, but of physics itself. In the *standard Big Bang model*, we have an FLRW universe with a positive cosmological constant and cold dark matter.

We close this subsection with another interesting theorem regarding FLRW universes [111, 158, 285]:

Ehlers-Geren-Sachs Theorem. *If, in a given universe, all freely falling observers measure the cosmic background radiation to have exactly the same properties in all directions (that is, they measure the background radiation to be isotropic), then that universe is an isotropic and homogeneous FLRW spacetime.*

Proof. Let us prove [111] the theorem using the kinetic theory model of matter with spacetime containing a system of particles all having (for simplicity) the same proper mass $m \geq 0$. The metric g_{ab} in (2.8) (without the Λ term) will thus be thought of as the macroscopic gravitational potential generated by all the particles, each particle moving as a test particle in this average field except during collisions.

Two cases are necessary for the proof.

- CASE A: collisions are neglected ;
- CASE B: collisional equilibrium (detailed balance) is assumed.

Assuming $f(x,p)$ to be the one-particle distribution function defined on the $7-D$ manifold of pairs (x,p) with x a spacetime point and p its tangent vector such that $p^2 = -m^2$. Then the function f determines the EMT:

$$T_{ab}(x) = \int_{P_m(x)} p_a p_b f(x,p) dP_m \ , \tag{2.49}$$

where $P_m(x)$ denotes the mass hyperboloid $p^2 = -m^2$ in the tangent space of the spacetime at x and dP_m denotes its Lorentz-invariant measure. Both cases A and B above satisfy the Liouville condition [292]:

$$f[x(s), p(s)] = \text{constant along timelike } (m > 0) \text{ or lightlike } (m = 0) \text{ geodesic } \{x(s), p(s)\} \ . \tag{2.50}$$

Considering isotropic solutions for (2.8), (2.49),(2.50) enables us to define a timelike unit vector $u^a(x)$ such that f takes the form

$$f(x,p) = h(x, -u(x).p) \ , \tag{2.51}$$

h being some real-valued function. Then the isotropy of f with respect to u^a can be shown [111] to imply that the EMT takes the form (2.21) and the distribution functiont f should be of the form

$$f(x,p) = \frac{1}{4\pi} g(a(t)^2 p^2) \ , \tag{2.52}$$

where $p^2 = h_{ab}p^a p^b$ defines the 3-momentum of a particle in the frame of u^a and g is a real-valued function, and that the resulting metric should be of the form (2.15).

□

An extension to the theorem (*almost EGS*) has been shown to hold to perturbed models, i.e., *almost isotropy* of the CMB implies *almost homogeneity and isotropy* of the Universe [285]. We will see in the next chapter that this theorem has also been extended to $f(R)$ theories of gravity.

2.2.3 Cosmography

Lack of accurate cosmological data and the fact that the Universe is expanding makes cosmography, the science of measuring the 'distance' between two observed cosmological objects or events, one of the most challenging subfields in cosmology [4, 88, 166]. The expansion of the Universe causes the comoving distances between any two objects to constantly change. We often use such directly observable quantities as the luminosity of a quasar, the redshift of a galaxy, or the angular size of the CMB power spectrum acoustic peaks to indirectly measure another quantity not directly observable, but mathematically calculable, such as the comoving coordinates of the quasar or the galaxy.

In this subsection we will discuss some of the most basic concepts related to distances and ages in cosmology in general and in the FLRW context in particular.

Cosmological distance measures (and ages) largely exploit the fact that light travels on radial null geodesics $x^a(\lambda)$ in spacetime [121, 166] with tangent vectors

$$k^a = \frac{dx^a}{d\lambda} \ , \tag{2.53}$$

satisfying the (null) conditions

$$k^a{}_{;b}k^b = 0, \ \ k^a k_a = 0. \tag{2.54}$$

If a photon is emitted at a time t_e , the comoving radial distance $r(t_e, t_o)$ it travels before it is received by an observer (receiver) at a later time t_0 is given, according to the metric defined by Eqn (2.15),

$$ds^2 = 0, \tag{2.55}$$

with

$$d\theta = 0 = d\phi, \tag{2.56}$$

and hence

$$r(t_e, t_o) = \int_{t_e}^{t_o} \frac{dt}{a} = \int_{a_e}^{a_o} \frac{da}{a\dot{a}}. \tag{2.57}$$

One of the most important parameters in cosmography is the Hubble constant H_0. This is because the age and the size of the observable region of the Universe scale with the value of the present-day rate of expnsion, as will be indicated shortly. The proportionality between the recession speed v and the distance D between two cosmological objects in an expanding universe is given by Eqn (2.20) with the exact value of H_0 yet to be determined beyond any reasonable doubt. The most recent measurement of the local Hubble parameter performed by considering recession velocities of objects around us is [207, 261]

$$H_0 = 73 \pm 2.4\text{km}s^{-1}Mpc^{-1} , \tag{2.58}$$

whereas the 9-year WMAP analysis gives [162, 207]

$$H_0 = 68.65 \pm 0.93\text{km}s^{-1}Mpc^{-1} , \tag{2.59}$$

and Planck's most recent results [251] give a lower bound of

$$H_0 = 67.3 \pm 1.2\text{km}s^{-1}Mpc^{-1} . \tag{2.60}$$

The corresponding Hubble time given by $\tau_H = \frac{1}{H_0}$ is the time taken for light to traverse a Hubble distance

$$D_H = \frac{c}{H_0} . \tag{2.61}$$

Thus these quantities set the scale of the Universe with the quantities $\tau_H \simeq 1.2 - 1.5 \times 10^{10}$ years, $D_H \simeq 1.2 - 1.5 \times 10^{26}\text{m} \simeq 3700 - 4700\text{Mpc}$ usually normalized to the geometric units $\tau_H = D_H = c = 1$.

Redshift

One of the most prevalent concepts and measures in cosmology and astrophysics, the redshift z of an object emitting a wavelength λ_e (and a frequency ν_e) and observed with wavelength λ_o (and a corresponding frequency ν_o) is defined to be the fractional Doppler shift of its emitted light (photons) due to its radial motion [121, 166]:

$$z \equiv \frac{\lambda_o}{\lambda_e} - 1 = \frac{\nu_e}{\nu_o} - 1. \tag{2.62}$$

In general, the redshift of a moving object in an expanding universe can be given by

$$1 + z = (1 + z_c)(1 + z_v), \tag{2.63}$$

where z_v is the redshift due to the local peculiar motion of the object whereas z_c is the cosmological redshift due to the expansion of the Universe given in terms of the scale factor as

$$1 + z_c = \frac{a(t_0)}{a(t_e)}. \tag{2.64}$$

For comoving objects, we have $z_v = 0$ from which follows $z_c = z$.

Distance Types

There are a number of confusing, and at times overlapping, concepts of distance measures in cosmology. Here we list some of the most commonly used ones in everyday cosmological applications.

The *proper distance* D_p is defined by [97]:

$$D_p = \int dt = \int_0^z \frac{dz'}{(1+z')H(z')} = \frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')h(z')}\,, \tag{2.65}$$

where $h(z) = H(z)/H0$ is an expansion parameter normalized by the Hubble constant. In the FLRW model with Λ, we have

$$h(z) = \sqrt{\Omega_r(1+z)^4 + \Omega_d(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}\,. \tag{2.66}$$

The *line-of-sight comoving distance* D_c between two nearby objects in the Universe is the constant distance between them at any epoch if the two objects are moving with the Hubble flow, and is defined by [97, 166]

$$D_c = \int \frac{cdt}{a} = D_H \int_0^z \frac{dz'}{h(z')}. \tag{2.67}$$

The *transverse comoving distance* D_M measures the distance between two events at the same redshift but separated by a certain angle in the sky and is mathematically defined by

$$D_M = \begin{cases} D_H \frac{1}{\Omega_k} \sinh(\sqrt{\Omega_k} D_c/D_H) & \text{for } \Omega_k > 0, \\ D_c & \text{for } \Omega_k = 0, \\ D_H \frac{1}{|\Omega_k|} \sinh(\sqrt{|\Omega_k|} D_c/D_H) & \text{for } \Omega_k < 0. \end{cases}$$

The *angular diameter distance* D_A is one of the most important and widely used distances in cosmology. It defines the ratio of an object's transverse physical size to its radian angular size; that is to say:

$$D_A = \frac{\ell}{\Delta\theta}. \tag{2.68}$$

There exists an interesting relationship between the angular diameter distance and the comoving distance given by

$$D_M = (1+z)D_A. \tag{2.69}$$

The *luminosity distance* D_L, defined as

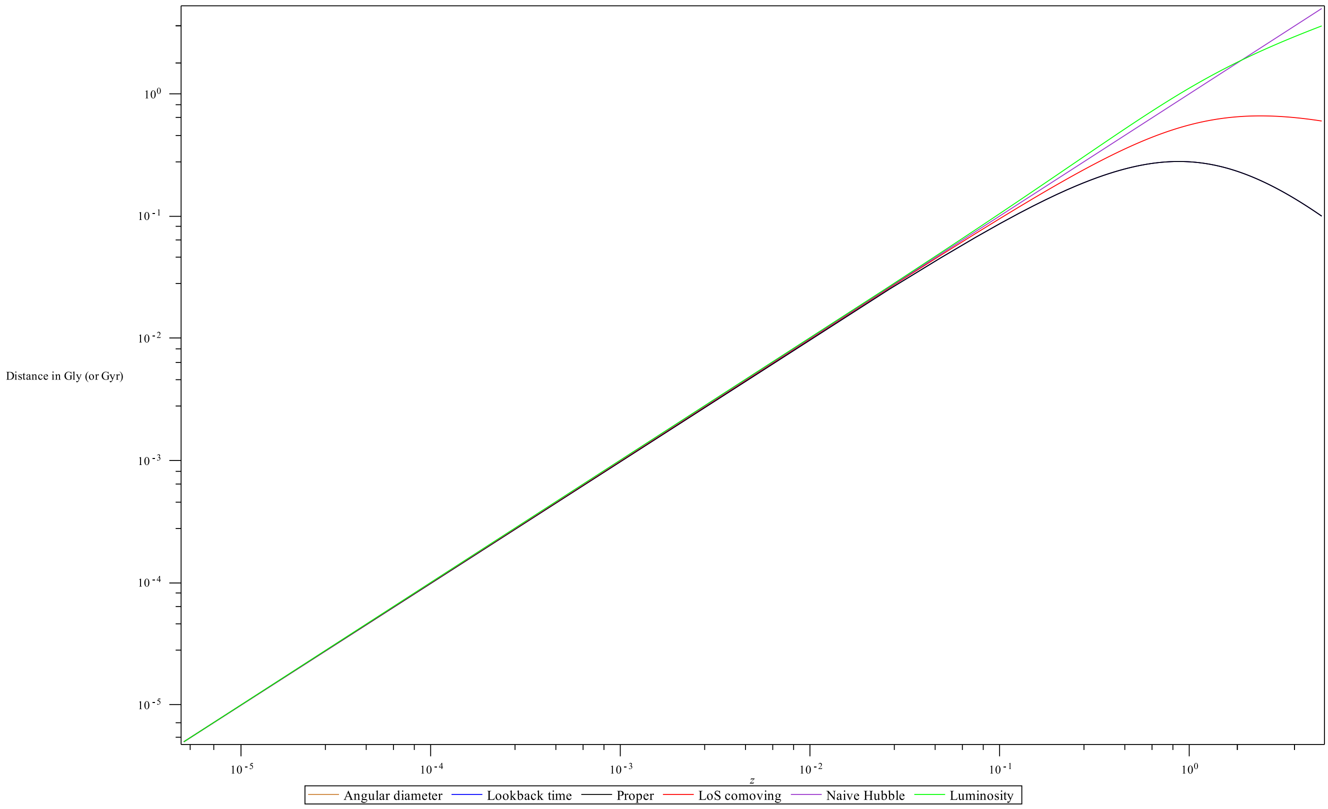

Figure 2.1: Distance measures of the nearby observable universe in the ΛCDM cosmology for $\Omega_{d0} = 0.3175, \Omega_{r0} \simeq 9.266. \times 10^{-5}, \Omega_{\Lambda 0} \simeq 0.6824, \Omega_{k0} \simeq 0.0000$. Geometrized units have been used with $\tilde{h} = 0.0100$. Note that the angular diameter distance and the light travel distance (or the lookback time) coincide in this plot.

$$D_L \equiv \sqrt{\frac{L}{4\pi f}}, \tag{2.70}$$

relates two bolometric quantities, the *luminosity* L and the *flux* f of a distant object such as a supernova. It is one of the most commonly used distance in cosmology.

Here also there is a simple mathematical relation between this quantity and the comoving and angular diameter distances:

$$D_L = (1+z)D_M = (1+z)^2 D_A. \tag{2.71}$$

Sometimes we might be interested in predicting the evolutionary properties of objects at high redshift. This can be achieved by taking the difference between the present age of the Universe t_0 and the age t_e of the Universe when the photons were emitted as measured by a hypothetical observer attached to the object. This difference in time gives us what is known as the *lookback time* t_L of the object. Mathematically this is given as

$$t_L = t_H \int_0^z \frac{dz'}{(1+z')h(z')}. \tag{2.72}$$

The *light travel distance LTD* is the distance the emitted photons travel during the lookback time:

$$LTD = ct_L. \tag{2.73}$$

Horizons in Cosmology

The finite speed of light has serious consequences in physics. It restricts causal relationships and due to the fact that causal effects cannot propagate faster than light, the features of cosmological structure formation and our observational knowledge of the cosmos are highly constrained [121, 125]. We can only influence or be influenced by regions inside our past null cone; in short, there are regions of the Universe beyond which we have no access. The boundary separating the accessible part of the Universe from the inaccessible is called a *horizon*. There two main types of horizons in cosmology.

The *particle horizon*

$$\chi_{ph} = \int_0^{t_0} \frac{dt'}{a(t')} \tag{2.74}$$

is the maximum distance particles could move to an observer during the Universe's period of existence. In other words, this is the largest region of spacetime we could have probed so far. The physical distance to the matter comprising this horizon is

$$D_{ph} = a(t_0)\chi_{ph}. \tag{2.75}$$

This horizon exists for all FLRW spacetimes for all matter (dust) and radiation, since (2.74) converges in those cases. This horizon always grows, and once matter enters the horizon, it never leaves [121]. Apart from limiting causality (and hence cosmic structure), the particle horizon can also set limits on what is testable in the Universe. It has been found that in a perturbed FLRW Universe, once a causal contact has taken place, it remains there for as long as the Universe exists [121].

The *event horizon*.

$$\chi_{eh} = \int_{t_0}^{t_{\inf}} \frac{dt'}{a(t')} \tag{2.76}$$

represents the largest comoving distance that light emitted now can ever reach an observer any time in the future. It sets the maximum extent to the particle horizon, and is said to exist if the integral in (2.76) does not diverge. One common application of event horizons is in the general relativistic description of blackholes, where the escape velocity of the blackhole inside the horizon is superluminal. Light emitted from beyond the horizon can never reach the observer and time itself stops at the boundary. Within the horizon, all light-like paths, and hence all paths in the forward lightcones of particles within the horizon, are warped so as to fall farther into the hole.

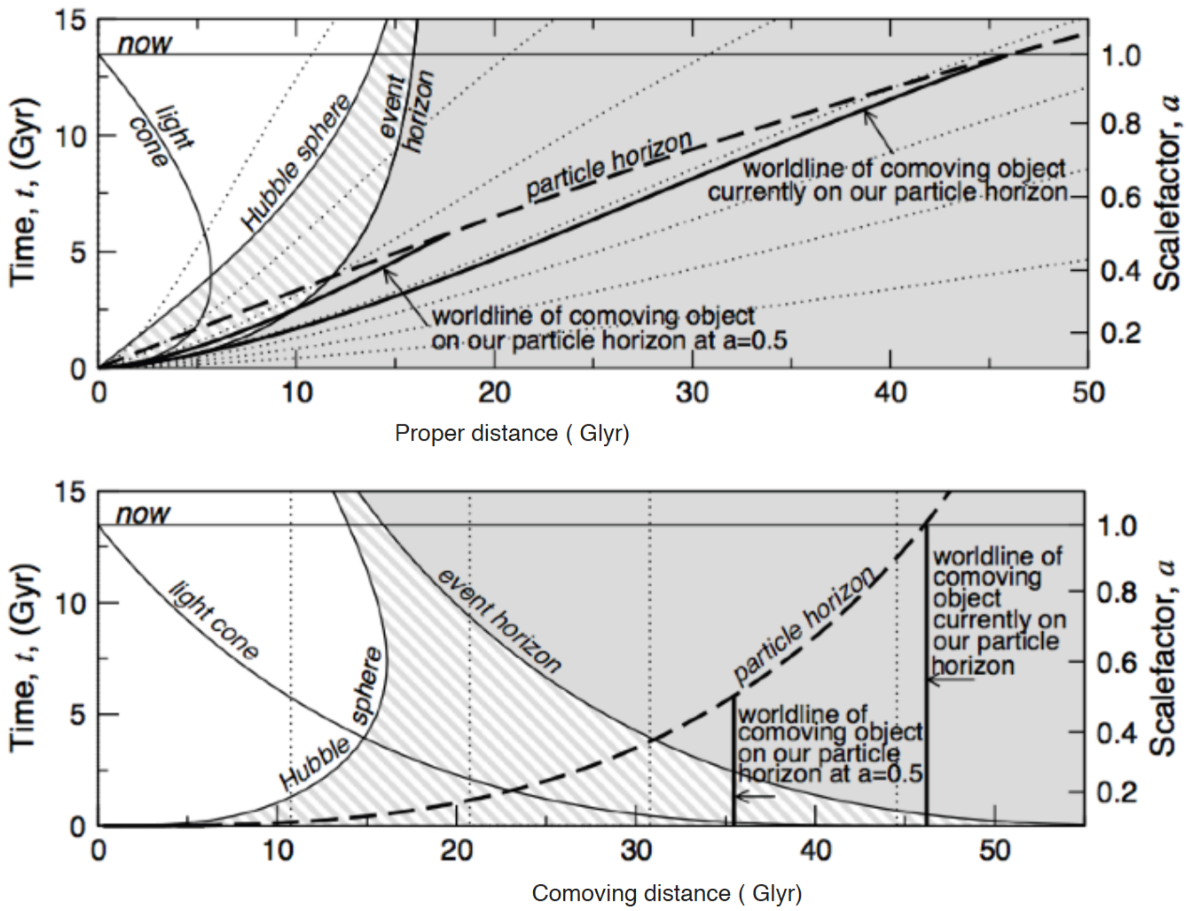

Figure 2.2: Horizons in ΛCDM cosmology [88, 250] for $\Omega_{m0} = 0.3, \Omega_{\Lambda 0} = 0.7, \tilde{h} = 0.7$. Top panel shows proper distance $D_p = a\chi_{ph}$ and bottom panel shows comoving distance $D_c = a_0\chi_{ph}$.

The *comoving Hubble radius* $\lambda_H = 1/aH$ determines the relevant physical scales for local causal influences in an expanding universe. This radius increases during any standard evolutionary history of the Universe (such as the radiation-dominated and dust-dominated epochs).

2.3 Shortcomings of the Standard Model

The Hot Big Bang model is by far the most successful cosmological paradigm. It beautifully explains key cosmic phenomena such as the expansion of the Universe, the origin of the CMB, the synthesis of light elements and the formation of galaxies and large-scale structure. But it also leaves many serious puzzles unanswered. Some of these puzzles have their origins in the early universe, whereas others emerge only in late-time cosmology. In this section, we explore some of the most serious shortcomings of the standard Big Bang cosmology [4, 121, 154, 165, 173, 197, 243, 250].

2.3.1 The Horizon Problem

This is the problem associated with different causally disconnected regions in the Universe sharing similar physical properties such as temperature.

As we have seen earlier, (2.74) converges since $a \propto t^{\frac{1}{2}}$ in the early radiation-dominated epoch of the Universe and at later times the Universe enters the dust-dominated phase, in which case [243]

$$D_{ph} \simeq \frac{6000}{\sqrt{\Omega(z)}} \tilde{h}^{-1} Mpc. \tag{2.77}$$

This implies that at last scattering the particle horizon was only ∼100 Mpc in size and subtending an angle of ∼ 1 degree in the sky. However, this is in contradiction with the large number of causally disconnected patches we see on the CMB sky, all at the same temperature.

2.3.2 The Flatness Problem

This is a problem related to the fine-tuning of the initial conditions which would otherwise have greatly affected the geometry and expansion history of the Universe. If the matter content of the Universe comprises only dust and radiation, the evolution of the curvature density parameter

$$\begin{aligned} \Omega_k &\equiv -\frac{c^2 k}{a^2 H^2} = 1 - \Omega_m - \Omega_\Lambda = \left[\frac{(1+z)H_0}{H(z)}\right]^2 \\ &= \frac{\Omega_{k0}}{(1+z)\Omega_{d0} + (1+z)^2\Omega_{r0} + (1+z)^{-2}\Omega_{\Lambda 0} + \Omega_{k0}}, \end{aligned} \tag{2.78}$$

can be shown to be [165]

$$\dot{\Omega}_k = 2\Omega_k H q = \Omega_k H(\Omega_d + 2\Omega_r - 2\Omega_\Lambda). \tag{2.79}$$

In the absence of a cosmological constant, we see that $\Omega_d + 2\Omega_r > 0$. If at high enough redshift (i.e., at an early cosmic time) Ω_k were slightly different from zero, then the spatial curvature would rapidly evolve away from the spatially flat case, i.e., Ω_k would either approach 1 in the open case or diverge to $-\infty$ in the closed universe case. But a positive Λ term would dominate the dust and radiation terms at some finite cosmic time such that Ω_k is *fine-tuned* to 0. The above relations show that no matter how much $\Omega_{k,0}$ is different from zero, Ω_k at high z could not have been significantly far from zero, the deviation being only 1 in 10^{60} at the Planck epoch, for example. This is a significant fine-tuning that the Hot Big Bang model alone does not address.

2.3.3 The Structure Problem

Although the Universe on the largest scales is assumed to be homogeneous, the homogeneity is not absolute. The real universe is clumpy. The question that the *structure* *problem* poses is then: what brought about the matter clumping that finally led to cosmic structures like galaxies and clusters? It is generally believed that inhomogeneities must have existed in the primordial matter to account for the structures we observe today. According to perturbation theory, any small inhomogeneities in the primordial matter rapidly grow into large ones through gravitational self-interaction. This implies that an extreme smoothness in the primordial matter must have existed for these inhomogeneities of galactic scale to exist at present. If we extrapolate further to 10^{-45}s after the Big Bang, then an almost perfect smoothness, but not quite an absolute smoothness, must be assumed to have existed. Why the primordial matter must have been so smooth is not accounted for in the standard Hot Big Bang (FLRW) model. The structure problem is also known as the *smoothness problem* or the *homogeneity problem* [173, 243].

2.3.4 The Anti-matter Problem

This problem is as follows [243]: at high enough temperatures $kT \geq m_p c^2$, where m_p is the mass of a proton, there are roughly equal numbers of photons (γ), protons (p) and antiprotons ($\bar{p}$) in equilibrium, whereas the ratios today stand at $N_p/N_\gamma \sim 10^{-9}$ and $N_{\bar{p}}/N_p \sim 0$. Since baryon number is a conserved quantity, it would then necessarily imply that $N_p/N_{\bar{p}} = 1 + O(10^{-9})$ during baryogenesis. In the Standard Model, the source of this initial asymmetry is still unknown, and is referred to as the anti-matter problem or the *baryon asymmetry problem*.

2.3.5 The Magnetic Monopole Problem

At very high energies ($\sim 10^{16} GeV$), local symmetries under grand unified theoretic symmetry groups spontaneously break to the gauge symmetry of the Standard Model (of particle physics) under the group $SU(3) \times SU(2) \times U(1)$. An implication of this is that if the Universe were very hot at early times, which it was in the Hot Big Bang model, a large number of heavy, stable magnetic monopoles would be produced, and should be detected observationally. But no experiment or observation has detected a magnetic monopole to this day, a problem known in cosmology as the magnetic-monopole problem or the *exotic relics problem*.

2.3.6 Dark Matter and Dark Energy

Dark matter and dark energy collectively account for the dark side of the Universe, i.e., the part of the Universe that is not in the luminous (baryonic) form.

Recent advances in precision cosmology have enabled us to determine the energy and matter content of the Universe, using techniques such as measurement of the temperature fluctuations in the CMB, distance-luminosity relations analyses in supernovae, and analyses of large scale structure and Big Bang nucleosynthesis. If the standard Big Bang model is the correct gravitational theory of cosmology, then the matter-energy contents of the Universe comprise the following [30, 85, 139, 181, 251, 280]:

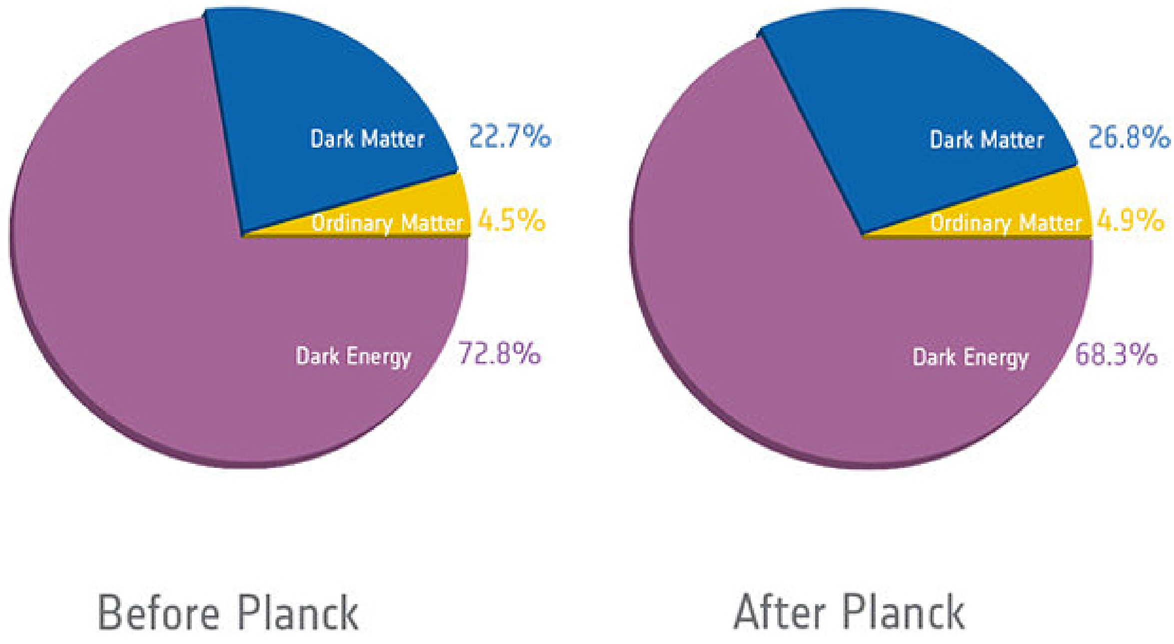

Figure 2.3: The energy content of the Universe then and now. Credit: Google Images/arstechnica.com.

- $\Omega_m \sim 0.315$, i.e., $\sim 31.5\%$ of the total energy of the Universe exists in the form of non-relativistic matter, of which only a tiny fraction ($\Omega_b \sim 0.049$) is known to exist in baryonic matter form, whereas the remaining ($\sim 85\%$ of) matter is not as yet properly understood and hence is thought to exist in the form of *dark matter*. It was first postulated by Oort in 1932 to account for the orbital velocities of stars in the Milky Way and Zwicky [311] in the following year to account for his observation of evidence of missing mass in the orbital velocities of galaxies in clusters.

  This missing matter is believed to exist in two forms: non-relativistic Cold Dark Matter ($\Omega_{CDM} \sim 0.268$) and non-relativistic hot dark matter ($\Omega_{HDM} < 0.0152$). Although no direct observation has been made (in which case it would no longer be called *dark* matter), its indirect gravitational effects on visible matter, radiation and the large scale structure of the Universe are becoming more and more convincing evidences of its existence.

- $\Omega_\Lambda \sim 0.685$, i.e., by far the largest portion ($\sim 68.5\%$) is believed to exist in an exotic, unclustered and invisible form of cosmic stuff that permeates all of space, and is known as *dark energy*. Another unknown in the cosmic pie, it was discovered when, in 1998, observational data from supernovae hinted at an accelerating expansion of the present Universe. According to (2.27) and

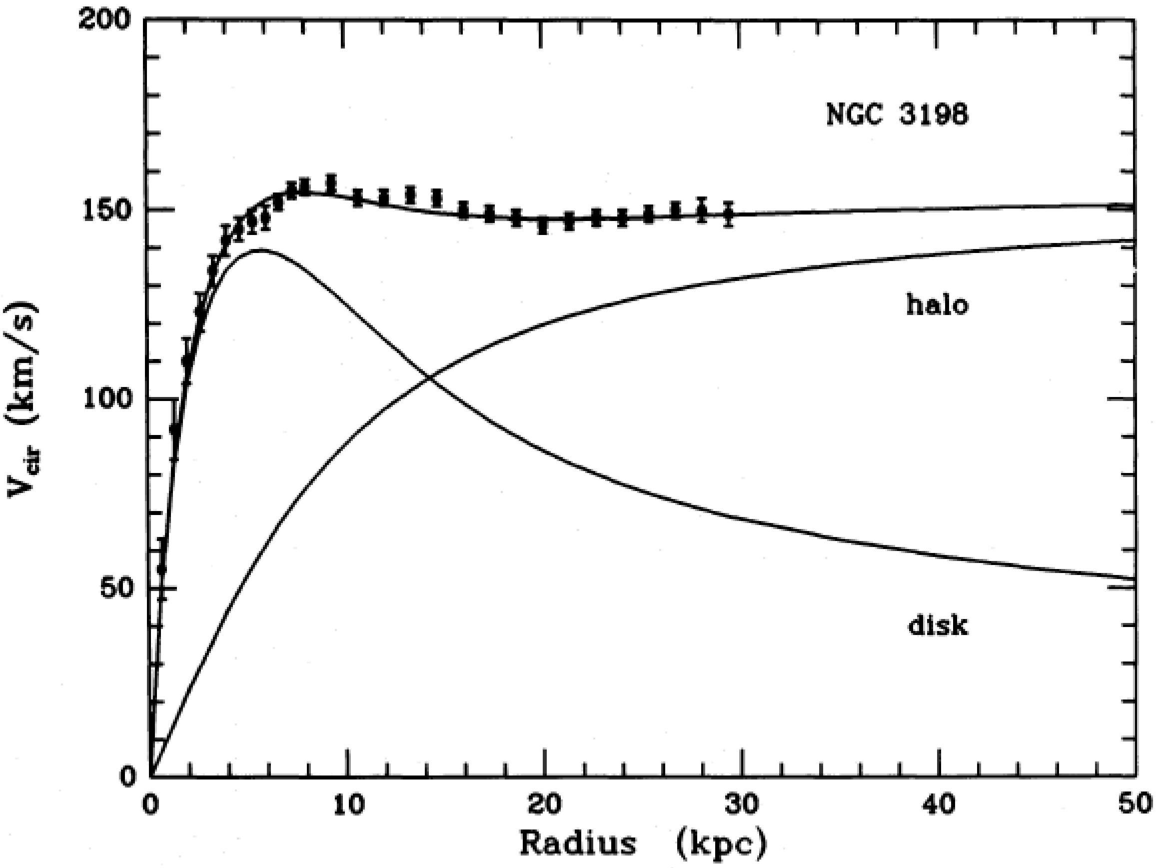

Figure 2.4: Rotation curves of a typical spiral galaxy (NGC 3198): predicted (disk and extended dark matter halo models) and observed (with error bars) [7]. The best fit is obtained for an exponential disk model with maximum mass. Figure reproduced by permission of the AAS.

(2.30), this would be possible if the Universe is dominated by a component "fluid" with negative pressure. Ironically, this called for the reinstatement of the cosmological constant Λ as the new component fluid with $w_\Lambda = -1$. A cosmological constant dominating at late times can cause cosmic acceleration and make the Universe enter an irreversible de Sitter phase, but this model has two long-standing problems of its own [306], viz.:

- the cosmological constant problem,
- the coincidence problem.

The cosmological constant problem refers to the huge ($\sim 10^{120}$ orders of magnitude) discrepancy between the "observed" value of Λ responsible for cosmic acceleration and that predicted by quantum field theoretic arguments for the energy of the quantum vacuum at Planckian scales

The total fractional energy density is close to 1.0 (or is precisely 1.0 if the Universe is taken to be perfectly flat) at the present time when we are here to observe it, after about 13.82 billion years of expansion when it was always greater than 1.0. Since Ω_Λ is the only constant component, it is natural to be curious about why Λ is so finely tuned as to be dominant only now. This is the essence of the so called coincidence problem. One rather philosophical solution to this is the *Anthropic Principle*: we see the Universe the way it is because we exist [24, 157].

Dark energy is probably the most extensively speculated candidate in recent years, but it is merely a phenomenological addendum whose existence has not been predicted from the Big Bang/inflationary cosmology. Its dynamical nature is hardly understood, and none of the variant models put forward (such as the *decaying* Λ, *quintessence*, *k-essence*, etc.) have been convincingly viable so far.

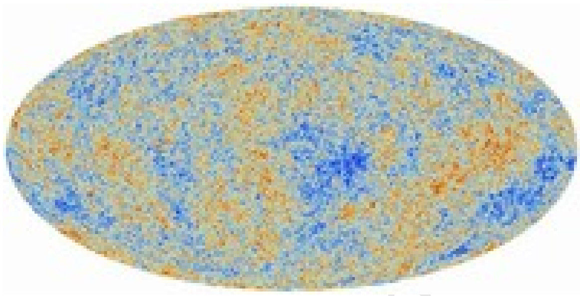

Figure 2.5: The Cosmic Microwave Background Radiation. Credit: ESA (2013).

2.4 The Way Out

The cosmological problems discussed above fall under two broad categories: those of, or originating in, the early universe (see sections 2.3.1, 2.3.2, 2.3.5) and those of late-time cosmology (see 2.3.6). In this section we explore some of the best explanations put forward as improvements or alternatives to the standard general relativistic Big Bang model.

2.4.1 Solution by Inflation

To tackle problems of the first kind, i.e., the horizon, flatness, structure and magnetic monopole problems, a special epoch of exponential expansion, known as *inflation*, was proposed in the early 1980s [154]. This epoch is characterized by a decreasing comoving Hubble radius, i.e.,

$$\frac{d(1/aH)}{dt} < 0 \Leftrightarrow \ddot{a} > 0. \tag{2.80}$$

This means that the Hubble radius would be swallowed by the outpacing accelerated growth of the expansion. As a consequence, all physical conditions become correlated on scales much larger than the Hubble radius [154, 194], thus *smoothing* out the primordial matter fluctuations along the way [4, 26, 184, 193, 198, 250].

The inflationary theory of cosmic evolution starts by assuming that at some point in the early universe, the matter energy density was dominated by some form of matter, called a *scalar field* ϕ with a negative pressure. In the absence of the cosmological constant, the Raychaudhuri equation (2.27) reduces to

$$\ddot{a} = -\frac{1}{6}(\mu + 3p)a \tag{2.81}$$

which means that

$$p < -\frac{1}{3}\mu \tag{2.82}$$

in order for inflation to occur. Thus the Friedmann equation (2.30) reads

$$\frac{\dot{a}^2}{a^2} = \frac{1}{3}\mu - \frac{k}{a^2}. \tag{2.83}$$

Since the scale factor must increase faster than $a(t) \propto t$, the curvature term becomes negligible.

The Lagrangian for the scalar field is given by

$$\mathcal{L} = \frac{1}{2}g^{ab}(\partial_a\phi)(\partial_b\phi) - V(\phi)\ . \tag{2.84}$$

This gives the equation of motion for the scalar field (the Euler-Lagrange equation) as:

$$\Box^2\phi + \frac{dV}{d\phi} = 0, \tag{2.85}$$

where $\Box^2 \equiv \nabla^a\nabla_a = g^{ab}\nabla_a\nabla_b$ is the covariant d'Alembertian operator.

Treating ϕ as a perfect fluid with negligible spatial variations, the EMT can be shown to be of the form

$$T_{ab} = (\partial_a\phi)(\partial_b\phi) - g_{ab}\left[\frac{1}{2}(\partial_c\phi)(\partial^c\phi) - V(\phi)\right]\ . \tag{2.86}$$

The energy density and the pressure associated with the scalar field can then be defined by

$$\mu_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi),$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2.87}$$

Rewriting the equation of motion for the scalar field Eqn (2.85), one can show that the Klein-Gordon equation becomes

$$\dot{\mu}_\phi + 3(\mu_\phi + p_\phi)H = 0 \Rightarrow \ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{2.88}$$

This, coupled with the Friedmann equation in the scalar field-dominated universe

$$H^2 = \frac{1}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] , \tag{2.89}$$

completely describes the evolution of the scalar field and the expansion during this epoch. The condition (2.82) and (2.87) will constrain ϕ and V such that for inflation to occur $\dot{\phi}^2 < V(\phi)$.

If one makes the further approximation, called the *slow-roll approximation*

$$\dot{\phi}^2 \ll V(\phi) , \tag{2.90}$$

then Eqns (2.88) and (2.89) become analytically solvable. Based on this approximation, we can rewrite the inflation equations to obtain the following relations:

$$3H\dot{\phi} = -\frac{dV}{d\phi}, \tag{2.91}$$

$$H^2 = \frac{1}{3}V(\phi), \tag{2.92}$$

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2. \tag{2.93}$$

Equation (2.93) is not an independent equation on its own, but tells us how the rates of change of the Hubble parameter and the scalar field are related. These equations further tell us that the scale factor grows exponentially, keeping the Hubble parameter *constant*:

$$a(t) \propto e^{Ht} \Rightarrow a(t) \propto e^{\sqrt{\frac{1}{3}V(\phi)}t}. \tag{2.94}$$

If the Universe followed the standard radiation-dominated epoch after inflation, then

$$a(t) \propto t^{\frac{1}{2}} \propto \frac{1}{T}, \tag{2.95}$$

where T is a measure of the typical particle energy: $E \sim k_B T$. This means that

$$\frac{a_{inf}}{a_0} \sim \left(\frac{t_{inf}}{t_0}\right)^{\frac{1}{2}} \sim \frac{T_0}{T_{inf}}. \tag{2.96}$$

CMB analyses show that at present $T_0 \sim 3$K and we have seen earlier that $t_0 \sim \frac{1}{H_0} \sim 10^{18}$s. Using (2.78) we can write

$$\frac{\Omega_{k,inf}}{\Omega_{k0}} = \left(\frac{H_0}{H_{inf}}\right)^2 \left(\frac{a_0}{a_{inf}}\right)^2 \sim \frac{t_{inf}}{t_0}. \tag{2.97}$$

Inflation is generally thought to have occurred somewhere between the Planck era and the period of GUT phase transition, which means that according to particle physics, the ratio (2.97) lies somewhere between 10^{-60} and 10^{-54}. Thus it is possible that that an extreme fine-tuning could, in principle, have occurred for Ω_k to attain its present value ($-0.5 < \Omega_{k0} < 0.5$). The implication of this is that the scale factor could have grown by a factor of about $10^{27} - 10^{30}$ or about 60-70 e-foldings during inflation, thus solving the flatness problem.

Once again, if the Universe underwent a radiation-dominated like expansion during its earliest stages, the particle horizon at inflation would be

$$D_{ph,inf} = 2ct_{inf}, \tag{2.98}$$

resulting in a size of a causally connected region about $10^{-33} - 10^{-27}$m across, in sharp contrast to the $10^{-3}m - 1m$ wide region obtainable from (2.97) [165]. This is in accordordance with the previous discussion that the scale factor must have grown by about 60-70 e-foldings, and hence solves the horizon problem.

The supercooling of the Universe occurs at the inflationary phase transition and this suppresses the production of magnetic monopoles.

Inflation is an extremely short period scenario ($\sim 10^{-36}s - \sim 10^{-32}s$) after the Big Bang and it ends when the scalar field is converted into radiation, a process called *reheating*. It is this reheating that lead the Universe to the Hot Big Bang epoch.

2.4.2 Inhomogeneous Cosmological Models and Backreaction

We have seen in the previous sub-section that the inflationary scenario only solves a subset of the shortcomings of the Standard Model. Although inflation solves a number of problems, it also has pressing issues that need to be addressed, such as its origin and ending mechanisms. We have also discussed in previous sections that modern cosmology is built upon a universe where more than 95% of its matter/energy content is yet to be known. The FLRW Big Bang scenario by itself is a problem, as symmetry (and hence all the laws of physics) break at the Big Bang: no one knows what happened exactly at the Big Bang and "before". Nevertheless, the Standard Model is by far the best fit to nearly all cosmological data and will most likely remain so until a rigorous scrutiny comes up with a brand new cosmological/gravitational theory. This consensus picture of Cosmology is widely known as the *Concordance Model* or *ΛCDM Cosmology*.

Our interpretation of cosmic acceleration from observations of the dimming of distant supernovae is only valid if we assume spatial homogeneity and isotropy on large scales [117, 137, 185, 257]. This assumption, largely motivated by the observed near isotropy of the CMB and the Cosmological/Copernican Principle[5], is being challenged

[5]The Copernican Principle is the underlying cosmosophical assumption that humans are not privileged observers of the Universe, and hence that neither the Earth nor the Milky Way occupies a special place in the Universe. Some authors use the *Cosmological Principle* as a generalization of the Copernican Principle. It states: when viewed on *sufficiently large* scales, matter is homogeneously and isotropically distributed.

as suspicions grow that the FLRW geometry may not be the right geometry for all scales [75]. An implication of the Cosmological Principle is that on large enough scales, FLRW assumes an implicit averaging of the nonlinear EFEs to smooth out (and hence ignore the dynamical effects of) the small-scale inhomogeneities [119].

But what if the Cosmological Principle is not true? Well, then the Universe will be inherently inhomogeneous and only inhomogeneous cosmological models, such as the Lemaître-Tolman-Bondi (LTB), Szekeres and Kantowski-Sachs models can best describe it [143, 176, 269, 289, 290, 295] (also see [183] for detailed historical overview of these and more inhomogeneous models).

On the other hand, if we treat inhomogeneities around a FLRW background and the cosmological dynamics is described by backreaction from averaging the nonlinear EFEs, we run into the problem of finding an appropriate averaging procedure due to the absence of a fixed background spacetime and the nonlinearity of the field equations in GR. This means that the averaged field equations

$$\langle G_{ab} + \Lambda g_{ab} \rangle \neq \langle T_{ab} \rangle \ , \tag{2.99}$$

and taking derivatives on functions Φ do not, in general, commute :

$$\partial_i \langle \Phi \rangle \neq \langle \partial_i \Phi \rangle \ , \tag{2.100}$$

where the angular brackets here indicate averaging over a certain scale.

Recently, there have been a number of attempts to develop averaging procedures for scalars and tensors [47, 185, 242, 288, 308]. In the *Buchert formalism* of scalar averaging [47], scalar quantities are considered volume-averaged over the domain of inhomogeneity:

$$\langle \Phi \rangle \equiv \frac{1}{V_D} \int_D \Phi J d^3 X \ , \quad J \equiv \sqrt{det(g_{ij})}, i, j = 1, 2, 3 \tag{2.101}$$

where the volume of the domain is given by

$$V_D \equiv \int_D J d^3 X \tag{2.102}$$

and a dimensionless scale factor is defined in terms of the the expanding volume normalized by the volume of the initial domain V_{D0} as

$$a_D(t) = \sqrt[3]{\frac{V_D}{V_{D0}}} \ . \tag{2.103}$$

Spatial averaging and temporal evolution do not commute, and the relation between these two operations can be shown to be

$$\partial_t \langle \Phi \rangle_D - \langle \partial_t \Phi \rangle_D = \langle \Phi \Theta \rangle_D - \langle \Phi \rangle_D \langle \Theta \rangle_D \ , \tag{2.104}$$

where $\langle \Theta \rangle_D = 3 \frac{\partial_t a_D}{a_D}$. In this formalism, the averaged Friedmann and Raychaudhuri equations are given by

$$3\left(\frac{\dot{a}_D}{a_D}\right)^2 = \mu_D + \Lambda - \frac{1}{2}\left(Q_D + \langle\tilde{R}\rangle\right) , \tag{2.105}$$

$$3\left(\frac{\ddot{a}_D}{a_D}\right) = -\frac{1}{2}\langle\mu\rangle_D + \Lambda + Q_D , \tag{2.106}$$

where the back reaction term Q_D is defined as

$$Q_D \equiv 2\langle II\rangle_D - \frac{2}{3}\langle I\rangle_D^2 , \quad II \equiv \frac{1}{3}\Theta^2 - \sigma^2, \quad I \equiv \Theta . \tag{2.107}$$

The σ^2 term is the shear scalar that describes the distortion of fluid flow. From the addition of the backreaction Q_D in Eqns (2.105) and (2.106), we observe that one can, at least in principle, obtain acceleration terms from the averaging process. Whether there can actually be enough backreaction to drive the recently observed cosmic acceleration is an issue of debate to date, but can be settled through future observational probes [186]. Another persisting issue of averaged field equations is the issue of closure: the cosmological evolution equations do not form a closed system of equations because of the noncommutativity of spatial averaging and temporal evolution.

2.4.3 Modifying the Gravitational Action

Perhaps one of the hottest topics in contemporary Cosmology in light of the cosmological dark sector is the search for a new (*modified* or *alternative*) gravitational physics. Historically, there have been several proposed "modified" gravity models most of which fall under the following broad categories [52, 78, 90, 277].

Theories of Gravity with Extra Fields

One can think of GR as a 4-dimensional theory of gravity where matter in the form of a single rank-2 tensor field (or a massless spin-2 particle) is coupled with gravity. Thus the generalized alternative of such a matter-mediated gravitational force theory would be to include extra scalar, vector, tensor or higher-rank fields in a way that the effect of the added fields be suppressed on scales where GR is highly constrained. The following are considered to be gravitational theories with additional fields:

- Scalar-Tensor theories
    - * Brans-Dicke theories (BD)
- Einstein-Æther theories
    - * Modified Newtonian Dynamics (MOND)
- Bimetric theories
- Tensor-Vector-Scalar theories (TeVeS).

Higher-dimensional Theories of Gravity

Einstein's formulation of GR intrinsically assumes space and time to form part of a curved $3+1$-dimensional Riemannian manifold. Over the years, however, there have been several higher-dimensional gravitational theories proposed as alternatives to GR:

- Kaluza-Klein (KK) theories
- Braneworld models
- Randall-Sundrum (RS) models
- Dvali-Gabadadze-Porrati (DGP) gravity
- (Einstein)Gauß-Bonnet (GB) gravity.

Table 2.1: Summary of Gravitaional Models and their Lagrangians:

Model	Action	Matter Lagrangian
GR	$\frac{1}{2}\int d^4x\sqrt{-g}\,(R-2\Lambda)$	$\mathcal{L}_m(g_{ab},\psi)$
BD	$\frac{1}{2}\int d^4x\sqrt{-g}\left(\phi R-\frac{\omega\partial_a\phi\partial^a\phi}{\phi}\right)$	$\mathcal{L}_m(g_{ab},\psi)$
MOND	$\frac{1}{2}\int d^4x\sqrt{-g}\left(R+2\mathcal{L}(g^{ab},A^b)\right)$	$\mathcal{L}_m(g^{ab},\psi)$
TeVeS	$\frac{1}{2}\int d^4x\,(\mathcal{L}_g+\mathcal{L}_s+\mathcal{L}_v)$	$\mathcal{L}_m(g_{ab},\psi)$
$f(R)$	$\frac{1}{2}\int d^4x\sqrt{-g}f(R)$	$\mathcal{L}_m(g_{ab},\psi)$ metric $\mathcal{L}_m(g_{ab},\psi)$ Palatini $\mathcal{L}_m(g_{ab},{\Gamma^a}_{bc},\psi)$ metric-affine
DGP	$\frac{1}{2\kappa_{(5)}}\int d^4x\sqrt{-\tilde{g}}\tilde{R}+\frac{1}{2}\int d^4x\sqrt{-g}R$	$\mathcal{L}_m^{brane}(g_{ab},\psi)$
GB	$\int d^Dx\sqrt{-g}\left(R^2-4R^{ab}R_{ab}+R^{abcd}R_{abcd}\right)$	$\mathcal{L}_m(g_{ab},\psi)$
Galileons	$\frac{1}{2}\int d^4x\sqrt{-g}\left(R+\sum_i^5 c_i\mathcal{L}_i\right)$	$\mathcal{L}_m(f(\pi)g_{ab},\psi_i)$
HL	$\frac{1}{2}\int dtd^Dx\sqrt{-g}\left((\dot{\Phi})^2-\frac{1}{4}\Phi(\nabla^2)^2\Phi\right)$	$\mathcal{L}_m(g_{ij},N,N_i,\psi)$
RS	$\frac{1}{2\kappa_{(5)}}\int d^5x\sqrt{-g_{(5)}}\left(R-2\Lambda_{(5)}\right)-\int d^4x\sqrt{-g}\sigma$	
KK	$\frac{1}{2\kappa_D}\int d^Dx\sqrt{-g_D}R_D$	$\mathcal{L}_m(g_{ab},\psi)$

where ψ is a matter field,Φ is an exotic scalar field, ω is the Dicke coupling constant, A^a is a spacetime 4-vector field, $N^i(x,t)$ is a shift vector, $N(t)$ is a homogeneous lapse function, π is a galleon field, σ is brane tension. $\mathcal{L}_g$, $\mathcal{L}_s$ and $\mathcal{L}_g$ are the actions for the metric (tensor), scalar and vector fields, respectively. κ_D is the bare gravitational constant in D-diemsnions. $\tilde{g}$ and $\tilde{R}$ represent the determinant of the spacetime metric in D-dimensions and its corresponding Ricci scalar.

Higher-derivative Theories of Gravity

GR is a second-order theory of gravity, i.e., the field equations of GR have at most second-order derivatives in the metric. Higher-order theories of gravity are those extended theories that allow more than second-order derivatives of the metric in the field equations. Although these kinds of generalizations do generally tend to suffer from instabilities, they are excellent candidates for attempts in renormalizing gravity. The most common of these theories include:

- Theories with Ricci and Riemann curvatures in the action
- Hořava-Lifshitz gravity
- Galileons
- $f(R)$ theories: theories with a generic function of the Ricci scalar in the action Lagrangian.

Investigations in this thesis fall under the last category; and our focus henceforth will be on $f(R)$ models of gravitation. Table 2.1 summarizes some of the most common alternative gravitational models and their *actions*.

2.5 Conclusion

For almost a century now, Einstein's General Relativity theory has stood the test of rivalry and experimentation and remains to be the best theoretical framework upon which contemporary astrophysics and cosmology are anchored. Among other things, it consistently explains the existence of blackholes and other compact objects, gravitational lensing and gravitational time dilation and redshift, as well as most of the cosmological phenomena in the observable universe, such as cosmic expansion and structure formation and the origin of the CMB and the light elements. On the other hand, two main loopholes stand out: on the theoretical side, no one knows how to reconcile GR with quantum physics. Hence a completely consistent description of quantum gravity is still off limits. On the phenomenological side, the recent discovery of cosmic acceleration, and the existence of dark energy by implication, has become the biggest puzzle in cosmology, followed by the presence of dark matter which also still remains unexplained by standard GR.

This cosmic dilemma might mean two things: either the standard model based on isotropy and homogeneity is not correct (and precision cosmology plays a big role here) or that the theory of gravity must somehow be generalized, modified or completely changed.

Chapter 3

Review of $f(R)$ Theories

> Science never solves a problem without creating ten more.
>
> George Bernard Shaw

3.1 Introduction

We have seen in Chapter (2) that despite the remarkable progress made towards a complete understanding of the Universe through GR as the theoretical foundation, observational evidences show that we need to go a long way before celebrating the completeness of our understanding.

Ever since the formulation of GR, there have been curiosities and questions about its uniqueness and completeness (e.g., [109]). For example, a study of higher-order theories of gravity has been considered since Weyl [307]. One of the most widely explored alternatives to GR in the context of the late time accelerated expansion of the Universe are Fourth Order Gravity (FOG) theories. These are a class of higher-order gravity models that attempt to address the shortcomings of GR in the infrared (IR) and ultraviolet (UV) ranges, i.e., very low and very high energy scales [38, 52]. They are generally obtained by including higher order curvature invariants in the Einstein-Hilbert action (2.2), or by making the action non-linear in the Ricci curvature R and/or contain terms involving combinations of derivatives of R, in which case the models are known as $f(R)$ *theories of gravity*.

First proposed by Buchdal [46], $f(R)$ theories gained more popularity after further developments by Starobinsky [282] and later following the realization of the discrepancy between theory and observation [4, 17, 27, 32, 42, 43, 48, 50, 51, 58, 65, 67, 77, 78, 90, 107, 135, 136, 177, 191, 204, 220, 223, 225, 237, 254–256, 265, 272, 277–279, 281, 283, 294, 310]. These theories are the main focus of study in this thesis.

The generalized Einstein-Hilbert action (2.2) in the $f(R)$-gravity framework is given by

$$\mathcal{A}_{f(R)} = \frac{1}{2}\int d^4x\sqrt{-g}\,[f(R) + 2\mathcal{L}_m], \tag{3.1}$$

the idea being that, because of the arbitrary function introduced in the Lagrangian, there is more freedom to explain the observed cosmic acceleration and large scale structure formation without the inclusion of exotic matter and energy. However, not all functional forms of these models can be viable cosmological models: a wide range of them can be ruled out based on observations, cosmological and astrophysical, while others can be rejected because of theoretical pathologies.

Using the least action, a generalization of the EFEs (2.8)[1] can be derived in three ways [135]: in the *metric*, *Palatini* or *metric-affine* formalisms.

The metric (second order) formalism

In this formalism, the metric g_{ab} is the only independent variable with respect to which the action (3.1) is varied to derive the field equations:

$$f'G_{ab} = f'(R_{ab} - \frac{1}{2}g_{ab}R) = T^m{}_{ab} + \frac{1}{2}g_{ab}(f - Rf') + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f', \tag{3.2}$$

where

$$f \equiv f(R), \qquad f' \equiv \frac{df}{dR}, \qquad T^m{}_{ab} \equiv \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{ab}}. \tag{3.3}$$

It is due to the introduction of fourth order derivatives of the metric in the last two terms of the RHS of (3.2) that this formalism is sometimes referred to as a fourth-order theory of gravity (FOG).

The Palatini ("first-order") formalism

The Palatini formalism treats both the metric and the affine connection $\Gamma^a{}_{bc}$ as independent variables. Hence the field equations are derived by varying the action with respect to both the metric and the affine connections. In GR it results in the same field equations as those obtained via the metric formalism, but this no longer holds for $f(R)$ theories whose Lagrangians are no longer linear:

$$G_{ab} = \frac{T_{ab}}{f'} - \frac{1}{2}(R - \frac{f}{f'})g_{ab} + \frac{1}{f'}(\nabla_a\nabla_b - g_{ab}\Box)f' - \frac{3}{2(f')^2}\left[\nabla_a f'\nabla_b f' - \frac{1}{2}g_{ab}\nabla_c f'\nabla^c f'\right]. \tag{3.4}$$

We note that there are no second order covariant derivatives of f', and hence the Palatini formalism is sometimes known as a first order approach.

[1] The logic here is that the Λ should be dropped, and whatever comes out as a result of the deviation from GR can be taken as an *effective* cosmological constant term.

The metric-affine formalism

The matter part of the action (3.1) depends explicitly on the affine connection and hence introduces a torsion associated with matter. The theory is not yet a well-explored one.

From now on, the metric variation of (3.1) is assumed unless mentioned otherwise. Thus let us rewrite the generalized field equations (3.2) in a more compact form as

$$G_{ab} = \tilde{T}^{m}{}_{ab} + T^{R}{}_{ab} = T_{ab}, \tag{3.5}$$

where

$$\tilde{T}^{m}_{ab} = \frac{T^{m}{}_{ab}}{f'}, \quad \text{and} \quad T^{R}{}_{ab} = \frac{1}{f'}\left[\frac{1}{2}g_{ab}(f - Rf') + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f'\right] \tag{3.6}$$

are the effective EMTs for standard matter and *curvature* treated as a *fluid*, respectively, and T_{ab} is the EMT of the *total fluid*.

Since the energy-momentum of standard matter and that of the total fluid are conserved, i.e.,

$$T^{m;b}{}_{ab} = 0, \quad T^{;b}{}_{ab} = 0\ , \tag{3.7}$$

it is evident to see that $\tilde{T}^{m}{}_{ab}$ and $T^{R}{}_{ab}$ are not individually conserved[2] [58]:

$$\tilde{T}^{m;b}{}_{ab} = \frac{T^{m;b}{}_{ab}}{f'} - \frac{f''}{f'^2}T^{m}{}_{ab}R^{;b}, \tag{3.8}$$

$$T^{R;b}{}_{ab} = \frac{f''}{f'^2}\tilde{T}^{m}{}_{ab}R^{;b}. \tag{3.9}$$

It should be pointed out at this stage that the structure of the $f(R)$ field equations is by no means unique, i.e., there are different ways of writing the field equations, leading to different definitions of the density and pressure of the curvature fluid, and hence different definitions for the effective equation of state for the curvature term [214, 278]. One thing that remains the same, however, is the total equation of state.

3.2 Background (Thermo)dynamics of $f(R)$ Gravity

The effective total energy-momentum tensor for a cosmic medium described by $f(R)$ gravity

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab} \tag{3.10}$$

sources the following thermodynamical quantities:

$$\mu = \tilde{\mu}_m + \mu_R\ ,$$

[2] f', f'', f''',... abbreviate $\partial^n f/(\partial R)^n$ for $n = 1, 2, 3, ..$ respectively.

$$p = \tilde{p}_m + p_R \ ,$$
$$q_a = \tilde{q}^m_a + q^R_a \ , ,$$
$$\pi_{ab} = \tilde{\pi}^m_{ab} + \pi^R_{ab}, \tag{3.11}$$

with

$$\tilde{\mu}_m = \frac{\mu_m}{f'}, \quad \tilde{p}_m = \frac{p_m}{f'}, \quad \tilde{q}^m_a = \frac{q_a}{f'}, \quad \tilde{\pi}^m_{ab} = \frac{\pi^m_{ab}}{f'}. \tag{3.12}$$

In a perfect fluid cosmological medium, the quantities q^m_a and π^m_{ab} both vanish and the effective background (zeroth-order) energy density and isotropic pressure of an FLRW universe [3] are given by

$$\mu_R = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R}\right], \tag{3.13}$$
$$p_R = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\Theta f''\dot{R}\right], \tag{3.14}$$

with an effective equation of state [54] [4]

$$w_R \equiv \frac{3f - 3Rf' + 6f''\ddot{R} + 6f''\dot{R}^2 + 4\Theta f''\dot{R}}{-3f + 3Rf' - 6\Theta f''\dot{R}} \ . \tag{3.15}$$

The energy conservation (continuity) equations for the matter and curvature-fluid components read

$$\dot{\mu}_m = -\Theta(\mu_m + p_m), \tag{3.16}$$
$$\dot{\mu}_R = -\Theta(\mu_R + p_R) + \mu_m\frac{f''\dot{R}}{f'^2} \ , \tag{3.17}$$

thus showing energy exchange between the two components due to the coupling between these equations.

The Friedmann and Raychaudhuri equations for these theories are generalized as

$$\Theta^2 = 3(\tilde{\mu}_m + \mu_R) - \tfrac{3}{2}\tilde{R} = 3\frac{\mu_m}{f'} + \frac{3}{2}\left(R - \frac{f}{f'}\right) - 3\Theta\dot{R}\frac{f''}{f'} - \frac{9k}{a^2} \ , \tag{3.18}$$
$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\tilde{\mu}_m + 3\tilde{p}_m) - \tfrac{1}{2}(\mu_R + 3p_R)$$
$$= -\frac{1}{3}\Theta^2 - \frac{1}{2f'}\left(2\mu_m - f - 2\Theta\dot{R}f''\right) \ , \tag{3.19}$$

where the Ricci scalar is given by

[3]The EGS theorem (2.2.2) has been generalized for $f(R)$ theories in [134, 262].

[4]This definition is correct if the structure of the modified field equations is as that given in (3.2). But, as mentioned earlier, there are other ways of defining the effective equation of state w_R in a way that it mimics the equation of state of dark energy [189, 208, 214, 227].

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 6\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right). \tag{3.20}$$

and

$$\tilde{R} = 2\left(\mu - \frac{1}{3}\Theta^2\right) = 6k/a^2 \tag{3.21}$$

is the 3-Ricci (curvature) scalar.

3.3 $f(R)$ as Scalar-Tensor Theory of Gravity

An interesting aspect of $f(R)$ theories of gravity is their proven equivalence with Scalar-Tensor theories of gravity [142, 275]. This equivalence is shown by recasting the $f(R)$ action (3.1) into one with a standard matter non-minimally coupled with a classical scalar field ϕ

$$\mathcal{A}_\phi = \int d^4x\sqrt{-g}\left[\frac{1}{2}F(\phi(R)) + \mathcal{L}_m\right]. \tag{3.22}$$

where

$$\phi \equiv f' - 1\ , \tag{3.23}$$

effectively making equations (3.2) the field equations of a classical canonical scalar field ϕ with a potential $V(\phi)$ given by

$$(1+\phi)G_{ab} = T^m_{ab} + \frac{1}{2}g_{ab}(f - (1+\phi)R) + \nabla_b\nabla_a\phi - g_{ab}\nabla_c\nabla^c\phi\ , \tag{3.24}$$

with the EMT of the scalar field defined as

$$T^\phi_{ab} = \frac{1}{1+\phi}\left[\frac{1}{2}g_{ab}(f - (1+\phi)R) + \nabla_b\nabla_a\phi - g_{ab}\nabla_c\nabla^c\phi\right]. \tag{3.25}$$

The scalar field ϕ satisfies the Klein-Gordon equation

$$\nabla_a\nabla^a\phi - \frac{1}{3}\left[2f - (1+\phi)R + (\mu_m - 3p_m)\right] = 0\ , \tag{3.26}$$

which, using the identification

$$\frac{dV}{d\phi} \equiv \frac{1}{3}\left[2f - (1+\phi)R\right] = \frac{dV}{dR}\frac{dR}{d\phi}\ , \tag{3.27}$$

and the driving term coming from the trace equation

$$\frac{1}{3}T^a{}_a = \frac{T}{3} = \frac{1}{3}(\mu_m - 3p_m)\ , \tag{3.28}$$

can be rewritten as

$$\nabla_a \nabla^a \phi - \frac{dV}{d\phi} - \frac{1}{3}(\mu_m - 3p_m) = 0 \; . \tag{3.29}$$

In terms of the Ricci scalar, one gets the scalar potential evolving as

$$\frac{dV}{dR} = \frac{1}{3}\left[2f - (1+\phi)R\right] f'' \; . \tag{3.30}$$

The scalar field equivalent decomposition of the the energy momentum tensor of the scalar field will give the corresponding thermodynamical quantities for the scalar field and the background Friedmann, Raychaudhuri and Klein-Gordon equations can be shown to be

$$\Theta^2 = 3\mu - \frac{3\tilde{R}}{2} = 3\mu - \frac{9K}{a^2} = 3\left[\frac{\mu_m}{1+\phi} + \mu_\phi\right] - \frac{9K}{a^2} \; , \tag{3.31}$$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p) = -\frac{1}{3}\Theta^2 - \frac{1}{2}\left[\frac{(1+3w)}{1+\phi}\mu_m + \mu_\phi + 3p_\phi\right] \; , \tag{3.32}$$

$$\ddot{\phi} + \Theta\dot{\phi} + \frac{dV}{d\phi} - \frac{1}{3}\left[\mu_m - 3p_m\right] = 0 \; . \tag{3.33}$$

These equations, together with the continuity equation and the prescription for the equation of state for matter, form a closed system of cosmological equations in Scalar-Tensor gravity.

3.4 Conditions for Viability of $f(R)$ Models

Higher-order theories of gravity such as $f(R)$ do generally have enough freedom to produce any kind of cosmic background evolution history by a suitable choice of the defining action functional. Given the large number of $f(R)$ models proposed to explain some aspect of cosmological phenomena or another [8,53,91,135,210,223,252,276,298], it is important to gauge their validity based on some general criteria. The criteria include, inter alia, that the expansion history possess a matter-dominated epoch followed by a late-time accelerated expansion. Moreover, the matter-domination epoch should be long enough to allow for structure formation and that the effective equation of state be such that $w_{eff} \simeq 0$ in order to match the observations of the diameter distance of acoustic peaks of CMB anisotropies; i.e., it has to expand as $a(t) \sim t^{2/3}$.

For any class of $f(R)$ gravity, the following are binding conditions on the Lagrangian for consistent cosmological viability on all scales [91, 132, 133, 168, 235, 252]:

1. For the effective Newtonian gravitational constant $G_{eff} \equiv \frac{G}{f'}$ to not change sign,

$$f' > 0 \;\; \forall R \; . \tag{3.34}$$

This ensures that the graviton energy is positive and hence that gravity remain attractive. The implications of $f' < 0$ are understood to be the Universe quickly breaking isotropy and homogeneity [217, 232, 252, 297] and the graviton turning

into a ghost particle [153, 252].
2. For stable matter-dominated and high-curvature cosmological regimes [100, 252],

$$f'' > 0 \quad \forall R \gg f'' \; . \tag{3.35}$$

A scalar-tensor theoretic and quantum mechanical interpretation of this would ensure that the scalaron remain nontachyonic.

3. The tight observational constraints coming from Big Bang nucleosynthesis (BBN) and the CMB dictate that the early universe was governed by a GR-like law of gravitation, i.e.,

$$\lim_{R\to\infty} \frac{f(R)}{R} = 1 \Rightarrow f' < 1 \; , \tag{3.36}$$

implying that, together with condition (2), f' must monotonically asymptote to 1 from below.

A less relaxed, albeit controversial [72, 91, 168, 236, 252], condition is that

4. $f' - 1$ be very small at recent epochs, i.e.,

$$|f' - 1| \ll 1 \tag{3.37}$$

and is not a necessary condition for the ongoing cosmic acceleration.

In [216], it has been shown that

5. For a stable late-time de Sitter-type expansion the ζ parameter defined in Chapter (5) is given (for $\xi = -2$) by,

$$0 < \frac{1}{\zeta} < 1 \Rightarrow \frac{f'}{f''} > R \; \forall R > 0 \; . \tag{3.38}$$

Note that the first part of condition (5) immediately follows from conditions (1) and (2) as well.

3.5 Some $f(R)$ Models

Although much of the attention has recently shifted to explaining late-time cosmic acceleration, previous $f(R)$ considerations included studies of singularity-free cosmological models and cosmic inflation [21, 78, 225–227, 282]. Starobinsky [282] showed that R^2 corrections in the standard GR gravitational action, i.e., Lagrangians of the form

$$f(R) = R + \beta R^2 \tag{3.39}$$

can produce an early de Sitter phase of expansion. In this model $\beta \equiv \frac{1}{6M^2}$ and M represents a characterisitic mass scale.

Requiring that slow-roll inflation occur in the regions of the inflaton field $\phi \gg M_{Pl}$

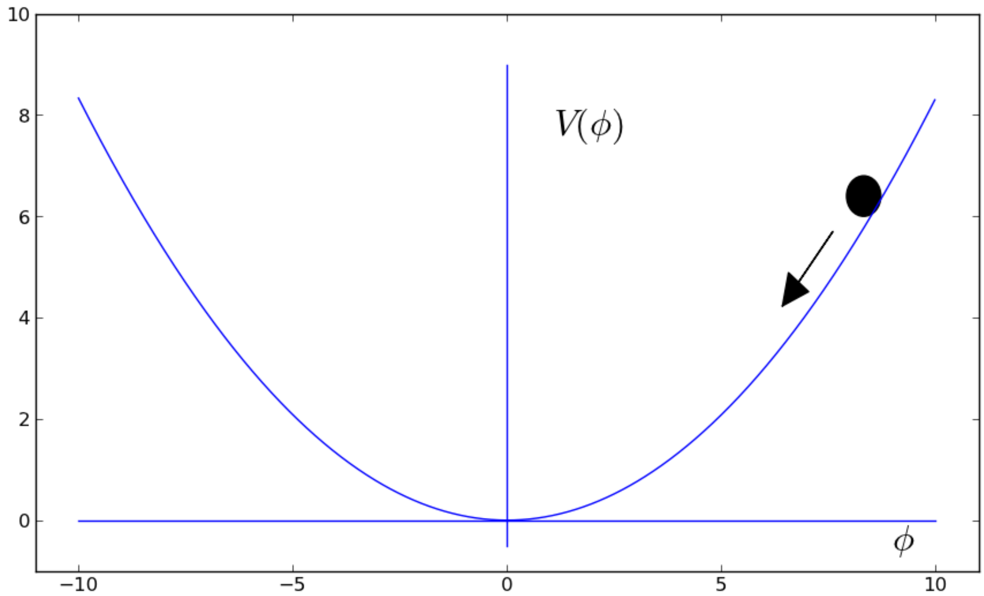

Figure 3.1: A roll-down potential for the scalar-field equivalent of the $R + \beta R^2$ $f(R)$ model. Note the scalar field rolling down to the minimum of the potential well. For this figure $\beta = 1$ has been used.

gives the slow-roll parameter of

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \simeq \frac{1}{36\beta H^2} \tag{3.40}$$

and

$$N \simeq \frac{1}{2\epsilon} \tag{3.41}$$

e-foldings whereas the viability of reheating (around $\phi \simeq 0$) requires that the potential be given by

$$V(\phi) \simeq \frac{1}{12\beta}\phi^2 \ . \tag{3.42}$$

Carroll et al [65] proposed one of the earliest geometrical alternatives to dark energy to explain late-time cosmic speed-up by introduction a Lagrangian of the form

$$f(R) = R - \frac{\beta^{2(n+1)}}{R^n} \ , \tag{3.43}$$

where β is a constant with dimensions of mass. In vacuo, the total effective fluid is due to curvature and hence a power-law accelerated expansion requires

$$w_R = -1 + \frac{2(n+2)}{3(2n+1)(n+1)} < -\frac{1}{3} \,, \tag{3.44}$$

and can be achieved with carefully chosen values of β and n.

Although this model has now been ruled out as a viable model due to cosmological and pathological considerations [8, 9, 28, 68, 74, 78, 100, 130–132, 236, 267, 273], it has opened a new wave of investigations into other parametrizations of $f(R)$-gravity models.

Here we briefly discuss some of the $f(R)$ models that have been studied under different contexts in this work.

3.5.1 R^n Models

These are probably the simplest and most widely studied form of higher order $f(R)$ gravitational theories whose Lagrangian densities are given by

$$f(R) = \beta R^n \,, \tag{3.45}$$

where $\beta = \beta(n)$ is a running coupling constant such that $\beta = 1$ for GR (when $n = 1$).

These models can also be reparametrized in a dimensionless way as

$$f(R) = \beta H_0^2 (R/H_0^2)^n = H_0^2(\beta r^n) \,, \tag{3.46}$$

where in this case $\beta = \beta(n)$ is an a-dimensional coupling constant, H_0 is the ΛCDM value of the Hubble parameter today, and

$$r \equiv \frac{R}{H_0^2} \,, \tag{3.47}$$

is a dimensionless curvature scalar.

For this class of models the cosmological equations associated with a FLRW universe are particularly easy to analyse. A dynamical systems analysis (DSA) (see Chapter (5) for discussions on DSA) of R^n gravity was presented in [61] where it was shown that for a universe dominated by a single-component matter fluid the model has a Friedmann-like transient solution

$$a = a_0 t^{\frac{2n}{3(1+w)}} \tag{3.48}$$

before it enters the de Sitter-like, accelerated expansion phase. In this case, we obtain the following expressions for the expansion, the Ricci scalar, the curvature fluid energy density, the curvature fluid pressure and the effective matter energy density respectively:

$$\Theta = \frac{2n}{(1+w)t}, \tag{3.49}$$

$$R = \frac{4n\left[4n - 3(1+w)\right]}{3(1+w)^2t^2}, \tag{3.50}$$

$$\mu_R = \frac{2(n-1)\left[2n(3w+5) - 3(1+w)\right]}{3(1+w)^2t^2}, \tag{3.51}$$

$$p_R = \frac{2(n-1)\left[n(6w^2+8w-2) - 3w(1+w)\right]}{3(1+w)^2t^2}, \tag{3.52}$$

$$\mu_m = \left(\frac{3}{4}\right)^{1-n} n\beta \left(\frac{n(4n-3(1+w))}{(1+w)^2t^2}\right)^{n-1} \frac{4n^2 - 2(n-1)\left[2n(3w+5) - 3(1+w)\right]}{3(1+w)^2t^2} \tag{3.53}$$

3.5.2 $\alpha R + \beta R^n$

These are models whose Lagrangian densities include power-law corrections to the standard GR action

$$f(R) = \alpha R + \beta R^n\ . \tag{3.54}$$

It is interesting to note that this is a generalization of both the GR and the R^n actions since $\alpha = 1,\ \beta = 0$ reduces to GR and $\alpha = 0$ reduces to the R^n case. Using the parametrization in (3.47) we can also write these models as

$$f(R) = H_0^2\left(\alpha r + \beta r^n\right)\ , \tag{3.55}$$

a way of writing we will find more convenient in later chapters. This toy model is currently gaining popularity as an alternative model of gravitation within the context of early universe (inflationary) and late time (dark energy) cosmologies [23, 57, 65, 212, 282].

3.5.3 Starobinsky Models

These are models of the form [55, 78, 283]

$$f(R) = R + \beta R_c\left[\left(1 + R^2/R_c^2\right)^{-n} - 1\right]\ , \tag{3.56}$$

with positive free parameters $\beta,\ n$. R_c parametrizes the curvature scale such that its magnitude is of the order of the present-day value of the effective cosmological constant. One can show that in the limiting extreme curvature regimes:

$$\lim_{R/R_c\to 0} f(R) = R\ , \tag{3.57}$$

$$\lim_{R/R_c\to\infty} f(R) \simeq R - \beta R_c \equiv R - 2\Lambda\ , \tag{3.58}$$

meaning that at low-curvature regimes, there appears no cosmological constant in the model whereas there is an effective cosmological constant term $\Lambda \equiv \beta R_c/2$ that can, in principle, mimic dark energy at late times. In fact, it has been shown in [283] that for some limiting values of n and $R/R_c \gg 1$ in the de Sitter solution $R = constant$, one can obtain a cosmic expansion history indistinguishable from that in the Concordance Model.

3.5.4 Hu-Sawicki Models

Hu and Sawicki introduced [55, 78, 168] Lagrangian densities of the form

$$f(R) = R - \frac{\beta_1 m^2 \left(R/m^2\right)^n}{1 + \beta_2 \left(R/m^2\right)^n} \,, \tag{3.59}$$

where $\beta_1,\ \beta_2$ and n are the free parameters of the model and $m^2 = \frac{\mu_m}{3}$ is a curvature scale that depends on the value of the matter energy density. It is interesting to note that this model also introduces no cosmological constant (Λ-term) for low curvature regimes, since

$$\lim_{R/m^2 \to 0} f(R) = R \,,$$

whereas for high curvature regimes, one can recover an effective cosmological constant at the present epoch because in this limiting case

$$\lim_{R/m^2 \to \infty} f(R) \simeq R - \frac{\beta_1}{\beta_2} m^2 + \frac{\beta_1}{\beta_2^2} m^2 \left(\frac{R}{m^2}\right)^{-n} \,.$$

3.5.5 Appleby-Battye Models

These are models that can be parametrized as [14, 15, 78]

$$f(R) = R + R_c \log\left[e^{-\beta} + \left(1 - e^{-\beta}\right) e^{-R/R_c}\right] \,, \tag{3.60}$$

where $\beta > 0$ is a dimensionless constant and R_c scales as the present-day value of Λ. These models can, like the previous two, be shown to mimic ΛCDM for large R and avoid the cosmological constant in the limiting $R = 0$ regime:

$$\lim_{R/R_c \to 0} f(R) = R \,, \tag{3.61}$$

$$\lim_{R/Rc \to \infty} f(R) \simeq R - \beta R_c \,. \tag{3.62}$$

3.6 Conclusion

Although the earliest $f(R)$ generalizations of the Einstein gravitational action were theoretical considerations such as the treatment of initial cosmic singularities and inflationary expansion histories, the recent observational discovery of late-time cosmic speed-up has raised the motivation for studying $f(R)$ theories to a whole new level. In this exciting era of precision cosmology, existing and upcoming observational probes should be able to distinguish observationally viable models and rule out those whose predictions do not fall within the current cosmological paradigm.

Chapter 4

$f(R)$ Theories in the $1+3$ Covariant Formalism

> Nothing exists except atoms and empty space; everything else is opinion.
>
> Democritus

4.1 Introduction

One of the toughest tasks in cosmology is to understand how large scale structures in the Universe form(ed). As we have seen in Chapter (2), the Universe is not perfectly smooth: primordial fluctuations seeded the formation of structures on large scales and hence the real Universe is slightly lumpy. Theories of cosmological perturbations tell us how these small fluctuations grow and form the large scale structures (galaxies, clusters and superclusters) in the real Universe. In the study of $f(R)$-gravity theories, this requires extending the standard gauge-invariant theory of cosmological perturbations for GR to $f(R)$ gravity. This has been done using both the metric based approach [28, 93, 167, 213] originally developed for GR by Lifshitz [195], Bardeen [22], and Kodama and Sasaki [179], and the $1+3$ covariant approach [4, 5, 58] developed by Ehlers [110], Hawking [156], Olson [238], and Ellis and Bruni [128].

The metric formalism is based on the foliation of the background spacetime with hypersurfaces and perturbing away from it. It is a non-local, linear theory which requires that the metric be specified from the start, i.e., it is coordinate-dependent. Since it is a linear theory, nonlinear effects are not accounted for. The main shortcoming of this approach, however, is in handling the unphysical gauge modes that are inherent to the theory [22, 33, 41, 44, 99, 101, 103, 104, 118, 128, 170, 172, 179, 215, 240, 244, 246, 305].

The covariant formalism is a way of describing spacetime via covariantly defined variables with respect to a partial frame formalism such as $1+3$ or $1+1+2$. It is a suitable method to describe physics and geometry by tensor quantities and relations valid in all coordinate systems. It is a local, covariant theory based on threading spacetimes with frames. This approach differs from the standard one in that it starts from the theory and reduces to linearities in a particular background. Nonlinearities can be accommodated, but the main advantage of this approach is that no unphysical gauge modes appear here.

4.1.1 The 1+3 Covariant Approach

In this approach, a fundamental observer slices spacetime into time and space (hence $1+3$ to indicate the number of dimensions involved in each slice) to investigate deviations from homogeneity and isotropy of the Universe. It provides an alternative description of spacetime in terms of scalars, 3-vectors and projected symmetric trace-free (PSTF) 3-tensors and their corresponding equations using the Ricci and Bianchi identities [36, 106, 126].

4.1.2 Covariant Variables

The average motion of a cosmological fluid at a point can always be represented by a family of preferred worldlines in spacetime, with a uniquely defined average 4-velocity with respect to fundamental observers associated with the worldlines given as

$$u^a = \frac{dx^a}{d\tau}, \qquad u_a u^a = -1, \tag{4.1}$$

where τ is proper time measured along the worldlines [126]. For any u^a, there exist unique projection tensors

$$U^a{}_b = -u^a u_b \Rightarrow U^a{}_c U^c{}_b = U^a{}_b,\ U^a{}_a = 1,\ U_{ab}u^b = u_a \tag{4.2}$$

$$h_{ab} = g_{ab} + u_a u_b \Rightarrow h^a{}_c h^c{}_b = h^a{}_b,\ h^a{}_a = 3, h_{ab}u^b = 0. \tag{4.3}$$

$U^a{}_b$ projects *along* the 4-velocity vector u^a whereas h_{ab} projects the metric properties of the instantaneous restspaces of observers *orthogonal* to u^a. The volume element for the 3-restspaces orthogonal to u^a is defined by

$$\eta_{abc} = u^d \eta_{dabc} \Rightarrow \eta_{abc} = \eta_{[abc]},\ \eta_{abc}u^c = 0, \tag{4.4}$$

where η_{abcd} is the 4-dimensional volume element such that

$$\eta_{abcd} = \eta_{[abcd]} = 2\eta_{ab[c}u_{d]} - 2u_{[a}\eta_{b]cd}. \tag{4.5}$$

In particular, $\eta_{0123} = \sqrt{|det\ g_{ab}|}$.
η_{abc} satisfies the following identities [36]:

$$\eta^{abc}\eta_{def} = 3! h^{[a}{}_d h^b{}_e h^{c]}{}_f\ , \tag{4.6}$$

$$\eta^{abc}\eta_{cef} = 2! h^{[a}{}_e h^{b]}{}_f\ , \tag{4.7}$$

$$\eta^{abc}\eta_{bcf} = 2!h^{a}{}_{f} \ , \tag{4.8}$$
$$\eta^{abc}\eta_{abc} = 3! \ . \tag{4.9}$$

Since it is a time-space split formalism, we define a covariant time derivative *along* the fundamental worldlines

$$\dot{T}^{a...b}{}_{c...d} \equiv u^{e}\nabla_{e}T^{a...b}{}_{c...d} \tag{4.10}$$

and a fully *orthogonally* projected covariant derivative

$$\tilde{\nabla}_{e}T^{a...b}{}_{c...d} \equiv h^{f}{}_{e}h^{a}{}_{g}...h^{b}{}_{i}h^{t}{}_{c}...h^{m}{}_{d}\nabla_{f}T^{g...i}{}_{t...m} \tag{4.11}$$

for any tensor $T^{a...b}{}_{c...d}$. It is worth mentioning here that $\tilde{\nabla}$ is not the same as the 3-dimensional (spatial) covariant derivative ${}^{3}\nabla$ unless u^{a} is vorticity-free, a consequence of the Frobenius' Theorem [141]. Orthogonal projections of vectors, the orthogonally PSTF part of tensors, and orthogonal projections of covariant time derivatives along u^{a} (known as *'Fermi derivatives'*) are denoted by angular brackets as follows:

$$v^{\langle a\rangle} = h^{a}{}_{b}v^{b}, \qquad T^{\langle ab\rangle} = \left[h^{(a}{}_{c}h^{b)}{}_{d} - \frac{1}{3}h^{ab}h_{cd}\right]T^{cd}, \tag{4.12}$$
$$\dot{v}^{\langle a\rangle} = h^{a}{}_{b}\dot{v}^{b}, \qquad \dot{T}^{\langle ab\rangle} = \left[h^{(a}{}_{c}h^{b)}{}_{d} - \frac{1}{3}h^{ab}h_{cd}\right]\dot{T}^{cd}. \tag{4.13}$$

4.1.3 Kinematic Quantities

These are the quantities that tell us about the overall spacetime kinematics, i.e., the expansion, shear and vorticity of the fundamental worldlines, obtained by splitting $\nabla_{a}u_{b}$ into its irreducible parts defined by their symmetry properties:

$$\nabla_{a}u_{b} = \tilde{\nabla}_{a}u_{b} - u_{a}\dot{u}_{b} = \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab} - u_{a}\dot{u}_{b} \ . \tag{4.14}$$

The volume rate of expansion of the fluid

$$\Theta \equiv \tilde{\nabla}_{a}u^{a} = h^{ab}\nabla_{a}u_{b} \tag{4.15}$$

is related to the linear Hubble expansion by (2.24). The symmetric, trace-free rate of shear tensor is defined as

$$\sigma_{ab} \equiv \tilde{\nabla}_{\langle a}u_{b\rangle} = h^{c}{}_{a}h^{d}{}_{b}\nabla_{(c}u_{d)} - \frac{1}{3}\Theta h_{ab} \tag{4.16}$$

and has the properties

$$\sigma_{ab} = \sigma_{(ab)}, \ \ \sigma_{ab}u^{b} = 0, \ \ \sigma^{a}{}_{a} = 0 \tag{4.17}$$

and describes the rate of distortion of the fluid flow; and

$$\omega_{ab} \equiv \tilde{\nabla}_{[a}u_{b]} = h^{c}{}_{a}h^{d}{}_{b}\nabla_{[d}u_{c]} \tag{4.18}$$

is the skew-symmetric vorticity tensor such that

$$\omega_{ab} = \omega_{[ab]},\ \omega_{ab}u^{b} = 0\ , \tag{4.19}$$

describing the rotation of the fluid relative to a non-rotating (Fermi-propagated) frame. The vorticity vector ω^a is defined to be

$$\omega^{a} \equiv \frac{1}{2}\eta^{abc}\omega_{bc} = \frac{1}{2}\eta^{abc}\nabla_{[b}u_{c]} \Rightarrow \omega_{a}u^{a} = 0,\ \omega_{ab} = \eta_{abc}\omega^{c}. \tag{4.20}$$

The following definitions are frequently used in the literature:

$$\sigma^{2} \equiv \frac{1}{2}\sigma_{ab}\sigma^{ab} \geq 0, \qquad \omega^{2} \equiv \frac{1}{2}\omega_{ab}\omega^{ab} = \omega_{a}\omega^{a} \geq 0. \tag{4.21}$$

The relativistic acceleration vector

$$a_{a} \equiv \dot{u}_{a} = u_{a;b}u^{b} \tag{4.22}$$

represents the effects of non-gravitational forces (such as pressure) and vanishes for a particle moving only under gravitational or inertial forces [106].

Matter Description

The EMT specifies the matter-energy content of the Universe and can be decomposed along the fluid flow lines

$$T_{ab} = \mu u_{a}u_{b} + q_{a}u_{b} + u_{a}q_{b} + ph_{ab} + \pi_{ab}, \tag{4.23}$$

where

$$\mu = T_{ab}u^{a}u^{b}, \qquad q^{a} = -T_{bc}u^{b}h^{ca}, \qquad p = \tfrac{1}{3}(T_{ab}h^{ab}), \qquad \pi_{ab} = T_{cd}h^{c}{}_{\langle a}h^{d}{}_{b\rangle} \tag{4.24}$$

are the total relativistic energy density with respect to u^a, the relativistic momentum density (energy flux), the relativistic isotropic pressure and the trace-free anisotropic pressure of the fluid as defined in (3.11). The generalized (non-linearlized) curvature contributions are given by

$$\mu^{R} = \frac{1}{f'}\left[\frac{1}{2}(Rf'-f) - \Theta f''\dot{R} + f''\tilde{\nabla}^{2}R\right]\ , \tag{4.25}$$

$$p^{R} = \frac{1}{f'}\left[\frac{1}{2}(f-Rf') + f''\ddot{R} + f'''\dot{R}^{2} + \frac{2}{3}\left(\Theta f''\dot{R} - f''\tilde{\nabla}^{2}R - f'''\tilde{\nabla}^{a}R\tilde{\nabla}_{a}R\right)\right]\ , \tag{4.26}$$

$$q_a^R = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \frac{1}{3}f''\Theta\tilde{\nabla}_a R\right] , \tag{4.27}$$

$$\pi_{ab}^R = \frac{1}{f'}\left[f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R - \sigma_{ab}\dot{R}\right] . \tag{4.28}$$

The trace of the EMT above is given by

$$T = T^a{}_a = 3p - \mu. \tag{4.29}$$

We note that

$$q_a u^a = 0 \qquad \pi^a{}_a = 0 \qquad \pi_{ab} = \pi_{(ab)}, \qquad \pi_{ab}u^b = 0. \tag{4.30}$$

In a perfect cosmological fluid both q^a abd π_{ab} vanish, and the equation of state $p = p(\mu, s)$, where s is the entropy density of the fluid, characterizes the thermodynamics of the fluid.

The following energy conditions generally put constraints on μ and p in any cosmological model [277]:

- the Null Energy Condition (NEC)

$$\mu + p \geq 0 , \tag{4.31}$$

- the Weak Energy Condition (WEC):

$$\mu \geq 0, \mu + p \geq 0 , \tag{4.32}$$

- the Strong Energy Condition (SEC):

$$\mu + p \geq 0, \mu + 3p \geq 0 , \tag{4.33}$$

- Dominant Energy Condition (DEC):

$$\mu \geq |p| . \tag{4.34}$$

Inflationary models typically violate the SEC. The isentropic speed of sound (*acoustic speed*) of the fluid is defined by

$$c_s^2 = (\partial p/\partial\mu)_{s=const}. \tag{4.35}$$

In order for matter stability and causality to be preserved the acoustic speed should be constrained as

$$0 \leq c_s^2 \leq 1. \tag{4.36}$$

Causality breaks beyond $c_s^2 = 1$ and matter with $c_s^2 < 0$ is unstable.

4.1.4 Geometry of Spacetime: the Ricci and Bianchi Identities

The general field equations (2.4) and the corresponding $f(R)$ ones (3.2) determine the trace part of the gravitational field at each point in spacetime from the matter at that point. They are obtained using the Ricci and the Bianchi identities.

The Ricci identities

The curvature of spacetime is represented by the Riemann tensor R_{abcd} via the Ricci identities:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)u_c = R_{abc}{}^{d}u_d. \tag{4.37}$$

These identities tell us about the non-commutativity of the second covariant derivatives of a dual vector (the 4-velocity in this particular case). The Riemann tensor has the following symmetry properties :

$$R_{abcd} = R_{[ab][cd]} = R_{cdab}, \qquad R_{a[bcd]} = 0, \tag{4.38}$$

which result in its 20 independent components. The tensor can be split into its trace part given by the *Ricci tensor* $R_{ab} = R^{c}{}_{acb}$ (10 components) and the trace-free part defined to be the *Weyl tensor* or *conformal curvature tensor* C_{abcd} (10 components):

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{R}{3}g^{[a}{}_{[c}g^{b]}{}_{d]}. \tag{4.39}$$

Because it is trace-free,

$$C^{a}{}_{bad} = 0 \tag{4.40}$$

and can be split into its "electric" and "magnetic" parts, E_{ab} and H_{ab} respectively given by

$$E_{ab} \equiv C_{agbh}u^g u^h, \qquad H_{ab} = \tfrac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d. \tag{4.41}$$

These tensors are each symmetric and trace-free in the local rest frame (LRF) of u^a:

$$\begin{aligned} &E_{ab} = E_{(ab)}, \qquad H_{ab} = H_{(ab)},\\ &E^{a}{}_{a} = 0, \qquad\quad H^{a}{}_{a} = 0,\\ &E_{ab}u^b = 0, \qquad H_{ab}u^b = 0. \end{aligned} \tag{4.42}$$

Using these tensors the Weyl tensor can be rewritten as

$$C_{abcd} = (\eta_{abpq}\eta_{cdrs} + g_{abpq}g_{cdrs})u^p u^r E^{qs} + (\eta_{abpq}g_{cdrs} + g_{abpq}\eta_{cdrs})u^p u^r H^{qs}, \tag{4.43}$$

where

$$g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}. \tag{4.44}$$

H_{ab} and E_{ab} represent the free gravitational field, enabling gravitational action at a distance (tidal forces, gravitational waves), and influence the motion of matter and radiation through the geodesic deviation for timelike and null vectors respectively [36].

Whereas the magnetic part does not have a Newtonian analogue, the electric part does, and can be given by

$$E_{\alpha\beta} = \phi_{,\alpha\beta} - \tfrac{1}{3}h_{\alpha\beta}\phi^{\delta}{}_{,\delta}\ , \tag{4.45}$$

where ϕ represents the Newtonian gravitational potential [106].

The Bianchi Identities

The Bianchi identities read

$$\nabla_{[a}R_{bc]d}{}^{e} = 0. \tag{4.46}$$

The twice-contracted Riemann tensor yields the Ricci scalar $R \equiv R^{a}{}_{a}$. Using this and contracting twice (4.46) we obtain

$$\nabla_a R_c{}^{a} + \nabla_b R_c{}^{b} - \nabla_c R = 0 \Leftrightarrow \nabla^{a}G_{ab} = 0. \tag{4.47}$$

The *trace equation*

$$R = \mu - 3p \tag{4.48}$$

can be obtained by substituting the trace of the EMT into the trace of the EFEs (2.8). Using this result we can write out the $1+3$-split of the Ricci tensor

$$R_{ab} = \tfrac{1}{2}(\mu+3p)u_a u_b + \tfrac{1}{2}(\mu-p)h_{ab} + 2u_{(a}g_{b)} + \pi_{ab}. \tag{4.49}$$

Spatial Gradients

The spatial gradient (orthogonal to u^a) of any scalar function f in the LRF of fundamental observers O_u is defined as

$$f_a = \tilde{\nabla}_a f. \tag{4.50}$$

In particular we define here the gradients of the thermodynamical quantities μ, p, Θ [106]

$$X_a = \tilde{\nabla}_a\mu, \qquad Y_a = \tilde{\nabla}_a p, \qquad Z_a = \tilde{\nabla}_a\Theta\ , \tag{4.51}$$

and the divergence of the acceleration vector and its spatial gradient are given by

$$A = a^{a}{}_{;a}, \qquad A_a = \tilde{\nabla}_a A\ . \tag{4.52}$$

4.1.5 Propagation Equations

A complete description of any arbitrary spacetime using the covariant approach requires the irreducible sets of the geometrical and thermodynamical quantities

$$\{\Theta, \sigma_{ab}, \omega_{ab}, \dot{u}^a, E_{ab}, H_{ab}\},$$
$$\{\mu, p, q_a, \pi_{ab}\}\ , \tag{4.53}$$

with a prescribed equation of state of the cosmological fluid. The evolution equations for the kinematic quantities are obtained by separating the parallel projected part of the Ricci identity for the fundamental timelike 4-velocity u^a into trace, symmetric trace-free and skew-symmetric parts.

The Raychaudhuri equation

This is the basic equation of gravitational attraction and gives the propagation equation for the expansion

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - 2\omega_a\omega^a - \tilde{\nabla}^a\dot{u}_a + \dot{u}_a\dot{u}^a + \tfrac{1}{2}(\mu + 3p) = 0, \tag{4.54}$$

which can be rewritten split into its GR and non-GR contributions as

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - 2\omega_a\omega^a - \tilde{\nabla}^a\dot{u}_a + \dot{u}_a\dot{u}^a + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) = -\tfrac{1}{2}(\mu^R + 3p^R)\,. \tag{4.55}$$

The combined $(\mu+3p)$-term in this equation represents the active gravitational mass density; the pressure term appears as a general-relativistic effect. A non-negative $(\mu_R + 3p_R)$ acts as a cosmological constant with repulsive force and hence tends to speed up the expansion; the vorticity tends to hold the matter apart while the shear tends to cause contraction. The divergence of the acceleration represents spatial pressure gradients and affects the average distance of the worldlines through its divergence.

The shear propagation equation

The twice-projected symmetric part of the Ricci identities (4.37) yields the evolution equation for the shear:

$$\dot{\sigma}^{\langle ab\rangle} - \tilde{\nabla}^{\langle a}\dot{u}^{b\rangle} = -\tfrac{2}{3}\Theta\sigma^{ab} + \dot{u}^{\langle a}\dot{u}^{b\rangle} - \sigma^{\langle a}{}_c\sigma^{b\rangle c} - \omega^{\langle a}\omega^{b\rangle} - (E^{ab} - \tfrac{1}{2}\pi^{ab}). \tag{4.56}$$

It shows how the anisotropic pressure π_{ab} and the "electric" part of the Weyl tensor E_{ab} induce distortion in the surrounding fluid flow.

The vorticity propagation equation

This equation is obtained from the twice-projected skew-symmetric part of the Ricci identities and is given by

$$\dot{\omega}^{\langle a\rangle} - \tfrac{1}{2}\eta^{abc}\tilde{\nabla}_b\dot{u}_c = \tfrac{2}{3}\Theta\omega^a + \sigma^a{}_b\omega^b. \tag{4.57}$$

4.1.6 Constraint Equations

These equations are also obtained from the Ricci identity, through orthogonal projection, indices contraction and by taking the PSTF parts. These equations do not involve time derivatives of the kinematic quantities.

The shear constraint

This equation shows how the energy flux vector q^a controls the spatial gradients of Θ, ω_{ab} and σ_{ab}:

$$\tilde{\nabla}_b\sigma^{ab} - \tfrac{2}{3}\tilde{\nabla}^a\Theta + \eta^{abc}[\tilde{\nabla}_b\omega_c + 2\dot{u}_b\omega_c] + q^a = 0. \tag{4.58}$$

The vorticity constraint equation

$$\tilde{\nabla}_a\omega^a - (\dot{u}_a\omega^a) = 0. \tag{4.59}$$

This equation is also known as the *vorticity divergence identity* [126]. In the linear regime, this equation shows the decoupling of vector and tensor modes from the scalar modes.

The H_{ab} constraint equation

This constraint characterizes the "magnetic" Weyl tensor as being constructed from the distortion of the vorticity and the curl of the shear,

$$H^{ab} + 2\dot{u}^{\langle a}\omega^{b\rangle} + \tilde{\nabla}^{\langle a}\omega^{b\rangle} - \eta^{cd\langle a}\tilde{\nabla}_c\sigma^{b\rangle}{}_d = 0\ , \tag{4.60}$$

where the "curl" is defined by the expression $(curl\ \sigma)^{ab} = \eta^{cd\langle a}\tilde{\nabla}_c\sigma^{b\rangle}{}_d$.

The twice-contracted Bianchi identities

The twice-contracted Bianchi identities (4.47) give us two important conservation equations. Projecting the identity along u^a results in the energy conservation equation

$$\dot{\mu} + \tilde{\nabla}^a q_a = -\Theta(\mu+p) - 2\dot{u}_a q^a - \sigma^a{}_b\pi^b{}_a, \tag{4.61}$$

whereas orthogonal projection with respect to u^a leads to the momentum conservation equation

$$\dot{q}^{\langle a\rangle} + \tilde{\nabla}^a p + \tilde{\nabla}_b\pi^{ab} = -\frac{4}{3}\Theta q^a - \sigma^a{}_b q^b - (\mu+p)\dot{u}^a - \dot{u}_b\pi^{ab} - \eta^{abc}\omega_b q_c. \tag{4.62}$$

4.1.7 Maxwell-like Gravitational Equations

Using the Weyl tensor and the Bianchi identities we can write

$$\nabla_a C_{bcd}{}^a + \nabla_{[}(R_{c]d} - \tfrac{1}{6}Rg_{c]d}) = 0. \tag{4.63}$$

Covariant decomposition of these identities will produce the following evolution and constraint equations :

The $\dot{E}$-equation

$$\begin{aligned}
&\dot{E}^{\langle ab\rangle} + \tfrac{1}{2}\dot{\pi}^{\langle ab\rangle} - curl H^{ab} + \tfrac{1}{2}\tilde{\nabla}^{\langle a}q^{b\rangle} \\
&= -\tfrac{1}{2}(\mu+p)\sigma^{ab} - \Theta(E^{ab} + \tfrac{1}{6}\pi^{ab}) + 3\sigma^{\langle a}{}_{c}(E^{b\rangle c} - \tfrac{1}{6}\pi^{b\rangle c}) - \dot{u}^{\langle a}q^{b\rangle} \\
&+ \eta^{cd\langle}\left[2\dot{u}_c H^{b\rangle}{}_{d} + \omega_c(E^{b\rangle}{}_{d} + \tfrac{1}{2}\pi^{b\rangle}{}_{d})\right]. \qquad (4.64)
\end{aligned}$$

The $\dot{H}$-equation

$$\begin{aligned}
\dot{H}^{\langle ab\rangle} + curl E^{ab} - \tfrac{1}{2}curl\pi^{ab} &= -\Theta H^{ab} + 3\sigma^{\langle a}{}_{c}H^{b\rangle c} + \tfrac{3}{2}\omega^{\langle a}q^{b\rangle} \\
&- \eta^{cd\langle a}\left[2\dot{u}_c E^{b\rangle}{}_{d} - \tfrac{1}{2}\sigma^{b\rangle}{}_{c}q_d - \omega_c H^{b\rangle}{}_{d}\right]. \qquad (4.65)
\end{aligned}$$

These propagation equations describe gravitational radiation and are the analogues to the Maxwell's equations for electromagnetic radiation [4, 106].

The Divergence of E equation

$$\tilde{\nabla}_a(E^{ab} + \tfrac{1}{2}\pi^{ab}) - \tfrac{1}{3}\tilde{\nabla}^a\mu + \tfrac{1}{3}\Theta q^a - \tfrac{1}{2}\sigma^a{}_b q^b - 3\omega H^{ab} - \eta^{abc}[\sigma_{bd}H^d{}_c - \tfrac{3}{2}\omega_b q_c] = 0. \qquad (4.66)$$

The Divergence of H equation

$$\tilde{\nabla}_b H^{ab} + (\mu+p)\omega^a + 3\omega_b(E^{ab} - \tfrac{1}{6}\pi^{ab}) + \eta^{abc}\left[\tfrac{1}{2}\tilde{\nabla}_b q_c + \sigma_{bd}(E^d{}_c + \tfrac{1}{2}\pi^d{}_c)\right]. \qquad (4.67)$$

Eqns(4.66) and (4.67) are constraint equations sourced by the spatial gradient of the energy density and the vorticity, respectively. The first equation shows how scalar modes are coupled to the divergence of the electric Weyl tensor while the second one shows how vector (vorticity) modes are coupled to the divergence of the magnetic Weyl tensor [4, 36]. [1]

4.2 Gauge-invariant Perturbation Theory

Although the Universe on large scales is almost homogeneous and isotropic, and is well described by the FLRW background model, deviations from symmetry (homogeneity and isotropy) start to become more and more apparent as we go to smaller and smaller scales. In the standard cosmological model, such deviations are usually accounted for by perturbing away from the FLRW background [4, 36, 106].

Let us consider a background spacetime $(\bar{\mathcal{M}}, \bar{g}_{ab})$ in the homogeneous, isotropic universe and the real, physical spacetime $(\mathcal{M}, g_{ab})$. Quantifying the deviation of the physical from the background spacetime requires a mapping $\psi : \bar{\mathcal{M}} \to \mathcal{M}$ which

[1] A complete list of the covariant consistency relations have been presented in [299].

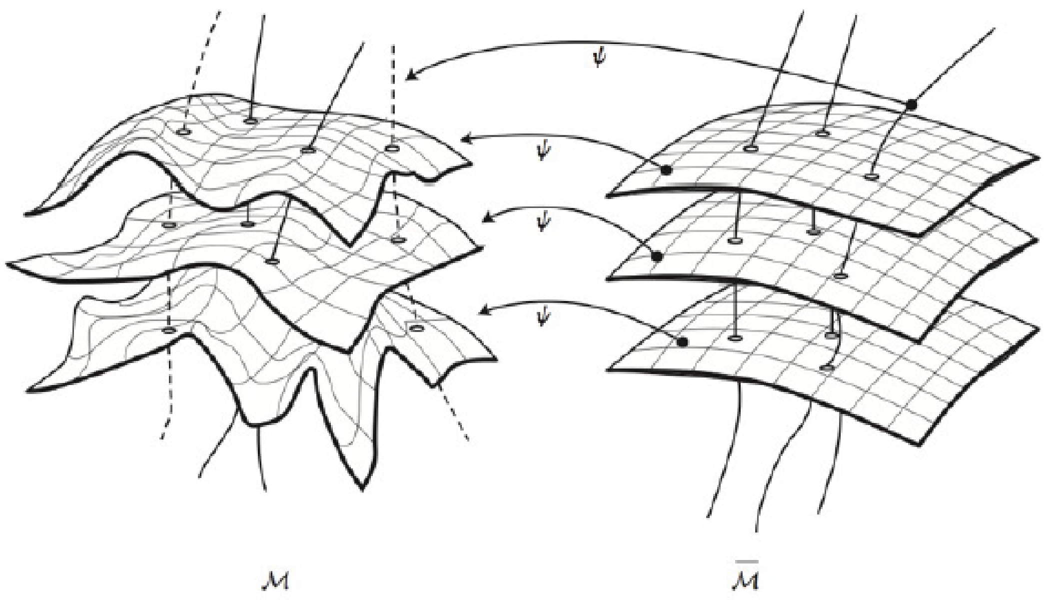

Figure 4.1: The gauge issue: how does the mapping occur between quatities in the background spacetime $\bar{\mathcal{M}}$ and those in the perturbed spacetime $\mathcal{M}$? Picture taken from [250].

identifies points in the background $\bar{\mathcal{M}}$ with their corresponding points in $\mathcal{M}$ such that $\bar{g}_{ab} \to g_{ab} = \bar{g}_{ab} + \delta g_{ab}$. Given a physical quantity Q on $\mathcal{M}$ and the corresponding quantity $\bar{Q}$ on $\bar{\mathcal{M}}$, the perturbation δQ of Q at a point $p \in \mathcal{M}$ is defined as

$$\delta Q(p) = Q(p) - \bar{Q}(\psi^{-1}(p)). \tag{4.68}$$

The perturbation δQ is usually taken to be small, but can be assigned to take any value at the point p by altering the mapping function ψ: there is no *a priori* reason for choosing a particular mapping over another.The freedom of choice of the mapping between $\bar{\mathcal{M}}$ and $\mathcal{M}$ is called the *gauge freedom*. Any change of the mapping ψ which leaves the background manifold $\bar{\mathcal{M}}$ unchanged is called a *gauge transformation*. Gauge transformations reflect the freedom of choosing different coordinates $\{x^a\}$ on the manifold $\mathcal{M}$, i.e.,

$$x^a \to \tilde{x}^a = x^a + \epsilon^a(x) \tag{4.69}$$

for an arbitrary infinitesimal vector field $\epsilon^a(x)$. Thus in the new correspondence $\tilde{\psi}$,the perturbation becomes

$$\delta\tilde{Q}(p) = Q(p) - \bar{Q}(\tilde{\psi}^{-1}(p)). \tag{4.70}$$

Taking the difference of (4.68) and (4.70) we get

$$\Delta Q(p) \equiv \delta\tilde{Q}(p) - \delta\bar{Q}(p) = \bar{Q}(\bar{\psi}^{-1}(p)) - \bar{Q}(\tilde{\psi}^{-1}(p)). \tag{4.71}$$

This difference is a pure gauge artifact which leads to unphysical modes. Such unphysical modes should, therefore, be identified and eliminated when they arise. The *gauge problem* is a persistent issue in the standard (metric) cosmological perturbation theory in GR, and has generally been the prime motivation for the covariant, gauge-invariant formulation of cosmological perturbations.

The Stewart-Walker Lemma

We can notice from Eqn (4.71) that Q is *gauge-invariant* if it vanishes in the background spacetime $\bar{\mathcal{M}}$. In general, the Stewart-Walker Lemma states that the linear perturbation δQ of a tensorial quantity $\bar{Q}$ on the background spacetime $(\bar{\mathcal{M}}, \bar{g})$ is gauge-invariant (GI) if and only if one of the following holds:

1. $\bar{Q}$ vanishes;
2. $\bar{Q}$ is a constant scalar;
3. $\bar{Q}$ is a constant linear combination of products of Kronecker deltas.

4.3 Linearized Field Equations about FLRW Background

In the $1+3$-covariant perturbation theory, the quantities that vanish in the background spacetime are considered to be first order and are automatically gauge-invariant [44, 45, 102, 104, 123, 128, 201, 203] by virtue of the Stewart and Walker lemma [284]. In this work, we consider the background to be FLRW where the Hubble scale sets the characteristic scale of the perturbations. In the perturbed spacetime the standard matter is considered to be a perfect fluid with the Energy Momentum tensor given by (3.10). Furthermore, we assume standard matter to have a barotropic linear equation of state $p_m = w\mu_m$ satisfying the Weak and Dominant energy conditions:

$$\mu_m > 0 \;\; ; \;\; \mu_m + p_m > 0 \;\; ; \;\; \mu_m \geq |p_m| \, . \tag{4.72}$$

Since the matter is a perfect fluid the heat flux (q^m_a) and the anisotropic stress (π^m_{ab}) vanish in the perturbed spacetime.

For the curvature fluid the linearised thermodynamic quantities are given as

$$\mu_R = \frac{1}{f'}\left[\tfrac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R\right] \, , \tag{4.73}$$

$$p_R = \frac{1}{f'}\left[\tfrac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2\right.$$

$$+\tfrac{2}{3}\left(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R\right)\Big] \ , \tag{4.74}$$

$$q_a^R = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \tfrac{1}{3}f''\Theta\tilde{\nabla}_a R\right], \tag{4.75}$$

$$\pi_{ab}^R = \frac{1}{f'}f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R. \tag{4.76}$$

With the conditions above, the linearised field equations are then given by:

Propagation Equations

$$\dot{\Theta} - \tilde{\nabla}_a A^a = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu + 3p) \ , \tag{4.77}$$

$$\dot{\omega}^{\langle a\rangle} - \tfrac{1}{2}\eta^{abc}\tilde{\nabla}_b A_c = -\tfrac{2}{3}\Theta\omega^a \ , \tag{4.78}$$

$$\dot{\sigma}_{\langle ab\rangle} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tilde{\nabla}_{\langle a}\dot{u}_{b\rangle} + \tfrac{1}{2}\pi_{ab} \ . \tag{4.79}$$

$$\dot{E}^{\langle ab\rangle} - \eta^{cd\langle a}\tilde{\nabla}_c H_d^{\rangle b} = -\Theta E^{ab} - \tfrac{1}{2}\dot{\pi}^{ab} - \frac{1}{2}(\mu + p)\,\sigma^{ab} - \tfrac{1}{2}\tilde{\nabla}^{\langle a}q^{b\rangle} - \tfrac{1}{6}\Theta\pi^{ab} \ , \tag{4.80}$$

$$\dot{H}^{\langle ab\rangle} + \eta^{cd\langle a}\tilde{\nabla}_c E_d^{\rangle b} = -\Theta H^{ab} + \tfrac{1}{2}\eta^{cd\langle a}\tilde{\nabla}_c\pi_d^{\rangle b} \ , \tag{4.81}$$

$$\dot{\mu}_m = -(\mu_m + p_m)\Theta, \tag{4.82}$$

$$\dot{\mu} + \tilde{\nabla}^a q_a = -(\mu + p)\Theta; \tag{4.83}$$

$$\dot{q}^{\langle a\rangle} + \tilde{\nabla}^a p + \tilde{\nabla}_b\pi^{ab} = -\frac{4}{3}\Theta q^a - (\mu + p)\dot{u}^a. \tag{4.84}$$

Constraint Equations

$$(C_1)^a := \tilde{\nabla}_b\sigma^{ab} + \frac{2}{3}\tilde{\nabla}^a\Theta - \eta^{abc}\tilde{\nabla}_b\omega_c - q_R^a = 0 \ , \tag{4.85}$$

$$(C_2) := \tilde{\nabla}^a\omega_a = 0 \ , \tag{4.86}$$

$$(C_3)^{ab} := \eta^{cd(a}\tilde{\nabla}_c\sigma^{b)}{}_d - H^{ab} + \tilde{\nabla}^{\langle a}\omega^{b\rangle} = 0 \ . \tag{4.87}$$

$$(C_5)^a := \tilde{\nabla}_b E^{ab} + \tfrac{1}{2}\tilde{\nabla}_b\pi_R^{ab} - \tfrac{1}{3}\tilde{\nabla}^a\mu + \tfrac{1}{3}\Theta q_R^a = 0 \ , \tag{4.88}$$

$$(C_6)^a := \tilde{\nabla}_b H^{ab} + (\mu + p)\omega^a + \tfrac{1}{2}\eta^{abc}\tilde{\nabla}_b q_c^R = 0 \ . \tag{4.89}$$

Since most of the cosmological analysis done in this dissertation is focused on applications of first-order cosmological perturbations, in the coming chapters we will use the linearized equations discussed in this section extensively.

Chapter 5

Dynamical Systems in $f(R)$ Cosmology

Nulla est lex quae omnia regit.
(There is no law governing all things.)

Giordano Bruno

5.1 Introduction

We have seen in previous chapters that, over the last 14 billion years, the Universe has undergone different epochs of evolutionary history: from the quantum-gravity through inflationary and CMB to the present-day galactic epoch. For a physical system such as the Universe that evolves in time, a qualitative study of the evolution can be achieved using the Dynamical Systems Approach (DSA) [304].

The study of dynamical systems in cosmology has recently become a popular undertaking [1,73,81,82,84,120,150,152,164,231,291,304], but the theory of dynamical systems goes back to the work of Poincaré [155, 163, 192, 248, 253]. Poincaré proposed using topological and geometrical methods to determine properties of the set of all solutions to a differential equation (DE), viewed as orbits (trajectories) in a state space, instead of trying to find a particular exact solution to that DE. The formal mathematical development of the theory, however, came only years later following works by Birkhoff, Andronov, Pontryagin, Smale and others.

Although it comes at the cost of highly idealized assumptions (such as the perfect-fluid assumptions of the matter content) about the physical processes in the Universe, the DSA in cosmology has proved itself to be a very useful tool in obtaining qualitative information about the evolution of general classes of cosmological models, homogenous or otherwise. It is a useful tool in analyzing the range of initial conditions and possible trajectories of evolution that are compatible with current observational data.

The main purpose of this chapter is to give a a bird's-eye-view of the concepts of the dynamical systems theory and to introduce its applications to cosmology, particularly to some aspects of the work presented in this thesis. More rigorous treatments of the theoretical aspects of the theory can be found in [1, 84, 155, 163, 192, 248, 286, 304] (and the references therein).

Assumption: The state of a physical system (the Universe, in our case) at an instant in time can be described by an *element* $\mathbf{x}$ of a *state space* X, which may be finite dimensional ($\mathbb{R}^n$, or a non-trivial differentiable manifold) or infinite dimensional (a *function space*). The evolution of the system is then described by an autonomous differential equation on X:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in X \ , \tag{5.1}$$

where $f : X \mapsto X$. For a finite-dimensional X, (5.1) is an *autonomous* system of ODEs, but if X is infinite-dimensional (function space), the resulting system would be an autonomous set of PDEs.

In this work, we will focus on the application of the DSA to rewrite the (modified) Einstein field equations as first-order ODEs on $X = \mathbb{R}^n$, $\mathbf{x} = (x_1, .., x_n)^T$, the solution curves of which partition $\mathbb{R}^n$ into orbits [40, 84, 304] and on the analysis of the infinite-dimensional state space associated with cosmological perturbations (to be presented in Chapter 6) by studying the finite dimensional state space with the approximating ODEs that arise from harmonic decomposition.

5.2 Dynamical Systems Dictionary

Writing out (5.1) in full, we can see that the autonomous system can be given by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, ..., x_n) \ , \\ \dot{x}_2 &= f_2(x_1, ..., x_n) \ , \\ &\ \ . \\ &\ \ . \\ &\ \ . \\ \dot{x}_n &= f_n(x_1, ..., x_n) \ , \end{aligned} \tag{5.2}$$

where $x_i \in \mathbb{R}$ is an element of the *phase space* $(x_1, ..., x_n)$, and $f_i : \mathbb{R} \mapsto \mathbb{R}$ for $i = 1, ..., n$.

Given the initial conditions $\mathbf{x}_0 = \mathbf{x}(t_0)$ at time t_0, the set of points $(t, \mathbf{x}(t)), t \in \mathbb{R}$ which solve the system (5.2) forms the *trajectory* or *solution curve* of the dynamical system passing through $\mathbf{x}_0$ whereas the *phase flow* is the set of all trajectories obtained by varying t_0 and $\mathbf{x}_0$ in the domain. An *orbit* or a *phase curve* constitutes the set of points $\mathbf{x}(t)$ which solve the system given $\mathbf{x}(t_0) = \mathbf{x}_0$.

A *fixed*, a *critical* or an *equilibrium* point of the dynamical system is the point $\mathbf{x}_*$ that solves

$$\dot{\mathbf{x}} = 0 \; , \tag{5.3}$$

making it the simplest kind of an orbit. If a system starts from equilibrium at $\mathbf{x}_*$, it will remain there forever.

An important aspect of equilibrium solutions is their stability, which can be summarized as [1] [1]

1. If all solutions that start *close* to an equilibrium converge to the equilibrium asymptotically as $t \to \infty$, then the equilibrium is *asymptotically flat* ;
2. If all solutions in a sufficiently small neighborhood of the equilibrium remain close to the equilibrium point, the equilibrium is *stable*;
3. If every neighborhood of the equilibrium point contains solutions arbitrarily close to the equilibrium point and leaves the neighborhood, the equilibrium point is *unstable*.

If the dynamical system (5.1) is nonlinear (and most nonlinear systems are not analytically solvable) with $f(\mathbf{x}_*) = \mathbf{0}$ an equilibrium point, then for a small perturbation $\delta = \mathbf{x} - \mathbf{x}_*$ away from $\mathbf{x}_*$, we have

$$\dot{\delta} = \frac{d}{dt}(\mathbf{x} - \mathbf{x}_*) = \dot{\mathbf{x}}_* = f(\mathbf{x}) = f(\mathbf{x}_* + \delta) \; , \tag{5.4}$$

which, using Taylor expansion, yields

$$f(\mathbf{x}_* + \delta) = f(\mathbf{x}_*) + \delta Df(\mathbf{x}_*) + O(\delta^2) \; . \tag{5.5}$$

This means that for quadratically small terms in δ, we can *linearize* the nonlinear system around x_* and write

$$\dot{\delta} \simeq \delta Df(\mathbf{x}_*) \Rightarrow \dot{\mathbf{x}} = \mathbf{J}\mathbf{x} \; , \tag{5.6}$$

where

$$\mathbf{J} = Df(\mathbf{x}_*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}_*}$$

is the Jacobian matrix evaluated at the fixed points. Knowing the eigenvalues of the Jacobian matrix at the equilibrium points can tell us about the behaviour of the dynamical system near the equilibrium points:

- if the eigenvalues are complex, the equilibrium points are either centers (i.e., neutrally stable) or spirals (i.e., exponentially decaying/stable or exponentially

[1] A detailed analysis of dynamical systems, linearizations and equilibrium points and stability conditions can be found in [286].

growing/unstable). If all the eigenvalues are purely imaginary, then we have periodic fixed point solutions.

- if the eigenvalues all have real part with a negative magnitude, then the system is stable.
- if the eigenvalues are real and have opposite signs, then the point is a saddle point.
- if any eigenvalue has a real part with a positive magnitude, then the point is unstable.
- if the largest real part of the eigenvalues is zero, the stability of the system cannot be determined based on the Jacobian matrix.

5.3 Dynamical Variables

The fact that, in (5.2), $x_i \in \mathbb{R}$ and $f_i : \mathbb{R} \mapsto \mathbb{R}$ tells us a great deal about the kind of cosmological parameters we have to deal with when applying the dynamical systems formalism to the field equations: the (modified) EFEs should be purely algebraic and hence dimensionless. This can be achieved by rewriting the field equations in terms of expansion-normalized variables [3, 60, 64, 84, 161, 302–304]. This is because we are interested in the evolution of the various physical and geometrical quantities relative to the overall rate of expansion of the Universe, or the *Hubble scalar* $H = \frac{1}{3}\Theta$.

Rewriting (3.13), (3.14) and the background trace equation (4.48) in terms of H

$$\mu_R = \frac{1}{f'}\left[\tfrac{1}{2}(Rf' - f) - 3Hf''\dot{R}\right] ,$$
$$p_R = \frac{1}{f'}\left[\tfrac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + 2Hf''\dot{R}\right] , \tag{5.7}$$

$$3f''\ddot{R} + 3\dot{R}^2 f''' + 9H\dot{R}f'' - \mu_m - Rf' + 2f + 3p_m = 0 , \tag{5.8}$$

one can show that the Friedman (3.18) and Raychaudhuri (3.19) equations can be simplified as

$$H^2 = \frac{R}{6} - \frac{f}{6f'} - \frac{\dot{R}Hf''}{f'} + \frac{\mu_m}{3f'} - \frac{k}{a^2} , \tag{5.9}$$
$$\dot{H} = -H^2 - \frac{1}{6f'}\left[2\mu_m - f - 6\dot{R}Hf''\right] . \tag{5.10}$$

In terms of the deceleration parameter, the Ricci scalar (3.20) is given by

$$R = 6H^2(1 - q) . \tag{5.11}$$

If we introduce the dimensionless dynamical variables [6, 64]

$$x \equiv \frac{\dot{R}f''}{f'H} , \qquad y \equiv \frac{R}{6H^2} , \qquad \chi \equiv \frac{f}{6f'H^2} ,$$

$$\Omega_m \equiv \frac{\mu_m}{3f'H^2}\;, \qquad K \equiv \frac{k}{a^2H^2}\;, \qquad h \equiv \frac{H}{H_0}\;, \tag{5.12}$$

and define the dimensionless time variable

$$N \equiv |\ln a| \Rightarrow df/dN = \dot{f}/|H|\;, \tag{5.13}$$

then the cosmological system of equations (5.9)-(5.10) is equivalent to the autonomous dynamical system

$$\begin{aligned}
\frac{dx}{dN} &= \varepsilon\left[-x^2 - x(y-K+1) + 2y - 4\chi + (1-3w)\Omega_m\right]\;, \\
\frac{dy}{dN} &= y\varepsilon(x\zeta + 2K - 2y + 4)\;, \\
\frac{d\chi}{dN} &= \chi\varepsilon\left(2K - x - 2y + 4\right) + xy\zeta\;, \\
\frac{d\Omega_m}{dN} &= \Omega_m\varepsilon\left(2K - x - 2y - 3w + 1\right)\;, \\
\frac{dK}{dN} &= 2K\varepsilon\left(K - y + 1\right)\;.
\end{aligned} \tag{5.14}$$

where $\varepsilon \equiv \frac{|H|}{H}$ and the auxiliary variable ζ is defined as [8, 64]

$$\zeta \equiv \frac{f'}{Rf''}\;, \tag{5.15}$$

and is the key term that characterizes the specific $f(R)$ model under consideration.

The dimensionality of the resultant system (5.14) can be reduced further using the Friedmann constraint

$$1 + x - y + \chi - \Omega_m + K = 0\;, \tag{5.16}$$

but is always ≥ 3 for fourth-order gravity. The evolution of the Hubble parameter can then be determined by writing (3.20) in terms of the DS variables:

$$\frac{\mathrm{d}h}{\mathrm{d}N} = h\varepsilon(y - K - 2)\;. \tag{5.17}$$

Furthermore the deceleration parameter can be determined directly from y using:

$$q = 1 - y\;. \tag{5.18}$$

For an expanding universe, $\varepsilon = 1$, and thus the corresponding DS equations in redshift space using

$$df/dN = -(1+z)df/dz\;, \tag{5.19}$$

become

$$
\begin{aligned}
(1+z)\frac{dx}{dz} &= x^2 + x(y-K+1) - 2y + 4\chi + (3w-1)\Omega_m \ , \\
(1+z)\frac{dy}{dz} &= y(2y - x\zeta - 2K - 4) \ , \\
(1+z)\frac{d\chi}{dz} &= \chi\left(x + 2y - 2K - 4\right) - xy\zeta \ , \\
(1+z)\frac{d\Omega_m}{dz} &= \Omega_m\left(x + 2y - 2K + 3w - 1\right) \ , \\
(1+z)\frac{dK}{dz} &= 2K\left(y - K - 1\right)
\end{aligned}
\tag{5.20}
$$

Similarly, the Raychaudhuri equation (5.17) becomes

$$
(1+z)\frac{dh}{dz} = h\left(K - y + 2\right) \ . \tag{5.21}
$$

If we use (5.16) to eliminate one of the variables, say K, from (5.20), then the reduced system can be given by

$$
\begin{aligned}
(1+z)\frac{dx}{dz} &= 2x^2 + x(\chi+2) - 2y + 4\chi + (3w - x - 1)\Omega_m \ , \\
(1+z)\frac{dy}{dz} &= y\left[x(2-\zeta) + 2(\chi - \Omega_m - 1)\right] \ , \\
(1+z)\frac{d\chi}{dz} &= \chi\left(2\chi + 3x - 2\Omega_m - 2\right) - xy\zeta \ , \\
(1+z)\frac{d\Omega_m}{dz} &= \Omega_m\left(3x + 2\chi - 2\Omega_m + 3w + 1\right) \ ,
\end{aligned}
\tag{5.22}
$$

with

$$
K = y + \Omega_m - x - \chi - 1 \ , \tag{5.23}
$$

and

$$
(1+z)\frac{dh}{dz} = -h\left(x + \chi - \Omega_m - 1\right) \ . \tag{5.24}
$$

An important step of the phase space analysis lies in expressing ζ in terms of the dynamical variables, since it is only then that the system (5.20) will be closed [58]. It is clear from (5.15) that ζ depends only on R. Thus if one can write $R = R(x, y, \chi, \Omega_m)$, which is usually the harder part, then obtaining $\zeta = \zeta(x, y, \chi, \Omega_m)$ becomes straightforward. Following [8, 64], we can achieve this by defining another auxiliary, albeit intermediate, dimensionless variable

$$
\xi \equiv -\frac{Rf'}{f} \tag{5.25}
$$

such that in terms of the dynamical variables

$$\xi = -\frac{y}{\chi} . \tag{5.26}$$

Inverting (5.26) to find R in terms of y and χ will in turn enable us to write $\zeta = \zeta(y, \chi)$, thus closing the autonomous system (5.20). It should be noted here, however, that since not all $f(R)$ Lagrangians are invertible, and since ζ could have a non-trivial domain, be divergent or not in the class $\mathbb{C}^1$, this method of phase space analysis is not a trivial procedure.

Once the fixed points of the system (5.22) are calculated (which, in general, is easy to do), the exact cosmological solutions [2] corresponding to these points can be found using the Raychaudhuri and continuity equations (5.10), (4.82) and rewriting

$$\dot{H} = \gamma H^2, \qquad \gamma \equiv x_* + \chi_* - \Omega_{m,*} - 1 , \tag{5.27}$$

$$\dot{\mu}_m = \frac{3(1+w)}{\gamma t}\mu_m . \tag{5.28}$$

The general solutions of this (decoupled) system are given by

$$a = a_0(t - t_0)^{\frac{t}{\gamma}} , \tag{5.29}$$

$$\mu_m = a_0(t - t_0)^{\frac{3(1+w)}{\gamma}} \tag{5.30}$$

provided $\gamma \neq 0$, whereas for $\gamma = 0$,

$$\dot{H} = 0 \tag{5.31}$$

describes a static or a de Sitter solution.

One can determine where in phase space the present time ($z = 0$) is by determining the coordinates of the point corresponding to where H and q take their ΛCDM values.

5.4 Cosmological Dynamics of Some $f(R)$ Models

In this section, we will apply the DS analysis to two types of $f(R)$ Lagrangians that will be used in later chapters of this thesis. The limit of flat ($K = 0$) dust models that we will focus on later are sub-classes of the general solutions to be presented here.

5.4.1 The case of $f(R) = \beta R^n$

From the dynamical systems analysis point of view, R^n models are one of the most widely investigated toy models [6,8,57,60,64]. In dimensionless variables, these models

[2] It is worth mentioning here, however, that not all the solutions found this way are physically meaningful and cosmological considerations should involve a delicate handling.

can be reparametrised as

$$f(R) = R^n = H_0^2 r^n \tag{5.32}$$

where the normalized Ricci scalar, making use of (5.11), is given by

$$r \equiv R/H_0^2 = 6h^2(1-q)\ . \tag{5.33}$$

The parameter ζ for this type of Lagrangian can be shown to be the constant

$$\zeta = \frac{1}{n-1}\ . \tag{5.34}$$

Another interesting aspect of these models is that the dynamical variables y and χ are no longer independent since $y = n\chi$.This means that the phase space of R^n-gravity is contained in the subspace $y = n\chi$ of the general phase space described by (5.22) [57]. Thus, eliminating one of the variables, say χ, the reduced autonomous system becomes

$$\begin{aligned}
(1+z)\frac{dh}{dz} &= h\,(2-y)\ ,\\
(1+z)\frac{dx}{dz} &= 2x^2 + x\left(\frac{2n+y}{n}\right) + 2y\left(\frac{2-n}{n}\right) + (3w-x-1)\Omega_m\ ,\\
(1+z)\frac{dy}{dz} &= y\left[x\left(\frac{2n-3}{n-1}\right) + 2\left(\frac{y}{n} - \Omega_m - 1\right)\right]\ ,\\
(1+z)\frac{d\Omega_m}{dz} &= \Omega_m\left(3x + \frac{2y}{n} - 2\Omega_m + 3w + 1\right)\ .
\end{aligned} \tag{5.35}$$

Table 5.1: Coordinates of the fixed points for the model $f(R) = \beta R^n$. The superscript "*" represents a double solution. The point $\mathcal{B}$ is a double solution for $n = 0, 2$ [57, 60].

Point	Coordinates (x, y, χ, Ω)	K
$\mathcal{A}^*$	$(0, 0, 0, 0)$	-1
$\mathcal{B}$	$(-1, 0, 0, 0)$	0
$\mathcal{C}$	$(-1-3w, 0, 0, -1-3w)$	-1
$\mathcal{D}$	$(1-3w, 0, 0, 2-3w)$	0
$\mathcal{E}$	$(2(1-n), 2n(n-1), 2(n-1), 0)$	$2n(n-1)-1$
$\mathcal{F}$	$\left(\frac{2(n-2)}{2n-1}, \frac{(4n-5)n}{2n^2-3n+1}, \frac{4n-5}{2n^2-3n+1}, 0\right)$	0
$\mathcal{G}$	$\left(-\frac{3(n-1)(w+1)}{n}, \frac{4n-3(w-1)}{2n}, \frac{4n-3(w-1)}{2n^2}, \frac{-2(3w+4)n^2+(9w+13)n-3(w+1)}{2n^2}\right)$	0

The limiting FLRW flat case is obtained by considering only those solutions for which $K = 0$. This subspace is an invariant sub-manifold, meaning an orbit with an initial condition $K = 0$ will always remain in the subspace, and orbits with initial condition $K \neq 0$ can approach the $K = 0$ hyperplane only asymptotically [64].

The corresponding cosmological solutions and the stability[3] of the fixed points are shown below [1, 57, 60].

Table 5.2: Solutions associated to the fixed points of the $f(R) = \beta R^n$ model .

Point	Scale Factor	
$\mathcal{A}$	$a = a_0(t - t_0)$	$\mu_m = 0$
$\mathcal{B}$	$a = a_0(t - t_0)^{1/2}$ (only for $n = 3/2$)	$\mu_m = 0$
$\mathcal{C}$	$a = a_0(t - t_0)$	$\mu_m = 0$
$\mathcal{D}$	$a = a_0(t - t_0)^{1/2}$ (only for $n = 3/2$)	$\mu_m = 0$
$\mathcal{E}$	$\begin{cases} a = \frac{kt}{2n^2-2n-1} & \text{if} \quad k \neq 0 \\ a = a_0 t & \text{if} \quad k = 0 \end{cases}$	$\mu_m = 0$
$\mathcal{F}$	$a = a_0\, t^{\frac{(1-n)(2n-1)}{n-2}}$	$\mu_m = 0$
$\mathcal{G}$	$a = a_0\, t^{\frac{2n}{3(1+w)}}$	$\mu_m = \mu_{m,0} t^{-2n}$
	$\mu_{m,0} = (-1)^n 3^{-n} 2^{2n-1} n^n (1+w)^{-2n} \times$ $(4n - 3(1+w))^{n-1}[2n^2(4+3w) - n(13+9w) + 3(1+w)]$	

We can see that there exists a solution obtained for spatially flat, matter dominated R^n gravity given by $a(t) = a_0 t^{\frac{2n}{3(1+w)}}$ (which is a (Friedmann-type) saddle point $\mathcal{G}$ of the corresponding dynamical system for these theories) that eventually tends towards the (de Sitter-type)late-time attractor $\mathcal{F}$ with $a = a_0 t^{\frac{(1-n)(2n-1)}{n-2}}$. In [2, 3, 59, 60, 64, 147, 148, 188] it was shown that if $1.36 < n < 1.5$, $\mathcal{G}$and $\mathcal{F}$ respectively represent a decelerated matter dominated solution and late-time power-law acceleration.

[3]The stability of the fixed points is obtained by linearizing the system (5.22)

Table 5.3: Stability of the fixed points for R^n-gravity with dust and radiation as the matter single-fluid sources. The terms "spiral$^+$" and "saddle-focus" denote pure attractive focus-nodes and the unstable focus-nodes, respectively.

	$n<\frac{1}{2}(1-\sqrt{3})$	$\frac{1}{2}(1-\sqrt{3})<n<0$	$0<n<1/2$	$1/2<n<1$
$\mathcal{A}$	saddle	saddle	saddle	saddle
$\mathcal{B}$	repellor	repellor	repellor	repellor
$\mathcal{C}$	saddle	saddle	saddle	saddle
$\mathcal{D}$	saddle	saddle	saddle	saddle
$\mathcal{E}$	saddle	attractor	spiral	spiral
$\mathcal{F}$	attractor	saddle	saddle	attractor

	$1<n<5/4$	$5/4<n<4/3$	$4/3<n<\frac{1}{2}(1+\sqrt{3})$	$n>\frac{1}{2}(1+\sqrt{3})$
$\mathcal{A}$	saddle	saddle	saddle	saddle
$\mathcal{B}$	saddle	repellor	repellor	repellor
$\mathcal{C}$	saddle	saddle	saddle	saddle
$\mathcal{D}$	saddle	saddle	saddle	saddle
$\mathcal{E}$	spiral	spiral	attractor	saddle
$\mathcal{F}$	repellor	saddle	saddle	attractor

$\mathcal{G}$	$n\lesssim 0.33$	$0.33\lesssim n\lesssim 0.35$	$0.35\lesssim n\lesssim 0.37$	$0.37\lesssim n\lesssim 0.71$	$0.71\lesssim n\lesssim 1$
$w=0$	saddle	saddle-focus	saddle-focus	saddle-focus	saddle
$w=1/3$	saddle	saddle	saddle-focus	saddle-focus	saddle-focus

	$1\lesssim n\lesssim 1.220$	$1.220\lesssim n\lesssim 1.223$	$1.223\lesssim n\lesssim 1.224$	$1.224\lesssim n\lesssim 1.28$
$w=0$	saddle-focus	saddle-focus	saddle-focus	saddle-focus
$w=1/3$	saddle-focus	saddle-focus	saddle-focus	saddle-focus

	$1.28\lesssim n\lesssim 1.32$	$1.32\lesssim n\lesssim 1.47$	$1.47\lesssim n\lesssim 1.50$	$n\gtrsim 1.50$
$w=0$	saddle-focus	saddle	saddle	saddle
$w=1/3$	saddle	saddle	saddle	saddle

5.4.2 The case of $f(R)=H_0^2\left(\alpha r+\beta r^n\right)$

For this type of Lagrangian, the characteristic ζ parameter can be shown to be

$$\zeta=-\frac{y}{n(\chi-y)}\;, \tag{5.36}$$

and hence the system of equations (5.22) can be recast as

$$(1+z)\frac{dx}{dz}=2x^2+x(\chi+2)-2y+4\chi+(3w-x-1)\Omega_m\;,$$
$$(1+z)\frac{dy}{dz}=y\left[x\left(2+\frac{y}{n(\chi-y)}\right)+2(\chi-\Omega_m-1)\right]\;,$$

$$(1+z)\frac{d\chi}{dz} = \chi\left(2\chi + 3x - 2\Omega_m - 2\right) + \frac{xy^2}{n(\chi - y)} ,$$
$$(1+z)\frac{d\Omega_m}{dz} = \Omega_m\left(3x + 2\chi - 2\Omega_m + 3w + 1\right) . \tag{5.37}$$

Table 5.4: Coordinates of the fixed points for $\alpha R + \beta R^n$ gravity.

Point	Coordinates (x, y, χ, Ω)	K
$\mathcal{A}$	$(0,0,0,0)$	-1
$\mathcal{B}$	$(-1,0,0,0)$	0
$\mathcal{C}$	$(-1-3w, 0, 0, -1-3w)$	-1
$\mathcal{D}$	$(1-3w, 0, 0, 2-3w)$	0
$\mathcal{E}$	$(0,2,1,0)$	0
$\mathcal{F}$	$(2,0,-2,0)$	-1
$\mathcal{G}$	$(4,0,-5,0)$	0
$\mathcal{H}$	$(2(1-n), 2n(n-1), 2(1-n), 0)$	$2n(n-1)-1$
$\mathcal{I}$	$\left(\frac{2(n-2)}{2n-1}, \frac{(5-4n)n}{2n^2-3n+1}, \frac{5-4n}{2n^2-3n+1}, 0\right)$	0
$\mathcal{L}$	$\left(-\frac{3(n-1)(w+1)}{n}, \frac{-4n+3w+3}{2n}, \frac{-4n+3w+3}{2n^2}, \frac{-2(3w+4)n^2+(9w+13)n-3(w+1)}{2n^2}\right)$	0

Table 5.5: Solutions associated to the fixed points of the $\alpha R + \beta R^n$ model.

Point	Scale Factor	Energy Density
$\mathcal{A}$	$a = a_0\left(t-t_0\right)$	0
$\mathcal{B}$	$a = a_0\left(t-t_0\right)^{1/2}$	0
$\mathcal{C}$	$a = a_0\left(t-t_0\right)$	0
$\mathcal{D}$	$a = a_0\left(t-t_0\right)^{1/2}$	0
$\mathcal{E}^*$	$\begin{cases} a = a_0, \\ a = a_0 \exp\left[\pm 2\sqrt{3}\alpha^\gamma (2-3n)^\gamma (t-t_0)\right], \end{cases}$ $\gamma = \frac{1}{2(1-n)}$	0
$\mathcal{F}$	$a = \left(t-t_0\right)$	0
$\mathcal{G}$	$a = a_0\left(t-t_0\right)^{1/2}$	0
$\mathcal{H}$	$a = \sqrt{1-2n(n-1)}\left(t-t_0\right)$	0
$\mathcal{I}^*$	$a = a_0\left(t-t_0\right)^{\frac{2n^2-3n+1}{2-n}}$	$\mu^m = \mu_{m\,0} t^{-\frac{3\left(2n^2-3n+1\right)(w+1)}{n-2}}$
$\mathcal{L}$	$a = a_0\left(t-t_0\right)^{\frac{2n}{3(w+1)}}$	$\mu_m = \mu_{m,0}(t-t_0)^{2p}$

An interesting result of the analysis for this model is that there exists a fixed point $\mathcal{L}$ that corresponds to a transient Friedmann-type behavior just as in the R^n case for exactly the same values of the dynamical system variables.

Table 5.6: The stability of the fixed points in the $\alpha R + \beta R^n$ model.

Point	Stability
$\mathcal{B}$	saddle
$\mathcal{B}$	$\begin{cases} \text{repellor} & 0 < w < 2/3 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{C}$	saddle
$\mathcal{D}$	$\begin{cases} \text{repellor} & 2/3 < w < 1 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{E}$	$\begin{cases} \text{attractor} & \frac{32}{25} \leq n < 2 \\ \text{spiral}^{+} & 0 < n < \frac{32}{25} \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{F}$	saddle
$\mathcal{G}$	saddle
$\mathcal{H}$	$\begin{cases} \text{attractor} & \frac{1}{2}(1-\sqrt{3}) < n \leq 0 \\ \text{spiral}^{+} & 0 < n < 1 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{I}$	$\begin{cases} \text{attractor} & n < \frac{1}{2}(1-\sqrt{3}) \cup n > 2, \\ \text{repeller} & \begin{cases} 1 < n < \frac{5}{4}, (w = 0, 1/3), \\ 1 < n < \frac{1}{14}(11+\sqrt{37}), (w = 1) \end{cases} \\ \text{saddle} & \text{otherwise,} \end{cases}$
$\mathcal{L}$	$\begin{cases} w = 0, 1/3 & \text{saddle,} \\ w = 1 & \begin{cases} \text{repellor} & 1 < n\frac{1}{14}\left(11+\sqrt{37}\right), \\ \text{saddle} & \text{otherwise} \end{cases} \end{cases}$

PART II

This part of the thesis focuses on cosmological applications of the topics discussed in chapters 3-5. In particular, we apply the $1+3$ covariant decomposition and cosmological dynamical systems techniques in the analysis of perturbations in multi-component cosmic fluids where the underlying gravitational dynamics is described by $f(R)$ gravitation and the background spacetime is FLRW.

According to the current understanding of large scale structure (LSS) formation, quantum fluctuations in the very early universe were amplified during the inflationary phase of expansion. These amplified quantum fluctuations became the very small density fluctuations - the observed fluctuations in the CMB power spectra, which in turn are related to the matter power spectra that we can observe today - that seeded the growth of large scale structure (the hierarchical structure of stars, star clusters, galaxies, galaxy clusters and superclusters, voids, walls and filaments [124, 244]). The clustering patterns of these large scale structures encode imprints of the distribution of photons and baryons in the form of *baryon acoustic oscillations* (BAOs), caused by the propagation of sound waves in the relativistic plasma of the early universe [39, 114, 245, 287]. This makes BAOs excellent standard ruler candidates in mapping out distance-redshift relations, which in turn can give us strong constraints on the models of gravitation that allow compatible cosmic expansion histories.

In light of this, we present a completely general set of perturbation equations for a multi-component fluid system for any $f(R)$ theory of gravity in Chapter 6 and use these equations for a cosmological medium with radiation and dust as matter fluid sources in Chapter 7. There we employ the dynamical systems approach to analyze the background solutions and use these background solutions to analyze the matter power spectra for R^n models and compare the constraints arising from the full shape of the SDSS matter power spectra and the ones obtained using BAO data.

Chapter 6

Cosmological Perturbations in $f(R)$

> If what appears little be universally despised, nothing greater can be attained; for all that is great was at first little, and rose to its present bulk by gradual accessions and accumulated labours.
>
> Samuel Johnson

6.1 Introduction

The current understanding of large scale structure formation is that the dynamical evolution of small density perturbations, seeded in the early universe, led to the large-scale structure we see today [22, 31, 33, 41, 89, 99, 179, 205, 215, 240, 244, 246]. As mentioned earlier, a detailed analysis of the background FLRW models using dynamical system techniques has shown that there exist classes of $f(R)$ theories which admit a transient decelerated expansion phase during which structure formation can take place, followed by a dark energy-like era which drives the present cosmological acceleration (see [92, 132, 210, 223, 267, 276] among many others).

An important feature of these $f(R)$ theories of gravity is that the field equations can be written in a way which makes it easy to compare with GR. This is done by moving all the higher-order corrections to the curvature onto the RHS of the field equations and defining an "effective" source term, known as the *curvature fluid*. Once this has been done the strong energy condition can be easily violated and this gives rise to a curvature fluid-driven period of late-time acceleration. Unfortunately this comes at the cost of having to study a considerably more complex set of field equations, making it difficult to obtain both exact and numerical solutions which can be compared with observations. Many studies of the expansion history of $f(R)$ gravity have been performed using a range of strategies for numerically integrating the cosmological equations and these studies have highlighted, among other things, how sensitive results are to initial conditions, the presence of rip singularities and oscillations in the deceleration and snap parameters, and the existence and stability of the Einstein Static models and bounce solutions [15,16,18–20,62,69,70,94,96,138, 151,214,226,259,266].

These difficulties can be reduced somewhat by using the theory of dynamical systems discussed in the previous chapter, which, with careful choice of dynamical variables, provides a relatively simple method for obtaining exact solutions (via the equilibrium points of the system) and a description of the global dynamics of these models for a given $f(R)$ theory.

Another useful (but more limited) approach is to assume that the expansion history of the Universe is known exactly, and to invert the field equations to deduce what class of $f(R)$ theories give rise to this particular cosmological evolution [92]. This has recently been done for exact power-law solutions for the scale factor, corresponding to phases of cosmic evolution when the energy density is dominated by a perfect fluid. It was found that such expansion histories only exist for modifications of the type R^n [146]. It was also shown in [105] that the only $f(R)$ theory of gravity that admits an exact ΛCDM expansion history is standard General Relativity with a positive cosmological constant. If one wants to obtain this behaviour of the scale factor for more general functions of R, additional degrees of freedom need to be added to the matter sector. A more extensive analysis of *reconstruction methods* has been carried out in [79,115,224,228–230] to obtain theories which give an approximate description of deceleration-acceleration transitions in cosmology and also in [63] where the reconstruction method was based on standard cosmic parameters instead of specifying the time evolution of the scale factor.

Because it is possible to find background expansion histories which are consistent with the standard ΛCDM model, it is necessary to investigate the growth of structure in order to break this degeneracy. This requires extending the standard theory of cosmological perturbations for GR to $f(R)$ gravity. This has been done using both the metric based approach [28,93,167,213][1] originally developed for GR by Bardeen [22] and the $1+3$ covariant approach first introduced, among other things, to analyze the evolution of linear perturbations of Friedmann-Lemaître-Robertson-Walker (FLRW) models in GR by Ellis and Bruni [1,4,12,28,44,45,68,101,104,106,128,169,171,172, 252,283,298]. For example, in [10,12,58], evolution equations were obtained for scalar and tensor perturbations for $f(R)$ gravity and applied to the spatially flat, matter dominated solution of R^n gravity given by $a(t) = a_0 t^{\frac{2n}{3(1+w)}}$ (which is a saddle point G of the corresponding dynamical system for these theories).

In [58] the evolution of scalar perturbations of FLRW models in fourth order gravity was developed for single barotropic fluids using the 1+3 covariant approach.

[1]It has been proved [93] that when dust matter scalar cosmological perturbations are studied in the metric formalism, $f(R)$ theories, even mimicking the standard cosmological expansion, provide a different matter power spectrum from that predicted by the ΛCDM model [95].

The solutions of the perturbation equations on large-scales showed that a decelerated phase is not necessarily required to form large scale structures. This divergence from the standard GR provides us with a distinguishable signature of fourth order theories, which can be tested against observations.

Since the Universe consists of a mixture of fluids, a complete treatment of perturbations in fourth order theories requires taking this into account. The aim of this chapter is, therefore, to present a general framework for studying multi-fluid cosmological perturbations with a completely general equation of state in a generic $f(R)$ theory of gravity, using the $1+3$-covariant approach.

The standard perturbation theory based on the metric formalism has disadvantages when it comes to extracting physical information from the perturbation variables. For example, it requires a complete specification of the correspondence between the lumpy, perturbed universe and the background spacetime. In other words, this approach is *gauge-dependent*. The covariant formalism, on the other hand, is a fluid approach, which, when applied to cosmological perturbations, leaves no unphysical modes in the evolution of the fluctuations: it requires no prior metric specification and is *gauge-invariant* by construction.

Fluid flow vector u^a is uniquely defined as the future directed timelike eigenvector of the Ricci tensor: $u^a = \frac{dx^a}{d\tau}$, where $x^a(\tau)$ describes the worldline of the fluid in terms of the proper time τ. In our multi-fluid picture, it corresponds to the normal to the surface of homogeneity. In this formalism, the covariant convective and spatial covariant derivatives on a scalar function X are adapted, respectively, from Eqns (4.10) and (4.11):

$$\dot{X} = u_a \nabla^a X, \qquad \tilde{\nabla}_a X = h_a{}^b \nabla_b X \ , \tag{6.1}$$

and the geometry of the flow lines is determined by the kinematics of u^a:

$$\nabla_b u_a = \tilde{\nabla}_b u_b - a_a u_b \ , \tag{6.2}$$

$$\tilde{\nabla}_b u_a = \tfrac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab} \ . \tag{6.3}$$

From (6.2) and (6.3) we obtain Eqn (4.14), the key equation relating the kinematic quantities.

The evolution of the expansion is given by the Raychaudhuri equation (given here for the FLRW background) :

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \frac{1}{2}(\mu + 3p) = 0 \ , \tag{6.4}$$

where μ and p hold for the total energy density and isotropic pressure respectively. This equation together with the equation of state $p = p(\mu, s)$, the energy conservation equation

$$\dot{\mu} + \Theta(\mu + p) = 0 \ , \tag{6.5}$$

and the Friedmann equation

$$\Theta^2 + \frac{9K}{a^2} - 3\mu = 0 \ , \tag{6.6}$$

form a closed system of equations and completely characterize the kinematics of the background cosmological model.

6.2 Matter Description

6.2.1 Effective Total Energy-Momentum Tensor

The thermodynamical description of a relativistic fluid is dictated by the total energy momentum tensor $\mathcal{T}_{ab} \equiv T^{total}{}_{ab}$, the particle flux N^a and the entropy flux S^a of the system. Whereas T^{ab} and S^a always satisfy respectively the conservation of 4-momentum and the second law of thermodynamics, namely

$$\mathcal{T}^{ab}{}_{;b} = 0 \;, \quad S^a{}_{;a} \geq 0 \;, \tag{6.7}$$

the condition on the particle flux conservation, i.e., $N^a{}_{;a} = 0$, is not binding.

The total energy-momentum tensor in a general frame is sourced by μ, p, the energy flux $q_{\langle a\rangle}$, and the anisotropic pressure $\pi_{\langle ab\rangle}$:

$$\mathcal{T}_{ab} = \mu u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab} = \tilde{\mathcal{T}}^m_{ab} + T^R_{ab} \;, \tag{6.8}$$

with $\tilde{\mathcal{T}}^m_{ab} = \frac{\mathcal{T}^m_{ab}}{f'}$, $\mathcal{T}^m_{ab}$ being the total EMT of standard matter given by (6.15). It defines our thermodynamical quantities:

$$\mu^{tot} = \mathcal{T}_{ab}u^a u^b = \tilde{\mu}_m + \mu_R \;, \tag{6.9}$$
$$p^{tot} = \tfrac{1}{3}\mathcal{T}_{ab}h^{ab} = \tilde{p}_m + p_R \;, \tag{6.10}$$
$$q^{tot}_a = -\mathcal{T}_{bc}h^b{}_a u^c = \tilde{q}^m_a + q^R_a , \tag{6.11}$$
$$\pi^{tot}_{ab} = \mathcal{T}_{cd}h^c_{\langle a}h^d_{b\rangle} = \tilde{\pi}^m_{ab} + \pi^R_{ab} \;, \tag{6.12}$$

where in this chapter $\tilde{\mu}_m = \frac{\mu_m}{f'}$, etc., are the effective (total) thermodynamic quantities of matter.

If we impose the SEC $\mathcal{T}_{ab}V^aV^b \geq 0$ for all timelike vectors V^a, then $\mathcal{T}_{ab}$ will have a unique unit timelike vector u^a_E ($u^a_E u^E_a = -1$). Another timelike vector u^a_N can be defined along the flux N^a_N, i.e., $u^a_N = \frac{N^a}{\sqrt{-NbN^b}}$.

For a perfect fluid (or an unperturbed fluid in the background space), u^a_E, u^a_N and S^a are all parallel [104] and a unique hydrodynamic 4-velocity u^a can be defined for the fluid flow, in which case

$$\mathcal{T}_{ab} = \mu u_a u_b + p h_{ab} \;, \quad N^a = n u^a, \quad S^a = s u^a \;, \tag{6.13}$$

where μ and p are related by the equation of state $p = p(\mu, s)$. $n = -N^a u_a$ and $s = -S_a u^a$ define the particle and entropy densities respectively in the local rest frame of an observer attached to u^a.

We can also decompose the EMT with respect to another frame, say n^a, but in this case we need to introduce a particle drift $\tilde{j}^a = \tilde{h}^a{}_b N^a$ [44, 104, 178].

If the fluid is imperfect, the fluid hydrodynamic 4-velocity is no longer unique and our EMT will take the more general form given above Eqn (8.2) and the particle flux includes a *drift* term:

$$N^a = nu^a + j^a \ . \tag{6.14}$$

Choosing the relevant frame is a crucial step in the covariant formulation of perturbation theories [4], since u^a is the velocity of fundamental observers in the Universe.

In the particle frame $u^a = u^a_N$, called the Eckart choice, an observer $O_{u=u_N}$ sees no particle drift and hence $j^a = j^a_N = 0$. If, on the other hand, we consider the energy frame $u^a = u^a_E$, also known as the Landau choice, an observer O_{u_E} measures no energy flux ($q_a = q^E_a = 0$) along the flow line and the EMT takes the form (6.13).

For multi-component matter fluids we have

$$\mathcal{T}^m_{ab} = \sum_i T^i_{ab} \ , \tag{6.15}$$

where

$$T^i_{ab} = \mu_i u^i_a u^i_b + p_i h^i_{ab} + q^i_a u^i_b + q^i_b u^i_a + \pi^i_{ab} \ , \tag{6.16}$$
$$h^i_{ab} = g_{ab} + u^i_a u^i_b \ , \tag{6.17}$$
$$N^a_i = n_i u^a_i + j^a_i \ , \tag{6.18}$$

u^i_a being the normalized fluid 4-velocity vector for the i^{th} component, $u^a_i u^i_a = -1$, which we can fix by either choosing the energy frame $u^a_i = u^a_{Ei}$ thereby setting $q^a_i = q^a_{Ei} = 0$, or the particle frame $u^a_i = u^a_{Ni}$ for which $j^a_i = j^a_{Ni} = 0$ for that component. The velocity of the i^{th} fluid component relative to the fundamental observer O_u is defined to be

$$V^a_i \equiv u^a_i - u^a \ . \tag{6.19}$$

$V^a_i \neq 0$ for *tilted*, inhomogeneous cosmological media whereas the special case where u^a_i coincides with u^a describes an *untilted* homogeneous cosmological medium. Decomposition of the total matter EMT with respect to the 4-velocity u^a gives the following thermodynamical quantities:

$$\mu_m = \mathcal{T}^m_{ab} u^a u^b = \textstyle\sum_{i=1}^N \mu_i \ , \tag{6.20}$$
$$p_m = \tfrac{1}{3}\mathcal{T}^m_{ab} h^{ab} = \textstyle\sum_{i=1}^N p_i \ , \tag{6.21}$$
$$q^m_a = -\mathcal{T}^m_{bc} h^b_a u^c = \textstyle\sum_{i=1}^N (\mu_i + p_i) V^i_a \ , \tag{6.22}$$
$$\pi^m_{ab} = T^m_{cd} h^c_{\langle a} h^d_{b\rangle} = 0 \quad \text{(to first order)}. \tag{6.23}$$

Decomposing the EMT of the curvature fluid in a similar fashion yields the corresponding thermodynamical quantities (4.73)-(4.76).

In the background FLRW universe, $V^a_i = 0$ and all perfect fluid components have the same 4-velocity. By applying the Stewart-Walker Lemma [106, 284], we can show that V^a_i is a first-order gauge-invariant (GI) quantity. If we choose the fluid flow vector u^a to coincide with the energy frame u^a_E (see Fig.1 above), then exact FLRW models will be characterized by vanishing shear and vorticity of u^a and all spatial

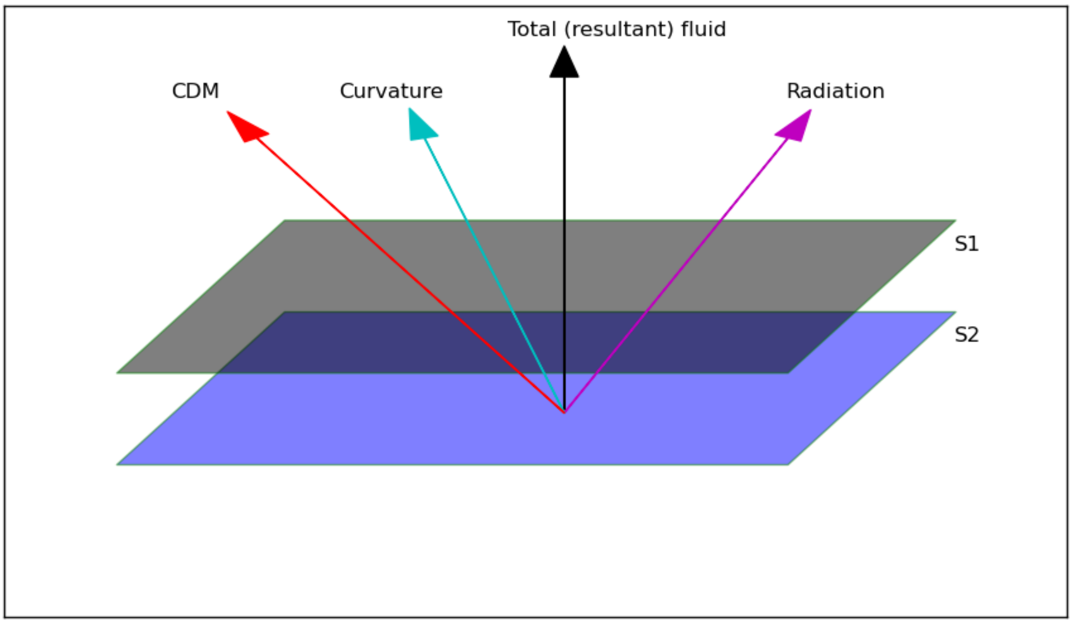

Figure 6.1: The Multi-fluid diagram: The different arrows show the unit time-like four-velocity vectors at different hyper-surfaces S1 and S2. The vectors do not coincide at the perturbative level.

gradients orthogonal to u^a of any scalar quantity [44]:

$$\sigma_{ab} = \omega_{ab} = 0, \qquad \tilde{\nabla}_a X = 0 \; . \tag{6.24}$$

It then follows that, since

$$X_a = \tilde{\nabla}_a \mu = 0, \quad Y_a = \tilde{\nabla}_a p, \quad Z_a = \tilde{\nabla}_a \Theta = 0 \tag{6.25}$$

in the background, then $\mu = \mu(t)$, $p = p(t)$ and $\Theta = \Theta(t)$. This necessitates the energy momentum tensor having the perfect fluid form, and hence the vanishing of the anisotropic pressure π_{ab} and the energy flux q_a.

6.2.2 Inhomogeneity Variables for the Total Matter

The key variables characterizing the inhomogeneities of matter are defined as

$$\begin{aligned} D^m_a &= a\frac{\tilde{\nabla}_a \mu_m}{\mu_m} \; , & Y_a &= \tilde{\nabla}_a p_m, \\ Z_a &= a\tilde{\nabla}_a \Theta, & C_a &= a\tilde{\nabla}_a \tilde{R}, \\ \varepsilon_a &= \frac{a}{p_m}\left(\frac{\partial p}{\partial s}\right)\tilde{\nabla}_a s \; , & A &= a^a{}_{;a} = \tilde{\nabla}_a a^a \; , \\ A_a &= \tilde{\nabla}_a A \; , & Q &= q^a{}_{;a} \simeq \tilde{\nabla}_a q \; , \end{aligned} \tag{6.26}$$

where $a \equiv a(t)$ here is the usual FLRW cosmological scale factor. D_a^m and Z_a define the comoving fractional density gradient and comoving gradient of the expansion respectively and can in principle be measured observationally [44]. The relation

$$p\varepsilon_a = \sum_i p_i \varepsilon_a^i + \frac{1}{2}\sum_{i,j} \frac{h_i h_j}{h}(c_{si}^2 - c_{sj}^2) S_a^{ij} \tag{6.27}$$

defines the dimensionless variable ε_a that quantifies entropy perturbations in the total fluid. We have defined the shorthand $h \equiv \mu_m + p_m$ for the total matter fluid and $h_i \equiv \mu_i + p_i$ for the component matter fluids. w and c_s^2 denote the effective barotropic equation of state and speed of sound of the total matter fluid, respectively, and are defined by

$$w \equiv \frac{p_m}{\mu_m}, \qquad c_s^2 \equiv \frac{\partial p_m}{\partial \mu_m} , \tag{6.28}$$

whereas for each component matter fluid, these two quantities are given by $w_i \equiv p_i/\mu_i$ and $c_{si}^2 \equiv \partial p_i / \partial \mu_i$.

6.2.3 Matter Inhomogeneity Variables for the Components

The variables characterizing inhomogeneities of matter for the i^{th}- component fluid are defined as

$$D_a^i = a\frac{\tilde{\nabla}_a \mu_i}{\mu_i} , \qquad Y_a^i = \tilde{\nabla}_a p_i , \qquad \varepsilon_a^i = \frac{a}{p^i}\left(\frac{\partial p^i}{\partial s_i}\right)\tilde{\nabla}_a s_i . \tag{6.29}$$

In near-perfect fluid analyses such as the present one, ε_a^i is often taken to be negligible. Thus in subsequent discussions all terms containing this quantity are dropped.

6.2.4 Curvature Fluid Variables

The information about our deviation from standard GR is carried by the following dimensionless gradient quantities :

$$\mathcal{R}_a = a\tilde{\nabla}_a R , \qquad \Re_a = a\tilde{\nabla}_a \dot{R} . \tag{6.30}$$

These variables describe the inhomogeneities in the Ricci scalar. Finally, the velocity of the curvature fluid is defined, following [45], by

$$V_a^R = -\frac{\tilde{\nabla}_a R}{\dot{R}} , \tag{6.31}$$

provided that $\dot{R} \neq 0$. Cases with constant scalar configurations or pathologic $f(R)$ models with points of inflection in $R(t)$ are excluded from this analysis.

6.3 Equations in the Energy Frame

We now derive the time evolution of the perturbations to linear order in the energy frame of the matter, i.e., in the frame where $u^a = u^a_m$.

6.3.1 Total Fluid Equations

These equations characterize the temporal fluctuations of inhomogeneities in a generic perfect cosmological fluid with an equation of state evolving as $\dot{w} = (1+w)(w-c_s^2)\Theta$. They are the following:

$$\dot{D}_a + (1+w)Z_a - w\Theta D_a = 0 \,, \tag{6.32}$$

$$\begin{aligned}
\dot{Z}_a &- \left(\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta\right) Z_a - \left[\frac{(1+3w)c_s^2 - 2(1+w)}{2(1+w)}\frac{\mu_m}{f'} + \frac{2c_s^2\Theta^2 + 3c_s^2(\mu_R + 3p_R)}{6(1+w)}\right. \\
&\left. + \frac{c_s^2}{1+w}\frac{2K}{a^2}\right] D^m_a - \left[\frac{2f'\Theta^2 + 3(1+3w)\mu_m + 3f'(\mu_R + 3p_R)}{6f'(1+w)} + \frac{1}{1+w}\frac{2K}{a^2}\right] w\varepsilon_a \\
&- \Theta\frac{f''}{f'}\Re_a - \left[\frac{1}{2} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta\left(\frac{f''}{f'}\right)^2 + \dot{R}\Theta\frac{f'''}{f'} + \frac{2K}{a^2}\frac{f''}{f'}\right]\mathcal{R}_a \\
&+ \frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}_a + \frac{c_s^2\tilde{\nabla}^2 D^m_a}{1+w} + \frac{w\tilde{\nabla}^2\varepsilon_a}{1+w} = 0 \,,
\end{aligned} \tag{6.33}$$

$$\dot{\mathcal{R}}_a - \Re_a + \dot{R}\left[\frac{c_s^2}{1+w}D_a + \frac{w}{1+w}\varepsilon_a\right] = 0 \,, \tag{6.34}$$

$$\begin{aligned}
\dot{\Re}_a &+ \left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\Re_a - \left[\frac{(1-3c_s^2)\mu_m}{3f''} - \frac{c_s^2}{1+w}\ddot{R}\right] D_a + \left(\frac{w\mu_m}{f''} + \frac{w}{1+w}\ddot{R}\right)\varepsilon_a \\
&+ \dot{R}Z_a + \left(\frac{2K}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(iv)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}_a - \tilde{\nabla}^2\mathcal{R}_a = 0 \,.
\end{aligned} \tag{6.35}$$

The covariant, GI gradient C_a gives, to linear order,

$$\begin{aligned}
\frac{C_a}{a^2} \quad &+ \left(\tfrac{4}{3}\Theta + 2\tfrac{\dot{R}f''}{f'}\right) Z_a - 2\tfrac{\mu_m}{f'}D^m_a + 2\Theta\tfrac{f''}{f'}\Re_a - 2\tfrac{f''}{f'}\tilde{\nabla}^2\mathcal{R}_a \\
&+ \left[2\Theta\dot{R}\tfrac{f'''}{f'} - \tfrac{f''}{f'}\left(\tfrac{f}{f'} - 2\tfrac{\mu_m}{f'} + 2\dot{R}\Theta\tfrac{f''}{f'} + \tfrac{4K}{a^2}\right)\right]\mathcal{R}_a = 0 \,.
\end{aligned} \tag{6.36}$$

This variable quantifies the spatial variation in the 3-curvature and is a geometrically natural quantity useful in the long-wavelength analysis of our perturbation equations. The evolution equation of this quantity is given [56, 58] by

$$\dot{C}_a \quad = 2K\left\{\tfrac{3f'C_a}{a^2(2\Theta f' + 3\dot{R}f'')} + D^m_a\left[\tfrac{8w\Theta}{3(1+w)} - \tfrac{2f'\Theta^2 - 6f'\mu_R}{2\Theta f' + 3\dot{R}f''}\right] - \tfrac{6f''\tilde{\nabla}^2\mathcal{R}_a}{2\Theta f' + 3\dot{R}f''}\right.$$

$$
-\left[\frac{a^2(\Theta f''-3\dot{R}f''')}{3f'}\,\frac{12\dot{R}\Theta f'f'''-2f''\left(3f-2\Theta^2f'+6\Theta^2\mu_R+6\dot{R}\Theta f''\right)}{2\Theta f'+3\dot{R}f''}\right]\mathcal{R}_a
$$
$$
+\left[\frac{6\Theta f''}{2\Theta f'+3\dot{R}f''}+\frac{f''}{f'}\right]\Re_a\Bigg\}+K^2\left[\frac{36f''\mathcal{R}_a}{a^2(2\Theta f'+3\dot{R}f'')}-\frac{36f'D^m_a}{a^2(2\Theta f'+3\dot{R}f'')}\right]
$$
$$
+\tfrac{2}{3}\tilde{\nabla}^2\left\{\tfrac{2wa^2\Theta}{(1+w)}D^m_a+\tfrac{a^2}{f'}\left[3f''\Re_a-(\Theta f''-3\dot{R}f'')\mathcal{R}_a\right]\right\}\;. \tag{6.37}
$$

6.3.2 Component Equations

These are the equations that describe the evolution dynamics of the individual fluid component fluctuations. For the component matter and velocity fluctuations, these are given by

$$
\dot{D}^i_a-(w_i-c^2_{si})\Theta D^i_a+(1+w_i)Z_a=\frac{1}{\mu_i h}h_i\Theta(c^2_s\mu D_a+p\varepsilon_a)
$$
$$
-a(1+w_i)\tilde{\nabla}_a\tilde{\nabla}^bV^i_b\;, \tag{6.38}
$$
$$
\dot{V}^i_a-\left(c^2_{si}-\frac{1}{3}\right)\Theta V^i_a=\frac{1}{ahh_i}\left(h_ic^2_s\mu D_a+h_ip\varepsilon_a-hc^2_{si}\mu^iD^i_a\right)\;. \tag{6.39}
$$

We note that the equations involving the gradients of the inhomogeneities in the expansion and curvature variables $(Z_a,\mathcal{R}_a,\Re_a,C_a)$ remain the same as in the total fluid equations (6.33)-(6.37). This is to be expected since these quantities are global intrinsic properties of the spacetime itself rather than of the individual components of matter in the fluid.

6.3.3 Relative Equations

Let us now define the variables that relate features of pairs of the different components of the fluid, and derive their governing evolution equations. These relative variables depend only on the choice of the individual velocities, not on the choice of the overall frame.

$$
S^{ij}_a\equiv\frac{\mu_iD^i_a}{h_i}-\frac{\mu_jD^j_a}{h_j}\;,\qquad V^{ij}_a\equiv V^i_a-V^j_a\;. \tag{6.40}
$$

These are the quantities that allow us to distinguish between adiabatic and isothermal perturbations [104, 179].

The derivation of the evolution equations for the above quantities is straightforward and yields

$$
\dot{V}^{ij}_a-(c^2_{sj}-\tfrac{1}{3})\Theta V^{ij}_a-(c^2_{si}-c^2_{sj})\Theta V^i_a=-\frac{1}{ah_i}(c^2_{si}-c^2_{sj})\mu_iD^m_i
$$
$$
-\frac{1}{a}c^2_{sj}S^{ij}_a\;, \tag{6.41}
$$
$$
\dot{S}^{ij}_a+a\tilde{\nabla}_a\tilde{\nabla}^bV^{ij}_b=0\;. \tag{6.42}
$$

6.4 Scalar Equations

The quantities we have considered so far contain both a scalar and a vector (solenoidal) part. Structure formation on cosmological scales is believed to follow spherical clustering and therefore we present here the spherically symmetric, scalar density perturbations obtained by taking the divergence of the gradient quantities. In so doing, we first apply a *local decomposition*

$$a\tilde{\nabla}_b X_a = X_{ab} = \tfrac{1}{3}h_{ab}X + \Sigma^X_{ab} + X_{[ab]} \ , \tag{6.43}$$

where $\Sigma^X_{ab} = X_{(ab)} - \frac{1}{3}h_{ab}X$ describes shear whereas $X_{[ab]}$ describes the vorticity. Vorticity and shear describe the rotation and distortion of the density gradient field, respectively. The above decomposition extracts the scalar part of the perturbation vectorial gradients. Accordingly, when extracting the scalar contribution the vorticity term vanishes [118].

6.4.1 Scalar Variables

On the basis of the above decomposition scheme, our scalar variables are:

$$\begin{aligned}
&\Delta_m = a\tilde{\nabla}^a D^m_a \ , \qquad Z = a\tilde{\nabla}^a Z_a \ , \qquad C = a\tilde{\nabla}^a C_a \ , \qquad \mathcal{R} = a\tilde{\nabla}^a \mathcal{R}_a \ , \\
&\Re = a\tilde{\nabla}^a \Re_a \ , \qquad \varepsilon = a\tilde{\nabla}^a \varepsilon_a \ , \qquad \Delta^i_m = a\tilde{\nabla}^a D^i_a \ , \qquad V_i = a\tilde{\nabla}^a V^i_a \ , \\
&S_{ij} = a\tilde{\nabla}^a S^{ij}_a \ , \qquad V_{ij} = a\tilde{\nabla}^a V^{ij}_a \ .
\end{aligned} \tag{6.44}$$

The scalar variables describing the total fluid will thus evolve according to

$$\dot{\Delta}_m + (1+w)Z - w\Theta\Delta_m = 0 \ , \tag{6.45}$$

$$\begin{aligned}
&\dot{Z} - \left(\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta\right)Z - \left[\frac{1}{2} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta\left(\frac{f''}{f'}\right)^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R} \\
&+\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R} - \left[\frac{(1+3w)c_s^2 - 2(1+w)}{2(1+w)}\frac{\mu_m}{f'} + \frac{2c_s^2\Theta^2 + 3c_s^2(\mu_R + 3p_R)}{6(1+w)}\right]\Delta_m \\
&-\left[\frac{2f'\Theta^2 + 3(1+3w)\mu_m + 3f'(\mu_R + 3p_R)}{6f'(1+w)}\right]w\varepsilon - \Theta\frac{f''}{f'}\Re + \frac{c_s^2}{1+w}\tilde{\nabla}^2\Delta_m \\
&+\frac{w}{1+w}\tilde{\nabla}^2\varepsilon = 0 \ ,
\end{aligned} \tag{6.46}$$

$$\dot{\mathcal{R}} - \Re + \dot{R}\left(\frac{c_s^2}{1+w}\Delta_m + \frac{w}{1+w}\varepsilon\right) = 0 \ , \tag{6.47}$$

$$\begin{aligned}
&\dot{\Re} + \left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\Re + \left(\ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(iv)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R} - \tilde{\nabla}^2\mathcal{R} \\
&+\dot{R}Z - \left[\frac{(1-3c_s^2)\mu_m}{3f''} - \frac{c_s^2}{1+w}\ddot{R}\right]\Delta_m + \left[\frac{w\mu_m}{f''} + \frac{w}{1+w}\ddot{R}\right]\varepsilon = 0 \ ,
\end{aligned} \tag{6.48}$$

$$\frac{C}{a^2} + \left(\frac{4}{3}\Theta + 2\frac{\dot{R}f''}{f'}\right)Z + \left[2\Theta\dot{R}\frac{f'''}{f'} - \frac{f''}{f'}\left(\frac{f}{f'} - 2\frac{\mu_m}{f'} + 2\dot{R}\Theta\frac{f''}{f'}\right)\right]\mathcal{R}$$

$$-2\frac{\mu_m}{f'}\Delta_m + 2\Theta\frac{f''}{f'}\Re - 2\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R} = 0\ . \tag{6.49}$$

The evolution of the constraint equation is given by

$$\begin{aligned}\dot{C} = K\Bigg\{&\frac{6f'C}{a^2(2\Theta f' + 3\dot{R}f'')} + \Delta_m\left[\frac{16w\Theta}{3(1+w)} - \frac{4f'\Theta^2 - 12f'\mu_R}{2\Theta f' + 3\dot{R}f''}\right] - \frac{12f''\tilde{\nabla}^2\mathcal{R}}{2\Theta f' + 3\dot{R}f''}\\ &- \left[\frac{2a^2(\Theta f'' - 3\dot{R}f''')}{3f'}\frac{12\dot{R}\Theta f'f''' - 2f''\left(3f - 2\Theta^2 f' + 6\Theta^2\mu_R + 6\dot{R}\Theta f''\right)}{2\Theta f' + 3\dot{R}f''}\right]\mathcal{R}\\ &+ \left(\frac{12\Theta f''}{2\Theta f' + 3\dot{R}f''} + \frac{2f''}{f'}\right)\Re\Bigg\} + K^2\left[\frac{36f''\mathcal{R}}{a^2(2\Theta f' + 3\dot{R}f'')} - \frac{36f'\Delta m}{a^2(2\Theta f' + 3\dot{R}f'')}\right]\\ &+\tilde{\nabla}^2\left[\frac{4wa^2\Theta}{3(1+w)}\Delta_m + \frac{2a^2f''}{f'}\Re - \frac{2a^2(\Theta f'' - 3\dot{R}f'')}{3f'}\mathcal{R}\right]\ .\end{aligned} \tag{6.50}$$

For the scalar variables describing component inhomogeneities and interactions in the fluid, the evolution equations are given by

$$\begin{aligned}&\dot{\Delta}^i_m - \Theta(w_i - c^2_{si})\Delta^i_m + (1+w_i)Z = \frac{1+w_i}{1+w}\left(c^2_s\Theta\Delta_m + w\Theta\varepsilon\right)\\ &\quad - a(1+w_i)\tilde{\nabla}^2 V_i\ ,\end{aligned} \tag{6.51}$$

$$\dot{V}_i - \left(c^2_{si} - \frac{1}{3}\right)\Theta V_i = \frac{1}{ahh_i}\left(h_i c^2_s\mu\Delta_m + h_i p\varepsilon - hc^2_{si}\mu^i\Delta^i_m\right)\ , \tag{6.52}$$

$$\begin{aligned}&\dot{V}_{ij} - \left(c^2_{sj} - \frac{1}{3}\right)\Theta V_{ij} - (c^2_{si} - c^2_{sj})\Theta V_i = -\frac{1}{ah_i}(c^2_{si} - c^2_{sj})\mu_i\Delta^i_m\\ &\quad - \frac{1}{a}c^2_{sj}S_{ij}\ ,\end{aligned} \tag{6.53}$$

$$\dot{S}_{ij} + a\tilde{\nabla}^2 V_{ij} = 0\ . \tag{6.54}$$

6.4.2 Second-order Equations

The above first order equations (6.49)-(6.54) can be reduced to a set of linearly independent second order equations. This has the advantage of simplifying the equations and making comparisons to GR more transparent [58]:

$$\begin{aligned}&\ddot{\Delta}_m + \left[(c^2_s + \frac{2}{3} - 2w)\Theta - \dot{R}\frac{f''}{f'}\right]\dot{\Delta}_m - c^2_s\tilde{\nabla}^2\Delta_m + \Bigg[\left(\frac{3}{2}w^2 + 5c^2_s - 4w - 1\right)\frac{\mu_m}{f'}\\ &+\frac{1}{2}(3w - 5c^2_s)\frac{f}{f'} + (c^2_s - w)\left(2R - 4\dot{R}\Theta\frac{f''}{f'} - \frac{12K}{a^2}\right)\Bigg]\Delta_m - (1+w)\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}\\ &= \frac{1+w}{2}\left[-1 + (f - 2\mu_m + 2\dot{R}\Theta f'')\frac{f''}{f'^2} - 2\dot{R}\Theta\left(\frac{f''}{f'}\right)^2 - 2\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}\end{aligned}$$

$$-(1+w)\Theta\frac{f''}{f'}\dot{\mathcal{R}}-\left(2\frac{\mu_m}{f'}+\frac{R}{2}-\frac{f}{f'}-\dot{R}\Theta\frac{f''}{f'}-\frac{3K}{a^2}\right)w\varepsilon+w\tilde{\nabla}^2\varepsilon\;, \tag{6.55}$$

$$\begin{aligned}&\ddot{\mathcal{R}}+\left(2\dot{R}\frac{f'''}{f''}+\Theta\right)\dot{\mathcal{R}}+\left(\ddot{R}\frac{f'''}{f''}+\dot{R}^2\frac{f^{(iv)}}{f''}+\dot{R}\Theta\frac{f'''}{f''}+\frac{1}{3}\frac{f'}{f''}-\frac{R}{3}\right)\mathcal{R}\\&+\frac{c_s^2-1}{1+w}\dot{R}\dot{\Delta}_m+\left\{\frac{(3c_s^2-1)\mu_m}{3f''}+\frac{w+c_s^2}{1+w}\dot{R}\Theta+\frac{2c_s^2}{1+w}\left(\ddot{R}+\dot{R}^2\frac{f'''}{f''}\right)\right.\\&\left.+\frac{\dot{R}}{1+w}\left[\dot{c_s^2}+c_s^2(c_s^2-w)\Theta\right]\right\}\Delta_m-\tilde{\nabla}^2\mathcal{R}+\frac{w}{1+w}\dot{R}\dot{\varepsilon}\\&+\left[\frac{w\mu_m}{f''}+\frac{2w-c_s^2}{1+w}\dot{R}\Theta+\frac{2w}{1+w}\left(\ddot{R}+\dot{R}^2\frac{f'''}{f''}\right)\right]\varepsilon=0\;,\end{aligned} \tag{6.56}$$

$$\begin{aligned}&\ddot{\Delta}_i+\left(\frac{2}{3}-w_i\right)\Theta\dot{\Delta}_i-\frac{1+w_i}{1+w}\left[\dot{R}\frac{f''}{f'}+(c_s^2-c_{si}^2)\Theta\right]\dot{\Delta}_m-(1+w_i)\frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}\\&-\frac{1+w_i}{1+w}\left[(1+w)\frac{\mu_m}{f'}-\left(2\frac{\mu_m}{f'}-\frac{f}{f'}-2\Theta\dot{R}\frac{f''}{f'}\right)c_s^2+\dot{c_s^2}\Theta\right.\\&\left.+(c_s^2-c_{si}^2)(c_s^2-w)\Theta^2-(c_s^2+w)\dot{R}\Theta\frac{f''}{f'}\right]\Delta_m-\frac{1+w_i}{1+w}w\Theta\dot{\varepsilon}-c_{si}^2\tilde{\nabla}^2\Delta_i\\&+(1+w_i)\Theta\frac{f''}{f'}\dot{\mathcal{R}}+(1+w_i)\left[\frac{1}{2}-\frac{1}{2}\frac{ff''}{f'^2}+\frac{f''\mu_m}{f'^2}-\dot{R}\Theta\left(\frac{f''}{f'}\right)^2+\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}\\&-\frac{1+w_i}{1+w}\left[(w-c_s^2-c_{si}^2w)\Theta^2-w\left(2\frac{\mu_m}{f'}-\frac{f}{f'}-\dot{R}\Theta\frac{f''}{f'}\right)\right]\varepsilon=0\;.\end{aligned} \tag{6.57}$$

The second order equations (6.55)-(6.57) governing the propagation of the entropy perturbations for a general ε (or S_{ij}) are in general very complicated and consequently we will present them for specific (radiation-dust) applications in section 6.6.

6.5 Harmonic Analysis

The above evolution equations can be thought of as a coupled system of harmonic oscillators of the form

$$\ddot{X}+A\dot{X}+BX=C(Y,\dot{Y})\;, \tag{6.58}$$

where the second term from the left represents the friction (damping) term, the third one the restoring force term, while C represents the source forcing term. A key assumption in the analysis of the equation here is that we can apply the separation of variables technique

$$X(x,t)=X(\vec{x})X(t)\;,\qquad Y(x,t)=Y(\vec{x})Y(t)\;, \tag{6.59}$$

and write

$$X=\sum_k X^k(t)Q_k(\vec{x})\ ,\qquad Y=\sum_k Y^k(t)Q_k(\vec{x})\ ,\tag{6.60}$$

where $Q_k(x)$ are the eigenfunctions of the covariantly defined spatial Laplace-Beltrami operator [106, 128, 144] on an almost FLRW space-time:

$$\tilde{\nabla}^2 Q=-\frac{k^2}{a^2}Q\ .\tag{6.61}$$

Here $k=\frac{2\pi a}{\lambda}$ is the order of the harmonic and $\dot{Q}_k(\vec{x})=0$ (Q is covariantly constant). In this way the evolution equations and the constraint equation can be converted into ordinary differential equations for each mode. After harmonic decomposition the first order total and component fluid equations (6.49)-(6.54) can be rewritten in the following form:

$$\dot{\Delta}^k_m+(1+w)Z^k-w\Theta\Delta^k_m=0\ ,\tag{6.62}$$

$$\begin{aligned}\dot{Z}^k&-\left(\dot{R}\frac{f''}{f'}-\frac{2}{3}\Theta\right)Z^k\\&-\left[\frac{(1+3w)c_s^2-2(1+w)}{2(1+w)}\frac{\mu_m}{f'}+\frac{2c_s^2\Theta^2+3c_s^2(\mu_R+3p_R)}{6(1+w)}+\frac{c_s^2}{1+w}\frac{k^2}{a^2}\right]\Delta^k_m\\&-\left[\frac{2f'\Theta^2+3(1+3w)\mu_m+3f'(\mu_R+3p_R)}{6f'(1+w)}+\frac{1}{1+w}\frac{k^2}{a^2}\right]w\varepsilon^k-\Theta\frac{f''}{f'}\Re^k\\&-\left[\frac{1}{2}+\frac{k^2}{a^2}\frac{f''}{f'}-\frac{1}{2}\frac{ff''}{f'^2}+\frac{f''\mu_m}{f'^2}-\dot{R}\Theta\left(\frac{f''}{f'}\right)^2+\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k=0,\end{aligned}\tag{6.63}$$

$$\dot{\mathcal{R}}^k-\Re^k+\dot{R}\left(\frac{c_s^2}{1+w}\Delta^k_m+\frac{w}{1+w}\varepsilon^k\right)=0\ ,\tag{6.64}$$

$$\begin{aligned}\dot{\Re}^k&+\left(2\dot{R}\frac{f'''}{f''}+\Theta\right)\Re^k+\left[\frac{(1-3c_s^2)\mu_m}{3f''}-\frac{c_s^2}{1+w}\ddot{R}\right]\Delta^k_m+\left[\frac{w\mu_m}{f''}+\frac{w}{1+w}\ddot{R}\right]\varepsilon^k\\&+\dot{R}Z^k+\left(\frac{k^2}{a^2}+\ddot{R}\frac{f'''}{f''}+\dot{R}^2\frac{f^{(iv)}}{f''}+\dot{R}\Theta\frac{f'''}{f''}+\frac{1}{3}\frac{f'}{f''}-\frac{R}{3}\right)\mathcal{R}^k=0\ ,\end{aligned}\tag{6.65}$$

$$\begin{aligned}\frac{C^k}{a^2}&+\left(\frac{4}{3}\Theta+2\frac{\dot{R}f''}{f'}\right)Z^k-2\frac{\mu_m}{f'}\Delta^k_m\\&+\left[2\Theta\dot{R}\frac{f'''}{f'}-\frac{f''}{f'}\left(\frac{f}{f'}-2\frac{\mu_m}{f'}+2\dot{R}\Theta\frac{f''}{f'}-2\frac{k^2}{a^2}\right)\right]\mathcal{R}+2\Theta\frac{f''}{f'}\Re^k=0\ ,\end{aligned}\tag{6.66}$$

$$\begin{aligned}\dot{C}^k&=K\left[\frac{36K(f''\mathcal{R}^k-f'\Delta^k_m)+6f'C^k}{a^2(2\Theta f'+3\dot{R}f'')}\right]+K\left(\frac{12\Theta f''}{2\Theta f'+3\dot{R}f''}+\frac{2f''}{f'}\right)\Re\\&+K\left\{\Delta^k_m\left[\frac{16w\Theta}{3(1+w)}-\frac{4f'\Theta^2-12f'\mu_R}{2\Theta f'+3\dot{R}f''}\right]+\frac{12f''}{2\Theta f'+3\dot{R}f''}\frac{k^2}{a^2}\mathcal{R}^k\right.\end{aligned}$$

$$
-\frac{2\mathcal{R}^k a^2(\Theta f''-3\dot{R}f''')}{3f'}\frac{12\dot{R}\Theta f' f'''-2f''\left(3f-2f'(\Theta^2-3\mu_R)+6\dot{R}\Theta f''\right)}{2\Theta f'+3\dot{R}f''}\Bigg\}
$$
$$
-\frac{k^2}{a^2}\left[\frac{4wa^2\Theta}{3(1+w)}\Delta_m+\frac{2a^2f''}{f'}\Re^k-\frac{2a^2(\Theta f''-3\dot{R}f'')}{3f'}\mathcal{R}^k\right] , \tag{6.67}
$$

$$
\dot{\Delta}_i^k-(w_i-c_{si}^2)\Theta\Delta_i^k+(1+w_i)Z^k=\frac{1+w_i}{1+w}\left(c_s^2\Delta_m^k+w\varepsilon^k\right)\Theta
$$
$$
+(1+w_i)\frac{k^2}{a}V_i , \tag{6.68}
$$
$$
\dot{V}_i^k-\left(c_{si}^2-\frac{1}{3}\right)\Theta V_i^k=\frac{1}{ahh_i}\left(h_ic_s^2\mu\Delta_m+h_ip\varepsilon^k-hc_{si}^2\mu^i\Delta_i^k\right) , \tag{6.69}
$$
$$
\dot{V}_{ij}^k-\left(c_{sj}^2-\frac{1}{3}\right)\Theta V_{ij}^k-(c_{si}^2-c_{sj}^2)\Theta V_i^k=-\frac{1}{ah_i}(c_{si}^2-c_{sj}^2)\mu_i\Delta_i^k
$$
$$
-\frac{1}{a}c_{sj}^2S_{ij}^k , \tag{6.70}
$$
$$
\dot{S}_{ij}^k-\frac{k^2}{a}V_{ij}^k=0 . \tag{6.71}
$$

The form and use of Eqn (6.67) will be more transparent when we discuss the long-wavelength limits of our perturbations for radiation and dust backgrounds. The harmonically decomposed set of second-order equations (6.55)-(6.57) will become

$$
\ddot{\Delta}_m^k+\left[(c_s^2+\frac{2}{3}-2w)\Theta-\dot{R}\frac{f''}{f'}\right]\dot{\Delta}_m^k+\left[\left(\frac{3}{2}w^2+5c_s^2-4w-1\right)\frac{\mu_m}{f'}\right.
$$
$$
\left.+\frac{1}{2}(3w-5c_s^2)\frac{f}{f'}+(c_s^2-w)\left(2R-4\dot{R}\Theta\frac{f''}{f'}-\frac{12K}{a^2}\right)+c_s^2\frac{k^2}{a^2}\right]\Delta_m^k
$$
$$
=\frac{1+w}{2}\left[-1-\frac{2k^2}{a^2}\frac{f''}{f'}+(f-2\mu_m+2\dot{R}\Theta f'')\frac{f''}{f'^2}-2\dot{R}\Theta\left((\frac{f''}{f'})^2+\frac{f'''}{f'}\right)\right]\mathcal{R}^k
$$
$$
-(1+w)\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k-\left[2\frac{\mu_m}{f'}+\frac{R}{2}-\frac{f}{f'}-\dot{R}\Theta\frac{f''}{f'}-\frac{3K}{a^2}+\frac{k^2}{a^2}\right]w\varepsilon^k , \tag{6.72}
$$

$$
\ddot{\mathcal{R}}^k+\left(2\dot{R}\frac{f'''}{f''}+\Theta\right)\dot{\mathcal{R}}^k+\left(\frac{k^2}{a^2}+\ddot{R}\frac{f'''}{f''}+\dot{R}^2\frac{f^{(iv)}}{f''}+\dot{R}\Theta\frac{f'''}{f''}+\frac{1}{3}\frac{f'}{f''}-\frac{R}{3}\right)\mathcal{R}^k
$$
$$
+\frac{c_s^2-1}{1+w}\dot{R}\dot{\Delta}_m^k+\left[\frac{w\mu_m}{f''}+\frac{2w-c_s^2}{1+w}\dot{R}\Theta+\frac{w}{1+w}\left(2\ddot{R}+2\dot{R}^2\frac{f'''}{f''}\right)\right]\varepsilon^k
$$
$$
+\left\{\frac{(3c_s^2-1)\mu_m}{3f''}+\frac{w+c_s^2}{1+w}\dot{R}\Theta+\frac{c_s^2}{1+w}\left(2\ddot{R}+2\dot{R}^2\frac{f'''}{f''}\right)\right.
$$
$$
\left.+\frac{\dot{R}}{1+w}\left[\dot{c_s^2}+c_s^2(c_s^2-w)\Theta\right]\right\}\Delta_m^k+\frac{w}{1+w}\dot{R}\dot{\varepsilon}^k=0 , \tag{6.73}
$$

$$\begin{aligned}
&\ddot{\Delta}_i^k + (\frac{2}{3} - w_i)\Theta\dot{\Delta}_i^k - \frac{1+w_i}{1+w}\left[\dot{R}\frac{f''}{f'} + (c_s^2 - c_{si}^2)\,\Theta\right]\dot{\Delta}^k + (1+w_i)\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k\\
&- \frac{1+w_i}{1+w}\Bigg[(1+w)\frac{\mu_m}{f'} - \left(2\frac{\mu_m}{f'} - \frac{f}{f'} - 2\Theta\dot{R}\frac{f''}{f'}\right)c_s^2 + \dot{c}_s^2\Theta\\
&+ (c_s^2 - c_{si}^2)(c_s^2 - w)\Theta^2 - (c_s^2 + w)\dot{R}\Theta\frac{f''}{f'}\Bigg]\Delta^k - \frac{1+w_i}{1+w}w\Theta\dot{\varepsilon}^k\\
&- \frac{1+w_i}{1+w}\left[(w - c_s^2 - c_{si}^2 w)\Theta^2 - w\left(2\frac{\mu_m}{f'} - \frac{f}{f'} - \dot{R}\Theta\frac{f''}{f'}\right)\right]\varepsilon^k + c_{si}^2\frac{k^2}{a^2}\Delta_i^k\\
&+ (1+w_i)\left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta\left(\frac{f''}{f'}\right)^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k = 0\,.
\end{aligned} \tag{6.74}$$

As can be seen, this second order set of equations is not closed. For a two-component fluid the entropy and velocity perturbations equations are given by

$$\ddot{S}_{ij}^k = \frac{k^2}{a}\dot{V}_{ij} - \frac{k^2}{3a}\Theta V_{ij}\,, \tag{6.75}$$

$$\begin{aligned}
\ddot{V}_{ij}^k &= \left(c_z^2 - \frac{1}{3}\right)\Theta\dot{V}_{ij} + \frac{c_{si}^2 - c_{sj}^2}{a(1+w)}\left(\frac{1}{3} + w - c_s^2\right)\Theta\Delta_m - \frac{c_z^2}{a}\dot{S}_{ij} + \frac{c_z^2\Theta - 3\dot{c}_z^2}{3a}S_{ij}\\
&+ \left\{\dot{c}_z^2\Theta - \left(c_z^2 - \frac{1}{3}\right)\left[\frac{1}{3}\Theta^2 + \frac{1}{2}(1+3w)\frac{\mu_m}{f'} + \frac{1}{2}(\mu_R + 3p_R)\right]\right\}V_{ij} - \frac{c_{si}^2 - c_{sj}^2}{a(1+w)}\dot{\Delta}_m\,.
\end{aligned} \tag{6.76}$$

Since Eqns (6.75) and (6.76) are not linearly independent equations, we can choose either one of them to close our system of second order equations (6.72-6.74).

6.6 Perturbations in a Radiation - Dust Universe

6.6.1 Background Setup

Now that we have derived the equations for perturbations of a general multi-fluid system, we consider an application of the equations for a cosmological medium containing a non-interacting radiation-dust mixture and described by a flat FLRW spacetime. Since our component fluids satisfy the conservation equations separately, we write

$$\dot{\mu}_d + \Theta\mu_d = 0\,, \tag{6.77}$$

$$\dot{\mu}_r + \frac{4}{3}\Theta\mu_r = 0\,. \tag{6.78}$$

The general equation of state w for such a radiation-dust mixture is given by

$$w = \frac{p_m}{\mu_m} = \frac{p_d + p_r}{\mu_d + \mu_r} = \frac{1}{3}\frac{\mu_r}{\mu_d + \mu_r} \tag{6.79}$$

and the speed of sound in the mixture is

$$c_s^2 = \frac{\dot{p}_m}{\dot{\mu}_m} = \frac{4\mu_r}{3(3\mu_d + 4\mu_r)} \ . \tag{6.80}$$

In general, since we do not have an explicit expression of the Hubble parameter H and the curvature scalar R as functions of the scale factor a in generic $f(R)$ gravity theories, an exact multi-fluid background solution is not available and numerical solutions need to be obtained. A first step towards this will be addressed in Chapter 7, where we consider a radiation-dust fluid system for R^n models.

Here we will confine our discussion to R^n models and look for solutions in the short-wavelength and long-wavelength approximations for perturbations deep in the radiation and dust dominated epochs. During these epochs, since one fluid is negligible with respect to the other, we can use the exact single fluid background transient solution for R^n models given by $a = a_{eq}(t/t_{eq})^{\frac{2n}{3(1+w)}}$ where a_{eq} is the scale factor at the time of radiation-dust equality t_{eq} and will henceforth be normalized to unity.

In R^n models we recall that the expressions for the expansion, the Ricci scalar, the curvature fluid energy density, the curvature fluid pressure and the effective matter energy density at the transient solutions are given by Eqns (3.49)-(3.53).

6.6.2 Total Fluid Equations

Upon expanding Eqn (6.27) for a mixture of dust and radiation, we obtain

$$p_m\varepsilon = -\frac{4\mu_d\mu_r}{3(3\mu_d + 4\mu_r)}S_{dr} \ , \tag{6.81}$$

and hence

$$\varepsilon = -\frac{4\mu_d}{3\mu_d + 4\mu_r}S_{dr} \ . \tag{6.82}$$

We can thus readily derive the evolution equation for ε as follows

$$\dot{\varepsilon} = -\frac{16\mu_d\mu_r\Theta}{3(3\mu_d + 4\mu_r)^2}S_{dr} - \frac{4\mu_d}{3\mu_d + 4\mu_r}\dot{S}_{dr} = -4c_z^2c_s^2\Theta S_{dr} - 4c_z^2\dot{S}_{dr} \ . \tag{6.83}$$

Using these relations and applying the general total fluid second order equations to the radiation-dust mixture yields

$$\begin{aligned}\ddot{\Delta}_m^k \ &+ \left[\left(c_s^2 + \tfrac{2}{3} - 2w\right)\Theta - \dot{R}\tfrac{f''}{f'}\right]\dot{\Delta}_m^k - 4wc_z^2\left[2\tfrac{\mu_m}{f'} + \tfrac{R}{2} - \tfrac{f}{f'} - \dot{R}\Theta\tfrac{f''}{f'} + \tfrac{k^2}{a^2}\right]S_{dr}^k \\ &+ \left[\left(\tfrac{3}{2}w^2 + 5c_s^2 - 4w - 1\right)\tfrac{\mu_m}{f'} + \tfrac{1}{2}(3w - 5c_s^2)\tfrac{f}{f'} + (c_s^2 - w)\left(2R - 4\dot{R}\Theta\tfrac{f''}{f'}\right)\right. \\ &\left.+c_s^2\tfrac{k^2}{a^2}\right]\Delta_m^k - \tfrac{1+w}{2}\left[-1 - \tfrac{2k^2}{a^2}\tfrac{f''}{f'} + (f - 2\mu_m + 2\dot{R}\Theta f'')\tfrac{f''}{f'^2} - 2\dot{R}\Theta\left(\tfrac{f''}{f'}\right)^2\right. \\ &\left.-2\dot{R}\Theta\tfrac{f'''}{f'}\right]\mathcal{R}^k + (1+w)\Theta\tfrac{f''}{f'}\dot{\mathcal{R}}^k = 0 \ ,\end{aligned} \tag{6.84}$$

$$\begin{aligned}
\ddot{\mathcal{R}}^k \quad & + \left(2\dot{R}\tfrac{f'''}{f''}+\Theta\right)\dot{\mathcal{R}}^k + \left[\tfrac{k^2}{a^2}+\ddot{R}\tfrac{f'''}{f''}+\dot{R}^2\tfrac{f^{(iv)}}{f''}+\dot{R}\Theta\tfrac{f'''}{f''}+\tfrac{1}{3}\tfrac{f'}{f''}-\tfrac{R}{3}\right]\mathcal{R}^k \\
& +\tfrac{c_s^2-1}{1+w}\dot{R}\dot{\Delta}_m^k + \left\{\tfrac{(3c_s^2-1)\mu_m}{3f''}+\tfrac{w+c_s^2}{1+w}\dot{R}\Theta+\tfrac{c_s^2}{1+w}\left(2\ddot{R}+2\dot{R}^2\tfrac{f'''}{f''}\right)\right. \\
& \left.+\tfrac{\dot{R}}{1+w}\left[\dot{c_s^2}+c_s^2(c_s^2-w)\Theta\right]\right\}\Delta_m^k - 4wc_z^2\left[\tfrac{2}{1+w}\left(\ddot{R}+\dot{R}\Theta+\dot{R}^2\tfrac{f'''}{f''}\right)\right. \\
& \left.+\tfrac{\mu_m}{f''}\right]S_{dr}^k - \tfrac{4w}{1+w}c_z^2\dot{R}\dot{S}_{dr}^k = 0\;, \qquad (6.85)
\end{aligned}$$

$$\ddot{S}_{dr}^k + \left(c_s^2+\tfrac{1}{3}\right)\Theta\dot{S}_{dr}^k + \tfrac{k^2}{a^2}c_z^2 S_{dr}^k - \tfrac{k^2}{a^2}\left(c_z^2+\tfrac{3}{4}c_s^2\right)\Delta_m^k = 0\;, \qquad (6.86)$$

where

$$\Delta_m = \frac{\mu_d\Delta_d+\mu_r\Delta_r}{\mu_d+\mu_r}\;, \qquad S_{dr} = \Delta_d - \tfrac{3}{4}\Delta_r\;, \qquad c_z^2 \equiv \frac{1}{h}\left(h_r c_{sd}^2 + h_d c_{sr}^2\right)\;. \qquad (6.87)$$

6.6.3 Component Equations

The perturbations of the density gradients of the radiation component of the fluid are described by the propagation equation

$$\begin{aligned}
\ddot{\Delta}_r^k &+ \left\{\left[\frac{1}{3}-\frac{4}{3(1+w)}\left(\frac{3w\mu_d}{3\mu_d+4\mu_r}+\frac{(c_s^2-\frac{1}{3})\mu_r}{\mu_d+\mu_r}\right)\right]\Theta - \frac{4\mu_r\dot{R}f''/f'}{3(1+w)(\mu_d+\mu_r)}\right\}\dot{\Delta}_r^k \\
&+\frac{4\mu_d}{3(1+w)}\left[\left(\frac{4w}{3\mu_d+4\mu_r}-\frac{c_s^2-\frac{1}{3}}{\mu_d+\mu_r}\right)\Theta - \frac{\dot{R}f''}{(\mu_d+\mu_r)f'}\right]\dot{\Delta}_d^k \\
&+\frac{4}{3(1+w)}\left[\frac{k^2}{3a^2}+\left(\frac{(w-c_s^2)\mu_r}{\mu_d+\mu_r}-\frac{3w\mu_d}{3\mu_d+4\mu_r}+\frac{\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right)\dot{R}\Theta\frac{f''}{f'}\right. \\
&-\left(\frac{4\mu_d\mu_r}{(3\mu_d+4\mu_r)^2}\frac{3w\mu_d+(3w-1)\mu_r}{3(\mu_d+\mu_r)}-\frac{3(c_s^2-\frac{2w}{3})\mu_d}{3\mu_d+4\mu_r}+\frac{(c_s^2-\frac{1}{3})(c_s^2-w)\mu_r}{\mu_d+\mu_r}\right. \\
&\left.-\frac{(c_s^2-\frac{1}{3})\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right)\Theta^2-\left(\frac{(1+w-2c_s^2)\,\mu_r}{\mu_d+\mu_r}-\frac{6w\mu_d}{3\mu_d+4\mu_r}\right)\frac{\mu_d+\mu_r}{f'} \\
&\left.-\left(\frac{3w\mu_d}{3\mu_d+4\mu_r}+\frac{c_s^2\mu_r}{\mu_d+\mu_r}\right)\frac{f}{f'}\right]\Delta_r^k + \frac{4}{3(1+w)}\left[\left(\frac{(w-c_s^2)\mu_d}{\mu_d+\mu_r}+\frac{4w\mu_d}{3\mu_d+4\mu_r}\right.\right. \\
&\left.-\frac{\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right)\dot{R}\Theta\frac{f''}{f'}+\left(\frac{4\mu_d\mu_r}{3(3\mu_d+4\mu_r)^2}\frac{4w\mu_r+(4w+1)\mu_d}{\mu_d+\mu_r}-\frac{(c_s^2-\frac{1}{3})\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right. \\
&\left.-\frac{4(c_s^2-\frac{2w}{3})\mu_d}{3\mu_d+4\mu_r}-\frac{(c_s^2-\frac{1}{3})(c_s^2-w)\mu_d}{\mu_d+\mu_r}\right)\Theta^2+\left(\frac{4w\mu_d}{3\mu_d+4\mu_r}-\frac{c_s^2\mu_d}{\mu_d+\mu_r}\right)\frac{f}{f'} \\
&\left.-\left(\frac{(1+w-2c_s^2)\,\mu_d}{\mu_d+\mu_r}+\frac{8w\mu_d}{3\mu_d+4\mu_r}\right)\frac{\mu_d+\mu_r}{f'}\right]\Delta_d^k+\frac{4}{3}\Theta\frac{f''}{f'}\dot{\mathcal{R}}^k \\
&+\frac{4}{3}\left[\frac{1}{2}+\frac{k^2}{a^2}\frac{f''}{f'}-\frac{1}{2}\frac{ff''}{f'^2}+\frac{f''(\mu_r+\mu_d)}{f'^2}-\dot{R}\Theta\left(\frac{f''}{f'}\right)^2+\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k = 0\;.
\end{aligned}$$

(6.88)

Similarly the propagation equation of the dust density gradient is given by

$$
\begin{aligned}
&\ddot{\Delta}^k_d + \left[\left(\frac{2}{3} + \frac{\mu_d}{1+w}(\frac{4w}{3\mu_d+4\mu_r} - \frac{c_s^2}{\mu_d+\mu_r})\right)\Theta - \frac{\mu_d}{(1+w)(\mu_d+\mu_r)}\dot{R}\frac{f''}{f'}\right]\dot{\Delta}^k_d \\
&\quad - \frac{1}{1+w}\left[\left(\frac{3w\mu_d}{3\mu_d+4\mu_r} + \frac{c_s^2\mu_r}{\mu_d+\mu_r}\right)\Theta + \frac{\mu_r}{\mu_d+\mu_r}\dot{R}\frac{f''}{f'}\right]\dot{\Delta}^k_r \\
&\quad + \frac{\mu_d}{1+w}\left[\left(\frac{w-c_s^2}{\mu_d+\mu_r} + \frac{4w}{3\mu_d+4\mu_r} - \frac{\mu_r}{3(\mu_d+\mu_r)^2}\right)\dot{R}\Theta\frac{f''}{f'} + \left(\frac{4\mu_r}{3(3\mu_d+4\mu_r)^2}\times\right.\right. \\
&\quad \left.\frac{4w\mu_r+(4w+1)\mu_d}{\mu_d+\mu_r} - \frac{4(c_s^2-w)}{3\mu_d+4\mu_r} - \frac{(c_s^2-w)c_s^2}{\mu_d+\mu_r} - \frac{c_s^2\mu_r}{3(\mu_d+\mu_r)^2}\right)\Theta^2 \\
&\quad \left. - \left(\frac{1+w-2c_s^2}{\mu_d+\mu_r} + \frac{8w}{3\mu_d+4\mu_r}\right)\frac{\mu_d+\mu_r}{f'} + \left(\frac{4w}{3\mu_d+4\mu_r} - \frac{c_s^2}{\mu_d+\mu_r}\right)\frac{f}{f'}\right]\Delta^k_d \\
&\quad + \frac{1}{1+w}\left[\left(\frac{(w-c_s^2)\mu_r}{\mu_d+\mu_r} - \frac{3w\mu_d}{3\mu_d+4\mu_r} - \frac{\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right)\dot{R}\Theta\frac{f''}{f'} + \left(\frac{c_s^2\mu_d\mu_r}{3(\mu_d+\mu_r)^2}\right.\right. \\
&\quad \left. + \frac{3(c_s^2-w)\mu_d}{3\mu_d+4\mu_r} - \frac{(c_s^2-w)c_s^2\mu_r}{\mu_d+\mu_r} + \frac{4\mu_d\mu_r}{(3\mu_d+4\mu_r)^2}\frac{(1-3w)\mu_r - 3w\mu_d}{3(\mu_d+\mu_r)}\right)\Theta^2 \\
&\quad - \left(\frac{3w\mu_d}{3\mu_d+4\mu_r} + \frac{c_s^2\mu_r}{\mu_d+\mu_r}\right)\frac{f}{f'} - \left(\frac{(1+w-2c_s^2)\,\mu_r}{\mu_d+\mu_r} - \frac{6w\mu_d}{3\mu_d+4\mu_r}\right)\times \\
&\quad \left.\frac{\mu_d+\mu_r}{f'}\right]\Delta^k_r + \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''(\mu_r+\mu_d)}{f'^2} - \dot{R}\Theta\left(\frac{f''}{f'}\right)^2\right. \\
&\quad \left. + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k + \Theta\frac{f''}{f'}\dot{\mathcal{R}}^k = 0\,. \qquad (6.89)
\end{aligned}
$$

In terms of the component perturbation variables of section 6.2 we can rewrite the propagation equation for the curvature fluid gradient as

$$
\begin{aligned}
&\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\dot{\mathcal{R}}^k + \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(iv)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}^k \\
&+ \left(\frac{c_s^2-1}{1+w}\frac{\mu_d}{\mu_d+\mu_r} - \frac{4wc_z^2}{1+w}\right)\dot{R}\dot{\Delta}_d^k + \left(\frac{c_s^2-1}{1+w}\frac{\mu_r}{\mu_d+\mu_r} + \frac{3wc_z^2}{1+w}\right)\dot{R}\dot{\Delta}_r^k \\
&+ \Bigg\{\Bigg[(3c_s^2-1)\frac{\mu_d+\mu_r}{3f''} + \frac{w+c_s^2}{1+w}\dot{R}\Theta + \frac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right) \\
&+ \frac{\dot{R}}{1+w}\left(\dot{c_s^2} + c_s^2(c_s^2-w)\Theta\right)\Bigg]\frac{\mu_d}{\mu_d+\mu_r} - 4wc_z^2\Bigg[\frac{\mu_d+\mu_r}{f''} + \frac{2}{1+w}\Bigg(\ddot{R} + \dot{R}\Theta \\
&+ \dot{R}^2\frac{f'''}{f''}\Bigg)\Bigg] + \frac{c_s^2-1}{3(1+w)}\frac{\mu_d\mu_r}{(\mu_d+\mu_r)^2}\dot{R}\Theta\Bigg\}\Delta_d^k + \Bigg\{\Bigg[(3c_s^2-1)\frac{\mu_d+\mu_r}{3f''} \\
&+ \frac{w+c_s^2}{1+w}\dot{R}\Theta + \frac{c_s^2}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''}\right) + \frac{\dot{R}}{1+w}\left(\dot{c_s^2} + c_s^2(c_s^2-w)\Theta\right)\Bigg]\frac{\mu_r}{\mu_d+\mu_r} \\
&+ 3wc_z^2\left[\frac{\mu_d+\mu_r}{f''} + \frac{2}{1+w}\left(\ddot{R} + \dot{R}\Theta + \dot{R}^2\frac{f'''}{f''}\right)\right] \\
&- \frac{c_s^2-1}{3(1+w)}\frac{\mu_d\mu_r}{(\mu_d+\mu_r)^2}\dot{R}\Theta\Bigg\}\Delta_r^k = 0\,. \qquad (6.90)
\end{aligned}
$$

6.7 Short-wavelength Solutions

In this section we will study the evolution of the short-wavelength modes, i.e., large values of the wave number k, by using the equations presented in section 6 valid for a radiation and dust mixture. The general results will then be considered for R^n models and a proposal for a *quasi-static* approximation for the matter perturbations will be introduced for both radiation and dust dominated epochs. In that approximation, widely used in the literature [68, 252, 283, 298], all the time derivative terms for the gravitational potentials are discarded, and only those including density perturbations are kept. The decoupling process for the equations involved is therefore highly simplified. Nonetheless, when this approximation was used to study fourth-order gravity theories in the metric formalism, it was proved to be too aggressive and a more detailed analysis is required [28, 93].

6.7.1 Perturbations in the Radiation-dominated Epoch

Let us now look at the case where the characteristic size of the fluid inhomogeneities is much less than the Jeans length for the radiation fluid but is still larger than the mean free path of the photon, i.e., $\lambda \ll \lambda_H \ll \lambda_J$. Similar investigation has been made by [101] for the case of GR. Note that the scale dependence of the perturbations equations is not trivial as can be seen in [12].

Here we assume that we can neglect the interaction between the component fluids and that the radiation energy density can be taken as *almost* homogeneous, meaning $\Delta_r \approx 0$.

This amounts to studying dust and curvature fluctuations on a homogeneous radiation background, whereby radiation affects the growth of the dust fluctuations by speeding up the cosmic expansion [104] for which the equations are given by

$$\dot{\Delta}^k_d + Z^k = \frac{\Theta}{h}\left(c^2_s\mu\Delta^k_m + p_m\varepsilon^k\right) + a\left(\frac{k^2}{a^2}\right)V^k_d \,, \tag{6.91}$$

$$\begin{aligned}
&\dot{Z}^k - (\dot{R}\frac{f''}{f'} - \frac{2}{3}\Theta)Z^k - \left[\frac{(2c^2_s - w - 1)}{(1+w)}\frac{\mu_m}{f'} - \frac{c^2_s}{(1+w)}\left(\frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'}\right)\right]\Delta^k_m \\
&- \frac{w}{(1+w)}\left[2\frac{\mu_m}{f'} + \frac{R}{2} - \frac{f}{f'} - 2\dot{R}\Theta\frac{f''}{f'}\right]\varepsilon^k - \frac{1}{h}\left(\frac{k^2}{a^2}\right)\left(c^2_s\mu_m\Delta^k_m + p\varepsilon^k\right) + \Theta\frac{f''}{f'}\Re^k \\
&- \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_m}{f'^2} - \dot{R}\Theta\left(\frac{f''}{f'}\right)^2 + \dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k = 0\,,
\end{aligned} \tag{6.92}$$

$$\dot{V}^k_d + \frac{1}{3}\Theta V^k_d = \frac{1}{ah}\left(c^2_s\mu\Delta_m + p\varepsilon\right)\,, \tag{6.93}$$

$$\dot{V}^k_{dr} - \left(c^2_z - \frac{1}{3}\right)\Theta V^k_{dr} = \frac{1}{3ah}\mu\Delta^k_m - \frac{1}{a}c^2_z S^k_{dr}\,, \tag{6.94}$$

$$\dot{\mathcal{R}}^k = \Re^k - \frac{\dot{R}}{h}\left(c^2_s\mu_m\Delta^k_m + p_m\varepsilon^k\right)\,, \tag{6.95}$$

$$\begin{aligned}
\dot{\Re}^k = &-\left(2\dot{R}\frac{f'''}{f''} + \Theta\right)\Re^k - \dot{R}Z^k + \frac{\mu_m}{3f''}\Delta^k_m - \frac{1}{f''}\left(c^2_s\mu_m\Delta^k_m + p\varepsilon^k\right) \\
&- \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(iv)}}{f''} + \dot{R}\Theta\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}^k.
\end{aligned} \tag{6.96}$$

Since $\Delta_r \ll \Delta_d$ we have

$$c^2_s\mu\Delta^k_m + p\varepsilon^k = \frac{1}{3}\mu_r\Delta^k_r \approx 0\,, \tag{6.97}$$

and so

$$S^k_{dr} \approx \Delta^k_d\,. \tag{6.98}$$

In these limits the above set of equations (6.91-6.96) can be rewritten as

$$\dot{\Delta}^k_d + Z^k - a\left(\frac{k^2}{a^2}\right)V^k_d = 0\,, \tag{6.99}$$

$$\begin{aligned}
&\dot{Z}^k - \left(\dot{R}\frac{f''}{f'} - 2H\right)Z^k + \frac{\mu_d}{f'}\Delta^k_d - \Theta\frac{f''}{f'}\Re^k \\
&- \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_r}{f'^2} - 3H\dot{R}\left(\frac{f''}{f'}\right)^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0\,,
\end{aligned} \tag{6.100}$$

$$\dot{V}^k_d + HV^k_d = 0\,, \tag{6.101}$$

$$\dot{V}^k_{dr} + \frac{4}{3}\frac{\mu_r}{h}HV^k_{dr} = 0 \;, \tag{6.102}$$

$$\dot{\mathcal{R}}^k = \Re^k, \tag{6.103}$$

$$\dot{\Re}^k = -\left(2\dot{R}\frac{f'''}{f''} + 3H\right)\Re^k - \dot{R}Z^k + \frac{\mu_d}{3f''}\Delta^k_d - \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(iv)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{1}{3}\frac{f'}{f''} - \frac{R}{3}\right)\mathcal{R}^k \;. \tag{6.104}$$

From Eqns (6.99)-(6.104) we obtain the following two second order differential equations:

$$\ddot{\Delta}^k_d + \left(2H - \frac{3\dot{R}f''}{4f'}\frac{\mu_d}{\mu_r}\right)\dot{\Delta}^k_d - \frac{\mu_d}{f'}\Delta^k_d + 3H\frac{f''}{f'}\dot{\mathcal{R}}^k + \left[\frac{1}{2} + \frac{k^2}{a^2}\frac{f''}{f'} - \frac{1}{2}\frac{ff''}{f'^2} + \frac{f''\mu_r}{f'^2} - 3H\dot{R}\left(\frac{f''}{f'}\right)^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0 \;, \tag{6.105}$$

$$\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''} + 3H\right)\dot{\mathcal{R}}^k - \frac{3\dot{R}\mu_d}{4\mu_r}\dot{\Delta}^k_d - \frac{\mu_d}{3f''}\Delta^k_d + \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(iv)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right)\mathcal{R}^k = 0 \;. \tag{6.106}$$

In terms of magnitude, H and $\dot{R}f''/f'$ behave similarly for R^n models, whereas $\mu_d \ll \mu_r$, implying that curvature and radiation fluids effectively dominate the fluctuation dynamics. In effect, terms like $\mu_d\Delta^k_d$ merely sub-dominate in the curvature-radiation-dust mixture. Hence we can safely approximate the above equations by

$$\ddot{\Delta}^k_d + 2H\dot{\Delta}^k_d + 3H\frac{f''}{f'}\dot{\mathcal{R}}^k + \left[\frac{1}{2}\left(1 - \frac{ff''}{f'^2}\right) + \frac{k^2}{a^2}\frac{f''}{f'} + \frac{f''\mu_r}{f'^2} - 3H\dot{R}\left(\frac{f''}{f'}\right)^2 + 3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k = 0 \;, \tag{6.107}$$

$$\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''} + 3H\right)\dot{\mathcal{R}}^k + \left(\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(iv)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right)\mathcal{R}^k = 0 \;. \tag{6.108}$$

These two equations tell us that, deep in the radiation-dominated era, the curvature fluctuations are decoupled from the matter in the second order equations.

GR is a specific example of the generalized R^n models where $n = 1$. In this limit, Eqns (6.107) and (6.107) reduce to

$$\ddot{\Delta}^k_d + 2H\dot{\Delta}^k_d + \frac{1}{2}\mathcal{R}^k = 0 \;, \tag{6.109}$$

$$\mathcal{R}^k = 0 \;, \tag{6.110}$$

thus yielding the standard GR equation for the density contrast in a radiation background

$$\ddot{\Delta}^k_d + \frac{1}{t}\dot{\Delta}^k_d = 0 \,, \tag{6.111}$$

whose general solution is given by

$$\Delta^k_d(t) = C_1 + C_2 \ln t \,. \tag{6.112}$$

with $C_{1,2}$ arbitrary constants. For $n \neq 1$, with $w = \frac{1}{3}$ in the radiation-dominated epoch, Eqns (6.107) and (6.107) take the following forms:

$$\ddot{\Delta}^k_d + \tfrac{n}{t}\dot{\Delta}^k_d + \tfrac{t}{2}\dot{\mathcal{R}}^k + \left[\tfrac{12-5n}{4} + \tfrac{n}{12}\left(\tfrac{\lambda_H}{\lambda}\right)^2_{eq} t^{2-n}\right]\mathcal{R}^k = 0 \,, \tag{6.113}$$

$$\ddot{\mathcal{R}}^k - \left(\tfrac{5n-16}{2t}\right)\dot{\mathcal{R}}^k + \left[\tfrac{n^2}{4}\left(\tfrac{\lambda_H}{\lambda}\right)^2_{eq} t^{-n} - \tfrac{6(n-2)}{t^2}\right]\mathcal{R}^k = 0 \,, \tag{6.114}$$

where we have used the fact that $\frac{k^2}{a^2} = \frac{n^2}{4}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{-n}$ with normalized time $t_{eq} = 1$ at the time of *radiation-matter equality*.

Quasi-static Analysis

In general the system of equations (6.113)-(6.114) yield Bessel hypergeometric type analytic solutions. However, since we are dealing with small scales we can take a *quasi-static approximation*, where the time variations in $\mathcal{R}^k$ are treated as negligible, i.e., $\ddot{\mathcal{R}}^k \simeq 0$ and $\dot{\mathcal{R}}^k \simeq 0$. In this scheme the overall dynamics of the density perturbations leads to the simplified, k- independent, equation

$$\ddot{\Delta}^k_d + \frac{n}{t}\dot{\Delta}^k_d = 0 \,. \tag{6.115}$$

This equation admits the general solution

$$\Delta^k_d(t) = C_1 + C_2 t^{1-n} \,. \tag{6.116}$$

On small scales, radiation suppresses the growth of fluctuations as they enter the horizon before radiation-dust equality, and dust (baryon) self-gravitation is not yet strong enough to offset the cosmic expansion. This is because the expansion scale factor grows faster than the perturbation amplitudes do. The phenomenon is known in the literature as the *Mészáros effect* [211].

It is clear from the above analysis that the Mészáros effect puts a constraint on the value of n in R^n gravity. To do so, all we need do is determine the allowed values of n for which the perturbation amplitudes grow slower than the expansion in the radiation dominated era, i.e.,

$$\frac{d}{dt}\left[\frac{\Delta^k_d(t)}{a(t)}\right] \propto \frac{d}{dt}\left[\frac{t^{1-n}}{t^{\frac{n}{2}}}\right] < 0 \Rightarrow 1 - \frac{3n}{2} < 0 \,. \tag{6.117}$$

This means that only values of $n > \frac{2}{3}$ give a growth rate compatible with the Mészáros effect.

In figure 2, we plot the normalized dust density contrast $\delta(t) \equiv \Delta_m(t)/\Delta_{eq}$ in the radiation-dominated epoch.

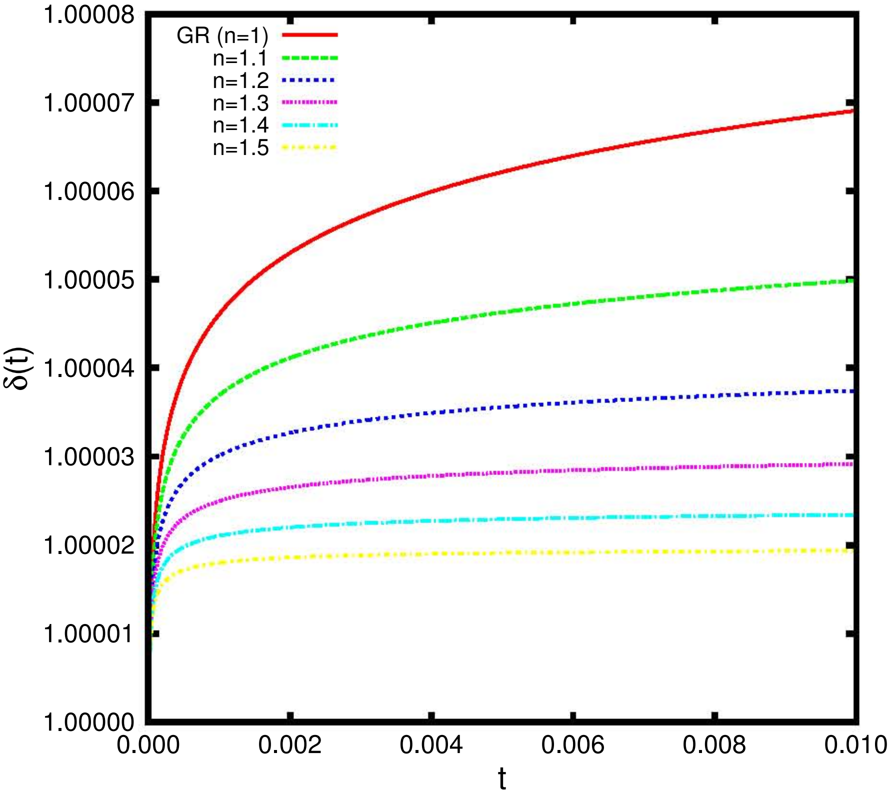

Figure 6.2: Dust growth factor in the radiation dominated epoch for R^n models: The plots show the growth factor obtained by solving numerically the full system of equations (6.113) and (6.114) for scale $\lambda_H = 100\lambda$ and the quasi-static solution (6.115) for $n = 1.5, 1.4, 1.3, 1.2, 1.1$ from bottom to top. The top-most plot corresponds to GR ($n = 1$). It can be seen that quasi-static results are quite close to those of the full system for the stated values of n, only slightly (but invisibly) lower in the plots. For values of $\lambda_H > 100\lambda$ the growth factor appears to be insensitive to scale showing the convenience of introducing the *quasi-static* approximation.

6.7.2 Perturbations in the Dust-dominated Epoch

During this epoch of the universe the dust energy density is the dominant component in the two-fluid dynamics and all order-of-magnitude approximations go in line with the assumption that $\mu_d >> \mu_r$. The equations (6.91)-(6.96) will, upon imposing the short-wavelength assumptions (6.97),(6.98), become

$$\dot{\Delta}^k_d + Z^k + a\tilde{\nabla}^2 V^k_d = 0\ , \tag{6.118}$$

$$\dot{Z}^k - \left(\dot{R}\frac{f''}{f'} - 2H\right) Z^k + \frac{\mu_d}{f'}\Delta^k_d - \Theta\frac{f''}{f'}\Re^k$$

$$-\left[\frac{1}{2}+\frac{k^2 f''}{a^2 f'}-\frac{f f''}{2f'^2}+\frac{f''\mu_d}{f'^2}-3H\dot{R}\left(\frac{f''}{f'}\right)^2+3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k=0\,, \tag{6.119}$$

$$\dot{V}_d^k+HV_d^k=0\,, \tag{6.120}$$

$$\dot{V}_{dr}^k+\frac{4}{3}\frac{\mu_r}{h}HV_{dr}^k=0\Rightarrow\dot{V}_{dr}^k+\frac{4\mu_r}{3\mu_d}HV_{dr}^k=0\,, \tag{6.121}$$

$$\dot{\mathcal{R}}^k=\Re^k\,, \tag{6.122}$$

$$\dot{\Re}^k=-\left(2\dot{R}\frac{f'''}{f''}+3H\right)\Re^k-\dot{R}Z^k+\frac{\mu_d}{3f''}\Delta_d^k$$
$$-\left[\frac{k^2}{a^2}+(3H\dot{R}+\ddot{R})\frac{f'''}{f''}+\dot{R}^2\frac{f^{(iv)}}{f''}+\frac{f'}{3f''}-\frac{R}{3}\right]\mathcal{R}^k\,. \tag{6.123}$$

The resulting set of second order equations is therefore

$$\ddot{\Delta}_d^k+\left(2H-\dot{R}\frac{f''}{f'}\right)\dot{\Delta}_d^k-\frac{\mu_d}{f'}\Delta_d^k+3H\frac{f''}{f'}\dot{\mathcal{R}}^k$$
$$+\left[\frac{1}{2}+\frac{k^2 f''}{a^2 f'}-\frac{f f''}{2f'^2}+\frac{f''\mu_d}{f'^2}-3H\dot{R}\left(\frac{f''}{f'}\right)^2+3H\dot{R}\frac{f'''}{f'}\right]\mathcal{R}^k=0\,, \tag{6.124}$$

$$\ddot{\mathcal{R}}^k+\left(2\dot{R}\frac{f'''}{f''}+3H\right)\dot{\mathcal{R}}^k-\dot{R}\dot{\Delta}_d^k-\frac{\mu_d}{3f''}\Delta_d^k$$
$$+\left[\frac{k^2}{a^2}+(3H\dot{R}+\ddot{R})\frac{f'''}{f''}+\dot{R}^2\frac{f^{(iv)}}{f''}+\frac{f'}{3f''}-\frac{R}{3}\right]\mathcal{R}^k=0\,. \tag{6.125}$$

As can be observed, these two equations differ from their counterparts in the radiation-dominated epoch in that the curvature perturbations are *not* decoupled from that of matter in the system of equations.

The limiting GR perturbation equations for (6.124) and (6.125) in this epoch are given by

$$\ddot{\Delta}_d^k+2H\dot{\Delta}_d^k-\mu_d\Delta_d^k+\frac{1}{2}\mathcal{R}^k=0\,, \tag{6.126}$$

$$-\frac{\mu_d}{3}\Delta_d^k+\frac{1}{3}\mathcal{R}^k=0\,, \tag{6.127}$$

and combine to give the equation

$$\ddot{\Delta}_d^k+\frac{4}{3t}\dot{\Delta}_d^k-\frac{2}{3t^2}\Delta_d^k=0\,. \tag{6.128}$$

This equation admits the well known solution

$$\Delta_d^k(t)=C_1t^{-1}+C_2t^{\frac{2}{3}}\,. \tag{6.129}$$

For R^n models, Eqns (6.124, 6.125) take the form

$$\ddot{\Delta}^k_d + \left(\frac{10n-6}{3t}\right)\dot{\Delta}^k_d + \frac{2(8n^2-13n+3)}{3t^2}\Delta^k_d + \frac{3(n-1)}{2(4n-3)}t\dot{\mathcal{R}}^k + \left[\frac{n(n-1)}{3(4n-3)}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{2-\frac{4n}{3}} + \frac{27n^2-8n^3-18n}{2n(4n-3)}\right]\mathcal{R}^k = 0 \,, \tag{6.130}$$

$$\ddot{\mathcal{R}}^k + \left\{\frac{8n\left[n(8n-13)+3\right](4n-3)}{27(n-1)t^4}\right\}\Delta^k_d + \frac{8n(4n-3)}{3t^3}\dot{\Delta}^k_d + \frac{8-2n}{t}\dot{\mathcal{R}}^k + \left\{\frac{4n^2}{9}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{-\frac{4n}{3}} - \frac{2\left[n(8n+5)-69\right]+54}{9(n-1)t^2}\right\}\mathcal{R}^k = 0 \,, \tag{6.131}$$

where $\frac{k^2}{a^2} = \frac{4n^2}{9}\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} t^{-\frac{4n}{3}}$ during this epoch.

Quasi-static Analysis

In the quasi-static limit with $\left(\frac{\lambda_H}{\lambda}\right)^2_{eq} >> 1$ we get a single second order k-scale independent equation

$$\ddot{\Delta}^k_d + \frac{4n}{3t}\dot{\Delta}^k_d + \left[\frac{4(8n^2-13n+3)}{9t^2}\right]\Delta^k_d = 0 \,, \tag{6.132}$$

the solution of which is given by

$$\Delta^k_d(t) = C_1 t^{\alpha_+} + C_2 t^{\alpha_-} \,, \tag{6.133}$$

where $\alpha_\pm = -\frac{2n}{3} + \frac{1}{2} \pm \frac{\sqrt{-112n^2+184n-39}}{6}$. The coefficients $C_{1,2}$ can be determined by imposing initial conditions.

At $t = t_{eq} = 1$ we have

$$\Delta^k_{(d)\,eq} \equiv \Delta^k_{(d)}(t_{eq}) = C_1 + C_2 \,, \tag{6.134}$$

and differentiating (6.133) gives

$$\dot{\Delta}^k_d(t) = \alpha_+ C_1 t^{\alpha_+ -1} + \alpha_- C_2 t^{\alpha_- -1} \,, \tag{6.135}$$

which, at equality, will give

$$\dot{\Delta}^k_{(d)\,eq} \equiv \dot{\Delta}^k_{(d)}(t_{eq}) = \alpha_+ C_1 + \alpha_- C_2 \,. \tag{6.136}$$

Solving (6.134) and (6.136) simultaneously we obtain

$$C_{1,2} = \frac{\pm\dot{\Delta}^k_{(d)\,eq} \mp \alpha_\mp \Delta^k_{(d)\,eq}}{\alpha_+ - \alpha_-} \,. \tag{6.137}$$

Fig. 3 shows the evolution of the density perturbations $\delta(t) \equiv \Delta^k_{(d)}(t)/\Delta^k_{(d)eq}$ in time

(t from 1 onwards, where $t = t_{eq} = 1$ is the normalized time at equality) for the above linearly independent solutions, $C_{1,2}$ having been obtained by setting $\Delta^k_{(d)eq} = 10^{-5}$ and $\dot{\Delta}^k_{(d)eq} = 10^{-5}$.

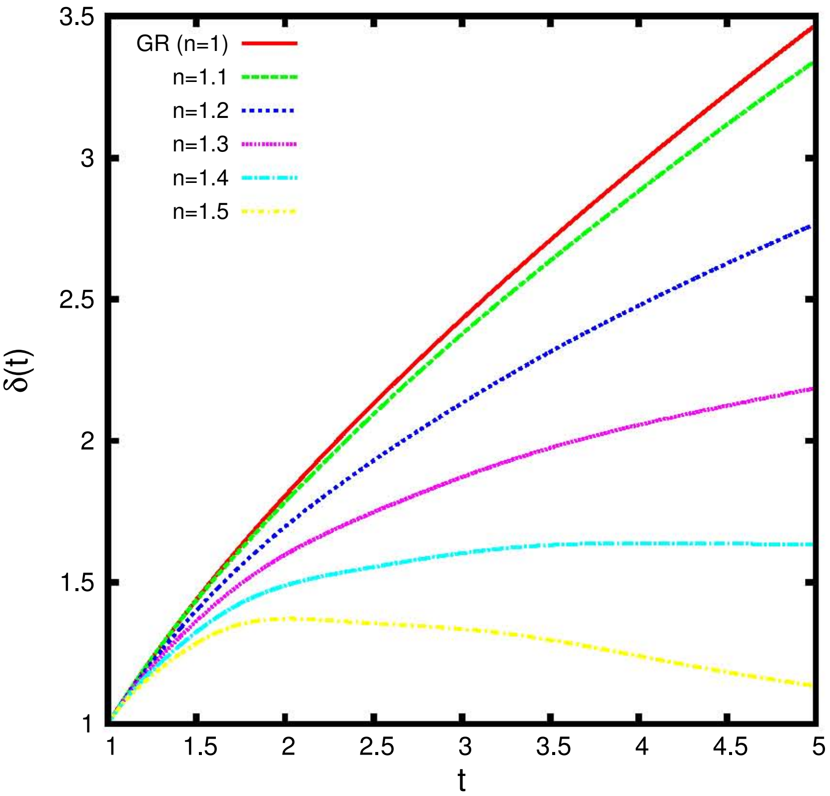

Figure 6.3: Dust growth factor in the dust dominated epoch for R^n models: The plots show the growth factor obtained by solving numerically the full system equations (6.130) and (6.131) for scale $\lambda_H = 100\lambda$ and the quasi-static analytic solution (6.132) . It can be seen that quasi-static results are indistinguishable from the full results for $n = 1.5, 1.4, 1.3, 1.2, 1.1$ from bottom to top, with the full system solutions slightly higher than those of the quasi-static approximation. It can also be seen that for higher values of n the growth factor increases more slowly till a critical value of n somewhere between 1.4 & 1.5 where the growth factor becomes a decreasing function of time. Note the $n = 1$ case (GR) is presented on the topmost plot.

6.8 Long-wavelength Solutions

For specific intervals of n, a set of initial conditions gives rise to cosmic histories which include a transient decelerated phase which evolves towards an accelerated phase. Structure formation takes place during the transient regime [58].

In this section we analyze the evolution of scalar perturbations during this phase, in the *long-wavelength limit*. In this limit the wavenumber k is so small that $\lambda = \frac{2\pi a}{k} \gg \lambda_H$, i.e., $\frac{k^2}{a^2 H^2} \ll 1$. All Laplacian terms can therefore be neglected and spatially flat ($K = 0$) backgrounds guarantee the conservation of C, i.e., $\dot{C}^k = 0$. In this work we are considering only adiabatic perturbations, i.e. $S_{ij}=0$ and hence, for a radiation-dust mixture, the equation for the evolution of entropy perturbations

$$\dot{S}_{dr}+a\tilde{\nabla}^2V_{dr}=0\;. \tag{6.138}$$

implies that

$$V_{dr}=0\;. \tag{6.139}$$

And from this and the equation

$$\dot{V}_{dr}-\left(c_z^2-\frac{1}{3}\right)\Theta V_{dr}=-\frac{1}{ah}(c_{sd}^2-c_{sr}^2)\mu\Delta_m-\frac{1}{a}c_z^2S_{dr}\;. \tag{6.140}$$

follows

$$(c_{sd}^2-c_{sr}^2)\mu\Delta_m=0\;. \tag{6.141}$$

We therefore have the following system of equations:

$$\dot{\Delta}_m+(1+w)Z-w\Theta\Delta_m=0\;, \tag{6.142}$$

$$\dot{Z}-\left(\dot{R}\frac{f''}{f'}-\frac{2}{3}\Theta\right)Z-\left[\frac{(2c_s^2-w-1)}{(1+w)}\frac{\mu_m}{f'}+\frac{c_s^2}{(1+w)}\left(\frac{R}{2}-\frac{f}{f'}-2\dot{R}\Theta\frac{f''}{f'}\right)\right]\Delta_m$$
$$-\Theta\frac{f''}{f'}\Re-\left[\frac{1}{2}-\frac{1}{2}\frac{ff''}{f'^2}+\frac{f''\mu_m}{f'^2}-\dot{R}\Theta\left(\frac{f''}{f'}\right)^2+\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}=0\;, \tag{6.143}$$

$$\dot{\mathcal{R}}-\Re+\frac{c_s^2}{1+w}\dot{R}\Delta_m=0\;, \tag{6.144}$$

$$\dot{\Re}+\left(2\dot{R}\frac{f'''}{f''}+\Theta\right)\Re+\dot{R}Z+\frac{(3c_s^2-1)\mu_m}{3f''}\Delta_m$$
$$+\left[(\dot{R}\Theta+\ddot{R})\frac{f'''}{f''}+\dot{R}^2\frac{f^{(iv)}}{f''}+\frac{f'}{3f''}-\frac{R}{3}\right]\mathcal{R}=0\;,$$
$$\frac{C_0}{a^2}+\left(\frac{4}{3}\Theta+2\frac{\dot{R}f''}{f'}\right)Z-2\frac{\mu_m}{f'}\Delta_m+\left[2\dot{R}\Theta\frac{f'''}{f'}-\frac{f''}{f'}\left(\frac{f}{f'}-2\frac{\mu_m}{f'}+2\dot{R}\Theta\frac{f''}{f'}\right)\right]\mathcal{R}$$
$$+2\Theta\frac{f''}{f'}\Re=0, \tag{6.145}$$

C_0 being the conserved value for the quantity C.

In terms of the background R^n solutions and making use of the conservation of C the above equations can be rewritten as

$$\dot{\Delta}_m=\left[\frac{1+w-2n}{1+w}-\frac{6(n-1)n}{n+3(n-1)w-3}\right]\frac{\Delta_m}{t}$$
$$-\frac{3(1+w)^2}{4a_0^2\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]}t^{1-\frac{4n}{3(1+w)}}C_0$$

$$
\begin{aligned}
&- \frac{9(n-1)(1+w)^3 t^2}{4\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]} t^2 \Re \\
&+ \left\{ \frac{3(n-1)(1+w)^2 \left[n(6w+8)-15(1+w)\right]}{4\left[n+3(n-1)w-3\right]\left[4n-3(1+w)\right]} \right\} t\mathcal{R} \,, \qquad (6.146)
\end{aligned}
$$

$$
\dot{\mathcal{R}} = \Re + \frac{8nc_s^2\left[4n-3(1+w)\right]}{3(1+w)^3}\frac{\Delta_m}{t^3} \,, \qquad (6.147)
$$

$$
\begin{aligned}
\dot{\Re} &= -2\left[\frac{(n-4)+2(n-2)w}{(1+w)t} - \frac{3n(n-1)}{n+3w(n-1)-3}\right]\Re \\
&+ \frac{2n(4n-3w-3)}{(1+w)\left[n+3(n-1)w-3\right]} C_0 t^{-\frac{4n}{3(1+w)}-2} \\
&-2\left[\frac{9n(n-2)(n-1)}{n+3(n-1)w-3} + 2n^2 - 7n - \frac{3n^2(9n-26)+57n}{9(1+w)(n-1)} - \frac{8n^2(n-2)}{9(1+w)^2(n-1)}\right. \\
&\left. +6\right]\frac{\mathcal{R}}{t^2} + 16n\frac{\Delta_m}{t^4}\frac{\left[4n-3(1+w)\right]\left[4n+3(n-1)w-3\right]}{27(n-1)(1+w)^4\left[n+3(n-1)w-3\right]} \times \\
&\left[\left(9w(1+w)+8\right)n^2 - \left(27w^2+24w+13\right)n + 3(1+w)(1+6w)\right] \,. \qquad (6.148)
\end{aligned}
$$

6.8.1 Perturbations in the Radiation-dominated Epoch

The second order set of equations governing the dynamics of density perturbations in this epoch is given by

$$
\begin{aligned}
&\ddot{\Delta}_r^k + \frac{n(9n-14)+4}{2(n-2)t}\dot{\Delta}_r^k + \frac{n\left[n(n(19n-54)+58)-32\right]+8}{2(n-2)^2t^2}\Delta_r^k \\
&+ \frac{2\left[n(3n-4)+2\right]}{3(n-2)^2} t\dot{\mathcal{R}}^k - \frac{n(15n-22)+14}{3(n-2)}\mathcal{R}^k + \frac{4(n^2-1)}{3(n-2)^2}t^{-n}C_0 = 0, \qquad (6.149)
\end{aligned}
$$

$$
\begin{aligned}
&\ddot{\mathcal{R}}^k - \frac{n(11n-32)+32}{2(n-2)t}\dot{\mathcal{R}}^k + \frac{3\left[n(5n-9)+8\right]}{2t^2}\mathcal{R}^k - \frac{3n\left[n(n-3)+2\right]}{2(n-2)t^3}\dot{\Delta}_r^k \\
&- \frac{3n(n-1)\left[n(19n-28)+4\right]}{4(n-2)t^4}\Delta_r^k - \frac{3n(n-1)}{(n-2)}t^{-(n+2)}C_0 = 0 \,. \qquad (6.150)
\end{aligned}
$$

Making use of the conservation of C, we can eliminate $\dot{\mathcal{R}}^k$ and $\mathcal{R}^k$ quantities in favour of Δ_r^k (and its derivatives) and C_o. This way we can get a decoupled third order k-scale independent equation for Δ_r^k:

$$
\dddot{\Delta}_r^k - \frac{n-5}{t}\ddot{\Delta}_r^k + \frac{24n-19n^2+8}{4t^2}\dot{\Delta}_r^k + \frac{(n-2)\left[5n^2-8n+2\right]}{2t^3}\Delta_r^k - \frac{(12-7n)C_0}{3t^{(n+1)}} = 0 \,.
$$

This equation admits the general solution

$$
\Delta_r^k(t) = C_1 t^{\frac{n}{2}-1} + C_2 t^{\beta_+} + C_3 t^{\beta_-} + C_4 t^{2-n} \,. \qquad (6.151)
$$

where $C_{1,2,3}$ are arbitrary integration constants to be evaluated from initial conditions with

$$C_4 \equiv \frac{2\,(24-14n)\,C_0}{9(7n^3-18n^2+16)} \tag{6.152}$$

and

$$\beta_{\pm} \equiv -\frac{1}{2}+\frac{n}{4}\pm\frac{\sqrt{3(81n^2-44n+12)}}{4}\,. \tag{6.153}$$

Provided that the initial values of Δ^k_r, $\dot{\Delta}^k_r$, $\ddot{\Delta}^k_r$ and C_0 are known at $t_{eq}=1$, the integration constants can be determined since

$$\begin{aligned}
\Delta^k_{(r)eq} &= C_1+C_2+C_3+C_4\,,\\
\dot{\Delta}^k_{(r)eq} &= \left(\tfrac{n-2}{2}\right)C_1+C_2\beta_+ + C_3\beta_- +(2-n)C_4\,,\\
\ddot{\Delta}^k_{(r)eq} &= \left[\tfrac{(n-2)(n-4)}{4}\right]C_1+C_2\beta_+(\beta_+-1)\\
&\quad +C_3\beta_-(\beta_--1)+(2-n)(1-n)C_4\,.
\end{aligned} \tag{6.154}$$

We do not present $C_{1,2,3}$ explicitly for the sake of simplicity.

6.8.2 Perturbations in the Dust-dominated Epoch

Proceeding in a similar fashion for the dust dominated, long-wavelength regime gives the second order evolution equations given by

$$\begin{aligned}
&\ddot{\Delta}^k_d+\frac{n(8n-13)+3}{(n-3)t}\dot{\Delta}^k_d+\frac{[n(8n-13)+3]\,[n(16n-15)+9]}{3(n-3)^2t^2}\Delta^k_d\\
&+\frac{3(n-1)\,[n(16n-15)+9]}{4(n-3)^2(4n-3)}t\dot{\mathcal{R}}^k-\frac{n\,[(n(16n(8n-31)+711)-540]+189}{4(n-3)^2(4n-3)}\mathcal{R}^k\\
&-\frac{n\,(27+54n-56n^2)-27}{4(n-3)^2(4n-3)}t^{-\frac{4n}{3}}C_0=0\,,
\end{aligned} \tag{6.155}$$

$$\begin{aligned}
&\ddot{\mathcal{R}}^k-\frac{4(n-1)\,[n(2n-5)+6]}{[n(n-4)+3)\,t}\dot{\mathcal{R}}^k+\frac{4\,[n\,(n(2n(16n-65)+213)-198)+81]}{9\,[n(n-4)+3]\,t^2}\mathcal{R}^k\\
&-\frac{16n(3-4n)^2\,[n(8n-13)+3]}{27\,[(n-4)n+3]\,t^4}\Delta^k_d-\frac{2n\,[n(4n-7)+3]}{n(n-4)+3}t^{-(n+2)}C_0=0\,,
\end{aligned} \tag{6.156}$$

which reduce to a single third order evolution equation for the density perturbations given as

$$\begin{aligned}
&\dddot{\Delta}^k_d+\frac{5}{t}\ddot{\Delta}^k_d-\frac{2\,[n\,(4n(8n-19)+33)+9]}{9(n-1)t^2}\dot{\Delta}^k_d-\frac{2(4n-3)\,[n(8n-13)+3]}{9(n-1)t^3}\Delta^k_d\\
&-\frac{(12n^2-31n+18)}{6(n-1)\,t^{n+1}}C_0=0\,,
\end{aligned} \tag{6.157}$$

which is a third order decoupled k-scale independent equation. The general solution of (6.157) is given by

$$\Delta_d^k(t) = C_1 t^{-1} + C_2 t^{\gamma_+} + C_3 t^{\gamma_-} + C_4 t^{2-\frac{4n}{3}} \;, \tag{6.158}$$

where $C_{1,2,3}$ are arbitrary integration constants to be evaluated from initial conditions and

$$C_4 \equiv \frac{9\,(12n^2 - 31n + 18)\,C_0}{8(48n^4 - 184n^3 + 159n^2 + 63n - 81)} \tag{6.159}$$

together with

$$\gamma_\pm \equiv -\frac{1}{2} \mp \frac{1}{6}\sqrt{\frac{256n^3 - 608n^2 + 417n - 81}{n-1}}\;. \tag{6.160}$$

As in the radiation epoch, the integration constants $C_{1,2,3}$ can be determined from the initial values of Δ_d^k, $\dot{\Delta}_d^k$, $\ddot{\Delta}_d^k$ and C_0 known at $t_{eq} = 1$ as follows:

$$\begin{aligned} \Delta^k_{(d)eq} &= C_1 + C_2 + C_3 + C_4, \\ \dot{\Delta}^k_{(d)eq} &= -C_1 + C_2\gamma_+ + C_3\gamma_- + \left(\tfrac{6-4n}{3}\right) C_4, \\ \ddot{\Delta}^k_{(d)eq} &= 2C_1 + C_2\gamma_+(\gamma_+ - 1) \\ &\quad + C_3\gamma_-(\gamma_- - 1) + \tfrac{(6-4n)(3-4n)}{9} C_4 \;. \end{aligned} \tag{6.161}$$

Once again, for the sake of simplicity, we do not present them explicitly here .

It turns out that in the adiabatic limit, the long-wavelength solutions of the growth factor both in the radiation and dust epochs are exactly the same as those found in [58].

6.9 Conclusions

In this chapter we have presented a detailed analysis of the $(1+3)$- covariant and gauge-invariant theory of cosmological perturbations in situations where the Universe is described by a multi-component fluid, with a general equation of state parameter for an arbitrary $f(R)$ theory of gravity. The linearized evolution equations of the density and curvature perturbations of such a universe have been derived for both the fluid components and the total matter, relative to the energy frame. We have then taken the background transient solutions of R^n gravity for a two-fluid system dominated respectively by radiation and CDM (dust) and obtained solutions in both the short and long-wavelength approximations. These solutions are important both when testing a full numerical implementation of these equations, and for generating the complete matter power spectrum for $f(R)$ gravity theories with a realistic background cosmological expansion history. We also found that for R^n gravity to be consistent with the Mészáros effect, the parameter n needs to satisfy $n > 2/3$.

We have also given a new covariant characterization of the quasi-static approximation and used this to show that on small scales this approximation is valid for values of n in the neighborhood of 1, i.e., it agrees well with a numerical integration of the full set of equations for the given set of initial conditions. This is the first time such a quasi-static analysis has been presented in a fully covariant way both for radiation and dust universes. It provides the foundation for detailed comparison with what is found using the metric formalisms. As for the full computation of the power spectra, a first attempt will be presented in the next chapter, where a full integration of the radiation-dust fluid system is carried out with the background expansion history obtained using the DSA presented in the previous chapter.

Chapter 7

Large Scale Structure Constraints for $f(R)$ Theories of Gravity

The great tragedy of science -
the slaying of a beautiful
hypothesis by an ugly fact.

Thomas Huxley

7.1 Introduction

Although a lot has been done in the study of $f(R)$ models of gravity as natural alternatives to the cosmic background expansion history, the study will not be complete without a closer look at the perturbed universe and, for example, a comparison of the matter power spectrum predictions made by these theories with available data. This is mainly because it is possible to find many background expansion histories which are consistent with the standard ΛCDM model, making it necessary to investigate the growth of structure in order to break this degeneracy. In this chapter, we study the predicted power spectra using both a dynamical systems approach for the background and solving for the matter perturbations without using the quasi-static approximation, comparing the theoretical results with the several Sloan Digital Sky Survey (SDSS) data available. The importance of studying the first order perturbed equations by assuming the correct background evolution and the relevance of the initial conditions are also stressed. We determine the statistical significance in relation to the observational data and demonstrate their conflict with existing observations.

In Chapter 6 we have seen how the matter dominated solution of R^n gravity given by the saddle point G of the dynamical system given in Chapter 5 has been applied to the study of the evolution equations of scalar perturbations.

The results obtained demonstrate that the evolution of scalar perturbations is determined by a fourth order differential equation rather than a second order one, so that the evolution of density fluctuations contains, in general, four modes rather than two and can give rise to a more complex evolution than is obtained in GR. It was also shown that the perturbations depend on the scale for any value of the equation of state parameter of standard matter (while in GR the evolution of the dust perturbations is not scale dependent) and that there is a characteristic scale-dependent signature in the matter power spectrum [13]. It was also found that the growth of large density fluctuations can also occur in backgrounds in which the expansion rate is increasing in time. This surprising result is strikingly different from what one finds in GR and could provide a strong constraint on some $f(R)$ theories using the integrated Sachs-Wolfe (ISW) effect and the matter power spectrum [95].

These features can be interpreted by comparing the system of fourth order equations which produced them, with those governing scalar perturbations for two interacting fluids in GR [104], because they have the same structure, i.e., there are friction and source terms due to the interaction of the two effective fluids. On very large and on very small scales, the system of equations become independent of k, so that the evolution of the perturbations does not change as a function of scale and the power spectrum is consequently scale invariant. On intermediate scales the interaction between the two fluids is maximized, and the curvature fluid acts as a relativistic component whose pressure is responsible for the oscillations and the dissipation of the small scale perturbations, in the same way in which the photons operate in a baryon-photon system. This suggests that the variables describing the fluctuations in the curvature fluid can be interpreted as representing the modes associated with the additional scalar degree of freedom typical of $f(R)$-gravity. In this sense the spectrum can be explained physically as a consequence of the interaction between these scalar modes and standard matter and result in a considerable loss of power for a relatively small variation of the parameter n.

The results described above were obtained assuming the simple background model G and can therefore only provide hints about what features to expect in the matter power spectrum when realistic expansion histories are considered. In order to test the robustness of the results presented in [5, 11, 12, 58], the complete expansion history of the background FLRW universe for R^n gravity, which resembles the ΛCDM model, needs to be found. This is achieved by integrating the dynamical systems equations so that an orbit representing a cosmic history passes close to the matter dominated point G and eventually tends towards the late-time attractor C discussed in Chapter 5. The equations for Δ_m and $\mathcal{R}$ can then be solved for this background to obtain a matter power spectrum which can be directly compared with the one found for ΛCDM.

This chapter is organized as follows: In Section 7.2 we give the basic equations needed to study cosmological perturbations in $f(R)$ gravity using the covariant approach. In Section 7.3 we discuss the cosmology of R^n gravity and describe their main features by recasting the cosmological equations as an autonomous system of first order equations. We then integrate these equations for different values of the parameter n using initial conditions in the radiation-dominated epoch which have Hubble and deceleration parameters equal to their ΛCDM values and we find that it is impossible to have cosmic histories that simultaneously have present- day values of the Hubble and deceleration parameters close to their ΛCDM values today. For such initial conditions, we compare the cosmological background evolution with baryon acoustic oscillations (BAO) data in Section 7.4. We also find that the BAO analysis corroborates the inviability of these models at the cosmological evolution level. In Section 7.5 we determine the matter power spectra for the expansion histories given in 7.3 and compare them with what is obtained by integrating the perturbation equations for the matter dominated solution G. We find that although the broad large and small scale features of the power spectrum are largely the same as in [13], the scale-dependent features are no longer present when the complete background expansion history is considered. We then use the observed matter power spectrum based on the luminous red galaxies (LRG) in the Sloan Digital Sky Survey (SDSS) [293] and the DR9 CMASS galaxy sample observed by the Sloan Digital Sky Survey (SDSS)-III [264] to directly constrain this class of $f(R)$ theories of gravity. We find that the models considered give power spectra in the SDSS-III wavenumber interval, which are in good agreement with the available data for the recent DR9 CMASS sample.

Finally in Section 7.6 we discuss the results and give an outline of future work to be done.

7.2 Density Perturbations in $f(R)$ Revisited

Using the scalar gradient variables defined in (6.44)(rewritten in an equivalent way unto first order)

$$\Delta_m = \frac{a^2}{\mu_m} D^2 \mu_m , \quad Z = 3a^2 D^2 H , \quad C = S^4 D^2 \tilde{R} ,$$
$$\mathcal{R} = a^2 D^2 R , \quad \Re = a^2 D^2 \dot{R} , \tag{7.1}$$

that completely characterize the evolution of density perturbations, and (6.61) we can expand every first order quantity in the above equations, so for example in the case of Δ_m we have

$$\Delta_m(t, \mathbf{x}) = \sum \Delta_m^{(k)}(t) \, Q^{(k)}(\mathbf{x}) , \tag{7.2}$$

where $\sum$ stands for both a summation over a discrete index or an integration over a continuous one. In this way, it can be shown that the pair of second order equations describing the k^{th} mode for density perturbations in $f(R)$ gravity are given by[1]:

$$\ddot{\Delta}_m^k - \left[(3w-2)\,H + \frac{\dot{R}f''}{f'}\right]\dot{\Delta}_m^k + \left[w\frac{k^2}{a^2} + (w-1)\frac{\mu_m}{f'} - w\frac{f}{f'}\right]\Delta_m^k$$
$$= \frac{1+w}{2}\left[-1 - \frac{2k^2}{a^2}\frac{f''}{f'} + (f - 2\mu_m + 6\dot{R}Hf'')\frac{f''}{f'^2} - 6\dot{R}H\frac{f'''}{f'}\right]\mathcal{R} - \frac{3(1+w)}{f'}Hf''\dot{\mathcal{R}}^k,$$
$$\ddot{\mathcal{R}}^k + \left(2\dot{R}\frac{f'''}{f''} + 3H\right)\dot{\mathcal{R}}^k + \left[\frac{k^2}{a^2} + \ddot{R}\frac{f'''}{f''} + \dot{R}^2\frac{f^{(iv)}}{f''} + 3H\dot{R}\frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3}\right]\mathcal{R}^k$$

[1]These are single-fluid equivalents of the more complicated Eqns (6.72) and (6.73) where $w = c_s^2$.

$$= -\left[\frac{1}{3}(3w-1)\frac{\mu_m}{f''} + \frac{w}{1+w}\left(2\ddot{R} + 2\dot{R}^2\frac{f'''}{f''} + 6\dot{R}H\right)\right]\Delta_m^k + \frac{1-w}{1+w}\dot{R}\dot{\Delta}_m^k \,. \qquad (7.3)$$

Already on super-Hubble scales, $k/aH \ll 1$, a number of important features can be found which allow one to differentiate from what is obtained in GR. Firstly, it is clear that the evolution of density perturbations is determined by a *fourth order* differential equation rather than a second order one. This implies that the evolution of the density fluctuations contains, in general, four modes rather that two and can give rise to a more complex evolution than the one of GR. Secondly, the perturbations are found to depend on the scale for any equation of state for standard matter (while in GR the evolution of the CDM perturbations are scale-invariant). This means that even for dust, the evolution of super-horizon and sub-horizon perturbations are different. Thirdly, it is found that the growth of large density fluctuations can also occur in backgrounds in which the expansion rate is increasing in time. This is in striking contrast with what one finds in GR and would lead to a time-varying gravitational potential, putting tight constraints on the ISW for these models.

Let us now turn to the case of a general wave mode k. One of the most instructive ways of understanding the details of the evolution of density perturbations for a general k is to compute the *matter transfer function* $T(k)$, defined as the ratio of the power spectrum at an initial time t_i and a later time t and given by the relation $P(k,t) = T(k,t,t_i)P(k,t_i)$, where $\mathbf{k}$ is a wavevector characterizing the Fourier component of the solutions of (7.3) and $T(\mathbf{k}) = T(k)$ because of isotropy in the distribution of the perturbations [80, 250]. This quantity tells us how the fluctuations of matter depend on the wavenumber at a specific time and carries information about the amplitude of the perturbations (but not on their spatial structure). In GR, the transfer function on large scales is constant, while on small scales it is suppressed in comparison with the large scales (i.e., modes which entered the horizon during the radiation era) [240, 241]. In the case of pure dust in GR the transfer function is scale invariant. Substituting the details of the background, the values of the parameter n, the barotropic factor w and the wavenumber k into (7.3) one is able to obtain $T(k)$ numerically.

One can easily see from expressions (7.3) that the matter power-spectrum in $f(R)$ gravity theories is further processed after equality and would differ from the standard ΛCDM power spectrum $P_k^{\Lambda\mathrm{CDM}}$ when evaluated today. The latter is widely assumed to represent accurately the evolution of perturbations till radiation-matter equality. Prior to that the effects of any modification to the usual Concordance Model needs to be negligible in order to preserve the cosmological standard model predictions in the radiations-dominated epoch, such as the primordial light element abundances during Big Bang Nucleosynthesis (BBN).

Therefore, these two power spectra, when evaluated today, would be related linearly by a transfer function $T(k)$ given by

$$P_k^{f(R)} = T(k)P_k^{\Lambda\mathrm{CDM}}|_{eq} \qquad (7.4)$$

where $T(k) \propto |\Delta_m^k|^2_{today}$ and Δ_m^k is obtained from the system of equations (7.3).

On linear scales, $P_k^{f(R)}$ will in general depend on both the $f(R)$-model and the scale k, therefore differing from the ΛCDM model, where it is scale-invariant.

In the GR limit: $f(R) = R$, (7.3) reduces to the standard equations for the evolution density perturbations in GR:

$$\ddot{\Delta}_m^k - (3w-2)\,H\dot{\Delta}_m^k + \left[w\frac{k^2}{a^2} + \left(\frac{-1+2w-3w^2}{2}\right)\mu_m\right]\Delta_m^k = 0, \tag{7.5}$$

$$\mathcal{R}^k = (1-3w)\mu_m\Delta_m^k, \tag{7.6}$$

and one can easily see that the linear evolution of CDM density perturbations for sub-Hubble ($k \gg aH$) scales in ΛCDM is given by the well-known result:

$$\Delta_m^{\prime\prime k} + \mathcal{H}\Delta_m^{\prime k} - \frac{1}{2}a^2\mu_m\Delta_m^k = 0\,, \tag{7.7}$$

where $\mathcal{H} = a'/a$ and prime (only for this equation) denotes derivative with respect to conformal time.

Notice that according to (7.7) the evolution of the Fourier modes does not depend upon k. This means that for ΛCDM on sub-Hubble scales, once the density contrast starts to grow after matter-radiation equality, evolution only changes the overall normalisation of the matter power-spectrum $P(k)$, but not its shape.

7.3 Determining the Expansion History for R^n-gravity

To proceed, we need to fix our theory of gravity. As indicated in previous chapters, the simplest and most widely studied form of $f(R)$ gravitational theories is $f(R) = \beta H_0^2(R/H_0^2)^n$. For this class of models the cosmological equations associated with a FLRW universe are particularly easy to analyze.

However the aim of this investigation is to show that studies of different $f(R)$-gravity models that share similar background expansion history with ΛCDM can in principle provide useful constraints on the viability of these models via the power spectra of matter density perturbations they produce, and with the help of BAOs as standard rulers of known geometrical information.

If we apply the implementation of the DSA in this class of flat, dust-radiation (but dust-dominated) models, we can define the dimensionless variables [60]:

$$x = \frac{\dot{R}(n-1)}{HR},\quad y = \frac{R(1-n)}{6nH^2},$$
$$\Omega_d = \frac{\mu_d}{3n\alpha H^2 R^{n-1}},\quad \Omega_r = \frac{\mu_r}{3n\alpha H^2 R^{n-1}}\,. \tag{7.8}$$

In terms of these variables, the Friedmann equation (5.16) takes the form

$$1 + x + y - \Omega_d - \Omega_r = 0\,. \tag{7.9}$$

The autonomous system of ordinary differential equations can be given by differentiating:

$$\begin{aligned}
-(z+1)\frac{\mathrm{d}x}{\mathrm{d}z} &= -x - x^2 + \frac{(4-2n+nx)y}{n-1} + \Omega_d\,,\\
-(z+1)\frac{\mathrm{d}y}{\mathrm{d}z} &= 4y + \frac{(x+2ny)y}{n-1}\,,\\
-(z+1)\frac{\mathrm{d}\Omega_d}{\mathrm{d}z} &= \left(1 - x + \frac{2ny}{n-1}\right)\Omega_d\\
-(z+1)\frac{\mathrm{d}\Omega_r}{\mathrm{d}z} &= \left(-x + \frac{2ny}{n-1}\right)\Omega_r\,.
\end{aligned} \tag{7.10}$$

whose dimensionality can be reduced further using (7.9). The evolution of the Hubble parameter can then be determined by writing (3.20) in terms of the DS variables:

$$(1+z)\frac{\mathrm{d}h}{\mathrm{d}z} = \frac{h(2+ny)}{n-1}\,. \tag{7.11}$$

Furthermore the deceleration parameter can be determined directly from y:

$$q = \frac{ny}{(n-1)} + 1\,. \tag{7.12}$$

If n lies in the range $1.36 < n < 1.5$, the two cosmologically interesting fixed point exact solutions of these equations, C and G respectively, represent decelerated and accelerated phases of the Universe with positive energy density.

With this in mind, let us integrate (7.10), by fixing the initial conditions for the DS variables (7.8) to be identical to their ΛCDM values in the radiation-dominated era (at a redshift $z_0 = 6000$), and determine the expansion history for R^n models with eight different values for the exponent $n > 1$ between $n = 1.1$ and 1.4 in order to allow the possibility of late-time acceleration. In this way we can determine for which values of n we obtain present-day values for $q(z)$ and $H(z)$ which are consistent with the ΛCDM model.

It is clear from the results in Table 7.1 that it is not possible for R^n gravity to admit FLRW cosmic histories that simultaneously have present-day values of the Hubble and deceleration parameters which are close to their ΛCDM values today, if initial conditions are chosen in order that the expansion history is close to the ΛCDM model at early times. Even without considering LSS data, this already put severe constraints on these models.

n	1.1	1.2	1.27	1.29	1.3	1.31	1.33	1.4
h_0	0.65	0.75	0.94	0.99	1.44	2.43	7.34	159.67
q_0	0.39	0.20	0.10	0.25	0.36	0.35	0.22	−0.17

Table 7.1: Present-day values of the Hubble $h_0 \equiv H(\text{today})/H_0$ and deceleration (q_0) parameters for the R^n models under consideration. H_0 corresponds to the ΛCDM Hubble parameter value today. Only $n = 1.4$ provides acceleration at the present time, whereas $n = 1.29$ gives the closest value for h_0 to ΛCDM. With regard to the χ^2 analysis for BAO to be studied in the Section 7.4 $n = 1.29$ provided the best value ($\chi^2_{\rm BAO} = 16.11$) but well above the one provided by ΛCDM ($\chi^2_{\rm BAO} = 4.51$). The remaining values of exponent n give $\chi^2_{\rm BAO}$ values showing incorrect fits to BAO data.

7.4 BAO Constraints

As standard rulers, BAO constraints provide an ideal arena for the analysis of cosmic expansion history. This is mainly because these oscillations correspond to a preferred length scale in the early universe that can be predicted from CMB measurements [39]. Some relevant quantities for these analyses are the comoving distance from an observer to some redshift z which is given by

$$r(z) = \frac{1}{H_0}\int_0^z \frac{\mathrm{d}z}{h(z)}\,, \tag{7.13}$$

and the scaled distance to recombination, the comoving sound horizon at recombination and the dilation scale respectively as given by [187]

$$R = H_0\sqrt{\Omega_{0d}}\, r(z_{\rm CMB})\,, \tag{7.14}$$

$$r_s(z_{\rm CMB}) = \frac{1}{H_0}\int_{\infty}^{z_{\rm CMB}} \frac{c_s(z)}{h(z)}\,\mathrm{d}z\,, \tag{7.15}$$

$$D_V(z_{\rm BAO}) = \left[\left(\int_0^{z_{\rm BAO}} \frac{\mathrm{d}z}{H(z)}\right)^2 \frac{z_{\rm BAO}}{H(z_{\rm BAO})}\right]^{1/3} \tag{7.16}$$

where $c_s(z) = \left[3\left(1+\frac{\bar{R}_b}{1+z}\right)\right]^{-1/2}$ is the sound speed of the photon-baryon relativistic plasma with photon-baryon density ratio

$$\bar{R}_b = \frac{3}{4}\frac{\Omega_b\tilde{h}^2}{\Omega_\gamma\tilde{h}^2} = 3.15\times10^4\,\Omega_b\tilde{h}^2\left(\frac{T_{\rm CMB}}{2.7\,\mathrm{K}}\right)^{-4}. \tag{7.17}$$

where we have used the Planck result of $\tilde{h} = 0.6711$ [251] for this analysis, as well as $z_{CMB} = 1021.44$.

Following the methods presented in [187] for a flat prior we apply the maximum likelihood method using the numerically generated datapoints for the different n values considered, corresponding to recent measurements [206] of the 6dF Galaxy

Survey at $z = 0.1$ [37], the SDSS DR7 at $z = 0.2,\ 0.35$ [239, 247], the WiggleZ at $z = 0.44, 0.60, 0.73$ [39]: and thus define

$$\mathbf{X_{BAO}} = \begin{pmatrix} \frac{r_s(z_{\rm CMB})}{D_V(0.106)} - 0.336 \\ \frac{r_s(z_{\rm CMB})}{D_V(0.2)} - 0.1905 \\ \frac{r_s(z_{\rm CMB})}{D_V(0.35)} - 0.1097 \\ \frac{r_s(z_{\rm CMB})}{D_V(0.44)} - 0.0916 \\ \frac{r_s(z_{\rm CMB})}{D_V(0.6)} - 0.0726 \\ \frac{r_s(z_{\rm CMB})}{D_V(0.73)} - 0.0592 \end{pmatrix} \tag{7.18}$$

to calculate the χ^2 from the BAO as

$$\chi^2_{\rm BAO} = \mathbf{X_{BAO}}^{\mathbf{T}}\ \mathbf{C_{BAO}}^{-1}\ \mathbf{X_{BAO}}\ , \tag{7.19}$$

where $\mathbf{C_{BAO}}^{-1}$ corresponds to the inverse covariance matrix as given in [39]. The results found for the models under study showed that the $\chi^2_{\rm BAO}$ analysis proves that the cosmological evolution as provided by the models under study cannot achieve the goodness-of-fit of ΛCDM ($\chi^2_{\rm BAO} = 4.51$) and that only for $n = 1.29$ ($\chi^2_{\rm BAO} = 16.11$) the fit to BAO data can be considered of the same order of magnitude, though much bigger, than ΛCDM.[2]

7.5 The Matter Power Spectrum and SDSS Constraints

Let us now turn to the matter power spectrum.

Taking the dominant component to be dust, the system (7.3) can be written in terms of the dynamical variables:

$$\begin{aligned} &\frac{(-1+n)^2(1+z)}{2ny}\hat{\mathcal{R}}^{k\prime} - \frac{3h^2\left[(-1+n)(1+(-2+n)\Omega_d)+(-2+n)y\right]+\hat{k}^2(-1+n)^2(1+z)^2}{6h^2ny}\hat{\mathcal{R}}^k \\ &\quad + h^2(1+z)^2\Delta_m^{k\prime\prime} + h^2\frac{\left[(-1+n)\Omega_d+y\right](1+z)}{n-1}\Delta_m^{k\prime} - 3h^2\Omega_d\Delta_m^k = 0, \\ &\hat{\mathcal{R}}^{k\prime\prime} - \frac{\left[4-4\Omega_d+4y+n\left(-2+2\Omega_d-3y\right)\right]}{(-1+n)(1+z)}\hat{\mathcal{R}}^{k\prime} \\ &\quad + \left\{\frac{\hat{k}^2}{h^2} + \frac{(-2+n)\left[-\Omega_d^2-(-1+y)^2+\Omega_d(1+n+2y)\right]}{(-1+n)^2(1+z)^2}\right\}\hat{\mathcal{R}}^k \end{aligned}$$

[2] A more detailed analysis of different cosmological models requires using χ^2 per degree-of-freedom, rather than just χ^2. But since the R^n model under discussion here has the same degree of freedom as ΛCDM, the χ^2 and χ^2 per degree-of-freedom analyses are equivalent.

n exponent	1.1	1.2	1.27	1.29	1.3	1.31	1.33	1.4
χ^2	15.1394	13.1839	13.0184	13.0093	13.0104	13.0098	13.0102	13.0128
σ exclusion	1.9849	1.4086	1.3486	1.3452	1.3457	1.3454	1.3456	1.3465
% suppression	29.5	7.94	4.75	4.45	4.37	4.33	4.35	4.47

Table 7.2: Fits to the SDSS 2006 data for R^n cosmology by using set of initial conditions **III**: eight different values of exponent n were investigated from $n = 1.1$ to 1.4. Values for χ^2 and the confidence region σ are presented in the second and third rows respectively. The data to be fitted by the theoretical spectra are taken from [293] and normalisation from WMAP7 was imposed for all the studied models. The fit provided by ΛCDM ($\chi^2 = 11.1996$) is not improved by any of these parameter values. The final row gives the suppression in the overall initial amplitude required to get the best fits. For all the values, this suppression turns out to be smaller than 30% and is therefore in the experimental uncertainty interval for this quantity. One can see that the best fit corresponds to the value $n = 1.29$ with a suppression of 4.45% and good fits are also obtained for $n = 1.3$, 1.31 and 1.33 with similar suppressions.

$$+\frac{6h^2ny\,(1-\Omega_d+y)}{(-1+n)^2(1+z)}\Delta_m^{k'}+\frac{6h^2n\Omega_d y}{(-1+n)^2(1+z)^2}\Delta_m^k=0\,, \tag{7.20}$$

where prime denotes derivative with respect to redshift and the dimensionless quantities $\hat{\mathcal{R}}^k = \mathcal{R}^k/H_0^2$ and $\hat{k} = k/H_0$ have been introduced. Note that equations (7.20) are valid only for $n \neq 1$.

SDSS correlation data from LRG have been used to test the predictions from the ΛCDM power spectrum obtained from linear perturbation theory with WMAP cosmological data to high accuracy ($\chi^2 \approx 11.2$, degrees of freedom $(d.o.f.) = 14$) [293]. In what follows, we will do the same for this class of $f(R)$ theories of gravity.

To do this we first determine the cosmological background evolution as described in the previous section and then use these results to solve the system of equations (7.20) in order to obtain the density contrast today. Then, by applying expression (7.4) to these results, one can obtain the fully processed power spectra $P_k^{f(R)}$ for the above models, which can be compared to the ΛCDM predictions and the LRG data.

Before proceeding, let us mention that three sets of different initial conditions were considered for the system (7.20) in order to determine how sensitive the final processed power spectrum is to changes in these values:

- **I**: $\Delta_m^k|_0 = \hat{\mathcal{R}}^k|_0 = 10^{-5}$, $\Delta_m^{k'}|_0 = \hat{\mathcal{R}}^{k'}|_0 = 10^{-5}$,
- **II**: $\Delta_m^k|_0 = \hat{\mathcal{R}}^k|_0 = 10^{-5}$, $\Delta_m^{k'}|_0 = \hat{\mathcal{R}}^{k'}|_0 = 10^{-8}$,
- **III**: $\Delta_m^k|_0 = \hat{\mathcal{R}}^k|_0 = 10^{-5}$, $\Delta_m^{k'}|_0 = \hat{\mathcal{R}}^{k'}|_0 = 0$,

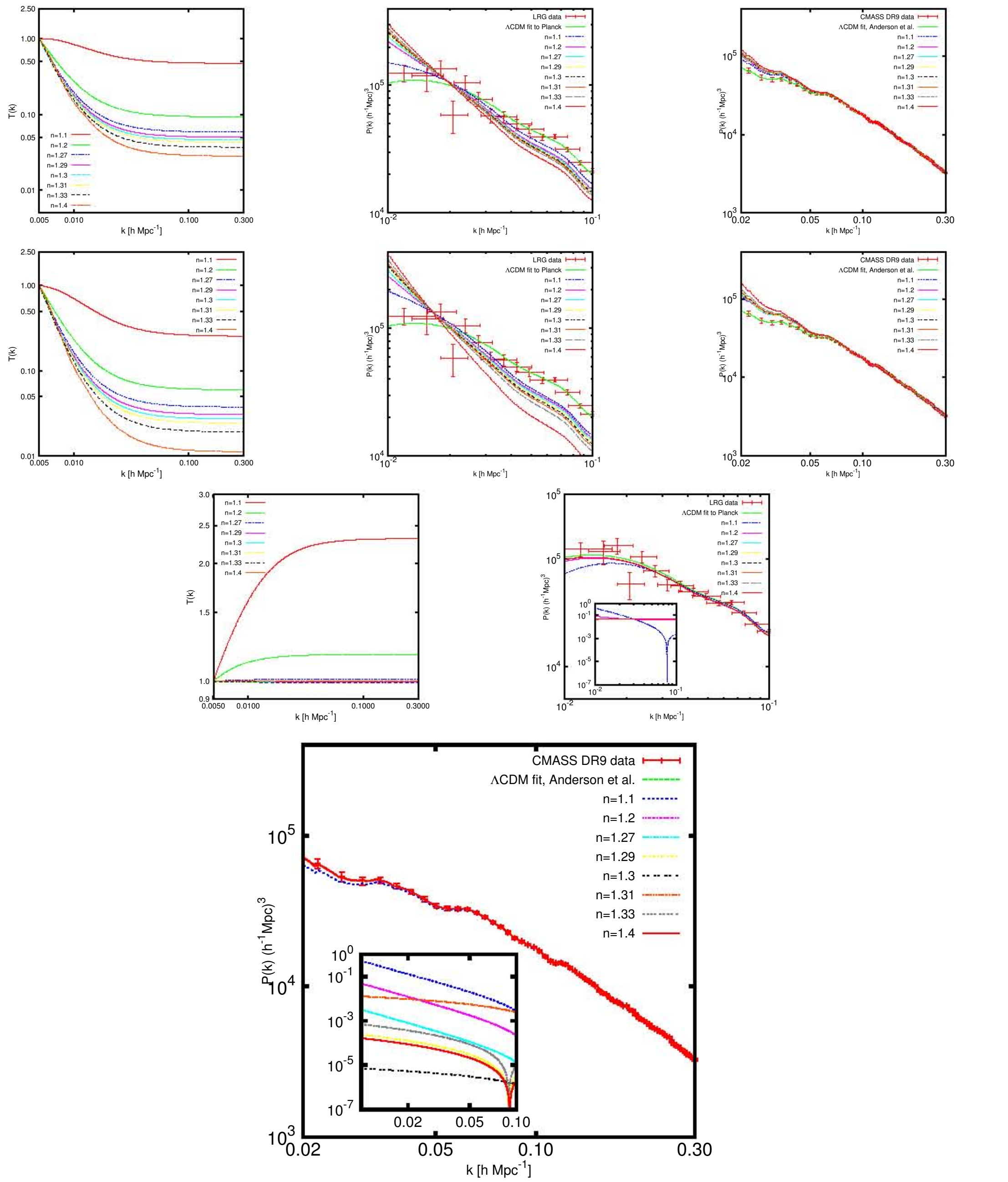

Figure 7.1: The left panels show the transfer function $T(k) = |\Delta_k/\Delta_k^{\Lambda\mathrm{CDM}}(z = 2000)|^2$ evaluated today ($z = 0$) for wavenumber k (in $h\,\mathrm{Mpc}^{-1}$ units) in the range 0.005 to 0.3 for the initial conditions sets **I**, **II** and **III** as described in the bulk of this investigation. The transfer functions $T(k)$ on left panels have been normalized in such a way that the curves coincide on large scales. On the central and right panels we present the corresponding linear matter power-spectra $P(k)$ for ΛCDM and R^n models for $n = 1.1$, 1.2, 1.27, 1.29, 1.3, 1.31, 1.33 and 1.4. Data correspond to SDSS 2006 [293] (central panel) and SDSS-III data [264] (right panel) respectively. All the power spectra were assumed to have an arbitrary overall normalisation at the scale $k = 0.01\,h\,\mathrm{Mpc}^{-1}$ (central panel) and $k = 0.02\,h\,\mathrm{Mpc}^{-1}$ (right panel) in order to find the best fit to the data. Conditions **I** and **II** lead to power spectra in complete disagreement with the observed data. Conditions **III**, due to the almost flatness of the spectra in the range covered by data present a good fit to the data. On the bottom panels (central and right) we show in a window the relative discrepancy between the ΛCDM and the R^n fits power-spectra for every studied exponent. For SDSS 2006 data, the smallest discrepancy in scales $k > 3 \times 10^{-2}\,h\,\mathrm{Mpc}^{-1}$ happens for $n = 1.1$ whereas for smaller scales, all the remaining values of n provide similar relative error around 5×10^{-2}. For DR9 SDSS-III data, the smallest discrepancy (order 10^{-5}) happens for $n = 1.3$ in the whole scale-range despite some punctual values of k where other exponents may present smaller relative errors with respect to ΛCDM. whereas for smaller scales, all the remaining values of n provide similar relative error around 5×10^{-2}.

n exponent	1.1	1.2	1.27	1.29	1.3	1.31	1.33	1.4
χ^2	4.5463	1.0507	1.0366	1.0357	1.0355	1.0458	1.0360	1.0357
σ exclusion	1.874	0.123	0.0316	0.012	0.002	0.101	0.020	0.001
% suppression	13	1.5	0.1	0.01	0.001	1	0.04	0.009

Table 7.3: Fits to the SDSS CMASS DR9 data for R^n cosmology by using set of initial conditions **III**: eight different values of exponent n were investigated from $n = 1.1$ to 1.4. Values for χ^2 and the confidence region σ are presented in the second and third rows respectively. The data to be fitted by the theoretical spectra are taken from [264]. The fit provided by ΛCDM ($\chi^2 = 61.1/59 \approx 1.03559$) is slightly improved by the $n = 1.3$ parameter value. The final row gives the suppression in the overall initial amplitude required to get the best fits. For all the values, this suppression turns out to be smaller than 15% and is therefore in the experimental uncertainty interval for this quantity. For the best fit $n = 1.3$ the corresponding suppression is 10^{-3}% and very good fits are also obtained for $n = 1.27$, 1.29, 1.33 and 1.4 with similar suppressions.

where the subscript 0 refers to the initial redshift $z_0 = 2000$. The choice of sets **I** and **II** as initial conditions for the system (7.3) can be understood as providing scale-invariant initial conditions for the variables Δ^k_m and $\hat{\mathcal{R}}^k$ and their first derivatives are all taken to be small (but non-zero) at the initial redshift. On the other hand, in set **III** we set the first derivatives of Δ^k_m and $\hat{\mathcal{R}}^k$ to zero at $z = z_0$. This choice has important consequences for the obtained spectra.

For each value of n, we present in Fig. 7.1 both the transfer functions and the processed power spectra for all the initial conditions sets just mentioned.

One can see that for these models the transfer functions have a nearly-flat plateau on large scales [13] regardless of the set of initial conditions. On intermediate scales, however, the density contrast behaviour and its amplitudes today depend both upon the value of n and the initial conditions. The left panels in Fig. 7.1 clearly illustrate this.

Thus, by using expression (7.4) for the transfer function, one can obtain the processed power spectra and compare these theoretical results with the LRG data. We find that initial conditions **I** and **II** are not able to provide good fits to the SDSS data due to the fact that in the required scales ($0.01 - 0.1\,\mathrm{h\ Mpc^{-1}}$) the spectra are not flat but decrease with the wavenumber k (see the left panels in Fig. 7.1). On the other hand, initial conditions **III** give power spectra which are in good agreement with the LRG data. This is due to the almost-flat transfer function in the LRG data range. Note however that the initial amplitude was assumed to be a free parameter which was determined to achieve the best fit.

For this set of initial conditions, Table 7.2 give the χ^2 ($d.o.f. = 14$) analysis for the eight studied R^n models when their respective spectra evolutions are fitted to the SDSS data, assumed to be non-correlated. We also include the value for the confidence regions σ with respect to ΛCDM, as well as the overall amplitude suppression in the initial scale ($k = 0.01\,\mathrm{h\ Mpc^{-1}}$) to get the best fits after a Least Square method analysis. It is clear that none of the R^n models under consideration acquire the same goodness-of- fit as the ΛCDM model as seen in Table 7.2 in the σ exclusion regions.

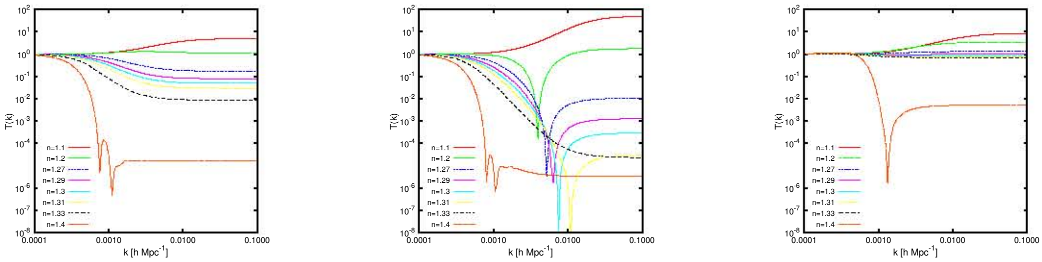

Figure 7.2: Saddle-point analysis: scale dependence of the transfer function $T(k) = |\Delta_k/\Delta_k^{\Lambda\mathrm{CDM}}(z = 2000)|^2$ evaluated today ($z = 0$) for wavenumber k (in $h\,\mathrm{Mpc}^{-1}$ units) in the range 0.0001 to 0.1 for the initial same conditions **I**, **II** and **III** used in the rest of this investigation. The transfer functions have been normalized in such a way that the curves coincide on large scales. Both on very large and on very small scales, the Δ becomes k-independent, so that the evolution of the perturbations does not change as a function of scale and the transfer function is consequently scale invariant. On intermediate scales the curvature fluid causes the oscillations. The result is a considerable loss of power for a relatively small variation of the parameter n [56]. As before the shape and amplitudes (i.e. the increasing or decreasing behaviour with k) of the transfer functions depend on the initial conditions **I** (left), **II** (center) and **III** (right). For example, when $n = 1.27$ we have decreasing behaviour for **I** and **II** but increasing behaviour for **III**.

For completeness and in order to emphasize the importance of using the complete expansion history for the background, we have also given $|\Delta_m^k|^2$ for models whose background evolution is given by the exact saddle point solution G in the case of dust ($w = 0$). We do this for the same parameter values n and initial conditions **I**, **II** and **III** as shown in Fig. 7.2. These results agree with previous investigations [13] which showed that when this background scale factor is assumed, the spectrum is composed of three parts corresponding to three different evolution regimes for the perturbations. In this scenario, on intermediate scales, the interaction between the two fluids (dust and curvature) is maximized and the curvature fluid acts as a relativistic component whose pressure is responsible for the oscillations and the dissipation of the small scale perturbations, in the same way in which the photons operate in a baryon-photon system. If we compare these results to the left panel in Fig. 7.1, we conclude that the scale-dependent features in Fig. 7.2 are washed out when the complete background expansion history is considered. The main large and small scale features of the power spectrum found in [13], however, are retained.

7.6 Conclusions

In this chapter we presented a complete analysis of the background and matter perturbations for one of the most widely studied modified gravity theories: R^n gravity with $n \gtrsim 1$. Both the cosmological background evolution and linear perturbation equations were solved by combining the dynamical systems approach for the background and using the $1 + 3$ covariant approach to evolve the matter perturbations, without assuming any intermediate (quasi-static) approximation.

We solved the background equations for different values of the parameter n using initial conditions in the radiation dominated epoch, with Hubble and deceleration parameters equal to their ΛCDM values. For such initial conditions, we performed a baryon acoustic oscillations analysis. By using this tool we found that it is impossible to obtain fits as good as ΛCDM. We also proved the impossibility of having cosmic histories that simultaneously have present day values of these cosmological parameters close to their ΛCDM values today. In fact, of the ten models considered, only $n = 1.4$ provided a negative deceleration parameter today, but gave a Hubble parameter completely incompatible with its observed value, while values around $n = 1.29$ gave the closest value for the present-day Hubble parameter to ΛCDM, but exhibited no late time acceleration. The value $n = 1.29$ provided the best χ^2 when its cosmological evolution is compared with BAO data, but is well above that of the ΛCDM one.

We then used the observed matter power spectrum based on both luminous red galaxies (2006) and the DR9 CMASS galaxy sample in the Sloan Digital Sky Survey to further constrain these models. For the studied exponents, we found that all the models gave rise to almost-flat transfer functions in the Sloan wavenumber interval, provided very special initial conditions are chosen. In this case the best fit to the data for 2006 data was found for the value $n = 1.29$ with a suppression of 4.45%. Good fits and good fits were also obtained for $n = 1.3$, 1.31 and 1.33. The exponent $n = 1.4$ (the only one providing acceleration today) required a suppression slightly bigger (4.47%). With regard to DR9 2012 data and partially thanks to the accuracy in this catalogue, most of the studied R^n models provided good fits to the data. $n = 1.3$ with a suppression of 10^{-3}% proved to be the best fit slightly improved by ΛCDM . Other exponents (1.27, 1.29, 1.33 and 1.4) also provided good fits with slightly bigger suppressions.

Regardless of the Large Structure Constraints none of the studied exponents were however able to fit the baryon acoustic oscillations data as well as the ΛCDM model and the obtained χ^2 were much bigger than the best-fit model as provided by ΛCDM. It is clear from this analysis that R^n gravity does not successfully meet any of the cosmology requirements for it to be considered as a viable alternative to the standard model. This work does, however, illustrate in depth the utility of our approach and it should be possible to use these techniques with the most updated available data to constrain which $f(R)$ theories remain consistent with current data, even if they are indistinguishable from the ΛCDM model either at the level of the FLRW background or cosmological perturbations. It is also a well-known result [77] the solar system tests are problematic to R^n models.

PART III

The expansion, rotation and shear of time-like geodesic congruences (see Eqn (4.14)) are encoded in the differential properties of the geodesics, which in turn are related to the physical interpretations we give to the gravitational field equations. If we look at the Raychaudhuri equation (4.54), we see that shear and vorticity play important roles in the expansion dynamics of cosmological fluid models [83, 87, 122, 124, 160] and in the way distant matter can influence the local gravitational field. In particular, if the equation is subject to the SEC (4.33), then we see that non-accelerating irrotational fluids undergo gravitational collapse. On the other hand, collapse can be prevented if the acceleration and vorticity terms dominate over the shear. The natural question to ask, then, is: what happens in the absence of vanishing shear?

There have been numerous studies on the role of shear in GR, and the special nature of shear-free cases in particular. In [145] Gödel showed that shear-free time-like geodesics of some spatially homogeneous universes cannot expand and rotate simultaneously, a result later generalized by Ellis [129] to include inhomogeneous cases of shear-free time-like geodesics. Goldberg and Sachs, on the other hand, showed [149] that shear-free null geodesic congruences *in vacuo* require an algebraically special Weyl tensor, a result later generalized by Robinson and Schild [263] to include non-vanishing, but special, forms of the Ricci tensor.

An interesting aspect of these shear-free solutions is that they do not hold in Newtonian gravitation theory [122, 218, 219] although Newtonian theory is a limiting case of GR under special circumstances, namely at low-speed relative motion of matter with no gravito-magnetic effects (vanishing magnetic part of the Weyl tensor) and hence no gravitational waves. In the Newtonian formulation of cosmological models, one considers potentials rather than forces in the dynamical evolution of spacetime, and a generalized concept of acceleration is introduced to represent the combined effects of gravitation and inertia [124]. Thus, Newtonian cosmologies are a generalization of the Newtonian Theory (NT) of gravity and are usually refered to as "quasi-Newtonian", rather than strictly Newtonian, formulations.

In this part of the thesis, we investigate a class of shear-free perfect fluid cosmological models in the context of $f(R)$ gravity, focusing on generalizing earlier GR results for time-like geodesics. Chapter 8 generalizes the Ellis theorem on shear-free dust models linearized about a FLRW background , where we show that there are a class of $f(R)$ models that are closer to NT than GR is, i.e., models in which the time-like geodesic dust flows can both rotate and expand simultaneously. Chapter 9 generalizes the potential and acceleration terms of the quasi-Newtonian GR formulation of integrable dust models about a linearized FLRW background.

Chapter 8

Shear - free Perturbations in $f(R)$

> You cannot apply mathematics as long as words still becloud reality.
>
> Hermann Weyl

8.1 Introduction

Due to the highly non-linear nature of the field equations for $f(R)$ theories, difficult conceptual and technical issues arise which need to be resolved in order to uncover the detailed physics of these theories. Consequently, it is crucial to develop new methods which are able to assist in resolving these problems. The DSA approach to cosmology and the $1+3$-covariant approach have proved very useful in contributing to our understanding of how the astrophysics and cosmology of these theories differ from what is found in GR. Most of the work thus far has focussed on the dynamics of homogenous cosmological models [9, 60, 64, 105, 146, 188], the linear growth of large scale structure [5, 28, 58, 273] and on finding exact solutions which describe the gravitational field of stars and compact objects [76, 233].

Of particular importance is understanding the relationship between the Newtonian and relativistic limits of $f(R)$ which is important in describing the dynamics of non-linear fluid flows in such theories. This is relevant both in the physics of gravitational collapse and the late (non-linear) stages of structure formation [209].

Central to all of these problems is the differential properties of time-like geodesics which describe the fluid flow in cosmology. In general, the kinematics of such fluid flows are described by the expansion Θ, shear (or distortion) σ_{ab}, rotation ω^c, and acceleration A_a of the four-velocity field u^a tangent to the fluid flow lines. Their governing equations are obtained by contracting the Ricci identities (applied to u^a) along and orthogonal to u^a, which determine how they couple to gravity [127].

The delicate relationship between the kinematic quantities in Newtonian and relativistic fluid flows in GR is most strikingly seen in a remarkable result first obtained by Ellis in 1967 [129]. In this work it was found that if the four velocity vector field of a barotropic perfect fluid with vanishing pressure is shear-free, then either the expansion or the rotation of the fluid vanishes. This is a purely local result to which no corresponding Newtonian equivalent appears to hold, as counter-examples can be explicitly constructed [159]. It is therefore interesting to ask whether such a result holds in the more general setting of $f(R)$.

As a first step towards this goal, we examine whether this result holds in situations where the hydrodynamic and gravitational equations have been linearized about a FLRW background [44, 45, 102, 104, 118, 128]. These *almost FLRW* models can be thought of as lying somewhere between the full non-linear situation and Newtonian theory, at least in the cosmological context, and therefore an analysis of the theorem in this context could shed some light on the generality of the result in $f(R)$. Indeed, since it has already been shown in earlier work that cosmologies with a bounce occur more naturally in such theories [62], one might expect a somewhat weaker version of the theorem to emerge.

We show that if the 3-curvature vanishes, then the result of [129] can always be avoided for vacuum universes. We also demonstrate there is at least one physically realistic non-vacuum case in which both rotation and expansion are simultaneously possible.

8.2 Linearized $f(R)$ Field Equations about FLRW Background Revisited

We have seen in Chapter 4 that the irreducible covariant decomposition of the derivative of the time-like vector u^a is given by

$$\nabla_a u_b = -A_a u_b + \tfrac{1}{3}h_{ab}\Theta + \sigma_{ab} + \eta_{abc}\omega^c, \tag{8.1}$$

where $A_a = \dot{u}_a$ is the acceleration, $\Theta = \tilde{\nabla}_a u^a$ is the expansion, $\sigma_{ab} = \tilde{\nabla}_{\langle a}u_{b\rangle}$ is the shear tensor and $\omega^a = \eta^{abc}\tilde{\nabla}_b u_c$ is the vorticity vector.

If we consider the background spacetime to be FLRW where the Hubble scale sets the characteristic scale of the perturbations, and in the perturbed spacetime the standard matter is considered to be a perfect fluid with the EMT given by

$$T^m_{ab} = (\mu_m + p_m)u_a u_b + p_m g_{ab} \ , \tag{8.2}$$

such that standard matter has a barotropic linear equation of state $p_m = w\mu_m$ then the heat flux (q^m_a) and the anisotropic stress (π^m_{ab}) vanish in the perturbed spacetime. In addition, since we consider shear-free perturbations, the shear tensor σ_{ab} vanishes identically.

The linearised thermodynamic quantities of the curvature fluid are as given by Eqns (4.73)- (4.76). Moreover, the linearised field equations are also given in Section

4.3.

We note that the constraints $(C_1)^a$, (C_2), $(C_3)^{ab}$, $(C_5)^a$ and $(C_6)^a$ are the constraints of Einstein field equations for general matter motion and are shown to be consistently *time propagated* along u^a locally in GR. However, the conditions $\sigma_{ab}=0$ and $q^a_m=0$ in Eqns (4.79) and (4.84) result in two new constraints $(C_0)^{ab}$ and $(C_4)^a$ respectively given by

$$(C_0)^{ab} := E^{ab} - \tilde\nabla^{\langle a}A^{b\rangle} - \tfrac{1}{2}\pi^{ab}_R = 0\ , \tag{8.3}$$

$$(C_4)^a := \tilde\nabla^a p_m + (\mu_m + p_m)A^a = 0\ . \tag{8.4}$$

In what follows we will use the following linearized commutation relations for shear-free congruences. For any scalar ϕ

$$\begin{aligned}[\tilde\nabla_a\tilde\nabla_b - \tilde\nabla_b\tilde\nabla_a]\phi &= 2\eta_{abc}\omega^c\dot\phi\ ,\\ \eta^{abc}\tilde\nabla_b\tilde\nabla_c\phi &= 2\omega^a\dot\phi\ .\end{aligned} \tag{8.5}$$

If the gradient of the scalar is of the first order, we have

$$[\tilde\nabla^a\tilde\nabla_b\tilde\nabla_a - \tilde\nabla_b\tilde\nabla^2]\phi = \tfrac{1}{3}\tilde R\tilde\nabla_b\phi \tag{8.6}$$

and

$$[\tilde\nabla^2\tilde\nabla_b - \tilde\nabla_b\tilde\nabla^2]\phi = \tfrac{1}{3}\tilde R\tilde\nabla_b\phi + 2\eta_{dbc}\tilde\nabla^d(\omega^c\dot\phi)\ . \tag{8.7}$$

Also, for any first order 3-vector $V^a = V^{\langle a\rangle}$, we have

$$[\tilde\nabla^a\tilde\nabla_b - \tilde\nabla_b\tilde\nabla^a]V_a = \tfrac{1}{3}\tilde R h^a{}_{[a}V_{b]}\ , \tag{8.8}$$

$$h^a{}_c h^d{}_b(\tilde\nabla_d V^c)^{\cdot} = \tilde\nabla_b \dot V^{\langle a\rangle} - \tfrac{1}{3}\Theta\tilde\nabla_b V^a, \tag{8.9}$$

$$h^a{}_c(\tilde\nabla^2 V^c)^{\cdot} = \tilde\nabla_b(\tilde\nabla^{\langle b}V^{a\rangle})^{\cdot} - \tfrac{1}{3}\Theta\tilde\nabla^2 V^a. \tag{8.10}$$

8.3 Consistency of the new constraints

We have already seen that the conditions of shear-free perturbations together with the matter being described by a perfect fluid in the perturbed spacetime, gives the new constraints $(C_0)^{ab}$ and $(C_4)^a$ respectively. To check their compatibility with the existing constraints of Einstein's field equations, we substitute $(C_0)_{bd}$ into $(C_5)_b$ to obtain

$$\tilde\nabla^d\tilde\nabla_{\langle b}A_{d\rangle} - \tfrac{1}{3}\tilde\nabla_b\mu + \tilde\nabla^d\pi^R_{bd} + \tfrac{1}{3}\Theta q^R_b = 0\ . \tag{8.11}$$

Now from the constraint $(C_4)_b$ we have

$$A_b = -\frac{w}{w+1}\tilde{\nabla}_b\phi \,, \tag{8.12}$$

where $\phi = \ln(\mu_m)$. Using equation (8.12) in (8.11) we get the constraint

$$\frac{w}{w+1}\tilde{\nabla}^d\tilde{\nabla}_{\langle b}\tilde{\nabla}_{d\rangle}\phi + \tfrac{1}{3}\tilde{\nabla}_b\mu - \tilde{\nabla}^d\pi^R_{bd} - \tfrac{1}{3}\Theta q^R_b = 0. \tag{8.13}$$

We note that for the new constraints to be compatible with the existing ones the above constraint must be satisfied. To check the spatial consistency of the above constraint on any initial hypersurface we take the curl of (8.13) to get

$$\begin{aligned}&\frac{w}{w+1}\eta^{acb}\tilde{\nabla}_c\tilde{\nabla}^d\tilde{\nabla}_{\langle b}\tilde{\nabla}_{d\rangle}\phi + \tfrac{1}{3}\eta^{acb}\tilde{\nabla}_c\tilde{\nabla}_b\mu \\ &-\eta^{acb}\tilde{\nabla}_c\tilde{\nabla}^d\pi^R_{bd} - \tfrac{1}{3}\Theta\eta^{acb}\tilde{\nabla}_c q^R_b = 0 \,,\end{aligned} \tag{8.14}$$

which, on using (8.5), gives

$$\begin{aligned}&\frac{w}{w+1}\eta^{acb}\tilde{\nabla}_c\tilde{\nabla}^d\tilde{\nabla}_{\langle b}\tilde{\nabla}_{d\rangle}\phi + \tfrac{2}{3}\omega^a\dot{\mu} \\ &+\tfrac{1}{3}\Theta\eta^{acb}\tilde{\nabla}_c\left[\frac{f'''}{f'}\dot{R}\tilde{\nabla}_b R + \frac{f''}{f'}\tilde{\nabla}_b\dot{R} - \frac{\Theta f''}{3f'}\tilde{\nabla}_b R\right] \\ &-\eta^{acb}\tilde{\nabla}_c\tilde{\nabla}^d\left[\frac{f''}{f'}\tilde{\nabla}_{\langle b}\tilde{\nabla}_{d\rangle}R\right] = 0.\end{aligned} \tag{8.15}$$

Breaking the PSTF part according to equation (4.12), using the commutators (8.6), (8.7) and keeping only terms up to first order, we have:

$$\begin{aligned}&\frac{w}{w+1}\eta^{acb}\left[\tfrac{2}{3}\tilde{\nabla}_c\tilde{\nabla}_b\tilde{\nabla}^2\phi + \tfrac{1}{3}\tilde{R}\tilde{\nabla}_c\tilde{\nabla}_b\phi + \dot{\phi}\eta_{dbk}\tilde{\nabla}_c\tilde{\nabla}^d\omega^k\right] \\ &+\tfrac{2}{3}\omega^a\dot{\mu} - \frac{f''}{f'}\eta^{acb}\tilde{\nabla}_c\tilde{\nabla}^d\tilde{\nabla}_{\langle b}\tilde{\nabla}_{d\rangle}R + \frac{f'''\Theta\dot{R}}{3f'}\eta^{acb}\tilde{\nabla}_c\tilde{\nabla}_b R \\ &+\frac{f''\Theta}{3f'}\eta^{acb}\tilde{\nabla}_c\tilde{\nabla}_b\dot{R} - \frac{f''\Theta^2}{9f'}\eta^{acb}\tilde{\nabla}_c\tilde{\nabla}_b R = 0.\end{aligned} \tag{8.16}$$

Again using (8.5) and (8.10) in the above equation and linearizing we get

$$\begin{aligned}&\frac{w}{w+1}\left[\tfrac{2}{3}\tilde{R}\omega^a\dot{\phi} - \dot{\phi}\tilde{\nabla}_k\tilde{\nabla}^a\omega^k + \dot{\phi}\tilde{\nabla}^2\omega^a\right] + \tfrac{2}{3}\omega^a\dot{\mu} \\ &-\frac{f''}{f'}\eta^{acb}\left[\tfrac{2}{3}\tilde{\nabla}_c\tilde{\nabla}_b\tilde{\nabla}^2 R + \tfrac{1}{3}\tilde{R}\tilde{\nabla}_c\tilde{\nabla}_b R + \dot{R}\eta_{dbk}\tilde{\nabla}_c\tilde{\nabla}^d w^k\right] \\ &+\frac{2\omega^a}{3f'}\left[f'''\Theta\dot{R}^2 - \tfrac{1}{3}f''\Theta^2\dot{R} + f''\Theta\ddot{R}\right] = 0.\end{aligned} \tag{8.17}$$

This can also be written as

$$\frac{w}{w+1}\left[\tfrac{2}{3}\tilde{R}\omega^a\dot{\phi} - \dot{\phi}\tilde{\nabla}_k\tilde{\nabla}^a\omega^k + \dot{\phi}\tilde{\nabla}^2\omega^a\right] + \tfrac{2}{3}\omega^a\dot{\mu}$$
$$-\dot{R}\frac{f''}{f'}\left[\tfrac{2}{3}\tilde{R}\omega^a - \tilde{\nabla}_k\tilde{\nabla}^a\omega^k + \tilde{\nabla}^2\omega^a\right]$$
$$+\frac{2\omega^a}{3f'}\left[f'''\Theta\dot{R}^2 - \tfrac{1}{3}f''\Theta^2\dot{R} + f''\Theta\ddot{R}\right] = 0. \tag{8.18}$$

Now, from relation (8.7) and using (4.86) we know that

$$\tilde{\nabla}_k\tilde{\nabla}^a\omega^k = \tfrac{1}{3}\tilde{R}\omega^a\ , \tag{8.19}$$

and from (4.82) we have

$$\dot{\phi} = -(1+w)\Theta\ . \tag{8.20}$$

Thus rearranging terms gives

$$w\Theta\left[\frac{\tilde{R}}{3}\omega^a + \tilde{\nabla}^2\omega^a\right] + \tfrac{2}{3}(\mu+p)\Theta\omega^a + \dot{R}\frac{f''}{f'}\left[\frac{\tilde{R}}{3}\omega^a\right.$$
$$\left.+\tilde{\nabla}^2\omega^a\right] - \frac{2\Theta\omega^a}{3f'}\left[f'''\dot{R}^2 - \tfrac{1}{3}f''\Theta\dot{R} + f''\ddot{R}\right] = 0\ . \tag{8.21}$$

Since

$$\mu + p = \frac{(1+w)\mu_m}{f'} - \frac{f''\dot{R}\Theta}{3f'} + \frac{f''\ddot{R}}{f'}$$
$$+\frac{f'''\dot{R}^2}{f'} + \frac{f''}{3f'}\tilde{\nabla}^2 R, \tag{8.22}$$

the above equation, to linear order, simplifies to

$$w\Theta\left[\frac{\tilde{R}}{3}\omega^a + \tilde{\nabla}^2\omega^a\right] + \frac{2(1+w)\mu_m\Theta\omega^a}{3f'}$$
$$+\dot{R}\frac{f''}{f'}\left[\tfrac{1}{3}\tilde{R}\omega^a + \tilde{\nabla}^2\omega^a\right] = 0. \tag{8.23}$$

Further manipulation leads to

$$\omega^a\left[\left(\frac{w\Theta}{3} + \frac{\dot{R}f''}{3f'}\right)\tilde{R} + \frac{2(1+w)\mu_m\Theta}{3f'}\right]$$
$$+\left(\frac{\dot{R}f''}{f'} + w\Theta\right)\tilde{\nabla}^2\omega^a = 0. \tag{8.24}$$

We know that in terms of the scale factor $a(t)$ of a FLRW spacetime, the expansion, acceleration, jerk and snap parameters are defined by the following relations:

$$\Theta = 3\frac{\dot{a}}{a}\,, \qquad q = -\frac{\ddot{a}a}{\dot{a}^2}, \tag{8.25}$$

$$j = \frac{\dddot{a}\,a^2}{\dot{a}^3}\,, \qquad s = \frac{a^3}{\dot{a}^4}\frac{d^4a}{dt^4}. \tag{8.26}$$

From the above equations we can easily see that the time propagations of these quantities can be written as

$$\dot{\Theta} = -\frac{1}{3}\Theta^2(1+q)\,, \tag{8.27}$$

$$\dot{q} = -\frac{1}{3}\Theta\left(j-q-2q^2\right)\,, \tag{8.28}$$

$$\ddot{\Theta} = \frac{1}{9}\Theta^3\left(2+3q+j\right)\,, \tag{8.29}$$

$$\dot{j} = \frac{1}{3}\Theta\left(s+2j+3qj\right)\,, \tag{8.30}$$

$$\ddot{q} = -\frac{1}{9}\Theta^2\left[s+2j-3q^2+6qj-6q^3\right]\,. \tag{8.31}$$

Then, the Ricci scalar R is given by

$$R = \frac{2}{3}\Theta^2(1-q)+\tilde{R} \tag{8.32}$$

and hence

$$\dot{R} = \frac{2}{3}\Theta Q\,, \tag{8.33}$$

where

$$Q = \frac{1}{3}\Theta^2(j-q-2)+\tilde{R}, \tag{8.34}$$

$$\dot{Q} = \tfrac{1}{9}\Theta\left[(4+5q+j+jq+s)\Theta^2+6\tilde{R}\right]\,. \tag{8.35}$$

This means that we can rewrite (8.24) as

$$\tfrac{2}{3}\Theta\Bigg\{\omega^a\left[\left(\frac{w}{2}+\frac{f''}{3f'}Q\right)\tilde{R}+\frac{(1+w)\mu_m}{f'}\right] + \left[\frac{f''}{f'}Q+\frac{3w}{2}\right]\tilde{\nabla}^2\omega^a\Bigg\} = 0\,. \tag{8.36}$$

We can see from this equation that spatial consistency requires the vanishing of either Θ or the terms in the curly brackets.

To check for temporal consistency of the new constraint (8.36) we take its time evolution which can be written as

$$\omega^a \Bigg\{ \Theta \left[\frac{(1-w)P}{3}\tilde{R} + \frac{(1+w)}{f'}\frac{(3w+5)f' + 4f''Q}{6f'}\mu_m \right] + \frac{\dot{P}}{P}\left[(\frac{1+w}{f'})\mu_m \right] \Bigg\} = 0 \,, \tag{8.37}$$

where we have used

$$\dot{\omega}^a = (w - \tfrac{2}{3})\Theta\omega^a \tag{8.38}$$

and

$$(\tilde{\nabla}^2\omega^a)^{\cdot} = \frac{(3w-5)}{6}\Theta\tilde{\nabla}^2\omega^a + \frac{(w-1)}{6}\Theta\tilde{R}\omega^a \,. \tag{8.39}$$

We have also defined

$$P \equiv \frac{f''}{f'}Q + \frac{3w}{2} \,. \tag{8.40}$$

From (8.35), we can write

$$\dot{P} = Z\Theta \,, \tag{8.41}$$

where

$$Z = \frac{2}{3}\left(\frac{f'''}{f'} - (\frac{f''}{f'})^2 \right) Q^2 + \frac{f''}{9f'}\left((4 + 5q + j + jq + s)\Theta^2 + 6\tilde{R} \right) . \tag{8.42}$$

Equation (8.36) can then be rewritten as

$$\Theta\omega^a \Bigg\{ \left[\frac{(1-w)P}{3}\tilde{R} + \frac{(1+w)}{f'}\frac{(3w+5)f' + 4f''Q}{6f'}\mu_m \right] + \frac{Z}{P}\left[(\frac{1+w}{f'})\mu_m \right] \Bigg\} = 0 \,. \tag{8.43}$$

It follows that for the new constraints to be spatially and temporally consistent we must have either $\omega^a\Theta = 0$, or the expression in the curly brackets must vanish. It is interesting to see whether there exist solutions of a given $f(R)$ theory of gravity which can avoid the Ellis condition.

From (8.43), it is easy to see that if the 3-curvature vanishes, then the result of [234] can always be avoided for vacuum universes ($\mu_m = 0$). This implies that *a shear-free, spatially flat vacuum universe in any $f(R)$ theory can rotate and expand simultaneously in the linearized regime.*

The non-vacuum case is more difficult to analyze in general; however, as we will see below, there does exist at least one non-trivial case which does violate the Ellis condition.

For a flat Milne universe, where the matter energy density is given by $\mu_m = \frac{\mu_0}{a^{3(1+w)}}$, we have

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2, \tag{8.44}$$
$$R = \tfrac{2}{3}\Theta^2, \tag{8.45}$$
$$a(R) = \frac{1}{\sqrt{R}}, \tag{8.46}$$
$$\dot{R} = -\sqrt{\tfrac{2}{3}}R^{\frac{3}{2}}\,. \tag{8.47}$$

Substituting these quantities into the Friedmann equation

$$\frac{1}{3}\Theta^2 = \frac{1}{f'}\left[\mu_m + \frac{Rf' - f}{2} - \Theta\dot{R}f''\right]\,, \tag{8.48}$$

one gets

$$-R^2\frac{d^2f(R)}{dR^2} + \frac{f(R)}{2} - \frac{\mu_0}{a(R)^{3(1+w)}} = 0\,, \tag{8.49}$$

which has the following general solution:

$$f(R) = C_1R^{\frac{1+\sqrt{3}}{2}} + C_2R^{\frac{1-\sqrt{3}}{2}} - \frac{4\mu_0}{1+12w+9w^2}R^{\frac{3(1+w)}{2}}. \tag{8.50}$$

Let us only consider the particular solution (the last term of the above equation), which is an R^n - theory of gravity. Now, if we look at (8.43), for the corresponding flat Milne universe in R^n gravity, the term in the curly brackets reduces to

$$\frac{(1+w)\mu_m}{6f'}\left[3w + 9 - 4n\right] = 0. \tag{8.51}$$

Comparing solutions (8.51) and the particular solution of (8.50) (with $n = 3(1+w)/2$) we get $w = 1$ if $\mu_m \neq 0$. In other words, *for a stiff fluid in R^3 gravity, there exists a flat Milne-universe solution which can rotate and expand simultaneously at the level of linearised perturbation theory.*

8.4 Discussion and Conclusion

In this chapter we have considered shear-free fluid flows in $f(R)$ gravity in situations where the hydrodynamic and gravitational equations have been linearized about a FLRW background. We showed that if the 3-curvature vanishes, then the result of [129, 271] can always be avoided for vacuum universes. We also demonstrated there is at least one physically realistic non-vacuum case in which both rotation and expansion are simultaneously possible. This suggests that there are situations where linearized fourth-order gravity shares properties with Newtonian theory not valid in GR.

A generalization of the results obtained in this chapter for the full nonlinear regime of any perfect fluid cosmic medium is left for future investigation.

Chapter 9

Quasi-Newtonian Perturbations in $f(R)$-gravity

Consistency is contrary to nature, contrary to life. The only completely consistent people are dead.

Aldous Huxley

9.1 Introduction

Despite there being no proper Newtonian limit for GR on cosmological scales, recent works on so-called *quasi-Newtonian* cosmologies [200, 202, 300] have shown that gravitational physics (such as analysis of nonlinear collapse and structure formation using the Zeldovich approximation [309]) can be studied to a good approximation. In [300], it has been shown that non-linear quasi-Newtonian cosmologies are generally covariantly inconsistent, with the exception of some special cases such as the FLRW solutions. This inconsistency is due to the imposition of a shear-free and irrotational congruence condition for all dynamical evolutions of matter, resulting in a vanishing Weyl curvature [296], and hence ruling out gravitational radiation [301]. This in turn puts strict constraints on the gravitational field [180, 200, 274, 301]. Nor are the covariantly linearized regimes of such models consistent, despite expectations that terms arising from the evolution of the constraints would be removed [1].

Based on the ansatz for the evolution of the gravitational potential introduced by van Elst and Ellis [300] and Maartens' generalization [200], we give $f(R)$ extensions of the GR integrability conditions for the linearized models about the FLRW background and derive the evolution of the velocity and density perturbations in the comoving (Lagrangian) and quasi-Newtonian (Eulerian) [25, 200] frames.

[1] In the standard (metric) perturbation theory, however, there are no such expectations of integrability conditions resulting from shear-free conditions [200].

For a given choice of the 4-velocity vector field u^a, the Ehlers-Ellis covariant approach [110, 116] can only be employed on fully covariant quantities and equations with transparent physical and geometrical meaning [200]. In such a treatment, the dynamics, kinematics and gravito-electromagnetics of the FLRW background is characterized, respectively, by

$$\tilde{\nabla}_a\mu_m = 0 = \tilde{\nabla}_a p_m \ , \quad q^m_a = 0 \ , \quad \pi^m_{ab} = 0 \ , \tag{9.1}$$

$$\tilde{\nabla}_a\Theta = 0 \ , \quad A_a = 0 = \omega_a \ , \quad \sigma_{ab} = 0 \ , \tag{9.2}$$

$$E_{ab} = 0 = H_{ab} \ ; \tag{9.3}$$

whereas the linearized evolution equations are given by

$$\dot{\mu}_m = -\mu_m\Theta - \tilde{\nabla}^a q^m{}_a \ , \tag{9.4}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu + 3p) + \tilde{\nabla}_a A^a \ , \tag{9.5}$$

$$\dot{q}^m{}_a = -\frac{4}{3}\Theta q^m{}_a - \mu_m A_a \ , \tag{9.6}$$

$$\dot{\omega}^{\langle a\rangle} = -\tfrac{2}{3}\Theta\omega^a - \tfrac{1}{2}\eta^{abc}\tilde{\nabla}_b A_c \ , \tag{9.7}$$

$$\dot{\sigma}_{ab} = -\frac{2}{3}\Theta\sigma_{ab} - E_{ab} + \frac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a}A_{b\rangle} \ , \tag{9.8}$$

$$\dot{E}^{\langle ab\rangle} = \eta^{cd\langle a}\tilde{\nabla}_c H^{\rangle b}_d - \Theta E^{ab} - \tfrac{1}{2}\dot{\pi}^{ab} - \tfrac{1}{2}\tilde{\nabla}^{\langle a}q^{b\rangle} - \tfrac{1}{6}\Theta\pi^{ab} \ , \tag{9.9}$$

$$\dot{H}^{\langle ab\rangle} = -\Theta H^{ab} - \eta^{cd\langle a}\tilde{\nabla}_c E^{\rangle b}_d + \tfrac{1}{2}\eta^{cd\langle a}\tilde{\nabla}_c\pi^{\rangle b}_d \ , \tag{9.10}$$

constrained by the following linearized equations:

$$(C_0)^{ab} := E^{ab} - \tilde{\nabla}^{\langle a}A^{b\rangle} - \tfrac{1}{2}\pi^{ab} = 0 \ , \tag{9.11}$$

$$(C_1)^a := \tilde{\nabla}_b\sigma^{ab} - \eta^{abc}\tilde{\nabla}_b\omega_c - \frac{2}{3}\tilde{\nabla}^a\Theta - q^a = 0 \ , \tag{9.12}$$

$$(C_2) := \tilde{\nabla}^a\omega_a = 0 \ , \tag{9.13}$$

$$(C_3)^{ab} := \eta_{cd(}\tilde{\nabla}^c\sigma_{b)}{}^d + \tilde{\nabla}^{\langle a}\omega^{b\rangle} - H^{ab} = 0 \ , \tag{9.14}$$

$$(C_5)^a := \tilde{\nabla}_b E^{ab} + \tfrac{1}{2}\tilde{\nabla}_b\pi^{ab} - \tfrac{1}{3}\tilde{\nabla}^a\mu + \tfrac{1}{3}\Theta q^a = 0 \ , \tag{9.15}$$

$$(C_6)^a := \tilde{\nabla}_b H^{ab} + (\mu + p)\omega^a + \tfrac{1}{2}\eta^{abc}\tilde{\nabla}_b q_c = 0 \ . \tag{9.16}$$

Since these equations arise from the Ricci identity for u^a, one expects the dynamic, kinematic and gravity-electromagnetic quantities to undergo transformations if a different 4-velocity $\tilde{u}^a$ is chosen. In addition, quasi-Newtonian cosmological models are almost-FLRW dust universes with a congruence observers with irrotational, shear-free 4velocity u^a non-relativistic comoving observers, one can choose the comoving 4-velocity $\tilde{u}^a$ to be given by the linearized form [200, 300]

$$\tilde{u}^a = u^a + v^a \ , \tag{9.17}$$

where v^a is the nonrelativistic ("peculiar") velocity and vanishes in the background.

Thus in the quasi-Newtonian frame,

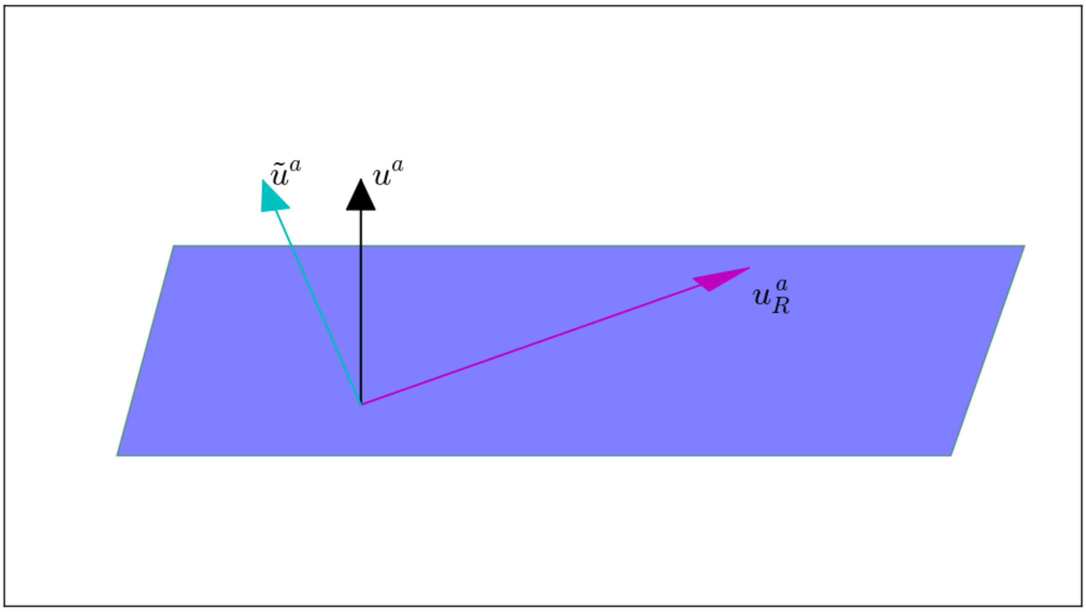

Figure 9.1: The quasi-Newtonian frame.

$$p_m = 0, \quad q_a^m = \mu_m v_a, \quad \pi_{ab}^m = 0 \,, \tag{9.18}$$
$$\omega_a = 0, \quad \sigma_{ab} = 0 \,. \tag{9.19}$$

with a non-vanishing matter energy flux (momentum density) q_a^m due to the tilting of the quasi-Newtonian frame relative to the coming one and hence a net particle flux. Note, however, that the isotropic and anisotropic pressures of matter vanish in the linear order because they are second-order effects of the relative motion[2].

The shear-free and irrotational condition (9.19) and the gravito-electromagnetic (GEM) constraint (9.12) result in the *"silent"* constraint

$$H_{ab} = 0 \,, \tag{9.20}$$

thus avoiding gravitational radiation, and in turn showing that q_a^m (as well as v_a) is irrotational, i.e.,

$$\eta_{abc}\tilde{\nabla}^b v^c = 0 = \eta_{abc}\tilde{\nabla}^b q_m^c \,. \tag{9.21}$$

It follows, therefore, that, for a vanishing vorticity, there exists a velocity potential

[2]A generalized nonlinear transformation of quantities between the two frames is given in [200].

ψ [35, 200] such that

$$v_a = \tilde{\nabla}_a \psi \ . \tag{9.22}$$

9.2 Integrability Conditions

A constraint equation $C^A = 0$ is said to *evolve consistently* with the evolution equations [199, 200, 202] if

$$\dot{C}^A = F^A{}_B C^B + G^A{}_{Ba} \tilde{\nabla}^a C^B \ , \tag{9.23}$$

where F and G are quantities that depend on the kinematic, dynamic and GEM quantities but not their derivatives. It has been shown [200, 301] that the nonlinear models are generally inconsistent if the silent constraint (9.20) is imposed, but that the linear models are consistent. Thus, in order to find the integrability conditions of the quasi-Newtonian cosmologies, it suffices to show, using the appropriate transformations between the two frames, that these cosmologies are a subclass of the linearized silent models.

The following shows the corresponding mapping of the linearised kinematic, dynamic and GEM quantities between the comoving and quasi-Newtonian frames:

$$\tilde{\Theta} = \Theta + \tilde{\nabla}^a v_a \tag{9.24}$$
$$\tilde{A}_a = A_a + \dot{v}_a + \frac{1}{3}\Theta v_a \ , \tag{9.25}$$
$$\tilde{\omega}_a = \omega_a - \frac{1}{2}\eta_{abc}\tilde{\nabla}^b v^c \ , \tag{9.26}$$
$$\tilde{\sigma}_{ab} = \sigma_{ab} + \tilde{\nabla}_{\langle a} v_{b\rangle} \ , \tag{9.27}$$
$$\tilde{\mu} = \mu \ , \ \tilde{p} = p \ , \ \tilde{\pi}_{ab} = \pi_{ab} \ , \ \tilde{q}^R_a = q^R_a \ , \tag{9.28}$$
$$\tilde{q}^m_a = q^m_a - (\mu_m + p_m) v_a \ , \tag{9.29}$$
$$\tilde{E}_{ab} = E_{ab} \ , \quad \tilde{H}_{ab} = H_{ab} \ . \tag{9.30}$$

Using Eqns (9.18)-(9.30) one can covariantly describe linearised silent universe models by the following equations:

$$\tilde{p} = 0, \quad \tilde{q}^m_a = 0, \quad \tilde{\pi}^m_{ab} = 0 \ , \tag{9.31}$$
$$\tilde{A}_a = 0, \quad \omega_a = 0, \quad \tilde{\sigma}_{ab} = \tilde{\nabla}_{\langle a} v_{b\rangle} \ , \tag{9.32}$$
$$\tilde{E}_{ab} = E_{ab}, \quad \tilde{H}_{ab} = 0 \ . \tag{9.33}$$

Note the special restriction on the form of the shear, the very fact that gives the quasi-Newtonian models their integrability conditions [200].

For shear-free dust spacetimes of which quasi-Newtonian models are a subclass, we see that Eqn (9.8) reduces to a new constraint due to the vanishing of the shear in the quasi-Newtonian frame:

$$\mathcal{E}_{ab} \equiv E_{ab} - \frac{1}{2}\pi^R_{ab} - \tilde{\nabla}_{\langle a} A_{b\rangle} = 0 \ . \tag{9.34}$$

From (9.7) and using the identity that, for any scalar φ,

$$\eta^{abc}\tilde{\nabla}_b\tilde{\nabla}_c\varphi = -2\dot{\varphi}\omega_a \tag{9.35}$$

we can see that, for an irrotational model,

$$\eta^{abc}\tilde{\nabla}_b A_c = 0 \implies A_a \equiv \tilde{\nabla}_a\varphi\ . \tag{9.36}$$

Thus φ here is the acceleration potential and is the covariant relativistic generalisation of the Newtonian potential.

9.2.1 First integrability condition

Differentiating Eqn (9.34) with respect to cosmic time t, and using Eqns (9.9) and (9.12), it can be shown that $\mathcal{E}_{ab}$ evolves consistently if

$$\tilde{\nabla}_{<a}\tilde{\nabla}_{b>}\left(\dot{\varphi}+\frac{1}{3}\Theta+\frac{f''}{f'}\dot{R}\right)+\left(\dot{\varphi}+\frac{1}{3}\Theta+\frac{f''}{f'}\dot{R}\right)\tilde{\nabla}_{<a}\tilde{\nabla}_{b>}\varphi = 0, \tag{9.37}$$

where the identity

$$\left(\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\varphi\right)^{\cdot} = \tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\dot{\varphi}+\left(\dot{\varphi}-\frac{2}{3}\Theta\right)\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\varphi \tag{9.38}$$

has been employed.

Eqn (9.37) is the *First Integrability Condition* (FIC) for quasi-Newtonian cosmologies in fourth-order gravity and is a generalisation of the one obtained in [200]. The modified van Elst - Ellis condition [200, 300] is thus generalised to

$$\dot{\varphi}+\frac{1}{3}\Theta = -\frac{f''}{f'}\dot{R}\ . \tag{9.39}$$

An important consequence of this condition is the evolution equation of the 4-acceleration which can be derived from the gradient of (9.39) and using the scalar commutation

$$(\tilde{\nabla}_a X)^{\cdot} = \tilde{\nabla}_a\dot{X}-\frac{1}{3}\Theta\tilde{\nabla}_a X+\dot{X}A_a \tag{9.40}$$

along with the shear - free constraint (9.12)

$$q_a = \frac{q^m_a}{f'}+q^R_a = \frac{\mu_m}{f'}v_a-\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R+f''\tilde{\nabla}_a\dot{R}-\tfrac{1}{3}f''\Theta\tilde{\nabla}_a R\right] = \frac{2}{3}\tilde{\nabla}_a\Theta\ , \tag{9.41}$$

and is given by

$$\dot{A}_a+\left(\frac{2}{3}\Theta+\frac{f''}{f'}\dot{R}\right)A_a = -\frac{1}{2f'}\left\{\mu_m v_a+\left[\left(f'''-\frac{2f''^2}{f'}\right)\dot{R}+\frac{1}{3}\Theta f''\right]\tilde{\nabla}_a R+f''\tilde{\nabla}_a\dot{R}\right\}\ . \tag{9.42}$$

9.2.2 Second integrability condition

To check for the consistency of the constraint (9.34) on any spatial hyper-surface, let us take the divergence of $\mathcal{E}_{ab}$, using the identity for projected vectors A_a:

$$\tilde{\nabla}^b\tilde{\nabla}_{\langle a}A_{b\rangle} = \frac{1}{2}\tilde{\nabla}^2 A_a + \frac{1}{6}\tilde{\nabla}_a(\tilde{\nabla}^c A_c) + \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)A_a \,, \tag{9.43}$$

which, after some simplification, becomes

$$\tilde{\nabla}^b\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\varphi = \frac{2}{3}\tilde{\nabla}_a(\tilde{\nabla}^2\varphi) + \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\tilde{\nabla}_a\varphi \,. \tag{9.44}$$

Using Eqns (9.12) and (9.14), (9.44) can be re-written as

$$\tilde{\nabla}_a\mu - \frac{2}{3}\Theta\tilde{\nabla}_a\Theta - 2\tilde{\nabla}_a(\tilde{\nabla}^2\varphi) - 2\left(\mu - \frac{1}{3}\Theta^2\right)\tilde{\nabla}_a\varphi - 2\frac{f''}{f'}\tilde{\nabla}_a(\tilde{\nabla}^2 R) - 2\frac{f''}{f'}\left(\mu - \frac{1}{3}\Theta^2\right)\tilde{\nabla}_a R = 0 \,. \tag{9.45}$$

To first order, one can simplify (9.45) and the *Second Integrability Condition* (SIC) becomes

$$\tilde{\nabla}_a\mu_m - \left(f''\dot{R} + \frac{2}{3}\Theta f'\right)\tilde{\nabla}_a\Theta - 2f'\tilde{\nabla}^2(\tilde{\nabla}_a\varphi) - 2\left(\mu_m + \frac{Rf'}{2} - \frac{f}{2} - \Theta f''\dot{R} - \frac{1}{3}f'\Theta^2\right)\tilde{\nabla}_a\varphi$$
$$-\Theta f''\tilde{\nabla}_a\dot{R} + \left(\frac{ff''}{2f'} - \frac{\mu_m f''}{f'} + \Theta\dot{R}\frac{f''^2}{f'} - \dot{R}\Theta f'''\right)\tilde{\nabla}_a R - f''\tilde{\nabla}^2(\tilde{\nabla}_a R) = 0 \,, \tag{9.46}$$

which, upon splitting GR and non-GR contributions, becomes

$$\tilde{\nabla}_a\mu_m - \tfrac{2}{3}\Theta f'\tilde{\nabla}_a\Theta - 2f'\tilde{\nabla}^2(\tilde{\nabla}_a\varphi) - 2\left(\mu_m - \tfrac{1}{3}f'\Theta^2\right)\tilde{\nabla}_a\varphi = 2\left(\tfrac{Rf'}{2} - \tfrac{f}{2} - \Theta f''\dot{R}\right)\tilde{\nabla}_a\varphi$$
$$+f''\dot{R}\tilde{\nabla}_a\Theta + \Theta f''\tilde{\nabla}_a\dot{R} - \left(\tfrac{f''f}{2f'} - \tfrac{f''\mu_m}{f'} + \dot{R}\Theta\tfrac{f''^2}{f'} - \dot{R}\Theta f'''\right)\tilde{\nabla}_a R + f''\tilde{\nabla}^2(\tilde{\nabla}_a R) \,. \tag{9.47}$$

For GR, the RHS of Eqn (9.47) vanishes, thus recovering Maartens' result [200] for the SIC.

Although this result is, in general, independent of the FIC (as is also the case in GR [200]), one can show that the second condition is identically satisfied if one uses the modified van Elst-Ellis solution (9.39).

Evolution of the van Elst-Ellis condition results in the covariant *modified Poisson equation* in $f(R)$:

$$\tilde{\nabla}^2\varphi = \frac{1}{2f'}\mu_m - (3\ddot{\varphi} + \Theta\dot{\varphi}) + \frac{1}{2f'}\left[f - Rf' - \Theta f''\dot{R} - 3f''\ddot{R} - 3f'''\dot{R}^2 + 6\dot{R}^2\frac{f''^2}{f'} - f''\tilde{\nabla}^2 R\right] \,. \tag{9.48}$$

It can also be shown from (9.39) and (9.41) that the peculiar velocity

$$v_a = -\frac{1}{\mu_m}\left[2f'\tilde{\nabla}_a\dot{\varphi} + f''\tilde{\nabla}_a\dot{R} + \left(f'''\dot{R} - f''\dot{\varphi} - \frac{3f''^2}{f'}\dot{R}\right)\tilde{\nabla}_a R\right], \tag{9.49}$$

and that it evolves according to the equation

$$\dot{v}_a + \frac{1}{3}\Theta v_a = -A_a \tag{9.50}$$

by virtue of Eqns (9.4) and (9.6).

The coupling of v_a and A_a in Eqn (9.50) can be broken at the second order of evolution of the peculiar velocity which can be given, using the Raychaudhuri (9.5) and and flat Friedmann (6.6) equations in (9.50), as

$$\ddot{v}_a + \left(\Theta + \dot{R}\frac{f''}{f'}\right)\dot{v}_a + \left[\frac{1}{9}\Theta^2 - \frac{1}{6f'}\left(5\mu_m - f - 4\dot{R}\Theta f''\right)\right]v_a - \frac{f''}{2f'}\tilde{\nabla}_a\dot{R}$$
$$+\left(\dot{R}\frac{f''^2}{f'^2} - \frac{f'''}{2f'} - \Theta\frac{f''}{6f'}\right)\tilde{\nabla}_a R = 0\,. \tag{9.51}$$

Now let us use (9.49) in (9.50) to obtain the equation

$$2f'\tilde{\nabla}_a\ddot{\varphi} + 2\left(f''\dot{R} + f'\Theta\right)\tilde{\nabla}_a\dot{\varphi} - \left(\mu_m - 2f'\ddot{\varphi} - f''\ddot{R} - \dot{R}^2 f''' + \dot{\varphi}f'' + 3\dot{R}^2\frac{f''^2}{f'}\right)\tilde{\nabla}_a\varphi$$
$$+ f''\tilde{\nabla}_a\ddot{R} + \left(2f'''\dot{R} + \Theta f'' - f''\dot{\varphi} - 3\dot{R}\frac{f''^2}{f'}\right)\tilde{\nabla}_a\dot{R} + \Big[\Theta\dot{R}f''' - \Theta\dot{\varphi}f'' - f''\ddot{\varphi}$$
$$-6\dot{R}^2\frac{f'''f''}{f'} + 3\dot{R}^2\frac{f''^3}{f'^2} + f^{(iv)}\dot{R}^2 + f'''\ddot{R} - f'''\dot{\varphi}\dot{R} - 3\ddot{R}\frac{f''^2}{f'} - 3\Theta\dot{R}\frac{f''^2}{f'}\Big]\tilde{\nabla}_a R = 0\,. \tag{9.52}$$

Using Eqns (9.5) and (9.39) one can show that

$$\ddot{\varphi} = \frac{1}{9}\Theta^2 + \frac{1}{6f'}\left[\mu_m + f - Rf' + \Theta f''\dot{R} - 3f''\ddot{R} - 3f'''\dot{R}^2 + 6\frac{f''^2}{f'}\dot{R}^2 - f''\tilde{\nabla}^2 R\right] - \frac{1}{3}\tilde{\nabla}^2\varphi\,, \tag{9.53}$$

and hence that

$$2f'\tilde{\nabla}_a\ddot{\varphi} + 2f'\Theta\tilde{\nabla}_a\dot{\varphi} - \frac{2}{3}\left(\mu_m - \frac{1}{3}\Theta^2 f'\right)\tilde{\nabla}_a\varphi + 2f''\dot{R}\tilde{\nabla}_a\dot{\varphi} + f''\tilde{\nabla}_a\ddot{R}$$
$$+\frac{1}{3}\left[f - Rf' + 2\Theta f''\dot{R}\right]\tilde{\nabla}_a\varphi + \left(2f'''\dot{R} + \frac{4}{3}\Theta f'' - 2\dot{R}\frac{f''^2}{f'}\right)\tilde{\nabla}_a\dot{R}$$
$$+\Big[\frac{7ff''}{6} - \frac{\mu_m f''}{3f'} + \frac{4}{3}\Theta\dot{R}f''' - 2\dot{R}^2\frac{f'''f''}{f'} + 2\dot{R}^2\frac{f''^3}{f'^2} + f^{(iv)}\dot{R}^2 + f'''\ddot{R} - \frac{1}{3}Rf''$$

$$-\frac{1}{3}\dot{R}\Theta\frac{f''^2}{f'}\Big]\,\tilde{\nabla}_a R = 0\,. \tag{9.54}$$

Taking the gradient of (9.48) and using (9.39) one obtains

$$2f'\tilde{\nabla}_a(\nabla^2\varphi) = \tilde{\nabla}_a\mu_m - 6f'\tilde{\nabla}_a\ddot{\varphi} - 2f'\Theta\tilde{\nabla}_a\dot{\varphi} - \left(2f'\dot{\varphi} + \dot{R}f''\right)\tilde{\nabla}_a\Theta - 3f''\tilde{\nabla}_a\ddot{R}$$
$$+\left[12\dot{R}\frac{f''^2}{f'} - 6\dot{R}f''' - \Theta f''\right]\tilde{\nabla}_a\dot{R} + \left[12\dot{R}^2\frac{f'''f''}{f'} - 6\dot{R}^2\frac{f''^3}{f'^2} - \dot{R}\Theta f''' - 3\dot{R}^2 f^{(iv)} - 3\ddot{R}f'''\right.$$
$$\left. -2\dot{\varphi}\Theta f'' - 6f''\ddot{\varphi} - Rf''\right]\tilde{\nabla}_a R - f''\tilde{\nabla}^2(\tilde{\nabla}_a R)\,. \tag{9.55}$$

The aim is to compare this equation with (9.47): If one uses (9.55) in (9.47), the second integrability condition becomes

$$6f'\tilde{\nabla}_a\ddot{\varphi} \quad +6\left(f''\dot{R} + f'\Theta\right)\tilde{\nabla}_a\dot{\varphi} - \left[2\mu_m - \frac{2}{3}\Theta^2 f' + Rf' - f - 2\dot{R}\Theta f''\right]\tilde{\nabla}_a\varphi + 3f''\tilde{\nabla}_a\ddot{R}$$
$$+\left(6f'''\dot{R} + 4\Theta f'' - 6\dot{R}\frac{f''^2}{f'}\right)\tilde{\nabla}_a\dot{R} + \left[\frac{7ff''}{2f'} - \frac{f''}{f'}\mu_m - \dot{R}\Theta\frac{f''^2}{f'} + 3f^{(iv)}\dot{R}^2 + 4\dot{R}\Theta f'''\right.$$
$$\left. -6\dot{R}^2\frac{f'''f''}{f'} + 6\dot{R}^2\frac{f''^3}{f'^2} + 3\ddot{R}f''' - Rf''\right]\tilde{\nabla}_a R = 0\,, \tag{9.56}$$

which is identically satisfied by virtue of Eqn (9.54).

9.3 Perturbations

On very small (sub-Hubble) scales, analysis of gravitational instabilities of nonrelativistic matter (and hence the formation cosmic structure) can be effectively carried out within the (quasi-)Newtonian framework. In the previous section, we showed that the integrability condition arising from the shear-free gauge implies a propagation equation for the acceleration, a result not found in the standard non-covariant treatment. In this section, we show how one can obtain the velocity and density perturbations via this propagation equation, thus generalizing GR results obtained in [200].

9.3.1 Gradient Variables

In the following, we define the velocity inhomogeneities of the matter and curvature fluids and the comoving acceleration as

$$V^R_a = \frac{a\tilde{\nabla}_a R}{\dot{R}}, \qquad \Psi_a = \frac{a\tilde{\nabla}_a\dot{R}}{\dot{R}}, \qquad V^m_a = av_a\,, \qquad \mathcal{A}_a = aA_a\,. \tag{9.57}$$

The variables characterizing inhomogeneities in the energy density and background expansion are as defined in (6.26). We will also use the following commutation relations (obtained from (9.40)) between spatial and temporal derivatives:

$$a\tilde{\nabla}_a\dot{X} = (a\tilde{\nabla}_a X)^{\cdot} - \dot{X}\mathcal{A}_a \tag{9.58}$$

$$a\tilde{\nabla}_a\ddot{X} = (a\tilde{\nabla}_a X)^{\cdot\cdot} - \ddot{X}\mathcal{A}_a - \dot{X}\dot{\mathcal{A}}_a. \tag{9.59}$$

The comoving rate of the expansion gradient Z_a can be given by

$$Z_a = \frac{3\mu_m}{2f'}V_a^m + \left(\frac{1}{2}\dot{R}\Theta f'' - \frac{3\dot{R}^2 f'''}{2f'}\right)V_a^R - \frac{3f''}{2f'}\dot{R}\Psi_a. \tag{9.60}$$

9.3.2 Evolution Equations

The system of equations governing the evolutions of the fluctuations defined earlier are given by

$$\dot{D}_a^m + \Theta\mathcal{A}_a + \frac{3\mu_m}{2f'}V_a^m - \frac{3f''\dot{R}}{2f'}\Psi_a + \left(\frac{f''\dot{R}\Theta}{2f'} - \frac{3f'''\dot{R}^2}{2f'}\right)V_a^R + \tilde{\nabla}^2 V_a^m = 0, \tag{9.61}$$

$$\dot{\mathcal{A}}_a + \left(\frac{1}{3}\Theta + \frac{f''}{f'}\dot{R}\right)\mathcal{A}_a + \frac{f''}{2f'}\dot{R}\Psi_a + \left(\frac{f'''}{2f'}\dot{R}^2 + \frac{1}{6f'}\Theta f''\dot{R} - \left(\frac{f''}{f'}\right)^2\dot{R}^2\right)V_a^R$$
$$+ \frac{1}{2f'}\mu_m V_a = 0, \tag{9.62}$$

$$\dot{V}_a^R + \frac{\ddot{R}}{\dot{R}}V_a^R - \Psi_a - \mathcal{A}_a = 0, \tag{9.63}$$

$$\dot{V}_a^m + \mathcal{A}_a = 0, \tag{9.64}$$

$$\dot{\Psi}_a + \left(\frac{\ddot{R}}{\dot{R}} + \frac{2\dot{R}f'''}{f''} - \frac{3f''}{2f'}\dot{R} + \Theta\right)\Psi_a - \frac{\ddot{R}}{\dot{R}}\mathcal{A}_a + \frac{\mu_m}{3\dot{R}f''}D_a^m + \frac{3\mu_m}{2f'}V_a^m$$
$$+ \left(\ddot{R}\frac{f'''}{f''} - \frac{R}{3} + \frac{f'}{3f''} + \dot{R}^2\frac{f''''}{f''} + \frac{\dot{R}\Theta f'''}{f''} - \frac{3\dot{R}^2 f'''}{2f'} + \frac{\dot{R}\Theta f''}{2f'}\right)V_a^R$$
$$- \tilde{\nabla}^2 V_a^R = 0. \tag{9.65}$$

9.3.3 Second Order Equations

One can show that Eqn (9.51) can be recast into the equation for the comoving variable V_a and is given by

$$\ddot{V}^m_a+\left(\frac{1}{3}\Theta+\dot{R}\frac{f''}{f'}\right)\dot{V}^m_a-\frac{1}{2f'}\mu_m V^m_a-\dot{R}\frac{f''}{2f'}\Psi_a+\left(\dot{R}^2\frac{f''^2}{f'^2}-\dot{R}\frac{f'''}{2f'}-\dot{R}\Theta\frac{f''}{6f'}\right)V^R_a=0\,. \tag{9.66}$$

If one time propagates Eqn (9.61), the resulting second order evolution equation for the matter density perturbations will be

$$\begin{aligned}&f'\ddot{D}^m_a+\dot{R}f''\dot{D}^m_a-\frac{\mu_m}{2}D^m_a-\Theta f'\ddot{V}^m_a+\left(\frac{1}{3}\Theta^2 f'-\frac{1}{2}\dot{R}\Theta f''+\frac{5}{2}\mu_m-\frac{f}{2}+\frac{9}{2}\dot{R}^2 f'''\right.\\&\left.+\frac{3}{2}\ddot{R}f''-\frac{9}{4}\dot{R}^2\frac{f''^2}{f'}\right)\dot{V}^m_a+f'\tilde{\nabla}^2\dot{V}^m_a+\left(\frac{9}{4}\frac{\dot{R}f''}{f'}-\frac{3}{2}\Theta\right)\mu_m V^m_a\\&+\left(\dot{R}f''-\frac{2}{3}\Theta^2 f'\right)\tilde{\nabla}^2 V^m_a+\left(3\dot{R}^2 f'''-\frac{9}{4}\dot{R}^2\frac{f''^2}{f'}+\dot{R}\Theta f''\right)\dot{V}^R_a-\frac{3}{2}\dot{R}f''\tilde{\nabla}^2\dot{V}^R_a\\&+\left(6\dot{R}\ddot{R}f'''+3\dot{R}^3 f''''+\ddot{R}\Theta f''+\frac{1}{6}\Theta^2 f''+\frac{1}{2}\frac{\dot{R}f''\mu_m}{f'}-\frac{\dot{R}ff''}{4f'}-\frac{1}{2}\dot{R}^2\Theta\frac{f''^2}{f'}\right.\\&\left.+\dot{R}^2\Theta f'''-\frac{9}{4}\dot{R}\ddot{R}\frac{f''^2}{f'}-\frac{1}{2}R\dot{R}f''+\frac{1}{2}\dot{R}f'-\frac{9}{4}\dot{R}^3\frac{f'''f''}{f'}+\frac{3}{4}\dot{R}^2\Theta\frac{f''^2}{f'}\right)V^R_a\\&\quad+\dot{R}\Theta f''\tilde{\nabla}^2 V^R_a=0\,.\end{aligned} \tag{9.67}$$

On the other hand, if one uses Eqns (9.65) and (9.62) in the time- propagated equation of Eqn (9.63), one obtains

$$\begin{aligned}&\ddot{V}^R_a-\left(2\dot{R}\frac{f'''}{f''}-\frac{3}{2}\frac{\dot{R}f''}{f'}+\Theta\right)\dot{V}^R_a-\left(2\frac{\ddot{R}^2}{\dot{R}^2}-\frac{3}{2}\ddot{R}\frac{f''}{f}+\Theta\frac{\ddot{R}}{\dot{R}}-\frac{\dddot{R}}{\dot{R}}+3\ddot{R}\frac{f'''}{f''}-\frac{R}{3}\right.\\&\left.+\frac{f'}{3f''}+\dot{R}^2\frac{f''''}{f''}+\frac{\dot{R}\Theta f'''}{f''}-\frac{3\dot{R}^2 f'''}{2f'}+\frac{\dot{R}\Theta f''}{2f'}\right)V^R_a+\tilde{\nabla}^2 V^R_a+\ddot{V}^m_a\\&-\left(\frac{\ddot{R}}{\dot{R}}+\frac{2\dot{R}f'''}{f''}-\frac{3f''}{2f'}\dot{R}+\Theta\right)\dot{V}^m_a-\frac{3\mu_m}{2f'}V^m_a+\frac{\mu_m}{3\dot{R}f''}D^m_a=0\,.\end{aligned} \tag{9.68}$$

(9.69)

Using Eqns (9.64) and (9.62), the second order evolution equation for the peculiar matter velocity perturbations can be given by

$$\ddot{V}^m_a+\left(\frac{1}{3}\Theta+\dot{R}\frac{f''}{f'}\right)\dot{V}^m_a-\frac{1}{2f'}\mu_m V^m_a-\dot{R}\frac{f''}{2f'}\Psi_a+\left(\dot{R}^2\frac{f''^2}{f'^2}-\dot{R}\frac{f'''}{2f'}-\dot{R}\Theta\frac{f''}{6f'}\right)V^R_a=0\,, \tag{9.70}$$

in perfect agreement with Eqn (9.66).

Finally to form a closed system of second-order evolution equations, one can rewrite the above equation as

$$\ddot{V}^m_a+\left(\frac{1}{3}\Theta+\dot{R}\frac{f''}{2f'}\right)\dot{V}^m_a-\frac{1}{2f'}\mu_m V^m_a-\dot{R}\frac{f''}{2f'}\dot{V}^R_a+\left(\dot{R}^2\frac{f''^2}{f'^2}-\dot{R}\frac{f'''}{2f'}-\dot{R}\Theta\frac{f''}{6f'}-\ddot{R}\frac{f''}{2f'}\right)V^R_a=0\ . \tag{9.71}$$

9.4 Scalar Perturbations

Taking the divergence of the first order evolution equations obtained earlier, one can extract only the scalar parts and re-write the evolution equations as:

$$\dot{\Delta}^m+\Theta\mathcal{A}+\frac{3\mu_m}{2f'}V^m-\frac{3f''\dot{R}}{2f'}\Psi+\left(\frac{3f'''\dot{R}^2}{2f'}-\frac{f''\dot{R}\Theta}{2f'}\right)V^R+\tilde{\nabla}^2V^m=0, \tag{9.72}$$

$$\dot{\mathcal{A}}+\left(\frac{1}{3}\Theta+\frac{f''}{f'}\dot{R}\right)\mathcal{A}+\frac{f''}{2f'}\dot{R}\Psi+\frac{1}{2f'}\mu_m V^m+\left(\frac{f'''}{2f'}\dot{R}^2+\frac{1}{6f'}\Theta f''\dot{R}-\left(\frac{f''}{f'}\right)^2\dot{R}^2\right)V^R=0, \tag{9.73}$$

$$\dot{V}^R+\frac{\ddot{R}}{\dot{R}}V^R-\Psi-\mathcal{A}=0, \tag{9.74}$$

$$\dot{V}^m+\mathcal{A}=0, \tag{9.75}$$

$$\dot{\Psi}+\left(\frac{\ddot{R}}{\dot{R}}+\frac{2\dot{R}f'''}{f''}-\frac{3f''}{2f'}\dot{R}+\Theta\right)\Psi-\frac{\ddot{R}}{\dot{R}}\mathcal{A}+\frac{\mu_m}{3\dot{R}f''}\Delta^m+\frac{3\mu_m}{2f'}V^m$$
$$+\left(\ddot{R}\frac{f'''}{f''}-\frac{R}{3}+\frac{f'}{3f''}+\dot{R}^2\frac{f''''}{f''}+\frac{\dot{R}\Theta f'''}{f''}-\frac{3\dot{R}^2f'''}{2f'}+\frac{\dot{R}\Theta f''}{2f'}\right)V^R-\tilde{\nabla}^2V^R=0. \tag{9.76}$$

The harmonically decomposed set of these equations becomes

$$\dot{\Delta}^k_m+\Theta\mathcal{A}^k+\left(\frac{3\mu_m}{2f'}-\frac{k^2}{a^2}\right)V^k_m-\frac{3f''\dot{R}}{2f'}\Psi^k+\left(\frac{3f'''\dot{R}^2}{2f'}-\frac{f''\dot{R}\Theta}{2f'}\right)V^k_R=0, \tag{9.77}$$

$$\dot{\mathcal{A}}^k+\left(\frac{1}{3}\Theta+\frac{f''}{f'}\dot{R}\right)\mathcal{A}^k+\frac{f''}{2f'}\dot{R}\Psi^k+\frac{1}{2f'}\mu_m V^k_m+\left(\frac{f'''}{2f'}\dot{R}^2+\frac{1}{6f'}\Theta f''\dot{R}-\left(\frac{f''}{f'}\right)^2\dot{R}^2\right)V^k_R=0, \tag{9.78}$$

$$\dot{V}^k_R+\frac{\ddot{R}}{\dot{R}}V^k_R-\Psi^k-\mathcal{A}^k=0, \tag{9.79}$$

$$\dot{V}^k_m+\mathcal{A}^k=0, \tag{9.80}$$

$$\dot{\Psi}^k+\left(\frac{\ddot{R}}{\dot{R}}+\frac{2\dot{R}f'''}{f''}-\frac{3f''}{2f'}\dot{R}+\Theta\right)\Psi^k-\frac{\ddot{R}}{\dot{R}}\mathcal{A}^k+\frac{\mu_m}{3\dot{R}f''}\Delta^k_m+\frac{3}{2f'}\mu_m V^k_m$$

$$+\left(\ddot{R}\frac{f'''}{f''}-\frac{R}{3}+\frac{f'}{3f''}+\dot{R}^2\frac{f''''}{f''}+\frac{\dot{R}\Theta f'''}{f''}-\frac{3\dot{R}^2f'''}{2f'}+\frac{\dot{R}\Theta f''}{2f'}+\frac{k^2}{a^2}\right)V_R^k=0. \quad (9.81)$$

9.5 Solutions

We first find the solutions for $\tilde{\Delta}_m$, $\tilde{V}_R$ and $\tilde{\Psi}$ in the comoving frame and employ the transformation to the quasi-Newtonian frame

$$\tilde{D}_a f=\tilde{\nabla}_a f+\dot{f}v_a. \quad (9.82)$$

The comoving perturbation variables are defined as

$$\tilde{D}_a^m=\frac{a}{\mu_m}\tilde{D}_a\mu_m=D_a^m-\Theta V_a^m, \quad (9.83)$$

$$\tilde{Z}_a=a\tilde{D}_a\tilde{\Theta}=Z_a+\tilde{\nabla}^2V_a^m-\left(\frac{1}{3}\Theta^2+\frac{\mu_m}{f'}-\frac{f}{2f'}-\dot{R}\Theta\frac{f''}{f'}\right)V_a^m,$$

$$\tilde{C}_a=a\tilde{D}_a\tilde{R}\quad (\text{since}\quad \tilde{R}=0\quad \text{for}\quad \text{flat}\quad \text{FLRW}), \quad (9.84)$$

$$\mathcal{R}_a=a\tilde{D}_aR=\dot{R}V_a^R+\dot{R}V_a^m, \quad (9.85)$$

$$\Re_a=a\tilde{D}_a\dot{R}=\dot{R}\Psi_a+\ddot{R}V^m. \quad (9.86)$$

One can also define the new comoving variables

$$\tilde{V}_a^R\equiv\frac{a\tilde{D}_aR}{\dot{R}},\qquad \tilde{\Psi}_a\equiv\frac{a\tilde{D}_a\dot{R}}{\dot{R}}\,. \quad (9.87)$$

The set of evolution equations of the perturbations and their constraint in the comoving frame are the single-fluid, dust adaptations of the equations given by (6.62)-(6.66):

$$\dot{\tilde{\Delta}}_m^k=-\tilde{Z}^k, \quad (9.88)$$

$$\dot{\tilde{Z}}^k=(\dot{R}\frac{f''}{f'}-\frac{2}{3}\Theta)\tilde{Z}^k-\frac{\mu_m}{f'}\tilde{\Delta}_m^k+\Theta\frac{f''}{f'}\Re^k$$
$$+\left[\frac{1}{2}+\frac{k^2}{a^2}\frac{f''}{f'}-\frac{1}{2}\frac{ff''}{f'^2}+\frac{f''\mu_m}{f'^2}-\dot{R}\Theta(\frac{f''}{f'})^2+\dot{R}\Theta\frac{f'''}{f'}\right]\mathcal{R}^k, \quad (9.89)$$

$$\dot{\mathcal{R}}^k=\Re^k, \quad (9.90)$$

$$\dot{\Re}^k=-\left(\Theta+2\dot{R}\frac{f'''}{f''}\right)\Re^k-\dot{R}\tilde{Z}^k+\frac{\mu_m}{3f''}\tilde{\Delta}_m^k$$
$$-\left[\frac{k^2}{a^2}+\ddot{R}\frac{f'''}{f''}+\dot{R}^2\frac{f^{(4)}}{f''}+\Theta\dot{R}\frac{f'''}{f''}+\frac{f'}{3f''}-\frac{R}{3}\right]\mathcal{R}^k, \quad (9.91)$$

$$\frac{\tilde{C}^k}{a^2}+\left[\frac{4\Theta}{3}+\frac{2\dot{R}f''}{f'}\right]\tilde{Z}^k-2\frac{\mu_m}{f'}\tilde{\Delta}_m^k$$
$$+\left[2\Theta\dot{R}\frac{f'''}{f'}-\frac{f''}{f'}(\frac{f}{f'}-\frac{2\mu_m}{f'}+2\dot{R}\Theta\frac{f''}{f'}-2\frac{k^2}{a^2})\right]\mathcal{R}^k+\frac{2\Theta f''}{f'}\Re^k=0\,.$$

(9.92)

Since there are generally no analytic solutions for the system of Eqns (9.88)-(9.91) for even the simplest models of non-GR $f(R)$, a full numerical integration is required to fully understand the effect of peculiar velocities on large scale structure formation. This can be done either in the comoving frame (where there are fewer equations to integrate) followed by the transformation of solutions according to (9.82), or directly in the quasi-Newtonian frame, depending on whichever method produces a numerically more stable set of solutions. This is left for a future investigation.

9.6 Results and Discussion

In this chapter, we have shown the existence of an integrability condition for generic $f(R)$ gravity models based on a linearized covariant consistency analysis of dust universes in the shear-free hypersurfaces (longitudinal) gauge originally developed for GR [200, 300]. We have also derived the velocity and density perturbation equations resulting from the propagation of the acceleration equation that in turn is a result of the generalized van Elst- Ellis condition.

As in GR, the velocity perturbations are scale-independent and hence have effect on all scales. However, the density perturbations, as expected, are scale dependent. Unlike the GR case, the equations are too complicated to admit analytic solutions, even for the simplest cases of $f(R)$ around a flat FLRW background. A careful numerical analysis can, in principle, give us a closer insight into such important areas of interest as the Zeldovich approximation for cosmological perturbations of some viable $f(R)$ models.

As for the nonlinear models, it would be interesting to check if the covariant analysis results in consistent cosmologies and any integrability conditions.

Chapter 10

Conclusions and Future Outlook

Experience seems to most of us to lead to conclusions, but empiricism has sworn never to draw them.

George Santayana

It has now become common knowledge that the Universe is in a state of expansion. The recent (observational) discovery of the accelerated rate of this expansion has, however, come as a surprise to cosmologists, mainly because the standard Einstein's field equations do not predict such a phenomenon. In this thesis we have presented an alternative approach to the theory of gravitation and shown that if the gravitational action is chosen to include generic functions of the Ricci scalar, an accelerated expansion can indeed be accounted for. This is due to the extra degree of freedom that comes as a result of generalizing General Relativity.

The whole thesis is divided into three parts, **Part I** covered the literature review and the general theoretical foundation of what would follow as cosmological applications of perturbations in **Part II**. **Part III** focus on a class of shear-free perfect fluid cosmological models in $f(R)$ gravity.

In Chapter 2, we presented some basic background information about the formulation of General Relativity, its historical evolution as a tool to the study of modern cosmology, and its shortcomings in the context of the Concordance Model. Based on these shortcomings, we motivated towards the end of this chapter the need for looking at other alternative gravitational theories that potentially demystify [arguably] the greatest riddle of modern science: the unexplained accelerated cosmic expansion.

Chapter 3 focused on a review of one such alternative, namely $f(R)$ theories of gravitation. We presented the generalized forms of the field equations of GR in Chapter 2 in the framework of these new gravitational theories. We also outlined the general conditions that any $f(R)$ model should satisfy for it to be a viable candidate for explaining observations on all cosmological scales.

In Chapter 4 we gave an overview of the 1+3 covariant spacetime splitting formalism as applied to $f(R)$ theories and gave the propagation and constraint equations relating the kinematical quantities arising from the covariant decomposition, first for the full, nonlinear case and then in the linearized regime.

In Chapter 5 we gave a short overview of the dynamical systems approach to differential equations and their applications to cosmology. We defined dimensionless dynamical variables whose evolution equations with respect to normalized (dimensionless) time will form an autonomous system of cosmological equations. We indicated that, although solving this system is easier than solving the actual cosmological equations in non-dimensionless form, it comes at the cost of oversimplification and hence, the fixed point solutions of the dynamical systems may not necessarily represent physically meaningful cosmological solutions. Finally, we applied the DSA techniques to analyze two types of $f(R)$ models: the R^n and $R + R^n$-type ones. We showed that there exist fixed point solutions that correspond to transient Friedmann-type solutions which eventually evolve to de Sitter-type attractor solutions.

In Chapter 6, we studied the evolution of scalar cosmological perturbations in the $1 + 3$ covariant gauge-invariant formalism for generic $f(R)$ theories of gravity. After giving a complete set of equations describing the evolution of matter and curvature fluctuations for a multi-fluid cosmological medium, we used these equations to analyze perturbations in a radiation-dust fluid described by barotropic equations of state and solved the perturbation equations around a background solution of R^n gravity. In particular, we studied exact solutions for scales much smaller and much larger than the Hubble radius and showed that in order to have a growth rate compatible with the Mészáros effect one has to constrain n such that $n > 2/3$. It is also in this chapter that we introduced a covariant characterization of the so-called quasi-static approximation and showed its agreement with the standard (metric) approach on sub-Hubble scales.

In Chapter 7 we used the dynamical systems approach presented in Chapter 5 and the perturbations analyzed in Chapter 6 to look at the BAO geometrical constraints and predicted matter power spectra for dust-dominated epoch perturbations. This involved solving the background cosmological equations for R^n in the dust-dominated epoch, where initially, the Hubble and deceleration parameters are set to equal that predicted by ΛCDM. It turned out that it is impossible to produce consistent background cosmic histories for such initial conditions and for the values of n used in the model investigated. To further constrain these models, we compared the theoretical power spectra produced by these models with the SDSS power spectra of luminous red galaxies and DR9 2012 CMASS galaxy sample in the SDSS-III data. With regard to DR9 2012 data, most of the studied R^n models provided good fits to the data $n = 1.3$, with a suppression of $10^{-3}\%$ being the best fit slightly improved by ΛCDM . Other exponents (1.27, 1.29, 1.33 and 1.4) also provided good fits with slightly bigger suppressions. Regardless of the Large Structure Constraints none of the studied exponents were, however, able to fit the baryon acoustic oscillations data as well as the ΛCDM model. The obtained χ^2 were consequently much bigger than the best-fit model as provided by ΛCDM. We found that, for very special initial conditions, the power spectra are almost flat in the Sloan wavenumber range, but that none of the models studied were able to fit the observed data as well as the ΛCDM model, thus putting question marks on the viability of such models.

In Chapter 8 we looked at fully covariant shear-free perturbations in generic $f(R)$ gravity and showed that there exists a class of linearized $f(R)$ models for which a shear-free, almost FLRW universe can expand rand rotate at the same time. In particular, we found that for vacuum universes with vanishing 3-curvature, the Ellis shear-free conjecture can always be avoided. Moreover, we also showed that there is at least one physically realistic non-vacuum case (R^3 Milne-type models) in which simultaneous expansion and rotation are possible. The investigation of generalized shear-free fluid flows for such models will not be complete until the full, nonlinear regime is considered, and this is left for future work.

In Chapter 9 we studied covariant consistency relations in linearized, shear-free $f(R)$ gravity and proved the existence of an integrability condition for generic dust models known as *quasi-Newtonian* models. We derived the velocity perturbations and acceleration evolution equations that arise from the derived integrability conditions. Finally, we presented the full set of equations that completely describe the evolution of density, acceleration, velocity and curvature fluctuations. These very complicated coupled ODEs are unlikely to admit any analytic solutions on small scales, and a numerical integration and the implications of the analysis are left for future work, as are the nonlinear consistency analysis and the study of nonlinear collapse and structure formation in the Zeldovich approximation in these models.

Overall we have shown that apart from closing in on the missing link between theoretical and observational cosmology, one can also address a number of other, albeit 'less' intriguing, issues of cosmological and astrophysical interest within the context of higher order theories of gravity. With a number of observational probes up and coming, more accurate measurements of the main cosmological parameters should enable us to tightly constrain the different modified models of gravity in general and $f(R)$ theories in particular, based on the robustness of the theoretical predictions.

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