Stochastic Modelling of Financial Markets with Differential Information

F.V. Damiani
Department of Mathematics and Applied Mathematics
University of Cape Town

supervised by
Dr. Peter Ouwehand

Dissertation

presented to the Faculty of Science
of the University of Cape Town

in partial fulfillment of the requirements of the degree of
M.Sc. in the Mathematics of Finance

May 22, 2008

Finally, in Chapter 6 we specialize some of our earlier assumptions in an attempt to provide models that are a bit closer to reality. There are two major concerns. The first is to describe models in which the dynamics are influenced by the presence of a large insider. This is done by assuming that the market coefficients are stochastic processes that are dependent both on time *and* on the insider's portfolio; that is, whereas we previously had $\mu(t)$ and $r(t)$, we now assume $\mu(t, \pi)$ and $r(t, \pi)$. The effect of this π-dependence on the insider's optimal portfolio problem is studied, and we also consider the behaviour of a small honest trader in such an insider-influenced market. The chapter includes the study of a model in which the insider influences the drift continuously through information that the honest trader only receives δ time units after the insider. The second concern of Chapter 6 is to study conditions under which the is optimal utility is finite for both the honest trader and the insider. We reconsider an example of Chapter 5, in which the insider's additional information is given by $G = W_T + \varepsilon$, where ε represents an independent source of Gaussian noise. The assumption of distorted information is modified by introducing time dependence into the noise term, which rectifies the problem of the insider's additional information not improving as time progresses. We close the chapter with the study of optimizing an insider portfolio under certain constraints that penalize the insider for engaging in heavy volumes of trade.

Abstract

Many of the fundamental results in mathematical finance are based on the assumption that all traders have access to exactly the same information, usually assumed to be the filtration generated by the history of stock prices or the history of the underlying Brownian motion. In the last fifteen years or so, many articles in the financial mathematics literature have been concerned with using techniques of stochastic calculus to model financial markets in which different traders have access to different levels of information.

This thesis aims to provide a coherent account of the various approaches that have been used to model financial markets with heterogeneously informed agents. Part I of the thesis presents a description of the two branches of stochastic analysis that are required to understand the financial models: namely, *Malliavin's calculus* and *enlargements of filtrations*. Part II applies the mathematical results of Part I to provide a comprehensive presentation of the various financial models that have been introduced in the literature.

Contents

1 **Introduction** **1**

I Mathematical background 8

2 **Malliavin calculus** **9**
- 2.1 The Wiener-Ito chaos expansion . . . 9
- 2.2 The Malliavin derivative . . . 14
- 2.3 The Skorohod integral . . . 18
- 2.4 The Clark-Ocone formula . . . 20
- 2.5 The forward integral . . . 21

3 **Enlargement of filtrations** **26**
- 3.1 An introductory example . . . 26
- 3.2 Initial enlargements for general semimartingales . . . 28
- 3.3 Martingale representation theorems under initial enlargements . . . 31
- 3.4 Initial enlargements in a Brownian world . . . 33
  - 3.4.1 The role of Malliavin's calculus . . . 33
  - 3.4.2 Yor's approach . . . 37

II Financial applications 40

4 **Development of the market model** **41**
- 4.1 Formulation of the optimal portfolio problem . . . 42
- 4.2 Characterization of the optimal portfolio . . . 44
- 4.3 The partial information case . . . 49
- 4.4 The inside information case . . . 51
- 4.5 A link between optimal portfolios and semimartingales . . . 55

5 **Additional insider utility under initial enlargements** **57**
- 5.1 Additional utility via the information drift . . . 58

5.2 Entropy representation of additional utility . . . 63
5.2.1 Examples . . . 65
5.3 A word about arbitrage . . . 67

6 Adjustments to the model 69
6.1 The influence of a "large" insider . . . 69
6.1.1 An honest trader in a market influenced by a large insider . 72
6.1.2 Continuous flow of large-insider information . . . 74
6.2 Independent noise, insider friction and the absence of arbitrage . . . 77
6.2.1 Diminishing independent noise . . . 78
6.2.2 Penalty functions . . . 80

7 Conclusion 84

Chapter 1

Introduction

The history of financial mathematics has demonstrated the mathematical arena of stochastic analysis to be of tremendous power in describing, understanding and developing many aspects of modern finance. Two particularly important applications are the fields of derivative pricing and portfolio optimization. As a parallel to the obvious advantages offered to finance by stochastic analysis, the tangible "real" examples offered by the world of finance afford mathematicians and probability theorists many opportunities to better understand some of the abstract ideas contained in their field. Some years ago, the abstract notion of an *Ito integral* found its living counterpart in the *gains process* of a trader on a financial market. The existence of a *martingale representation theorem* on a particular probability space was then found to be an ideal mathematical expression of the notion of *completeness* on a financial market; that is, whether any contingent claim may be replicated by a portfolio of stocks and bonds. At the heart of financial mathematics lies the First Fundamental Theorem, which relates the *absence of arbitrage* on a market to the existence of an *equivalent martingale measure* on a given probability space. The few examples just listed should indicate that the abstract mathematical field of stochastic analysis has found a suitable worldly stage in the rapidly developing field of mathematical finance.

In recent years, several additional branches of stochastic analysis have found their way into mathematical finance via the study of financial markets with differential information. An assumption of much of the standard theory of mathematical finance, often referred to as *strong form efficiency*, is that all traders are *homogeneously informed*: all agents acting on the market have access to exactly the same information, which is typically assumed to consist of knowledge of the collection of past values of the stock prices. This information is modelled mathematically by the *filtration* $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$, where each $\mathcal{F}_t$ represents information about the evolution of the stock price up to time t:

$$\mathcal{F}_t = \sigma(S_s; s \leq t).$$

The filtration $\mathbb{F}$ is often referred to as the *public information flow*. The problem with the assumption of homogeneous information is that in reality the multitude of agents acting on a financial market generally have access to different sources of information; indeed, it is rare for an investment decision to be based purely on the price history of a particular stock. In this thesis we consider a market with *heterogeneously informed agents*. An agent who possesses information that is strictly larger than the public filtration is generally referred to as an *insider*. It should be noted that the term "insider" does not necessarily imply that the agent is partaking in any illegal activity; we merely use the term to describe someone whose information filtration is larger than the public filtration. Suppose, for example, that, in addition to the market filtration $\mathbb{F}$. the insider also has access to $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$, which denotes a filtration that is not contained in the market filtration. We *enlarge* $\mathbb{F}$ by $\mathbb{H}$ to obtain the *insider filtration* $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$, where $\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \vee \mathcal{H}_s$.

A large proportion of the academic literature on insider trading is concerned with the scenerio in which the insider possesses *initial information*. That is. the insider possesses knowledge of a future random variable G from the beginning of the trading interval, and G doesn't change as time progresses. A typical (rather unrealistic) example is when G represents the terminal value of a stock price S_T. In this case we have $\mathbb{H} = \sigma(S_T)$, so that the insider's filtration is given by $\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_t \vee \sigma(S_T)$. The fact that the insider possesses this extra information results in a different perception of the price process. Accordingly, this enhanced level of perception allows the insider to make better choices when making investment decisions. Mathematically. we say that the portfolio process π, which denotes the proportion of wealth invested in the risky asset, must be *adapted* to the enlarged filtration $\mathbb{G}$. A reasonable assumption is that an investor will choose a portfolio π that maximizes his expected utility, which is a function of the underlying value process. A self-financing portfolio is characterized by the requirement that the value of the trader's portfolio is given by $V_t = V_0 + \int_0^t \pi_u dS_u$, which means that we need to be able to compute the stochastic integral of the portfolio process π with respect to the stock price process S. It is at this point that the basic theory of Ito integration used in standard mathematical finance. in which agents are homogeneously informed, fails to provide an adequate solution. The problem is that the integrator S is assumed to be a semimartingale with respect to the public filtration $\mathbb{F}$, but the integrand π is adapted to $\mathbb{G}$, which is larger than $\mathbb{F}$. The general Ito integral would only make sense if the integrand S were a semimartingale with respect to $\mathbb{G}$. The central problem of the mathematics of insider trading is essentially to make sense of integrals of the above form; i.e. integrals in which the integrand is adapted to a filtration that is larger than the market filtration.

Broadly speaking, two fundamental mathematical approaches have been used to tackle the problem of non-adapted or *anticipating* integrands. The first approach involves defining an extension of the Ito integral that accounts for non-adapted integrands. The field of *Malliavin's calculus* includes the study of an object known as the *Skorohod integral*, which is well-defined for non-adapted processes and which coincides with the ordinary Ito integral for adapted integrands. The Skorohod integral is closely related to another object known as the *forward integral*, which is also well-defined for non-adapted processes (see Chapter 2). The forward integral has become a standard tool in the modelling of financial markets with heterogeneously informed agents, since the gains process $\int \pi_u dS_u$ makes perfect sense if we interpret it as the forward integral $\int \pi_u d^- S_u$. The theory of stochastic calculus with anticipating integrands has been rigorously developed in Nualart and Pardoux [33] and Russo and Valois [38], [39], [40].

The second approach involves finding conditions under which the price process S remains a semimartingale with respect to the enlarged filtration $\mathbb{G}$. If this is case then, since π is adapted to $\mathbb{G}$, the object $\int \pi_u dS_u$ qualifies as an Ito integral, and no anticipating calculus is necessary. The task of determining when $\mathbb{F}$-semimartingales remain semimartingales with respect to a strictly larger filtration $\mathbb{G}$ falls within a branch of stochastic analysis known as *enlargements of filtrations*. A quantity of great importance in applications is the *information drift*, which is defined as the process μ^G such that an $\mathbb{F}$-semimartingale S has the following decomposition with respect to the filtration $\mathbb{G}$:

$$S_t = \tilde{M}_t + \int_0^t \mu_s^G ds$$

where $\tilde{M}_t$ is a $\mathbb{G}$-martingale. The theory of enlargements of filtrations was initiated in a paper by K. Ito [22], when he showed that a standard Brownian motion remains a semimartingale when its natural filtration is enlarged by $\sigma(B_1)$, B_1 being a future value of the Brownian motion. In the early 1980's, the study of enlargements (or *grossissement de filtrations*) was greatly furthered in a series of papers by the French school of probability theorists (including Jeulin, Yor, Jacod and others).

We shall now present a brief overview of the historical development of the two basic approaches.

The use of enlargement techniques to model differentially informed markets was initiated in the seminal paper by Pikovsky and Karatzas [36].[1] Their paper studies the problem of optimizing the terminal logarithmic utility of an insider who has access to initial knowledge of an $\mathcal{F}_1$-measurable random variable G. Initial

[1]Other methods had previously been used to study differentially informed markets (see Kyle [27], Back [6] and Duffie and Huang [15]) but [36] seems to have been the first to apply the powerful enlargement techniques of Jacod et al, effectively initiating the standard approach in the literature.

enlargement techniques are applied to several examples of G, including the cases where $G = W(1)$ (terminal value of the Brownian motion) and $G = \lambda W(1) + (1-\lambda)\epsilon$ (where ϵ is a Gaussian random variable). Several key results were first introduced in this paper, including the idea that the insider's additional utility (that is, the difference between the expected terminal utility of the insider and that of the honest trader) may be expressed as the energy of the information drift, given by $\mathbb{E}[\int(\mu_s^G)^2 ds]$. The paper also relates the additional utility to the *relative entropy of the honest trader's probability measure with respect to the insider's probability measure* (see Chapter 5 for details). These results paved the way for the publication of a number of important papers in the years to follow, including Grorud and Pontier [17]. [18]. Amendinger, Imkeller and Schweizer [3] and Amendinger [2], all of which use the basic approach of initial enlargements to solve problems concerning the optimization of insider utility.

Amendinger. Imkeller and Schweizer [3] generalized many of the results of Pikovsky and Karatzas in solving the optimal portfolio problem for an insider in a general incomplete market, with dynamics given by $dS_t = S_t(\alpha_t d\langle M\rangle_t + M_t)$. Their paper presents simple conditions on the additional information G for the finiteness of the insider's additional utility, and proceeds to expand on the entropy representation first suggested in [36]. Amendinger [2] acts as a follow up to [3], in that it applies the ideas of that paper to the problem of portfolio optimization of an insider in a market that is *complete* for the honest trader. In order to extend the completeness property to the insider, the paper begins with a study of when the existence of martingale representation theorems on $\mathbb{F}$ implies their existence on $\mathbb{G}$.

An interesting approach that combines the enlargement approach with techniques from Malliavin's calculus is presented in Imkeller, Pontier and Weiss [21] and Imkeller [20]. In addition to studying the problem of optimizing the insider's expected logarithmic utility, these papers include extensive analysis of free lunch and arbitrage possibilities for the insider. A key feature of these papers is the development of a type of Malliavin calculus for measure-valued martingales on the Wiener space. The market is assumed to be driven by a standard $\mathbb{F}$-Brownian motion W; and the measure-valued Malliavin calculus is used to derive an expression for the information drift associated with the semimartingale decomposition of W with respect to the enlarged filtration $\mathbb{G}$.

The application of anticipating stochastic calculus to the problem of insider trading began much more recently in the paper by Biagini and Øksendal [9]. Their approach differs from the then-typical enlargement approach in that it is not assumed that the underlying Brownian motion is a semimartingale from the insider's perspective. Instead, the stock price dynamics are modelled by $dS(t) = S(t)[\mu(t)dt + \sigma(t)d^{-}W(t)]$, where the integral implied by the second term on the right denotes a *forward* integral. In addition it is assumed that the market is anticipating, in the sense that the coefficients are adapted to a filtration that is larger than that generated by the underlying Brownian motion. The main result shows that if an optimal insider portfolio exists in this framework, *then* the Brownian motion driving the stock price remains a semimartingale in the insider's filtration. This is essentially a converse of most of the results obtained in other papers which find the optimal utility under an *a priori* semimartingale assumption. Other notable features include the assumption of a generalized utility function rather than the usual logarithmic utility. Also, no particular assumptions are made about the insider's filtration, other than that it is larger than the honest trader's filtration. This means that many of the results apply for general enlargements rather than being specialized to initial enlargements.

Shortly after the publication of the Biagini and Øksendal paper, another paper dealing with the use anticipating calculus was published by Leon, Navarro and Nualart [28]. In their study, it is assumed that the insider knows the value of an $\mathcal{F}_T$-measurable random variable L. Malliavin calculus and forward integration are then used to optimize the insider's expected logarithmic utility. Although the paper uses techniques of forward integration, the ideas are essentially related to the enlargement of filtrations approach: in order to solve the insider's optimal portfolio problem, they assume the existence of a specified process $h(t)$ which ensures that $W(t) - \int_0^t h(s)ds$ is a martingale in the insider's enlarged filtration.

Øksendal and Sulem [35] study an anticipative market, driven by a Lévy process, from the point of view of a trader who observes *less* information than the rest of the market. The problem of studying an insider was thus replaced by the problem of a *partially informed trader*.

The next major development in the use of anticipative calculus models was in the study of markets that are influenced by so-called *large* insiders. This means that the market coefficients are actually influenced by the trading behaviour of the insider. Kohatsu-Higa and Sulem [26] is a comprehensive paper that generalizes the results of [9] and [35] in the sense that it considers an anticipating financial market without making any initial assumptions about the relationship between the trader's information and the information generated by the market. This generality allows for the study of both insiders and partially informed traders. A novel feature of the paper is that it explicitly examines the case in which the drift depends on an anticipating random variable, which is interpreted as a result of insider influence. Other papers that study insider-influenced markets include Kohatsu-Higa and Sulem [25] and Di Nunno, et al [14].

This dissertation is organized as follows:

In Chapter 2 we present a mathematical overview of the Malliavin calculus. We begin with a description of the decomposition of the Wiener space by the Ito chaos expansion. We proceed by developing the Malliavin derivative and its adjoint, the

Skorohod integral. This is followed by a presentation of the celebrated Clark-Ocone formula. We close the chapter with a section on forward integration, which includes the basic definition of the forward integral, along with several key results. As this chapter is intended as background material for the financial applications of Part II, some of the proofs have been omitted and original sources have been referenced when necessary.

In Chapter 3 we turn to a review of the theory of enlargements of filtrations. A systematic mathematical development is sacrificed to some extent in favour of the presentation of key results that will aid us in the mathematical models of later chapters. The bulk of this chapter deals with the study of initial enlargements, and we are particularly concerned with deriving expressions for the *information drift*. After discussing enlargements in a general semimartingale setting we turn to the more specialized study of enlargements in a Brownian world.

In Chapter 4 we turn our attention to the development of financial models. We begin by proposing dynamics for a general anticipative market with differentially informed traders, in which we modify standard assumptions such as the self-financing condition and the stock price s.d.e. so that the usual Brownian term is replaced by a forward integral term. Here it is assumed that $\mathbb{F}$ is the filtration generated by the underlying Brownian motion and that the coefficients of the stock price process are adapted to a larger filtration $\mathbb{G}$. We assume that a trader acting on the market possesses information given by a general filtration $\mathbb{H}$, which may be smaller than $\mathbb{F}$ (partially informed trader) or larger than $\mathbb{G}$ (insider). The problem of maximizing the expected terminal logarithmic utility is then solved for this general trader. The chapter proceeds by studying separately the cases of partial information and inside information using the anticipating calculus framework. The chapter concludes with an important result that links the existence of an optimal insider portfolio to existence of a semimartingale decomposition of the underlying Brownian motion in the insider's filtration. This motivates the necessity of solving the optimal portfolio problem of an insider via techniques from enlargements of filtrations.

Taking our cue from the final result of Chapter 4, in Chapter 5 we study the problem of portfolio optimization for an insider using the method of enlargements of filtrations. Specifically, we concentrate on initial enlargements, i.e. the insider acquires all additional information at the beginning of the trading interval. Here it is found that the additional expected logarithmic utility of the insider is given by $\mathbb{E}[\int \mu^2 ds]$. Motivated by the difficulty of calculating explicitly the information drift μ for an arbitrary type of additional information we proceed to find an entropy representation of the additional utility that doesn't depend on μ. The chapter closes with some results about the ability of an insider to exercise arbitrage, which depends on the relationship between the conditional laws of the additional information and the law of the additional information.

Part I

Mathematical background

Chapter 2

Malliavin calculus

Malliavin calculus, also known as the stochastic calculus of variations, is a type of infinite-dimensional differential calculus defined on the Wiener space. The theory was initiated in 1974, when P. Malliavin used it to provide a probabilistic proof of Hörmander's theorem [29]. Since then it has developed substantially and has been used to define differentiation and integration operators on random variables. The differentation operator, which we will study in Section 2.2, is known as the Malliavin derivative and the integration operator, which we will study in Section 2.3, is known as the Skorohod integral. A result that has had particular relevance in mathematical finance is known as the Clark-Ocone representation formula. If a random variable X has the Ito representation $X = \mathbb{E}[X] + \int \phi(\omega, s)dW_s$, then the Clark-Ocone formula gives us a means of expicitly finding the integrand ϕ; this will be dealt with in Section 2.4. In Section 2.5 we present some results from the theory of *anticipating stochastic calculus*, which is closely related to traditional Malliavin calculus in that it is also concerned with the study of anticipating integrands.

Since this chapter is intended as background material for the financial applications of Part II, some of the proofs have been omitted and original sources have been referenced when necessary.

2.1 The Wiener-Ito chaos expansion

The Wiener-Ito chaos expansion allows one to express elements of a square-integrable Hilbert space in terms of a particular orthogonal basis. The elements of the decomposition can be expressed as *iterated Ito integrals* (see Eq. (2.5)). This particular representation of the random elements allows for very convenient expressions of the Malliavin derivative and Skorohod integral (see Theorems 2.2.8 and 2.3.5 respectively). Let us begin by defining and stating some properties of the so-called iterated Ito integrals, since these objects form the basis of the chaos expansion. This section is largely based on [34].

We will work on a measure space $(\Omega, \mathcal{F}, P)$ equipped with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the Brownian motion W.

We define $\Delta_n(t)$ to be the portion of the n-dimensional box $[0,t]^n$ given by

$$\Delta_n(t) = \{(x_1, \ldots, x_n) \in [0,t]^n; 0 \le x_1 \le \ldots \le x_n \le t\}. \tag{2.1}$$

Now if $f = f(t_1, \ldots, t_n) \in L^2(\Delta_n(T))$ (that is, $\int_{\Delta_n(T)} f^2(t_1, \ldots, t_n)dt_1 \ldots dt_n < \infty$) then the associated n-fold iterated Ito integral is defined as

$$\begin{aligned} J_n(f) &:= \int_0^T \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_2} f(t_1, \ldots, t_n) dW(t_1)dW(t_2)\ldots dW(t_n) \\ &= \int_{\Delta_n(T)} f(t_1, \ldots, t_n) dW^{\otimes n} \end{aligned} \tag{2.2}$$

where $dW^{\otimes n} := dW(t_1)dW(t_2)\ldots dW(t_n)$. Using the Ito isometry and the fact that the expectation of an Ito integral is zero we can show

Theorem 2.1.1. ([34] Eq. (1.7))

$$\begin{aligned} E[J_n(f)J_m(g)] &= 0 && \text{if } n \neq m \\ E[J_n(f)J_m(g)] &= (f,g)_{L^2(\Delta_n(T))} && \text{if } n = m \end{aligned}$$

Proof. Suppose $n = m$. We have

$$\begin{aligned} \mathbb{E}[J_n^2(f)] &= \mathbb{E}\left[\left\{\int_0^T \int_0^{t_n} \cdots \int_0^{t_2} h(t_1, \ldots, t_n) dW(t_1)\ldots dW(t_n)\right\}^2\right] \\ &= \int_0^T \mathbb{E}\left[\left\{\int_0^{t_n} \cdots \int_0^{t_2} h(t_1, \ldots, t_n) dW(t_1)\ldots dW(t_{n-1})\right\}^2\right] dt_n \\ &= \ldots \\ &= \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} h^2(t_1, \ldots, t_n) dt_1 \ldots dt_n = \|h\|^2_{L^2(\Delta_n(T))} \end{aligned}$$

by repeated applications of the Ito isometry. Using essentially the same procedure, along with the fact that $\mathbb{E}[\int f dW] = 0$ for deterministic f, it is easy to show that

$$\mathbb{E}[J_n(f)J_m(g)] = 0 \quad (n \neq m)$$

□

Corollary 2.1.2. *The map $f \to J_n(f)$, $L^2(\Delta_n(T)) \to L^2(\Omega)$ is an isometry.*

It is appropriate at this point to introduce some ideas concerning symmetric functions. We begin with some simple definitions. Firstly, a real symmetric function $f : [0,T]^n \to \mathbb{R}$ is one that satisfies

$$f(t_1, \dots, t_n) = f(t_{\sigma(1)}, \dots, t_{\sigma(n)}) \tag{2.3}$$

for all permutations σ of $(1, \dots, n)$. The symmetrization $\tilde{f}$ of a function $f : [0,T]^n \to \mathbb{R}$ is given by

$$\tilde{f}(t_1, \dots, t_n) = \frac{1}{n!} \sum_{\sigma \in S} f(t_{\sigma(1)}, \dots, t_{\sigma(n)}) \tag{2.4}$$

where S denotes the group all permutations of the first n integers.

Remark 2.1.3. Using this symmetrization we can extend any function f defined on the wedge $\Delta_n(T)$ to a symmetric function $\tilde{f}$ defined on the whole box $[0,T]^n$. We simply define f to be zero on all points outside $\Delta_n(T)$ and define $\tilde{f}$ to be the symmetrization as defined above. (Note that it is necessary to define f to be zero outside the wedge since it is not otherwise defined in this region.)

The space of symmetric functions in $L^2([0,T]^n)$ will be denoted by $\tilde{L}^2([0,T]^n)$. By considering the number of permutations of n objects we can reason that there are $n!$ sets of shape $\Delta_n(T)$ that fill the box $[0,T]^n$ so that we have

$$\|\tilde{f}\|^2_{L^2([0,T]^n)} = n! \, \|f\|^2_{L^2(\Delta_n(T))}$$

For $f \in \tilde{L}^2([0,T]^n)$ we define $I_n(f)$ by

$$I_n(f) := n! J_n(f) \tag{2.5}$$

Theorem 2.1.4.

$$\begin{aligned} \mathbb{E}[I_n(\tilde{f}) I_m(\tilde{g})] &= 0 && \text{if } n \neq m \\ \mathbb{E}[I_n(\tilde{f}) I_m(\tilde{g})] &= n!(\tilde{f}, \tilde{g})_{L^2([0,T]^n)} && \text{if } n = m \end{aligned}$$

Proof. Using Th. 2.1.1 and Eq. (2.5) we obtain

$$\mathbb{E}[I_n^2(f)] = \mathbb{E}[(n!)^2 J_n^2(f)] = (n!)^2 \, \|f\|^2_{L^2(\Delta_n(T))} = n! \, \|f\|^2_{L^2([0,T]^n)}$$

The $n \neq m$ case follows immediately from Theorem 2.1.1, since $\mathbb{E}[I_n(\tilde{f}) I_m(\tilde{g})] = (n!)^2 \mathbb{E}[J_n(\tilde{f}) J_m(\tilde{g})] = 0$. □

In order to find a decomposition of the probability space $(\Omega, \mathcal{F}, P)$ it is necessary to present some results concerning the *Hermite polynomials*. We follow [30]. Let (f, g) denote the inner product defined on the Hilbert space $L^2(\Omega)$:

$$(f, g) = \int_\Omega f(\omega) g(\omega) dP(\omega).$$

The *annihilation* operator is simply defined as

$$(\partial x)(t) = x'(t).$$

where $x : \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function. The *creation* operator ∂^* is then defined as the adjoint of ∂. On a one-dimensional Gaussian Hilbert space[1], with probability measure $d\gamma = 1/\sqrt{2\pi}\exp(-x^2/2)dx$, we have

$$(x, y) = \mathbb{E}[x(t)y(t)] = \int_{\mathbb{R}} x(t)y(t)d\gamma(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x(t)y(t)e^{-t^2/2}dt$$

and

$$\begin{aligned}(\partial x, y) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\partial x)(y(t)e^{-t^2/2})dt = -\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x\partial(y(t)e^{-t^2/2})dt \quad \text{(by ibp)} \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x(-\partial y + ty)e^{-t^2/2}dt = (x, \partial^* y)\end{aligned}$$

where $\partial^* x = -\partial x + tx$. Two useful identities follow from the relationship $(\partial x, y) = (x, \partial^* y)$:

$$\partial\partial^* - \partial^*\partial = 1 \tag{2.6}$$

$$\partial(\partial^*)^n - (\partial^*)^n\partial = n(\partial^*)^{n-1} \tag{2.7}$$

We now define the Hermite polnomials by

$$H_n(t) := (\partial^*)^n(1) \quad (n = 0, 1, 2, \ldots) \tag{2.8}$$

Applying the identity (2.7) to the constant function 1 we obtain (see [30])

$$H_n' = nH_{n-1}$$

Moreover, it is not hard to show that

$$(H_m, H_n) = ((\partial^*)^m 1, H_n) = (1, \partial^m H_n) = \mathbb{E}[\partial^m H_n]$$

so that $(H_m, H_n) = 0$ for $m < n$. Since the inner product is symmetric we have that $H_m \perp H_n$ for $m \neq n$. It is also clear from the definition that $H_n(t)$ is of degree n with leading term t^n so that

$$||H_n||^2 = (H_n, H_n) = \mathbb{E}[\partial^n H_n] = n!$$

This finally leads us to conclude that

$$\left\{\frac{1}{\sqrt{n!}}H_n,\ n = 0, 1, 2, \ldots\right\}$$

is an orthonormal set in $L^2(\gamma)$. It can in fact be shown that it constitutes an orthonormal basis of $L^2(\gamma)$. We conclude this brief study of Hermite polynomials with some interesting results.

[1]Recall that a Gaussian Hilbert space is one whose elements are centred Gaussian random variables; i.e. for each random variable X, there is a $\sigma > 0$ such that $X \sim N(0, \sigma^2)$.

Proposition 2.1.5. *Any C^∞-function x with $\partial^n x \in L^2(\gamma)$ for all $n \in \mathbb{N}$ can be written as*

$$x = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}[\partial^n x] H_n \tag{2.9}$$

Proof. Using the orthonormal basis we have, for $x \in L^2(\gamma)$

$$x = \sum_{n=0}^{\infty} \frac{1}{n!}(x, H_n) H_n$$

If all derivatives are in $L^2(\gamma)$ we also have

$$(x, H_n) = (x, (\partial^*)^n 1) = (\partial^n x, 1) = \mathbb{E}[\partial^n x]$$

□

Corollary 2.1.6.

$$\exp\left(ct - \frac{1}{2}c^2\right) = \sum_{n=0}^{\infty} \frac{c^n}{n!} H_n(t) \tag{2.10}$$

Proof. Applying Proposition 2.1.5 to $x(t) = e^{ct}$ yields

$$e^{ct} = \mathbb{E}[e^{cu}] \sum_{n=0}^{\infty} \frac{c^n}{n!} H_n(t)$$

We then use

$$\mathbb{E}[e^{cu}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{cu} e^{-u^2/2} du = e^{c^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(u-c)^2/2} du = e^{c^2/2}$$

to obtain

$$\exp\left(ct - \frac{1}{2}c^2\right) = \sum_{n=0}^{\infty} \frac{c^n}{n!} H_n(t)$$

□

We will also require the following result, which provides a relation between the iterated Ito integral and the Hermite polynomial:

Theorem 2.1.7. *For $g \in L^2([0,T])$ we have*

$$n! J_n(g^{\otimes n}) = \|g\|^n \, H_n\left(\frac{W(g)}{\|g\|}\right) \tag{2.11}$$

where $g^{\otimes n}(t_1, t_2, \ldots, t_n) := \prod_{i=1}^n g(t_i)$.

Proof. See Appendix of [8]. □

We now present the major theorem of this section:

Theorem 2.1.8. *If $\phi \in L^2(\Omega)$ is an $\mathcal{F}_T$-measurable random variable, then there exists a unique sequence of symmetric deterministic functions $f_n \in \tilde{L}^2([0,T]^n)$ such that*

$$\phi(\omega) = \sum_{n=0}^{\infty} I_n(f_n) \tag{2.12}$$

Furthermore, ϕ satisfies

$$\|\phi\|^2_{L^2(\Omega)} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2([0,T]^n)} \tag{2.13}$$

Proof. See Theorem 1.1 of [34]. □

2.2 The Malliavin derivative

Recall that $L^2([0,T])$ denotes the Hilbert space of square integrable deterministic functions $h : [0,T] \to \mathbb{R}$. For $h \in L^2([0,T])$ we define

$$W(h) := \int_0^T h(t) dW_t \tag{2.14}$$

Let $C_p^\infty(\mathbb{R}^n)$ denote the set of all infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives have polynomial growth. The class of *smooth* random variables, which we denote by $\mathcal{S}$, consists of all random variables of the form

$$F = f(W(h_1), \dots, W(h_n)) \tag{2.15}$$

where $f \in C_p^\infty(\mathbb{R}^n)$ and $h_1, \dots, h_n \in L^2([0,T])$.

For $F \in \mathcal{S}$ we define the derivative of F by

$$DF := \sum_{i=1}^{n} \partial_i f(W(h_1), \dots, W(h_n)) h_i \tag{2.16}$$

Remark 2.2.1. In what follows we will sometimes write $D_t F$ for the derivative at a particular time t. That is, the value of the derivative at the point t is given by

$$D_t F = \sum_{i=1}^{n} \partial_i f(W(h_1), \dots, W(h_n)) h_i(t)$$

Note, however, that since $h \in L^2([0,T])$, $h(t)$ is only well-defined for *a.a.* t.

We denote by $\mathcal{P}_{1,2}$ the set of all smooth random variables for which such a derivative exists. (See [32] for a way to interpret DF as a directional derivative.) Using Eq. (2.16) it is easy to prove the following properties:

Proposition 2.2.2. *For $F, G \in \mathcal{P}_{1,2}$ and a continuously differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ we have*

1. *Linearity:* $D(\alpha F + \beta G) = \alpha DF + \beta DG$
2. *Chain rule:* $D(f(F)) = f'(F)DF$
3. *Product rule:* $D(FG) = (DF)G + F(DG)$

Remark 2.2.3. Note that if $F \in \mathcal{P}_{1,2}$ (i.e. $F = f(W(h_1), \ldots, W(h_n))$ where f is an nth degree polynomial), then we can orthonormalize the h_i's via the Gram-Schmidt procedure. (i.e. We let $e_1 = h_1/\|h_1\|$, $e_2 = (h_2 - (e_1, h_2)e_1)/\|(h_2 - (e_1, h_2)e_1\|, \ldots)$ and write the h_i's in terms of the orthonormal e_i's to obtain $F = \bar{f}(W(e_1), \ldots, W(e_n))$ where $\bar{f}$ is a new nth degree polynomial.) We can therefore assume that the h_i's are orthonormal when it is convenient.

The following integration-by-parts formulae are useful:

Theorem 2.2.4. *Let $h \in L^2([0,T])$, $F \in \mathcal{P}_{1,2}$. Then*

$$E[(DF, h)_{L^2([0,T])}] = E[FW(h)] \tag{2.17}$$

Proof. We follow [32]. We can normalize the arguments of F along the lines of the preceding remark (normalizing Eq. (2.17) such that h is of norm one) to obtain

$$F = f(W(e_1), \ldots, W(e_n))$$

where $f \in C_p^\infty$, $h = e_1$, and the e_i's are orthonormal elements. We have

$$\begin{aligned}
\mathbb{E}[(DF, h)_H] &= \mathbb{E}[(\sum_i \partial_i f(W(e_1), \ldots, W(e_n))e_i, e_1)] \\
&= \mathbb{E}[(\sum_i (\partial_i f)(e_i, e_1)_{L^2([0,T])}] = \mathbb{E}[\partial_1 f] \\
&= \int_{\mathbb{R}^n} \partial_1 f(x) d\gamma_n(x)
\end{aligned}$$

where

$$d\gamma_n = \frac{1}{(2\pi)^{n/2}} \exp(-|x|^2/2) dx$$

Thus

$$\begin{aligned}\mathbb{E}[(DF,h)_H] &= -\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)\partial_1 e^{-|x|^2/2}dx \\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)x_1 e^{-|x|^2/2}dx \\ &= \int_{\mathbb{R}^n} f(x)x_1 d\gamma_n(x) \\ &= \mathbb{E}[f(x)x_1] = \mathbb{E}[FW(e_1)] = \mathbb{E}[FW(h)]\end{aligned}$$

□

Corollary 2.2.5. *For $h \in L^2([0,T])$, $F, G \in \mathcal{P}_{1,2}$ we have*

$$E[G(DF,h)_{L^2([0,T])}] = E[(-FDG,h)_{L^2([0,T])} + (FG)W(h)] \tag{2.18}$$

Proof. Apply Theorem 2.2.4 to the product FG and use the product rule for the derivative. □

Let us now define a norm on $\mathcal{P}_{1,2}$ by

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|DF\|_{L^2([0,T]\times\Omega)} \tag{2.19}$$

Since the family $\mathcal{P}_{1,2}$ is not closed with respect to the above norm (see [8] p.15), we define $\mathbb{D}_{1,2}$ to be the closure of $\mathcal{P}_{1,2}$ with respect to $\|\cdot\|_{1,2}$. For a convergent sequence $F_n \to F$ in $L^2(\Omega)$ it would be natural to define the derivative of F by

$$DF := \lim_{n\to\infty} DF_n. \tag{2.20}$$

Before we do this, however, we need to know whether D is a *closable* operator, i.e. whether

$$F_n \to F, DF_n \to L_1, G_n \to F, DG_n \to L_2 \Rightarrow L_1 = L_2. \tag{2.21}$$

Closability would imply that the definition suggested by Eq. (2.20) is unique. As it happens, the derivative operator is closable by the following theorem (See [34] Thm 4.11. [8] Thm 1.4.1):

Theorem 2.2.6. *The derivative operator $D : \mathcal{P}_{1,2} \to L^2([0,T]\times\Omega)$ is closable.*

Proof. Suppose $(F_n)_{n\in\mathbb{N}}$ is a sequence of smooth random variables (i.e. $F_n \in \mathcal{P}_{1,2}$ for each n) such that $F_n \to 0$ in $L^2(\Omega)$ and $DF_n \to F$ in $L^2([0,T],\Omega)$. Then, by the integration-by-parts formula of Eq. (2.17) we have

$$\mathbb{E}[(DF_n,h)] = \mathbb{E}[F_nW(h)]$$

where $h \in L^2([0,T])$. Now $\lim_{n\to\infty} \mathbb{E}[DF_n, h] = \mathbb{E}[(\lim_{n\to\infty} DF_n, h)] = \mathbb{E}[(DF, h)]$ for the LHS and $\lim_{n\to\infty} \mathbb{E}[F_n W(h)] = \mathbb{E}[FW(h)] = 0$ for the RHS. Thus we have

$$\mathbb{E}[(DF, h)] = 0 \quad \text{for all } h \in L^2([0,T])$$

from which it follows that $DF = 0$. □

We are now able to formally define the Malliavin derivative as the closure of the previously defined derivative operator (see Def. 4.13 of [34]):

Definition 2.2.7. *Let $F \in \mathbb{D}_{1,2}$, i.e. there is a sequence $(F_n)_{n\in\mathbb{N}}$ with each $F_i \in \mathcal{P}_{1,2}$ such that $F_n \to F$ in $L^2(\Omega)$ and $(D_t F_n)_{n\in\mathbb{N}}$ converges in $L^2([0,T]\times\Omega)$. The Malliavin derivative is then defined as*

$$DF = \lim_{n\to\infty} DF_n \tag{2.22}$$

We conclude this section by presenting a particularly useful result concerning the connection between the Malliavin derivative and the Wiener-Ito chaos expansion. Let us first consider how the chaos expansion may be applied to stochastic processes. Suppose that $u(t,\omega) \in L^2([0,T]\times\Omega)$ is a stochastic process. As long as

$$u(t,\cdot) \text{ is } \mathcal{F}_T - \text{measurable for all } t \in [0,T] \tag{2.23}$$

and

$$\mathbb{E}[u^2(t,\omega)] < \infty \text{ for all } t \in [0,T] \tag{2.24}$$

we can apply the chaos expansion to $u(t,\cdot)$ for each $t \in [0,T]$. That is, for each t we obtain a sequence of functions $f_{n,t}(t_1,\ldots,t_n) \in \tilde{L}^2(\mathbb{R}^n)$ such that

$$u(t,\omega) = \sum_{n=0}^{\infty} I_n(f_{n,t}(\cdot)) \tag{2.25}$$

In what follows we will write $\tilde{f}_n(\cdot,t) = \tilde{f}_n(t_1,\ldots,t_{n-1},t)$ so that $\tilde{f}_n$ is symmetric with respect to its first $n-1$ variables.

Theorem 2.2.8. *Let $F \in \mathbb{D}_{1,2}$ and let $F = \sum_{n=0}^{\infty} I_n(\tilde{f}_n)$ be the chaos expansion of F. Then*

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(\tilde{f}_n(\cdot,t)). \tag{2.26}$$

Moreover we have $F \in \mathbb{D}_{1,2}$ iff

$$\sum_{n=1}^{\infty} n n! \|\tilde{f}_n\|^2_{L^2([0,T]^n)} < \infty \tag{2.27}$$

Proof. See Theorem 4.18 of [34]. □

2.3 The Skorohod integral

Recall that the Malliavin derivative is an operator that takes square integrable random variables to square integrable processes: we have

$$D : \mathbb{D}_{1,2} \subset L^2(\Omega) \to L^2([0,T] \times \Omega) \tag{2.28}$$

We would like to define an integral operator δ on a subset $\mathbb{D}^*_{1,2} \subset L^2([0,T]\times\Omega)$ that 'reverses' the derivative operator in some sense. That is, it takes square integrable processes back to square integrable random variables:

$$\delta : \mathbb{D}^*_{1,2} \subset L^2([0,T] \times \Omega) \to L^2(\Omega) \tag{2.29}$$

We will thus define the so-called *Skorohod integral* as the adjoint of the derivative operator (i.e. $(F, \delta(u))_{L^2(\Omega)} = (D_t F, u)_{L^2([0,T]\times\Omega)}$). We define the space of Skorohod integrable random processes by

$$\mathbb{D}^*_{1,2} = \{u \in L^2([0,T] \times \Omega) | \exists c \in \mathbb{R} \text{ such that } |(DF, u)| \leq c||F||_{L^2(\Omega)}, \forall F \in \mathbb{D}_{1,2}\} \tag{2.30}$$

where c is a constant that is independent of F. We can now define the Skorohod integral δ as follows (see [8] for a discussion of existence and uniqueness of δ):

Definition 2.3.1. *If $u \in \mathbb{D}^*_{1,2}$, then $\delta(u)$, known as the Skorohod integral of u, is the unique element of $L^2(\Omega)$ that satisfies*

$$\mathbb{E}[(DF, u)_{L^2[0,T]}] = \mathbb{E}[F\delta(u)] \tag{2.31}$$

for any $F \in \mathbb{D}_{1,2}$.

Remark 2.3.2. Another common notation for the integral is given by

$$\delta(u) = \int_0^T u(t, \omega)\delta W \tag{2.32}$$

and the space of Skorohod integrable processes if often denoted by Dom δ.

Remark 2.3.3. Note that taking $F = 1$ in Def. 2.3.1 gives $E[\delta(u)] = 0$, as is the case for Ito integrals.

We next present a type of integration-by-parts formula that is very useful in other areas of Malliavin's calculus:

Theorem 2.3.4. *Let $F \in \mathbb{D}_{1,2}$, $u(t, \omega) \in \mathbb{D}^*_{1,2}$ and $Fu \in L^2([0,T] \times \Omega)$. Then $Fu \in \mathbb{D}^*_{1,2}$ and*

$$\delta(Fu) = F\delta(u) - (DF, u)_{L^2([0,T])} \tag{2.33}$$

Proof. See Proposition 1.3.3 of [32]. □

Just as we did for the Malliavin derivative, we will now find a representation of the Skorohod integral in terms of the Wiener-Ito chaos expansion. This is useful when it comes to actually computing the integral of a particular random variable and also aids us in proving other interesting properties of the integral. In Eq. (2.35) of the following result, $\tilde{f}_{n+1}$ refers to the symmetrization of $f_{n,t} = f_n(t_1, \ldots, t_n, t_{n+1})$, where we have set $t = t_{n+1}$.

Theorem 2.3.5. *Suppose $u \in L^2([0,T] \times \Omega)$ has the chaos expansion*

$$u(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)) \tag{2.34}$$

Then

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_{n+1}) \tag{2.35}$$

provided the series converges in $L^2(\Omega)$.

Proof. See Proposition 1.3.7 of [32]. □

Remark 2.3.6. In some sources, the Skorohod integral is *defined* in terms of the chaos expansion. We have not chosen this approach as it is difficult to see why the operator actually represents an *integral* using such a definition.

The above chaos expansion representation can be used to show that the Ito integral coincides with the Skorohod integral when the integrand is $\mathcal{F}_t$-adapted. In this sense, we can regard the Skorohod integral as a generalization of the Ito integral that is well-defined for non-adapted integrands. Before we present this result, we state an intermediate result that characterizes the $\mathcal{F}_t$-adaptedness of the terms in the chaos expansion.

Lemma 2.3.7. (Lemma 2.5 of [34]) *Let the square integrable process $u(t,\omega) \in L^2(\Omega)$ be $\mathcal{F}_T$-measurable for all $t \in [0,T]$, with chaos expansion given by*

$$u(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t))$$

for each $t \in [0,T]$. Then $u(t,\omega)$ is $\mathcal{F}_t$-adapted iff

$$f_n(t_1, \cdots, t_n, t) = 0 \quad \text{if} \quad t < \max_{1 \le i \le n} t_i \tag{2.36}$$

Theorem 2.3.8. (Theorem 2.6 of [34]) *Let $u(t,\omega)$ be an $\mathcal{F}_t$-adapted process for $t \in [0,T]$ such that*

$$\mathbb{E}\left[\int_0^T u^2(t,\omega)dt\right] < \infty$$

Then u is Skorohod integrable and

$$\int_0^T u(t,\omega)\delta W(t) = \int_0^T u(t,\omega)dW(t) \tag{2.37}$$

2.4 The Clark-Ocone formula

Recall that the Ito representation theorem states that any square integrable $\mathcal{F}_T$-measurable random variable $F \in L^2(\Omega)$ can be expressed as the Ito integral of a unique $\mathcal{F}_t$-adapted process $f(t,\omega)$:

$$F(\omega) = E[F] + \int_0^T f(t,\omega)dW(t) \tag{2.38}$$

Unfortunately we cannot calculate the form of the process $f(t,\omega)$ using elementary stochastic analysis. In order to find such an expression we turn to Malliavin's calculus, which enables us to find f provided F is Malliavin differentiable.

The following results (see [34] Prop. 5.5, Prop. 5.6 and Cor. 5.7 for proofs) are useful for what follows:

Theorem 2.4.1. *Let $f_n \in \tilde{L}^2([0,T]^n)$. Then*

$$\mathbb{E}[I_n(f_n)|\mathcal{F}_t] = I_n[f_n 1_{[0,t]}^{\otimes n}] \tag{2.39}$$

Proposition 2.4.2. *If $F \in \mathbb{D}_{1,2}$ then $\mathbb{E}[F|\mathcal{F}_s]1_{[0,s]}(t) \in \mathbb{D}_{1,2}$ and*

$$D_t(\mathbb{E}[F|\mathcal{F}_s]) = \mathbb{E}[D_tF|\mathcal{F}_s]1_{[0,s]}(t)$$

Corollary 2.4.3. *Let $u(s,\omega)$ be an $\mathcal{F}_s$-adapted process such that $u(s,\cdot) \in \mathbb{D}_{1,2}$ for all s. Then $D_tu(s,\omega)$ is $\mathcal{F}_s$-adapted for all t and*

$$D_tu(s,\omega) = 0 \quad \text{for } t > s$$

Our intuition from ordinary (non-stochastic) integration suggests that the integrand f from Eq. (2.38) should correspond to the "derivative" of F. The Clark-Ocone formula clarifies this intuition:

Theorem 2.4.4. *Let $F \in \mathbb{D}_{1,2}$ and suppose that W is an $\mathbb{F}$-Brownian motion. Then*

$$F = E[F] + \int_0^T E[D_tF|\mathcal{F}_t]dW_t \tag{2.40}$$

Proof. We represent F using the chaos expansion $F = \sum_{n=0}^{\infty} I_n(f_n)$ where $f_n \in \tilde{L}^2([0,T]^n)$ for each n. We have

$$\begin{aligned}
\int_0^T \mathbb{E}[D_tF|\mathcal{F}_t]dW(t) &= \int_0^T \mathbb{E}\left[D_t \sum_{n=0}^{\infty} I_n(f_n)|\mathcal{F}_t\right] dW(t) \\
&= \int_0^T \mathbb{E}\left[\sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t))|\mathcal{F}_t\right] dW(t)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \sum_{n=1}^{\infty} n\mathbb{E}[I_{n-1}(f_n(\cdot,t))|\mathcal{F}_t]dW(t) \\
&= \int_0^T \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t)1_{[0,t]}^{\otimes(n-1)}(\cdot))dW(t) \\
&= \int_0^T \sum_{n=1}^{\infty} n(n-1)!J_{n-1}(f_n(\cdot,t)1_{[0,t]}^{\otimes(n-1)}(\cdot))dW(t) \\
&= \sum_{n=1}^{\infty} n!J_n(f_n(\cdot)) = \sum_{n=1}^{\infty} I_n(f_n) = \sum_{n=0}^{\infty} I_n(f_n) - I_0(f_0) \\
&= F - \mathbb{E}[F]
\end{aligned}
$$

□

2.5 The forward integral

As we have seen in Section 2.3, the Skorohod integral coincides with the usual Ito integral for adapted integrands, and can thus be viewed as a generalization of the Ito integral. That is, if the process $u(t,\omega)$ is adapted and Skorohod integrable we have

$$\int_0^T u(t,\omega)\delta W(t) = \int_0^T u(t,\omega)dW(t)$$

Let us consider a common scenario in which the necessity of integrating a non-adapted process arises. We consider a setup in which a Brownian motion W is adapted to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ and a process ϕ_t is adapted to a strictly larger filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ (so that $\mathcal{G}_t \supset \mathcal{F}_t$ for each t). We would like to be able to interpret the integral given by

$$\int_0^T \phi_t dW_t$$

It cannot be understood as an Ito integral since the integrand ϕ_t is not adapted to the filtration generated by the Brownian motion. We already know that the Skohorod integral is defined for non-adapted integrands, and we shall now consider a closely related object known as the *forward integral*. The forward integral also allows for non-adapted integrands, and is particularly useful in the financial modelling of heterogeneously informed traders, as we shall see in later chapters. The mathematical theory of forward integration was largely developed in [33], [38], [39] and [40]. The approach we shall take is along the lines of [9], [19], [26] and [35].

Definition 2.5.1. *Let $\phi : [0,T] \times \Omega \to \mathbb{R}$ be a measurable process that is not necessarily adapted. The forward integral of ϕ with respect to W is defined by*

$$\int_0^T \phi(t,\omega)d^-W(t) = \lim_{\epsilon\to 0}\int_0^T \phi(t,\omega)\frac{W(t+\epsilon)-W(t)}{\epsilon}dt \tag{2.41}$$

if the limit exists in probability, in which case ϕ is said to be forward integrable in the weak sense.

The following property, which doesn't in general hold for Skorohod integrals, follows immediately from the above definition:

Lemma 2.5.2. *Suppose that the process ϕ is forward integrable and that G is a random variable. Then*

$$\int_0^T G\phi(t)d^-W(t) = G\int_0^T \phi(t)d^-W(t)$$

When dealing with càglàd stochastic processes, the forward integral has the following intuitive representation as a limit of Riemann sums:

Theorem 2.5.3. *Let $\phi : [0,T] \times \Omega \to \mathbb{R}$ be forward integrable and càglàd. Then*

$$\int_0^T \phi(t,\omega)d^-W(t) = \lim_{\Delta t\to 0}\sum_{j=0}^{N-1}\phi(t_j)\cdot\Delta W(t_j) \tag{2.42}$$

where the limit is taken in probability. Here $\Delta t = \max_{j=0,\dots,N-1}(t_{j+1}-t_j)$ and $\Delta W(t_j) = W(t_{j+1}) - W(t_j)$ for all $j = 0, 1, \dots, N-1$, and $0 = t_0 < t_1 < \dots < t_N = T$ is a partition of $[0,T]$.

Proof. Since any càglàd process can be approximated pointwise in ω and uniformly in t by a simple stochastic process, we may assume that ϕ is simple:

$$\phi(t,\omega) =: \phi(t) = \sum_{j=0}^{N-1}\phi(t_j)1_{(t_j,t_{j+1}]}(t)$$

Applying Def. 2.5.1 to the function ϕ above, we obtain

$$\begin{aligned}\int_0^T \phi(s)d^-W(s) &= \lim_{\epsilon\to 0}\int_0^T \phi(t)\frac{W(t+\epsilon)-W(t)}{\epsilon}dt \\ &= \sum_{j=0}^{N-1}\phi(t_j)\lim_{\epsilon\to 0}\int_{t_j}^{t_{j+1}}\frac{W(t+\epsilon)-W(t)}{\epsilon}dt \\ &= \sum_{j=0}^{N-1}\phi(t_j)\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_j}^{t_{j+1}}(\int_{u-\epsilon}^{u} dt)dW_u\end{aligned}$$

$$= \sum_{j=0}^{N-1} \phi(t_j) \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_j}^{t_{j+1}} \epsilon dW_u$$
$$= \sum_{j=0}^{N-1} \phi(t_j)(W(t_{j+1}) - W(t_j))$$

where the third line follows by considering the integration region as $\epsilon \to 0$ and applying a stochastic Fubini theorem. □

Remark 2.5.4. The above theorem motivates the use of the forward integral as a gains process in financial applications.

Another method that is often applied to solve stochastic integrals with non-adapted integrands involves a branch of stochastic analysis known as *enlargements of filtrations*, which will be described in more detail in Chapter 3. Again we consider the scenario described at the beginning of this section. Roughly speaking, the enlargement of filtrations approach involves finding conditions under which the $\mathcal{F}_t$-adapted Brownian motion W remains a semimartingale with respect to the larger filtration $\mathbb{G}$.

Theorem 2.5.5. *Let $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ denote a general filtration. Suppose that W is a standard Brownian motion on $(\Omega, \mathcal{F}, P)$ and that W is a semimartingale with respect to $\mathbb{G}$. Furthermore, suppose that the measurable process ϕ is adapted to $\mathbb{G}$ so that $\int_0^T \phi(t)dW(t)$ exists as an Ito integral. Then ϕ is forward integrable and we have*

$$\int_0^T \phi(t)dW(t) = \int_0^T \varphi(t)d^-W(t) \tag{2.43}$$

The above result essentially tells us that, whenever the enlargement of filtrations method can be applied to evaluate an integral with an anticipating integrand, it gives the same result as the forward integral (see the beginning of Chapter 5 for further details).

In order to equip ourselves to deal with some of the applications in later chapters, we will often work with the following modified definition of the forward integral:

Definition 2.5.6. *Let $\phi : [0,T] \times \Omega \to \mathbb{R}$ be a measurable process that is not necessarily adapted. We say that ϕ is forward integrable in the strong sense if*

$$\lim_{\epsilon \to 0} \int_0^T \phi(t,\omega) \frac{W(t+\epsilon) - W(t)}{\epsilon} dt$$

exists as a limit in $L^1(\Omega)$.

Note that forward integrability in the strong sense clearly implies forward integrability in the weak sense.

We shall now turn our attention to finding a relationship between the Skohorod integral and the forward integral.

Theorem 2.5.7. *Suppose that the measurable process $\phi : [0,T] \times \Omega \to \mathbb{R}$ is such that $\phi(t) \in \mathbb{D}_{1,2}$ for each $t \in [0,T]$ and*

$$\|\phi\|^2_{\mathbb{L}_{1,2}} := \mathbb{E}\left[\int_0^T |\phi(t)|^2 dt + \int_0^T \int_0^T |D_u\phi(t)|^2 du dt\right] < \infty$$

Furthermore, suppose that

$$D_{t^+}\phi(t) := \lim_{s \to t^+} D_s\phi(t)$$

exists for a.a. $t \in [0,T]$. Then ϕ is forward integrable (in the strong sense) and

$$\int_0^T \phi(t) d^- W(t) = \int_0^T \phi(t) \delta W(t) + \int_0^T D_{t^+}\phi(t) dt \tag{2.44}$$

Proof. See Lemma 2.2 of [26] for a sketch of the proof. □

Since the expectation of the Skorohod integral is zero (from Remark 2.3.6), we also have

Corollary 2.5.8. *If ϕ satisfies the conditions of the above theorem, then*

$$\mathbb{E}\left[\int_0^T \phi(t) d^- W(t)\right] = \mathbb{E}\left[\int_0^T D_{t^+}\phi(t) dt\right]$$

We conclude this section with some elementary results on *forward processes* (see [19]), in which we present a method of solution that is analogous to Ito's formula for ordinary diffusions.

Definition 2.5.9. *A (1-dimensional) forward process is a stochastic process $X(t) = X(t,\omega)$ of the form*

$$X(t) = x + \int_0^t u(s) ds + \int_0^t v(s) d^- W(s), \quad t \in [0,T] \tag{2.45}$$

where $u(s,\omega)$ and $v(s,\omega)$ are stochastic processes such that

$$\int_0^T |u(s)| ds < \infty \quad P - a.s.$$

and v is forward integrable.

We will generally use the following shorthand notation for a forward process:

$$d^-X(t) = u(t)dt + v(t)d^-W(t)$$

Theorem 2.5.10. *(Ito's formula for forward processes) Let*

$$d^-X(t) = u(t)dt + v(t)d^-W(t)$$

be a forward process. Let $f \in C^{1,2}([0,T] \times \mathbb{R})$ *and define*

$$Y(t) = f(t, X(t))$$

Then Y *is a forward process with*

$$dY(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))d^-X(t) + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(t, X(t))v^2(t)dt \qquad (2.46)$$

The forward Ito formula can be applied to a forward stochastic differential equation to obtain the following result:

Corollary 2.5.11. *Let* $u(t)$ *and* $v(t)$ *be measurable processes such that* $\int_0^t(|u(s)| + v^2(s))ds < \infty$ $P-a.s.$ *for* $t \in [0,T]$ *and* v *is forward integrable. Then the forward s.d.e.*

$$dX(t) = X(t)[u(t)dt + v(t)d^-W(t)]; \quad X(0) = x > 0$$

has the unique solution

$$X(t) = x\exp\left(\int_0^t (u(s) - \frac{1}{2}v^2(s))ds + \int_0^t v(s)d^-W(s)\right) \qquad (2.47)$$

Chapter 3

Enlargement of filtrations

Recall that Theorem 2.5.5 of the previous chapter stated that, if an $\mathbb{F}$-Brownian motion W can be expressed as a semimartingale with respect to a larger filtration $\mathbb{G}$, then the forward integral of a measurable $\mathcal{G}_t$-adapted process ϕ is given by

$$\int_0^T \phi(t)d^-W(t) = \int_0^T \phi(t)dW(t)$$

Let us assume that there exists a measurable process μ such that W has a semimartingale decomposition with respect to $\mathbb{G}$ of the form

$$W(t) = \tilde{W}(t) + \int_0^t \mu(t)dt$$

where $\tilde{W}$ is a $\mathbb{G}$-Brownian motion by Lévy's characterization theorem. Then we can naturally define

$$\int_0^T \phi(t)dW(t) = \int_0^T \phi(t)d\tilde{W}(t) + \int_0^T \phi(t)\mu(s)ds$$

so that each term on the RHS is well-defined in ordinary stochastic calculus.

Motivated by the above method of solving stochastic integrals with non-adapted integrands, this chapter aims to determine conditions under which $\mathbb{F}$-semimartingales remain semimartingales with respect to an enlarged filtration $\mathbb{G}$. We will be particularly concerned with finding the semimartingale decompositions in the enlarged filtration.

3.1 An introductory example

Perhaps the simplest example of an initial enlargement, which often features in the financial literature, is the case in which a Brownian filtration is enlarged by a future value of the Brownian motion. This oft-quoted result will be dealt with below as an introductory example (see [37] Ch. 6). Here $(\Omega, \mathcal{F}, P)$ is a probability space equipped with the filtration $\mathbb{F}$.

Theorem 3.1.1. (Anticipation of $W(T)$) *Suppose that W is an $\mathbb{F}$-Brownian motion and that $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is the smallest filtration such that W_1 is $\mathcal{G}_0$-measurable and $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \geq 0$. Then W is a $\mathcal{G}$-semimartingale. In this case*

$$M_t = W_t - \int_0^t \frac{W_1 - W_s}{1-s} ds \tag{3.1}$$

is a $\mathbb{G}$-martingale.

Proof. We follow [37]. Since W is a Brownian motion we have $\mathbb{E}[{W_t}^2] = t < \infty$ for all $t > 0$ and $\mathbb{E}[W_t] = 0$. We also know that W is an $\mathbb{F}$-martingale. Let $0 \leq s < t \leq 1$ be rational numbers such that $s = j/n$ and $t = k/n$. Now set

$$Y_i = W_{\frac{i+1}{n}} - W_{\frac{i}{n}} \tag{3.2}$$

We have the telescoping sums $\sum_{i=j}^{n-1} Y_i = W_{\frac{n}{n}} - W_{\frac{j}{n}} = W_1 - W_s$ and $\sum_{i=j}^{k-1} Y_i = W_t - W_s$. Thus we can write

$$\begin{aligned}
\mathbb{E}[W_t - W_s | W_1 - W_s] = \mathbb{E}\left[\sum_{i=j}^{k-1} Y_i | \sum_{i=j}^{n-1} Y_i\right] &= \sum_{i'=j}^{k-1} \mathbb{E}\left[Y_{i'} | \sum_{i=j}^{n-1} Y_i\right] \\
&= \sum_{i'=j}^{k-1} \left\{\frac{1}{n-j} \sum_{i=j}^{n-1} Y_i\right\} \\
&= \frac{k-j}{n-j} \sum_{i=j}^{n-1} Y_i \\
&= \frac{t-s}{1-s}(W_1 - W_s)
\end{aligned}$$

By the independence of the Brownian increments we then have $\mathbb{E}[W_t - W_s | \mathcal{G}_s] = \mathbb{E}[W_t - W_s | W_1 - W_s] = \frac{t-s}{1-s}(W_1 - W_s)$ for all rationals $0 \leq s < t \leq 1$. Since the W_t's are uniformly integrable and since the paths of W are right-continuous, we can take limits to show that the preceding result is also true for all reals.

$$\begin{aligned}
\mathbb{E}[M_t - M_s | \mathcal{G}_s] &= \mathbb{E}\left[(W_t - W_s) - \int_s^t \frac{W_1 - W_u}{1-u} du | \mathcal{G}_s\right] \\
&= \frac{t-s}{1-s}(W_1 - W_s) - \int_s^t \frac{1}{1-u} \mathbb{E}[W_1 - W_u | \mathcal{G}_s] du
\end{aligned}$$

by Fubini's theorem for conditional expectations. But note that

$$\begin{aligned}\mathbb{E}[W_1 - W_u|\mathcal{G}_s] &= \mathbb{E}[(W_1 - W_u) - (W_u - W_s)|\mathcal{G}_s] \\ &= W_1 - W_s - \frac{u-s}{1-s}(W_1 - W_s) \\ &= \frac{1-u}{1-s}(W_1 - W_s)\end{aligned}$$

So we have $\mathbb{E}[M_t - M_s|\mathcal{G}_s] = \frac{t-s}{1-s}(W_1 - W_s) - \int_s^t \frac{1}{1-s}(W_1 - W_s)du = 0$, from which it follows that M is a $\mathbb{G}$-martingale. □

3.2 Initial enlargements for general semimartingales

This section is largely based on [3]. Let $(\Omega, \mathcal{F}, P)$ be a probability space that is equipped with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. Let $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ denote the enlargement of $\mathbb{F}$ by an $\mathcal{F}_T$-measurable random variable X, where X takes values in the Polish space $(U, \mathcal{U})$. We have

$$\mathcal{G}_t = \bigcap_{s>t}(\mathcal{F}_s \vee \sigma(X)) \tag{3.3}$$

We begin by presenting two hypotheses that will be of great relevance throughout this chapter. They are both related to the conditional distributions of the additional information G given $\mathcal{F}_t$:

Hypothesis (AC)[1]: (absolute continuity) there is a σ-finite measure η on $(U, \mathcal{U})$ such that, for all $t \in [0, T)$,

$$P[G \in \cdot|\mathcal{F}_t] << \eta(\cdot) \quad \text{for } P-a.a.\ \omega \in \Omega \tag{3.4}$$

Hypothesis (E): (equivalence) there is a σ-finite measure η on $(U, \mathcal{U})$ such that, for all $t \in [0, T)$,

$$P[G \in \cdot|\mathcal{F}_t] \sim \eta(\cdot) \quad \text{for } P-a.a.\ \omega \in \Omega \tag{3.5}$$

We shall use the following notation: $\mathbb{F}^0 := (\mathcal{F}_t)_{t\in[0,T)}$ and $\mathbb{G}^0 := (\mathcal{G}_t)_{t\in[0,T)}$ (i.e. the right end-point is excluded); we also use $\tilde{\Omega} := \Omega \times U$, $\tilde{\mathcal{F}}_t := \bigcap_{\epsilon>0}(\mathcal{F}_{t+\epsilon} \otimes \mathcal{U})$ and $\tilde{\mathbb{F}}^0 := (\tilde{\mathcal{F}}_t)_{t\in[0,T)}$. Let us also recall the definitions of optional and predictable σ-fields.

[1] This first hypothesis is often referred to as Jacod's condition.

Definition 3.2.1. *The optional σ-algebra of $\mathbb{F}^0$, denoted by $\mathcal{O}(\mathbb{F}^0)$, is the σ-algebra, defined on $[0,T)\times\Omega$, that is generated by all $\mathcal{F}_t$-adapted càdlàg processes.*

Definition 3.2.2. *The predictable σ-algebra of $\mathbb{F}^0$, denoted by $\mathcal{P}(\mathbb{F}^0)$, is the σ-algebra, defined on $[0,T)\times\Omega$, that is generated by all $\mathcal{F}_t$-adapted left continuous processes.*

Supposing that Hypothesis (AC) holds, it is natural to investigate the properties of the density process implied by the absolute continuity assumption. The following lemma, originally formulated in Jacod [23], elucidates some of these properties:

Lemma 3.2.3. *There exists a non-negative càdlàg $\mathcal{O}(\tilde{\mathbb{F}}^0)$-measurable process $(\omega,l,t)\mapsto q_t^l(\omega)$ such that*

1. *for all $l\in U$, q^l is an $\mathbb{F}^0$-martingale, the processes q^l and q^l_- are strictly positive on $[[0,T^l))$, and $q^l=0$ on $[[T^l,T))$, where*
$$T^l := \inf\{t\geq 0 | q^l_{t-}=0\}\wedge T \tag{3.6}$$
2. *for all $t\in[0,T)$, the measure $q_t^l(\cdot)\eta(dl)$ on $(U,\mathcal{U})$ is a version of the conditional distribution $P[G\in dl|\mathcal{F}_t]$,*

In fact it is straightforward to show that the measure η of (AC) can be taken to be the *law* of G. We have

$$\int_B P[G\in dl] = P[G\in B] = P[G\in B|\mathcal{F}_0] = \int_B q_0^l\eta(dl)$$

That is, $q_0^l\eta(dl) = P[G\in dl]$. This means we can write

$$P[G\in B|\mathcal{F}_t](\omega) = \int_B p_t^l(\omega)P[G\in dl]$$

where
$$p_t^l(\omega) := \frac{q_t^l(\omega)}{q_0^l(\omega)} \tag{3.7}$$

Taking η to be the law of G thus amounts to using the p defined above for the process q of Lemma 3.2.3.

Let M be a continuous local $\mathbb{F}$-martingale. If we assume that (AC) holds, it can be shown that M is a semimartingale in the filtration $\mathbb{G}^0$. This is expressed in the following theorem, also originally from [23], in which the semimartingale decomposition of M is related to the density process q^l:

Theorem 3.2.4. *Suppose that (AC) holds and that M is a continuous local $\mathbb{F}$-martingale. Then there exists a $\mathcal{P}(\tilde{\mathbb{F}}^0)$-measurable function $(\omega, l, t) \mapsto \mu_t^l(\omega)$ with covariation*

$$\langle q^l, M \rangle = \int \mu^l q_-^l d\langle M \rangle \tag{3.8}$$

such that

1. $\int_0^t |\mu_s^G| d\langle M \rangle_s < \infty$ *$P-a.s.$ for all $t \in [0, T)$, and*
2. *M is a $\mathbb{G}^0$-semimartingale with decomposition*

$$M_t = \tilde{M}_t + \int_0^t \mu_s^G d\langle M \rangle_s \tag{3.9}$$

where $\tilde{M}$ is a continuous local $\mathbb{G}^0$-martingale.

Remark 3.2.5. Note the relevance to financial mathematics: under the assumption of no-arbitrage, the discounted price process of a stock $(S_t)_{t\geq 0}$ is a local martingale in the market filtration. The above theorem thus tells us that S_t remains a semimartingale from the point of view of a trader with additional information given by G, so that the gains process $\int_0^t \theta_u dS_u$ of the insider makes sense as an Ito integral. (Bear in mind, however, that the theorem only holds for $t \in [0, T)$, so that the final trading time T is excluded.)

We proceed by taking η to be the law of G, thus using p of Eq. (3.7) as our density process. Since $p^l > 0$ $P-a.s.$ on $[[0, T))$ by Lemma 3.2.3, the process $1/p^G$ is finite on the interval $[0, T)$. In the following theorem we shall define a new probability measure $\tilde{P}_t$ on $(\Omega, \mathcal{G}_t)$, such that $\tilde{P}_t = P$ on $\mathcal{F}_t$, and $\mathcal{F}_t$ and $\sigma(G)$ are independent under $\tilde{P}_t$. For this reason, $\tilde{P}_t$ is often referred to as a decoupling measure. It will be necessary for this result to assume Hypothesis (E): that the conditional laws of G given $\mathcal{F}_t$ are equivalent to the law of G for each $t \in [0, T)$.

Theorem 3.2.6. *If Hypothesis (E) holds then*

1. $\frac{1}{p^G}$ *is a $\mathbb{G}^0$-martingale.*
2. *For $t \in [0, T)$, the probability measure defined by*

$$\tilde{P}_t(A) := \int_A \frac{1}{p_t^G} P[G \in dl] \quad \text{for } A \in \mathcal{G}_t \tag{3.10}$$

has the following properties:

(a) $\mathcal{F}_t$ and $\sigma(G)$ are independent under $\tilde{P}_t$
(b) $\tilde{P}_t = P$ on $(\Omega, \mathcal{F}_t)$.

Proof. See Proposition 2.3 of [3] or Proposition 1.6 of [1]. □

Due to the content of the following theorem, we refer to the measure $\tilde{P}_t$ as the $[0, t]$-*martingale preserving measure under initial enlargement.*

Theorem 3.2.7. *Suppose that (E) holds. Then, for* $t \in [0, T)$, *we have that any (local)* $(P, \mathbb{F})$-*martingale* M *on* $[0, t]$ *is a (local)* $(\tilde{P}_t, \mathbb{G})$-*martingale on* $[0, t]$.

Proof. See Proposition 2.5 of [3] or Proposition 1.7 of [1]. □

Until now we have assumed that M is a continuous local $(P, \mathbb{F})$-martingale. Instead, let us now consider a general $\mathcal{F}_t$-adapted stochastic process S and assume the existence of a measure $Q \sim P$ on $(\Omega, \mathcal{F}_T)$ such that S is a local $(Q, \mathbb{F})$-martingale. Let $Z^{\mathbb{F}}$ denote the density process of Q with respect to P. We then have the following result, which follows directly from Theorem 3.2.6:

Corollary 3.2.8. ([2] Th. 3.1) *If Hypothesis (E) holds then*

1. $Z^{\mathbb{G}} := \frac{Z^{\mathbb{F}}}{p^G}$ *is a* $\mathbb{G}^0$-*martingale.*

2. *For* $t \in [0, T)$ *the* martingale preserving probability measure, *defined by*

$$\tilde{Q}_t(A) := \int_A \frac{Z^{\mathbb{F}}}{p_t^G} P[G \in dl] \quad \text{for } A \in \mathcal{G}_t \tag{3.11}$$

has the following properties:

(a) $\mathcal{F}_t$ *and* $\sigma(G)$ *are independent under* $\tilde{Q}_t$
(b) $\tilde{Q}_t = Q$ *on* $(\Omega, \mathcal{F}_t)$ *and* $\tilde{Q}_t = P$ *on* $(\Omega, \sigma(G))$.

Corollary 3.2.9. *If (E) holds, then any (local)* $(Q, \mathbb{F})$-*martingale* S *on* $[0, t]$ *is a (local)* $(\tilde{Q}_t, \mathbb{G})$-*martingale on* $[0, t]$, *with* $\tilde{Q}_t$ *defined by Eq. (3.11).*

This important result tells us that the local martingale property is preserved under the initial enlargement as long as we simultaneously change from the measure Q to the decoupling measure $\tilde{Q}$.

3.3 Martingale representation theorems under initial enlargements

We shall now concentrate our efforts on finding explicit representations for the processes p^l and $1/p^G$. Specifically, it will be shown that, under Hypothesis (E), p^l and $1/p^G$ can be expressed as Dolean's exponentials.

Proposition 3.3.1.

1. *Suppose that Hypothesis (E) holds. Then there is a local $\mathbb{G}^0$-martingale $\tilde{N}$, with $\tilde{N}_0 = 0$, which is orthogonal to $\tilde{M}$ and such that*

$$\frac{1}{p_t^G} = \mathcal{E}\left(-\int \mu_s^G d\tilde{M}_s + \tilde{N}\right)_t \tag{3.12}$$

2. *Fix $l \in U$. If $p_{T^l}^l > 0$ $P-a.s.$, then there exists a local $\mathbb{F}^0$-martingale N^l, with $\tilde{N}_0^l = 0$, which is orthogonal to M and such that*

$$p_t^l = \mathcal{E}\left(\int \mu_s^l dM_s + N^l\right)_t \tag{3.13}$$

Proof. See Proposition 2.9 of [3]. □

We conclude this short section with a result that enables us to express the process $\tilde{N}$ in the above proposition in terms of the process L, under the assumption that the density p^l is continuous and strictly positive for all $l \in U$. This enables us to remove one of the unknowns in the above representations.

Corollary 3.3.2. *If p^l is continuous and strictly positive for all $l \in U$, then*

$$\frac{1}{p_t^G} = \mathcal{E}\left(-\int \mu_s^G d\tilde{M}_s - N^G + \langle N^G \rangle\right)_t \tag{3.14}$$

That is, we have $\tilde{N}_t = -N_t^G + \langle N^G \rangle_t$.

Proof. See Corollary 2.10 of [3]. □

Remark 3.3.3. Note that Corollary 3.3.2 clearly implies that if the density process can be expressed as $p^l = \mathcal{E}(\int \mu^l dM)$ then

$$\frac{1}{p_t^G} = \mathcal{E}\left(-\int \mu^G d\tilde{M}\right)_t$$

and we don't have to worry about finding N or $\tilde{N}$. We shall see in Chapter 5 how this type of representation may be interpreted as a completeness assumption in finance.

3.4 Initial enlargements in a Brownian world

Until now we have been concerned with the question of whether continuous local martingales remain martingales in an enlarged filtration. In this section, we assume that our probability space is equipped with the filtration $\mathbb{F}$ generated by the Brownian motion W. We will consider whether W remains a semimartingale in the initially enlarged filtration $\mathbb{G}$. We shall see that, in a Brownian world, we are able to derive some convenient expressions for the information drift associated with the enlargement.

3.4.1 The role of Malliavin's calculus

The results in this section are based on [20] and [21], in which Peter Imkeller and others study the problem of initial enlargements in a Brownian setting. After developing a measure-valued Malliavin calculus, the associated Clark-Ocone formula is used to find an expression for the information drift which doesn't depend on the reference measure η of Hypothesis (AC). This is coupled with an extension of Hypothesis (AC) which also doesn't depend on η. Our treatment is largely heuristic.

We work with a probability space $(\Omega, \mathcal{F}, P)$ that supports a Brownian motion W with natural filtration $\mathbb{F}$. Again $\mathbb{G}$ denotes the filtration obtained by enlarging $\mathbb{F}$ with the $\mathcal{F}_T$-measurable random variable G. For now, we assume that Hypothesis (AC) holds and denote by p^l the density process of the conditional laws with respect to the law of G, as in the previous section. It has been shown previously that p^l is an $\mathbb{F}$-martingale, so that we may invoke the martingale representation theorem to write

$$p_t^l = p_0^l + \int_0^t k_u^l dW_u \tag{3.15}$$

where k^l is $\mathcal{F}_t$-adapted. Our objective at this point is to find an expression for the information drift in terms of the kernels k.

Theorem 3.4.1. *Suppose that Hypothesis (AC) holds. Then W is a semimartingale in the enlarged filtration $\mathbb{G}$ with decomposition*

$$W_t = \tilde{W}_t + \int_0^t \mu_s^G ds$$

where $\tilde{W}$ is a $\mathbb{G}$-Brownian motion and

$$\mu_t^G = \frac{k_t^l}{p_t^l}|_{l=G} = \frac{\frac{d}{dt}\langle p^l, W\rangle_t}{p_t^l}|_{l=G} \tag{3.16}$$

as long as μ_t^G satisfies

$$\int_0^T |\mu_s^G| ds < \infty \quad P-a.s. \tag{3.17}$$

Proof. See Theorem 3.2 of [20]. □

The above representation enables us to determine the semimartingale decomposition of W in $\mathbb{G}$ for situations in which (AC) is satisfied. We need to bear in mind, however, that while (AC) is sufficient to guarantee that W is a semimartingale in $\mathbb{G}$, it is not necessary. That is, if it is found that Hypothesis (AC) does *not* hold for a particular G, then we still cannot rule out the possibility that W is a semimartingale in $\mathbb{G}$. It is shown in [20], [21] and [31] that if the additional information is given by the supremum of the Brownian motion over $[0, T]$ (i.e. $G = \sup_{t \in [0,T]} W_t$), then in fact Hypothesis (AC) is *not* satisfied.

We shall now concern ourselves with trying to find an extension of Hypothesis (AC) in order to be able to study forms of G for which (AC) isn't true. For this we need to recall the Clark-Ocone formula of Theorem 2.4.4. Under certain conditions on G it can be shown that the density process of Eq. (3.7) may be expressed as

$$p_t^l = p_0^l + \int_0^t D_u p_u^l dW_u \tag{3.18}$$

i.e. the kernel from Eq. (3.15) is given by $k_t^l = D_t p_t^l$. This allows us to rewrite the information drift as

$$\mu_t^G = \frac{k_t^l}{p_t^l}|_{l=G} = \frac{D_t p_t^l}{p_t^l}|_{l=G} = D_t \ln p_t^l|_{l=G} \tag{3.19}$$

Now under the assumption of certain smoothness properties, allowing for the interchange of the derivative operators D_t and d/dt, we would like to be able to write

$$\frac{D_t p_t^l}{p_t^l} = \frac{D_t \frac{dP_t^G}{dP^G}}{\frac{dP_t^G}{dP^G}} = \frac{dD_t P_t^G}{dP_t^G} \tag{3.20}$$

Recall that Jacod's condition requires the conditional laws to be absolutely continuous with respect to a measure η, where η *may* be taken as the law of G. But η is not in fact *required* to be the law of G and in some instances a simpler measure may be chosen. It can be seen that the expression for the information drift given by Eq. (3.20) does not depend on η in any way; in this sense it is more general than (AC) and should be applicable to a wider class of G. We should note however that, although it adds more generality to Jacod's condition, Eq. (3.20) introduces some new problems that need to be tackled. Note that $D_t P_t^G$ and P_t^G are in fact *measure-valued* random variables. In order to understand how $D_t P_t^G$ functions as a measure-valued random variable, and to study the question of whether $D_t P_t^G$ is equivalent or absolutely continuous with respect to P_t^G, we need some understanding of a *measure-valued Malliavin's calculus.* We provide a brief survey of key features.

Let $\mathcal{M}$ denote the space of signed measures on $(\mathbb{B}, \mathcal{B}(\mathbb{R}))$. The total variation of a signed measure $\mu \in \mathcal{M}$ is defined by

$$|\mu|(A) := \sup\left\{\sum_n |\mu(A_n)| : A_n \text{ disjoint with } \bigcup_n A_n = A\right\}$$

Note that $\mathcal{M}$ becomes a Banach space when equipped with the supremum norm, given by

$$\|\mu\| = \sup\left\{\int f d\mu : f \in C_b(\mathbb{R}), \|f\| \leq 1\right\}$$

For $\mu \in \mathcal{M}$, $f \in C_b(\mathbb{R})$ we will write $\langle \mu, f\rangle$ instead of $\int f d\mu$. It can in fact be shown that $\mathcal{M}$ is separable, from which it follows that $\mathcal{M}$ can be embedded into the space of real-valued sequences $\mathbb{R}^{\mathbb{N}}$. [20] and [21] construct such an embedding as follows: they fix a countable dense subset $\{f_i : i \in I\}$ of $C_b(\mathbb{R})$ and define the map

$$\begin{aligned} \Phi : \mathcal{M} &\rightarrow \mathbb{R}^{\mathbb{N}} \\ \mu &\mapsto (\langle \mu, f_i\rangle)_{i\in\mathbb{N}} \end{aligned}$$

Recall that in Chapter 2 we studied Malliavin calculus for real-valued random variables. The map Φ is the key to adapting the definitions and results of Chapter 2 so that they can be applied to measure-valued random variables. For $F : \Omega \rightarrow \mathcal{M}$ (i.e. each $F(\omega)$ is a signed measure) we have that each $\langle F, f_i\rangle$ is a real-valued random variable, so that $D\langle F, f_i\rangle$ is already understood provided $\langle F, f_i\rangle \in \mathbb{D}_{1,2}$ (see Def. 2.2.7). A natural way to define the measure-valued Malliavin derivative would thus be to use the inverse map Φ^{-1}:

$$DF = \Phi^{-1}\left((D\langle F, f_i\rangle)_{i\in\mathbb{N}}\right) \tag{3.21}$$

Similarly, we could define the Ito integral of the measure-valued stochastic process $(F_t)_{t\geq 0}$ by

$$\int_0^T F_t dW_t = \Phi^{-1}\left(\left(\int_0^T \langle F_t, f_i\rangle dW_t\right)_{i\in\mathbb{N}}\right) \tag{3.22}$$

Remark 3.4.2. [20] and [21] do not actually *define* the derivative by Eq. (3.21). Instead, they define a set of measure-valued *smooth cylinder functions* $\mathcal{S}(\mathcal{M})$ analogously to the way we have defined $\mathcal{S}$ for real-valued random variables in Eq. (2.15). They then define a derivative and a norm on $\mathcal{S}(\mathcal{M})$ that are analogous to Equations (2.16) and (2.19) respectively. The space $\mathbb{D}_{1,2}(\mathcal{M})$ is then defined as the closure of $\mathcal{S}(\mathcal{M})$ with respect to the norm.

The most important feature for our purposes is that a version of the Clark-Ocone formula holds in the measure-valued Malliavin calculus. We have

Theorem 3.4.3. *Let $F \in \mathbb{D}_{1,2}(\mathcal{M})$. Then*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_s F|\mathcal{F}_s]dW_s \tag{3.23}$$

Proof. See Theorem A.1 of [21]. □

This enables us to find a new formulation of the information drift which is based on the measure-valued Malliavin derivative of the conditional law of G:

Theorem 3.4.4. *Suppose that*

$$\sup_{f \in C_b(\mathbb{R}), \|f\| \leq 1} \mathbb{E}\left[\int_0^t \langle D_s P_s^G(\cdot, dx), f\rangle^2 ds\right] < \infty \tag{3.24}$$

Define

$$\mu_t^G := \frac{dD_t P_t^G(\cdot, dl)}{dP_t^G(\cdot, dl)}(l) \tag{3.25}$$

and suppose that $\int_0^t |\mu_s^G| ds < \infty$ P-a.s.. Then the $\mathbb{F}$-Brownian motion W is a $\mathbb{G}$-semimartingale with decomposition

$$W_t = \tilde{W}_t + \int_0^t \mu_s^G ds$$

where $\tilde{W}$ is a $\mathbb{G}$-Brownian motion.

Proof. See Theorem 2.1 of [21]. □

Recall that the motivation behind the development of the measure-valued Malliavin calculus was to extend our previous results to enable us to study forms of inside information G which don't satisfy Hypothesis (AC). The following results, taken from [21], give an indication of the additional generality achieved.

Theorem 3.4.5. *Suppose that*

$$\mathbb{E}\left[\int_0^T (\mu_t^G)^2 dt\right] < \infty$$

where μ_t^G is as defined in Eq. (3.25). Then Hypothesis (AC) holds.

Theorem 3.4.6. *Suppose that*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T (\mu_t^G)^2 dt\right)\right] < \infty$$

where μ_t^G is as defined in Eq. (3.25). Then Hypothesis (E) holds.

3.4.2 Yor's approach

In this section we obtain a useful decomposition formula based on the results of [31] and [42], using an approach that is not explicitly based on Jacod's criterion.

Let X be an $\mathcal{F}_\infty$-measurable random variable and let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Borel function. As usual, we denote by $\mathbb{G}$ the initial enlargement of $\mathbb{F}$ by X. Let the process $\lambda(f)$ be a continuous version of the $\mathbb{F}$-martingale $(\mathbb{E}[f(X)|\mathcal{F}_t])_{t\geq 0}$. Due to the martingale representation theorem, there is a unique predictable process $\dot{\lambda}(f)$ such that

$$\lambda_t(f) = \mathbb{E}[f(X)] + \int_0^t \dot{\lambda}_u(f) dW_u$$

Let $\lambda_t(dx)$ denote the family of regular conditional distributions of X with respect to $\mathcal{F}_t$. We then have

$$\lambda_t(f) = \int f(x)\lambda_t(dx)$$

We continue by making the following assumption:

Assumption 3.4.7. There is a family of measures $\dot{\lambda}_t(dx)$ such that

$$\dot{\lambda}_t(f) = \int f(x)\dot{\lambda}_t(dx) \quad t - a.e.$$

Remark 3.4.8. In fact, Section 6.3 of [30] shows that the above assumption is true if the conditional law process satisfies a weak condition of differentiability. Specifically, Theorem 6.7 of the afore-mentioned text uses measure-valued Malliavin calculus to prove that Assumption 3.4.7 is true as long as the conditional law process satisfies $\lambda_t(dx) \in \mathbb{D}_{1,2}(\mathcal{M})$.

In what follows, we shall also make two further assumptions:

Assumption 3.4.9.

$$\dot{\lambda}_t(dx) << \lambda_t(dx) \quad dt \times dP - a.e.$$

with

$$\rho(x,t) := \frac{\dot{\lambda}_t(dx)}{\lambda_t(dx)}$$

Assumption 3.4.10.

$$\int_0^t |\rho(X,u)||d\langle M, W\rangle_u| < \infty \quad a.s.\ for\ t \geq 0 \tag{3.26}$$

We shall now proceed to derive a semimartingale decomposition formula for an $\mathbb{F}$-martingale M.

Theorem 3.4.11. *Assume that Assumptions 3.4.7, 3.4.9 and 3.4.10 are satisfied. Then, if M is an $(\mathbb{F}, P)$-martingale, there exists a $(\mathbb{G}, P)$-local martingale $\tilde{M}$ such that*

$$M = \tilde{M} + \int_0^{\cdot} \rho(X, u) d\langle M, W\rangle_u \tag{3.27}$$

We obtain the following obvious result by setting $M = W$ and using Levy's characterization theorem:

Corollary 3.4.12. *Under the assumptions of Theorem 3.4.11, for any $\mathbb{F}$-Brownian motion W there is a $\mathbb{G}$-Brownian motion $\tilde{W}$ such that W decomposes as*

$$W = \tilde{W} + \int_0^{\cdot} \rho(X, s) ds \tag{3.28}$$

provided that $\int_0^t |\rho(X, s)| ds < \infty$ a.s. for all $t \geq 0$.

We now show that the assumptions of Theorem 3.4.11 hold true if the regular conditional probabilities have densities (with respect to Lebesgue measure) of a specified form. This result will allow us to calculate decomposition formulae for certain important examples.

Corollary 3.4.13. *Assume that $\lambda_t(dx) = \phi(t, x)dx$, where ϕ may be written as*

$$\phi(t, x) = \phi(0, x) \exp\left(\int_0^t \rho(x, s) dW_s - \frac{1}{2}\int_0^t \rho(x, s)^2 ds\right) \tag{3.29}$$

Then we have $\dot{\lambda}_t(dx) = \rho(x, t)\lambda_t(dx)$, so that the decomposition of Theorem 3.4.11 applies.

We are now in a position to generalize the example $X = W_T$ presented at the beginning of this chapter (we follow 12.1.2 of [42]).

Example 3.4.14. *We consider an enlargement by the random variable $X = \int_0^\infty \psi(s) dW_s$, where ψ is a deterministic square-integrable function. Now*

$$\mathbb{E}[\int_0^\infty \psi(s) dW(s) | \mathcal{F}_t] = \int_0^t \psi(s) dW(s)$$

and

$$\begin{aligned} \mathbb{E}[(X - \mathbb{E}[X])^2 | \mathcal{F}_t] &= \mathbb{E}[(\int_t^\infty \psi(s) dW(s))^2 | \mathcal{F}_t] \\ &= \mathbb{E}[(\int_t^\infty \psi(s) dW(s))^2] = \int_t^\infty \psi^2(s) ds \end{aligned}$$

so that, conditionally on $\mathcal{F}_t$, X is Gaussian with mean and variance given by $m_t = \int_0^t \psi(s)dW(s)$ and $\sigma_t^2 = \int_t^\infty \psi^2(s)ds$. We would like to find $\rho(x,s)$ such that

$$\frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{(x-m_t)^2}{2\sigma_t^2}\right) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-x^2/(2\sigma_0^2)} \exp\left(\int_0^t \rho(x,s)dW_s - \frac{1}{2}\int_0^t \rho(x,s)^2 ds\right)$$

If we denote the LHS by M_t and apply Ito's formula we obtain $dM_t = \rho(x,t)M_t dW_t$, where

$$\rho(x,t) = \frac{x-m_t}{\sigma_t^2}\psi(t)$$

and the well-known solution is given by

$$M_t = M_0 \exp\left(\int_0^t \rho(x,s)dW_s - \frac{1}{2}\int_0^t \rho^2(x,s)ds\right)$$

which is what we wanted. Under suitable integrability conditions we can then apply Corollary 3.4.13 to obtain

$$W_t = \tilde{W}_t + \int_0^t \frac{\psi(s)}{\sigma_s^2}\left(\int_s^\infty \psi(u)dW_u\right) ds \tag{3.30}$$

where $\tilde{W}$ is a $\mathbb{G}$-Brownian motion.

Note that we can use this to retrieve the result obtained in Theorem 3.1.1 by setting $\psi(t) = 1_{[0,T]}$, in which case $X = \int_0^\infty \psi(t)dW_t = W_T$, and

$$W_t = \tilde{W}_t + \int_0^{t\wedge T} \frac{W_T - W_s}{T-s} ds$$

as before.

Part II

Financial applications

Chapter 4

Development of the market model

We have mentioned that anticipating calculus and enlargements of filtrations both play a role in the modelling of markets with differential information. In this chapter we shall mainly focus on the tools of anticipating calculus to build a general market model which accounts for the existence of heterogeneously informed traders. We shall deal not only with insiders, but also with *partially informed traders* who have less information at their disposal than the typical honest trader. This chapter is largely based on the results of [9], [26], [28] and [35]. We begin by citing some important motivations for using the forward integral in the modelling of insider trading (see [9]).

(i) The forward integral can be interpreted as the limit of Riemann sums for càglàd integrands (see Theorem 2.5.3).

(ii) Consider the simple buy-and-hold trading strategy given by $\pi(t,\omega) = 1_{[\tau_1,\tau_2]}(t)$, where τ_1 and τ_2 are random times. That is, the trader invests all money in stocks between the times τ_1 and τ_2 and puts all of the money in a bank account otherwise. Using Theorem 2.5.3, it is straightforward to show that the gains from trade are given by

$$G = \int_0^T \pi(t)d^-S(t) = \lim_{\Delta t_j \to 0} \sum \pi(t_j)\Delta S(t_j) = \int_{\tau_1}^{\tau_2} dS(t) = S(\tau_2) - S(\tau_1)$$

Clearly this is what we expect from the buy-hold-strategy, i.e. the gains are given by the change in price between τ_1 and τ_2.

(iii) It is known that the no-arbitrage property is closely related to the assumption that the stock price is a semimartingale (see Delbaen and Schachermeyer [12]). If the stock price happens to be a semimartingale with respect to the insider's filtration $\mathbb{H}$ then Theorem 2.5.5 tells us that

$$\int_0^T \pi(t)dS(t) = \int_0^T \pi(t)d^-S(t)$$

where π (assumed to be $\mathcal{H}_t$-adapted) denotes the proportion of wealth invested by the insider in the risky asset. Thus the forward integral coincides with the ordinary stochastic integral in the general semimartingale framework and we can always interpret the gains process as $\int_0^T \pi(t)d^- S(t)$.

4.1 Formulation of the optimal portfolio problem

As usual, we work on a probability space $(\Omega, \mathcal{F}, P)$ equipped with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ generated by the Brownian motion $(W_t)_{t\in[0,T]}$. Let $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ be another filtration such that

$$\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F} \qquad \text{for all } t \in [0,T] \tag{4.1}$$

where T is a fixed time that denotes the end of the trading interval. For simplicity we consider a market with one riskless asset and only one risky asset. That is, it is possible to invest in a bond, with dynamics given by

$$dB(t) = r(t)B(t)dt; \qquad B(0) = 1 \tag{4.2}$$

and a stock, with dynamics given by

$$dS(t) = S(t)[\mu(t)dt + \sigma(t)d^- W(t)]; \qquad S(0) > 0 \tag{4.3}$$

In order to model an anticipating environment, in which the above coefficients are not necessarily adapted to the Brownian filtration $\mathbb{F}$. we make the following assumption:

Assumption 4.1.1.

1. $r(t)$, $\mu(t)$ and $\sigma(t)$ are $\mathcal{G}_t$-adapted
2. $\mathbb{E}[\int_0^T \{|r(t)| + |\mu(t)| + \sigma^2(t)]\}dt < \infty \ P - a.s.$

The Brownian term in Eq. (4.3) represents a forward integral since the volatility $\sigma(t)$ is not necessarily adapted to the Brownian filtration $\mathbb{F}$ (and we have made no assumption that W remains a semimartingale in $\mathbb{G}$).

Remark 4.1.2. In [9] it is noted that the anticipating term "models a market which is influenced by large investors with inside information". The standard meaning of the term *large* in the related literature signifies that the actions of the insider have an influence over the price dynamics. We shall see in Chapter 6 how the coefficients may be anticipating if there is a large insider acting on the market.

Let the filtration $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ denote the filtration available to the investor. In accordance with the approach taken in [26], we do not assume any specific relationship between $\mathbb{H}$ and the filtrations $\mathbb{F}$ and $\mathbb{G}$. Thus we have a general setup in which it is not yet specified whether the investor is an insider or a partially informed trader.

As usual, we characterize the investment strategy by the proportion of wealth invested in the stock, denoted by $\pi(t)$. The wealth process of the portfolio is given by

$$V(t) = p_0(t)B(t) + p_1(t)S(t) \tag{4.4}$$

where p_0 and p_1 denote the number of bonds and stocks held in the portfolio. We will impose a condition that is analogous to the usual self-financing condition, given by

$$dV(t) = p_0(t)dB(t) + p_1(t)dS(t) \tag{4.5}$$

Using $X(t) = e^{-\int_0^t r(s)ds}V(t)$ and $\pi(t) = p_1(t)S(t)/V(t)$ yields the following equation for the discounted wealth process:

$$dX(t) = X(t)[(\mu(t) - r(t))\pi(t)dt + \pi(t)\sigma(t)d^-W(t)] \tag{4.6}$$

It is appropriate at this point to define the set of admissible portfolios $\mathcal{A}_{\mathbb{H}}$. Mathematically, $\mathcal{A}_{\mathbb{H}}$ is essentially just a set of $\mathcal{H}_t$-adapted processes such that Eq. (4.6) has a strong $\mathcal{H}_t$-adapted solution.

Definition 4.1.3. (Admissible portfolios) $\mathcal{A}_{\mathbb{H}}$ *denotes the class of portfolios* $\pi = \{\pi_t, t \in [0,T]\}$ *such that*

1. π *is càglàd and* $\mathcal{H}_t$*-adapted*
2. π *satisfies*

$$\mathbb{E}\left[\int_0^T \left(|\mu(t) - r(t)||\pi(t)| + \sigma^2(t)\pi^2(t)\right) dt\right] < \infty$$

3. *the product* $\pi\sigma$ *is càglàd and forward integrable.*

If $\pi \in \mathcal{A}_{\mathbb{H}}$, then the forward s.d.e. given by Eq. (4.6) can be solved using Corollary 2.5.11, which yields the terminal value of the discounted wealth process to be

$$X(T) = X(0)\exp\Big(\int_0^T \left[(\mu(t) - r(t))\pi(t) - \frac{1}{2}\pi^2(t)\sigma^2(t)\right] dt + \int_0^T \pi(t)\sigma(t)d^-W(t)\Big) \tag{4.7}$$

We assume that the investor will choose his portfolio with the aim of maximizing his expected terminal utility. Mathematically, it is generally required that a utility function is continuously differentiable, concave and non-decreasing:

Definition 4.1.4. *A utility function is a strictly increasing, strictly concave and twice continuously differentiable function*

$$U : (0, \infty) \to \mathbb{R}$$

which satisfies

$$\lim_{x \to \infty} U'(x) = 0, \quad \lim_{x \to 0^+} U'(x) = \infty$$

In this thesis we shall only consider the logarithmic utility function, which is the most widely used and natural choice in the literature. Financially, this choice is appropriate since it represents a trader's returns. Using a logarithmic utility function is also mathematically convenient as it removes the abundance of exponential functions that arise in the solution of the s.d.e.[1] We now define

$$\begin{aligned} J(\pi) &:= \mathbb{E}[\log X(T)] - \log X(0) \\ &= \mathbb{E}\left[\int_0^T \left[(\mu(t) - r(t))\pi(t) - \frac{1}{2}\pi^2(t)\sigma^2(t)\right] dt + \int_0^T \pi(t)\sigma(t) d^- W_t\right] \end{aligned} \tag{4.8}$$

That is, $J(\pi)$ represents the expected change in logarithmic utility due to the investment strategy π. The task of finding an optimal portfolio can thus be expressed as follows (see [9] Problem 3.3):

PROBLEM 4.1.5. *Find* $\pi^* \in \mathcal{A}_{\mathbb{H}}$ *and* $J(\pi^*) \in \mathbb{R}$ *such that*

$$J(\pi^*) = \sup_{\pi \in \mathcal{A}_{\mathbb{H}}} J(\pi) \tag{4.9}$$

π^* *is referred to as the* optimal portfolio *(if it exists).*

In the following section we present several conditions on a portfolio $\pi \in \mathcal{A}_{\mathbb{H}}$ that are equivalent to π being an optimal portfolio. We also use one of these characterizations to find explicit formulae for both π^* and $J(\pi^*)$.

4.2 Characterization of the optimal portfolio

We begin with the characterization theorem of [26], which provides several conditions on an admissible portfolio π that are equivalent to π being an optimal portfolio.

Theorem 4.2.1. (Characterization Theorem) *The following statements are equivalent:*

[1] Refer to [9] for a treatment of the generalized utility function and see Section 13 of [4] for a discussion of the pertinence of using a logarithmic utility function.

(i) *An optimal portfolio* $\pi^* \in \mathcal{A}_{\mathbb{H}}$ *exists for Problem (4.1.5) ;*
(ii) *There exists* $\pi^* \in \mathcal{A}_{\mathbb{H}}$ *such that*

$$M_{\pi^*}(t) := \mathbb{E}\left[\int_0^t (\mu(s) - r(s) - \sigma^2(s)\pi^*(s))ds + \int_0^t \sigma(s)d^- W(s)|\mathcal{H}_t\right] \tag{4.10}$$

is an $\mathbb{H}$*-martingale;*
(iii) *The function*

$$s \mapsto \mathbb{E}\left[\int_0^s \sigma(u)d^- W(u)|\mathcal{H}_t\right]; \quad s > t \tag{4.11}$$

is absolutely continuous and there exists $\pi^* \in \mathcal{A}_{\mathbb{E}}$ *such that for a.a.* t, ω

$$\frac{d}{ds}\mathbb{E}\left[\int_0^s \sigma(u)d^- W(u)|\mathcal{H}_t\right] = -\mathbb{E}[\mu(s) - r(s) - \sigma^2(s)\pi^*(s)|\mathcal{H}_t] \tag{4.12}$$

Proof. We follow [26]. (i)⇒(ii): Suppose (i) holds. Then we have

$$J(\pi^*) \geq J(\pi^* + \delta\beta)$$

for all $\beta \in \mathcal{A}_{\mathbb{H}}$ and $\delta \in \mathbb{R}$. Since $J(\cdot)$ attains its maximum value at π^*, we have

$$\frac{d}{d\delta}J(\pi^* + \delta\beta)|_{\delta=0} = 0$$

Using the definition of J and performing simple differentiation, we thus have

$$\mathbb{E}\left[\int_0^T (\mu(u) - r(u) - \sigma^2(u)\pi^*(u))\beta(u)du + \int_0^T \beta(u)\sigma(u)d^- W(u)\right] = 0 \tag{4.13}$$

for all $\beta \in \mathcal{A}_{\mathbb{H}}$. Let us choose

$$\beta(u) = \beta_0(t)1_{[t,s]}(u)$$

where $0 \leq t \leq s \leq T$ and $\beta_0(t)$ is $\mathcal{H}_t$-measurable and bounded. Inserting this $\beta(u)$ into Eq. (4.13) gives

$$\mathbb{E}\left[\left(\int_t^s (\mu(u) - r(u) - \sigma^2(u)\pi^*(u))du + \int_t^s \sigma(u)d^- W(u)\right)\beta_0(t)\right] = 0 \tag{4.14}$$

from which we obtain

$$\mathbb{E}\left[\left(\int_t^s (\mu(u) - r(u) - \sigma^2(u)\pi^*(u))du + \int_t^s \sigma(u)d^- W(u)\right)|\mathcal{H}_t\right] = 0 \tag{4.15}$$

since Eq. (4.14) holds for all $\mathcal{H}_t$-measurable $\beta_0(t)$. This can be written as

$$\mathbb{E}[K_{\pi^*}(s) - K_{\pi^*}(t)|\mathcal{H}_t] = 0$$

where we define

$$K_{\pi^*}(t) := \int_0^t (\mu(u) - r(u) - \sigma^2(u)\pi^*(u))du + \int_0^t \sigma(u)d^- B(u)$$

For $s \geq t$ we thus obtain

$$\begin{aligned}
\mathbb{E}[M_{\pi^*}(s)|\mathcal{H}_t] &= \mathbb{E}[\mathbb{E}[K_{\pi^*}(s)|\mathcal{H}_s]|\mathcal{H}_t] \\
&= \mathbb{E}[K_{\pi^*}(s)|\mathcal{H}_t] \quad \text{(tower property)} \\
&= \mathbb{E}[K_{\pi^*}(t)|\mathcal{H}_t] = M_{\pi^*}(t)
\end{aligned}$$

Thus $M_{\pi^*}(t)$ is an $\mathbb{H}$-martingale and we have (ii).
(ii)⇒(iii): Suppose (ii) holds. Then, for $s \geq t$, we have

$$\begin{aligned}
\mathbb{E}[K_{\pi^*}(s)|\mathcal{H}_t] &= \mathbb{E}[\mathbb{E}[K_{\pi^*}(s)|\mathcal{H}_s]|\mathcal{H}_t] \\
&= \mathbb{E}[M_{\pi^*}(s)|\mathcal{H}_t] = M_{\pi^*}(t) \quad \text{(by (ii))} \\
&= \mathbb{E}[K_{\pi^*}(t)|\mathcal{H}_t]
\end{aligned}$$

Working backworks, this clearly means that Eq. (4.15) holds, which can be written as

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^s (\mu(u) - r(u) - \sigma^2(u)\pi^*(u))du + \int_0^s \sigma(u)d^- W(u)\right) |\mathcal{H}_t\right] \\
&= \mathbb{E}\left[\left(\int_0^t (\mu(u) - r(u) - \sigma^2(u)\pi^*(u))du + \int_0^t \sigma(u)d^- W(u)\right) |\mathcal{H}_t\right]
\end{aligned}$$

so that, after differentiating both sides w.r.t. s, we obtain (iii).
(iii)⇒(i): Suppose (iii) holds. Integrating Eq. (4.12) gives Eq. (4.15) and hence Eq. (4.14). We partition the interval $[0, T]$ into $p_n = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ and introduce the sequence of $\mathcal{H}_{t_i}$-measurable and bounded random variables $\beta_i(t_i)$, where $i \in 0, 1, \ldots, n-1$. Since Eq. (4.14) holds for each $\beta_i(t_i)$, we have

$$\begin{aligned}
0 &= \sum_{i=0}^{n-1} \mathbb{E}\left[\left(\int_{t_i}^{t_{i+1}} (\mu(u) - r(u) - \sigma^2(u)\pi^*(u))du + \int_{t_i}^{t_{i+1}} \sigma(u)d^- W(u)\right) \beta_i(t_i)\right] \\
&= \sum_{i=0}^{n-1} \mathbb{E}\Big[\int_0^T (\mu(u) - r(u) - \sigma^2(u)\pi^*(u))\beta_i(t_i)1_{(t_i,t_{i+1}]}(u)du \\
&\qquad + \int_0^T \sigma(u)\beta_i(t_i)1_{(t_i,t_{i+1}]}(u)d^- W(u)\Big] \\
&= \mathbb{E}\left[\int_0^T (\mu(u) - r(u) - \sigma^2(u)\pi^*(u))\beta_n(u)du + \int_0^T \beta_n(u)\sigma(u)d^- W(u)\right]
\end{aligned}$$

so that Eq. (4.13) is satisfied for $\beta_n(u) := \sum_{i=0}^{n-1} \beta_i(t_i) 1_{(t_i,t_{i+1}]}(u)$. Now since any cadlag process may be approximated by such a β_n, we have that Eq. (4.13) is true for all $\beta \in \mathcal{A}_{\mathbb{H}}$ by a density argument. From this it follows that the directional derivative of J ar π^* w.r.t. the direction β is zero:

$$D_\beta J(\pi^*) := \lim_{r \to 0} \frac{J(\pi^* + r\beta) - J(\pi^*)}{r} = 0 \quad for\ all\ \beta \in \mathcal{A}_{\mathbb{H}} \tag{4.16}$$

Since $\log x$ is concave, we have that $J(\pi^*)$ is a concave function of π^*:

$$J(\lambda\alpha + (1-\lambda)\beta) \geq \lambda J(\alpha) + (1-\lambda)J(\beta)$$

where $\lambda \in [0,1]$ and $\alpha, \beta \in \mathcal{A}_{\mathbb{H}}$. Thus, for $\epsilon \in (0,1)$ we can write

$$\begin{aligned} J(\alpha + \epsilon\beta) - J(\alpha) &= J\left((1-\epsilon)\frac{\alpha}{1-\epsilon} + \epsilon\beta\right) - J(\alpha) \\ &\geq (1-\epsilon)J\left(\frac{\alpha}{1-\epsilon}\right) + \epsilon J(\beta) - J(\alpha) \\ &= J(\frac{\alpha}{1-\epsilon}) - J(\alpha) + \epsilon\left(J(\beta) - J\left(\frac{\alpha}{1-\epsilon}\right)\right) \end{aligned} \tag{4.17}$$

Writing $1 + \eta = \frac{1}{1-\epsilon}$ we have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(J\left(\frac{\alpha}{1-\epsilon}\right) - J(\alpha)\right) = \lim_{\eta \to 0} \frac{1+\eta}{\eta}(J(\alpha + \eta\alpha) - J(\alpha)) = D_\alpha J(\alpha)$$

which we can apply to Eq. (4.17) to obtain

$$\begin{aligned} D_\beta J(\alpha) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}(J(\alpha + \epsilon\beta) - J(\alpha)) \\ &\geq \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(J(\frac{\alpha}{1-\epsilon}) - J(\alpha) + \epsilon\left(J(\beta) - J\left(\frac{\alpha}{1-\epsilon}\right)\right)\right) \\ &= D_\alpha J(\alpha) - J(\alpha) + J(\beta) \end{aligned}$$

Substituting $\alpha = \pi^*$ and using the fact that $D_\beta J(\pi^*) = 0$ for all $\beta \in \mathcal{A}_{\mathbb{H}}$ (from Eq. (4.16) we finally obtain

$$J(\beta) - J(\pi^*) \leq 0 \quad for\ all\ \beta \in \mathcal{A}_{\mathbb{H}}$$

so that π^* is indeed optimal. □

A particularly useful application of the characterization theorem is that statement (iii) suggests a neat formula for the optimal portfolio. We have

Corollary 4.2.2. *If an optimal strategy exists then it satisfies*

$$\pi^*(t)\mathbb{E}[\sigma^2(t)|\mathcal{H}_t] = \mathbb{E}[\mu(t) - r(t)|\mathcal{H}_t] + a(t) \tag{4.18}$$

where

$$a(t) := \lim_{h\to 0^+} \frac{1}{h}\mathbb{E}\left[\int_t^{t+h} \sigma(s)d^-W(s)|\mathcal{H}_t\right] \tag{4.19}$$

Proof. This follows directly from the definition of the derivative and Theorem 4.2.1. Since $\pi^*(t)$ is $\mathcal{H}_t$-measurable we have

$$\pi^*(t)\mathbb{E}[\sigma^2(t)|\mathcal{H}_t] = \mathbb{E}\left[\mu(t) - r(t)|\mathcal{H}_t\right] + a(t)$$

where

$$\begin{aligned} a(t) &:= \frac{d}{ds}\mathbb{E}\left[\int_0^s \sigma(u)d^-W(u)|\mathcal{H}_t\right]\Big|_{s=t} \\ &= \lim_{h\to 0^+}\frac{1}{h}\left(\mathbb{E}\left[\int_0^{s+h}\sigma(u)d^-W(u)|\mathcal{H}_t\right] - \mathbb{E}\left[\int_0^s \sigma(u)d^-W(u)|\mathcal{H}_t\right]\right)\Big|_{s=t} \\ &= \lim_{h\to 0^+}\frac{1}{h}\mathbb{E}\left[\int_s^{s+h}\sigma(u)d^-W(u)|\mathcal{H}_t\right]\Big|_{s=t} \\ &= \lim_{h\to 0^+}\frac{1}{h}\mathbb{E}\left[\int_t^{t+h}\sigma(u)d^-W(u)|\mathcal{H}_t\right] \end{aligned}$$

□

Using the above expression, it is fairly straightforward to compute the value of $J(\pi^*)$ by substitution[2] (see [26]):

Theorem 4.2.3. *If an optimal portfolio π^* exists and $\sigma(t) \neq 0$ for a.a. (t,ω), then $J(\pi^*)$ is given by*

$$\begin{aligned} J(\pi^*) = \mathbb{E}\Big[\int_0^T \Big\{&\frac{1}{2}\frac{\mathbb{E}[\mu(s) - r(s)|\mathcal{H}_s]^2}{\mathbb{E}[\sigma^2(s)|\mathcal{H}_s]} - \frac{1}{2}\frac{a^2(s)}{\mathbb{E}[\sigma^2(s)|\mathcal{H}_s]} \\ &+ D_{s^+}\left(\sigma(s)\frac{\mathbb{E}[\mu(s) - r(s)|\mathcal{H}_s] + a(s)}{\mathbb{E}[\sigma^2(s)|\mathcal{H}_s]}\right)\Big\}ds\Big] \end{aligned} \tag{4.20}$$

Recall that we have not yet made any assumption about the level of the investor's filtration $\mathbb{H}$. Thus, the expressions for π^* and $J(\pi^*)$ are general in the sense that they can be applied for either partially informed traders *or* insiders, or, indeed, regular traders on an insider-free market. In order to specialize our expressions for π^* and $J(\pi^*)$ to particular types of trader, we essentially only need to find $a(t)$. In order to give a simple illustration of the method, we will recover the usual Merton solution in the following section. We proceed by solving the optimal portfolio problem for partially informed traders and insiders respectively.

[2]The D_{s^+} term comes about as a result of Corollary 2.5.8

4.3 The partial information case

We shall now derive an expression for $a(t)$ for the case in which $\mathcal{H}_t \subseteq \mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \in [0, T]$. This models an anticipating market (since the coefficients are adapted to $\mathcal{G}_t \supseteq \mathcal{F}_t$) in which the trader's filtration contains less information than the filtration $\mathbb{F}$ generated by the Brownian motion W.

Proposition 4.3.1. *Suppose $\mathcal{H}_t \subseteq \mathcal{F}_t \subseteq \mathcal{G}_t$ and σ satisfies the conditions of Theorem 2.5.7. Then*

$$a(t) = \mathbb{E}[D_{t+}\sigma(t)|\mathcal{H}_t] \tag{4.21}$$

so that

$$\pi^*(t) = \frac{\mathbb{E}[\mu(t) - r(t) + D_{t+}\sigma(t)|\mathcal{H}_t]}{\mathbb{E}[\sigma^2(t)|\mathcal{H}_t]}. \tag{4.22}$$

as long as $\pi^* \in \mathcal{A}_{\mathbb{H}}$. Furthermore, if $\sigma(t) \neq 0$ for a.a. (t, ω), we have

$$J(\pi^*) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left\{\frac{\mathbb{E}[\mu(s) - r(s) + D_{s+}\sigma(s)|\mathcal{H}_s]^2}{\mathbb{E}[\sigma^2(s)|\mathcal{H}_s]}\right\} ds\right] \tag{4.23}$$

Proof. The expression for π^* follows from Corollary 4.2.2, where $a(t) = \mathbb{E}[D_{t+}|\mathcal{H}_t]$ from Corollary 2.5.8. The expression for $J(\pi^*)$ follows from Theorem 4.2.3. □

Example 4.3.2. *Suppose that $\mathcal{H}_t = \mathcal{F}_t = \mathcal{G}_t$. This models the assumption in basic mathematical finance that all traders base their portfolio decisions on a common flow of information. Since $\sigma(t)$ is adapted to $\mathcal{F}_t$ we have $D_t\sigma(s) = 0$ for all $t > s$ by Corollary 2.4.3 and thus $a(t) = D_{t+}\sigma(t) = 0$. This gives*

$$\pi^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)} \tag{4.24}$$

and

$$J(\pi^*) = \mathbb{E}\left[\int_0^T \frac{1}{2}\left(\frac{\mu(t) - r(t)}{\sigma(t)}\right)^2 dt\right] \tag{4.25}$$

which is the well-known solution of the Merton problem.

Example 4.3.3. (Delayed noise - [35] Example 2.14) *Suppose $\mathcal{H}_t = \mathcal{F}_t$ and $\mathcal{G}_t = \mathcal{F}_{t+\delta}$ for some $\delta > 0$. Let $\mu(t)$ and $r(t)$ be bounded and $\mathcal{G}_t$-measurable and take*

$$\sigma(s) = \exp(W(s + \delta)) \tag{4.26}$$

This models a partial information scenario in the sense that the trader does not know what the volatility $\sigma(t)$ is at time t. Since the partially informed trader only observes $W(t)$ at time t and not $W(t + \delta)$, he cannot know the volatility at any time - he must wait δ time units to know what the noise is. For this reason it is *referred to as delayed noise.*

The Malliavin chain rule gives

$$D_t\sigma(s) = \exp(W(s+\delta))D_tW(s+\delta)$$

We have

$$W(s+\delta) = \int_0^{s+\delta} 1dW(\tau) = \int_0^T 1_{[0,s+\delta]}(\tau)dW(\tau)$$

so that

$$D_tW(s+\delta) = 1_{[0,s+\delta]}(t)$$

and

$$\lim_{t\to s+} D_tW(s+\delta) = 1$$

We thus have that $D_{s+}\sigma(s) = \exp(W(s+\delta)) = \sigma(s)$. *By Eq. (4.22) we then have*

$$\pi_\delta^*(s) = \frac{\mathbb{E}[\mu(s) - r(s) + \sigma(s)|\mathcal{F}_s]}{E[\sigma^2(s)|\mathcal{F}_s]} \tag{4.27}$$

with the expected gain in utility given by Eq. (4.23) to be

$$J(\pi_\delta^*) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left\{\frac{\mathbb{E}[\mu(s) - r(s) + \sigma(s)|\mathcal{F}_s]^2}{\mathbb{E}[\sigma^2(s)|\mathcal{F}_s]}\right\} ds\right]$$

But note that setting $\delta = 0$ *corresponds to* $\mathcal{H}_t = \mathcal{F}_t = \mathcal{G}_t$ *as in the previous example, resulting in the curious fact that*

$$\lim_{\delta\to 0} \pi_\delta^*(s) = \frac{\mu(s) - r(s) + \sigma(s)}{\sigma^2(s)} \neq \pi_0^*(s)$$

and

$$\lim_{\delta\to 0} J(\pi_\delta^*) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left(\frac{\mu(s) - r(s)}{\sigma(s)} + 1\right)^2 ds\right] \neq J(\pi_0^*)$$

Øksendal and Sulem [35] interpret this to mean that "any positive delay, no matter how small, has a substantial effect on the control problem".

Remark 4.3.4. Note that the above example cannot be solved by the enlargement of filtrations technique. This is because the $\mathbb{F}$-Brownian motion W doesn't remain a semimartingale in the enlarged filtration $\mathbb{G}$, where $\mathcal{G}_t = \mathcal{F}_{t+\delta}$ (see Ex. 78 of [24] and Prop. 12 of [25]).

4.4 The inside information case

We shall now consider the case when the trader's filtration is larger than both $\mathbb{F}$ and $\mathbb{G}$:

$$\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{H}_t \qquad \textit{for all } t \in [0, T] \tag{4.28}$$

This models the perpective of an *insider* - a trader who possesses information that is vaster than the market knowledge. It is often assumed in the literature that the insider possesses, from the start of the trading interval, knowledge about an $\mathcal{F}_T$-measurable random variable G. In this case the insider's filtration may be expressed as

$$\mathcal{H}_t = \mathcal{G}_t \vee \sigma(G) \tag{4.29}$$

One of the shortcomings of this *initial information* approach is that the insider's knowledge doesn't improve as time passes. This problem is investigated in [11], in which the authors develop the theory of *dynamical* evolution of information, where the distortion goes to zero as the terminal time approaches. We will develop this topic further in Chapter 6.

For the time being we shall not make any assumptions about the insider's information other than that it is larger than the honest trader's information. That is, we aren't assuming an initial enlargement or any other specific details about how $\mathbb{H}$ is obtained; the insider's filtration is still quite general.

Example 4.4.1. *(Insider strategy) Suppose $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{H}_t$ for all t. Then, since the coefficients are $\mathcal{H}_t$-adapted, we clearly have*

$$\pi^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)} + \frac{a(t)}{\sigma^2(t)} \tag{4.30}$$

We recognize the above as a Merton-type solution plus an additional term that is due to the extra information of the insider. Of course the solution isn't particularly useful unless we can solve for $a(t)$. It is difficult to obtain a general expression for $a(t)$ because the solution may vary greatly according to the *type* of extra information. In order to obtain more explicit results we will consider the initial information case. That is, the insider possesses knowledge of an $\mathcal{F}_T$-measurable random variable G from the beginning of the trading interval. We follow Section 3 of [28]. We assume for the remainder of this chapter that $\mathcal{F}_t = \mathcal{G}_t \subseteq \mathcal{H}_t$ for all $t \in [0, T]$, so that the coefficients are adapted to the natural Brownian filtration. We also have

$$\mathcal{H}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(G) \tag{4.31}$$

We shall find the optimal portfolio by using results from Malliavin's calculus and forward integration. Although we shall see that the result is related to the enlargement of filtrations technique, which is covered in greater detail in Chapter 5, we include it here due to its reliance on anticipating calculus, which is the focus of this chapter. We begin by presenting two hypotheses which ensure that all relevant integrals exist.

Hypothesis 4.4.2. For a.a. $s \in [0, T]$, the process

$$I_\cdot(s, G) := 1_{[s,T]}(\cdot) 1_{\{\int_s^t (D_u G)^2 ds > 0\}} \frac{(D_s G)(D_\cdot G)}{\int_s^T (D_u G)^2 ds} \tag{4.32}$$

belongs to Dom δ and there exists a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable process $h = \{h_t(y) : t \in [0, T], y \in \mathbb{R}\}$ such that[3]

$$\mathbb{E}\left[\int_0^T I_u(s, G) dW_u | \mathcal{H}_s\right] = h_s(G) \tag{4.33}$$

Hypothesis 4.4.3. For $t \in [0, T]$ the processes $|r|^{1/2}$, $(\mu - r)/\sigma$ and $h(G)$ all belong to $L^2([0, t] \times \Omega)$.

We now derive a general expression for the optimal portfolio for the initial information case.

Theorem 4.4.4. *Under Hypotheses 4.4.2 and 4.4.3 we have*

$$\mathbb{E}[|\log X_t^\pi|] < \infty \qquad \textit{for all } t < T, \pi \in \mathcal{A}_{\mathbb{H}} \tag{4.34}$$

Furthermore, if

$$\pi_t^*(y) = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{h_t(y)}{\sigma_t} \tag{4.35}$$

belongs to $\mathcal{A}_{\mathbb{H}}$, then $\pi^*(y)$ is an optimal portfolio for Problem 4.1.5.

Proof. For notational simplicity we will use $B_s^\pi(y) := (\mu_s - r_s)\pi_s(y)$ and $\sigma_s^\pi := \sigma_s \pi_s(y)$ for all $s \in [0, T]$ and $y \in \mathbb{R}$. We have

$$\begin{aligned} J(\pi) &= \mathbb{E}\left[\int_0^T \left[(\mu(t) - r(t))\pi(t) - \frac{1}{2}\pi^2(t)\sigma^2(t)\right] dt + \int_0^T \pi(t)\sigma(t) d^- W(t)\right] \\ &= \mathbb{E}\left[\int_0^T \left(B_s^\pi(y) - \frac{1}{2}(\sigma_s^\pi(G))^2\right) ds + \int_0^T \sigma_s^\pi d^- W_s\right] \\ &= \mathbb{E}\left[\int_0^T \left(B_s^\pi(y) - \frac{1}{2}(\sigma_s^\pi(G))^2\right) ds + \int_0^T \sigma_s^\pi \delta W_s + \int_0^T (\sigma_s^\pi)'(G) D_s G ds\right] \end{aligned}$$

[3] Here $\mathcal{P}$ denotes the predictable σ-algebra of $[0, T] \times \Omega$ and $y \in \mathbb{R}$ denotes the realized value of the extra information G.

where we have use the Malliavin chain rule and the relation between the forward and Skorohod integrals in the last step. Note that

$$\begin{aligned}
& \mathbb{E}\left[\int_0^t \int_s^T I_\theta(s,t) D_\theta(\sigma_s^\pi(G)) d\theta ds\right] \\
= & \mathbb{E}\left[\int_0^t \int_s^T I_\theta(s,t)(\sigma_s^\pi)'(G) D_\theta G d\theta ds\right] \\
= & \mathbb{E}\left[\int_0^t \int_s^T 1_{[s,T]}(\theta) 1_{\{\int_s^t (D_u G)^2 ds>0\}} \frac{(D_s G)(D_\theta G)}{\int_s^T (D_u G)^2 ds)} (\sigma_s^\pi)'(G) D_\theta G d\theta ds\right] \\
= & \mathbb{E}\left[\int_0^t (\sigma_s^\pi)'(G) D_s G (\int_s^T (D_u G)^2 ds)^{-1} \int_s^T (D_\theta G)^2 d\theta ds\right] \\
= & \mathbb{E}\left[\int_0^t (\sigma_s^\pi)'(G) D_s G ds\right]
\end{aligned}$$

So we have

$$\mathbb{E}\left[\int_0^t (\sigma_s^\pi)'(G) D_s G ds\right] = \mathbb{E}\left[\int_0^t \int_s^T I_\theta(s,t) D_\theta(\sigma_s^\pi(G)) d\theta ds\right] \tag{4.36}$$

Using the integration-by-parts formula of Malliavin's calculus (see Theorem 2.2.4) we obtain

$$\mathbb{E}\left[\int_s^T I_\theta(s,t) D_\theta(\sigma_s^\pi(G)) d\theta ds\right] = \mathbb{E}\left[\sigma_s^\pi(G) \int_s^T I_\theta(s,G) dW_\theta\right] \tag{4.37}$$

Using the above identity and the tower property yields

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \int_s^T I_\theta(s,t) D_\theta(\sigma_s^\pi(G)) d\theta ds\right] &= \mathbb{E}\left[\int_0^t \sigma_s^\pi(G) \mathbb{E}\left[\int_s^T I_\theta(s,G) dW_\theta | \mathcal{H}_s\right] ds\right] \\
&= \mathbb{E}\left[\int_0^t \sigma_s^\pi(G) h_s(G) ds\right]
\end{aligned}$$

We have shown that

$$\mathbb{E}\left[\int_0^t (\sigma_s^\pi)'(G) D_s G ds\right] = \mathbb{E}\left[\int_0^t \sigma_s^\pi(G) h_s(G) ds\right] \tag{4.38}$$

which we substitute back into our expression for $J(\pi)$ to obtain

$$\begin{aligned}
J(\pi) &= \mathbb{E}\left[\int_0^T \left(B_s^\pi(G) - \frac{1}{2}(\sigma_s^\pi(G))^2 + \sigma_s^\pi(G) h_s(G)\right) ds\right] \\
&= \mathbb{E}\left[\int_0^T \left((\mu_s - r_s)\pi_s(y) - \frac{1}{2}\sigma_s^2 \pi_s(y)^2 + \sigma_s \pi_s(y) h_s(G)\right) ds\right]
\end{aligned}$$

In order to find the optimal portfolio π^*, we need to find the $\pi \in \mathcal{A}_{\mathbb{H}}$ that maximizes J. From the above expression, this simply amounts to maximizing the value of the integrand $(\mu_s - r_s)\pi_s(y) - \frac{1}{2}\sigma_s^2\pi_s(y)^2 + \sigma_s\pi_s(y)h_s(G)$ over $\pi_s(y)$. Simple differentiation yields

$$\pi_s^*(y) = \frac{\mu_s - r_s}{\sigma_s^2} + \frac{h_s(G)}{\sigma_s} \tag{4.39}$$

and we are done. □

Note that our value for the additional term $a(t)$ for the insider case can now be expressed as

$$a(t) = \sigma_t h_t(G) \tag{4.40}$$

It is important at this point to observe the connection between the above approach and the enlargement of filtrations approach. Recall that a particular concern of the study of enlargements is to be able to determine whether an $\mathcal{F}_t$-Brownian motion W is a semimartingale in an enlarged filtration $\mathbb{H}$. in which case the anticipating integral reduces to the classical integral. If W_t can expressed in the form

$$W(t) = \tilde{W}(t) + \int_0^t h_s(G)ds \tag{4.41}$$

where $\tilde{W}$ is an $\mathbb{H}$-Brownian motion, then we call h the *information drift.* If we substitute this decomposition into the original expression for $J(\pi)$ it is fairly straightforward to obtain the expression for the optimal portfolio given by Eq. (4.35) using ordinary Ito calculus (see Chapter 5). The preceding theorem thus suggests that, under the given hypotheses, there is a semimartingale decomposition of the Brownian motion W of the form given by Eq. (4.41), where the information drift $h(G)$ is given by Eq. (4.33).

In fact, Theorem 4.4.4 gives a particular instance in which we can obtain an expression for the information drift via the existence of a Malliavin *integration-by-parts (ibp) formula* (see [24]).

Definition 4.4.5. *Let X be a random variable and Y a random process. We say that there is an integration by parts formula for the pair (X, Y) in $[t, T]$ if there exists a random variable $H_{t,T}(X, Y) \in L^2(\Omega)$ such that for any bounded differentiable function $f \in C_b^1$ and $A \in \mathcal{F}_t$ we have*

$$\mathbb{E}[f'(X)Y; A] = \mathbb{E}[f(X)H_{t,T}(X, Y); A]$$

Here $H_{t,T}$ is referred to as the weight associated with the ibp formula.

Theorem 4.4.6. *Let X be a random variable such that an ibp formula exists for (X, D_uX) in $[u, T]$. Then W is a semimartingale in the initially enlarged filtration $\{\mathcal{H}_s; s \le t\} := \{\mathcal{F}_s \vee \sigma(X); s \le T\}$ with decomposition given by*

$$W(t) = \tilde{W}(t) + \int_0^t \mathbb{E}[H_{u,T}(X, D_uX)|\mathcal{H}_u]du \tag{4.42}$$

where $\tilde{W}$ is an $\mathbb{H}$-Brownian motion.

Proof. See Theorem 38 of [24]. □

In the following section we shall further explore the link between the forward integral approach and the enlargement of filtrations approach.

4.5 A link between optimal portfolios and semimartingales

It has already been mentioned that the forward integral approach to finding an optimal portfolio is not necessary if the honest trader's Brownian motion W is a semimartingale in the insider's filtration. If this is the case we may simply use the semimartingale decomposition of W and compute the ordinary Ito integral - no anticipations are involved. One justification of the forward integral approach is that it applies whether or not W is a semimartingale from the insider's perpective; but are there any practical situations in which the forward integral approach yields useful results where the enlargement approach fails? When considering scenarios for which the optimization problem is soluble for the insider, the answer would appear to be *no*, in light of the following result: *if an optimal insider portfolio exists for Problem 4.1.5, then a semimartingale decomposition of the form given by Eq. (4.41) exists, where the information drift is closely related to the optimal portfolio.* This theorem is the main result of [9], where it is proved for general utility functions. This section will be based on their presentation, which we simplify for the specific case of the logarithmic utility function.

In what follows, we continue to work under the assumption that $\mathcal{H}_t \supseteq \mathcal{G}_t \supseteq \mathcal{F}_t$ for each t (we don't assume anything about the type of insider information, i.e. whether it is *initial* or progressive, or exact or distorted). We immediately present the main result of [9], since much of the preliminary work has already been done in the proof of Theorem 4.2.1, in which we presented a characterization theorem for optimal portfolios.

Theorem 4.5.1. *(i) A process $\pi^*(t)$ is an optimal strategy iff the process*

$$K_\pi(t) = \int_0^t (\mu(s) - r(s) - \sigma^2(s)\pi^*(s))ds + \int_0^t \sigma(s)d^-W(s) \tag{4.43}$$

is an $\mathbb{H}$-martingale;
(ii) If an optimal portfolio exists, then the process

$$N(t) = \int_0^t \sigma(s)d^-W(s) \tag{4.44}$$

is an $\mathbb{H}$*-semimartingale;*
(iii) If an optimal portfolio exists and

$$\sigma(s) \neq 0 \; for \; a.a. \; (s, \omega) \in [0, T] \times \Omega \tag{4.45}$$

then W *is an* $\mathbb{H}$*-semimartingale.*

Proof. (i) has been proved as an intermediate step of the proof of (i)⇒(ii) of Th. 4.2.1.
(ii) We have $N(t) = K_\pi(t) - \int_0^t (\mu(s) - r(s) - \sigma^2(s)\pi^*(s))ds$. It is thus the sum of a martingale and a process of finite variation, and so $N(t)$ is an $\mathbb{H}$-semimartingale.
(iii) By (ii), the process $N(t)$ is an $\mathbb{H}$-semimartingale. Thus, if $\sigma(s) \neq 0$. we know that

$$W(t) = \int_0^t \sigma^{-1}(s)\sigma(s)d^- W(s) = \int_0^t \sigma^{-1}(s)dN(s)$$

is also an $\mathbb{H}$-semimartingale. □

Remark 4.5.2. Although the above proof only applies for logarithmic utility functions, the result still holds true for generalized utility functions. in which it is only assumed that

$$U : [0, \infty) \longrightarrow [-\infty, \infty)$$

is continuously differentiable on $(0, \infty)$. See [9] for the complete proof.

Chapter 5

Additional insider utility under initial enlargements

In the previous chapter we studied a market model in which the traders possess varying levels of information. In this chapter we specialize the information of the trader being studied. Here we only consider an insider who acquires all additional information, denoted by G, at the beginning of the trading interval. More specifically, we apply the techniques of initial enlargements of filtrations to solve the problem of optimizing the insider's terminal logarithmic utility. The enlargement approach is partly motivated by the final result of Chapter 4, in which we demonstrated that a the stock price is a semimartingale in the insider's filtration whenever the optimal portfolio problem is soluble. The maximal utility of the insider is then compared with the maximal utility of the honest trader in order to quantify the value of the additional information.

The *initial information* approach can (roughly) be classified into two types. The first type is when G represents *exact* knowledge of the terminal value of a random variable (such as the stock price or the range of the stock price). The second type is when G represents *distorted* knowledge of a future random variable; that is, the insider possesses knowledge of the final value of a random variable (e.g. the stock price) but this knowledge is distorted by a source of independent noise. We shall see that when the insider possesses exact advance knowledge his utility is infinite, and when this knowledge is distorted by noise the utility may be finite.

The market is modelled by the probability space $(\Omega, \mathcal{F}, P)$ equipped with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, where $\mathbb{F}$ satisfies the usual conditions and $\mathcal{F}_0$ is trivial. The filtration $\mathbb{F}$ represents the information available to the honest trader. We will not use the anticipative model of Chapter 4, in which the coefficients were adapted to a filtration $\mathbb{G}$ larger than $\mathbb{F}$. Instead we will make the standard assumption that the coefficients are adapted to the honest trader's filtration $\mathbb{F}$. We assume that the insider has access to the $\mathcal{F}_T$-measurable random variable G, and $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$

will denote the insider's information filtration, where

$$\mathcal{G}_t = \bigcap_{\epsilon>0} (\mathcal{F}_{t+\epsilon} \vee \sigma(G))$$

We shall also assume the existence of an equivalent (local) martingale measure Q for the honest trader, from which it follows that the market is arbitrage-free for the honest trader.

5.1 Additional utility via the information drift

This section uses the ideas of Chapter 3 to find an expression for the insider's additional logarithmic utility. It is largely based on [3].

We will assume that the stock price process evolves according to the following s.d.e.:

$$dS_t = S_t(\alpha_t d\langle M\rangle_t + dM_t) \tag{5.1}$$

where M is a local $\mathbb{F}$-martingale and α satisfies

$$\mathbb{E}\left[\int_0^T \alpha_s^2 d\langle M\rangle\right] < \infty \tag{5.2}$$

There is also a riskless asset but we shall assume for simplicity that the interest rate has been normalized to zero.

Remark 5.1.1. Recall that in Chapter 4, we assumed that the market was driven by a Brownian motion W: $dS_t = S_t(\mu_t dt + \sigma_t dW_t)$. In this section we assume a slightly more general model in which the stock price is driven by a general continuous local $\mathbb{F}$-martingale M. We will return to the Brownian model shortly, but we begin with the model given by Eq. (5.1) as the added generality requires very little additional work.

In Chapter 4 we discussed conditions under which a local $\mathbb{F}$-martingale remains a semimartingale in an enlarged filtration. In order to proceed we assume that Hypothesis (E) holds, so that it is indeed possible to find a semimartingale decomposition of M in the enlarged filtration $\mathbb{G}^0$. (Recall that $\mathbb{G}^0 := (\mathcal{G}_t)_{t\in[0,T)}$, i.e. the terminal time T is excluded.) M is thus a $\mathbb{G}^0$-semimartingale with decomposition

$$M = \tilde{M} + \int_0^t \mu_t^G d\langle M\rangle_s \tag{5.3}$$

where $\tilde{M}$ is a local $\mathbb{G}^0$-martingale. It will be assumed throughout this section that the information drift μ^G satisfies

$$\mathbb{E}\left[\int_0^T (\mu_s^G)^2 d\langle M\rangle_s\right] < \infty \tag{5.4}$$

Substituting expression 5.3 for M into Eq. (5.1) yields the following expression for the stock price dynamics from the insider's point of view:

$$dS_t = S_t((\alpha_t + \mu_t^G)d\langle M\rangle_t + d\tilde{M}_t) \tag{5.5}$$

As in Section 4.1, we denote by π the proportion of wealth invested in the stock. We recall the self-financing condition for the discounted wealth process $V(t,\pi)$:

$$\frac{dV_t}{V_t} = \pi_t \frac{dS_t}{S_t} \tag{5.6}$$

Definition 5.1.2. *(Admissible portfolios) The set of admissible portfolios, denoted by $\mathcal{A}_{\mathbb{G}}$, is the set of $\mathbb{G}$-predictable processes π such that*

1. $\int_0^t \pi_s^2 d\langle M\rangle_s < \infty \quad P-a.s.$
2. *The discounted wealth process $V(t,\pi)$ satisfies the self-financing condition (Eq. (5.6)) with $V_0(\pi) = x > 0$*
3. $\mathbb{E}[\log V(t,\pi)] > -\infty$

Remark 5.1.3. Note that condition 3 means that we require $V(t,\pi)$ to be a.s. strictly positive for all t and π.

It is straightforward to solve Eq. (5.6) by Ito's formula to obtain

$$V_t(\pi) = V_0\mathcal{E}\left(\int_0^t \pi_s dM_s + \int_0^t \pi_s \alpha_s d\langle M\rangle_s\right) \tag{5.7}$$

which can be expressed as

$$V_t(\pi) = V_0\mathcal{E}\left(\int_0^t \pi_s d\tilde{M}_s + \int_0^t \pi_s(\alpha_s + \mu_s^G) d\langle M\rangle_s\right) \tag{5.8}$$

by substituting Eq. (5.3) into Eq. (5.7), thus giving the value from the insider's point of view.

We are now in a position to solve the problem of maximizing the expected logarithmic utility of an investor acting on the market defined above. For the insider we have

$$\begin{aligned}
\log V_t(\pi) &= \log V_0 + \int_0^t \pi_s d\tilde{M}_s + \int_0^t (\alpha_s + \mu_s^G - \frac{1}{2}\pi_s)\pi_s d\langle M\rangle_s \\
&= \log V_0 + \int_0^t \pi_s d\tilde{M}_s + \frac{1}{2}\int_0^t (\alpha_s + \mu_s^G)^2 d\langle M\rangle_s \\
&\quad -\frac{1}{2}\int_0^t (\alpha_s + \mu_s^G - \pi_s)^2 d\langle M\rangle_s.
\end{aligned} \tag{5.9}$$

Now if the local $\mathbb{G}^0$ martingale $\int_0^t \pi_s d\tilde{M}_s$ were a true martingale, its expectation would be zero and it is clear that the portfolio maximizing the above expression would be given by $\pi_s^* = \alpha_s + \mu_s^G$. We could then substitute π_s^* into $\mathbb{E}[V_t(\pi)]$ in order to obtain the insider's maximal expected utility. Taking $\mu_s^G = 0$ could similarly be used to obtain the honest trader's optimal portfolio and maximal expected utility.

Unfortunately we do not necessarily have that $\mathbb{E}[\int_0^t \pi_s^2 d\langle M\rangle_s < \infty]$, so that $\int_0^t \pi_s d\tilde{M}_s$ is not necessarily a true martingale. This problem can be tackled by introducing the *minimal martingale density processes*:

Definition 5.1.4. *The $\mathbb{F}$-minimal martingale density process $Z^{\mathbb{F}} = (Z_t^{\mathbb{F}})_{t\in[0,T]}$ is defined by*

$$Z_t^{\mathbb{F}} := \mathcal{E}\Big(-\int \alpha_s dM_s\Big)_t$$

and the $\mathbb{G}$-minimal martingale density process $Z^{\mathbb{G}} = (Z_t^{\mathbb{G}})_{t\in[0,T]}$ is defined by

$$Z_t^{\mathbb{G}} := \mathcal{E}\Big(-\int (\alpha_s + \mu_s^G) dM_s\Big)_t$$

Remark 5.1.5. Recall that in Section 3.2, $Z^{\mathbb{F}}$ denoted the density process of Q with respect to P, where Q was an EMM for the honest investor. We also had $Z^{\mathbb{G}} = Z^{\mathbb{F}}/p_t^G$, where p^G denotes the density process arising from Hypothesis (E). It is proved in [1] that the definitions from Section 3.2 coincide with the expressions in Def. 5.1.4 under the assumption that the market is *complete.*

We will need the following result:

Proposition 5.1.6.

1. *The processes $Z^{\mathbb{F}}S$ and $Z^{\mathbb{F}}V(\pi)$ with $\pi \in \mathcal{A}_{\mathbb{F}}$ are local $\mathbb{F}$-martingales on $[0,T]$.*

2. *The processes $Z^{\mathbb{G}}S$ and $Z^{\mathbb{G}}V(\pi)$ with $\pi \in \mathcal{A}_{\mathbb{G}}$ are local $\mathbb{G}_0$-martingales on $[0,T)$. If $\tilde{M}$ is a local $\mathbb{G}$-martingale, this extends to the interval $[0,T]$.*

Proof. See [3] Proposition 2.4. □

We can now take advantage of the above result to find explicit expressions for the optimal portfolios and maximal expected utilities of the traders acting on our market. What follows is the major result of this section.

Theorem 5.1.7.

1. *An optimal strategy up to time $t \in [0,T]$ for the honest trader is given by*

$$\pi_s^* = \alpha_s \tag{5.10}$$

and the associated maximal expected logarithmic utility is given by

$$\mathbb{E}[\log V_t(\pi^*)] = \log V_0 + \frac{1}{2}\mathbb{E}\left[\int_0^t \alpha^2 d\langle M\rangle_s\right]. \tag{5.11}$$

2. *An optimal strategy up to time $t \in [0, T)$ for the insider is given by*

$$\pi_s^* = \alpha_s + \mu_s^G \tag{5.12}$$

and the associated maximal expected logarithmic utility is given by

$$\mathbb{E}[\log V_t(\pi^*)] = \log V_0 + \frac{1}{2}\mathbb{E}\left[\int_0^t \alpha^2 d\langle M\rangle_s\right] + \frac{1}{2}\mathbb{E}\left[\int_0^t (\mu_s^G)^2 d\langle M\rangle_s\right]. \tag{5.13}$$

If $\tilde{M}$ is a local $\mathbb{G}^0$-martingale then Eq. (5.12) and Eq. (5.13) hold for $t \in [0, T]$.

Proof. We follow [3]. We shall use the following inequality for concave C^1 functions f such that the derivative of f' has inverse I:

$$f(a) \le f(I(b)) - b(I(b) - a) \qquad \textit{for all } a, b \in \mathbb{R}$$

For a fixed time $t \in [0, T)$ and admissible portfolio $\pi \in \mathcal{A}_{\mathbb{G}}$ we set $f(\cdot) = \log(\cdot)$, $a = V_t(\pi)$ and $b = yZ_t^{\mathbb{G}}$ for some constant $y > 0$, thus obtaining

$$\begin{aligned} \log(V_t(\pi)) &\le \log\frac{1}{yZ_t^{\mathbb{G}}} - yZ_t^{\mathbb{G}}\left(\frac{1}{yZ_t^{\mathbb{G}}} - V_t(\pi)\right) \\ &= -\log y - \log Z_t^{\mathbb{G}} - 1 + yZ_t^{\mathbb{G}}V_t(\pi) \end{aligned}$$

By Prop. 5.1.6, $Z_t^{\mathbb{G}}V_t(\pi)$ is a non-negative local $\mathbb{G}^0$-martingale, and hence a $\mathbb{G}^0$-supermartingale with initial value $x = V_0(\pi)$ (since $Z_0^{\mathbb{G}} = 1$). Thus $\mathbb{E}[Z_t^{\mathbb{G}}V_t(\pi)] \le x$ and we have

$$\mathbb{E}[\log(V_t(\pi))] \le -\log y - \mathbb{E}[\log Z_t^{\mathbb{G}}] - 1 + yx \tag{5.14}$$

In order to find an optimal portfolio we thus need to find $\pi \in \mathcal{A}_{\mathbb{G}}$ and $y > 0$ such that equality holds in (5.14). Choosing $\pi = \pi^* = \alpha + \mu^G$ and $y = 1/x$ and substituting into Eq. (5.9) yields

$$\begin{aligned} \log(V_t(\pi)) &= \log x + \int_0^t \pi_s d\tilde{M}_s + \frac{1}{2}\int_0^t (\alpha_s + \mu_s^G)^2 d\langle M\rangle_s \\ &= -\log y - \log Z_t^{\mathbb{G}} \end{aligned}$$

by the definition of $Z_t^{\mathbb{G}}$. Thus equality holds for (5.14) using the given choices of π and y. Finally $\pi^* \in \mathcal{A}_{\mathbb{G}}$ by (5.2), (5.4) and the fact that both y and $Z_t^{\mathbb{G}}$ are strictly positive (meaning that $\mathbb{E}[\log V(t, \pi)] > -\infty$). We have thus obtained Eq. (5.12).

In order to obtain Eq. (5.13) we note that, by Doob's inequality and the integrability condition of Eq. (5.2), the processes $\int \alpha_s dM_s$ and $\int \alpha_s d\tilde{M}_s$ are $\mathbb{F}$- and $\mathbb{G}$-martingales respectively. Using the decomposition $M = \tilde{M} + \int \mu_s^G d\langle M\rangle_s$ we thus have

$$\mathbb{E}\left[\int_0^t \alpha_s \mu_s^G d\langle \tilde{M}\rangle_s\right] = \mathbb{E}\left[\int_0^t \alpha_s dM_s - \int_0^t \alpha_s d\tilde{M}_s\right] = 0$$

Substituting Eq. (5.12) into Eq. (5.9) and taking expectations thus yields

$$\begin{aligned}\mathbb{E}[\log V_t(\pi^*)] &= \log x + \mathbb{E}\left[\int_0^t (\alpha_s + \mu_s^G) d\tilde{M}_s\right] + \frac{1}{2}\mathbb{E}\left[\int_0^t (\alpha_s^2 + 2\alpha_s\mu_s^G + (\mu_s^G)^2) d\langle M\rangle_s\right] \\ &= \log x + \frac{1}{2}\mathbb{E}\left[\int_0^t \alpha_s^2 d\langle M\rangle_s\right] + \frac{1}{2}\mathbb{E}\left[\int_0^t (\mu_s^G)^2 d\langle M\rangle_s\right]\end{aligned}$$

The optimal portfolio and maximal expected utility for the honest trader easily follow by taking the information drift μ^G to be zero. □

Definition 5.1.8. *The insider's additional expected logarithmic utility up to time t is defined by*

$$U^a(t) = \max_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[\log V_t(\pi)] - \max_{\pi \in \mathcal{A}_{\mathbb{F}}} \mathbb{E}[\log V_t(\pi)]$$

It is clear from the preceding result that the insider's additional expected logarithmic utility (called the *utility gain* in [3]) up to time $t \in [0, T]$ is given by

$$U^a(t) = \frac{1}{2}\mathbb{E}[\int_0^t (\mu_s^G)^2 d\langle M\rangle_s] \tag{5.15}$$

We have thus obtained a neat expression for the insider's additional utility in terms of the information drift μ^G. We also have the following result if we work within a Brownian framework:

Corollary 5.1.9. *Suppose that the stock price dynamics are given by*

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t) \tag{5.16}$$

and that the insider information G consists of some future knowledge about the Brownian motion W. If W is a $\mathbb{G}$-semimartingale with decomposition $W = \tilde{W} + \int \mu_s^W ds$ then the optimal portfolio for the insider at time t is given by

$$\pi_t^* = \frac{\mu_t + \sigma_t \mu_t^W}{\sigma_t^2} \tag{5.17}$$

and the corresponding additional logarithmic utility is given by

$$U^a(t) = \frac{1}{2}\mathbb{E}[\int_0^t (\mu_s^W)^2 ds] \tag{5.18}$$

Example 5.1.10. *Suppose that the stock price follows the usual Brownian dynamics given by Eq. (5.16) and the insider knows the terminal value* W_T. *Then* $\mu_s^W = (W_T - W_t)/(T-t)$ *by Theorem 3.1.1 and, by Eq. (5.18), the expected additional utility of the insider at time* t *is given by*

$$\begin{aligned} U^a(t) &= \frac{1}{2}\mathbb{E}\left[\int_0^t (\mu_s^G)^2 ds\right] = \frac{1}{2}\int_0^t \frac{E[(W_T - W_s)^2]}{(T-s)^2} ds \\ &= \frac{1}{2}\int_0^t \frac{1}{T-s} ds = \frac{1}{2}\log\frac{T}{T-t} \end{aligned}$$

which clearly becomes infinite as $t \to T$.

5.2 Entropy representation of additional utility

The problem with the expression derived for U^a is that, even if we know that a semimartingale decomposition exists for M, it is often unfeasible to find an explicit expression for the associated information drift. Thus Eq. (5.15) is impossible to evaluate in many cases. In order to remedy this problem, we shall turn our attention to an alternative method that is based on finding an entropy representation. The entropy respresentation depends on the conditional density process of the extra information G and does not require explicit computation of the information drift.

We assume throughout this section that the conditional distributions of G given $\mathcal{F}_t$ are equivalent to the law of G for all $t \in [0,T)$. Let p^l denote the density process of $P_t^G := P[G \in dl|\mathcal{F}_t]$ with respect to $P^G := P[G \in dl]$, where $l \in U$.

According to Proposition 3.3.1 of Section 3.3 we can express $1/p^G$ as

$$\frac{1}{p_t^G} = \mathcal{E}\left(-\int \mu_s^G d\tilde{M}_s + \tilde{N}\right)_t \tag{5.19}$$

where $\tilde{N}$ is a continuous local $\mathbb{G}^0$-martingale orthogonal to $\tilde{M}$ and null at $t=0$. (Recall that the local $\mathbb{G}^0$-martingale $\tilde{M}$ is given by $\tilde{M} = M - \int \mu^G d\langle M\rangle$.) Taking logarithms on both sides then yields

$$-\log p_t^G = -\int_0^t \mu_s^G d\tilde{M}_s + \tilde{N}_t - \frac{1}{2}\int_0^t (\mu_s^G)^2 d\langle M\rangle_s - \frac{1}{2}\langle\tilde{N}\rangle_t \tag{5.20}$$

Note that $\int_0^t \mu_s^G d\tilde{M}_s$ is a $\mathbb{G}$-martingale on $[0,T]$ by Eq. (5.4) and Doob's inequality. Now if $\tilde{N}$ was also a true martingale we would have $\mathbb{E}[\tilde{N}_t] = 0$, so that

$$U^a(t) = \mathbb{E}\left[\frac{1}{2}\int_0^t (\mu_s^G)^2 d\langle M\rangle_s\right] = \mathbb{E}[\log p_t^G] - \frac{1}{2}\mathbb{E}[\langle\tilde{N}\rangle_t]$$

Recall that the martingale-preserving measure $\tilde{P}_t$ of Section 3.2 was defined by

$$\tilde{P}_t(A) = \int_A \frac{1}{p_t^G} dP \quad for\ A \in \mathcal{G}_t$$

so that $p_t^G = dP/d\tilde{P}_t$. We thus have

$$\begin{aligned} U^a(t) + \frac{1}{2}\mathbb{E}[\langle \tilde{N} \rangle_t] &= \mathbb{E}[\log p_t^G] \\ &= \mathbb{E}[\log \frac{dP}{d\tilde{P}_t}] \end{aligned}$$

For two probability measures P and Q defined on a measure space $(\Omega, \mathcal{F})$ the relative entropy of P with respect to Q is defined as

$$H_{\mathcal{F}}(P|Q) := \mathbb{E}_P \left[\log \frac{dP}{dQ} \right] \tag{5.21}$$

This allows us to write the utility gain at time t in terms of the relative entropy of P with respect to $\tilde{P}_t$:

$$U^a(t) + \frac{1}{2}\mathbb{E}[\langle \tilde{N} \rangle_t] = H_{\mathcal{G}_t}(P|\tilde{P}_t)$$

We should note that there are two clear problems that need to be dealt with. Firstly we assumed that $\tilde{N}$ was a true martingale in order to obtain the above result. Fortunately this is not a necessary assumption; the result is shown to be true under the more general assumption that $\tilde{N}$ is a local $\mathbb{G}^0$-martingale via a localization procedure (see Theorem 3.7 of [3]). Another problem is that this $\tilde{N}$ appears in the expression for U^a at all. One of our motivations for finding a new formulation of the additional utility was that Eq. (5.15) was dependent on the information drift of M, which is often difficult or impossible to evaluate. But now it seems we have to evaluate $\tilde{N}$, which is just as tricky. We are relieved of this difficulty if there is a particular martingale representation theorem for the honest trader's filtration $\mathbb{F}$, in which case $\tilde{N}$ is zero. Indeed, this is the case under the assumption that the market is complete, as mentioned in Remark 5.1.5.

Remark 5.2.1. Even in the case where $\tilde{N} \neq 0$. the entropy representation is still useful as it gives an upper bound for the insider's additional utility.

Theorem 5.2.2. *If $p^l = \mathcal{E}(\int \mu^l dM)$ for each $l \in U$, then the insider's additional expected logarithmic utility up to time $t \in [0, T)$ is given by*

$$U^a(t) = H_{\mathcal{G}_t}(P|\tilde{P}_t) = \mathbb{E}[\log p_t^G] \tag{5.22}$$

and up to time T it is given by

$$U^a(T) = \lim_{t \to T} H_{\mathcal{G}_t}(P|\tilde{P}_t) = \lim_{t \to T} \mathbb{E}[\log p_t^G] \tag{5.23}$$

Proof. Eq. (5.22) follows from the discussion above since $\tilde{N} = 0$ due to the expression for p^l. Eq. (5.23) follows by monotone convergence. □

5.2.1 Examples

We shall now use the entropy characterization of Theorem 5.2.2 to calculate the insider's additional utility for some specific examples of extra information. We shall first consider the case in which the extra information G is a discrete random variable; i.e. it can only take on a countable set of values. A good example would be if the insider knows whether the value of the terminal Brownian motion lies within a certain interval. That is, he knows the value of $G = 1_{[a,b]}(W_T)$ where $-\infty < a < b < \infty$. We will also consider the case in which the insider has distorted information about a future random variable. Specifically we will calculate the additional utility for the case $G = \lambda W_T + (1-\lambda)\epsilon$, where ϵ is a centred normal random variable that is independent of W.

In order to proceed, we restate the following assumptions about the local martingale M that drives the market:

Assumption 5.2.3.

1. M is a continuous local $\mathbb{F}$-martingale.
2. $\mathbb{E}[\int_0^T \alpha_s^2 d\langle M\rangle] < \infty$.
3. There is a $\mathcal{P}(\mathbb{F}^0) \otimes \mathcal{U}$-measurable process μ_t^l such that $\tilde{M} = M - \int \mu^G d\langle M\rangle$ is a continuous local $\mathbb{G}^0$-martingale.
4. $\mathbb{E}[\int_0^t (\mu_s^G)^2 d\langle M\rangle] < \infty$ for all $t \in [0, T)$.
5. $\frac{1}{p_t^G} = \mathcal{E}(-\int \mu_s^G d\tilde{M}_s)_t$.

Example 5.2.4. *Let us reconsider the case in the which $G = W_T$ using our new entropy formulation. In order to calculate $\mathbb{E}[a_T]$ we need to find the density process of the conditional laws P_t^G with respect to the law of G. Conditional on $\mathcal{F}_t$, W_T follows a normal distribution with mean W_t and variance $T-t$. The density of P_t^G with respect to Lebesgue measure is thus given by*

$$q_t^l = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{(l-W_t)^2}{2(T-t)}\right)$$

and the density of P_t^G with respect to Lebesgue measure is just q_0^l, so that the density of P_t^G with respect to P^G is

$$p_t^G = \frac{q_t^l}{q_0^l} = \sqrt{\frac{T}{T-t}} \exp\left(-\frac{(l-W_t)^2}{2(T-t)} + \frac{l^2}{2T}\right)$$

The additional logarithmic utility for $t < T$ is thus given by

$$\begin{aligned} U^a(t) &= \mathbb{E}[\log p_t^G] = \mathbb{E}\left[\frac{1}{2}\log\frac{T}{T-t} - \frac{l^2 - 2lW_t + W_t^2}{2(T-t)} + \frac{l^2}{2T}\right] \\ &= \frac{1}{2}\log\frac{T}{T-t} \end{aligned}$$

since $\mathbb{E}[l^2] = \mathbb{E}[W_T^2] = T$, $\mathbb{E}[lW_t] = T \wedge t = t$ and $\mathbb{E}[W_t^2] = t$. This correponds with the result obtained in Example 5.1.10

Example 5.2.5. *Suppose that, instead of knowing the exact terminal value of W_T, the insider knows a value that is deformed by an independent source of Gaussian noise. More precisely, suppose that the extra information is given by*

$$G = \lambda W_T + (1-\lambda)\epsilon \tag{5.24}$$

where $\lambda \in [0,1)$ and ϵ is a centred normal random variable that is independent of W. Now since $W_T \sim N(W_t, T-t)$ conditional on $\mathcal{F}_t$ and $\epsilon \sim N(0,\sigma^2)$ for some $\sigma^2 > 0$, we have that, conditional on $\mathcal{F}_t$, G follows a normal distribution with mean $m_t = \lambda W_t$ and variance $\Sigma^2 = \lambda^2(T-t) + (1-\lambda)^2\sigma^2$. The density of P_t^G with respect to Lebesgue measure is thus given by

$$q_t^l = \frac{1}{\sqrt{2\pi(\lambda^2(T-t)+(1-\lambda)^2\sigma^2)}}\exp\left(\frac{-(l-\lambda W_t)^2}{2\pi(\lambda^2(T-t)+(1-\lambda)^2\sigma^2)}\right)$$

and the density of P^G is simply given by q_0^l. It is thus clear that $P_t^G << P^G$ for all $t \in [0,T]$ with density

$$\begin{aligned} p_t^G &:= \frac{dP_t^G}{dP^G} = \frac{dP_t^G}{d\mu}\frac{d\mu}{dP_t^G} = \frac{q_t^l}{q_0^l} \\ &= \sqrt{\frac{\lambda^2 T + (1-\lambda)^2\sigma^2}{\lambda^2(T-t)+(1-\lambda)^2\sigma^2}}\exp\left(\frac{-(l-\lambda W_t)^2}{2\pi(\lambda^2(T-t)+(1-\lambda)^2\sigma^2)}\right. \\ &\qquad \left. + \frac{l^2}{2\pi(\lambda^2 T + (1-\lambda)^2\sigma^2)}\right) \end{aligned}$$

Theorem 5.2.2 then implies that the additional expected utility is given by

$$U^a(t) = \mathbb{E}[\log p_t^G] = \frac{1}{2}\log\left(\frac{\lambda^2 T + (1-\lambda)^2\sigma^2}{\lambda^2(T-t)+(1-\lambda)^2\sigma^2}\right)$$

since the expectation of the argument of the exponential factor of p_t^G is easily shown to be zero. Taking the limit as $t \to T$ finally gives us the expected additional terminal utility:

$$U^a(T) = \frac{1}{2}\log\left(\frac{\lambda^2 T + (1-\lambda)^2\sigma^2}{(1-\lambda)^2\sigma^2}\right) \tag{5.25}$$

Remark 5.2.6. We see that the insider does not obtain infinite utility if the information is distorted with $\lambda < 1$. Note that taking $\lambda = 1$, which corresponds to exact knowledge of W_T, leads to infinite additional utility. Indeed this is what we expect when the insider has exact knowledge of a future random variable.

Let us now turn our attention to the case where G is a discrete random variable. For the remaining examples we will use the following definitions of entropy for discrete random variables:

Definition 5.2.7. *The entropy of a discrete random variable G is defined by*

$$H(G) := -\sum_{l \in U} P(G = l) \log P(G = l) \tag{5.26}$$

The conditional entropy of a discrete random variable G is defined by

$$H(G|\mathcal{F}_t) := -\mathbb{E}\left[\sum_{l \in U} P(G = l|\mathcal{F}_t) \log P(G = l|\mathcal{F}_t)\right] \tag{5.27}$$

Theorem 5.2.8. ***(Atomic G)*** *Suppose that G is a discrete random variable with finite entropy. Then*

$$U^a(t) = H(G) - H(G|\mathcal{F}_t) \quad t \in [0, T] \tag{5.28}$$

Example 5.2.9. *Suppose that the insider knows whether the terminal value of Brownian motion falls within a certain range:*

$$G = 1_{[a,b]}(W_T)$$

By Theorem 5.2.8 we then have

$$\begin{aligned} U^a(T) = H(G) &= -\sum_{l \in \{0,1\}} P(G = l) \log P(G = l) \\ &= -P[W_T \notin (a,b)] \log P[W_T \notin (a,b)] - P[W_T \in (a,b)] \log P[W_T \in (a,b)] \\ &= -P \log P - (1 - P) \log(1 - P) \end{aligned}$$

where $P := P[W_T \in (a, b)] = \Phi(b/\sqrt{T}) - \Phi(a/\sqrt{T})$.

5.3 A word about arbitrage

Recall that the First Fundamental Theorem of Mathematical Finance states that the existence of an *equivalent (local) martingale measure* for a particular market model implies that the model is arbitrage free. Here we briefly consider whether the existence of an EMM with respect to one filtration (that of the honest trader) implies the existence of an EMM with respect to a larger filtration (that of the insider). That is, we will study a sufficient condition under which the absence of arbitrage for the honest trader extends to absence of arbitrage for the insider.

Theorem 5.3.1. *Suppose Hypothesis (E) of Section 3.2 holds. If there is an EMM on* $(\Omega, \mathcal{F}_T)$, *then there is an EMM on* $(\Omega, \mathcal{G}_t)$ *for all* $t \in [0, T)$, *so that there are no arbitrage possibilities for the insider in the interval* $[0, T)$.

Proof. Let Q be an EMM on $(\Omega, \mathcal{F}_T)$. If Hypothesis (E) holds then, by Corollary 3.2.8, we can define the measure $\tilde{Q}_t$ as in Eq. (3.11) for each $t \in [0, T)$, such that $\tilde{Q}_t = Q \sim P$ on $(\Omega, \mathcal{F}_t)$ and $\tilde{Q}_t = P$ on $(\Omega, \sigma(G))$. So clearly $\tilde{Q}_t \sim Q$ on $\mathcal{G}_t$ for all $t \in [0, T)$. We also have that the stock price S is a local $(\tilde{Q}_t, \mathbb{G})$-martingale on $[0, t]$ by Corollary 3.2.9. Thus $\tilde{Q}_t$ is an EMM for S on $(\Omega, \mathbb{G})$. □

Free lunch in a Brownian market

In concluding this chapter, we recall our brief discussion of measure-valued Malliavin calculus in Chapter 3. There we mentioned that the ideas of the measure-valued calculus could be used to obtain generalizations of Hypotheses (AC) and (E) (see Theorems 3.4.5 and 3.4.6 respectively). The paper by Imkeller, Pontier and Weisz [21] makes use of the generalized hypotheses to study the question of whether the insider possesses arbitrage opportunities in a Brownian market. The paper presents a number of results describing conditions on the inside information, as well as on the market coefficients, that lead to arbitrage. [20] and [21] both deal extensively with the case in which the insider has advance knowledge of the maximum of the stock price process; indeed, it is found that if $G = \sup_{t \in [0,T]} S_t$ then arbitrage opportunities do exist for the insider. Since many of the mathematical ideas are beyond the scope of this dissertation, we refer the reader to the original papers for further details.

Chapter 6

Adjustments to the model

In this chapter we will specialize some of our earlier assumptions in order to build models that are more representative of reality. We will concern ourselves with two broad objectives. The first is to describe models in which the market dynamics are influenced by the presence of a large insider. This is done by assuming that the market coefficients are stochastic processes that are dependent both on time *and* on the insider's portfolio π. The effect of this π-dependence will lead us to re-evaluate the insider's optimal portfolio problem. We shall also examine the behaviour of a small honest trader in such an insider-influenced market. Our second objective in this chapter is to study conditions under which the market is arbitrage-free for both the honest trader and the insider. We will consider an example in which the insider's additional information is given by $X_t = W_T + \varepsilon_t$, where ε_t is a source of Brownian noise that decreases with time. We will also study the problem of optimizing an insider portfolio under constraints that penalize the insider for engaging in heavy volumes of trade.

6.1 The influence of a "large" insider

A *large* trader is a financial agent who is so influential in the market that his trading behaviour has an impact on the market dynamics. This may occur, for instance, when a wealthy agent with inside information engages in trading volumes of such magnitude that the price of the asset being traded changes as a result. Since a trader's behaviour is essentially characterized by his portfolio strategy, we can model a market with a large insider (as in [14]) by assuming that the drift coefficients have some dependence on the insider strategy π. Recall that in Chapter 4 we introduced the notion of an 'anticipating market', in which the market coefficients are adapted to a filtration $\mathbb{G}$ that is larger than the public filtration $\mathbb{F}$. We are now able to understand how this may be the case if the coefficients depend on the insider portfolio, which is adapted to a filtration that is larger than $\mathbb{F}$.

Our framework is essentially the same as in Chapter 4, apart from the fact that we now assume that the drift processes are dependent on $\pi(t)$. We thus work on a probability space $(\Omega, \mathcal{F}, P)$ equipped with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ generated by the Brownian motion $(W_t)_{t\in[0,T]}$. We consider the problem of portfolio optimization from the point of view of the large insider, who observes the filtration $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ with $\mathcal{H}_t \supseteq \mathcal{F}_t$ for all $t \in [0,T]$. We generalize our original dynamics as follows:

$$dB(t) = r(t, \pi(t))B(t)dt; \qquad B(0) = 1 \tag{6.1}$$

and

$$dS(t) = S(t)[\mu(t, \pi(t))dt + \sigma(t)d^{-}W(t)]; \qquad S(0) > 0 \tag{6.2}$$

where the stochastic coefficients $r(t,\pi)$ and $\sigma(t,\pi)$ are measurable càglàd processes that are adapted to $\mathbb{F}$ for each constant π (see [14] page 6). We assume that these coefficients satisfy the integrability condition of Assumption 4.1.1. As before the dynamics of the discounted wealth process under the self-financing condition are given by the s.d.e.

$$dX(t) = X(t)[(\mu(t,\pi) - r(t,\pi))\pi(t)dt + \pi(t)\sigma(t)d^{-}W(t)]$$

which has the solution

$$X(T) = X(0)\exp\Big(\int_0^T \left[(\mu(t,\pi) - r(t,\pi))\pi(t) - \frac{1}{2}\pi^2(t)\sigma^2(t)\right]dt + \int_0^T \pi(t)\sigma(t)d^{-}W(t)\Big)$$

We will now solve Problem 4.1.5 in our new setting. The set of admissible insider portfolios $\mathcal{A}_{\mathbb{H}}$ is as defined in Def. 4.1.3, apart from the fact that μ and r are now dependent on π. Let us recall the first part of the Characterization Theorem of Section 4.2, which states that the existence of an optimal portfolio π^* implies that the process defined by

$$M_{\pi^*}(t) := \mathbb{E}\left[\int_0^t (\mu(s) - r(s) - \sigma^2(s)\pi^*(s))ds + \int_0^t \sigma(s)d^{-}W(s)|\mathcal{H}_t\right]$$

is an $\mathbb{H}$-martingale. In proving this we used the fact that, for an optimal portfolio $\pi^* \in \mathcal{A}_{\mathbb{H}}$, we have

$$\frac{d}{d\delta}J(\pi^* + \delta\beta)|_{\delta=0} = 0$$

for all $\beta \in \mathcal{A}_{\mathbb{H}}$ and $\delta \in \mathbb{R}$. In order to calculate the derivative we didn't need to be concerned with differentiating the coefficients since there was no dependence on π. For the large insider case however, we need to include terms such as

$$r'(t,\pi) := \frac{d}{d\pi}r(t,\pi) \text{ and } \mu'(t,\pi) := \frac{d}{d\pi}\mu(t,\pi)$$

Specifically, we have that

$$\begin{aligned} 0 &= \frac{d}{d\delta} J(\pi^* + \delta\beta)|_{\delta=0} \\ &= \mathbb{E}\Big[\int_0^T \beta(t)[(\mu(t,\pi^*) - r(t,\pi^*)) + (\mu'(t,\pi^*) - r'(t,\pi^*))\pi^* - \sigma^2(t)\pi^*]dt \\ &\quad + \int_0^T \beta(t)\sigma(t)d^- W(t)\Big] \end{aligned}$$

If we now continue directly along the lines of the proof of Theorem 4.2.1, we obtain the following result [14]:

Theorem 6.1.1. *i) If a stochastic process $\pi \in \mathcal{A}_{\mathbb{H}}$ is an optimal strategy, then*

$$M_\pi(t) := \mathbb{E}[\int_0^T [(\mu(t,\pi) - r(t,\pi)) + (\mu'(t,\pi) - r'(t,\pi))\pi - \pi\sigma(t)^2]dt + \int_0^T \sigma(t)d^- W(t)] \tag{6.3}$$

is an $\mathcal{H}$-martingale.

ii) Conversely, if the function $g(\delta) := \mathbb{E}[\log X^{(\pi+\delta\beta)}(T)]$ is concave for each $\beta \in \mathcal{A}_{\mathbb{H}}$ and $M_\pi(t)$ is an $\mathbb{H}$-martingale, then π is optimal.

Note that since $\log x$ is a concave increasing function, and the composition of a concave increasing function with a concave function is itself concave, we see that a sufficient condition for $g(\delta)$ to be concave is that the map

$$\Lambda(s) : \pi \longrightarrow (\mu(t,\pi) - r(t,\pi))\pi - \frac{1}{2}\sigma^2(s)\pi^2 \tag{6.4}$$

is concave for all $s \in [0,T]$. It is thus sufficient that the drift terms $\mu(s,\cdot)$ and $r(s,\cdot)$ are twice continuously differentiable for all $s \in [0,T]$ and that

$$(\mu''(t,\pi) - r''(t,\pi))\pi + 2(\mu'(t,\pi) - r'(t,\pi)) - \sigma^2(s) \leq 0 \tag{6.5}$$

for all $s \in [0,T]$ and $\pi \in \mathcal{A}_{\mathbb{H}}$.

In order to obtain some explicit results, let us assume that the Brownian motion W is a semimartingale in the enlarged filtration $\mathbb{H}$ with information drift $\alpha(t)$; that is, $W(t) = \tilde{W}(t) + \int_0^t \alpha(s)ds$. We then have the following theorem:

Theorem 6.1.2. *Suppose the function $\Lambda(t)$ of Eq. (6.4) is concave for all $t \in [0,T]$. A portfolio $\pi \in \mathcal{A}_{\mathbb{H}}$ is optimal if and only if*

$$[\mu'(t,\pi(t)) - r'(t,\pi(t))]\,\pi(t) + [\mu(t,\pi(t)) - r(t,\pi(t))] - \sigma^2(t)\pi(t) + \sigma(t)\alpha(t) = 0 \tag{6.6}$$

Let us revisit the simple case in which the insider knows the final value of the Brownian motion. That is, we take $\sigma(t) \neq 0$ and $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(W_{T_0})$ where $T_0 > T$. The following result follows immediately from substituting the expression for the associated information drift into Eq. (6.1.2):

Corollary 6.1.3. *Suppose the insider's filtration is given by* $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(W_{T_0})$ *and that* $\Lambda(t)$ *is concave for all* $t \in [0, T]$. *A portfolio* $\pi \in \mathcal{A}_{\mathbb{H}}$ *is optimal if and only if*

$$(\mu'(t, \pi(t)) - r'(t, \pi(t)))\pi(t) + (\mu(t, \pi(t)) - r(t, \pi(t))) - \sigma^2(t)\pi(t) + \sigma(t)\frac{W(T_0) - W(t)}{T_0 - t} = 0 \qquad (6.7)$$

We present below what is perhaps the simplest large insider model, in which the drift is modelled to be linearly dependent on the portfolio of the insider:

Corollary 6.1.4. *Suppose that the conditions of Theorem 6.1.2 hold. Suppose also that* $r(t, \pi) = r(t)$ *and*

$$\mu(t, \pi) = \mu_0(t) + b(t)\pi \qquad (6.8)$$

where $b(t)$ *and* $\mu_0(t)$ *are adapted to* $\mathbb{F}$ *and* $0 \leq b(t) \leq \frac{1}{2}\sigma^2(t)$. *Then the optimal insider strategy* π^* *is given by*

$$\pi^*(t) = \frac{\mu_0(t) - r(t) + \sigma(t)\alpha(t)}{\sigma^2(t) - 2b(t)} \qquad (6.9)$$

Proof. Since r doesn't depend on π we have $r'(t, \pi) = 0$ and of course $\mu'(t, \pi) = b(t)$. Therefore the function Λ of Eq. (6.4) is concave by Eq. (6.5) and π^* as defined in Eq. (6.9) is optimal by Theorem 6.1.2. □

We have shown above how a large insider may take advantage of his privileged position to find an optimal investment strategy, but an equally important consideration is how a small investor may benefit from simply being aware that a large trader exists in the market. How can a small honest investor utilize such knowledge? This is dealt with in the next section.

6.1.1 An honest trader in a market influenced by a large insider

We begin by stating some of the modelling assumptions used in [26], [25] and [24]. In our study of partial information (see Section 4.3) we assumed that the trader's filtration satisfied $\mathcal{H}_t \subseteq \mathcal{F}_t$ for all t, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by the Brownian motion. We shall now abandon this assumption in favour of a more realistic assumption: the small trader's filtration $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ is generated by the stock price process, so that

$$\mathcal{H}_t = \sigma(S(s); 0 \leq s \leq t) \tag{6.10}$$

Remark 6.1.5. When the coefficients are constant, knowledge about the stock price process is equivalent to knowledge about the underlying Brownian motion. In most cases, however, this assumption is simply not realistic and the small trader observes the evolution of the stock price rather than the Brownian motion.

Note that the quadratic variation of S is given by

$$\langle S\rangle_t = \int_0^t \sigma^2(s)S^2(s)ds; \quad 0 \leq t \leq T \tag{6.11}$$

from which it follows that the volatility $\sigma(t)$ is adapted to $\mathbb{H}$.

In Corollary 6.1.4 it was shown that, if the stock price is modelled by

$$dS(t) = S(t)[(\mu + b\pi(t))dt + \sigma d^- W(t)] \tag{6.12}$$

(i.e. the drift depends linearly on the insider's portfolio), and if we assume that $W(t)$ is a $\mathbb{G}$-semimartingale with information drift $\alpha(t)$, then the optimal portfolio of the insider is given by

$$\pi^*(t) = \frac{\mu - r + \sigma\alpha(t)}{\sigma^2 - 2b}$$

If the honest trader (with filtration given by Eq. (6.10)) suspects that the insider influences the market in the above manner, he could use the following model:

$$dS(t) = S(t)\left[\mathbb{E}[\mu + b\pi^*(t)|\mathcal{H}_t]dt + \sigma d\tilde{W}(t)\right] \tag{6.13}$$

where $\tilde{W}(t)$ is an $\mathbb{H}$-Brownian motion. In this case no forward integrals or enlargements are necessary and we simply use the Merton solution (see Example 4.3.2) to compute the expression for the honest trader's optimal portfolio:

$$\begin{aligned}
\tilde{\pi}^*(t) &= \frac{\mu(t) - r(t)}{\sigma^2(t)} \\
&= \frac{\mathbb{E}[\mu + b\pi^*(t)|\mathcal{H}_t] - r}{\sigma^2} \\
&= \frac{\mu - r}{\sigma^2} + \frac{b}{\sigma^2}\mathbb{E}[\pi^*(t)|\mathcal{H}_t] \\
&= \frac{\mu - r}{\sigma^2} + \frac{b}{\sigma^2}\mathbb{E}\left[\frac{\mu - r + \sigma\alpha(t)}{\sigma^2 - 2b}|\mathcal{H}_t\right] \\
&= \frac{(\mu - r)(\sigma^2 - b)}{\sigma^2(\sigma^2 - 2b)} + \frac{b}{\sigma(\sigma^2 - 2b)}\mathbb{E}[\alpha(t)|\mathcal{H}_t]
\end{aligned}$$

The drift from Eq. (6.13) can be substituted into Eq. (4.24) to obtain the optimal utility gain for the honest trader using the model of Eq. (6.13):

$$J(\tilde{\pi}^*) = \frac{(\mu - r)^2(\sigma^2 - b)^2}{2\sigma^2(\sigma^2 - 2b)^2} t + \frac{b^2}{2(\sigma^2 - 2b)^2} \int_0^t \mathbb{E}\Big[\mathbb{E}[\alpha(s)|\mathcal{H}_s]^2\Big] ds$$

Note that the cross-terms from Eq. (4.24) involving $\mathbb{E}[\alpha(t)|\mathcal{H}_t]$ are zero since

$$\mathbb{E}\left[\int \mathbb{E}[\alpha(s)|\mathcal{H}_s]ds\right] = \mathbb{E}\left[\int \alpha(s)ds\right] = \mathbb{E}\left[\int dW(s) - \int d\tilde{W}(s)\right] = 0.$$

It is claimed in Kohatsu [24] that the actual utility of the honest trader using the portfolio $\tilde{\pi}^*$, derived from the true model of Eq. (6.12), is given by

$$J(\tilde{\pi}^*) + \frac{b}{\sigma^2 - 2b} \int_0^t \mathbb{E}\Big[\mathbb{E}[\alpha(s)|\mathcal{H}_s]^2\Big] ds$$

This result is interpreted by [24] to mean that "if you make more money than you expect with your models it may be because there is a large investor exercising an influence on the price". Since we have certain reservations about this interpretation we refer to Section 11 of [24] and Section 6 of [26] for further discussions about the role of a small trader in an insider-influenced market.

6.1.2 Continuous flow of large-insider information

Here we continue along the lines of the previous section. in that we consider a model in which the drift is anticipating and depends on a future value of the underlying Brownian motion. The novel feature of this section is that the drift does not only depend on a single future value of the Brownian motion, but rather it depends continuously on a stream of future values. The drift 'foresees' the Brownian motion by δ time units. This is modelled by setting

$$\mu(t) = \mu + bW(t + \delta)$$

so that the market model is now given by

$$dS(t) = S(t)[(\mu + bW(t + \delta))dt + \sigma dW(t)] \tag{6.14}$$

For now we shall assume that $\delta \geq T$ to simplify calculations. We still assume that the honest trader observes the stock price filtration, so that $\mathcal{H}_t = \sigma(S(s); s \leq t)$. The solution of Eq. (6.14) is given by

$$\begin{aligned} S(t) &= S(0) \exp\left((\mu - \frac{1}{2}\sigma^2)t + b\int_0^t W(s + \delta)ds + \sigma W(t)\right) \\ &= S(0) \exp\left((\mu - \frac{1}{2}\sigma^2)t + b\int_\delta^{t+\delta} W(s)ds + \sigma W(t)\right) \end{aligned}$$

so we have

$$\mathcal{H}_t = \sigma\left(b\int_\delta^{s+\delta} W(u)du + \sigma W(s); s \le t\right) \tag{6.15}$$

In order to find the honest trader's optimal portfolio we will compute $a(t)$, as defined in Eq. (4.19). The method of solution involves finding the covariance between the underlying Brownian motion and the process that generates the honest trader's filtration, which leads to a second order p.d.e. which we then solve as an intermediate step in finding the honest trader's optimal portfolio.

Theorem 6.1.6. *Suppose $\delta \ge T$ and define $Y(t) := b\int_\delta^{t+\delta} W(r)dr + \sigma W(t)$ (i.e. $\mathcal{H}_t = \sigma(Y(s); s \le t)$). Then*

$$a(t) = \sigma b M \int_0^t g(t,u)dY(u) \tag{6.16}$$

and

$$\mathbb{E}[W(t+\delta)|\mathcal{H}_t] = (b(t+\delta)+\sigma)M\int_0^t g(t,u)dY(u) \tag{6.17}$$

where

$$M = M(t) = \sigma^{-1}\left[(bs+2\sigma)\left(e^{\frac{2bt}{\sigma}}-1\right)+\sigma\left(e^{\frac{2bt}{\sigma}}+1\right)\right]^{-1}$$

and

$$g(t,u) = e^{\frac{b}{\sigma}(2t-u)} + e^{\frac{b}{\sigma}u}$$

Proof. We follow [26]. It is clear that Y is a Gaussian process. This allows us to write

$$\mathbb{E}[W(s)|\mathcal{H}_t] = \int_0^t h(s,t,u)dY(u)$$

where h is a deterministic function. For $\nu \le t \le T$ and $s \le T$ we obtain

$$\begin{aligned}
\mathbb{E}[W(s)Y(\nu)] &= \mathbb{E}[W(s)b\int_\delta^{\nu+\delta} W(r)dr] + \mathbb{E}[W(s)\sigma W(\nu)] \\
&= b\int_\delta^{\nu+\delta} \mathbb{E}[W(s)W(r)]dr + \sigma(s\wedge\nu) \\
&= b\int_\delta^{\nu+\delta} s dr + \sigma(s\wedge\nu) \\
&= bs\nu + \sigma(s\wedge\nu) \qquad (6.18)
\end{aligned}$$

where we have used the common expression for the covariance of two Brownian motions and the fact that $\nu \le T \le \delta$. We also obtain

$$\begin{aligned}
\mathbb{E}\left[\int_0^t h(s,t,u)dY(u)Y(\nu)\right] &= b^2\int_0^t\int_0^\nu h(s,t,\theta_1)(\theta_1\wedge\theta_2+\delta)d\theta_2 d\theta_1 \\
&+ 2b\sigma\nu\int_0^t h(s,t,\theta)d\theta + \sigma^2\int_0^\nu h(s,t,\theta)d\theta \qquad (6.19)
\end{aligned}$$

Note that

$$\begin{aligned} \mathbb{E}\left[\int_0^t h(s,t,u)dY(u)Y(\nu)\right] &= \mathbb{E}[\mathbb{E}[W(s)|\mathcal{H}_t]Y(\nu)] \\ &= \mathbb{E}[\mathbb{E}[W(s)Y(\nu)|\mathcal{H}_t]] = E[W(s)Y(\nu)] \end{aligned}$$

where we have used the fact that $Y(\nu)$ is $\mathcal{H}_t$-measurable since $\nu \leq t$. Thus the two expressions (6.18) and (6.19) are equal. Next, we obtain a second order partial differential equation by partially differentiating the equality w.r.t. ν three times. The first derivative yields

$$bs + \sigma 1_{[0,s]}(\nu) = b^2 \int_0^t h(s,t,\theta)(\theta \wedge \nu + \delta)d\theta + 2b\sigma \int_0^t h(s,t,\theta)d\theta + \sigma^2 h(s,t,\nu)$$

the second derivative yields

$$0 = b^2 \int_\nu^t h(s,t,\theta)d\theta + \sigma^2 \frac{\partial h}{\partial \nu}(s,t,\nu)$$

and the third derivative yields

$$-b^2 h(s,t,u) + \sigma^2 \frac{\partial^2 h}{\partial u^2}(s,t,u) = 0 \tag{6.20}$$

The solution is easily found to be

$$h(s,t,u) = C_1(s,t)e^{-\frac{b}{\sigma}u} + C_2(s,t)e^{\frac{b}{\sigma}u} \tag{6.21}$$

Note that by setting $\nu = 0$ after differentiating the equality once and setting $\nu = t$ after the second differentiation we obtain the initial conditions $bs + \sigma = b(b\delta + 2\sigma)\int_0^t h(s,t,\theta)d\theta + \sigma^2 h(s,t,0)$ and $\frac{\partial h}{\partial u}(s,t,t) = 0$. Substituting Eq. (6.21) into the second condition yields

$$C_1(s,t) = e^{\frac{2bt}{\sigma}} C_2(s,t) \tag{6.22}$$

and using the first condition yields

$$C_2(s,t) = \sigma^{-1}(bs+\sigma)\left[(bs+2\sigma)\left(e^{\frac{2bt}{\sigma}} - 1\right) + \sigma\left(e^{\frac{2bt}{\sigma}} + 1\right)\right]^{-1} \tag{6.23}$$

so that Eq. (6.21) can be written as

$$h(s,t,u) = (bs+\sigma)Mg(t,u) \tag{6.24}$$

It is now straightforward to find $a(t)$:

$$\begin{aligned} a(t) &= \lim_{h\to 0^+}\frac{1}{h}\mathbb{E}[\int_h^{t+h}\sigma(s)d^-W(s)|\mathcal{H}_t] \\ &= \sigma\lim_{h\to 0^+}\frac{1}{h}\mathbb{E}[W(t+h)-W(t)|\mathcal{H}_t] \\ &= \sigma\lim_{h\to 0^+}\frac{1}{h}\int_0^t (h(t+h,t,u)-H(t,t,u))dY(u) \\ &= \sigma\lim_{h\to 0^+}\frac{1}{h}\int_0^t \left[(b(t+h)+\sigma)Mg(t,u)-(bt+\sigma)Mg(t,u)\right]dY(u) \\ &= bM\int_0^t g(t,y)dY(u) \end{aligned}$$

Of course we also have $\mathbb{E}[W(t+\delta)|\mathcal{H}_t] = (b(t+\delta)+\sigma)M\int_0^t g(t,u)dY(u)$ by the same type of calculation. Finally, using Corollary 4.2.2, we obtain

$$\begin{aligned} \pi^*(t) &= \frac{1}{\sigma^2}\left[\mathbb{E}[\mu(t)-r|\mathcal{H}_t]+a(t)\right] \\ &= \frac{\mu-r}{\sigma^2}+\mathbb{E}[W(t+\delta)|\mathcal{H}_t]+a(t) \\ &= \frac{\mu-r}{\sigma^2}+\frac{M}{\sigma^2}(b(t+\delta+1)+\sigma)\int_0^t g(t,u)dY(u) \end{aligned}$$

using $\mu(t)=\mu+W(t+\delta)$ along with the expressions that were found for $a(t)$ and $\mathbb{E}[W(t+\delta)|\mathcal{H}_t]$. □

The above problem has only been solved for the case in which the drift is influenced by information that the honest trader only perceives $\delta \geq T$ time units in the future. The interesting case where $\delta < T$ has been studied in Theorem 18 of [25].

6.2 Independent noise, insider friction and the absence of arbitrage

It is certainly possible to find situations in which a small honest trader has finite optimal utility (and hence no arbitrage) in a market with an insider, but we have seen that it is always the case that the insider can exercise arbitrage when in possession of exact information about future prices. Naturally, we are interested in the possibility of a heterogeneously informed market that is completely free of arbitrage; that is, neither the honest trader nor the insider may exercise arbitrage. We have seen that Hypothesis (E) of Chapter 5 is sufficient to guarantee that no-arbitrage for the honest trader implies no-arbitrage for the insider. Unfortunately, when the inside information is exact, this result is only true for the period $[0, T)$ (i.e. the terminal time is excluded). One example for which it was found that the market is arbitrage-free for the entire period $[0, T]$ is when the insider has initial information distorted by a source of Gaussian noise: $G = \lambda W(T) + (1-\lambda)\epsilon$. The problem with this approach is that the investor's information doesn't improve as the trading interval draws to a close. In this section we explore several other possibilities in order to construct models which are free of arbitrage for both honest traders and insiders.

For the remainder of this chapter, we will only consider the behaviour of the insider. We will also drop the assumption of anticipating coefficients, so that all coefficients are $\mathcal{F}_t$-adapted. From now on $\mathbb{G}$ will be used to denote the insider's filtration. Also, we will no longer assume that the insider is a large trader.

6.2.1 Diminishing independent noise

We have mentioned that the initial enlargement approach used to model distorted inside information described in Example 5.2.5 is unrealistic in the sense that the information held by the insider does not improve as $t \to T$ (i.e. as the trading interval draws to a close). In the last few years several articles have been published that attempt to rectify this problem by describing models in which the magnitude of the blurring term decreases as $t \to T$, which means that the insider's knowledge improves as the end of the trading interval draws nearer. We will study the model proposed in [24] and [25].

We assume that the insider's extra information changes continuously, and is given by

$$X(t) = W_T + W'((T-t)^\alpha)$$

where W' is a Brownian motion that is independent of W. The insider's filtration is given by $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$, where

$$\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(X_u; u \leq s)$$

We begin by finding the semimartingale decomposition of W in the enlarged filtration:

Theorem 6.2.1. *W is a semimartingale in $\mathbb{G}$ with decomposition*

$$W_t = \tilde{W}_t + \int_0^t \frac{X(s) - W(s)}{T - s + (T-s)^\alpha} ds \tag{6.25}$$

where $\tilde{W}$ is a $\mathbb{G}$-Brownian motion.

Proof. Firstly, we note that elements of the insider's filtration $\mathbb{G}$ can be reexpressed as

$$\mathbb{G} = \mathcal{F}_s \vee \sigma(X(s) - W_s) \vee \sigma(W'((T-s)^\alpha) - W'((T-v)^\alpha) : v \leq s)$$

where $s \leq u \leq t$. We thus have

$$\begin{aligned} & \mathbb{E}[W_t - W_u | \mathcal{G}_s] \\ = \; & \mathbb{E}[W_t - W_u | \mathcal{F}_s \vee \sigma(X(s) - W_s) \vee \sigma(W'((T-s)^\alpha) - W'((T-v)^\alpha) : v \leq s)] \\ = \; & \mathbb{E}[W_t - W_u | X(s) - W_s] \end{aligned}$$

due to the independence of $W_t - W_u$ and the σ-algebras $\mathcal{F}_s$ and $\sigma(W'((T-s)^\alpha) - W'((T-v)^\alpha) : v \leq s)$. Also, since $W_t - W_u$ and $X(t) - W_s$ are both Gaussian with mean zero we can use the fact that

$$\mathbb{E}[X|Y] = \frac{\text{Cov}(X,Y)}{\text{Var}(Y)} Y$$

for a mean zero Gaussian random vector (X, Y). Therefore we obtain

$$\begin{aligned} \mathbb{E}[W_t - W_s | \mathcal{G}_s] &= \frac{\text{Cov}(W_t - W_u, W_T + W'((T-s)^\alpha) - W_s)}{\text{Var}(W_T + W'((T-s)^\alpha) - W_s)} (X(s) - W_s) \\ &= \frac{t-u}{T - s + (T-s)^\alpha} (X(s) - W_s) \end{aligned}$$

using the linearity properties of the covariance function. Similarly, we obtain

$$\begin{aligned} & \mathbb{E}[W'((T-u)^\alpha) | \mathcal{G}_s] \\ = \; & \frac{\text{Cov}(W'((T-u)^\alpha), W_T + W'((T-s)^\alpha) - W_s)}{\text{Var}(W_T + W'((T-s)^\alpha) - W_s)} (X(s) - W_s) \\ = \; & \frac{(T-u)^\alpha}{T - s + (T-s)^\alpha} (X(s) - W_s) \end{aligned}$$

Using Eq. (6.25), we now compute $\mathbb{E}[\tilde{W}_t - \tilde{W}_s | \mathcal{G}_s]$ to check that $\tilde{W}$ is a martingale.

$$\begin{aligned} & \mathbb{E}[\tilde{W}_t - \tilde{W}_s | \mathcal{G}_s] \\ = \; & \mathbb{E}\left[W_t - W_s - \int_s^t \frac{X(u) - W_u}{T - u + (T-u)^\alpha} du \middle| \mathcal{G}_s \right] \\ = \; & \frac{t-u}{T - s + (T-s)^\alpha} (X(s) - W_s) - \int_s^t \frac{\mathbb{E}[W_T + W'((T-u)^\alpha) - W_u | \mathcal{G}_s]}{T - u + (T-u)^\alpha} du \\ = \; & \frac{t-u}{T - s + (T-s)^\alpha} (X(s) - W_s) \\ & - \int_s^t \frac{1}{T - u + (T-u)^\alpha} \cdot \frac{T - u + (T-u)^\alpha}{T - s + (T-s)^\alpha} (X(s) - W_s) du = 0 \end{aligned}$$

We thus have that $\tilde{W}$ is a $\mathbb{G}$-Brownian motion by Lévy's characterization. □

Now, we recall from Section 5.1 that the additional expected logarithmic utility at time t of an insider with information drift μ^G is given by

$$U^a(t) = \frac{1}{2}\mathbb{E}\left[\int_0^t (\mu_s^G)^2 ds\right]$$

From this we obtain

$$\begin{aligned} U^a(T) &= \frac{1}{2}\mathbb{E}\left[\int_0^T \left(\frac{X(s) - W(s)}{T - s + (T-s)^\alpha}\right)^2 ds\right] \\ &= \frac{1}{2}\int_0^T \frac{\mathbb{E}[X^2(s)] - 2\mathbb{E}[X(s)W(s)] + \mathbb{E}[W^2(s)]}{(T - s + (T-s)^\alpha)^2} ds \\ &= \frac{1}{2}\int_0^T \frac{(T + (T-s)^\alpha) - 2s + s}{(T - s + (T-s)^\alpha)^2} ds \\ &= \frac{1}{2}\int_0^T \frac{1}{T - s + (T-s)^\alpha} ds \end{aligned}$$

It follows that the insider's additional expected utility is finite as long as $\alpha < 1$. Finally, this means that there are no arbitrage opportunities for the insider in the above model. We have thus found an arbitrage-free model that is more realistic than the distorted information model proposed in Example 5.2.5 of Chapter 5, since the insider's information improves as time passes.

6.2.2 Penalty functions

Another way to constuct an arbitrage-free market is to assume that the insider's behaviour is subject to certain constraints, so that the range of admissible insider strategies is effectively narrowed down. In Section 5 of Pikovsky and Karatzas [36] the insider's portfolio is constrained to exist in a certain subset K of $\mathbb{R}$ and it is shown that the optimization problem is finite iff K is compact. In Hu and Øksendal [19] the optimization problem itself is modified by subtracting a term from the utility, which models the assumption that an insider is penalized for conspicuously large portfolio fluctuations or large volumes of trade.

In an attempt to design a model in which the optimal insider portfolio is finite, we follow [19] and introduce the concept of a penalty function $\mathbb{Q} : \mathcal{A}_{\mathbb{G}} \to \mathcal{A}_{\mathbb{G}}$. $\mathbb{Q}$ is a linear operator that acts on the portfolio π with the effect of providing a quantitative measure of the volume of trade or the fluctuations of the portfolio. For example we could take

$$\mathbb{Q}\pi(t) = \lambda(t)\pi(t)$$

where λ is a weighting function. This would quantify the volume of trade. Taking

$$\mathbb{Q}\pi(t) = \lambda(t)\pi'(t)$$

where $\pi'(t) = \frac{d\pi(t)}{dt}$ would quantify the trade fluctuations. These penalty functions are then used to modify Problem 4.1.5 of Section 4.1 as follows:

PROBLEM 6.2.2. *Find* $\pi^* \in \mathcal{A}_{\mathbb{G}}$ *and* $J_{pen}(\pi^*) \in \mathbb{R}$ *such that*

$$J_{pen}(\pi^*) = \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} J_{pen}(\pi) \tag{6.26}$$

where

$$J_{pen}(\pi) = J(\pi) - \frac{1}{2}\mathbb{E}\left[\int_0^T |\mathbb{Q}\pi(t)|^2 dt\right] \tag{6.27}$$

and J *is as defined in Eq.(4.8).*

In the above problem, $\mathcal{A}_{\mathbb{G}}$ is a slight modification of the set of admissible portfolios described in Def. 4.1.3:

Definition 6.2.3. $\mathcal{A}_{\mathbb{G}}$ *is the set of all processes* π *such that the conditions of Definition 4.1.3 hold and also*

$$\mathbb{E}\left[\int_0^T |\mathbb{Q}\pi(t)|^2 dt\right] < \infty \tag{6.28}$$

Problem 6.2.2 can now be solved by a method similar to that of Problem 4.1.5; i.e. by a variational calculus technique. We have

$$\begin{aligned} J_{pen}(\pi) \;=\; & \mathbb{E}\Big[\int_0^T \Big\{(\mu(t) - r(t))\pi(t) - \frac{1}{2}\sigma^2(t)\pi^2(t)\Big\} dt + \int_0^T \sigma(t)\pi(t)d^-W(t) \\ & -\frac{1}{2}\int_0^T |\mathbb{Q}\pi(t)|^2 dt\Big] \end{aligned} \tag{6.29}$$

Suppose π^* is optimal for Problem 6.2.2. Then we have

$$J(\pi^*) \geq J(\pi^* + \delta\beta)$$

for all $\beta \in \mathcal{A}_{\mathbb{G}}$ and $\delta \in \mathbb{R}$. Since $J(\cdot)$ attains its maximum value at π^*, we have

$$\frac{d}{d\delta}J(\pi^* + \delta\beta)|_{\delta=0} = 0$$

Using the definition of J_{pen} and performing simple differentiation, we have

$$\begin{aligned} \mathbb{E}\Big[\int_0^T (\mu(u) - r(u) - \sigma^2(u)\pi^*(u))\beta(u)du + \int_0^T \beta(u)\sigma(u)d^-W(u) \\ - \int_0^T \mathbb{Q}\pi(u)\mathbb{Q}\beta(u)du\Big] = 0 \end{aligned} \tag{6.30}$$

for all $\beta \in \mathcal{A}_{\mathbb{G}}$. Next we define $\mathbb{Q}^*$ to be the adjoint operator of $\mathbb{Q}$ in $L^2([0,T]\times\Omega)$:

$$\mathbb{E}\left[\int_0^T a(t)(\mathbb{Q}b)(t)dt\right] = \mathbb{E}\left[\int_0^T (\mathbb{Q}^*a)(t)b(t)dt\right] \tag{6.31}$$

so that

$$\mathbb{E}\left[\int_0^T \mathbb{Q}\pi(u)\mathbb{Q}\beta(u)du\right] = \mathbb{E}\left[\int_0^T (\mathbb{Q}^*\mathbb{Q}\pi(u))\,\beta(u)du\right] \tag{6.32}$$

and Eq. (6.30) can be rewritten as

$$\mathbb{E}\left[\int_0^T (\mu(u) - r(u) - \sigma^2(u)\pi^*(u) - \mathbb{Q}^*\mathbb{Q}\pi^*(u))\beta(u)du + \int_0^T \beta(u)\sigma(u)d^-W(u)\right] = 0$$

If we now choose $\beta(u) = \beta_0(t)1_{[t,s]}(u)$, where $0 \leq t \leq s \leq T$ and $\beta_0(t)$ is $\mathcal{G}_t$-measurable and bounded, it is easy to follow the exact method of proof of Theorem 4.2.1 to obtain the following result:

$$M(t) := \mathbb{E}\left[\int_0^T (\mu(u) - r(u) - \sigma^2(u)\pi^*(u) - \mathbb{E}[\mathbb{Q}^*\mathbb{Q}\pi^*(u)|\mathcal{G}_u])du + \int_0^T \sigma(u)d^-W(u)\right]$$

is a $\mathbb{G}$-martingale. This can be rewritten as

$$dW(t) = \sigma^{-1}(t)dM(t) - \sigma^{-1}(t)\{\mu(t) - r(t) - \sigma^2(t)\pi^*(t) - \mathbb{E}[\mathbb{Q}^*\mathbb{Q}\pi^*(t)|\mathcal{G}_t]\}dt \tag{6.33}$$

where the forward integral has been replaced by an ordinary integral since $\sigma(t)$ is adapted to the Brownian filtration $\mathcal{F}_t$. It is clear that the first term on the right of the equality is a $\mathbb{G}$-martingale since $M(t)$ is a $\mathbb{G}$-martingale, and the second term is of finite variation, so that we have obtained

Theorem 6.2.4. *If an optimal insider portfolio exists for Problem 6.2.2 then $W(t)$ is a $\mathbb{G}$-semimartingale.*

Note that this is analogous to the result of Theorem 4.5.1 of Chapter 4. Our next task is to find an explicit expression for the optimal insider portfolio. Let us assume that there is a semimartingale decomposition of $W(t)$ of the form

$$W(t) = \sigma^{-1}(t)M(t) + \int_0^t \alpha(s)ds$$

By comparing the above with Eq. (6.33) we obtain

$$\int_0^t \alpha(s)ds = -\int_0^t \sigma^{-1}(s)\{\mu(s) - r(s) - \sigma^2(s)\pi^*(s) - \mathbb{E}[\mathbb{Q}^*\mathbb{Q}\pi^*(s)|\mathcal{G}_s]\}ds$$

and finally

$$\sigma^2(t)\pi(t) - \mathbb{E}[\mathbb{Q}^*\mathbb{Q}\pi(t)|\mathcal{G}_t] = \mu(t) - r(t) + \sigma(t)\alpha(t) \tag{6.34}$$

Theorem 6.2.5. *Suppose that an optimal insider portfolio π^* exists for Problem 6.2.2 and that $W(t)$ can be expressed in the form given by Eq. (6.33). Then π^* is the solution of Eq. (6.34).*

We shall now apply the above result to an example in which the penalty function serves to penalize the insider for engaging in large volumes of trade.

Example 6.2.6. *Suppose the penalty function is given by*

$$\mathbb{Q}\pi(t) = \lambda(t)\sigma(t)\pi(t)$$

where $\lambda(t) \geq 0$ is deterministic. (The factor of $\sigma(t)$ is included to simplify the subsequent calculations.) We clearly have $\mathbb{Q}(\pi) = \mathbb{Q}^*(\pi)$ so that, using Eq. (6.34), the solution of the optimal portfolio problem is given by

$$\pi^*(t) = \frac{\mu(t) - r(t) + \sigma(t)\alpha(t)}{\sigma^2(t)[1 + \lambda^2(t)]}$$

as long as this π^* is admissible. We can substitute the above expression for π^* into the expression for the insider value given by Eq. (6.29) to obtain

$$\begin{aligned} J_{pen}(\pi^*) &= \mathbb{E}\left[\int_0^T \left((\mu(t) - r(t) + \sigma(t)\alpha(t))\pi^*(t) - \frac{1}{2}(1 + \lambda^2(t))(\pi^*)^2(t)\right)\right] dt \\ &= \frac{1}{2}\mathbb{E}\left[\int_0^T \frac{(\mu(t) - r(t) + \sigma(t)\alpha(t))^2}{\sigma^2(t)(1 + \lambda^2(t))} dt\right] \end{aligned}$$

where we have used $W(t) = \tilde{W}(t) + \int_0^t \alpha(s)ds$ and $\mathbb{E}[\int \ldots d\tilde{W}(t)] = 0$ to eliminate the forward integral term.

Let us take this example further by considering the case in which the insider knows the terminal value of the Brownian motion. We then have $\alpha(t) = (W(T) - W(t))/(T - t)$, so that

$$J_{pen}(\pi^*) = \frac{1}{2}\mathbb{E}\left[\int_0^T \frac{1}{1 + \lambda^2(t)}\left(\frac{\mu(t) - r(t)}{\sigma(t)} + \frac{W(T) - W(t)}{T - t}\right)^2 dt\right]$$

In [19] it is further assumed that $\sigma(t) \geq \sigma_0 > 0$ and

$$\lambda(t) = (T - t)^{-\beta}$$

where $\beta > 0$, from which it follows that the maximal expected change in utility over $[0, T]$ is finite:

$$J_{pen} \leq C_1 + C_2 \int_0^T (T - t)^{-1+2\beta} dt < \infty$$

We have thus constructed a (somewhat artificial) situation in which the value problem for the insider, who possesses exact information about the future, is finite.

Chapter 7

Conclusion

The intention behind this dissertation was to provide a systematic account of the various mathematical approaches that have been used to model heterogeneously informed markets in the recent literature. We have distinguished between two principal approaches that are used in such models; namely, anticipative stochastic calculus (including Malliavin's calculus) and enlargement of filtrations. By and large, the studies in the literature tend to focus on one or other of these general approaches, and few resources exist that develop both approaches with equal rigour.[1] In Part I of this project, we have given broad surveys of the relevent mathematical results, covering both anticipating calculus *and* enlargements of filtrations. In Part II we have studied how each approach may be applied to the study of heterogeneously informed investors.

Broadly speaking, we have found that the formalism of anticipative calculus is a natural tool in the construction of a general market model that accomodates for such heterogeneously informed agents. In modelling the stock price dynamics by

$$dS(t) = S(t)[\mu(t)dt + \sigma(t)d^{-}W(t)]$$

we have the corresponding discounted value process

$$dX(t) = X(t)[(\mu(t) - r(t))\pi(t)dt + \sigma(t)\pi(t)d^{-}W(t)]$$

The strength of the above formulation is that we do not need the *a priori* semimartingale assumption required by the enlargement of filtrations approach, since the forward integral does not require that W is a semimartingale w.r.t. the trader's filtration. In addition, the anticipative calculus framework may be applied to model an anticipating market, in which the coefficients of the dynamics are adapted to a filtration that is larger than the market filtration. We have seen that such a

[1]The results of [20] and [21] may be considered exceptions in some sense as they use Malliavin's calculus to obtain an enlargement formula for the insider.

scenario may occur when a "large" trader influences the price dynamics. In Chapter 4 we have studied the afore-mentioned anticipating market, solving the problem of optimizing the expected logarithmic utility for both partially informed traders and insiders. In the current academic literature, the problems of anticipative coefficients and partially informed traders have not been tackled using an enlargement approach. Also, the assumption required by the enlargement approach that the price process is a semimartingle w.r.t. the insider's filtration is often difficult or impossible to verify. Indeed, even when it can be verified, the semimartingale decomposition is very specific to the type of extra information available, which prevents the enlargement approach from achieving the level of generality of the forward integral approach.

On the other hand, we have seen in Theorem 4.5.1 that if an optimal insider portfolio exists then a semimartingale decomposition does indeed exist for W. In Theorem 2.5.5 of Chapter 2 we found that if this is the case then, for an anticipating process ϕ, the forward integral coincides with the ordinary stochastic integral:

$$\int_0^T \phi(t)d^-W(t) = \int_0^T \varphi(t)dW(t)$$

This led us, in Chapter 5, to solve the problem of portfolio maximization for an insider by enlargement techniques. As a result we found that, although the enlargement framework could be viewed as being less general than the anticipative calculus framework, it is essential when we desire explicit results for specific examples of additional information.[2]

In Chapter 6 we saw how various adjustments could be made to the models of earlier chapters in order to build models that aim to better represent reality. We have seen how a market with "large" traders may be modelled, and how certain features such as dynamic time-dependent additional information or trade restrictions can lead to finite values for the optimal portfolio problem. In spite of these adjustments however, we should note that many of the examples considered in the literature are rather artificial. In light of this, we should perhaps be cautious in overemphasizing the current "real world" applicability of some of these results.

That being said, it is clear that the mathematical techniques described in this thesis are, for the most part, very well-suited to model markets with differentially informed agents. We should bear in mind that most of the models have only appeared in the last decade or so, and with some further development we can fully

[2]When we say that the anticipative calculus approach is more general than the enlargement approach we should clarify that we are comparing the two approaches in a Brownian setting. The enlargement approach may be considered more general in another sense; that is, we have developed enlargement techniques for a quite general semimartingale model (i.e. not necessarily a Brownian model) whereas we have only developed forward integrals with respect to Brownian motion.

expect this exciting new field to find considerable applicability in the world of finance.

Further developments

Due to the sheer number of papers that now exist on the topic of differential information, it has been necessary to be somewhat selective in our presentation. One notable example has been in our treatment of enlargements, where we have largely restricted ourselves to the case of initial enlargements.[3] We have mentioned that this is somewhat unrealistic in the sense that the insider's additional information doesn't improve as time progresses. Some notable papers that employ a more general enlargement approach include Ankirchner [4], Ankirchner, Dereich and Imkeller [5] and Corcuera, et al [11].

We have also assumed, for the majority of this thesis, that the market is driven by a continuous Brownian motion.[4] Although this is a fairly typical assumption in mathematical finance, a lot of interest has recently been generated in modelling markets that are driven by Lévy processes, which allow for the possibility of jumps in the stock price. The assumption of an underlying Lévy process driving the market has been addressed in several articles on insider trading, including di Nunno, et al [13], [14] and Øksendal and Sulem [35].

One final feature that has not been studied in this thesis is the concept of *weak information*. We have seen in several examples that an insider may obtain infinite expected utility when in possession of exact information about the future. We resolved this problem to some extent by considering examples in which the insider's information is blurred by a noise term. Another approach, which is perhaps more realistic, is to assume that instead of having direct knowledge of a future random variable G, the insider instead knows the *law* of G under the market probability. This approach, which leads to finite utility for the insider, has been studied in Baudoin [7].

[3]Chapter 5 was exclusively concerned with initial enlargements, but recall that we studied an example of a dynamical enlargement in Chapter 6.

[4]Recall, however, that we dealt with a more general semimartingale model in Chapter 5.

Bibliography

[1] J. Amendinger, *Initial Enlargement of Filtrations and Additional Information in Financial Markets.* PhD thesis, Humboldt University, Berlin, (1999).

[2] J. Amendinger, Martingale representation theorems for initially enlarged filtrations. *Stochastic Processes and Their Applications*, 89:101-116, (2000).

[3] J. Amendinger, P. Imkeller and M. Schweizer, Additional logarithmic utility of an insider. *Stochastic Processes and Their Applications*, 75:263-286, (1998).

[4] S. Ankirchner, *Information and Semimartingales.* PhD Thesis, Humboldt University. Berlin, (2005).

[5] S. Ankirchner, S. Dereich and P. Imkeller, The Shannon information of filtrations and the additional logarithmic utility of insiders. *The Annals of Probability*, 34:743-778, (2006).

[6] K. Back, Insider trading in continuous time. *Review of Financial Studies*, 5:387-409, (1992).

[7] F. Baudoin, *Modelling anticipations in financial markets.* In Paris-Princeton Lectures on Mathematical Finance 2002. Lecture Notes in Mathematics 1814, Springer-Verlag, Berlin, (2003).

[8] R. Becker, *A course in Malliavin calculus.* From University of Cape Town seminar on Malliavin calculus and Lévy processes, (2007).

[9] F. Biagini and B. Øksendal, A general stochastic calculus approach to insider trading. *Applied Mathematics and Optimization.* 52:167-181, (2005).

[10] T. Björk, *Arbitrage Theory in Continuous Time.* Oxford University Press, (2004).

[11] J. M. Corcuera, P. Imkeller, A. Kohatsu-Higa and D. Nualart, Additional utility of insiders with imperfect dynamical information. *Finance and Stochastics*, 8:437-450, (2003).

[12] F. Delbaen and W. Schachermayer, A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300:463-520, (1994).

[13] G. Di Nunno, T. Meyer-Brandis, B. Øksendal and F. Proske. Optimal portfolio for an insider in a market driven by Levy processes. *Quantitative Finance*, 6:83-94, (2006).

[14] G. Di Nunno, A. Kohatsu-Higa, T. Meyer-Brandis, B. Øksendal, F. Proske and A. Sulem. Optimal portfolio for a "large" insider in a market driven by Levy processes. University of Oslo, Norway, Preprint Series in Pure Mathematics, 30, (2005).

[15] D. Duffie and C. Huang, Multiperiod security markets with differential information. *Journal of Mathematical Economics*, 15:283-303, (1986).

[16] H. Föllmer and P. Imkeller, Anticipation Cancelled by a Girsanov Transformation: A Paradox on Wiener Space. *Annales de l'Institut Henri Poincare*, 29:569-586, (1993).

[17] A. Grorud and M. Pontier, Comment détecter le délit d'initiés? *C.R. Acad. Sci. Paris Sér. I Math.* 324:1137-1142, (1997).

[18] A. Grorud and M. Pontier, Insider trading in a continuous time market model. *International Journal of Theoretical Finance*, 1:331-347, (1998).

[19] Y. Hu and B. Øksendal, Optimal Smooth Portfolio Selection for An Insider. *Journal of Applied Probability*, 44:742-752, (2007).

[20] P. Imkeller. Malliavin's calculus in insider models: additional utility and free lunches. *Mathematical Finance*, 13:153-169, (2003).

[21] P. Imkeller, M. Pontier and F. Weisz, Free lunch and arbitrage possibilities in a financial market with an insider. *Stochastic Processes and Their Applications*, 92:103-130, (2003).

[22] K. Ito. Extension of stochastic integrals. *Proc. Int. Symp. on Stochastic Differential Equations*, 95-109, Wiley, (1978).

[23] J. Jacod. *Grossissement de filtrations: exemples et applications.* LNM 1118. Springer, (1985).

[24] A. Kohatsu-Higa, *Insider models with finite utility.* Lecture notes, (2005).

[25] A. Kohatsu-Higa and A. Sulem, A large trader-insider model. Proceedings of the Ritsumeikan international symposium, March 2005, World Scientific.

[26] A. Kohatsu-Higa and A. Sulem, Utility maximization in an insider influenced market. *Mathematical Finance*, 16:153-179, (2006).

[27] A. Kyle, Continuous auctions and insider trading. *Econometrica*, 53:1315-1335, (1985).

[28] J.A. Leon, R. Navarro and D. Nualart. An anticipating calculus approach to the utility maximization of an insider. *Mathematical Finance*, 13:171-185, (2003).

[29] P. Malliavin, Stochastic calculus of variations and hypoelliptic operators. *Proc. Int. Symp. on Stochastic Differential Equations*, 195-263. Wiley, (1976).

[30] P. Malliavin and A. Thalmeyer, *Stochastic Calculus of Variations in Mathematical Finance*. Springer, (2006).

[31] R. Mansuy and M. Yor, *Random Times and Enlargements of Filtrations in a Brownian Setting*. Springer, (2006).

[32] D. Nualart, *The Malliavin Calculus and Related Topics*. Springer, (2006).

[33] D. Nualart and E. Pardoux, Stochastic calculus with anticipating integrands. *Probability Theory and Related Fields*, 78:535-581, (1988).

[34] B. Øksendal, *An Introduction to Malliavin Calculus with Applications to Economics*. Lecture Notes, Norwegian School of Economics and Business Administration, (1996).

[35] B. Øksendal and A. Sulem, Partial observation control in an anticipating environment. *Russian Mathematical Surveys*, 59:355-375, (2004).

[36] I. Pikovsky and I. Karatzas, Anticipative portfolio optimization. *Adv. in Applied Probability*, 28:1095-1122, (1996).

[37] P. Protter, *Stochastic Integration and Differential Equations*. Springer, (2004).

[38] F. Russo and P. Vallois, Forward, backward and symmetric stochastic integration. *Probability Theory and Related Fields*, 97:403-421, (1993).

[39] F. Russo and P. Vallois, The generalized covariation process and Ito formula. *Stochastic Processes and their Applications*, 59:81-104. (1995).

[40] F. Russo and P. Vallois, Stochastic calculus with respect to continuous finite quadratic variation processes. *Stochastics and Stochastic Reports*, 70:1-40, (2000).

[41] M. Steele, *Stochastic Calculus and Financial Applications*. Springer, (2001).

[42] M. Yor, *Some Aspects of Brownian Motion, Part II: Some Recent Martingale Problems.* Lectures in Mathematics ETH. Birkhäuser, (1997).