University of Cape Town

Department of Mathematics

FUNCTION ALGEBRAS, RESTRICTION SETS AND MEASURES

by

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A thesis prepared under the supervision
of Professor W. Kotzé for the degree of
Doctor of Philosophy in Mathematics.

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In accordance with the regulations governing the award of the degree of Ph.D. at the University of Cape Town, the candidate presents the following summary of the contents of the thesis to indicate in what way it constitutes a contribution to knowledge.

SUMMARY

Chapter I is preliminary and contains no new results.

Chapter II deals, in turn, with the maximal ideal space, the essential set, the Shilov boundary, p-sets, A-convex hulls, Maximal antisymmetric sets, Gleason parts, a new decomposition {P}-parts and the relating of these results to the general closed boundary, X .

Chapter III gives some characterizations of a function algebra and introduces the concepts of normal and clopen measures.

Chapter IV characterizes essential, analytic, pervasive and maximal function algebras and integral domains. It also relates some of these to normal and clopen measures.

Chapter V gives bibliographic references and includes new results on w-interpolation sets and Glicksberg's closed restriction theorem.

The contribution of this work to knowledge should not be regarded as definitive or complete in a certain area but rather as one which opens up a number of avenues for investigation. This has arisen from our study of the basic structures of function algebras, such as p-sets, A-convex hulls and decompositions of the underlying space from a largely measure theoretic point of view. This study has resulted in a number of new results on a fairly broad front. More particularly it has made a contribution to our understanding of Gleason parts and the essential set of a function algebra. We briefly list some of the areas in which new results have been obtained :

Several characterizations of Gleason parts; characterizations of the essential set; characterizations of C(X) in terms of the p-set topology and in terms of relationships between functions and annihilating measures of A ; characterizations of essential, analytic, pervasive and maximal function algebras and integral domains; a study of normal and clopen annihilating measures and their relation to pervasive function algebras; results regarding closed restriction sets and w-interpolation sets; strengthening of Glicksberg's closed restriction theorem.

INTRODUCTION

The main objective of this thesis is to emphasize the use of measure theoretic techniques in the development and study of the subject of function algebras. A little reflection will show that a large proportion of the structures and results relating to function algebras are, in fact special types of restriction sets or are expressible in terms of such restriction sets. e.g. peak sets; A-convex hulls; the Bishop decomposition into maximal antisymmetric sets. Since measures are set functions on the underlying space, one would expect to find significant relationships between the measures associated with function algebras, such as representing measures and annihilating measures, and the various restriction sets associated with function algebras. It is this approach, which evolves in this natural fashion, that leads to our title: Function Algebras, Restriction Sets and Measures.

We do not claim that the measure theoretic approach is the most natural or efficient way of tackling the subject. Indeed, it is clear that a combination of all of the available techniques constitutes the most economic path of exploration. Nevertheless we feel that this somewhat uncommon approach with its consequent shifts in emphasis is well rewarded. For example, it leads, via Glicksberg's p-set criterion to a shift in emphasis from peak set structure to p-set structure. This in turn leads directly to a new approximation condition expressible as a regularity condition on the p-set topology. (See 2.2.8 and 2.5.2.13.)

Another major advantage of this different approach is that it reveals new insights into known structures and also helps to pinpoint areas which would profit from further investigation. Examples of this are the characterizations given in terms of representing measures and annihilating measures (see 2.5.1.10 and 2.5.1.23). These certainly played a part in modifying our "intuitive picture" of Gleason parts. In addition, both of these results, together with others, e.g. those relating to the Choquet boundary (see 3.2.1) and completely singular measures (see 3.2.6) shed light on the differing roles played by those annihilating measures which are atomic at certain points and those which are purely non-atomic. It is our feeling that this is an area of investigation where much can be learned. Another such area of possible investigation is the question of the significance of the number $\frac{1}{2}$, which has made an appearance in several places, often in connection with measures; e.g. the second Gleason part characterization (2.5.1.23 and prior to that, the result 2.5.1.18); a characterization of interpolation sets (see Gamelin [26]) and an approximation condition related to ε-normal function algebras (see Bade and Curtis [4]).

Having spent some time on explanation and justification of our particular approach, we shall now look more closely at the structure and content of this work. The labelling system is a numerical one, i.e. $\alpha.\beta.\gamma.\delta.$ ... where the first digit, α , gives the chapter number. The second digit, β , usually represents a subsection of a given chapter where a new concept or structure is introduced. This is particularly true of chapter two where each subsection is given a descriptive title, e.g. p-sets, decompositions, etc. The general process is that a concept is introduced, initially explored and then related fairly systematically to most of the previously introduced concepts. At levels γ,δ and lower the reasons for the grouping together of any particular level becomes progressively less well-defined and these, in fact, often consist of individual results or definitions. In some cases, however, level γ may still represent a relatively large and coherent body of information, as may be seen in subsection 2.4, for example. Each chapter and a number of the subsections are introduced by a brief comment giving some indication of the content, intention and points of interest of the relevant sections.

The first chapter is preliminary to the body of the thesis and indicates fairly specifically the area of familiarity which would be a useful adjoint to reading this work. It contains no new material but serves as a convenient source of reference. Several well known abbreviations are used without special mention. Among these are : iff = "if and only if" and the quantifiers : $\forall$ = "for all" and $\exists$ = "there exists". The symbol [ ] generally directs the reader to a reference in the bibliography.

Chapter two is the longest chapter of the thesis and the substantial part of it is set in the context $X = M_A$, the maximal ideal space. Our justification for this approach is given at the beginning of the chapter where M_A is introduced. The next restriction sets dealt with are, in turn, the essential set and the Shilov boundary. The measure theoretic bias is evident at this early stage. The following structure examined is that of p-sets. As mentioned earlier, considerably more emphasis is placed on the p-set structure than that of peak sets, the latter being used more as a tool for developing the former. Together with a brief mention of antisymmetric sets, these provide the means for outlining the basic structure of M_A in terms of the essential set, the Shilov boundary, the Choquet boundary and the maximal point sets of antisymmetry and the relationships of all these to the underlying topology. Some interesting approximation conditions come to light in this area.

Although the subsection on A-convex hulls is short, only introducing the subject and relating it to p-sets and antisymmetric sets, extensive reference is made to this concept in later subsections. The next subsection, 2.4, is a large one and is divided into three parts: The first part introduces the concept of closed restriction sets, giving particular attention to measure theoretic formulations and relating them to earlier subsections. The second part may be considered as inessential (as explained in the text) in that many of its results are superceded in the following part. It does, however, when taken with the third part, provide considerable insight to the nature of the essential set and gives rise to some new results which are mentioned in chapter five. The third part initially developes the Local Maximum Modulus principle in terms of p-sets, providing the tool whereby we are able to relate the results of the second part to the essential set. Two characterizations of the essential set appear as a byproduct of this process.

The subsection after this is entitled "Decompositions" and deals with three decompositions: Maximal antisymmetric sets, Gleason parts and a new decomposition which is described in terms of p-sets. Apart from this decomposition which is of considerable interest and raises some open questions, there are several other noteworthy features in this subsection: Firstly the intimate relationship between Gleason parts and measures is revealed, culminating in characterizations of Gleason parts in terms of representing and annihilating measures. Secondly the emergence of p-sets as a powerful tool for examining these three decompositions; and thirdly, in the last part, the work on the p-set topology in which a characterization of $C(X)$ is given in terms of a regularity condition on the p-set topology.

The final subsection, 2.6, looks at the more general situation of any closed boundary, X, rather than the whole maximal ideal space. The first and most significant result of this subsection is that which gives the correspondence between p-sets on X and p-sets on M_A. Satisfyingly, from our point of view, this is achieved by using measure theoretic properties of p-sets and A-convex hulls. Much flows from this result. Some indication is given of this, in particular the relationship between the essential set on X and the essential set on M_A. A brief review is given of the previous results of the chapter, indicating those which may be stated in this more general context. Greater attention is paid to the subject of decompositions, where consideration of the general case leads to a strengthening of one of the characterizations previously obtained for Gleason parts.

The third chapter is principally concerned with the set of annihilating measures of a function algebra. The first subsection begins with a look at the real parts of a function algebra and the establishing of some notation and some approximation conditions, but is mainly concerned with characterizing function algebras in predominantly measure theoretic terms. A number of variations and refinements of the initial characterization are given and these, in turn, lead to several new approximation conditions. This leads naturally to the second subsection which focuses on the role of purely non-atomic annihilating measures and relation to p-points, completely singular measures, Gleason parts and extreme points of the unit ball of $A^{\perp}$. The third and fourth subsections deal with two conditions on $A^{\perp}$ which we call "normal" and "clopen" respectively. We outline a number of interesting properties of these two conditions, particularly in the case of normal measures. It is found, in fact, that there are marked similarities between the two. The second condition is dealt with fairly briefly at this stage, but more extensively in the following chapter where it is found to occur naturally in the context of pervasive function algebras.

Chapter four may be considered as an application of the approach and results of the previous chapters. The vehicle for this application is the well known sequence of implications : A is maximal essential $\Rightarrow A$ is pervasive $\Rightarrow A$ is analytic $\Rightarrow A$ is an integral domain $\Rightarrow$ A is anti-symmetric $\Rightarrow A$ is essential.

Characterizations, in measure theoretic terms, for each of these types of function algebra are obtained with varying degrees of success. The initial numbering of the subsections corresponds, more or less, to the treatment of each of these types in turn. Analytic and pervasive function algebras are dealt with quite extensively. In the case of analytic function algebras, an interesting comparison with essential function algebras comes to light. It is also found that the condition "$A^{\perp}$ is normal" and "$A^{\perp}$ is clopen", considered in the previous chapter, have some significance here. The characterizations of pervasive function algebras are particularly pleasing and the condition "$A^{\perp}$ is clopen" is found to play a considerable role, being, in fact, a necessary condition. Maximal function algebras are also dealt with at some length, applying the techniques of subsection 3.1. The next subsection brings many of these results together in proving, to a large extent, the sequence of implications mentioned above. The final subsection deals briefly with the concept of weak analyticity, relating it to some of the results previously established.

Chapter five briefly reviews the first four chapters, giving fairly detailed reference to various relevant papers and texts. (The bibliography which follows, consisting only of those items to which direct reference is made, comprises only a portion of the background reading done). A number of new results which have some connection with the previous chapters are listed here. Outlines of proofs are given in some cases. Among these are results relating to the Shilov boundary of A_E, w-interpolation sets, characterizations of the essential set, the set of points in X which are not strongly bounded and generalizations of Glicksberg's theorem on Closed Restrictions. We also ask the reader to excuse the style of this chapter which may appear a little immodest. It has been written in this way to assist the examiners of this work. Indeed, one of the concomitants of producing this work has been a growing realization, on the part of the author, of the debt owed by any one mathematician to those who have gone before.

ACKNOWLEDGEMENTS

My first thanks go to my supervisor, Professor W Kotzé, who has on various occasions played the roles of tutor, advisor, encourager, coercer, manager and friend, according to my needs at the time. His tutelage I admire, his scholarship I respect and his friendship I value.

My thanks also to Professor K O Househam, both for financial support, in the form of Junior Lectureship, and for his interest in and support of my career.

Of those who have helped with the typing and reproduction of the thesis I am especially grateful to Mrs Fouquet who has lent her expertise to the bulk of the typing, despite considerable pressure of work.

I must particularly thank my wife for her patience and support over the last few years.

Lastly, though perhaps firstly in a causal sense, I thank my parents whose ideal of education for me was fulfilled, I see in retrospect, by considerable sacrifice on their part. More importantly, they have consistently expressed more faith in my abilities than I have sometimes had in myself.

CHAPTER ONE

PRELIMINARIES

1.0 In this first chapter are listed a number of the basic definitions and results of which use is made later in this work. This list should by no means be regarded as being exhaustive, either as a prelude to the subject of function algebras or as a list of prerequisites for reading this thesis. We have endeavoured, however, to include all those bits of information to which reference is frequently made; giving them in a form suitable for our purposes and giving relevant sources in the literature. A secondary objective is to establish a body of common notational usage which we shall apply fairly consistently throughout the thesis. This chapter then, does not form an integral part of the thesis, but is included for the convenience of both the reader and the author.

1.1 We shall use the following notation throughout :

X - a compact, Hausdorff, topological space.

$C(X)$ - set of all continuous, complex-valued functions on X .

$C_R(X)$ - set of all continuous, real-valued functions on X .

We regard $C(X)$ as an algebra under point-wise operations. It is known that with respect to the uniform norm : $\|f\| = \sup_{x \in X} |f(x)|$ for $f \in C(X)$, any closed subalgebra of $C(X)$ becomes a Banach Algebra.

1.2 Definition

A is a function algebra on X if and only if :

(i) A is a uniformly closed subalgebra of $C(X)$.

(ii) A separates points of X ; i.e. for every $x,y \in X$ with $x \neq y$ there is $f \in A$ such that $f(x) \neq f(y)$.

(iii) A contains the constants, or equivalently, $1 \in A$.

The term "Uniform Algebra" is often used synonymously. We shall consistently use the symbol A to denote a function algebra.

(see e.g. [16] or [24])

1.3 Of obvious significance to the subject of function algebras is the **Stone-Weierstrass Theorem.**

If A is a subalgebra of $C(X)$ with the properties :

(i) $1 \in A$.

(ii) A separates point of X .

(iii) A is closed under complex conjugation.

Then A is uniformly dense in $C(X)$.

(see [45], 17 Theorem 1)

1.4.1 A signed measure is an extended, real-valued countably additive set function on the σ-ring of a measurable space which is zero on the empty set and assumes at most one of the values $+\infty$ or $-\infty$.

1.4.2 A complex measure is a set function $\mu = \mu_1 + i\mu_2$ where the μ_i are signed measures on a given σ-ring.

1.4.3 A Borel set in X is a member of the σ-ring generated by all the compact subsets of X.

1.4.4 A Borel measure is a measure defined on the σ-ring of Borel sets which is finite on every compact set.

1.4.5 A Borel set E is regular with respect to a positive Borel measure μ if $\mu(E)$ is simultaneously equal to the infimum of μ's values on open sets containing E and the supremum of its values on compact subsets of E. μ is regular if every Borel set is regular with respect to μ.

1.4.6 **Jordan Decomposition Theorem**

Any signed measure, μ, can be expressed as $\mu = \mu^+ - \mu^-$ where μ^+ and μ^- are positive measures.

(see [5], 8.5)

1.4.7 A signed, regular Borel measure is a signed measure whose positive and negative parts (cf 1.4.6) are regular Borel measures.

1.4.8 A complex, regular Borel measure is a complex measure whose real and imaginary parts are signed, regular Borel measures.

Hereafter, "measure", unqualified, shall mean "finite, complex, regular, Borel measure".

1.4.9 $|\mu|$; the total variation measure: Let E be a Borel set and let π_E denote the class of finite, pairwise disjoint collections $E_1, \ldots, E_n$ of measurable subsets of E.

Then $|\mu|(E) = \sup_{\pi_E} \{ \sum_{i=1}^{n} |\mu(E_i)| \}$.

1.4.10 δ_x ; the unit point mass at x defined by

$$\delta_x(E) = \begin{cases} 0 ; & x \notin E \\ 1 ; & x \in E \end{cases} .$$

1.4.11 supp μ ; the support of μ : The complement of the union of all open sets U such that $|\mu|(U) = 0$. We shall say that "μ is supported by F" if supp $\mu \subset F$ and we shall say that "μ is carried by F" if $\mu_F = \mu$ where μ_F is the set function defined by $\mu_F(E) = \mu(E \cap F)$ for every Borel set E .

1.5.1 **Proposition**

μ is a regular Borel measure $\Longleftrightarrow$ $|\mu|$ is a regular Borel measure.

(see [38], 2.3(f))

1.5.2 We denote by $M(X)$ the set of all Borel measures on X .

1.5.3 $\|\mu\| = |\mu|(X)$ defines a norm on $M(X)$.

(see [38], 2.3(e))

1.6 Let B be a linear subspace of $C(X)$ and B^* the space of continuous linear functionals on B .

1.6.1 μ is a complex representing measure for $\phi \in B^*$ if $\int f d\mu = \phi(f)$ for every $f \in B$.

1.6.2 μ is a representing measure for $\phi \in B^*$ if $\int f d\mu = \phi(f)$ for every $f \in B$ and $\|\phi\| = \|\mu\|$.

1.6.3 $B^\perp$ is the set of annihilating measures for B . i.e. all $\mu \in M(X)$ such that $\int f d\mu = 0$ for all $f \in B$.

1.7.1 Each $\phi \in B^*$ has a representing measure (which must be a probability measure).

(see [38], 2.11(c),(d) or [16], pg 80)

1.7.2 The closure of B, $\overline{B} = \{f \in C(X) : \int f d\mu = 0$ for all $\mu \in B^{\perp}\}$.

(see [38], 2.11(c) or [16] pg 80)

1.7.3 The closure of B, $\overline{B} = C(X) \Longleftrightarrow B^{\perp} = \{0\}$.

(from [38], 2.8(d) and 2.11(c))

1.8.1 We say measure ν is absolutely continuous with respect to measure μ if $|\mu|(E) = 0 \Rightarrow \nu(E) = 0$ for all Borel sets E. We denote this: $\nu \ll \mu$.

1.8.2 We say two measures μ and ν are mutually singular if there are two disjoint sets A and B such that $A \cup B = X$ and for every measurable set E, $A \cap E$ and $B \cap E$ are measurable and $|\mu|(A \cap E) = |\nu|(B \cap E) = 0$. We denote this $\nu \perp \mu$.

1.8.3 If μ is a complex representing measure for $\phi \in A^*$, then there is a representing measure ν for ϕ such that $\nu \ll \mu$.

(see [38], 2.17(e) or [24] pg.33)

1.9.1 If μ is a measure and f is $|\mu|$ - integrable then the set function defined by $(f\mu)(E) = \int_E f d\mu$ for every Borel set, E, is a measure. We denote this measure by $f\mu$.

(see [38], 2.13 or [16], pg 82)

1.9.2 $|f\mu| = |f||\mu|$

1.9.3 $f \in A$ and $\mu \in A^{\perp} \Rightarrow f\mu \in A^{\perp}$

(see [16] pg.82)

1.10.1 We denote by M_A the space of maximal ideals of A.

1.10.2 It turns out that X can be homeomorphically embedded in M_A, a compact Hausdorff space, and that A can be considered to be a function algebra in $C(M_A)$.

(see [46], §11-§13 or [41], Chap 4)

1.11 The Shilov Idempotent Theorem

If F is a clopen subset of M_A, then the characteristic function of F, χ_F is in A (considered as a function algebra on M_A).

(see [24], pg 88, Cor. 6.5)

1.12.1 A closed boundary for A is a closed subset of M_A which is such that every $f \in A$ attains its maximum modulus on that set.

We shall usually use the symbol X, to denote such a set. It will become apparent that this choice does not clash with our established usage of X.

1.12.2 A minimal closed boundary for A is called a Shilov boundary for A.

1.12.3 There exists a unique Shilov boundary for A.

(see [46] §13)

We shall denote this unique set by the symbol, Γ_A.

1.12.4 For any $x \in X$ we can form the linear functional ϕ_x defined by $\phi_x(f) = f(x)$ for any $f \in A$.

If $\mu \in M(X)$ is a representing measure for ϕ_x we use the phrases: " μ is a representing measure for x" or " μ represents x".

Clearly δ_x represents x .

1.12.5 We call the set, $\{x \in X : x$ has a unique representing measure$\}$, the Choquet boundary for A . We denote the Choquet boundary by the symbol $\text{ch}\,A$.

1.12.6 Let F be a closed subset of X , then the following are equivalent :

(i) $F \supset \Gamma_A$

(ii) Every $x \in X$ has a representing measure which is supported on F .

(see [41], pg 49, Thm. 6)

The relationships between A, M_A, X, Γ_A and $\text{ch}\,A$ will be dealt with more fully in Chapter Two.

1.13.1 A subset K of X is a peak set for A if there exists $f \in A$ such that :

$$f(x) = 1 \quad \text{for} \quad x \in K \quad \text{and} \quad |f(x)| < 1 \quad \text{for} \quad x \notin K$$

Any such f is said to peak on K .

$x \in X$ is called a peak point if the set $\{x\}$ is a peak set.

1.13.2 A subset K of X is a p-set if it is the intersection of a collection of peak sets. $x \in X$ is called a p-point if the set $\{x\}$ is a p-set.

1.13.3 A p-set K is a peak set if and only if it is a G_δ-set i.e. a countable intersection of open sets.

1.14.1 Urysohn's Lemma

If F_0, F_1 are disjoint closed subsets of X, then there exists $f \in C_R(X)$ such that $f(F_0) = 0$, $f(F_1) = 1$ and $f(X) \subset [0,1]$.

(see e.g. [46], §2)

1.14.2 By Urysohn function we shall mean a function of the type f in 1.14.1.

1.15 Glicksberg's p-set Criterion

A closed subset E of X is a p-set if and only if $\mu_E \in A^\perp$ for every $\mu \in A^\perp$.

(see [28])

1.16.1 The p-sets for A are the closed sets for a topology on X which we shall call the p-set topology for A.

1.16.2 The p-set topology is T_2 if and only if $A = C(X)$.

(see [16], pg 113)

1.17 The Gleason parts of A are the equivalence classes of M_A defined by the equivalence relation : $x \sim y$ if and only if $\|x - y\| < 2$.

(see [16], pgs. 128-129)

1.18.1 If $x, y \in M_A$ have representing measures on X which are not mutually singular, then x and y are in the same Gleason part.

1.18.2 If x,y are in the same Gleason part, then there exist constants $0 < a \leqslant b < \infty$ and a Borel function h on X such that $a \leqslant h \leqslant b$ and representing measures μ and ν for x and y respectively such that $\mu = h\nu$.

1.18.3 If x,y are in the same Gleason part and μ is a representing measure for x , then there is a representing measure, λ , for y such that $\mu \ll \lambda$.

1.18.4 If x,y are in different Gleason parts, then there are disjoint Borel sets E_x and E_y such that every representing measure for x (respectively y) is carried by E_x (respectively E_y). In particular, every representing measure for x is singular to every representing measure for y .

(see [24], pgs. 143-144)

1.19.1 We denote by M_x the set of all representing measures for $x \in X$.

1.19.2 We say that a measure μ is M_x-singular if μ is carried by some Borel set F such that $\lambda(F) = 0$ for every $\lambda \in M_x$. Such an F is called an M_x-null set.

1.19.3 A measure μ is M_x-absolutely continuous if μ vanishes on all M_x-null sets. We denote this by $\mu \ll M_x$.

1.19.4 Lebesgue Decomposition of μ :

For any measure μ and any set of probability measures M , there is a unique Lebesgue decomposition of μ with respect to M . i.e.

$$\mu = \mu_F + \mu_{F'} \text{, where } \mu_F \text{ is M-singular and } \mu_{F'} \ll M \text{ .}$$

(see e.g. [38], 6.1(c))

1.20 Abstract F and M Riesz Theorem

If $\mu \in A^{\perp}$ and $\mu = \mu_F + \mu_{F'}$ is the Lebesgue decomposition of μ with respect to M_x, any $x \in M_A$, then μ_F, $\mu_{F'} \in A^{\perp}$.

(see [24], pg. 44, 7.6)

1.21.1 We say that μ is completely singular if μ is M_x-singular for every $x \in M_A$.

1.21.2 Decomposition Theorem for Orthogonal Measures

Let $\{x_i\}$ be a subset of M_A containing exactly one point from each Gleason part of M_A. Let $\mu \in A^{\perp}$ and μ_i the absolutely continuous component of μ with respect to M_{x_i}. Then $\mu_i \in A^{\perp}$ and the μ_i are mutually singular.

Furthermore : (a) $\mu = \mu_s + \sum\limits_i \mu_i$

(b) $\|\mu\| = \|\mu_s\| + \sum\limits_i \|\mu_i\|$

(c) $\mu_s \in A^{\perp}$ and μ_s is completely singular.

(see [24], pg. 145, 2.3)

1.22 The Local Maximum Modulus Principle

We give three versions of this well-known Theorem.

1.22.1 A closed $K \subset M_A$ is a local peak set for A on M_A if there is an open $U \supset K$ and $f \in A$ such that $f(x) = 1$ on K and $|f(x)| < 1$ on $U \sim K$.

1.22.2 If x is a local peak point for A on M_A, then x is a peak point for A on M_A.

1.22.3 Every local peak set for A on M_A is a peak set for A on M_A .

1.22.4 For any non-empty open $U \subset M_A \sim \Gamma_A$ we have :

(a) $bd(U)$ is non-empty

(b) $\sup_{x \in U} |f(x)| = \sup_{x \in bd(U)} |f(x)|$ for every $f \in A$.

Here $bd(U)$ denotes the topological boundary of U .

(see [50] and [41] pgs. 193-194)

1.23.1 $K \subset X$ is a set of antisymmetry for A if every $f \in A$ which is real-valued on K is also constant on K .

1.23.2 Every set of antisymmetry is contained in a maximal set of antisymmetry.

1.23.3 If $\mu \in (bA^{\perp})^e$, the set of extreme points of the unit ball of $A^{\perp}$, then supp μ is antisymmetric for A .

(see [28], Lemma 2.1)

1.23.4 Every maximal set of antisymmetry is a p-set.

(see [28], Lemma 2.3)

1.23.5 $\{K\}$, the family of maximal sets of antisymmetry, forms a decomposition of X into pair-wise disjoint closed sets with the property that :

(i) $A_{|K}$ is an antisymmetric function algebra for every $K \in \{K\}$

(ii) $f \in C(X)$ and $f_{|K} \in A_{|K}$ for every $K \in \{K\}$ implies that $f \in A$.

(see [14])

1.24 The essential set is the zero set of the largest ideal of $C(X)$ which is contained in A. We shall denote this throughout by the symbol E.

(see [8] or [41], 4.4)

1.25.1 A is an essential function algebra if $E = X$.

1.25.2 A is antisymmetric if X is a set of antisymmetry.

1.25.3 A is an Integral Domain if for every $f,g \in A \sim \{0\}$ we have $fg \neq 0$.

1.25.4 A is analytic on X if the zero set of every $f \in A \sim \{0\}$ has empty interior.

1.25.5 A is pervasive on X if for every proper closed $F \subset X$ we have $A_{|F}$ dense in $C(F)$.

1.25.6 A is maximal on X if for any function algebra B such that $A \subset B \subset C(X)$, we have either $A = B$ or $B = C(X)$.

1.25.7 A is normal on X if for any closed, disjoint $F_0, F_1 \subset X$ there is $f \in A$ such that $f(F_0) = 0$ and $f(F_1) = 1$.

1.25.8 A is approximately normal on X if for any closed, disjoint $F_0, F_1 \subset X$ and $\varepsilon > 0$, there is $f \in A$ such that $|f(x)| < \varepsilon$ for $x \in F_0$ and $|f(x) - 1| < \varepsilon$ for $x \in F_1$.

1.26 We have the following sequence of implications :

A is essential and maximal on X

$\Rightarrow$ A is pervasive on X

$\Rightarrow$ A is analytic on X

$\Rightarrow$ A is an Integral Domain

$\Rightarrow$ A is antisymmetric

$\Rightarrow$ A is essential.

(see [41])

CHAPTER TWO

RESTRICTION SETS

2.1 Some Special Restriction Sets

2.1.1 It may seem a little strange to deal with the maximal ideal space, M_A , of a function algebra, A , under the heading "special restriction sets". Nevertheless, this seems an appropriate place in which to outline some basic properties of M_A and to establish the setting for our initial discussions. One may consider two different approaches when studying A and its underlying space, X :

Firstly, one may define A relative to its underlying space, X . One then discovers associated structures, M_A and Γ_A , the Shilov boundary, which are related to X and which have many interesting and useful properties. This is the approach adopted by most of the texts and it has obvious advantages, particularly when one is dealing with specific examples of function algebras.

Secondly, one may regard A as a commutative Banach algebra, independent, in a sense, of the underlying space, X . One then finds that there are two spaces, M_A and Γ_A , which are naturally associated with A in that they are, respectively, the largest and smallest spaces on which A can be realized as that particular Banach algebra, in a sense which will be outlined below. The underlying space may then be regarded as any one of (possibly many) closed boundaries of A ; i.e. any space, X , closed in M_A such that $\Gamma_A \subset X \subset M_A$. The X with which one initially defined A just happens to be one of these.

Our approach shall be, to a large extent, the latter. We hope to demonstrate that this, combined with measure theoretic techniques, leads to an elegant development of the subject and leads to particular insights into the relationship between the structures built up on the "whole" space, M_A and the corresponding structures on any closed boundary, X .

2.1.2 Let A be a function algebra on X . For $h \in M_A$ and $f \in A$ define $\hat{f}(h) = h(f)$. Let $\hat{A} = \{\hat{f}: f \in A\}$, then it turns out that $\hat{A}$ is a subalgebra of $C(M_A)$, where M_A has the weak topology defined by the functions in $\hat{A}$. If $x \in X$, then ϕ_x defined by $\phi_x(f) = f(x)$ is a complex homomorphism on A , hence associated with a maximal ideal of A . Thus we can say $\phi_x \in M_A$. A full treatment of this appears, for instance, in [46] §11 - §13. The The notation used here is used in [41] chapter 4 in which also appear the following two theorems.

2.1.3 Theorem

Let A be a function algebra on X , then $\phi : X \to M_A$, defined by $x \to \phi_x$, is a homeomorphism from X onto a compact subset of M_A . The Gelfand mapping : $f \to \hat{f}$ is an algebraic isomorphism and an isometry from A onto $\hat{A}$. Also, $\hat{A}$ is a function algebra on M_A.

2.1.4 Theorem

Let A be a function algebra on X. If we identify X with its image in M_A (cf. 2.1.3), we may regard $\hat{f}$ as a continuous extension with preservation of norm of f to M_A.

[for proofs see [41], 4.1; Thms 1 and 2]

2.1.5 Making use of these results we can see that M_A is the largest space upon which A can be realized as a function algebra in the following sense:

Proposition

Let A and A' be function algebras on X and X' respectively. Say that there is an isometric algebraic isomorphism from A onto A'. Then there exists an homeomorphism from X' onto a compact subset of M_A.

Outline of Proof Clearly there is a $1-1$ correspondence between the maximal ideals of A and those of A', hence between M_A and $M_{A'}$. At the same time, by 2.1.3 we have an isometric algebraic isomorphism between $\hat{A}$ and $\hat{A}'$. It can then be seen that the correspondence between M_A and $M_{A'}$ is given by $h \Leftrightarrow h'$ iff $(\hat{f}(h) = 0 \Leftrightarrow \hat{f}'(h') = 0 \ \forall\ \hat{f} \in \hat{A})$. From this we deduce that if $h \Leftrightarrow h'$ then $\hat{f}(h) = \hat{f}'(h')$, for all $\hat{f} \in \hat{A}$ and hence that the correspondence between M_A and $M_{A'}$ is an homeomorphism. Thus the homeomorphism $\phi' : X' \to M_{A'}$ (cf. 2.1.3) can be used to construct the required homeomorphism from X' into M_A .

2.1.6 On the other hand, let $\Gamma_{\hat{A}}$ be the Shilov boundary of $\hat{A}$ in M_A (cf. 1.12.3). It then turns out that Γ_A may be regarded as the smallest space on which A can be realized as a function algebra in a sense similar to that of 2.1.5.

Proposition

Let A and A' be function algebras on X and X' respectively. If there is an isometric algebraic isomorphism from A onto A' , then there exists an homeomorphism from Γ_A onto a compact subset of X' .

Outline of Proof : As in the proof of 2.1.5 we develope an isomorphism between $\hat{A}$ and $\hat{A}'$ and an homeomorphism between M_A and $M_{A'}$. As before we obtain an homeomorphism from X' onto a compact subset, X'' of M_A . Now, since $h \Leftrightarrow h' \Rightarrow \hat{f}(h) = \hat{f}'(h')$ (see 2.1.5) we can say that the restriction algebra $\hat{A}|_{X''}$ is isometrically isomorphic to $\hat{A}'|_{X'}$ which, by 2.1.4 is isometrically isomorphic to $\hat{A}'$ which we know is isometrically isomorphic to $\hat{A}$ on M_A .
Thus $\hat{A}|_{X''}$ isometrically isomorphic to $\hat{A}$ which means that X'' is a closed boundary for $\hat{A}$. Now, if the unique minimality of Γ_A is not to be contradicted, we must have $\Gamma_A \subset X''$.

2.1.7 <u>Note</u>: The condition "isometric" in 2.1.5 and 2.1.6

can be considered redundant by virtue of [41] 4.1 ; Theorem 3 .

2.1.8 It can be seen from the preceeding discussion that M_A and Γ_A are two "natural" settings for the function algebra A . Indeed, a little further reflection on the above results will show that M_A and Γ_A are quite independent of the underlying space, X as long as we are dealing with the same algebra A , in the sense of algebraic isomorphism. This is a question which we will approach with different techniques at a later stage. In the meantime we shall regard this as sufficient justification for limiting our preliminary investigations to the case $X = M_A$. (i.e. in sections 2.1 to 2.5). Later, having developed some useful constructions we shall return to a consideration of the general closed boundary, X . To avoid increasing complexity in notation we shall speak, perhaps rather loosely, of the function algebra A on M_A (or A on X as the case may be) and will avoid use of the symbols $\hat{A}$ or $\hat{f}$. From this point the symbols A , Γ_A , chA and E shall stand for a function algebra A on M_A , the Shilov boundary of A on M_A , the Choquet boundary of A on M_A , and the essential set of A on M_A respectively. $A^{\perp}$ shall be the set of measures (see 1.6.3) on M_A which annihilate A .

2.1.9 The next important restriction set which we consider is the essential set, E . This was introduced by

Bear in [8] where he defines it as follows:

E is that minimal closed subset of X such that, if $f \in C(X)$ and f is zero on E, then $f \in A$.

He shows, in effect, that this is equivalent to the definition that we have given in 1.24 in terms of ideals. We would like to characterize E in terms of the set $A^{\perp}$. The following rather elegant and useful statement is such a characterization.

2.1.10 Proposition

E is that minimal closed subset of M_A which supports every $\mu \in A^{\perp}$.

Proof : Let $\{S_i \, ; \, i \in I\}$ be the family of closed supporting sets; i.e. $\operatorname{supp} \mu \subset S_i \, ; \, \forall \, \mu \in A^{\perp}$ and $i \in I$. Clearly we can obtain a minimal closed supporting set, S, by intersection. Take any $f \in C(M_A)$ such that $f|_S = 0$. Clearly $\int f d\mu = 0 \, , \, \forall \, \mu \in A^{\perp}$. Thus $f \in A$. By 2.1.9 we have $E \subset S$.

On the other hand, take set P, open and Q, closed such that $Q \subset P \subset M_A \sim E$. This is possible, since M_A, being compact and Hausdorff, is normal. Let $g \in C(M_A)$ be a Urysohn function such that $g|_Q = 1$ and $g|_{M_A \sim P} = 0$. Then, by 2.1.9, $g \in A$. Now, for any $\mu \in A^{\perp}$, we can choose a decreasing sequence $\{P_n\}$ of open sets and a corresponding sequence $\{g_n\}$ of Urysohn functions in A such that $g_n \to \chi_Q$ pointwise a.e. $(|\mu|)$. The regularity of μ is needed for this. Then, by use of the Lebesgue Dominated convergence theorem we have:

$$\mu(Q) = \int_Q d\mu = \int \chi_Q d\mu = \int \lim g_n d\mu = \lim \int g_n d\mu = 0$$

This is true for every $\mu \in A^\perp$ and for every closed set disjoint from E. Let B be any Borel set disjoint from E. By the regularity of μ, $\mu(B) = \sup_Q \{\mu(Q) : B \subset Q \text{, closed}\} = 0$. Thus $|\mu|(M_A \sim E) = 0$ for any $\mu \in A^\perp$ and supp $\mu \subset E$, for all $\mu \in A^\perp$. Since E is closed, $S \subset E$. Thus $S = E$ as required.

2.1.11 It is an interesting exercise to define the essential set in terms of the characterization given in 2.1.10. Then, using measure theoretic techniques, one can easily obtain most of the commonly used properties of E. In order to prove some of the examples following we make use of results which, for convenience, only appear later in this work, although they are quite independent of these results. They can be found in [26], Theorem 1 and Corollary.

2.1.11.1 If $f \in C(M_A)$ and $f_{|E} = 0$, then $f \in A$.

Proof: This was proved in the first part of the proof 2.1.10.

2.1.11.2 If $f \in C(M_A)$ and $f_{|E} \in A_{|E}$, then $f \in A$.

Proof: Since $\mu_{|E} = \mu$ for every $\mu \in A^\perp$ we have

$f_{|E} \in A_{|E} \Rightarrow \int_E f d\mu = 0 \ \forall \mu \in A^\perp$. Thus

$\int f d\mu = 0, \ \forall \mu \in A^\perp$. Hence $f \in A$ as required

2.1.11.3 If F is closed and such that

$(f_{|F} = 0 \Rightarrow f \in A, \ \forall f \in C(M_A))$, then $F \supset E$.

Proof: This was proved in the second part of the proof of 2.1.10.

2.1.11.4 $E = \phi \Leftrightarrow A = C(M_A)$.

Proof: $E = \phi \Leftrightarrow A^{\perp} = \{0\} \Leftrightarrow A = C(M_A)$ (see 1.7.3) .

2.1.11.5 If F is a closed set, disjoint from E , then $A_{|F} = C(F)$.

Proof: As $\mu_{|F} = 0 \ \forall \ \mu \in A^{\perp}$ we apply 2.4.1.2 to obtain the result.

2.1.11.6 If $f \in A$ and $\text{supp } f = \overline{\{x : f(x) \neq 0\}}$ is such that $A_{|\text{supp } f}$ is dense in $C(\text{supp } f)$, then $E \subset Z(f) = \{x : f(x) = 0\}$.

Proof: Take any $\mu \in A^{\perp}$ then $f\mu \in A^{\perp}$ (see 1.9.3). Clearly $\text{supp } f\mu \subset \text{supp } f$. Thus $f\mu \in (A_{|\text{supp } f})^{\perp}$. Since $(\overline{A_{|\text{supp } f}}) = C(\text{supp } f)$, we have $f\mu = 0$ (see 1.7.3). Set $F_n = \{x : |f(x)| \geqslant \frac{1}{n}\}$ $n \in \mathbb{N}$. Then $0 = |f\mu| \ (F_n) \geqslant \frac{1}{n} |\mu| \ (F_n)$ (see 1.9.2) . Thus $|\mu|(F_n) = 0 \ \forall \ n \in \mathbb{N}$. Since $\{x : f(x) \neq 0\} = \overset{\infty}{\underset{1}{\cup}} F_n$, we have, by regularity of $|\mu|$ (see 1.5.1), that $|\mu|(\{x : f(x) \neq 0\}) = 0$. Clearly then $\text{supp } \mu \subset Z(f)$. Since this is true for every $\mu \in A^{\perp}$, we have, by 2.1.10 that $E \subset Z(f)$.

2.1.11.7 $A_{|E} = (\overline{A_{|E}})$(the uniform closure of $A_{|E}$) and $A_{|E} \neq C(E)$, if A is a proper function algebra.

Proof: By 2.1.10 we have $(A_{|E})^{\perp} = A^{\perp}$. Since $A^{\perp} \neq \{0\}$ we have, by 1.7.3 , $A_{|E} \neq C(E)$. The closure of $A_{|E}$

follows from 2.4.1.3.

2.1.12 The results 2.1.11.5 and 2.1.11.7 indicate the importance of E. The first seems to indicate that the behaviour of A outside of E is, firstly, rather uninteresting and, secondly, substantially the same for all function algebras. So we see that most of the properties which distinguish a function algebra, i.e. its essential properties, are determined by the essential set. This, together with the second result, indicates that, to a large extent, the study of any function algebra, A may be reduced to the study of the function algebra $A_{|E}$ which is clearly an essential function algebra. In view of this we give some further characterizations of E which may be regarded as corollaries to 2.1.10.

2.1.13 Proposition

$$E = \overline{\left(\bigcup_{\mu \in A^{\perp}} \operatorname{supp} \mu \right)}$$

Proof: Let $F = \overline{\left(\bigcup_{\mu \in A^{\perp}} \operatorname{supp} \mu \right)}$. Since F is closed and supports every $\mu \in A^{\perp}$, we have $E \subset F$ (by 2.1.10). On the other hand, E supports every $\mu \in A^{\perp}$. Thus $\bigcup_{\mu \in A^{\perp}} \operatorname{supp} \mu \subset E$. Since E is closed, we have $F \subset E$. Thus $F = E$ as required.

2.1.14 Proposition

Let $(b\,A^{\perp})^e$ denote the set of extreme points in the unit ball of $A^{\perp}$,

$$\text{then } E = \overline{\left(\bigcup_{\mu\in(bA^{\perp})^e} \operatorname{supp}\mu\right)} .$$

Proof: Firstly, since $(b\,A^{\perp})^e \subset A^{\perp}$, we have by 2.1.13 that $\overline{\left(\bigcup_{\mu\in(bA^{\perp})^e} \operatorname{supp}\mu\right)} \subset E$. Now, to obtain the reverse inclusion, let S be the convex hull of $(b\,A^{\perp})^e$. So S consists of elements of the form: $\alpha_1 t_1 + \alpha_2 t_2 + \ldots. + \alpha_n t_n$ where $t_i \in (bA^{\perp})^e$; $1 \leqslant i \leqslant n$ and $\alpha_i \in [0,1]$ for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^{n} \alpha_i = 1$. Let $\bigcup_{\mu\in S} \operatorname{supp}\mu = P$. We claim that for any $\mu \in \bar{S}$ (the weak* closure of S), $\operatorname{supp}\mu \subset \bar{P}$, closure of P in M_A. To see this choose any closed F disjoint from $\bar{P}$ and a Urysohn function f such that $f_{|F} = 1$ and $f_{|\bar{P}} = 0$. Now choose a neighbourhood of μ as follows:

$$U(\mu,f,\varepsilon) = \{\nu \in M(M_A) : |\int f d\nu - \int f d\mu| < \varepsilon\} \text{ where } \varepsilon > 0.$$

If $\nu \in S$, then $\int f d\nu = 0$. Since $U(\mu,f,\varepsilon)$ must contain such measures, we have $|\int f d\mu| < \varepsilon$. Choose a sequence of Urysohn functions, $\{f_n\}$, each similar to f such that $f_n \to \chi_F$ a.e. $(|\mu|)$. Then using the Lebesgue Dominated convergence Theorem we obtain $|\mu(F)| < \varepsilon$. Since ε can be chosen to be arbitrarily small we have $\mu(F) = 0$. As this is true for each such closed set F, we may, by using the regularity of μ in an argument similar to that in the proof of

2.1.10, show that $|\mu|(M_A \sim \bar{P}) = 0$. Thus $\operatorname{supp} \mu \subset \bar{P}$, establishing our claim.

Now we may say:

$$E = \overline{\left(\bigcup_{\mu \in A^\perp} \operatorname{supp} \mu \right)} \quad \text{(by 2.1.13)}$$

$$= \overline{\left(\bigcup_{\mu \in bA^\perp} \operatorname{supp} \mu \right)}$$, since clearly non-zero scalar multiplication does not affect supports of measures.

$$= \overline{\left(\bigcup_{\mu \in \bar{S}} \operatorname{supp} \mu \right)}$$, by the Krein-Milman Theorem. (see [45] §3 ; 9 Thm 1)

$$= \overline{\left(\bigcup_{\mu \in S} \operatorname{supp} \mu \right)}$$, established in the claim above.

$$= \overline{\left(\bigcup_{\mu \in (bA^\perp)^e} \operatorname{supp} \mu \right)}$$, since clearly a linear combination of measures will have a support contained in the union of the relevant supports.

This gives the required result.

2.1.15.1 Proposition

If $x \in M_A$ then : $x \in E$ iff $\forall$ neighbourhood U of x, $\exists\ \mu \in A^\perp$ st. $|\mu|(U) > 0$.

Proof: If $x \notin E$, then, by 2.1.10, one can find an open $U \ni x$ such that $|\mu|(U) = 0\ \forall\ \mu \in A^\perp$. If $x \in E$ and there exists an open $U \ni x$ such that $|\mu|(U) = 0\ \forall\ \mu \in A^\perp$, then $E \sim U$ is a closed set which supports all $\mu \in A^\perp$ (by 2.1.13). This contradicts the minimality of E in 2.1.10.

2.1.15.2 Proposition

If $x \in M_A$, then : $x \in E$ iff for every open $U \ni x$, there is $\mu \in (bA^{\perp})^e$ such that $|\mu|(U) > 0$.

Proof: As for the previous proposition, with reference to 2.1.14 in the appropriate place.

2.1.16 We shall now leave E for a while and state a simple result which will be very useful later.

2.1.16.1 Firstly we shall establish the use of the following notation:

(i) $I = M_A \sim \Gamma_A$

(ii) Let $U \subset M_A$, then A_U is the uniform closure of $A_{|\bar{U}}$ in $C(\bar{U})$. Note that A_U is a function algebra.

2.1.16.2 Proposition

For every $x \in I$ there is a real-valued $\mu \in A^{\perp}$ of the form:

$\mu = \delta_x - m_x$ where m_x is a representing measure for x which is supported on Γ_A .

Proof: An easy consequence of 1.12.6.

2.1.17 Corollary

If $f \in A$ is such that $\left\{\begin{matrix} f \text{ is real-valued} \\ f \geqslant 0 \\ f > 0 \\ \mathrm{Re} f \geqslant 0 \\ \mathrm{Re} f > 0 \\ f \text{ is constant} \end{matrix}\right\}$ on Γ_A,

then $\left\{\begin{matrix} f \text{ is real-valued} \\ f \geqslant 0 \\ f > 0 \\ \mathrm{Re} f \geqslant 0 \\ \mathrm{Re} f > 0 \\ f \text{ is constant} \end{matrix}\right\}$ on M_A.

Proof: A simple application of 2.1.16.2.

2.1.18 Corollary

For every $x \in I$ and every open U such that $x \in U \subset I$, there is a real-valued $\mu \in A^{\perp}$ of the form: $\mu = \delta_x - m_x$, where m_x is a representing measure for x which is supported on $\mathrm{bd}U$, the topological boundary of U in M_A.

Proof: The Local Maximum Modulus Principle (see 1.22.4) tells us that $\Gamma_{A_U} \subset \mathrm{bd}U$. Now, bearing in mind that $(A_U)^{\perp} \subset A^{\perp}$ we apply 2.1.16.2.

2.1.19 Corollary

If a closed $F \subset I$ is such that its interior $F^0 \neq \phi$, then $A_F \neq C(F)$.

Proof: By 2.1.18 $(A_F)^{\perp} \neq \{0\}$. Thus $A_F \neq C(F)$. (see 1.7.3).

2.2 p - Sets

2.2.1 It is clear that Glicksberg's p-set Criterion (see 1.15) can be used as a basis for defining p-sets. One of the advantages of this will become clearer in the next section where we shall introduce some new terminology which represents a slight refinement in the concept of a p-set. One of the several disadvantages of this approach is that one tends to lose sight of the distinction between p-sets and peak sets (the distinction being in this case the addition of a G_δ - condition on the set). Our policy, however, will be to place less emphasis than is usual on peak sets and more on p-sets. Peak sets will mainly be used as a vehicle for developing p-set structure which we consider to be more natural and widely applicable. We shall set down some of the basic results relating to p-sets and antisymmetric sets and shall relate these particularly to the restriction sets dealt with in 2.1. They will also be useful for the development of later results.

2.2.2 We shall now give definitions in terms of annihilating measures which in fact produce a classification of all the closed subsets of M_A . We shall later give a rather elegant characterization of pervasive function algebras in terms of this classification.

Let P and Q be closed subsets of M_A , then we define:

P is a strong p-set for A iff $|\mu|(P) = 0 \ \forall \ \mu \in A^\perp$

P is a weak p-set for A iff $\mu \in A^\perp \Rightarrow \mu_{|P} \in A^\perp$

Q is a weak q-set for A iff $\exists \ \mu \in A^\perp$ such that $\mu_{|Q} \notin A^\perp$

Q is a strong q-set for A iff $\mu \in A^\perp \Rightarrow \mu_{|Q} \notin A^\perp \sim \{0\}$ and Q is a weak q-set for A .

We tend to use the phrases " F is a weak p-set (or q-set) for A" exclusively, in the sense that this indicates that F is not a strong p-set (or q-set) for A . Where there is no doubt about the relevant function algebra, we shall omit "for A" .

2.2.3 It is clear that every closed subset of M_A will fall into one of these categories and that the following two implications hold.

P is a strong p-set $\Rightarrow$ P is a weak p-set

Q is a strong q-set $\Rightarrow$ q is a weak q-set.

We shall frequently lapse into the colloquialism "F is a p-set (or q-set)" meaning that F could be either a strong or a weak p-set (or q-set). It is easy to see that the notion of a strong p-set coincides with that of a "peak interpolation set in the weak sense". If in addition there is a G_δ - condition then we have a "peak interpolation set for A". For this terminology, see [16] pages 110-111.

2.2.4 We list several immediate consequences of the definition in 2.2.2.

2.2.4.1 If P is finite, then P is either a strong p-set or a strong q-set.

Proof: P is clearly closed since M_A is Hausdorff. Let $P = \{x_1, x_2, \ldots, x_n\}$. If $\mu \in A^\perp$ and $|\mu|(P) > 0$, then there is at least one i ; $1 \leqslant i \leqslant n$ such that $\mu\{x_i\} \neq 0$. Since A has the separation property we can construct $f \in A$ such that $f(x_i) = 1$ and $f(x_j) = 0$; $j \neq i$ and $1 \leqslant j \leqslant n$. Then $\int_P f d\mu = \mu\{x_i\} \neq 0$ and $\mu_P \notin A^\perp$. As this is true for every such $\mu \in A^\perp$, we see that P is a strong q-set. On the other hand, if no such $\mu \in A^\perp$ exists, P is a strong p-set.

2.2.4.2 M_A is a strong p-set for A iff $A = C(M_A)$.

2.2.4.3 Any closed set containing E, in particular E itself is a weak p-set. (where $E \neq \phi$)

Proof: Make use of 2.1.10.

2.2.4.4 If $E \neq \phi$, then E is not finite.

Proof: Using 2.2.4.3 and 2.2.4.1.

2.2.4.5 If P is a strong p-set, then $P \cap E$ is nowhere dense in M_A.

Proof: Certainly $P \cap E$ is closed. Let $U = (P \cap E)^0$, the interior of $P \cap E$. Since $U \subset P$ we have $|\mu|(U) = 0 \ \forall \mu \in A^\perp$. But then $E \sim U$ is a closed set supporting every $\mu \in A^\perp$. This contradicts the minimality of E (see 2.1.10). Thus $(P \cap E)^0 = \phi$, giving the result.

2.2.5 Proposition

If P is a p-set for A and F is a p-set for A_p, then F is a p-set for A.

Proof: Since $(A_p)^\perp \subset A^\perp$, this follows from the definition.

2.2.5.1 The result 2.2.5 can clearly be strengthened in the following ways:

If P is a peak set for A and F is a peak set for A_p, then F is a peak set for A

If P is a weak p-set for A and F is a weak p-set for A_p, then F is a weak p-set for A

If P is a weak p-set for A and F is a strong p-set for A_p, then F is a strong p-set for A

If P is a strong p-set for A and F is closed in P, then F is a strong p-set for A

2.2.6 For some of the following results we need the usual definition of p-sets (see 1.13.2) which we can obtain from our definition (2.2.2) via the Glicksberg criterion.

Proposition

If $f \in A$ and $P = \{x : f(x) = \|f\|\}$, then P is a peak set.

Proof: We may assume that $\|f\| = 1$. Then $g = \frac{1}{2}(1 + f)$ peaks on P and $g \in A$.

2.2.7 Proposition

If P is a p-set and not a singleton, then there exists $L \subsetneq P$ such that L is a p-set.

Proof: Since A separates points we can find $g \in A$ which is non-constant on P. We can assume $\|g\|_P = 1$ and $g(x) = 1$ for some $x \in P$. Set $L = \{x \in P : g(x) = 1\}$. By 2.2.6, L is a peak set for A_P. By 2.2.5 L is a p-set for A.

2.2.8 Proposition

Every p-set contains a p-point.

Proof: Let F be a p-set which is not a singleton. Let $\{P_i\}$ be the family of proper p-sets for A which are contained in F. By 2.2.7, $\{P_i\}$ is non-empty. Form a partial order on $\{P_i\}$ by inclusion of sets and let $\{P_n\}$ be a subfamily of $\{P_i\}$, linearly ordered by inclusion. Then $P = \cap P_n$ is a p-set and is clearly a lower bound for $\{P_n\}$. By Zorn's Lemma, $\{P_i\}$ must contain a minimal element. By 2.2.7 any such minimal element must be a p-point.

2.2.9 Proposition

If P is a p-set, then $\mu_{M_A \sim P} \in A^\perp \;\; \forall \mu \in A^\perp$.

Proof: This follows easily from 2.2.2.

2.2.10 Proposition

If P is a p-set and $x \in P$, then P contains the support of every representing measure for x.

Proof: Let m_x be a representing measure for x. Assume that $m_x(M_A \sim P) \neq 0$, i.e. that P does not support m_x. Now $\mu = \delta_x - m_x \in A^\perp$ but $\mu_{M_A \sim P}$ being a real-valued negative measure is not in $A^\perp$. This contradicts 2.2.9.

2.2.11 Proposition

Any proper closed $F \supset \Gamma_A$ is not a p-set.

Proof: If $f \in A$ and $f_{|F} = 1$ then $f_{|\Gamma_A} = 1$. Now by 2.1.17, $f = 1$. Thus F is not a p-set. (cf. 1.13.1).

2.2.12 Proposition

M_A is an antisymmetric set $\Longleftrightarrow$ Γ_A is an antisymmetric set.

Proof: We make repeated use of 2.1.17:

$\Rightarrow$: f is real-valued on Γ_A $\Rightarrow$ f is real-valued on M_A

$\Rightarrow$ f is constant on M_A $\Rightarrow$ f is constant on Γ_A.

$\Leftarrow$: f is real-valued on M_A $\Rightarrow$ f is real-valued on Γ_A

$\Rightarrow$ f is constant on Γ_A $\Rightarrow$ f is constant on M_A.

2.2.13 Proposition

M_A is an antisymmetric set $\Longleftrightarrow$ any $F \supset \Gamma_A$ is an antisymmetric set.

Proof: As the proof for 2.2.12.

2.2.14 We have now developed machinery which enables us to set down many of the relationships between various important restriction sets, some of which have been mentioned in 2.1. We shall deal with these :

E , the Essential set.

P , the set of peak points.

$\mathrm{ch}A$, the Choquet boundary.

Γ_A , the Shilov boundary.

S , the set of maximal point sets of antisymmetry.

2.2.15 Proposition

$\mathrm{ch}A$ is precisely the set of p-points.

Proof: We have $x \in \mathrm{ch}A \iff x$ has a unique representing measure

$\iff \delta_x$ is the only representing measure for x

$\iff \mu\{x\} = 0 \quad \forall\ \mu \in A^\perp$

To prove this last equivalence:

$\Leftarrow$: Say there is a representing measure m_x for x with $m_x \neq \delta_x$ then $m_x - \delta_x = \mu \in A^\perp$ and $\mu\{x\} \neq 0$

$\Rightarrow$: Say that there exists $\mu \in A^\perp$ such that $\mu\{x\} \neq 0$. Choose $\mu\{x\} = 1$. Then $\delta_x - \mu$ is a complex representing measure for x which is zero on x . By 1.8.3, we have m_x a representing measure for x such that $m_x \ll \delta_x - \mu$. i.e. $m_x\{x\} = 0$.

But the condition $\mu\{x\} = 0 \ \forall\ \mu \in A^\perp$ is equivalent to "x is a p-point".

2.2.16 Proposition

chA is a boundary. (cf. 1.12.5)

Proof: By 2.2.6, the maximal set of any $f \in A$ contains a peak set

By 2.2.8, the maximal set of any $f \in A$ contains a p-point

By 2.2.15, the maximal set of any $f \in A$ meets chA.

Thus chA is a boundary.

2.2.17.1 Proposition

Every peak set intersects with Γ_A.

Proof: Any peak set, P, is the maximal set of some $f \in A$. Thus $P \cap \Gamma_A \neq \phi$.

2.2.17.2 Proposition

Every p-set intersects with Γ_A.

Proof: Let F be a p-set and $F = \bigcap_{i \in I} F_i$, where each F_i is a peak set and I is some index set. By 2.2.17.1 $\{F_i \cap \Gamma_A\}$ is a family of non-empty closed sets. We know that $\bigcap_1^n F_i$ is a peak set for any finite subset $\{1,2,\ldots,n\}$ of I. Thus, by 2.2.17.1, $\bigcap_1^n (F_i \cap \Gamma_A) = (\bigcap_1^n F_i) \cap \Gamma_A) \neq \phi$ and we see that the family $\{F_i \cap \Gamma_A ; i \in I\}$ has the finite intersection property. By the compactness of M_A, $F \cap \Gamma_A = (\bigcap_{i \in I} F_i) \cap \Gamma_A = \bigcap_{i \in I} (F_i \cap \Gamma) \neq \phi$, which is the required result.

2.2.17.3 Corollary

$\mathrm{ch}A \subset \Gamma_A$

Proof: By 2.2.17.2, every p-point is in Γ_A. Then apply 2.2.15 .

2.2.18 Proposition

$\Gamma_A = \overline{\mathrm{ch}A}$, the closure in M_A of $\mathrm{ch}A$.

Proof: By 2.2.16 and 2.2.17.3 $\overline{\mathrm{ch}A}$ is a boundary and $\overline{\mathrm{ch}A} \subset \Gamma_A$. By the minimality of Γ_A (see 1.12.2) ; $\overline{\mathrm{ch}A} = \Gamma_A$.

2.2.19 Proposition

If P is a proper p-set, then there exist p-points, x, y such that $x \in P$ and $y \notin P$.

Proof: By 2.2.8 we obtain $x \in P$. By 2.2.11, P does not contain Γ_A . By 2.2.15 and 2.2.18, any closed set which does not contain Γ_A , does not contain certain p-points. From these we can choose $y \notin P$.

2.2.20 Proposition

If P_i ; $1 \leqslant i \leqslant n$ are p-sets and $P = \bigcup_1^n P_i \supset \mathrm{ch}A$, then $P = M_A$.

Proof: P is a p-set which contains $\mathrm{ch}A$. By 2.2.18 $P \supset \Gamma_A$. By 2.2.11 $P = M_A$.

2.2.21 Proposition

If P_i ; $i \in \mathbb{N}$ are p-sets and $P = \bigcup_1^\infty P_i \supset \mathrm{ch}A$ and P is closed, then $P = M_A$.

Proof: Noting that P is a p-set (see [24] page 59 Cor 12.8), we proceed as in 2.2.20.

2.2.22 Proposition

$E \cap \mathrm{ch}A \neq \phi$ and $E \cap \Gamma_A \neq \phi$.

Proof: By 2.2.4.3 , E is a p-set . Now apply 2.2.8 and 2.2.15. The second result follows from the first by application of 2.2.17.3.

2.2.23 Proposition

$M_A \sim E \subset \mathrm{ch}A$.

Proof: If $x \notin E$, then $\mu\{x\} = 0$, $\forall\, \mu \in A^{\perp}$ (by 2.1.10) . Thus x is a p-point. By 2.2.15, $x \in \mathrm{ch}A$.

2.2.24.1 In order to place S we use the following result:

If F is a set of antisymmetry which is not a point, then $F \subset E$.

Proof: Take any x , $y \in F$ and assume that $x \notin E$. There is a Urysohn function f such that $f(x) = 1$ and $f_{|E\cup\{y\}} = 0$. By 2.1.11.1 we have $f \in A$. Since f is real-valued and clearly not constant this contradicts the antisymmetry of F . So our assumption that $x \notin E$ is incorrect and we have $F \subset E$.

2.2.24.2 Corollary

E contains $\overline{M_A \sim S}$, the closure of $M_A \sim S$.

Proof: By 2.2.24.1 $x \in M_A \sim E \Rightarrow x \in S$. Now use the fact that E is closed.

2.2.25 Proposition

$S \subset \mathrm{ch}A$.

Proof: This follows from the fact that every maximal set of antisymmetry is a p-set. See [41] 7.1 ; Theorem 2.

2.2.26 Proposition

$E = \overline{M_A \sim S}$.

Proof: We have one inclusion $\overline{M_A \sim S} \subset E$ by 2.2.24.2. Now for any $\mu \in (bA^{\perp})^e$, $\mathrm{supp}\,\mu$ is an antisymmetric set and as such, is contained in some maximal set of antisymmetry which is clearly not a singleton. (see 1.23.2 and 1.23.3). Thus $\mathrm{supp}\,\mu \subset M_A \sim S$.

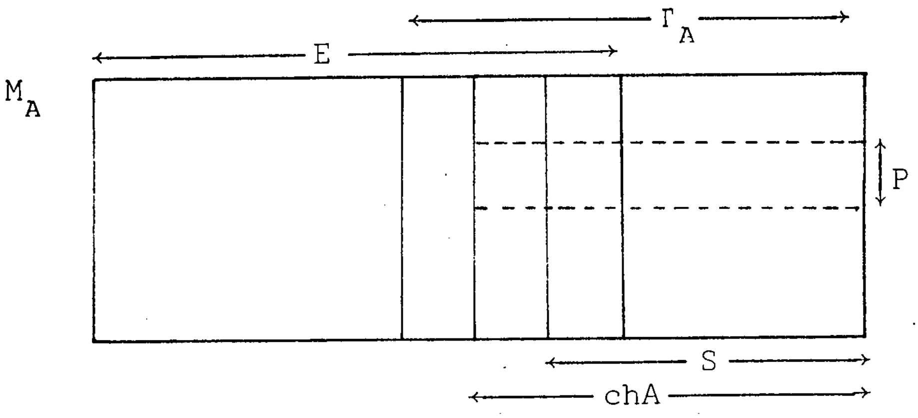
So we have $\bigcup_{\mu \in (bA^{\perp})^e} \mathrm{supp}\,\mu \subset M_A \sim S$

$$\Rightarrow \overline{\left(\bigcup_{\mu \in (bA^{\perp})^e} \mathrm{supp}\,\mu\right)} \subset \overline{M_A \sim S}$$

$\Rightarrow E \subset \overline{M_A \sim S}$, by 2.1.14 .

This inclusion then gives the desired result.

2.2.27 It may be useful to represent these relationships between the basic restriction sets graphically, as follows:



We set down various remarks concerning these:

2.2.27.1 $E \cup S = M_A$ (by 2.2.26)

2.2.27.2 $(\Gamma_A \sim chA)^0 = \phi = (E \cap S)^0$ (by 2.2.18 and 2.2.26 respectively).

2.2.27.3 $A = C(M_A) \iff S = M_A$.

Proof: We have $A = C(M_A) \iff E = \phi$ (see 2.1.11.4)

$\iff S = M_A$ (by 2.2.26)

2.2.27.4 We can strengthen 2.2.27.3 as follows:

$A = C(M_A) \iff chA = S$.

Proof:

$\Rightarrow$: $A = C(M_A) \Rightarrow chA = M_A = S$. This is easily seen by looking at Urysohn functions.

$\Leftarrow$: Assume $A \neq C(M_A)$. Let $\{K\}$ be the decomposition of M_A into maximal sets of antisymmetry. Choose any $K \in \{K\}$ such that $K \notin S$ (i.e. K is not a singleton). We know that K is a p-set (see 1.23.4). By 2.2.8 and 2.2.15, there exists $x \in K$ such that $x \in chA$. Since $\{K\}$ is a disjoint cover we know that $x \notin S$. Thus $\mathrm{ch}A \neq S$.

2.2.27.5 Corollary

$A = C(M_A) \iff \Gamma_A = S$.

2.2.27.6 Let U be open in M_A;

then $U \subset S \iff |\mu|(U) = 0 \quad \forall\, \mu \in A^\perp$.

Proof: $\Rightarrow$: $U \subset S \Rightarrow U \cap (M_A \sim S) = \phi$

$\Rightarrow U \cap (\overline{M_A \sim S}) = \phi$, since U is open

$\Rightarrow U \cap E = \phi$, by 2.2.26

$\Rightarrow |\mu|(U) = 0, \quad \forall\, \mu \in A^\perp$ by 2.1.10.

$\Leftarrow$: On the other hand, U open and

$|\mu|(U) = 0, \ \forall\, \mu \in A^\perp$

$\Rightarrow U \cap \operatorname{supp} \mu = \phi, \ \forall\, \mu \in A^\perp$

$$U \cap \left(\bigcup_{\mu \in A^\perp} \operatorname{supp} \mu \right) = \phi$$

$$\Rightarrow U \cap \overline{\left(\bigcup_{\mu \in A^\perp} \operatorname{supp} \mu \right)} = \phi, \text{ since } U \text{ is open}$$

$\Rightarrow U \subset M_A \sim E$, by 2.1.13

$\Rightarrow U \subset S$ by 2.2.26.

2.2.27.7 $E \cap \Gamma_A = \phi \iff A = C(M_A)$

Proof: We know that E is a p-set (see 2.2.4.3). Thus $E \cap \Gamma_A \neq \phi$ (see 2.2.17.2). The only possible exception to this is if $E = \phi \iff A = C(M_A)$. (see 2.1.11.4).

2.2.27.8 We can strengthen the previous result as follows:

$E \cap \mathrm{ch}A = \phi \iff A = C(M_A)$.

Proof: E is a p-set. Now by 2.2.8 and 2.2.15 there exists $x \in E$ such that $x \in \mathrm{ch}A$. Again, the only exception is $E = \phi \iff A = C(M_A)$.

2.3 A - convex hulls

2.3.1 Definition

For any $U \subset M_A$, the A-convex hull of U is $\{a \in M_A : |f(a)| \leqslant \sup_{x \in U} |f(x)| \; ; \forall f \in A\}$.
We shall denote this $\tilde{U}$. We say that U is A-convex if $U = \tilde{U}$.
We note that $\tilde{U} = \tilde{\tilde{U}}$ and that $\tilde{U}$ is closed.

2.3.2 $\tilde{U}$ is the maximal ideal space of A_U (see [41] 7.2, theorem 7.)
We may be led to expect from this result that the study of A-convex hulls will be particularly relevant in our context, i.e. $X = M_A$. We shall see later that this is the case.

2.3.3 Proposition

If F is closed then $F \supset \Gamma_{A_{\tilde{F}}}$, the Shilov boundary of $A_{\tilde{F}}$.

Proof: Let $f \in A_{\tilde{F}}$, then there is a sequence $\{f_i\} \subset A$ such that $f_i \to f$ uniformly on $\tilde{F}$. By the definition of 2.3.1 each $f_i|_{\tilde{F}}$ attains its maximum modulus on F. Assume that f attains its maximum modulus, M, at some $x \in \tilde{F} \sim F$ and that $|f|_F| < M - \varepsilon$ for some $\varepsilon > 0$. Then, for large enough i, f_i is such that $|f_i(x)| > M - \varepsilon/4$ and $\|f_i\|_F < M - 3\varepsilon/4$. This contradicts an earlier statement, so f must attain its maximum modulus on F, as required.

2.3.4 We are now able to give, for a closed F, a characterization of the A-convex hull, $\tilde{F}$, in measure theoretic terms.

If F is closed, then $x \in \tilde{F}$ iff there is a representing measure, m_x, for x such that

$$\operatorname{supp} m_x \subset F .$$

Proof: This is a consequence of 2.3.1, 2.3.2, 2.3.3 and 1.12.6. From these we obtain :

If $x \in \tilde{F}$, then there is a representing measure m_x for x on $A_{\tilde{F}}$ with $\operatorname{supp} m_x \subset F$. Then m_x also represents x on A.

On the other hand :

If $x \notin \tilde{F}$, then there exists $f \in A$ such that $f(x) = 1$ and $\|f\|_F < 1$ (since $F \subset \tilde{F}$). If m_x is any representing measure for x on A, then $\int fdm_x = 1$ but $\int_F fdm_x < \|f\|_F . m_x(F) < 1$.

Thus $\operatorname{supp} m_x \not\subset F$. This completes the proof.

2.3.4.1 Using 2.3.4 we are able to produce a further measure theoretic characterization of $\tilde{F}$, this time in terms of annihilating measures.

Proposition

If F is closed, then $x \in \tilde{F}$ iff

either $x \in F$

or there is $\mu \in bA^{\perp}$ such that

$\mu\{x\} = \frac{1}{2}$ and $\operatorname{supp}\mu \subset F \cup \{x\}$.

Proof: Clearly $F \subset \tilde{F}$. If $x \in \tilde{F} \sim F$ we can form $\mu = \frac{1}{2}(\delta_x - m_x)$ where m_x is the representing measure obtained by 2.3.4. and is such that $m_x \neq \delta_x$. It is easily seen that μ satisfies the conditions of this result.

On the other hand, assume that we have such a μ for some $x \notin F$. Then $\delta_x - 2\mu$ is a complex representing measure for x which is supported on F. By 1.8.3 we have m_x representing x such that $\operatorname{supp} m_x \subset F$. By 2.3.4 , $x \in \tilde{F}$ . . .

2.3.5 Proposition

Every p-set is A-convex.

Proof: Let P be a p-set. Now, if $y \notin P$ and there is a representing measure, m_y, for y with $\operatorname{supp} m_y \subset P$, then we can form $\mu \in A^{\perp}$ where $\mu = \delta_y - m_y$. But then $\mu_P = -m_y \notin A^{\perp}$, which contradicts the fact that P is a p-set. Thus $\tilde{P} \subset P$ by 2.3.4. Since, clearly, $P \subset \tilde{P}$, we have the desired result. (Alternatively we could use 2.3.4.1 in a similar way.)

2.3.6 Proposition

If F is closed and $f \in A$ is such that

$$\left\{\begin{array}{l} f \text{ is real-valued} \\ f \geqslant 0 \\ f > 0 \\ \mathrm{Re} f \geqslant 0 \\ \mathrm{Re} f > 0 \\ f \text{ is constant} \end{array}\right\} \text{ on } F\text{, then } \left\{\begin{array}{l} f \text{ is real-valued} \\ f \geqslant 0 \\ f > 0 \\ \mathrm{Re} f \geqslant 0 \\ \mathrm{Re} f > 0 \\ f \text{ is constant} \end{array}\right\} \text{ on } \tilde{F}$$

Proof: We make use of 2.3.2 , 2.3.3 and 2.1.17.

2.3.7 Proposition

A closed set P is antisymmetric iff $\tilde{P}$ is antisymmetric.

Proof: $f \in A$ is real-valued on $\tilde{P}$

$\Rightarrow f$ is real-valued on $P \Rightarrow f$ is constant on P (hypothesis)

$\Rightarrow f$ is constant on $\tilde{P}$ (by 2.3.6)

On the other hand :

$f \in A$ is real-valued on P

$\Rightarrow f$ is real-valued on $\tilde{P}$ (by 2.3.6)

$\Rightarrow f$ is constant on $\tilde{P}$ (by hypothesis)

$\Rightarrow f$ is constant on P .

2.3.8 Corollary

Every maximal set of antisymmetry is A-convex .

Proof: Let K be a maximal set of antisymmetry. By 2.3.7, $\tilde{K}$ is antisymmetric. By the maximality of K, $\tilde{K} = K$ as required.

2.4 C R S - sets

2.4.1.1 Definition

A closed set F is a CRS-set (i.e. a closed restriction set) in A if $A_{|F} = A_F$. A closed set F is an interpolation set if $A_{|F} = C(F)$. If there is no doubt about the function algebra involved, we may simply say : F is a CRS-set.

2.4.1.2 We have a measure theoretic characterization of CRS-sets and interpolation sets. These results appear in [26] ; Theorem 1 and Corollary. We note that these results have been proved in the context of the general closed boundary, X, rather than in our context, M_A. It will be convenient at this point to conserve much of this generality and, in order to conform to our established notation we shall write $A^{\perp}_{X} = A^{\perp} \cap M(X)$, i.e. all the annihilating measures for A which are supported on X. Since $(A_F)^{\perp} \subset A^{\perp}$ we may write $(A_F)^{\perp}_{F} = A^{\perp}_{F}$.

Let kF be the ideal of all $f \in A$ which are zero on the closed set F. We denote by $A/_{kF}$ the quotient space of A modulo that ideal.

Define an operator $T_F : A^{\perp} \to M(F)/_{A^{\perp}_{F}}$ by $T_F(m) = m_F + A^{\perp}_{F}$, where F is a closed set.

Now we have :

(a) $\|T_F\| < 1$ iff F is CRS

(b) $\|T_F\| \leqslant \frac{1}{2}$ iff $A_{|F}$ is isometric with $A/_{kF}$

(c) F is an interpolation set iff there is $t < 1$ such that $\|m_F\| \leqslant t\|m\|$, $\forall\ m \in A^{\perp}$.

(d) $A/_{kF}$ is isometric with $C(F)$ iff $\|m_F\| \leqslant \|m\|/_2$, $\forall\ m \in A^{\perp}$.

2.4.1.3 We can write results (a) and (b) above more explicitly in terms of measures. We proceed by examining T_F :

$$\|T_F\| = \sup_{m\in A^{\perp}\sim\{0\}} \frac{\|T_F(m)\|}{\|m\|}$$

$$= \sup_{m\in A^{\perp}\sim\{0\}} \frac{\|m_F + A_F^{\perp}\|}{\|m\|}, \text{ (by definition)}$$

$$= \sup_{m\in A^{\perp}\sim\{0\}} \frac{\inf_{\nu \in A_F^{\perp}} \|m_F + \nu\|}{\|m\|}.$$

(from the definition of the norm on the quotient space.)

It is clear that T_F is norm decreasing because

$$\|T_F\| \leqslant \sup_{m\in A^{\perp}\sim\{0\}} \frac{\|m_F\|}{\|m\|}, \text{ since } 0 \in A_F^{\perp}$$

$$\leqslant 1$$

Thus results (a) and (b) above can be written:

(i) F is CRS if

$$\sup_{m \in A^{\perp} \sim \{0\}} \frac{\inf_{\nu \in A_F^{\perp}} \| m_F + \nu \|}{\| m \|} < 1$$

(ii) $A/_{kF}$ is isometric with $A_{|F}$ iff

$$\sup_{m \in A^{\perp} \sim \{0\}} \frac{\inf_{\nu \in A_F^{\perp}} \| m_F + \nu \|}{\| m \|} \leqslant \tfrac{1}{2}$$

2.4.1.4 Proposition

Every p-set is a CRS-set. In fact, if P is a p-set, then $A_{|P}$ is isometric with $A/_{kP}$.

Proof:

We have $$\sup_{m \in A^{\perp} \sim \{0\}} \frac{\inf_{\nu \in A_P^{\perp}} \| m_P + \nu \|}{\| m \|}$$

$$= \sup_{m \in A^{\perp} \sim \{0\}} \frac{0}{\| m \|}$$, since P is a p-set we have $m_P \in A_P^{\perp}$, $\forall\, m \in A^{\perp}$

$$= 0$$

This satisfies the conditions for both (i) and (ii) in 2.4.1.3 and we obtain the desired result.

2.4.1.5 The proof of the previous result can be used to give another measure theoretic characterization of p-sets.

F is a p-set iff $T = 0$ i.e. F is a p-set iff

$$\sup_{m \in A^{\perp} \sim \{0\}} \frac{\inf_{\nu \in A_F^{\perp}} \| m_F + \nu \|}{\|m\|} = 0 \, .$$

Proof: The proof of the forward implication arises in the proof of 2.4.1.4. For the reverse implication we note that:

$$\sup_{m \in A^{\perp} \sim \{0\}} \frac{\inf_{\nu \in A_F^{\perp}} \| m_F + \nu \|}{\|m\|} = 0 \Rightarrow \inf_{\nu \in A_F^{\perp}} \| m_F + \nu \| = 0 \quad \forall\, m \in A^{\perp}.$$

Thus there is a sequence $\{\nu_n\}_m \subset A_F^{\perp}$ such that $\| m_F - \nu_n \| < \frac{1}{n}$, $\forall\, n \in \mathbb{N}$. Such a sequence exists for each $m \in A^{\perp}$.

Let $f \in A$, then
$$\left| \int f dm_F \right| = \left| \int f dm_F - \int f d\nu_n \right| \, ,$$
since $\nu_n \in A_F^{\perp}$
$\Rightarrow \nu_n \in A^{\perp}$, $\forall\, n \in \mathbb{N}$.
$$= \left| \int f d(m_F - \nu_n) \right|$$
$$< \|f\| \circ \frac{1}{n} \, .$$

As this is true for every $n \in \mathbb{N}$ and every $f \in A$, we have $m_F \in A^{\perp}$. Since the same process holds for every $m \in A^{\perp}$ we have : $m \in A^{\perp} \Rightarrow m_F \in A^{\perp}$. Thus F is a p-set.

2.4.1.6 Some remarks about CRS-sets.

2.4.1.6.1 Any closed $F \supset E$ is CRS. In particular E is CRS .

Proof: This follows from 2.2.4.3 and 2.4.1.4.

2.4.1.6.2 Any closed $F \subset M_A \sim E$ is CRS.

Proof: For any such F and $\mu \in A^{\perp}$, we have $\mu_F = 0$ (by 2.1.10). Now 2.4.1.3 (i) or 2.4.1.4 will yield the result.

2.4.1.6.3 Any closed $F \supset \Gamma_A$ is CRS. In particular, Γ_A is CRS.

Proof: Say we have a sequence $\{f_i\} \subset A$ such that $f_{i|F} \to f \in C(F)$. Thus $f_{i|F}$ is a Cauchy sequence in $A_{|F}$, i.e. $\|f_n - f_m\|_F \to 0$ as $m,n \to \infty$. But $F \supset \Gamma_A$ and each $f_n - f_m \in A$, achieves its maximum modulus on F. So we have $\|f_n - f_m\|_{M_A} \to 0$ as $m,n \to \infty$. Since A is closed $f_i \to \tilde{f} \in A$ and clearly $\tilde{f}_{|F} = f$.

2.4.1.6.4 Any finite set is CRS .

Proof: Let $F = \{x_1, x_2, \ldots x_n\}$. Since the relative topology on F is discrete, $C(F)$ is the set of all possible functions $f : F \to \mathbb{C}$. On the other hand, arguing as in 2.2.4.1 we can, for any given $g \in C(F)$, construct $f \in A$ such that $f_{|F} = g$.

2.4.1.7 Proposition

If F is CRS in A and closed $P \subset F$ is CRS in A_F, then P is CRS in A.

Proof: Take $f_1 \in A_P$, then $\exists\, f_2 \in A_F$ such that $f_{2|P} = f_1$, since P is CRS in A_F. But, since F is CRS in A, there exists $f_3 \in A$ such that $f_{3|F} = f_2$. Clearly $f_{3|P} = f_1$ as required.

2.4.1.8 Proposition

Let P be CRS and F be closed and such that $\Gamma_{A_F} \subset P \subset F$, then F is CRS.

Proof: Clearly we have $\Gamma_{A_P} \subset \Gamma_{A_F}$. By 2.4.1.6.3, Γ_{A_F} is CRS in A_P. By 2.4.1.7, Γ_{A_F} is CRS in A. Now choose any $f \in A_F$, then there is a sequence $\{f_n\} \subset A$ such that $\|f_n - f\|_F \to 0$. Then clearly $\|f_n - f\|_{\Gamma_{A_F}} \to 0$. Thus $f_{|\Gamma_{A_F}} \in A_{\Gamma_{A_F}}$. But Γ_{A_F} is CRS, thus $f_{|\Gamma_{A_F}} \in A_{|\Gamma_{A_F}}$, i.e. there is $\tilde{f} \in A$ such that $\tilde{f}_{|\Gamma_{A_F}} = f_{|\Gamma_{A_F}}$. So we see that $f - \tilde{f}_{|F} \in A_F$ and $f - \tilde{f}_{|F} = 0$ on Γ_{A_F}. Thus $f = \tilde{f}_{|F}$ as required

2.4.1.9 Proposition

Let F be CRS and P be closed and such that $\Gamma_{A_F} \subset P \subset F$. Then P is CRS.

Proof: By 2.4.1.6.3, P is CRS in A_F. By 2.4.1.7, P is CRS in A.

2.4.1.10 Corollary

Let F and P be closed and such that $\Gamma_{A_F} \subset P \subset F$, then F is CRS iff P is CRS.

Proof: This follows immediately from 2.4.1.8 and 2.4.1.9.

2.4.1.11 We now look at some relationships between CRS sets and some of the constructions we have already investigated, e.g. antisymmetric sets, A-convex hulls and p-sets.

Proposition

F is CRS iff $\tilde{F}$ is CRS.

Proof: By 2.3.3 we have $\Gamma_{A_{\tilde{F}}} \subset F \subset \tilde{F}$. Now apply 2.4.1.10.

2.4.1.12 Proposition

Let F be CRS and P be closed and such that $P \subset F$, then P is anti-symmetric with respect to A_F iff P is antisymmetric with respect to A .

Proof:

$\Rightarrow$: Let $f \in A$ be real-valued on P , then $f_{|F} \in A_F$ and is real-valued on P , hence constant on P . Thus f is constant on P .

$\Leftarrow$: Let $f \in A_F$ be real-valued on P ; Since F is CRS we have $\tilde{f} \in A$ such that $\tilde{f}_{|F} = f$. Thus $\tilde{f}$ is real-valued on P , hence constant on P (by hypothesis). Thus f is constant on P .

2.4.1.13 Proposition

Let F be CRS and P be closed and such that $P \subset F$, then P is antisymmetric with respect to A_F iff $\tilde{P}$ is antisymmetric with respect to A .

Proof: P is antisymmetric with respect to A_F

$\iff$ P is antisymmetric with respect to A (by 2.4.1.12)

$\iff$ $\tilde{P}$ is antisymmetric with respect to A (by 2.3.7)

2.4.1.14 Proposition

Let F be CRS and P be a p-set, then $P \cap \tilde{F}$ is a p-set for A_F in $\tilde{F}$.

Proof: Bearing in mind that $\tilde{F}$ is the maximal ideal space for A_F (see 2.3.2) we recall that $(A_F)^{\perp}$ denotes all the measures on $\tilde{F}$ which annihilate A_F. It is easily seen that $(A_F)^{\perp} = \{\mu \in A^{\perp}$ such that $\operatorname{supp}\mu \subset \tilde{F}\}$. Now let $\mu \in (A_F)^{\perp}$, then we have $\mu_{P\cap\tilde{F}} = (\mu_{\tilde{F}})_P = \mu_P$ (since $\operatorname{supp}\mu \subset \tilde{F}$). But $\mu_P \in (A_F)^{\perp}$ since P is a p-set and $\operatorname{supp}\mu_P \subset \tilde{F}$. Thus $\mu \in (A_F)^{\perp} \Rightarrow \mu_{P\cap\tilde{F}} \in (A_F)^{\perp}$ which gives the result.

2.4.1.15 Proposition

Let F be CRS and A-convex and such that $F = F_1 \cup F_2$, where the F_i $(i = 1,2)$ are disjoint closed sets. Then there is $f \in A$ such that $f_{|F_1} = 0$ and $f_{|F_2} = 1$.

Proof: $M_{A_F} = F$ (by 2.3.2) and since the F_i $(i = 1,2)$ are relatively clopen in F, we know, by Shilov's Idempotent theorem (see 1.11) that there is $\tilde{f} \in A_F$ such that $\tilde{f}_{|F_1} = 0$ and $\tilde{f}_{|F_2} = 1$. Since F is CRS, there is $f \in A$ such that $f_{|F} = \tilde{f}$.

2.4.1.16 Corollary

If $E = F_1 \cup F_2$, where F_i $(i = 1,2)$ are disjoint closed sets, then there is $f \in A$ such that $f_{|F_1} = 0$ and $f_{|F_2} = 1$.

Proof: E is a p-set (see 2.2.4.3), hence is A-convex (see 2.3.5) and CRS (see 2.4.1.4). Now use 2.4.1.15.

2.4.1.17 Corollary

Let P_1 and P_2 be disjoint p-sets, then there is $f \in A$ such that $f_{|P_1} = 0$ and $f_{|P_2} = 1$.

Proof: We know that $P = P_1 \cup P_2$ is a p-set, hence A-convex (see 2.3.5) and CRS (see 2.4.1.4).

2.4.1.18 Corollary

Let K_1 and K_2 be distinct maximal sets of antisymmetry, then there is $f \in A$ such that $f_{|K_1} = 0$ and $f_{|K_2} = 1$.

Proof: The K_i $(i = 1,2)$ are p-sets (see 1.23.4). Now use 2.4.1.17.

2.4.2 This section (i.e. the 2.4.2 series) does not form an integral part of our development of the subject but can be regarded rather as a kind of mathematical interlude of interest for its own sake. This is particularly so as many of these results are later superceded by more direct methods (see the 2.4.3 series). On the other hand, it then becomes apparent, in retrospect, that this section has given a number of valuable insights into the nature of the essential set, E. Our starting point is the characterization of interpolation sets given in 2.4.1.2 (c). This enables us to define a class of sets which has the property that any closed set disjoint from any member of the class is an interpolation set.

2.4.2.1 Definition

For $0 < \alpha \leqslant 1$ let F_α be a closed set such that $\frac{|\mu|(F_\alpha)}{\|\mu\|} \geqslant \alpha$, for all $\mu \in A^\perp \sim \{0\}$. Equivalently, for $0 < \alpha \leqslant 1$ let F_α be a closed set such that $|\mu|(F_\alpha) \geqslant \alpha$ for all $\mu \in A^\perp$ such that $\|\mu\| = 1$.

We denote by $\{F_\alpha\}$, the family of all such sets for a particular α.

2.4.2.2 Proposition

$\{F_\alpha\}$ has a minimal element.

Proof: $\{F_\alpha\}$ is non-empty since $E \in \{F_\alpha\}$, $\forall\alpha$ such that $0 < \alpha \leqslant 1$. (see 2.1.10). Set inclusion is a partial order on $\{F_\alpha\}$. Let $\{P_i\}$ be any chain in $\{F_\alpha\}$. Clearly $\{P_i\}$ has the finite intersection property. Thus, by the compactness of M_A, $P = \bigcap_i P_i$ is non-empty and is clearly closed. To show that P is a lower bound for the chain $\{P_i\}$, we need only show that $P \in \{F_\alpha\}$. Assume that $P \notin \{F_\alpha\}$. Then there is $\mu \in A^\perp$ with $\|\mu\| = 1$ and $|\mu|(P) = \beta < \alpha$. Then

$|\alpha|(P_i \sim P) \geqslant \alpha - \beta$ for all i . But $\{P_i \sim P\} \xrightarrow[\infty]{i} \phi$. This contradiction shows that $P \in \{F_\alpha\}$. Now, by Zorn's Lemma, $\{F_\alpha\}$ has a minimal element.

2.4.2.3 Notation: We denote by C_α one of the minimal elements of $\{F_\alpha\}$.

2.4.2.4 Proposition

$\{F_1\} = \{F\colon F$ is closed and $F \supset E\}$. Also C_1 is unique and $C_1 = E$.

Proof: Trivial (see 2.1.10).

2.4.2.5 Proposition

Any particular F_α contains some C_β for β such that $0 < \beta \leqslant \alpha$.

Proof: Take a particular $F^*_\alpha \in \{F_\alpha\}$. Let $\{F_\alpha\}^*$ be all the elements of $\{F_\alpha\}$ which are contained in F^*_α . Now, reasoning as in the proof of 2.4.2.2 (only replacing E by F^*_α) we obtain a minimal element of $\{F_\alpha\}^*$ which is clearly also minimal for $\{F_\alpha\}$. Furthermore, we note that $F_\alpha \in \{F_\beta\}$ for any α, β such that $0 < \beta \leqslant \alpha$.

2.4.2.6 Proposition

Let P be closed and such that $P \cap F_\alpha = \phi$ for some F_α , then $A_{|P} = C(P)$.

Proof: This is an immediate consequence of 2.4.1.2 (c) and the definition 2.4.2.1.

2.4.2.7 Proposition

Let P be a weak p-set. Then $P \cap F_\alpha \neq \phi$, for any F_α.

Proof: We have $\mu \in A^\perp \sim \{0\}$ such that $\operatorname{supp} \mu \subset P$. (see 2.2.2). By 2.4.2.1 such a P must intersect any F_α.

2.4.2.8 Corollary

Any F_α intersects every non-trivial maximal set of anti-symmetry.

Proof: It is known that any maximal antisymmetric set, K, is a p-set (see 1.23.4) and that $A_{|K}$ is an antisymmetric algebra (see 1.23.5) and hence that K is the essential set for $A_{|K}$ (if K is non-trivial) (see 1.26). Thus we have $\mu \in (A_K)^\perp$ with $\operatorname{supp} \mu \subset K$ and $\mu \neq 0$. Clearly $\mu \in A^\perp$. Thus K is a weak p-set. Now use 2.4.2.7.

2.4.2.9 Proposition

For any F_α : either $E \subset F_\alpha$ or F_α is not a p-set.

Proof: Clearly $F_\alpha \cap E \neq \phi$ (from 2.2.4.3 and 2.4.2.7). Say $E \sim F_\alpha \neq \phi$. Assume that $|\mu|(E \sim F_\alpha) = 0$, $\forall\, \mu \in A^\perp$. Then $F_\alpha \cap E$ supports every $\mu \in A^\perp$, contradicting the minimality of E (2.1.10). Thus there exists $\mu \in A^\perp$ such that $|\mu|(E \sim F_\alpha) \neq 0$.

Now if F_α is a p-set, then $\mu_{E\sim F_\alpha} \in A^\perp \sim \{0\}$. However, since $|\mu_{E\sim F_\alpha}|(F_\alpha) = 0$, this contradicts 2.4.2.1.

2.4.2.10 Proposition

For any F_α and any $\mu \in A^\perp \sim \{0\}$, we have $F_\alpha \cap \operatorname{supp} \mu \neq \phi$.

Proof: Trivial.

2.4.2.11 Proposition

For any C_α , we have : $C_\alpha \subseteq E$.

Proof: Trivial, since, clearly, $(F_\alpha \cap E) \in \{F_\alpha\}$ for any F_α .

2.4.2.12 Proposition

For any F_α we have, $F_\alpha \cap \Gamma_A \neq \phi$.

Proof: Assume $A_{\Gamma_A} = C(\Gamma_A)$. By a suitable choice of Urysohn function on Γ_A and using 2.1.17 we see that $x \in \Gamma_A \Rightarrow x \in S$. This contradicts 2.2.27.5 (since A is a proper subalgebra of $C(M_A)$). Thus we have $A|_{\Gamma_A} \neq C(\Gamma_A)$. Now use 2.4.2.6.

2.4.2.13 Proposition

If P is a p-set such that $E \not\subset P$, then $F_\alpha \cap (M_A \sim P) \neq \phi$, for any F_α .

Proof: By 2.2.9, $\mu \in A^\perp \Rightarrow \mu_{M_A \sim P} \in A^\perp$. If $\mu_{M_A\sim P} \neq 0$, then $F_\alpha \cap (M_A \sim P) \neq \phi$ by 2.4.2.1. On the other hand if $\mu_{M_A \sim P} = 0$, $\forall\ \mu \in A^{\perp}$, then $P \supset E$ by 2.1.10. This contradicts the hypothesis and establishes the result.

2.4.2.14 Proposition

No isolated point of M_A is in C_α , for any C_α . Equivalently, every C_α is a perfect set.

Proof: Let x be isolated in M_A . By Shilov's Idempotent Theorem (see 1.11), x is a p-point (in fact, x is a peak point). Thus $\mu\{x\} = 0$, $\forall\ \mu \in A^{\perp}$. Now, if $x \in C_\alpha$, then $C_\alpha \sim \{x\}$ is also closed, contradicting the minimality of C_α .

2.4.2.15 Proposition

Let C_α^1 and C_α^2 be minimal sets for a particular $0 < \alpha \leqslant 1$. Then $C_\alpha^1 \cap C_\alpha^2 \neq \phi$.

Proof: Assume that $C_\alpha^1 \cap C_\alpha^2 = \phi$ and that every $\mu \in A^{\perp}$ is supported by $C_\alpha^1 \cup C_\alpha^2$. Then we have $C_\alpha^1 \cup C_\alpha^2 \supset E$ (by 2.1.10). On the other hand, by the minimality of the C_α^i $(i = 1,2)$, any open U intersecting either of the C_α^i $(i = 1,2)$ has $|\mu|(U) > 0$ for some $\mu \in A^{\perp}$. Thus, by 2.1.15.1, $C_\alpha^1 \cup C_\alpha^2 \subset E$. So we have $C_\alpha^1 \cup C_\alpha^2 = E$. Now we can apply 2.4.1.16 to obtain $f \in A$ such that $f_{|C_\alpha^1} = 1$ and $f_{|C_\alpha^2} = 0$. Then $\forall\ \mu \in A^{\perp} \sim \{0\}$ we have $f\mu \in A^{\perp}$ (see 1.9.3). In fact, since it is clear that $f\mu$ is non-zero on C_α^1 , we have $f\mu \in A^{\perp} \sim \{0\}$. On the other hand $|f\mu|(C_\alpha^2) = 0$.

This contradiction to the definition of C^2_α (see 2.4.2.1) means that we must assume that there exists $\mu \in A^\perp$ such that $|\mu|\{M_A \sim (C^1_\alpha \cup C^2_\alpha)\} > 0$. By the regularity of μ, choose a closed $Q \subset M_A \sim (C^1_\alpha \cup C^2_\alpha)$ such that $|\mu|(Q) = C > 0$. Since C^1_α is disjoint from $Q \cup C^2_\alpha$, we know by 2.4.2.6 that $A_{|Q\cup C^2_\alpha} = C(Q \cup C^2_\alpha)$. So we have $g \in A$ such that $g_{|Q} = 1$ and $g_{|C^2_\alpha} = 0$. Reasoning as before, we have $g\mu \in A^\perp$, $g\mu$ is non-zero on Q and $|g\mu|(C^2_\alpha) = 0$. This contradiction forces us to conclude that $C^1_\alpha \cap C^2_\alpha \neq \phi$ as required.

2.4.2.16 Corollary

For any F_α and F_β we have $F_\alpha \cap F_\beta \neq \phi$.

Proof: Say that $\beta \geqslant \alpha$, then, by 2.4.2.5, F_α contains some C^1_α and F_β contains some C^2_α. Now by 2.4.2.15, $F_\alpha \cap F_\beta \neq \phi$.

2.4.2.17 Proposition

For any F_α and F_β where $\frac{1}{2} < \alpha \leqslant \beta \leqslant 1$, there is $F_\gamma \subset F_\alpha \cap F_\beta$ where $\gamma \geqslant 2\alpha - 1$.

Proof: For any $\mu \in A^\perp \sim \{0\}$ and $\|\mu\| = 1$ we have $|\mu|(F_\alpha) \geqslant \alpha$ and $|\mu|(F_\beta) \geqslant \alpha$. But since $|\mu|(F_\alpha \cup F_\beta) \leqslant 1$ we have $|\mu|(F_\alpha \cap F_\beta) \geqslant 2\alpha - 1$ as required.

2.4.2.18 Proposition

For any F_α, where $\alpha \geqslant \frac{1}{2}$, and any closed set P such that $P \cap F_\alpha = \phi$, we have $A_{|P} = C(P)$ and $A/_{kP}$ is isometric with $C(P)$.

Proof: This is a consequence of 2.4.1.2 (d).

2.4.2.19 Proposition

Any F_α has the property :

$f \in A$ and $f_{|F_\alpha} = 0 \Rightarrow f_{|E} = 0$.

Proof: Since $f_{|F_\alpha} = 0$ we must have $\|f\mu\| = 0$ for every $\mu \in A^\perp$ or else the definition of F_α is contradicted. Now set $U_f = \{x: f(x) \neq 0\}$. Clearly U_f is open and, reasoning as in the proof of 2.1.11.6, we see that $|\mu|(U_f) = 0$ for all $\mu \in A^\perp$. Thus U_f is disjoint from E, or else the minimality of E is contradicted (see 2.1.10). This gives the result.

2.4.2.20 Corollary

For any F_α such that $F_\alpha \subset E$ we have :

$f_{|E} = 0$ iff $f_{|F_\alpha} = 0$, for all $f \in A$.

2.4.2.21 Corollary

$f_{|C_\alpha} = 0$ iff $f_{|E} = 0$, for any $f \in A$ and any C_α.

Proof: This follows from 2.4.2.11 and 2.4.2.20.

2.4.2.22 Proposition

$A = C(M_A)$ iff there is some C_α which is finite.

Proof: $A = C(M_A) \Rightarrow E = \phi$ (by 2.1.11.4) $\Rightarrow C_\alpha = \phi$ (by 2.4.2.11). This gives the forward implication. For the reverse implication, we note again that $C_\alpha \subseteq E$ (by 2.4.2.11). Let $y \in E \sim C_\alpha$. Reasoning as in 2.2.4.1 we can construct $f \in A$ such that $f(y) = 1$ and $f|_{C_\alpha} = 0$. This contradicts 2.4.2.21. Thus we have $E = C_\alpha$. Now by 2.2.4.4 and 2.1.11.4 we have $A = C(M_A)$.

2.4.2.23 Proposition

A nested sequence $\{C_{1/n} : n \in \mathbb{N}\}$ can be formed. Then $F = \bigcap_{\mathbb{N}} C_{1/n}$ is non-empty for such a sequence.

Proof: The first statement follows from 2.4.2.5. Clearly $\{C_{1/n}\}$ has the finite intersection property. By the compactness of M_A, F is non-empty.

2.4.2.24 Proposition

Let F be as in 2.4.2.23. Then $F \cap P \neq \phi$ for any weak p-set, P.

Proof: By 2.4.2.7, $P \cap C_{1/n} \neq \phi$ for all $n \in \mathbb{N}$. Thus $\{P \cap C_{1/n}\}$ is a nested sequence of closed sets with the finite intersection property. Since M_A is compact, $\bigcap_{\mathbb{N}} (P \cap C_{1/n}) \neq \phi$.

Thus $F \cap P = (\bigcap_{\mathbb{N}} C_{1/n}) \cap P = \bigcap_{\mathbb{N}} (P \cap C_{1/n}) \neq \phi$ as required.

2.4.2.25 Proposition

Let F be as in 2.4.2.23. Then $F \cap K \neq \phi$ for every non-trivial maximal set of antisymmetry, K.

Proof: By 2.4.2.8, $K \cap C_{1/n} \neq \phi$ for all $n \in \mathbb{N}$. Now argue as in the proof of 2.4.2.24.

2.4.2.26 Proposition

Let F be as in 2.4.2.23. Then $F \cap \operatorname{supp} \mu \neq \phi$, for all $\mu \in A^{\perp} \sim \{0\}$.

Proof: We know that $\operatorname{supp} \mu \cap C_{1/n} \neq \phi$, $\forall\, n \in \mathbb{N}$. Now argue as in the proof of 2.4.2.24.

2.4.2.27 Proposition

Let F be as in 2.4.2.23 and P be a closed set such that $P \cap F = \phi$, then we have $A_{|P} = C(P)$.

Proof: The sets $\{M_A \sim C_{1/n}\}$ form an open cover for P. Since P is compact we can choose a finite subcover $\{M_A \sim C_{1/n_i}\}_{i=1}^{i=k}$. Then, since $\{C_{1/n}\}$ is a nested sequence we have $\bigcap_{i=1}^{k} C_{1/n_i} = C_{1/n_j}$ where $n_j = \max\{n_1, n_2, \ldots, n_k\}$. Thus we have $C_{1/n_j} \cap P = \phi$ and we can apply 2.4.2.6.

2.4.2.28 In this section we see how the previous results of 2.4.2 can be considerably strengthened by the imposition of an approximate normality condition on A (see 1.2.5.8). In the following section (i.e. 2.4.3 series) we shall see, by means of other methods, that the approximate normality condition is in fact not necessary for obtaining these strengthened results.

2.4.2.28.1 Proposition

A is approximately normal $\Rightarrow E \subset F_\alpha$ for any F_α .

Proof: Assume that $E \sim F_\alpha \neq \phi$. Then, if the minimality of E is not to be contradicted (see 2.1.10) there must be $\mu \in A^\perp$ such that $|\mu|(E \sim F_\alpha) > 0$. We may assume without loss of generality that $\|\mu\| = 1$. Now, using the regularity of $|\mu|$ (see 1.5.1) choose a closed set $Q \subset E \sim F_\alpha$ such that $|\mu|(Q) = c > 0$. By the approximate normality of A , we can, for any given $\varepsilon > 0$, choose $f \in A$ such that $|f| < \varepsilon$ on F_α and $|1-f| < \varepsilon$ on Q . It is easily seen that $f\mu \in A^\perp \sim \{0\}$ (see 1.9.3). Now choose $\varepsilon < \dfrac{\alpha c}{1 + \alpha c}$,

then $\dfrac{|f\mu|(F_\alpha)}{\|f\mu\|} < \dfrac{\varepsilon}{\|f\mu\|}$ since $|f\mu|(F_\alpha) = (|f||\mu|)(F_\alpha) < \varepsilon|\mu|(F_\alpha) < \varepsilon$

$< \dfrac{\varepsilon}{(1-\varepsilon)c}$ since $\|f\mu\| > |f\mu|(Q) \geqslant (|f||\mu|)(Q) > (1-\varepsilon)c$

$< \alpha$ since $\dfrac{\varepsilon}{(1-\varepsilon)c}$ clearly decreases as $\varepsilon > 0$ decreases.

But this contradicts the definition of F_α , thus establishing the result.

2.4.2.28.2 Corollary

A is approximately normal $\Rightarrow \{F_\alpha\} = \{F_\beta\}$ for all α, β such that $0 < \alpha < \beta \leqslant 1$.

Proof: Both collections clearly consist of all closed sets containing E .

2.4.2.28.3 Corollary

A is approximately normal $\Rightarrow C_\alpha$ is unique and $C_\alpha = E$.

Proof: By 2.4.2.28.1, $E \subset C_\alpha$ for any C_α . Clearly, also $E \in \{F_\alpha\}$ for all α .

2.4.3 In this section we shall show that the results of 2.4.2.28 do, in fact, hold for a general function algebra, A . Useful in approaching these results is the Local Maximimum Modulus principle, of which we give a version expressible in terms of p-sets. These results, together with the insights gained in 2.4.2 enable us to develope some characterizations of the essential set, E.

2.4.3.1 Proposition

$F_\alpha \supset I = M_A \sim \Gamma_A$ for all F_α .

Proof: Say that $x \in I \sim F_\alpha$. Since M_A is normal, being compact and Hausdorff, we can choose a closed neighbourhood V of x such that $V \subset I$ and $V \cap F_\alpha = \phi$. By the Local Maximum Modulus Principle, $\Gamma_{A_V} \subset \text{bd } V$. (see 1.22.4). Now by 2.1.18 we have $\mu \in A^\perp \sim \{0\}$ of the form $\mu = \delta_x - m_x$ where m_x is a representing measure for x which is supported on bd V . Thus $|\mu|(F_\alpha) = 0$, in contradiction to the definition of F_α .

2.4.3.2. Firstly we put down the notion of a local p-set in a way which arises naturally from the definition of a local peak set (see 1.22.1).

Definition:

A closed set F is a local p-set for A if there is an open set $U \supset F$ and if for every $x \in U \sim F$ there is $f_x \in A$ such that $Q = \{y: f_x(y) = 1\} \cap U$ contains F but not x and that $|f_x(y)| < 1$ for $y \in U \sim Q$.

It is easily seen that this is equivalent to the following definition:

A closed set F is a local p-set for A if there is an open set $U \supset F$ and a collection of local peak sets for A, defined by U, whose intersection is F.

2.4.3.3 **Theorem**

F is a local p-set for $A \Longleftrightarrow F$ is a p-set for A.

Proof: Let F, U, x and f_x be as in the definition 2.4.3.2. Choose an open set U_1 such that $F \subset U_1 \subset \overline{U}_1 \subset U$. The sets, $\{y: f_x(y) \neq 1\}$; $\forall\, x \in U \sim F$, form an open cover for bd U_1. Since bd U_1 is a compact set we can choose a finite subcover from these sets and form $f = f_1 \ f_2 \ \ldots \ f_n$ from the corresponding functions, f_i. Then $F_0 = \{y: f(y) = 1\} \cap \overline{U}_1$ is a closed set and is such that $F \subset F_0 \subset U_1$. Furthermore, $|f(y)| < 1$ for $y \in \overline{U}_1 \sim F_0$. Thus F_0 is a local peak set for A and by the Local Maximum Modulus Principle (see 1.22.3) is a peak set for A. Now, for any $x \in U \sim F$, the set $U \sim \{x\}$ is an open set containing F and satisfying the conditions of the definition. Thus we can choose an open U_1 as before such that $F \subset U_1 \subset \overline{U}_1 \subset U \sim \{x\}$. As before we construct a peak set F_x (previously labelled F_0) such that $F \subset F_x \subset U_1$. Then clearly $F = \bigcap_{x \in U \sim F} F_x$ is a p-set, thus establishing the forward implication. The reverse implication is trivially confirmed.

2.4.3.4 **Note:** We have in fact established a formulation of the Local Maximum Modulus Principle, since we have :

(F is a local peak set for $A \Longleftrightarrow F$ is a peak set for A)

iff (F is a local p-set for $A \Longleftrightarrow F$ is a p-set for A).

Proof: The forward implication is proved in 2.4.3.3. The reverse implication is trivially true.

2.4.3.5 We are now able to strengthen the result 2.4.3.1.

Proposition

$F_\alpha \supset M_A \sim \text{ch}\, A$ for all F_α.
or equivalently: For any F_α and any $x \notin F_\alpha$, x is a p-point.

Proof: The equivalence of the two statements is easily seen by using 2.2.15. Let $x \notin F_\alpha$. Then we have an open $U \ni x$ such that $\overline{U} \cap F_\alpha = \phi$. By 2.4.2.6 we have $A_{|\overline{U}} = C(\overline{U})$. By use of suitable Urysohn functions on

U it is seen that $\{x\}$ and U satisfy the conditions of 2.4.3.2, i.e. that $\{x\}$ is a local p-set. Now, by 2.4.3.3, x is a p-point as required.

2.4.3.6 Proposition

For any F_α we have $F_\alpha = \tilde{F}_\alpha$; i.e. F_α is A-convex.

Proof: $x \notin F_\alpha \Rightarrow x \in \text{ch } A$ (by 2.4.3.5)

$\Rightarrow \delta_x$ is the only representing measure for x . (definition of ch A .)

$\Rightarrow x \notin \tilde{F}_\alpha$ (by 2.3.4).

Since we know that $F_\alpha \subset \tilde{F}_\alpha$, we have $F_\alpha = \tilde{F}_\alpha$.

2.4.3.7 Proposition

$x \in M_A$ has a closed neighbourhood V such that $A_{|V} = C(V)$ iff $x \notin F_\alpha$ for some F_α .

Proof: $\Rightarrow$: Let $\mu \in A^\perp$ such that $\|\mu\| = 1$. Then by 2.4.1.2 (c) we have $|\mu|(V) < t$ for some t such that $0 \leqslant t < 1$. Then clearly $|\mu|(\overline{M_A \sim V}) \geqslant 1 - t$, for all $\mu \in A^\perp$ such that $\|\mu\| = 1$. So we have $\overline{M_A \sim V} \in \{F_{1-t}\}$ and $x \notin \overline{M_A \sim V}$

$\Leftarrow$: If $x \notin F_\alpha$, then there is a closed neighbourhood V of x such that $V \cap F_\alpha = \phi$. Now by 2.4.2.6, $A_{|V} = C(V)$ as required.

2.4.3.8 Corollary

Let $P = \{x \in M_A : \exists$ a closed neighbourhood V of x with $A_{|V} = C(V)\}$ and let $P' = M_A \sim P$. Then $P' = \cap F_\alpha$; the intersection taken over all possible F_α .

Proof: This is an immediate consequence of 2.4.3.7.

2.4.3.9 Proposition

$E \subset F_\alpha$, for any F_α .

Proof: We have $M_A \sim \text{ch } A \subset F_\alpha$, for all F_α (by 2.4.3.5) and $P' \subset F_\alpha$, for all F_α (by 2.4.3.8; definition of P' given there). Thus $P' \cup (M_A \sim \text{ch } A) \subset F_\alpha$ for all F_α . But, by [43] ; thm 1, we have $E = P' \cup (M_A \sim \text{ch } A)$.

2.4.3.10 Proposition

$E = P'$ (where P' is as in 2.4.3.8).

Proof: We have $E \in \{F_\alpha\}$ for any α such that $0 < \alpha \leqslant 1$. This, together with 2.4.3.9 gives $E = \cap F_\alpha$, the intersection taken over all possible F_α . Now by 2.4.3.8 we have $E = P'$.

2.4.3.11 With these results in mind we make some remarks about the results of 2.4.2.

2.4.3.11.1 $\{F_\alpha\} = \{F_\beta\}$, for all α, β such that $0 < \alpha < \beta \leqslant 1$.

2.4.3.11.2 C_α is unique and $C_\alpha = E$ for all possible α .

2.4.3.11.3 In the 2.4.2 series the proofs of 7, 12, 13, 15, 16, 19, 20, 21 and 22 are much easier.

2.4.3.11.4 For 8 and 10 we note that F_α contains every non-trivial maximal set of anti-symmetry and that F_α contains $\text{supp } \mu$, for all $\mu \in A^\perp$.

2.4.3.11.5 We may say $\gamma = 1$ in 17 and $\alpha = \frac{1}{2}$ is not needed in 18.

2.4.3.11.6 Note that $F = E$ in 23 to 27.

2.4.3.11.7 The results of 28 hold without the approximate normality condition.

2.4.3.12 Proposition

For every open set U such that $U \cap E \neq \phi$, there is a sequence of measures $\{\mu_n\} \subset A^{\perp}$ with $\|\mu_n\| = 1$, for all $n \in \mathbb{N}$ such that $|\mu_n|(U) = |\mu_n|(U \cap E) \geqslant 1 - \frac{1}{n}$, $\forall\ n \in \mathbb{N}$.

Proof: Assume that this does not hold. We see that $E \sim U$ is closed set and there is some t with $0 < t \leqslant 1$ such that ; $\mu \in A^{\perp}$ and $\|\mu\| = 1 \Rightarrow |\mu|(E \sim U) \geqslant t$. This means that $E \sim U \in \{F_t\}$. By 2.4.3.9, $E \subset E \sim U$, contradicting our assumption.

2.4.3.13 Corollary

For every closed set V such that $V^0 \cap E \neq \phi$ we have $A_{|V} \neq C(V)$.

Proof: From 2.4.3.12 and 2.4.1.2 (c).

2.4.3.14 Corollary

For any closed set V we have :

$$\begin{aligned} A_{|V} = C(V) &\Rightarrow V^0 \cap E = \phi \\ &\Rightarrow (V \cap E)^0 = \phi \\ &\Rightarrow V \cap E \text{ is nowhere dense.} \end{aligned}$$

2.4.3.15 Proposition

E is a perfect set.

Proof: This follows from 2.4.2.14 and 2.4.3.11.2.

2.4.3.16 We can modify the characterization of E which is given in 2.1.15.1 as follows :

Proposition

$x \in E$ iff (for every neighbourhood U of x , there is a sequence of measures $\{\mu_n\} \subset A^{\perp}$ with $\|\mu_n\| = 1$ and such that $|\mu_n|(U) \geq 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$.)

Proof: This follows from 2.1.15.1 and 2.4.3.12

2.5 Decompositions

2.5.1.1 Any family of sets which covers M_A and is pairwise disjoint we shall call a decomposition of M_A . We do not require the sets in this family to be closed. It is easily seen that such a decomposition naturally gives rise to an equivalence relation on M_A and one would expect that such decompositions could be characterized in terms of equivalence relations. Two well-known decompositions are the maximal sets of antisymmetry, which we shall denote by $\{K\}$, and the Gleason parts, which we shall denote by $\{G\}$. (see 1.23.5 and 1.17). We shall also introduce a third decomposition defined in terms of p-sets, which is shown to lie between these two. Although we show that it is distinct from $\{K\}$ we have not as yet been able to show whether or not it coincides with $\{G\}$. The main results of this section are characterizations of $\{G\}$ in terms of representing measures and annihilating measures. We shall write $\{D_1\} \leqslant \{D_2\}$ meaning that the decomposition $\{D_1\}$ is finer than the decomposition $\{D_2\}$ in the sense that $D_1 \in \{D_1\} \Rightarrow D_1 \subset D_2$ for some $D_2 \in \{D_2\}$.

2.5.1.2 Proposition

If $x, y \in M_A$ are separated by a p-set, i.e. if there is a p-set containing one of these points but not the other, then there are disjoint Borel sets F_1 and F_2 such that every representing measure for x is carried by F_1 and every representing measure for y is carried by F_2.

More precisely, we can say :

If $x, y \in M_A$ are such that there is a p-set, P, with $x \in P$ and $y \notin P$, then every measure for x is supported on P and every measure for y is carried by $M_A \sim P$.

Proof: We shall limit our proof to the second statement which clearly implies the first. By 2.2.10 we know that P supports every representing measure for x. Let m_y be a representing measure for y, other than δ_y. (Clearly δ_y is carried by $M_A \sim P$). Then $\mu = \delta_y - m_y \in A^\perp$. Since P is a p-set we have

$\mu_P = -(m_y)_P \in A^{\perp}$. Since m_y is a positive measure, this can only be true if $-m_y|_P = 0$. Thus m_y is carried by $M_A \sim P$ as required.

2.5.1.3 Corollary

If $x,y \in M_A$ are separated by a p-set, then every representing measure for x is mutually singular with every representing measure for y .

2.5.1.4 Proposition

The relation "$x \sim y$ iff there is no p-set separating x and y" is an equivalence relation on M_A .

Proof: The reflexivity and commutativity are trivially confirmed. To show that the relation is transitive, let $x \sim y$ and $y \sim z$. Then, if there is a p-set separating x and z it would also separate either x and y or y and z in contradiction to the hypothesis.

2.5.1.5 Proposition

If $x,y \in M_A$ are separated by a p-set, then x and y are in different Gleason parts. i.e. if $x \not\sim y$ in the equivalence relation of 2.5.1.4, then x and y are in different Gleason parts.

Proof: If x and y are in the same Gleason part then they have representing measures which are not mutually singular (see 1.18.2). Application of 2.5.1.3 now gives the result.

2.5.1.6 Definition

Denote by $\{P\}$, the decomposition of M_A formed by the equivalence classes of the relation defined in 2.5.1.4.

2.5.1.7 Proposition

We have $\{G\} \leqslant \{P\} \leqslant \{K\}$ (cf. 2.5.1.1).

Proof: The first inequality follows from 2.5.1.5. The second inequality is a consequence of the fact that every $K \in \{K\}$ is a p-set. (see 1.23.4).

2.5.1.8 Proposition

If x,y are in the same Gleason part, then there is a representing measure for y which is non-zero on x .

Proof: We know, by 1.18.3, that if m_x is a representing measure for x , then there is a representing measure, m_y , for y such that $m_x \ll m_y$. Now, in place of m_x choose δ_x , then the corresponding measure, m_y is non-zero at x as required.

2.5.1.9 Proposition

If $x,y \in M_A$ are in different Gleason parts, then $m_y\{x\} = 0$ for every representing measure, m_y , of y .

Proof: This is an immediate consequence of 1.18.4.

2.5.1.10 We are now able to give a characterization of Gleason parts in terms of representing measures.

Theorem

$x,y \in M_A$ are in the same Gleason part iff there is a representing measure for y which is non-zero on x.

Proof: The forward implication is given by 2.5.1.8 and the reverse implication by 2.5.1.9.

2.5.1.11 Corollary

The relation "$x \sim y$ iff there is a representing measure for y which is non-zero on x" is an equivalence relation.

Proof: By 2.5.1.10, this relation defines the Gleason decomposition into equivalence classes (see 1.17). Thus it must be an equivalence relation.

2.5.1.12 Corollary

(i) If there is a representing measure for x which is non-zero on y, then there is a representing measure for y which is non-zero on x.

(ii) If there is a representing measure of x which is non-zero on y and a representing measure for y which is non-zero on z, then there is a representing measure for x which is non-zero on z.

Proof: These are just statements of the commutativity and transitivity, respectively, of the equivalence relation which is assured by 2.5.1.11.

2.5.1.13 Proposition

Every p-point is a trivial Gleason part (i.e. a one-point Gleason Part).

Proof: x is a p-point $\Longleftrightarrow \delta_x$ is the only representing measure for x (see 2.2.15)

$\Rightarrow x$ is a one-point Gleason part (by 2.5.1.10).

2.5.1.14 Proposition

x is a one-point Gleason part iff every representing measure for x can be atomic only at x.

Proof: This follows from 2.5.1.10 and 2.5.1.12.

2.5.1.15 Proposition

x is a trivial Gleason part iff every representing measure for every $y \in M_A \sim \{x\}$ is zero on x.

Proof: This follows from 2.5.1.10 and 2.5.1.12.

2.5.1.16 Proposition

$x, y \in M_A$ are in the same Gleason part iff either $x = y$ or there is a representing measure for y which is zero on y and non-zero on x.

Proof: We use 2.5.1.10 and need only show that the representing measure for y can be chosen to be zero on y. Let m_y be the representing measure ensured by 2.5.1.10 and let $m_y(y) = c$ where $0 < c < 1$. Then form a new measure $\mu = \dfrac{m_y - c\delta_y}{1 - c}$. It is easily seen that this is a representing measure for y with the desired properties.

2.5.1.16.1 **Remark:** Clearly results 2.5.1.11 and 2.5.1.12 could be slightly modified by making use of 2.5.1.16.

2.5.1.17 We now establish some measure theoretic results which enable us to develope another characterization of Gleason parts.

Proposition

If m_x is a complex representing measure for $x \in M_A$, then $\|m_x\| \geqslant 1$.

Proof: We have $\|m_x\| = \int d|m_x| \geqslant |\int dm_x| = 1$.

2.5.1.18 **Proposition**

For every $\mu \in bA^\perp$ we have $|\mu|\{x\} \leqslant \frac{1}{2}$, for all $x \in M_A$.

Proof: Clearly we may, without loss of generality, consider $\mu \in A^\perp$ such that $\|\mu\| = 1$ and assume that $\mu(x) = \alpha > \frac{1}{2}$. Then construct $m_x = \delta_x - \mu$ a complex representing measure for x. Clearly we have $m_x(x) = 1 - \alpha$ and $(\mu - m_x)(x) = (2\mu - \delta_x)(x) = 2\alpha - 1$.

Now we have : $\|m_x\| = \|\mu\| - (\mu - m_x)(x) = 1 - (2\alpha - 1)$

$= 2(1-\alpha) < 1$ since $\alpha > \frac{1}{2}$.

This contradicts 2.5.1.17 and establishes the result.

2.5.1.19 **Proposition**

If $x,y \in M_A$ with $x \neq y$ are in the same Gleason part, then there is $\mu \in bA^\perp$ such that $|\mu|(x) + |\mu|(y) > \frac{1}{2}$.

Proof: By 2.5.1.16 we have a representing measure, m_y, for y such that $m_y(y) = 0$ and $m_y(x) > 0$. Now form $\mu = \frac{1}{2}(\delta_y - m_y)$. It is easily seen that $\|\mu\| = 1$ $|\mu|(x) + |\mu|(y) > \frac{1}{2}$ (since $|\mu|(y) = \frac{1}{2}$ and $|\mu|(x) > 0$)

2.5.1.20 Proposition

If $x,y \in M_A$ with $x \neq y$ are in the same Gleason part, then for every closed set F such that $\{x,y\} \subset F$ we have: A/k_F is not isometric with $C(F)$.

Proof: The existence of the annihilating measure ensured by 2.5.1.19 contradicts the necessary condition given in 2.4.1.2(d).

2.5.1.21 It can be shown that the condition given in 2.5.1.19 bears some relation to the $\{P\}$-decomposition. More exactly:

Proposition

If $x,y \in M_A$ are separated by a p-set, P, then for every $\mu \in bA^{\perp}$ we have $|\mu|(x) + |\mu|(y) \leqslant \frac{1}{2}$.

Proof: We may consider $\mu \in A^{\perp}$ such that $\|\mu\| = 1$. Say that $x \in P$, $|\mu|(x) = \alpha$ and $|\mu|(y) = \beta$. Since P is a p-set we have $\mu_P \in A^{\perp}$ and $\mu_{M_A \sim P} \in A^{\perp}$ (see 2.2.9). Let $\|\mu_P\| = a \neq 0$ and $\|\mu_{M_A \sim P}\| = b \neq 0$. Then we have $a + b = 1$ (since the norm is additive over mutually singular measures). Also, by 2.5.1.18 we have $\frac{\alpha}{a} \leqslant \frac{1}{2}$ and $\frac{\beta}{b} \leqslant \frac{1}{2}$. This gives: $2\alpha \leqslant a$ and $2\beta \leqslant b \Rightarrow 2(\alpha+\beta) \leqslant a + b = 1$. Thus $\alpha + \beta \leqslant \frac{1}{2}$ as required. If either of a or b is zero then the corresponding α or β is zero and the result follows from 2.5.1.18.

2.5.1.22 Proposition

If $x, y \in M_A$ are in different Gleason parts, then for every $\mu \in bA^{\perp}$ we have $|\mu|(x) + |\mu|(y) \leqslant \frac{1}{2}$.

Proof: Take $\mu \in A^{\perp}$ with $\|\mu\| = 1$ and let $|\mu|(x) = a \neq 0$ and $|\mu|(y) = b \neq 0$. [If either a or b is zero, then the result follows from 2.5.1.18.] Now we use the measure decomposition result which is derived from the abstract F. and M. Riesz theorem (see [24] pg 145 Thm 2.3) in order to write $\mu = \mu_x + \mu_y + \mu_P$ where μ_x, μ_y and μ_P are mutually singular, μ_x is absolutely continuous with respect to M_x, the set of representing measures for x, and μ_y is absolutely continuous with respect to M_y. Furthermore we have μ_x, μ_y and μ_P in $A^{\perp}$. Since μ_y and μ_P are each supported on a null-set of M_x (i.e. a set which has zero measure for each measure in M_x) and since $\delta_x \in M_x$ we see that $\mu_y(x) = \mu_P(x) = 0$. Thus $|\mu_x|(x) = a$ and, similarly, $|\mu_y|(y) = b$. Now let $\|\mu_x\| = \alpha$ and $\|\mu_y\| = \beta$ (clearly α and β are non-zero). Clearly $\alpha + \beta \leqslant \|\mu\| = 1$. By 2.5.1.18, we have $\frac{a}{\alpha} \leqslant \frac{1}{2}$ and $\frac{b}{\beta} \leqslant 1/2$. Arguing as in 2.5.1.21 we obtain $a + b \leqslant \frac{1}{2}$ as required.

2.5.1.23 We have now established another measure theoretic characterization for Gleason parts.

Theorem

$x,y \in M_A$ are in the same Gleason part iff

$$\begin{cases} \text{either } x = y \\ \text{or there is } \mu \in bA^{\perp} \text{ with } |\mu|\{x,y\} > \frac{1}{2} . \end{cases}$$

Proof: The forward implication is given by 2.5.1.19 and the reverse implication is given by 2.5.1.22.

2.5.1.24 Proposition

If for distinct points $x,y,z \in M_A$ we have $\mu,\nu \in bA^{\perp}$ such that $|\mu|\{x,y\} > \frac{1}{2}$ and $|\nu|\{y,z\} > \frac{1}{2}$, then there is $\eta \in bA^{\perp}$ such that $|\eta|\{x,z\} > \frac{1}{2}$.

Proof: The condition given in 2.5.1.23, since it defines Gleason parts, is an equivalence relation (cf. 2.5.1.11). The transitivity of this relation gives our result.

2.5.1.25 In 2.5.1.7 we established the basic relationship between $\{G\}$, $\{P\}$ and $\{K\}$. We shall now make some remarks which will help to throw further light on this relationship.

2.5.1.26 Proposition

For any p-set, F, and any $\{P\}$-part, P, we have either $P \subset F$ or $P \cap F = \phi$.

Proof: This is clear since no p-set can separate points of P. (see 2.5.1.6).

2.5.1.27 Proposition

For any p-set, F, and any Gleason part, G, we have either $G \subset F$ or $G \cap F = \phi$.

Proof: This is clear since no p-set can separate points of G. (see 2.5.1.5).

2.5.1.28 Proposition

Any non-trivial $\{P\}$-part cannot be a p-set.

Proof: Let the $\{P\}$-part, P, be a p-set. By 2.2.8, P contains a p-point. If P is non-trivial, this contradicts 2.5.1.26.

2.5.1.29 Proposition

No non-trivial Gleason part is a p-set.

Proof: Let $G \in \{G\}$ be a p-set. By 2.2.8, G contains a p-point. If G is non-trivial, this contradicts 2.5.1.27 (or 2.5.1.13).

2.5.1.30 Proposition

$\{P\} \subsetneq \{K\}$, in fact for any $P \in \{P\}$ and $K \in \{K\}$, we have either $P \cap K = \phi$ or $P \subsetneq K$ or $P = K =$ a p-point, if $A \neq C(M_A)$.

Proof: Firstly we note that K is a p-set (see 1.23.4). The second statement then follows from 2.5.1.26 and 2.5.1.28. The first statement follows from the second and the fact that there must be some non-trivial K (if $A \neq C(M_A)$). (cf. 2.4.2.8)

2.5.1.31 Proposition

$\{P\} = \{K\} \iff \{G\} = \{K\} \iff A = C(M_A)$.

Proof: This follows from 2.5.1.30 and 2.5.1.13. This situation occurs iff each $K \in \{K\}$ is a singleton.

2.5.1.32 Proposition

The smallest p-set containing x coincides with the smallest p-set containing P_x, where P_x is the $\{P\}$-part containing x.

Proof: Let F be the smallest p-set containing x and Q the smallest p-set containing P_x. Clearly $F \subseteq Q$. By 2.5.1.26, $Q \subseteq F$ as required.

2.5.1.33 Proposition

The smallest p-set containing x coincides with the smallest p-set containing G, where G is the Gleason part containing x.

Proof: This is a direct consequence of 2.5.1.32 and 2.5.1.7.

2.5.1.34 Proposition

If P_1, P_2 are distinct $\{P\}$-parts, then there is a p-set containing one but disjoint from the other.

Proof: Let F_1 be the smallest p-set containing P_1. Then, by 2.5.1.26, either $P_2 \cap F_1 = \phi$ which gives the result, or $P_2 \subset F_1$. If the latter case, let F_2 be the smallest p-set containing P_2. By 2.5.1.26, either $F_2 \cap P_1 = \phi$, which gives the result, or $P_1 \subset F_2$. But if $P_1 \subset F_2$ there is no p-set separating points in P_1 and P_2 (using also 2.5.1.32). This contradiction gives the result.

2.5.1.35 Proposition

$\{G\} = \{P\}$ iff for any two distinct Gleason parts there is a p-set containing one and disjoint from the other.

Proof: The forward implication follows from 2.5.1.34 and the reverse implication follows from 2.5.1.6 and 2.5.1.7.

2.5.1.36 Proposition

If M_A has a unique non-trivial Gleason part G_1 and every $x \notin G_1$ is a one point $\{P\}$-part, then $\{G\} = \{P\}$.

Proof: This follows easily from the definitions.

2.5.1.37 We now look briefly at the relation between CRS sets and our three basic decompositions.

Proposition

Let F be a CRS-set. Let $\{K\}_A$ be a maximal anti-symmetric decomposition of M_A with respect to A and let $\{K\}_{A_F}$ be the maximal anti-symmetric decomposition of $\tilde{F}$ with respect to A_F. Then

$\{K\}_{A_F} \leqslant \{K \cap \tilde{F}\ ,\ K \in \{K\}_A\}$.

Proof: Let P be closed and such that $P \subset \tilde{F}$ and P is antisymmetric with respect to A_F . Now using 2.4.1.11 and 2.4.1.12 we have that P is antisymmetric with respect to $A_{\tilde{F}}$. Hence, that P is antisymmetric with respect to A . Thus $P \subset K$ for some $K \in \{K\}_A$. Then certainly $P \subset K \cap \tilde{F}$ as required.

2.5.1.38 Proposition

Let $F \subset M_A$ be a CRS-set. Let $\{P\}_{A_F}$ be the $\{P\}$-decomposition of $\tilde{F}$ with respect to A_F and let $\{P\}_A$ be the $\{P\}$-decomposition of M_A with respect to A. (see 2.5.1.6). Then $\{P\}_{A_F} \leqslant \{P \cap \tilde{F} , P \in \{P\}_A\}$.

Proof: Let $x,y \in \tilde{F}$ be separated by the set $P \subset M_A$ where P is a p-set for A. By 2.4.1.14 we know that $P \cap \tilde{F}$ is a p-set for $A_{|F}$. Clearly $P \cap \tilde{F}$ separates x and y. This gives the result.

2.5.1.39 Proposition

Let $F \subset M_A$ be a CRS-set. Let $\{G\}_{A_F}$ be the collection of Gleason parts of $\tilde{F}$ with respect to A_F and let $\{G\}_A$ be the collection of Gleason parts of M_A with respect to A. Then $\{G\}_{A_F} \leqslant \{G \cap \tilde{F} , G \in \{G\}_A\}$.

Proof: Let $x,y \in \tilde{F}$ be in different Gleason parts of M_A. By 2.5.1.22 we have $|\mu|\{x,y\} \leqslant \frac{1}{2}$ for every $\mu \in bA^{\perp}$. Since $(A_F)^{\perp} \subset A^{\perp}$ we have $|\mu|\{x,y\} \leqslant \frac{1}{2}$ for every $\mu \in b(A_F)^{\perp}$. Then, by 2.5.1.19, x and y are in different $\{G\}_{A_F}$-parts.

2.5.2 We know from 1.16.1 that the p-sets behave like closed sets in the sense of closure under finite unions and arbitary intersections. Since M_A and ϕ are clearly p-sets we see that the collection of p-sets is actually the set of closed sets of a topology on M_A. We shall call this the p-set topology and denote it by $\mathscr{P}$. Making use of [17] we shall investigate the separation and regularity conditions associated with this topology. For convenience, we list some definitions.

2.5.2.1 A topology is T_0 if for any two points in the space there is an open set containing one but not the other..

2.5.2.2 A topology is T_1 if for any two points in the space there is for each an open set containing that point but not the other.

2.5.2.3 A topology is T_2 if for any two points in the space there are two disjoint open sets, each containing one of the points.

2.5.2.4 A topology is R_0 if for any two points in the space their closures either coincide or are disjoint.

2.5.2.5 A topology is R_1 if for any two points in the space their closures either coincide or are entirely separated (in the sense that there are disjoint open sets each containing one of the closures.)

2.5.2.6 We shall express many of our results in terms of the decompositions which we investigated earlier.

Proposition

$\mathcal{P}$ is T_0 iff every $\{P\}$-part is a singleton.

Proof: We have :

$\mathcal{P}$ is $T_0 \iff$ For every $x,y \in M_A$ there is an open set containing one but not the other.

$\iff$ For every $x,y \in M_A$ there is a p-set containing one but not the other.

$\iff$ Every pair $x,y \in M_A$ are separated by a p-set.

$\iff$ Every $x,y \in M_A$ are in different $\{P\}$-parts.

$\iff$ Every $\{P\}$-part is a singleton.

2.5.2.7 Corollary

$\mathcal{P}$ is $T_0 \Rightarrow$ Every Gleason part is trivial.

Proof: Follows easily from 2.5.2.6 and 2.5.1.7.

2.5.2.8 Proposition

$\mathcal{P}$ is R_0 iff every $\{P\}$-part is a p-point.

Proof: $\mathcal{P}$ is $R_0 \iff$ For every $x,y \in M_A$, the p-closures of x and y either coincide or are disjoint.

Now let P_x be the p-closure of x . By 2.2.8, P_x contains a p-point which we shall call y . Then the p-closure of y , P_y , neither coincides with P_x , nor is disjoint from P_x unless $y = x$. So we have, $\mathcal{P}$ is R_0 implies that every point is a p-point. Hence every $\{P\}$-part is a p-point. The converse is trivial since then the p-closures of x and y are simply $\{x\}$ and $\{y\}$ respectively.

2.5.2.9 Corollary

$\mathcal{P}$ is $R_0 \iff \mathrm{ch}\, A = M_A$.

2.5.2.10 Proposition

If $\mathcal{P}$ is R_0 , then $\mathcal{P}$ is T_0 .

Proof: By 2.5.2.8, every {P}-part is a singleton. By 2.5.2.6, $\mathcal{P}$ is T_0 .

2.5.2.11 Proposition

$\mathcal{P}$ is $R_0 \Longleftrightarrow \mathcal{P}$ is T_1 .

Proof: By 2.5.2.10, $\mathcal{P}$ is R_0 and T_0 . By [17] §3, (R_0 and T_0) $\Longleftrightarrow T_1$.

2.5.2.12 Proposition

$\mathcal{P}$ is $R_1 \Longleftrightarrow \mathcal{P}$ is T_2 .

Proof: In [17] §3 we find that (R_1 and T_1) $\Longleftrightarrow T_2$. This gives the reverse implication. On the other hand, if $\mathcal{P}$ is R_1 , then by 2.5.2.11, $\mathcal{P}$ is also T_1 . Then the Davis result tells us that $\mathcal{P}$ is T_2 .

2.5.2.13 Corollary

$\mathcal{P}$ is $R_1 \Longleftrightarrow A = C(M_A)$.

Proof: It is known that $A = C(M_A) \Longleftrightarrow \mathcal{P}$ is T_2 (see 1.16.2). Now apply 2.5.2.12.

2.5.2.14 Corollary

$A = C(M_A)$ iff for every $x_1, x_2 \in M_A$ the following holds for $i = 1, 2$ and $j = 3 - i$.

If there is a p-set $P_{x_i} \ni x_i$ such that $x_j \notin P_{x_i}$, then there are p-sets P_{x_j} , Q_{x_i} and Q_{x_j} such that $x_j \in P_{x_j}$; $Q_{x_i} \cap P_{x_j} = \phi = Q_{x_j} \cap P_{x_i}$ and $Q_{x_i} \cup Q_{x_j} = M_A$.

Proof: We firstly note that, either P_{x_i} exists for some $i = 1,2$ or the p-closures of x_1 and x_2 coincide. If the former, then the existence of the Q_{x_i}, $i = 1,2$ ensure that the $M_A \sim Q_{x_i}$, $i = 1,2$ are disjoint open sets entirely separating the p-closures of x_1 and x_2. This tells us that $\mathcal{P}$ is R_1. We then apply 2.5.2.13.

2.5.2.15 Corollary

$A \neq C(M_A) \iff$ There exist $x,y \in M_A$ such that their p-closures are not entirely separated (in $\mathcal{P}$) and that they do not coincide.

$\iff$ There exist $x,y \in M_A$ with p-closures P_x and P_y respectively such that : $P_x \neq P_y$ and for all p-sets Q_x and Q_y such that $Q_x \cap P_y = \phi = Q_y \cap P_x$ we have $Q_x \cup Q_y \neq M_A$.

Proof: Both equivalences are simply restatements of 2.5.2.13 We make use of arguments in 2.5.2.14.

2.5.2.16 Remark

From [17] §3 we have, for general topological spaces, the basic relationship (T_n and R_n) $\iff T_{n+1}$ between the separation conditions, T_n, and the regularity conditions, R_n. In view of 2.5.2.10 we have shown that for any p-set topology: $R_n \iff T_{n+1}$. Of course, since we know that $A = C(M_A) \iff \mathcal{P}$ is T_2, the fact becomes relatively uninteresting for $n \geqslant 2$.

2.6 General Closed Boundaries

2.6.1 In this section we shall set down results relating to a function algebra on any closed boundary, X . We recall from the discussions in sections 2.1.1 to 2.1.8 that X may be regarded as any set, closed in M_A , such that $\Gamma_A \subset X \subset M_A$. For a large proportion of the results in the previous sections, proved mostly in the context $X = M_A$, completely analogous statements hold for the general X . In most cases the analogous statements and proofs can be obtained by superimposing on the existing statements and proofs the following system of notation, which we introduce in order to avoid confusion with the system that we have already established (cf. 2.1.8). In the following, a closed or open set, without qualification, shall mean one which is closed or open in X . We shall replace M_A by X ; $A^\perp$ by $A^\perp_X$ where $A^\perp_X = A^\perp \cap M(X)$ (cf. 2.4.1.2); E by E_X , the essential set for A on X in the sense of 1.24; M_x by $M_{x|X}$ i.e. the set of all representing measures for x which are supported on X ; I by I_X where $I_X = X \sim \Gamma_A$ and a p-set or peak set in X shall be in the sense of 1.13 which, since Glicksberg's p-set criterion (see 1.15) is proved independently of X , is analogous to the sense of 2.2.2.

It will be seen that the cases where analogous results do not hold are usually those whose proofs are based either on Shilov's Idempotent Theorem (see 1.11) or the

Local Maximum Modulus Principle (see 1.22). The reason for this can easily be seen to lie in the fact that X can often be chosen to include a set, U, open in X but not open in M_A. In such a case the analogous result may not hold without the imposition of stronger conditions; conditions which, in fact, ensure the applicability of Shilov's Idempotent Theorem or the Local Maximum Modulus Principle.

We shall begin by giving various results showing some of the relations between structures on M_A and the analogous structures on X. The first and major of these will be 2.6.2 which gives the relationships between p-sets on M_A and p-sets on X. From this a number of useful results arise very easily, among them, the fact that Γ_A is independent of X. Thus it will be unnecessary for us to introduce the notation, Γ_{A_X} for the Shilov boundary of A on X. This body of results also enables us to prove quite easily many of the results analogous to those of earlier sections. We do not, however, intend to devote too much detail to these, but shall merely list those results which hold for general X, by analogy or otherwise.

2.6.2 The following result gives us considerable insight when comparing the p-set structures on M_A and on X.

Theorem

$P \subset X$ and P is a p-set for A on X iff there is a unique P', a p-set for A on M_A, such that $P = P' \cap X$. Furthermore $P' = \tilde{P}$.

Proof:

$\Leftarrow$: Let P' be a p-set for A on M_A. Since $A_X^{\perp} \subset A^{\perp}$, it is clear from Glicksberg's p-set criterion that $P' \cap X$ is a p-set for A on X. Note also that, by 2.2.17.2, if $P' \neq \phi$ then $P' \cap X \neq \phi$.

$\Rightarrow$: If P is a p-set for A on X, then for every $x \in X \sim P$ there is $f_x \in A$ on X such that $\|f_x\|_X = 1$; $f_x = 1$ on P and $|f_x(x)| < 1$. Now using 2.1.3 and 2.1.4 we have unique extension, $\tilde{f}_x \in A$ on M_A, of f_x such that $\|\tilde{f}_x\| = 1$. By 2.2.6 we know that the set $P_x = \{y \in M_A : \tilde{f}_x(y) = 1\}$ is a peak set in M_A which contains P but not x. In fact, by 2.3.6, P_x contains $\tilde{P}$. Now set $P' = \bigcap_{x \in X \sim P} P_x$. Clearly P' is a p-set in M_A such that $P' \cap X = P$ and also $\tilde{P} \subset P'$. We shall now show that $P' = \tilde{P}$, thereby establishing the uniqueness of P' and completing the proof. Let $y \in P' \sim \tilde{P}$. Since $P' \cap X = P$ it is clear that $y \notin X$ and hence that $y \in I$. Using 1.12.6 we have a representing measure, m_y, for y such that $m_y \neq \delta_y$ and $\operatorname{supp} m_y \subset \Gamma_A$. On the other hand, by 2.3.4, $\operatorname{supp} m_y \not\subset P$; i.e. $m_y\{\Gamma_A \sim P\} > 0$. Now we can construct $\mu = m_y - \delta_y$. Clearly $\mu \in A^{\perp}$ and $\mu_{M_A \sim P'} = (m_y)_{\Gamma_A \sim P}$ which is a positive measure. By 2.2.9 this contradicts the fact that P' is a p-set and hence the existence of such a y.

2.6.3. We can make the result of 2.6.2 more specific in several ways which we give together with some other results in the following

2.6.3.1 Proposition

P is a peak set for A on X iff there is a unique peak set P' for A on M_A such that $P = P' \cap X$. Furthermore, $P' = \tilde{P}$.

Proof:

$\Leftarrow$: Let P' be a peak set in M_A. Arguing as in 2.6.2, we know that $P' \cap X$ is a p-set in X. Furthermore, since P' is a G_δ-set (see 1.13.3) we know that $P' \cap X$ is G_δ in X, i.e. in the relative topology. Thus $P' \cap X$ is a peak set in X.

$\Rightarrow$: Arguing as in 2.6.2, the peaking function, f, which defines P in X can be extended to $\tilde{f} \in A$ on M_A which defines a peak set P' in M_A such that $P = P' \cap X$. As before, $P' = \tilde{P}$.

2.6.3.2 Proposition

If P' is a strong p-set in M_A, then the corresponding p-set in X, i.e. P (cf. 2.6.2) is a strong p-set in X and $P = P'$.

Proof: If P' is a strong p-set, then $\mu\{x\} = 0$ for every $x \in P'$ and every $\mu \in A^\perp$. Thus every point of P' is a p-point. Thus $P' \subset \text{ch}A$ (by 2.2.15). Thus $P' \subset \Gamma_A$ (by 2.2.18). Thus $P' \subset X$. Now, by 2.6.2, $P' = P$.

2.6.3.3 Proposition

If P is a weak p-set for A on X, then the corresponding p-set, P' in M_A is a weak p-set in M_A (cf. 2.6.2).

Proof: If there is $\mu \in A_X^\perp \sim \{0\}$ with $\operatorname{supp} \mu \subset P$ then there is $\mu \in A^\perp \sim \{0\}$ with $\operatorname{supp} \mu \subset P'$. This is trivial since $P \subset P'$ and $A_X^\perp \subset A^\perp$.

2.6.3.4 Proposition

The p-topology on X (generated by the p-sets of A on X) is the relative topology on X with respect to the p-topology on M_A (generated by the p-sets of A on M_A).

Proof: A set $P \subset X$ is closed in the p-topology on X

$\Longleftrightarrow$ P is a p-set for A on X

$\Longleftrightarrow$ $P = P' \cap X$ where P' is a p-set for A on M_A (by 2.6.3).

$\Longleftrightarrow$ P is relatively closed in X with respect to the p-topology on M_A.

2.6.3.5 Comments

(i) If two p-sets differ, then they differ on Γ_A

(ii) There is a one-to-one correspondence between the p-sets in M_A and the p-sets in any closed boundary.

(iii) There is a one-to-one correspondence between the peak sets in M_A and the peak sets in any closed boundary.

(iv) Consequently: There is a one-to-one correspondence between the collections of p-sets of any two closed boundaries for A.

(v) There is a one-to-one correspondence between the collections of peak sets of any two closed boundaries for A.

(vi) For any two closed boundaries, X and Y of A such that $Y \subsetneq X \subset M_A$, Y is not a p-set for A on X.

Proof: If Y is a p-set in X, then $\tilde{Y}$ is a p-set in M_A (by 2.6.2) and $Y = X \cap \tilde{Y}$. But it is clear that $\tilde{Y} = M_A$ (since $\Gamma_A \subset \tilde{Y}$) and thus that $Y \neq X \cap \tilde{Y}$. This contradiction gives the result.

2.6.4 Proposition

chA is independent of the closed boundary X, which we choose as the underlying space for A.

Proof: By 2.2.15, chA is the collection of p-points for A. We then note that :

x is a p-point for A on $X \Longleftrightarrow x$ is a p-point for A on M_A.

For the forward implication we use 2.6.2, noting that $\{x\} = \{\tilde{x}\}$ for any singleton, $\{x\}$. The reverse implication follows from 2.6.3.2.

2.6.5 Proposition

If F is a p-set for A on M_A and F is not a singleton then $F \cap X$ is not a singleton for every closed boundary X.

Proof: This follows from 2.6.2 and 2.6.4.

2.6.6 Proposition

Γ_A is independent of the closed boundary, X, which we choose as the underlying space for A.

Proof: This follows from 2.6.4 and 2.2.18.

2.6.7 We shall now briefly review the results of previous sections and indicate in some of the cases whether or not they hold for the general closed boundary, X.

We note that the definition of the essential set given in 1.24 is applicable to any closed boundary, X. Bearing this in mind and using the single change of notation detailed in 2.6.1 we can rewrite the definitions, statements and proofs of the first section of results relating to the essential set (i.e. 2.1.9 to 2.1.15.2).

The results 2.1.16 to 2.1.17 may validly undergo a similar transformation, whereas 2.1.18 and 2.1.19 do not hold in the general case.

2.6.8 In view of the brief note on p-sets given in 2.6.1 it is easily seen that the statements and proofs of sections 2.2.1 to 2.2.4.5 carry through without any real problems.

2.6.9 It will be convenient at this point to make use of 2.6.2 in order to prove some results relating to decompositions.

2.6.9.1 Definition

$\{P\}_X$ is the decomposition of X produced by the equivalence relation : " $x \sim y$ iff there is a p-set of A on X which separates x and y " (c.f. 2.5.1.6).

2.6.9.2 Proposition

$\{P\}_X = \{P \cap X \,;\, P \in \{P\}\}$, for any closed boundary, X , of A .

Proof: By 2.6.2 it is easily seen that any $x,y \in X$ are separated by P , a p-set in X iff x and y are separated by the corresponding p-set, P' , in M_A .

2.6.9.3 Definition

$\{K\}_X$ is the decomposition of X consisting of subsets of X which are maximal antisymmetric sets with respect to A on X .

2.6.9.4 Proposition

$\{K\}_X = \{K \cap X \,,\, K \in \{K\}\}$, for any closed boundary, X , of A .

Proof: Let $K \in \{K\}$ and let $f \in A$ be real-valued on $K \cap X$. By 2.3.6 and 2.6.2, f is real-valued on K , hence constant on K and obviously constant on $K \cap X$. This establishes that $K \cap X$ is a set of antisymmetry. Now choose $K_1 \in \{K\}_X$ such that $K_1 \cap K \neq \phi$. If $K_1 \sim (K \cap X) \neq \phi$, then it is easily seen that $K_1 \cup K$ is a set of antisymmetry which contradicts the maximality of K . Thus $K_1 \subset K \cap X$. Now, by the maximality of K_1 , we have $K_1 = K \cap X$ as required.

2.6.9.5 Proposition

$\{x\}$ is maximal antisymmetric set for A on X $\Longleftrightarrow$ $\{x\}$ is a maximal antisymmetric set for A on M_A .

Proof: This follows from 2.6.9.4 and the fact that, since each $K \in \{K\}$ is a p-set in M_A (see 1.23.4) we know by 2.2.15 and 2.2.18 that any singleton $K \in \{K\}$ lies inside Γ_A , hence inside X .

Remark: We have shown here that the set S , of maximal antisymmetric points, is independent of X , the closed boundary. This gives a further notational simplification $S_X = S$, in our generalizing of previous results.

2.6.10 The remaining results of section 2.2 are now seen to carry through reasonably easily. We do, however, make the following comments.

2.6.10.1 The statement of 2.2.5 would now read :

If P is a p-set for A on X and F is a p-set for A_P on P, then F is a p-set for A on X .

The following variant is also easily proved :

If P is a p-set for A on X and F is a p-set for A_P on $\tilde{P}$, then F is a p-set for A on M_A .

(Here we could make use of 2.6.2).

2.6.10.2 A simpler alternative proof for the generalization of 2.2.8 can be devised, using 2.6.2, 2.2.8 and 2.6.4.

2.6.10.3 In the new statement of 2.2.10 "every representing measure for x" would read "every $m \in M_x|_X$" or "every representing measure for x which is supported on X".

2.6.11 Having established in 2.6.8 that E_X is a p-set in X we are now able to use the results of 2.6.9 in order to formulate the relationship between E and E_X.

Theorem

We have $E_X = E \cap X$ and $\tilde{E}_X = E$ for every closed boundary, X.

Proof: Since $A_X^{\perp} \subset A^{\perp}$, the generalized result of 2.1.13 makes it clear that $E_X \subset E \cap X$. Since E_X is a p-set in X, we have, by 2.6.2, $\tilde{E}_X = E'$ the unique corresponding p-set in M_A. Clearly $E' \subset E$. Let $y \in X \sim E_X$, by the generalized result of 2.2.26 we know that $\{y\}$ is a maximal antisymmetric set in X. By 2.6.9.5, $\{y\}$ is a maximal antisymmetric set in M_A. Now let $x \in I \sim E'$ and let $K \in \{K\}$ be such that $x \in K$. Since $x \notin \Gamma_A$, K is clearly not a singleton (by 1.23.4 and 2.2.15). Thus $K \cap (X \sim E_X) = \phi$ since every $y \in X \sim E_X$ is a maximal set of antisymmetry. Now we have distinct p-sets in M_A, namely E' and $E' \cup K$ (by 1.23.4 again) whose intersections with X coincide. This contradicts 2.6.2 (or, more precisely, the note 2.6.3.5(i).). Thus we have $I \sim E' = \phi$. Thus $M_A \sim E' = X \sim E_X$ (using also 2.6.2), is open in

M_A and consists entirely of maximal antisymmetric point-sets. By 2.2.26, $E \subset E'$. This gives us $E = E'$ as required.

2.6.12 We shall briefly pick out only some of the results of section 2.4. Some comment on the general boundary has already been made in 2.4.1.2. The results of 2.4.1.2 to 2.4.1.11 all carry through. In the latter one many replace $\tilde{F}$ by $\tilde{F} \cap X$. The analogy of 2.4.1.15 will not hold without stronger conditions. However, the statements of 2.4.1.17 and 2.4.1.18 do hold, but with proofs based on 2.6.2.. Most of section 2.4.2 does hold with the exception of 2.4.2.14 and possibly 2.4.2.15 and dependent results. Most of the relevant results of 2.4.3 do not carry through. The results of 2.4.3.7 and 2.4.3.8, however, can be generalized.

2.6.13 We have already done some work on decompositions in section 2.6.9 and we shall carry on to consider only the Gleason parts in relation to X, and at the same time shall strengthen some of the earlier results.

If we examine the proof of 2.5.1.8 we find, firstly, that it also holds in the context of the general boundary, X (bearing in mind that the sets of representing measures are now those which are supported on X) and secondly, that this result may be strengthened as follows :

Proposition

Let X be any closed boundary for A and let $x,y \in X$ be in the same Gleason part of M_A. Then there is a representing measure, m_x, for x which is non-zero on y and such that $\operatorname{supp} m_x \subset X$.

This observation leads to the following sequence of results (i.e. 2.6.14 to 2.6.19) including a natural way of defining the Gleason decomposition of X.

2.6.14 Definition

We say that the points $x,y \in X$ are in the same Gleason part of X if there is a representing measure, supported on X, for one which is non-zero on the other.

2.6.15 Proposition

Let $\{G\}$ be the set of Gleason parts of M_A and $\{G\}_X$ the set of Gleason parts for X, a closed boundary of A. Then $\{G\}_X = \{G \cap X : G \in \{G\}$ and $G \cap X \neq \phi\}$.

Proof: If $x,y \in X$ are in the same $\{G\}$-part, then by 2.6.13, they are in the same $\{G\}_X$-part. If x,y are in the same $\{G\}_X$-part, then by 2.5.1.9, they are in the same $\{G\}$-part.

Note that this result validates the definition of 2.6.14 in the sense that we now know that it does in fact define a decomposition (since $\{G\}$ is a decomposition).

2.6.16 Proposition

The points $x,y \in M_A$ are in the same Gleason part iff there is a representing measure, m_x, for x which is non-zero on y and such that $\operatorname{supp} m_x \subset \{x\} \cup \{y\} \cup \Gamma_A$.

Proof: We make use of the results 2.5.1.10 and 2.6.13 and consider the boundary, $X = \{x\} \cup \{y\} \cup \Gamma_A$.

2.6.17 Proposition

The points $x,y \in M_A$ are in the same Gleason part iff either $x = y$ or there is a representing measure, m_x, for x which is non-zero on y and is such that $\operatorname{supp} m_x \subset \{y\} \cup \Gamma_A$.

Proof: The reverse implication follows from 2.6.16. The forward implication follows from 2.6.16 and the argument which appears in the proof of 2.5.1.16.

2.6.18 Proposition

The points $x,y \in M_A$ are in the same Gleason part $\Longleftrightarrow$ either $x = y$ or there is $\mu \in bA^{\perp}$ such that $|\mu|\{x,y\} > \frac{1}{2}$ and $\operatorname{supp} \mu \subset \{x\} \cup \{y\} \cup \Gamma_A$.

Proof: The reverse implication follows from 2.5.1.23. The forward implication follows from 2.6.17 and the argument which appears in the proof of 2.5.1.19.

2.6.19 Proposition

$x_1, x_2, \ldots, x_n \in M_A$ are in the same Gleason part iff there is some i, $1 \leqslant i \leqslant n$ and a representing measure m_i for x_i, non-zero on x_j, $j \neq i$ and $1 \leqslant j \leqslant n$ and such that $\operatorname{supp} m_i \subset \bigcup_{j \neq i} \{x_j\} \cup \Gamma_A$.

Proof: If the right hand side of the equivalence holds, then by 2.6.17, x_i is in the same Gleason part as x_j for all $j \neq i$. Thus $x_1, \ldots, x_n$ are in the same Gleason part giving the reverse implication. To establish the forward implication, we proceed by an induction argument. By 2.6.17, the statement holds for $n = 2$. Assume that it is true for $n = k$. So there is $x_i \in \{x_1, \ldots, x_k\}$ and a representing measure m_i for x_i such that $m_i\{x_j\} > 0$ for all $j \neq i$ and $\operatorname{supp} m_i \subset \bigcup_{i \neq j} \{x_j\} \cup \Gamma_A$. Now we can form a new representing measure for x_i, namely $\nu_i = \alpha\delta_{x_i} + \beta m_i$ where $\alpha + \beta = 1$ and $\alpha, \beta > 0$. Clearly ν_i is non-zero on all x_i; $1 \leqslant i \leqslant k$. Consider $n = k + 1$; Since x_{k+1} is in the same part as x_i, we have, by 1.18.3, a representing measure ν_{k+1} for x_{k+1} such that $\nu_i \ll \nu_{k+1}$. Thus ν_{k+1} is non-zero on each $x_1, \ldots, x_k$. Now, arguing as in 2.5.1.16 we obtain m_{k+1}, a representing measure for x_{k+1} such that $m_{k+1}(x_i) > 0$ for $1 \leqslant i \leqslant k$ and $m_{k+1}(x_{k+1}) = 0$. Since this whole argument could have been carried out on the closed boundary $X = \{x_1\} \cup \{x_2\} \cup \ldots \cup \{x_{k+1}\} \cup \Gamma_A$, we may assume that $\operatorname{supp} m_{k+1} \subset \{x_1\} \cup \{x_2\} \cup \ldots \cup \{x_k\} \cup \Gamma_A$. The Principle of Mathematical Induction now gives the required result.

2.6.20 Proposition

If there is $x \in M_A \sim X$ with a unique representing measure on X, then $\Gamma_A \not\subset X^0$, the interior of X.

Proof: If $\Gamma_A \subset X^0$ then $\{bd(M_A \sim X)\} \cap \Gamma_A = \phi$. Now, by 1.12.6, x has a representing measure on Γ_A and, by 1.22.4, the Local Maximum Modulus Principle, x has a representing measure on $bd(M_A \sim X)$. This contradicts the uniqueness.

CHAPTER THREE

MEASURES

3.1 As seen in the previous chapter, considerable use can be made of measure theoretic techniques in the study of function algebras. A natural question which arises is whether it is possible to characterize a function algebra entirely in terms of measures. More precisely: given a compact Hausdorff space, X and a space of Borel measures on X, say B, can one place such conditions on B, purely in terms of the relationships between the measures in B and the underlying space X, that the associated subspace, B', of $C(X)$, i.e. all the functions in $C(X)$ which are annihilated by all the members of B, will be a function algebra.

We are not aware of any such formulation and have not, as yet, been able to produce one. We present here a characterization which only partly fulfills these conditions, in that some of the conditions imposed on $A^{\perp}$ are expressible in terms of the functions in A. These are, however, of some interest and lead to several approximation conditions which we set down in the first subsection (i.e. the 3.1 series). The second subsection notes some of the properties of $A^{\perp}$ and the last two look at particular conditions imposed on $A^{\perp}$, which, as we shall find in the next chapter, are particularly relevant to analytic and pervasive function algebras.

3.1.1 Before characterizing a function algebra we shall briefly look at some relationships between $A_X^\perp$ and the real parts of A. We introduce some notation :

$\mathrm{Re}A = \{\mathrm{Re}f : f \in A\}$, where $\mathrm{Re}f$ is the real part of the function f.

$A_R = \{f \in A : f \in C_R(X)\}$, i.e. the real-valued functions which are in A.

$\overline{A} = \{\overline{f} : f \in A\}$, where $\overline{f}$ is the complex conjugate of f.

[Note that the symbol $\overline{A}$ has been used to represent topological closure. Our context of usage of the two, however, shall preclude any confusion, particularly since we know A to be uniformly closed.]

Similarly $\overline{A_X^\perp} = \{\overline{\mu} : \mu \in A_X^\perp\}$, where $\overline{\mu}$ is the complex conjugate of μ.

$\mathrm{core}\ A = A \cap \overline{A}$ and $\mathrm{core}\ A_X^\perp = A^\perp \cap \overline{A_X^\perp}$.

A notation such as $(\mathrm{core}\ A)_X^\perp$ would refer to the annihilating measures for core A which are supported by X.

3.1.2.1 **Proposition**

$\mathrm{Re}A \subset A \iff A = C(X)$.

Proof : $\mathrm{Re}A \subset A \iff A = \overline{A}$. This completes the Stone-

$\iff A = C(X)$ Weierstrass requirements (see 1.3).

3.1.2.2 **Proposition**

$A = \mathrm{core}\ A \iff \mathrm{core}\ A$ separates points of $X \iff A = C(X)$.

Proof: $A = \text{core } A \iff A = \overline{A} \iff A = C(X)$, by the Stone-Weierstrass Theorem. core A is certainly a uniformly closed subalgebra which contains the constants. (Being the intersection of two such algebras, A and $\overline{A}$). The separation of the points of X is then the outstanding Stone-Weierstrass requirement. This gives the second equivalence.

3.1.2.3 Proposition

$A_R = \text{Re}(\text{core } A)$.

Proof: $f \in A_R \iff f \in \text{core } A$ and f is real-valued $\iff f \in \text{Re}(\text{core } A)$; noting that core A is closed under complex conjugation.

3.1.2.4 Proposition

Core $A = A_R + iA_R$.

Proof: Noting that core A is closed under complex conjugation, this follows from 3.1.2.3.

3.1.2.5 Proposition

$(A_R)^{\perp}_{X} = (\text{core } A)^{\perp}_{X}$.

Proof: $\mu \perp A_R \Rightarrow \mu \perp \text{core } A$, by 3.1.2.4.

On the other hand, say $\mu \perp \text{core } A$. Then $\int f d\mu = 0$, for all $f \in \text{core } A$.

$\Rightarrow \int \bar{f} d\mu = 0$, $\forall\, f \in \text{core } A$ (core A being closed under complex conjugation)

$\Rightarrow \int \frac{f + \bar{f}}{2} d\mu = 0$, $\forall\, f \in \text{core } A \Rightarrow \int g d\mu = 0$, $\forall\, g \in \text{Re}(\text{core } A)$

$\Rightarrow \int g d\mu = 0$, $\forall\, g \in A_R$ (by 3.1.2.3) $\Rightarrow \mu \in (A_R)^{\perp}_{X}$ as required.

3.1.2.6 Proposition

$(\mathrm{Re}A)^{\perp}_{X} = \text{core } A^{\perp}_{X}$.

Proof: Let $\mu \in \text{core } A^{\perp}_{X}$ and $\mu = \mu_1 + i\mu_2$. Then

$\mu_i \in \text{core } A^{\perp}_{X}$; $i = 1,2 \Rightarrow \mu_i \in A^{\perp}_{X}$; $i = 1,2$

$\Rightarrow \mu_i \perp \mathrm{Re}f$, $\forall f \in A$ and $i = 1,2$ (since the μ_i are real-valued)

$\Rightarrow \mu_i \in (\mathrm{Re}A)^{\perp}_{X}$; $i = 1,2 \Rightarrow \mu \in (\mathrm{Re}A)^{\perp}_{X}$ as required.

On the other hand : Say $\mu \in (\mathrm{Re}A)^{\perp}_{X}$ and $\mu = \mu_1 + i\mu_2$.

Then $\mu_i \in (\mathrm{Re}A)^{\perp}_{X}$; $i = 1,2 \Rightarrow \mu_i \in A^{\perp}_{X}$; $i = 1,2$

$\Rightarrow \mu_1 + i\mu_2 \in A^{\perp}_{X}$ and $\mu_1 - i\mu_2 \in A^{\perp}_{X} \Rightarrow \mu \in \text{core } A^{\perp}_{X}$ as required.

3.1.2.7 Proposition

$A^{\perp}_{X} = \text{core } A^{\perp}_{X} \Longleftrightarrow A = C(X)$.

Proof: The reverse implication is trivial since $C(X)^{\perp} = \{0\}$.

The forward implication : We have $\mu = \mu_1 + i\mu_2 \in A^{\perp}_{X}$

$\Rightarrow \bar{\mu} \in A^{\perp}_{X} \Rightarrow \mu_i \in A^{\perp}_{X}$; $i = 1,2$

$\Rightarrow [f = f_1 + if_2 \in A \Rightarrow f_i \in A$; $i = 1,2]$ since each μ_i annihilates each f_i

$\Rightarrow \mathrm{Re}A \subset A \Rightarrow A = C(X)$ (by 3.1.2.1) .

3.1.2.8 Proposition

$A^{\perp}_{X} = (\text{core } A)^{\perp}_{X} \Longleftrightarrow A = C(X)$.

Proof: The reverse implication is trivial since $C(X) = \text{core } C(X)$.

The forward implication : We have $f \in A \Rightarrow \int f d\mu = 0$, $\forall \mu \in (\text{core } A)^{\perp}_{X}$

By 1.7.2 we know that f is in the closure of core A.

But core A is closed (see proof of 3.1.2.2).

Thus $A = \text{core } A$.

Now, by 3.1.2.2, $A = C(X)$ as required.

3.1.2.9 Proposition

$$(\text{core } A)^{\perp}_{X} = \text{core } A^{\perp}_{X} \Longleftrightarrow A = C(X) .$$

Proof: $\Rightarrow$: Since $(\text{core } A)^{\perp}_{X} \supset A^{\perp}_{X} \supset \text{core } A^{\perp}_{X}$ we have $A^{\perp}_{X} = \text{core } A^{\perp}_{X}$ and $A^{\perp}_{X} = (\text{core } A)^{\perp}_{X}$.

Thus by either 3.1.2.7 or 3.1.2.8 we have $A = C(X)$.

$\Leftarrow$: core $C(X) = C(X)$ and $C(X)^{\perp} = \{0\}$. Thus $(\text{core } C(X))^{\perp} = \{0\} = \text{core } C(X)^{\perp}$ as required.

3.1.3 Here is the characterization of a function algebra to which reference is made in 3.1.

Proposition

Let B be a linear subspace of $C(X)$ and let $B^{\perp}_{X}$ be the set of annihilating measures for B . Then B is a function algebra if and only if the following conditions hold :

(a) For every $f \in C(X) \sim B$ there is $\mu \in B^{\perp}_{X}$ such that $\int f d\mu \neq 0$

(b) $\mu(X) = 0$ for every $\mu \in B^{\perp}_{X}$

(c) If $\mu \in B^{\perp}_{X}$ and $f \in B$, then $f\mu \in B^{\perp}_{X}$

(d) $\delta_x - \delta_y \notin B^{\perp}_{X}$ for every pair $x,y \in X$ such that $x \neq y$.

Proof: Necessity :

(a) B is closed; Thus by 1.7.2 we have property (a). (Refering to 1.2).

(b) B contains the constants. Thus $\int d\mu = 0$ for any $\mu \in B_X^\perp$. Hence we have property (b).

(c) B is an algebra. Thus for every pair $f,g \in B$ and every $\mu \in B$ we have $\int fg\, d\mu = 0$ (since $fg \in B$). Thus $\int g d f\mu = \int g(f d\mu) = 0$ for every $g \in B$. So $f\mu \in B_X^\perp$ for every $f \in B$ and $\mu \in B_X^\perp$, giving property (c).

(d) B separates points. So for every pair $x,y \in X$ with $x \neq y$ we have $f \in B$ such that $f(x) = 1$ and $f(y) = 0$. Thus $\int f d(\delta_x - \delta_y) = 1$ and $\delta_x - \delta_y \notin B_X^\perp$ as required.

Sufficiency :

(i) Condition (a) gives the closure of B in the norm topology as a consequence of the Hahn-Banach Theorem. (More precisely see [51], 5.19).

(ii) Condition (b) gives $\int 1 d\mu = 0$ for every $\mu \in B_X^\perp$. By property (a), $1 \in B$. So B contains the constants.

(iii) Let $f,g \in B$. Then $\int fg d\mu = \int f d(g\mu) = 0$, for every $\mu \in B_X^\perp$, by condition (c). Now, using condition (a), we have $fg \in B$ and conclude that B is an algebra.

(iv) Say there is a pair $x,y \in X$ with $x \neq y$ such that $f(x) = f(y)$ for every $f \in B$. Then $\int f d(\delta_x - \delta_y) = 0$ for every $f \in B$. Thus $\delta_x - \delta_y \in B_X^\perp$. This contradicts condition (d) and we conclude that B separates points of X.

3.1.4 In the next four subsections we shall restate the conditions of 3.1.3 in various ways and shall investigate the effects of strengthening those conditions or of adding further conditions.

Proposition

We may replace condition (a) of 3.1.3 with :

(a_1) For every $f \in C(X) \sim B$ there is $\mu \in B^{\perp}_X$ such that $f\mu \notin B^{\perp}_X$.

Proof: (a) $\Rightarrow$ (a_1) : We have $\int df\mu = \int fd\mu \neq 0$. Thus $f\mu(X) \neq 0$ and $f\mu \notin B^{\perp}_X$ (by condition (b) of 3.1.3).

$(a_1) \Rightarrow$ (a) : We have $f\mu \notin B^{\perp}_X$. Thus there is $g \in B$ such that $\int gd(f\mu) \neq 0$. Since $\int gd(f\mu) = \int gfd\mu = \int fd(g\mu)$ we have $\int fd(g\mu) \neq 0$. But, by condition (c) of 3.1.3 we have $g\mu \in B^{\perp}_X$. So we have some $\nu \in B^{\perp}$ (i.e. $\nu = g\mu$) such that $\int fd\nu \neq 0$.

3.1.5 We may now combine conditions (a_1) and (c) in order to restate the characterization as follows :

Proposition

Let B be a linear subspace of $C(X)$ with corresponding set $B^{\perp}_X$. Then B is a function algebra if and only if all of the following conditions hold :

(a_2) $\mu(X) = 0$ for every $\mu \in B^{\perp}_X$.

(b_2) For every pair $x,y \in X$ with $x \neq y$ we have $\delta_x - \delta_y \notin B^{\perp}_X$.

(c_2) $[f \in B \Longleftrightarrow f(B^{\perp}_X) \subset B^{\perp}_X]$ for every $f \in C(X)$.

[By $f(B^{\perp}_X)$ we denote the set of measures $\{f\mu\colon \mu \in B^{\perp}_X\}$]

Proof: Since $(a_2) \Longleftrightarrow (b)$ of 3.1.3 and $(b_2) \Longleftrightarrow (d)$ of 3.1.3 it remains only to show that $(c_2) \Longleftrightarrow [(a)$ and $(c)]$. It is easily seen that the forward implication in (c_2) is equivalent to (c) whereas the reverse implication is equivalent to (a_1) of 3.1.4. By 3.1.4, $(a_1) \Longleftrightarrow (a)$.

3.1.6 Proposition

We may replace the condition (c) of 3.1.3 with the condition :

(c_1) $\int fgd\mu = 0$ for every pair $f,g \in B$ and for every $\mu \in B_X^\perp$.

Proof: $(c) \Rightarrow (c_1)$: For every $\mu \in B_X^\perp$ and every $g \in B$ we have $g\mu \in B_X^\perp$. Thus $\int fgd\mu = \int fd(g\mu) = 0$, for all $f \in B$.

$(c_1) \Rightarrow (c)$: Since $\int fd(g\mu) = \int fgd\mu = 0$ for every $f \in B$, we have $g\mu \in B_X^\perp$. As this is true for every $g \in B$ and $\mu \in B_X^\perp$, we have condition (c).

3.1.7 We may now combine conditions (a) and (c_1) in order to restate the conditions of 3.1.3 in a much shorter form :

Theorem

Let B be a linear subspace of $C(X)$ with corresponding set $B_X^\perp$. Then B is a function algebra if and only if the following conditions hold :

(a_3) For every pair $x,y \in X$ with $x \neq y$ we have $\delta_x - \delta_y \notin B_X^\perp$.

(b_3) $[f \in B \Longleftrightarrow \int fgd\mu = 0$ for every $\mu \in B_X^\perp$ and every $g \in B]$ for every $f \in C(X)$.

Proof: Since $(a_3) \Leftrightarrow (d)$ of 3.1.3 it remains only to show that $(b_3) \Leftrightarrow [(a)$ and (b) and $(c)]$

$\Rightarrow$: It is easy to see that the forward implication of (b_3) is equivalent to (c_1) which, by 3.1.6 is equivalent to (c). Consider $f \in C(X)$ such that $\int f d\mu = 0$ for every $\mu \in B_X^\perp$. Then by (c), which we have obtained above from (b_3), we have : $\int fgd\mu = \int fd(g\mu) = 0$ for every $g \in B$ and $\mu \in B_X^\perp$. The reverse implication of (b_3) now tells us that $f \in B$ and this establishes (a). Now consider $f = 1$. Clearly this satisfies the right hand side of (b_3) which then gives $1 \in B$. From this we obtain (b) (see (b) in proof of 3.1.3).

$\Leftarrow$: As remarked above, (c) gives the forward implication of (b_3). We know that $[(a)$ and $(b)] \Rightarrow 1 \in B$ (see (ii) in proof of 3.1.3). If $f \in C(X)$ satisfies the right hand side of (b_3) we can substitute $g = 1$ and have: $\int f d\mu = 0$ for all $\mu \in B_X^\perp$. Now, by (a), $f \in B$, thus establishing the reverse implication of (b_3).

3.1.8 Proposition

Let B be a linear subspace of $C(X)$ with corresponding set $B_X^\perp$. Then $B = C(X)$ if and only if the following conditions hold :

(a_3) as in 3.1.7.

(b_3) as in 3.1.7.

(c_3) $\mu \in B_X^\perp \Rightarrow \bar{\mu} \in B_X^\perp$.

Proof: (a_3) and (b_3) ensure that B is a function algebra (by 3.1.7). (c_3) is simply a restatement of $B_X^{\perp} = \text{core } B_X^{\perp}$. The result now follows from 3.1.2.7.

3.1.9. We now consider strengthening condition (c_2) of 3.1.5 and arrive at another characterization of $C(X)$.

Proposition

Let B be a linear subspace of $C(X)$ with corresponding set $B_X^{\perp}$. Then $B = C(X)$ if and only if the following conditions hold :

(a_4) $\mu(X) = 0$ for every $\mu \in B_X^{\perp}$.

(b_4) $[f \in B \iff f(B_X^{\perp}) = \{0\}]$ for all $f \in C(X)$.

Proof: We have $[(a_4)$ and $(b_4)] \Rightarrow [(a_2)$ and $(c_2)]$ of (3.1.5)
$\Rightarrow [(b)$ and $(a)]$ of 3.1.3 (by 3.1.5)
$\Rightarrow 1 \in B$ (by (ii) in proof 3.1.3)

Now substitute $f = 1$ in (b_4) to obtain $B_X^{\perp} = \{0\}$. Thus $B = C(X)$, establishing the reverse implication. The forward implication is trivial since $C(X)^{\perp} = \{0\}$.

3.1.10 An approximation result was perhaps to be expected from a constraint as strong as that of (b_4) above. The same result, however, follows from the following, seemingly much weaker, set of conditions :

Proposition

Let B be a linear subspace of $C(X)$ with corresponding set $B_X^{\perp}$. Then $B = C(X)$ if and only if the following conditions hold :

(a_5) $\mu(X) = 0$ for every $\mu \in B_X^\perp$

(b_5) For every pair $x,y \in X$ with $x \neq y$ we have $\delta_x - \delta_y \notin B_X^\perp$

(c_5) $[f \in B \Longleftrightarrow f(B_X^\perp) \subset \text{core } B_X^\perp]$ for every $f \in C(X)$.

Proof: Clearly $[(a_5)$ and (b_5) and $(c_5)] \Rightarrow [(a_2)$ and (b_2) and $(c_2)]$. So, by 3.1.5, B is a function algebra and $1 \in B$. Substituting $f = 1$ in (c_5) we obtain : $B_X^\perp \subset \text{core } B_X^\perp$. Thus $B_X^\perp = \text{core } B_X^\perp$. By 3.1.2.7, $B = C(X)$ establishing reverse implication. The forward implication is trivial since $C(X)^\perp = \{0\} = \text{core } C(X)^\perp$.

3.2 We now go on to set down some of the relationships between annihilating measures and the underlying space X . The first of these is perhaps most conveniently expressible in terms of the characterizations of the previous subsection.

3.2.1 Proposition

Let B be a subspace of $C(X)$ with corresponding set $B_X^\perp$. Then B is a function algebra whose Choquet boundary coincides with X if and only if the following conditions hold :

(a_6) Every $\mu \in B_X^\perp$ is non-atomic

(b_6) $[f \in B \Longleftrightarrow \int fg\,d\mu = 0$ for every $\mu \in B_X^\perp$ and every $g \in B\,]$ for every $f \in C(X)$.

Note: A measure is non-atomic if it has value zero on every singleton.

Proof: Certainly $(a_6) \Rightarrow (a_3)$ (of 3.1.7) and thus B is a function algebra (by 3.1.7).

Now $x \in \text{ch}B \Longleftrightarrow x$ is a p-point (by 2.2.15)

$\Longleftrightarrow \mu\{x\} = 0$ for all $\mu \in B_X^\perp$

$\Longleftrightarrow$ Every $\mu \in B_X^\perp$ is non-atomic at x

$\Longleftrightarrow x \in X$ (by (a_6)).

3.2.2 Proposition

If $\mu \in A^\perp$ is completely singular (see 1.21.1), then μ is purely non-atomic.

Proof: We know that μ is M_x-singular for every $x \in M_A$. Since $\delta_x \in M_x$, μ is carried by $M_A \sim \{x\}$. Thus $\mu\{x\} = 0$. As this is true for every $x \in M_A$, we have the required result.

3.2.3 Proposition

If $\mu \in A^\perp$ is completely singular and P is a Gleason part, then $|\mu|(P) = 0$.

Proof: By 3.2.2, it is only necessary to consider non-trivial Gleason parts, P. Say $x \in P$, then μ must be carried by an M_x-null set (see 1.19.2). Call this set F. Then $x \notin F$ since $\delta_x \in M_x$. But, for any $y \in P$ there is $m_x \in M_x$ such that $m_x(y) > 0$ (see 2.5.1.8). Thus $y \notin F$ and we have $P \cap F = \phi$. Thus $|\mu|(P) = 0$ as required.

3.2.4 Proposition

Every $\mu \in A^\perp$ is M_x-singular with respect to every p-point x.

Proof: Say we have some $\mu \in A^{\perp}$ which is not M_x-singular with respect to some p-point x. Then for every Borel set, F, such that $\mu_F = \mu$ we have some $m_x \in M_x$ such that $m_x(F) \neq 0$. But, since x is a p-point, we have $M_x = \{\delta_x\}$ (see 2.2.15). Thus $x \in F$ for every such set, F. i.e. if F is a Borel set such that $\mu_F = \mu$, then $F \sim \{x\}$ is such that $\mu_{F\sim\{x\}} \neq \mu$. Thus $\mu\{x\} \neq 0$ and x is not a p-point, contradicting our assumption and completing the proof.

3.2.5 Proposition

If $M_A = \text{ch}A$, then every $\mu \in A^{\perp}$ is completely singular.

Proof: By 2.2.15, every $x \in M_A$ is a p-point. We then apply 3.2.4.

3.2.6 Proposition

$M_A = \text{ch}A \Longleftrightarrow$ every $\mu \in A^{\perp}$ is completely singular $\Longleftrightarrow$ every $\mu \in A^{\perp}$ is purely non-atomic.

Proof: $M_A = \text{ch}A \Rightarrow$ every $\mu \in A^{\perp}$ is completely singular (by 3.2.5)

$\Rightarrow$ every $\mu \in A^{\perp}$ is purely non-atomic (by 3.2.2)

$\Rightarrow M_A = \text{ch}A$ (by 3.2.1)

3.2.7 Proposition

If $\mu \in A^{\perp}$ is M_x-singular for some $x \in M_A$, then $\mu\{x\} = 0$.

Proof: This is, in fact, proved in the proof of 3.2.2.

3.2.8 Proposition

Let $\mu \in (bA_X^{\perp})^e$ and $\mu = \mu_1 + \mu_2$ where $\mu_i \neq 0$ and $\mu_i \in A_X^{\perp}$; $i = 1,2$; then $\|\mu_1\| + \|\mu_2\| > 1$.

Proof: Let $\|\mu_1\| = \alpha$ and $\|\mu_2\| = \beta$. Assume $\alpha + \beta = 1$. We have $\mu = \alpha\left(\frac{\mu_1}{\alpha}\right) + \beta\left(\frac{\mu_2}{\beta}\right)$ where $\left\|\frac{\mu_1}{\alpha}\right\| = \left\|\frac{\mu_2}{\beta}\right\| = 1$. Since μ is extremal we have $\alpha = 1$ or $\beta = 1$ and thus $\mu = \mu_i$ some i , contradicting the conditions of the statement.

3.2.8.1 Corollary

No $\mu \in (bA_X^{\perp})^e$ can be split into two mutually singular annihilating measures.

Proof: Say $\mu = \mu_1 + \mu_2$ where the μ_i are mutually singular and $\mu_i \in A_X^{\perp}$, $i = 1,2$. Then certainly $\|\mu\| = \|\mu_1\| + \|\mu_2\| = 1$, contradicting 3.2.8.

3.2.9 Proposition

Let $\mu \in (bA_X^{\perp})^e$ and let P be a p-set in X , then either $P \cap \operatorname{supp} \mu = \phi$ or $P \supset \operatorname{supp} \mu$ or $|\mu|(P \cap \operatorname{supp} \mu) = 0$.

Proof: Say $|\mu|(P \cap \operatorname{supp} \mu) \neq 0$ and $P \not\supset \operatorname{supp} \mu$. Then clearly $\mu_P \neq \mu$, but we know that $\mu_P \in A_X^{\perp}$. So we have $\mu = \mu_P + \mu_{X\sim P}$ in contradiction to 3.2.8.1.

3.2.10 Proposition

Let $\mu \in (bA^{\perp})^e$, then either μ is M_x-absolutely continuous (see 1.19.3) for some $x \in M_A$ or μ is completely singular.

Proof: This follows from 3.2.8.1 and [24] chapter VI, Theorem 2.3 which is derived from the abstract F. and M. Riesz Theorem (see 1.20 and 1.21.2).

3.3 We shall now investigate the effects of placing upon $A^{\perp}$ a condition which appears to be stronger than condition (a_6) of 3.2.1.

3.3.1 Definition

A measure μ on M_A is normal if $|\mu|(F) = 0$ for every Borel set which is nowhere dense in M_A. We say that $A_X^{\perp}$ is normal if every $\mu \in A_X^{\perp}$ is normal.

3.3.2 Proposition

If the measure μ is normal, then μ is non-atomic at any non-isolated point of M_A.

Proof: This follows trivially from the definition.

3.3.3 Proposition

If $\mu \in A^{\perp}$ is normal, then μ is purely non-atomic.

Proof: By Shilov's Idempotent Theorem (see 1.11) we know that any point x which is isolated in M_A, is a p-point for A. Thus $\mu\{x\} = 0$. The result now follows from 3.3.2.

3.3.4 Proposition

If $A_X^{\perp}$ is normal, then $\mathrm{ch}A = X$.

Proof: $A_X^\perp$ is normal $\Rightarrow$ every $\mu \in A_X^\perp$ is purely non-atomic (by 3.3.3)

$\Rightarrow X = \text{ch}A$ (by 3.2.1).

Note that in the following, closure and interior are taken with respect to M_A .

3.3.5 Proposition

If μ is normal, then $\text{supp}\,\mu = \overline{(\text{supp}\,\mu)^0}$; i.e. the closure of the interior of supp μ .

Proof: Clearly $\overline{(\text{supp}\,\mu)^0} \subset \text{supp}\,\mu$. It is easily seen that $\text{supp}\,\mu \sim (\text{supp}\,\mu)^0$ is nowhere dense in M_A . Thus $F = \text{supp}\,\mu \sim \overline{(\text{supp}\,\mu)^0}$ is nowhere dense in M_A and $|\mu|(F) = 0$. Now by the minimality of supp μ we have the required result.

3.3.6 Corollary

If μ is normal, then $(\text{supp}\,\mu)^0$ carries μ .

Proof: Since $F = \text{supp}\,\mu \sim (\text{supp}\,\mu)^0$ is nowhere dense in M_A we have $|\mu|(F) = 0$. The result follows from this.

3.3.7 Proposition

If $A_X^\perp$ is normal and P is a p-set in X , then $\overline{X \sim P}$ is a p-set in X .

Proof: We firstly note that $F = \overline{X \sim P} \sim (X \sim P)$ is nowhere dense in M_A . Thus $|\mu|(F) = 0$ for any $\mu \in A_X^\perp$. Thus $\mu_{\overline{X \sim P}} = \mu_{X \sim P}$ for all $\mu \in A_X^\perp$. But, in using a general boundary version of 2.2.9 (i.e.

(i.e. with X replacing M_A etc.), we know that $\mu_{X\sim P} \in A_X^{\perp}$ for every $\mu \in A_X^{\perp}$. The result now follows.

3.3.8 Proposition

If $A_X^{\perp}$ is normal and P is a p-set in X, then any closed set F such that $P^0 \subset F \subset P$ is a p-set in X.

Proof: By 3.3.6, $\mu_{P^0} = \mu_P$ for any $\mu \in A_X^{\perp}$. Since, clearly, $F^0 = P^0$ we have for every $\mu \in A_X^{\perp}$ that : $\mu_F = \mu_{F^0} = \mu_{P^0} = \mu_P \in A_X^{\perp}$, since P is a p-set.

3.3.9 Proposition

If $A_X^{\perp}$ is normal and P is a p-set in X, then any closed set F such that $P \subset F$ and $F \sim P$ is nowhere dense in M_A, is a p-set in X.

Proof: Since $F \sim P$ is nowhere dense in M_A, we have $\mu_F = \mu_P \in A_X^{\perp}$ for every $\mu \in A_X^{\perp}$. This gives the result.

3.3.10 Proposition

If $A_X^{\perp}$ is normal and P is a closed set, then : P is a p-set in $X \Longleftrightarrow \overline{X \sim P}$ is a p-set in X.

Proof: The forward implication is given in 3.3.7. For the reverse implication we note firstly that $X \sim \overline{(\overline{X \sim P})} \subset P$ and secondly that $P \sim \overline{(X \sim (\overline{X \sim P})}$ is nowhere dense in M_A. Since, by 3.3.7, $\overline{X \sim (\overline{X \sim P})}$ is a p-set in X, the result now follows from 3.3.9

3.3.11 Proposition

If $A_X^\perp$ is normal, then $P^0 \neq \phi$ for any weak p-set, P. In particular, E_X is not nowhere dense.

Proof: From the definition: $P^0 = \phi \Rightarrow |\mu|(P) = 0$ for every $\mu \in A_X^\perp$. Thus P is a strong p-set, unless $P^0 \neq \phi$. The second statement holds since E_X is a weak p-set, by 2.2.4.3 (the general closed boundary version).

3.3.12 In fact we may characterize $A^\perp$-normality in this way.

Proposition

$A_X^\perp$ is normal if and only if every closed $F \subset X$ such that F is nowhere dense in M_A is a strong p-set in X.

Proof: This follows easily from the definition (i.e. 3.3.1).

3.3.13 Proposition

For any closed $F \subset M_A$ we have:

$A^\perp$ is normal $\Rightarrow A^\perp|_F$ is normal $\Rightarrow (A_F)^\perp$ is normal.

Proof: This follows easily from the fact that any set nowhere dense in F is also nowhere dense in M_A.

3.3.14 The previous result leads us to consider a weaker condition on the measures. For a closed boundary $X \subset M_A$ we define :

A measure μ on X is normal in X if $|\mu|(F) = 0$ for every Borel set which is nowhere dense in X. We say that $A_X^\perp$ is normal in X if every $\mu \in A_X^\perp$ is normal in X.

It is clear that : μ is normal $\Rightarrow$ μ is normal in X . Now by replacing "normal" with "normal in X" and making other natural changes we may state results analogous to the previous results of this subsection. The results analogous to results 3.3.5 to 3.3.13 will hold whereas those analogous to 3.3.2 to 3.3.4 will not hold in general.

3.4 We shall now briefly introduce a further condition on the annihilating measures. This will be dealt with at greater length in the following chapter where it will be found to be particularly relevant.

3.4.1 Definition

A measure μ on M_A is clopen if supp μ is clopen in M_A .

We say that $A_X^{\perp}$ is clopen if every $\mu \in A_X^{\perp}$ is clopen.

3.4.2 Proposition

If $A_X^{\perp}$ is clopen, then $P^0 \neq \phi$ for every weak p-set P in X . In particular, E_X is not nowhere dense in M_A .

Proof: This follows easily from the definition.

3.4.3 Proposition

If $A_X^{\perp}$ is clopen and $F \subset X$ is closed, then $A_F^{\perp}$ is clopen.

Proof: This is trivial since $A_F^{\perp} \subset A_X^{\perp}$.

3.4.4 Proposition

If a measure μ is clopen, then $\operatorname{supp}\mu = \overline{(\operatorname{supp}\mu)^0}$.

Proof: This follows trivially from the definition. Note that interior is taken with respect to M_A .

3.4.5 Proposition

If $A^\perp$ is clopen, then $M_A = \Gamma_A$.

Proof: Let $x \in M_A \sim \Gamma_A$. Choose a closed neighbourhood, V , of x such that $x \in V \subset M_A \sim \Gamma_A$. By the Local Maximum Modulus Principal there is a representing measure, m_x , for x such that $\operatorname{supp} m_x \subset \operatorname{bd}V$. Thus we can form $\mu = \delta_x - m_x \in A^\perp$ with $\operatorname{supp}\mu \subset \{x\} \cup \operatorname{bd}V$. Since, by hypothesis, $\operatorname{supp}\mu$ is clopen in M_A , we know by the Shilov Idempotent Theorem that $\operatorname{supp}\mu \cap \Gamma_A \neq \phi$. This contradiction means that there is no $x \in M_A \sim \Gamma_A$, thus establishing the result.

3.4.6 Proposition

If $A^\perp$ is clopen and a closed $F \subset M_A$ contains no clopen sets, then $A_F = C(F)$.

Proof: By 3.4.3, $A_F^\perp$ is clopen. Since no $\mu \in A_F^\perp$ is supported on F , we have $A_F^\perp = \{0\}$. This gives the result.

CHAPTER FOUR

SOME SPECIAL FUNCTION ALGEBRAS

4.0 Continuing our emphasis on measure theoretic techniques we shall now examine several different types of function algebra. More specifically we shall look at essential function algebras, antisymmetric function algebras, Integral Domains, Analytic function algebras, Pervasive function algebras and Maximal function algebras (see 1.25). In some cases we find characterizations of these purely in terms of the annihilating measures and the functions in the algebra. These produce varying degrees of insight to the corresponding type of function algebra, the least successful being the antisymmetric case. This approach, however, does lead to an interesting comparison between essential function algebras and analytic function algebras which appears in 4.3.4. Furthermore, they also facilitate to a large degree the proving of the well-known chain of implications which links these varying types of function algebra (see 1.26). We shall to some extent link this to the work of previous chapters, referring particularly to the types of p-sets given in 2.2.2 and using at some length the conditions on the set of annihilating measures which were introduced in Chapter Three.

4.1.1 We consider firstly the case of the essential function algebra. We say that A is essential on X if $E_X = X$.

By virtue of 2.6.11 and 2.2.27.1 it is clear that :

A is essential on X if and only if A is essential on M_A .

From this it is clear that, for a function algebra, the property of being essential is independent of the choice of the underlying space, X . Thus it will be unambiguous to use the phrase : "A is essential".

4.1.2 **Proposition**

A function algebra, A on X is essential if and only if for every proper open $B \subset X$ there is $\mu \in A_X^\perp$ such that $|\mu|(B) \neq 0$.

Proof: To prove the reverse implication, assume that A is not essential. Let $B = X \sim E_X$. Clearly B is open in X . By 2.1.13 and 2.6.7 we have $\left(\bigcup_{\mu \in A_X^\perp} \operatorname{supp} \mu\right) \subset E_X$. Thus we have $|\mu|(B) = 0$ for every $\mu \in A^\perp$. For the forward implication assume that there is a proper $B \subset X$ such that $|\mu|(B) = 0$ for every $\mu \in A_X^\perp$. Thus $B \cap \left(\bigcup_{\mu \in A_X^\perp} \operatorname{supp} \mu\right) = \phi$ and by 2.1.13 and 2.6.7 $E_X \neq X$. Thus A is not essential. This establishes the result.

4.1.3 Here are three further conditions, all equivalent, which characterize an essential A .

Proposition

A on X is essential $\Longleftrightarrow$ for every $f \in A \sim \{0\}$, there is $\mu \in A_X^\perp$ such that $\operatorname{supp} f\mu \neq \phi$

$\Longleftrightarrow$ for every $f \in A \sim \{0\}$, there is $\mu \in A_X^\perp$ such that $f\mu \neq 0$

$\Longleftrightarrow$ for every $f \in A \sim \{0\}$, there is $\mu \in A_X^\perp$ such that $|f\mu|(X) \neq 0$.

Proof: The equivalence of the three conditions is trivially shown. We shall prove the first listed.

$\Leftarrow$: Assume that A is not essential. By 4.1.2 there is a proper open $U \subset X$ such that $|\mu|(U) = 0$ for all $\mu \in A^\perp$. Then any Uryshohn function, $f \in C(X)$ which is zero off U will be annihilated by all $\mu \in A_X^\perp$. It follows that such an f is in $A \sim \{0\}$. But for every $\mu \in A_X^\perp$, $\operatorname{supp} f\mu = \phi$, contradicting the right hand side of the equivalence.

$\Rightarrow$: Assume that we have $f \in A \sim \{0\}$ such that $\operatorname{supp} f\mu = \phi$ for every $\mu \in A_X^\perp$. The set $U = \{x \in X : |f(x)| > 0\}$ is a non-empty open set. Now for every $\mu \in A_X^\perp$ we have :

$$0 = |f\mu|(X) = (|f||\mu|)(X) \text{ , see 1.9.2}$$

$$\geqslant (|f||\mu|)(U) \text{ , since } |f||\mu| \text{ is positive.}$$

Form the closed set $F_n = \{x \in X : |f(x)| \geq \frac{1}{n}\}$, $n \in \mathbb{N}$. So we have $0 = (|f||\mu|)(U) \geq (|f||\mu|)(F_n) \geq \frac{1}{n} |\mu|(F_n)$. Thus $|\mu|(F_n) = 0$ for all $n \in \mathbb{N}$. Since for any closed subset P of U we can find F_n such that $P \subset F_n$ (simply choose n such that $\frac{1}{n} \leq \inf_{x \in P} |f(x)|$) we have by the regularity of μ that $|\mu|(U) = 0$, (using also 1.5.1) for every $\mu \in A_X^{\perp}$. Thus A is not essential by 4.1.2.

4.1.4 Proposition

If A on X is essential, then every strong p-set for A is nowhere dense in X .

Proof: This follows immediately from 2.2.4.5 and 2.6.8.

4.2 Our consideration of antisymmetric function algebras is very brief and the characterizations set down here are really just restatements of the usual definition. We say that A is antisymmetric on X if X is a set of antisymmetry for A . By 2.3.7 and the fact that $\tilde{X} = M_A$ we can say :

> A is antisymmetric on X if and only if A is antisymmetric on M_A .

Analogously to 4.1.1 we see that the property of antisymmetry is independent of X and we shall freely say : "A is antisymmetric" .

4.2.1 Proposition

A is antisymmetric $\Longleftrightarrow A_R = \{\text{real constants}\}$.
$\Longleftrightarrow$ core $A = \{\text{constant functions}\}$
$\Longleftrightarrow (\text{core } A)^{\perp}_{X} = \{\mu : \mu(X) = 0\}$.

Proof: These statements are easily checked, using the terminology of 3.1.1.

4.2.2 Proposition

A is antisymmetric if and only if for every non-constant $f \in C_R(X)$ there is $\mu \in A^{\perp}_{X}$ such that $\int f d\mu \neq 0$.

Proof: This is an easy consequence of the definition (see 1.25.2) and the condition 3.1.3(a).

4.2.3 The definition of antisymmetry for A can clearly be stated in terms of the decomposition $\{K\}_X$, of maximal sets of antisymmetry. We can obtain sufficient conditions for A to be antisymmetric in terms of the other two decompositions that we have examined.

Proposition

There exists a Gleason part of X which is dense in X .
$\Rightarrow$ there exists a $\{P\}_X$-part of X which is dense in X .
$\Rightarrow$ A is antisymmetric.

Proof: This follows easily from 2.5.1.7 and results 2.5.1.37 - 2.5.1.39, noting in addition that each $K \in \{K\}_X$ is closed.

4.2.4 Before proceeding to a consideration of analytic function algebras we shall give one characterization of an Integral Domain, which, it will be noted, bears marked similarities to the characterization of an essential function algebra given in 4.1.3. That we may write unambiguously " A is an Integral Domain" is a consequence of the following : A is an Integral Domain on X if and only if A is an Integral Domain on M_A. This in turn follows from the fact that : f,g , $fg \in A \sim \{0\}$ for A on X if and only if f,g , $fg \in A \sim \{0\}$ for A on M_A. This is so since $X \supset \Gamma_A$ on which each $f \in A$ attains its maximum modulus.

Proposition

A is an Integral Domain on X

$\iff$ for every $f,g \in A \sim \{0\}$ there is $\mu \in A_X^\perp$ such that $\operatorname{supp} fg\mu \neq \phi$

$\iff$ for every $f,g \in A \sim \{0\}$ there is $\mu \in A_X^\perp$ such that $fg\mu \neq 0$

$\iff$ for every $f,g \in A \sim \{0\}$ there is $\mu \in A_X^\perp$ such that $|fg\mu|(X) \neq 0$.

Proof: The equivalence of the three statements is trivially confirmed. If the right hand side holds then clearly $fg \neq 0$ for every pair $f,g \in A \sim \{0\}$. Thus A is an integral domain. In order to prove the forward implication we shall proceed by showing that $\sim$RHS $\Rightarrow$ $\sim$LHS . Assume that there exists a pair

$f,g \in A \sim \{0\}$ such that $\operatorname{supp} fg\mu = \phi$ for every $\mu \in A_X^{\perp}$. Consider the open set $U = \{x \in X : |fg(x)| > 0\}$. Now for every $\mu \in A_X^{\perp}$ we have :

$$0 = |fg\mu|(X) = |f||g||\mu|(X) \geqslant |f||g||\mu|(U) \quad \text{(see 1.9.2)}$$

Form the closed set $F_n = \{x \in X : |fg(x)| \geqslant \frac{1}{n}\}$. Then $0 = |f||g||\mu|(U) \geqslant |f||g||\mu|(F_n) \geqslant \frac{1}{n}\,|\mu|(F_n)$. Thus $|\mu|(F_n) = 0$, for every $n \in \mathbb{N}$. Since we can choose F_n to contain any closed subset F of U (simply choose n such that $\frac{1}{n} \leqslant \inf_{x \in F} |fg(x)|$) we have, by the regularity of μ, and that of $|\mu|$ (see 1.5.1), that $|\mu|(U) = 0$ for every $\mu \in A_X^{\perp}$. We now have three possibilities for U ; either $U = \phi$ in which case $fg = 0$ and A is not an Integral Domain, or U is a singleton, $\{x\}$, in which case the characteristic function $\chi_{\{x\}}$ is continuous on X and is clearly annihilated by each $\mu \in A_X^{\perp}$. Thus $\chi_{\{x\}} \in A$. Now set $f' = \chi_{\{x\}}$ and choose any $g' \in A \sim \{0\}$ such that $g'(x) = 0$. Then the fact that $f'g' = 0$ would indicate that A is not an Integral Domain. Thirdly, if we have $x,y \in U$ with $x \neq y$ we can choose disjoint open neighbourhoods U_x and U_y of x and y respectively such that $U_x \subset U$ and $U_y \subset U$. Now choose two Urysohn functions $\tilde{f},\tilde{g} \in C_R(X)$ such that $\tilde{f}(x) = 1$ and $\tilde{f}_{X \sim U_x} = 0$ and $\tilde{g}(y) = 1$ and $\tilde{g}_{|X \sim U_y} = 0$. Clearly $\tilde{f}$ and $\tilde{g}$ are both annihilated by every $\mu \in A_X^{\perp}$ and thus $\tilde{f},\tilde{g} \in A \sim \{0\}$. Clearly also, $\tilde{f}\tilde{g} = 0$. Thus A is not an Integral Domain. This completes the proof.

4.3 We now give several characterizations of analytic function algebras in terms of relationships between the annihilating measures, the functions of the algebra and the underlying topology. These bring to light an interesting relationship between essential function algebras and analytic function algebras. We then examine some necessary conditions and some sufficient conditions of analyticity of a function algebra. We find that the added imposition on these of some of the measure conditions studied in Chapter Three can yield equivalences.

4.3.1 Proposition

Let P be a proper open subset of X and $f \in C(X) \sim \{0\}$, then :

A is an analytic function algebra on X if and only if for every pair (P,f) with $f_{|P} = 0$, there is $\mu \in A_X^{\perp}$ such that $\int f d\mu \neq 0$.

Proof:

$\Rightarrow$: If A is analytic, then any such $f \in C(X) \sim \{0\}$ which is zero on a proper open subset P of X, does not belong to A. Thus, by 3.1.3(a), there is $\mu \in A_X^{\perp}$ such that $\int f d\mu \neq 0$.

$\Rightarrow$: If A is not analytic, there is $f \in A \sim \{0\}$ which is zero on some open subset, P of X. This pair (P,f) then has the required properties but $\int f d\mu = 0$ for all $\mu \in A_X^{\perp}$ since $f \in A$.

4.3.2 Proposition

Let P be a proper open subset of X and $f \in A \sim \{0\}$, then :

A is an analytic function algebra on X if and only if for every pair (P,f) there is $\mu \in A_X^\perp$ such that $\operatorname{supp} f\mu_P \neq \phi$.

Proof:

$\Leftarrow$: If A is not analytic there is $f \in A \sim \{0\}$ whose zero set contains a proper open set, P . Then clearly, for this pair (P,f) , $\operatorname{supp} f\mu_P = \phi$ for every $\mu \in A_X^\perp$

$\Rightarrow$: Let A be analytic and let P be a proper open subset of X and $f \in A \sim \{0\}$. Now set $F = \{x : f(x) \neq 0\}$. Clearly F is a non-empty open set and $U = F \cap P$ is non-empty since A is analytic. Now assume that $|\mu|(U) = 0$ for every $\mu \in A_X^\perp$. If U is a singleton then χ_U , the characteristic function of U , is continuous on X and is clearly annihilated by every $\mu \in A_X^\perp$. Thus $\chi_U \in A \sim \{0\}$ and A is not analytic, contradicting our hypothesis. If $x,y \in U$ with $x \neq y$, then we can choose disjoint open neighbourhoods U_x and U_y of x and y respectively, each of which is contained in U . We now choose a Urysohn function $f \in C(X)$ such that $f(x) = 1$ and $f|_{U_y \cup (X\sim U)} = 0$. Clearly $\int f d\mu = 0$ for every $\mu \in A_X^\perp$ and thus $f \in A \sim \{0\}$. But f is zero on the non-empty open set U_y and A is not analytic. So we see that the assumption that $|\mu|(U) = 0$ for every $\mu \in A_X^\perp$

contradicts the hypothesis that A is analytic. So there exists some $\mu \in A_X^{\perp}$ such that $|\mu|(U) \neq 0$. By the regularity of μ and hence of $|\mu|$ (see 1.5.1) we have a closed $Q \subset U$ such that $|\mu|(Q) \neq 0$. Since Q is compact and f continuous, $|f|$ attains its minimum on Q . Say $|f|_{|Q} > \varepsilon$ for some $\varepsilon > 0$. Then we have $|f\mu|(P) = |f||\mu|(P) \geqslant |f||\mu|(Q) > \varepsilon\,|\mu|(Q) > 0$ (see 1.9.2). Thus $\operatorname{supp} f\mu_P \neq \phi$ as required.

4.3.3 Let $\mathcal{B}$ be an open base for the topology on X , then we may restate 4.3.2 as follows.

Proposition

Let $B \in \mathcal{B}$ be a proper subset of X and let $f \in A \sim \{0\}$, then :

A is an analytic function algebra on X if and only if for every pair (B,f) there is $\mu \in A_X^{\perp}$ such that $\operatorname{supp} f\mu_B \neq \phi$.

Proof:

⇐: For any open P as in 4.3.2 there is $B \in \mathcal{B}$ with $\phi \neq B \subset P$ (since $\mathcal{B}$ is a base for the topology). By the right hand side of the above we have $\mu \in A_X^{\perp}$ such that $\operatorname{supp} f\mu_B \neq \phi$. Clearly then, $\operatorname{supp} f\mu_P \neq \phi$. This establishes the right hand side of 4.3.2 which implies that A is analytic.

⇒: This follows immediately from 4.3.2 since each proper $B \in \mathcal{B}$ satisfies the requirements for P of 4.3.2.

4.3.4 Comment

Compare result 4.3.3 with that of 4.1.3. It seems that if we take the property which defines an essential function algebra and make it hold within every proper base open set, for any given base for the topology on X, then we arrive at a condition defining an analytic function algebra. Thus it would make sense to think of an analytic function algebra as being "topologically essential".

4.3.5 We now look at some necessary conditions for a function algebra to be analytic in X.

Proposition

If A is analytic in X, then for every triple (U,μ,f) such that U is a proper open subset of X, $\mu \in A_X^\perp$ and $f \in A \sim \{0\}$ we have $U \not\subset (\text{supp}\ \mu \sim \text{supp}\ f\mu)$

or equivalently :

If A is analytic in X, then $\text{supp}\ \mu \sim \text{supp}\ f\mu$ has empty interior in X for every $\mu \in A_X^\perp$ and every $f \in A \sim \{0\}$.

Proof: The equivalence of the two necessary conditions given is easily seen. We shall prove the first by showing that $\sim\text{RHS} \Rightarrow \sim\text{LHS}$.

Say that there exists such a triple (U,μ,f) such that $U \subset (\text{supp}\ \mu \sim \text{supp}\ f\mu)$. Let $V = \{x \in X : f(x) \neq 0\}$. Assume that $V \cap U \neq \phi$. Now since $U \subset \text{supp}\ \mu$ we have $|\mu|(U_i) \neq 0$ for every open $U_i \subset U$. On the other hand, since $U \cap \text{supp}\ f\mu = \phi$ we have

$|f\mu|(U_i) = 0$ for every $U_i \subset U$. In particular, let $U_1 = V \cap U$, then we have $|f\mu|(U_1) = 0 \Rightarrow |f||\mu|(U_1) = 0$. Now consider any closed $F \subset U_1$. Since f is continuous on F, a compact set, it attains its minimum absolute value, say $\varepsilon > 0$, on F. Thus we have : $0 = |f||\mu|(U_1) \geqslant |f||\mu|(F) \geqslant \varepsilon|\mu|(F) \Rightarrow |\mu|(F) = 0$. Since this is true for every closed $F \subset U_1$, we have by the regularity of μ, and hence of $|\mu|$, that $|\mu|(U_1) = 0$. This contradiction invalidates our assumption that $V \cap U \neq \phi$. But the alternative, i.e. $V \cap U = \phi$, means that the zero set of f contains U. Thus A is not analytic.

4.3.6 Proposition

If A is analytic in X, then every proper p-set for A in X is nowhere dense in X.

Proof: If A is analytic then the interior of any zero set of any function in $A \sim \{0\}$ is empty. Since zero sets are closed, each proper zero set is nowhere dense in X. Now every proper p-set is contained in some proper zero-set. To see this, note first that every proper peak set is a zero set and secondly that every proper p-set is an intersection of proper peak sets. This gives the result.

4.3.7 Proposition

If A is analytic in X, then for every proper closed $P \subset X$ such that $P^0 \neq \phi$ there is a pair (f,μ) with $\mu \in A_X^\perp$ and $f \in A$ such that $\int_P f d\mu \neq 0$.

Proof: If P is closed with non-empty interior, then, by 4.3.6, P is not a p-set for A. Thus there is $\mu \in A_X^{\perp}$ such that $\mu_P \notin A_X^{\perp}$. Thus there is $f \in A$ such that $\int f d\mu_P \neq 0$. This gives $\int_P f d\mu \neq 0$ as required.

4.3.8 We shall now look at some sufficient conditions for a function algebra to be analytic.

Proposition

If for every pair (μ, f) with $\mu \in A_X^{\perp}$ and $f \in A \sim \{0\}$ we have $\text{supp}\ \mu = \text{supp}\ f\mu$, then A is an analytic function algebra in X.

Proof: Firstly we claim that this is a sufficient condition for A to be an essential function algebra. If A is a proper function algebra then there is $\mu \in A_X^{\perp}$ with $\text{supp}\ \mu \neq \phi$. So by the given condition, $\text{supp}\ f\mu \neq \phi$ for every $f \in A \sim \{0\}$. We can rephrase this as follows : For every $f \in A \sim \{0\}$, there is $\mu \in A_X^{\perp}$ such that $\text{supp}\ f\mu \neq \phi$. But this is simply the condition characterizing an essential function algebra which is given in 4.1.3. This establishes the claim. Now let U be a proper open subset of X. Since A is essential we can use the characterization of 4.1.2 to say that there exists some $\nu \in A_X^{\perp}$ such that $|\nu|(U) \neq 0$. Thus $\text{supp}\ \nu \cap U \neq \phi$ and by our hypothesis, $\text{supp}\ f\nu \cap U \neq \phi$ for all $f \in A \sim \{0\}$. We can rephrase all this as follows : For every pair

(U,f) , proper open $U \subset X$ and $f \in A \sim \{0\}$, there is $\nu \in A_X^\perp$ such that $\operatorname{supp} f\nu_U \neq \phi$. This is just a statement of the characterization of an analytic function algebra given in 4.3.2. This concludes the proof.

4.3.9 It is interesting to compare the second formulation of the condition of 4.3.5 to the condition given in 4.3.8. They are related in the following way :

[condition of 4.3.8] ⇒ A is analytic ⇒ [condition of 4.3.5]

So, in a sense, an analytic function algebra is "bracketed between" these two conditions. We find that it is sufficient to add to the condition of 4.3.5 the existence of an annihilating measure whose support is the whole space in order to obtain another sufficient condition for A to be an analytic function algebra. It is not clear, however, whether or not this additional condition is also necessary. A formal statement and proof follow:

Proposition

If there is $\nu \in A_X^\perp$ such that $\operatorname{supp} \nu = X$ and if for every $\mu \in A_X^\perp$ and $f \in A \sim \{0\}$ we have $(\operatorname{supp} \mu \sim \operatorname{supp} f\mu)^0 = \phi$, then A is an analytic function algebra.

Proof: We proceed by showing that ~RHS ⇒ ~LHS.

Say there exists an open $U \subset X$ and $f \in A \sim \{0\}$ such that $f|_U = 0$. Clearly we have $|f\mu|(U) = |f||\mu|(U) = 0$, for all $\mu \in A_X^\perp$. Thus $U \cap \operatorname{supp} f\mu = \phi$ for all $\mu \in A_X^\perp$. In particular $U \cap \operatorname{supp} f\nu = \phi$. Thus $U \subset (\operatorname{supp} \nu \sim \operatorname{supp} f\nu)$. This negates the statement of the LHS.

4.3.10 Now we shall consider the effect of setting the condition, $A_X^{\perp}$ is normal, on the annhilating measures. In fact, we can use the weaker condition : $A_X^{\perp}$ is normal in X . We modify the proof of 4.3.5 in order to obtain a new result.

Proposition

If $A_X^{\perp}$ is normal in X , then A is analytic implies that $\operatorname{supp} \mu \sim \operatorname{supp} f\mu$ is nowhere dense for every $\mu \in A_X^{\perp}$ and every $f \in A \sim \{0\}$.

Proof: We shall proceed analogously to the proof of 4.3.5 by showing that if $A_X^{\perp}$ is normal, then the existence of $f \in A \sim \{0\}$ and $\mu \in A_X^{\perp}$ such that $\operatorname{supp} \mu \sim \operatorname{supp} f\mu$ is not nowhere dense implies that A is not analytic. Assume that there is a triple (U,μ,f) with U a proper open subset of X , $\mu \in A_X^{\perp}$ and $f \in A \sim \{0\}$, such that $U \subset \overline{\operatorname{supp} \mu \sim \operatorname{supp} f\mu}$. Since $U \subset \operatorname{supp} \mu$ we have $|\mu|(U_i) \neq 0$ for every open $U_i \subset U$. Let $S = \overline{\operatorname{supp} \mu \sim \operatorname{supp} f\mu} \sim (\operatorname{supp} \mu \sim \operatorname{supp} f\mu)$. It is fairly easy to see that S is a closed, nowhere dense set in X . Thus $|\nu|(S) = 0$ for every $\nu \in A_X^{\perp}$, since $A_X^{\perp}$ is normal. So we have $|f\mu|(U_i) = |f\mu|(U_i \sim S) + |f\mu|(U_i \cap S)$. The first term on the right hand side of this equality is zero since $(U_i \sim S) \cap \operatorname{supp} f\mu = \phi$ and the second term is zero since $|\mu|(S) = 0$. Thus $|f\mu|(U_i) = 0$ for every open $U_i \subset U$. Now set $V = \{x \in X : f(x) \neq 0\}$ as before, assume that $V \cap U \neq \phi$ and set $U_1 = V \cap U$. From this point the proof proceeds as in 4.3.5.

4.3.11 Proposition

If $A_X^\perp$ is normal in X, then A is analytic if and only if $\operatorname{supp} \mu = \operatorname{supp} f\mu$ for every $\mu \in A_X^\perp$ and every $f \in A \sim \{0\}$.

Proof: Let $A_X^\perp$ be normal and A be analytic. Then, by 4.3.10, $\operatorname{supp} \mu \sim \operatorname{supp} f\mu$ is nowhere dense for any $\mu \in A_X^\perp$ and $f \in A \sim \{0\}$. Thus $|\mu|(\operatorname{supp} \mu \sim \operatorname{supp} f\mu) = 0$ for all $\mu \in A_X^\perp$ and $f \in A \sim \{0\}$. By the minimality of the support we have $\operatorname{supp} \mu = \operatorname{supp} f\mu$ as required. The proof is completed with a reference to 4.3.8.

4.3.12 We obtain a similar result to the previous one by requiring that the annihilating measures be clopen.

Proposition

If $A_X^\perp$ is clopen, then A is analytic if and only if $\operatorname{supp} \mu = \operatorname{supp} f\mu$ for every $\mu \in A_X^\perp$ and every $f \in A \sim \{0\}$.

Proof: Since $A_X^\perp$ is clopen we know that $\operatorname{supp} \mu$ and $\operatorname{supp} f\mu$ are both clopen. Thus $\operatorname{supp} \mu \sim \operatorname{supp} f\mu$ is clopen. But this contradicts 4.3.5 unless $\operatorname{supp} \mu \sim \operatorname{supp} f\mu = \phi$. So we obtain $\operatorname{supp} \mu = \operatorname{supp} f\mu$ as required. The proof is completed by a reference to 4.3.8.

4.4

In this subsection we shall deal with pervasive function algebras. After giving several characterizations of a pervasive function algebra we shall look at the relationships between pervasive function algebras and the condition : $A_X^\perp$ is clopen.

4.4.1 Proposition

A is a pervasive function algebra on X if and only if for every pair (P,μ) where P is a proper closed subset of X and $\mu \in A_X^\perp$ such that $|\mu|(P) \neq 0$, we have the existence of some $f \in A$ such that $\int_P f d\mu \neq 0$.

Proof:

$\Rightarrow$: Let A be pervasive on X and let P be a proper closed subset of X. We know that $A_P = C(P)$. Thus $A_P^\perp = \{0\}$. Let $\mu \in A_X^\perp$ be such that $|\mu|(P) \neq 0$. Then clearly $\mu_P \notin A_X^\perp$ because $\mu_P \in A_X^\perp \Rightarrow \mu_P \in A_P^\perp \Rightarrow \mu_P = 0$, contradicting $|\mu|(P) \neq 0$. But if $\mu_P \notin A_X^\perp$ then there is $f \in A$ such that $\int f d\mu_P \neq 0 \Rightarrow \int_P f d\mu \neq 0$. As this is true for each such $\mu \in A_X^\perp$ (i.e. each μ non-zero on P), we have the statement of the right hand side.

$\Leftarrow$: Firstly we note that $A_P^\perp \subset A_X^\perp$. Now, if the right hand side is true and $\mu \in A_P^\perp$, then there is $f \in A$ such that $\int_P f d\mu \neq 0 \Rightarrow \int f d\mu \neq 0$ (since $\mu_P = \mu$). This contradicts the fact that $\mu \in A_X^\perp$ unless $\mu = 0$. Thus $A_P^\perp = \{0\}$ and $A_P = C(P)$. As this is true for every proper closed $P \subset X$, A is pervasive in X.

4.4.2 From the previous result we can derive a characterization of a pervasive function algebra purely in terms of the underlying space and the annihilating measures.

Proposition

A is pervasive on X if and only if $\operatorname{supp} \mu = X$ for every $\mu \in A_X^\perp$.

Proof:

$\Rightarrow$: We show that $\sim$RHS $\Rightarrow$ $\sim$LHS : Say we have $\mu \in A_X^\perp \sim \{0\}$ such that $\operatorname{supp} \mu \neq X$. Let $P = \operatorname{supp} \mu$. Clearly, for every $f \in A$, we have $\int_P f d\mu = \int f d\mu = 0$. Thus the pair (P,μ) contradicts the characterization given in 4.4.1.

$\Leftarrow$: We show that $\sim$LHS $\Rightarrow$ $\sim$RHS : Assume that A is not a pervasive function algebra. Then by 4.4.1 there is a proper closed $P \subset X$ and $\mu \in A_X^\perp$ with $|\mu|(P) \neq 0$ such that $\int_P f d\mu = 0$, for every $f \in A$. Thus $\int f d\mu_P = 0$ for every $f \in A$. Clearly then, $\mu_P \in A^\perp$. Since $\operatorname{supp} \mu_P \subseteq P$, this contradicts the RHS as required.

4.4.3 Our final characterization of a pervasive function algebra is given in terms of p-sets and q-sets (see 2.2.2). This is also derived from that given in 4.4.1.

Proposition

A is pervasive on X if and only if every proper closed subset of X is either a strong p-set in X or a strong q-set in X .

Proof:

$\Rightarrow$: Let A be a pervasive function algebra on X and let P be proper closed subset of X . Then either $|\mu|(P) = 0$ for every $\mu \in A_X^\perp$, in which case P is a strong p-set, or there is some $\mu \in A_X^\perp$ with $|\mu|(P) \neq 0$. By 4.4.1 we know that there exists $f \in A$

such that $\int_P f d\mu \neq 0$. Thus $\mu_P \notin A_X^\perp$. Since this is true for every such μ (i.e. non-zero on P), we see that P is a strong q-set.

$\Leftarrow$: Arguing in the same way for the reverse implication, let P be as before. Then either P is a strong p-set, in which case $|\mu|(P) = 0$ for all $\mu \in A_X^\perp$ or P is a strong q-set, in which case, for every $\mu \in A_X^\perp$ such that $|\mu|(P) \neq 0$ (and such μ do exist), we have $\mu_P \notin A_X^\perp$. Thus, for every such μ , there is $f \in A$ such that $\int f d\mu_P \neq 0 \Rightarrow \int_P f d\mu \neq 0$. This is just a statement of the right hand side of the characterization of 4.4.1. Thus A is pervasive on X as required.

4.4.4 In relating pervasive function algebras to the decompositions dealt with in the 2.5 series, it is well-known, and we shall deal with it in the 4.6 series, that if A is pervasive, then X is antisymmetric. In other words, there is only one $\{K\}_X$-part, namely X . The following result relates pervasive function algebras to $\{P\}_X$-parts.

Proposition

If A is pervasive on X , then there exists at most one $\{P\}_X$-part which is not a p-point.

Proof: Firstly note that every point of a strong p-set is a p-point. Now assume that there are two non-p-point $\{P\}_X$-parts containing x and y respectively. Clearly x and y are not p-points and we may say, without loss of generality that $y \notin P_x$, the smallest p-set containing x . By our initial comment, P_x is not a strong p-set. This contradicts the characterization of 4.4.3, negating our assumption.

4.4.5 For the remainder of this subsection we shall consider the condition : $A_X^{\perp}$ is clopen. By virtue of definition 3.4.1 we find that we can most naturally deal with the case $X = M_A$.

Proposition

$A^{\perp}$ is clopen and M_A is connected $\Rightarrow$ A is pervasive on M_A .

Proof: The only possible support of any $\mu \in A^{\perp}$ is M_A . The result follows from 4.4.2.

4.4.6 **Proposition**

If $A^{\perp}$ is clopen and F is a component of M_A , then either A_F is pervasive or $A_F = C(F)$.

Proof: Firstly we note that, in a compact space, the component containing a point x is precisely the intersection of all clopen sets containing x . This is easily confirmed. Thus we can write $F = \bigcap_{i \in I} F_i$ where each F_i is a clopen set and $F_i \supset F$. Now by Shilov's Idempotent Theorem (see 1.11), it is clear that each F_i is a p-set. Thus F is a p-set. The result then follows from 3.4.3 and 4.4.5.

4.4.7 **Proposition**

Let $A^{\perp}$ be clopen and let K be a maximal set of antisymmetry of A in M_A , then either K is a point set or a component of M_A . Furthermore, if K is non-trivial, then K is clopen.

Proof: Let K be any non-trivial maximal antisymmetric set for A. Then K is a p-set (see 1.23.4) and A_K is antisymmetric. Thus we have $\mu \in A^{\perp}$ such that $\mu_K \in A^{\perp} \sim \{0\}$ (since A_K is a proper subalgebra of $C(K)$). Furthermore, K is connected (since it is antisymmetric - this is easily shown.) Since $A^{\perp}$ is clopen we must have $K \subset \operatorname{supp} \mu_K$. Thus $\operatorname{supp} \mu_K = K$. So K is a component of M_A and is also clopen in M_A.

4.4.8 Lemma

If A is a normal function algebra, then A is not an Integral Domain (see 1.25.7).

Proof: Let $x,y \in X$ such that $x \neq y$. Since X is compact and T_2 we can choose disjoint closed neighbourhoods, U_x and U_y, of x and y respectively. Since A is normal we can choose $f,g \in A \sim \{0\}$ such that $f(x) = 1$ and $f_{|X \sim U_x} = 0$ and $g(y) = 1$ and $g_{|X \sim U_y} = 0$. Clearly $fg = 0$ and A is not an Integral Domain.

4.4.9 Corollary

If A is normal on X, then A is not analytic on X and A is not pervasive on X.

This follows immediately from 4.4.8 and 1.26.

4.4.10 Proposition

If $A^{\perp}$ is clopen and A is normal, then $A = C(M_A)$.

Proof: Let F be a non-trivial maximal set of antisymmetry Then A_F is pervasive (by 4.4.6) and normal. This contradicts 4.4.9. So we conclude that all the $\{K\}$-parts of M_A are trivial and the result follows from 1.23.5.

4.4.11 Proposition

If $A^\perp$ is clopen, then : M_A is connected

$\iff$ A is pervasive on M_A

$\iff$ A is analytic on M_A

$\iff$ A is an Integral Domain

$\iff$ A is antisymmetric.

Proof: M_A is connected $\Rightarrow$ A is pervasive on M_A (by 4.4.5).

$\Rightarrow$ A is analytic $\Rightarrow$ A is an Integral Domain $\Rightarrow$ A is antisymmetric (by 1.26)

$\Rightarrow$ M_A is connected (by 4.4.7 or easily from 1.11).

4.4.12 Proposition

If $A^\perp$ is clopen, then there is at most one non p-point $\{P\}$-part in each component.

Proof: Bearing in mind that any component F is a p-set and hence that any $\{P\}_{A_F}$-part is also a $\{P\}$-part for A we make use of 4.4.6 and 4.4.4.

4.4.13 Proposition

If $A^{\perp}$ is clopen, then for every p-set, P , and every $K \in \{K\}$, either $K \subset P$ or $K \cap P$ is a nowhere dense strong p-set.

Proof: The possibility $K \cap P = \phi$ is of course included in the second alternative. It follows from 4.4.7 that either K is a p-point, in which case the result is easily verified, or K is a nontrivial component of M_A . If this is the case then A_K is pervasive (by 4.4.6) hence essential (by 1.26). Since K is a p-set, $\mu_{K \cap P} \in A^{\perp}$ for all $\mu \in A^{\perp}$. By hypothesis, either $\operatorname{supp} \mu_{K \cap P} = K$ for some $\mu \in A^{\perp}$, in which case $K \subset P$, or $\operatorname{supp} \mu_{K \cap P} = \phi$ for every $\mu \in A^{\perp}$, in which case $K \cap P$ is a strong p-set. The fact that $K \cap P$ is nowhere dense follows from 4.1.4.

4.4.14 Proposition

If A is pervasive on M_A , then $A^{\perp}$ is clopen.

Proof: This is an immediate consequence of 4.4.2.

4.4.15 Proposition

If M_A is connected, then A is pervasive on M_A if and only if $A^{\perp}$ is clopen.

Proof: This follows from 4.4.5 and 4.4.14.

4.4.16 Proposition

If $A^{\perp}$ is clopen, then E is the union of all the non-trivial maximal sets of antisymmetry in M_A .

Proof: This follows from 2.2.26 and 4.4.7.

4.4.17 Proposition

If $A^{\perp}$ is clopen and M_A is connected, then $A^{\perp}$ is normal if and only if the strong p-sets in M_A are precisely the closed, nowhere dense sets in M_A .

Proof: The reverse implication of the equivalence follows from 3.3.12. If A is antisymmetric then the forward implication of the equivalence follows from 4.4.13 and 3.3.12. The fact that A is antisymmetric follows from 4.4.5 and 1.26.

4.5 In this subsection we shall give some sufficient conditions for a function algebra to be maximal (see 1.25.6) and a characterization of a maximal function algebra.

4.5.1 We refer to the characterization of a function algebra given in 3.1.3 and consider strengthening condition (a) of that result in the following way :

(a_7) For every $f \in C(X) \sim B$ and every $\mu \in B_X^{\perp} \sim \{0\}$ we have $\int f d\mu \neq 0$. We now state this formally :

Proposition

Let B be a linear subspace of $C(X)$ with corresponding set $B_X^{\perp}$. B is a maximal function algebra on X if the following conditions hold :

(a_7) (above) and (b),(c),(d) of 3.1.3.

Proof: B is certainly a function algebra since, clearly, $(a_7) \Rightarrow (a)$ and we can apply 3.1.3. Consider the linear space $\langle B,f\rangle$ generated by B and any $f \in C(X) \sim B$. Clearly $\langle B,f\rangle_X^\perp \subset B_X^\perp$. However, any $\mu \in B_X^\perp \sim \{0\}$ is not in $\langle B,f\rangle_X^\perp$ since $\int f d\mu \neq 0$. Thus $\langle B,f\rangle_X^\perp = \{0\}$ and $\langle B,f\rangle$ is dense in $C(X)$. Thus any function algebra which properly contains B, being closed, coincides with $C(X)$.

4.5.2 We can obtain a sufficient condition similar to 4.5.1 by modifying condition (a) of 3.1.3 in a different way and replacing it with :

(a_8) For every $f \in C(X) \sim B$ and for every $\mu \in B_X^\perp \sim \{0\}$ we have $f\mu \notin B_X^\perp$. We state this more fully as follows :

Proposition

Let B be a linear subspace of $C(X)$ with corresponding set $B_X^\perp$. B is a maximal function algebra on X if the following conditions hold : (a_8) (above) and (b), (c), (d) of 3.1.3.

Proof: B is certainly a function algebra since, clearly $(a_8) \Rightarrow (a_1)$ and we can apply 3.1.4. Take any $f \in C(X) \sim B$ and let $[B,f]$ be the function algebra generated by B and f i.e. $[B,f]$ is the intersection of all function algebras which contain B and f. (This will again be a function algebra since it contains B, thus separating points of X). Clearly

$[B,f]_X^\perp \subset B_X^\perp$ but any $\mu \in B_X^\perp \sim \{0\}$ is not in $[B,f]_X^\perp$ since, by (a_8), $f\mu \notin B_X^\perp$ (thus violating condition (c) of 3.1.3 applied to the function algebra $[B,f]$). So we have $[B,f]_X^\perp = \{0\}$ and $[B,f] = C(X)$. Since this is true for every $f \in C(X) \sim B$, we see that B is maximal.

4.5.3 One suspects that the sufficient conditions given in 4.5.1 and 4.5.2 are too strong to be necessary. We will in fact show that the following two, apparently weaker conditions can be used in characterizations of maximal function algebras. Again let B be a linear subspace of $C(X)$ with corresponding set $B_X^\perp$.

(a_9) For every $\mu \in B_X^\perp \sim \{0\}$ and for every $f \in C(X) \sim B$, there is $g_m \in B$ and $m \in \mathbb{N}$ such that $\int g_m f^m d\mu \neq 0$.

(a_{10}) For every $\mu \in B_X^\perp \sim \{0\}$ and for every $f \in C(X) \sim B$, there is $g_m \in B$ and $m \in \mathbb{N}$ such that $g_m f^m \mu \notin B_X^\perp$.

Before proceeding, we need two results regarding these conditions :

4.5.3.1 Lemma

If $1 \in B$ and (c) (of 3.1.3) hold, then $(a_9) \iff (a_{10})$.

Proof:

$\Rightarrow$: Since $\int g_m f^m d\mu = \int 1.d(g_m f^m \mu)$ we see that condition (a_9), giving $\int g_m f^m d\mu \neq 0$, and $1 \in B$ imply that the corresponding measure $g_m f^m \mu$ does not annihilate B. This gives the forward implication.

$\Leftarrow$: We show that : $\sim(a_9) \Rightarrow \sim(a_{10})$. Say there exist $\mu \in B_X^{\perp} \sim \{0\}$ and $f \in C(X) \sim B$ such that $\int g_m f^m d\mu = 0$ for all $g_m \in B$ and all $m \in \mathbb{N}$. Then $f^m\mu \in B_X^{\perp}$ for all $m \in \mathbb{N}$. By (c), $g_m f^m \mu \in B_X^{\perp}$ for all $m \in \mathbb{N}$ and all $g_m \in B$ as required.

4.5.3.2 Lemma

If conditions (a_9) (above) and (c) (of 3.1.3) hold, then condition (a) (of 3.1.3) holds.

Proof: Using (c) we set $\nu = g_m\mu$ and know that $\nu \in B_X^{\perp}$. We can then obtain from (a_9) the following: For every $f \in C(X) \sim B$ we have $\nu \in B_X^{\perp}$ and $m \in \mathbb{N}$ such that $\int f^m d\nu \neq 0$. Now assume that (a) does not hold. Thus for some $f \in C(X) \sim B$ we have $\int f d\mu = 0$ for every $\mu \in B_X^{\perp}$. But then for any $g \in B$ and $\mu \in B_X^{\perp}$ we have $\int fg d\mu = 0$ (by (c)). Thus $\int g d(f\mu) = 0$. Thus $f\mu \in B_X^{\perp}$. Now if $\int f^m d\nu \neq 0$ (as obtained above from (a_9)), then $\int f^{m-1} d(f\nu) \neq 0$. Setting $\nu_1 = f\nu \in B_X^{\perp}$ and carrying on in this way we obtain $\nu_{m-1} \in B_X^{\perp}$ which does not annihilate f . This contradicts our assumption and completes the proof.

4.5.4 Proposition

Let B be a linear subspace of $C(X)$ with corresponding set $B_X^{\perp}$. Then B is a maximal function algebra on X if and only if the following conditions hold: (a_9) as in 4.5.3 and (b), (c), (d) of 3.1.3.

Proof:

$\Rightarrow$: Take any $f \in C(X) \sim B$ and consider the set of functions

$$B_1 = \{g_0 + g_1 f + g_2 f^2 + \ldots + g_n f^n \mid n \in \mathbb{N} \ ; \quad g_i \in B \quad 0 \leqslant i \leqslant n\} \ .$$

It is easily seen that, since B is an algebra, B_1 is an algebra containing B. Since B is maximal we know that the closure of B_1 coincides with $C(X)$. Thus $(B_1)^{\perp}_{X} = \{0\}$. Thus for every $0 \neq \mu \in B^{\perp}_{X}$ there is $h = g_0 + g_1 f + \ldots + g_n f^n$ in B_1 such that $\int h d\mu \neq 0$. It follows that for every $\mu \in B^{\perp}_{X} \sim \{0\}$, there is $m \in \mathbb{N}$ and $g_m \in B$ such that $\int g_m f^m d\mu \neq 0$. As this is true for any $f \in C(X) \sim B$, we have condition (a_9). Since B is a function algebra we obtain conditions (b), (c) and (d) from 3.1.3. This completes the forward implication.

$\Leftarrow$: Having conditions (a_9) and (c) together, we have, by 4.5.3.2, condition (a). Then, having conditions (a), (b), (c), (d) together, we know, by 3.1.3 that B is a function algebra. Let B_2 be any function algebra properly containing B. Then if B_2 contains $f \in C(X) \sim B$ it is clear that $B_2 \supset B_1$, the algebra constructed above. But, by condition (a_9) we see that any $\mu \in B^{\perp}_{X}$ does not annihilate B_2 since μ does not annihilate the function $g_m f^m \in B_2$. Since also $(B_2)^{\perp}_{X} \subset B^{\perp}_{X}$ we can say $(B_2)^{\perp}_{X} = \{0\}$. Thus $B_2 = C(X)$ and B is maximal as required.

4.5.5 We may obtain a similar characterization using the condition (a_{10}).

Proof: It will be necessary and sufficient to show that $(c_{11}) \iff [(a_9)$ and $(c)]$ (of 4.5.4).

$\Rightarrow$: The reverse implication in (c_{11}) is a statement of (a_9); and the forward implication of (c_{11}) with $m = 1$ is a statement of (c_1) (see 3.1.6). By 3.1.6 this gives the condition (c) of 3.1.3 as required.

$\Leftarrow$: The reverse implication of (c_{11}) is given by (a_9). For the forward implication of (c_{11}) we obtain from (a_9) and (c) the conditions (a) and (c) (using 4.5.3.2). From (a) and (c) we see, as in the proof of 3.1.3, that B is an algebra. So, for any $f,g \in B$ and $m \in \mathbb{N}$ we have $gf^m \in B$. Thus $\int gf^m d\mu = 0$ for every $\mu \in B_X^\perp$. This proves the forward implication.

Thus, since the given conditions are equivalent to (a_9), (b), (c) and (d) of 4.5.4, we have the result, by 4.5.4.

4.5.7 Proposition

Let B be a linear subspace of $C(X)$ with corresponding set $B_X^\perp$. Then B is a maximal function algebra on X if and only if the following conditions hold :

(a_{11}), (b_{11}) (as in 4.5.6) and

(c_{12}) For every $f \in C(X)$ and every $\mu \in B_X^\perp \sim \{0\}$ we have $[f \in B \iff gf^m\mu \in B_X^\perp$ for every $g \in B$ and every $m \in \mathbb{N}]$.

Proof: It will be sufficient to show that if $1 \in B$, then $(c_{12}) \iff [(a_{10})$ and $(c)]$ (of 4.5.5).

$\Rightarrow$: The reverse implication of (c_{12}) is, in fact, a statement of (a_{10}) and the forward implication of (c_{12}), with $g = 1$ and $m = 1$ is a statement of (c).

$\Leftarrow$: The reverse implication of (c_{12}) is given by (a_{10}). For the forward implication of (c_{12}) we have : (a_{10}) and (c) $\Rightarrow$ (a_9) and (c) (using 4.5.3.1 and since $1 \in B$) $\Rightarrow$ (a) and (c) (by 4.5.3.2) $\Rightarrow$ B is an algebra (as in the proof of 3.1.3). Thus, for every $f,g \in B$ and $m \in \mathbb{N}$ we have $gf^m \in B$. One more application of condition (c) now establishes the forward implication of (c_{12}).

The fact that $1 \in B$ follows from (a_{11}) (see proof of 3.1.3). Thus the conditions (a_{11}), (b_{11}) and (c_{12}) are equivalent the conditions (a_{10}), (b), (c) and (d) of 4.5.5 and the result follows from 4.5.5.

4.5.8 Proposition

A function algebra A on X is maximal if and only if $A_{|E_X}$ is maximal.

Proof: By 2.4.1.6 we know that $A_{|E_X}$ is a function algebra so that the statement is meaningful. Now, since E_X supports each $\mu \in A_X^{\perp}$ (see 2.1.10 and 2.6.7) it is easily seen that the conditions of, for example, 4.5.4, will hold on X if and only if they hold on E_X . This establishes the result.

4.6 In this subsection we shall try to demonstrate, to some degree, the utility of the characterizations of the various types of function algebra dealt with in this chapter by giving proofs for most of the links in the following well-known chain of implications which connects these function algebras (of 1.2.6). We have :

A is a maximal essential function algebra on X

$\Rightarrow$ A is pervasive on X

$\Rightarrow$ A is analytic on X

$\Rightarrow$ A is an Integral Domain

$\Rightarrow$ A is antisymmetric

$\Rightarrow$ A is essential

The reader is probably aware that our characterization of the antisymmetric function algebra is perhaps least satisfying in terms of the framework which we have set for ourselves i.e. "measure theoretic" characterizations which help to give new insights into the nature of the construction. It is probably for this reason that we have been unable to find a new simple proof in terms of these characterizations for the step :

A is an Integral Domain $\Rightarrow$ A is antisymmetric.

All the other stated implications, however, easily yield simple proofs which we shall outline below. In fact, most of the implications implicit in the above chain, similarly yield simple proofs in terms of the given characterizations. The reader may wish to verify this for himself. It is our feeling that further investigation, within this framework, of the antisymmetric case, may provide some useful insights. There is a brief mention, in the latter part of this subsection of the concept of weak analyticity and its relation to some of these function algebras and some of the ideas, e.g.decompositions and clopen measures, with which we have dealt.

4.6.1 Proposition

If A is a maximal essential function algebra on X, then A is pervasive on X.

Proof: Let A be a maximal essential function algebra and let $\mu \in A_X^\perp \sim \{0\}$. Say $\operatorname{supp} \mu = P$. Take any $f \in C(X)$ such that $f_{|P} = 0$. Clearly then $\int gf^m d\mu = 0$ for every $g \in A$ and for every $m \in \mathbb{N}$. By 4.5.4 we know that $f \in A$. In other words the closed set, P, is such that if $f \in C(X)$ and $f_{|P} = 0$, then $f \in A$. By 2.1.11.3 (in the general boundary case) we have $P \supset E_X$. Since A is essential, $E_X = X$, thus $P = X$. As this is true for each $\mu \in A_X^\perp \sim \{0\}$, the characterization of 4.4.2 tells us that A is pervasive.

4.6.2 Proposition

If A is pervasive on X, then A is analytic on X.

Proof: Let A be pervasive and $f \in A \sim \{0\}$. Set $V = \{x: |f(x)| > \varepsilon\}$ with $\varepsilon > 0$ such that $V \neq \phi$. Since V is open we have $|\mu|(V) \neq 0$ for any $\mu \in A_X^\perp \sim \{0\}$, (by 4.4.2). Thus $|f\mu|(V) = |f||\mu|(V) \geqslant \varepsilon|\mu|(V) > 0$ for every $\mu \in A_X^\perp \sim \{0\}$. From this and 3.1.3 we have $f\mu \in A_X^\perp \sim \{0\}$ and hence by 4.4.2 that $\operatorname{supp} f\mu = X$ for every $f \in A \sim \{0\}$ and every $\mu \in A_X^\perp \sim \{0\}$. Clearly then, $|f\mu|(P) \neq 0$ for any non-empty open $P \subset X$. Thus we can state that for every $f \in A \sim \{0\}$ and every non-empty open $P \subset X$, there is $\mu \in A_X^\perp$

(in fact, every $\mu \in A_X^{\perp} \sim \{0\}$) such that $\operatorname{supp} f\mu_P \neq \phi$. By 4.3.2, A is analytic on X .

4.6.3 Proposition

If A is analytic on X , then A is an Integral Domain.

Proof: Let A be analytic on X and let $f,g \in A \sim \{0\}$. Set $V = \{x \in X : f(x) \neq 0\}$. Since V is a non-empty open set we have by 4.3.2 that there is $\mu \in A_X^{\perp}$ such that $\operatorname{supp} g\mu_V \neq \phi$. Clearly then, g is not zero on all of V . Thus $fg \neq 0$ and A is an Integral Domain.

4.6.4 Proposition

If A is antisymmetric, then A is an essential function algebra.

Proof: We show that : ~RHS ⇒ ~LHS .

If A is not an essential function algebra then, by 4.1.2, there is a proper open $U \subset X$ such that $|\mu|(U) = 0$ for all $\mu \in A_X^{\perp}$. Let f be any non-zero Urysohn function which is zero off U . Clearly $\int f d\mu = 0$ for all $\mu \in A_X^{\perp}$. This contradicts the condition of 4.2.2, establishing the result.

4.6.5 Proposition

If A is a maximal function algebra on X , then A_{E_X} is pervasive on E_X .

Proof: From 4.5.8 it is easily seen that A_{E_X} is a maximal essential function algebra. The result then follows from 4.6.1.

4.6.6 The remaining few results of this chapter rely heavily on [2]. We shall briefly outline the definitions and results (without proof) of which we shall make use.

4.6.6.1 Definition

A closed set $F \subset X$ is weakly analytic for A if every peak set for A_F either coincides with F, or is nowhere dense in F.

4.6.6.2 The definition given in 4.6.6.1 is that given in [2]. We modify it slightly, expressing it in terms of p-sets.

Lemma

A closed set $F \subset X$ is weakly analytic for A if and only if every p-set for A_F either coincides with F or is nowhere dense in F.

Proof: The forward implication holds since any subset of a nowhere dense set is nowhere dense and the reverse implication since every peak set is also a p-set.

4.6.6.3 Lemma

Every singleton in M_A is weakly analytic for A.

Proof: This follows easily from the definition (4.6.6.1).

4.6.6.4 Lemma

Every weakly analytic set for A is antisymmetric for A . (from [2], Lemma 2).

4.6.6.5 Definition

A decomposition of X is a pairwise disjoint, closed cover of X .

(Note that the term is used in a more specialized way than we have hitherto.)

4.6.6.6 Lemma

Let $\mathcal{R}$ be a family of subsets of X . Among all decompositions $\mathcal{K}$ that satisfy the condition : each $R \in \mathcal{R}$ is contained in some $K \in \mathcal{K}$, there exists a finest one. We denote this finest one by $\mathcal{K}(\mathcal{R})$. (see [2], Lemma 4).

4.6.6.7 Theorem

(i) Every weakly analytic set is contained in a maximal one, and the family $\mathcal{R}$ of all maximal weakly analytic sets covers X .

(ii) $\{K_2\}_X \leq \{K\}_X$ where $\{K_2\}_X = \mathcal{K}(\mathcal{R})$ and $\{K\}$ is the collection of maximal sets of antisymmetry. (see 2.6.9.3).

(For an outline of the proof, see [2], Theorem IA).

4.6.7 Proposition

If A is analytic on X , then X is a weakly analytic set for A .

Proof: This follows immediately from 4.3.6 and 4.6.6.2.

4.6.8 Proposition

If $A^{\perp}$ is clopen, then $\{K_2\} = \{K\}$.

(Note that we are using the notation of 4.6.6.7 (ii) where it is understood that $X = M_A$.)

Proof: Let $K \in \{K\}$. By 4.4.7, either K is a singleton, in which case it is weakly analytic (by 4.6.6.3) or K is a component. If K is a non-trivial component, then by 4.4.6, A_K is pervasive. So by 4.6.2, A_K is analytic on K . Now by 4.6.7, K is weakly analytic for A . By 4.6.6.7 (i), $K \subset K_2$ for some $K_2 \in \{K_2\}$. Thus $\{K\} \leqslant \{K_2\}$. By 4.6.6.7, $\{K\} = \{K_2\}$ as required.

4.6.9 Proposition

If $A^{\perp}$ is clopen, then : M_A is weakly analytic for A

$\iff$ A is antisymmetric

$\iff$ A is pervasive on M_A

$\iff$ A is analytic on M_A

$\iff$ A is an Integral Domain.

Proof: If M_A is weakly analytic for A , then M_A is antisymmetric. (by 4.6.6.4). If A is antisymmetric, then M_A is weakly analytic for A (by 4.6.8). This establishes the first equivalence. The remaining equivalences follow from 4.4.11.

CHAPTER FIVE

BIBLIOGRAPHIC NOTES AND OTHER COMMENTS

5.0 The rather broad scope of this chapter's title covers several functions. Firstly it contains fairly detailed reference to source material for the ideas set out in the previous chapters. We hasten to add that, in using the word "source" we do not claim to systematically trace the origins of the various results, but rather to provide a good starting point for the reader who is interested in deeper study of that particular area. With regard to this, brief historical sketches of the development of particular results appear in several places. Secondly, a number of results associated with the previous material are given, whose inclusion would either have disturbed the integrity of the previous chapters or, it must be admitted, was not chronologically convenient. Thirdly, there are comments of an almost personal slant, on the part of the author, where some points of interest or difficulty are mentioned as well as areas which it is felt would fruitfully yield to further investigation, even though, at present, the connections seen are only tenuous.

5.1.1 Our definition of a function algebra is taken from Browder [16]. This coincides with the definition of a uniform algebra given in Stout [53] and in Gamelin [24]. Stout, in fact, defines a function algebra in a slightly wider sense which does not, however, coincide with the definition that we use in the context of a compact underlying space, since the condition of uniform closure is missing. The predecessor of these notions is probably the "sup-norm algebra" as defined by Hoffman in [34].

5.1.2 The significance of the Stone-Weierstrass Theorem to our definition of a function algebra is obvious and, as can be expected, this theorem plays a basic part in the development of the theory and refers in particular to a number of approximation conditions; i.e. conditions under which $A = C(X)$. This is treated in detail in Nachbin [45]. Of considerable interest in regard to this are also De Branges [18], Bishop [14] whose work is immanently applicable to our subject and Mullins [44] who produces a rather ingenious converse theorem.

5.1.3 The basic Measure theoretic notions can be found in Bartle [5] and Halmos [31]. In progressing to the case of complex measures the work of Taylor [55] is useful, while the application of these to the development of the necessary integration theory may be adapted from Halmos. The introduction of these structures to the context of function algebra is, for instance, set out in chapter II of Browder. It is evident at this stage, though not mentioned in our preliminary chapter, that the Riesz Representation Theorem forms a cornerstone of this part of the theory. A treatment of this, and of course Borel sets, may be found in Rudin [51] or, from a more "measure theoretic" point of view, in Kendal [38]. In fact, all of these sources are brought together in [38], giving a development of the techniques which is particularly suited to our needs.

5.1.4 An exposition of the Maximal Ideal Space and the Shilov boundary appear in Naimark [46] while Leibowitz [41] lists some of the consequences which are relevant to this subject.

5.1.5 The definition of peak sets is taken from Gamelin [24], where the notation "p-sets" is also introduced. Urysohn's Lemma is of fundamental importance and is of particular use in dealing with p-sets and the essential set of a function algebra.

5.1.6 Glicksberg's p-set Criterion is one of the most powerful tools in the study of function algebras, especially in the approach that we have taken. Some of the initial results in this area were for peak sets and peak points, e.g. in Bishop [13]. Glicksberg's result immediately shifts the attention to p-sets.

5.1.7 The idea of Gleason parts originated with Gleason [27], whose definition is given in 1.1.17. There are several equivalent, and perhaps more convenient, definitions but these do not seem to arise in as natural a manner as Gleason's. The treatment given in Browder or Gamelin [24], largely based on the work of Bishop [15], is sufficient for our needs. We should perhaps draw attention to the difference between "supported by" and "carried by" as set out in 1.4.11. This has some importance in the statement of 1.18.4 where Gamelin's version, in our terminology, would be incorrect. A great deal of work has been done on Gleason parts. In particular we mention Bear [7], [10], [11] and mainly [12] where the previous results are mostly brought together. Much work has also been done in relating Gleason parts to analytic structure and Dirichlet and Logmodular algebras, for instance, by Wermer and Hoffman. We have not however touched on this area at all.

5.1.8 Subsections 1.19 to 1.21 appear in Glicksberg [30], who generalizes the classical F. and M. Riesz theorem, and in Gamelin [24]. Their work is the culmination of work by, among others, Forelli [23] and Ahern [1]. A further generalization is given by Konig and Seever [40] who also introduce the notion of a band of measures. This interesting idea is further developed in Gamelin [25]. It is our feeling that, in conjunction with some of the results of this work, it may be refined into an even more useful tool.

5.1.9 The Local Maximum Modulus Theorem first appear in Rossi [50] and several equivalent versions exist, given for example, in Leibowitz. The importance of result 1.23.5 has already been mentioned and is due to Bishop [14]. Much work has also been done in this area by

Glicksberg [28]. The essential set of a function algebra was introduced by Bear [8] and a great deal of work has been done on this. The work of Mullins [42] and [43] is of interest here. The function algebras of 1.25 and the relationships of 1.26 are well known and may be found in Leibowitz or Hoffman and Singer [35].

5.2.1 The comments of 2.1.5 and 2.1.6 are probably well known. They are mentioned, for instance, by Detraz [20]. The proofs outlined are our own.

5.2.2 The characterizations of E given in 2.1.13 and 2.1.14 are probably fairly well known; 2.1.13 is mentioned in Browder and appears as an exercise in Leibowitz. Leibowitz proves 2.1.14 using representing measures whereas our approach is in terms of annihilating measures. Results 2.1.10 and 2.1.15 may be regarded as variants and consequences of these. The proofs given are all our own. The properties given in 2.1.11 are well known, but the proofs given in terms of annihilating measures are our own. In particular, the proof of 2.1.11.6, which appears in Bear's original paper [8], is simplified by means of measure theoretic techniques.

5.2.3 The result of 2.1.17 is well known (see e.g. Leibowitz), whereas we have not seen explicitly stated the results of 2.1.16, 2.1.18 and 2.1.19.

5.3.1 The refinement in the definition of p-sets given in 2.2.2 is our own and its correlation with existing nomenclature has already been mentioned. The results 2.2.3 and 2.2.13 are probably mostly known, in one form or another. We have only seen one reference to 2.2.8 in the literature, i.e. in Suciu [54]. Result 2.2.10 also appears in Suciu's book in his study of "absorbant sets" and is also mentioned in Hayashi [32]. Of the non-trivial proofs in this section, most are our own, with the exception of 2.2.6 which appears in Browder [16]. The proofs of 2.2.7 and 2.2.8 are adapted from analogous proofs in Browder.

5.3.2 The results of 2.2.15 to 2.2.26, culminating in the diagram of 2.2.27 are, on the whole, well-known, though we do not recall seeing 2.2.17.2 and 2.2.19 to 2.2.22 explicitly stated anywhere. The result 2.2.17.2, and consequently the second half of 2.2.22 do however arise as a corollary to a more general result in Suciu which says that the intersection of an absorbant set (of which p-sets are examples) and a determining set (of which Γ_A is an example) is non-empty. The treatment of the subject bears some similarity to that given in Browder but the proofs are mainly our own, with the exception of the results immediately leading to 2.2.26 which are taken substantially from Leibowitz. Some of the results of 2.2.27 are known, but 2.2.27.4, 2.2.27.5 and 2.2.27.8 in particular are our own.

5.4.1 The results of 2.3 are all fairly well-known, with the exception of 2.3.4.1 which is our own. It is easy to see, making use of 2.3.4, that the Zarisky-closure mentioned by Detraz in [19] and [20] corresponds to A-convexity in the sense that, for any $K \subset M_A$:

> K is Zarisky closed if and only if K is A-convex.

5.5.1 The formulations and proofs of 2.4.1.3 and 2.4.1.5 are our own. Result 2.4.1.5 can also easily be seen to follow from Theorem 2 and the corollary to Theorem 5 of Gamelin's paper [26]. The results of 2.4.1.6 to 2.4.1.12 are probably well-known with the exception of 2.4.1.8 and 2.4.1.10 which we have not seen in the literature. Likewise, 2.4.1.13 and 2.4.1.18 are our own. Detraz [19] gives some counter-examples concerning the union and intersection of CRS sets.

5.5.2 All the results and proofs of the 2.4.2 series are our own. Closely allied to our construction of the F_α sets is the concept of a w-interpolation set dealt with by Ishikawa, Tomiyama and Wada in [36]. In bringing the two together, several new results arise which we shall briefly list here, without giving details of the proofs.

5.5.2.1 Firstly we shall establish some notation which will be useful later as well.

$\lambda = E_X \cap \Gamma_A$; By the results of 2.6, particularly 2.6.11, we know that λ is independent of X .

$\lambda_1 = \Gamma_{A_E}$, i.e. the Shilov boundary of A_E .

5.5.2.2 The sets λ and λ_1 are of interest in themselves and, before proceeding, we shall look at some of their properties without giving details of proof.

5.5.2.2.1 λ is CRS .

Proof: We know that Γ_A is CRS, and thus that A_{Γ_A} is a function algebra. By 2.6.11, λ is the essential set for A_{Γ_A}, i.e. $\lambda = E_{\Gamma_A}$. Thus λ is CRS in A_{Γ_A} and hence is CRS in A .

5.5.2.2.2 λ_1 is CRS .

Proof: Since $\lambda = \Gamma_{A_E}$, λ_1 is CRS in A_E . Since E , itself, is CRS, the result follows.

5.5.2.2.3 $\lambda_1 \subset \lambda$

Proof: By Glicksberg's p-set criterion and 2.6.4 we have : x is a p-point for A_E if and only if x is a p-point for A .

Now by 2.2.15 and 2.2.18, the result follows.

According to Wada [56], the inclusion is sometimes strict. Some of the above results are mentioned in [36] and [56].

5.5.2.2.4 A_λ is an essential function algebra.

Proof: Applying 2.6.11 to the function algebra A_E .

5.5.2.2.5 A_{λ_1} is an essential function algebra.

Proof: Applying 2.6.11 to the function algebra A_λ .

5.5.2.2.6 We have : λ_1 = the Shilov boundary of A_E
= the Shilov boundary of A_λ
= the Shilov boundary of A_{λ_1} .

5.5.2.2.7 $\tilde{\lambda}_1 = \tilde{\lambda} = E$.

5.5.2.2.8 λ_1 is independant of X .

5.5.2.2.9 The following approximation conditions are easily established :

$A = C(X)$ if and only if $A_\lambda = C(\lambda)$
if and only if $A_{\lambda_1} = C(\lambda_1)$.

5.5.2.2.10 **Proposition**

There is a one-to-one correspondence between the p-sets of A_E and those of A_λ

There is a one-to-one correspondence between the p-sets of A_λ and those of A_{λ_1} .

Consequently: There is a one-to-one correspondence between the p-sets of A_{E_X} and those of A_{λ_1} for any X .

Proof: These all follow from an application of 2.6.2. The correspondence is given in each case, as before, in terms of A-convex hulls.

5.5.2.2.11 Corollary

There is a one-to-one correspondence between the p-sets of A_{λ_1} on λ_1 and the p-sets of A on X which intersect with λ.

5.5.2.2.12 Corollary

The p-sets of A_{λ_1} on $\tilde{\lambda}_1$ are precisely the p-sets of A on M_A which are contained in E.

5.5.2.2.13 Corollary

The set $\lambda \sim \lambda_1$ contains no p-sets (of A or of $A\lambda$)

5.5.2.2.14 Proposition

If $F \subset X$ is such that $F \cap \lambda \neq \phi$, then :
F is a maximal set of antisymmetry for A on X if and only if $F \cap \lambda$ is a maximal set of antisymmetry for A_λ if and only if $F \cap \lambda_1$ is a maximal set of antisymmetry for A_{λ_1}.

Proof: This follows from the above results and 2.6.9.4.

5.5.2.3 The significance of the set P (defined in 2.4.3.8) to the set E was first pointed out by Mullins [42] in the context $X = M_A$ and with the condition that X be metric. The metric condition was dropped in [36] where the result $E = M_A \sim P$ is obtained. This was obtained in a different way in 2.4.3.10 where we use a later result of Mullins [43] in which, noticing that P is dependent upon X (we shall denote this P_X), he obtains a characterization of E_X. There are several concomitant approximation conditions of some interest developed in [36], [42] and [43]. We may use this second result of Mullins, i.e. $E_X = (X \sim P_X) \cup (X \sim \mathrm{ch}A)$ together with the results of 5.5.2.2 to obtain some new results, for example :

5.5.2.3.1 Proposition

If x is a p-point for A , then :

$$x \notin E_X \text{ if and only if } x \in P_X$$

$$\text{and} \quad x \notin \lambda_1 \text{ if and only if } x \in P_X .$$

This enables us to write Mullins' result in another form :

5.5.2.3.2 Proposition

$E_X = (X \sim P_X) \dot{\cup} (X \sim \Gamma_A) \cup (\lambda \sim \lambda_1)$.

The reader may verify these for himself.

5.5.2.4 We return now to the question of w-interpolation sets.

5.5.2.4.1 Notation

Let $F = \cap F_\alpha$, the intersection taken over all possible F_α , as described in 2.4.2.

(Note that the F_α's and F itself are dependent on on X ; We shall write F_X).

5.5.2.4.2 By 2.4.3.8 and the comments of 2.6.12 we have a characterization of P_X as follows :

$$P_X = X \sim F_X \quad , \text{ for all possible } X .$$

One could apply this to 5.5.2.3.2 to obtain another characterization of E_X .

5.5.2.4.3 $X \sim F_\alpha$ is a w-interpolation set for all X and for all F_α .

Proof: This follows from a generalization of 2.4.2.6.

5.5.2.4.4 **Proposition**

P_X is the largest w-interpolation set for A on X .

Proof: If $x \in G$, for some w-interpolation set G , then we have a closed neighbourhood V_x of x such that $x \in V_x \subset G$. Clearly $C(V_x) = A_{|V_x}$ and thus $x \in P_X$. The fact that P_X is itself a w-interpolation set follows from 5.5.2.4.3 and 5.5.2.4.2 and a simple compactness argument.

5.5.2.4.5 Now we can apply Thm. 1 of [36] to obtain :

$$\lambda_1 \subset F_X \text{ for any closed boundary } X .$$

5.5.2.4.6 The corollary to Thm. 1 of [36] can now be expressed in our terms as follows :

$$E_X = \lambda_1 \Rightarrow X \sim E_X \text{ is the largest w-interpolation set ,}$$

or equivalently :

$$E_X = \lambda_1 \Rightarrow E_X \text{ is the smallest } F_\alpha \text{ set.}$$

5.5.2.4.7 In view of the above results, the condition of the corollary to Thm. 1 of [36] can be seen as a sufficient condition to have $E_X = X \sim P_X$. Another such condition appears in Thm. 2 of [36] or in 2.4.3.10 and is simply $X = M_A$ (as indicated in 5.5.2.3).

5.5.3 In the 2.4.3 series we make use of the Local Maximum Modulus Principle expressed in terms of p-sets. We have not seen this formulation in the literature. The other results of this subsection are our own, with the exception of 2.4.3.15 which is fairly well known, though the given proof is new.

5.5.4 Any treatment of CRS sets should perhaps mention the well-known theorem of Glicksberg which appears in [29]. This states :

Every closed $F \subset X$ is CRS if and only if $A = C(X)$.

In view of the one-to-one correspondence between the regular Borel measures (defined on closed subsets of X) and the regular Baire measures (defined on closed G_δ subsets of X) (see e.g. Halmos), one would suspect that certain approximation conditions, expressible in terms of closed sets, may equally well be expressible in terms of closed G_δ -sets. This turns out to be the case for Glicksberg's Theorem. The author, in [39], gives the result:

Every closed G_δ-set $F \subset X$ is CRS if and only if $A = C(X)$.

It is proved by a modification of a proof for Glicksberg's theorem which appears in Stout [53] and is based on some ideas of Katznelson [37]. The results leading up to this proof, together with other related results, are listed here without proof. Unless otherwise specified, we have not seen them in the literature.

5.5.5.1 Firstly we can refine the concept of boundedness which appears in [37] as follows :

A is weakly bounded on a set $V \subset X$ if there is a constant $C_V > 0$ such that if F is a closed subset of V and $[f]$ is an idempotent in $A_{|kF}$, then $\|[f]\| \leqslant C_V$.

A is strongly bounded on a set $V \subset X$ if A is weakly bounded on some open set U such that $V \subset U \subset X$.

Where there is no confusion we shall lapse in the colloquillism of "weakly bounded sets" and "strongly bounded sets".

5.5.5.2 If every closed G_δ-set $F \subset X$ is CRS, then A is a normal function algebra on X .

5.5.5.3 If A is a normal function algebra on X and is weakly bounded on open subsets V_1 and V_2 of X , then any closed $F \subset V_1 \cup V_2$ is strongly bounded.

5.5.5.4 If every closed G_δ-set, $F \subset X$ is CRS, then T , the set of points of X which are not strongly bounded, is finite.

The proofs of these three are analogous to those appearing in the proof given by Stout, though some further ingenuity is required in places.

5.5.6 Arising from the results of 5.5.5 are some further results of interest, particularly relating to the set T of 5.5.5.4.

5.5.6.1 **Proposition**

$T \subset E_X$, for any function algebra.

Proof: There is a fairly simple proof of this using Urysohn functions.

5.5.6.2 **Proposition**

If A is a normal function algebra on X, then $T = E_X$.

There are a number of outstanding questions about this set T; e.g. What is its relationship to λ, λ_1, P_X and E_X in general? Is it independent of X?

5.5.6.3 From the previous result one can easily see that :

If A is a normal function algebra, then :

$$\begin{aligned} A = C(X) &\iff T = \phi \\ &\iff X \text{ is strongly bounded} \\ &\iff X \text{ is weakly bounded.} \end{aligned}$$

This result, however, appears in a much more general form in Bade and Curtis [4] and one wonders if the result 5.5.6.2 may not also hold in the case of ε-normal function algebras (with $\varepsilon < \frac{1}{2}$).

5.5.7 Glicksberg's closed restriction theorem may be generalized in another direction. We do this by noting, as Wada [56] does, that a closed $F \subset X$ is an interpolation set if and only if every closed $F_0 \subset F$ is a CRS set for A. Ewer [22] also does some work in this direction. The next step, however, is to choose a restriction set for A which we know not to be an interpolation set for a proper function algebra A. Imposition of the Glicksberg condition on this set will then give us an approximation condition. So we have :

5.5.7.1 Every proper closed subset of Γ_A or E_X or λ or λ_1 is CRS if and only if $A = C(X)$.

5.5.7.2 Noting that λ_1 is contained in each of these sets, we may combine this with the result of [38] to obtain the most general version :

Every proper closed G_δ subset of λ_1 is CRS if and only if $A = C(X)$.

5.5.7.3 Other related results may be found in Sidney and Stout [52] and Natzitz [47].

5.6.1 With a few exceptions (as pointed out below), the results and proofs of the 2.5.1 series are mainly our own. 2.5.1.13 is, of course, well known.

5.6.2 Part of result 2.5.1.7 can be associated with Thm.2.3 of Hayashi [32] (so, incidentally, can 2.2.10). There are a number of similarities between Hayashi's paper and this thesis, some of which we shall later point out.

5.6.3 Some of the results of Bear [7] and the more general result, e.g. in [41] Ex 4.5 (1) can be seen to be an easy consequence of 2.5.1.27 and 2.2.6.

5.6.4 The result 2.5.1.20 is given in Oberlin [49] in the form:

> Any interpolation set of type 1 intersects each Gleason part at most once.

Oberlin proves a converse result in the case of countable closed sets. We can obtain a much simpler proof to Oberlin's converse result in the following way: Let F be a countable closed set with at most one point in each Gleason part. We then use 3.2.3; 3.2.10 and 2.5.1.10 to say : For every $\mu \in (b\ A^{\perp})^e$, either $|\mu|(F) = 0$ or $\mu_F = c\delta_x$ for some $x \in F$ and $c \in \mathbb{C}$. We then use 2.5.1.18 and 2.4.1.2(d) to obtain the result.

5.6.5 Some versions of 2.5.1.37 and 2.5.1.39 are probably known.

5.7 The work of the 2.5.2 series is mainly our own, reference being made, of course, to Davis [17]. The result 2.2.8 is fairly crucial here. Related to this area is a conjecture by Bishop (see the end of Gamelin's paper [26]) known as the peak point conjecture. A counter-example was given to this some time ago by B. Cole (see appendix to [41]).

5.8.1 Probably the most significant result of subsection 2.6 is that of 2.6.2. The only place that this appears in the literature, to our knowledge, is in Hayashi [32]. It does seem a little surprising that this was not noticed before since it might be obtained, for instance, from Thm. 2.17 in Suciu and 7.2; Thm. 11 of Leibowitz. Indeed, Detraz, even earlier, came very close to this result in [20]. Our proof of this result has some common points with the proof of Hayashi and the reader will have noticed considerable similarity between the approach of Hayashi's paper and that of this work, which both rely largely on measure theoretic techniques. By our proofs, comments and results of 2.6.3 and 2.6.4 and the extensive use of 2.6.2 in the remainder of this subsection for generalizing some of the earlier results, we feel that we have placed it in a better perspective.

5.8.2 Of the remaining results, 2.6.9.1 and 2.6.9.2 are our own. 2.6.9.4 and 2.6.9.5 are well known (see e.g. Nishizawa [48] or Leibowitz), as are forms of 2.6.11. (see e.g. Leibowitz or, earlier, a result of R.S. Pierce and some associated results in Bear [9]). The proofs given for these are our own. The results 2.6.13 to 2.6.19, mainly strengthening earlier results on Gleason parts are our own.

5.9.1 The notation of 3.1.1 is in general use, except for the "core" notation which is our own. The results of 3.1.2 are simple and well known with the exception, perhaps, of the approximation conditions 3.1.2.7 to 3.1.2.9.

5.9.2 The results and proofs of the 3.2 series are our own except for 3.2.9 which can be found in Hayashi and 3.2.10 which can be found in Glicksberg [30]. The proof of 3.2.8 is based on a proof in Glicksberg [28] which in turn is based on an idea of De Branges [18].

5.10 The idea of a normal measure is adapted from the definition given in Bade [3]. It must be confessed that the author, for a short while, laboured under the illusion that : μ is normal $\Rightarrow$ μ is clopen (in the sense of 3.3 and 3.4). This may or may not be true, but he did not realize at first that Bade was working in the context of Stonian spaces. His disillusionment, however, served as the spur to formulate the notion of clopen measures which is a fruitful one, having, as it turns out, a natural setting in pervasive function algebras. With the exception of one or two initial results, such as 3.3.5, which are taken from [3], the work of sections 3.3 and 3.4 is our own.

5.11.1 The comments and results of 4.1.1, 4.1.2, 4.2.1, 4.2.2 and 4.5.8 are probably well known. Result 4.4.2 appears in Leibowitz, but the proof given is our own. 4.3.3 however, was suggested to us by Professor Kotzé, as a refinement of 4.3.2.

5.11.2 We notice that a result of Batikjan [6], which is as follows :

Let A be an essential maximal function algebra and $h \in C(X)$;

If there exist $m_0 \in \mathbb{N}$ and $f \in A \sim \{0\}$ such that $fh^m \in A$ for all $m \geqslant m_0$, then $h \in A$.

is obtainable from 4.5.5 (specifically condition (a_{10})) and 4.1.3. We must mention that we have not actually seen Batikjan's paper, but have only read the review.

5.11.3 As stated, the implications given in 4.6 to 4.6.5 are well known. (see e.g. Hoffman and Singer [35]). The proofs given, however, are our own.

5.11.4 In the last subsection, results 4.6.7 to 4.6.9 are our own. Worth mentioning at this point is a paper by Ellis [21]. It is felt that the concept of weakly prime sets introduced there, apart from having similarities to weakly analytic sets, warrant further study for the following reason :

> If A is an analytic function algebra on X , then X is weakly prime.

This follows easily from 4.3.6.

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