SPHERICALLY SYMMETRIC OBSERVATIONAL COSMOLOGY

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Contents

Preface to Part I

This thesis forms part of a comprehensive series of papers aimed at characterizing in detail the way in which cosmological observations can be used to directly determine the geometry of cosmological space-time, and critically examine the limits of observational cosmology. The program was initiated by G.F.R. Ellis and co-workers and is subdivided into four major areas:

(i) paper I deals with cosmographic observations (Ellis et al. 1985, Maartens 1980); i.e. one proceeds without the assumption of particular dynamical laws that determine the very large-scale structure of space-time;

(ii) ideal cosmological observations (Mel et al. 1985, Nel 1980) where one examines the large-scale structure of space-time assuming that Einstein's field equations govern this structure.

(iii) paper II considers observational cosmology in view of realistic limitations on possible observations (Maartens et al. 1985).

(iv) paper III investigates "nearly Robertson-Walker space-times" (Stoeger et al. 1985).

This thesis is an extension of the paper by Stoeger and Ellis (1984) which forms the basis for the fourth current phase of the program. Stoeger and Ellis examine spherically symmetric cosmologies from an observational viewpoint. They determine optimal observationally based coordinates and a related tetrad, and use these to establish the field equations. They then proceed to examine integration of the exact equations, in the spatially homogeneous case, for the curvature case $K = +1$. Further, a function $W(r)$ - representing the total energy density- is found for the general spherically symmetric integration, following the procedure employed by Bondi (1947).

Aim

This thesis aims at :

(i) extending the integration of the exact equations for the spatially homogeneous case, to all three curvature cases, viz.

$$K = (+1, 0, -1).$$

(ii) investigating the possibility of determining the observable quantities for the general curvature case in terms of a generating function, and the implications of the existance of such a function in FLRW universes.

(iii) examination of the function W(r) in the context of the optimal coordinate choice utilized here; establishing the conditions for its validity in FLRW universes and its dependence on the generating function found in (ii).

Approach and Motivation

It is assumed that the matter content of the universe is a pressure-free perfect fluid ("dust"). The field equations have been previously integrated (see e.g. Tolman 1934, Bondi 1947, Ellis 1967). The essential new feature of the approach is its strong observational base: we choose coordinates and a tetrad that relate as closely as possible to observations made on the past light cone of a central observer, as well as to the fluid. Thus the equations obtained are more difficult to examine and integrate than when purely fluid based coordinates are used. However, for a possible direct confrontation of the space-time geometry with possible astronomical observations made by central observer, the difficulties encountered are believed inherent and therefore essential. The thesis therefore extends the frontiers of observational cosmology in pursuing the limits to verification in cosmology (Ellis 1980a), and forms the basis for further papers examining axially symmetric and general perturbations about FLRW universes.

The idea of maintaining cosmology as a directly observationally based subject stems from the paper by Kristian and Sachs (1966) who examined in detail local space-time geometry and its related field equations ; and Ellis (1975, 1979a,b, 1984a,b; et al. 1985 and references there in). This series strives toward an integrated observationally based cosmological framework which may eventually lead to an overall cosmological theory (see Ellis 1984a).

Acknowledgements

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I Introduction to Part I

1.1 Outline

• The first chapter summarizes the salient features within the framework of this thesis. It then continues to review current understanding and motivation of spatially homogeneous FLRW cosmological models.

Chapter two introduces the fundamental assumptions and notation employed, and the kinematics of perfect fluid. It further introduces observational coordinates and observational quantities in spherically symmetric space times with specific application to FLRW universes.

In chapter three fluid ray-tetrad formalism is reviewed and related to observational coordinates. The metric is then investigated with central conditions as constraints. Observational coordinates are then fashioned.

Chapter four applies the commutator relations and rotation coefficients, and discusses the geometrical interpretation of the non-zero commutation coefficients.

The fifth chapter determines the Jacobi identities, field equations and contracted Bianchi identities for a spherically symmetric perfect fluid. It then, following the introduction of naturally occurring quantities, reconstructs the equations to be expressed as a set of radial and time derivative equations.

Chapter six introduces the conditions for spatial homogeneity and investigates the possibilities arising for isotropic fluid expansion. The consequences of spatial homogeneity and isotropy are then structured into the equations, thereby requiring spherically symmetric dust-filled space-time to be a FLRW universe. The field equations are integrated in a coordinate free manner and then investigated in the light of possible coordinate choices. Once the observational coordinate choice is made the observable quantities are determined for all three possible curvature cases.

In the seventh chapter the dust field equations on the past light cone for

FLRW initial data are explicitly integrated in terms of observational data. Integration along and off the light cone is carried out, incorporating the optimal coordinate choice, again for k={+1,0,-1}. Further, the general curvature case is investigated.

Finally, chapter- eight carries out the spherically symmetric integration in terms of a function W(r) adapted to observational coordinates. The geometrical interpretation of this function and the conditions for its validity in FLRW universes are investigated.

1.2 Friedmann LeMaitre Robertson Walker Models

We follow, in part, Hawking and Ellis (1973) in reviewing FLRW universe models. We are interested in finding a suitable representation of the observable universe in a general relativistic frame work. The Copernican principle (Bondi 1960) sets our ideology in maintaining that our position in the universe is not a specially distinguished one. When viewed on a suitable scale, this principle implies approximate spatial homogeneity, by which we mean that there exist spacelike 3-surfaces, any point of which is equivalent to any other point on the same surface. By a suitable scale we mean a scale large enough to exclude local irregularities such as stars and galaxies (i.e. a smooth, moving substratum; e.g. Rowan-Robinson 1981, Ellis and Sciama 1972).

Homogeneity however, is difficult to test by observation, as the separation between the fundamental observer (comoving with the fluid) and distant objects is not a simple measurable quantity. One is faced within our past light cone, even at relatively small redshift values, with evolution effects, the "lumpy" nature of the real universe and so on; at substantial distances source evolution must be taken into account (Ellis 1979a, 1980a, et al. 1978). This restriction may be eased, in principle, by observing isotropies (i.e. seeing if universe looks the same in different directions).

Isotropy and observations

Observational investigations of isotropy conducted so far support the conclusion that the universe is approximatelly spherically symmetric about us. Evidence for large scale isotropy arises either from matter or from radiation isotropy (to within ~3% and ~0.03% respectively) and is obtained via :

(a) distribution of galaxies and their apparent magnitudes and redshifts (see e.g., Hubble (1934b,1936), Sandage 1972a, Sandage et al. 1972).

(b) extragalactic radio sources (at epochs much later than .n1; see e.g. Holden 1966, Hughes et al 1967, Ellis & Baldwin 1984).

(c) cosmic microwave background radiation on very large scale (nI500),(Penzias & Wilson 1965, Boughn et al. 1971) which places severe limits on any an isotropic models of the universe (for review see Peebles 1971, Sciama 1971).

(d) x-ray background in the 2-20 keV band (in direction away from galactic plane, to within 1%; see e.g., Warwick et al. 1980).

In the case in which the universe is isotropic about every spacetime point, the Copernican principle may be interpreted as stating that the universe is approximately spherically symmetric about every point. Cosmological models describing such universes were originally derived by Friedmann (1922), LeMaitre (1927), Robertson (1929) and Walker (1935), and are called FLRW universe models. Walker (1944) showed that exact spherical symmetry about every point implies spatial homogeneity of the universe, whose surfaces are spacelike 3-surfaces of constant curvature. We conclude therefore, that FLRW models are a good approximation to the large scale geometry of space-time in the observable region.

The Isotropy Dispute

The question remains as to why the universe should be isotropic. Mach's principle which relates the local inertial frame to the large scale distribution of matter in the universe and forbids the arbitrary rotation between these, prescribes it to be a consequence of the initial conditions of the universe,(see e.g., Brans et al. 1961). However, as shown by Bode] (1952), such a rotation is permissible in general relativity.

In Misner's chaotic cosmology (Misner 1968, 1969) the universe started off highly anisotropic and inhomogeneous but somehow evolved into isotropy and homogeneity on the large scale. Hawking and Collins (1973a,b) however, argue in favour of an isotropic universe, claiming that we would not be here were the universe not isotropic, since galaxies would not be able to form in an anisotropic universe. Thus the universe is isotropic because we are here (also see Carter 1974).

McCrea (1968) promotes the idea of degradation of information, reaching us from remote parts of the universe, by the redshift, thus introducing a principle of uncertainty. i.e. our ability to observe homogeneity and isotropy deteriorates with increasing distances.

If the idea of hierarchical ordering of clusters and superclusters of galaxies is correct, it is possible that no averaging scale exists. This idea is demoted by Zwicky et al. (1966) who maintain a smooth universe existance on a scale larger than the average distance between clusters; it is claimed true by Abell (1967) and de Vaucouleurs (1970), and by Kiang (1961) (clusters occur on all scales). The latter is recently supported by work on the galaxy covariance function by Peebles (1980) which suggests the formation of galaxies before clusters followed by aggregation into clusters by mutual gravitational action.

There emerges therefore the questionable definition of a mean space density (for a review see de Vaucouleurs 1970) on a cosmological scale. Further, no clear criteria for isotropy have been established (Ellis 1980a). In particular, CMWB radiation isotropy does not imply spherical symmetry about the central observer (Ellis et al. 1978).

Alternatives to Spatial Homogeneity

Different approaches to spatial homogeneity are:

(i) the universe is spherically symmetric but inhomogeneous,

(ii) the universe is completely chaotic and infinite; life evolves only where local expansion, homogeneity and isotropy exist (Anthropic principle),

(iii) the universe is small, finite and expanding. Apparent homogeneity and isotropy are guaranteed in such a finite universe. This class includes, through identification, FRW universe models of compact space sections (for general discussion see Ellis 1779a).

(iv) the universe is spherically symmetric and static, with two centres, our **galaxy** being near one of the centres (Ellis et al. 1978).

Observational conditions for FRW universes

However, for the purposes of this thesis, on the basis of current observational evidence for isotropy discussed above, we shall treat the large scale structure of the universe as spherically symmetric about every point. In terms of realistic observations, what one therefore requires for spherically symmetric space-time and matter distributions are the conditions:

(i) background radiation must be isotropic;

(ii) (magnitude,redshift and (number count, redshift) relations must be isotropic. Then the area distance r and energy density V are functions of the redshift z only: $r=r(z)$, $V=V(z)$; in addition,

(iii) no galactic proper motions must occur <=> u = u = 0;

(iv) no observational distortions take place. Hence image distortion resulting from ambiguous apparent shapes of observed objects will not be considered.

In practice, conditions (i) and (ii) can be checked (as discussed above). Condition (iii) cannot be checked, because on a cosmological scale no proper motions can be detected. Condition (iv) cannot be checked either, since no 'standard shapes' are known (Ellis 1980b).

We note, in conclusion, present upper limits on the shear σ, and the rotation ω, derived by direct observations:

$$\sigma_o \lesssim 1/4\ \theta_o\ ,\quad \omega_o \lesssim 1/3\ \theta_o$$

and by direct limits due to CMWB

$$|\sigma_o| \lesssim 10^{-3}\theta_o\ ,\ |\omega_o| \lesssim 10^{-3}\ \theta_o$$

(for a recent review see Tolman et al. 1984).

2 Geometry and Physical Fields

2.1 Fundamental Assumptions

Spacetime is a manifold (M,g), where M is a connected 4-dimensional Hausdorff C^0 manifold, g is a Lorentz metric of signature 2 on M. The covariant derivative of the metric is $g_{ab;c}=0$ (see Hawking and Ellis 1973, Ehlers 1973).

We assume the Copernican principle to hold (Bondi 1960):"We do not occupy a privileged position in space-time". This implies firstly that local physical laws, which have been experimentally determined (and govern the physical fields present in spacetime), may be extrapolated to hold on a large scale. Secondly, considering local astronomical observations, our view of the universe is not a preferred picture. This permits the assumption that the ordering of matter into stars, star clusters, galaxies, clusters of galaxies and interstellar matter, as determined by local astronomical observations, also holds at large distances, which leads to the usual "fluid approximation".

The fields on M which describe the matter energy content of spacetime obey tensor equations; the fields may be represented by a symmetric total energy-momentum tensor T, which satisfies on M the conservation equations:

$$T^{ab}{}_{;b}=0 \tag{2.1}$$

(see Hawkino and Ellis 1973, Ehlers 1973).

The gravitational field is represented by the spacetime metric itself. The curvature is dynamically related to the matter-energy content by the non-linear Einstein field equations (for vanishing cosmological constant).

$$G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab} = T_{ab} \tag{2.2}$$

where R_{ab} is the ricci tensor of g, and $R=R^a{}_a$ is the Ricci scalar. Then the contracted Bianchi identities $G^{ab}{}_{;b}=0$ show that (2.1) are the integrability conditions of (2.2).

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We assume that galaxies may be regarded as particles of a cosmological fluid and clusters of galaxies form the dominant component of the matter-energy content of spacetime the average motion of the galaxies is represented by a smooth timelike (normalized) vector field **u** (the 4-velocity)

$$u^a u_a = -1 \tag{2.3}$$

and the energy-momentum tensor takes a perfect fluid form:

$$T_{ab} = (\mu + p) u_a u_b + p g_{ab} \tag{2.4}$$

(Ellis 1971. Weinberg 1972) where V is the energy density and p the isotropic pressure of the galactic -fluid (a perfect fluid is characterized by negligible heat conduction and viscosity). For pressureless energy-momentum tensor, the matter content of the universe is called dust.

We assume that the universe is so large that we are unable to move off our local galactic world line (but cf. small universe proposal, e.g. Gott 1980, Ellis et al. 1979). i.e. we view the universe from effectively a single space-time point- "here and now" (Ellis 1975).

Electromagnetic signals received along an observer world-line from distant galactic world-lines propagate along null geodesics (rays; see sec. 2.6, geometric optics approximation) and are directly observable (e.g. radio to x-ray observations; Ehlers 1971).

Other observations (element abundance, age of clusters etc.) are considered as indirect observations not restricting the detailed spacetime structure but restrain the integral cosmological model (for a comprehensive review see Ellis et al. 1985).

2.2 Notation

The metric tensor g_{ab} has signature (- + + +). Covariant differentiation in the X^i direction is $\nabla_X = {}_{;k}X^k$; partial differentiation in the X^i direction is ${}_{,k}X^k$. A vector is regarded as a directional derivative.

The commutator of the vectors X,Y is [X,Y] defined by $[X,Y]f \equiv X(Yf) - Y(Xf)$, i.e. $[X,Y]f = (Y^j{}_{,i}X^i - X^j{}_{,i}Y^i) \times (\partial f/\partial x^i) = (L_X Y)f$, where $L_X Y$ is the Lie derivative of Y with respect to X ; this is the vector field yielding the difference between the vector field Y, and the vector field produced if Y is "dragged along" by X (see e.g., Carmeli 1977). This gives the commutator a useful geometric interpretation closely related to integrability conditions.

A set of vectors $\{p_a\}$ (labelled by the index a) that are orthonormal at each point is called a tetrad. The notation a_a is used to emphasize the action of these vectors as directional derivatives : $8_a f=e_a(f)$. i,j,k.., are coordinate indices, and a,b,c... are tetrad indices. Coordinate indices and tetrad indices run from 0 to 3; '0' denotes timelike, so 1,2.3 denote spacelike. Greek indices run over 1,2,3.

Round brackets denote symmetrized indices, and square brackets denote skew-symmetrized indices (see Schouten 1954).

The Ricci tensor is defined by $R_{ab}\equiv R^c{}_{acb}$, the curvature scalar by $R=R^a{}_a$. The metric scalar product is denoted by a dot $X\cdot Y=X^iY_i=g_{ij}X^iY^j$ (see Ellis 1967). The rate of change of any tensor X , as measured by an observer with 4-velocity u , is $X^{\cdot}=\nabla_u X$. The projection of a tensor $S_{abc...}$ is denoted by $(S_{abc...})_\perp$. Finally, the skew symmetric tensor η is defined by

$$\eta^{abcd}=4!\; e_0{}^{[a} e_1{}^{b} e_2{}^{c} e_3{}^{d]} \;,\quad \eta^{abcd}=\eta^{[abcd]} \;;$$

has tetrad components $\eta^{0123}=1=-\eta_{0123}$.

2.3 Kinematics of a Timelike Congruence and Perfect Fluids

We follow Ellis (1971,1973) in summarizing results.

2.3.1 Projection Tensor

For a space time filled with dust, the fluid flow vector u^a determines the tensor

$$h_{ab} = g_{ab} + u_a u_b \qquad (2.5)$$

which, at each point, projects into the instantaneous rest space of an observer moving with 4-velocity u^a. This tensor obeys

$$h_a{}^b h_b{}^c = h_a{}^c \ ; \ h_b{}^a u^b = 0 \ ; \ h_a{}^a = 3$$

2.3.2 Kinematic Quantities

The tensors θ_{ab}, ω_{ab}, $\dot{u}_a$ and scalars θ, σ, ω are defined by

$$\dot{u}_a = u_{a;b}u^b$$

$$u_{a;b} = \omega_{ab} + \theta_{ab} - \dot{u}_a u_b \qquad (2.6b)$$

where

$$\omega_{ab} = \omega_{[ab]} \ , \ \omega_{ab}u^b = 0 \ . \ \theta_{ab} = \theta_{(ab)} \ , \ \theta_{ab}u^b = 0 \ ; \qquad (2.6c)$$

$$\sigma_{ab} = \omega_{ab} - 1/3\,\theta h_{ab} \ , \ \theta = u^a{}_{;a} \ , \ \sigma^2 = \tfrac{1}{2}\sigma_{ab}\sigma^{ab} \ , \ \omega^2 = \tfrac{1}{2}\omega_{ab}\omega^{ab} \ .$$

W_{ab} is the expansion tensor

W_{ab} is the skew vorticity tensor

D_{ab} is the symmetric, trace-free skew tensor

u'_a is the acceleration

$ is the expansion scalar

0 is the shear scalar

The vorticity vector $_w{}^a$ is defined by

$$\omega^a = \tfrac{1}{2}\eta^{abcd}u_b\omega_{cd} \ , \ \omega_{tu} = \eta_{tuas}\omega^a u^s \ ; \qquad (2.6d)$$

the vector W^a is in the rest space of u^a which defines the instantaneous axis of rotation of the fluid due to vorticity. The kinematic quantities govern to first order the evolution of a small sphere of fluid. From (2.3), (2.6a) => $u'u^a = 0$. Also, the rate of change of volume gV enclosed by a Group of neighbouring particles is

$$(\delta V)^\cdot/\delta V = \theta \qquad (2.7)$$

We note that the scalars 0, W (2.6c) vanish iff the corresponding tensors vanish.

BY projecting the covariant derivative Vu and decomposing, one obtains in terms of the Quantities defined

$$u_{c;d}h_a{}^c h_b{}^d = \omega_{ab}+\sigma_{ab}+{}^1\!/\!_3\,\theta h_{ab}-\dot{u}_a u_b \qquad (2.8a)$$

hence

$$u_{[a;b]} = \omega_{ab} \qquad (2.8b)$$

For the observer on one fluid particle observing a neighbouring fluid particle at distance δl in direction e_a ($e_a e^a = 1$, $e_a u^a = 0$), the rate of change of distance and direction of neighbouring particle relative to the observer are given, respectively, by

$$(\delta l)^\cdot/\delta l = \theta_{ab}e^a e^b + {}^1\!/\!_3\,\theta \qquad (2.9)$$

$$h_a{}^b(e_b)^\cdot = (\omega^b{}_a+\theta^b{}_a-(\sigma_{cd}e^c e^d)h_a{}^b)e_b \qquad (2.10)$$

2.3.3 Conservation Equations

The contracted Bianchi identities (2.1) for a perfect fluid are

$$\dot{\mu}+(\mu+p)\theta=0 \ , \ (\mu+p)\dot{u}^c+h^{cb}p_{;b}=0 \qquad (2.11)$$

which are respectively the energy and momentum conservation equation.

For dust, (1.9) reduce to

$$\dot{\mu}+\mu\theta=0 \ , \ \dot{u}_a=0 \qquad (2.12)$$

i.e. the world lines of dust are geodesic, and the mass of any particle of the fluid is conserved.

2.3.4 Hubble Constant and Deceleration Parameter

The Hubble constant may be defined in terms of the isotropic volume expansion 0, and a representative length 1 (using (2.7)), at any time t, by

$$H = l^{\cdot}/l = {}^{1}/{}_{3}\theta,\qquad (2.13a)$$

and the deceleration parameter by

$$q = -(l^{\cdot\cdot}/l)(1/H^{2}).\qquad (2.13b)$$

i.e. H is the direction averaged rate-of change of distance of clusters of galaxies as a function of proper time (units: time) ; a represents the rate of expansion (dimensionless).

2.3.5 Equations of State

To include the physics in the current context, one must specify the properties of T_{ab} which, in the case of perfect fluid, reduce to equations restricting p and 1¹. In addition to the general restrictions

$$\mu+p > 0\ ,\quad \mu+3p > 0\qquad (2.14a)$$

(i.e. the relativistic inertial mass density of the fluid, and the energy density of matter are positive) one imposes stability of fluid against local mechanical instability (lower limit) and an upper limit on the screed of sound v_{s}, not increasing the speed of light c :

$$v_{s}/c = dp/d\mu \Rightarrow 0 \leq dp/d\mu \leq 1\qquad (2.14b)$$

(see Israel 1960, Curtis 1950). The idealized equation of state is

$$p = (\gamma-1)\mu \Leftrightarrow p = p(\mu)\qquad (2.14c)$$

which, together with (1.4b) may be combined to show

$$0 \leq (\gamma-1) \leq 1\ ,\ 1 \leq \gamma \leq 2.\qquad (2.14d)$$

Equations (2.14a) ensure that the conservation equations (2.11) have physical solutions, and that the acceleration is always away from a high pressure region towards a neighbouring low-pressure region; and that compression of fluid increases its energy density. From eqs (2.14c) one can deduce that there is only one independent thermodynamic variable along each world line, here chosen to be A. Thus, statement (3.17) of Ellis (1971) holds:

" The change of thermodynamic state along the flow lines of a perfect fluid is determined (for a given equation of state) by the average length l alone".

2.3.6 Field Equations

From (2.2) Einstein's field equations may be written as

$$R_{ab} = T_{ab} - \tfrac{1}{2} T g_{ab} \tag{2.15}$$

These equations determine the trace part of the gravitational field algebraically at each point of space-time from the matter content at each point; the 10 field equations for a perfect fluid are then

$$R_{ab} u^a u^b = \tfrac{1}{2}(\mu + 3p) \quad \text{, 1 field equation} \tag{2.16a}$$

$$R_{ab} u^a h^b{}_c = 0 \quad \text{, 3 field equations} \tag{2.16b}$$

$$R_{ab} h^a{}_c h^b{}_d = \tfrac{1}{2}(\mu - p) h_{cd} \quad \text{, 6 field equations} \tag{2.16c}$$

The curvature tensor is defined by

$$u_{a;dc} - u_{a;cd} \equiv u^b R_{badc} \; ; \tag{2.17}$$

multiplying by u^c, taking the trace and using (2.16b) yields the propagation equation for θ,

$$\dot{\theta} + \tfrac{1}{3}\theta^2 + 2(\sigma^2 - \omega^2) - \dot{u}^a{}_{;a} + \tfrac{1}{2}(\mu + 3p) = 0 \, , \tag{2.18a}$$

better known as the Raychaudhuri equation (see Raychaudhuri 1955, Ehlers 1961). Differentiation of (2.7) allows (1.18a) to be written in the form

$$3\ddot{l}/l = 2(\omega^2 - \sigma^2) + \dot{u}^a{}_{;a} - \tfrac{1}{2}(\mu + 3p) \, , \tag{2.18b}$$

showing the repulsive (positive) and attractive (negative) forces contributing to the determination of $\ddot{l}$.

Upon multiplication of (2.17) by $g^{ac} h^b{}_e$ and substitution from (2.16b) one obtains

$$h^a{}_b(\omega^{bc}{}_{;c} - \sigma^{bc}{}_{;c} + \tfrac{2}{3}\theta^{;b}) + (\omega^a{}_b + \sigma^a{}_b)\dot{u}^b = 0 \tag{2.18c}$$

The symmetric, trace free parts of (2.17) are (using (2.18a))

$$h_a{}^f h_b{}^g (\sigma_{fg})^{\cdot} - h_a{}^f h_b{}^g \dot{u}_{(f;g)} - \dot{u}_a \dot{u}_b + \omega_a \omega_b + \sigma_{af}\sigma^f{}_b + \tfrac{2}{3}\theta\sigma_{ab} +$$
$$+ h_{ab}(-\tfrac{1}{3}\omega^2 - \tfrac{2}{3}\sigma^2 + \tfrac{1}{3}\dot{u}^c{}_{;c}) + E_{ab} = 0 \tag{2.18d}$$

where $E_{ab} = C_{abcd} u^b u^d$, C_{abcd} is the Weyl tensor (trace free part of R_{abcd}) and the gravitational field E_{ab} induces shear in the fluid flow lines.

Equations (2.18a,c,d) are 9 of the 10 general relativity field equations; the remaining field equation is a consequence of one of the four first integrals which exist, when all the other condition are satisfied, as a consequence of the contracted Bianchi identities (2.11).

2.4 Friedmann LeMaitre Robertson Walker Models

These are the general relativistic models which are locally isotropic about every point of space-time <=> there is no preferred direction , which by (2.10), implies zero vorticity tensor; for isotropic expansion the generalized Hubble law (2.9) implies $\sigma_{ab} = 0$. Thus, no distortion of fluid flow lines arise, and no change of position in the sky of neighbouring clusters of galaxies with respect to a local inertial rest frame occurs. Equations (2.6d), (2.8b) therefore show that zero rotation $\Leftrightarrow u_{[a}u_{b;c]} = 0$. $\Rightarrow \exists$ locally functions such that $u_a = g t_{,a}$. i.e. the case $\omega = 0$ is the condition for local existance of 3-surfaces in space time (the surfaces t=constant) orthogonal to the 4-velocity vector fields. These simultaneous surfaces, for all the fluid observers, define a cosmic time coordinate (the function t) determined by the fluid flow. For $\dot{u}^a = 0$, the time coordinate can be locally normalized to measure proper time along each world line. The momentum conservation equation (2.11) implies $h_a{}^b p_{;b} = 0$; then the equation of state $p = p(\mu)$ implies $h_a{}^b \mu_{;b} = 0$,which is the condition for exact isotropy of number counts and radiation distribution. The surfaces (μ= constant) are therefore orthogonal to the fluid 4-velocity vector u_a (which in turn $\Rightarrow \omega_a = 0$). With these conditions (2.18c) shows $h_a{}^b \theta_{;b} = 0$. These models can therefore be characterized by the condition

$$\omega_a = \sigma_{ab} = 0 = \dot{u}_a \qquad (2.19a)$$

which implies the further conditions

$$\mu = \mu(t) \ , \ p = p(t) \ , \ \theta(t) \ ; \qquad (2.19b)$$

then l(t) is the radius function R(t) commonly used in describing the RW space-

Equations (2.18a,c,d) are 9 of the 10 general relativity field equations; the remaining field equation is a consequence of one of the four first integrals which exist, when all the other condition are satisfied, as a consequence of the contracted Bianchi identities (2.11).

2.4 Friedmann LeMaitre Robertson Walker Models

These are the general relativistic models which are locally isotropic about every point of space-time (=) there is no preferred direction , which by (2.10), implies zero vorticity tensor; for isotropic expansion the generalized Hubble law (2.9) implies $\sigma_{ab} = 0$. Thus, no distortion of fluid flow lines arise, and no change of position in the sky of neighbouring clusters of galaxies with respect to a local inertial rest frame occurs. Equations (2.6d), (2.8b) therefore show that zero rotation $\Leftrightarrow u_{[a}u_{b;c]} = 0. \Rightarrow \exists$ locally functions such that $u_a = gt_{,a}$. i.e. the case $\omega = 0$ is the condition for local existance of 3-surfaces in space time (the surfaces t=constant) orthogonal to the 4-velocity vector fields. These simultaneous surfaces, for all the fluid observers, define a cosmic time coordinate (the function t) determined by the fluid flow. For $\dot{u}^a = 0$, the time coordinate can be locally normalized to measure proper time along each world line. The momentum conservation equation (2.11) implies $h_a{}^b p_{;b} = 0$; then the equation of state $p = p(\mu)$ implies $h_a{}^b \mu_{;b} = 0$,which is the condition for exact isotropy of number counts and radiation distribution. The surfaces (μ= constant) are therefore orthogonal to the fluid 4-velocity vector u_a (which in turn $\Rightarrow \omega_a =$ 0). With these conditions (2.18c) shows $h_a{}^b \theta_{;b} = 0$. These models can therefore be characterized by the condition

$$\omega_a = \sigma_{ab} = 0 = \dot{u}_a \qquad (2.19a)$$

which implies the further conditions

$$\mu = \mu(t) \ , \ p = p(t) \ , \ \theta(t) \ ; \qquad (2.19b)$$

then l(t) is the radius function R(t) commonly used in describing the RW spacetimes.

Defining a spatial curvature tensor $R^3{}_{abcd}$ by a spatial vector v^a ($v^a u_a = 0$)

$$\nabla^3{}_c \nabla^3{}_d v_a - \nabla^3{}_d \nabla^3{}_c v_a = R^3{}_{abcd} v^b \qquad (2.19a)$$

one can now (for $\omega = 0$) obtain, using (1.8), the Gauss Codacci equation (Schouten 1954)

$$^3R_{abcd} = (R_{abcd})_\perp - \theta_{ca}\theta_{db} + \theta_{bc}\theta_{da} \qquad (2.19b)$$

Contracting, and using (1.5), (1.16c) one obtains the spatial Ricci tensor

$$^3R_{ab} = h_a{}^f h_b{}^g (u^\cdot{}_{(f;g)} - (l^3 \sigma_{fg})^\cdot / l^3) + u^\cdot{}_a u^\cdot{}_b +$$
$$+ {}^2/{}_3 h_{ab}(-{}^1/{}_3\theta^2 + \sigma^2 - {}^1\!/\!{}_2 u^{\cdot c}{}_{;c} + \mu) \qquad (2.19c)$$

Further contraction results in the Ricci scalar in 3-dimensions

$$^3R = 2(\sigma^2 - {}^1/{}_3\theta^2 + \mu) \qquad (2.19d)$$

The trace free part of (2.19c) now replaces (2.18d), and the trace (eq 1.19d) is the tenth field equation.

Equation (2.19c) may be re-expressed in differential geometric terms (Ehlers 1961): if, in a 3-space (g=const), the curvature of the 2-surface formed by all geodesics through the point x orthogonal to the direction e_a ($e^a e_a = 1$, $e^a u_a = 0$) at that point is $K^*(x,e^a)$, then (for RW universe models)

$$K^*(x,e^a) = {}^1/{}_3(\mu - {}^1/{}_3\theta^2) = K(t), \qquad (2.20a)$$

showing that the 3-surfaces orthogonal to u^a are isotropic at each point, and so are 3-spaces of constant curvature K(t).

The Raychaudhuri equation (2.18b) may now be integrated to obtain the Friedmann equation

$$-{}^1/{}_3\theta^2 + \mu = \text{const}/R^2, \qquad (2.20b)$$

for $l^\cdot \neq 0$. Comparing (2.19d) and (2.20a) yields $^3R = 6K^*$, which, using (2.20b), results in

$$^3R = 2(\mu - 3R^{\cdot 2}/R^2) = 6K^* = 6K/R^2, \qquad (2.20c)$$

where we can normalize k, such that k=(1, 0, -1), by rescaling R(t).

The remaining non trivial equations are

$\mu' + (\mu+p)\, 3R'/R = 0$ Conservation of energy eq (2.21a)

$3R'^2/R^2 - \mu = -3K/R^2$ Friedmann eq (2.21b)

$p = (\gamma-1)\mu$ State eq (2.21c)

2.5 Coordinates

It can be shown from (2.19), (2.20a) that there exist coordinates $[t, x^\nu]$ such that the metric takes the form (see Rindler 1969)

$$dS^2 = -dt^2 + h_{\mu\nu} \;,\; h_{\mu\nu} = R^2(t) f_{\mu\nu}(x^\sigma)\, dx^\mu dx^\nu \tag{2.22a}$$
$$= -dt^2 + R^2(t) d\sigma^2$$

where $d\sigma^2$ is the metric of 3-space of constant curvature, $K = (1,\ 0,\ -1)$ (see e.g. Heckmann and Schuking 1959). We note that the curvature of $h_{\mu\nu}$ is K^* ; curvature of $f_{\mu\nu}$ is K . Coordinates may be chosen to be comoving so that

$$dS^2 = dr^2 + f^2(r)(d\theta^2 + \sin^2\theta\, d\phi^2),$$

where
$$f(r) = \begin{cases} \sin r \\ r \\ \sinh r \end{cases} \quad \text{if } K = \begin{cases} +1 \\ 0 \\ -1 \end{cases} \tag{2.22c}$$

(see, e.g. Misner et al. 1973). The coordinate r runs from 0 to ∞ if $K = 0$ or -1, but runs from 0 to 2π if $K=+1$. When $K = 0$ or -1, the 3-spaces are diffeomorphic to R^3 and so are 'infinite', but when $K = 1$, they are diffeomorphic to a 3-sphere S^3 and so are compact (closed or 'finite' ; we shall not make topological identifications for $K \leq 0$ (Heckmann et al. 1962) which spoils the global isotropy of the model, but retains its homogeneity and local isotropy).

The metric $d\sigma^2$ may be written in alternative forms on choosing different coordinates (e.g. Rindler 1969).

2.6 Observations in FLRW Models

2.6.1 Geometric Option Approximation

Electromagnetic signals received along an observer world-line from distant galactic world lines propagate along null geodesics (rays; the curves whose tangent vector field is k^a). This follows from the geometric optic approximation to the solution of Maxwell's field equations. Hence, we find

$$k^a k_a = 0 \ , \Rightarrow k^a k_{a;b} = 0 \tag{2.23}$$

2.6.2 Redshifts

An observer moving with 4-velocity u^a will receive signals, emitted at proper time interval $d\tau_1$ at source, in a proper time interval $d\tau_2$ at observer. This time dilatation effect is then $d\tau_2/d\tau_1 = (k_a u^a)_1/(k_b u^b)_2$. In particular, observed frequencies ν of light or radio waves are related by $\nu_1/\nu_2 = (k_a u^a)_1/(k_b u^b)_2$.

The redshift z of a source as measured by an observer is defined in terms of wavelength by

$$z = (\lambda_{obs} - \lambda_{emit})/\lambda_{emit} = \Delta\lambda/\lambda_{emit} \tag{2.24a}$$

Hence, we find

$$1+z = \lambda_{obs}/\lambda_{emit} = \nu_{emit}/\nu_{obs}$$

and therefore

$$1+z = (u^a k_a)_{emit}/(u^b k_b)_{obs} \tag{2.24b}$$

which determines the redshift from the 4-velocity vectors $u^a|_{obs}$, $u^a|_{emit}$ and from the tangent vector k^a to the null geodesic. We further note, that upon using (2.8a), (2.9), (2.23), the change in redshift along the null geodesic may be written as (Ehlers 1961)

$$d\lambda/\lambda = (dl)^{\cdot} + (\dot{u}^a n_a)\ dl \tag{2.24c}$$

where n^a is the unit vector in the direction of the projection of k^a into the rest space of the observer; the redshift is thus split into a radial Doppler part (the first term) and a gravitational part (the second term), the latter vanishing for FLRW universe models.

In a FLRW model, whose metric is (2.22), since by (2.24c) the redshift is radial in character, it may be shown that

$$1 + z = R_{obs}/R_{emit} = d\tau_{obs}/d\tau_{emit} \; ; \qquad (2.24d)$$

i.e. the redshift is a measure of the expansion that has taken place (not necessarily a uniform expansion).

2.6.3 Area Distance

Consider a bundle of null geodesics diverging from a radiation source, with cross sectional area dS perpendicular to the propagation vector k at a point. Define a galaxy area distance r_G of the source from the galaxy by

$$dS_G = r_G^2 d\Omega_G \; , \qquad (2.25a)$$

where $d\Omega$ is the solid angle subtended by a bundle of null geodesics diverging from the point considered. Since $d\Omega_G$ is unmeasurable, r_G is not a measurable quantity. We therefore define an observable (in principle) quantity, the observer area distance r_o (also called "corrected luminosity distance" in Kristian and Sachs (1966)), by

$$dS_o = r_o^2 \; d\Omega_o \; , \qquad (2.25b)$$

which we can find, if the solid angle subtended by some object is measurable, and whose cross sectional area can be found from astrophysical considerations. The geometric result known as the Reciprocity Theorem relates the area distances r_o and r_G by

$$r_G^2 = r_o^2 \; (1+z)^2 \qquad (2.25c)$$

In a RW universe, by metric (2.22) it can be shown that

$$dS_G = R^2(t_o) f^2(r) \; d\Omega_G \qquad (2.26a)$$

where t_o is the time of observation, and $d\Omega_G = \sin\theta \; d\theta \; d\phi$. Hence, by (2.25a), (2.26a)

$$r_G^2 = R^2(t_o) f^2(r) \qquad (2.26b)$$

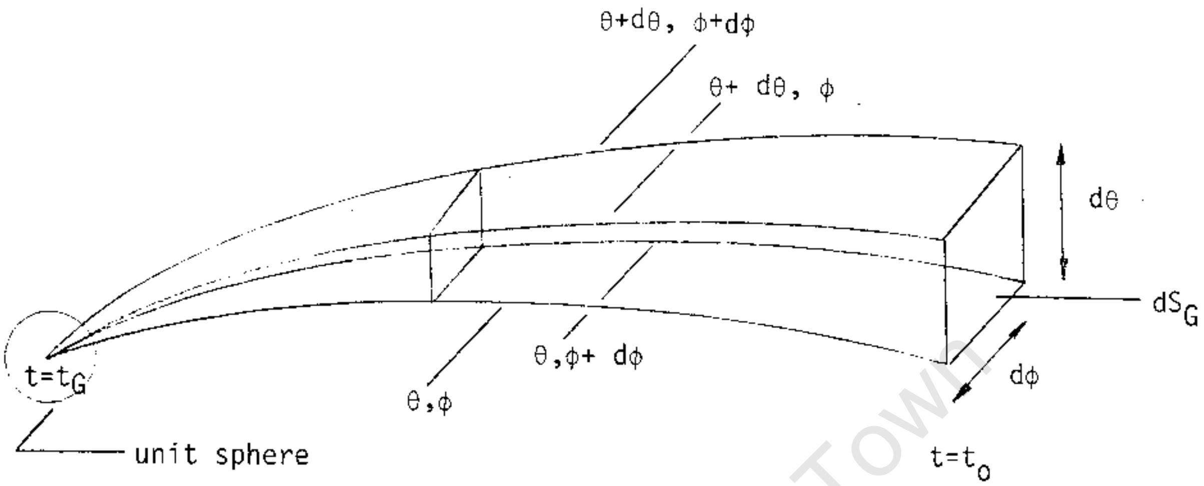

Figure 1. Galaxy area distance for a Robertson Walker Universe

Hence, from (2.25c), (2.26b), using (2.24d) one finds

$$r_o = R(t_o)/(1+z)\ f(r) \qquad (2.27a)$$

Again, using the metric (2.22) along a null geodesic ($dS^2 = 0 = d\theta = d\phi$) one obtains

$$r = \int_{R_e}^{R_o} dR/(RR^{\cdot}) \qquad (2.27b)$$

where r 's +he coordinate distance between the galaxy and the observer and 0. Thence we can substitute from the Friedmann ec to obtain c: r_0

We evaluate $r_o(Z)$, when pressure free matter is the dominant energy component in the universe (as is believed to be at recent times; but see part II) in the subsection.

2.6.4 **Present Time Evaluation of Observable Parameters, for Dust**

The non-trlvial equations together with the Hubble constant, H, and deceleration parameter G_o, may be evaluated at the present time, to yield the observable area distance r_o, in terms of observable parameters (in principle). The procedure is outlined below.

The conservation eq (2.21a) for dust (p= 0) results in

$$\mu = M/R^3 = \mu_o R_o^{\,3}/R^3 \qquad (2.28a)$$

Using eqs (2.13) yields

$$\mu_o = 6H_o^{\,2}\, q_o \;; \qquad (2.28b)$$

hence, substituting into the Friedmann eq (2.21b), one obtains

$$k/R_o^{\,2} = H_o^{\,2}\ (2q_o - 1) \qquad (2.28c)$$

i.e.

$$k \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} \iff q_o \begin{cases} > \frac{1}{2} \\ = \frac{1}{2} \\ < \frac{1}{2} \end{cases}$$

Eq (2.21b) gives an expression for the rate of change of R(t) ,

$$R^{\cdot} = R\ (\mu/3 - k/R^2) \qquad (2.28d)$$

Eq (2.27a) therefore, upon integration of (2.27b), using eqs (2.28) yields

$$r_o = (\ H_o q_o^{\,2} (1+z)^2\ )^{-1}\ \{\ q_o z + (q_o - 1)((1+2q_o z)^{\frac{1}{2}} - 1)\ \} \qquad (2.29a)$$

when $q_o \neq 0$, and

$$r_o = (2H_o)^{-1}\ \{\ 1 - (1+z)^{-2} \} \qquad (2.29a)$$

when $q_o = 0$ (see Mattig 1958).

2.6.5 Number Counts

Let an observer on a central world line C count a number dN of sources between redshifts z and z + dz in a solid angle dO , where z corresponds to the radial position r. Let F be the selection function, that is, completeness of the count (i.e. the fraction of sources in the volume that are counted is F). The radial proper distance measured by a comoving observer at the source is dl, while the cross-sectional area dS at the source of light rays spanning a solid angle d(at the central observer is given by eq (2.25b). Let the time coordinate be w. Hence

$$dN = F\ n(w_o, r)\ r_o^{\ 2}\ d\Omega\ dl/dz\ dz \qquad (2.30a)$$

where n(w,r) is the number density of sources at radial distance (w,r). Then, for M, the relativistic energy density per source counted, the total energy density $\mu(w_o, r)$ at (w_o, r) is

$$\mu = Mn = M_o(z)\ (dz/dl) \qquad (2.30b)$$

where

$$M_o(z) = (M/F)\ (1/d\Omega)\ (1/r_o^{\ 2})\ (dN/dz) \qquad (2.30d)$$

is in principle observable (see for expanded discussion subsection 3.5).

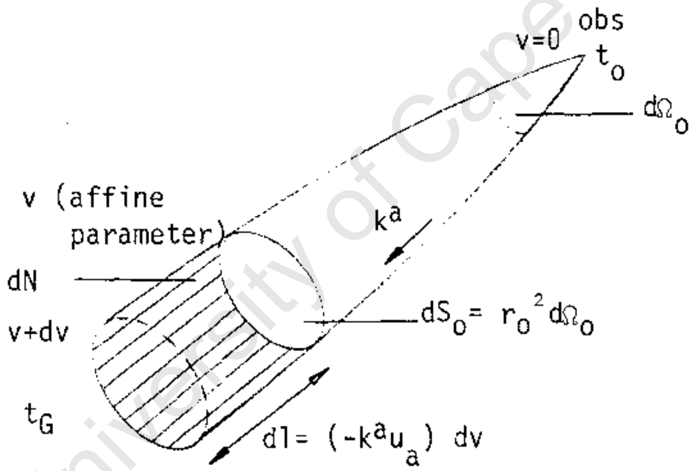

Figure 2. A section (v, v + dv) of a bundle of null geodesics which substends a solid angle $d\Omega_o$ at the observer (adapted from Ellis 1971, Fig. 8)

2.6.6 Magnitute-Redshift Relation

In the context of astronomical measurement, the apparent magnitude m of a source is defined by

$$m = -2.5\ \log_{10} F + \text{const.} \qquad (2.31a)$$

where the flux of radiation received, F, may be shown to be given in terms of the source luminosity L (using (2.25)) by

$$F = (L/4\pi)\,(1/r_o^{\,2})\,(1+z)^{-4} \qquad (2.31b)$$

So (2.31a) may be rewritten as

$$m = -2.5\,\log_{10}L + 5\log_{10}(r_o(1+z)^2) + \text{const.} \qquad (2.31c)$$

2.6.7 Null Cone Observations in a RW Model

For a RN model, filled with dust (and with A= 0), we may summarize the observable quantities as follows:

i) no proper motion or distortion effect is observed (see eqs <2.10),(2.23))

ii) the redshift, observed for any distant sources in the model, is given by (2.24b,d)

iii) the area distance r_o for any =source is given by (2.29)

the number N of sources observed upto a redshift z in any direction is obtained by integrating (2.30a)

v) the magnitude of a source is given by (2.31c)

Note that in the general case the specific intensity of radiation from any given source, and of background radiation may be additionally obtainable.

3 Spherical Symmetry : Observational Coordinates and Tetrad

3.1 Fluid Tetrad Formalism

We review the use of tetrad techniques briefly (see Maartens 1980 for a more complete set of references). The techniques have been applied in the study of both vacuum and fluid-filled spacetimes. Orthonormal tetrads adapted to fluid flow (0 = u) were applied to the kinematic classification and exact solutions of certain spacetimes (see Heckmann and Schucking 1962, Ellis 1964,1967, Stewart and Ellis 1968. Ellis and MacCallum 1969, King and Ellis 1973, MacCallum 1979 for an overall review); pseudo-orthonormal (null) tetrads have aided vacuum and radiation spacetime analysis (see Newman and Penrose 1962, Geroch et al. 1973, and for a review see Carmeli 1977).

The motivation for a choice of a particular tetrad is its natural geometrical suitability for the situation investigated. Observational cosmology is geometrically characterized by a timelike congruence (modelling the galactic motion), and the null congruence (modelling the photons received, from galactic sources, at the observer galaxy C -belonging to the timelike congruence). The formalism is therefore designed to eliminate the unsuitability of the fluid tetrad to describe electromagnetic signals, and the null tetrad at describing galactic motion.

3.1.1 Tetrad Formalism in General

We follow Heckmann and Schucking (1962) in reviewing the tetrad formalism in general.

The set of vectors te a) are orthonormal and linearly independent at each point; their scalar products with respect to the metric g are constant, i.e.

$$e_a(g_{ab}) = 0 \ , \text{ where } g_{ab} = g(e_a, e_b) = e_a \cdot e_b \tag{3.1}$$

The affine connection is described by the Ricci rotation coefficients,

$$\Gamma_{abc} = e_a \cdot \nabla_{e_b} e_c \ , \ \Gamma^a{}_{bc} = g^{ad}\Gamma_{dbc} \tag{3.2a}$$

where $g^{ab}g_{bc} = \delta^a{}_c$ determines the tetrad metric tensor components g^{ab}.

The commutator relations is given by

$$[\mathbf{e}_a, \mathbf{e}_b] = \gamma^c{}_{ab}\mathbf{e}_c \qquad (3.2b)$$

where $\gamma^a{}_{bc}$ is called the 'object of anholonomicity' (Schouten 1954) as they determine the integrability properties of the four congruences of curves corresponding to the four vector fields $\{\mathbf{e}_a\}$; and the commutator coefficients are skew : $\gamma^c{}_{ab} = \gamma^c{}_{[ab]}$; $\gamma_{abc} = g_{ad}\gamma^d{}_{bc}$. Due to vanishing torsion

$$\gamma^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb} \qquad (3.3a)$$

while the Christoffel relations (3.1), (3.2a) show that

$$\partial_b g_{ab} = \Gamma_{abc} + \Gamma_{cba} = 0, \qquad (3.3b)$$

since the tetrad frame g_{ab} are constants. Relations (3.3) can be solved for Γ_{abc} giving

$$\Gamma_{abc} = \tfrac{1}{2}\,(\gamma_{abc} + \gamma_{acb} - \gamma_{bca}) \qquad (3.3c)$$

The Jacobi identities for $\{\mathbf{e}_a\}$, which are equivalent to the cyclic identities $R^a{}_{[bcd]} = 0$, obeyed by the Riemann tensor, are

$$\begin{pmatrix} f \\ dcb \end{pmatrix} : \ \partial_{[d}\gamma^f{}_{cb]} + \gamma^g{}_{[dc}\gamma^f{}_{b]g} = 0 \ , \qquad (3.4)$$

where $\partial_a f = L_{e_a}(f)$.

The curvature tensor $R^a{}_{bcd}$ satisfies the Ricci identity (cf. (2.19a))

$$v^a{}_{;cd} - v^a{}_{;dc} = v^e R_e{}^a{}_{cd}$$

valid for any differentiable vector field v. Choosing v to be a basis vector $\mathbf{e}_f$ and explicitly calculating the covariant derivatives, one finds the tetrad components of the Riemann tensor relative to the basis $\mathbf{e}_a$:

$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{db} - \partial_d\Gamma^a{}_{cb} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb} + \Gamma^a{}_{eb}\gamma^e{}_{dc} \ ,$$

which upon contraction give the Ricci tensor:

$$-R_{bd} = \partial_d\Gamma^c{}_{cb} - \partial_c\Gamma^c{}_{db} - \Gamma^c{}_{cg}\Gamma^g{}_{db} + \Gamma^g{}_{cb}\Gamma^c{}_{gd} \qquad (3.5)$$

In the tetrad formalism, the 24 $\gamma^a{}_{bc}$ are the field variables, obeying the 10 field equations

$$-R_{ab} + \tfrac{1}{2}Rg_{ab} = -T_{ab} \,, \tag{3.6}$$

and the 16 identities (3.4). Equivalently, by equations (3.3a), (3.3c), one may choose $\Gamma^a{}_{bc}$ as the variables, as Γ_{abc} and γ_{abc} are linear combinations of each other.

The virtue of the tetrad formalism over the usual coordinate-component approach to the field equations (in which the g_{ij} , $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$, are the 10 field variables), is that the field equations are first order differential equations in the variables $\gamma^a{}_{bc}$ (or Γ_{abc}).

The draw back is that there are more variables, and more equations (the Jacobi identities (3.4)) to be satisfied. It may be shown on contracting (3.4) that the expression (3.5) for R_{bd} is in fact symmetric (see e.g. Roque 1985).

3.1.2 General Procedure

The general procedure used can be summarized as follows:

Given $\mathbf{e}_a = e_a{}^i\, \partial/\partial x^i$, one finds $e^a{}_i$, where $e_a{}^i e_i{}^b = \delta_a{}^b$, whence

$$g_{ij} = e^a{}_i e^b{}_j\, g_{ab} \,, \tag{3.7}$$

is determined (Ellis 1967).

Having determined the commutator coefficients using the commutator relations (3.2b) one finds Γ_{abc} from(3.3c). The Jacobi identities for the commutator coefficients are determined using (3.4), and form the integrability conditions for (3.2b).

The tetrad components of the field equations can then be written in terms of the rotation coefficients through (3.5) and (3.6).

3.2 Obseravational Coordinates and Tetrad

3.2.1 Observational Coordinates

We consider a spherically symmetric spacetime given in observational coordinates. These are $(x_a) = (w, r, \theta, \phi)$ (cf. subsect. 2.5; Temple 1938, Kristian and Sachs 1966). Where,

w   labels the past light cones of events along the central world line C ;

r   measures distance down the past light cone from C ; and

θ,ϕ are angular coordinates based on C .

3.2.2 Fluid-Ray Tetrad

Associated with the coordinates we introduce a fluid-ray tetrad (see Maartens 1980) $(\mathbf{e}_a) = (u, \mathbf{K}, \mathbf{m}, \mathbf{m}^+)$. Here,

u   is the normalized fluid 4-velocity ($u \cdot u = -1$); i.e. it is the (unit) time like vector field defining the galactic motion.

K   is a normalized ray vector, i.e. it is parallel to the affinely parametrized null vector field **k** ($k_a = w_{,a}$) generating the past null cones through C but normalized such that $K \cdot u = +1$; hence $K \cdot K = 0$.

m, m^+ are complex conjugate vectors ($^+$ denotes comlex conjugation) which are null ($m \cdot m = 0$), have unit scalar product relative to each other ($m \cdot m^+ = +1$), and are orthogonal to u and K ($u \cdot \mathbf{m} = 0 = K \cdot m$).

The tetrad and coordinates are related to each other by component functions A(w,r), B(w,r), C(w,r) as follows:

$$\mathbf{e}_0 = u = (A(w,r))^{-1}\, \partial/\partial w \tag{3.8a}$$

$$\mathbf{e}_1 = K = (B(w,r))^{-1}\, \partial/\partial r \tag{3.8b}$$

$$\mathbf{e}_2 = \mathbf{m} = 2^{-1/2}(C(w,r))^{-1}\, \{\partial/\partial\theta - (i/\sin\theta)\partial/\partial\phi\},\ \mathbf{e}_3 = \mathbf{m}^+ \tag{3.8c}$$

This specifies that the radial coordinate r is chosen to be a fluid comoving coordinate ($r_{,a}u^a = 0$); so the fluid flow lines with tangent vector **u** are the lines (r,θ,ϕ constant) with w a time (but not necessarily a proper time) parameter along these lines. The null geodesics centred on C with tangent vectors k, K are the lines (w,θ,ϕ constant) with r a distance parameter along these lines. Due to the spherical symmetry of the spacetime, it is possible to have θ,ϕ comoving both for the fluid and the null rays.

3.3 The Metric

The scalar products above show that the metric tensor has tetrad components g_{ab} given by

$g_{00}= -1$, $g_{01}= g_{10}= 1$, $g_{23}= g_{32}= 1$, the rest vanishing.

i.e.

$$g_{ab}= \left(\begin{array}{cc|cc} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \tag{3.9a}$$

The metric components g_{23}, g_{32} are directly observable if the intrinsic size and shape of the observed object are known, for these just relate the actual size of the object to the corresponding observed angles. In this case, spherical symmetry implies vanishing distortion effect caused by anisotropic curvature, and knowledge of the focusing effect caused by isotropic curvature expressed through the observer area distance r_0. We note that the distortion effect cannot be detected since objects of "standard shapes" are not clearly defined in the universe (Ellis 1980).

Now g_{ij} is given by (3.7), where

$$e_a{}^i= \begin{pmatrix} A^{-1} & 0 & 0 & 0 \\ 0 & B^{-1} & 0 & 0 \\ 0 & 0 & 2^{-½}C^{-1} & -2^{-½}C^{-1}\,i/\sin\theta \\ 0 & 0 & 2^{-½}C^{-1} & 2^{-½}C^{-1}\,i/\sin\theta \end{pmatrix} \tag{3.9b}$$

$$e^a{}_i= \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 2^{-½}C & 2^{-½}C \\ 0 & 0 & 2^{-½}Ci\sin\theta & -2^{-½}Ci\sin\theta \end{pmatrix} \tag{3.9c}$$

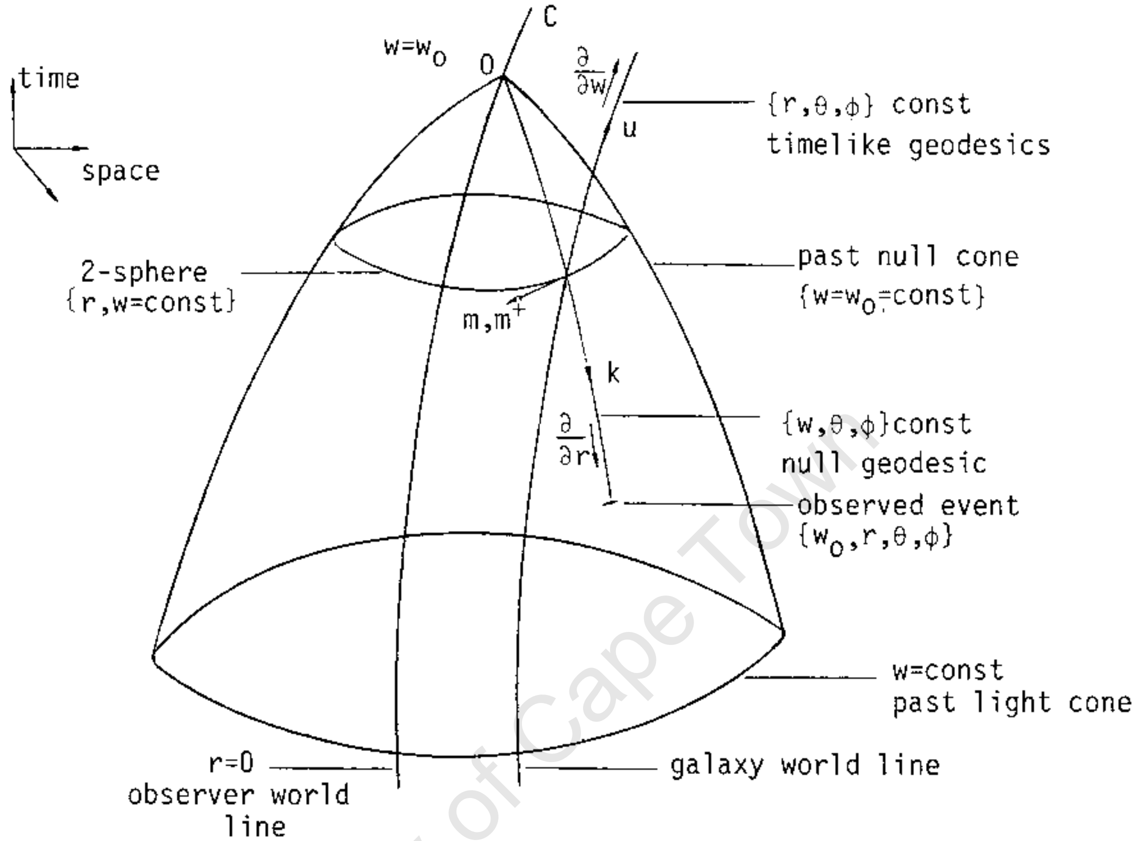

Figure 3. Fluid ray tetrad; and observational coordinates {w,r,e,} based on the world line C of an observer 0 characterizing the time of observation w, the direction of observation S,T, and the distance r to the galaxy observed, down the null geodesic joining these events.

It then follows from (3.8), (3.9) that the metric components in terms o+ these coordinates are divan

dS = -A (w.r) dW2 + 2A(w.r) B(w,r)Hw-",r f D2 (w.r))CO2 f sin (3.10)

3.4 Central Conditions

The metric: clearly symmetric about world l ine C at r= 0), provided suitable central conditions hold (c+. procedure used Marasse et al. 1962, and as adapted by Ellis et al. 1985; and c+. Temple 1938).

These are that

$$(r \to 0) \Rightarrow (C(w,r) \to 0 \; ; \; \partial C(w,r)/\partial r \to B(w,0) \neq 0 \; ; \; A(w,0) \neq 0) \qquad (3.11)$$

$C(w,r) \to 0$ as unit sphere area is $4\pi C^2(w,r)$.

$\partial C(w,r)/\partial r \to B(w,0) \neq 0$ since radius of unit sphere is $B(w,r)dr$ approaching Euclidean geometry as $r \to 0$.

$A(w,0) \neq 0$ clearly as $r \to 0$, since $w=w_o$ at the central observer.

The coordinate freedom remaining is

(a) choice of the time parameter w along the central world line C; once chosen on this line, it is determined on the other world-lines by the condition that {w=constant} are the past light cones of points on C. This freedom gives one the ability to arbitrarily specify $A(w,0)$ ($\neq 0$).

(b) choice of the radial parameter r on an initial light cone ($w=w_o$); once chosen on the null cone, it is determined on all the other because it is a fluid comoving coordinate and so is "dragged along" with the fluid. This freedom gives one the ability to arbitrarily specify $B(w_o,r)$ ($\neq 0$).

3.5 Observable Quantities

Finally, in terms of these coordinates the basic observable quantities for an observer on C are:

(1) **Redshift**

The redshift z measured at time w_o on the central line C for a source at radial distance r is given by the relation (cf. subsect. 2.6.2, in particular eq (2.24d)):

$$1+z = A(w_o,0)/(A(w_o,r)). \qquad (3.12a)$$

(2) **Area Distance**

The area distance r_o measured at time w_o on the central line C for a source at radial distance r is given by (cf. subsect. 2.6.3)

$$r_o = C(w_o, r), \quad (3.12b)$$

provided the central condition (3.11) holds (determining the relation between C and B for small values of r). The area distance may , for example, be found directly by measuring the solid angle subtended by the image of an object of known size (see fig.1). Alternatively it can be determined by measuring the observed luminosity of a source of known intrinsic luminosity (see Kristian and Sachs 1966, Ellis 1971). In practice however, both methods depend on the detection limit (see Ellis et al. 1985) which determines the observed image boundary, and so necessitates source brightness distribution and spectrum knowledge as a priori (see Ellis and Perry 1979, Ellis et al. 1984).

(3) Number Counts

The radial proper distance dl (see Fig.2, subsect. 2.6.4) measured by a comoving observer at the source as corresponding to dr is dl= Bdr. Following subsect 2.6.4, eq (2.30a), using (3.12a), becomes

$$dN = Fn(w_o, r)\ B(w_o, r)\ C^2(w_o, r)\ d\Omega\ dr/dz\ dz$$

Then the total energy density $\mu(w_o, r)$ at (w_o, r), (eq (2.30b)) is

$$\mu = Mn = M_o(z)(dz/dr)\ (1/B(w_o, r)) \quad (3.12c)$$

where

$$M_o(z) = (M/F)\ (1/d\Omega)\ (1/r_o^2)\ (dN/dz) \quad (3.12d)$$

which is in principle observable. The term r_o^2 has been included for convenience in the definition of M; alternatively it could be moved from (3.12d) to (3.12c) at the cost of making the integration of the field equations more difficult.

We encounter difficulties in realistically determining M, since many other contributions apart from visible matter may occur;this is the problem of evaluating the mass to light ratio.

Both F and M will in general depend on z; in practice one can estimate F from a knowledge of the galactic brightness distribution and spectrum, the area distance r and redshift z, and the detection limit (see Ellis, Perry et al. 1984).

Further, the Doppler shifts caused by random velocities of galaxies prevent one from directly using observed redshifts as a measure of distance down the light cone. One must therefore allow for a possible Doppler component to be added to the observed redshift in order to estimate the cosmological redshift.

Hence, number counts might be better expressed in terms of area distances, but both observationally as well as mathematically one is faced with difficulties; i.e. measuring r_o accurately and obtaining simple expressions for $M_o(r_o)$ is in practice difficult.

4 Commutators and Rotation Coefficients

We follow the procedure developed by Stoeger and Ellis (1984) in the ensuing chapters. We use the notation for commutators and rotation coefficients introduced by Maartens (1980, p. 3.1.6 ;cf. subsect. 3.1).

4.1 Commutator Relations

Upon substitution of relations (3.8) into the commutators relations (3.2b), one finds when comparing with Maartens (1980, eq (3.1.14))

$$a = e = h = q = j = c = g = 0 \qquad (4.1a)$$

(see Appendix I for a full list of definitions of the general rotation coefficients).

One further finds that the remaining commutator coefficients are real; they are given by

$$n = \tfrac{1}{2}A'/(AB) \ , \ b = \dot{C}/(AC) \ , \ f = C'/(BC), \qquad (4.1b)$$

$$s = 2^{-\frac{1}{2}}\cot\theta/C \ , \ l = (\dot{B} + A')/(2AB) \qquad (4.1c)$$

where we employ the notation : for any function $g(w,r)$, $\dot{g} = \partial g/\partial w$, $g' = \partial g/\partial r$.

In terms of these quantities the commutator relations are

$$[u,K] = 2nu + 2(n-l)K \qquad (4.2a)$$

$$[u,m] = - bm \qquad (4.2b)$$

$$[K,m] = - fm \qquad (4.2c)$$

$$[m,m^+] = sm - sm^+ \qquad (4.2d)$$

(see Appendix I for the general fluid-ray commutator relations).

i.e. the non zero commutator coefficients are:

$$\gamma^0{}_{01} = 2n \ , \ \gamma^1{}_{01} = 2(n-l)$$
$$\gamma^2{}_{02} = -b$$
$$\gamma^2{}_{12} = -f \qquad (4.2e)$$
$$\gamma^2{}_{23} = s \ , \ \gamma^3{}_{23} = -s$$
$$\gamma^3{}_{03} = -b$$

$\gamma^3{}_{13} = -f$

4.2 Rotation Coefficients

The commutator coefficients uniquely determine the rotation coefficients $\Gamma^a{}_{bc}$, Γ_{abd} which are defined by the relation (3.2a).

The non-zero components of g^{ab} are,

$g^{01} = g^{10} = 1$, $g^{11} = 1$, $g^{23} = g^{32} = 1$

i.e.
$$g^{ab} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Since the tetrad frame g_{ab} are constants, relations (3.3) and in particular (3.3c) for Γ_{abc} hold. This is the tetrad form of the Christoffel relations. From (4.2), (3.3c) the non zero rotation coefficients $\Gamma^a{}_{bc}$ are,

$$\Gamma^0{}_{00} = \Gamma^1{}_{00} = -\Gamma^1{}_{01} = 2l \qquad (4.3a)$$

$$\Gamma^0{}_{10} = \Gamma^1{}_{10} = -\Gamma^1{}_{11} = -2n \qquad (4.3b)$$

$$\Gamma^2{}_{20} = b,\ \Gamma^2{}_{21} = -\Gamma^0{}_{23} = f \qquad (4.3c)$$

$$\Gamma^2{}_{22} = -\Gamma^3{}_{23} = s,\ \Gamma^1{}_{23} = -b- f \qquad (4.3d)$$

$$\Gamma^3{}_{30} = b,\ \Gamma^3{}_{31} = -\Gamma^0{}_{32} = f \qquad (4.3e)$$

$$\Gamma^2{}_{32} = -\Gamma^3{}_{33} = -s,\ \Gamma^1{}_{32} = -b- f \qquad (4.3f)$$

4.3 Geometrical Interpretation

The definition of the quantities $\Gamma^a{}_{bc}$ shows that

$$\nabla_{\mathbf{e}_b}(\mathbf{e}_c) = \Gamma^a{}_{bc}\, \mathbf{e}_a$$

One can therefore read off from this immediately the geometric meaning of the non-zero commutation coefficients. Both n and b represent components of the expansion of the fluid:

b   is the expansion component in the transverse direction (orthogonal to

K), and

(-2n) is the expansion component in the radial direction (θ_{11})

(b+2n) is the fluid shear σ, so the fluid expands isotropically iff (b+2n)= 0

(eq (2.6) shows $\sigma^2 = ½ (\theta_{ab}\theta^{ab} - 1/3\,\theta^2) = 1/3(2n+b)^2$, trace free)

(2b-2n) is the volume expansion of the fluid θ.

($\theta_{ab} = u_{a;b} = \Gamma_{boa}$; $\theta = u^a{}_{;a} = \Gamma^a{}_{ao}$, using (2.6b,c)).

l determines the fluid acceleration $\dot{u}^a$ (2.6a) ; the fluid moves geodesically iff l= 0.

The fluid vorticity $\omega_{ab} = \omega_{[ab]}$ is necessarily zero; it is a direct consequence of the spherical symmetry (eq (2.6c), (2.19a)).

f determines the expansion of the radial null geodesics ($f = k^a{}_{;a} = \Gamma^a{}_{a1}$); note that this quantity will diverge at the origin because of the central conditions (3.11). These geodesics are necessarily rotation-free and distortion-free (from (4.3), but also from direct symmetry).

s represents the standard spherical-polar relations of the coordinates θ,ϕ on the 2-sphere {w,r constant} with area $4\pi C^2(w,r)$.

5 Component Function Radial and Time Derivative Equations

5.1 Jacobi Identities and Field Equations

Jacobi Identities

The commutator coefficients must satisfy the Jacobi identities (3.4). From Maartens (1980, pp. 3.1.7 and A3.1), when the coefficients are as in (4.2), the non trivial such identities are

$$\begin{pmatrix}2\\123\end{pmatrix} \qquad Ds = -fs \tag{5.1a}$$

$$\begin{pmatrix}2\\023\end{pmatrix} \qquad \Delta s = -bs \tag{5.1b}$$

$$\begin{pmatrix}2\\021\end{pmatrix} \qquad \Delta - Db = 2(n-1)f + 2nb \tag{5.1c}$$

where we follow Maartens in using the operator notation: $(\Delta,\ D,\ \delta,\ \delta^+)$ are the vector directional derivatives $(u\ ,\ K,\ m,\ m^+)$. That is, by (3.8), for any function g ,

$$\Delta g = A^{-1}\ \partial g/\partial w = A^{-1} g^{\cdot}\ ;\ Dg = B^{-1}\ \partial g/\partial r = B^{-1} g'\ ; \tag{5.2a}$$

$$\delta g = 2^{-½} C^{-1}\ (\partial g/\partial\theta - (i/\sin\theta)\partial g/\partial\phi) \tag{5.2b}$$

The δ operator may only be applied non-trivially to s, resulting in $\delta s = -¼C^{-2}\operatorname{cosec}^2\theta$. Substitution from (4.1) shows equations (5.1) are identically satisfied (as expected, for they are the integrability conditions for (4.1); see Appendix I for the general case of Jacobi identities).

Rotation Coefficients

The tetrad components of the field equations can be written in terms of the rotation coefficients (combining eqs (3.5), (3.6)):

$$\begin{aligned} -R_{bd} &= \partial_d\Gamma^a{}_{ab} - \partial_a\Gamma^a{}_{db} - \Gamma^a{}_{ae}\Gamma^e{}_{db} + \Gamma^e{}_{ab}\Gamma^a{}_{ed} \\ &= -T_{bd} + ½T^c{}_c g_{bd} \end{aligned} \tag{5.3}$$

where the contracted matter tensor for a perfect fluid (upon using (2.3)) is

$$T^a{}_a = -R^a{}_a = -\mu + 3p \tag{5.4a}$$

We assume the matter tensor T_{ab} is that of a perfect fluid as given by (2.4), where

$$u^a = (1,\ 0,\ 0,\ 0)\ ,\ u_a = g_{ab}u^b = (-1,\ 1,\ 0,\ 0)\ ;$$

the vanishing of components u^2, u^3 indicates that there are no proper motions (these cannot be detected in a cosmological context since such transverse velocities are too small to measure directly, Ellis 1980a,b).

Thus, the Ricci scalar may be written as

$$-R_{bd} = -(\mu+p)\ u_b u_d\ -\tfrac{1}{2}(\mu-p)\ g_{bd} \qquad (5.4b)$$

Then from thr rotation coefficients (4.2), the non-trivial field equations are

$$-R_{00} = -(\mu+3p)/2 = \Delta(2b-2n) - D(2l) - 2l(4n+2b+2f) + 4n^2 + 2b^2, \qquad (5.5a)$$

$$-R_{01} = (\mu+3p)/2 = D(2l+2b) + \Delta(2n) + 2n(4l+2b-2n+2f) + 2bf, \qquad (5.5b)$$

$$-R_{11}/2 = -(\mu+p)/2 = D(f) - 2fn + f^2, \qquad (5.5c)$$

$$-R_{23} = (p-\mu)/2 = \Delta f + D(b+f) + 2(\delta s+s^2) + 2f(l+2b+f) + 2nb \qquad (5.5d)$$

Note that $R_{02} = R_{12} = 0$ are consistent with spherical symmetry. [$R_{02} = 0$, $R_{12} = 0$, imply respectively, time derivatives and radial derivatives of component functions-in angular coordinate directions θ, ϕ- are equal].

The field equations agree with those of Maartens for the case p= 0 (pp. 3.1.8 and A3.2; see Appendix I for the general field equations), except that there appears to be an overall sign error in those equations; the error has been corrected here (i.e. replace $\mu \to -\mu$ in Maartens equations to obtain present ones). Hence,

$$R_{ab} = \begin{pmatrix} \tfrac{1}{2}(\mu+3p) & -\tfrac{1}{2}(\mu+3p) & 0 & 0 \\ -\tfrac{1}{2}(\mu+3p) & (\mu+p) & 0 & 0 \\ 0 & 0 & 0 & \tfrac{1}{2}(\mu-p) \\ 0 & 0 & \tfrac{1}{2}(\mu-p) & 0 \end{pmatrix} \qquad (5.5e)$$

$$= \begin{pmatrix} \frac{1}{2}\mu & -\frac{1}{2}\mu & 0 & 0 \\ -\frac{1}{2}\mu & \mu & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\mu \\ 0 & 0 & \frac{1}{2}\mu & 0 \end{pmatrix} \qquad (5.5f)$$

for pressure free matter ("dust").

Contracted Bianchi Identities

Finally, the contracted Bianchi identities (2.11), which are the integrability conditions for the field equations (5.5), take the form

$$\Delta\mu + (\mu+p)\,(2l-2n+2b) = 0 \qquad (5.6a)$$

$$(\mu+p)\,2l + (\dot{p}+p') = 0 \qquad (5.6b)$$

viz, the energy and momentum conservation equations respectively.

Dust Solutions

In the case of pressure free matter ("dust"), $p=0$, $\mu \neq 0$, and (5.6b) shows

$$l = 0, \qquad (7.7)$$

which is the condition for geodesic fluid flow. We shall assume this restriction in the rest of this thesis. Then (4.1c) shows

$$\dot{B} + A' = 0. \qquad (5.8)$$

Thus, the field equations (5.5), Jacobi identities (5.1) and contracted Bianchi identities (5.6) for pressure free matter are

$$-R_{00} = -\mu/2 = \Delta(2b-2n) + 4n^2 + 2b^2 \qquad (5.5a')$$

$$-R_{01} = \mu/2 = D(2b) + 2n(2b-2n+2f) + 2bf \qquad (5.5b')$$

$$-R_{11}/2 = -\mu/2 = D(f) - 2fn + f^2 \qquad (5.5c')$$

$$-R_{23} = -\mu/2 = \Delta(f) + D(b+f) + 2(\delta s+s^2) + 2f(2b+f) + 2nb \qquad (5.5d')$$

$$\Delta\mu = 2\mu(n-b) \qquad (5.6a')$$

$$Ds = -fs \qquad (5.1a')$$

$$\Delta s = -bs \qquad (5.1b')$$

$$\Delta f - Db = 2n(f+b) \qquad (5.1c')$$

Examining eqs (5.5a′-d′) we find that the quantity s appears only in the last equation, and thus we define for convenience the auxiliary quantity r by the relation

$$r= \delta s+ s^2= -\tfrac{1}{2}C^{-2}(w,r) \ , \ \delta s= -\tfrac{1}{2}C^{-2}\operatorname{cosec}^2\theta \qquad (5.9a)$$

Applying the commutators $[\Delta,\delta]$ and $[D,\delta]$ to s together with (4.2b,c) and the Jacobi identities (5.1a,b) it follows that

$$\Delta r= -2br \ , \ Dr= -2fr \ , \ \delta r= 0 \ , \qquad (5.9b)$$

the last following from the commutator $[\Delta,\delta]$ applied to b and eq (5.11c). Then the spherical derivatives of all quantities in these equations vanish, as expected from the spherical symmetry:

$$\delta n= \delta b= \delta f= \delta\mu= 0 = \delta r \qquad (5.9c)$$

5.2 **Radial and Time Derivative Equations**

The Jacobi identities, field equations, and contracted Bianchi identities involve only first derivatives and sums of products of the rotation coefficients. One can take linear combinations of these equations to give an equivalent set of equations explicitly for the first derivatives. To express these simply, it is convenient to introduce a further auxiliary quantity ω , defined by

$$\omega= r+ bf+ \tfrac{1}{2}f^2 \qquad (5.9d)$$

The set of equations to be satisfied can now be written as a set of radial equations:

$$Df= 2nf - f^2- \mu/2 \ , \qquad (5.10a)$$

$$D\omega= -3f\omega- (\mu/2)\ (b+f) \ , \qquad (5.10b)$$

$$Db= -\omega- 2n(f+b)- bf \ . \qquad (5.10c)$$

and a set of time derivative equations:

$$\Delta f = -\omega - bf \ , \qquad (5.11a)$$

$$\Delta\omega = -3b\omega \ , \qquad (5.11b)$$

$$\Delta b = \omega - b^2 \ , \qquad (5.11c)$$

$$\Delta n = \omega + 2n^2 + \mu/4 \ , \qquad (7.11d)$$

$$\Delta\mu = 2\mu(n-b) \ ; \qquad (5.11e)$$

the spherical derivatives all vanish:

$$\delta n = \delta b = \delta f = \delta\mu = \delta\omega = 0 \ . \qquad (5.11f)$$

One can now check the consistency of these equations by applying the commutator identity

$$\Delta(Dg) - D(\Delta g) = 2n(\Delta g + Dg) \qquad (5.12)$$

(valid for any function $g(w,r)$) in turn to f, ω, and b. In each case, on using eqs (5.10) and (5.11) repeatedly, one obtains an identity.Note that applying the commutator identities $[\Delta,\delta]$, $[D,\delta]$ and $[\delta,\delta^+]$ to f, ω and b is trivially satisfied.

The freedom to specify the radial physics and geometry is evident in the feature that no equations are given for the radial derivatives of μ or n, and no restrictions on these quantities are implied by our consistency check; so they may be freely specified as functions of r on an initial surface $(w= w_0)$, and the other equations then used consistently to determine the time evolution of the physical situation so specified to other surfaces $(w= \text{const})$.

On using eqs (4.1), it is easy to write these equations out explicitly in terms of the functions A, B, C defined in (3.8) (determining the metric tensor directly, see (3.10)). The time derivative eqs (5.11b,e) for μ and ω may be immediately integrated to get

$$\mu(w,r) = \mu_0(r)\ B^{-1}(w,r)\ C^{-2}(w,r) \qquad (5.13a)$$

$$\omega(w,r) = \omega_0(r)\ C^{-3}(w,r) \qquad (5.13b)$$

where we have set for convenience $B(0,r) = C(0,r) = 1$.

The definition (5.9d) of ω then becomes

$$\Delta f = -\omega - bf \ , \qquad (5.11a)$$

$$\Delta\omega = -3b\omega \ , \qquad (5.11b)$$

$$\Delta b = \omega - b^2 \ , \qquad (5.11c)$$

$$\Delta n = \omega + 2n^2 + \mu/4 \ , \qquad (7.11d)$$

$$\Delta\mu = 2\mu(n-b) \ ; \qquad (5.11e)$$

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$$\mu(w,r) = \mu_0(r)\ B^{-1}(w,r)\ C^{-2}(w,r) \qquad (5.13a)$$

$$\omega(w,r) = \omega_0(r)\ C^{-3}(w,r) \qquad (5.13b)$$

where we have set for convenience $B(0,r) = C(0,r) = 1$.

The definition (5.9d) of ω then becomes

$$\omega(w,r) = \omega_o(r)/C^3 = -1/(2C^2) + (\dot{C}/AC)(C'/BC) + \tfrac{1}{2}(C'/BC)^2 \qquad (5.14a)$$

The radial equations (5.10) become

Df : $$C''/C = (C'/C)\,(A'/A + B'/B) - \tfrac{1}{2}B^2\mu, \qquad (5.14b)$$

Dω : $$\omega_o' = -\tfrac{1}{2}\mu_o\,(\dot{C}/A + C'/B), \qquad (5.14c)$$

Db : $$\dot{C}'/C = (\dot{B}/B)\,(C'/C) - \omega AB\ ; \qquad (5.14d)$$

the time derivative equation (5.11a) again yields (5.14d); and the remaining time derivative equations (5.11c,d) give

Δb : $$\ddot{C}/C = (\dot{C}/C)\,(\dot{A}/A) + \omega A^2\ , \qquad (5.15a)$$

Δn : $$\ddot{B}/B = (\dot{B}/B)\,(\dot{A}/A) - 2\omega A^2 - \tfrac{1}{2}\mu A^2. \qquad (5.15b)$$

The equations we have to satisfy in solving for A, B, C are (5.8), (5.13-5.15); however eq (5.14c) does not have to be explicitly integrated, as this is a trivial equation (it will be satisfied automatically if all other equations are satisfied, because of the consistency of the equations; which we have checked).

Because of (5.8), $A'' = -\ddot{B}$, which is given by (5.15b); so

$$A''/B = -(\dot{B}/B)\,(\dot{A}/A) + 2\omega A^2 + \tfrac{1}{2}\mu A^2 \qquad (5.15c)$$

which is a consequence of the other equations but whose later significance makes it worth noting.

Finally, we note in passing that eqs (5.14d), (5.15a-c) share a common term ωA^2 which, for manipulation reasons, may be written as

$$\omega A^2 = (A/b)\xi - A^2/(2C^2)\ ,\ \xi = (A/B)\,(C'/C)\,((\dot{C}/C) + (A/2B)\,(C'/C)).$$

6 Observational Coordinates for Robertson-Walker Spacetimes

6.1 Spherically Symmetric Conditions for FLRW Universes

The vector $e \equiv \mathbf{u} + K$ is a spatial vector which points radially out from C and is orthogonal to u. Thus any function $g(w,r)$ is measured by an observer moving with 4-velocity u to be spatially homogeneous if it satisfies the equations

$$\Delta g + Dg = 0 \ , \ \delta g = 0.$$

The FRW models are characterized within the spherically symmetric dust-filled universes in the previous section by spatial homogeneity. Thus we add to the equations to be satisfied the restriction

$$\Delta\mu + D\mu = 0 \ , \qquad (6.1)$$

which, with (5.9c), specifies the energy density of matter in the universe is spatially homogeneous.

By (5.11e) this immediatelly gives the equation

$$D\mu = -2\mu(n-b) \qquad (6.2a)$$

for $\Delta\mu$. Taking the commutator (5.12) for μ now gives an equation for Dn, which can be written in the form

$$D(n-b) = -2n^2 - b^2 - \tfrac{1}{4}\mu = -\Delta(n-b), \qquad (6.2b)$$

showing the (n-b), which is just the fluid expansion, is spatially homogeneous. Applying the commutator (5.12) now to (n-b) shows

$$(f+b)\ (2n+b) = 0. \qquad (6.2c)$$

The first possibility $(f+b) = 0$, gives a consistent solution of the field equations. This solution is the Kantowski-Sachs (1966; also see Collins 1977) spatially homogeneous anisotropic universe. It cannot be spherically symmetric about a world-line C, for at the centre of symmetry we must have $f \to \infty$ (f is the divergence of the null rays centred on C); this would imply $b \to \infty$, which implies a curvature singularity (as b is the transverse component of the fluid expansion). Thus these spaces do not lie in the class considered (we will not be able to satisfy the central conditions (3.11)).

The second possibility is $(2n+b)=0$, which is just the condition that the fluid expansion is isotropic (see subsect. 4.3). Applying the operators Δ, D to $(2n+b)$ now gives a further condition : $3\omega+\mu/2=0$. This condition may also be obtained by applying the commutator $[\Delta,D]$ to n for isotropic fluid expansion. Defining the quantities α, β by

$$\alpha\equiv 3\omega+\tfrac{1}{2}\mu, \tag{6.3a}$$

$$\beta\equiv 2n+b, \tag{6.3b}$$

we now have $\alpha=\beta=0$. The derivatives of these equations are now identically satisfied. On the other hand the definition (6.3) and equations (5.11,12) show that

$$\Delta\alpha=\alpha(\beta-3b)-3\omega\beta, \tag{6.4a}$$

$$\Delta\beta=\alpha+\beta(\beta-2b), \tag{6.4b}$$

$$\Delta\omega+D\omega=-\alpha(b+f), \tag{6.4c}$$

$$\Delta b+Db=-\beta(b+f). \tag{6.4d}$$

The first two equations show that if μ, n are specified on an initial surface $(w=w_0)$ such that $\alpha=\beta=0$ there, then $\Delta\alpha$, $\Delta\beta$ vanish at all times; and the second pair show that then ω and b, and so μ and n, are spatially homogeneous for an observer moving with 4-velocity **u**. We note that

$$\Delta f+Df=f\beta-\alpha+2r, \tag{6.4e}$$

which, when $\alpha=\beta=0$, is non vanishing as anticipated for expansion in the radial null direction.

The field equations to be satisfied now take the form, upon using equations (5.10,11) and (6.3)

radial equations:

$$Df=3\omega-f(b+f)+\beta f-\alpha \tag{5.10a$'$}$$

$$D\omega=3\omega b-\alpha(b+f) \tag{5.10b$'$}$$

$$Db=-\omega+b^2-\beta(b+f) \tag{5.10c$'$}$$

time derivative equations:

$$\Delta f= -\omega- bf \qquad (5.11a')$$

$$\Delta \omega= -3b\omega \qquad (5.11b')$$

$$\Delta b= \omega- b^2 \qquad (5.11c')$$

$$\Delta \alpha= \alpha(\beta- 3b)- 3\omega\beta \qquad (6.4a')$$

$$\Delta \beta= \alpha+ \beta(\beta- 2b) \qquad (6.4b')$$

where $D\mu= 3\mu b-\mu\beta= -\Delta\mu$, $Dn= -\frac{1}{2}Db= -\Delta n$ are satisfied automatically within the field equations.

All the equations are now consistent. The condition $\alpha= 0 =\beta$ therefore guarantees the fluid motion to be geodesic, vorticity free, and shear free so these are the standard spatially homogeneous and isotropic (FLRW) universe models (Ehlers 1973, Ellis 1973). Hence a spherically symmetric dust filled space-time is a FLRW universe if and only if $\alpha= \beta= 0$.

6.2 Component Functional Forms

We now aim to integrate the field equations in a gauge independent manner, i.e. we will not use the remaining coordinate freedom to simplify the integration. Then when the integration is completed, we will examine the various possible coordinate choices that might simplify the metric and solution of the field equations. This study proves an indispensible aid in integrating the FLRW universes from null initial data, as done in the following section, and so provides the basis on which we can integrate the perturbed field equations from null initial data for geometries near to those of FLRW universes.

We assume FLRW conditions, that is,

$$\alpha= \beta= 0, \qquad (6.5a)$$

$$\Delta\omega+ D\omega= 0,\ \Delta\mu+ D\mu= 0,\ \Delta n+ Dn=0,\ \Delta b+ Db=0, \qquad (6.5b)$$

$$\Delta f+ Df= 2r \qquad (6.5c)$$

First we apply restrictions (6.5) to determine the forms of the functions

A, B. C. Equations (6.5a) together with (4.1), (5.8) and (5.13) show that

$$\dot{C}/C = \dot{B}/B, \quad C/B = 3\omega_o(r)/(-\mu_o(r)/2).$$

i.e. condition for isotropy implies that the time rate of change of C varies in direct relation to that of B. Thus, there is a function g(r) such that

$$C(w,r) = g(r)\, B(w,r) \tag{6.6a}$$

$$3\omega_o(r) = -\tfrac{1}{2}\mu_o(r) g(r) \tag{6.6b}$$

Define t(w,r) such that it is a solution of the equation

$$\Delta t + Dt = 0. \ (\Leftrightarrow) \ (1/A) \ \partial t/\partial w + (1/B) \ \partial t/\partial r = 0. \tag{6.7a}$$

Then the spatial homogeneity of 6), P, n and b (eq (6.5b)) shows

$$\omega = \omega(t), \ \mu = \mu(t), \ n = n(t), \ b = b(t)$$

But

$$\omega = \omega_o(r) \ g^{-3}(r) \ B^{-3}(w,r),$$

where r and t are independent functions (i.e. p(r)= q(t) => both p and q are constant, by (6.7a)). Hence there are functions a(r), R(t) such that

$$B(r,w) = a(r) \ R(t) \ ; \ a(r) = (\omega_o(r)/g^3(r))^{1/3}, \ R(t) = (1/\omega(t))^{1/3} \tag{6.7b}$$

$$\omega_o(r) = W g^3(r) a^3(r) \ , \ W = \omega(t) R^3(t) = \text{const.} \tag{6.7c}$$

Similarly the spatial homogeneity of V (eq (6.5b)) shows that

$$\mu_o(r) = M g^2(r) a^3(r) \ , \ M = \mu(t) R^3(t) = \text{const.} \tag{6.8a}$$

Then (6.6b) becomes

$$3 W = -\tfrac{1}{2} M. \tag{6.8b}$$

Finally, homogeneity of b (eq (6.5b); which is the same as homogeneity of n due to isotropy condition (6.3b)) and (4.1) shows

$$(1/A(w,r)) \ \ t/ \ w = k(t) \ , \ k(t) \equiv b(t) R(t) \ (dR(t)/dt)^{-1} \ ;$$

that is, there is a function k(t) such that

$$\dot{t} = \partial t/\partial w = A(w,r) k(t) \tag{6.9a}$$

Now (6.7a) shows that

$$t' = \partial t/\partial r = -a(r) \ k(t) \ R(t) \ . \tag{6.9b}$$

The integrability condition guaranteeing consistency of (6.9a,b) is

$$t^{\cdot\prime} = t^{\prime\cdot} \iff (1/A(w,r))\,\partial A(w,r)/\partial r = -a(r)k(t)\, dR(t)/dt = -a(r)b(t)R(t) \quad (6.10a)$$

which is also the condition (5.8) when (6.7b), (6.9a) hold. Using (6.9b) this can be rewritten as

$$(1/A(w,r)\,\partial A(w,r)/\partial r = (1/R(t))\, dR(t)/dt\ \partial t/\partial r. \quad (6.10b)$$

which can now be integrated to show: there is a function $p(w)$ such that

$$A(w,r) = p(w)R(t) \quad (6.11)$$

Equation (6.7a) can now be rewritten as

$$(1/p(w))\ \partial t/\partial w + (1/a(r))\ \partial t/\partial r = 0\ , \quad (6.12a)$$

which can be integrated to show:

$$t = t(\eta),\ \eta = \int p(w)dw - \int a(r)dr\ , \quad (6.12b)$$

where by (6.9a) the function $t(\eta)$ is related to $k(t)$ by the equation

$$k(t) = dt(\eta)/d\eta\ (1/R(t))\ , \quad (6.12c)$$

which is consistent with (6.9b).

Next we consider the field equations. First, substituting from (6.6-10) into (5.14a) shows

$$k^2(dR(t)/dt)^2 + K - (\ M/3)R^{-1} = 0\ , \quad (6.13)$$

the Friedmann equation (cf. eq (2.21b)) for the FRW universe, where the constant K is related to the functions $a(r)$, $g(r)$ by

$$((ag)')^2 - a^2 = -Ka^4g^2 \quad (6.14)$$

Secondly, substituting into (5.14d) shows that

$$k^2d^2R/dt^2 + k\ dk/dt\ dR/dt + (\ M/6)R^{-2} = 0\ , \quad (6.15)$$

the Raychaudhuri equation (cf. eq (2.18a)) for this universe; differentiating (6.13) (with respect to t) shows that (6.15) is always satisfied when (6.13) is, provided $dR/dt \neq 0$.

Equations (5.15a), (5.15b) are identical in view of (6.5a), (6.6a); they both again give (6.15). Finally (5.14b), after use of (6.15), reduces to

$$a(a''g + a'g' + ag'') - g(a')^2 = -Kga^4. \quad (6.16)$$

This equation is identically satisfied in view of (6.14). Hence from (6.14) and (6.16) we obtain the equation:

$$(a''/a + g''/g) = (a'/a + g'/g)^2 + a'/a(a'/a - g'/g) - 1/g^2 ,$$

which is always satisfied when (6.14) and (6.16) are, but due to its independence of K will only yield solutions for the case K= 0.

In summary, when using observational coordinates (3.8) we have a FLRW (i.e. p= 0, isotropic) universe if and only if

$$A(w,r) = p(w)\, R(t),\ B(w,r) = a(r)\, R(t),\ C(w,r) = g(r)a(r)\, R(t) \qquad (6.17a)$$

where t is determined by (6.12b) ; p(w), a(r) and t(η) are arbitrary C^2 functions subject only to the restrictions

$$p(w) \neq 0,\ a(r) \neq 0,\ dt(\eta)/d\eta \neq 0 \ ; \qquad (6.17b)$$

g(r) is determined from a(r) (or vice versa) by eq (6.14), or equivalently eq (6.16), with the initial conditions

$$g(0) = 0,\ g'(0) = 1 \qquad (6.17c)$$

(so that the central conditions (3.11) are satisfied); and R(t) obeys eq (6.13), or equivalently (6.15), with K(t) determined from t(η) by eq (6.12c). K is an arbitrary constant, and by (3.10), (6.17a) the metric form is

$$dS^2 = R^2(t)\ (-p^2(w)\ dw^2 + 2p(w)a(r)\ dw\ dr + a^2(r)g^2(r)\ (d\theta^2 + \sin^2\ d\phi^2)\ ).$$

Finally proper time τ along the fluid flow lines is related to the coordinates by the relation

$$u^i = dx^i/d\tau = (1/(p(w)R(t)))\ \partial x^i/\partial w$$

which shows, using (6.9a), (6.12c)

$$dt/d\tau = (1/R)\ dt/d\eta = k(t). \qquad (6.18)$$

6.3 Coordinate Freedom

The remaining coordinate freedom is now easily described :

(a) the freedom to choose the time coordinate w along the central line C is represented by the arbitrariness of the function $p(w)$;

(b) the freedom to choose the radial coordinate r on an initial surface $\{w=w_0\}$ is represented by the arbitrariness of the function $a(r)$;

(c) the freedom to choose the spatially homogeneous time function t is represented by the arbitrariness of the function $t(\eta)$.

The only restrictions are that there be C^2 funcions satisfying (6.17b).

We can now examine various coordinate choices and their consequences. There are two kinds of choices one can make; we look at these in turn.

The first kind rests on optimal choice of the coordinate r in an initial surface $\{w=w_0\}$ to simplify observational relations. The preferred such choice is

(1) Choose $r=r_0$, the area distance. By (3.12b) and (6.17a) this is achieved by setting

$$r=C(W_0,r) \Leftrightarrow a(r)= rg^{-1}(r)R^{-1}(t(w_0,r)) \tag{6.19a}$$

with t given by (6.12b), and suitable choices of $p(w)$, $t(\eta)$. Three problems arise: (i) because of the indirect determination of t via $a(r)$ (see (6.12b)), solving (6.19a) for $a(r)$ is non-trivial; (ii) as $a(r)$ will then be non-trivial, solving (6.14) or (6.16) for $g(r)$ will be difficult; (iii) although r will be the area distance on $\{w=w_0\}$, this relation will not be preserved in an expanding universe, because r is a fluid comoving coordinate. Thus r will have no obvious meaning for general values of w.

(2) Similarly, one can set $r=z$, the redshift. By (3.12a) and (6.17a) this is achieved by setting

$$r= A(w_0,0)/A(w_0,r)- 1 \Leftrightarrow r= R(t(w_0,0))/R(t(w_0,r))-1 \tag{6.19b}$$

which implies a suitable choice of $a(r)$ (i.e. $a(r)$ is given by (6.19a,b)); this suffers from the same problems (i)-(iii) as the previous choice.

(3) One can choose to make r an affine parameter v on the initial surface $(w=w_0)$; this is achieved by setting

$$A(w_0,r)\ B(w_0,r)= 1 \iff a(r)= p^{-1}(w_0)R^{-2}(t(w_0,r)). \qquad (6.19c)$$

The same problems (i)-(iii) arise in this case too.

The second kind of choice strives to simplify the relations between the functions A,B,C directly by suitable choice of r. Three clearly suitable choices arise:

(4) Set $a(r)=1$, corresponding to choice of r as a comoving distance paremeter in the surfaces (t= const). Then we can choose p(w) to set

$$A(w,r)= B(w,r) \iff a(r)= 1 = p(w) \qquad (6.20a)$$

(5) Set $g(r)=1$, corresponding to choice of comoving isothermal coordinates. Then

$$B(w,r)= C(w,r) \iff g(r)= 1 \qquad (6.20b)$$

In this caes, (6.14) is solved for a(r) instead of g(r).

(6) Set $g(r)a(r)=r$, corresponding to choice of r as a comoving area coordinate. Then

$$C(w,r)= rR(t) \iff a(r)= rg^{-1}(r) \qquad (6.20c)$$

The respective line element for choices (4)-(6) are:

(4): $$dS^2= R^2(w-r)\ (-dw^2+ 2dwdr+ f^2(r)\ d\Omega^2)$$

(5): $$dS^2= R^2(w-r)\ (-p^2\ dw^2+ 2apdwdr+ a^2 d\Omega^2)$$

(6): $$dS^2= R^2(w-r)\ (-p^2\ dw^2+ 2pr/g\ dwdr+ r^2 d\Omega^2)$$

where unit sphere is $d\Omega^2= d\theta^2+ \sin^2\theta\ d\phi^2$.

6.4 Coordinate Choice and Observable Quantities

Choices (4)-(6) correspond to the standard three choices of radial coordinates in FLRW universe models (see e.g. Rindler 1969; cf. subsect 2.5); all can be completed easily. We choose choice (4), which leads to slightly simpler functional forms than the others (although it has the disadvantage of requiring different functions for K positive, negative, and zero).

Our simplest observational coordinate choice then is implemented as follows. If $K \neq 0$, rescale $R(t)$ to set $K = (-1, 0, +1)$ (cf. subsect. 2.4). Then set

$$p(w) = 1,\ a(r) = 1,\ t(\eta) = \eta \Rightarrow t = w - r. \tag{6.21a}$$

We then have

$$A(w,r) = R(t),\ B(w,r) = R(t),\ C(w,r) = g(r)\,R(t), \tag{6.21b}$$

and $k(t) = 1/R(t)$ by (6.19c); where (6.14) or (6.16) shows

$$g(r) = (\sinh r,\ r,\ \sin r) \text{ if } K = (-1, 0, +1) \tag{6.21c}$$

respectively (cf. (2.22c)). The remaining field equation (6.13) then takes the form

$$R^{-2}(dR/dt)^2 + K - (M/3R) = 0, \tag{6.22a}$$

which may easily be integrated in terms of the parameter t which is related to proper time τ along the fluid flow lines by the relation (6.18) (see (2.27b)):

$$\tau = \int R(t)dt. \tag{6.22b}$$

For simplicity we shall here only give details for the cases $K = +1$; the other two cases ($K = 0$ and $K = -1$) will be tabulated below; where results are identical for all 3 cases it will be noted so. Then the solutions are

$$R(t) = Q_o(1 - \cos t), \tag{6.23a}$$

$$\tau(t) = Q_o(t - \sin t), \tag{6.23b}$$

where the integration constants are chosen so that $R(0) = 0 = \tau(0)$; that is, the origin of the universe is at the origin of both coordinate time t and proper time τ. Evaluating from these equations H_o and q_o at the present time t_o, we find

$$H_o \equiv R^{-1}\ dR/d\tau|_o = (Q_o)^{-1} \sin t_o (1 - \cos t_o)^{-2},$$

$$q_o \equiv -(1/R)d^2R/d\tau^2|_o\ (1/H_o^2) = (1 - \cos t_o)\sin^{-2} t_o,$$

so inverting,

$$\cos t_o = (1 - q_o)/q_o,\ \sin t_o = (2q_o - 1)^{1/2}/q_o\ , \tag{6.24a}$$

$$Q_o = (H_o)^{-1} q_o (2q_o - 1)^{-3/2}. \tag{6.24b}$$

These show that

$$R_o \equiv R(t_o) = (H_o)^{-1}(2q_o-1)^{-½}, \qquad (6.24c)$$

$$t_o = \text{arc cos}((1-q_o)/q_o) \;, \qquad (6.24d)$$

$$\tau_o = \tau(t_o) = Q_o \;\{\text{arc cos}((1-q_o)/q_o) - (2q_o-1)^{½}/q_o\} \;,$$

are the present values of R, t, .T respectively; these are all unique as we have normalized away all significant coordinate freedom.

Finally, because $\Lambda = 0$ (see (2.28b)),

$$\mu_o = 6H_o^2 q_o, \qquad (6.25a)$$

relates the energy density and q_o, so

$$\mu = \mu_o(1+z)^3 = 6H_o^2 q_o(1+z)^3 \;; \qquad (6.25b)$$

this is the same for all three cases.

The observable quantities in all three cases in these universes follow from (3.12) and the above results. We wish to have these both in terms of the coordinate r, and in observational terms. Ideally we would give them in terms of r_o, but determining $z(r_o)$ seems difficult. Therefore as a second best we express them in terms of z.

Let an observer at time w_o measure properties of a galaxy situated at coordinate distance r. Then the observation is made at time $t_o = w_o$ and the galaxy emits the light observed at time

$$t_o = w_o - r = t_o - r \equiv \Phi. \qquad (6.26)$$

where t_o is given by (6.24d). By (3.12a), (6.19b) and (6.24c) the redshift z is given by

$$1+z(r) = ((2q_o-1)/q_o)\;(1-\cos\Phi)^{-1} \qquad (6.27a)$$

This yields the consistency check, z(0)= 0. From this we find

$$dz/dr = ((2q_o-1)/q_o)\sin\Phi(1-\cos\Phi)^{-2} \qquad (6.27b)$$

By (3.12b) the area distance is

$$r_o(r) = Q_o \sin r(1-\cos\Phi). \qquad (6.28)$$

In terms of z, eliminating r from (6.27b), (6.28) by use of (6.27a) we find

$$dr(z)/dz= ((2q_o-1)/(2q_oz+1))^{½}(1+z)^{-1} \qquad (6.29a)$$

$$r_o(z)= H_o^{-1}q_o^{-2}(1+z)^{-2}(q_oz+ (1-q_o)(1- (2q_oz+ 1)^{½})) \qquad (6.29b)$$

or

$$z^2B+ z(2B-q_o)+ x(A-1)+ B=0, \quad B=r_oH_oq_o^2, \quad x=1-q_o$$

and

$$dr_o(z)/dz=(H_oq_o^2(1+z)^3)^{-1} \; (3q_o-2-q_oz+ x(3q_oz+2-q_o)/(1+2q_oz)^{½}).$$

Clearly, z may not be expressed as a function of r_o in a simple manner. We note (see Ellis et al. 1985) that in RW, r_o is not a monotonic function; and in spherically symmetric spacetimes the redshift is not monotonic (Ellis et al. 1978). Also (3.12) and (6.25b) show

$$M_o(z)= 6H_oq_o(1+z) \; (2q_oz+1)^{-½}. \qquad (6.29c)$$

We shall now summarize the results for the three curvature cases K= (+1,0,-1). upon examination of the results one finds certain terms which repeatedly occur, for each case; these terms are therefore defined for convenience as follows

$$x\equiv 1-q_o, \quad D\equiv 1+z, \quad B\equiv r_oH_oq_o^2, \quad A\equiv (2q_oz+1)^{½}, \qquad (6.30)$$

Further, these are terms which recur but change from case to case systematically (e.g. Misner et al. 1973 recognize these changes for cases K= +1).

TABLE 1

Recurring terms for K= (+1, 0, -1)

K= +1	K= -1	K= 0	K(general)
$C\equiv(2q_o-1)^{½}$	$C_-\equiv(1-2q_o)^{½}$	$t_o/2$	$C(K)$
$p\equiv1-\cos\phi$	$h\equiv\cosh\phi-1$	$\phi^2/2$	$\xi(\phi)$
$g\equiv\sin\phi$	$g_-\equiv\sinh\phi$	ϕ	$g(\phi)$
$m\equiv\cos^{-1}(x/q_o)$	$m_-\equiv\cosh^{-1}(x/q_o)$	-	

Column 4 thus defines a general term for all three cases. We list the results in terms of these parameters below (for a more expanded tabulation and examination of the general curvature case see chapter 7).

TABLE 2

Obervable Quantities

	K= +1	K= -1	K= 0	K(general)
q_o	$>\frac{1}{2}$	$<\frac{1}{2}$	$=\frac{1}{2}$	
$\cos(h)t_o$	x/q_o		-	
$\sin(h)t_o$	C/q_o	C_-/q_o	-	
Q_o	$H_o^{-1} q_o C^{-3}$	$H_o^{-1} q_o C_-^{-3}$	$H_o^{-1} 4/t^3$	$H_o^{-1} q_o C^{-3}(K)$
R_o	$H_o^{-1} C^{-1}$	$H_o^{-1} C_-^{-1}$	$H_o^{-1} 2/t$	$H_o^{-1} C^{-1}(K)$
t_o	m	m_-	$(4H_o^{-1} Q_o^{-1})^{1/3}$	
τ_o	$q_o H_o^{-1} C^{-3}(m - C/q_o)$	$q_o H_o^{-1} C_-^{-3}(-m_- + C_-/q_o)$	$2/3H_o^{-1}$	
$1+z(r)$	$C^2/q_o p^{-1}$	$C_-^2/q_o h^{-1}$	t^2/ϕ^2	$C(K)/(q_o \xi(\phi))$
dz/dr	$C^2/q_o g p^{-2}$	$C_-^2/q_o g_- h^{-2}$	$2t^2/\phi^3$	$C^2(K)/(Q_o \xi(\phi))$
r	$w-\cos^{-1}((z+\cos w)/D)$	$w-\cosh^{-1}((z+\cos w)/D)$	$t[(D^{\frac{1}{2}}-1)/D^{\frac{1}{2}}]$	
$r_o(r)$	$Q_o \sin r\ p$	$Q_o \sinh r\ h$	$Q_o r \phi^2/2$	$Q_o g(r)\ \xi(\phi)$
$r_o(z)$	$H_o^{-1} q_o^{-2} D^{-2}\ \{q_o z + x(1-A)\}$		$2H_o^{-1} D^{-2}\{D-D^{\frac{1}{2}}\}$	
$M_o(z)$	$6H_o q_o D A^{-1}$		$3H_o D^{-\frac{1}{2}}$	
$z(r)$	$z^2 B + z(2B-q_o) + x(A-1) + B = 0$			
dr_o/dz	$(H_o q_o^2 D^3)^{-1}\ \{(3q_o-2-q_o z) + x(3q_o z+2-q_o)A^{-1}\}$		$-2(H_o D^3)^{-1}\{D-3/2D^{\frac{1}{2}}\}$	

These are the observational relations needed in the next chapter.

Finally, it is apparent that the FLRW universes and their observational relations are complex in these observational coordinates. Do there exist other coordinates than those considered above, that will simplify the situation?

There are two main possibilities. First, a new time coordinate w may be considered which defines the past light cone when $w=w_o$, but then measures proper time along the fluid flow lines (see e.g. Bondi 1947 as examined in chapter 8). Thus w does not define a null surface almost always, so while simplifying one metric component, another (the magnitude of the gradient of w) will appear; in addition, the direct interaction with the observations is then lost.

Secondly, a coordinate r that is geometrically meaningful (e.g. an area distance) can be considered, but such a coordinate is not fluid comovinq. The 4-velocity u will have as a consequence a second non-zero component and the relation of fluid expansion to coordinate or rotation coefficients will be complicated.

It is concluded that although it is possible that one of the alternatives might be a better choice of coordinates, it is believed that the difficulties encountered are intrinsic to the way in which the light cone relates to spatial homogeneity. Other choices suffer from the non invertibility of relation (6.29b) to obtain the explicit expresssion for $z=z(r_o)$ (see table for quartic expression of $z(r_o)$), in order to rewrite the observational relations in terms of r intead of z. As such our present notion is that the coordinates used here are the simplest.

7 Integration of FLRW Universes for Observational Data

7.1 Integration Scheme

The field equations for spherically symmetric dust, the optimal observational coordinate choice for FLRW universes, and observational relations in those universes have now been established. Our objective now is to construct the universes from observational data, i.e. we seek to integrate explicitly the dust equations on the past light cone utilizing FLRW initial data.

We pursue the integration from observational data first along the light cone (the radial equations) and then off the light cone (the propagation equations). The optimal coordinate choice, as discussed in the previous chapter, is attained by choosing the radial coordinate r such that B initially satisfies $B(w_o, r) = A(w_0, r)$; and the time coordinate w such that A satisfies $A(w,0) = B(w,0)$ on the central world line C. These initial conditions will result in $A(w,r) = B(w,r)$ everywhere, so regaining the optimal coordinates discussed above.

The observational quantities are $A(w_o, r)$, determined by the redshift z (see (3.12a)); $C(w_o, r)$, which is the area distance $r_o(z)$ (see (3.12b)); and $M_o(z)$, the mass function derived from number counts (see (3.12d)). The basic integration scheme applies to all three curvature cases and is as follows:

Radial Integration

(a) integrate equation (5.14b), the Null Raychaudhuri Equation, down the initial null cone ($w = w_o$) to determine z(r) there from the observable relations $r_o(z)$, $M_o(z)$. Then substitute from z(r) to re-express the observational relations in terms of r instead of z.

(b) calculate $B'(w_o, r)$ from equation (5.8).

(c) integrate equation (5.14d) down the initial null cone to determine $C'(w_o, r)$.

(d) evaluate $\omega_o(r)$ from equation (5.14a).

Time Integration

Integrate the system of equations (5.15) off the initial null cone,

(e) determining co and V from equations (5.13).

(f) determining A'(w,r) by integrating (5.15c) from the central line C down successive light cones.

(g) integrating (5.15a) and (5.15b), which determine C" and 8" respectively, along the matter world lines.

Essentially, we know

$$\{A,\ A',\ B,\ B',\ B'',\ C,\ C',\ C'',\ \omega_o(r),\ \mu_o(r)\}$$

on each null cone $\{w= w_1\}$; this is sufficient to determine the same set of quantities on the next null cone $\{w= w_2\}$, so enabling us to determine the solution stepwise off $\{w= w_o\}$.

One should note here that in principle we do not have sufficient data to carry out this integration to the future, as incoming gravitational waves could destroy the spherical symmetry. However as long as spherical symmetry remains, we do indeed have sufficient data to make this prediction. Thus our procedure implicitly assumes the no-interfe•ence condition: incoming data to the future of the past light cone ($w= w_o$) does not destroy the spherical symmetry of the spacetime.

The Robertson-Walker initial data on the past light cone ($w= w_o$) are the $r_o(z)$ and $M_o(z)$ relations given in equations (6.29b,c). Our task is to construct the FLRW universes from that data. (Nel 1980, Integrated the radial equations in the Newman-Penrose formalism, and hence showed that this initial data would indeed give a FLRW universe. However he did not carry out the time integration; this would have been difficult using the NP formalism, based on null directions only).

7.2 Radial Integration

STEP 1: The Null Raychaudhuri Equation

(a) The first step (as in the integration carried out by Nel 1980) is to integrate equation (5.14b) to determine $z(r)$. We assume $r_o(z)$ and $M_o(z)$ are known from astronomical measurements; then

$$(r_o)' = dr_o/dz\; z' \;,\; (r_o)'' = d^2r_o/dz^2\,(z')^2 + dr_o/dz\; z'' \;.$$

where, as before, $g' \equiv \partial g/\partial r$. By our coordinate choice, on the initial surface,

$$A(w_o,r) = B(w_o,r) \Rightarrow A'(w_o,r) = B'(w_o,r).$$

Using these results and equations (3.12), (5.14b) can be rewritten

$$(dr_o/dz)\, z'' + (d^2r_o/dz^2 + 2(1+z)^{-1} dr_o/dz)(z')^2 = -(A_o/2(r_o/(1+z))M_o(z))z'$$

$$= -3q_o/(2q_o-1)\;(1+z)r_o$$

where $A_o \equiv A(w_o,0)$. This equation may be shown to result in the expression (6.29b) obtained for r_o. Rewriting this as

$$d(z'\, dr_o/dz\; (1+z)^2)/dr = -\tfrac{1}{2}A_o r_o (1+z) M_o(z)\; dz/dr,$$

one can integrate, using the central conditions $dC/dr|_o = B|_o = A|_o$, to obtain

$$dz/dr = A_o (dr_o/dz)^{-1} (1+z)^{-2} (\; 1 - \tfrac{1}{2} \int (1+z) r_o(z) M_o(z)\; dz). \qquad (7.1)$$

where we let

$$I = \int_0^z (1+z) r_o(z) M_o(z)\; dz$$

using (6.29b,c); hence

$$I = 2[1-(dr_o/dz)(1+2q_o z)^{\frac{1}{2}} H_o q_o^2\; (1+z)^3]$$

i.e. the completed integral is of the form $dz/dr = g(z)$. One then integrates $dr/dz = 1/g(z)$ to determine $r(z)$ with $r(0) = 0$, and finally inverts $r(z)$ to find $z(r)$.

Upto now the integration was general. For illustrative purposes we now pursue the case $K = +1$ only; a summary of all three curvature cases will follow thereafter. In the case of FLRW initial data substituting for $r(z)$ and $M_o(z)$ into (7.1) from (6.29b,c), setting $A_o = R_o$ given by (6.24c), and integrating yields

$$dz/dr = (1+z)(1+2q_o z)^{1/2}(2q_o-1)^{-1/2}. \qquad (7.2)$$

But this is the same as (6.29a); one can therefore integrate to obtain the z(r) relation (6.27a) in the case $2q_o > 1$. Thus,

$$r(z) = 2\tan^{-1}[(2q_o-1)^{1/2}((1+2q_oz)^{1/2}-1)/((2q_o-1)+(1+2q_oz)^{1/2})]$$

$$z(r) = (1/2q_o)[(2q_o-1)\tan^2(\psi+\tfrac{1}{2}r(z))-1] \ , \ \tan\psi = (2q_o-1)^{-1/2}$$

Then one can substitute this result into $r_o(z)$ to obtain (6.28). Now (3.12a,b) show

$$A(w_o,r) = B(w_o,r) = Q_o(1-\cos\Phi) = R(\Phi) \qquad (7.3a)$$

$$C(w_o,r) = Q_o\sin r\ (1-\cos\Phi) = g(r)\ R(\Phi) \qquad (7.3b)$$

where $Q_o = H_o^{-1}q_o(2q_o-1)^{-3/2}$ (see (6.24b)) and $\Phi = t_o - r$ (see (6.26)). Also (6.25b) and (5.13a) together with (6.29a), (6.27b) show

$$\mu_o(r) = 6Q_o\sin^2 r$$
$$\Rightarrow \mu(w_o,r) = 6/(Q_o^2(1-\cos\Phi)^3) = 6H_o^2q_o(1+z)^3 \qquad (7.3c)$$

STEP 2: The Other Radial Equations

(b) Equation (7.3a) immediately gives $A'(w_o,r)$, so (3.9) gives $B'(w_o,r)$:

$$B'(w_o,r) = Q_o\sin\Phi \qquad (7.4a)$$

(c) After some manipulation, remembering that $\Phi' = -1$ (eq (6.21a)), and applying (7.3a,b), equation (5.14d) can be rewritten in the form

$$\{C'\sin r(1-\cos\Phi)\}' = Q_o\{\sin^2 r\ \sin\Phi\ (1-\cos\Phi)\}'.$$

With the initial conditions (3.11), $C'_o=0$, this integrates to give

$$C'(w_o,r) = Q_o\sin r\ \sin\Phi \qquad (7.4b)$$

This is consistent with (7.3b). Equations (7.4a,b) imply

$$C'/C = \{\sin\Phi\ /(1-\cos\Phi)\} = B'/B \qquad (7.4c)$$

showing that the fluid expansion is isotropic on $\{w = w_o\}$. That is, the quantity β (see (4.3b)) is initially zero (iff $C'/C = B'/B$, which is identically satisfied here).

(d) Finally equation (5.14a) is now easily evaluated to give

$$\omega_o(r) = -Q_o \sin^3 r$$
$$\omega(w_o, r) = -1/(Q_o^2(1-\cos\phi)^3) \qquad (7.4d)$$
$$= -H_o^2 q_o (1+z)^3$$

which is consistent with (6.27a). Comparing with (7.3c) and (6.7), (6.8) shows that the quantity a (see eq (4.3a)) is initially zero. This is consistent with the integration carried but in (6.22a) resulting in

$$Q_o = M/6 \qquad (7.4e)$$

Thus , as both a and P vanish initially, the observational data has uniquely determined that the spacetime is initially a FLRW spacetime (see eqs (6.4); and cf. Nel 1980).

Eq (5.14c) is now automatically satisfied, providing a check on our result, i.e. $\omega_o' = -3Q_o \sin^2 r \cos r$.

7.3 Time Integration

Step 3: Time Integration

(e) Substituting for w_o, g_o one can write (5.15a,b) in the form

$$u' = (A'/A - B'/B - C'/C)u + (3Q_o \sin^2 r A^2 C^{-3})v,$$
$$v' = (C/B)u,$$

where $u = C'/C - B'/B$, $v = (C/B) - \sin r$. The unique solution such that $u=v=0$ initially is $u = v = 0$. Thus we have

$$C(w,r) = \sin r \, B(w,r) \qquad (7.5a)$$

(this is v= 0, which then implies u= 0). The initial isotropy of the expansion (7.4c) and relation between B and C (determined by (5.3)) guarantee that both these relations hold at all times. Then we have

$$(C'/C = B'/B) \Rightarrow (C''/A = B''/B) \Rightarrow (3\omega_o = \tfrac{1}{2}\mu_o \sin r) \qquad (7.5b)$$

everywhere (the last following from (5.15)) since (C' '/C - B"/B)= W: r; u+ A a ; which is just the statement:

$$\alpha(w,r) = \beta(w,r) = 0 \qquad (7.5c)$$

-our previous characterization of FLRW models (see (6.3,4)).

(f) From (5.15c) and (5.14d) we can obtain the equations

$$p' = \dot{B}s + A'/A\, p - \omega ABq,$$

$$p = \dot{q},$$

where $p \equiv \dot{B} - \dot{A}$, $q \equiv B - A$, $s \equiv B'/B - A'/A$. Now our boundary conditions set $p=q=s=0$ on $\{w = w_o\}$ and on $\{r=0\}$. Thus for all three cases the solution for isotropy is $p = q = s = 0$ everywhere: $A(w,r) = B(w,r)$. Now (5.8) shows $A(w,r)$ is a function only of $(w-r)$. Calling this function $R(w-r)$, we have

$$A(w,r) = B(w,r) = R(w-r). \tag{7.6}$$

(g) Finally eq (3.15b) can be integrated with the initial conditions (7.3) to obtain

$$A(w,r) = B(w,r) = Q_o(1-\cos(w-r)), \tag{7.7a}$$

$$C(w,r) = Q_o \sin r\ (1-\sin(w-r)), \tag{7.7b}$$

which is in agreement with (7.4a,b).

Thus we have regained the solutions of chapter 6, given in the optimal coordinates, by our integration from data on and off the light cone.

7.4 Summary of Observational Quantities

We summarize the results of the integration procedure for all three cases. Upon examination of the results, the same recurring expressions appear as in subsect. 6.4 (see table 1). In addition, we find that a potential $\xi(\Phi)$ exists, given by

$$K = \begin{Bmatrix} +1 \\ -1 \\ 0 \end{Bmatrix} \Rightarrow \quad \xi(\Phi) = \begin{Bmatrix} 1-\cos\Phi \\ -1+\cosh\Phi \\ \Phi^2/2 \end{Bmatrix} \Rightarrow \xi'(\Phi) = \begin{Bmatrix} \sin\Phi \\ \sinh\Phi \\ \Phi \end{Bmatrix} = g(\Phi) \quad \Rightarrow \xi''(\Phi) = \begin{Bmatrix} \cos\Phi \\ \cosh\Phi \\ 1 \end{Bmatrix} \tag{7.8a}$$

where $g(\Phi)$ is as defined in (6.19c). Further, for $K = \{+1, -1, 0\}$,

$$\tan(\phi) = \left\{ \begin{matrix} \tan\phi \\ \tanh\phi \\ \phi \end{matrix} \right\} = \xi'(\phi)/\xi''(\phi) \qquad (7.8b)$$

and recall C(K) as defined in table 1; and as given, for K= +1, by eq (6.24a) from which $C = q_o \sin t_o$; similarly, $C_- = q_o \sinh t_o$ (for K= −1), $t_o/2 = ½t_o$ (for K=0). Hence

$$C(K) = q_o \xi'(t_o) = q_o g(t_o) \qquad (7.8c)$$

We can therefore rewrite all the observable quantities in terms of the potential function ξ. The results for the general case are tabulated below (see Appendix II for a full list of results for individual cases K= (+1, −1, 0)). The function ξ is, by (7.3a) and (6.23)

$$\xi(\phi) = R(\phi)/Q_o \qquad (7.9)$$

which is as anticipated, since the thermodynamic state along the flow lines of a perfect fluid is determined (for a given equation of state) by the average length, which we have normalized to be R(ϕ), alone (Ellis 1971; cf. subsect 2.3.5).

TABLE 3

Observational Results for the General Case K, in terms of a Potential ξ(ϕ)

$r(z)$	$2\xi'/\xi''\ [q_o\xi'(t_o)(A-1)/(C^2+A)]$
$z(r)$	$1/(2q_o)\ \{[(1+(C^2/(q_o\xi'(t_o)\xi'(r(z)/2))/(1-1/(q_o\xi'(t_o))\ \xi'(r(z)/2)$ $/\xi''(r(z)/2))]-1\}$
dz/dr	$DA\ (q_o\xi'(t_o))^{-1}$
$B(w_o, r)$	$Q_o\xi(\phi)$
$C(w_o, r)$	$Q_o\xi'(r)\xi(\phi)$
Q_o	$H_o^{-1} q_o^{-2} (\xi'(t_o))^{-3}$

$\mu_o(r)$	$6Q_o \xi'^2(r)$
$\mu(w_o,r)$	$6Q_o^{-2}\xi^{-3}(\phi) = 6H_o^2 q_o(1+z)^3$
$\dot{B}(w_o,r)$	$Q_o \xi'(\phi)$
$\dot{C}(w_o,r)$	$Q_o \xi'(r)\xi'(\phi)$
$\dot{C}/C = \dot{B}/B$	$\xi'(\phi)/\xi(\phi)$
$\omega_o(r)$	$-Q_o \xi'^3(r)$
$\omega(w_o,r)$	$-Q_o^{-2}\xi^{-3}(\phi) = -H_o^2 q_o(1+z)^3$
M	$6Q_o$
W	$-Q_o$
$C(w_o,r)$	$\xi'(r)\ B(w_o,r)$
$\ddot{B}(w_o,r)$	$Q_o \xi''(\phi)$
$\ddot{C}(w_o,r)$	$Q_o \xi'(r)\xi''(\phi)$
$3\omega_o$	$\frac{1}{2}\mu_o \xi'(r)$
$A'(w_o,r)$	$-Q_o \xi'(\phi)$
$C'(w_o,r)$	$Q_o[\xi''(r)\xi(\phi) - \xi'(r)\xi'(\phi)]$
$\dot{C}'(w_o,r)$	$Q_o \xi'(\phi - r)$
R_o	$H_o^{-1}(q_o \xi'(t_o))^{-1}$
$1+ z(r)$	$q_o^{-2}(\xi'(t_o))^{-1}\xi(\phi)$
$M_o(z)$	$6H_o q_o D A^{-1}$
$r_o(z)$	$H_o^{-1} q_o^{-2} D^{-2}\{q_o z + x(1-A)\}$
dr_o/dz	$H_o^{-1} q_o^{-2} D^{-3}\{(3q_o - 2 - q_o z) + x(3q_o z + 2 - q_o)/A^{\frac{1}{2}}\}$

8 General Spherically Symmetric Integration

8.1 Existance of a Potential and Resulting Field Equations

The general scheme of integration laid out in section 5 will hold for any spherically symmetric dust model, as well as FLRW universes. In particular, the coordinate conditions imposed (setting A = B initially) can be retained, and eq (7.1) can then be used to determine the relation between z and a distance parameter r which will turn out to be the FRW parameter r used in chapter 7, should the universe be a FLRW universe.

However there is a very elegant integration of these universes laid out by Bondi (1947), following earlier work by Tolman (1934), Oppenheimer, and others, in which the integration in standard coordinates is greatly simplified by the existence of a potential U. We seek to find in a natural way a similar quantity in our observational coordinates for these spaces. It is indeed possible, as will now be demonstrated.

The key feature is that either by eliminating to from (5.14d), (5.15a) and integrating, or directly by inspection of (5.14c), there is a function W(r) such that

$$W(r)= \dot{C}/A + C'/B \tag{8.1}$$

This is analogous to Bondi's eq (21). The existance of such a potential is a consequence of the field equations (5.3) (for verification see Appendix in Bondi's paper). Substituting into (5.14c) shows

$$\omega_o'(r)= -\tfrac{1}{2}\mu_o(r)W(r) \tag{8.2a}$$

which integrates to give

$$\omega_o(r)= -\tfrac{1}{2} \int \mu_o(r)W(r)\, dr. \tag{8.2b}$$

Also substituting (8.1) into (5.14a) gives

$$\omega_o(r)/C= (W^2(r) - 1)/2 - \tfrac{1}{2} (\dot{C})^2/A^2, \tag{8.3}$$

as required by (5.14b) and which also satisfies (5.15b,c), and supplies us with the equation of motion; i.e. the time evolution of C is governed by a "Friedmann equation" with ω_0 as the mass term and (W^2-1) as the curvature term; this form of the equation shows how the relativistic mass is affected by the total energy through the curvature. (8.2) and (8.3) together are the analogues of Bondi's equation (22). Taking the radial derivative of (8.1) shows

$$W'(r)= (C'/A)(B'/A) - (1/C)(\mu/2 + \omega_0 B/C). \qquad (8.4)$$

Finally, the geodesic condition (5.8) will hold:

$$B' + A' = 0, \qquad (8.5)$$

as it is implemented in (5.14b) and (5.15c). Thus (8.1,2) satisfy (5.14d), (5.15a) and (5.14c); (5.14a) is satisfied by (8.1); the radial derivative of (8.1) gives (8.4), and (8.5) holds.

We proceed to tabulate W(r) for the coordinate choice investigated previously. For K= (+1, -1, 0),

$$W(r)= C'/A + C'/B = \left\{\begin{array}{l} \cos r \\ \cosh r \\ 1 \end{array}\right\} = g'(r) \qquad (8.1')$$

$$\omega_0{}'(r)= -\tfrac{1}{2}\mu_0(r)W(r)= -3Q_0\left\{\begin{array}{l} \sin^2 r\ \cos r \\ \sinh^2\ \cosh r \\ r^2 \end{array}\right\} \qquad (8.2a')$$

$$\omega_0(r)= -\tfrac{1}{2}\int \mu_0(r)W(r)\,dr = -Q_0\left\{\begin{array}{l} \sin^3 r \\ \sinh^3 r \\ r^3 \end{array}\right\} \qquad (8.2b')$$

$$\left[\begin{array}{l}\text{MASS}= \omega_0 \\ \text{TERM}\end{array}\right] = \left[\begin{array}{l}\text{CURVATURE}= W^2- 1 \\ \text{TERM}\end{array}\right] - \left[R'^2/R^2=H_0{}^2\right] \qquad (8.3')$$

$\omega_0(r)/C$	=	$(W^2(r) -1)/2$	$- \tfrac{1}{2}(C')^2/A^2$	
$\sin^2 r/(1-\cos\phi)$	=	$-\sin^2 r/2$	$- \tfrac{1}{2}((\sin r\ \sin\phi)/(1-\cos\phi))^2$	(K= +1)
$\sinh^2 r/(-1+\cosh\phi)$	=	$\sinh^2 r/2$	$- \tfrac{1}{2}((\sinh r\ \sinh\phi)/(-1+\cosh\phi))^2$	(K= -1)
$r^2/(\phi^2/2)$	=	0	$- \tfrac{1}{2}(r\phi/(\phi^2/2))^2$	(K= 0)

$$W'(r)= \begin{cases} \sin r \\ \sinh r \\ 0 \end{cases} \qquad (8.4')$$

$$\Leftrightarrow W^2 - 1 = \begin{cases} -\sin^2 r \\ \sinh^2 r \\ 0 \end{cases} = \begin{pmatrix} -\sin r \\ \sinh r \\ 0 \end{pmatrix} \begin{pmatrix} \sin r \\ \sinh r \\ 0 \end{pmatrix} = W'(r)\ g(r)$$

$$B' + A' = 0 \qquad (8.5')$$

Geometrical Interpretation of W(r)

W(r) represents the total energy density and related to the curvature of 3-space as shown in eq (8.3). In addition, W(r) is also the ratio of effective to invariant mass

By (8.1), 4(0)= 1 for all three curvature cases as required by the central conditions (3.11), and since C'(w,0) = 0. However, at r 0 the three cases differ:

(a) for K=0, W(r) remains unity in accordance with flat 3-space;

(b) for K= +1, W(r) is symmetric about the central observer as required by spherical symmetry, but

(i) for K= -1 bound by a lower limit 1 with no upper limit,

(ii) for K= +1 bound by an upper limit 1 and lower limit -1.

Cases (i) and (ii) are sympathetic with eq (41) of Bondi (eq (26) of the same paper defines the dynamic equation with reversed sign notation).

The sign of W therefore)differentiates one curvature case from another; i.e.

$$K= \begin{cases} +1 \\ 0 \\ -1 \end{cases} \Leftrightarrow W(r \neq 0) \begin{cases} < 0 \\ =0 \\ > 0 \end{cases}$$

Further, since C'(w,r) is a continuous function, then W must be continuous and hence cannot change sign without vanishing at some r. Examining eq (8.1) it is clear that such a change of sign occurs only for positive curvature case. In Bondi's coordinate system W= 0 represents an impenetrable barrier, since ds (see metric. eq (1) and definition of W(r), eq (33) of Bondi) for any dr. However, in our coordinate system, for K= +1 both A and B are bound :

$$(t= r + \pi) \quad 2Q_o \geq A= B \geq 0 \quad (t = r)$$

Hence it is deduced that W= 0 corresponds to C' + C'= 0, which by the geodesic flow condition (8.5) implies g/(r)= cosr. To determine the physical condition this corresponds to, we examine the commutator coefficients (2.1) and rotation coefficients (2.4) and find

$$W= 0 \Leftrightarrow C\,(b+f) = 0. \Rightarrow (b+f)= 0,\ t\ R\ ;\ C= 0,\ t= \pi/2.$$

It can be shown that (b+4)= 0 for all real values of t including t= A/2, when C=0. Thus (b+f)= 0 is the solution for all t. But this is just the

arguments as in subsect. 6.1 therefore follow showing that such a solution cannot be spherically symmetric about a world line C and is therefore excluded from the set of solutions required here. i.e. for K= +1

$$W(r)= \cos r\ ,\ r \neq E\ \pi/2\ ,\ E= 1,\ 3\ ;$$

where E can take on only the integral values indicated since r runs only from 0 to 21t if K= 1 (cf. subsect. 2.5). So W(r) can change sign approaching r= E1/2 from below or above excluding this value. The set of solutions for all three cases is now satisfied everywhere for spherical symmetry, without ambiguity.

We note that by (7.3a) and the geodesic flow eq (8.5), W(r) as given by (8.1) is independent of the component functions A or B and depends entirely on the first derivative of the function g(r) (as given by (6.21c)).
Eqs (8.2a',b') provide a consistency check with 'previous results (see Table 2) as well as verifying that the energy density û(w,r) is that given by (6.25).
Eqs (8.3') reinstate the curvature for each case while (8.4') sheds light on the

relation between the curvature term and the function g(r) explored previously (see (6.21c)), related by the first derivative of W(r).
Further, W(r) is, by (7.9),

$$W(r) = R''(r)/Q_o$$

which shows how R(r) is introduced into eqs (8.1-4), as stipulated *by* the Friedmann eq (8.3) with $W_o(r)$ as given by (8.2b).

To summarize, W(•) may be determined for geodesic fluid flow (obeying (8.5)) by

(i) the function g(r), i.e. $W(r) = g'(r)$; $W^2(r) - 1 = W'(r)\, g(r)$, the latter evaluating simultaneously the curvature term where $g(r) = R'(r)/Q_o$; or alternatively

(ii) the Friedmann eq (8.3) for a given mass term and known component functions.
We note that the second alternative requires knowledge of the resulting functions A, B and C after integration of the Friedmann equation as demonstrated in chapter 6, and their derivatives (cf. eqs (29), (30) Bondi). However, the first method is independent of their form and therefore requires only the appropriate coordinate choice determining g(•).

8.2 Verification of Field Equations

We row have therefore a set of equations based on 14(r) which guarantee that all the Einstein field equations are satisfied. We shall prove this in the rest of this section.

First, we define a set of quantities $E_i(w,r)$ by the following relations:

$$E_1 = \omega_o(r)/C^3 + 1/(2C^2) - (C'/AC)(C'/BC) - ¼(C'/BC)^2, \qquad (8.6a)$$

$$E_2 = C''/C - (C'/C)(A'/A + B'/B) + ¼B\mu_o/C^2, \qquad (8.6b)$$

$$E_3 = \omega_o' + ¼\mu_o(C'/A + C'/B), \qquad (8.6c)$$

$$E_4 = \dot{C}'/C - (\dot{B}/B)(C'/C) + \omega_o AB/C^3, \tag{8.6d}$$

$$E_5 = \ddot{C}/C - (\dot{C}/C)(\dot{A}/A) - \omega_o A^2/C^3, \tag{8.6e}$$

$$E_6 = \ddot{B}/B - (\dot{B}/B)(\dot{A}/A) + 2\omega A^2 + ½\mu_o A^2/BC^2, \tag{8.6f}$$

and the quantity G(w,r) by

$$G = \dot{B} + A'. \tag{8.7}$$

Then, from chapter 5, Einstein's equations hold for a spherically symmetric dust-filled space-time if and only if

$$E_1 = E_2 = E_3 = E_4 = E_5 = E_6 = G = 0. \tag{8.8}$$

Next, we define a set of quantities $F_J(w,r)$ by the following relations:

$$F_1 = W(r) - \dot{C}/A - C'/B, \tag{8.9a}$$

$$F_2 = \omega_o(r)/C - (W^2(r) - 1)/2 + ½(\dot{C})^2/A^2, \tag{8.9b}$$

$$F_3 = W'(r) - (\dot{C}/A)(\dot{B}/A) + (1/C)(\mu_o/2 + \omega_o B/C), \tag{8.9c}$$

$$F_4 = \omega_o'(r) + ½\mu_o(r)W(r) \; ; \tag{8.9d}$$

then the relations (8.1) to (8.4). defined at the beginning of this chapter hold if and only if

$$F_1 = F_2 = F_3 = F_4 = 0.$$

The following set of algebraic and differential identities between these quantities follow from the definitions above:

Time derivative equations:

$$F_2 + (\dot{C}/A + C'/B)F_1 + ½(F_1)^2 = C^2 E_1, \tag{8.10a}$$

$$F_4 - (\mu_o/2)F_1 = E_3, \tag{8.10b}$$

$$(F_1)^{\cdot} = -(C/A)E_5 - (C/B)E_4, \tag{8.10c}$$

$$(F_2)^{\cdot} = (\dot{C}C/A^2)E_5, \tag{8.10d}$$

$$(F_3)^{\cdot} = -(\dot{B}/A)(C/A)E_5 - (\dot{C}/A)(B/A)E_6. \tag{8.10e}$$

Radial derivative equations:

$$(F_1)' - F_3 = -(C/A)E_4 - (C/B)E_2 + G((C/A)^2 - C'/AB), \qquad (8.10f)$$

$$(F_2)' - (1/C)F_4 + WF_3 + (\dot{C}\dot{B}/A^2 - \omega_0 B/C^2)F_1 = (C\dot{C}/A^2)E_4 - ((\dot{C})^2/A^3)G. \qquad (8.10g)$$

One can immediately see that $(F_2 = 0)$ =) $(E_5 = 0)$ (by (6.10d)), so

$$(F_1 = F_2 = 0) \Rightarrow (E_1 = E_4 = E_5 = 0)$$

from (8.10c,d); then <8.10g) shows

$$(F_1 = F_2 = F_4 = G = 0) \Rightarrow (F_3 = 0).$$

Now the remaining equations easily show

$$(F_1 = F_2 = F_4 = G = 0) \Rightarrow (E_1 = E_2 = E_3 = E_4 = E_5 = E_6 = 0).$$

Thus we see that

$$(F_1 = F_2 = F_4 = G = 0) \Rightarrow (\text{Equations (8.8) hold}). \qquad (8.11)$$

That is, if in a spherically symmetric dust-filled space-time described in observational coordinates,

(a) the geodesic condition (8.5) holds:

$$\dot{B} + A' = 0 \ ;$$

(b) equation (8.1) defines a quantity W(r) :

$$W(r) = \dot{C}/A + C'/B \ ;$$

and

(c) the time evolution of C (w,r) is governed by the Friedmann Equation (8.3):

$$\omega_0(r)/C = (W^2(r) - 1)/2 - \tfrac{1}{2}(\dot{C})^2/A^2,$$

where the mass term $\omega_0(r)$ is given by the integral (8.2b):

$$\omega_0(r) = -\tfrac{1}{2}\int \mu_0(r)W(r)\, dr \ ;$$

then

equation (8.4) is identically true, and all the field equations are satisfied.

9 Summary of Conclusions to Part I

We have examined in detail general spherically symmetric and FLRW integration of Einstein's field equations. These are directly related to observable quantities for a perfect pressureless fluid and based on observational coordinates as well as a fluid-ray tetrad.

The condition for a geodesic fluid flow and fluid isotropic expansion were determined. FLRW models, characterized within the spherically symmetric dust filled universes by spatial homogeneity, were explored and the conditions for their existence determined.

The relation between observational functions was investigated and determined using observational coordinates for FLRW universe. The coordinate freedom was studied and various coordinate choices probed, establishing the simplest observational coordinate choice, resulting in three curvature cases. A general curvature case was developed encompassing all three curvature possibilities and determining the variation in terms, from one case to another (including the case $K= 0$).

Integration determined explicitly the observable quantities in terms of the Hubble constant, deceleration parameter and the redshift. It was concluded that the area distance- redshift relation is generally nontrivial. The complication encountered here is due to the nonsimple manner in which the light cone relates to spatial homogeneity.

Integration on (the radial equations) and off (the propagation equations) the light cone, using the optimal coordinate choice for FLRW universes and observational relations in those universes, regained and verified the previous solutions. The general curvature case was expanded and determined. It was found that there exists a function g($) generating all the observational quantities ; this function is the scaled radius of the universe.

The general spherically symmetric scheme of integration determined a function W(r) analogous to that found by Bondi. The function was determined in terms of observational quantities previously established; its general form was found and it was concluded that for a given coordinate choice and under conditions of geodesic fluid flow such a function depends only on the first derivative of the curvature function g(r) or the second derivative of the radius of the universe R(r). It was then proved that the Einstein field equations in this case are satisfied.

The procedure employed requires an extension to fluids with nonvanishing pressure and the resulting observational quantities. Can the integration be performed in this case and does it result in consistency as in the current case?

The work carried out here seeks to prove the uniqueness of the solutions of the field equations, and naturally leads to the question of stability of these solutions; it thus forms the basis to examination on axially symmetric and general perturbations about FLRW univeres.

10 Introduction to Part II

One of the major goals of relativistic cosmology, is the determination of the deceleration parameter q_o characterizing a 'best fit' Friedmann-LeMaitre-Robertson-Walker (FLRW) cosmological model of the universe (Sandage 1961, 1968). Observations of the deceleration of the redshift rate can in principle give the energy content of the universe, and we can obtain the components of the metric tensor by observations. Evaluation of q_o, for a given equation of state, will lead to a determination of the spatial curvature k/R_o^2 of FLRW universe models, where k is the sign of this curvature. If k=+1 the universe will of necessity be spatially closed and will have a finite future history ahead of it, recollapsing to a singularity in the future; while for k=0 or -1, its natural topology will be open (it will be "spatially infinite") and it will expand forever. However, the value of q_o has remained elusive despite tremendeous effort put into the search (see e.g. Gunn 1978, Tammann et al. 1980, Sandage et al. 1982), because of the many observational problems and the difficulties caused by the unsolved question of the nature and amount of galactic evolution (see e.g. R. Ellis 1983, Ostriker 1978, Freeman 1981, Norman 1983). Best current limits from direct observations of distant galaxies are in the range 0.1q_o 1.

This issue has acquired a new interest in recent years because the currently popular "inflationary universe" scenario strongly suggests the "naturalness" of k=0 presently (i.e. an Einstein-de Sitter universe model; see Guth 1981, Linde 1982, Albrecht et al. 1982) and therefore makes a definite prediction about q_o to explain formation of galaxies and clusters of galaxies, an initial matter density and metric tensor fluctuation is invoked resulting in q_o value lying within better than one part in 10^{4} of the critical value. However, all current astrophysical estimates indicate that there is sufficient matter present, detected either by the radiation it emits, or by its dynamical effects, to contribute for a $(1_{o}$ value of at most 0.11.0.05 (Peebles 1980a). However, all methods utilized are only sensitive to mass which clumps on scales 100 Mpc, and would not have revealed the presence of mass which is smoothly distributed out to Scales)>100 Mpc. Such observational techniques further rely upon the assumption that galaxies provide a good tracer of mass. The condition for the critical value seems to be unfulfilled. This apparent conflict may be resolved if the mass density of the universe today were dominated by:

(i) a cosmological constant which by definition is spatially constant, or by

(ii) relativistic particles produced by the recent decay of a massive relic particle species; such particles are necessarily, by virtue of their high speed, smooth on all scales upto the present horizon.

There are problems with either proposal (Turner et al. 1984, Peebles. 1984). Thus a large industry has arisen finding and promoting different ways that significant matter densities could reside in non-luminous, virtually indetectable forms of matter (see e.g. Primack et al 1983). Whether or not the inflationary universe proposal is correct, it is possible that the universe is dominated by relativistic particles at recent times (Discus et al. 1978). However if any of these forms of matter are actually present, this must be reflected in space-time curvature, and so on the one hand in the evolution of the scale function R(t) of the universe, and on the other hand, in the observational (area-distance, redshift) relation or equivalently in the (magnitude, redshift) relation. This implies the possibility of determining $(1_{o}$ either by comparing age limits of stars with the age of the universe, or by determining the redshift of matter in the universe. Thus the nature of observational relations in these circumstances is of interest.

We consider here only the case when the cosmological constant vanishes.

Aim

The second part of this thesis aims at:

(i) derivation and examination of the (area distance, redshift) relation for relativistic matter.

(ii) determination and investigation of the redshift of refocusing of null geodesics caused by the gravitational effect of matter in the universe, for both pressure-free and relativistic matter where the effects of evolution are considered.

(iii) determination of the (area distance, redshift) relation for stiff matter.

(iv) derivation and examination of the generalized equation of state for both pressure-free and relativistic matter and any given combination of either of these components.

(v) determination of the hyperbolic parametrization of the equations of state considered.

Outline

In chapter eleven the critical values of q for relativistic and pressure-free matter are determined. The influence of the equation of state for each case is then considered.

Chapter twelve reviews the observational determination of g_{o}. It then considers determination of cl_{o} through age limits on globular clusters.

The thirteenth chapter determines the observer area distance for p=0 and p='/3 and then derives the redshift of refocusing for both equations of state. The effects of evolution are then considered. Further, the two cases are quantitatively compared in terms of the (magnitude, redshift) relation.

Chapter fourteen determines the observer area distance for a general fluid mixture composed of any degree of either pressure-free or relativistic matter. It then evaluates the observer area distance for stiff matter. Finally it explores the hyperbolic parametrization of the equations of state considered.

11 **Field Equations**

11.1 The Critical Values of B_o

Evaluating the Friedmann equation (cf. eq (2.21b))

$$(R^{\cdot}/R)^2 = (8\pi G\mu/3) - K/R^2 \tag{11.1}$$

at the present time t_0, it follows that the Hubble constant $H_0 \equiv (R^{\cdot}/R)\big|_0$ and density parameter $\Omega_0 = 8\pi G\mu_0/(3H_0^2)$ are related to the present value R_0 of the radius function R(t) by

$$K/R_0^2 = H_0^2(\Omega_0 - 1) \; ; \tag{11.2}$$

Evaluating the Raychaudhuri equation

$$3R^{\cdot\cdot}/R + (4\pi G)(\mu + 3p) = 0$$

at the present time shows that the deceleration parameter q_0 defined by $q_0 = -(R^{\cdot\cdot}/R)\big|_0 H_0^{-2}$ is given by

$$q_0 = \tfrac{1}{2}\Omega_0(1 + 3p_0/\mu_0). \tag{11.3}$$

It is normally assumed that the pressure is zero at recent times; then

$$p_0 = 0, \quad \Omega_0 = 2q_0, \quad K/R_0^2 = H_0^2(2q_0 - 1) \tag{11.4a}$$

So the critical value separating the cases 1(=41 and K=-1 is $q_o=1/2$, and the inflationary prediction is that the value of g_o should be extremely close to 1/2. A crucial cosmological test is the independent measurement of n_o from the local mean density and the measurement of q_o from direct observation of the deceleration, to see whether in fact $0_o= 2q_o$ (Gunn 1978).

However, if the universe is dominated at recent times by relativistic particles, then

$$p_0 = \mu_0/3 \;, \quad \Omega_0 = q_0 \;, \quad K/R_0^2 = H_0^2(q_0 - 1). \tag{11.4b}$$

So the critical values separating the cases 1(=+1 and I(=-1 is $q_o=1$, and the inflationary universe prediction is that the value of q_o should be extremely close to I.

11.2 The Influence of the Equation of State

Indirect determination of (1$_0$ from c via eq (11.3) suffers from the variety of ways in which significant matter densities could reside in non-luminous, virtually indetectable forms of matter. However, presence of any of these forms of matter must be reflected directly in space-time curvature, and so on the one hand in the evolution of the scale function R(t) of the universe, and on the other, in the observational (area-distance, redshift) relation or equivalently in the (magnitude, redshift) relation. Determination of 0_0 from either of these relations will detect a uniform distribution of matter (unlike local dynamical tests such as determination of galactic rotation curves, which only detect inhomogeneities). Further, the contribution of matter of any kind (baryonic, non-baryonic, photinos, etc) to 0_0 will be taken into account. This is particularly attractive when considering the possibilities of dark matter residing in galactic halos and clusters of galaxies, since current understanding dictates such dark matter to be of a non-baryonic nature, in the form of neutrinos, gravitons, axions (corresponding to hot, warm and cold free streaming of elementary particles respectively on a cosmological scale; see Primack et al. 1983).

The equation of state relating the pressure p and energy density A enters these relations through the conservation equations

$$\mu' + 3(\mu+p)\, R'/R = 0$$

showing that in the case of pressure-free matter,

$$p=0,\ \mu=M/R^3 \ \ (M= \mu_0 R_0{}^3 = \text{constant}). \qquad (11.5a)$$

whereas in the case of relativistic matter,

$$p= \mu/3,\ \mu= M/R^4 \ \ (M= \mu_0 R_0{}^4 = \text{constant}) \qquad (11.5b)$$

We consider the effect of these equations of state on the age of the universe (subsect. 12.2) and the area-distance relations (chapter 13).

12 Determination of the Deceleration Parameter q_o

12.1 Observational Determination of q_o

We review the observational determination of g_o (see Bandage and Tammann 1982, and for a comprehensive review Fang et al. 1982). The methods of evaluation of g_o fall into three groups:

The first group determines g_o by evaluating the mean mass density in the universe. Groups 2 and 3 do so directly from the (angular diameter, redshift) and (apparent magnitude, redshift) relations of extragalactic objects (the 0-z and m-z methods).

Group 1: Mean Density of the Universe

Determination of the mean density μ_o or 0_o is attempted in three ways: direct evaluation of the galactic density, indirect evaluation of the galactic density from deviations from the Hubble flow and evaluation of the baryon density from the abundance of deuterium.

1) Direct evaluation of the galactic density - by measuring the mass to light ratio. There is evidence for massive halos (see e.g. Bosma 1978); however, the implied mass increase may not be more than a factor of 2. The results show

$$\Omega_G = 0.05$$

where 0_G is the density parameter corresponding to the galactic components in ûo (Yahil et al. 1980a).

2) Indirect evaluation of the galactic density from deviations from the Hubble flow - by observations of the local supergalaxy centred on the Virgo cluster. Measurements of 21 cm and infrared, and isotropy of CMWB radiation yield

$$0.3 < \Omega_G < 0.5$$

(Yahil et al. 1980b). Since this method measures all baryonic and non-baryonic matter which is clustered, the contribution by dark matter is seen to rise with scale. This suggests that dark matter is not as clustered as the galaxies are, and as such its scale of homogeneity is larger than the local supergalaxy size. Hence, the bounds on Ω_G can be regarded as lower bounds to the total mass density Ω_0.

From a statistical viewpoint, the entire galaxian population is regarded as a field of fluctuating density in random motion with respect to the Hubble flow. Usage of the cosmic energy theorem (Davis and Peebles 1977) and application of the second virial theorem (Peebles 1976) to observational data (Peebles 1979) yields

$$0.2 \lesssim \Omega_G \lesssim 0.6$$

showing agreement between the statistical approach results and those from the local supergalaxy (see Peebles 1984 and references therein).

3) The abundance of deuterium used to evaluate baryon density. The yield of light isotopes ^2D, ^3He, ^4He and ^7Li in the primeval fireball give a determination of the present Ω_0 contained in baryons. In particular, the yield of D is quite sensitive to baryon density. Its observed abundance, with a moderate allowance for depletion in stars, gives a relative abundance of $X_D < 3.5 \times 10^{-5}$ which requires (Yang et al. 1979)

$$0.04 \lesssim \Omega_B \lesssim 0.08.$$

Thus it is evident that Ω_B is smaller than Ω_G deduced from the local supergalaxy, which implies that a large part of the mass in the local supergalaxy cannot be in the form of baryons, and that it could be in the form of primeval black-holes, neutrinos, or gravitational waves.

Group 2: Angular Diameter-Redshift Relation

For an object of linear size 1, eqs (13.2) and (13.3) give a relation between its apparent angular size a and redshift z. This relation is quantitatively different from the simple inverse relation

$$\langle\alpha\rangle \propto \langle z\rangle^{-1} \qquad (12.1)$$

in that a does not decrease indefinitely with increasing z, rather it reaches a minimum at some definite value of z depending on q_0 and increases thereafter. It is therefore apparent that if we can identify a group of objects with the same linear size, q_o can be found from <13.2,3). This concept has been applied to:

1. extended radio sources with resolved components,
2. compact radio sources,
3. individual galaxies in clusters, and
4. galaxy clusters as units.

The main problem here is the uncertainty in the linear size of objects.

I. **Extended radio sources with resolved components**

Both radio galaxies and radio quasars are used. The obtained observed relation is (12.1) (e.g. Miley 1971) where the largest z used ranges 2.01z2.9 (e.g. Grueff et al. 1977). This simple inverse relation is unphysical within the standard model. The possible reasons are:

(i) evolution in the linear size, and

(ii) observational selection.

Wills (1979) pointed out that if the size evolution is of the form

$$l = l_n (1+z)^{-k}$$

then (13.2,3) will become (12.1) if the evolution is of that form with l<=2; then the observed relation (12.1) implies g_o= 1.

Selection effects can influence the LAS (Largest Angular Separation) values systematically at larger redshifts <Longair 1978).

2. **Compact radio sources**

Hewish et al. (1974) measured the angular sizes of compact sources or compact components <the 'hot spot') and found that <12.1) was not followed. By fitting (13.2,3) to their plots, they found

$$0.5 < q_o < 2.0.$$

However, their sample, although not plagued by selection effects, is too small for a firm conclusion.

3. Individual galaxies inside clusters

Baum (1972) using his 'photographic galaxy synthesizer' technique in the range 0.045210.46 found (1_o-0.3+ 0.2. However, Sandage (1972) found the usual isophotal diameters varied as z^{-1} giving q_o-1 . The difference may be attributed to an evolution effect, which affects the isophotal diameter, but not the metric diameter.

4. Angular size of galaxy clusters

Following the definition of Noonan (1972) for a cluster size, Bruzual and Spinrad (1978) find, for z upto 0.95, g_o= 0.25± 0.5.

To summarize,

Table 4

Various determinations of q_o using the a-z relation

Author	Object type	Largest z	q_o
Hewish et al. (1974)	compact radio sources	1.6	0.5-2.0
Baum (1972)	cluster galaxies	0.46	0.3±0.2
Sandage (1972)	brightest cluster galaxies	0.46	~+1
Bruzual & Spinrad (1978)	rich clusters	0.95	0.25±0.5

In conclusion, the evolution for any size parameter must be understood before any result can be properly interpreted.

Group 3: Apparent Magnitude-Redshift Relation

The problem facing us here is the identification of objects of the same luminosity. The possibilities which arise are:

1. The first ranked galaxies in clusters

These are the brightest members of clusters. Assuming that these first ranked galaxies have about the same luninosity, Bandage *et* al. (1976) find q_o=1.0.0.3.

2. Various subsets of quasars

It is only at large redshifts that different g_o predictions are distinguishable; hence quasars are a natural choice. Using luminosity indicators (i.e. standard candles; Kiang and Cheng 1982) the results obtained are 1.15%17.9, with a weighted mean of q_o=1.85, and a +20 range of 1.05%13.5. Thus the best estimate q_o=1.85 is more than 40 above the critical value of 0.5. These derivations were all made on the assumption of zero luminosity evolution, as shown by Hawking and Stewart (1981) to hold beyond a redshift of 0.5.

Summary of Results

Figure 4 (adapted from Fang et al. 1982) summarizes the various determinations of q_o made so far. The results show a large dispersion. However, the differences are systematic and depend on the method used.

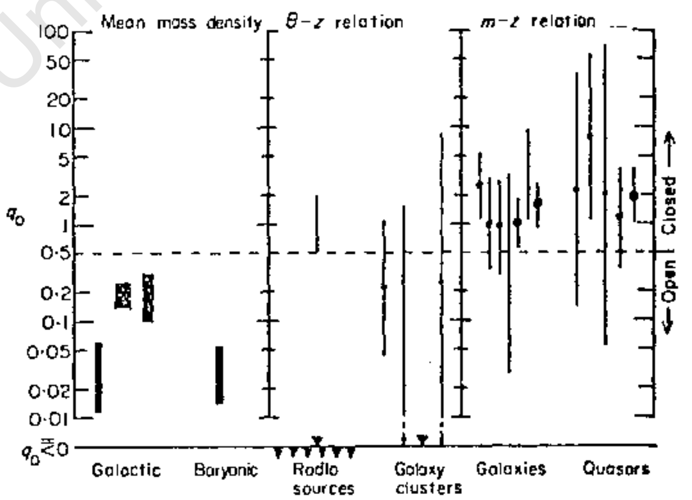

Figure 4. The various determinations of q_o depend strongly on the methods used. Lengths of error bars and symbols correspond to +20 (after Fang et al. 1982).

The Hubble diagram <i.e. the m-z method) consistently gives $q_o>1/2$, the mean mass density gives $q_o<1/2$, while the a-z method follows mostly the 'Euclidean' relation (a)a $(z)^{-1}$.

In the a-z and adz methods, no evolution in luminosity and linear size was assumed, respectively; while in the mean mass density method all matter was assumed to reside in galaxies, clusters or supergalaxies without a more uniform distribution of matter. The problem is lack of knowledge of luminosity evolution and evolution effects. Further, the mass to light ratio increases with system size. This suggests the existence of non-baryonic matter in the form of neutrinos or gravitational waves. Thus the existence of a quasi-uniform, non-luminous, non-baryonic mass distribution is favoured over luminosity decrease, because it takes cognizance, rather then ignores, the mean density results.

Moreover, recently it has been suggested by Lyubimov et al. (1980) that neutrinos have a small rest mass of between 14 and 46 eV. If so 99 percent of the mass in the universe will be provided by neutrinos, and they will certainly result in $q_o>$. , i.e. a closed, pressureless universe.

12.2 Age Omits and qo

The problem of measuring H_o has been recently reviewed by Sersic <1982), Tammann et al. (1980), Gunn (1978), van den Bergh (1975) and in IAU colloquium 37 (1977) (see in particular papers by Tully and Fisher, and Gunn). We summarize below the significant methods utilized in determining Ho.

Table 5

Summary of Data on Hubble Constant

Author	Method	H_o (km/s/Mpc)
1 Kirschner and Kwan (1974)	Supernova expansion ("Baade-Wesselink")	60±15

2	Sandage and Tammann (Babcock 1971)	Diameters of HII regions	72^{+30}_{-20}
3	van den Bergh (1960a,b)	Luminosity classification of galaxies	93^{+18}_{-15}
4	van den Bergh (1969)	Brightest globular clusters in galaxies	97^{+19}_{-16}
5	Roberts (1969)	mass-to-light ratios of galaxies	105^{+33}_{-26}
6	Palomar Sky Survey	Third brightest members of small clusters of galaxies	126^{+48}_{-35}
7	Kowal (1969)	Comparison of galactic and extra-galactic supernovae	69^{+41}_{-25}
8	Holmberg (1958)	Surface brightness of galaxies	89^{+46}_{-30}
9	Heidmann (1969)	Diameters of galaxies	89^{+46}_{-30}
10	van den Bergh (1972)	Brightest stars in galaxies	105^{+61}_{-39}
11	Sandage (1972b)	Dwarf galaxy colours	167^{+59}_{-43}
12	van den Bergh (1972)	Supernovae rate	95±36
Weighted mean			93±7

van den Bergh finds H_o=93±7 km/s/Mpc and understandably states that past experience teaches that little trust has to be placed in the small formal errors obtained for H_o.

All methods used were almost exclusively based on distances smaller than 20 Mpc. The value of H_o=93±7 km/s/Mpc therefore refers to a volume of space that contains the Virgo cluster and other groupings constituting the local supercluster. More recently, Tammann et al. <1980) find that H_o=50 km/s/Mpc. We conclude from the observational data that the Hubble constant is comprised within the bounds

$$50 \text{ Km/s/Mpc} < H_0 < 100 \text{ Km/s/Mpc}.$$

These bounds may be narrowed in the forseeable future by the envisaged Hubble Space Telescope (HST) (Longair 1985).

Globular Cluster Age Limits

The evolution of R(t) determines the age of the universe in terms of the Hubble constant. This puts limits on g_o that already cause severe problems for the inflationary proposal unless A#0 (Turner et al. 1984, Peebles 1984). For example, recent work by Penny and Dickens (1984) has obtained age limits of 161.2 Gyrs for the globular cluster NGC 6752. The age of the clusters must be less than the age of the universe.

If it were possible to make independent determination of H_o and t_o, the relation $H_o t_o = F(g_o)$ (Weinberg 1972) would allow us to estimate the value of g_o.

Table 6

Values of $H_o t_o$

q_o	0	0.1	0.2	0.5	0.8	1.0
$H_o t_o$	1	0.85	0.78	0.67	0.60	0.57
k		-1		0		+1

Actually, we can only set a lower bound for t_o, so this procedure provides only an upper limit.

Therefore in a low density universe (Er,O) with vanishing cosmological constant, the age of the universe is just less than $(H_o)^{d1}$, while in an inflationary universe with vanishing pressure (g_o=0.5), the age of the universe is just less than <2/3)$(H_o)^{-1}$.

Sandage (1982) proposed to add 0.2(Ho)d1 to the age of globular clusters to allow for the formation of galactic nuclei. The results for H_o are tabulated below.

Table 7

Bounds on H_o

q_o	H_o^{-1} (Gyr)	$+0.2\ H_o^{-1}$ 0	$+0.2\ H_o^{-1}$ 0.5	H_o (km/s/Mpc) 0	H_o (km/s/Mpc) 0.5	$+0.2\ H_o^{-1}$ 0	$+0.2\ H_o^{-1}$ 0.5
Globular cluster							
Age limit (Gyr)	16	20	34	63	42	50	29
Upper bound on $(H_o)^{-1}$	18	22.5	39	56	37	44	26
Lower bound on $(H_o)^{-1}$	14	17.5	30	71	48	57	33

So for example, the lower bound of 14 Gyr implies an upper limit of 70 km/s/Mpc on the Hubble constant for the low density universe; but in an inflationary universe the implied upper limit on the Hubble constant is 48 km/s/Mpc. If the lower bound on ages could be lowered to 16 Gyr or 18 Gyr, the inflationary universe upper bounds on the Hubble constant would be 42 km/s/Mpc and 37km/s/Mpc respectively, low enough to suggest conflict with Hubble constant measurements, which presently indicate values of 50 km/s/Mpc and over. Thus, provided the cosmological constant Λ vanishes, the age limits would then give evidence that the universe must be a low-density universe (when the corresponding upper bounds on the Hubble constant would be 63 km/s/Mpc and 56km/s/Mpc respectively, still comfortably compatible with the present Hubble constant estimates). In following Sandage (1982), H_o >17.5 and H_o<57 km/s/Mpc in the low density case; whereas in the critical density case H_o^{-1}>30 and H_o<30km/s/Mpc. Thus on this assumption the present lower age limit of 14 Gyr already provides evidence against the critical density case, as the present estimates of the Hubble constant indicate.

The inequality is strengthened if the universe is dominated by relativistic particles for then the age of an inflationary universe (q_o=1) is just less than $)^{-1}$ (Turner et al. 1984) and the implication is H_o<21 km/s/Mpc. The contradiction with present Hubble constant estimates is heightened if one accepts age estimates such as 18±2 Gyr (Sandage 1983). Thus if the cosmological constant **Λ** vanishes at recent times, the age limits suggest, whether p=0 or p=1/3, that the universe must be a lowddensity universe, contradicting the inflationary assumption (cf. Biome and Priester 1984).

The usage of age to determine limits on q_o has the disadvantage of relying on a determination of the Hubble constant. We point out here that measurement of the redshift at which the angular diameter of a suitable class of objects appears to be a minimum in a FLRW universe (Sandage 1961, Rowan-Robinson 1981) gives a simple observational test for q_o which might become feasible in the near future and is independent of the Hubble constant. Looking for the redshift of this minimum has several advantages over other more conventional ways of seeking to determine q_o directly from astronomical observations.

13 **Angular Diameters**

13.1 **Minimum Apparent Angles**

Let light be emitted at time t_o by an object of linear dimensions 1 in a FLRW universe and received by an observer at time t_{oe}. The observed redshift is then given by

$$1 + z = R(t_o)/R(t_e) \tag{13.1}$$

while the observed angular diameter a of the object, assumed to be lying perpendicular to the line of sight, is given by

$$1 = \alpha r_o. \tag{13.2}$$

Here r_o, the "observer area distance" (Ellis 1971), is given by

$$r_o = R(t_e)f(u), \quad u \equiv \int dR/R\dot{R} \tag{13.3}$$

where f(r)= {sinr, r, sinhr} if k= {+1, 0, -1} respectively and the integral is taken from $R(t_o)$ to $R(t_e)$.

It follows from equations (03.1,3), (11.1,4a,5a)) that when the pressure p and cosmological constant **A** are zero, r_o is determined in terms of the Hubble constant H_o and deceleration parameter q_o <q_o)0) by the expression (Mattig 1959, Sandage 1961)

$$r_o(H_o,q_o,z) = (H_o q_o^2(1+z)^2)^{-1}\{q_o z + (q_o - 1)[(1+2q_o z)^{1/2} - 1]\}. \tag{13.4a}$$

We have integrated equations (03.1,3), (11.1,4b,5b)) to obtain the corresponding relation in the case of relativistic matter, obtaining

$$r_o(H_o,q_o,z) = (H_o q_o(1+z)^2)^{-1}\{[q_o(1+z)^2 - (q_o - 1)]^{1/2} - 1\}. \tag{13.4b}$$

Equations (13.4a), <13.4b) are exact analytic expressions for the area distance in these two cases.

It has long been known that as one moves such a rigid object away in a FLRW universe, it will be seen to attain a minimum angular size and then to increase in apparent size again as it is moved still further away (Hoyle 1960, Sandage 1961); this corresponds to a refocusing of our past light cone because of the

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while the observed angular diameter a of the object, assumed to be lying perpendicular to the line of sight, is given by

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Here r_0, the "observer area distance" (Ellis 1971), is given by

$$r_o = R(t_e)f(u), \quad u \equiv \int dR/RR^{\cdot} \tag{13.3}$$

where f(r)= (sinr, r, sinhr) if k= (+1, 0, -1) respectively and the integral is taken from $R(t_o)$ to $R(t_e)$.

It follows from equations (03.1,3), (11.1,4a,5a)) that when the pressure p and cosmological constant **A are zero**, r_o is determined in terms of the Hubble constant H_o and deceleration parameter q_o (q_o>0) by the expression (Mattig 1959, Sandage 1961)

$$r_o(H_o,q_o,z) = (H_o q_o^2 (1+z)^2)^{-1}(q_o z + (q_o-1)[(1+2q_o z)^{1/2}-1]). \tag{13.4a}$$

We have integrated equations ((13.1,3), (11.1,4b,5b)) to obtain the corresponding relation in the case of relativistic matter, obtaining

$$r_o(H_o,q_o,z) = (H_o q_o (1+z)^2)^{-1}([q_o(1+z)^2 - (q_o-1)]^{1/2} - 1). \tag{13.4b}$$

Equations (13.4a), (13.4b) are exact analytic expressions for the area distance in these two cases.

It has long been known that as one moves such a rigid object away in a FLRW universe, it will be seen to attain a minimum angular size and then to increase in apparent size again as it is moved still further away (Hoyle 1960, Sandage 1961); this corresponds to a refocusing of our past light cone because of the gravitational attraction of the matter it contains (Hawking and Ellis 1968, Ellis 1971). Considering the variation of the angle a with z, (11.1) shows that

$$d\alpha/dz = \alpha(l^{-1}dl/dz - r_n^{-1}dr_n/dz). \tag{13.5}$$

Setting da./dz= 0 in (13.5) and substituting from (13.4a) shows that the apparent angular size of a class of objects with intrinsic size 1(z) is a minimum when

$$x\{[2+(1+z)e](2q_0z+1) - q_0(1+z)\} = (2q_0z+1)^{1/2}\{(3q_0-2-q_0z) - (1+z)(q_0z-x)e\} \tag{13.6}$$

where

$$e(z) \equiv l^{-1}\, dl/dz \;\Leftrightarrow\; l = e^{ez}$$

is a measure of the rate of change of size of the object at the time the light was emitted, and X= q_o-1.

For each value of q_o and function e(z), we regard (13.6) as an equation for z^*, the value of z for which the observed metric angular diameter of a class of objects of scale 1(z) is a minimum. For a given e, an inflection point on a q_o-z curve occurs. It is found where $dq_o/dz^*|_e = 0$.

Define a function

$$f(q_0, z, e) = x\{[2+De]A^2 - q_0D\} - A\{(3q_0-2-q_0z) - D(q_0z-x)e\}$$

where $D \equiv 1+z$, $A \equiv (1+2q_0z)^{1/2}$.

Then the condition $dq_0/dz^*|_e = 0$ is satisfied when $f=0$, $\partial f/\partial z|_e = 0$. This may be shown to result in the q_0-z^* relation

$$\{3xq_0 + q_0[A-CA^{-1}]\}/\{xF + q_0A^{-1}DE + AG\} + \{AC - x[2A^2 - q_0D]\}/\{xDA^2 + ADE\} = 0 \tag{13.7}$$

where $C \equiv 3q_0-2-q_0z^*$, $E \equiv q_0z^*-x$, $F \equiv 1+2q_0+4q_0z^*$, $G \equiv q_0+2q_0z^*-x$, and e is given by (13.6).

A BASIC program (Kantaris and Howden 1983) was implemented on a KAYPRO microcomputer to solve equation (13.6) for any of the quantities z^*, q_o, or e(z), when the other two are known; and equation (13.7) for either q_o or z^*. It was used to investigate the nature of the solutions of (13.6) and (13.7). We consider first the case when the class of objects observed are not changing in scale size (i.e. e= 0), and then the effect of the systematic evolution in source size.

Objects of Fixed Scale Size

The values of z^* are plotted in Figure 5 as a function of q_o; the curve for e=0 is the heavy curve. When q_o takes the critical value of 1/2, the minimum angular diameter occurs at $z^* = 5/4$ (assuming there is no source evolution). If z^* is less then 5/4, the universe is a high density universe which will collapse to the future; in particular if g_o= 1 then $z^* = 1$. If z^* is greater than 5/4, the universe is a low density universe that will expand forever; in particular if q_o= 0.1 then $z^* = 2.2$ and if $q_o = 0.02$ then $z^* = 3.97$. If the inflationary universe picture is correct, z^* must be equal to 5/4 to within better than one part in 10^4.

The proposal would be to search specifically for a minimum apparent size in a uniform class of objects lying at redshifts between 1 and 4, and particularly at redshifts between 1 and 1.5 . One must choose objects visible at those distances, with measurable redshifts, and with a directly measurable ("metric rod") linear size that is reasonably uniform for the class of objects. One would use suitable statistical tests to look for the value z^* of z corresponding to a minimum observed angular diameter for that class of objects, and determine if this value was approximately 1.25 or considerably greater. The sensitivity of the test is indicated by the curves in Figure 6: the inflationary case (g_o= 0.5) is the <e=0) curve in Figure 6a, and the lowddensity (q_o= 0.02) case is the (e=0) curve in Figure 6b. The statistical problem would be to distinguish between these two curves for the class of objects considered.

Three particular points need emphasis here. First, the value of z^* is independent of the Hubble constant H_o; so one does not have to solve the problem of the distance scale before using this as a test for q_o. Second, one only needs observations of the chosen class of sources near z^* .Thus one does not need to relate them to nearby "standard candles". The requirement is that at the epoch they are observed, they are reasonably standard in scale. Third, determining q_o by measuring the redshift z^* of refocusing will detect a uniform distribution of matter (unlike local dynamical tests such as determination of galactic rotation curves, which only detect inhomogeneities). Further, the procedure will detect matter of any kind (baryonic, non-baryonic, axions, photinos, etc), as long as its active gravitational effect is attractive. We currently believe that all known or postulated matter obeys this restriction.

What kind of objects could be observed in such a test? Galaxies are now being detected at redshifts close to those envisaged here, but spirals or ellipticals could not be used directly for the proposed test because one can only measure isophotal rather than metric diameters in these cases (Stock and schucking 1957, sandage 1961). However if one could detect barred galaxies at these distances then the apparent size from the centre of the galaxy to the bar could provide the appropriate angular measurement. It is just possible such measurements could be feasible by interferometric means <AAS Bulletin 1984, Chown 1985) - provided there are indeed barred galaxies at these redshifts. However it is likely that the variation in the diameters of the bars is such as to make them unusable for this purpose (A.P. Fairall, G.F.R. Ellis, private communication).

Clusters of galaxies could possibly provide suitable objects to use in searching for the minimum apparent angular diameter, but the analogue of the isophotal effect could make this also impracticable. However it is possible that careful analysis of the variation of the covariance function <Peebles 1980b) with galaxy redshift could reveal the value of z^*.

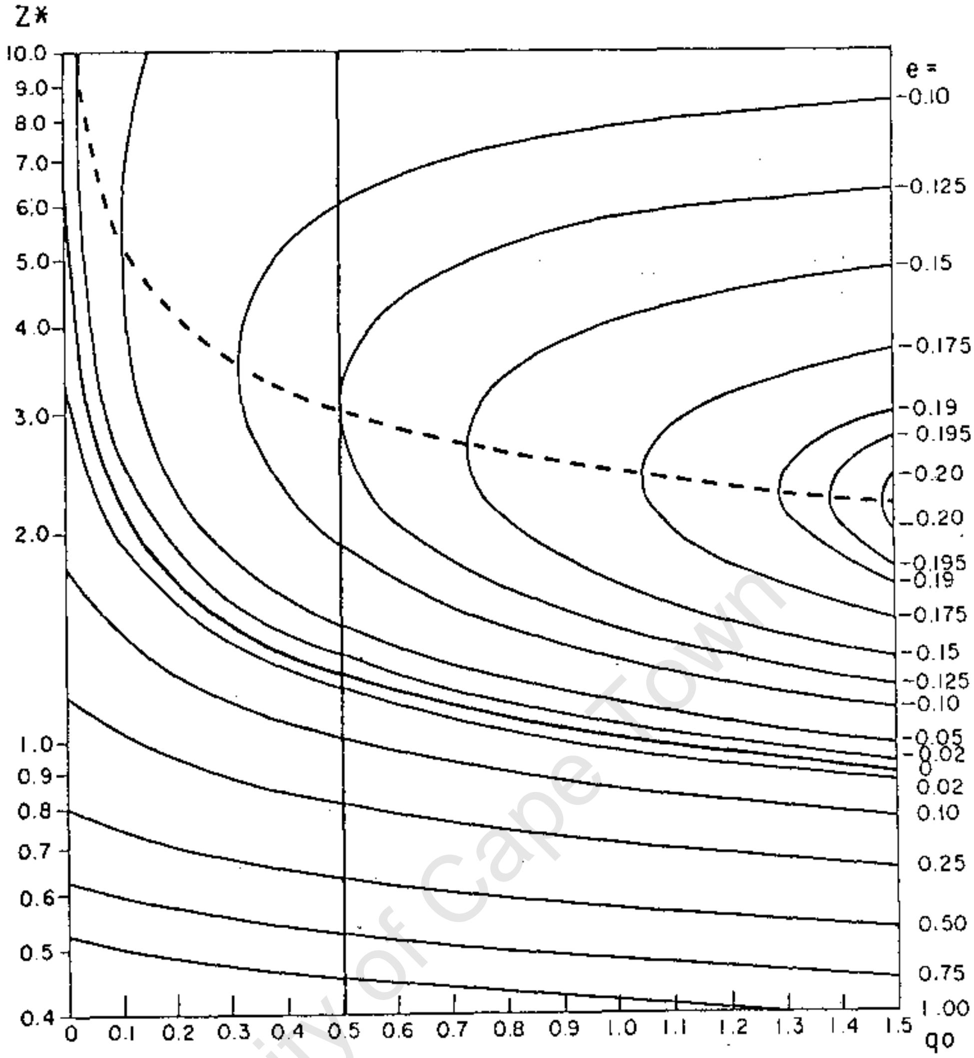

Figure 5

The redshift z^* at which refocusing takes place, and so linear scales are observed to have a minimum apparent angular diameter, as a function of the deceleration parameter g_o in a FLRW universe with vanishing pressure and zero cosmological constant. The parameter e labelling the curves denotes the rate of change of scale of the object observed at the redshift value z^*, positive values corresponding to a decrease in size at the time of observation and negative values to an object that is expanding at the time of observation. The heavy curve (e=0) is the case of objects of fixed scale size at that time. The vertical heavy line (q_o=.) is the case of an inflationary universe. When e is negative, ther will also occur an observed maximum in apparent size at a higher redshift than the minimum; the dotted curve separates the z^* value for apparent minima (below this curve) from those for the maxima (above it).

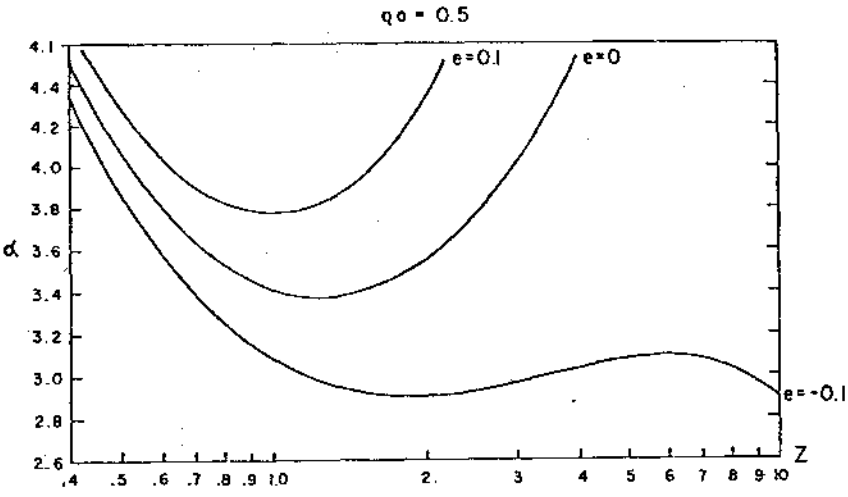

Figure 6a

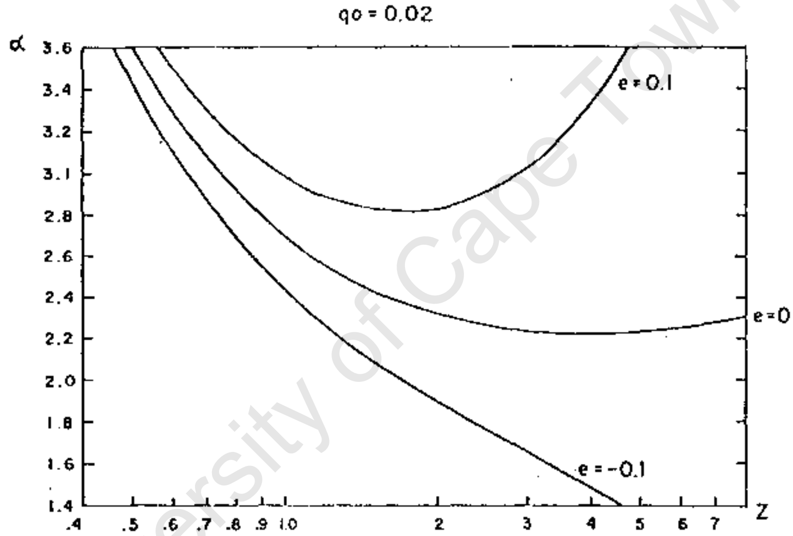

Figure 6b

Figure 6

6(a) Apparent angular size of an object at different redshifts in a critical density universe (q_o=16) for e=0, e=d0.1 and e=+0.1 . In the latter case there is an observed angular maximum at a higher redshift than the minimum. Note that it is not implied that any physical objects evolve at constant values of e over their entire histories; the curves refer rather to the evolutionary parameter e at the time of observation.

6(b) The same as in (a), but for a low density (q_o=0.02) universe. The observational problem is to determine whether or not this is a better fit to what we observe than (a), which is the inflationary universe case.

Quasi-stellar objects are easily detectable at the desired redshifts, but do not provide the required angular measurement. Probably the most promising class of objects are radio sources, which often have very clear measurable angular sizes. While studies have been done on the variation of angular sizes of radio sources with redshift (see e.g. Allington-Smith 1984) the minimum value z^* does not appear to have been explicitly searched for. One would need to measure angular sizes for a class of radio sources that seem physically similar, and with redshifts in the desired range. The problems to be dealt with would be the statistical range in the sizes of the objects concerned, and the question of whether these sizes were systematically varying (at the redshift z^*) or not. One would try to find a class of sources for which the variation parameter e was close to zero; then a straight statistical test would determine z^* and so g_o. The current Very Long Baseline Arrays (VLBA) project <Gordon 1985) may make such a measurement feasible.

Evolutionary Effects

If evolution effects in the scale size of the object at the time of observation are estimated to be significant, a somewhat more complex analysis is needed. Curves for different values of the evolution parameter e are shown in Figure 5.

Positive values of e correspond to an objects that were larger in the past (as z increases to the past) and so are decreasing in size at the time of observation For such objects the minimum apparent angular diameter occurs at a smaller value z^* than in the case of no evolution. Thus, for example, if the universe were filled with matter at precisely the critical density (so q_o=1/2), then the turn-around occurs at z^* = 1 if e= 0.1 and at z^* = 0.45 if e= 1.

Negative values of e correspond to objects that are expanding in size at the time of observation. For such objects the minimum apparent angular diameters occurs at a larger value z^* than in the case of no evolution. Thus for example if the universe were filled with matter at precisely the critical density <so o = 1/2), then the turn-around occurs at $z^* = 1.91$ if e= -0.1 . In this case, a second turn around (and so a maximum angular diameter) occurs at $z^* = 5.94$. This is shown in the curve in Figure 6a, and is apparent in Figure 5 where each curve of constant (negative) evolution parameter e intersects curves of constant o twice. Note that this effect is purely due to the changing size of the family of objects viewed; the past light cone of the observer continues contracting here, as shown by the fact that objects that are not evolving (e=0) do not have such a maximum. In the case of the critical density universe, there is a critical evolutionary value of e= -0.125: for objects expanding faster than this value, no minimum angular diameter is observed at all; if they are expanding slower, both a minimum and a maximum are observed. (Note that we are not supposing that any physical objects expand at this rate for all times; but rather that if this is the value of the evolutionary parameter at the time of observation, the angular size varies near that event as indicated here). For the critical value e= -0.125, there is neither a minimum nor a maximum but rather a point of inflection in the observed angular diameters at $z^* = 3$. The dashed line denotes the values of z^* and qo for which there occurs points of inflection; the curves plotted above this line represent values of z^* where the observed angular diameter is a maximum, whereas below this line they represent values where it is a minimum.

Through the possible evolutionary effects, angular diameters in an inflationary universe can partially mimic the angular minima in other universes. Thus an evolution e= -0.114 in the critical density universe will give a minimum angular diameter at the same redshift as a fairly low-density ($g_o = 0.1$) universe with no evolution; and an evolution e= +0.103 in a critical density universe will give a minimum angular diameter at the same redshift in a high density

(Wheeler-Sandage) universe with o = 1.0 and no evolution. On the other hand, if the density is low (o = 0.02), then no amount of evolution in a critical density universe can give a minimum angular diameter at the relevant redshift (e= -0.112 will indeed give a critical angular diameter at the appropriate redshift, but it will be a maximum and not a minimum !). In fact a minimum angular diameter at a redshift grater than 3 can only occur in a universe with less than the critical density (i.e. o < 0.5), no matter what the evolution may be. Note that if a family of objects were oscillating in size with increasing distance (and so redshift) this would correspond to an oscillatory variation of the value of e at constant o (e= 0 corresponding to a maximum or minimum linear size). The apparent angular change resulting from such a variation in size would result in detectable angular maxima and minima as redshift increases nearby, but sufficiently far out a final maximum would be detected (the possible values of z^* then lie above the dashed line, so no observed angular minima can occur).

Clearly one would wish to find a class of objects for measurement that were not expanding significantly at the time corresponding to z^*; if they were contracting, so much the better (for the turn-around could occur at much smaller redshifts). In any case, whether they are expanding or contracting the test has the advantage that one does not need to know the integrated evolution in source size from the present time but only rate of change of size near the redshift z^*. Further, given the value of e and o, z^* is independent of the Hubble constant even when the class of objects examined is evolving in size; thus again one does not have to evaluate the Hubble constant in order to determine the deceleration parameter.

Further Problems

As always, one would have to consider very carefully the problem of selection effects in the class of sources observed. This problem would be minimized somewhat in the case envisaged because one would only need a uniformity of selection of sources over a restricted range of redshift (near the turn-around value z^*). Nevertheless one would need careful consideration of such effects in the class of objects considered.

If matter is distributed in a lumpy rather than a smooth way with a fraction of the mean density in the form of intergalactic matter, the analysis becomes more complex because of local lensing effects (e.g. Dyer and Roeder 1973, 1974). However if selection effects do not consequently vary too significantly in the vicinity of the redshift value z^* (i.e. for inhomogeneities of less than galactic dimensions), one can argue that on average, the result should be the same as in a uniform universe when we observe over the entire sky (Weinberg 1976). Thus the major effect would be to make the statistical analysis yet more complex.

If the effective equation of state of the universe at late times were different, e.g. if there were some source of significant cosmological pressure, a new analysis would be required as then (13.4b) would replace (13.4a). Alternatively, a presently non-zero cosmological constant A is certainly possible, although one can raise various philosophical grounds for supposing it not to be so. The basic formula for the analysis, replacing <13.4a) is known (Kaufman 1971); but we do not pursue this case further here.

13.2 Observed Magnitudes

The respective (magnitude, redshift) relations are obtained from the area distance relation (13.4a,b) by substituting into

$$m = -2.5\log_{10}L + 5\log_{10}r_0(1+z)^2 + \text{const}, \qquad (13.8)$$

where L is the intrinsic luminosity of the galaxy observed. Particularly simple cases result when o= 1; both when p=0 and when p=P/3 the resulting relation is

$$m = -2.5\log_{10}L - 5\log_{10}H_0 + 5\log_{10}z + \text{const.} \qquad (13.9)$$

Sandage has given numerical values of ((13.4a), (13.8)) for the pressure free case for a variety of values of o and z, and plotted the corresponding curves (Sandage 1961). Using the same constants, we have determined the values of (,13.4b), (13.8)) for the same values of q_o (q_o* 0) and z. To our surprise, out to a redshift of 2 there is an indetectably small difference between the two sets of results (some examples of values obtained are given in Table 8; from <13.9) above, exactly the same results will be obtained in both cases when

for matter in the universe directly from observations of galaxy diameters or magnitudes for z<2.

Table 8

(Magnitude, Redshift) Relation for Non-Relativistic and Relativistic Matter

		q_0= 0.02			q_0= 0.5			q_0= 1	
z	log cz	M1	M2	DM	M1	M2	DM	M1=M2	DM
0.001	2.477	5.116	5.269	-.152	5.266	5.266	0.000	5.266	0
0.005	3.176	8.743	8.764	-.021	8.763	8.763	0.000	8.761	0
0.01	3.477	10.311	10.276	+.035	10.271	10.271	0.000	10.266	0
0.05	4.176	13.818	13.813	+.005	13.787	13.787	0.000	13.761	0
0.1	4.477	15.372	15.370	+.003	15.317	15.317	0.000	15.266	0
0.25	4.875	17.507	17.505	+.002	17.373	17.372	0.000	17.153	0
0.5	5.176	19.233	19.232	+.001	18.970	18.966	+.004	18.761	0
0.75	5.352	20.313	20.311	+.002	19.924	19.914	+.010	19.641	0

1.0	5.477	21.118	21.115	+.003	20.610	20.593	+.017	20.266	0
1.25	5.574	21.768	21.762	+.006	21.146	21.120	+.026	20.751	0
1.5	5.653	22.316	22.307	+.010	21.587	21.552	+.036	21.146	0
2.0	5.778	23.213	23.194	+.019	22.287	22.231	+.055	21.771	0
3.0	5.954	24.542	24.494	+.048	23.276	23.183	+.094	22.652	0
5.0	6.176	26.308	26.174	+.134	24.523	24.365	+.158	23.761	0
10.0	6.477	28.809	28.392	+.417	26.199	25.937	+.262	25.266	0

Table 8:

Apparent Magnitude M1 for the case of non-relativistic matter and M2 for the case of relativistic matter in a FLRW universe model, for different values of redshift z and deceleration parameter o (using the same constants as Sandage 1961). DM is the difference between MI and M2.

While the (magnitude, redshift) curves obtained in these two cases for any particular value of q_o will be indistinguishable, from <11.4a), (11.4b) the significance of the value of qo determined from such curves will be very different. Thus even if we can overcome the many observational problems (see e.g. Gunn 1978, Tammann et al. 1980, Sandage et al. 1982), and the difficulties caused by the unsolved question of the nature and amount of galactic evolution (cf. Fang et al. 1982), a direct determination of the value of q_o from astronomical observations up to $z \sim 2$ will lead to an ambiguous situation. For example, if we eventually determine $q_o=-1/2$, this could be support for the inflationary universe proposal (if non-relativistic particles dominate with p=0 at the present time and $q_o=4$ is the critical value), or a disproof of the proposal (if relativistic particles dominate with p=1 1/3 and $q_o=1$ is the critical value). Similarly if we were to prove $q_o=1$, this could be a disproof of the inflationary universe proposal <if the universe is recently dominated by non-

relativistic matter), or evidence for it of the recent universe is dominated by relativistic particles).

13.3 **Minimum Apparent Angles for Relativistic Matter**

Equation (13.4b) may be utilized, together with (13.5), to obtained the minimum apparent angular size for relativistic matter, given by

$$A(A-1)[2 + e(1+z)] = q_o(1+z)^2 \tag{13.10}$$

where $A \equiv [q_o(1+z)^2 - x]^{1/2}$. Following the procedure employed previously for p=0, the inflection point on a q_o-z curve for a given e is given by

$$B^2(2A-1) = A^2(A-1)(B + 2A(A-1)) \tag{13.11}$$

where $B= (1_o(1+z)^2$. In agreement with subsect. 13.2, it is found, when compairing <13.7) and (13.10) that for q_o=1, the minimum apparent angular sizes for p=0 and $p=1^{1}/3$ are identical, for all evolution rates, e.

Objects of Fixed Scale Size

The values of z^* are plotted in Figure 7 as a function of q_o; the curve for e=0 is the heavy curve. When q_o takes the critical value 1, the minimum angular diameter occurs at z^*=1 (as for p=0). Here, z^* less than 1 corresponds to a high density universe which will collapse to the future; in particular if q_o=1.5 then z^*=0.91 . If z^* is greater than 1, the universe is a low density universe that will expand forever; in particular if q_o=0.1 then z^*=1.89 and if q_o=0.02 then z^*=3.02. To fulfill the inflationary universe picture predictions, z^* must be equal to 1 to within better than one part in 10^4.

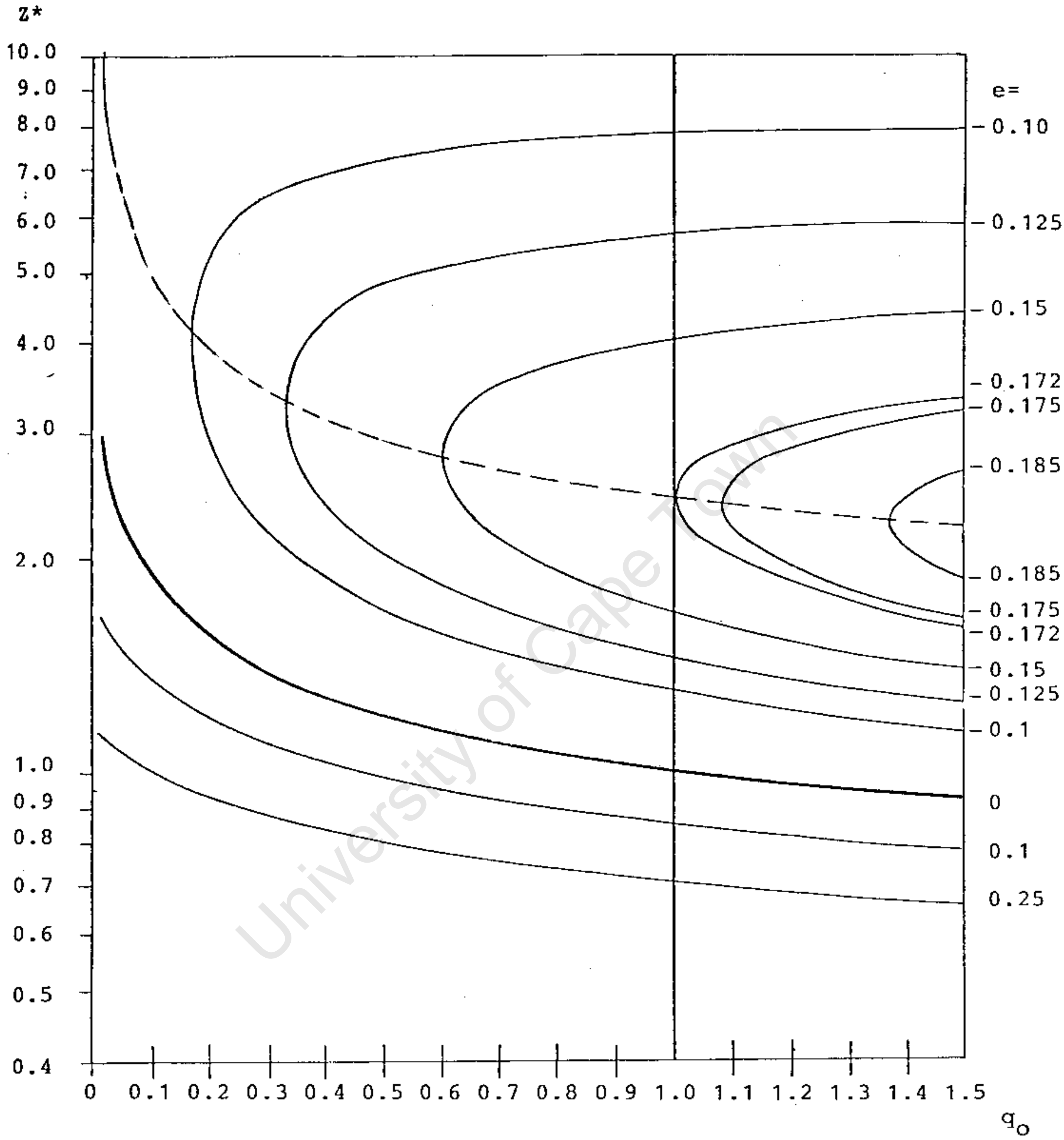

Figure 7

As in Figure 5, but for relativistic matter (p=g/3). The vertical heavy line for the inflationary universe is at q_o=1.

Evolutionary Effects

Evolution effects in the scale size of an object at the time of observation are plotted in Figure 7. The analysis follows the lines of the non-relativistic case with the crucial difference in the interpretation of the results, since here the critical density case is q_o=1. Thus, for example, for q_o=1 the turn-around for positive values of e occurs at z^* =0.84 if e=0.1 and at z^* =0.70 if e=.0.25; for negative values of e the turn around occurs at z^*=1.30 if e=-0.1. As for p=0, a second turn-around (and so a maximum angular diameter) occurs at z^* =7.70 . In the critical density universe, the critical evolutionary value is e= -0.172.

Angular diameter in an inflationary universe can again partially mimic the angular minima in other universes. Thus an evolution e= -0.163 in a critical density universe will give a minimum angular diameter at the same redshift as in a fairly low density (q_o=0.1) universe with no evolution; and an evolution e= +0.054 in a critical density universe will give a minimum angular diameter at the same redshift as a high density universe with q_o=1.5 and no evolution. On the other hand, if the density is low (q_o=0.02), then no amount of evolution in a critical density universe can give a minimum angular diameter at the relevant redshift (e= -0.166 will indeed give a critical angular diameter at the appropriate redshift, but it will be a maximum and not a minimum!). A minimum angular diameter at a redshift greater than 2.414 can only occur in a universe with less than the critical density (i.e. q_o<1.0), no matter what the evolution may be.

Comparison of Relativistic and Non-Relativistic Angular Diameters

As shown in subsect. 13.2 for the (magnitude, redshift) relation, out to a redshift of 2 observationally there is an indetectably small difference between the apparent angles for p=0 and p=$1^1/3$. From equations (13.2), (13.4a,b) the ratio of non-relativistic to relativistic apparent angles is given by

$$\alpha_{p=0}/\alpha_{p=\mu/3} = q_o\{[q_o(1+z)^2 - (q_o-1)]^{1/2} - 1\}/\{q_o z + (q_o-1)[(1+2q_o z)^{1/2} - 1]\} \qquad (13.12)$$

which is independent of evolutionary effects. We tabulate here the difference (in percent) between the two cases, where E defines powers of 10.

Table 9

(Minimum Apparent Angle, Redshift) Percental Difference for Non-Relativistic and Relativistic Matter

	q_o					
z	0.02	0.2	0.5	1.0	1.5	3
0.001	-5 E-5	-1 E-5	-2 E-6	0	0	0
0.01	-1 E-4	-2 E-6	-3.2E-6	0	+9 E-6	+7 E-5
0.1	-2.3E-4	-1.8E-3	-2.6E-3	0	+6.4E-3	+3.9E-2
0.25	-3.4E-3	-2.5E-2	-3.2E-2	0	+6.2E-2	+0.30
0.5	-2.4E-2	-0.16	-0.18	0	+0.27	+1.10
0.75	-7.2E-2	-0.43	-0.45	0	+0.56	+2.07
1.0	-0.15	-0.83	-0.79	0	+0.89	+3.10
1.25	-0.27	-1.35	-1.19	0	+1.23	+4.13
1.5	-0.43	-1.95	-1.62	0	+1.58	+5.14
2.0	-0.87	-3.31	-2.51	0	+2.24	+7.04
3.0	-2.16	-6.28	-4.23	0	+3.41	+10.37
5.0	-5.99	-11.75	-7.02	0	+5.24	+15.54
10.0	-17.46	-20.88	-11.36	0	+8.08	+23.80

Comparison of Tables 8 and 9 reveals however, that observations of galaxy diameters is a more sensitive detection method between these rival cases. For example, at quasar redshift of 3, a low density universe (q_o=0.02) will detect -2.16% metric diameter difference, but only +0.048 magnitude difference at a magnitude of 24.5, which represents a 0.2% detection difference. For q_o=0.5, at the same redshift, angular difference is -4.23% while magnitude difference is +.094 (or 0.40%) at a magnitude of 23.2 . We note that the non-relativistic matter consistently yields smaller diameters, when compared to the relativistic matter, for $o_{<}1$. The diameter difference increases with decreasing density of universe (i.e. decreasing q_o) upto a certain q_o (evaluated by differentiating (13.12) with respect to q_o and equating to zero; i.e. an inflection point) below which the difference continues to decrease. The situation is reversed for q_o>1. Non relativistic matter results in larger angular diameters, the difference increasing with q_o. Equation (13.12) is plotted below (Figure 8).

Evolutionary effects tend to accentuate the difference between the two cases with increasing (towards positive) values of e, i.e. towards objects whose rate of decrease in size at the time of observations is larger. As for equation (13.12) (which is independent of e) the difference follows the same pattern, and by (13.4a, b) will be enhanced by a factor of $e^{ez}(1+z)^{2}$. The angular diameter difference is plotted for q_o=0.5 and q_o=0.02 below (Figures 9a, 9b respectively). The proposal would therefore be to detect the angular diameter difference between the non-relativistic and relativistic matter in classes of objects which are decreasing in size at the time of observation, at the current measurable redshift range (i.e. z2-4). For a given class of objects, the discrimination between the equations of state p=0 and $p=1\,^{1}/3$ for the matter in the universe will depend entirely on o .

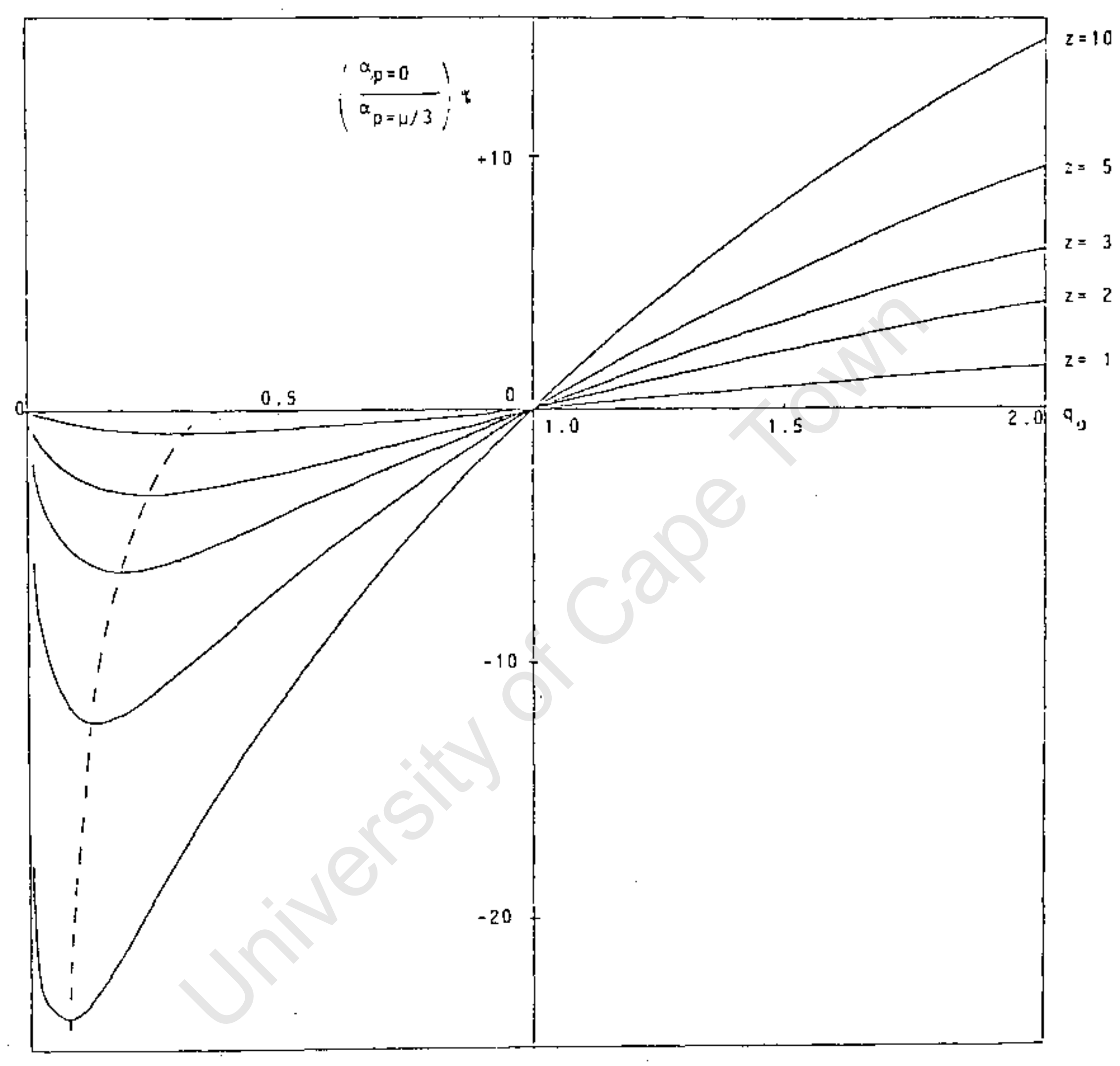

Figure 8

(Apparent angle, redshift) percental difference between non-relativistic and relativistic matter as a function of the deceleration parameter q_o in a FLRW universe with zero cosmological constant. The dashed curve seperates the increasing negative difference from the decreasing negative difference.

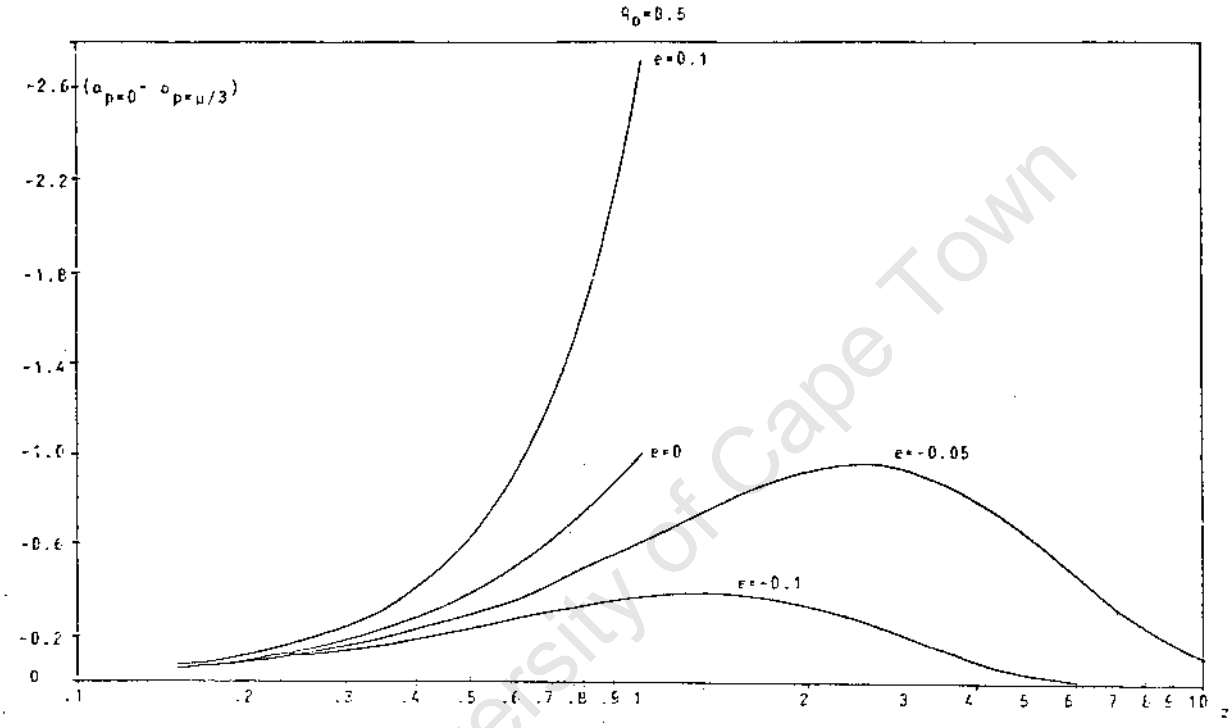

Figure 9a

Figure 9

(9a) Angular diameter difference between non-relativistic and relativistic matter, for qo=$_0$.5, and for e=0.1, e=0, e=-0.05, e=-0.1. In the latter two cases there is a maximum difference.

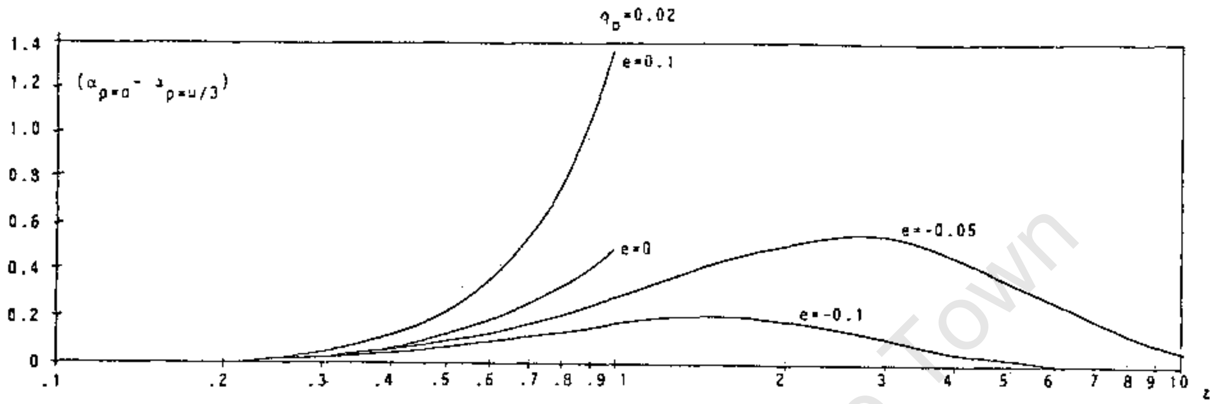

Figure 9b

Figure 9

(9b) The same as in (a), but for low density ($q_\circ$=0.02) universe.

14 Observer Area Distance for Various Equations of State

14.1 Observer Area Distance for a General Fluid Mixture

For both pressure-free matter and relativistic matter, the conservation equation

$$\mu^{\cdot} + 3(\mu+p)R^{\cdot}/R = 0$$

results in

$$\mu = M/R^3 + W/R^4 \quad (M = \mu_o R_o^3 = \text{const.}\ ,\ W = \mu_o R_o^4 = \text{const.}). \qquad (14.1)$$

From equations ((11.1), (11.3), (14.1)), which all apply to the general case of pressure free matter and relativistic matter, the curvature **k** is given by

$$k/R_o^2 = [M/3 + W/3R_o]/R_o^3 - H_o^2. \qquad (14.2)$$

Equation (11.3) may be shown to yield the present value of the energy density:

$$p=0 \Rightarrow \mu_o = 6H_o^2 q_o \ ; \quad p=\mu/3 \Rightarrow \mu_o = 3H_o^2 q_o \ , \qquad (14.3)$$

which together with (14.2) results in the evaluation of the constants M and W for the special cases:

$$p=0 \Rightarrow M/3 = 2q_o H_o^2 R_o^3 \ , \qquad (14.4a)$$

$$p=\mu/3 \Rightarrow W/3 = q_o H_o^2 R_o^4 \ . \qquad (14.4b)$$

Integration of <11.1) and (14.1) utilizing (13.3) and (14.2) yields the observer area distance,

$$r_o = [(B+A^2)(1+z)^2]^{-1} \ ((R_o^2 - AR_o)[B(1+z)^2 + 2R_o A(1+z) - R_o^2]^{1/2} - (R_o^2 - AR_o(1+z)) [B+2R_o A - R_o^2]^{1/2}) \qquad (14.5a)$$

where M/3= 2A, W/3= B. Defining W/M= C =) B/2A = C, we may rewrite (14.5a) in terms of A as a free parameter and C as a representative ratio of relativistic to pressure free matter:

$$r_o = [(2AC+A^2)(1+z)^2]^{-1} \ ((R_o^2 - AR_o)[2AC(1+z)^2 + 2AR_o(1+z) - R_o^2]^{1/2} - (R_o^2 - AR_o(1+z)) [2AC + sAR_o - R_o^2]^{1/2}). \qquad (14.5b)$$

Evaluating equations <14.5) for 8=0 <pressure-free matter) using (14.4a) results in

$$r_o = [H_o q_o^2 (1+z)^2]^{-1} \{q_o + (q_o-1)[(1+2q_o z)^{1/2} - 1]\} \tag{14.6a}$$

which is identical to (13.4a); and for A=0 (relativistic matter) using (14.4b) yields

$$r_o = [H_o q_o (1+z)^2]^{-1} \{[q_o(1+z)^2 - (q_o-1)]^{1/2} - 1\} \tag{14.6b}$$

which is identical to <13.4b). Thus equation (14.5a) (or (14.5b)) is the general fluid mixture equation.

Equation <11.3) may be rewritten in terms of A and C :

$$C = (R_o/2A)[H_o^2 q_o R_o^3 - A] \tag{14.7a}$$

$$\Rightarrow 6H_o^2 q_o = (6A/R_o^3)(1+C/R_o) + 6AC/R_o^4 \;;$$

Equation (14.2) can be written in the form

$$k/R_o^2 + H_o^2 = (2A/R_o^3)(1+C/R_o). \tag{14.7b}$$

Combining (14.7a) and (14.7b) we obtain a quartic for R_o in terms of A and C:

$$H_o^2(1-2q_o) + k/R_o^2 + 2AC/R_o^4 = 0; \tag{14.8a}$$

or a cubic in terms of A only:

$$H_o^2(1-q_o)R_o^3 + kR_o - A = 0. \tag{14.8b}$$

(equivalence of (14.8a), (14.8b) follows from 14.7a).

So, for example, $B=0 \Rightarrow A = q_o H_o^2 R_o^3 \Rightarrow k/R_o^2 = H_o^2(2q_o-1)$; $A=0 \Rightarrow k/R_o^2 = H_o^2(q_o-1)$ as required for pressure-free and relativistic matter respectively.

For the general case, let

$$A = fH_o^2 q_o R_o^3 \Rightarrow B = H_o^2 q_o R_o^4 (1-f) \;, \tag{14.9a}$$

$$0 \le f \le 1 \tag{14.9b}$$

where f is a constant describing the amount of pressure in the system. Thus, f=1 describes pressure-free matter (M/6 =A=0), and f=0 describes relativistic matter

(W/3 =B=O). Equation (14.8b) becomes an explicit equation for R $_o$,

$$k/R_o^2 = H_o^2[q_o(1+f) -1] \tag{14.10a}$$

$$\Rightarrow \quad k \overset{>}{\underset{<}{=}} 0 \quad \Leftrightarrow \quad q_o \overset{>}{\underset{<}{=}} 1/(1+f). \tag{14.10b}$$

i.e. k=0 =) o= (4 -) 1) ; k=+1 =) q_o)1/2, k=-1 =) qo<$_1$ are the extreme bounds in each of the curvature cases on qo for any fluid mixture obeying the equation of state (14.1) (and f as given by (14.9)). The observer area distance given by (14.5a) is composed of the following terms:

$$(B+A^2)(1+z)^2 = R_o^6 H_o^4 q_o[(q_o-1) +f], \tag{14.11a}$$

$$R_o^2-AR_o = R_o^4 H_o^2[q_o-1], \tag{14.11b}$$

$$\{B(1+z)^2 + 2R_oA(1+z)-R_o^2\}^{1/2} = \{R_o^4 H_o^2[q_oz(2+z(1-f))+1]\}^{1/2}, \tag{14.11c}$$

$$R_o^2-AR_o(1+z) = R_o^4 H_o^2[q_o(1-fz) -1], \tag{14.11d}$$

$$\{B+2R_oA-R_o^2\}^{1/2} = \{R_o^4 H_o^2\}^{1/2}. \tag{14.11e}$$

We notice that f appears only in terms which involve the redshift z.

Hence,

$$r_o = \{H_o q_o[q_o-(1-f)](1+z)^2\}^{-1}\{(q_o-1)[q_oz(2+z(1-f))+1]^{1/2} - [q_o(1-fz)-1]\} \tag{14.12}$$

Which is the general observer area distance equation for a fluid mixture with f given by (14.9).

14.2 **Stiff Matter**

For a stiff matter the equation of state is p= ^{1} . The conservation equation

$$\dot{\mu} + (\mu+p)3\dot{R}/R = 0$$

therefore shows

$$p=\mu \ , \ \mu = S/R^6 \quad (S = \mu_o R_o^6 = \text{const.}). \tag{14.13}$$

From equation {(11.1), (11.3), <14.13)), the curvature k is given by

$$k/R_o^{\ 2} = H_o^{\ 2}(q_o/2 - 1) \tag{14.14a}$$

$$\Rightarrow \quad k \gtreqless 0 \quad \Leftrightarrow \quad q_o \gtreqless 2. \tag{14.14b}$$

Integration of (11.1) and (14.13) utilizing <13.3) and (14.14) yields

$$u = \begin{cases} \tfrac{1}{2}\sin^{-1}(AR^2)\,]_{R_e}^{R_o} \\ \tfrac{1}{2}(AR^2)\,]_{R_e}^{R_o} \\ \tfrac{1}{2}\sinh^{-1}(AR^2)\,]_{R_e}^{R_o} \end{cases} \quad \text{if} \quad k = \begin{cases} +1 \\ 0 \\ +1 \end{cases} \tag{14.15}$$

where $A \equiv [H_o(q_o/2)^{½}R_o^{\ 3}]^{-1}$.

The factor of 1/2 occuring in each case in equations (14.15) necessitates the usage of the trigonometric relations for half angles. After manipulation, the resulting observer area distance is

$$r_o = ½\,[(q_o/2)^{½}H_o(1+z)^2]^{-1}\{D^{½}-D^{-½}\} \tag{14.16a}$$

where $D \equiv (B+C)/((q_o/2)^{½}+1)$, $B \equiv (q_o/2)^{½}(1+z)^2$, $C \equiv [B^2-(q_o/2 - 1)]^{½}$. For $k=0$ (i.e. for $q_o=2$) this expression reduces to

$$r_o = ½\,[(1+z)^3 H_o]^{-1}\{(1+z)^2-1\}. \tag{14.16b}$$

The form of r_o for $p=1$ as given by (14.16a) allows one to rewrite r_o as

$$r_o = [H_o(q_o/2)^{½}(1+z)^2]^{-1}\ \sinh\alpha \tag{14.17}$$

where a= 1/2 In D. This form of r_o eases the calculations of r_o to a great extent, as well as accentuating the geometrical form of r_o. Thus, for example, $q_o=2 \Rightarrow B=(1+z)^2$, $B+C= 2(1+z)^2 \Rightarrow D=(1+z)$. $\Rightarrow$ a= ln(1+z) and $r_o = 0.1481/H_o$. r_o for

$B+C= (o/2)^{½}+1$, which results in either of the conditions

$$z=0 \quad \text{or} \quad q_o=0.$$

14.3 Observer Area Distance for Different Equations of State in terms of Hyperbolic Sine

The hyperbolic form of $r_o|_{p=\mu}$ was shown to be mathematically and geometrically advantageous. It therefore encourages us to examine $r_o|_{p=0}$ and $r_o|_{p=\mu/3}$ and determine the possibility of extracting their hyperbolic form. Investigation of (13.4a) and (13.4b) revealed that this is indeed possible. Thus we can write

$$r_o|_{p=0}= [H_oq_o^2(1+z)^2]^{-1}A^{½}2\sinh\beta \qquad (14.18a)$$

where $\beta= ½ \ln A$, $A= 2+q_o(z-1)+ (q_o-1)(1+2q_oz)^{½}$; or

$$r_o|_{p=0}= [H_oq_o^2(1+z)^2]^{-1}(q_oz(2q_o-1) +(1-q_o)(1+2q_oz)^{3/4}2\sinh\gamma) \quad (14.18b)$$

where $\gamma= ¼ \ln(1+2q_oz)$. The latter expression (14.18b) is easily reduced, for $q_o=½$, to

$$r_o|_{p=0}|_{q_o=½}= 4/[H_o(1+z)^{5/4}] \sinh\gamma , \gamma_{q=½}= ¼ \ln(1+z). \qquad (14.18c)$$

and

$$r_o|_{p=\mu/3}= [H_oq_o^2(1+z)^2]^{-1}B^{¼}2\sinh\delta \qquad (14.19a)$$

where $\delta= ¼ \ln B$, $B= q_o(1+z)^2- (q_o-1)$; for $q_o=1$ this expression becomes

$$r_o|_{p=\mu/3}|_{q_o=1}= [H_o(1+z)^{3/2}]^{-1}2\sinh\delta , \delta_{q=1}= ½ \ln(1+z). \qquad (14.9b)$$

For the general expression (14.12), we find

$$r_o= [H_oq_o(q_o-(1-f))(1+z)^2]^{-1}A^{½}2\cosh\alpha , \alpha= ½ \ln A , A= (q_o-1)[q_oz(2+z(1-f))+1]^{½}- -q_o(1-fz). \qquad (14.20a)$$

or

$$r_o= [H_oq_o(q_o-(1-f))(1+z)^2]^{-1}((q_ofz)+ (q_o-1)B^{¼}2\sinh\beta) , \beta= ¼\ln B ,$$

$$B = q_o z(2+z(1-f))+1. \qquad (14.20b)$$

For $q_o=½$, (14.20b) reduces to

$$r_o = [H_o ½(f-½)(1+z)^2]^{-1}((½fz) - ½B^{½}2\sinh\beta) \;,\; B = z + ½z^2(1-f)+1 \;, \qquad (14.21a)$$

which, for f=1 (i.e. p=0), yields

$$r_o = 2/[H_o(1+z)^2] \{z - (1+z)^{½}2\sinh(½\ln(1+z))\}. \qquad (14.21b)$$

which agrees with (14.18c).

For $q_o=1$, (14.20b) becomes

$$r_o = z/[H_o(1+z)^2] \qquad (14.21c)$$

which is consistent with (14.19a), and remarkably is independent of f!. i.e. for

relativistic matter (as related to each other by (14.9)). We conclude from (14.20) that a mixture of pressure-free and relativistic matter (according to (14.9)) has:

(i) symmetric properties as represented by the hyperbolic cosine term in (14.20a) with symmetry at a=0 (satisfied at q_o=0=z); and deviation from symmetry expressed by the terms proceeding the hyperbolic cosine.

(ii) asymmetry as expressed in (14.20b) by the hyperbolic sine term (satisfied at z=0). Further, (14.21b,c) yield the relation between the critical observer area distance for pressure-free matter and relativistic matter:

$$r_o|_{q=½}|_{f=1} = 2r_o|_{q=1} - 4/[H_o(1+z)^{7/4}] \sinh(½\ln(1+z)) \qquad (14.14d)$$

15 **Summary of Conclusions to Part II**

The selection of a suitable class of sources would determine if the proposed measurement of the turn-around redshift z^* is in fact feasible or not. The introduction of new technologies such as the space telescope and optical interferometers at least raise hope that the measurement might be feasible at the redshift range needed <about I to 1.5). Interpretation of the result will then depend on estimates of the source linear size evolution at the time corresponding to the redshift z^*. While this will undoubtedly present problems, they may be no worse than those in the various other methods where inability to reliably estimate source evolution has prevented a good measurement of the deceleration parameter o. It is certainly worth trying this test in view of the decisive nature of the predictions made once the evolutionary parameter has been estimated; and particularly in view of the present apparent contradiction between the "high density" predictions of the inflationary universe model, and the apparent low density suggested by our currently best estimates of Qo from local astronomical studies.

A determination of the equation of state of matter in the universe at recent times - that is, whether the universe is dominated by relativistic or non-relativistic matter - is an indispensible component in any proposal to use astronomical observations to determine if the universe is hyperbolic, elliptic, or flat; for the deductions to be made from the value of qo depend crucially on this feature. Unfortunately the direct observational relations themselves will give no useful guidance, for z<2, on what this equation of state is. However tests based on the age of the universe are fairly sensitive to the equation of state.

Observations of the metric diameter constitute a more sensitive test than magnitude observations for the determination of the equation of state at the current observable range (z=2-4). This is further accentuated for objects which are decreasing in size at the time of observation. For a given class of objects, the discrimination between the equation of state will depend entirely on g_o.

For a General fluid mixture and stiff matter, the hyperbolic form of the observer area distance derived is both mathematically (from a manipulation point of view) and geometrically (from an interpretation view point) advantageous. Stiff matter is completely asymmetric at $q_o=0=z$. The general fluid mixture is symmetric at $q_o=0=z$; and has an asymmetry at z=0.
For $q_o=1$ the observer area distance is independent of the equation of state.

We have only considered here the case when A=0. Analytic expressions for the area distance are Known for the case of a dust-filled universe with non-zero cosmological constant (Kaufman 1971). Because of age problems, this may be the only possibility for the inflationary universe idea, whatever the equation of state of matter (Blome and Priester 1984).

Appendix I : Fluid-Ray Formalism : General Case

The generalised commutator relations (4.2) and rotation coefficients (4.3) for an arbitrary null vector field k :

Commutator Relations

$$[u,K] = (n+n^+)u + (n+n^+-l-l^+)K + (q^+-j^+)m + (q-j)m^+$$
$$[u,m] = (h-q)u + (h-q-a)K + (l-l^+-b)m - cm^+$$
$$[K,m] = -eu - (e+h+j)K + (n-n^+-f)m - gm^+$$
$$[m,m^+] = (f^+-f)u + (b^+-b+f^+-f)K + s^+m - sm^+$$

Rotation Coefficients

$$a = \Gamma_{200} = \gamma^0{}_{02} - \gamma^1{}_{02}$$
$$b = \Gamma_{320} = \tfrac{1}{2}\{-\gamma^2{}_{02} - \gamma^3{}_{03} + \gamma^0{}_{23} - \gamma^1{}_{23}\}$$
$$c = \Gamma_{220} = -\gamma^3{}_{02}$$
$$e = \Gamma_{211} = -\gamma^0{}_{12}$$
$$f = \Gamma_{321} = \tfrac{1}{2}\{-\gamma^2{}_{12} - \gamma^0{}_{23} - \gamma^3{}_{13}\}$$
$$g = \Gamma_{221} = -\gamma^3{}_{12}$$
$$h = \Gamma_{021} = \tfrac{1}{2}\{\gamma^0{}_{12} - \gamma^1{}_{12} + \gamma^0{}_{02} + \gamma^3{}_{01}\}$$
$$j = \Gamma_{210} = \tfrac{1}{2}\{-\gamma^3{}_{01} + \gamma^0{}_{12} - \gamma^1{}_{12} - \gamma^0{}_{02}\}$$
$$l = \tfrac{1}{2}(\Gamma_{100} + \Gamma_{203}) = \tfrac{1}{4}\{2\gamma^0{}_{01} - \gamma^1{}_{01} + \gamma^3{}_{03} - \gamma^2{}_{02} - \gamma^0{}_{23} + \gamma^1{}_{23}\}$$
$$n = \tfrac{1}{2}(\Gamma_{011} + \Gamma_{312}) = \tfrac{1}{4}\{2\gamma^0{}_{01} - \gamma^2{}_{12} - \gamma^3{}_{13} - \gamma^0{}_{23}\}$$
$$q = \Gamma_{201} = \tfrac{1}{2}\{\gamma^3{}_{01} - \gamma^0{}_{02} + \gamma^0{}_{12} - \gamma^1{}_{12}\}$$
$$s = \Gamma_{322} = -\gamma^3{}_{23}$$

The Jacobi identities and the Ricci tensor which generalize (5.1) and (5.5) respectively, for the case when K and u are not necessarily geodesic:

Jacobi Identities

$$ {}^{0}_{123}\quad D(f^+-f)+\delta e^+-\delta^+ e=(n+n^+-f-f^+)(f^+-f)+(s^++q^+-j^+-e^+-2h^+)e-(s+q-j-e-2h)e^+ $$

$$ {}^{1}_{123}\quad D(b-b^++f-f^+)+\delta(e^++h^++j^+)-\delta^+(e+h+j)=-(f+f^+)(b^+-b+f^+-f)-s(e^++h^++j^+)+ $$
$$ +s^+(e+h+j)+(f^+-f)(n+n^+-l-l^+)+e(a^++q^+-h^+)-e^+(a+q-h) $$

$$ {}^{2}_{123}\quad Ds^++\delta g^++\delta^+(n-n^+-f)=(n^+-n-f^+)s^++(e+h+j-s)g^++(e^++h^++j^+)(n-n^+-f)+ec^++ $$
$$ +e^+(l^+-l-b)+(f^+-f)(q^+-j^+) $$

$$ {}^{0}_{012}\quad -\Delta e+D(q-h)+\delta(n+n^+)=(b+2l)e+(n-n^+-f^+)(q-h)+(h+j)(n+n^+)+ce^++g(h^+-q^+)+ $$
$$ +(f-f^+)(q-j) $$

$$ {}^{0}_{023}\quad \Delta(f^+-f)+\delta(q^+-h^+)-\delta^+(q-h)=-(b+b^++n+n^+)(f^+-f)-(e+s)(q^+-h^+)+(e^++s^+)(q-h)+ $$
$$ +(b-b^+)(n+n^+)+e^+(a+q)-e(a^++q^+) $$

$$ {}^{1}_{012}\quad -\Delta(e+h+j)-D(h-q-a)+\delta(n+n^+-l-l^+)=(b+l+l^+)(e+h+j)+c(e^++h^++j^+)+ $$
$$ +(f+2n^+)(h-q-a)+g(h^+-q^+-a^+)+(q-h)(n+n^+-l-l^+)+(j-q)(b^+-b-f+f^+) $$

$$ {}^{2}_{023}\quad \Delta s^++\delta c^++\delta^+(l^+-l-b)=(l-l^+-b^+)s^++(q-h-s)c^++(q^+-h^+)(l^+-l-b)+ $$
$$ +(b-b^++f-f^+)(q^+-j^+)+g^+(a+q-h)+(f-n+n^+)(h^+-a^+-q^+) $$

$$ {}^{2}_{021}\quad \Delta(f-n+n^+)+D(l^+-l-b)+\delta(j^+-q^+)=(n+n^+-l-l^+)(f-n+n^+)-(n+n^+)(l^+-l-b)+ $$
$$ +(e+j+q)(j^+-q^+)+s^+(q-j)+gc^+-g^+c $$

$$ {}^{2}_{031}\quad \Delta g^+-Dc^++\delta^+(j^+-q^+)=(b-b^++l-3l^++n+n^+)g^+-(f-f^+-3n+n^+)c^++ $$
$$ +(e^++j^++q^++s^+)(j^+-q^+) $$

$$ {}^{1}_{023}\quad \Delta(f^+-f+b^+-b)-\delta(h^+-a^+-q^+)+\delta^+(h-a-q)=(l+l^+-n-n^+-b-b^+)(f^+-f+b^+-b)+ $$
$$ +(e+2h+j-q+s)(h^+-a^+-q^+)-(2h^++j^+-q^++s^+)(h-a-q)-e^+(h-a-q). $$

Ricci Tensor

$$-R_{00} = \Delta(b+b^{+}-n-n^{+}) - D(l+l^{+}) - \delta a^{+} - \delta^{+}a - (l+l^{+})(b+b^{+}) + (n+n^{+}-2l-2l^{+})(n+n^{+}) +$$
$$+ b^{2} + b^{+2} - (f+f^{+})(l+l^{+}) + a^{+}(e-2h+j+q-s) + a(e^{+}-2h^{+}+j^{+}+q^{+}-s^{+}) -$$
$$- 2hj^{+} - 2h^{+}j + 2cc^{+}$$

$$-R_{01} = \Delta(n+n^{+}) + D(l+l^{+}+b+b^{+}) - \delta j^{+} - \delta^{+}j + (b+b^{+})(n+n^{+}) + 2jj^{+} + jq^{+} + j^{+}q +$$
$$+ 2(l+l^{+})(n+n^{+}) + ej^{+} + e^{+}j + (n+n^{+})(f+f^{+}) - sj^{+} - s^{+}j - hq^{+} - h^{+}q -$$
$$- (n+n^{+})^{2} - he^{+} - h^{+}e + hj^{+} + h^{+}j + bf + b^{+}f^{+} + gc^{+} + g^{+}c$$

$$-R_{02} = \Delta h + Dh + \delta(b^{+}+l+l^{+}-n-n^{+}) - \delta^{+}c + h(b+b^{+}+f+f^{+}+2l-n+n^{+}) - (e+q)(l+l^{+}) +$$
$$+ (n+n^{+})(a+e+j+q) + b(e+j+q) - af^{+} + c(e^{+}+q^{+}-2s^{+}) - j(b^{+}+f^{+}) - g(a^{+}+j^{+})$$

$$-R_{11} = D(f+f^{+}) - \delta e^{+} - \delta^{+}e + e(j^{+}+2h^{+}+q^{+}-s^{+}) + e^{+}(j+2h+q-s) + 2ee^{+} -$$
$$+ (n+n^{+})(f+f^{+}) + f^{2} + f^{+2} + 2gg^{+}$$

$$-R_{12} = -Dh + \delta(n+n^{+}) + \delta f^{+} - \delta^{+}g - (2n^{+}+f+f^{+})h - (E+q)f^{+} + (q+j+e)f - j(n+n^{+}) +$$
$$+ (j^{+}-2s^{+})g + e(l+l^{+}-b^{+}-n-n^{+}) - ce^{+}$$

$$-R_{22} = \Delta g + D(c+g) - \delta(e+j+q) + g(2l-l^{+}+b+b^{+}+f+f^{+}+2n^{+}-2n) + c(3n^{+}-n+f+f^{+}) +$$
$$+ (2e+s)(j+q) + (2a+s)e + e^{2} + j^{2} + q^{2}$$

$$-R_{23} = \Delta f + D(b^{+}+f^{+}) + \delta s^{+} + \delta^{+}(s-e-j-q) + f^{+}(2l^{+}+n-n^{+}+b+b^{+}+f+f^{+}) + b^{+}(f+f^{+}+2n) +$$
$$+ s^{+}(2s-e-j-q) + e(a^{+}+e^{+}+j^{+}+q^{+}) + (e^{+}+j^{+})j + (e^{+}+q^{+})q +$$
$$+ f(l-l^{+}-n+n^{+}) - (n-n^{+})b + ae^{+}.$$

Appendix II : Observable Quantities for the Three Curvature Cases

With reference to subsect. 7.4 and in particular table 3, the three curvature cases K={+1, -1, 0) are tabulated below, using the definition of table 1 (subsect. 6.4) repeated here for clarity.

	$K = +1$	$K = -1$	$K = 0$
$r(z)$	$2\tan^{-1}[C(A-1)/(C^2+A)]$	$2\tanh^{-1}[C_-(A-1)/(C^2+A)]$	$t(1-D^{-1/2})$
$z(r)$	$1/(2q_0)[C^2\tan^2(\psi+\frac{1}{2}r(z))-1]$ $\psi = C^{-1}$	$1/(2q_0)[C_-^2\tanh^2(\psi-\frac{1}{2}r(z))-1]$ $\psi = C_-^{-1}$	$t^2/\phi^2 - 1$
dz/dr	$D\ A\ C^{-1}$	$D\ A\ C_-^{-1}$	$2t^2/\phi^3$
$A(w_0,r)$ $=B(w_0,r)$	$Q_0\ p$	$Q_0\ h$	$Q_0\ \phi^2/2$
$C(w_0,r)$	$Q_0\sin r\ p$	$Q_0\sinh r\ h$	$Q_0 r\ \phi^2/2$
Q_0	$H_0^{-1} q_0 C^{-3}$	$H_0^{-1} q_0 C_-^{-3}$	$4H_0^{-1} t^{-3}$
$\mu_0(r)$	$6Q_0\sin^2 r$	$6Q_0\sinh^2 r$	$6Q_0 r^2$
$\mu(w_0,r)$	$6Q_0^{-2}\ p^{-3}$	$6Q_0^{-2}\ h^{-3}$	$6Q_0^{-2}(\phi/2)^{-3}$
		$6\ H_0^2\ q_0\ (1+z)^3$	
$B'(w_0,r)$	$Q_0\ g$	$Q_0 g_-$	$Q_0\ \phi$
$C'(w_0,r)$	$Q_0\sin r\ g$	$Q_0\sinh r\ g_-$	$Q_0 r\ \phi$
C'/C $=B'/B$	g/p	g_-/h	$2/\phi$
$\omega_0(r)$	$-Q_0\sin^3 r$	$-Q_0\sinh^3 r$	$-Q_0 r^3$
$\omega(w_0,r)$	$-Q_0^{-2}\ p^{-3}$	$-Q_0^{-2}\ h^{-3}$	$-Q_0^{-2}\ (\phi/2)^{-3}$
		$Q_0/A^3(w_0,r) = -H_0^2 q_0 (1+z)^3$	
M		$6\ Q_0$	
W		$-\ Q_0$	
$C(w_0,r)$	$\sin r\ B(w_0,r)$	$\sinh r\ B(w_0,r)$	$r\ B(w_0,r)$
$B''(w_0,r)$	$Q_0\cos\phi$	$Q_0\cosh\phi$	Q_0

$C^{\cdot\cdot}(w_0,r)$	$Q_0 \sin r \cos\phi$	$Q_0 \sinh r \cosh\phi$	$Q_0\, r$
$3\omega_0$	$\frac{1}{2}\mu_0 \sin r$	$\frac{1}{2}\mu_0 \sinh r$	$\frac{1}{2}\mu_0 r$
$A'(w_0,r)$	$-Q_0 g$	$-Q_0 g_-$	$-Q_0\phi$
$C'(w_0,r)$	$Q_0[\cos r-\cos(\phi-r)]$	$Q_0[-\cosh r+\cosh(\phi-r)]$	$Q_0[\phi^2/2 -\phi r]$
$C^{\cdot\prime}(w_0,r)$	$Q_0 \sin(\phi-r)$	$Q_0 \sinh(\phi-r)$	$Q_0(\phi-r)$
R_0	$H_0^{-1}(2q_0-1)^{-\frac{1}{2}}$	$H_0^{-1}(1-2q_0)^{-\frac{1}{2}}$	$H_0^{-1}(t/2)^{-1}$
$1+z(r)$	$C^2 q_0^{-1} p^{-1}$	$C_-^2 q_0^{-1} h^{-1}$	t^2/ϕ^2
$M_0(z)$	$6 H_0 q_0 D A^{-1}$		$3H_0 D^{\frac{1}{2}}$
$r_0(z)$	$H_0^{-1} q_0^{-2} D^{-2}\{q_0 z+ x(1-A)\}$		$2H_0^{-1}D^{-2}\{D-D^{\frac{1}{2}}\}$
dr_0/dz	$(H_0 q_0^2 D^3)^{-1}\{(3q_0-2-q_0 z)+ x(3q_0 z+2-q_0)A^{-1}\}$		$-2(H_0 D^3)^{-1}\{D - 3/2\, D^{\frac{1}{2}}\}$

where C , C_- , p , h , g and g_- are defined as follws:

$K = +1$	$K = -1$	$K = 0$
$C = (2q_0-1)^{\frac{1}{2}}$	$C_- = (1-2q_0)^{\frac{1}{2}}$	$t_0/2$
$p = 1-\cos\phi$	$h = \cosh\phi - 1$	$\phi^2/2$
$g = \sin\phi$	$g_- = \sinh\phi$	ϕ

Notation Reference List

Symbols are listed alphabetically, according to the first letter in the symbol. Abbreviations follow thereafter.

A, B, C	component function
b	fluid expansion component in radial direction
C	central observer world line
c	speed of light
D, A, g	covariant operators of fluid ray tetrad
dS_o, d(	observer cross sectional area; solid angle
t_o	set of orthogonal vectors
F	selection effect; flux of radiation received
f	expansion of radial null geodesics
g	spacetime metric
H	Hubble constant
h	projection tensor
K, K	normalized ray vector; normalized curvature
k	affinely parametrized null vector field
L	source luminosity
l	fluid acceleration; distance
	spacetime manifold; relativistic energy density per source counted
m	apparent magnitude
N	number of sources
n	number density; fluid expansion component in radial direction
n^a	unit vector in direction of propogation of k
p	isotropic pressure
q	deceleration parameter
R	normalized radius of universe; Ricci scalar

✓	distance measure down past light cone from C
r_o	observer area distance
s	spherical polar relations of coordinates 0, 0 on 2-sphere
T	energy momentum tensor
t	time coordinate
u	normalized fluid 4-velocity
u*	fluid acceleration
v_5	speed of sound
w	past light cones of events along C
z	redshift
	object of anholonomicity
X	wavelength
V	total energy density
0	isotropic expansion
0, 4	angular coordinates based on C
	shear
	proper time
	vorticity
A	cosmological constant
	Ricci rotation coefficients

Abbreviations

CMWB	cosmic microwave background
emit	emitter
eq	equation
G	galaxy

o, obs	observer
subsect.	subsection

References

Abell, G.O.(1967). Astrophys. J. 72, 288.

Albrecht, A. and Steinhardt, P.J.(1982). Phys. Rev. Lett. 48, 1220

Allington-Smith, J.R.(1984). MNRAS 210, 611.

American Astronomical Society.(1984). Bulletin, AAS, 16, No.3.

Baum, W.A.(1972). IAU Symp.44.

Biome, H.J. and Priester, W.(1984). Naturwissenschaften 71, 528.

Bondi, H.(1947). MNRAS 107, 410.

(1960). Cosmology (Cambridge).

Bosma, A.(1978). PhD thesis, Groningen.

Boughn, S.P., Fran, D.M. and Partridge, R.8.(1971). Astrophys. J. 165, 439.

Brans, C. and Dicke, R.H.(1961). Phys. Rev. 124, 925.

Bruzual, A.G. and Spinrad, H.(1978). Astrophys. J. 220, 1.

Carmeli, M.(1977). Group Theory and General Relativity (McGraw-Hill).

Carter, B.(1974). In: Confrontation of Cosmological Theories with Observational Data, ed. M. Longair, I.A.U. Symp. 63 (D. Reidel, Dordrecht).

Chown, M.(1985). In: New Scientist, Vol. 105, No. 1441, 29.

Collins, C.B.(1977). J. Math. Phys. Vol. 18, No.11, 2116.

Curtis, A.R.(1950). Proc. Roy. Soc. A200, 248.

Davis, M. and Peebles, P.J.E.(1977). Astrophys. J. Suppl., 34, 425.

de Vaucouleurs, G.(1970). Science 167, 1203.

Dicus, D.A., Kolb, E.W. and Teplitz, V.L.(1978). Astrophys. J. 221, 327.

Dyer, C.C. and Roeder, R.C.(1973). Astrophys. J. 180, L31.

(1974). Astrophys. J. 189, 167.

Ehlers, J.(1961). Abh. Akad. Wiss. and Lit. Mainz, Math. Nat Kl, No. 11.

(1971). In: General Relativity and Cosmology; Ed. R.K. Sachs (academic Press, London).

(1973). In: Relativity, Astrophysics and Cosmology; Ed. W. Israel (D.Reidel).

Ellis, G.F.R.(1964). Ph.D. Thesis, Univ. Cambridge (unpublished).

(1967). JMP 8, No. 5, 1171

(1971). In: General Relativity and Cosmology; Ed. R.K.Sachs (Academic Press, London).

(1973). In: Cargese Lectures in Physics, Vol.6; Ed. E. Schatzmann (Gordon Breach).

(1975). Qu. Journ. Roy. Ast. Soc. 16, 245.

(1979a). Gen. Rel. and Grav. 11, No. 4, 281.

(1979b). S.A. Journ. Sci. 75, No. 12, 529.

(1980a). Ann. New York Acad. Sci. Vol. 366, 130.

(1980b). S.A. Journ. Sci. 76, No.11, 504.

(1984a). Invited Lecture, GR10, Padua, Italy 1983. Eds. A. Pascolini et al.

(1984b).In: Physical Science and History of Physics; Eds. R.S. Cohen and M.W. Wartofsky, 93 (D.Reidel).

and Baldwin, J.E.(1984). MNRAS Vol. 206, 377.

Maartens, R.M. and Nel, S.D.(1978). MNRAS 184, 439.

(1985). Rev. phys. Letters. North Noland, Amsterdam.

and MacCallum, M.A.H.(1969). CMP 12, 108.

and Perry, J.J.(1979). MNRAS 187, 357.

, Perry, J.J. and Sievers, A.W.(1984). Astron. Journ.Vol 89, No. 8.

and Sciama, D.W.(1972).In: General Relativity; Ed. L. O'Raifeartaigh. Clarendon Press, Oxford.

Ellis, R.(1983). In: The Origin and Evolution of Galaxies; eds. B.J.T. Jones and

J.E. Jones and J.E. Jones. Reidel, 255-293.

Fang, L.Z., Kiang, T., Cheng, F.H. and Hu, F.X.(1982). Quart. Journ. R. Ast. Soc. 23, 363.

Freeman, K.C.(1981). In The Structure and Evolution of Normal Galaxies; eds. S.M. Fall and D. Lynden-Bell. Cambridge Univ. Press, Cambridge.

Friedmann, A.(1922). Z. Phys. 10, 377.

Geroch, R., Held, A. and Penrose, R.(1973). JMP 14, 874.

Godel, K.(1952). Proc. Int. Cong. Math. Vol. 1, 175; Eds. L.M. Groves, E. Hills and Smith, R.A. Am. Mathematical Soc. Providence, Rhode Island.

Gordon, M.A.(1985). In: Sky & Telescope. Vol. 69, No. 6, 487.

Gott, J.R.(1980). MNRAS 193, 153.

Grueff, G., Schiavo-Campo, P., Vigotti, M. and Zanni, M.(1977). Astro. Astrophys. 60, 321.

Gunn, J.E.(1977). Colloq.37, CNRS No.263, 183.

(1978). In: Observational Cosmology; ed. A. Maeder, Swiss Society of Astronomy and Astrophysics, 1-124.

Guth, A.H.(1981). Phys. Rev. D23, 347.

Hawking, M.R.S. and Stewart, N.J.(1981). Astrophys. J., 251, 1.

Hawking, S.W. and Collins, C.B.(1973a). MNRAS 162, 307

(1973b). Ap.J. 180, 317.

and Ellis, G.F.R.(1968). Astrophys. Journ. 152, 25.

(1973). The Large Scale Structure of Space Time. (Cambridge).

Heckmann, O. and Shucking, E.(1959). Handbuch der Physik; ed. S.Flugge, Vol. 53,489 (Berlin).

(1962). In: Gravitation; ed. L. Witten.

Newish, A., Readhead, A.C.S. and Duffett-Smith, D.J.(1974). Nature, 252-657.

Hoyle, F.(1960). In: Proceedings of the Enrico Fermi Physics School, Course XX.

Academic Press, New York, 141.

Hubble, E.P.(1934b). Astrophys. J. 79, 8.

(1936). Astrophys. J. 84, 517.

Israel, W.(1960). Proc. Roy. Soc. A259, 129.

Kantaris, N. and Howden, P.F.(1983). The Universal Equation Solver. Sigma Technical Press.

Kantowski,R. and Sachs, R.K.(1966). J.Math. Phys. Vol. 7, No. 3, 443.

Kaufman, S.E.(1971). Astron. Journ. 76, 751; 77, 254.

Kiang, T.(1961). MNRAS 122, 263.

and Cheng, F.H.(1982). J. Astrophys. Astr., submitted.

King, A.R. and Ellis, G.F.R.(1973). CMP 31, 209.

Kristian, J. and Sachs, R.K.(1966). Astrophys. J. 143, 379.

LeMaitre, G.(1927). Ann. Soc. Sci. Bruxelles. 47A, 49.

Linde, A.D.(1982). Phys. Lett. 108B, 389.

Longair, M.S.(1978). In: Observational Cosmology; ed. A. Maeder, SSAA.

(1985). In: Sky & Telescope. Vol. 69, No. 4, 306.

Lyubimov, V.A., Norikov, E.G., Nozik, V.Z., Tretyakov, E.F. and Kozik, V.S.(1980). Phys. Lett. B., 94, 266.

Maartens, R.(1980). Ph.D. Thesis, Univ. of Cape Town.

, Ellis, G.F.R. and Nel, S.D.(1985). Observational Cosmology II. "Realistic Data"; in preperation.

MacCallum, M.A.H.(1973). In Cargese Lectures in Physics. Vol. 6; Eds. E. Schatzman (Gordon Breach).

(1979). In: General Relativity; Ed. S.W. Hawking, W. Israel. Cambridge.

Mannasse, F.K. and Misner, C.W.(1963). Journ. Math. Phys. 4, 735.

Mattig, W.(1958). Ast. Nach. 284, 109.

(1959). Ast. Nach.285, I.

Miley, G.K.(1971). MNRAS 152, 477.

Misner, C.W.(1968). Astrophys. J. 151, 431.

(1969). Phys. Rev. Lett. 22, 1071.

, Thorne, K.S. and Wheeler, J.A.(1973). Gravitation, Freeman, San Francisco.

McCrea, W.H.(1968). Science 160, 1295.

Nel, S.D.(1980). Ph.D. thesis, Univ. of Cape Town.

,Stoeger, W., Ellis, G.F.R., Whitman, A.P. and Maartens, R.(1985). Review Phys. Letters. North Holland, Amsterdam.

Newman, E. and Penrose, R.(1962). JMP 3, 566.

Noonan, T.(1972). Astr. J., 77, 134.

Norman, C.N.(1983). In: Formation and Evolution of Galaxies and Large Scale Structure of the Universe; eds. J. Thanh Van. Reidel.

Ostriker, J.P.(1978). In: IAU Symp. No. 79, The Large Scale Structure of the Universe ; eds. M.S. Longair and J. Einasto. Reidel.

Peebles, P.J.E.(1971). Physical Cosmology. Princeton Univ. Press, Princeton, NJ.

(1976). Astrophys. Space. Sci., 45, 3.

(1979). Astr. J., 84, 730.

(1980a). In: Physical Cosmology ; ed. R. Balian, J. Audouze and D.N. Schramm. North Holland, 213-270.

(1980b). The large Scale Structure of the Universe. Princeton Univ. Press, Princeton, NJ.

(1984). Astrophys. J. 284, 439.

Penny, A.J. and Dickens, R.J.(1984). Reported in The South African Astronomical Observatory: Report for the Year Ending December 31, 1984.

Penzias, A.A. and Wilson, R.W.(1965). Astrophys. J. 142, 419.

Primack, J.R. and Blumenthal, G.R.(1983). In: Formation and Evolution of

Galaxies and Large Scale Structure in the Universe; eds. J. Audouze and J. Thanh Van. Reidel.

Raychaudhuri, A.(1955). Phys. Rev. 98, 1123.

Rindler, W.(1969). Essential Relativity. Van Nostrand Reinhold, NY.

Robertson, H.P.(1935). Astrophys. J. 82, 248.

(1936). Astrophys. J. 83, 257.

Roque, W.L. and Ellis, G.F.R.(1985). In preperation.

Rowan-Robinson, M.(1981). Cosmology. Oxford Univ. Press. Second Edition.

Sandage, A.(1961). Astrophys. J. 133, 355.

(1968). Observatory, 88, 91.

(1972a). In: Proc. of the Symp. on the Galaxy and the Distance Scale, Essex, England.

(1972b). Astrophys. J. 173, 485.

(1982). Astrophys. J. 252, 553.

(1983). Astron. Journ. 88, 1159.

, Kristian, J. and Westphal, J.A. (1976). Astrophys. J., 205, 688.

and Tammann, G.A.(1982). In: Astrophysical Cosmology ; ed. H.A. Bruck, G.V. Coyne, and M.S. Longair. Pontifical Academy of Sciences, 23-84.

Tammann G.A. and Hardy, E.(1972). Astrophys. J. 172, 253.

Schouten, J.(1954). Ricci Calculus (Berlin).

Sciama, D.W.(1971). Modern Cosmology. (Cambridge: Cambridge Univ. Press).

Sersic, J.L.(1982). Extragalactic Astronomy. Reidel.

Stewart, J.M. and Ellis, G.F.R.(1968). JMP 9, 1072.

Stock, J. and Schucking, E.(1957). Astronom. Journ. 62, 98.

Stoeger, W. and Ellis, G.F.R.(1984). Observational Cosmology IV: Spherically symmetric Perturbations, in preperations.

, Nel, S.D. and Ellis, G.F.R.(1985).Observational Cosmology III: Nearly Homogeneous and Isotropic Spacetimes, in preperations.

Tammann, G.A., Sandage, A. and Yahil, A.(1980). In: Physical Cosmology ; ed. R. Balian, J. Audouze and D.N. Schramm. North-Holland, 53-160.

Temple, G.(1938). Proc. Roy. Soc. A168, 122.

Tolman, R.C.(1934). Proc. Nat. Acad. of Science. Vol 20, 169.

Tolman, B.W. and Matzner, R.W.(1984). Proc. Roy. Soc. Lond. A392, 391.

Tully, R.B. and Fisher, J.R.(1977). Colloq. 37, CNRS. No. 263, 95.

Tuner, M.S., Steigman, G. and Krauss, L.M.(1984). Phys. Rev. Lett. 52, 2090.

van den Bergh, S.(1975). In: Structure and Evolution of Galaxies; eds. G. Setti. Reidel.

Walker, A.G.(1936). Proc. Lond Math. Soc. 42, 90.

(1944). J. Lond. Math. Soc. 19, 219.

Warwick, R.S., Pye J.P. and Fabian, A.C.(1980). MNRAS 190, 243.

Weinberg, S. (1972). Gravitation and Cosmology, Wiley, NY.

(1976). Astrophys. Journ. 208, L1.

Wills, D.(1979). Astrophys. J. Suppl. 39, 291.

Yahil, A., Sandage, A. and Tammann, G.A.(1980a). Ap.J. 242, 448.

(1980b). Physical Cosmology ; eds. R. Balian, J. Audouze and D.N. Schramm. North-Holland, 53-160.

Yang, J., Schramm, D.N., Steigman, G. and Rood, R.T.(1979). Ap. J., 227, 697.

Zeldovich, Ya.B. and Grischuk, L.P.(1984). MNRAS 207, 23P.

Zwicky, F. and Rudnick, K.(1966). Z. Astrophys. 64, 246.