Estimates for the Rate of Convergence of Finite Element Approximations of the Solution of a Time-Dependent Variational Inequality

by

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I hereby certify that this is my own original work and has not been submitted before for any other degree.

Signed by candidate

Candidate Date

To my parents

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Summary

The main aim of this thesis is to analyse two types of general finite element approximations to the solution of a time-dependent variational inequality.

The two types of approximations considered are the following :

1. semidiscrete approximations, in which only the spatial domain is discretised by finite elements;

2. fully discrete approximations, in which the spatial domain is again discretised by finite elements and, in addition, the time domain is discretised and the time-derivatives appearing in the variational inequality are approximated by backward differences.

Estimates of the error inherent in the above two types of approximations, in suitable Sobolev norms, are obtained; in particular, these estimates express the rate of convergence of successive finite element approximations to the solution of the variational inequality in terms of element size h and, where appropriate, in terms of the time step size k.

In addition, the above analysis is preceded by related results concerning the existence and uniqueness of the solution to the variational inequality and is followed by an application in elastoplasticity theory.

Contents

Chapter 1

Introduction

The aim of this thesis is to provide a qualitative and numerical analysis of an abstract time-dependent variational inequality of the second kind which arises as a generalisation of a problem in quasistatic elastoplasticity. However, because of its general nature, the model problem has wider application, as indicated in Chapter 7. application estimate for the rate of convergence of various internal approximations of the solution of the model problem, specialising this to finite element approximations for an application in elastoplasticity theory.

In the following section of this Chapter we provide some motivation for work in the *general field* of variational inequalities (VIs) by looking at some common sources of simpler VIs which are prototypes for the more general cases. Thereafter we briefly review the development of the mathematical theory of variational inequalities as well as the numerical analysis of various approximation schemes, with particular reference to work related to our investigation. We provide an outline of the work contained in this thesis in the final section of this Chapter.

1.1 MOTIVATION

Let X be a Banach space and f: $X \to \Re$ be a differentiable map. The problem of finding u such that

$$f(u) \leq f(v) \quad \text{for all } v \in X, \quad u \in X, \tag{1.1}$$

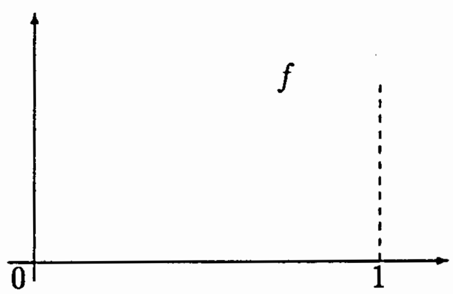
leads us to consider the following *equation* in X^* (the topological dual of X)

$$f'(u) = 0. \tag{1.2}$$

If we now consider the more general problem of finding u such that

$$f(u) \leq f(v) \quad \text{for all } v \in K, \quad u \in K, \tag{1.3}$$

where K is a proper convex subset of X (most of the terminology used in this introductory Chapter is defined in Chapter 2), then equation (1.2) may fail to hold. In particular, choosing $K = [0,1]$ and f as in the figure below :



0 is a point where f achieves its minimum, but $f'(0) \neq 0$. Although a solution u of problem (1.3) may fail to satisfy (1.2), that is

$$\langle f'(u), v \rangle = 0 \quad \text{for all } v \in K,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between X^* and X, we can prove that u satisfies

$$\langle f'(u), v - u \rangle \geq 0 \quad \text{for all } v \in K, \quad u \in K. \tag{1.4}$$

This assertion can be proved as follows : for all $v \in K$, let

$$\varphi(t) = f(u + t(v - u)) \quad \text{for all } t \in [0, 1].$$

Then from (1.3), we deduce

$$\varphi(t) \geq \varphi(0), \quad \text{for all } t \in [0, 1].$$

Hence,

$$0 \leq \lim_{t \to 0} \frac{\varphi(t) - \varphi(0)}{t} = \langle f'(u), v - u \rangle \quad \text{for all } v \in K.$$

Inequality (1.4) is termed a *Variational Inequality.*

Moreover, if we also assume that f is convex, then (1.4) characterises *exactly* the points of K for which (1.3) holds. To see this, we have (since then $\varphi(t)$ is also convex)

$$\begin{aligned} \varphi(1) &\geq \varphi(0) + \varphi'(0) \\ \iff & \\ f(v) &\geq f(u) + \langle f'(u), v - u \rangle \geq f(u) \quad \text{for all } v \in K. \end{aligned}$$

The above results are summarised in the following

Theorem 1.1 *Let $f\colon X \to \Re$ be convex and differentiable and K be a convex subset of X, then*

$$f(u) \leq f(v) \quad \text{for all } v \in K, \quad u \in K,$$

is equivalent to

$$\langle f'(u), v - u \rangle \geq 0 \quad \text{for all } v \in K, \quad u \in K.$$

Thus VIs arise naturally in variational problems on convex sets.

The VIs investigated in this thesis and those found in the numerous references contained herein are generalisations of (1.4); for example : let l be an element of X^* and A: $K \to X^*$. Find u such that

$$\langle Au, v-u\rangle \geq \langle l, v-u\rangle \quad \text{for all } v \in K, \quad u \in K. \tag{1.5}$$

We note an important special case of (1.5) where it is known that a minimum is indeed achieved : Let K be a non-empty closed convex set in a Hilbert space H with inner product $(\cdot,\cdot)$, associated norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$ and let l be an element of H. Then it is well known that there exists an unique element u in K, called the *projection* of l on K, such that

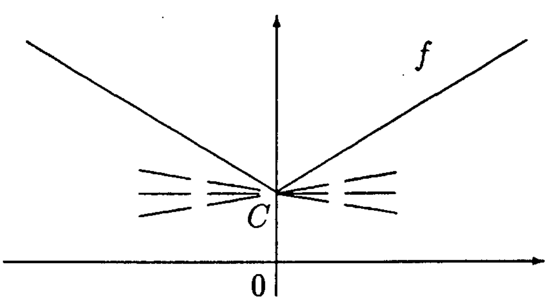
$$\|u-l\| \leq \|v-l\| \quad \text{for all } v \in K.$$

Moreover, u is the unique solution such that

$$(u, v-u) \geq \langle l, v-u\rangle \quad \text{for all } v \in K \tag{1.6}$$

(which can be obtained by applying Theorem 1.1, for example). Thus projection on closed convex sets in Hilbert spaces supply an extensive class of VIs. We shall later denote by P_K the mapping $l \mapsto u$ where u is the solution of (1.6).

For a second important source of VIs we again consider problem (1.1), but this time assuming that f is convex but *not differentiable.* In particular, choosing $X = \Re$ and f non-differentiable at the origin as shown in the figure below :



instead of the minimum condition $f'(0) = 0$, we have

$$0 \in \partial f(0) \tag{1.7}$$

where $\partial f(0)$ denotes the set of all slopes of straight lines through the point C. Condition (1.7) can be generalised to the fundamental notion of the *subdifferential*, which is characterised as solutions of VIs. Briefly, the subdifferential $\partial j(u)$ of the convex function j is given by

$$\left\{ u^* \in X^* \mid j(v) \geq j(u) + \langle u^*, v-u \rangle \right\} \quad \text{for all } v \in X. \tag{1.8}$$

We define this notion more precisely in Chapter 2. We motivate this definition by observing that on choosing u^* in (1.8) to be the slopes of all straight lines through the point C in the above figure, and also choosing $\langle \cdot, \cdot \rangle$ to be real multiplication, inequality (1.8) reduces to the condition (1.7). Convex but non-differentiable functionals (representing for example friction and dissipation effects, see Chapter 7) occur often in the mathematical formulation of physical problems.

In Chapter 3 we consider VIs of the form : Find u such that

$$a(u, v-u) \geq \langle l, v-u \rangle \quad \text{for all } v \in X, \quad u \in X, \tag{1.9}$$

where $a(\cdot, \cdot)$ is a bilinear form. Inequality (1.9) is a special case of (1.5), which can be seen if we denote by A the linear operator of $X \to X^*$ such that

$$a(u, v) = \langle Au, v \rangle \quad \text{for all } v \in X.$$

The reason we consider VIs of the form (1.9) is that the bilinear form $a(\cdot, \cdot)$ often occurs naturally in problems and has some physical interpretation associated with it.

Lastly we note that the above VIs are easily generalised to depend on the time-variable t, and hence we obtain VIs of evolution (see the model problem considered later), thus greatly extending their fields of application (see for example Chapter 7).

1.2 Review

The basis for the development of the theory of variational inequalities was the paper by Fichera [32] on the solution of the Signorini problem arising in elasticity theory. The foundations of the theory itself were later laid by Stampacchia [73], Lions and Stampacchia [53], and Brezis [6]; see also Hartman and Stampacchia [37].

The presentation adopted in this thesis draws on the works of Baiocchi and Capelo [4], Chipot [15], Glowinski [34] and Kinderlehrer and Stampacchia [46].

Some generalisations of the VIs considered so far will be considered later. One such is, for example, to find u such that

$$\langle Au, v-u\rangle + j(v) - j(u) \geq \langle l, v-u\rangle \quad \text{for all } v \in K, \quad u \in K, \tag{1.10}$$

where j is a convex, weakly lower semi-continuous function. Inequality (1.10) forms a prototype for *variational inequalities of the second kind*; for the development of this notion, see for example Moreau [60], Brezis [6] and Lions[50].

For the numerical solution and analysis of VIs see Glowinski [34], Glowinski, Lions and Tremolieres [35] and Hlaváček, Haslinger, Nečas and Lovíšek [40].

The model problem which is presented later in this thesis (after introducing the necessary preliminaries), is a time-dependent variational inequality of the second kind, which is similar to the standard Parabolic Variational Inequality (see Chapter 3), except for the important distinction that rate quantities occur in all of its terms. We will present results concerning existence and uniqueness of its solution, generalising the results of Reddy [67], and also numerically analyse general internal (Galerkin) approximation schemes. It is a typical feature of nonlinear problems such as VIs that general results are very scarce, and that error estimates for finite element approximations depend on the particular structure of the problem being investigated (see Glowinski [34]). It is therefore of some consolation that the model problem provides a mathematical model for more than just one application (see Chapter 7). Despite the tendency of work in nonlinear problems being very problem-specific, we note some previous work which is related to the current investigation.

The paper by Douglas and Dupont [25], although dealing with parabolic *equations*, has been a major influence on parabolic problems in general. This thesis has been influenced by their paper; indeed, we use some of their strategies in the numerical analysis of the semi-discrete internal approximation of the model problem.

Johnson [41], [43] studied a parabolic variational inequality of the first kind, arising as a mathematical model of the behaviour of an elastic-perfectly plastic body. He showed existence of a solution to the problem and also established an error estimate for finite element approximations of the stresses.

In [44], Johnson obtained an error estimate for the *mixed* finite element approximation for a parabolic variational inequality arising in quasistatic plasticity theory. In this mixed method the displacements and stresses are approximated independently using two finite dimensional spaces. This enables greater flexibility and allows both the displacements and stresses to be obtained directly. In Chapter 7 we consider the finite element (Galerkin) method for obtaining approximations of the model problem, which is suitably specialised to model the quasistatic behaviour of an elastoplastic body which undergoes kinematic hardening. Hlaváček [39] also proves an error estimate for a finite element solution for a problem in plasticity with strain-hardening, now using a mixed method to approximate the *stresses* and the *hardening parameters*. Again this is a parabolic variational inequality of the first kind.

Johnson [42] proves an error estimate for a parabolic variational inequality of the first kind arising from a Stefan problem. He uses a piecewise linear finite element discetisation in space and a backward differencing in time. Vuik [76] also studies the above variational inequality, but considers more general time-differencing schemes. We note that various parabolic variational inequalities arising in plasticity are analysed in Hlaváček, Haslinger, Nečas and Lovíšek [40].

All of the above studies differ from that undertaken in this thesis in that the model problem investigated here is a variational inequality *of the second kind*, and which also differs from the standard parabolic variational inequalities in that the rate quantities occur in *all of the terms* of the variational inequality. The problem of quasistatic linear visco-elasticity formulated and studied in Duvaut and Lions [26] closely resembles the structure of the model problem; however, the authors establish existence and uniqueness of its solution, but do not consider any approximation schemes. More general references on the approximation of various classes of variational inequalities are given in later chapters; here we have only listed some investigations more closely related to that of our model problem.

1.3 Overview

Before analysing any numerical approximation scheme, it is useful to know that there in fact exists a solution which is being approximated. The question of existence and uniqueness of the solution to the model problem (in a less general form) has been investigated by Reddy [68]; the results and analysis are summarised in Chapter 5 since the methods used have interesting parallels with the numerical analysis of the fully discrete finite element approximation of the solution, as well as with methods used in practice for computational purposes (see Reddy and Martin [70]).

As already briefly indicated, variational inequalities can arise in different ways, for example when minimising a differentiable functional over a convex subset of a Hilbert space or when minimising a non-differentiable functional over the whole space. The model variational inequality considered here is posed on a Hilbert space and contains a non-differentiable functional. An outline of the theory of variational inequalities is given in Chapter 3.

A widely used means of obtaining estimates of the error inherent in various finite element approximations is to reduce the question of determining the approximation error to one of determining the error inherent in the finite element interpolation of functions in some Sobolev space. This is the fundamental device used in obtaining estimates for the rate of convergence of the finite element approximations to the solution of the model problem.

This work is structured so as to make precise the above-mentioned concepts, theory and results.

Outline of this work

In Chapter 2 the mathematical preliminaries are presented which will be used later. In Chapter 3 we give a brief outline of some of the theory of variational inequalities, focussing especially on the existence and uniqueness results for elliptic variational inequalities, since these are subsequently used. Chapter 4 contains some of the ideas of the mathematical theory of finite elements, and presents the standard finite element interpolation error estimates.

The statement of the model problem and its qualitative analysis are the subjects of Chapter 5. In Chapter 6 we analyse the general semi-discrete internal approximation, obtained by approximating the solution by an element of a finite dimensional subspace of the Hilbert space over which the variational problem is posed, at each time t. We establish an inequality of Céa's lemma-type, which forms the basis of estimates of the error inherent in finite element approximations of the solution of the variational inequality for a given application. The fully-discrete internal approximations, in which the spatial domain is discretised as before and the time domain is additionally discretised by finite differences, is formulated and an estimate for the rate of convergence of the approximate solutions to that of the solution to the model problem is also given in Chapter 6.

In Chapter 7 we show how the model problem arises as a variational formulation of a model of the quasistatic behaviour of an elastoplastic body which undergoes kinematic hardening. We also note other fields of application for which the model problem provides a variational formulation. We consider the case of quasistatic elastoplasticity and, using the above results, establish existence and uniqueness of the solution and also obtain estimates for the rate of convergence of the semi-discrete and fully-discrete finite element approximations of the solution.

Finally, in Chapter 8, we discuss the results obtained and possible extensions of the research.

Chapter 2

Preliminaries

The starting point of our investigation is an abstract variational problem. The technique of using variational formulations of problems has proved to be a powerful tool for both their qualitative and numerical analysis. The variational formulation consists of relations between operators defined on an underlying function space. The purpose of this Chapter is to define those function spaces and types of operators which will be used in the formulation and analysis of the model problem.

Throughout this thesis, Ω will denote an open bounded domain in an n-dimensional Euclidean space $\Re^n$ with boundary $\partial\Omega$. We shall always assume that Ω is "nice", that is, generally simply connected, and that $\partial\Omega$ is "smooth"; by a smooth boundary, we will mean that $\partial\Omega$ is at least *Lipschitzian*, that is, $\partial\Omega$ can be represented as the union of a finite number of sets $\Phi^r = \{(y_1^r, \hat{\mathbf{y}}^r) \mid y_1^r = \phi^r(\hat{\mathbf{y}}^r),\ |\hat{\mathbf{y}}^r| < \epsilon\}$, where $\{\phi^r\}$ is a system of local Lipschitz-continuous coordinate maps, $(y_1^r, \hat{\mathbf{y}}^r)$, $\hat{\mathbf{y}}^r = (\hat{y}_2^r, \ldots, \hat{y}_n^r)$, is a local coordinate system and ϵ is a positive number. See, for example, Adams [1], Grisvard [33] or Nečas [62] for additional details. Points in $\Re^n$ will be denoted by $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and an element of volume by $dx = dx_1 dx_2 \ldots dx_n$.

Let u be a smooth function defined on Ω. We use *multi-index notation* to represent the derivatives of u; that is, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a n-tuple of nonnegative integers and set $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. Then by $D^\alpha u$ we shall mean the αth derivative of u defined by

$$D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}.$$

In this thesis we consider only Banach spaces over the real field $\Re$. The *Fréchet differential* of an operator $\mathcal{L}$ from a Banach space U into a Banach space V is defined as the operator $\delta\mathcal{L}(\xi,\eta)$, which is linear in η, for any η in U, such that

$$\lim_{\|\eta\|_U\to 0}\frac{1}{\|\eta\|_U}\|\mathcal{L}(\xi+\eta)-\mathcal{L}(\xi)-\delta\mathcal{L}(\xi,\eta)\|_V = 0. \tag{2.1}$$

Then $\delta\mathcal{L}(\xi,\cdot)$, regarded as a linear operator on η, is the *Fréchet derivative* of $\mathcal{L}$ at ξ. Similarly, by repeated use of (2.1), we can define Fréchet derivatives of all orders. In the case that $\mathcal{L}$ is a function $u(\mathbf{x})$ from $\Re^n$ to $\Re$, then the form of the Fréchet differential is particularly simple; the kth Fréchet differential of $u(\mathbf{x})$ is a symmetric k-linear mapping of $\Re^n\times\Re^n\times\cdots\times\Re^n$ (k times) into $\Re$ denoted by $\mathcal{D}^k u(\mathbf{x})$ satisfying

$$\mathcal{D}^k u(\mathbf{x})\cdot(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\ldots,\boldsymbol{\xi}_k) = \mathcal{D}^k u(\mathbf{x})\cdot(\boldsymbol{\xi}_{i_1},\boldsymbol{\xi}_{i_2},\ldots,\boldsymbol{\xi}_{i_k}). \tag{2.2}$$

Here $(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\ldots,\boldsymbol{\xi}_k)$ is any set of k vectors in $\Re^n$ and $(\boldsymbol{\xi}_{i_1},\boldsymbol{\xi}_{i_2},\ldots,\boldsymbol{\xi}_{i_k})$ denotes a rearrangement of these vectors for any permutation of the integers from 1 to k. The usual partial derivatives of $u(\mathbf{x})$ are then easily recovered from $\mathcal{D}^k u(\mathbf{x})$, $0\le m\le k$, as *directional derivatives*. For example, if $\Re^n$ is endowed with the orthonormal basis $(\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n)$, where

$$\mathbf{e}_i = (0,0,\ldots,0,1,0,\ldots,0),$$

with the nonzero entry occurring in the ith position, then

$$\begin{aligned}\frac{\partial u(\mathbf{x})}{\partial x_i} &= \mathcal{D}u(\mathbf{x})\cdot\mathbf{e}_i,\\ \frac{\partial^2 u(\mathbf{x})}{\partial x_i\partial x_j} &= \mathcal{D}^2 u(\mathbf{x})\cdot(\mathbf{e}_i,\mathbf{e}_j)\end{aligned}$$

and

$$D^\alpha u(\mathbf{x}) = \mathcal{D}^{|\alpha|}u(\mathbf{x})\cdot(\mathbf{e}_{\alpha_1},\mathbf{e}_{\alpha_2},\ldots\mathbf{e}_{\alpha_n}).$$

See Nashed [61] for an exhaustive treatment of Fréchet differentials.

We will make reference to the following function spaces:

$C^m(\Omega) =$ the linear space consisting of all functions u with partial derivatives $D^\alpha u$ of orders $0 \leq |\alpha| \leq m$ continuous on Ω, where m is some nonnegative integer.

$C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega) =$ the linear space of infinitely differentiable functions on Ω.

$C_0^m(\Omega) = \{v \in C^m(\Omega) \mid$ support of $v =$ the closure $\{\mathbf{x} \in \Omega \mid v(\mathbf{x}) \neq 0\}$ is compact in $\Omega\}$, $0 \leq m \leq \infty\}$.

$C^m(\bar{\Omega}) =$ the linear space of all functions u in $C^m(\Omega)$ for which $D^\alpha u$ is bounded and uniformly continuous on Ω for $0 \leq |\alpha| \leq m$. This space is a Banach space when equipped with the norm

$$\|u\|_{C^m(\bar{\Omega})} = \max_{0 \leq |\alpha| \leq m} \sup_{\mathbf{x} \in \Omega} |D^\alpha u(\mathbf{x})| . \tag{2.3}$$

$\bar{C}^m(\Omega) =$ the linear space of all functions u in $C^m(\Omega)$ for which $D^\alpha u$ is bounded on Ω for $0 \leq |\alpha| \leq m$. This space is larger that $C^m(\bar{\Omega})$ and is a Banach space when endowed with the norm (2.3).

$\mathcal{D}(\Omega) =$ the space of test functions defined on Ω, that is $C_0^\infty(\Omega)$ equipped with the usual locally convex topology.

$\mathcal{D}'(\Omega) =$ the space of distributions – the topological dual of $\mathcal{D}(\Omega)$ endowed with the strong dual topology.

$L^p(\Omega) =$ the space of equivalence classes of measurable functions v on Ω for which $\int_\Omega |v(\mathbf{x})|^p \, dx < \infty$, where Lebesgue integration is implied, endowed with the norm

$$\begin{aligned} \|v\|_{L^p(\Omega)} &= \left(\int_\Omega |v(\mathbf{x})|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty, \\ \|v\|_{L^\infty(\Omega)} &= \operatorname*{ess\,sup}_{\mathbf{x} \in \Omega} |v(\mathbf{x})| \, . \end{aligned} \tag{2.4}$$

The spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are locally convex linear topological spaces which are not metrizable; for further details see, for example, Lions and Magenes [52]. The spaces $L^p(\Omega)$ are Banach spaces, which are reflexive whenever $1 < p < \infty$, and $L^2(\Omega)$ is a Hilbert space when equipped with the inner product $(u, v)_{L^2(\Omega)} = \displaystyle\int_\Omega uv \, \mathrm{d}x$, where u and v are real-valued.

We will also deal with functions belonging to *Sobolev spaces.* Firstly, we recall that if $u \in \mathcal{D}'(\Omega)$, then the *distributional partial derivatives* of order α are the distributions $\mathrm{D}^\alpha u$ satisfying

$$\langle \mathrm{D}^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \phi \rangle = (-1)^{|\alpha|} \langle u, \frac{\partial^{|\alpha|} \phi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \rangle, \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

If $u \in \mathcal{D}'(\Omega)$ and there exists a locally integrable function $\hat{u}$ such that $\langle u, \phi \rangle = \displaystyle\int_\Omega \hat{u} \phi \, \mathrm{d}x$, for all $\phi \in \mathcal{D}(\Omega)$, then we identify u with $\hat{u}$ and do not distinguish between the distribution and the function which generates it.

We are now in a position to introduce the notion of Sobolev spaces. Let m be a non-negative integer and p be a real number satisfying $1 \leq p \leq \infty$. The Sobolev space $W^{m,p}(\Omega)$ of order (m, p) is the linear space of equivalence classes of functions in $L^p(\Omega)$ whose distributional partial derivatives of all order $|\alpha|$, for $0 \leq |\alpha| \leq m$, are also in $L^p(\Omega)$:

$$W^{m,p}(\Omega) = \{ v \mid \mathcal{D}^\alpha v \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m \} \, .$$

$W^{m,p}(\Omega)$ is equipped with the following norm:

$$\|u\|_{m,p,\Omega} = \left(\int_\Omega \sum_{|\alpha| \le m} |D^\alpha u(\mathbf{x})|^p \, dx \right)^{1/p}, \quad 1 \le p < \infty, \tag{2.5}$$

$$\|u\|_{m,\infty,\Omega} = \max_{0 \le |\alpha| \le m} \|D^\alpha u\|_{L^\infty(\Omega)}. \tag{2.6}$$

We will also use the following seminorm defined on $W^{m,p}(\Omega)$:

$$|u|_{m,p,\Omega} = \left(\int_\Omega \sum_{|\alpha| = m} |D^\alpha u(\mathbf{x})|^p \, dx \right)^{1/p}. \tag{2.7}$$

We will often only consider the special Sobolev spaces for which $p = 2$. Then we use the notation

$$H^m(\Omega) = W^{m,2}(\Omega).$$

The spaces $H^m(\Omega)$ are Hilbert spaces when equipped with the inner products

$$(u, v)_{m,\Omega} = \int_\Omega \sum_{|\alpha| \le m} D^\alpha u(\mathbf{x}) D^\alpha v(\mathbf{x}) \, dx$$

so that

$$\|u\|_{m,\Omega} = \|u\|_{m,2,\Omega} = [(u, u)_{m,\Omega}]^{1/2}.$$

When the domain of the function is contextually apparent, we use the simpler notation

$$\|\cdot\|_{m,p} = \|\cdot\|_{m,p,\Omega}, \qquad \|\cdot\|_m = \|\cdot\|_{m,\Omega}.$$

The completion of $\mathcal{D}(\Omega)$ with respect to the $\|\cdot\|_1$ norm is denoted by $H_0^1(\Omega)$. Elements of this space can be more concretely characterised by means of the *trace operator*; because of our smoothness assumptions about the boundary $\partial\Omega$, a superficial measure $d\gamma$ can be defined along it, so that we can consider the spaces $L^2(\partial\Omega)$, with norm denoted by $\|\cdot\|_{L^2(\partial\Omega)}$. Then it can be proved that there exists a constant

$C(\Omega)$ such that

$$\|v\|_{L^2(\partial\bar{\Omega})} \leq C(\Omega)\|v\|_1, \quad \text{for all } v \in C^\infty(\bar{\Omega}).$$

Since $H^1(\Omega)$ is the closure of $C^\infty(\bar{\Omega})$ with respect to the norm $\|\cdot\|_1$, there exists a continuous linear mapping tr: $v \in H^1(\Omega) \rightarrow \text{tr } v \in L^2(\partial\Omega)$, called the trace operator. We then have the following characterisation

$$H_0^1(\Omega) = \{v \in H^1(\Omega); \text{tr } v = 0 \text{ on } \partial\Omega\}.$$

An important aspect of Sobolev spaces is that they provide a useful means of quantifying the the degree of "smoothness" or regularity of functions; it is known, for example, which Sobolev spaces can be identified with spaces of continuous functions. These fundamental results form the content of the *Sobolev embedding theorem.* A normed space U is *embedded* in a normed space V (with norms $\|\cdot\|_U$ and $\|\cdot\|_V$, respectively) if

(i) U is a linear subspace of V and

(ii) the injection i of U into V is continuous.

Since i is linear, (ii) is satisfied iff there exists a constant $C > 0$ such that

$$\|u\|_V \leq C\|u\|_U \quad \text{for all } u \in U.$$

We denote these embeddings by

$$U \hookrightarrow V$$

and say that the embedding is *compact* if the injection i is a compact operator.

Theorem 2.1 (Sobolev Embedding Theorem) *Let $\Omega \subset \Re^n$ be Lipschitzian and Ω^k be a k-dimensional domain obtained by intersecting Ω with a k-dimensional hyperplane in $\Re^n$, for $1 \leq k \leq n$. Then, for nonnegative integers j and m, the following embeddings exist:*

(i) If $mp < n$ and $n - mp < k \leq n$, then

$$W^{m+j,p}(\Omega) \hookrightarrow W^{m,q}(\Omega^k)$$
$$\text{for } p \leq q \leq \frac{kp}{n - mp}$$

and this embedding is compact for $1 \leq p < \infty$.

(ii) If $mp = n$, then for any k such that $1 \leq k \leq n$,

$$W^{m+j,p}(\Omega) \hookrightarrow W^{m,q}(\Omega^k)$$
$$\text{for } p \leq q \leq \infty$$

and this embedding is compact for $1 \leq p < \infty$.

(iii) If $mp > n$,, then

$$W^{m+j,p}(\Omega) \hookrightarrow \bar{C}^j(\Omega) \tag{2.8}$$

and this embedding is compact for $1 \leq p < \infty$.

PROOF. See, for example, Adams [1]. □

We note, in particular, that from (2.8) elements of $W^{m,p}(\Omega)$ are continuous functions if $mp > n$.

We record an important property of the seminorm $|\cdot|_{m,p,\Omega}$ defined by (2.7) establishing its relationship to a norm on a quotient space which will later be of considerable theoretical value. Consider the quotient space

$$Q^{m,p}(\Omega) = W^{m,p}(\Omega)/\mathcal{P}_{m-1}(\Omega) \tag{2.9}$$

where $\mathcal{P}_{m-1}(\Omega)$ is the space of polynomials of degree $\leq m - 1$ on Ω. The elements of $Q^{m,p}(\Omega)$ are cosets $[v]$ of functions such that, for all $u, v \in W^{m,p}(\Omega)$

$$u \in [v] \rightarrow u - v \in \mathcal{P}_{m-1}(\Omega). \tag{2.10}$$

The natural norm on $Q^{m,p}(\Omega)$ is given by

$$\|[v]\|_{Q^{m,p}(\Omega)} = \inf_{f \in P_{m-1}(\Omega)} \|v + f\|_{m,p,\Omega}. \tag{2.11}$$

The following Theorem, due to Deny and Lions [24], asserts the equivalence of the seminorm $|\, v\, |_{m,p,\Omega}$ to the norm $\|[v]\|_{Q^{m,p}(\Omega)}$ defined on $W^{m,p}(\Omega)/\mathcal{P}_{m-1}(\Omega)$.

Theorem 2.2 *There exists a positive constant $C = C(\Omega)$ such that*

$$\|[v]\|_{Q^{m,p}(\Omega)} \leq C\, |\, v\, |_{m,p,\Omega} \tag{2.12}$$

for any $v \in W^{m,p}(\Omega)$.

PROOF. See, for example, Ciarlet [18] or Oden and Reddy [65]. □

Finally, we will also deal with "spaces of vector-valued functions". Let X be a Banach space and $0 < T < \infty$. Then

(i) the space $C^m([0,T];X)$, with m a nonnegative integer, consists of all continuous functions u: $[0,T] \to X$ that have continuous derivatives up to order m on $[0,T]$ with the norm

$$\|u\|_{C^m(0,T;H)} = \sum_{i=1}^{m} \max_{0 \leq t \leq T} \|u^{(i)}(t)\|_X, \tag{2.13}$$

where only the right-hand and the left-hand derivatives need exist at the boundary points $t = 0$ and $t = T$, respectively. We write $C([0,T],X)$ instead of $C^0([0,T],X)$;

(ii) the space $L^p(0,T;X)$ with $1 \leq p < \infty$ consists of all measurable functions u: $(0,T) \to X$ for which

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty; \tag{2.14}$$

(iii) the space $L^\infty(0,T;X)$ consists of all measurable functions u: $(0,T) \to X$ which are *essentially bounded*, that is, for which there exists a number B, called an *essential bound* of u, such that $\|u(t)\|_X \le B$ for almost all $t \in (0,T)$. We set

$$\|u\|_{L^\infty(0,T;H)} = \inf\{B : B \text{ is an essential bound of } u\}. \qquad (2.15)$$

We list some properties of these spaces in the following Theorem.

Theorem 2.3 *Let m be a nonnegative integer and $1 \le p \le \infty$. Let X and Y be real Banach spaces. Then:*

(i) $C^m([0,T];X)$ with the norm (2.13) is a Banach space.

(ii) $L^p(0,T;X)$ with the appropriate norm (2.14) or (2.15) is a Banach space if we identify functions that are equal almost everywhere on $(0,T)$.

(iii) If X is a Hilbert space with inner product $(\cdot,\cdot)_X$, then $L^2(0,T;X)$ is also a Hilbert space with the inner product

$$(u,v)_{L^2(0,T;X)} = \int_0^T (u(t),v(t))_X \, dt.$$

PROOF. See Zeidler [80]. □

If X is any Banach space, we denote its topological dual by X^*, and indicate the operation of an element $u^* \in X^*$ on an element $u \in X$ by $\langle u^*, u\rangle$.

Let X be a separable Banach space. Then the space $L^1(0,T;X)^*$ is separable and

$$L^1(0,T;X)^* = L^\infty(0,T;X^*).$$

By this we mean that there exists a bijective linear mapping $u^* \mapsto u$ from $L^1(0,T;X)^*$ onto $L^\infty(0,T;X^*)$ with

$$\langle u^*, v\rangle = \int_0^T \langle u(t),v(t)\rangle_X \, dt \quad \text{for all } v \in L^1(0,T,X)$$

and $\|u^*\|_{L^1(0,T;X)^*} = \|u\|_{L^\infty(0,T;X^*)}$; see Edwards [28] or Zeidler [80].

For any Hilbert space H, we define by $W^{1,2}(0,T;H)$ the space of functions $f \in L^2(0,T;H)$ such that $\dot{f} \in L^2(0,T;H)$, equipped with the norm

$$\|f\|^2_{W^{1,2}(0,T;H)} = \|f\|^2_{L^2(0,T;H)} + \|\dot{f}\|^2_{L^2(0,T;H)},$$

where $\dot{f}$ denotes the *generalised derivative* of f on $(0,T)$. We define $w = u^{(n)}$ to be the nth generalised derivative of the function u on $(0,T)$ iff

$$\int_0^T \phi^{(n)}(t)u(t)\mathrm{d}t = (-1)^n \int_0^T \phi(t)w(t)\mathrm{d}t \quad \text{for all } \phi \in C_0^\infty(0,T) \tag{2.16}$$

is valid. Note that these integrals are defined whenever $u, w \in L^1(0,T;H)$ (see, for example, Zeidler [80][page 418]). This generalised derivative is unique in the sense of the following Theorem.

Theorem 2.4 (Uniqueness of Generalised Derivative) *Let Y and Z be Banach spaces. Moreover, suppose that $u \in L^1(0,T;Y)$ and $v, w \in L^1(0,T;Z)$. If*

$$u^{(n)} = v \text{ and } u^{(n)} = w$$

in the sense of generalised derivatives, then we obtain $v(t) = w(t)$ almost everywhere on $(0,T)$, that is, $v = w$ in $L^1(0,T;Z)$.

Proof. See, for example, Zeidler [80][page 419]. □

We note that the generalised derivative does in fact generalise the notion of the derivative of a vector function of one real variable t (see Zeidler [79]), as can be seen by the following example.

Example. Let $u \in C^m([0,T],X)$, for m a nonnegative integer. Then the continuous mth derivative $u^{(m)}$: $[0,T] \to X$ is also the generalised mth derivative of u on $(0,T)$.

PROOF. This follows easily by successively using integration by parts; see Zeidler [80]. □

We record the fundamental inequality

$$\|f(t) - f(s)\|_H \leq \int_s^t \|\dot{f}(\tau)\|_H \mathrm{d}\tau, \tag{2.17}$$

which holds for $s < t$ and $f \in W^{1,2}(0, T; H)$ (see, for example, Zeidler [80]). We also have that $W^{1,2}(0, T; H) \subset C([0, T], H)$, with the embedding being continuous.

We now introduce two notions of convergence and associated Theorems which will later be of vital importance in the qualitative anaylis of the model problem.

A sequence (u_n) in the Banach space X is called *weakly convergent*, that is

$$u_n \rightharpoonup u \quad \text{as } n \to \infty,$$

iff

$$\langle f, u_n \rangle \to \langle f, u \rangle \quad \text{as } n \to \infty \text{ for all } \ f \in X^*.$$

Theorem 2.5 (Eberlein [27], Šmuljan [72]) *Each bounded sequence in a reflexive Banach space has a weakly convergent subsequence.*

PROOF. See Yosida [77]. □

Theorem 2.5 is useful when working with a reflexive Banach space. Another notion of convergence and an associated Theorem are useful when working with a Banach space which is not reflexive.

Let X be a Banach space. A sequence (f_n) in X^* is called *weakly* convergent*, that is

$$f_n \overset{*}{\rightharpoonup} f \quad \text{as } n \to \infty,$$

iff

$$\langle f_n, u\rangle \to \langle f, u\rangle \quad \text{as } n\to\infty \text{ for all } u \in X.$$

Theorem 2.6 *Let X be a separable Banach space. Then each bounded sequence (f_n) in X^* has a weakly* convergent subsequence.*

PROOF. See Zeidler [80]. □

We now define various types of operators. Recall that a set K in a vector space is called *convex* iff

$$u, v \in K \qquad \text{and} \qquad t \in [0,1] \qquad \text{imply} \qquad (1-t)u + tv \in K,$$

that is, if the points u and v belong to K, then so does the line segment joining them. A functional

$$j : K \to \Re$$

on a convex set K is *convex* iff

$$j((1-t)u + tv) \leq (1-t)j(u) + tj(v),$$

for all $t \in [0,1]$ and all u, $v \in K$.

Let j: $U \subset X \to \Re$ be a functional on the subset U of the Banach space X. Then j is called *lower semicontinuous* (l.s.c.) on U iff, for each sequence (u_n) in U,

$$u_n \to u \text{ as } n\to\infty \quad \text{implies} \quad j(u) \leq \liminf_{n\to\infty} j(u_n)$$

and j is called *weakly sequentially lower semicontinuous* iff

$$u_n \rightharpoonup u \text{ as } n\to\infty \quad \text{implies} \quad j(u) \leq \liminf_{n\to\infty} j(u_n).$$

We define an important generalisation of the concept of a derivative. Let j: $X \to [-\infty, \infty]$ be a functional on the Banach space X. The functional u^* in X^* is called a *subgradient* of j at the point u iff $j(u) \neq \pm\infty$ and

$$j(v) \geq j(u) + \langle u^*, v - u \rangle_X \quad \text{for all } v \in X.$$

The set of all subgradients of j at u is called the *subdifferential* $\partial j(u)$ at u. If no subgradients exist, then we set $\partial j(u) = \emptyset$.

We clarify some of the (standard) terminology used in this thesis: in general, the term *vector* is used (as above) to denote an element of a Banach (more generally vector) space; however, in specific applications, the term vector denotes a first order tensor quantity (representing displacement, for example) while the term tensor is used to denote a second order tensor (representing quantities such as the stress at a point in a continuum).

More notation will be introduced throughout the thesis, where required.

In this Chapter, we have not always introduced the notions and spaces in their greatest generality, and have also only listed their properties which will be of interest to us in the ensuing investigation; for additional details on properties of the Sobolev spaces described in this Chapter, see for example, Adams [1], Kufner, John and Fucik [48], Nečas [62] or Yosida [77]. For a detailed account of spaces of vector-valued functions, see Edwards [28] and Zeidler [80].

At the conclusion of this Chapter, we comment on the occurrence of the various function spaces just introduced in the formulation of problems in mechanics, their subsequent mathematical analysis and also in the numerical analysis of approximate solution methods. Continuum mechanics and variational formulations supply integral statements of physical principles of mechanics (for example, conservation of energy and variational principles of energy) which only make sense for certain classes of admissible functions. In addition to arising naturally in weak or variational statements of boundary value problems in continuum mechanics, Sobolev spaces provide a means of quantifying the concept of regularity; this is fundamental to the error analysis of finite element approximation methods since errors are naturally measured in some appropriate Sobolev norm and rates of convergence of the approximation solutions depend on the order of the space and on the regularity of the solution. This idea will be made more precise in Chapter 4 which contains an outline of finite element interpolation theory. The spaces of vector-valued functions play an important role in the formulation and analysis of time-dependent problems (problems of evolution) in mechanics; suppose that we seek some (solution) function $\mathbf{u}(\mathbf{x}, t)$ of the spatial variables $\mathbf{x}$ and the time t. For fixed t, $\mathbf{x} \rightarrow \mathbf{u}(\mathbf{x}, t)$ is then, by the above considerations, typically an element in a Banach (Sobolev) space X. With respect to t, we have a function $t \rightarrow \mathbf{u}(t) \equiv \mathbf{u}(\mathbf{x}, t)$, with values in X. Hence we seek the solution in an appropriate "Lesbesgue space of vector-valued functions," which has sufficient structure to enable a mathematical analysis of the problem.

Chapter 3

Variational Inequalities

3.1 Introduction

An important and useful class of nonlinear problems arising in fields such as mechanics and physics, for example, are formulated in terms of variational inequalities (see, for example, Duvaut and Lions [26], for some typical applications). For theoretical generalities on variational inequalities, see Lions and Stampacchia [53], Lions [50], Ekeland and Temam [29], Baiocchi and Capelo [4] and Kinderlehrer and Stampacchia [46].

We review here two general types of variational inequalities which have been widely investigated, namely :

(i) elliptic variational inequalities (EVI),

(ii) parabolic variational inequalities (PVI).

We will first recall some of the results concerning *existence* and *uniqueness* of solutions of EVI; PVI will be briefly considered later (since we do not later use the results concerning PVI we will simply list them for comparative purposes).

These results are recorded here since it is instructive to compare them with those obtained for the model variational inequality being investigated, and also since they are used later in the investigation.

3.2 Elliptic Variational Inequalities

In this Section we consider two classes of EVI, namely EVI of the *first kind* and EVI of the *second kind.*

We use the following *notation* :

- H : a real Hilbert space with scalar product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$;
- H^* : the dual space of H;
- a: $H \times H \to \Re$ is a bilinear, continuous and *H-elliptic* form on $H \times H$;

A bilinear form $a(\cdot,\cdot)$ is *H-elliptic* if there exists a constant $\alpha > 0$ such that $a(v,v) \geq \alpha\|v\|^2$, for all $v \in H$;

- l: $H \to \Re$ is a continuous linear functional;
- K : a closed, convex and nonempty subset of H;
- j: $H \to \bar{\Re} = \Re \cup \{\infty\}$ is a convex, lower semicontinuous (l.s.c) and proper functional ($j(\cdot)$ is proper if $j(v) > -\infty$, for all $v \in H$ and $j \not\equiv +\infty$).

EVI of the first kind

Problem ($\mathbf{EVI_1}$) *Find u such that*

$$a(u, v-u) \geq \langle l, v-u \rangle \qquad \text{for all } v \in K,\ u \in K. \tag{3.1}$$

Problem (EVI$_2$) *Find u such that*

$$a(u, v-u) + j(v) - j(u) \geq \langle l, v-u \rangle \qquad \text{for all } v \in H,\ u \in H. \tag{3.2}$$

REMARKS. The cases above are the simplest and most important; they are often used in the investigation of problems in which the variational inequality is of a more complicated type (see, for example the proof of Theorem 5.1).

Generalisations of the above problems, called *quasivariational inequalities* (QVI) are also studied; these arise, for instance, in problems of seepage through porous media (see, for example, Oden and Kikuchi [66]). A typical such problem is :
Find u such that

$$a(u, v-u) \geq \langle l, v-u \rangle \qquad \text{for all } v \in K(u), u \in K(u),$$

where $v \rightarrow K(v)$ is a family of closed, convex and nonempty subsets of H.

If $K = H$ and $j \equiv 0$, then problems (3.1) and (3.2) reduce to the classical linear variational equation

$$a(u, v) = \langle l, v \rangle \qquad \text{for all } v \in H,\ u \in H. \tag{3.3}$$

The variational equation (3.3) arises as a variational formulation of elliptic differential equations (see, for example, Reddy [67]).

The distinction between problems (3.1) and (3.2) is somewhat artificial since problem (3.1) can be considered as a special case of problem (3.2) by replacing $j(\cdot)$ in (3.2) by the *indicator functional* I_K of K defined by

$$I_K = \begin{cases} 0 & \text{if } v \in K \\ +\infty & \text{if } v \notin K. \end{cases}$$

In spite of this, (3.1) is often considered directly since it arises naturally, and also because it enables us to obtain geometrical insight into the problem. It is easily verified that I_K is a convex, l.s.c. and proper functional and that problem (3.1) is equivalent to the problem of finding u that satisfies

$$a(u, v-u) + I_K(v) - I_K(u) \geq \langle l, v-u \rangle \qquad \text{for all } v \in H,\ u \in H.$$

PHYSICAL APPLICATIONS. Although our primary interest in EVI is obtaining the theoretical existence and uniqueness results which are to follow, we briefly note here that EVI of the first and second kinds occur in mathematical models for the following (non-exhaustive!) list of problems:

- Contact problems (see Glowinski [34], Kikuchi and Oden [45]).
- Elasticity problems (see Ciarlet [18], Glowinski [34], Kikuchi and Oden [45]).
- Filtration of liquids in porous media (see Baiocchi [3], Cominicioli [22], Oden and Kikuchi [66]).
- Lubrication phenomena (see Cryer [23]).
- Two-dimensional irrotational flows of perfect fluids (see Brezis and Stampacchia [10], Brezis [9], and Ciavaldini and Tournemine [21]).
- Wake problems (see Bourgat and Duvaut [5]).

3.2.1 EXISTENCE AND UNIQUENESS RESULTS FOR EVI OF THE FIRST KIND

Theorem 3.1 (Lions and Stampacchia [53]) *The problem* (EVI_1) *has a unique solution.*

PROOF. We first prove the uniqueness and then the existence.

(1) *Uniqueness.* Let u_1 and u_2 be solutions of (3.1), then

$$a(u_1, v-u_1) \geq \langle l, v-u_1\rangle \quad \text{for all } v \in K, \quad u_1 \in K, \tag{3.4}$$

$$a(u_2, v-u_2) \geq \langle l, v-u_2\rangle \quad \text{for all } v \in K, \quad u_2 \in K. \tag{3.5}$$

Taking $v = u_2$ in (3.4), $v = u_1$ in (3.5), adding and using the H-ellipticity of $a(\cdot,\cdot)$ we obtain

$$\alpha\|u_2 - u_1\|^2 \leq a(u_2 - u_1, u_2 - u_1) \leq 0,$$

which proves that $u_1 = u_2$, since $\alpha > 0$.

(2) *Existence.* We follow Glowinski [34] who uses a generalisation of the proof used by Ciarlet (see, for example, [16]-[18]) for proving the Lax-Milgram lemma, that is, we reduce the problem (EVI_1) to a fixed point problem.

By the Riesz representation theorem for Hilbert spaces, there exists $A \in \mathcal{L}(H, H)$ and $z \in H$ such that

$$(Au, v) = a(u, v) \qquad \text{for all } u, v \in H \text{ and}$$

$$(z, v) = \langle l, v\rangle \qquad \text{for all } v \in H.$$

Then the problem (EVI_1) is equivalent to finding u such that

$$(u - \rho(Au - z) - u, v - u) \leq 0 \qquad \text{for all } v \in K, \quad u \in K, \quad \rho > 0. \tag{3.6}$$

This is equivalent to finding u such that

$$u = P_K(u - \rho(Au - z)) \text{ for some } \rho > 0,$$

where P_K denotes the *projection operator* from H to K in the $\|\cdot\|$ norm. Consider the mapping W_ρ: $H \to H$ defined by

$$W_\rho(v) = P_K(v - \rho(Av - z)).$$

Let $v_1, v_2 \in H$. Then since P_K is *non-expansive*, we have

$$\|W_\rho(v_1) - W_\rho(v_2)\|^2 \leq \|v_2 - v_1\|^2 + \rho^2\|A(v_2 - v_1)\|^2 - 2\rho a(v_2 - v_1, v_2 - v_1).$$

Hence we have

$$\|W_\rho(v_1) - W_\rho(v_2)\|^2 \leq (1 - 2\rho\alpha + \rho^2\|A\|^2)\|v_2 - v_1\|^2.$$

Thus W_ρ is a strict contraction mapping if $0 < \rho < 2\alpha/\|A\|^2$. By taking ρ in this range, we have a unique solution to the fixed-point problem which implies the existence of a solution for problem (EVI_1). □

REMARKS. If $K = H$, then Theorem 3.1 reduces to the Lax-Milgram lemma (see, for example, Ciarlet [16] - [18]).

If $a(\cdot,\cdot)$ is symmetric, then Theorem 3.1 can be proved using optimisation methods (see, for example, Céa [14]).

3.2.2 EXISTENCE AND UNIQUENESS RESULTS FOR EVI OF THE SECOND KIND

Theorem 3.2 (Lions and Stampacchia [53]) *Problem* (EVI_2) *has a unique solution.*

OUTLINE OF PROOF. As in Theorem 3.1, we shall first prove uniqueness and then existence.

(1) *Uniqueness.* Let u_1 and u_2 be two solutions of (3.1), we then have

$$a(u_1, v - u_1) + j(v) - j(u_1) \geq \langle l, v - u_1\rangle \qquad \text{for all } v \in H, \quad u_1 \in H, \tag{3.7}$$

$$a(u_2, v - u_2) + j(v) - j(u_2) \geq \langle l, v - u_2\rangle \qquad \text{for all } v \in H, \quad u_2 \in H, \tag{3.8}$$

Since $j(\cdot)$ is a proper functional, there exists $v_0 \in H$ such that $-\infty < j(v_0) < \infty$. Hence, for $i = 1, 2$,

$$-\infty < j(u_i) \leq j(v_0) - \langle l, v_0 - u_i \rangle + a(u_i, v_0 - u_i). \tag{3.9}$$

This shows that $j(u_i)$ is finite for $i = 1, 2$. Hence, by taking $v = u_2$ in (3.7) and $v = u_1$ in (3.8), and adding, we obtain

$$\alpha \|u_1 - u_2\|^2 \leq a(u_1 - u_2, u_1 - u_2) \leq 0.$$

Hence $u_1 = u_2$.

(2) *Existence.* We sketch only the main ideas of a proof; for the full details see, for example, Glowinski [34]. For each $u \in H$ and $\rho > 0$, we associate a problem (π_ρ^u) of type (EVI_2) defined as follows :

Find w such that

$$(w, v - w) + \rho j(v) - \rho j(w) \geq (u, v - w) + \rho \langle l, v - w \rangle - \rho a(u, v - w),$$
$$\text{for all } v \in H, \quad w \in H. \tag{π_ρ^u}$$

The advantage of considering this problem instead of problem (EVI_2) is that the bilinear form associated with (π_ρ^u) is the inner product of H which is symmetric. First we assume that (π_ρ^u) has a unique solution for all $u \in H$ and $\rho > 0$. For each ρ define the mapping f_ρ: $H \to H$ by $f_\rho(u) = w$, where w is the unique solution of (π_ρ^u). We then show that f_ρ is a uniformly strict contraction mapping for suitably chosen ρ. Let u_1, $u_2 \in H$ and $w_i = f_\rho(u_i), i = 1, 2$. Since $j(\cdot)$ is proper, we have $j(u_i)$ finite which can be proved as in (3.9). Therefore we have that

$$\begin{aligned}(w_1, w_2 - w_1) + \rho j(w_2) - \rho j(w_1) \;\; \geq \;\; & (u_1, w_2 - w_1) + \rho \langle l, w_2 - w_1 \rangle \\ & + \rho a(u_1, w_2 - w_1).\end{aligned} \tag{3.10}$$

$$(w_2, w_1 - w_2) + \rho j(w_1) - \rho j(w_2) \geq (u_2, w_1 - w_2) + \rho\langle l, w_1 - w_2\rangle - \rho a(u_2, w_1 - w_2). \tag{3.11}$$

Adding these inequalities we obtain

$$\begin{aligned} \|f_\rho(u_1) - f_\rho(u_2)\|^2 &= \|w_2 - w_1\|^2 \\ &\leq ((I - \rho A)(u_2 - u_1), w_2 - w_1) \\ &\leq \|I - \rho A\|\|u_2 - u_1\|\|w_2 - w_1\|. \end{aligned}$$

It is easy to show that $\|I - \rho A\| < 1$ if $0 < \rho < 2\alpha/\|A\|^2$. This proves that f_ρ is uniformly a strict contraction mapping and therefore has a unique fixed point u. This u is the solution of (EVI_2) since $f_\rho(u) = u$ implies

$$(u, v - u) + \rho j(v) - \rho j(u) \geq (u, v - u) + \rho\langle l, v - u\rangle - \rho a(u, v - u) \quad \text{for all } v \in H.$$

Therefore

$$a(u, v - u) + j(v) - j(u) \geq \langle l, v - u\rangle \quad \text{for all } v \in H.$$

Hence (EVI_2) has a unique solution. The existence and uniqueness of the solution of problem (π^u_ρ) follows from the following Lemma. □

Lemma 3.1 *Let b: $H \times H \to \Re$ be a continuous symmetric bilinear H-elliptic form with H-ellipticity constant β. Let $l \in H^*$ and j: $H \to \bar{\Re}$ be a convex, l.s.c. proper functional. Let $J(v) = \frac{1}{2}b(v, v) + j(v) - \langle l, v\rangle$. Then the minimization problem : Find u such that*

$$J(u) \leq J(v) \qquad \text{for all } v \in H, \quad u \in H,$$

has a unique solution which is characterised by

$$b(u, v - u) + j(v) - j(u) \geq \langle l, v - u\rangle \quad \text{for all } v \in H, \quad u \in H.$$

PROOF. This is a standard result; see, for example, Glowinski [34] for the proof. □

3.3 Parabolic Variational Inequalities

Formulation and Statement of the Main Results

Let H and V be two real Hilbert spaces with V dense in H. Assuming $H = H^*$, we have that $V \subset H \subset V^*$. The scalar product in H (respectively in V) and the corresponding norms are denoted by $(\cdot,\cdot), |\cdot|$ (respectively, $((\cdot,\cdot)), \|\cdot\|$). Also $\langle \cdot,\cdot \rangle$ denotes the duality pairing between V^* and V.

We now introduce:

- A time interval $[0, T]$ with $0 < T < \infty$, a bilinear form a: $V \times V \to \Re$, continuous and *elliptical* in the following sense: $\exists \alpha > 0$ and $\lambda \geq 0$ such that $a(u, v) + \lambda\, |v|^2 \geq \alpha \|v\|^2$ for all $v \in V$;
- $f \in L^2(0, T; V^*)$, $u^0 \in H$ (see Section 2 for the definition of $L^2(0, T; X)$);
- K : a closed convex nonempty subset of V;
- j: $V \to \bar{\Re}$ convex, proper, l.s.c.

We then consider the following two families of PVI:

PVI of the first kind

Find $u(t)$ such that

$$\left(\frac{\partial u}{\partial t}, v - u\right) + a(u, v - u) \geq \langle l, v - u \rangle \quad \text{for all } v \in K, \tag{3.12}$$

$$\text{for almost all } t \in (0, T),$$

$$u(t) \in V \text{ for almost all } t \in (0, T), \quad u(0) = u^0.$$

Find $u(t)$ such that

$$\left(\frac{\partial u}{\partial t}, v-u\right) + a(u, v-u) + j(v) - j(u) \geq \langle l, v-u\rangle \quad \text{for all } v \in K, \tag{3.13}$$

$$\text{for almost all } t \in (0,T),$$

$$u(t) \in V \text{ for almost all } t \in (0,T), \quad u(0) = u^0.$$

REMARK. If $K = V$ and $j \equiv 0$, then (3.12) and (3.13) reduce to the standard parabolic variational equation:

$$\left(\frac{\partial u}{\partial t}, v\right) + a(u,v) = \langle l, v-u\rangle \quad \text{for all } v \in K, \tag{3.14}$$

$$\text{for almost all } t \in (0,T),$$

$$u(t) \in V \text{ for almost all } t \in (0,T), \quad u(0) = u^0.$$

Under appropriate conditions on u^0, K and $j(\cdot)$, it is proved that (3.12) and (3.13) have unique solutions in $L^2(0,T;V) \cap C([0,T];H)$. For the proof we refer to Brezis [7], [8], Lions [50], and Duvaut and Lions [26]; see also Zeidler [78].

CHAPTER 4

FINITE ELEMENT APPROXIMATION AND INTERPOLATION

4.1 INTRODUCTION

In the following Chapters, after relating qualitative results concerning the existence and uniqueness of the solution to the model time-dependent variational inequality, we proceed to analyse its general internal approximation, that is, the infinite-dimensional function space over which the variational problem is posed is replaced by a finite dimensional subspace, and the solution to this new (approximate) problem is then sought. This approximate problem is in general amenable to solution since one can construct various methods to obtain the finite number of parameters which specify the approximate solution.

The predominant internal approximation method which is being widely used for both practical engineering applications and theoretical investigations is the (conforming) *finite element method* (FEM). The FEM is actually a collection of methods, including, for example, the Galerkin finite element method and the collocation finite element method. Since they are internal approximation methods, what they have in common is that the underlying space over which the problem is posed is some finite-dimensional subspace of the original space. The FEM is distinguished in that the (unknown) solution to the variational problem is approximated by simple functions (polynomials of low order) over subdivisions (the finite element mesh) of the domain

Ω – better approximations are obtained not by increasing the degree of the approximating polynomial, but by refining the mesh (this holds for the simpler h-version, in the p-version the order of the local polynomial approximants are increased). This has some important consequences; for example, it allows greater flexibility in the geometry of the domain than that required for obtaining analytic solutions and for implementing finite difference approximation schemes. Also, importantly for this investigation, since the mathematical basis of the FEM is sound, it is possible to obtain estimates of the error inherent in the finite element approximations. We will do this below in a general setting.

There are numerous books which list commonly used finite element spaces, for example Oden and Carey [64]. Also, for the computational aspects and implementation of the FEM, see for example Hinton and Owen [38] and Carey and Oden [13].

Zienkiewicz [83] contains an interesting account of the historical development of the FEM; for the general philosophical ideas underlying the FEM, and also its suitability for numerous applications, see for instance the paper by Oden [63].

4.2 Overview

We briefly motivate the study of finite element interpolation theory and then outline our development of the theory in this Chapter.

Suppose that we use finite element methods to obtain an approximation u_h to the solution u of some (variational) problem. We are then concerned with the *quality* (measured in some sense) of this approximation, and how this changes as we use more 'refined' approximations.

Very often, one can show that the (for example, Galerkin) finite element approximate solution satisfies the "near-best" approximation property (see, for example, Mitchell and Wait [59] and Reddy [67]), that is, one can show that the approximate solution u_h satisfies

$$\|u - u_h\| \leq C \inf_{v_h \in S_h} \|u - v_h\|, \tag{4.1}$$

where $\|\cdot\|$ is some appropriate Sobolev norm, S_h is some finite element space and C denotes a constant. Here the subscript h is a parameter which in a natural way measures how "refined" the approximation is. This is called the *near-best* approximation property because it is often not known whether the constant C is the lowest possible. If u is smooth enough, then we can construct the *finite element interpolant* $\tilde{u}_h$ of u, and bound the *finite element interpolation error* $\|u - \tilde{u}_h\|$ in terms of powers of the "refinement" parameter h. This provides an upper bound for the right hand side of estimate (4.1), and so we can then estimate the (asymptotic) rate of convergence of the finite element approximations in terms of the parameter h.

This interpolation error is also the basis for obtaining an estimate for the rate of convergence of finite element approximations to the solution of the variational inequality which serves as our model problem. We later outline the development of the standard finite element interpolation theory.

The central problem is the following: For a given function u belonging to a Sobolev space $W^{m,p}(\Omega)$, construct a finite element representation of u (which approximates u as closely as desired), and obtain estimates for the interpolation error for a given finite element mesh. We will see that because of the structure of the Sobolev spaces and the means of constructing the finite element interpolant, the problem of deriving an upper bound for the finite element approximation error $\|u - u_h\|$ is reduced to the problem of evaluating quantities such as $\|u - \tilde{u}_h\|$ *over each subdivision* of the domain.

This consideration motivates the structure of this Chapter: Section 4.3 provides the setting for a general theory of finite element interpolation; in Section 4.4 we make precise the idea of partitioning the domain into subdomains, and introduce the notion of (affine) equivalent families of finite elements, which plays an important role in the development of local finite element interpolation theory and its extension to global interpolation theory, which forms the subject of Section 4.5. Finally, in Section 4.7, we discuss some extensions of the theory outlined here.

A general theory of finite element interpolation has been developed by Ciarlet and Raviart [19] and [20] and further generalised by Ciarlet [17], [18]. Reddy [67] provides a very readable introduction to this theory for Lagrangian elements and its applications to finite element approximation error estimates for elliptic variational boundary value problems. See also Oden and Carey [64] and Oden and Reddy [65] for a detailed introduction. This Chapter draws on all of the above and follows Ciarlet [17], [18].

4.3 Some General Properties of Finite Elements

We provide the setting for a general theory of finite element interpolation. See, for example, Reddy [67] and Oden and Carey [64] for specific examples of finite elements which are used in practice.

Let Ω be an open bounded domain in $\Re^n$, with Lipshitzian boundary $\partial\Omega$. Let $u \in C^m(\bar{\Omega}), m \geq 0$. Construction of a finite element interpolant of u can be accomplished as follows:

Partitioning of $\bar{\Omega}$. We construct a partition $\mathcal{O}^h$ of $\bar{\Omega}$ by subdividing $\bar{\Omega}$ into a finite number E of subdomains $\bar{\Omega}_e \in \mathcal{O}^h$ (where h: $0 < h \leq 1$ denotes a parameter which is a measure of the 'size' of the subdomains of the partition), such that

(i) Each $\bar{\Omega}_e$ is closed and consists of non-empty interior Ω_e and a Lipshitzian boundary $\partial\Omega_e$.

(ii) $\bar{\Omega} = \bigcup_{e=1}^{E} \bar{\Omega}_e$.

(iii) $\Omega_e \cap \Omega_f = \emptyset \quad e \neq f$.

Local Interpolation. For each $\bar{\Omega}_e \in \mathcal{O}^h$, we introduce finite-dimensional spaces P_e spanned by linearly independent local interpolation functions $\{\psi_i^e\}_{i=1}^{N_e}$ of the points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

We approximate the restriction $u\,|_{\bar{\Omega}_e}$ of $u \in C^m(\bar{\Omega})$ by linear combinations of the form

$$u_h^e(\mathbf{x}) = \sum_{i=1}^{N_e} a_i^e \psi_i^e(\mathbf{x}) \quad \mathbf{x} \in \bar{\Omega}_e \tag{4.2}$$

where the coefficients a_i^e are usually taken to be the values of u and the values of various partial derivatives of u at a preassigned collection of points $\{\mathbf{b}_i^e\}_{i=1}^{m_e}$ within $\bar{\Omega}_e$. The coefficients a_i^e are called the values of *local degrees of freedom of element* $\bar{\Omega}_e$; the set D_e of local-degrees-of-freedom constitutes a set of continuous linear functionals on $C^m(\bar{\Omega}_e)$ (since the a_i^e depend linearly and continuously on u). The points $\{\mathbf{b}_i^e\}_{i=1}^{m_e}$ are called the *nodes* of element $\bar{\Omega}_e$.

In general, we require that for some k

$$\mathcal{P}_k(\bar{\Omega}_e) \subset P_e$$

where $\mathcal{P}_k(\bar{\Omega}_e)$ is the space of polynomials in $\mathbf{x}$ of degree $\leq k$ defined on $\bar{\Omega}_e$; that is, the functions ψ_i^e, $1 \leq i \leq N_e$, and nodal points $\mathbf{b}_i^e$, $1 \leq i \leq m_e$, are selected in such a way that linear combinations of the form (4.2) can be constructed which coincide with any polynomial of degree $\leq k$ on $\bar{\Omega}_e$.

Assembly. Global approximations are obtained by fitting together local approximations. The local interpolation functions ψ_i^e are designed so that common values of coefficients a_i^e at nodes common to adjacent elements produce a global representation of u. In this way, by matching together corresponding local interpolation functions, a system of M linearly independent basis functions $\{\phi\}_{i=1}^{M}$ is obtained.

Globally, we produce a representation of $u \in C^m(\bar{\Omega})$ of the form

$$u_h(\mathbf{x}) = \sum_{i=1}^{M} a_i(u)\phi_i(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega} \tag{4.3}$$

where the coefficients a_i are called the *global degrees of freedom* of the finite element approximation of u. The set D of global-degrees-of-freedom similarly consists of a set of M continuous linear functionals on $C^m(\bar{\Omega})$.

Being linearly independent, the collection $\{\phi_i\}_{i=1}^{M}$ provides a basis for a finite-dimensional subspace $S^h(\Omega)$ of $C^m(\bar{\Omega})$, referred to as a *finite element space*. By definition we have that

$$P_e = \{v_h|_{\bar{\Omega}_e}|\ v_h \in S^h(\Omega)\}. \tag{4.4}$$

By appropriate choice of element geometry, node location, degrees of freedom and local interpolation functions, the global basis functions $\{\phi_i\}$ can be constructed so that their derivatives of any order $r \geq 0$ are continuous in $\bar{\Omega}$. Then $S^h \subset C^r(\bar{\Omega})$, and the component elements which generate $S^h(\Omega)$ are referred to as *C^r-finite elements.*

Considering the above outline of properties of finite elements, we provide, following Ciarlet [16], a precise definition of a finite element:

A finite element in $\Re^n$ is a triple (G, D, P) where

(i) G is a nonempty closed subset of $\Re^n$ with a Lipshitzian boundary ∂G.

(ii) D is a finite set of linear functionals l_i, $1 \leq i \leq N_G$, defined on $C^\infty(G)$, the degrees of freedom of the element.

(iii) P is a space of functions defined on G, $P \subset C^\infty(G)$ such that for any real scalars α_i, $1 \leq i \leq N_G$, there exists a unique $\psi \in P$ such that $l_i(\psi) = \alpha_i$, $1 \leq i \leq N_G$; we then say that D is P-*unisolvent.* Equivalently, there exist N functions $\psi_i \in P, 1 \leq i \leq N$, which satisfy

$$l_i(\psi_j) = \delta_{ij} \quad 1 \leq i, j \leq N_G$$

which are called the basis functions of the finite element, since

$$v(\mathbf{x}) = \sum_{i=1}^{N_G} l_i(v)\psi_i(\mathbf{x}) \quad \text{for all } v \in P.$$

We now make a few remarks about this definition. When (G, D, P) is a member of a partition $\mathcal{O}^h$ of a given domain Ω, we set $G = \bar{\Omega}_e$ and write $(\bar{\Omega}_e, D_e, P_e)$. Also, for convenience, but conflicting with this definition, the set G is often referred to as a finite element.

The significance of P-unisolvence is that specifying N_G values of $p(\mathbf{x}) \in P_k(G)$ at the points $\{\mathbf{b}_i\}_{i=1}^{N_G}$ uniquely determines the polynomial $p(\mathbf{x})$.

We can use the basis functions $\{\psi_i\}_{i=1}^{N_G}$ of P and the linear functionals $\{l_i\}_{i=1}^{N_G}$ of D to construct P-interpolants of sufficiently smooth functions v, denoted by Πv or $\Pi_G v$, defined by the following conditions:

$$\Pi v \in P, \text{ and } l_i(\Pi v) = l_i(v) \quad 1 \leq i \leq N_G.$$

Therefore, the P-interpolate can be expressed as

$$\Pi v = \sum_{i=1}^{N_G} l_i(v)\psi_i.$$

We have thus defined a *P-interpolation operator* $\Pi : C^{\infty}(G) \rightarrow P$, also denoted by Π_G.

As mentioned previously, for $p \in C^{\infty}(\Omega)$, l_i are taken to be the values of p or the values of various partial derivatives of p at a preassigned collection of points (nodes) $\{\mathbf{b}_i^m\}$ within G. Using the Fréchet derivative, (see Chapter 2) with its compact expression of various partial derivatives, enables us to write down the general forms of the members of the sets of degrees of freedom:

$$\left.\begin{aligned} &l_i^0 : p \rightarrow p(\mathbf{b}_i^0), \\ &l_{ik}^1 : p \rightarrow \mathcal{D}p(\mathbf{b}_i^1) \cdot \boldsymbol{\xi}_{ik}^1, \\ &l_{ikl}^2 : p \rightarrow \mathcal{D}^2 p(\mathbf{b}_i^2) \cdot (\boldsymbol{\xi}_{ik}^2, \boldsymbol{\xi}_{il}^2), \end{aligned}\right\} \tag{4.5}$$

where the nodes $\mathbf{b}_i^r$, $r = 0, 1, 2$, belong to the finite element and the non zero vectors $\boldsymbol{\xi}_{ik}^1, \boldsymbol{\xi}_{ik}^2, \boldsymbol{\xi}_{il}^2$ are either constructed from the geometry of the finite element (for example, $\mathcal{D}p(\mathbf{b}_i) \cdot (\mathbf{b}_j - \mathbf{b}_i)$, etc.) or are fixed vectors of $\Re^n$ (for example, $\frac{\partial p}{\partial x_i}(\mathbf{b}_i)$ etc.). Conceivably, we could consider degrees of freedom which would be partial derivatives of arbitrarily high order, but these are seldom used in practice. It is convenient, however, to define the degrees of freedom over the space $C^{\infty}(\Omega)$ which does not explicitly depend on the specific form of the degrees of freedom to be considered. When all the degrees of freedom are of the form $l_i : p \rightarrow p(\mathbf{b}_i)$, then the associated finite element is called a *Lagrange finite element*; if at least one partial derivative occurs then the associated finite element is called a *Hermite finite element.*

We will next take a closer look at the nature of the *partitioning* of the domain Ω which allows us to extend properties from a finite element to the entire domain.

4.4 Equivalent Families of Finite Elements

Introduction

The question arises as to how we can usefully characterise a subdivision of the domain Ω into a family of finite elements (termed a finite element mesh) of a similar nature. The way this is done is to check whether each element of the family is equivalent in some sense to a single element, called the *master element.* By viewing each finite element in the family as the image of a map defined on the master element, we are able to express the finite element interpolation error over each element in terms of appropriate mesh parameters. If the family of finite elements is 'regular' then we can extend this local finite element interpolation error estimate to the whole domain Ω.

Affine families of finite elements

For two finite elements to be regarded as similar we would expect each point in one element to be in one-to-one correspondence with points in the other element. With this motivation, two finite elements $\hat{\Omega}$ and $\tilde{\Omega}$ are *equivalent* if there exists a unique invertible map F mapping points $\hat{\mathbf{x}} \in \hat{\Omega}$ onto points $\tilde{\mathbf{x}} \in \tilde{\Omega}$ such that $F(\hat{\mathbf{b}}_i) = \tilde{\mathbf{b}}_i$, $1 \leq i \leq \hat{m} = \tilde{m}$, where $\{\hat{\mathbf{b}}_i\}_{i=1}^{\hat{m}}$ and $\{\tilde{\mathbf{b}}\}_{i=1}^{\tilde{m}}$ are the nodal points of elements $\hat{\Omega}$ and $\tilde{\Omega}$, respectively. In the case that F is an *affine map*, that is, one which maps straight lines into straight lines, we say that the two elements are *affine equivalent.* When $\hat{\Omega}$ and $\tilde{\Omega}$ are affine equivalent, F takes on the form

$$F : \hat{\Omega} \to \tilde{\Omega}; \quad F(\hat{\mathbf{x}}) = \mathbf{T}\hat{\mathbf{x}} + \mathbf{c} = \tilde{\mathbf{x}}$$

$$F(\hat{\mathbf{b}}_i) = \tilde{\mathbf{b}}_i; \quad 1 \leq i \leq \hat{m} = \tilde{m},$$

where $\mathbf{T}$ is an invertible matrix and c is a translation vector in $\Re^n$. Once we have established a bijection $\hat{\mathbf{x}} \in \hat{G} \rightarrow \mathbf{x} = F_G(\hat{\mathbf{x}}) \in G$ between the points of the sets $\hat{G}$ and G, it is natural to associate the space

$$P_G = \left\{ \psi : G \rightarrow \Re \mid \psi = \hat{\psi} \circ F_G^{-1}, \;\; \hat{\psi} \in \hat{P} \right\} \tag{4.6}$$

with the space $\hat{P}$. We say that two finite elements $(\hat{G}, \hat{D}, \hat{P})$ and (G, D, P) with degrees of freedom of the form (4.5), are (affine) equivalent if there exists an invertible affine mapping

$$F : \hat{\mathbf{x}} \in \Re^n \rightarrow F(\hat{\mathbf{x}}) = \mathbf{T}\hat{\mathbf{x}} + \mathbf{c} \in \Re^n \tag{4.7}$$

such that the following relations hold:

$$G = F(\hat{G}); \tag{4.8}$$

$$\left.\begin{array}{l} \mathbf{b}_i^r = F(\hat{\mathbf{b}}_i^r), \quad r = 0, 1, 2, \\ \\ \boldsymbol{\xi}_{ik}^1 = \mathbf{T}\hat{\boldsymbol{\xi}}_{ik}^1, \quad \boldsymbol{\xi}_{ik}^2 = \mathbf{T}\hat{\boldsymbol{\xi}}_{ik}^2, \quad \boldsymbol{\xi}_{il}^2 = \mathbf{T}\hat{\boldsymbol{\xi}}_{il}^2, \end{array}\right\} \tag{4.9}$$

whenever the nodes $\mathbf{b}_i^r$ (respectively $\hat{\mathbf{b}}_i^r$) and the vectors $\boldsymbol{\xi}_{ik}^1$, $\boldsymbol{\xi}_{ik}^2$, $\boldsymbol{\xi}_{il}^2$ (respectively $\hat{\boldsymbol{\xi}}_{ik}^1$, $\hat{\boldsymbol{\xi}}_{ik}^2$, $\hat{\boldsymbol{\xi}}_{il}^2$) occur in the definition of the set D (respectively $\hat{D}$), and

$$P = \{\psi : G \rightarrow \Re; \;\; \psi = \hat{\psi} \circ F^{-1}, \hat{\psi} \in \hat{P}\}. \tag{4.10}$$

The concept of equivalent finite elements gives us a very useful way of thinking about families of finite elements:

(i) Let $(\hat{\Omega}, \hat{D}, \hat{P})$ be a master element whose geometry, degrees of freedom and interpolation functions are absolutely fixed.

(ii) Introduce a collection $\{F_e\}_{e=1}^{E}$ of invertible affine maps with $\mathrm{dom} F_e = \hat{\Omega}$, satisfying (4.7).

In this way a family of (affine) equivalent finite elements can be generated $\{\bar{\Omega}_e\}_{e=1}^{E}$, each member of which is affine equivalent to the master element.

When a finite element mesh contains elements of more than one type then additional master elements can be introduced. All the important mesh properties are now intrinsic properties of the collection of $\{F_e\}_{e=1}^{E}$.

We now point out an important property of the interpolation operators Π introduced earlier. Let $(\hat{\Omega}, \hat{D}, \hat{P})$ be a master element and $(\bar{\Omega}_e, D_e, P_e)$ an element affine equivalent to $(\hat{\Omega}, \hat{D}, \hat{P})$. On $\bar{\Omega}_e$, we recall, it is possible to construct a projection or interpolation operator Π_e such that

$$\Pi_e v = \sum_{i=1}^{N} l_i^e(v)\psi_i^e \tag{4.11}$$

where$\{\psi_i^e\}_{i=1}^{N}$ is a basis for P_e and $l_i^e \in D_i$, for any sufficiently smooth function $v : G \to \Re$.

Similarly, for the master element

$$\hat{\Pi}_{\hat{\Omega}}\hat{v} = \sum_{i=1}^{N} \hat{l}_i(\hat{v})\hat{\psi}_i \tag{4.12}$$

where $\{\hat{\psi}_i\}_{i=1}^{N}$ is a basis for $\hat{P}$ and $\hat{l}_i \in \hat{D}$, for any sufficiently smooth function $\hat{v} : \hat{G} \to \Re$. We record a fundamental relationship between the operators $\hat{\Pi}_{\hat{\Omega}}$ and Π_e in the following theorem.

Theorem 4.1 *Let $(\hat{G}, \hat{D}, \hat{P})$ and (G, D, P) be two equivalent finite elements. Then if $\hat{\psi}_i$, $1 \leq i \leq N$, are the basis functions of the finite element $\hat{G}$, then the functions ψ_i, $1 \leq i \leq N$, obtained by* (4.6) *are the basis functions of the finite element G. The interpolation operators Π and $\hat{\Pi}$ are such that*

$$\widehat{\Pi v} = \hat{\Pi}\hat{v} \tag{4.13}$$

for any sufficiently smooth functions $v\colon G \to \Re$ *and* $\hat{v}\colon \hat{G} \to \Re$ *associated by the correspondence*

$$v \rightarrow \hat{v}, \text{ where } v = \hat{v} \circ F^{-1}, \text{ that is } v(\mathbf{x}) = \hat{v}(\hat{\mathbf{x}}) \text{ for all } \mathbf{x} = F(\hat{\mathbf{x}}).$$

with a similar correspondence $\Pi v \rightarrow \widehat{\Pi v}$.

PROOF. See, for example, Ciarlet [17] for the proof of this Theorem, which follows easily by using expressions of the form (4.5) for the degrees of freedom of each finite element. □

Now we consider an assemblage of finite elements made up of a family of elements $(\bar{\Omega}_e, D_e, P_e)$, $\bar{\Omega}_e \in \mathcal{O}^h$ for $1 \leq i \leq E$, all of which are affine equivalent to a master element $(\hat{\Omega}, \hat{D}, \hat{P})$. If the *local interpolate* $\Pi_e v$ of a smooth function v is of the form (4.2) with degrees of freedom given by (4.5), that is,

$$\Pi_e v = \sum_i v(\mathbf{b}_i^0)\psi_i^0 + \sum_{i,k} \left[\mathcal{D}v(\mathbf{b}_i^1) \cdot \boldsymbol{\xi}_{ik}^1\right]\psi_{ik}^1 + \sum_{i,j,k} \left[\mathcal{D}^2 v(\mathbf{b}_i^2) \cdot (\boldsymbol{\xi}_{ij}^2, \boldsymbol{\xi}_{ik}^2)\right]\psi_{ijk}^2$$

(here we have suggestively rewritten the local interpolation functions $\{\psi_i^e\}$ so as to make their correspondence with the form of the degrees of freedom clear), then the *global interpolation* functions, ϕ_i in (4.3) are denoted here as

$$\phi_i^0, \quad \phi_{ij}^1, \quad \phi_{ijk}^2,$$

the derivatives of order $r = 0,\ 1,\ 2$ of which take on appropriate unit or zero values at the global nodal points $\{\mathbf{b}_i^0, \mathbf{b}_{ij}^1, \mathbf{b}_{ijk}^2\}$, respectively. Hence, with any sufficiently smooth function $v\colon \bar{\Omega} \to \Re$, we associate the S^h-interpolant $\Pi_h v$ such that

$$\Pi_h v = \sum_i v(\mathbf{b}_i^0)\phi_i^0 + \sum_{i,k} \left[\mathcal{D}v(\mathbf{b}_i^1) \cdot \boldsymbol{\xi}_{ik}^1\right]\phi_{ik}^1 + \sum_{i,j,k} \left[\mathcal{D}^2 v(\mathbf{b}_i^2) \cdot (\boldsymbol{\xi}_{ij}^2, \boldsymbol{\xi}_{ik}^2)\right]\phi_{ijk}^2. \quad (4.14)$$

While the indices i, j, k in (4.2) range over a set of values appropriate for a single element, those in (4.14) range over a larger set pertaining to the entire collection of assembled elements.

Moreover, the global degrees of freedom form a set D_h of continuous linear forms on $C^\infty(\bigcup_e \bar{\Omega}_e)$ satisfying

$$D_h\,|_{C^\infty(\bar{\Omega}_e)} = D_e. \tag{4.15}$$

By construction, we have that

$$\Pi_h\,|_e = \Pi_e, \;\; 1 \le i \le E. \tag{4.16}$$

Relation (4.16) is of fundamental importance in constructing a complete interpolation theory for finite elements, as will be seen in the ensuing development.

4.5 Interpolation Theory in Sobolev Spaces

We now come to an important theoretical aspect of finite elements, namely the finite element interpolation of functions in Sobolev spaces $W^{m,p}(\Omega)$, $m \ge 0$, $1 \le p \le \infty$. We have previously developed one way of constructing finite element representations of a function $u \in W^{m,p}(\Omega)$, namely: for some given finite element space $S^h(\Omega) \subset W^{m,p}(\Omega)$, we introduce the interpolation operator

$$\Pi_h : W^{m,p}(\Omega) \rightarrow S^h(\Omega)$$

satisfying

$$\Pi_h u = \sum_{i=1}^{M} l_i(u)\phi_i$$

where $\{\phi_i\}_{i=1}^M$ are the global basis functions for $S^h(\Omega)$, generated by the methods indicated earlier, and $\{l_i\}_{i=1}^M$ are the global degrees of freedom. Our goal is to determine the quality of the approximation $\Pi_h u$ of u and see how this varies as the mesh is refined. Achieving these goals depends on the character of the local interpolation operators $\Pi_e = \Pi_h\,|_e$. We now derive some properties of local projections for a typical finite element $(\bar{\Omega}_e, D_e, P_e)$ in a family of affine equivalent elements. First we introduce some useful mesh parameters:

$$\left.\begin{aligned} h &= \text{dia }(\Omega); \quad \hat{h} = \text{dia }(\hat{\Omega}), \\ \rho &= \sup\left\{\text{dia }(S); S \text{ is a sphere contained in } \Omega\right\} \\ \hat{\rho} &= \sup\left\{\text{dia }(\hat{S}); S \text{ is a sphere contained in } \hat{\Omega}\right\} \end{aligned}\right\} \tag{4.17}$$

We list some preliminary results relating $\|\mathbf{T}\|$ to the geometry of the element and relating seminorms of v on an element G which is affine-equivalent to a master element $\hat{G}$ under the affine map $F_G(\cdot) = \mathbf{T}(\cdot) + \mathbf{c}$. We can prove that

$$\|\mathbf{T}\|_{\Re^n} \leq h/\hat{\rho}, \quad \|\mathbf{T}^{-1}\|_{\Re^n} \leq \hat{h}/\rho, \tag{4.18}$$

$$|\hat{v}|_{m,p,\hat{G}} \leq C\|\mathbf{T}\|^m_{\Re^n} \, |\det(\mathbf{T})|^{-1/p} |v|_{m,p,G}, \tag{4.19}$$

$$|v|_{m,p,G} \leq C\|\mathbf{T}^{-1}\|^m_{\Re^n} \, |\det(\mathbf{T})|^{1/p} |\hat{v}|_{m,p,G}, \tag{4.20}$$

where $\|\cdot\|_{\Re^n}$ denotes the Euclidean norm of the matrices $\mathbf{T}$ and $\mathbf{T}^{-1}$,

$$|v|_{m,p,G} = \left\{\int_G \sum_{|\alpha|=m} |D^\alpha v|^p \, dx\right\}^{1/p}$$

is a seminorm on $W^{m,p}(G)$, C is a constant (independent of G, v, and $\hat{G}$), $\hat{v} = v \circ F$, etc.

For the proof of these assertions, see for example, Oden and Reddy [64], Ciarlet [17] or Reddy [67].

We now record a preliminary Lemma relating a property of the local interpolation error $v - \Pi v$.

Lemma 4.1 *Let $W^{k+1,p}(\Omega)$ be a Sobolev space continuously embedded in the Sobolev space $W^{m,q}(\Omega)$, that is*

$$W^{k+1,p}(\Omega) \hookrightarrow W^{m,q}(\Omega)$$

and let $\Pi \in \mathcal{L}(W^{k+1,p}(\Omega), W^{m,q}(\Omega))$ *be a continuous linear operator from* $W^{k+1,p}(\Omega)$ *to* $W^{m,q}(\Omega)$ *which preserves polynomials of degree* $\leq k$*; that is,*

$$\Pi w = w \qquad \text{for all } w \in \mathcal{P}_k(\Omega). \tag{4.21}$$

Then there exists a constant $C = C(\Omega)$ *such that for every* $v \in W^{k+1,p}(\Omega)$

$$| v - \Pi v |_{m,q,\Omega} \leq C(\Omega) \|I - \Pi\|_{\mathcal{L}(W^{k+1,p}(\Omega), W^{m,q}(\Omega))} \, | v |_{k+1,p,\Omega} \, . \tag{4.22}$$

PROOF. Here we follow Oden and Carey [64]. Let $w \in \mathcal{P}_k(\Omega)$. Then we have

$$v - \Pi v = v - \Pi v + w - \Pi w = (I - \Pi)(v + w)$$

for $v \in W^{k+1,p}(\Omega)$. Hence

$$\begin{aligned} | v - \Pi v |_{m,q,\Omega} &\leq \|v - \Pi v\|_{m,q,\Omega} \\ &= \|(I - \Pi)(v + w)\|_{m,q,\Omega} \\ &\leq \|I - \Pi\|_{\mathcal{L}(W^{k+1,p}(\Omega), W^{m,q}(\Omega))} \inf_{w \in \mathcal{P}_k(\Omega)} \|v + w\|_{k+1,p,\Omega}. \end{aligned} \tag{4.23}$$

The term $\inf\{\|v + w\|_{k+1,p,\Omega};\ w \in \mathcal{P}_k(\Omega)\}$ is the norm on the quotient space $Q^{k+1,p}(\Omega) = W^{k+1,p}(\Omega)/\mathcal{P}_k(\Omega)$. The result follows by (2.12). □

Now we come to the principal result of this section, the interpolation theorem for a finite element.

Theorem 4.2 *Let* $(\hat{\Omega}, \hat{D}, \hat{P})$ *be a finite element for which the set* $\hat{D}$ *of degrees of freedom involves the specification of derivatives of order* $s \geq 0$*. In addition, for* *positive integers m and k, let*

$$\left.\begin{aligned} &W^{k+1,p}(\hat{\Omega}) \hookrightarrow \hat{C}^{s}(\hat{\Omega}) \\ &W^{k+1,p}(\hat{\Omega}) \hookrightarrow W^{m,q}(\hat{\Omega}) \\ &P_k(\hat{\Omega}) \subset \hat{P} \subset W^{m,q} \end{aligned}\right\} \tag{4.24}$$

Then there exists a positive constant $C = C(\hat{\Omega}, \hat{D}, \hat{P})$ *depending only on the properties of* $(\hat{\Omega}, \hat{D}, \hat{P})$ *such that for all elements* $(\bar{\Omega}_e, D_e, P_e)$ *and all* $v \in W^{k+1,p}(\Omega_e)$ *we have*

$$| v - \Pi_e v |_{m,q,\Omega_e} \leq C(\hat{\Omega}, \hat{D}, \hat{P}) meas(\Omega_e)^{1/q-1/p} \frac{h_e^{k+1}}{\rho_e^m} | v |_{k+1,p,\Omega_e}, \tag{4.25}$$

where $\Pi_e v$ *denotes the* P_e*-interpolant of* v *and* $h = diam(\Omega_e)$ *and* $\rho_e = sup\{ dia(S) \mid S$ *is a sphere contained in* $\Omega_e \}$.

Proof. Here we follow Oden and Carey [64]. Since $\mathcal{P}_k(\hat{\Omega}) \subset \hat{P}$, and $\hat{D}$ is P-unisolvent, it follows that the $\hat{\Omega}$-interpolation operator $\hat{\Pi}$ satisfies

$$\hat{\Pi}\hat{w} = \hat{w}, \text{ for all } \hat{w} \in \mathcal{P}_k(\hat{\Omega}). \tag{4.26}$$

By the inclusions (4.24), we have that $\hat{\Pi} \in \mathcal{L}(W^{k+1,p}(\hat{\Omega}), W^{m,q}(\hat{\Omega}))$. By (4.24) and Lemma 4.1, we have

$$\begin{aligned} |\hat{v} - \hat{\Pi}\hat{v}|_{m,q,\hat{\Omega}} &\leq C(\hat{\Omega}) \|I - \Pi\|_{\mathcal{L}(W^{k+1,p}(\hat{\Omega}), W^{m,q}(\hat{\Omega}))} |v|_{k+1,p,\hat{\Omega}} \\ &= C(\hat{\Omega}, \hat{D}, \hat{P}) |\hat{v}|_{k+1,p,\hat{\Omega}} \end{aligned} \tag{4.27}$$

for every $\hat{v} \in W^{k+1,p}(\hat{\Omega})$. By Theorem 4.1 we have that

$$\hat{v} - \hat{\Pi}\hat{v} = v - \Pi_e v.$$

Hence, by (4.20) and (4.19), we have

$$| v - \Pi_e v |_{m,q,\Omega_e} \leq C \|\mathbf{T}_e^{-1}\|^m \mid \det (\mathbf{T}_e) \mid^{1/q} | \hat{v} - \hat{\Pi}\hat{v} |_{m,q,\hat{\Omega}_e} \tag{4.28}$$

and

$$| \hat{v} |_{k+1,p,\hat{\Omega}} \leq C \|\mathbf{T}_e\|^{k+1} \mid \det (\mathbf{T}_e) \mid^{-1/p} | v |_{k+1,p,\Omega_e} \tag{4.29}$$

where $\mathbf{T}_e$ is the invertible matrix occuring in the affine map $F : \hat{\Omega} \rightarrow \bar{\Omega}_e$. By combining (4.27), (4.28) and (4.29), we obtain

$$| v - \Pi_e v |_{m,q,\Omega_e} \leq C \|\mathbf{T}_e^{-1}\|^m \|\mathbf{T}_e\|^{k+1} \mid \det (\mathbf{T}_e) \mid^{1/q} | \det (\mathbf{T}_e) \mid^{-1/p} | v |_{k+1,p,\Omega_e} \tag{4.30}$$

By noting that $| \det (\mathbf{T}_e) | = \frac{\text{meas }(\Omega_e)}{\text{meas }(\hat{\Omega})}$ and using (4.18) we obtain the inequality (4.25).
□

We now remark on the inclusions (4.24) and its implications for the selection of finite elements with respect to global reqularity requirments.

(i) If D_e contains degrees of freedom involving derivatives of order s, then we require the local interpolation functions ψ to be in $\bar{C}^s(\Omega_e)$. By the Sobolev embedding theorem (Theorem 2.1), for $1 \leq p \leq \infty$, the inclusion $W^{k+1,p}(\hat{\Omega}) \hookrightarrow \bar{C}^s(\hat{\Omega})$ holds whenever

$$(k + 1 - s)p > n. \tag{4.31}$$

Inequality (4.31) usually holds for the standard finite element applications (see, for example, Oden and Carey [64], Ciarlet [16] and Reddy [67]), but should be checked to ensure the validity of the estimate (4.25).

(ii) The second inclusion in (4.24) determines the type of finite element one considers with respect to global continuity requirements. For example, for the case $p = q = 2$, and $m = 1$, by Theorem 2.1, $W^{k+1,p}(\hat{\Omega}) \hookrightarrow W^{m,q}(\hat{\Omega})$ holds for any n whenever $k \geq 1$.

(iii) Estimates such as (4.25) hold for many important families of elements which are not affine equivalent to a master element; these are termed *almost affine* elements (see, for example, Ciarlet [18]). Estimates of the type (4.25) also hold (with the seminorms replaced by norms) for curvilinear elements generated by nonaffine maps, called *isoparametric* elements (see, for example, Ciarlet [18]).

Estimate (4.25) reduces to a more useful result if we consider only *regular* families of finite elements. A family $\mathcal{F} = \{(\bar{\Omega}_e, D_e, P_e) : \bar{\Omega}_e \in \mathcal{O}^h, \quad 1 \le e \le E$ is called regular if

(i) F is an affine family.

(ii) There exists a constant $\sigma_0 > 0$ such that $h_e/\rho_e \le \sigma_0$, for all $\bar{\Omega}_e$ in the family $\mathcal{O}^h$.

(iii) The diameters of h_e approach zero.

Then we have

Corollary 4.1 *Let the conditions of Theorem 4.2 hold for a regular family of finite elements. Then there exists a constant $C > 0$ such that for all elements in the family and $v \in W^{k+1,p}(\Omega_e)$,*

$$| v - \Pi_e v |_{m,q,\Omega_e} \le C\big(h_e\big)^{n/q-n/p} h_e^{k+1-m} \, | v |_{k+1,p,\Omega_e} . \quad \square \tag{4.32}$$

We now consider *global* interpolation properties, using the gobal interpolation operator $\Pi_h : W^{k+1,p}(\Omega) \to S^h(\Omega) \subset W^{m,q}(\Omega)$ to approximate functions in $W^{k+1,p}(\Omega)$. We choose the sets of degrees-of-freedom so that each $\bar{\Omega}_e \in \mathcal{O}_h$ is at least a C^{m-1} finite element to guarantee the inclusion

$$S^h(\Omega) \subset W^{m,q}(\Omega). \tag{4.33}$$

Now suppose that

$$v \in W^{k+1,p}(\Omega) \text{ and } \Pi_e = \Pi \,|\, \Omega_e$$

and take $p = q$. Further, let

$$h = \max_{1 \le e \le E} h_e \tag{4.34}$$

Hence, from (4.32), we have that for regular families of finite elements

$$\begin{aligned} |v - \Pi_h v|^p_{m,p,\Omega} &= \sum_{e=1}^{E} |\, v|_{\bar{\Omega}_e} - \Pi v|_{\bar{\Omega}_e}|^p_{m,p,\Omega_e} \\ &\le \sum_{e=1}^{E} C\left(h_e^{k+1-m}\, |v|_{k+1,p,\Omega_e}\right)^p \\ &\le C\left(h^{k+1-m} \sum_{e=1}^{E} |v|_{k+1,p,\Omega_e}\right)^p \\ &= C\left(h^{k+1-m}\, |v|_{k+1,p,\Omega}\right)^p \end{aligned}$$

In summary, we have the following Theorem.

Theorem 4.3 *Let $\{(\bar{\Omega}_e, D_e, P_e) : \bar{\Omega}_e \in \mathcal{O}^h,\ 1 \le e \le E\}$ be a regular family of finite elements, each of which is affine equivalent to a master element $(\hat{\Omega}, \hat{D}, \hat{P})$; let $p = q$ and suppose that the conditions of Theorem 4.2 hold with $\Pi_e = \Pi_h\, |_e$, where Π_h is the global interpolation operator $\Pi_h : W^{k+1,p}(\Omega) \to S^h(\Omega) \subset W^{m,p}(\Omega)$. Then there exists a constant $C > 0$ such that for any $v \in W^{k+1,p}(\Omega)$,*

$$|\, v - \Pi_h v\, |_{m,p,\Omega} \le C h^{k+1-m}\, |\, v\, |_{k+1,p,\Omega} \quad \text{for all } v \in W^{k+1}(\Omega), \tag{4.35}$$

where h is the mesh parameter defined in (4.34). □

In the case of regular refinements, the boundedness of h/h_e as $h_e \to 0$ allows us to convert (4.32) immediately into estimates involving the norm of $v - \Pi_e v$ in $W^{m,p}(\Omega_e)$:

$$\|v - \Pi_e v\|^p_{m,p,\Omega_e} = \sum_{t=1}^{m} |\, v - \Pi_e v\, |^p_{t,p,\Omega}$$

$$\leq \sum_{t=1}^{m} C h_e^{p(k+1-t)} \mid v \mid_{k+1,p,\Omega_e}^{p}$$
$$\leq C h_e^{p(k+1-m)} \mid v \mid_{k+1,p,\Omega_e}^{p} .$$

Thus, for regular refinements, we have that

$$\|v - \Pi_e v\|_{m,p,\Omega_e} \leq C h_e^{k+1-m} \mid v \mid_{k+1,p,\Omega_e} \tag{4.36}$$

as $h_e \to 0$. Moreover, when the conditions of Theorem 4.3 hold, we also have globally,

$$\|v - \Pi v\|_{m,p,\Omega} \leq C h^{k+1-m} \mid v \mid_{k+1,p,\Omega} . \tag{4.37}$$

Very often we are concerned with the finite element approximation of functions in the Hilbert spaces $H^r(\Omega) = W^{r,2}(\Omega)$. In the following Section we summarise the standard finite element interpolation error estimates for this case.

4.6 Standard Interpolation Error Estimates

We describe some properties of family of finite element subspaces $H^h(\Omega)$ in view of the interpolation properties of the previous development.

(i) Let Ω be an open, bounded domain in $\Re^n$ with a Lipschitzian boundary $\partial\Omega$ and let $\{\mathcal{O}^h\}_{0<h\leq 1}$ be a family of partitions of Ω depending on the mesh parameter h.

(ii) For each h, let $\{(\bar{\Omega}_e, D_e, P_e), \quad \bar{\Omega}_e \in \mathcal{O}^h, \ 1 \leq e \leq E\}$ denote a family of finite elements, which, on assembly, lead to a set of global interpolation functions $\{\phi_i\}_{i=1}^{M_h}$, which serves as a basis for a finite dimensional linear space $H^h(\Omega)$.

(iii) Let Π_h denote the global interpolation operator corresponding to $\{\phi_i\}_{i=1}^{M_h}$, satisfying, for each h,

$$\left.\begin{aligned} &\Pi_h : H^{k+1}(\Omega) \to H^h(\Omega) \subset H^m(\Omega); \quad k+1 > m \geq 0 \\ &(\Pi_h v)\,|_{\bar{\Omega}_e} = \Pi_e(v\,|_{\bar{\Omega}_e}) \quad \text{for all } v \in H^{k+1}(\Omega) \\ &\Pi_h p = p \quad \text{for all } p \in \mathcal{P}_k(\Omega) \end{aligned}\right\} \tag{4.38}$$

(iv) By (ii), there corresponds to the family $\{\phi_i\}_{i=1}^{M_h}$ a family of subspaces $\{H^h(\Omega)\}_{i=1}^{M_h}$. This family is endowed with the following interpolation property: For every $v \in H^r(\Omega)$ and every h, there exists a constant $C > 0$ and an element $\tilde{v}_h \in H^h(\Omega)$ such that

$$\|v - \tilde{v}_h\|_{m,\Omega} \leq Ch^{\mu}\,|v|_{r,\Omega} \tag{4.39}$$

where

$$\mu = \min(k+1-m, r-m) \tag{4.40}$$

We remark on this last estimate: if the function v to be interpolated is sufficiently smooth that $r \geq k+1$, and if regular affine families of elements are used in constructing $H^h(\Omega)$, then estimate (4.39) follows from the estimate (4.36), with $\mu = k+1-m$ and $\tilde{v}_h = \Pi_h v$. If, on the other hand, $r < k+1$, then the term $|v|_{k+1}$ cannot in general be evaluated. We note, however, that the entire theory previously developed still holds on replacing $k+1$ by r.

4.7 Some Extensions

The previous analysis includes problems of approximation and interpolation of vector- and tensor-valued functions of $\Re^n$. In this case, the finite element subspace

$\mathbf{H}^h$ is a *product* of n or $n \times n$ identical subspaces H^h; equivalently, with each degree of freedom of a subspace H^h constructed as before, we associate n or $n \times n$ unknowns representing the components of the vector or tensor, respectively. See Zienkiewicz [82] for a discussion of the merits of various choices of finite elements in this context.

Chapter 5

Problem Statement and Qualitative Results

5.1 Introduction and Problem Statement

In this Chapter we introduce the general variational inequality which serves as our model problem for this investigation.

This time-dependent variational inequality closely resembles a parabolic variational inequality of the second kind, with the important distinction that the rate quantity occurs in the arguments of all the functionals occuring in the inequality.

The form of this variational inequality arises in the variational formulation of the quasistatic behaviour of an elastoplastic body which undergoes kinematic hardening. More details of this particular application of the model problem are supplied in Chapter 7. However, we now take as fundamental the following abstract variational problem.

Problem (P) *Given $l \in W^{1,2}(0,T;H^*)$ find the function $w : [0,T] \to H$ such that*

$$a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) - \langle l, z - \dot{w}(t) \rangle \geq 0, \tag{5.1}$$

$$\text{for all } z \in H, \quad \text{for almost all } t \in (0,T).$$

Here H denotes a Hilbert space, a: $H \times H \rightarrow \Re$ is a bilinear form which is assumed to be symmetric, bounded and H-elliptic, $l(t)$: $H \rightarrow \Re$ is a bounded linear functional, and j: $H \rightarrow \Re$ is a convex, positively homogeneous, nonnegative and continuous functional, which is *not* assumed to be differentiable.

This is a generalisation of a problem which was investigated by Reddy [68], who examined the questions of the existence and uniqueness of the solution to problem (P).

We note here that problem (P) is equivalent to the problem of finding the functions w: $[0,T] \rightarrow H$ and $w^*(t)$: $[0,T] \rightarrow H^*$ such that

$$a(w(t), z) + \langle w^*(t), w(t) \rangle = \langle l(t), z \rangle, \tag{5.2}$$

$$w^*(t) \in \partial j(\dot{w}(t)), \tag{5.3}$$

$$\text{for all } z \in H, \text{ for almost all } t \in (0,T),$$

where $\partial j(\dot{w}(t))$ denotes the subdifferential of $j(\cdot)$ at $\dot{w}(t)$.

From the definition of the subdifferential, we observe that the relation $w^*(\mathrm{t}) \in \partial j(\dot{w}(t))$ is equivalent to

$$\langle w^*(t), z \rangle \leq j(z), \quad \text{for all } z \in H \text{ and } \langle w^*(t), \dot{w}(t) \rangle = j(\dot{w}(t)). \tag{5.4}$$

An outstanding feature of the proofs of these qualitative results is that they employ discretisation methods which are closely related to those which are used in practice for computational purposes (see for example Reddy and Martin [70]). The method of proof has interesting parallels with the fully discretised approximations of problem (P), for which an estimate of the rate of convergence of the approximations is derived in Section 6.2.

An outline of the above-mentioned analysis follows, and the results are summarised by the final Theorem which is a generalisation of a result obtained by Reddy [68].

5.2 Existence and Uniqueness of the Solution

The analysis of this problem involves two stages : firstly discretising in time and establishing the existence of a family of solutions $\{w_n\}_{n=1}^N$ to the discrete problems. The second stage involves constructing a linear interpolate in time w_ϵ of the discrete solutions and finally showing that the limit, as the time step size ϵ approaches zero, of these interpolates is in fact the solution of problem (P).

The time-discretisation involves partitioning the time interval [0,T] by $0 = t_0 < t_1 \ldots < t_N = T$, where $t_n - t_{n-1} = \epsilon$. For given $l \in W^{1,2}(0,T;H^*)$, $l_n = l(t_n)$, and we define Δw_n to be the backward difference $w_n - w_{n-1}$ corresponding to a sequence $\{w_n\}_{n=0}^N$. We note that all of these quantities are well-defined by the embedding $W^{1,2}(0,T;X) \hookrightarrow C([0,T],X)$, for any Banach space X (see Chapter 2).

Existence. The proof of existence of a unique solution to problem (P) depends on establishing some preliminary lemmas which are used to show that the interpolates w_ϵ and their derivatives $\dot{w}_\epsilon$ are bounded (in $L^\infty(0,T;H)$ and $L^2(0,T;H)$, respectively) independently of the time-step ϵ. Consequently there exists a subsequence $\{w_{\epsilon_\mu}\}$ of $\{w_\epsilon\}$ such that as $\epsilon \rightarrow 0$,

$$\begin{aligned} w_{\epsilon_\mu} &\rightarrow w && \text{weakly star in } L^\infty(0,T;H) \\ \text{and} \quad \dot{w}_{\epsilon_\mu} &\rightarrow \dot{w} && \text{weakly in } L^2(0,T;H); \end{aligned}$$

these results follow by the properties of $L^\infty(0,T;H)$ and $L^2(0,T;H)$ established in Chapter 2, together with the fundamental Theorems 2.5 and 2.6.

The final step of the proof consists of showing that the function w obtained in this manner satisfies the variational inequality (5.1).

We start with the following

Lemma 5.1 *There exists a sequence* $\{w_n\}_{n=0}^N$ *in* H, *with* $w_0 = 0$, *such that*

$$a(w_n, z - \Delta w_n) + j(z) - j(\Delta w_n) - \langle l_n, z - \Delta w_n \rangle \geq 0 \tag{5.5}$$

for all $z \in H$, *for a given* $\{l_n\}_{n=0}^N \in H^*$. *Also, each solution* w_n *corresponding to a given* l_n *is unique and there exists a constant* C, *independent of* ϵ,*such that*

$$\|\Delta w_n\|_H \leq C \|\Delta l_n\|_{H^*} \tag{5.6}$$

PROOF. The inequality (5.5) may be rewritten as

$$a(\Delta w_n, z - \Delta w_n) + j(z) - j(\Delta w_n) \geq \langle l_n, z - \Delta w_n \rangle - a(w_{n-1}, z - \Delta w_n). \tag{5.7}$$

We proceed inductively. For $n = 1$ the problem (5.7) has a unique solution $\Delta w_n = w_1$ by Theorem 3.2, since by assumption the bilinear form $a(\cdot, \cdot)$ is continuous and H-elliptic, the functional $j(\cdot)$ is convex and continuous, and the functional defined by the righthand side of (5.7) is bounded and linear. Assuming now that the solution w_{n-1} is known, we similarly show the existence of the solution $w_n = \Delta w_n - w_{n-1}$. To derive the estimate (5.6), set $z = 0$ in (5.5) to get

$$a(\Delta w_n, \Delta w_n) \leq \langle \Delta l_n, \Delta w_n \rangle + a(w_{n-1}, \Delta w_n) - j(\Delta w_n) + \langle l_{n-1}, \Delta w_n \rangle. \tag{5.8}$$

We now show that $a(w_{n-1}, \Delta w_n) - j(\Delta w_n) + \langle l_{n-1}, \Delta w_n \rangle \leq 0$. By replacing n by $(n-1)$ and setting $z = \Delta w_{n-1} - \Delta w_n$ in (5.5) we obtain

$$\begin{aligned} 0 &\geq a(w_{n-1}, \Delta w_n) + \langle l_{n-1}, \Delta w_n \rangle + j(\Delta w_{n-1} - \Delta w_n) - j(\Delta w_n) \\ &\geq a(w_{n-1}, \Delta w_n) + \langle l_{n-1}, \Delta w_n \rangle - j(\Delta w_n); \end{aligned}$$

where we have used the convexity and positive homogeneity of $j(\cdot)$. Hence from (5.8) we obtain the inequality

$$a(\Delta w_n, \Delta w_n) \leq \langle \Delta l_n, \Delta w_n \rangle,$$

from which estimate (5.6) follows by the H-ellipticity of $a(\cdot,\cdot)$ and the boundedness of Δl_n. □

Lemma 5.2 *There exists constants C_1, $C_2 > 0$ such that*

$$\max_{1\leq n\leq N} \|w_n\|_H \leq C_1, \tag{5.9}$$

$$\sum_{n=1}^{N} \|\Delta w_n\|_H^2 \leq C_2\epsilon. \tag{5.10}$$

PROOF. The proof depends on Lemma 5.1 as well as the fundamental estimate (2.17) satisfied by elements of $W^{1,2}(0,T;H^*)$ (see Reddy [68], Lemma 3). □

We construct the linear interpolate w_ϵ of $\{w_n\}$ by setting

$$w_\epsilon(t) = w_{n-1} + \frac{\Delta w_n}{\epsilon}(t - t_{n-1})$$

for $t_{n-1} \leq t \leq t_n$. Clearly w_ϵ belongs to $L^\infty(0,T;H)$ while $\dot{w}_\epsilon \in L^2(0,T;H)$. The next step is to establish that w_ϵ satisfies the variational inequality

$$\begin{aligned} 0 \leq J_\epsilon \equiv & \int_0^T \left[a(w_\epsilon(t), z - \dot{w}_\epsilon(t)) + j(z) - j(\dot{w}_\epsilon(t)) - \langle l_\epsilon(t), z - \dot{w}_\epsilon(t)\rangle\right] dt \\ & + \tfrac{1}{2}a(w_N, z)\epsilon + \tfrac{1}{2}\langle l_N, z\rangle\epsilon + \tfrac{1}{2}c\epsilon\int_0^T \|\dot{l}(t)\|_{H^*}^2\, dt \end{aligned} \tag{5.11}$$

for all $z \in H$, a.e. in $(0,T)$, where $l_\epsilon(t)$ represents the linear interpolate of $\{l_n\}_{n=1}^N$ and c is the constant appearing in (5.6).

To show this we divide (5.5) throughout by ϵ , make use of the positive homogeneity of $j(\cdot)$, and replace the arbitrary z/ϵ by z. Finally we multiply throughout by ϵ and sum to obtain

$$\sum_{n=1}^{N} \epsilon[a(w_n, z - \delta w_n) + j(z) - j(\delta w_n) - \langle l_n, z - \delta w_n\rangle] \geq 0 \tag{5.12}$$

where $\delta w_n = \Delta w_n/\epsilon$. Now we have that

$$\begin{aligned}\sum_{n=1}^{N} a(w_n, z)\epsilon &= \sum_{n=1}^{N} \tfrac{1}{2}a(w_n + w_{n-1}, z)\epsilon + \tfrac{1}{2}a(w_N, z)\epsilon \\ &= \sum_{n=1}^{N} \tfrac{1}{2}a(w_n - w_{n-1}, z)\epsilon + \sum_{n=1}^{N} a(w_{n-1}, z)\epsilon + \tfrac{1}{2}a(w_N, z)\epsilon \\ &= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} a(w_{n-1} + (t - t_{n-1})\delta w_n, z)\mathrm{d}t + \tfrac{1}{2}a(w_N, z)\epsilon \\ &= \int_0^T a(w_\epsilon, z)\ \mathrm{d}t + \tfrac{1}{2}a(w_N, z)\epsilon.\end{aligned}$$

In the same way we find, after routine manipulations, that

$$\begin{aligned}\int_0^T a(w_\epsilon(t), \dot{w}_\epsilon(t))\ \mathrm{d}t &= \sum_{n=1}^{N} a(w_n, \delta w_n)\epsilon - \tfrac{1}{2}\sum_{n=1}^{N} a(\delta w_n, \delta w_n)\epsilon^2 \\ &\leq \sum_{n=1}^{N} a(w_n, \delta w_n)\epsilon.\end{aligned}$$

The terms involving $j(\cdot)$ are handled in a trivial way; this leaves the term involving l_n which becomes, after some manipulation,

$$\begin{aligned}-\sum_{n=1}^{N} \langle l_n, z - \delta w_n\rangle\epsilon &= -\int_0^T \langle l_\epsilon(t), z - \dot{w}_\epsilon(t)\rangle\ \mathrm{d}t \\ &\quad - \tfrac{1}{2}\langle l_N, z\rangle\epsilon + \tfrac{1}{2}\sum_{n=1}^{N} \langle \Delta l_n, \Delta w_n\rangle.\end{aligned}$$

But from the Schwarz inequality, (5.6) and the estimate (2.17), we have

$$\begin{aligned}\sum_{n=1}^{N} \langle \Delta l_n, \Delta w_n\rangle &\leq \sum_{n=1}^{N} \|\Delta l_n\|_{H^*} \|\Delta w_n\|_H \\ &\leq c\sum_{n=1}^{N} \|\Delta l_n\|_{H^*}^2 \\ &= c\epsilon\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\dot{l}(t)\|_{H^*}^2 dt \\ &= c\epsilon\int_0^T \|\dot{l}(t)\|_{H^*}^2\ \mathrm{d}t.\end{aligned}$$

Combining all of the above results, we obtain (5.11).

From (5.9) and (5.10) and the definition of w_ϵ we see by direct evaluation that

$$\|w_\epsilon\|_{L^\infty(0,T;H)} \leq C_1 \text{ and } \|\dot{w}_\epsilon\|_{L^2(0,T;H)} \leq C_2.$$

It follows that there exists a subsequence, denoted also by $\{w_\epsilon\}$, and a member w such that

$$w_\epsilon \overset{*}{\rightharpoonup} w \text{ in } L^\infty(0,T;H)$$

and

$$\dot{w}_\epsilon \rightharpoonup \dot{w} \text{ in } L^2(0,T;H)$$

as $\epsilon \to 0$.

It remains to show that w satisfies the variational inequality (5.11). We return to (5.5) and consider each of the terms appearing there. First, integrating by parts and using the fact that $w_\epsilon(0) = 0$, we obtain

$$\begin{aligned}\limsup_{\epsilon\to 0} -\int_0^T a(w_\epsilon(t), \dot{w}_\epsilon(t))\, dt &= -\liminf_{\epsilon\to 0} a(w_\epsilon(T), w_\epsilon(T)) \\ &\leq -a(w(T), w(T)) \\ &= -\int_0^T a(w(t), \dot{w}(t))\, dt.\end{aligned}$$

Next,

$$\limsup_{\epsilon\to 0} \int_0^T a(w_\epsilon(t), z)\, dt = \int_0^T a(w(t), z)\, dt$$

and

$$\begin{aligned}\limsup_{\epsilon\to 0} -\int_0^T j(\dot{w}_\epsilon(t))\, dt &= -\liminf_{\epsilon\to 0} \int_0^T j(\dot{w}_\epsilon(t))\, dt \\ &\leq -\int_0^T j(\dot{w}_\epsilon(t))\, dt\end{aligned}$$

by the weak sequential lower semicontinuity of j (which follows since j is both convex and continuous, see Zeidler [81]).

The three terms in J_ϵ which are coefficients of ϵ vanish in the limit as $\epsilon \to 0$ by virtue of the estimate (5.9) and the boundedness of l (recall from Chapter 2 that $l \in C(0,T;H^*)$).

This leaves the terms involving the approximation of $l_\epsilon(t)$ to the linear functional $l(t)$. By assumption and construction we have that $l, l_\epsilon \in L^2(0,T;H^*)$; furthermore, since for $t_{n-1} \le t \le t_n$ we have

$$\|l(t) - l_\epsilon(t)\|_{H^*} \le \|l(t) - l(t_{n-1})\|_{H^*} + \frac{|\, t - t_{n-1} \,|}{\epsilon}\|\Delta l_n\|_{H^*}$$

it follows that $l_\epsilon \to l$ in $L^2(0,T;H^*)$ as $\epsilon \to 0$.

Thus

$$\int_0^T \langle l_\epsilon(t), \dot{w}_\epsilon(t)\rangle \, \mathrm{d}t \to \int_0^T \langle l(t), \dot{w}(t)\rangle \, \mathrm{d}t$$

as $\epsilon \to 0$.

The groundwork is now complete; using the above results we have

$$\begin{aligned} 0 &\le \limsup_{\epsilon \to 0} J_\epsilon \\ &\le \int_0^T \left[a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) - \langle l(t), z - \dot{w}(t)\rangle \right] \, \mathrm{d}t. \end{aligned}$$

By a standard procedure (see for example Duvaut and Lions [26]), of passing to the pointwise inequality we find from (5.11) that w satisfies the variational inequality (5.1) a.e on $[0,T]$. □

Uniqueness. Suppose that problem (P) has two solutions, w_1 and w_2. Denote by Δw the difference $w_1 - w_2$. From (5.1), on setting $w = w_1$, $z = \dot{w}_2$ and then $w = w_2$, $z = \dot{w}_1$ respectively, we have

$$\begin{aligned} a(w_1, \Delta\dot{w}) + j(\dot{w}_1) - j(\dot{w}_2) &\le \langle l, \Delta\dot{w}\rangle, \\ -a(w_2, \Delta\dot{w}) + j(\dot{w}_2) - j(\dot{w}_1) &\le -\langle l, \Delta\dot{w}\rangle. \end{aligned}$$

Adding, we get

$$0 \leq a(\Delta w, \Delta \dot{w}) = \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} a(\Delta w, \Delta w).$$

Integration, the H-ellipticity of $a(\cdot,\cdot)$ and the initial conditions $w_1(0) = w_2(0) = 0$ together yield $w_2 = w_1$, as required. □

We summarise the above analysis in the following Theorem.

Theorem 5.1 (Existence and uniqueness) *For every $l \in W^{1,2}(0,T;H^*)$ there exists a unique solution w of problem (P) satisfying $w \in L^\infty(0,T;H)$ and $\dot{w} \in L^2(0,T;H)$ Furthermore, w: $[0,T] \to H$ is the solution to problem (P) if and only if there is a function $w^*(t)$: $[0,T] \to H^*$ such that*

$$a(w(t), z) + \langle w^*(t), z\rangle = \langle l(t), z\rangle, \tag{5.13}$$

$$w^*(t) \in \partial j(\dot{w}(t)), \tag{5.14}$$

for all $z \in H$, for almost all $t \in (0,T)$.

Chapter 6

Approximation and Numerical Analysis

In this Chapter we consider two general approximation procedures for the solution of the model problem (P). In both of these schemes, the Hilbert space H is replaced by a family of finite-dimensional subspaces H^h of H, where $h \in (0,1]$ is some parameter. We assume that as $h \to 0$ the subspaces H^h approach H in some suitable sense. In specific applications, we often take H^h to be a finite-element subspace of H. We do this in Chapter 7 where we make the above ideas more definite. In the semi-discrete approximation procedure only the space H is approximated by approximating the solution w of Problem (P) by an element w^h in H^h, while in the fully discrete approximation the time domain is also discretised.

For generalities on the numerical approximation of variational inequalities, see Falk [31], Glowinski, Lions and Trémolières [35], Strang [74], Brezzi, Hager and Raviart [11], [12], Oden and Kikuchi [66] and Lions [51].

We will make use of the following well-known *arithmetic-geometric mean inequality* : for any real numbers α, β and for $\epsilon > 0$

$$\alpha\beta \leq \epsilon\alpha^2 + (4\epsilon)^{-1}\beta^2. \tag{6.1}$$

In the ensuing analysis C denotes a generic constant, which is not necessarily the same at each occurrence.

We now pose the following problem :

Problem (P^h) *Given* $l \in W^{1,2}(0,T;H^*)$ *find the function* $w^h : [0,T] \to H^h$ *such that*

$$a(w^h(t), z^h - \dot{w}^h(t)) + j(z^h) - j(\dot{w}^h(t)) - \langle l(t), z^h - \dot{w}^h(t)\rangle \geq 0 \qquad (6.2)$$

$$\text{for all } z^h \in H^h, \text{ for almost all } t \in (0,T).$$

For any given h, the existence of a unique solution w^h to problem (P^h) follows from Theorem 5.1 with the Hilbert space H taken to be the space H^h.

The remainder of this Section is concerned with finding an estimate for the error $w - w^h$ which is due to the internal approximation; for this purpose we will make use of the existence of w^*: $[0,T] \to W^*$ which reduces the variational inequality (5.1) to the variational equality (5.2), at the expense of introducing the new variable w^*. We note that this technique was used by Han and Reddy [36] to prove the existence of a solution to a general mixed variational inequality.

The main result of this Section is the derivation of an inequality of Céa's lemma-type, which is the basis of error estimates for various finite element solutions.

We begin by taking z in (5.1) to be $\dot{w}^h(t)$: then

$$a(w(t), \dot{w}^h(t) - \dot{w}(t)) + j(\dot{w}^h(t)) - j(\dot{w}(t)) \geq \langle l(t), \dot{w}^h(t) - \dot{w}(t)\rangle; \qquad (6.3)$$

adding to (6.2) gives

$$a(w(t), \dot{w}^h(t) - \dot{w}(t)) + a(w^h(t), z^h - \dot{w}^h(t)) + j(z^h) - j(\dot{w}(t)) \geq \langle l(t), z^h - \dot{w}(t)\rangle. \qquad (6.4)$$

So we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a(w(t)-w^h(t),w(t)-w^h(t))\\
&\quad= a(w(t)-w^h(t),\dot{w}(t)-\dot{w}^h(t))\\
&\quad= a(w(t)-w^h(t),\dot{w}(t)-z^h)+a(w(t)-w^h(t),z^h-\dot{w}^h(t))\\
&\quad\leq a(w(t)-w^h(t),\dot{w}(t)-z^h)+a(w(t),z^h-\dot{w}^h(t))\\
&\qquad + a(w(t),\dot{w}^h(t)-\dot{w}(t))+j(z^h)-j(\dot{w}(t))-\langle l(t),z^h-\dot{w}(t)\rangle\\
&\quad\leq a(w(t)-w^h(t),\dot{w}(t)-z^h)+j(z^h-\dot{w}(t))-\langle w^*(t),z^h-\dot{w}(t)\rangle. \qquad (6.5)
\end{aligned}$$

In arriving at this result we have used :

(i) the relation (6.4),

(ii) the convexity and positive homogeneity of $j(\cdot)$,

(iii) the existence of $w^*(t)$, given by Theorem 5.1, which satisfies the variational equation (5.2) with z chosen to be $z^h-\dot{w}(t)$.

Now

$$\begin{aligned}
&a(w(t)-w^h(t),\dot{w}(t)-z^h)\\
&\quad\leq \mathrm{K}\|w(t)-w^h(t)\|_H\|\dot{w}(t)-z^h\|_H\\
&\quad\leq \mathrm{K}\epsilon\|w(t)-w^h(t)\|_H^2+\frac{\mathrm{K}}{4\epsilon}\|\dot{w}(t)-z^h\|_H^2\\
&\quad\leq \frac{\mathrm{K}\epsilon}{\alpha}\|w(t)-w^h(t)\|_a^2+\frac{\mathrm{K}}{4\epsilon}\|\dot{w}(t)-z^h\|_H^2\\
&\quad\leq \mathrm{C}\left(\|w(t)-w^h(t)\|_a^2+\|\dot{w}(t)-z^h\|_H^2\right), \qquad (6.6)
\end{aligned}$$

where we have used

(i) the estimate (6.1)

(ii) the continuity and H-ellipticity of the bilinear form $a(\cdot,\cdot)$;

here $\|\cdot\|_a$ denotes the (energy) norm associated with the bilinear form $a(\cdot,\cdot)$, $K > 0$ is a constant and $\alpha > 0$ is the H-ellipticity constant of the form $a(\cdot,\cdot)$. Also,

$$j(z^h - \dot{w}(t)) - \langle w^*(t), z^h - \dot{w}(t)\rangle \leq \mathrm{C}\|z^h - \dot{w}(t)\|_H, \tag{6.7}$$

by the boundedness of $j(\cdot)$, and estimate (5.4). Hence, the inequality (6.5) reduces to the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t) - w^h(t)\|_a^2 \leq \mathrm{C}\left(\|w(t) - w^h(t)\|_a^2 + \|\dot{w}(t) - z^h\|_H^2 + \|z^h - \dot{w}(t)\|_H\right). \tag{6.8}$$

Next we multiply inequality (6.8) throughout by $\mathrm{e}^{-\mathrm{C}t}$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-\mathrm{C}t}\|w(t) - w^h(t)\|_a{}^2\right] \leq \mathrm{C}\mathrm{e}^{-\mathrm{C}t}\left(\|\dot{w}(t) - z^h\|_H^2 + \|\dot{w}(t) - z^h\|_H\right). \tag{6.9}$$

For notational convenience, we rename the variable t in the inequality (6.9) to be s, and integrate from $s = 0$ to $s = t$, to obtain

$$\begin{aligned} \mathrm{e}^{-\mathrm{C}t}\|w(t) - w^h(t)\|_a^2 \;\leq\; & \|w(0) - w^h(0)\|_a^2 \\ & + \mathrm{C}\left(\int_0^t \mathrm{e}^{-\mathrm{C}t}\|\dot{w}(t) - z^h\|_H^2 \,\mathrm{d}s \right. \\ & \left. + \int_0^t \mathrm{e}^{-\mathrm{C}t}\|\dot{w}(t) - z^h\| \,\mathrm{d}s\right). \end{aligned} \tag{6.10}$$

The first term on the right hand side of inequality (6.10) is well defined since by Theorem 5.1 $w \in L^\infty(0,T;H)$ and $\dot{w} \in L^2(0,T;H)$, hence certainly $w \in W^{1,2}(0,T;H)$ which is continuously embedded in $C([0,T],H)$ (see Chapter 2). Similarly, by the remark immediately following the statement of problem (P), we also have that $w^h \in C([0,T],H)$.

Hence, since $e^{-Ct} \leq 1$ and e^{Ct} is bounded on $[0,T]$, we have

$$\|w(t) - w^h(t)\|_a^2 \leq \mathrm{C}\left(\|w(0) - w^h(0)\|_a^2 + \|\dot{w} - z^h\|_{L^2(0,T;H)}^2 + \|\dot{w} - z^h\|_{L^1(0,T;H)}\right) \tag{6.11}$$

so we have

$$\|w(t) - w^h(t)\|_a^2 \leq C\|w(0) - w^h(0)\|_a^2 + C \inf_{z^h \in H^h} \left\{ \|\dot{w} - z^h\|_{L^1(0,T;H)} + \|\dot{w} - z^h\|^2_{L^2(0,T;H)} \right\} \quad (6.12)$$

This inequality forms the basis of various finite element approximation error estimates (in specific applications) when the space H is precisely defined (usually H is some product Sobolev space); this is done in Chapter 7 for the case where Problem (P) is used to model a problem in quasistatic elastoplasticity.

The strategy used to obtain the above result was inspired by ideas contained in Douglas and Dupont [25], even though the problems and analyses differ greatly.

6.2 Fully-Discrete Internal Approximation and Error Estimate

We recall the model problem

Problem (P) Given $l \in W^{1,2}(0,T;H^*)$ find w: $(0,T) \to H$ such that

$$a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) - \langle l, z - \dot{w}(t)\rangle \geq 0$$

for all $z \in H$.

As before we introduce a finite-dimensional subspace H^h of H.

We also introduce a discretisation of the time-interval $I = [0,T]$. Let N be a positive integer, $k = N^{-1}T$, $t_n = nk$, $n = 0, 1, \ldots, N$ and $I_n = [t_{n-1}, t_n]$. We define $v_n = v(t_n)$, and $\delta v_n = (v_n - v_{n-1})/k$.

We now pose the following approximate variational inequality:

Problem ($\mathbf{P}^{hk}$) Find $w^{hk} \equiv \left\{w_n^{hk}\right\}_{n=0}^{N}$, where $w_n^{hk} \in H^h$ for $n = 0, \ldots, N$, such that

$$a(w_n^{hk}, z^h - \delta w_n^{hk}) + j(z^h) \; - j(\delta w_n^{hk}) \; - \langle l_n, z^h - \delta w_n^{hk}\rangle \; \geq 0 \qquad (6.13)$$

$$\text{for all } z^h \in H^h, \; n = 1, \ldots, N.$$

Note that the existence and uniqueness of w_n^{hk}, for $n = 0, \ldots, N$, follows by Lemma 5.1 (in this Chapter we use k to denote the time increment rather than ϵ (as used in Chapter 5) for the conventional (or psychological!) reason that the passage to the limit is not employed).

A PRIORI ERROR ESTIMATE

We assume that we have the solution w_n^{hk} $(n = 0, \ldots, N)$ of problem (6.13) and we estimate the error $w_n - w_n^{hk}$.

First, for $q = (q_1, \ldots, q_n)$, $\quad q_n \in H^h$, we define

$$\|q\|_{l^2(H)} = \left(\sum_{n=1}^{N} k\|q_n\|_H^2\right)^{1/2}.$$

We then have the following preliminary result :

Lemma 6.1 *There exists a positive constant C such that*

$$\|\delta w^{hk}\|_{l^2(H)} \leq C.$$

PROOF. This is the content of Lemma 5.2, since $\delta w^{hk} = \Delta w^{hk}/k$. $\square$

We define

$$\epsilon(h,k) = \inf_{\tau \in \mathcal{H}} \left\{ \|\delta w - \tau\|_{l^2(H)} \right\},$$

where $\mathcal{H} = \{\tau = (\tau_1, \dots, \tau_N) \mid \tau_n \in H^h, \quad n = 1, \dots, N\}$ and $\delta w \equiv \{\delta w_1, \dots, \delta w_N\}$. The quantity $\epsilon(h,k)$ is a measure of how well the exact solution w can be approximated by functions in H^h. The magnitude of $\epsilon(h,k)$ is actually determined by the approximation properties of H^h and by the regularity of the solution w.

Theorem 6.1 *Let w be the solution of problem (P) and w^{hk} be the solution of the fully-discrete problem (P^{hk})). Then for k sufficiently small we have*

$$\max_n \|w_n - w_n^{hk}\|_H \le C\big(\epsilon^{1/2}(h,k) + k^{1/4}\big). \tag{6.14}$$

PROOF. First of all we have

$$a(w(t), z - \dot w(t)) + j(z) - j(\dot w(t)) - \langle l(t), z - \dot w(t)\rangle \ge 0 \tag{6.15}$$
$$\text{for all } z \in H(t), \text{ for almost all } t \in I,$$

and

$$a(w_n^{hk}, z^h - \delta w_n^{hk}) + j(z^h) - j(\delta w_n^{hk}) - \langle l(t), z^h - \delta w_n^{hk}\rangle \ge 0 \tag{6.16}$$
$$\text{for all } z^h \in H^h(t_n).$$

We extend w^{hk} to the whole interval I by setting

$$w^{hk}(t) = w_{n-1}^{hk} - \delta w_n^{hk}(t - t_n)$$

for $t_{n-1} \le t \le t_n$. Then we easily check that $w^{hk}(t) \in L^\infty(0,T;H)$ and that $\dot w^{hk} \in L^2(0,T;H)$.

By integration we have the identity

$$a(w_n, \delta w_n^{hk} - \delta w_n) - \langle l_n, \delta w_n^{hk} - \delta w_n\rangle = \frac{1}{k}\int_{I_n} a(w_n, \dot w^{hk}(t) - \dot w(t)) - \langle l_n, \dot w^{hk}(t) - \dot w(t)\rangle \, dt. \tag{6.17}$$

Let us consider $\tau^h \in \mathcal{H}$ such that

$$\|\delta w - \tau^h\|_{l^2(H)} \leq 2\epsilon(h,k)$$

where as before $\delta w = \{\delta w_1, \dots, \delta w_N\}$, and set $z^h = \tau^h_n$ in (6.16). Thus we obtain

$$a(w_n^{hk}, \tau_n^h - \delta w_n^{hk}) + j(\tau_n^h) - j(\delta w_n^{hk}) - \langle l_n, \tau_n^h - \delta w_n^{hk}\rangle \geq 0. \tag{6.18}$$

By virtue of (6.17) and (6.18) we come to the following inequality for the error $e = w - w^{hk}$:

$$a(e_n, \delta e_n) \leq a(w_n^{hk}, \tau_n^h - \delta w_n) + j(\tau_n^h - \delta w_n) - \langle l_n, \tau_n^h - \delta w_n\rangle + |r_n| + j(\delta w_n), \tag{6.19}$$

where r_n is the right hand side of (6.17). Multiply this inequality by k and sum over n to obtain

$$\begin{aligned}\max_n \|e_n\|_H^2 \;\leq\; & \|w^{hk}\|_{l^2(H)}\|\tau^h - \delta w\|_{l^2(H)} + C\|\tau^h - \delta w\|_{l^2(H)} \\ & + C\|l\|_{l^2(H^*)}\|\tau^h - \delta w\|_{l^2(H)} + 2k\sum_{n=1}^{N}|r_n| + \sum_{n=1}^{N} j(\Delta w_n). \quad (6.20)\end{aligned}$$

We now give an estimate for r_n :

$$\begin{aligned}|r_n| \;\leq\; & Ck^{-1}\int_{I_n} \|w_n\|_H\|\dot{w}^{hk} - \dot{w}\|_H + \|l_n\|_{H^*}\|\dot{w}^{hk} - \dot{w}\|_H \,\mathrm{dt} \\ \leq\; & Ck^{-1}\int_{I_n} \|w_n\|_H(\|\dot{w}^{hk}\|_H + \|\dot{w}\|_H) + \|l_n\|_{H^*}(\|\dot{w}^{hk}\|_H + \|\dot{w}\|_H)\,\mathrm{dt} \\ \leq\; & Ck^{-1/2}\|w_n\|_H\left\{\left(\int_{I_n}\|\dot{w}^{hk}\|_H^2\,\mathrm{dt}\right)^{1/2} + \left(\int_{I_n}\|\dot{w}\|_H^2\,\mathrm{dt}\right)^{1/2}\right\} \\ & + Ck^{-1/2}\|l_n\|_{H^*}\left\{\left(\int_{I_n}\|\dot{w}^{hk}\|_H^2\,\mathrm{dt}\right)^{1/2} + \left(\int_{I_n}\|\dot{w}\|_H^2\,\mathrm{dt}\right)^{1/2}\right\}\end{aligned}$$

Also, by inequality (2.17) and the Schwarz inequality we have

$$\|w_n - w_{n-1}\|_H \;\leq\; \int_{I_n}\|\dot{w}(t)\|_H\,\mathrm{dt} \tag{6.21}$$

$$\leq\; k^{1/2}\left(\int_{I_n}\|\dot{w}(t)\|_H^2\,\mathrm{dt}\right)^{1/2} \tag{6.22}$$

so that

$$j(\Delta w_n) \leq C k^{1/2} \left(\int_{I_n} \|\dot{w}(t)\|_H^2 \,\mathrm{dt} \right)^{1/2} . \tag{6.23}$$

Let us substitute into (6.20). Thus we obtain the estimate

$$\begin{aligned}
\max_n \|e_n\|_H^2 \;\leq\; & C\|w^{hk}\|_{l^2(H)}\|\tau^h - \delta w\|_{l^2(H)} + C\|\tau^h - \delta w\|_{l^2(H)} \\
& + C\|l\|_{l^2(H^*)}\|\tau^h - \delta w\|_{l^2(H)} \\
& + C\left(\|w\|_{C([0,T],H)} + \|l\|_{C([0,T],H^*)}\right) k^{1/2} \left(\sum_{n=1}^{N} \left(\int_{I_n} \|\dot{w}^{hk}\|_H^2 \,\mathrm{d}t \right)^{1/2} \right. \\
& \left. + \int_{I_n} \|\dot{w}(t)\|_H^2 \,\mathrm{dt} \right)^{1/2} \Bigg) \\
& + C k^{1/2} \sum_{n=1}^{N} \left(\int_{I_n} \|\dot{w}(t)\|_H^2 \,\mathrm{dt} \right)^{1/2} \\
\leq\; & C\left(\epsilon(h,k) + k^{1/2}\right),
\end{aligned}$$

by taking into account the inequalities

$$\int_0^T \|\dot{w}^{hk}\|_H^2 \,\mathrm{dt} < \infty \text{ and } \int_0^T \|\dot{w}\|_H^2 \,\mathrm{dt} < \infty$$

and the fact that $\|w^{hk}\|_{l^2(H)}$ and $\|l\|_{l^2(H^*)}$ are uniformly bounded, which follows from the following Lemma.

Lemma 6.2 *There exists a constant C_3 independent of k, h such that*

$$\max_{1\leq n\leq N} \|w_n^{hk}\|_H \leq C_3 \textit{ and } \max_{1\leq n\leq N} \|l_n\|_{H^*} \leq C_3$$

PROOF. The proof is essentially the same as in Lemma 5.2. □

Hence we have that

$$\begin{aligned}
\|w^{hk}\|_{l^2(H)} \;=\; & \left(\sum_{n=1}^{N} k\|w_n^{hk}\|_H^2 \right)^{1/2} \\
\leq\; & \left(\sum_{n=1}^{N} k\left(\max_n\{\|w_n^{hk}\|_H\} \right)^2 \right)^{1/2} \\
\leq\; & T^{1/2} C_3 \leq C.
\end{aligned}$$

By the same argument we can show that $\|l\|_{l^2(H)} \leq C$. Hence the result. □

The above strategy was influenced by ideas contained in Hlaváček, Haslinger, Nečas and Lovíšek [40] and Johnson [43], who use quantities analogous to $\epsilon(h,k)$. Again the problems and analyses differ greatly, chiefly because of the presence of the non-differentiable functional j which occurs in the model problem.

CHAPTER 7

QUASISTATIC ELASTOPLASTICITY

7.1 INTRODUCTION

The plastic behaviour of a material is described in terms of rates of change of variables, such as plastic strain; therefore mathematical models of this behaviour contain rate quantities and are not simply boundary-value problems. In this investigation, however, processes are assumed to occur sufficiently slowly, so that inertial effects can be ignored (quasistatic behaviour). Therefore acceleration does not occur in the mathematical problem.

An advantage of the model presented here is that, unlike conventional formulations in elastoplasticity, (see Duvaut and Lions [26]), it is an extension of the standard displacement problem of linear elasticity since it reduces to this in the event of elastic behaviour of the body.

The model variational inequality has wider application than only quasistatic elastoplasticity, however, since there are close parallels with quasistatic problems of frictional contact of elastic bodies (see, for example Andersson [2], Klarbring, Mikelic and Shillor [47], and Martins and Oden [58]). The problems have a similar structure in that they are variational inequalities of a parabolic nature (which are not in standard form), with the inequality arising from the non-differentiability of a functional representing the internal dissipation.

These problems differ from those studied for example in Duvaut and Lions [26] and Johnson [41], in that while the latter are parabolic variational inequalities, they arise because the problems are posed on convex sets and not because of the non-differentiability of a functional.

7.2 Problem Formulation

We consider the initial-boundary value problem for quasistatic behaviour of an elastoplastic body which occupies a bounded domain $\Omega \subset \Re^d$ $(d \leq 3)$ with Lipschitz boundary Γ.

We assume that the plastic behaviour of the material can be described within the framework of a convex yield surface coupled with a normality law. We adopt the form of the flow law in which the dissipation function, rather than the yield function, is employed (see Eve, Reddy and Rockafellar [30], Martin and Reddy [56], Reddy and Griffin [69], and Reddy and Martin [69] for the advantages of this formulation; for a general reference on plasticity see, for example, Martin [55] and Lemaitre and Chaboche [49]).

The material is assumed to undergo linear kinematic hardening, which apart from representing realistic material behaviour, also allows a complete analysis within a Sobolev space framework, the case of perfect plasticity requiring special treatment (see, for example Reddy and Tomarelli [71] and Temam [75]). The model also assumes that there is no volume change accompanying plastic deformation.

We suppose that the system is initially at rest, undeformed and unstressed. A time-dependent field of body force $\mathbf{f}(t) = \mathbf{f}(\mathbf{x}, t)$ is given. We seek the displacement field $\mathbf{u}(t) = \mathbf{u}(\mathbf{x}, t)$ and the plastic strain field $\mathbf{p}(t) = \mathbf{p}(\mathbf{x}, t)$ which satisfy, for $0 \leq t \leq T$, the equilibrium equation

$$\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}(t), \mathbf{p}(t)) + \mathbf{f}(t) = \mathbf{0}, \tag{7.1}$$

the constitutive equations

$$\boldsymbol{\sigma}(\mathbf{u}(t), \mathbf{p}(t)) = \mathbf{C}(\boldsymbol{\epsilon}(\mathbf{u}(t)) - \mathbf{p}(t)) \,, \tag{7.2}$$

$$\boldsymbol{\sigma}^D(t) - \lambda \mathbf{p}(t) \in \partial D(\dot{\mathbf{p}}(t)), \tag{7.3}$$

the strain - displacement relation

$$\boldsymbol{\epsilon}(\mathbf{u}(t)) = \tfrac{1}{2}(\nabla \mathbf{u}(t) + (\nabla \mathbf{u}(t))^T), \tag{7.4}$$

and the condition of plastic incompressibility

$$tr\ \mathbf{p}(t) := \mathbf{I} \cdot \mathbf{p}(t) = p_{kk}(t) = 0, \tag{7.5}$$

where the summation convention is used, that is, any term in which the same index appears twice indicates summation with respect to this index, over its entire range.

Equations (7.1) - (7.5) are required to hold on Ω, and the following notation is used, with the dependence on time being understood :

$\boldsymbol{\sigma}$: stress tensor,

$\boldsymbol{\sigma}^D := \boldsymbol{\sigma} - \frac{1}{d}(tr\ \boldsymbol{\sigma})\mathbf{I}$: stress deviator,

$\boldsymbol{\epsilon}$: strain tensor,

$\mathbf{u}$: displacement vector,

$\mathbf{p}$: plastic strain tensor,

$\mathbf{C}$: fourth order tensor of elastic coefficients and

D : positively homogenous convex function, the dissipation function.

Thus D has the properties

$$D(\theta \mathbf{p} + (1-\theta)\mathbf{q}) \leq \theta D(\mathbf{p}) + (1-\theta)D(\mathbf{q}) , \qquad 0 < \theta < 1 \tag{7.6}$$

$$D(\alpha \mathbf{p}) = \alpha D(\mathbf{p}) , \qquad 0 > \alpha \in \Re, \text{ and } \mathbf{p}, \mathbf{q} \in M^d , \tag{7.7}$$

where M^d is the set of all symmetric $d \times d$ matrices. We assume also that

$$D(\mathbf{q}) \geq 0 , \qquad D(0) = 0 ,$$

the last assumptions being motivated by physical considerations (see Eve, Reddy and Rockafellar [30]).

We now comment briefly on the above equations. The equilibrium condition (7.1) is a statement of the balance of linear momentum, with the right hand side of (7.1) equal to zero since the acceleration is assumed to be negligible. We employ the common assumption that the (total) strain tensor $\boldsymbol{\epsilon}$ can be expressed as the sum of a plastic strain tensor $\mathbf{p}$ and an elastic strain tensor $\boldsymbol{\epsilon} - \mathbf{p}$. The constitutive equation (7.2) then models the elastic behaviour (by a generalised Hookes law) while condition (7.3) models its plastic behaviour. In the elastoplastic behaviour model we assume elastic behaviour of the solid for a range of stresses and then permit plastic, irreversible strains to take place when a threshold or yield value of stress is reached. The relation is generally written in terms of a *yield function*

$$\phi = \phi(\boldsymbol{\sigma})$$

which is negative when the material is elastic and zero when it is at yield. The yield function ϕ is assumed, on the basis of experimental evidence, to be convex. The dissipation function D is introduced as a convenient way of representing information concerning the yield surface. We note that the inclusion (7.3) holds if and only if

$\dot{\mathrm{p}}(t) \in N_K(\boldsymbol{\sigma} - \lambda \mathrm{p})$, where $N_K(\boldsymbol{\tau})$ denotes the normal cone to the yield surface at $\boldsymbol{\tau}$. Condition (7.3) thus describes the plastic flow law obeyed by the material.

A simple example of a dissipation function is that corresponding to the von Mises yield condition, for which

$$D(\mathbf{q}) = k \mid \mathbf{q} \mid = k\sqrt{q_{ij}q_{ij}}\ , \tag{7.8}$$

where k is a positive scalar. It is assumed that the material undergoes linear kinematic hardening and this is represented by the term $\lambda \mathrm{p}(t)$ appearing in (7.3). This is the back-stress, and λ is a scalar-valued hardening function (see, for example, Eve, Reddy and Rockafellar [30]). We assume that $\lambda \in L^\infty(\Omega)$, and that there exists a constant λ_0, such that

$$\lambda(\mathbf{x}) \geq \lambda_0 > 0\ , \quad \text{a.e. in } \Omega. \tag{7.9}$$

The elasticity tensor $\mathbf{C}$ has the symmetry properties

$$C_{ijkl} = C_{jikl} = C_{klij} = C_{ijkl}\ ,$$

and we assume that

$$C_{ijkl} \in L^\infty(\Omega)$$

and also that $\mathbf{C}$ is pointwise stable (see Marsden and Hughes [54]): there exists a constant $c_0 > 0$ such that

$$C_{ijkl}(\mathbf{x})\zeta_{ij}\zeta_{kl} \geq c_0 \mid \zeta \mid^2 \quad \text{for all } \zeta \in M^d\ , \ \text{a.e. in } \Omega. \tag{7.10}$$

Finally we take the boundary condition to be

$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma$$

and the initial conditions to be

$$\mathbf{u}(0) = \mathbf{0} \text{ and } \mathrm{p}(0) = \mathbf{0}.$$

We will seek the plastic strain in a class of traceless functions, that is, in a space $Q_0 = \{\mathbf{q} : tr\ \mathbf{q} = 0\}$ (Q_0 is defined more precisely in (7.12) below). Then (7.3) reads

$$D(\mathbf{q}) - D(\dot{\mathbf{p}}) - (\boldsymbol{\sigma}^D - h\mathbf{p}) \cdot (\mathbf{q} - \dot{\mathbf{p}}) \geq 0 \quad \text{for all } \mathbf{q} \in Q_0, \tag{7.11}$$

where the inner product on M^d is defined by $\mathbf{p} \cdot \mathbf{q} = p_{ij}q_{ij}$. From the definition of $\boldsymbol{\sigma}^D$ and Q, we have that $\boldsymbol{\sigma}^D \cdot \mathbf{q} = \boldsymbol{\sigma} \cdot \mathbf{q}$ for any $\mathbf{q} \in Q_0$, so that (7.11) (and (7.3)) may be replaced by

$$D(\mathbf{q}) - D(\dot{\mathbf{p}}) - (\boldsymbol{\sigma} - h\mathbf{p}) \cdot (\mathbf{q} - \dot{\mathbf{p}}) \geq 0 \quad \text{for all } \mathbf{q} \in Q_0.$$

We next define the spaces

$$V = H_0^1(\Omega)^d,$$

$$Q = \{\mathbf{q} = q_{ij}\ :\ q_{ij}\ \in\ L^2(\Omega)\ ,\ q_{ij} = q_{ji}\},$$

$$Q_0 = \{\mathbf{q} \in Q\ :\ \text{tr}\ \mathbf{q} = 0\}\ . \tag{7.12}$$

Both V and Q are Hilbert spaces with inner products given by

$$(\mathbf{u}, \mathbf{v})_V = \int_\Omega \frac{\partial u_i}{\partial v_j}\frac{\partial v_i}{\partial u_j}\, dx$$

and

$$(\mathbf{p}, \mathbf{q})_Q = \int_\Omega \mathbf{p} \cdot \mathbf{q}\, dx$$

and norms $\|\mathbf{v}\|_V = (\mathbf{v}, \mathbf{v})_V^{1/2}$, $\|\mathbf{q}\|_V = (\mathbf{q}, \mathbf{q})_V^{1/2}$. Furthermore , Q_0 is a closed subspace of Q.

We define the product space $W = V \times Q$, which is a Hilbert space with the inner product given by

$$(\mathbf{w}, \mathbf{z})_W = (\mathbf{u}, \mathbf{v})_V + (\mathbf{p}, \mathbf{q})_Q$$

and norm $\|\mathbf{z}\|_W = (\mathbf{z}, \mathbf{z})_W^{1/2}$, where $\mathbf{w} = (\mathbf{u}, \mathbf{p})$ and $\mathbf{z} = (\mathbf{v}, \mathbf{q})$.
We define the subspace $Z = V \times Q_0$ of W, which is closed in the norm $\|\cdot\|_W$.

We introduce the bilinear form a: $W \times W \to \Re$, where

$$\begin{aligned} a(\mathbf{w},\mathbf{z}) &= \int_\Omega \mathbf{C}(\epsilon(\mathbf{u}) - \mathbf{p}^D)\cdot(\epsilon(\mathbf{v}) - \mathbf{q}^D) + \lambda \mathbf{p}^D \cdot \mathbf{q}^D \, \mathrm{d}x \\ &= \int_\Omega C_{ijkl}(\epsilon_{ij}(\mathbf{u}) - p^D_{ij})(\epsilon_{kl}(\mathbf{v}) - q^D_{kl}) + \lambda p^D_{ij} q^D_{ij} \, \mathrm{d}x \, , \end{aligned} \tag{7.13}$$

the linear functional

$$l(t) : W \to \Re \qquad \langle l(t), \mathbf{z}\rangle = \int_\Omega \mathbf{f}(t)\cdot \mathbf{v} \, \mathrm{d}x \tag{7.14}$$

and the functional

$$j : W \to \Re \qquad j(\mathbf{z}) = \int_\Omega D(q(\mathbf{x})) \, \mathrm{d}x \tag{7.15}$$

where as before $\mathbf{w} = (\mathbf{u}, \mathbf{p})$ and $\mathbf{z} = (\mathbf{v}, \mathbf{q})$.

The functionals $l(t)$ and $j(\cdot)$ are easily shown to be bounded and, from the properties of D, $j(\cdot)$ is a convex, positively homogeneous, non-negative, continuous functional. Note however that $j(\cdot)$ is *not* differentiable.

We can now define the variational problem.

Problem ($\mathbf{P}_E$) Given $l(t) \in W^{1,2}(0,T; Z^*)$ find $\mathbf{w} = (\mathbf{u},\mathbf{p}) : (0,T) \to Z$ such that

$$a(\mathbf{w}(t), \mathbf{z} - \dot{\mathbf{w}}(t)) + j(\mathbf{z}) \; - j(\dot{\mathbf{w}}(t)) \; - \langle l, \mathbf{z} - \dot{\mathbf{w}}(t)\rangle \; \geq 0 \tag{7.16}$$

for all $\mathbf{z} \in Z$.

The *formal equivalence* of the variational problem (P_E) to that of the classical problem defined by (7.1) to (7.5) is readily established (see, for example, Reddy and Martin [70]). By this we mean that any sufficiently smooth solution of (7.16) is also a solution of (7.1) to (7.5) and conversely that any solution of (7.1) to (7.5) is also a solution of (7.16). However, we take as fundamental the more general variational problem (P_E), and investigate the qualitative problem of establishing existence and uniqueness of its solution as well as investigating the rate of convergence of both the semi-discrete and the fully-discrete finite element approximations of the solution.

In order to show the existence and uniqueness of a solution to problem (P_E), we show that the above bilinear form $a(\cdot,\cdot)$ also satisfies the conditions of Theorem 5.1, which may then be invoked to establish the required result. This is done in the following Lemma.

Lemma 7.1 *(a) The bilinear form a: $Z \times Z \to \Re$ is Z-elliptic in the sense that there exists a positive constant α such that*

$$a(\mathbf{z},\mathbf{z}) \geq \alpha \|\mathbf{z}\|_Z^2$$

for all $\mathbf{z} \in Z$.

(b) The bilinear form a: $Z \times Z \to \Re$ is continuous; that is, there exists a positive constant M such that

$$|\, a(\mathbf{w},\mathbf{z})\,| \leq M \|\mathbf{w}\|_Z \|\mathbf{z}\|_Z$$

for all $\mathbf{w}$, $\mathbf{z} \in Z$.

PROOF. (a) For any $\mathbf{z} = (\mathbf{v},\mathbf{q}) \in Z$,

$$\begin{aligned} a(\mathbf{z},\mathbf{z}) &\geq c_0 \int_\Omega |\,\epsilon(\mathbf{v}) - \mathbf{q}^D|^2 \; dx + \lambda_0 \int_\Omega |\mathbf{q}^D|^2 \; dx \\ &= c_0 \int_\Omega |\,\epsilon(\mathbf{v}) - \mathbf{q}\,|^2 \; dx + \lambda_0 \int_\Omega |\mathbf{q}|^2 \; dx \\ &= c_0 \int_\Omega \theta\,|\,\epsilon(\mathbf{v})\,|^2 + \left|\sqrt{1-\theta}\,\epsilon(\mathbf{v}) - \frac{1}{\sqrt{1-\theta}}\mathbf{q}\right|^2 + (\lambda_0 - \frac{1}{1-\theta})\,|\mathbf{q}|^2 \; dx \\ &\geq c_0 \int_\Omega \theta\,|\,\epsilon(\mathbf{v})\,|^2 + (\lambda_0 - \frac{1}{1-\theta})\,|\mathbf{q}|^2 \; dx \end{aligned}$$

for any $\theta \in (0,1)$; here we have used (7.9), (7.10) and the fact that $\mathbf{q}^D = \mathbf{q}$ for $\mathbf{q} \in Z$. The result follows by using Korn's inequality (see, for example, Duvaut and Lions [26]) and by choosing $\theta = \lambda_0/(2c_0 + \lambda_0)$.

(b) It follows from $C_{ijkl} \in L^\infty(\Omega)$ that

$$\left| \int_\Omega C_{ijkl} f_{ij} g_{kl} \, \mathrm{d}x \right| \leq M \|f\|_Q \|g\|_Q$$

for some $M > 0$, and for all $\mathbf{f}, \mathbf{g} \in Q$. Returning to the bilinear form $a(\mathbf{w}, \mathbf{z})$, where $\mathbf{w} = (\mathbf{u}, \mathbf{p})$ and $\mathbf{z} = (\mathbf{v}, \mathbf{q})$ we thus have

$$\begin{aligned} |a(\mathbf{w}, \mathbf{z})| &\leq M \left| \int_\Omega C_{ijkl}(u_{i,j} - p_{ij})(v_{k,l} - q_{kl}) + \lambda p_{ij} q_{ij} \, \mathrm{d}x \right| \qquad (7.17) \\ &\leq M \|\nabla \mathbf{u} - \mathbf{p}\|_Q \|\nabla \mathbf{v} - \mathbf{q}\|_Q + \Lambda \, |(\mathbf{p}, \mathbf{q})_Q| \\ &\leq M(\|\mathbf{u}\|_V + \|\mathbf{p}\|_Q)(\|\mathbf{v}\|_V + \|\mathbf{q}\|_Q) + \Lambda \|\mathbf{p}\|_Q \|\mathbf{q}\|_Q \\ &\leq \bar{M} \|\mathbf{w}\|_Z \|\mathbf{z}\|_Z \end{aligned}$$

(where $\Lambda > 0$ is a constant) using the simple inequality

$$a + b \leq \sqrt{2}(a^2 + b^2)^{1/2}, \qquad a, b \in \Re, \qquad (7.18)$$

and the fact that $\|\mathbf{p}\|_Q \leq \|\mathbf{w}\|_Z$. □

REMARK. We note here that the bilinear form defined by (7.13) is *not* W-elliptic, indeed, $a(\mathbf{z}, \mathbf{z}) = 0$ for any $\mathbf{z}$ of the form $\mathbf{z} = (0, \alpha I)$, where $\alpha \in \Re$.

Thus the bilinear form a: $Z \times Z \to \Re$ and the functionals $l(t)$: $Z \to \Re$ and j: $Z \to \Re$ satisfy the conditions of the Theorem 5.1 for $H = Z$, hence problem (P_E) has a unique solution. This is stated more precisely in the following Theorem.

Theorem 7.1 (Existence and Uniqueness) *Problem (P_E) has a unique solution* $\mathbf{w} = (\mathbf{u},\mathbf{p})$ *satisfying* $\mathbf{w} \in L^\infty(0, T; Z)$ *and* $\dot{\mathbf{w}} \in L^2(0, T; Z)$.

7.3 Semi-Discrete FE Approximation and Error Estimates

We now consider the semi-discrete finite element approximation of the solution to Problem (P_E). We replace Problem (P_E) by the following problem :

Problem ($\mathbf{P}_E^h$) Find $\mathbf{w}^h = (\mathbf{u}^h, \mathbf{p}^h) \in Z^h = V^h \times Q_0^h$ that satisfies

$$a(\mathbf{w}^h(t), \mathbf{z}^h - \dot{\mathbf{w}}^h(t)) + j(\mathbf{z}^h) - j(\dot{\mathbf{w}}^h(t)) - \langle l(t), \mathbf{z}^h - \dot{\mathbf{w}}^h(t)\rangle \geq 0 \tag{7.19}$$

$$\text{for all } \mathbf{z}^h \in Z^h, \text{ for almost all } t \in (0,T),$$

where Z^h is a suitable finite element subspace of Z, and h is a parameter measuring mesh size, as indicated in Chapter 4. We note that Problem (P_E^h) is a special case of Problem (P^h) with the operators $a(\cdot,\cdot)$, $j(\cdot)$ and $\langle l(t), \cdot\rangle$ defined by (7.13) - (7.15), with the Hilbert space H and its approximating subspace H^h now specialised to be Z and a suitable finite element subspace Z^h respectively. Hence, by the general internal approximation error estimate (6.12), we now have the following estimate for the finite element approximation error $\mathbf{w}$ - $\mathbf{w}^h$:

$$\begin{aligned} \sup_{0\leq t\leq T} \|\mathbf{w}(t) - \mathbf{w}^h(t)\|_a^2 \;&\leq\; C\|\mathbf{w}(0) - \mathbf{w}^h(0)\|_a^2 \\ &+ C \inf_{\mathbf{z}^h \in Z^h} \left\{ \|\dot{\mathbf{w}} - \mathbf{z}^h\|_{L^1(0,T;Z)} + \|\dot{\mathbf{w}} - \mathbf{z}^h\|_{L^2(0,T;Z)}^2 \right\}. \end{aligned} \tag{7.20}$$

We use this inequality of Céa's lemma-type, together with the finite element interpolation error estimates reviewed in Chapter 4, to obtain an estimate for the rate of convergence of a sequence of finite element solutions of the approximate problems (7.19) to that of the solution of problem (7.16), as the mesh size h is decreased.

The finite element space Z^h is assumed to be endowed with standard asymptotic properties, as indicated in Chapter 4 (see, also, for example Ciarlet [18] and Oden and Carey [64]). In particular, if the shape functions forming the basis of V^h (respectively Q_0^h) contain complete polynomials of degree $\leq k$ (respectively $\leq k-1$) and if a vector-valued function $\mathbf{w} = (\mathbf{u}, \mathbf{p})$ is given in

$$\mathbf{H}^m \equiv \left((H^m(\Omega))^d \cap V\right) \times \left((H^{m-1}(\Omega))^{d\times d} \cap Q_0\right), \; m > 1, \; d \leq 3,$$

then there exists an element $\bar{\mathbf{z}}^h \in Z^h$ such that

$$\|\mathbf{w} - \bar{\mathbf{z}}^h\|_{\mathbf{H}^s} \leq Ch^\mu \|\mathbf{w}\|_{\mathbf{H}^m}, \qquad \mu = \min(k+1-s, m-s), \tag{7.21}$$

where $\|\cdot\|_{\mathbf{H}^s}$ denotes the norm on the Sobolev space $\mathbf{H}^s$, for a given d, of order s. We have seen a construction of families of finite element subspaces which satisfy estimate (7.21) in Chapter 4.

By the definitions of the norms $\|\cdot\|_{L^1(0,T;Z)}$ and $\|\cdot\|_{L^2(0,T;Z)}$ and the estimate (7.21), we easily obtain the following estimate for the rate of convergence of the solutions to the semi-discrete problems (7.19) to that of the solution of problem (7.16).

Theorem 7.2 (Semi-Discrete FE Error Estimate) *Suppose that the solution* $\mathbf{w}$*of Problem* (P_E) *is of sufficient regularity that* $\dot{\mathbf{w}} \in L^2(0,T;\mathbf{H}^2)$ *and let the solution* $\mathbf{w}^h$ *of Problem* (P_E^h) *be such that*

$$\|\mathbf{w}^h(0) - \mathbf{w}(0)\|_Z \leq C_0 h^{1/2}. \tag{7.22}$$

Then there exists a constant C*, independent of* h*, such that*

$$\sup_{0\leq t\leq T} \|\mathbf{w}^h(t) - \mathbf{w}(t)\|_Z \leq Ch^{1/2}. \tag{7.23}$$

We remark that from Chapter 4 we can easily construct $\mathbf{w}^h(0)$ satisfying (7.22) (see Glowinski [34]).

For comparative purposes, we list some results obtained by Martins and Oden [57] for problems in elastodynamics with friction. The variational statement of the problem they consider is :

Find the function $t \rightarrow \mathbf{u}(t)$ of $(0,T] \rightarrow V$ such that for all $t \in (0,T]$ and for all $\mathbf{v} \in V$,

$$\langle \ddot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t)\rangle + a(\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) + j(\mathbf{v}) - j(\dot{\mathbf{u}}(t)) \geq \langle \psi(t), \mathbf{v} - \dot{\mathbf{u}}(t)\rangle \tag{7.24}$$

with the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{u}_1.$$

The inequality (7.24) is a variational statement of D'Alembert's principle of dynamic equilibrium, with the inequality holding rather than an equality since the friction functional j is non-differentiable.

Duvaut and Lions [26] have established sufficient conditions for the existence and uniqueness of solutions to problem (7.24).

All of the functionals occurring in (7.24) have similar properties to their analogues occurring in problem (P_E) and the "quasistatic" version of (7.24) would have the same form as problem (P_E). In the numerical analysis of its semi-discrete approximation (and in the qualitative analysis of the variational problem), the difficulties caused by the presence of the non-differentiable friction functional j are overcome by considering its *convex regularisation* j_ϵ (chosen such that $j_\epsilon \longrightarrow j$ as $\epsilon \longrightarrow 0$), which then reduces the problem to a variational *equality*. Under similar regularity assumptions on the solution $\mathbf{u}_\epsilon$ (and its derivative $\dot{\mathbf{u}}_\epsilon$) to the regularised problem, and with assumed initial appoximations of $O(h)$, the following semi-discrete error estimate is obtained.

There exist constants C, $\alpha > 0$, independent of h and t, such that for all $t \in [0, T]$,

$$\|\dot{\mathbf{u}}_\epsilon(t) - \dot{\mathbf{u}}_\epsilon^h\|_0^2 + \alpha \|\mathbf{u}_\epsilon(t) - \mathbf{u}_\epsilon^h(t)\|_1^2 \leq Ch^2,$$

which is half an order higher in the power of h than that obtained in the estimate (7.23). A loss in the rate of convergence due to non-differentiable terms is classical, see Glowinski [34].

We note that on implementing the finite element method in practice (see Chapter 4), problem (P_E^h) gives rise to a system of ordinary differential inequalities. When solving this numerically, the time domain is then also discretised, usually by some differencing scheme. In view of this we now consider the following fully-discrete finite element approximation of problem (P_E) :

Problem (P_E^{hk}) Find $\mathbf{w}^{hk} \equiv \{\mathbf{w}_n^{hk}\}_{n=0}^N$, where $\mathbf{w}_n^{hk} = (\mathbf{u}_n^{hk}, \mathbf{p}_n^{hk}) \in Z^h$ for $n = 0, \ldots, N$, such that

$$a(\mathbf{w}_n^{hk}, \mathbf{z}^h - \delta\mathbf{w}_n^{hk}) + j(\mathbf{z}^h) - j(\delta\mathbf{w}_n^{hk}) - \langle l_n, \mathbf{z}^h - \delta\mathbf{w}_n^{hk}\rangle \geq 0 \qquad (7.25)$$

$$\text{for all } \mathbf{z}^h \in Z^h,\ n = 1, \ldots, N.$$

As in the semi-discrete case we note that problem (P_E^{hk}) is a special case of the general problem (P^{hk}), hence it follows that there exists a solution $\mathbf{w}^{hk}$ of problem (P_E^{hk}) satisfying (by (6.14)) the following general error estimate :

$$\max_n \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_Z \leq C\big(\epsilon^{1/2}(h,k) + k^{1/4}\big), \qquad (7.26)$$

where in this context

$$\epsilon(h,k) = \inf_{\tilde{\mathbf{z}}^h \in \mathcal{Z}} \left\{ \|\delta\mathbf{w} - \tilde{\mathbf{z}}^h\|_{l^2(Z)} \right\},$$

where $\mathcal{Z} = \{\tilde{\mathbf{z}}^h = (\tilde{\mathbf{z}}_1^h, \ldots, \tilde{\mathbf{z}}_N^h) \mid \tilde{\mathbf{z}}_n^h \in Z^h,\ \ n = 1, \ldots, N\}$ and $\delta\mathbf{w} \equiv \{\delta\mathbf{w}_1, \ldots, \delta\mathbf{w}_N\}$.

From the definition of the norm $\|\cdot\|_{l^2(Z)}$, and by making use of the estimate (7.21) at each time t_n, we easily obtain the following error estimate :

Theorem 7.3 (Fully-Discrete FE Error Estimate) *Suppose that the solution* $\mathbf{w}$ *of Problem (P_E) is of sufficient regularity that* $\mathbf{w} \in L^2(0,T;\mathbf{H}^2)$, *then there exists* *a constant C, independent of h and k, such that*

$$\max_n \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_Z \leq C\big(h^{1/2} + k^{1/4}\big). \tag{7.27}$$

For comparative purposes, we note that Johnson [42], when considering a parabolic variational inequality *of the first kind*, and using a comparative norm, obtains an error estimate of $O(h + \log(k^{-1})^{1/4}k^{3/4})$. This error estimate is obtained when making a slightly restrictive assumption about the solution. Without this assumption, Johnson obtains an error estimate of $O(h + k^{1/2})$, which is of higher order than the estimate (7.27); this again confirms the classical loss in the rate of convergence due to the presence of non-differentiable terms.

Chapter 8

Further Research

We now discuss various possible extensions of the work discussed in this thesis.

The first opportunity arises when considering the motivation for the formulation of the model variational inequality: we observed that the behaviour of a general elasto-plastic body can be mathematically modelled by a momentum balance equation, constitutive laws, boundary and initial conditions; this problem can then be reformulated as a variational inequality. In this thesis we have undertaken a mathematical and numerical analysis of a subclass of this class of problems (in an abstract setting) in which quasistatic behaviour is assumed; that is, inertial effects are assumed to be negligible (corresponding to a slow loading process, differing in an essential way from the dynamic case) and so the acceleration term in the linear momentum balance equation is taken to be zero. This results in the parabolic nature of the variational inequality. Thus if we consider the more general problem in which quasistatic behaviour is *not* assumed, the resultant variational statement of the problem would be a hyperbolic variational inequality (not of standard form). This may then, in a similar manner to the model problem, be suitably abstracted and one could then attempt a mathematical and numerical analysis of the resulting problem. Also, more general boundary conditions could be considered, which would have immediate applications in contact problems in elastodynamics, for example. Finally, the linear kinematic hardening law used is the simplest possible; this could be extended to consider various nonlinear kinematic hardening laws as well as isotropic hardening (see, for example, Han and Reddy[36]).

A second avenue for further research arises when considering the actual implementation of (Galerkin) finite element methods for solving the model problem. Because it is either too costly, or simply because it is impossible to calculate the integrals over Ω which appear in (7.19) and (7.25), *numerical integration* is used for evaluating these integrals. For any finite element K belonging to the 'triangulation' $\mathcal{T}_h$, we introduce a quadrature formula over K:

$$\int_K \varphi(\mathbf{x})\mathrm{d}x \text{ is approximated by } \sum_{l=1}^{L} \omega_{l,K}\varphi(\mathbf{b}_{l,K}) \tag{8.1}$$

for some specified points $\mathbf{b}_{l,K} \in K$ and weights $\omega_{l,K} \in \Re$, $1 \leq l \leq L$. By using the quadrature formulas (8.1), we replace the semi-discrete problem (7.19) by the following one: Find a function $\mathbf{w}^h$ such that

$$a_h(\mathbf{w}^h(t), \mathbf{z}^h - \dot{\mathbf{w}}^h(t)) + j_h(\mathbf{z}^h) + j_h(\dot{\mathbf{w}}^h(t)) - \langle l_h(t), \mathbf{z}^h - \dot{\mathbf{w}}^h(t)\rangle \geq 0 \tag{8.2}$$
$$\text{for all } \mathbf{z}^h \in Z^h,$$
$$\text{for almost all } t \in (0,T)$$

where, for each $\mathbf{w}^h$, $\mathbf{z}^h \in Z^h$, we have

$$\begin{aligned} a_h(\mathbf{w}^h, \mathbf{z}^h) &= \sum_{K\in\mathcal{T}_h} \sum_{l=1}^{L} \omega_{l,K} a(\mathbf{w}^h(\mathbf{b}_{l,K}), \mathbf{z}^h(\mathbf{b}_{l,K})) \\ j_h(\mathbf{z}^h) &= \sum_{K\in\mathcal{T}_h} \sum_{l=1}^{L} \omega_{l,K} j(\mathbf{z}^h(\mathbf{b}_{l,K})) \\ \langle l_h(t), \mathbf{z}^h\rangle &= \sum_{K\in\mathcal{T}_h} \sum_{l=1}^{L} \omega_{l,K} \langle l(t), \mathbf{z}^h(\mathbf{b}_{l,K})\rangle. \end{aligned}$$

Then, for example, one could try to obtain an estimate for the error $\mathbf{w}^h - \mathbf{w}$, where $\mathbf{w}^h$ is the solution to problem (8.2). These considerations also apply to the time-discretised case.

Finally, it would be of interest to conduct some numerical experimentation using the schemes analysed in this thesis and then to use these to check the sharpness of the estimates obtained.

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