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**An investigation of the co-constitution of mathematics and learner identification in the pedagogic situations of schooling, with special reference to the teaching and learning of mathematics in a selection of grade 10 mathematics lessons at five schools in the Western Cape Province of South Africa**

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A minor dissertation submitted in partial fulfilment of the requirements for the award of the degree of Masters in Education

School of Education, Faculty of Humanities, University of Cape Town

September 2011

**DECLARATION**

This work has not been previously submitted in whole, or in part, for the award of any degree. It is my own work. Each significant contribution to, and quotation in, this dissertation from the work, or works, of other people has been attributed, and has been cited and referenced.

Signature: \_\_\_\_\_ Date: 7<sup>th</sup> September 2011



## Abstract

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This dissertation is an investigation of what is entailed in the co-constitution of mathematics and learner identification in the elaboration of school mathematics in a selection of grade 10 mathematics lessons in five working-class schools in the Western Cape Province of South Africa. The study is located within the broad framework of the sociology of education, specifically drawing on Bernstein's sociological theory of education and its application in the investigation of the relations between pedagogy and social justice. The specific problematic within which my study is located is the constitution of school mathematics in the pedagogic situations of schooling. My theoretical framework consists of the work of Davis (2009b, 2010a, 2010b, 2011a, 2011b) in conjunction with Lacan's (2006) psychoanalytic notions of the registers of the Real, the Imaginary and the Symbolic, Eco's (1979) notion of a model reader, and Bernstein's (1996) discussion of pedagogic discourse and the pedagogic device, which I use to fashion a set of resources for describing the co-constitution of school mathematics and learner identification in pedagogic situations. In my analysis I describe the operational activity making up fifteen grade 10 mathematics lessons selected for description and analysis. I use these descriptions of operational activity to discuss the realisation of content and the regulation of the learners in these lessons in order to explore the ways in which the extimate relation of the learner to the teacher, and the learner as obstacle to the reproduction of mathematics, appear in the exposition of mathematical content by teachers, and the implications of this for the co-constitution of mathematics and learner identification.

The results of this study show that (1) in most cases in these lessons the ways in which content is realised differs from the content indexed by a particular mathematical topic from the point of view of the mathematical body of knowledge; and (2) that there are instances where the ways in which content is realised also do not correspond with mathematical propositions, definitions, processes, rules and objects. I also found that (3) in most of these lessons, necessity is situated external to mathematics and the regulation of learners in these lessons is thus predominantly under the aspect of the Imaginary.

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# Chapter 1

## Introduction

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### ***1.1 Pedagogy and social justice***

Among the stated principles of the South African national curriculum are social transformation, a high level of knowledge and skills for all, and social justice. One of the goals of the curriculum is to ensure that “educational imbalances of the past are redressed, and that equal educational opportunities are provided for all sections of our population” (DoE, 2003: 3), and the introduction to the curriculum states that “social justice requires the empowerment of those sections of the population previously disempowered by the lack of knowledge and skills” (DoE, 2003: 3). But despite these stated intentions, recent South African Grade 12 results, notwithstanding the increasing pass rate, have been described as a “swindle” by Jansen (cited in Ramphele, 2009). Ramphele (2009) describes the education system as being in crisis and as failing the majority of children. She questions whether South African schools adequately prepare all learners for our society, and believes that our education system is “socially engineering the continuation of inequalities that leave the majority of poor black children behind” (Ramphele, 2009).

The gap in mathematics achievement between advantaged and disadvantaged schools and between learners of different social class and ‘race’ has been widely explored in recent years (Van der Berg, 2007; Fleisch, 2008; Reddy & Kanjee, 2007). Jaffer (2011b) highlights the way in which social class is entwined with ‘race’ in South Africa where the working-class is predominantly ‘African’ and ‘coloured’ and the middle-class still mostly ‘white’. In the Western Cape, only 9,1% of candidates in ex-Department of Education and Training (DET) schools (previously for ‘African’ learners) compared to 81,1% of candidates in ex-House of Assemblies (HoA) schools (previously mostly ‘white’ learners) scored 50% or more in the 2008 National Senior Certificate mathematics examination (Jaffer, 2011b). The results of the TIMSS 2003 also showed differences in the average achievement in mathematics of learners in schools categorized by ex-racial departments – the average scores of learners in ex-DET schools was almost half that of learners in ex-HoA schools (Reddy & Kanjee, 2007). Social class, and thus ‘race’, continues to be a strong predictor of success in school mathematics in South Africa.

Rose (2004) describes similar trends in Australia, where working-class learners are behind general standards in literacy and he believes that this reflects wider educational inequalities. He raises the concern that this stratification has not changed, despite moves towards more progressive pedagogy,

which he describes as the “apparent inertia of inequality” (Rose, 2004: 92), and considers it a consequence of the structure of educational systems which prepare privileged learners for university while relegating others to vocational or manual occupations (Rose, 2004). In England, Reay (2006) reports that the achievement gap between children from working and middle-class backgrounds has widened despite increased spending on education. She cites a study published in 2005 by the British Office for National Statistics which showed that social mobility in Britain steadily declined over the previous decade and that children from middle-class homes were 50% more likely to stay in education after 16 than their working-class counterparts.

The relations of curricula, school structure and pedagogy to social justice and equality in educational access, participation and outcomes have been widely explored, particularly within the field of the sociology of education. Bernstein’s sociological theory of education (1975, 1996, 1999) is one framework for exploring this, and has been the basis of numerous research studies within the field. My study, although falling within that broad framework, is not a full-blown sociological study as this literature suggests, but aims to explore the relation between teachers’ ideas of who learners are, as implied by their pedagogic practice, and the ways in which mathematics is constituted at a micro-level. Despite general concern within the field about the widening achievement gap between learners from different social classes, particularly in mathematics, not many questions are being asked about what is going on at the micro-level of the mathematical computations, such as what gets constituted as mathematics and how the learner of mathematics gets constituted in the pedagogic situation. My study aims to contribute at that level by taking a small step towards fuller descriptions of what is really going on at the level of the mathematics, through using and developing appropriate methodological resources to enable a micro-level understanding of the relations between mathematics and learner identification. I specifically focus on the way in which mathematics is constituted in response to teachers’ implied ideas about the learner, and the implications of that for the constitution of learner identification. My study investigates five cases in working-class schools, so for now I am not able to talk about the differences between middle and working-class settings, but am rather focusing on the development of methodological resources which could be used to investigate whether the disparities pointed out by the literature in this chapter appear at the level of the mathematics which is offered by teachers, and thus in the mathematics produced by learners.

## ***1.2 The co-constitution of mathematics and learner identification***

My study is a contribution to the development of methodological resources for the description and analysis of mathematics pedagogy emerging from research in five Western Cape schools. The

general problematic within which the study is located is that of the constitution of school mathematics in pedagogic situations. The specific aim of the study is to develop an analytical framework to describe the co-constitution of school mathematics and learner identification in pedagogic situations by analysing the operational activity that is entailed in the elaboration of school mathematics in a selection of grade 10 mathematics lessons in the five schools. The work of Davis (2009b, 2010a, 2010b, 2011a, 2011b) will be used in conjunction with Lacan's (2006) psychoanalytic notions of the Real, Imaginary and Symbolic registers, Eco's (1979) notion of a model reader, and Bernstein's (1996) discussion of pedagogic discourse and the pedagogic device to fashion a set of resources for describing the co-constitution of school mathematics and learner identification in pedagogic situations.

My research question is: *What is entailed in the co-constitution of mathematics and learner identification in the elaboration of school mathematics in a selection of grade 10 mathematics lessons in five working-class schools?*

I will approach the question by considering these sub-questions:

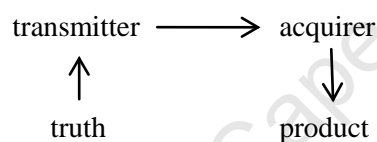
- What comes to be constituted as mathematics in the grade 10 mathematics lessons selected for description and analysis?
- Who is the learner implied by the operational resources required to work with the mathematics content as constituted in the pedagogic situation?
- Who is the learner implied by the pedagogic practices and the particular realisation of mathematics?
- What, if any, is the relation between these two learner identifications, and between the constitution of school mathematics and the constitution of learner identification?

In order to further develop my research question, I explore the relations between the teacher, the learner and knowledge, with reference to the work of Davis (2002, 2003, 2010c), as well as Bernstein's pedagogic device (1996) and Lacan's (1991) Discourse of the University.

### **1.3 The teacher, the learner and knowledge**

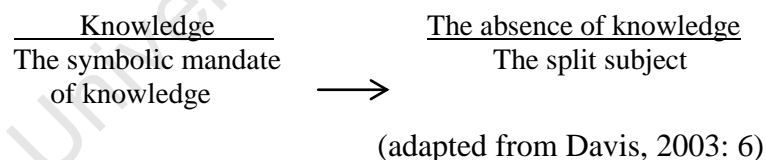
In his discussion of the transformation of knowledge into pedagogic communication Bernstein (1996) refers to the language device to introduce his concept of the pedagogic device. He describes the language device as a system of formal rules which govern speech and writing. The language device is activated by a meaning potential and results in communication, which has restricting or enhancing feedback on the meaning potential. In a similar way, the pedagogic device consists of

rules which regulate pedagogic communication by mediating between “the potential discourse available to be pedagogised” (Bernstein, 1996: 41) and what emerges as pedagogic communication. Both devices simply consist of a message flowing between two subjects, referred to by Bernstein (1996) as the *transmitter* and the *acquirer*, with a specific outcome or *product*. Davis (2002: 7) uses Lacanian terms to describe the transmitter and acquirer respectively as *subject-supposed-to-know* and *subject-supposed-not-to-know*, where the former is supposedly aligned with knowledge and the latter with ignorance. This assumes the existence of knowledge and ignorance, which Davis (2002) refers to as *subjectless knowledge* and *subjectless ignorance*. In his discussion of pedagogic communication, he explains that the absence of a “completely unified social consciousness” (Davis, 2003: 5) necessitates all communication, thus full pedagogic communication does not exist except as a regulative ideal. The notion of full pedagogic communication can be seen as a “moral ought-to-be” (Davis, 2003: 5). In light of this, Davis adds a fourth aspect to the flow of pedagogic communication prior to the transmitter, acquirer and product – that of truth, as “communication starts from the truth of its own failure and that truth then functions as its cause” (2003: 5).



**Figure 1.1** The flow of pedagogic communication

Using these four dimensions, Davis uses Lacan’s Discourse of the University<sup>1</sup> to yield the matrix in Figure 1.2.



**Figure 1.2** The opposition between knowledge and its absence

We can thus see that the pedagogic device and the Discourse of the University illustrate an opposition between knowledge and its absence on an overt level. Underlying this, the symbolic mandate of knowledge (Bourdieu, 1991), which is granted to the teacher, represents the truth, or condition of possibility of knowledge, and the split subject, typically the learner, represents the product of pedagogic communication. Davis (2003) highlights the way in which both Bernstein’s

<sup>1</sup> In Lacan’s (1991) theory of discourse, he develops a model which consists of four intrasubjective factors (knowledge, values, alienation and *jouissance* (enjoyment)) and produces four intersubjective effects (educating, governing, desiring and analyzing). He generates a fundamental relational matrix in which the top position is manifest and the bottom repressed, as discussed by Davis (2003). There are four discourses which emerge from the matrix, one of which is Discourse of the University, which emphasizes the social bond associated with knowledge and education.

(1996) pedagogic device and Lacan's (1991) Discourse of the University reveal the way in which pedagogic communication is focused on getting learners to encounter the field of knowledge and to be able to reproduce this knowledge. This stages the opposition between the field of production, representing the fullness of knowledge, and the site of reproduction, the learner, representing the absence of knowledge and the potential point at which knowledge breaks down<sup>2</sup>. The opposition can be described as an objective antagonism (Davis, personal communication), which seems paradoxical, but merely refers to an antagonism which is devoid of subjective elements, drawing on Žižek's (2008) discussion of subjective and objective violence<sup>3</sup>, and Benjamin's (1999) discussion of the objective contradiction associated with violence which is sanctioned by the law. The antagonism between knowledge and its absence can be described as a kind of objective violence due to the moral discourse it produces - "a cause becomes violent when it enters into moral relations" (Benjamin, 1999: 236). The basis of the moral discourse is the shift in focus from what 'is' (the production of knowledge) to what 'ought to be' (the reproduction of knowledge by the learner). This objective antagonism between knowledge and its absence is a useful place to start in developing an understanding of the relation between the constitution of school mathematics and the constitution of learner identification, as my study aims to do.

At the heart of this objective antagonism is the learner, who represents a potential point of breakdown of knowledge and, in Lacanian terms, the potential intrusion of the Real. For Lacan (2006), the reality of human beings is made up of three interconnected levels or registers<sup>4</sup>: the Imaginary, the Symbolic and the Real, described by Jameson as "sectors of experience" (Jameson, 1998: 82). I find Žižek's (2006) use of the example of a game of chess helpful in illustrating these registers:

This triad can be nicely illustrated by the game of chess. The rules one has to follow in order to play it are its symbolic dimension: from the purely formal symbolic standpoint, 'knight' is defined only by the moves this figure can make. This level is clearly different from the imaginary one, namely the way in which different pieces are shaped and characterized by their names (king, queen, knight), and it is easy to envisage a game with the same rules, but with a different imaginary, in which this figure would be called 'messenger' or 'runner' or

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<sup>2</sup>Describing the learner as representing the absence of knowledge, and thus as lacking in relation to knowledge, is not meant as a pathologising of the learner, but is rather descriptive of a structural arrangement which is a condition of pedagogy. Without an absence of knowledge, pedagogy is not necessary.

<sup>3</sup>Žižek describes subjective violence as "a perturbation of the "normal", peaceful state of things" (Žižek, 2008: 10), and objective violence as the "violence inherent to this "normal" state of things" (Žižek, 2008: 10).

<sup>4</sup>Lacan's description of these three registers arose out of his linguistic reading of psychoanalysis, which led him to claim that the unconscious behaves like a language, in that it has its own grammar and logic, in the words of Žižek, "it talks and thinks" (2006 : 3). This reading of psychoanalysis forms the basis of the three registers.

whatever. Finally, real is the entire complex set of contingent circumstances that affect the course of the game: the intelligence of the players, the unpredictable intrusions that may disconcert one player or directly cut the game short (Žižek, 2006: 8 – 9).

In order to deal with the learner as a potential point of breakdown of knowledge, both mathematics education research and pedagogy are obliged to construct the learner as facilitative rather than disruptive, using either Imaginary or Symbolic responses or a combination of the two. For example, the distinction between procedural and conceptual understanding, discussed in Chapter Two, can be seen as a response to the learner as Real within mathematics education theory. Another example is the idea of pedagogic content knowledge (Shulman, 1986). As described by Davis (2010c: 1), ideas<sup>5</sup> such as these “can be understood as an attempt to rethink knowledge in pedagogic situations in a manner that takes into account the potential destabilising effects on knowledge of the presence of the learner”. Pedagogy is also obliged to take into account the learner as Real, and I am interested in the ways in which this takes place and the implications of that for the co-constitution of mathematics and learner identification.

To lay a foundation for my study, which aims to investigate what is entailed in the co-constitution of mathematics and learner identification in the elaboration of school mathematics in a selection of grade 10 mathematics lessons in five working-class schools, I refer to literature on the procedural-conceptual distinction, a dominant theoretical distinction within mathematics education, in Chapter Two. I then explore the relation of pedagogy to teacher expectations or ideas about learners, which will lead me to a review of the notions of identity and identification in relation to mathematics teaching and learning, before developing my theoretical framework in Chapter Three. My theoretical framework draws on the work of Davis (2009b, 2010a, 2010b, 2011a, 2011b), as well as Lacan’s (2006) psychoanalytic notions of the registers of the Real, the Imaginary and the Symbolic, Eco’s (1979) notion of a model reader, and Bernstein’s (1996) discussion of pedagogic discourse and the pedagogic device to fashion a set of resources for describing the co-constitution of school mathematics and learner identification in pedagogic situations. The research design and analytical framework of my study are explained in Chapter Four, followed by a presentation and discussion of the results of the data production and analysis process in Chapters Five, Six and Seven, and conclusion in Chapter Eight. The appendices contain the actual production and analysis of data, which is too extensive to be included in the body of this project, so instead I have summarized and discussed the results, with examples from the analysis, in Chapters Five, Six and Seven.

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<sup>5</sup> Davis (2010c) also cites the various developmental models of learning as other examples of the way in which education theory is forced to take into account the learner as Real.

## Chapter 2

### A Review of the Literature

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I start this chapter by reviewing some of the extensive literature on the procedural-conceptual distinction in mathematics education, a dominant distinction within the field. I then discuss the relation of pedagogy to teacher expectations or ideas of who the learner is. This will lead me to a review of the notions of identity and identification in relation to mathematics teaching and learning as a precursor to introducing my theoretical framework Chapter Three.

#### ***2.1 The procedural-conceptual distinction***

A dominant and widely-used theoretical distinction within the field of mathematics education is the distinction between procedural and conceptual understanding in describing pedagogic activity. Due to its longevity and widespread use within the field, and despite recent refinements and criticisms (Kilpatrick, Swafford & Findell, 2001; Star, 2000, 2005), the procedural-conceptual distinction gives insight into how mathematics education thinks of the learner in relation to the constitution of mathematics, and is thus useful to my study, which aims to explore what is constituted as mathematics and what is simultaneously constituted with respect to learner identification in pedagogic situations. The distinction influences the way in which mathematics and the learner of mathematics are constituted for both researchers and teachers, and as such, is implicated in how teachers think about the relations between themselves, their learners and mathematics. A study of the co-constitution of mathematics and learner identification is thus not complete without acknowledging the widespread influence of this distinction.

It is within the field of the psychology of mathematics education that the distinction between procedural and conceptual understanding is most prevalent. Hiebert & Lefevre's (1986) description of the distinction is widely used within mathematics education research. They trace the development of the distinction in mathematics education, describing the different forms that have been used over the past century, beginning with the distinction between skills and understanding discussed by McLellan & Dewey (1895), and referring to the work of Scheffler (1965), who distinguished between 'knowing how' and 'knowing that'. Another influential theorist within this field is Skemp (1976), who draws on the work of Mellin-Olsen (1981), and distinguishes between instrumental and relational understanding<sup>6</sup>. Still within the psychology of mathematics education

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<sup>6</sup>In his discussion of this distinction, Skemp (1976) describes instrumental understanding as "rules without reasons" (1976: 22) and relational understanding as "knowing both what to do and why" (1976: 22). Skemp (1987) later

the work of Steinbring (1989), Ma (1999) and Rittle-Johnson & Siegler (2001) all refer to a dichotomy between these two forms of understanding.

Gray & Tall (1994), Dubinsky (1991) and Sfard & Linchevski (1994) engage with the distinction as more of a continuum. Gray & Tall's (1994) work arose out of their engagement with Skemp's three categories (1987). They describe three mathematical worlds – the conceptual-embodied world, the proceptual-symbolic world and the axiomatic-formal world, and explain the learning of mathematics as a movement through the three worlds. The way in which Dubinsky (1991) draws on the procedural-conceptual distinction is similar to Tall's work and arises from his interpretation of Piaget's constructivism. Traces of the procedural-conceptual distinction can also be found in sociology of education literature, one example being Dowling's (1998) discussion of procedural and principled knowledge<sup>7</sup>, which resonates with the procedural-conceptual distinction.

Despite the prevalence of the procedural-conceptual distinction within mathematics education research, Star (2000, 2005) questions the theory which relates conceptual and procedural knowledge with respect to two main criticisms. Firstly, most of the studies underlying the distinction were conducted in primary school contexts. Secondly, Star argues that procedural and conceptual knowledge are assessed and operationalised differently within the field of mathematics education. He believes that procedural knowledge is defined as less complex than conceptual knowledge. Related to this is the way in which the notion of a procedure is constructed in opposition to a concept, as if there is no concept attached to it. Thus it would seem that *procedural* is synonymous with *concept-less*. Vergnaud (1998) describes the emptying of procedural knowledge of all concepts as a “schizophrenic view of cognition” (Vergnaud, 1998: 173), because procedures cannot exist apart from concepts. Davis & Johnson (2008) believe that any procedural understanding of mathematics is underpinned by a foundational, conceptual ground, and that mathematical objects are always present, but not always accessible to learners due to the pedagogic strategies used. There will always be concepts attached to procedures, even if specific concepts seem to be absent, other concepts will be supporting a procedure and it is these which are masked through classifying understanding as procedural without further analysis.

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introduced a third category of logical understanding, which refers to “the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning” (Skemp, 1987: 166).

<sup>7</sup>Dowling (1998) refers to procedural knowledge as knowledge which minimizes connections between mathematical concepts and replaces definitions with instructions, whereas principled knowledge shows connective complexity and uses definitions to facilitate the expression of the regulating principles of school mathematics. This places more focus on the role of pedagogy in facilitating learner acquisition of principled (conceptual) knowledge.

Venkat & Adler (2010), in their investigation of a selection of episodes of mathematics teaching in South Africa, found that although all episodes could be described as involving a procedural approach, further analysis revealed key differences between them. They thus emphasise the need to “disaggregate procedural practice” (Venkat & Adler, 2010: 1). Recent work on the constitution of school mathematics in five working-class schools in the Western Cape (Arendse, 2011; Basbozkurt, 2010; Davis, 2009, 2010, 2011; Davis & Johnson 2007, 2008; Gripper, 2011; Jaffer, 2009, 2010, 2011) also suggests that there is a level of pedagogic activity which may not be captured by deploying the procedural-conceptual framework alone. Davis & Johnson (2007, 2008) and Jaffer (2010) found that the dominant supports for the evaluative criteria in these schools are procedural and iconic in nature. Thus, in terms of the procedural-conceptual distinction, the teaching and learning of mathematics in the schools can be classified as procedural. But in order to describe the constitution of mathematics, as is part of the purpose of my study, it is necessary to understand what is happening within the procedures used, rather than classifying the teaching and learning of mathematics as procedural without further analysis.

For the purpose of my study it is necessary to accept and analyse whatever emerges in the operational activity of school mathematics, even if not immediately recognizable as mathematics, as contributing to the constitution of mathematics in that pedagogic situation<sup>8</sup>, as described by Davis (2011a). This means not approaching the analysis with particular expectations or a priori categories, but rather with theoretical resources which act as filters for describing what emerges, so that the categories get constituted from the analysis rather than existing prior to it. I assume that the agents in the pedagogic context act intentionally and that pedagogic activity of teachers is not random, and thus that the ideas held by teachers about learners play a central role in shaping their pedagogy, as the following literature suggests.

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<sup>8</sup> Badiou’s (2005) work is helpful in understanding this - he adopts a strictly extensional rather than intensional stance. An extensional stance “defines the conditions for membership solely and strictly with reference to the set of those entities (whatever their nature) that fall within the relevant domain” (cf. Norris, 2009: 52), whereas an intensional stance defines conditions for membership based on “qualifying attributes or distinctive features that mark them out as fit candidates according to this or that (e.g. intuitive) criterion” (cf. Norris, 2009: 52). Also useful is Dowling’s (2009) comparison between a “forensics” approach to research, involving upfront claims or expectations about what should be discovered, and what he refers to as “constructive description”, which involves deduction from the theory and induction from the empirical, to constitute a set of theoretical propositions, or an organisational language which can be used to describe what emerges. An example of such an approach is Filloy, Puig & Rojano’s (2008: 6 - 7), who use a broad notion of mathematical sign systems (which they refer to as “mathematical systems of signs” instead of “systems of mathematical signs”) as a tool to analyse the mathematical texts produced by students, rather than having as their object of study an ideal text.

## ***2.2 The relation of pedagogy to ideas about the learner***

Many studies have sought to determine the effect on learners of teachers' values, beliefs and expectations. McDiarmid & Ball (1989) and Schmidt & Kennedy (1990) describe a study which examined teachers' and training teachers' beliefs about teaching, learning and their subject matter, and showed that teachers' beliefs influence pedagogical and content choices. Peterson, Fennema, Carpenter & Loeff (1989) found that teachers' beliefs about teaching and learning mathematics were associated not only with their pedagogical practices but also with what students learned. The work of Rosenthal & Jacobson (1968), and more recently Madon, Jussim & Eccles (1997) and Hinnant, O'Brien & Ghazarian (2009), shows that teacher expectations of learner academic performance may influence learner performance, which has been likened to a type of self-fulfilling prophecy – if teachers expect high performance from learners, they will receive it and vice versa. Rist (1970) carried out a longitudinal study with a group of 'African-American' children in an urban school to explore the process whereby teacher expectations give rise to the social organisation of a class, specifically focusing on the relation between teacher expectations of academic performance and learners' social status. His results suggested that academic achievement was highly correlated with social class and that teacher expectations played a significant role in this through their impact on the organisation of the classroom and differential treatment of learners of differing socioeconomic status. Reay (2006) describes similar studies which explored the images associated with working-class learners and how these contributed to the inadequate academic support offered to them.

Numerous studies within the sociology of education examine the relation between social class and learner performance, and the ways in which inequalities are reproduced, such as those focusing on differential distribution of knowledge (Walkerdine, 1988) and texts (Dowling, 1998) to learners from different social class backgrounds, or the way in which mathematical problem setting disadvantages working-class learners (Cooper & Dunne, 1998; Cooper & Harries, 2005). I focus here on studies which explore the ways in which pedagogy is implicit in the reproduction of social class as part of my survey of literature which relates pedagogy to ideas about the learner.

Hoadley's (2005) research in South African primary school mathematics classrooms explored variation in pedagogy across social class school settings to show how inequalities are potentially reproduced through pedagogic practices, using orientation to meaning as a central concept. Orientation to meaning refers to the transmission and acquisition of either more context-independent meanings (elaborated code) or more context-dependent meanings (restricted code), as

described by Bernstein (1990). Bernstein drew on the work of Luria (1976) and Holland (1981) to show that working-class children are more likely to enter school with a restricted, “community code” (Hoadley & Ensor, 2009: 2), whereas middle-class learners more easily acquire an elaborated “school code” (Hoadley & Ensor, 2009: 2) due to their socialisation at home. The two main sites of acquisition of this code are the home and the school (Bernstein, 1990), but not all to the same degree - Hoadley & Ensor draw on Bernstein’s proposition to explain that “a code, consonant with that of the school, which entails ways of thinking, reasoning and speaking required for school, is much more likely to be developed in a middle-class home than in a working-class home” (Hoadley & Ensor, 2009: 2). Hoadley’s (2005) study, conducted across both working and middle-class settings, found that these different meanings are reproduced through pedagogic practices in different social class settings.

Rose (2004) also examined unequal development of orientations to meaning through schooling, and the role of pedagogy in maintaining inequality, specifically in the area of literacy. He suggests that there is an underlying curriculum which is easily and tacitly accessed by children from highly literate communities, while disadvantaged children, from less literate communities are left behind. In light of studies such as these, which report the failure of schools to interrupt the restricted orientation to meaning of working-class students, Maton & Muller (2007: 16) describe an elaborated orientation to meaning as both “privileged and privileging” – it is the more privileged middle-class learners who are more likely to develop an elaborated code at home, and it is their possession of this code which privileges them at school – thus according to Maton & Muller (2007), code can potentially be seen as the root of the stratification in education systems across the world.

But despite persistent stratification, Bernstein (1990) believed that “it is certainly possible to create a visible pedagogy which would weaken the relation between social class and educational achievement” (Bernstein, 1990: 79). Bernstein’s concepts of classification and framing (1975), and subsequent research around these concepts, have enabled description of the effects of various pedagogic modalities with children from different social backgrounds and have allowed researchers to address why some groups of learners may not do as well as others in particular classrooms or schools. Maton & Muller (2007) refer to research which has shown that children from disadvantaged backgrounds struggle to recognise and apply the code needed for achievement at school. They describe the implications of this – the two options are to either match the underlying structuring principles of schools, curricula or pedagogy to the existing code of working-class learners, or to find ways of enabling learners to acquire the code which will lead to their success in a school context. The second option has been the aim of recent research around optimal pedagogy

for working-class learners, which suggests that a mixed pedagogy works best for working-class learners and that inequality may be reversed using different strengths of classification and framing for various aspects of pedagogic practice (Morais, 2002; Morais, Neves & Pires, 2004; Rose, 2004; Hoadley, 2007; Gamble & Hoadley, 2008; Lubienski, 2004). The first option described by Maton & Muller (2007) has been chosen by “well-intentioned but misguided educationists”, but, as they point out, the danger is that it gives disadvantaged children access to “lower-status forms of educational knowledge” (2007: 17).

The central premise of the literature cited here is that pedagogy plays a role in the reproduction of social class through the unequal distribution of content and thus knowledge to learners from different class backgrounds, and that this unequal distribution of knowledge through pedagogy is based on an implicitly held expectation or idea of who learners are. The studies cited here focus at the broader level of the distribution of content, but I am interested in what takes place at the more subtle, micro-level of the operational activity in the classroom, and the relations between the ideas about learners implied by teachers’ pedagogic practices and the ways in which mathematics is constituted. For now I am not able to determine whether the disparities pointed out by the literature appear at the micro-level of the operational activity as this study explores only working-class settings. But the purpose of my study is to use and develop methodological resources to investigate what gets constituted as mathematics at this level, and how the learner of mathematics gets constituted in the pedagogic situation. This leads me to a discussion of the identification of learners and teachers, in relation to knowledge, as a precursor to developing my theoretical framework in Chapter Three.

### ***2.3 Learner and teacher identification***

In order to discuss learner and teacher identification it is necessary to begin and end this section with some theory, in preparation for Chapter Three, for which I apologise. As a basis for this discussion I return to Lacan’s three registers introduced in Chapter One, focusing on the Imaginary register. For Lacan (2006), the Imaginary emerges during what he refers to as the mirror stage when an infant recognizes him/herself in the mirror, and identifies with his/her image. Lacan (2006) describes the image as that which reflects the subject’s distinct behaviours in unified images. According to Lacan, (2006), the image gives a child an imaginary mastery over his/her body before real mastery exists, and consequently, the child is captivated by the image and develops an ego concept in relation to it. The Imaginary register is thus characterized by the importance of perception - the visual, specular relation between the subject and an image outside of him/herself. Another central characteristic of the Imaginary is duality - Jameson (2006) describes the Imaginary

as a “dual system” (Jameson, 2006: 381) which correlates empirical realities, especially with respect to the relationship between the self and other. As Fink (1995) highlights, Imaginary relations are ego relations, “wherein everything is played out in terms of but one opposition: same and different” (Fink, 1995: 84).

In Žižek’s (1998) discussion of Hegel’s logic of essence, he points out that what Hegel calls identity “is not a simple self-equality of any notional determination ... but the identity of an essence which “stays the same” beyond the ever-changing flow of appearances” (1998: 74). The implication of this is that identity can only be determined through what makes it different. De Carneri (1998) discusses the traditional notion of identity as the relation of something to itself, as opposed to its relation with other entities, which is generally referred to as difference. She highlights the problem with this definition of identity – that of “reconciling the necessary predicate of identity – unity – with the split that the definition itself produces at the very moment it determines its concept” (De Carneri, 1998: 1). Psychoanalysis has refined the definition of identity to that which is formed through a dialectical process involving plurality and temporality, which leads to a definition of identification – “the operation through which the subject is constituted in time” (De Carneri, 1998: 1). Lacan defines identification as “the transformation that takes place in the subject when he assumes an image” (Lacan, 2006: 76). He describes the function of the mirror stage as the establishing of a relationship between an individual and his/her reality. But the individual sees an image of him/herself during the mirror stage which is not his/her ‘real self’, but a fantasy of him/herself. Lacan believes that the gap between the image and the ‘real self’ is never closed. Thus, for Lacan, identity is “necessarily an alienated state” (Luepnitz, 2003: 225), essential for functioning in the world, but also “radically unstable” (Luepnitz, 2003: 225). As Žižek (1989: 104) puts it, to “achieve self-identity, the subject must identify himself with the imaginary other, he must alienate himself—put his identity outside himself, so to speak, into the image of his double”. Based on this Lacanian description of identification, I turn to the context of education, focusing on the identity of the teacher and learner in relation to each other and to knowledge, before ending the chapter with a discussion of the notion of extimacy.

Recent work on teacher identity within mathematics education includes an exploration of the role of identity in negotiating the transition from learner to teacher in newly-qualified teachers (Jones, Brown, Hanley & McNamara, 2000), and the role of teachers’ perceptions of themselves when shifting from one teaching paradigm to another (Brown, Hanley, Darby & Calder, 2007). Their work suggests that teacher identity is a function of how teachers draw on elements from the different discourses surrounding them. In terms of the relations between teacher and learner

identity, Brown (2008) uses Lacan's work on identification and subjectivity to suggest that the relationship between teachers and learners is "co-formative, each seeking something from the other" (Brown, 2008: 12) and also draws attention to Lacan's discussions of the way in which the human subject is always incomplete and self-identifications are captured in a supposed image (Brown & England, 2005). The work of Brown & England (2005) draws on Lacan's emphasis on the subject's identifications with images of him/herself and his/her social relations, and in their exploration of researcher identity, "analysis of these identifications is privileged over any notion of encouraging movement to a harmonised *identity* through a process of analysis" (Brown & England, 2005: 5, italics in original). Brown & England (2005: 6) also raise the point that "identity" itself is fragmented, formed through a "disconnected amalgam of identifications", referring to Laclau & Mouffe's (2001) work on identity. Laclau & Mouffe (2001) explain that all identity is relational, and that every identity is over-determined due to the way in which "the presence of some objects in the others prevents any of their identities from being fixed" (2001: 104). This conception of identity can be used to explain why it is that the identity of the teacher cannot be fully acquired due to the presence of the learner, or as Brown suggests, that teachers and learners are "each seeking something from each other" (2008: 12), which resonates with Lacan's notion of extimacy.

Lacan's (1992) notion of extimacy, as discussed by Miller (1994), means that the exterior is present in the interior. Intimacy is defined as that which is most interior, but the notion of extimacy ascribes a quality of exteriority to that which is most interior. Extimacy is Lacan's way of describing the intimate that is radically Other. The notion of extimacy illustrates why it is that identity, in a Lacanian sense, has non-identity at its core – the positivity of a category is undermined at the core by its opposite. This is how the notion of extimacy can be used to construct relations between categories, in this case, between the category 'learner' and the category 'teacher', in relation to knowledge. As previously discussed, the learner represents the absence of knowledge and the potential intrusion of the Real, and as such prevents the full realisation of the self-identity of the teacher. But despite this, the learner is needed in order for the teacher's self-identity to be fully realised. The relation between the teacher and the learner is thus one of extimacy – it is external but intimate.

In relation to knowledge, the teacher mediates the encounter between the learner and knowledge. The necessary presence of the learner, who embodies the point at which knowledge is absent, means that the category 'teacher', in relation to knowledge, cannot arrive at full self-identity. The notion of extimacy is central to the relations between knowledge, the learner and the teacher, and as such, is a base for my theoretical framework, which aims to investigate what is entailed in the co-

constitution of mathematics and learner identification in the pedagogic situations of schooling. I am interested in the ways in which the extimate relations of the learner, the teacher and mathematics, and the learner as obstacle to the reproduction of mathematics, appear in the exposition of mathematical content by teachers, and the implications of this for learner identification. It is with this in mind that I present my theoretical framework in Chapter Three.

## Chapter 3

### Theoretical Framework

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The theoretical framework presented in this chapter has been selected to lay a foundation for the analytical framework of my study, which aims to investigate what is entailed in the co-constitution of mathematics and learner identification in the elaboration of school mathematics in a selection of grade 10 mathematics lessons in five working-class schools. The relation between the learner, the teacher and knowledge will be further developed in this chapter drawing on Bernstein's (1996) discussion of pedagogic discourse and the pedagogic device, Eco's concept of a model reader (1979), Lacan's (2006) psychoanalytic registers of the Imaginary, the Symbolic and the Real, and Davis' discussion of the regulation of pedagogic activity (2010a, 2011b).

#### **3.1 The pedagogic device and pedagogic discourse**

Bernstein's (1996) discussion of the pedagogic device relates these three categories - teacher, learner and knowledge - by its focus on who gets what knowledge, and how this takes place. He draws attention to the fact that knowledge is differentially distributed to different people through his discussion of the distributive rules of the device, which "distribute forms of consciousness through distributing different forms of knowledge" (Bernstein, 1996: 43). He also emphasises the centrality of evaluation, which "condenses the meaning of the whole device" (Bernstein, 1996: 50), and explains the purpose of pedagogic practice as the transmission of evaluative criteria. Evaluative criteria are the rules for regulating an activity and reveal what is realised as legitimate in particular pedagogic situations. Davis (2010c) extends Bernstein's discussion of evaluation by pointing out that evaluation by its very nature produces concern with what 'ought-to-be'. In order to understand what it is that 'ought-to-be', we can draw on the two levels of pedagogic discourse elaborated by Bernstein (1996) - instructional and regulative discourse. The instructional discourse refers to that which "creates specialised skills and their relationship to each other" (1996: 32), and the regulative discourse to a "moral discourse which creates order, relations and identity" (1996: 32). Davis (2010c) renders these two levels at which evaluation operates more precise through a Lacanian reading of Bernstein's pedagogic discourse, referring to regulative discourse as the level at which there is a particular view of who the learner should be, and instructional discourse as the level where what "ought-to-be" is the realisation of particular knowledge by the learner. Davis (2010c) explains that "what the *ought* of pedagogic evaluation proposes is a correlation of a pedagogic identity and a particular realisation of content" (Davis, 2010c: 1, italics in original). This raises the

question of whether a certain pedagogic identity is correlated with a certain realisation of content at each of these two levels of evaluation.

In order to better understand the way in which evaluation is implicated in the construction of the identity of the learner, I draw on Eco's (1979) notion of a model reader and Lacan's (2006) psychoanalytic registers of the Imaginary, the Symbolic and the Real.

### **3.2 Model reader to model learner**

In Eco's (1979) discussion of the relationship between a text and its reader, he describes *open texts* as those which do not allow any interpretation but can be used "only as the text wants you to use it" (1979: 9) while *closed texts* are "open to every possible interpretation" (1979: 8) and can be read in numerous ways.

Those texts that obsessively aim at arousing a precise response on the part of more or less precise empirical readers (be they children, soap opera addicts, doctors, law-abiding citizens, swingers, Presbyterians, farmers, middle-class women, scuba divers, effete snobs, or any other imaginable sociopsychological category) are in fact open to any possible 'aberrant' decoding. A text so immoderately 'open' to every possible interpretation will be called a closed one (Eco, 1979: 8).

[Open texts] work at their peak revolutions per minute only when each interpretation is reechoed by the others, and vice versa ... You cannot use the text as you want, but only as the text wants you to use it. An open text, however 'open' it be, cannot afford whatever interpretation. An open text outlines a 'closed' project of its Model Reader as a component of its structural strategy (Eco, 1979: 9).

Thus, with an open text, the reader is strictly defined by the organisation of the text. Eco defines a *model reader* as a model of a possible reader anticipated by the author of a particular text "supposedly able to deal interpretatively with the expressions in the same way as the author deals generatively with them." (1979: 7). He discusses the way in which texts select a model reader through the choice of linguistic code, literary style and specialisation indices. Some texts give explicit information about the model reader they presuppose through direct appeals; others through implicitly presupposing a "specific encyclopaedic competence" (1979: 7). Eco thus suggests that a well-organised text not only presupposes a model of competence coming from outside the text (the model reader) but also constructs this competence.

Although Eco (1979) refers to literary texts, and not to pedagogy, his notion of a model reader can be applied to pedagogic situations. Dowling (1998) draws on Eco's notion of a model reader in his discussion of the analysis of what he refers to as *pedagogic texts*. He describes a pedagogic text as "an utterance within the context of a pedagogic relationship which implicates a pedagogic subject and one or more pedagogic objects" (Dowling, 1998: 112). This is based on his use of Eco's discussion of textual strategies to interpret the categories of author and reader as the "products of the principled analysis of the text" (Dowling, 1998: 112), where Dowling interprets the author of the text as the pedagogic subject, and the reader as the pedagogic object, thus shifting the focus from the empirical author and reader – pedagogic subject and object – to the textual ones. My theoretical framework draws on Dowling's application of the notion of a model reader to pedagogic texts, but while Dowling's focus is the reproduction of ideology by a pedagogic text, I focus on the mathematical competence reproduced by a pedagogic text.

Eco's discussion of the way in which the semiotic resources detailed in a text imply and construct a specific competence can be likened to the way in which the evaluative criteria generated in a pedagogic text imply and attempt to structure a particular mathematical competence. Jaffer (2011a) applies Eco's notion of a model reader to pedagogic situations to develop the notion of a *model learner*. I use this notion as a way of describing how it is that pedagogy both presupposes and structures the mathematical competence of learners. The teacher anticipates or presupposes a certain kind of learner competence (the model learner), and I will suggest that this shapes the evaluative criteria generated in pedagogic contexts, which in turn structure the ways in which learners do mathematics. Thus any pedagogic activity implies a model learner, not to be confused with the actual learner, as highlighted by Dowling's (1998) distinction between textual and empirical subjects. I draw on the Lacanian psychoanalytic registers of the Imaginary and the Symbolic to describe the nature of the model learner implied by pedagogy.

### **3.3 The Real, the Imaginary and the Symbolic**

As introduced in Chapter One, Lacan's three registers form a central part of my theoretical framework. I have discussed the Imaginary register, which emerges during the mirror stage when the infant recognizes him/herself in the mirror, and identifies with his/her image. While the Imaginary refers to engagement with the Other<sup>9</sup> in terms of an image, the Symbolic refers to engagement with the Other in terms of the way in which the Other functions within a particular structure. The focus of the Symbolic register, which emerges at the onset of language, is on the

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<sup>9</sup> The big Other is described by Brown (2008) as "the network of symbolic structures and discourses that I inhabit ... see myself reflected in" (Brown, 2008: 232). Žižek describes it as the "anonymous symbolic order" (2006: 41).

internal constituents of this structure – it consists of the social, cultural and linguistic networks into which a child is born. Lacan (2006) extended the mirror stage by discussing its Symbolic aspect – the way in which a child is tied to his/her image by language affects his/her identity - identification with an image during the Imaginary register paves the way for a more Symbolic engagement with the world.

For Lacan (2006), the Symbolic represents the place available within the symbolic network, as distinct from the individual occupying that place. The way in which the symbolic network is structured enables the maintenance of this distinction between an individual and the place they occupy, as discussed by Žižek (1989), who notes that the difference between the Imaginary and the Symbolic is the difference between how we see ourselves and the place from which we are being observed. He also discusses the relation between Imaginary and Symbolic identification - “to put it simply, Imaginary identification is identification with the image in which we appear likeable to ourselves, with the image representing ‘what we would like to be’, and Symbolic identification, identification with the very place *from where* we are being observed, *from where* we look at ourselves so that we appear to ourselves likeable, worthy of love.” (Žižek, 1989: 116, italics original). This is a useful distinction for my study and I return to it later to discuss how I identify when the Imaginary and Symbolic registers are in play in the evaluative criteria which emerge in the pedagogic context.

The Symbolic cannot capture everything and will at some point fail, and it is at this point of breakdown that the Real, according to Lacan, emerges. Lacan introduced the Real register in response to that which cannot be or has not been symbolized – the Real “resists symbolization absolutely” (cited in Jameson, 1988: 104) and refers to existence outside of the Symbolic. Žižek (2006: 72) describes the Real as the “fissure within the symbolic network”. The presence of anomalies within the Symbolic<sup>10</sup>, which Fink (1995: 30) refers to as “kinks in the Symbolic”, point to the presence of the Real, and its influence on the Symbolic. In terms of education, as discussed in Chapters One and Two, the learner represents the potential point at which knowledge breaks down, and the potential intrusion of the Real. The learner’s lack in relation to knowledge can be construed in different ways. The learner can either be seen as lacking in knowledge but with the potential to fully grasp the content, or as incapable of grasping the content. Where the learner is considered incapable or inadequate in some way, education theory or pedagogy attempts to find alternate ways for learners to realise the content, i.e. to reproduce the mathematics. The responses of education

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<sup>10</sup>In his description of the Real, Fink (1995) uses as an example an infant’s body “before it comes under the sway of the symbolic order” (Fink, 1995: 24). He explains that although the Real gets progressively symbolized through a child’s life, a remainder will always persist alongside the Symbolic.

theory and pedagogy to the learner as Real can recruit the Imaginary or the Symbolic, or a combination of the two. I will return to this later, but for now Žižek's (2008) discussion of the three registers illustrates the interaction between Symbolic and Imaginary responses to the learner as Real and highlights the complex way in which they interconnect. He states that "the entire triad is reflected within each of its three elements (Žižek, 2008: xii) - within each register, all three registers are present, for example within the register of the Real, Žižek differentiates between the imaginary Real, the symbolic Real and the real Real. Thus the Imaginary and the Symbolic cannot be separated – we cannot study one without the other, or without the Real. But we can speak of a predominance of one register over another – when one overrides another, or is usurped by another. It is these instances which are of interest in my study. Davis (2010c) explains how we can draw on Žižek's (2008) explanation of the way in which the three registers can be thought of in terms of each other in order to describe phenomena "under the aspect' of either the Real, or the Imaginary, or the Symbolic, no matter where they are located" (Davis, 2010c: 3).

For my study, I have appropriated Lacan's three registers in the manner suggested by Davis (2010c) as analytic tools to investigate the co-constitution of mathematics and learner identification in pedagogic situations, in conjunction with the notion of a model learner derived from Eco's (1979) work. In order to refine my use of the three registers, I return to my earlier discussion of evaluation.

On one level, evaluation produces ideological formulations about the "kind of learner who is envisaged" (DoE, 2003: 4) – who the learner should be. The introduction to the South African National Curriculum Statement includes a list of what learners should be able to do, including "think creatively", "communicate effectively", "organise and manage themselves" and "demonstrate an understanding of the world as a set of related systems" (DoE, 2010: 3). The Principles and Standards for School Mathematics contains a vision for school mathematics that starts with the words "imagine a classroom, a school, or a school district where ..." followed by statements such as "students are flexible and resourceful problem solvers" and "they value mathematics and engage actively in learning it" (NCTM, 2000). These statements illustrate the Imaginary register in play. For my study, I take the Imaginary as representative of the image held by the teacher of the learner, as implied by their pedagogic practice. In relation to the constitution of mathematics, the Imaginary register can be identified through looking for instances when the evaluative criteria emerging within a particular pedagogic context situate necessity external to mathematics, through appealing to extra-mathematical factors such as iconic features of solutions, as opposed to symbolic aspects of mathematics, in an attempt to get learners to realise the content. This will be further developed in Chapter Four.

At another level, evaluative criteria specify mathematics in a particular way. This is the level of the Symbolic, the actual production of mathematics. For my study, I take the Symbolic to represent what learners need to acquire in terms of the symbolic aspects of mathematics. A section of content from mathematics can be described by Lacan's (1988) use of Aristotle's notion of an *automaton* – a smoothly functioning Symbolic machine. As discussed in Chapter One, the goal of pedagogy is the reproduction of knowledge by the learner, and due to the way in which the extimate relation of the learner and mathematics in pedagogic situations potentially disturbs the functioning of the Symbolic automaton of mathematical content, the teacher, as the mediator between the learner and mathematical knowledge, recontextualises and reconstitutes mathematical procedures in such a way as to re-assert the automaton and enable the reproduction of mathematical content by the learners. The way in which the teacher performs mathematical activity can either facilitate or disrupt this symbolic acquisition. In my study the mathematical activity of teachers will be inferred through analyzing their procedures in terms of the mathematical (or non-mathematical) objects which they manipulate and the manipulations themselves, which could be mathematical or unfamiliar operations. The framework for this analysis will be elaborated in Chapter Four, but now I introduce Davis' method for describing the operations and objects which make up mathematical activity, to enable a more detailed discussion of the constitution of mathematics in pedagogic situations.

### **3.4 Operations and their objects**

Davis (2010a) discusses the need for a more direct engagement with the mathematical activity of pedagogic situations than that generally offered within the field of mathematics education. This engagement can be realised by describing and analysing the operational features constituting such activity. The unit of analysis Davis introduces is an *evaluative event* (Davis, 2003, 2005, 2010b: 5), which is “composed of a sequence of pedagogic activity, starting with the presentation of specific content in some initial form, and concluding with the presentation of the realisation of the content in final form”. The evaluative event enables division of records of pedagogic situations (such as videos and transcripts) into segments based on the mathematical topic and the particular type of activity that teachers and learners are engaged in.

My study will be based on his method for describing the “foundational assumptions, objects and operations that ground school mathematics” (Davis, 2010a: 1). The method entails constructing descriptions of mathematical operations and objects and their inter-relations as they emerge and circulate in pedagogic situations. Davis (2010b: 21) reminds us of the definition of an operation:

a function of the form  $f: X_1 \times \dots \times X_k \rightarrow Y$ . The sets  $X_j$  are called the domains of the operation, the set  $Y$  is called the codomain of the operation, and the fixed non-negative integer  $k$  (the number of arguments) is called the type or arity of the operation.

Davis' emphasis on the necessarily functional nature of all mathematical operations, as defined by mathematics itself, is central to any discussion of the constitution of mathematics, and enables the identification of what he refers to *pseudo-operations* or *operation-like manipulations* - the production and manipulations of objects that cannot be covered by this definition, as they do not behave as functions. Describing the operational activity of a pedagogic situation in terms of operations and their objects enables a comparison with the operational resources in the mathematical body of knowledge, referred to as the *mathematics encyclopaedia*.

### 3.4.1 The mathematics encyclopaedia

As described by Mac Lane (1986: 409), the “development of Mathematics provides a tightly connected network of formal rules, concepts, and systems”. Davis uses the term *mathematics encyclopaedia* to describe this network – the mathematical body of knowledge. Mathematical knowledge has been generated within the field of production of mathematics, from which selections of content have been recontextualised into mathematics curricula, textbooks and pedagogy as a collection of specific mathematical topics. These topics are recontextualised from the field of production into the field of recontextualisation in order for learners to be able to reproduce the required content for each topic. I will discuss the notions of topic and content in more detail but I first highlight key features of the mathematics encyclopaedia.

Although the term mathematics encyclopaedia conveys uniformity within the mathematical body of knowledge, there is no one reference which contains this knowledge. There is also no consensus among mathematicians as to what should be included in the body of knowledge and how it should be structured. But a common feature of mathematics texts is a drive towards internal consistency and coherence, as seen in the work of Bourbaki and Euclid, amongst others. The internal logic of mathematics imposes itself on mathematics researchers, so that the subject itself is the encyclopaedia. Mac Lane (1986: 410) draws attention to this in his discussion of the rules, definitions, axioms and proofs which make up Mathematics:

The presentation of Mathematics is formal: Calculations are done following rules specified in advance; proofs are made from previous axioms and follow predetermined rules of inference; new concepts recognized as relevant are introduced by unambiguous definitions;

errors and disagreements are cleared up not by dispute but by appeal to the relevant rules. It is characteristic of any formal procedure that it makes no reference to the meaning or to the applications, but only to the form. The formalism may be imperfect and sketchy, but it carries with it perpetually the possibility of perfection. Because of these characteristics, Mathematics (within its limits) is absolutely precise and independent of persons.

Mac Lane (1986) also describes the centrality of the notion *function* to the organisation of mathematics. Davis (2010c) takes this further and highlights the fundamental feature of mathematics – “operations that populate mathematics are functions” (Davis, 2010c: 4). He emphasises the functional nature of operations because it is what brings stability to mathematics, due to the stable way in which functions behave. This essential property of mathematics cannot change as mathematics moves from the field of production to the fields of recontextualisation or reproduction.

Another feature of the encyclopaedia is the way in which the operational resources found there enable us to select procedures based on mathematical definitions, properties and axioms. As operations are functions, and the rules for any function are necessarily infinite, the effects of an operation can be arrived at in many different ways. There is seldom one specific procedure dictated by the operational resources for a particular topic, but instead access to the resources allows us to choose any procedure which obeys the properties and axioms relating to that topic, and all such procedures will lead to the same outcome. Access to the basic axiomatic features of mathematics thus gives a freedom which is suggestive of Eco’s notion of an *open text*, introduced earlier in this chapter, as an open text does not attempt to produce only one particular reading, but the readings it does produce converge to a particular interpretation. Mac Lane’s (1986) discussion of the rules for arithmetic illustrate this - he emphasizes that these rules are unambiguous, and that different calculations of the same sum or product, if carefully carried out, yield the same answer. It is thus the “austerity of the rules of arithmetic” which is the basis of their applicability (Mac Lane, 1986: 410). Badiou (2005) makes a related point in his discussion of deduction in relation to fidelity. He draws attention to the “richness and complexity” of mathematical thought in comparison to the “extreme poverty” (Badiou, 2005: 243) of the rules of mathematics. As explained by Norris (cf, 2009: 182), Badiou emphasizes the “remarkable contrast between the extreme poverty (or so it would appear) at the level of basic terms, structures or modes of canonically valid logical argument and the extraordinary range or creativity of which mathematics is somehow capable while nonetheless respecting the rules and constraints of that same logical regimen”. It is also important to note that failure of a particular procedure can never be attributed to the rules themselves (assuming

that they arise from mathematical definitions, properties and axioms), as illustrated by Mac Lane's (1986) discussion of the relation between the rules of arithmetic and the operation of counting, where he points out that "the very generality of the rules and their manifold prior uses means that accidental error is never laid at their doorstep ... the formal rules of arithmetic are a firm background for the occasionally faulty operation of counting" (Mac Lane, 1986: 410).

In light of these key features of the mathematics encyclopaedia, I return to the notions of topic and content. As mentioned earlier, selections of content from the mathematics encyclopaedia have been recontextualised into mathematics curricula, textbooks and pedagogy as a collection of mathematical topics. For each topic, the encyclopaedia proposes certain content – a particular set of objects and operations which can be drawn together, often in more than one way, to produce the same outcome. As highlighted, there is seldom one specific procedure dictated by the encyclopaedia for a particular topic. In this study I distinguish between two levels of topic – the intended and the realised topic of a pedagogic text<sup>11</sup>. I take the *intended topic* as the topic announced by the teacher in an evaluative event, together with the elements of the mathematics encyclopaedia (the content) which this topic recruits. The *realised topic* is the content which is taught and learnt (the objects and operations drawn together in the pedagogic situation), which can be read off the operational activity. A comparison of the intended topic with the realised topic enables us to assess whether the content indexed by the intended topic is realised, and whether the mathematical activity in the classroom corresponds with the mathematics encyclopaedia. In order to do so we need to focus on the nature of the objects operated on, as well as the objects which are the outcome of a particular operation, which will be elaborated in my analytical framework, but for now I turn to the regulative resources underlying pedagogic activity, which can be thought of as a type of ground, as discussed by Davis & Johnson (2007, 2008) and Davis (2010a, 2011b).

### 3.4.2 The regulation of mathematical activity

Davis & Johnson (2008) suggest that for learners to engage with the range of mathematical objects and operations indexed by mathematical expressions, those objects and operations need to be present as the ground on which procedures rest. They propose that this more fundamental grounding, which they refer to as *propositional ground* (Davis & Johnson, 2007, 2008; Davis, 2011b), underpins any proceduralising of mathematics, but that the fundamental mathematical objects and operations indexed by mathematical expressions are not always

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<sup>11</sup>This is similar to the distinction made between the intended-implemented-attained curriculum (see the intended-implemented-attained curriculum model used by Robitaille & Garden, 1989 and Travers & Westbury, 1989 amongst others), where the intended curriculum refers to official documents such as the National Curriculum Statements, the implemented curriculum to the texts for teaching and the attained curriculum to the competencies of learners.

accessible to learners due to the pedagogic strategies in play. In research carried out in the five schools they observed a widespread use of iconic and procedural resources which did not give learners access to the fundamental mathematical objects and operations of a propositional ground. Davis & Johnson (2007, 2008) and Davis (2010a, 2011b) describe the use of iconic resources in pedagogy as *iconic ground*, which involves regulation of the production of knowledge statements through reference to iconic similarity of expression. They also describe *algorithmic ground* – use of a ‘standard form’ and rules, which appeal to more than just iconic similarity and involve the selection of operations from a cluster commonly used in working through the particular type of procedure being dealt with. They found that these two types of ground function as the dominant supports for the evaluative criteria operating in the teaching and learning of mathematics in these schools. Another category described is *empirical ground*, which involves the regulation of mathematical activity by some kind of empirical test or measurement. Note that these categories should not be arranged hierarchically, but can operate simultaneously in pedagogic activity.

**Table 3.1** Categories of ground (Davis, 2011b)

Ground	Central regulative resource	Objects of central concern
Iconic	Iconic similarities and differences of expressions	Graphical and/or symbolic expressions treated as images
Empirical	Empirical testing of expressions	Graphical and/or symbolic expressions treated as ‘measurable’
Propositional	Knowledge of mathematical objects and propositional relations	Mathematical objects indexed by the axioms, definitions and propositions signified by expressions
Algorithmic	Meta-rules governing an algorithm	Operations commonly used within a particular algorithm, and their sequencing

These categories describe the primary regulative resource for a particular event or procedure. But as explained by Davis (2011b), although a description of the primary regulative resources is useful, it is necessary to first generate a description of the mathematical objects and operations in order to describe the detail of the operational activity. According to Davis (2011b), a description of the operational activity that unfolds in pedagogic situations names three things - the operations that emerge in that situation, the collections of objects over which these operations range, and the criteria governing the selection and sequencing of these operations. These criteria in turn indicate the ways in which the mathematical activity in a pedagogic situation is regulated, and, as discussed by Davis, Adler & Parker (2007: 37), “any evaluative act, implicitly or explicitly, has to appeal to some or other authorising ground in order to justify the selection of criteria”. Davis et al (2007) suggest that the nature of appeals made to authorizing or legitimating ground in order to authorise

the actions and statements of subjects reveals whether the Imaginary or the Symbolic are operative or whether one is rendered under the aspect of the other in a pedagogic situation, and thus whether Imaginary or Symbolic identifications are taking place.

This leads us to the notion of necessity, originating in Hegel's theory of judgement<sup>12</sup> (1923) and discussed by Žižek (1998, 2002) and Davis (2001, 2003, 2005). When appeals are made to extra-mathematical factors as authorising ground, then necessity is located external to the field of mathematics. Similarly, when appeals are made to mathematical propositions, definitions and processes as authorising ground, necessity is located within the field of mathematics. In order to develop this further, I return to Lacan's distinction between the Imaginary and the Symbolic introduced earlier in this chapter, and the way in which the symbolic network is structured to enable the maintenance of this distinction between an individual and the place they occupy. Necessity with respect to the Symbolic rests on maintaining this distinction. In the context of education, the teacher holds the symbolic mandate (Bourdieu, 1991), but only in so far as he/she observes the rules of mathematics. Thus necessity resides outside of the teacher and in the field of mathematics, with its propositions, axioms and definitions. As explained by Davis, Parker & Adler (2005: 2), "symbolic relations and symbolic identification are predicated on the social existence of a legitimating field external to the individual subject, including s/he who holds any particular mandate". But as soon as mathematical necessity is suspended, there is a collapsing of the distinction between the place and the occupant, which now converge, so that necessity resides with the teacher and the criteria he/she generates, rather than within the field of mathematics, for example when the evaluative criteria operating within a particular pedagogic situation appeal to extra-mathematical factors, as opposed to the symbolic aspects of mathematics. This renders the Symbolic under the aspect of the Imaginary.

I now give an example of the way in which the regulative resources underlying operational activity can appeal to extra-mathematical factors as authorising ground, situating necessity external to the field of mathematics and rendering the Symbolic under the aspect of the Imaginary.

The following extract is from one of the grade 10 lessons on number patterns, where a teacher explains why it is that minus seven plus five is equal to minus two, and not minus twelve, as some learners claim. He uses the examples in Figure 3.1 in his explanation.

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<sup>12</sup>In his theory of judgement Hegel (1923) describes four moments of judgement - the judgement of existence, of reflection, of necessity and of the notion.

$$\begin{array}{l} 7 + 5 = 12 \\ 7 - 5 = 2 \\ -7 + 5 = -2 \\ -7 - 5 = -12 \end{array}$$

**Figure 3.1** Adding integers

- Teacher: If the signs are the same what do you do? You take the common sign in both of them. In this case (points to  $-7 - 5$ ) common sign is what?
- Learners: Minus
- Teacher: Minus. So this is equal to minus (writes " $-7 - 5 = -$ "). And then what do you do? You add. Minus seven and minus five will give you minus?
- Learners: Twelve.
- Teacher: So if the signs are the same, what do you do? You take the common sign and then you add. If the signs are not the same what do you do? You subtract.
- Learners: Subtract.
- Teacher: But first you take the sign of the what? The sign of the bigger number. You look at the bigger number between the two and then you take the sign of the bigger number.

Transcript School P6 Grade 10 Lesson 1

In this extract, the teacher explains how to add and subtract integers. But he does not mention that the mathematical objects are integers, nor does he draw on the properties of integers<sup>13</sup>, but instead gives the learners a set of complicated rules to follow. His method involves separating the signs (+ and -) from the numerals (7 and 5), so that he is no longer working with integers but simply characters (+, -, 7, 5) which can be selected and combined according to his rules. This extract illustrates how the teacher's criteria do not draw on the operational resources of the mathematics encyclopaedia, but consist of unfamiliar operations and depend on the iconic resource of similarity and difference of signs. In this example, the authorizing ground which is appealed to is predominantly iconic and procedural, without any reference to the propositional ground underlying the procedure, thus situating necessity external to the field of mathematics with its propositions,

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<sup>13</sup> The operational resources needed to add integers are the properties of associativity and commutativity of addition over the integers (for example, the sum  $-2 + 7$  is equivalent to  $7 + (-2)$ ) and the existence of additive inverses (for example the sum  $-7 + 2$  can be written as  $-5 + -2 + 2$ , from which the answer of  $-5$  can easily be obtained).

processes and objects, and suggesting a rendering of the Symbolic under the aspect of the Imaginary. This example will be discussed in detail in Chapter Four, but at this point I briefly discuss the use of iconic and procedural resources, and their role in pedagogic situations.

Peirce discusses the use of the iconic in mathematics:

[...] thus, an algebraic formula is an icon, rendered such by the rules of commutation, association, and distribution of the symbols. It may seem at first glance that it is an arbitrary classification to call an algebraic expression an icon; that it might as well, or better, be regarded as a compound conventional sign [symbol]. But it is not so. Because a great distinguishing property of the icon is that by direct observation of it other truths concerning its object can be discovered than those which suffice to determine its construction. [...] This capacity of revealing unexpected truth is precisely that wherein the utility of algebraic formulae consists, so that the iconic character is the prevailing one (Peirce, 2.279: 158).

In this sense, the iconic is a useful tool in mathematics. But the iconic can be exploited in more destructive ways. Davis (2011b: 313) discusses what Fodor refers to as the “picture principle” - “If P is a picture of X, then parts of P are pictures of parts of X”. What this means is that icons can be broken down in whatever ways we choose. But mathematical expressions cannot be broken up in any way. This is the possible danger that the use of iconic ground presents – aberrations which result from the breaking up of mathematical expressions in ways which violate mathematical propositions, definitions, processes and rules. As Davis (2011b: 213) points out, iconic ground “is indexed by the—usually distorting—over-determining effect that arises when mathematical expressions are treated as a source of imagistic data for the purposes of mathematical processing.”

The use of procedural, or algorithmic, resources is a central feature of mathematics. Whitehead (in Davis & Johnson, 2008) comments on the practical use of operations which can be performed without thinking, such as when carrying out a familiar mathematical procedure. But in order to apply mathematical procedures, we need to be able to monitor our use of the procedures using knowledge of the mathematical propositions, processes and definitions that function as ground for the procedures. So the use of procedural ground without the supporting propositional ground can result in learners not having access to the appropriate mathematical propositions, definitions and processes in order to monitor their use of procedures. An example is the process of ‘cancellation’, commonly found in school mathematics. As pointed out by Davis & Johnson (2008), in order to

carry out the ‘cancellation’ of terms, we need to know when it is legitimate to do so, and thus we need access to the propositional ground underlying the process of ‘cancellation’.

What this discussion aims to highlight is that although the use of the iconic and the procedural in mathematics is not problematic in itself, the problem comes in when the dominant supports for the evaluative criteria in operation are iconic and procedural, without giving learners access to the corresponding propositional ground. For the purposes of my analysis, I consider appeals to the iconic and procedural as predominantly appeals to extra-mathematical factors, because although iconic and procedural resources can be successfully used in mathematics, on their own (without appeals to the appropriate mathematical propositions, processes and definitions) they are not sufficient to ground the production of mathematical statements. In addition to this, it has been found that the use of the iconic and the procedural as dominant supports for the evaluative criteria in these schools is not associated with appeals to fundamental mathematical propositions, definitions, processes and objects as authorizing ground (Davis & Johnson, 2008; Jaffer, 2010).

Analysis of the evaluative criteria operating in the grade 10 lessons under discussion in order to identify what is appealed to as authorizing ground, and thus where necessity is situated, enables the identification of whether the regulation of the learner is under the aspect of the Imaginary or the Symbolic. Central to this analysis is a focus on instances where the operational resources differ from those indexed by the particular mathematical topic, as seen in the example. Points of difference from the encyclopaedia are of interest because of what they reveal about the underlying principles at work, in a similar way to the operation of Freud’s notion of a *parapraxis*<sup>14</sup>, which can loosely be described as the emergence of something different from what is expected in a particular situation. This difference is interesting from a psychoanalytical point of view as it reveals what is going on beneath the surface. In a similar way, I look for pedagogic instances involving points of difference<sup>15</sup> from the mathematics encyclopaedia in order to get insight into the principles at work. My analysis will focus on the ways in which the operational activity differs from the encyclopaedia and what these differences reveal about the way in which the mathematical content is constituted, and the implications of that for the co-constitution of mathematics and learner identification.

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<sup>14</sup> Freud (1901) describes a *parapraxis* as the “way in which a name sometimes escapes one and a quite wrong substitute occurs to one in its place” (1901: ix). Freud (1901) claimed that “certain seemingly unintentional performances prove, if psycho-analytic methods of investigation are applied to them, to have valid motives and to be determined by motives unknown to consciousness” (1901: 239). Freud includes slips of the tongue, misreadings, forgetting things despite knowing better and chance actions in the notion of a *parapraxis*. Freud saw *parapraxes* as unintentional manifestations of mental activity, which he believed revealed hidden or suppressed motives and impulses. The analysis of *parapraxes* is central to Freud’s psycho-analytic method.

<sup>15</sup> These points of difference are not necessarily “slips” in the Freudian sense, as they are often consciously explicated ideas, but what is of interest is the way in which these ideas differ from the expected resources contained within the mathematics encyclopaedia.

To summarise this chapter, I outline the propositions which I have drawn upon and which form the basis of my analytical framework in Chapter Four.

### **3.5 Summary: Propositions**

In summarizing the propositions of my study I follow Davis (2005), who describes three types of propositions. The first are theoretical propositions, and the second are empirico-theoretical propositions which relate specifically to the empirical context, including general features of the context as well as analytical resources which have been developed. The third type of propositions is research hypotheses, which have been developed from the first two groups of propositions.

#### **3.5.1 Theoretical propositions**

I assume the existence of knowledge, teachers and learners, as well as their absences, as the foundation of my theoretical framework.

TP1: Any pedagogic situation involves at least *three relations*:

- The relation of the learner to the field of knowledge
- The relation of the teacher to the field of knowledge
- The relation of the teacher to the learner.

TP2: All three relations are characterized by incompleteness at the points of contact between the categories of teacher, learner and knowledge, which prevents each category from realizing its full self-identity. They are thus relations of *extimacy*.

TP3: The relation between the learner and knowledge specifically can be described as one of *objective antagonism*, as the learner can be described as the absence of knowledge, and thus as a potential obstacle to the reproduction of mathematics.

The teacher acts as mediator between the learner and the mathematical body of knowledge in order to enable the learner to reproduce the required mathematical content.

TP4: In their pedagogy, teachers are obliged to take into account the learner as a point of incompleteness. This shapes the evaluative criteria generated in the pedagogic situation.

The purpose of pedagogic practice is the transmission of evaluative criteria. Evaluative criteria are the rules for regulating an activity and reveal what is realised as legitimate in particular pedagogic

contexts. The evaluative criteria circulating in a pedagogic situation produce concern with what *'ought-to-be'* at the level of the mathematics produced by learners, as well as of the type of learner identity which is expected in that particular situation.

TP5: Pedagogy is necessarily evaluative.

TP6: Pedagogy implies a *model learner* due to the way in which the evaluative criteria both presuppose and structure the mathematical competence of the learner, which is similar to the way in which a text presupposes and structures the competence of its reader.

This leads us to draw on the interrelations between Lacan's three registers – the Imaginary, the Symbolic and the Real – in order to describe the response of the teacher to the extimate relations of the learner and knowledge.

TP7: We can describe the learner as a potential irruption of the *Real*, or point at which the Symbolic breaks down.

The evaluative criteria which are generated in response to the learner as Real can be thought of in terms of the Imaginary and Symbolic.

TP8: The Imaginary register is that which emerges during the mirror stage when the infant recognizes him/herself in the mirror, and identifies with his/her image. The Imaginary register is thus characterized by the importance of perception - the visual, specular relation between the subject and an image outside of him/herself.

TP9: The Symbolic register emerges at the onset of language and consists of social, cultural and linguistic networks.

The Imaginary and the Symbolic cannot be separated – we cannot study one without the other, or without the Real. But we can speak of a predominance of one register over another.

TP10: It is possible to think of the three registers in terms of each other in order to describe phenomena under the aspect of either the Real, or the Imaginary, or the Symbolic.

The next three theoretical propositions pertain to the nature of mathematical operations.

TP11: The operations that populate mathematics are functions, investing mathematics with stability at the level of its operations since functions have unique outputs for given inputs.

TP12: Removing the requirement that operations be functions removes the stability enjoyed by operations. Operation-like manipulations which are not functions can be described as pseudo-operations.

TP13: Just as with functions, it is possible to replace an operation by a rule that is composed of more than one operation, but which still produces the same output for any given input. This often happens in the pedagogic situations of schooling.

Selections of content from the mathematical body of knowledge, referred to as the mathematics encyclopaedia, are recontextualised into curricula, textbooks and pedagogy as specific topics.

TP14: The intended topic is that which is announced by the teacher, together with the content, or the elements of the mathematics encyclopaedia, which this topic recruits.

TP15: The realised topic consists of the objects and operations which are drawn together in the pedagogic context, and represents what is actually being taught and learnt.

The intended topic of a pedagogic text may or may not be realised in a pedagogic situation, and the realised topic may or may not correspond with the mathematics encyclopaedia. Differences between the intended and realised topics, and between the realised topic and the encyclopaedia can be identified through focusing on the objects and operations which are drawn together and make up the content which is realised.

TP16: In a similar way to the psychoanalytical use of the notion of a parapraxis, pedagogic instances involving such points of difference from the mathematics encyclopaedia are useful in order to get insight into the underlying principles at work.

### 3.5.2 Empirico-theoretical propositions

These are propositions derived from previous research pertaining to the cases used in my study. Based on research carried out in the five schools which participated in the first phase of the project (three of which participated in the second phase, from which the data for my study was obtained), Davis & Johnson (2007, 2008) found that:

ETP1: The teaching and learning of mathematics in the schools occurs mainly through the exposition of procedures, with worked examples being the primary pedagogical approach.

ETP2: On average in the schools, teachers and learners work through three or four problems per lesson, spending between nine and eleven minutes per problem.

Davis & Johnson (2007, 2008) also noted the absence of discussion of mathematical propositions, definitions, objects and processes. In the absence of this *propositional ground*, they identified two types of ground which functioned as dominant supports for the evaluative criteria operating in the teaching and learning of mathematics in these schools - *iconic ground* (which regulates the production of knowledge statements through reference to iconic similarity of expression) and *procedural or algorithmic ground* (Davis, 2011b) – the use of a “standard form” and rules, which appeals to more than iconic similarity as it involves the selection of operations from a cluster that are commonly used in working through the particular type of procedure being dealt with. These findings were confirmed by Jaffer (2010) in her analysis of grade 9 and 10 lessons in the second phase of the project.

ETP3: The dominant supports for the evaluative criteria in these schools are procedural and iconic in nature.

In their work in the five schools participating in the second phase of the project, Jaffer (2009, 2010a, 2010b), Basbozkurt (2010) and Davis (2010a, 2010b, 2011a) describe examples of instances where teachers use alternative operations to those indexed by the mathematical topics announced in the pedagogic situation in order to enable shifts in the domain operated over.

ETP4: The phenomenon of domain-shifting is a commonly occurring feature of pedagogic discourse in the five schools.

ETP5: The criteria used in the elaboration of mathematics in these schools incorporate a number of operations and operation-like manipulations that are not usually recognised as elements of topic-specific procedures for the solution of problems.

### **3.5.3 Research hypotheses**

Based on the theoretical and empirico-theoretical propositions, the following are the research hypotheses for my study:

RH1: I expect to find that there are cases in the analysed lessons when the content indexed by the intended or announced topic is not realised due to the drawing together of content (in the form of a set of objects and operations) which is not usually indexed by the intended topic from the point of view of the mathematics encyclopaedia, and as such, is a substitute for the indexed content.

RH2: I expect that the content may be realised in ways that do not correspond with the propositions, processes, rules and objects of the mathematics encyclopaedia, for example, due to the use of manipulations which are not functions and thus not operations.

RH3: In cases where the mathematics encyclopaedia is not functioning as a primary regulative resource, I expect that there will be other regulative resources appealed to as authorizing ground, amongst them procedural and iconic ones, and thus that necessity will be situated external to the field of mathematics.

Identification of the regulative resources which are appealed to as authorizing ground for criteria will enable the determination of where necessity is located and thus whether the Imaginary or Symbolic register is in play, or whether one is rendered under the aspect of the other, in the evaluative criteria in operation in a particular lesson.

Now that I have established the propositions of this study, I move to a discussion of the production and analysis of data.

## Chapter 4

### **A Framework for the Production and Analysis of Data**

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The central object of this chapter is the discussion of procedures for the production of data from the information archive consisting of the video-recorded grade 10 lessons from the five schools under consideration. I begin with a discussion of the schools, followed by an explanation of the data collection process. I then introduce my analytical framework, drawing on the theoretical framework and propositions outlined in Chapter Three.

#### **4.1 Research Design**

##### **4.1.1 The cases**

The five high schools on which my study is based are situated in the greater Cape Town area and are all currently participating in an education research and development project. The project is a partnership between a university, the Western Cape Education Department, and five carefully selected Dinaledi<sup>16</sup> schools. The project's first phase began in 2007 with the selection of five schools (P1, P2, P3, P4, P5). The second phase began in 2009, when schools P4 and P5 were replaced by schools P6 and P7.

The five schools currently participating in the project are referred to as P1, P2, P3, P6 and P7. Two of the schools are ex-DET schools situated in townships in greater Cape Town (P2, P3), the third is an ex-DET school situated in a Cape Town suburb (P6), the fourth is an ex-HoA school with an entire student population from 'black' and 'coloured' working-class families (P1), and the fifth is an ex-HoR school with a learner population comprising predominantly 'coloured' children from working-class families (P7).

As reported by Davis (2010a), most of the learners attending the schools live in areas that are among the lowest 20% in terms of socio-economic status (as measured by the City of Cape Town in 2001). Learners from these areas are referred to as working-class because the majority of working adults (20+) living in such areas are employed as semi-skilled and unskilled workers or labourers according to census data used by the City of Cape Town (2006, 2008 in Davis, 2010a). As explained in Chapter One, work in the sociology of education indicates that the social class membership of learners is an important variable in education research. But although the social class membership of learners in the schools is significant for the larger project, the purpose of my particular study does not require a direct engagement with social class as such because I am

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<sup>16</sup>Department of Education-supported schools with a Mathematics and Science focus.

drawing on methodological resources which are intended for use in the description and analysis of mathematics pedagogy in any educational context, irrespective of the social class membership of learners.

The 2008 Grade 12 Mathematics results of the five schools participating in the project show overall poor learner performance in the schools and can be found in Appendix Six.

#### **4.1.2 Data collection**

In February 2009 mathematics lessons in grades 8, 9 and 10 at the five schools were observed and video-recorded. An information archive consisting of observation notes, videos and transcripts of mathematics lessons in grades 8, 9 and 10 at each of the schools was created. For each grade, three consecutive lessons involving a single class were observed and video-recorded (except for two classes where only two lessons were observed, one being a double lesson). There are thus forty-three videos and transcripts of lessons across the five schools. Two cameras were used, one focusing on the teacher and the other on the activity of learners. The speech of teachers and learners captured in the video-records were transcribed and where necessary, translated from isiXhosa and Afrikaans to English.

The raw data for my study resides in the video-records of lessons, observation notes and transcripts collected from the grade 10 classes in the five schools – fifteen lessons in total, which I have analysed using the framework that follows. Prior to conducting the analysis outlined in this chapter, I list the topics making up the fifteen lessons and describe the way in which time is spent across the lessons in order to set the scene for the analysis which follows – this data is presented in the beginning of Chapter Five.

#### **4.2 Analytical framework**

My analysis aims to generate data that will enable me to make statements about the relations between learner identification and the constitution of mathematics in pedagogic situations. To that end, I need to identify when it is that the registers of the Imaginary and the Symbolic are operative in the evaluative criteria that emerge in the grade 10 mathematics lessons I analyse, which will allow a discussion of the co-constitution of mathematics and learner identification. Based on the theoretical framework developed in Chapter Three, there are two levels of data which need to be produced in order to enable such a discussion. The first level, primary data production, is a description of the operational activity making up a lesson, drawing on the analytical method introduced in Chapter Three. Once the primary data has been produced, the second level of data

production analyses the realisation of content and the regulation of learners. The rest of this chapter outlines the framework for the production of data at these two levels.

### 4.2.1 Primary data production

The first level of data production involves the generation of descriptions of operational activity. The central object of this description is to identify the operations being carried out and the objects that are being operated with. A description of the objects and operations in use will also identify the existence of operations and objects which are not familiar mathematical operations and objects that emerge in pedagogic situations. There are three components to the primary data production, based on the analytical method, which will be described now.

#### 4.2.1.1 Generation of evaluative events

Firstly, I segment each of the fifteen transcripts of the grade 10 lessons into evaluative events and sub-events. As introduced in Chapter Three, an evaluative event is “composed of a sequence of pedagogic activity, starting with the presentation of specific content in some initial form, and concluding with the presentation of the realisation of the content in final form” (Davis, 2010b: 5, 2003, 2005). The process of segmenting lessons involves identifying instances where specific mathematical content is initially presented to learners and instances where the realisation of that same content is presented in some final form. As found by Davis & Johnson (2008), the teaching and learning of mathematics in the schools occurs mainly through the exposition of procedures, elaborated through worked examples (ETP1). Thus in this empirical context, an evaluative event often consists of a teacher and learners working through one or more examples for the purposes of teaching and learning a topic-related procedure. With respect to mathematical definitions and propositions, although they are always implied, they are not always explicitly referred to in this context, as found by Davis & Johnson (2008). When segmenting lessons into evaluative events, I look out for topic-related procedures, definitions and propositions that are explicitly dealt with.

One lesson may consist of only one evaluative event or a number of events each containing sub-events. Sub-events are either instances where teachers digress from the topic due to interruptions in the pedagogic encounter, or they are separate portions of related mathematical content (often worked examples) which together make up the exposition of a particular procedure or topic. The notation used for an evaluative event is  $E_n$ , where the subscripted number  $n$  indicates that it is the  $n^{\text{th}}$  evaluative event of that particular lesson. The notation used for a sub-event is  $E_{n,m}$ , which represents the  $m^{\text{th}}$  sub-event of the  $n^{\text{th}}$  evaluative event. To illustrate the process of segmenting a lesson into evaluative events, I return to my earlier example from a lesson on number patterns, which can be divided into three evaluative events, the first of which contains three sub-events.

**Table 4.1** Segmenting a lesson into evaluative events

Time	Event	Activity	Type
00:00 – 21:00	E <sub>1</sub>	Finding the difference between successive terms of a number pattern.	Expository
00:00 – 05:30	E <sub>1.1</sub>	Finding the difference between the terms of the pattern 2; 5; 8.	Expository
05:30 – 07:00	E <sub>1.2</sub>	Finding the difference between the terms of the pattern -7; -2; 3; 8.	Expository
07:00 – 21:00	E <sub>1.3</sub>	Calculating $-7 + 5$ and $-7 - 5$ .	Expository
21:00 – 25:40	E <sub>2</sub>	Finding term fifty of the number pattern -7; -2; 3; 8.	Expository
25:40 – 48:00	E <sub>3</sub>	Learners work on a number patterns question from a grade 10 exam.	Exercise

Once a lesson has been segmented into evaluative events, I construct descriptions of operational activity in each event.

#### 4.2.1.2 Descriptions of operational activity

In order to describe the operational activity of teachers, the analytical method involves listing the transformations making up procedures (exposition of procedures through the elaboration of worked examples is the dominant pedagogy strategy in this empirical context, as described in Chapter Three), and identifying the domain(s)<sup>17</sup>, operations or operation-like manipulations and co-domain(s) implicated in each procedure.

As described by Hiebert & Lefevre (1986), procedures are step-by-step, sequentially ordered, deterministic instructions for solving a task. A procedure typically consists of a number of statements or expressions and the transformations involved in moving from one expression to the next. It is important to emphasise that a central feature of mathematical procedures is that the transformations which make up a procedure produce difference at the level of expression but preserve value at the level of identity. Each transformation has an input object(s), domain(s), output object(s), and co-domain(s). A mathematical operation must not change the determined value of the co-domain elements, despite changes at the level of expression. Analysis of the procedures used in school mathematics involves descriptions of the transformations making up each procedure, in order to ascertain whether identity at the level of value is preserved and to test for the essential property that all operations are functions (TP11).

<sup>17</sup> A feature of mathematics in the context of schools in general, and these schools in particular, is that although the field of reals is the domain underlying most school mathematics topics, the rationals is the set which is most commonly operated over in school mathematics, specifically the positive rationals. Apart from the occasional implicit contact with the reals, such as when dealing with square roots, the reals are seldom engaged with as conceptual entities in school mathematics. Hence in my analysis I usually refer to the rationals (or integers and natural numbers where appropriate) instead of the reals as the domain being operated over, even though the reals are the domain underlying school mathematics topics.

As an example, I return to the procedure for adding integers introduced in Chapter Three, and which seems to be a commonly used procedure in the empirical context. Let's examine the case of learners asked to compute  $-7 + 2$  as a specific example. One way of describing the transformations making up the procedure is as follows:

- (1) Separate  $-7$  from  $+2$ .
- (2) Separate the negative sign from  $7$  and the positive sign from  $2$ .
- (3) Compare the two natural numbers to decide which is the bigger and smaller of  $7$  and  $2$ .
- (4) Subtract the smaller number,  $2$ , from the larger number,  $7$ , to get an answer of  $5$ .
- (5) Append the sign of the larger number, which is a negative sign in this case, to  $5$  to get a final answer of  $-5$ .

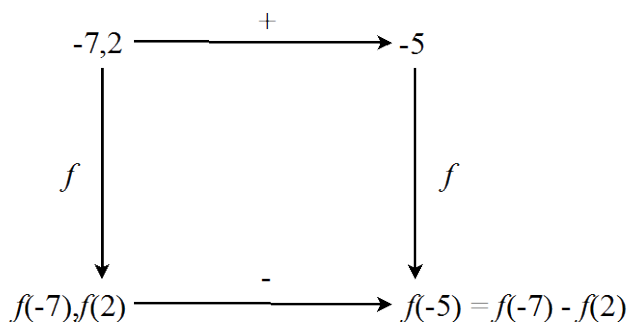
But this list of transformations does not capture what is happening at the level of value. In order for  $/7/$ <sup>18</sup> to be the 'bigger number',  $/-7 + 2/$  must be taken as an expression referring to natural or whole numbers, as pointed out by Davis (2010a). Thus to enable the performance of operations that require signs to be separated from the numerals, which together represent the integers  $-7$  and  $+2$  in the sum  $-7 + 2$ , an existential shift has taken place. Numbers themselves cannot serve as arguments to operations that either detach signs from or append them to numerals. Here we need to be very clear that an integer is not a natural number with a negative sign attached to it, which is what is implied by the procedure. Implicit to the procedure are existential shifts whereby the procedure shifts from using operations that take integers as arguments to others that take characters or symbols as arguments, so that signs can be freely detached from and put back together with numerals. Such existential shifts change the nature of objects at the level of value, but the changes are indistinguishable at the level of expression. That is, speaking operationally, the signifiers remain constant, but what is signified changes, even if only momentarily. Existential shifts such as these are not familiar operations found within the mathematics encyclopaedia, but the methodology used here requires that we accept whatever emerges operationally, whether familiar or unfamiliar, as participating in the constitution of mathematics in the local pedagogic situation.

In this example, it seems that the existential shift is motivated by a desire to operate over the domain of natural numbers, which is a domain of operation familiar to learners. The teacher's procedure starts with integers, shifts to natural numbers and operates on these, then returns to the

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<sup>18</sup>In my analysis a pair of forward slashes is used to indicate character strings (sequences of alphanumeric characters) following Davis (2010a), and deriving from a convention among scholars of semiotics to use a pair of forward slashes to emphasize a referral to the signifier, as opposed to the signified, or the meaning of a word.

domain of integers. It should be noted that there are obvious morphisms<sup>19</sup> that can do the work of shifting operations from the domain of integers to the domain of natural numbers. One such morphism maps addition over the integers to subtraction over the natural numbers by using the absolute value function. Let  $f(x) = |x|$  be the absolute value function. We can represent the particular morphism pertaining to the computation  $-7 + 2$  as in Figure 4.1.

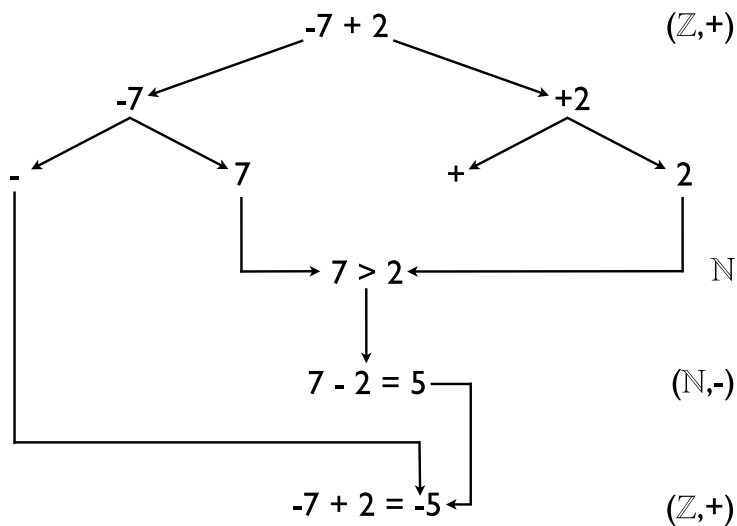


**Figure 4.1** A morphism mapping  $(\mathbb{Z}, +)$  to  $(\mathbb{N}, -)$

Completing the computation requires a mapping  $g(x) = -x$ , which maps  $f(-7) - f(2)$  to  $-5$ . The operations used in the procedure discussed here, however, achieve the same effect as the morphism described in Figure 4.1, but the work done by the mappings  $f$  and  $g$  in Figure 4.1 is achieved by operations that act directly on the symbols, so entailing existential shifts from numbers to characters. I am interested in how such existential shifts take place and what the implications of this are for the constitution of mathematics, and for my study particularly, the implications of this for the constitution of learner identification.

It is useful to illustrate operational activity using diagrams, such as the one in Figure 4.2 which Davis (2010a) constructed to illustrate the process of integer addition. These can be simplified to form existential maps, as in Figure 4.3, which summarise the description of operations and their objects, as well as links between various types of objects. They specifically reveal the points at which there are existential shifts.

<sup>19</sup> A morphism is a function which links two structures,  $f: (A, \circ) \mapsto f((A), \square)$  such that  $f(a_1) \square f(a_2) = f(a_1 \circ a_2) \forall a_1, a_2 \in A$ . A homomorphism is a morphism for which the function is many-to-one - it is a structure-preserving map between two algebraic structures (see Krause, 1969; Baker, Bruckheimer & Flegg, 1971).



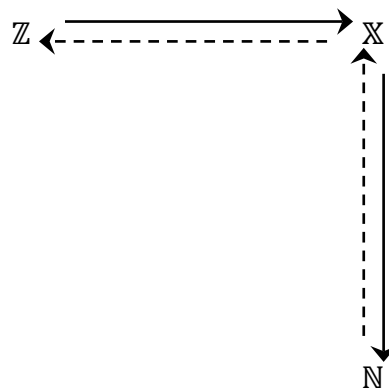
**Figure 4.2** A procedure for integer addition (Davis, 2010a)

In addition to constructing diagrams to give an overall picture of operational activity, I also record the input objects, domain(s), output objects, co-domain(s), operations for each procedure as follows, in order to show the detail of the operations or operation-like manipulations (TP12), as displayed in Table 4.2, where  $\mathbb{X}$  indicates the domain of character strings; as usual,  $\mathbb{Z}$  and  $\mathbb{N}$  indicate the integers and natural numbers.

**Table 4.2** Key transformations of a teacher's procedure for computing  $-7 + 2$ .

T	Input	Domain	Operation	Output	Codomain
1	$-7 + 2$	$\mathbb{Z}$	Existential shift	$/-7 + 2/$	$\mathbb{X}$
2	$/-7 + 2/$	$\mathbb{X}$	Sundering	$/-7/,/+2/$	$\mathbb{X}$
3	$/-7/,/+2/$	$\mathbb{X}$	Sundering	$/-/,/7/,/+/,/2/$	$\mathbb{X}$
4	$/7/,/2/$	$\mathbb{X}$	Existential shift	$7,2$	$\mathbb{N}$
5	$7,2$	$\mathbb{N}$	Ordering the numbers	$7 > 2$	$\mathbb{N}$
6	$7 - 2$	$\mathbb{N}$	Subtraction over $\mathbb{N}$	$5$	$\mathbb{N}$
6	$5$	$\mathbb{N}$	Existential shift	$/5/$	$\mathbb{X}$
8	$/-/,/5/$	$\mathbb{X}$	Concatenation	$/-5/$	$\mathbb{X}$
9	$/-5/$	$\mathbb{X}$	Existential shift	$-5$	$\mathbb{Z}$

Once the operational activity has been described in terms of inputs, outputs, objects and operations, the teacher's procedure can be compared to the elements of the mathematical encyclopaedia which are activated by the particular topic.



**Figure 4.3** Existential map based on integer addition procedure used to compute  $-7 + 2$

#### 4.2.1.3 Activation of the mathematics encyclopaedia

Once descriptions of operational activity have been generated and summarised in tables and diagrams, I describe the elements of the mathematics encyclopaedia<sup>20</sup> activated by the particular mathematical topic. As discussed in Chapter Three, Davis (2010c) highlights the fundamental feature of mathematics – “operations that populate mathematics are functions” (Davis, 2010c: 4). He emphasises this because it is this which brings stability to mathematics, due to the stable way in which functions behave (TP11). Davis (2010a) also emphasizes the need for mathematical procedures to preserve the level of value of an expression, despite transformations at the level of expression (where value refers to a particular element of the co-domain of the procedure). In other words, identity must be preserved at the level of value while difference is allowed and required at the level of expression. Any legitimate procedure involves transformations at the level of expression which do not alter the implicit value of the expression. Thus in order for a procedure to be valid, it must preserve value despite transformations at the level of the expression.

These essential properties of mathematics cannot change as mathematics moves from the field of production to the fields of recontextualisation or reproduction. It is this principle upon which Davis’ analytical method is based, and which enables comparison of the resources or elements within the mathematics encyclopaedia required for each topic within school mathematics, and the mathematics produced in the classroom. When identifying the operational resources within the mathematics encyclopaedia, the method entails focusing on descriptions of objects and processes and of relations between such, as well as propositions in the form of statements and formulae.

<sup>20</sup>For the purposes of my study, in order to identify the elements of the encyclopaedia in the field of production for the topics covered in the lessons under analysis, I use the work of Stewart & Tall (1977), Courant and Robbins (1941) and Mac Lane (1986), as well as reference books such as the Princeton Companion to Mathematics (2008). These outline the foundations and structure of the mathematical body of knowledge.

In addition to drawing on the elements of the encyclopaedia in the field of production, I also look at what is activated in the encyclopaedia at the level of recontextualisation. This is necessary in order to take into account the grade-specific constraints of the particular pedagogic situation. As described by Bernstein (1996), the recontextualising field consists of the official recontextualising field (ORF) as well as the pedagogic recontextualising field (PRF)<sup>21</sup>. For the purposes of my analysis, I consider the ORF to be represented by curriculum documents. As part of my analysis I thus consult the National Curriculum Statement for Further Education and Training Mathematics (DoE, 2003) in order to identify what is activated there for each mathematical topic encountered in the lessons under analysis. I also refer to the textbooks which are used, and examine what is activated in these texts with respect to the particular topics. The texts are an element of the PRF. Taken together, the curriculum documents and, where applicable, the texts used for teaching, give a general picture of the elements of the mathematics encyclopaedia activated within the curriculum for each topic<sup>22</sup>.

A key feature of the recontextualising field is the insertion of the learner into the presentation of mathematics (TP4). In the field of production of mathematical knowledge, there is no reference to the learner of mathematics, but when it comes to recontextualisation the mathematical content is tied to the learner. This is evident from general statements found in curricula, such as “Mathematics is a distinctly human activity practised by all cultures” (DoE, 2003: 9) and “in an ever-changing society, it is essential that all learners ... acquire a functioning knowledge of the Mathematics that empowers them to make sense of society” (DoE, 2003: 10), as well as specific statements about mathematical content, such as “learners should be able to ... calculate confidently and competently with and without calculators, and use rational and irrational numbers with understanding” (DoE, 2003: 10) and “learners will ... explore real-life and purely mathematical number patterns and problems which develop the ability to generalise, justify and prove” (DoE, 2003: 12). Many textbooks contain similar statements which incorporate the learner into descriptions of mathematical content. Thus learner behaviour and identity are entwined with mathematical content as presented in mathematics curricula and texts for teaching, which insert the learner as a central part of the story. I will return to this feature of the recontextualising field later as it relates to my focus on the relations between the constitution of mathematics and learner identification.

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<sup>21</sup> The ORF is “created and dominated by the state and its selected agents and ministries” (Bernstein, 1996: 33) and the PRF consists of “pedagogues in schools and colleges, and departments of education, specialised journals, private research foundations” (Bernstein, 1996: 33).

<sup>22</sup> This use of policy documents and texts for teaching is also reminiscent of the notions of the intended and implemented curricula, mentioned in Chapter Three, where the intended curriculum would refer to official documents such as the National Curriculum Statements and the implemented curriculum would refer to the texts for teaching.

To illustrate the process of activating the topic-specific elements from within the mathematics encyclopaedia at the levels of production and recontextualisation, I continue with the example of the addition of integers.

From the field of production, the operational resources needed in order to do these calculations are the properties of the integers. Multiplication over the integers,  $(\mathbb{Z}, \times)$ , and addition over the integers,  $(\mathbb{Z}, +)$ , have the operator properties listed in Table 4.3.

**Table 4.3** Operator properties of  $(\mathbb{Z}, +)$  and  $(\mathbb{Z}, \times)$

Axioms	Properties
$\forall a, b, c \in \mathbb{Z}, a + (b + c) = (a + b) + c$	Associativity of addition
$\forall a, b, c \in \mathbb{Z}, a \times (b \times c) = (a \times b) \times c$	Associativity of multiplication
$\forall a, b \in \mathbb{Z}, a + b = b + a$	Commutativity of addition
$\forall a, b \in \mathbb{Z}, a \times b = b \times a$	Commutativity of multiplication
$\forall a, b, c \in \mathbb{Z}, a \times (b + c) = (a \times b) + (a \times c)$	Distributivity of multiplication over addition
$0 \in \mathbb{Z}$ and for $\forall a \in \mathbb{Z}, a + 0 = a = 0 + a$	Existence of additive identity
$0 \neq 1$ and for $\forall a \in \mathbb{Z}, a \times 1 = a = 1 \times a$	Existence of multiplicative identity
$\forall a \in \mathbb{Z}, \exists (-a) \in \mathbb{Z}$ such that $a + (-a) = 0$	Existence of additive inverses

(adapted from Stewart & Tall, 1977)

These axioms are the operator resources which are available for the addition and subtraction of integers and are the ground upon which we would expect any procedure for adding or subtracting integers to rest.

From the field of recontextualisation, the first mention of integers in the General Education and Training Revised National Curriculum Statement for Mathematics is in a grade 7 assessment standard – “the learner counts forwards and backwards ... in integers for any intervals” (DoE, 2002: 68). For both grades 7 and 8, the curriculum list integers among numbers which the learner “recognises, classifies and represents ... in order to describe and compare” (DoE, 2002: 68). Another grade 7 assessment standard states that the learner “estimates and calculates by selecting and using operations appropriate to solving problems that involve ... multiple operations with integers” (DoE, 2002: 70). Integers are thus assumed knowledge by grade 10 level, and are referred to as such in FET assessment standards for grade 10, for example, learners should be able to “establish between which two integers any simple surd lies” (DoE, 2003: 16).

The grade 10 textbook used in this classroom revises integers – “the set of integers includes the whole numbers and negative numbers” (Laridon et al, 2005: 8). Number lines are central to the

textbook's discussion of integers and are used to show that "positive numbers are to the right of zero and the negative numbers are to the left of zero" (Laridon et al, 2005: 8). The notion of additive inverses is introduced through a discussion of integers and their "opposites", which are explained as being "the same distance from zero on the number line but in opposite directions" (Laridon et al, 2005: 8). A method for adding integers is given using a number line – "start at the first integer. If you are adding a positive integer, move to the right on the number line. If you are adding a negative integer, move to the left." (Laridon et al, 2005: 9). One of the examples done using the number line method is  $-3 + 7$ . The commutativity of addition over the integers is not mentioned as a possible operatory resource, where  $-3 + 7 = 7 + (-3)$ .

This example illustrates how I activate elements from within the mathematics encyclopaedia at the levels of production and recontextualisation in order to enable secondary data production.

#### 4.2.1.4 Summary of primary data production process

The following is a summary of the production of a description of the operational activity of a particular lesson:

- (i) Segment the lesson into evaluative events.
- (ii) Describe the operational activity of the teacher in terms of domain(s), operation(s) and co-domain(s) that are implicated in each procedure.
- (iii) Identify the stated mathematical topic of each event and describe the elements of the mathematics encyclopaedia activated by that particular topic within the fields of production (the mathematical body of knowledge) and recontextualisation (curriculum documents and textbooks).

### 4.2.2 Secondary data production

The object of the secondary data production of this study is to fashion resources which relate the description of operational activity generated during the primary data production process to the theoretical and analytical resources introduced in Chapter Three. The analytical tools introduced here are based on the theoretical framework developed in previous chapters. There are two levels of secondary data production – the first relating to the realisation of content and the second to the regulation of the learner.

#### 4.2.2.1 The realisation of content

In a similar way to Freud's use of the notion of parapraxis as an analytical tool (TP16), part of my analysis involves the identification of points of difference from the mathematics encyclopaedia, as explained in Chapter Three, in order to get insight into implicit features of the pedagogy. It is

important to emphasise once again that because of the way in which the basic axiomatic features of mathematics function as an open text, as discussed in Chapter Three, there are many ways of producing a particular mathematical result (TP13). But the object of this analysis is the identification of instances where something different from that which is indexed by the topic (from the point of view of the mathematics encyclopaedia) emerges in the pedagogic situation. This suggests the possibility of something different from the announced or intended topic being taught and learnt. As explained in Chapter Three, the intended topic is the topic announced by the teacher in a particular evaluative event, together with the elements of the mathematics encyclopaedia, the content, which this topic recruits (TP14). The realised topic is what is being taught and learnt, or what content is being drawn together in the pedagogic context, which can be read off the operational activity (TP15).

Once a description of the operational activity in a particular evaluative event has been produced, and the elements from the mathematics encyclopaedia which are activated at the levels of production and recontextualisation are identified, I ask the following questions about the objects and operations within the event in order to identify whether the operational activity differs in any way from the operational resources indexed by the particular topic in the mathematics encyclopaedia:

- (i) Are the manipulations making up the event functions and thus operations?
- (ii) Is identity at the level of value preserved when carrying out transformations?
- (iii) Do the objects being operated on conform to their namesakes in the encyclopaedia? For example, the manner in which the idea of *function* is used in school contexts often violates the idea, despite enabling learners to reach the correct solutions.
- (iv) Are the manipulations used familiar mathematical operations which are indexed by the topic? (Davis, 2010a, 2010b, 2010c). If the manipulations are unfamiliar, they may still be functions, and thus operations (if they have a unique, stable output for each input) or they may be pseudo-operations. For example, the pseudo-operation of *sundering*, as described by Davis (2009) and cited by Jaffer (2009), does not have unique outputs for each input and cannot be classified as a function.
- (v) Do the manipulations used in the event require domains of objects different from those which are mathematically necessary to the specific topic? (Basbozkurt, 2010, Davis, 2010a). For example, the way in which integer addition procedures involve shifting to natural numbers, as introduced earlier in this chapter and discussed in more detail here.

These reveal potential points of difference at two levels – firstly, between the intended topic (which activates elements of the mathematics encyclopaedia, as described in the previous section) and the realised topic. Secondly, between the realised topic and the encyclopaedia - if the realised topic differs from the intended topic, it may or may not correspond with the propositions, objects and processes of the mathematics encyclopaedia, as Figure 4.4 illustrates. Note that the third quadrant in the matrix does not exist in pedagogic contexts, as it represents the production of mathematics which does not exist within the encyclopaedia, and thus is excluded from pedagogic contexts which involve the recontextualisation and reproduction of mathematics.

		In relation to the intended topic	
		<i>corresponds with the intended topic</i>	<i>does not correspond with the intended topic (there is a shift in the topic)</i>
In relation to the encyclopaedia	<i>corresponds with the mathematics encyclopaedia</i>	I	II
	<i>does not correspond with the mathematics encyclopaedia</i>	III	IV

Figure 4.4 Matrix used to classify the realisation of content in an evaluative event

This analysis gives us a picture of the constitution of mathematics which emerges within a particular evaluative event.

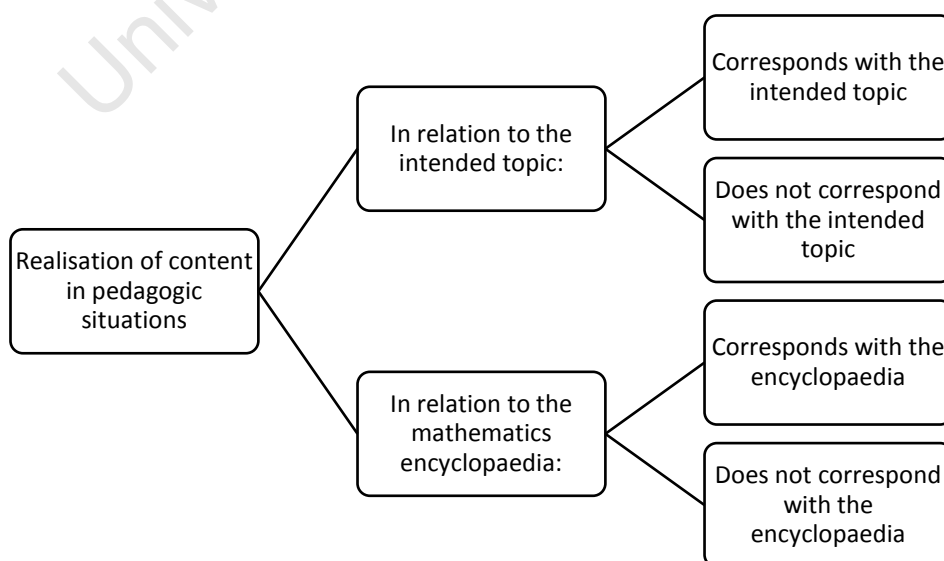


Figure 4.5 The realisation of content in pedagogic situations

To illustrate this process I return to the integer addition example. The operator properties of addition over the integers are not explicitly drawn on in the procedure outlined earlier in this chapter. In order to add two integers such as negative seven and two using the elements of the encyclopaedia which were outlined earlier, we could exploit the existence of additive inverses for all integers. This would involve replacing 2 with  $7 + (-5)$ , which would yield  $-7 + 7 + (-5)$ , from which the answer of  $-5$  is clearly obtained. This approach draws explicitly on the operator properties of addition over the integers.

When we compare the operational activity with the elements activated by the topic within the mathematics encyclopaedia we find some key differences. The objects being operated on are unfamiliar in that they are not usually associated with the topic of integer addition. There is a shift from the domain of the integers to the domain of the natural numbers, as seen in Figure 4.2. Although natural numbers are part of the set of integers, in this example and others in the context, the learners do not engage with them as integers but as something separate from integers, thus this is considered a shift in the domain (ETP4, Basbozkurt, 2010 and Davis, 2010a). In order to enable this shift in domain, the initial objects (integers) are changed into characters which can be taken apart and treated as natural numbers. The existential shift which has taken place enables the process of taking numbers apart into characters. This breaking apart is an unfamiliar manipulation which is not found in the mathematics encyclopaedia – that of *sundering* (discussed in detail in Jaffer, 2009). Although it is possible to describe *sundering* as a function in some situations (such as in the language of computer programming, where the primary resources are functions), in this case *sundering* cannot be described as an operation as it does not necessarily have a stable, unique output for any input, and thus is not a function. For example, the string  $/-7 + 2/$  could be *sundered* to yield any of these combinations of character strings:  $/-7/$  and  $/+2/$ ,  $/-/$  and  $/7 + 2/$ ,  $/-7 +/$  and  $/2/$ , or  $/-/$ ,  $/7/$ ,  $/+/$  and  $/2/$ . The concatenation which occurs when the negative sign is rejoined to the final answer (referred to as ‘giving the sign of the bigger number to the answer’), is more stable than *sundering*, as given two character strings,  $/-/$  and  $/5/$ , there is only one way of concatenating them, yielding  $/-5/$ <sup>23</sup>. Concatenation in this context is thus a function and therefore an operation, although not a familiar one when dealing with the topic of integer addition from the perspective of the mathematics encyclopaedia.

In summary, comparison of the procedure for adding integers with the elements from the mathematics encyclopaedia has shown a difference at two levels. Firstly, the content is realised in a

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<sup>23</sup> This is assuming that concatenation is an ordered operation, as if it is not an ordered operation then  $/-/$  and  $/5/$  can be joined in two ways:  $/-5/$  and  $/5-/$ . Thus concatenation is only unambiguous and stable if it is ordered.

way which does not correspond with the intended topic as natural numbers are being operated on instead of integers. This shift in domain from  $\mathbb{Z}$  to  $\mathbb{N}$  renders the procedure more abstract than it need be by acting on elements of  $\mathbb{Z}$  at a distance, as Davis (2010a) highlights. Secondly, the content is realised in a way that does not correspond with the mathematics encyclopaedia, as seen by the use of *sundering*, which is not a function in this context, indicating that the essential property of mathematics that all its operations be functions is not adhered to in this procedure. Thus this evaluative event would fall into quadrant IV in the matrix in Figure 4.4. Analysis of this procedure has shown that the propositional ground of the integers with their operatory properties is not the primary regulative resource for this procedure. This prompts the question of what it is that is regulating learners in carrying out this procedure, and why it is that these unfamiliar objects and operations are selected pedagogically to make up this particular procedure. Questions such as these form part of the next component of my analysis.

#### **4.2.2.2 The regulation of the learner**

In order to explore the regulation of the learner in a particular evaluative event, I refer to Davis' (2010a, 2011b) discussion of primary regulative orientations, Davis, Adler & Parker's (2007) discussion of appeals made to an authorising ground, and the notion of necessity, all introduced in Chapter Three, in order to identify what the evaluative criteria primarily appeal to during a particular event, and whether the regulation of learners is predominantly under the aspect of the Imaginary or the Symbolic (TP8 - 10). It is possible, and even likely, that there will be more than one regulative resource appealed to, and thus more than one register operative in the evaluative criteria generated within a particular evaluative event. But my analysis aims to identify what is primarily appealed to as authorising ground for the criteria in a particular evaluative event in order to determine whether the regulation of the learner in that event is predominantly under the aspect of the Imaginary or the Symbolic.

As an example of identifying the regulative resources which are appealed to as authorising ground for evaluative criteria, I introduce an evaluative event focused on the procedure for sketching a line using the "gradient-intercept method", in school P1. In this event the teacher appeals to the mathematical proposition that only two points are necessary in order to sketch a line as part of an explanation of why the "gradient intercept method" is easier and quicker than any other methods for sketching lines. His reason for referring to the gradient intercept method as easier is that "the modern people, they want to do things very quickly" – his explanation of why only two points are needed to sketch a line is related to speed and ease of execution as opposed to the definition of a line. This suggests that although the proposition that a unique line can be drawn through any two points is implicit in the teacher's explanation, the primary regulative resource underlying his

explanation is an appeal to ease and speed to encourage learners to use the gradient-intercept method – the primary regulation of the learner in this event thus situates necessity external to mathematics.

An example taken from the work of Venkat & Adler (2010: 6), also involving sketching lines, shows a similar privileging of method, but a different method is privileged. In this lesson, the teacher (Nash) has just taught the dual-intercept method:

Learner 2: Is this the simplest method sir?

Nash: The simplest method and the most accurate...

Learner 4: Compared to which one?

Nash: Compared to that one (points to the calculation of the previous question where the gradient-intercept method was used) because here if you make an error trying to write it in  $y$  form ... that means it now affects your graph ... whereas here (points to the calculations he has just done on the dual intercept method) you can go and check again ... you can substitute ... if I substitute for 2 in there (points to the  $x$  in  $3x - 2y = 6$ ) I should end up with 0.

This is interesting as Nash selects a different method as the simplest and most accurate – the dual intercept method, which he directly contrasts with the gradient-intercept method, the method privileged by the teacher in school P1. This illustrates the subjectivity involved in the privileging of particular methods – a method privileged by one teacher may be described by another as more complicated, but from the point of view of the mathematics encyclopaedia, both methods are equally valid and accurate. One method of sketching a line should not be “easier” or more accurate than another, as for any line only two points are required in order to sketch the line. This suggests that there is something else behind teachers’ privileging of particular methods. The privileging of one method, due to it being easier, quicker or more accurate than other methods could be seen as an attempt to appeal to learners on a subjective level so that they identify with the privileged method. In the example from School P1, it seems that the teacher wants the learners to identify with this method so that they will be able to draw the graph despite their lack of knowledge about the definition and properties of a line. The teacher is regulating the activity of the learners through privileging one method over another, rather than through the propositional ground underlying the sketching of a line, and thus situating necessity external to mathematics, which suggests that the primary regulation of the learner in this event recruits elements of the Imaginary.

Before any further discussion on the way in which I go about using the Imaginary and the Symbolic registers in my analysis, I highlight a central point raised in Chapter Three – the distinction between the place available within the symbolic network and the individual occupying that place. In terms of mathematics, we described the mathematics body of knowledge with its sections of mathematical content using Lacan's description of an automaton. This smoothly functioning machine can be likened to the symbolic network, and the place available within that network to the mathematical propositions, definitions and processes available to carry out a procedure. As the previous chapters have explained, the goal of pedagogy is the reproduction of knowledge by the learner, and due to the way in which the extimate relation of the learner to mathematics potentially disrupts the functioning of the Symbolic automaton of mathematical content, the teacher, as the mediator between the learner and mathematics, reconstitutes mathematical procedures so as to re-assert the automaton and enable the reproduction of mathematical content by learners (TP1 - 3). This is the crucial point at which my analysis aims to describe the nature of the evaluative criteria and what is appealed to as authorising ground for these criteria – the point at which the evaluative activity of pedagogy attempts to find a way of realising the intended content in response to the extimate relation of the learner and mathematics. I use Lacan's registers as analytical tools to describe the regulation of the learner at this point as under the aspect of either the Imaginary or the Symbolic.

In order to identify which register is predominantly at play in the regulation of the learner during an evaluative event, and thus in the constitution of learner identification, we can assume that if substitutions of content are made in place of content indexed by the intended topic from the point of view of the encyclopaedia, then the intended topic is no longer acting as a primary regulative resource. In addition to this, if the realised topic does not correspond with objects, processes and propositions of the mathematics encyclopaedia then the encyclopaedia itself is no longer operating as a primary regulative resource. If we examine the justifications and explanations for decisions, statements and actions, we may find that these appeal primarily to extra-mathematical factors (such as ease, impending examinations, iconic and procedural features of solutions) – situating necessity within criteria generated by the teacher which are external to mathematics. This suggests that the regulation of the learner is under the aspect of the Imaginary. But if regulative resources primarily appeal to the propositional ground associated with the intended topic, thus situating necessity internal to mathematics, then the regulation of the learner in the event can be classified as under the aspect of the Symbolic. Using the number of appeals made to different factors is not necessarily a final determinant, but gives an overall picture of the nature of the regulation of the learner in an event. In my analysis, I count the appeals made to different factors but I also analyse the appeals in terms of the emphasis placed on them by teachers and in relation to the way in which content is

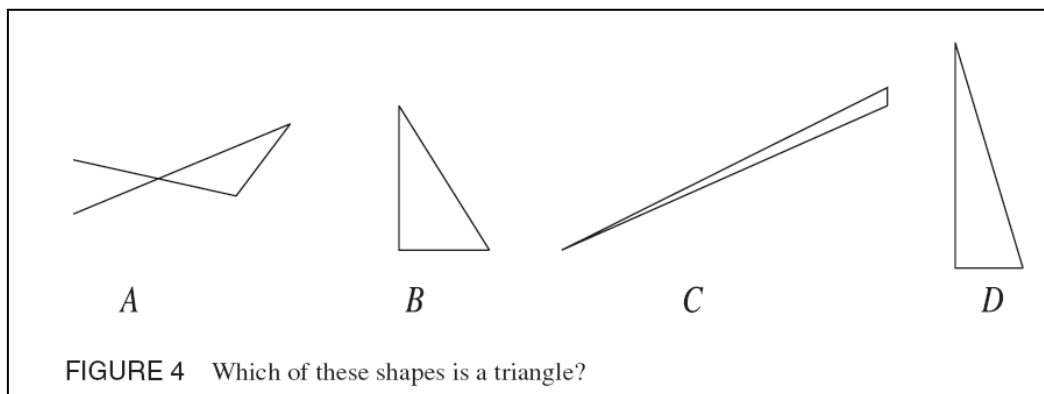
realised in order to determine where necessity is situated. Generally, when appeals are predominantly made to extra-mathematical factors as authorising ground, I conclude that necessity is situated external to mathematics and I describe the regulation of the learner as under the aspect of the Imaginary. When appeals are predominantly made to mathematical factors as authorising ground for criteria, then necessity is situated within mathematics and I describe the regulation as under the aspect of the Symbolic. There may be exceptions to this general principle, in which case I motivate these through further analysis of interactions between teachers and learners.

It can be argued that there are situations where necessity is situated external to mathematics, but where regulation is not necessarily Imaginary in nature. For example, instances where learners are distributed content which is not aligned with the topic as indexed by the encyclopaedia, resulting in Symbolic activity which is different from what is expected, but still Symbolic, and could be described as an alternate Symbolic system (Davis, personal communication). Despite this, I consider such instances as under the aspect of the Imaginary because at the point at which the substitution of content indexed by the topic with some other content takes place, it seems that the image of the learner, implied by the teacher's pedagogic practice, is regulating the replacements. This implied image suggests that the learner isn't capable of engaging with the intended content, so the teacher, in response to the extimate relation of the learner and mathematics in the pedagogic situation, transforms the intended topic into something which he/she expects the learners to be able to engage with. I describe this re-symbolisation of the content as under the aspect of the Imaginary because it is the implied image of the learner which is motivating and regulating the transformation.

To illustrate this, I complete the analysis for the integer addition example. In this event, appeals are predominantly made to the rules generated by the teacher ("so if the signs are the same, what do you do? You take the common sign and then you add. If the signs are not the same what do you do? You subtract ... But first you take the sign of the what? The sign of the bigger number") and not to the field of mathematics, as the propositional ground underlying the addition of integers is not appealed to, even implicitly, by the teacher, suggesting that necessity is situated external to mathematics. In addition to this, the way in which the teacher shifts the domain from integers to natural numbers, which learners are familiar with, suggests that the teacher does not expect the learners to be able to engage with the integers and replaces the content with something that the learners are expected to be able to engage with. The activity that results is Symbolic, but the re-symbolisation of content is in the direction of the Imaginary as the image of the learner implied by the pedagogy constitutes the learner as unable to engage with the integers and regulates the decision

made to replace or transform the content. The regulation of the learner in this event can thus be described as under the aspect of the Imaginary.

In contrast to this example, an extract (from Sfard, 2007) shows the operation of regulation which draws on the propositional ground underlying the topic. Prior to this exchange, the teacher had asked learners to identify which shapes in Figure 4.6 are triangles.



**Figure 4.6** Identifying triangles (extract from Sfard, 2007: 598)

One learner struggles to see that shape C is a triangle, and the teacher responds:

- Teacher: How do we know that a triangle ... whether a shape is a triangle? What did we say? What do we need in order to say that a shape is a triangle?
- Learner: Three points ... three vertices ... and ...
- Teacher: Three vertices and ...?
- Learner: Three sides.
- Teacher: And three sides. Good. If so, this triangle fits (*points to shape C*). Look, one side ... and here I have one long side, and here I have another long side. So we have a triangle here.
- Learner: And ... one vertex, and a second vertex, and a ... point?
- Teacher: Look here: one vertex, second vertex, third vertex.
- Learner: So it *is* a triangle.

(Sfard, 2007: 600, italics in original)

Here the evaluative criteria are focused on the propositional ground underlying the topic of triangles. The teacher explicitly appeals to the definition of a triangle in her explanation and then moves to the specific triangle in question, thus placing necessity within the field of mathematics. The evaluation operating in this situation does not transform the mathematical content in the

direction of the learner, but instead transforms the learner's notion of a "point" in the direction of the mathematical content, renaming it a "vertex". This suggests that regulation depends primarily on the Symbolic.

The analytical framework described in this chapter enables a discussion of the constitution of mathematics in the grade 10 mathematics lessons selected for description and analysis. My analysis of the realisation of content in the lessons enables me to describe the implied learner in terms of the operational resources required to work with the mathematics content as constituted in the pedagogic situation. My analysis of the regulation of the learner enables me to describe the implied learner in terms of the pedagogic practices. Once each event has been analysed in terms of the realisation of content and the regulation of the learner, I am in a position to discuss the relation between the constitution of mathematics and the constitution of learner identification. I conclude this chapter with a discussion of the reliability and validity of this study.

### ***4.3 Reliability and validity of my study***

The reliability of my study lies in its dependence on the essential properties of mathematics and the operational resources found in the mathematics encyclopaedia. These essential properties of mathematics cannot change as mathematics moves from the field of production to the fields of recontextualisation or reproduction, thus using them as a basis for my analysis offers stability and reliability to the results.

In terms of validity, my research cases consist of five schools whose learners are all from working-class backgrounds, thus I am not able to make conclusions about the co-constitution of mathematics and learner identification based on these cases alone. But my study does not aim to draw conclusions but to investigate what is entailed in the co-constitution of mathematics and learner identification in these schools, and to develop methodological resources which could be applied in other school settings. The validity of my study would be enhanced by triangulating the results through carrying out interviews with teachers and learners. This would be helpful to get insight into the reasons behind teachers pedagogical choices as well as the mathematical competence of the learners, which would strengthen my analysis. Due to the size of this project it was not possible to do this, but in a bigger project triangulation would enhance the analysis.

## Chapter 5

### Presentation of Results

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The next three chapters present and discuss the results of the data production and analysis. As discussed in Chapter Four, there are fifteen video-recorded grade 10 lessons in the information archive gathered from the five schools. I have analysed each lesson using the framework outlined in Chapter Four. As previously explained, the analysis of events is too extensive to include in the body of this project. Instead I summarise and discuss the results, giving examples, in this chapter and those which follow. The full analysis can be found in the appendices. I start this chapter with a brief description of the lessons in order to set the scene for the rest of the chapter and the next, focusing on the mathematical topics covered and the way in which time is used in the lessons. This is followed by a summary of the results of my analysis and, in Chapter Six, a discussion of these results. In Chapter Seven I focus on common pedagogic practices which have emerged from the analysis and discuss them in relation to the co-constitution of mathematics and learner identification.

#### **5.1 Describing the lessons**

##### **5.1.1 Mathematical topics and procedures**

As the elaboration of procedures, taught through the carrying out of worked examples, has been noted as the most common way of teaching in the five schools (Davis & Johnson, 2007, 2008; Jaffer, 2010), I have listed the primary stated mathematical topics or procedures in the fifteen analysed lessons in Table 5.1, showing how many times each procedure for a particular topic is carried out per school, as well as the number of times the correct solution is obtained and the number of errors occurring during the procedure. This refers to worked examples carried out on the board, either by teachers or by learners with teacher guidance.

From Table 5.1 we see that 93% of the solutions obtained are correct, and that 84% contain no errors. But this does not capture what is going on at the level of the operational activity. Before examining this more closely, I give an overview of how time is used in the fifteen lessons.

**Table 5.1** The procedures carried out in the fifteen grade 10 lessons

School	Mathematical topic/procedure	No of times carried out	No of times correct solution obtained	No of errors made during use of procedure <sup>24</sup>
P1	Solving linear inequalities	6	6	0
P1	Sketching lines	2	2	0
P2	Converting recurring decimals to common fractions	5	5	5
P2	Expanding exponential expressions	7	6	1
P3	Simplifying exponential expressions	8	7	1
P3	Simplifying exponential expressions involving factorizing	3	3	0
P6	Finding the difference between terms of a number pattern	2	2	0
P6	Finding the general term of a linear pattern	5	4	1
P6	Using the general term to find any term	3	3	0
P6	Solving for the dependent variable using the general term	1	0	1
P7	Simplifying an exponential expression	7	7	0
P7	Solving an exponential equation	8	8	0
TOTAL		57	53	9

### 5.1.2 Time use in the lessons

The mathematical topics making up the lessons are usually announced by the teacher at the start of an evaluative event, followed by the exposition of a procedure through a number of worked examples. A common practice in all lessons is working through solutions to homework or classwork questions – this is done by either calling learners to the board to write the solutions, which the teacher accepts or redoes with the class, or by the teacher working through the questions. The use of time in the fifteen lessons can be broken up into categories, following earlier work done by Davis & Johnson (2008).

**Exposition of mathematical principles** – this consists of time spent on the exposition of the mathematical ideas, principles, propositions and definitions that ground the procedures being rehearsed.

<sup>24</sup>This refers to the point at which an error is made. If the rest of the solution is correct based on the error, then there is only one error. But if another error is made later on in the solution, but the remainder of the solution after that second error is made is correct, there are two errors, etc – based on the principle of continuous accuracy.

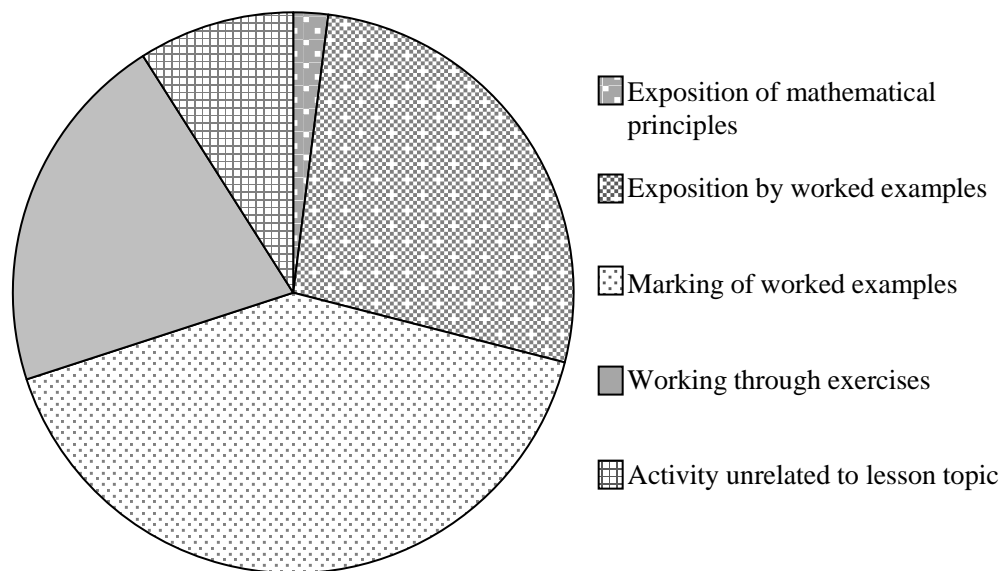
**Exposition by worked examples** – this consists of teachers working through examples which learners have not seen before (i.e. they have not been set as homework or classwork questions).

**Marking of worked examples** – this involves teachers calling learners to the board to work through or write the solutions to homework or classwork questions. If a question is incorrect the teacher either calls another learner to redo it or redoes it him/herself. In some lessons, teachers work through homework or classwork questions themselves.

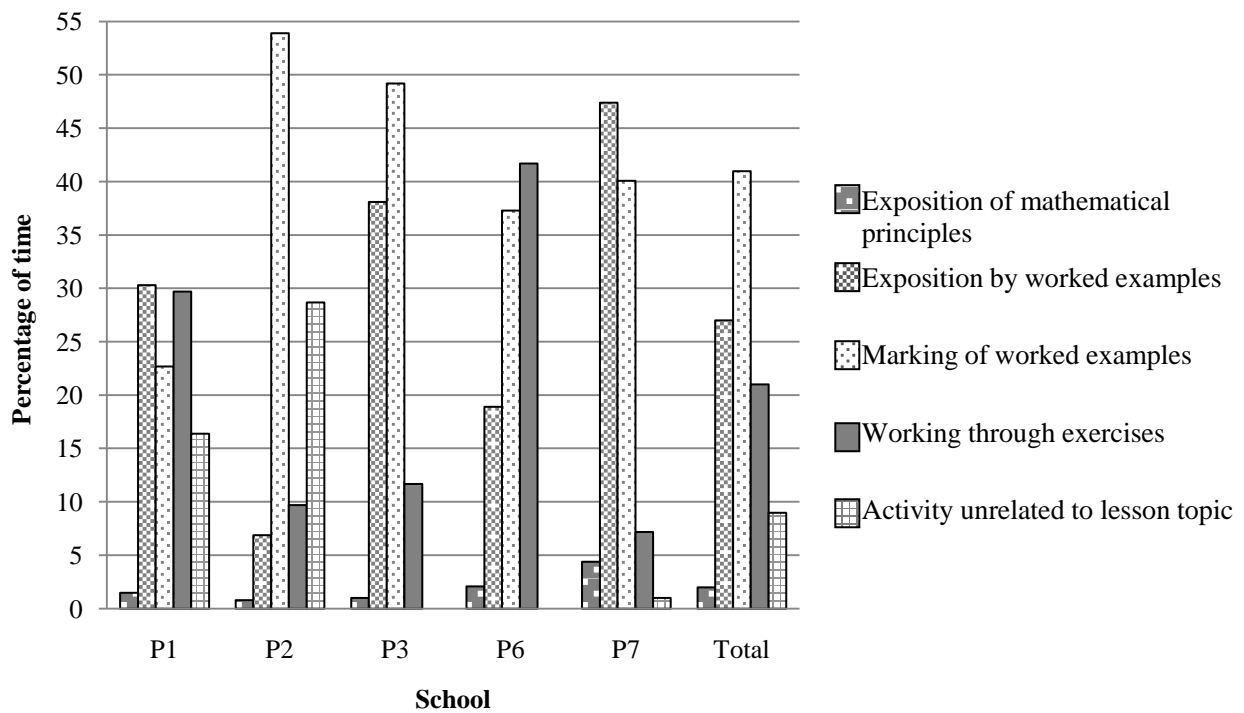
**Working through exercises** – this involves learners working on an exercise, alone or in groups. The teacher sometimes walks around the class observing, marking and answering questions.

**Activity unrelated to lesson topic** – this includes times when the teacher arrives late for the lesson, gives instructions, settles the class etc.

Figures 5.1 and 5.2 show overall time use in the fifteen lessons and time use per school respectively (the corresponding table is in Appendix Seven).



**Figure 5.1** Pie chart showing overall time use in the lessons



**Figure 5.2** Bar graph showing time use per school

From this data we see that in the fifteen lessons under analysis, content is primarily elaborated by the carrying out of procedures, either as exposition or marking of worked examples. These two categories make up 68% of the time use in the lessons. This corresponds with the findings of Davis & Johnson (2008) that mathematical teaching and learning happens “almost exclusively through the elaboration of worked examples to demonstrate the application of standard procedures” (Davis & Johnson, 2008: 2). The amount of time spent on the exposition of mathematical ideas, definitions and propositions (2%) suggests that the propositional ground underlying the elaborated content is either implicit or absent from the evaluative criteria generated in these schools, which has implications for the realisation of content.

## **5.2 The realisation of content**

My analysis compared the intended topic of each event with the operational activity at two levels in order to discuss the way in which content is realised. As previously explained, the intended topic is that which is announced by the teacher, together with the elements of the mathematics encyclopaedia (the content) which this topic recruits. The realised topic consists of the objects and operations which are drawn together in the pedagogic context, and represents what is actually being taught and learnt. If the realised topic differs from the intended topic, I also compare the realised topic with the propositions, processes and objects found in the mathematics encyclopaedia in order

to determine whether the realised topic still corresponds with the encyclopaedia. I classified each event according to the matrix in Figure 5.3.

<b>Realised topic</b>		<b>In relation to the intended topic</b>	
		<i>corresponds with the intended topic</i>	<i>does not correspond with the intended topic (there is a shift in the topic)</i>
<b>In relation to the encyclopaedia</b>	<i>corresponds with the mathematics encyclopaedia</i>	I	II
	<i>does not correspond with the mathematics encyclopaedia</i>	III	IV

**Figure 5.3** Matrix used to classify the realisation of content in an evaluative event

An event falling into quadrant I contains a recruitment of appropriate elements from the encyclopaedia, contains no shifts in the domain being operated over and does not violate any mathematical propositions, processes or objects, i.e. all operations used in the event are functions, identity is preserved at the level of value, although it may change at the level of expression, and there are no mathematical errors which are uncorrected. When one or more shifts in the domain take place in a particular event, or when the operations used are unfamiliar ones not usually indexed by the topic from the point of view of the encyclopaedia, I consider this an indication that the intended topic is not realised, thus classifying the event in either quadrant II or IV. If despite this, the operations are still functions, identity at the level of value is preserved and the realised content corresponds with mathematical propositions, processes and objects, I classify the event in quadrant II. But if the realised content does not correspond with mathematical propositions, processes and objects, for example if not all of the operations used are functions, or if identity at the level of value is not preserved, I classify the event as falling into quadrant IV.

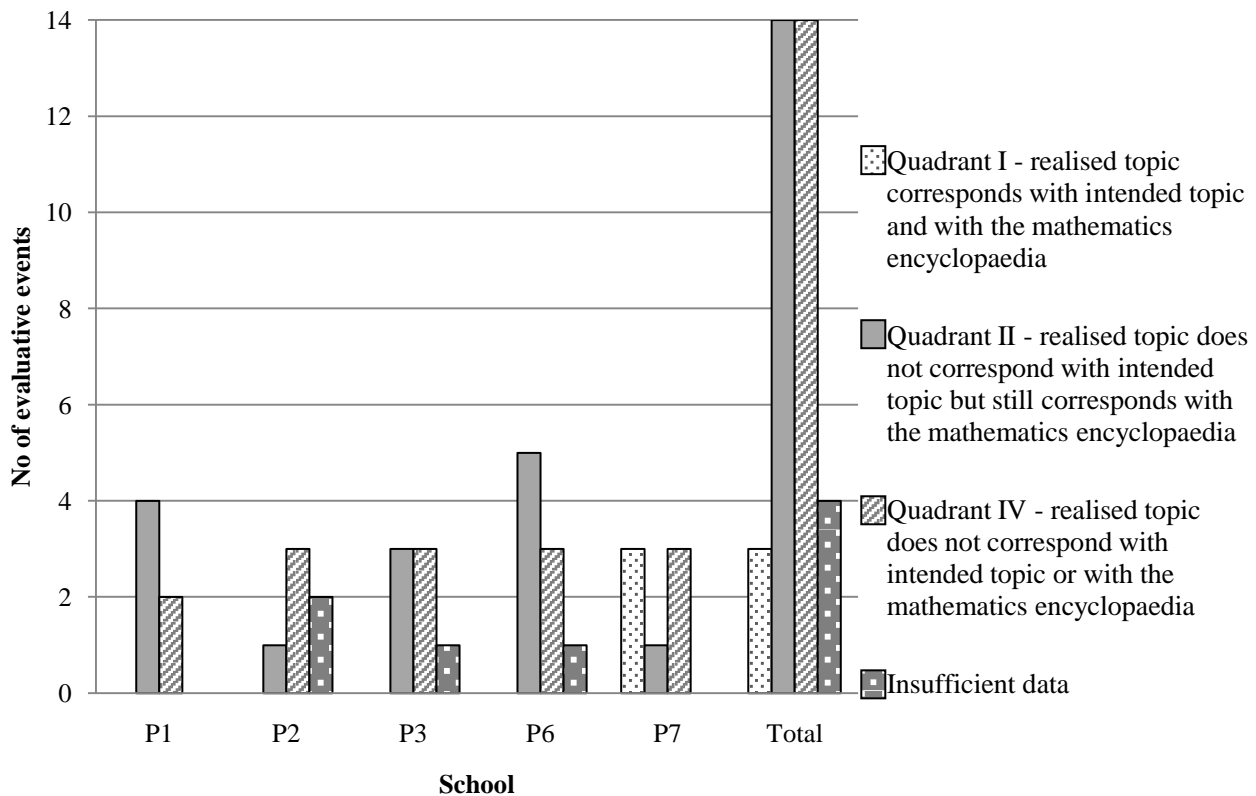
In events in which learners work on an exercise, with minimal interaction with the teacher, but we have access to examples of a few learners' work on the video-recording, I draw tentative conclusions about the realisation of content based on these examples. It has been noted in this context that the mathematics produced by learners within one class or school is similar, possibly due to the way in which teachers use a communalising pedagogic approach (Dowling & Brown, 2009; MacKay, 2009, 2010; Matobako, in press), thus making it possible to draw conclusions about the realisation of content in the event based on a few learners' work. It is possible that the learners whose work is shown represent exceptions to the mathematics produced by other learners, but this

is unlikely in this context where uniformity amongst learners is common. In a few evaluative events it is not possible to describe the way in which content is realised, as learners work on an exercise with little or no interaction with the teacher and we do not have access to the work of any learners. Such events are recorded as containing insufficient data. The results of this part of my analysis are recorded in Table 5.2 and Figure 5.4.

**Table 5.2** Classifying evaluative events in terms of the realisation of the topic

School	Lesson	No of EEs	No of EEs in quadrant:				Insufficient data
			I	II	III	IV	
P1	L1	2		1		1	
	L2	2		1		1	
	L3	2		2			
P2	L1	3				1	2
	L2	1				1	
	L3	2		1		1	
P3	L1	3		1		2	
	L2	2		1		1	
	L3	2		1			1
P6	L1	3		1		2	
	L2	3		2		1	
	L3	3		2			1
P7	L1	3	2	1			
	L2	2	1			1	
	L3	2				2	
TOTAL	15	35	3	14		14	4
	%		9	40		40	11

From these results we see that in 80% of evaluative events in the fifteen lessons, the realised topic does not correspond with the intended topic (quadrants II and IV), i.e. the content which emerges in the event differs from the content indexed by the topic from the point of view of the encyclopaedia. In 40% of evaluative events the realised topic still corresponds with the propositions, definitions and objects found in the mathematics encyclopaedia (quadrant II), and in 40% of events the realised topic does not correspond with the intended topic or the propositions, definitions and objects of the encyclopaedia (quadrant IV). In 9% of events the content which is realised corresponds with both the intended topic and the propositions, processes and objects in the mathematics encyclopaedia (quadrant I).



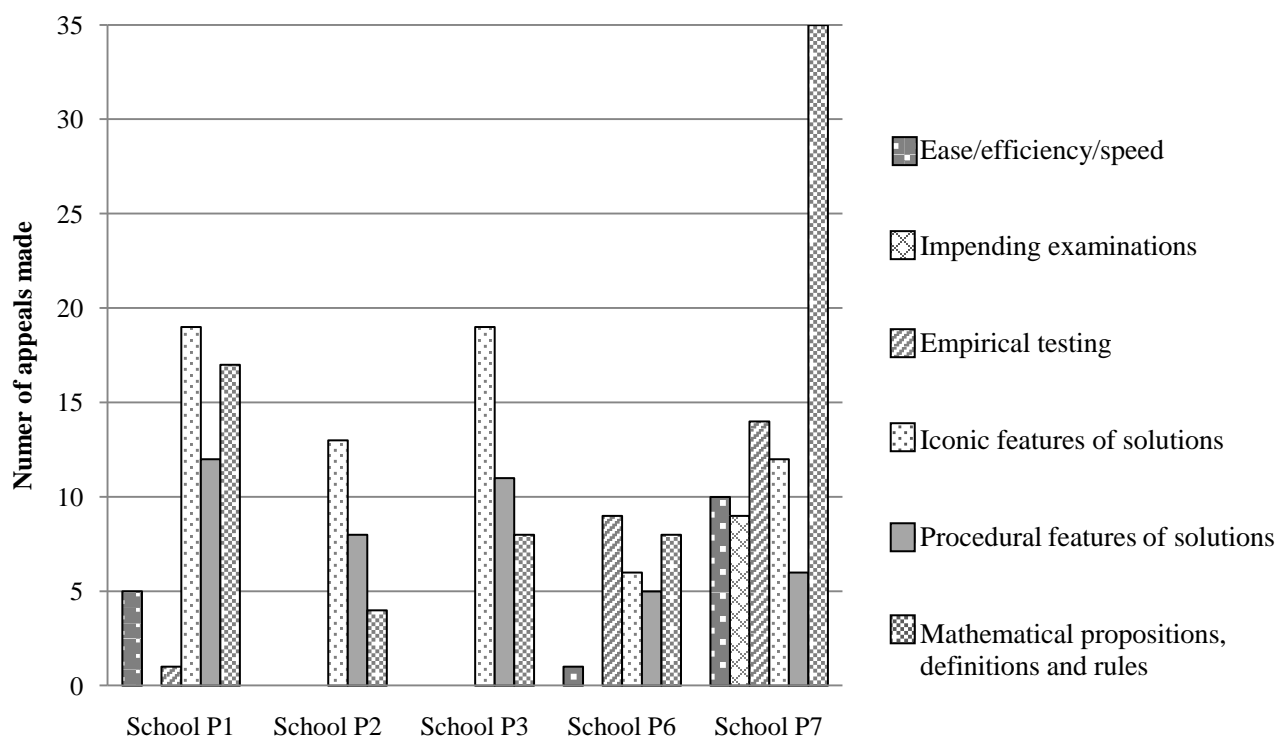
**Figure 5.4** Bar graph showing evaluative events in terms of the realisation of content

### 5.3 The regulation of the learner

In order to determine whether the regulation of the learner in each event is under the aspect of the Imaginary or the Symbolic, I analysed interactions between teachers and learners (such as instructions, motivations, explanations, justifications, and responses to learners' questions) to determine what is appealed to as authorising ground to explain or justify statements, actions and decisions and thus whether necessity is located internal or external to the field of mathematics. To do this, I counted the number of appeals made to different factors. I found that appeals were made to the following as authorising ground for criteria:

- Ease/speed/efficiency
- Impending examinations
- Procedural features of solutions
- Iconic features of solutions
- Empirical testing
- Mathematical propositions, processes, rules and objects

I counted the number of appeals made in each evaluative event to these categories. The results of this analysis per school are shown in Figure 5.5.



**Figure 5.5** Bar graph showing appeals made to authorizing ground

Where appeals to factors external to mathematics are predominantly used to authorise procedures and decisions, I conclude that necessity is located external to the field of mathematics and the regulation is classified as under the aspect of the Imaginary. Where appeals are predominantly made to mathematical propositions, processes and objects in order to authorise procedures and decisions, I conclude that necessity is located internal to mathematics and regulation of the learner is classified as under the aspect of the Symbolic, as discussed in previous chapters. In all events there are instances where appeals are made to more than one factor. In such cases, I looked at what is predominantly appealed to or decided upon in order to classify the regulation of learners in the event as predominantly under the aspect of the Imaginary or the Symbolic. For example, in an event in school P7 where the intended topic is the solving of exponential equations, the teacher appeals twice to the need for learners to memorise the prime table, which suggests that he may be situating necessity external to mathematics and encouraging learners to rely on rote learning rather than an understanding of prime factorisation. But despite this, his explanation of solving exponential equations repeatedly appeals to and is explicitly grounded on the mathematical principle of equality, and implicitly on the importance of maintaining identity at the level of value, despite changes at the level of expression. At one point a learner asks if they can ‘cancel’ the twos in the equation  $2^x = 2^4$ .

Learner: Two divided by two and on the right hand side also two divided by two and you cancel both.

Teacher: Okay. That is mathematically incorrect in a sense. If you have two  $x$  equal to something. Right? And two is a co-efficient then you can divide by two ... yes. Okay? But the two to the power of  $x$  equal to two to the power of four you can't divide by two. 'Cause if you divide by two just see what will happen here ... If you have two to the power of four and you divide by two. What's the answer? The twos won't cancel. It won't cancel. You'll have two to the power of four minus one. Isn't that correct? The law states ... the index law states that if you have the same top same bottom you must subtract your indices.

Transcript School P7 Grade 10 Lesson 1

Here, and at seven other points in the event, the teacher appeals to the mathematical propositions and processes underlying the procedure. I thus classified the regulation of the learner in this event as under the aspect of the Symbolic, despite the emphasis placed on memorising the prime table on two instances in the event.

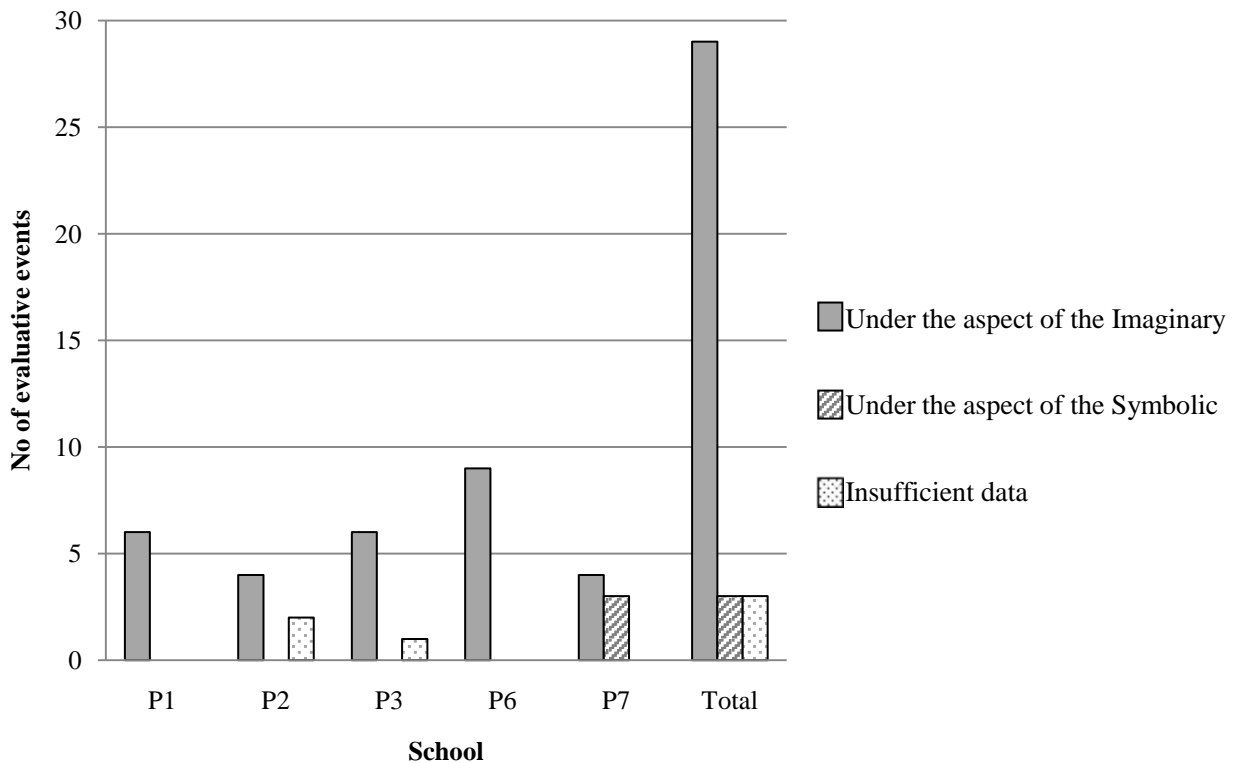
In events in which learners work on exercises without interaction with the teacher, I assume that learners are regulated by the teacher's activity from the previous event (usually the exposition of a procedure through worked examples, which learners are now carrying out). Thus although there is not much data for analysis in such events, my conclusion about the regulation of the learner will draw on the previous event's analysis. Table 5.3 and Figure 5.6 show the results of this part of my analysis.

From these results we see that in 82% of evaluative events in the fifteen lessons the regulation of the learner is predominantly under the aspect of the Imaginary, while in only 9% is the regulation under the aspect of the Symbolic. The other 9% contained insufficient data.

Now that I have presented a summary of the results of data production and analysis, I move on to a more detailed discussion of these results in the next two chapters.

**Table 5.3** Classifying evaluative events in terms of the regulation of the learner

School	Lesson	No of EEs	No of EEs in which regulation is under the aspect of the:		
			Imaginary	Symbolic	Insufficient data
P1	L1	2	2		
	L2	2	2		
	L3	2	2		
P2	L1	3	1		2
	L2	1	1		
	L3	2	2		
P3	L1	3	3		
	L2	2	2		
	L3	2	1		1
P6	L1	3	3		
	L2	3	3		
	L3	3	3		
P7	L1	3	1	2	
	L2	2	1	1	
	L3	2	2		
TOTAL	15	35	29	3	3
	%		82	9	9



**Figure 5.6** Bar graph showing evaluative events in terms of the regulation of the learner

## Chapter 6

### Discussion of results

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In this chapter I summarise and discuss the results for each school, including the topics covered, the way in which time is used, the realisation of content in comparison to the mathematics encyclopaedia, what is appealed to as authorizing ground, and whether the regulation of the learner is predominantly under the aspect of the Imaginary or the Symbolic.

#### 6.1 School P1

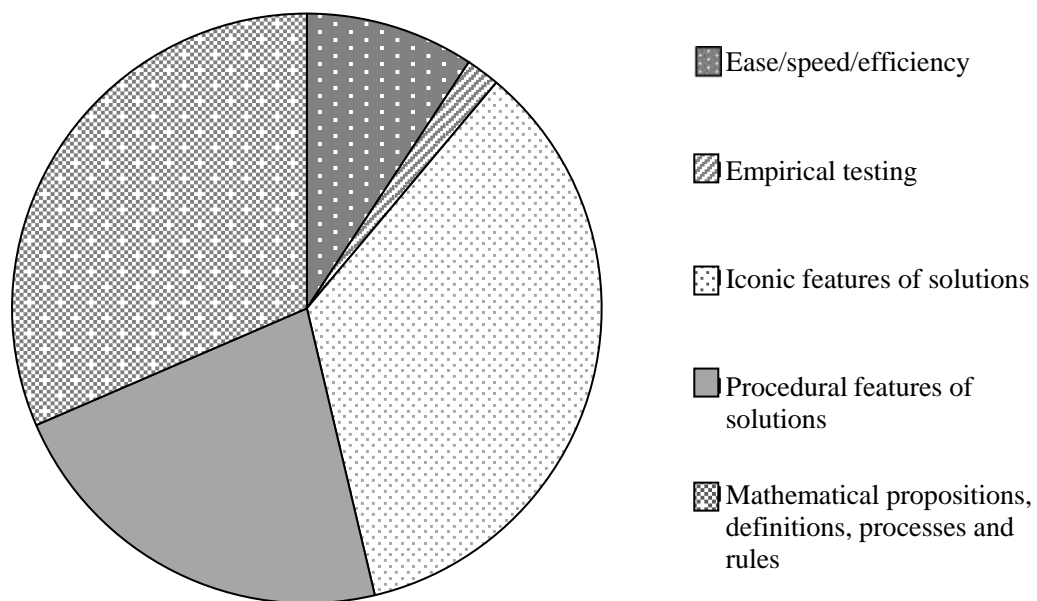
In school P1 under two minutes is spent across the lessons explicitly discussing mathematical principles, as seen in Figure 5.2. Most time is spent on the exposition of worked examples, followed by learners working through exercises. The two topics covered in the six events at school P1 are the solving of linear inequalities and the sketching of straight line graphs (using the gradient-intercept method). If we examine the results for school P1, we see that these topics are not realised in any events from the point of view of the encyclopaedia. In four events the way in which the topics are realised still corresponds with the encyclopaedia – these are all events in which the teacher carries out a procedure through worked examples. There are two events in which the realised topic does not align with the encyclopaedia – these are both events in which learners are practicing the procedure for solving linear inequalities which has been taught. Thus although the teacher's procedure does not violate the propositions or processes of the mathematics encyclopaedia, the learners' attempts to carry out the procedure do violate these propositions and processes - the teacher's procedure does not result in correct reproductions of mathematics by the learners. Analysis of the teacher's procedure for solving linear inequalities, found in lessons one and two, suggests that it functions as a closed text, which although intending to steer learners in a particular direction (the teacher insists on learners carrying out specific 'steps' in a particular order, as will be discussed in Chapter Seven), results in divergent interpretations by learners due to the way in which the elements of the encyclopaedia recruited by the topic of solving linear inequalities remain closed to the learners.

The teacher's privileging of one particular procedure for sketching lines in lesson three is discussed in Chapter Four. His procedure, although not recruiting the content usually indexed by the topic of straight lines from the point of view of the encyclopaedia, does not violate mathematical propositions, principles or definitions.

In school P1 there are 54 appeals made to an authorizing ground, with iconic or spatial features of solutions as the most common factor to which appeals are made (35%), as seen in Table 6.1 and Figure 6.1.

**Table 6.1** Appeals in School P1

School	Lesson	Evaluative event	Ease/speed/efficiency	Examinations	Empirical testing	Iconic or spatial features of solutions	Procedural features of solutions	Mathematical propositions, definitions, processes, rules and objects
P1	L1	EE1				8	3	7
		EE2			1	1	1	1
	L2	EE1				4	5	6
		EE2					1	
	L3	EE1	4			4	1	3
		EE2	1				2	1
Total			5		1	19	12	17
% (n=54)			9		2	35	22	32



**Figure 6.1** Appeals in School P1

It is interesting to note that although this teacher appeals to mathematical propositions, definitions, processes and rules 32% of the time, almost all of these are accompanied by a corresponding appeal to an extra-mathematical factor, such as iconic or procedural features of solutions. For example,

when explaining the procedure for solving linear inequalities the teacher refers to the need to “get rid of” numbers in order to “get  $x$  on its own” – an appeal to an iconic feature of the solution (that  $x$  will be ‘on its own’ on one side). But he also appeals implicitly to the mathematical principles of additive inverses and equality to explain how to ‘get rid’ of a particular number - “... so to get rid of the three, I need to subtract a three ... what I do on the left hand side I must also do on the right hand side”. The teacher thus uses both mathematical and non-mathematical language to describe what he is doing. But analysis of learners’ work in two events show that learners do not acquire the criteria for ‘getting rid of’ terms, and thus that they are predominantly regulated by the non-mathematical rather than the mathematical language. The teacher’s appeals to extra-mathematical factors exceed his appeals to mathematical factors in all events, showing that in all events in P1 necessity is situated external to mathematics and the regulation of the learner is under the aspect of the Imaginary.

## **6.2 School P2**

The two topics making up the lessons in school P2 are the conversion of recurring decimals to common fractions (lessons one and two) and the simplifying of exponential expressions (lesson three). The way in which both of these topics are realised differs from the intended topic from the point of view of the encyclopaedia in all evaluative events, and in 75% of events which contain sufficient data for analysis the way in which the topic is realised does not correspond with the propositions, definitions and processes found in the encyclopaedia either. These are all events consisting of the exposition or marking of worked examples. Analysis of these events suggests that the teacher and learners suspend mathematical operations in order to reach the expected solutions, as found in the textbook. So although in all but one of the worked examples in these events the correct solution is arrived at, the procedure violates mathematical propositions, definitions and processes. This is particularly evident in the procedure for converting a recurring decimal to a common fraction, where the teacher and the learners have a clear idea of what the solution should look like and suspend knowledge of subtraction in order to make sure that they reach the correct solution. The procedure also involves a change in the domain being operated over from recurring decimals to non-recurring decimals, as will be discussed in Chapter Six.

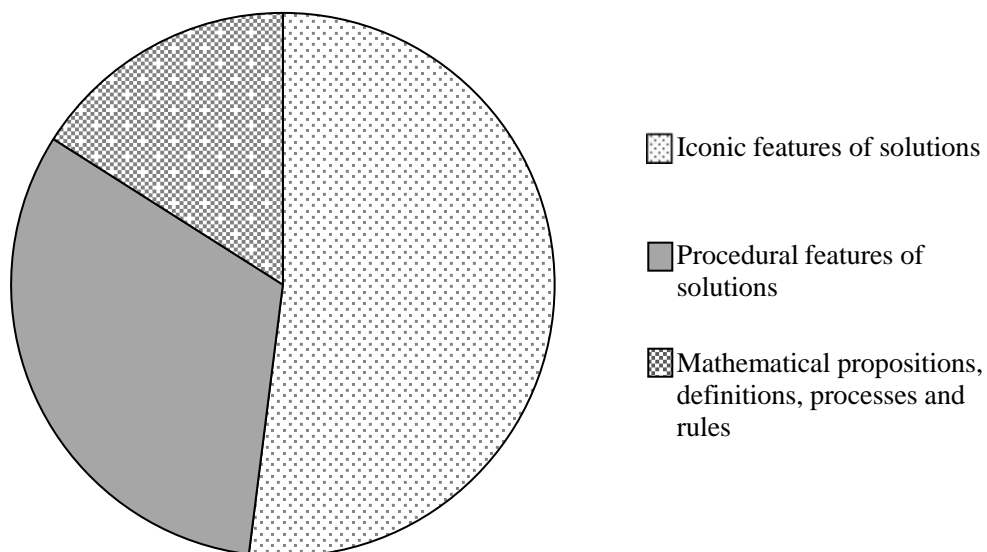
The one event in which the realised topic still corresponds with the encyclopaedia is in lesson three, where the teacher spends one minute on the exposition of mathematical principles related to the topic of exponents. This is the only time during the three lessons spent on the exposition of mathematical principles - P2 is one of the schools in which the least time is spent on the exposition

of mathematical principles, as seen in Figure 5.2. Most time is spent on the marking of worked examples.

School P2 is the school in which fewest appeals (only 25 appeals in total) are made to an authorizing ground to justify or explain decisions, statements or actions. 52% of these appeals are made to iconic or spatial features of solutions, with the layout of solutions in the textbook acting as a strong regulative resource in all lessons, followed by 32% to procedural features of solutions. The remaining 16% of appeals are made to mathematical propositions, definitions and rules, and appeals are predominantly made to extra-mathematical factors in all events.

**Table 6.2** Appeals in School P2

School	Lesson	Evaluative event	Ease/speed/efficiency	Examinations	Empirical testing	Iconic or spatial features of solutions	Procedural features of solutions	Mathematical propositions, definitions, processes, rules and objects
P2	L1	EE1						
		EE2				5	1	
		EE3						
	L2	EE1				7	4	2
	L3	EE1				1	2	2
		EE2					1	
Total						13	8	4
% (n=25)						52	32	16



**Figure 6.2** Appeals in School P2

Thus in all of the events analysed in School P2 the regulation of the learner is under the aspect of the Imaginary.

### **6.3 School P3**

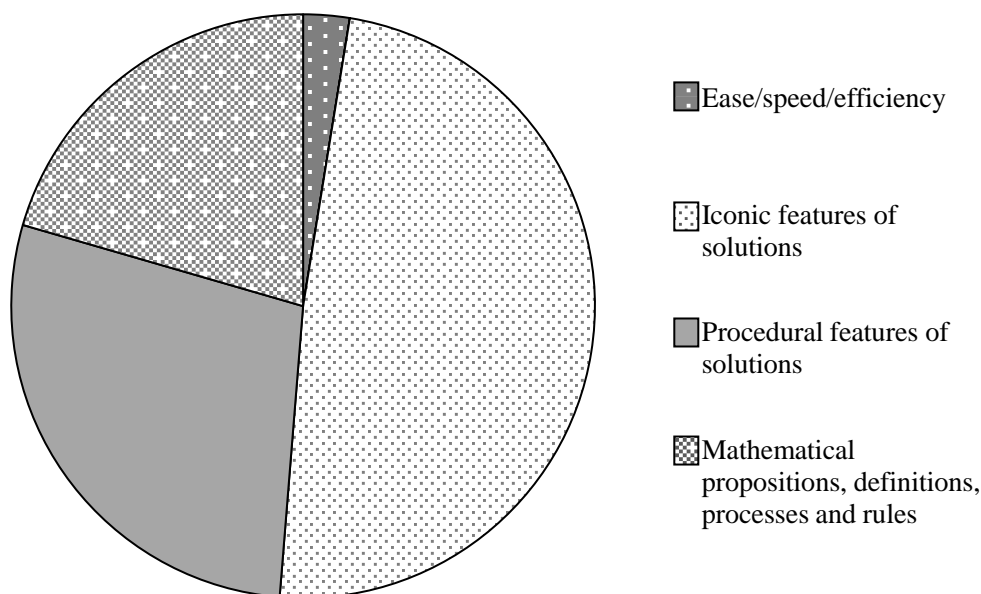
School P3 is one of the schools in which the least time is spent on the exposition of mathematical principles, as seen in Figure 5.2, and most time is spent on the marking of worked examples. The two topics covered in school P3 are the simplifying of exponential expressions involving multiplication and division of powers (lessons one and two) and the addition and subtraction of powers (lesson three). In school P3, once again these intended topics are not realised in any events. In addition to this, the realised topics in 50% of events do not correspond with the encyclopaedia. These events are mostly ones in which learners are working on an exercise. In a similar way to school P1, it seems that although the realised topic still corresponds with the encyclopaedia in the three events which consist of worked examples, when learners carry out the procedure which they have been taught, we see that they have not acquired the teacher's criteria for simplifying exponential expressions. The teacher's procedure, with its emphasis on the steps to be carried out, seems to function as a closed text, which is open to error.

Most of the appeals made to authorising ground in school P3 are made to iconic features of solutions (49%), followed by appeals made to procedural features of solutions (28%) and then mathematical features (21%). The number of appeals to mathematical propositions, definitions and rules (8 out of 39 appeals) seems high if we consider that so little time is spent on the exposition of mathematical principles in school P3. But these appeals are all made in the process of carrying out a procedure and the propositions, definitions and rules appealed to are not elaborated.

In all events analysed the number of appeals to extra-mathematical factors outweigh the appeals to mathematical factors, thus situating necessity external to mathematics (with the exception of one event containing insufficient data) and suggesting that the regulation of the learner is under the aspect of the Imaginary in all events.

**Table 6.3** Appeals in School P3

School	Lesson	Evaluative event	Ease/speed/efficiency	Examinations	Empirical testing	Iconic or spatial features of solutions	Procedural features of solutions	Mathematical propositions, definitions, processes, rules and objects
P3	L1	EE1					3	
		EE2				7	2	2
		EE3						
	L2	EE1				7	2	2
		EE2						
	L3	EE1	1			5	4	4
EE2								
Total			1			19	11	8
% (n=39)			3			49	28	21



**Figure 6.3** Appeals in School P3

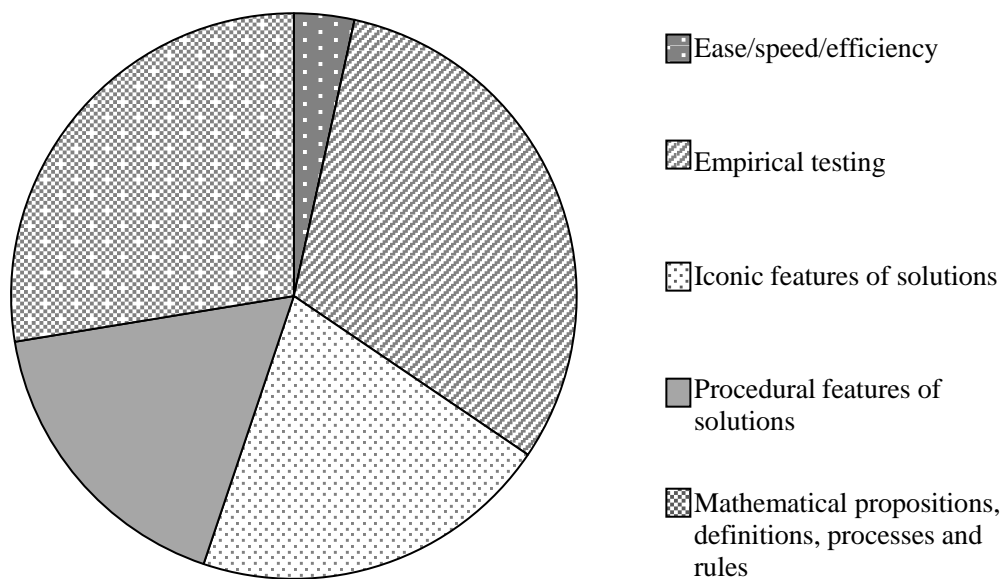
## 6.4 School P6

The overall topic of all three lessons in school P6 is linear number patterns, involving adding terms to a linear pattern, finding the general term, and substituting or solving to find a particular term. In terms of time use, school P6 has slightly more time spent on the exposition of mathematical principles compared to the previous three schools, but most of the time in the three lessons for school P6 involves learners working on exercises, followed by the marking of worked examples. In school P6 a similar picture to the previous three schools emerges – the content associated with the

intended topic of number patterns from the point of view of the encyclopaedia is not realised across all events. In 38% of events the way in which the topic is realised also does not correspond with the encyclopaedia. School P6 contains the second fewest appeals to an authorizing ground (29 in total), with the highest number of appeals made to empirical testing (31%) and then mathematical propositions, definitions, processes and rules (28%).

**Table 6.4** Appeals in School P6

School	Lesson	Evaluative event	Ease/speed/efficiency	Examinations	Empirical testing	Iconic or spatial features of solutions	Procedural features of solutions	Mathematical propositions, definitions, processes, rules and objects
P6	L1	EE1			3	3	1	2
		EE2					1	
		EE3						
	L2	EE1			1			
		EE2	1		3		2	2
		EE3				3		3
	L3	EE1			1			1
		EE2						
		EE3			1		1	
Total			1		9	6	5	8
% (n=29)			3		31	21	17	28



**Figure 6.4** Appeals in School P6

The number of appeals made to extra-mathematical factors is greater than the appeals made to mathematical factors in all events but two, suggesting that the regulation of the learner in these events is under the aspect of the Imaginary. In the two events (one in lesson two and the other in lesson three) where there are the same number of appeals made to mathematical and extra-mathematical factors, further analysis led me to conclude that in both events the regulation of the learner is also under the aspect of the Imaginary. Both involve the use of a procedure for finding the general term of a linear pattern which was taught in the first lesson. Because of the way in which the procedure for finding the general term of a linear pattern is constituted as algebraic and arithmetic manipulation, without drawing on the propositional ground of linear functions, as discussed in an example in Chapter Seven, as well as the repetition of the idea of ‘discovering’ the pattern (as seen in the number of appeals made to empirical testing across all lessons in P6) and thus an appeal to the empirical, necessity is situated outside of the field of mathematics and the regulation of the learner is also under the aspect of the Imaginary in these two events.

### **6.5 School P7**

The two topics found in school P7 are simplification of exponential expressions and solving of exponential equations. School P7 has more time spent on the exposition of mathematical principles than the other schools, as seen in Figure 5.2. Most time in school P7 is taken up by the exposition of worked examples, followed by the marking of worked examples. Analysis of the events in school P7 shows a slightly different picture to the others. In 43% of events the intended topic is realised. There is one event in which the intended topic is not realised, but the way in which the topic is realised still corresponds with the encyclopaedia. In the remaining three events in which the intended topic is not realised, the realised topic also does not correspond with the encyclopaedia.

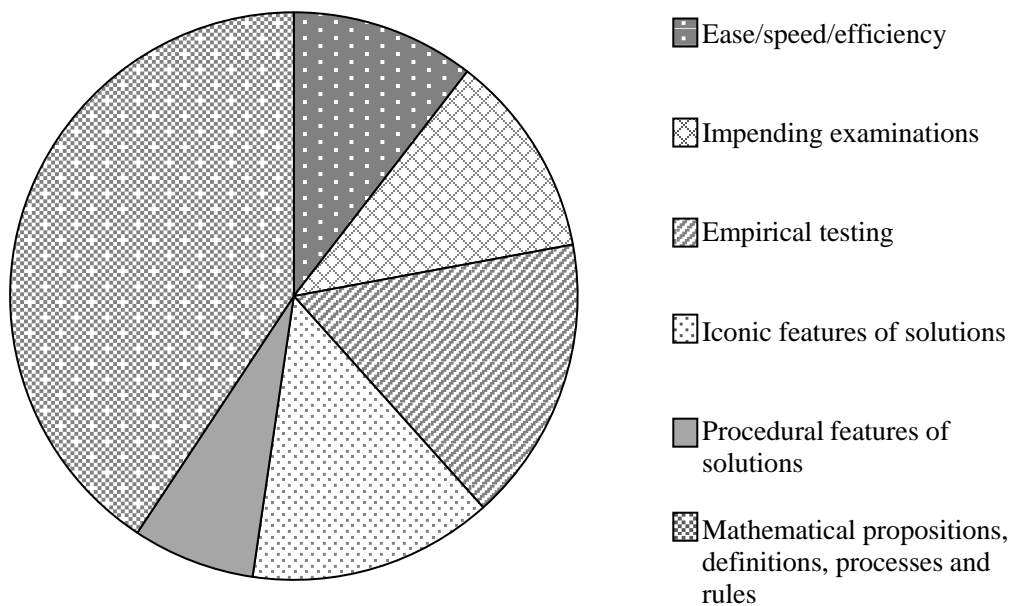
School P7 contains the most appeals to an authorizing ground – 86 in total, which range across all of the categories, as seen in the Table and Figure 6.5. Appeals to mathematical propositions, definitions, rules and processes make up the biggest proportion of appeals (41%), followed by empirical testing (16% - all of which are found in lessons two and three in the trial-and-improvement procedure taught for solving exponential equations).

In two of the seven evaluative events, appeals made to mathematical factors exceed appeals made to extra-mathematical factors, and necessity is situated within the field of mathematics, thus the regulation of the learner in these events is under the aspect of the Symbolic. In another event there are an equal number of appeals made to mathematical and extra-mathematical factors, but further analysis shows that necessity is situated within the field of mathematics and thus the regulation of

the learner in this event is also under the aspect of the Symbolic. Two of these three events are ones in which the intended topic is realised (quadrant I), the other is one in which the intended topic is not realised, but the realised topic still corresponds with the encyclopaedia (quadrant II). In the remaining four events the appeals to extra-mathematical factors outweigh the appeals to mathematical factors, and necessity is situated external to mathematics, thus the regulation of the learner is predominantly under the aspect of the Imaginary in these events.

**Table 6.5** Appeals in School P7

School	Lesson	Evaluative event	Ease/speed/efficiency	Examinations	Empirical testing	Iconic or spatial features of solutions	Procedural features of solutions	Mathematical propositions, definitions, processes, rules and objects
P7	L1	EE1	1	3		1		3
		EE2	2	1		1	1	5
		EE3	1			1		8
	L2	EE1	1	1		5	3	11
		EE2	1	1	4	2		5
	L3	EE1	3	3	9	1	2	3
		EE2			1	1		
	Total			9	10	14	12	6
% (n=86)			10	12	16	14	7	41



**Figure 6.5** Appeals in School P7

## Chapter 7

### Four pedagogic practices and their implied model learner

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My analysis of each school has shown that in most cases in the events analysed, necessity is based on factors external to mathematics and thus that the regulation of the learner in most events is predominantly under the aspect of the Imaginary. In this chapter I examine this in more detail through outlining four pedagogic practices which occur across the schools. Analysis suggests that the model learner implied by these practices is unable to engage with the propositional ground underlying the topic. The image of the learner implied by these pedagogic practices seems to function as a regulative resource which has a structuring effect on pedagogy through making available to the learner certain things rather than other things. I end the chapter with a discussion of the implications of this in relation to the co-constitution of mathematics and learner identification.

#### **7.1 Four pedagogic practices**

During my analysis of the fifteen lessons, the following pedagogic practices emerged in a number of lessons across the five schools. I discuss them here, with examples, in order to draw attention to the way in which the image of the learner implied by the pedagogic practices has a structuring effect on pedagogy, and the implications of this for the co-constitution of mathematics and learner identification.

##### **7.1.1 Privileging one method over another**

In fourteen of the events analysed, teachers privilege one method as being better than another, for various reasons. I gave an example of this in Chapter Four, where the teacher privileged the gradient-intercept method for sketching a line. Another example comes from a series of events in school P7 where the teacher introduces two methods for solving exponential equations when the two sides of the equation cannot be written as powers of the same base – trial-and-improvement and logarithms. After introducing both methods, the teacher instructs learners to carry out the trial-and-improvement method in the exams (“you can’t in the examination give us a solution that’s log of ... in the exams when we give you this question we will say by means of trial-and-improvement”) because “this is stated in our document, we get a document from the department, and it’s stated that we have to teach that particular method to you”. But despite saying this, the teacher seems to favour the logarithm method, describing it as a “shortcut” and as more accurate. He encourages learners to use the logarithm method, saying that “nothing stops you using your calculator ... you can use

logs”. This, and other appeals to the “log method” as more accurate, suggests that the only reason he teaches them the trial-and-improvement method is because it is stated in the curriculum, but he prefers the “log method”, possibly because he knows that the log method is more likely to get learners to the correct answer - it is almost fail-safe if the ‘steps’ are memorised. On the other hand the trial-and-improvement method is more open to errors and involves more need for learners to make decisions. We see a number of examples where learners select different answers as the closest approximation. As a result the teacher seems to regulate the learners by privileging the log method, although he does not make any reference to the mathematical principles and properties underlying logarithms.

In these examples, teachers privilege particular methods (using appeals to ease, accuracy, speed or impending examinations). But the privileged methods do not always involve fewer computations or transformations. It seems that the motivation for the selection and privileging of certain methods by teachers is to ensure that learners are able to get to the correct answers (to reproduce the mathematics) despite themselves. This pedagogic practice implies an image of learners as insufficient in some way, and in response teachers try to get learners to identify with and prefer certain methods – it seems that they are bending the mathematical content in the direction of the implied image of the learner.

### 7.1.2 Changes in the domain

I have already discussed the commonly occurring phenomenon of domain shifting in the context, specifically with reference to the integer addition example in Chapters Three and Four, which entailed existential shifts from the domain of the integers to the natural numbers. Another example comes from school P2 in a series of events on converting recurring decimals to common fractions. The teacher carries out the worked example shown in Figure 7.1.

We first see a change in the domain in the teacher’s first step, where she writes  $x = 0,7777$ , instead of  $0,7777\dots$ , which changes the object she is operating on from a recurring decimal to a terminating decimal. This has implications for the rest of the procedure, as she seems to use  $0,7$ ,  $0,77$  and  $0,7777$  interchangeably to represent the recurring decimal  $0,\dot{7}$ . Her final answer is correct despite her interchangeable use of  $0,7$ ,  $0,77$  and  $0,7777$  to represent  $0,\dot{7}$  and her errors related to this, specifically the way in which she violates the operation of subtraction by claiming that the answer of  $7,77 - 0,7$  is  $7$ , which is clearly incorrect. It seems she does this because she knows what the final answer should be and makes sure that she gets there - the initial and terminal points of the procedure have a stronger regulative effect on the ‘steps’ used by the teacher than mathematical

propositions, processes and operations, so much so that she allows suspension of an operation that is universally valid and stable.

recurring decimal fraction  
 e.g.  $0,7\bar{7}$   
 let decimal to be equal  $x$   
 ①  $x = 0,7777$   
 $10x = 0,77770$   
 ②  $10x = 7777$   
 $10x - x = 7777 - 0,7$   
 $9x = 7$

**Figure 7.1** Procedure for converting recurring decimals to common fractions

Changes in the domain occur in nine events, often with the goal of operating over the natural numbers instead of another domain less familiar to learners. The strategy of domain shifting may be motivated by the idea that learners are not capable of engaging with a domain (such as the integers or rationals), and so teachers introduce methods which shift the domain to something more familiar. This is another way in which teachers transform the mathematical content in the direction of the implied image of the learner.

### 7.1.3 Specifying the order in which operations must be carried out

In ten events the teacher insists on a specific order in which operations must be carried out during a procedure, which is not always mathematically necessary. When operating on the reals, for example, multiplication and addition are both associative, meaning that, while respecting the order of operations, we can enter an expression at any point and do not necessarily have to work from left to right in any order to achieve the same outcome. It is thus interesting to examine cases where teachers specify one order in which operations must be carried out.

A series of two lessons on solving linear inequalities in school P1 contains six worked examples in which the teacher elaborates a procedure for solving linear inequalities. The general principle on which his procedure is based is to “get rid or to get the constants or the numbers on one side and your  $x$ ’s or the variables on the other side” – “I must have  $x$  on its own.” It seems that the criterion in place for solving an inequality is to “have  $x$  on its own”, and in order to do that they need to “get

rid” of certain things. The teacher places particular emphasis on the order in which things must be ‘gotten rid of’. In an example,  $-5x + 2 \leq 12$ , the teacher asks “what is it that I need to get rid of, which is a term which is with the  $x$ ?” Some learners reply that it is two, others five, others negative five - this is interesting as it reveals that his criteria for ‘getting rid of’ are not clear.

Teacher: What must I have on this side alone? (*pointing to left of inequality*)

Learners:  $x$

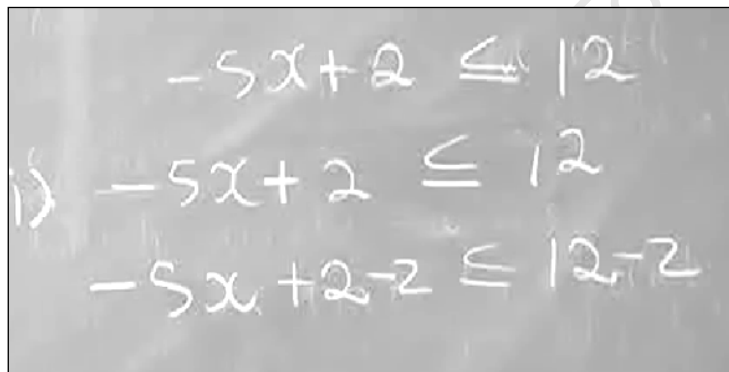
Teacher: Now what is together with the  $x$  that I need to get rid of?

Learners: Five. Minus five. Minus Two.

Teacher: The plus two first. Okay. So I need to get rid of the plus two by doing what?

Learners: Minus the two.

Transcript School P1 Grade 10 Lesson 1


$$\begin{array}{l} -5x + 2 \leq 12 \\ 1) -5x + 2 \leq 12 \\ -5x + 2 - 2 \leq 12 - 2 \end{array}$$

**Figure 7.2** ‘Getting rid’ of the plus two

In response to learners’ confusion, the teacher emphasizes that the plus two must be ‘gotten rid of’ first, which is not mathematically necessary. The emphasis placed on the order in which things must be ‘gotten rid of’ reveals a difference from the content indexed by the topic of solving linear inequalities, where any order of operations is possible due to the associativity of addition and multiplication over the reals. Although this difference does not involve any violation of mathematical propositions, processes or objects, it does not explicitly draw on the resources within the encyclopaedia, which functions as an open text, but instead functions as a closed text, which has the potential to produce aberrant readings, as seen in the work of a few learners when carrying out the procedure later in the lesson. In these learners’ work, which can be found in Appendix One, we see a learner subtracting instead of dividing in order to ‘get rid of’ a number, another learner just ‘getting rid of’ terms without doing anything to them, and a third learner ‘getting rid of’ terms without preserving identity at the level of value. We thus see confusion over whether to add, subtract or divide in order to ‘get rid of’ numbers, as well as the order in which this should be done.

“Getting rid of” is a context-dependent operation – it takes a very particular form depending on the context. In this lesson learners do not acquire the teacher’s criteria relating to the order in which things must be gotten rid of and the operations which need to be carried out in order to do so.

Specifying the order in which operations must be carried out is another instance where mathematical content is transformed in the direction of the implied image of the learner, based on the idea that learners are not capable of selecting appropriate operational resources from the encyclopaedia to carry out a procedure. Teachers attempt to pre-empt learner errors by specifying a particular order in which operations must be carried out. But instead of pre-empting errors, the way in which the procedure is constituted results in mathematical absences which require insertions by the learner - in the linear inequalities example, learners need to make decisions about what to “get rid of”, the order in which to “get rid of” numbers, and what operations are required in order to do so, rather than drawing on the underlying propositional ground to solve the inequality. The procedure functions as a closed text - open to error and preventing learners from acquiring the mathematical principles underlying the procedure.

#### **7.1.4 Using expression states as triggers for a procedure or parts of a procedure**

In nine events there seem to be particular expression states which serve as triggers for the use of a particular procedure, or for the next step in the procedure. It seems that teachers train learners to carry out a particular procedure or step when they see these expression states. As an example, I examine a procedure carried out in school P6. In a series of three lessons on the topic “number patterns”, there are four worked examples involving the same procedure for finding the general term which can be used to represent a particular sequence of numbers. In all four examples the given pattern is linear, which is never stated in the question. The learners are expected to approach the sequence of numbers inductively and empirically in order to determine which kind of pattern it is. In all examples the teacher leads them to conclude that the pattern is “linear” due to the presence of a “common difference” - he instructs learners to “investigate the difference. If the difference happens to be common then what do you know? That pattern is linear.” Once they have identified the pattern as linear, a particular procedure follows in order to find the general term. Here are the ‘steps’ of this process:

- 1) “Check the differences” to decide the type of sequence – if there’s a “common difference” between consecutive terms, the sequence is linear, if not it’s “another type of thing”.
- 2) **Because the sequence is linear**, write down the formula  $T_n = a + (n - 1)d$ .

3) Substitute the first term ( $a$ ) and common difference ( $d$ ).

Although this method is valid from the point of view of the mathematics encyclopaedia, the notion of a linear function, with its properties, is implicit, or even absent, in the ground of this procedure. The ground seems to be procedural in nature. The announced topic of 'linear number patterns' is taught in a procedural way without any reference to the nature of a linear function or its properties. In this procedure the expression 'linear' seems to be used as a trigger for the procedure from step 2 onwards, and not as descriptive of the type of relationship represented, and thus the mathematical treatment of such a relationship. It seems that the expression 'linear' acts as a trigger which regulates the procedure without recruiting the notion of a linear function with its properties. Instead, the procedure is based on algebraic and arithmetic manipulation, which seems to be something learners are familiar with. It thus seems that the teacher uses the pedagogic strategy of triggering to recontextualise the notion of linear patterns so that learners are able to reach the correct answers, despite their lack of knowledge about linear functions.

Triggering is an essential feature of thought, and the use of triggering as a pedagogic strategy is in itself not problematic. The issue here is what is being triggered. It seems that in these examples the triggering is implicated in the production of a closed text through the way in which the procedures which are triggered are constituted – the learners are trained to carry out particular procedures in specific ways and the propositional ground underlying the topic remains closed to them, resulting in openness to error. It seems that teachers use triggering to ensure that learners get to the correct answers, and that, once again, the image of the learners as insufficient or incapable of selecting and carrying out the correct procedure implied by the pedagogy has a regulative effect on the pedagogy.

## **7.2 Implications for the implied model learner**

The model learner implied by these pedagogic practices is one who is unable to engage with the mathematical content associated with the topic, so the content is reconstituted in such a way as to enable the reproduction of the content by the learners despite their lack of knowledge of the topic. In all of the examples given, I suggest that the teachers' implicitly held image of the learner as insufficient in some way, as implied by their pedagogic practices, has a regulative effect on their pedagogical choices (remembering that teachers' ideas about learners may be unconsciously held, as pointed out previously, as the psychoanalytic ideas used in my framework do not require agents to be conscious of the ideas they hold). The pedagogic practices discussed here are examples of the ways in which teachers reconstitute mathematical content in response to the extimate relations between the learner, the teacher and mathematics. Generally, teachers can take into account these

extimate relations, or the learner as obstacle to the reproduction of mathematics, in two different ways – either by transforming the mathematical content in the direction of their image of learners, or by transforming the learners in the direction of the field of mathematics. In the examples given in this chapter, the teachers try to find alternate ways for learners to realise the required content by bending the mathematical content towards the learners, in response to their presupposed model learner, which has implications for the mathematical competence of learners.

Something which I have not mentioned thus far is the mathematical competence of teachers, and the possibility that teachers are not familiar with the propositional ground underlying the topic and so uses strategies which avoid engaging with that ground. In such cases the mathematical content would thus be reconstituted in response to the extimacy of teachers, rather than learners, in relation to mathematics. This can only be determined through a more detailed study of the mathematical knowledge of teachers in the context, which is outside the scope of my project, but would be interesting for further study.

University of Cape Town

## Chapter 8

### Conclusion: the co-constitution of mathematics and learner identification

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This study set out to investigate what is entailed in the co-constitution of mathematics and learner identification in the pedagogic situations of schooling, with special reference to the teaching and learning of mathematics in a selection of grade 10 mathematics lessons at five schools in the Western Cape Province of South Africa. The study is located within the broad framework of the sociology of education, specifically drawing on Bernstein's sociological theory of education and its application in the investigation of the relations between pedagogy and social justice. The specific problematic within which my study is located is the constitution of school mathematics in the pedagogic situations of schooling. My theoretical framework consists of the work of Davis (2009b, 2010a, 2010b, 2011a, 2011b) in conjunction with Lacan's (2006) psychoanalytic notions of the Real, Imaginary and Symbolic registers, Eco's (1979) notion of a model reader, and Bernstein's (1996) discussion of pedagogic discourse and the pedagogic device, which I used to fashion a set of resources for describing the co-constitution of school mathematics and learner identification in pedagogic situations. In my analysis I described the operational activity making up fifteen grade 10 mathematics lessons selected for description and analysis. I used these descriptions of operational activity to discuss the realisation of content and the regulation of the learners in the lessons in order to explore the ways in which the extimate relations of the learner, the teacher and mathematics, and the learner as potential obstacle to the reproduction of mathematics, appear in the exposition of mathematical content by teachers, and the implications of this for the co-constitution of mathematics and learner identification.

In this concluding chapter I summarise the results of my analysis in relation to the research hypotheses of my study and with reference to the theoretical framework, followed by a discussion of the limitations and potential of this study.

#### **8.1 Summary of results in relation to research hypotheses**

##### **8.1.1 Research hypothesis 1**

*I expect to find that there are cases in the analysed lessons when the content indexed by the intended or announced topic is not realised due to the drawing together of content (in the form of a set of objects and operations) which is not usually indexed by the intended topic from the point of view of the mathematic encyclopaedia, and as such, is a substitute for the indexed content.*

In relation to my first research hypothesis, in 80% of events analysed I found differences between the content indexed by the intended topic from the point of view of the mathematics encyclopaedia and the way in which content which was drawn together to make up the realised topic. The content which was substituted for the content indexed by the topic did not necessarily contain fewer transformations, but analysis suggests that the substitutions of content were made in order to simplify things for learners. These substitutions still enable learners to reach the correct solutions in many cases but do not provide learners with access to the propositional ground underlying the topic. Further research would be useful in order to explore the underlying motivations for these substitutions in more detail, for example conducting teacher interviews.

### **8.1.2 Research hypothesis 2**

*I expect that the content may be realised in ways that do not correspond with the propositions, processes, rules and objects of the mathematics encyclopaedia, for example, due to the use of manipulations which are not functions and thus not operations.*

In relation to my second research hypothesis, in 40% of events analysed content is realised in ways that do not correspond with the propositions, processes, rules and objects of the mathematics encyclopaedia, either due to the use of manipulations which are not functions and thus not operations, or to changes at the level of identity when performing transformations, or other violations of mathematical propositions, definitions, processes, rules or objects. This result is concerning as these violations seem to take place without the teachers' knowledge, and are reinforced by the evaluative criteria in operation.

### **8.1.3. Research hypothesis 3**

*In cases where the mathematics encyclopaedia is not functioning as a primary regulative resource, I expect that there will be other regulative resources appealed to as authorizing ground, amongst them procedural and iconic ones, and thus that necessity will be situated external to the field of Mathematics.*

In relation to my third research hypothesis, I found that teachers appealed predominantly to extra-mathematical factors in regulating learners in 82% of events analysed. The common feature of these appeals is that they are associated with a bending of the mathematical content in the direction of the teachers' implied image of the learners as insufficient in some way, as discussed in Chapter Six. It seems that the image of the learners as insufficient in some way, implied by pedagogic practices, acts as a regulative resource in these events, resulting in appeals to extra-mathematical factors and substitutions of content. The point at which mathematical content is bent in the direction of the

image of the learner as insufficient in these events is the point at which the teacher reconstitutes the mathematics in response to the extimate relations of the learner and mathematics, as discussed in Chapters Two, Three and again in Chapter Six. At this point, the teacher can attempt to overcome the extimacy of the learner and reassert the Symbolic automaton to enable the reproduction of mathematics by the learner in two ways – either by transforming the mathematics in the direction of the image of the learner or by transforming the learner in the direction of the field of mathematics. Although the latter may at first glance appear to contradict the current learner-centred approach to teaching, my results suggest that this option is more likely to achieve the desired outcome of the reproduction of mathematics by learners. I return to Eco's notion of closed and open texts in order to elaborate what these two options involve.

## **8.2 A return to the theory**

Transforming the learner in the direction of the mathematical content resonates with Eco's (1979) notion of an open text, which does not allow any interpretation but can be used only as the text prescribes, thus constructing the competence of its reader. The pedagogic text constructs the mathematical competence of the learner through the generation of evaluative criteria which do not transform mathematical content but instead give learners access to the mathematical propositions, definitions and properties. The model learner implied by such a pedagogic text is one who is able to reproduce mathematical content using operational resources found in the mathematics encyclopaedia which enable the selection of procedures based on mathematical propositions, definitions and properties. Thus transforming the learner in the direction of mathematics situates necessity within the field of mathematics and recruits the Symbolic in regulating the learner, resulting in implicit Symbolic identifications.

In contrast, a closed text aims to generate a very precise response but is open to many different interpretations. A closed pedagogic text attempts to determine a specific outcome, despite the learner, and in order to do this bends the mathematical content in the direction of the learner. Such a text is open to aberrant readings and interpretations, as my analysis has revealed. The model learner implied by a closed pedagogic text is not necessarily able to select procedures based on the mathematical propositions, definitions and properties found in the encyclopaedia, but instead is only able to reproduce the content presented by the teacher, which in many cases requires insertions by the learner due to the foreclosing of the propositional ground underlying the topic, as seen in my results. The ways in which mathematics is transformed in the direction of the learner situate necessity external to mathematics and result in a rendering of the Symbolic nature of mathematics under the aspect of the Imaginary, and the generation of implicit Imaginary identifications.

As explained in Chapter Three, the teacher anticipates or presupposes a certain kind of learner competence, and my results suggest that the image of the learner implied by teachers' pedagogic practices shapes the evaluative criteria generated in pedagogic contexts, which in turn structure the way in which mathematics is reconstituted in response to the extimate relation of the learner and mathematics. This has implications for the mathematical competence of learners. Further research would be useful to examine the mathematics produced by learners in more detail in order to ascertain the degree to and manner in which teachers' reconstitution of mathematics structures learners' mathematical competence.

### ***8.3 Limitations and potential of my study***

As previously mentioned, something which my study has not addressed is the mathematical knowledge of teachers. My study focused on the relations between the learner, the teacher and mathematical knowledge, with specific focus on the relation between the learner and knowledge, with the teacher as mediator of this relation. Due to the size of the project I was not able to explore the relation between the teacher and mathematical knowledge, and the influence this has on the way in which mathematics is constituted in pedagogic situations. Another area which I did not have space to fully explore is the relation between teachers' pedagogical choices and the curriculum. As I mentioned in Chapter Four, the curriculum is part of the official recontextualising field, and as such has an influence on the way in which teachers constitute mathematics in the pedagogic context. It would have been productive to examine this further in order to get more insight into the factors which play a role in the constitution of mathematics in pedagogic situations.

Another limitation of my study, which I raised in Chapter Four, is that due to the size of this project I was not able to carry out triangulation of data. It would have been helpful to interview teachers and learners to strengthen my analysis, particularly to get more insight into the teacher's implied model learner and their pedagogical choices, as well as to the mathematical competence of the learners. Such interviews could be carried out in further research.

I began this study with a discussion of the relation between pedagogy and social justice issues, and I return to this in concluding. The literature I referred to in Chapters One and Two suggests that pedagogy is implicit in the reproduction of social class inequalities through the unequal distribution of content and thus knowledge to learners from different class backgrounds. My study focused at the micro-level of the operational activity in fifteen grade 10 mathematics lessons at five working-class schools in the Western Cape Province of South Africa to produce a picture of the co-

constitution of mathematics and learner identification in these schools. It would be dangerous to conclude that the picture which has emerged is as a result of learners in the schools being from working-class backgrounds. At this stage, we do not know what picture would emerge in other contexts, and thus cannot make any comparisons in terms of social class. But it would be interesting to apply my analytical framework across schools in different class settings or schools with learners from different class groups in order to explore whether the social class of learners (and teachers) is implicated in the co-constitution of mathematics and learner identification, and potentially to enable interventions at the pedagogic level in order to address social justice issues relating to the mathematics achievement gap between learners from different class backgrounds. Whether or not this would be productive remains to be discovered. But the main potential of my study is the generation of methodological tools which extend previous work and which need to be extended further.

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# Appendix 1: Analysis for School P1

## Primary data production P1 Lesson 1 EE1

### 1) Generating evaluative events

Table A1.1 Evaluative events P1 Lesson 1

Time	Evaluative event/sub-event	Activity	Type
00:00 – 21:46	E <sub>1</sub>	Worked examples for solving linear inequalities and representing solution on a number line	Expository
00:00 – 11:00	E <sub>1,1</sub>	Solving linear inequality $x + 3 < 2$ and representing solution on a number line	Expository
11:00 – 17:00	E <sub>1,2</sub>	Solving linear inequality $-5x + 2 \leq 12$ and representing solution on a number line	Expository
17:00 – 21:46	E <sub>1,3</sub>	Solving linear inequality $4(2x - 1) < 5x + 2$ and representing solution on a number line	Expository
21:46 – 40:49	E <sub>2</sub>	Working on exercise number 1	Exercise
32:59 – 34:46	E <sub>2,1</sub>	Explanation of rewriting $\frac{8}{3}$ as $2\frac{2}{3}$	Expository

### 2) Describing operational activity

This lesson is one of a series of two lessons on solving linear inequalities in school P1. In the teacher's introduction to the first example, he says: "a linear inequality ... now in linear equations, that inequality was replaced by an equal sign." This is the closest he gets to defining a linear inequality - that it is a linear equation with an inequality sign instead of an equal sign. This is also the only reference he makes to the relationship between linear equations and inequalities.

The general principle on which his procedure is based is to "get rid of or to get the constants or to the numbers on one side and your  $x$ 's or the variables on the other side". The following exchange reveals the way in which the teacher recontextualises the concept of solving an equation or inequality:

Teacher: What does that mean to you, K? To solve for  $x$ ?

Learner: To look for the correct answer, sir.

Teacher: Hey?

Learner: To solve, to look for the correct answer.

Teacher: Okay, what must I have?

Learner: Find the value of  $x$ .

Teacher: I must have  $x$  on its own. Then I have solved for  $x$ , right? So that is what we're going to do.

Transcript School P1 Grade 10 Lesson 1

His procedure for solving for  $x$  ('getting  $x$  on its own') is shown in the following extract:

Teacher: If I have  $x$  plus three and I want to solve for  $x$ , I want to get  $x$  alone on one side ... so to get rid of the three, I need to subtract a three. Are you with me?

Learners: Yes

Teacher: Right, to get rid of that three. But what I do on the left hand side, I must also do on the?

Learners: Right hand side.

Transcript School P1 Grade 10 Lesson 1

It seems that the criterion in place for solving an equation or inequality is to “have  $x$  on its own”, and in order to do that they need to “get rid” of certain things. In the second example,  $-5x + 2 \leq 12$  the teacher asks “what is it that I need to get rid of, um, which is a term which is with the  $x$ ?” Some learners reply that it is the two, others the five, others negative five - this is interesting as it reveals that his criteria for ‘getting rid of’ are not clear. The teacher tries to clarify it for them:

Teacher: What must I have on this side alone? (while pointing to the left hand side of the inequality)

Learners:  $x$

Teacher: Now what is together with the  $x$  that I need to get rid of?

Learners: Five. Minus five. Minus Two.

Teacher: The plus two first. Okay. So I need to get rid of the plus two by doing what?

Learners: Minus the two.

Transcript School P1 Grade 10 Lesson 1

$$-5x + 2 \leq 12$$

$$1) -5x + 2 \leq 12$$

$$-5x + 2 - 2 \leq 12 - 2$$

**Figure A1.1** ‘Getting rid’ of the plus two

The last part of this solution involves dividing by the coefficient of  $x$ , and the teacher approaches it like this:

Teacher: What happens to an inequality when I divide by a negative number?

Learners: The sign will change.

Teacher: The sign will change around so a less than or equal to become now

Learners: More than.

Teacher: Or greater than or equal to, okay? Do we all understand that?

Learners: Yes sir.

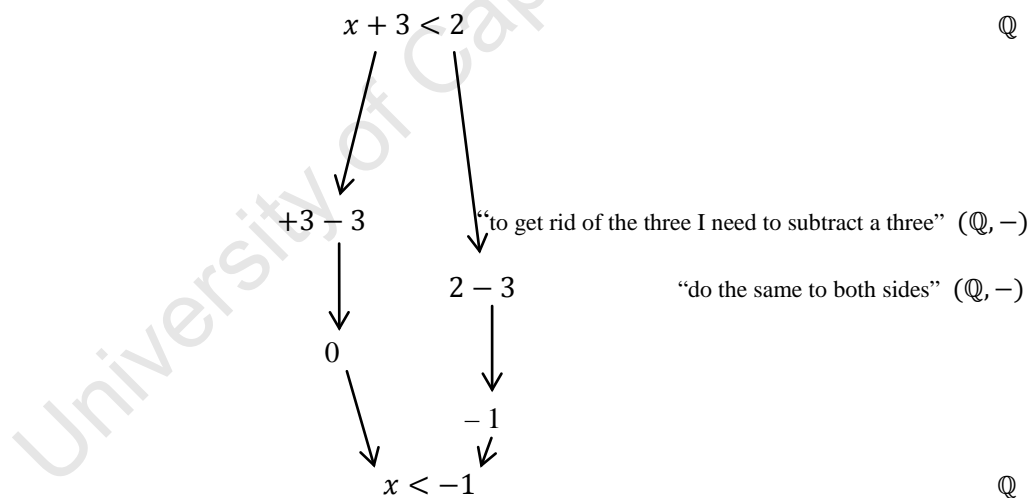
Teacher: Good, you all get it. So I indicate that now by divide by negative five (draws in a division line and a negative five), so then five  $x$  divide by negative five then, this is now important, so normally you divide, not the step after. The moment you do that your sign changes. Okay guys?

The teacher does not give any explanation for why it is that the signs “change”.

In addition to this, the teacher emphasizes a particular order in which things should be ‘gotten rid of’ – he insists on the learners ‘getting rid of’ the two first in the example described above, without giving a reason for this. This is interesting because mathematically it is not necessary to start by subtracting the two – any order of operations is possible due to the associativity of multiplication and addition over the reals. I will examine why it is that the teacher insists on a particular order of operations later.

Generally, the teacher uses non-mathematical language to describe the operations he is carrying out – such as “get rid of”, “remove” brackets and numbers, “put the numbers on the right”, “take” numbers to the “other side”, but he also refers to the equivalent mathematical operations of subtraction, addition, multiplication and division. But despite this, the domain of the reals remains implicit in his procedure as the operatory properties of multiplication and addition over the reals are not explicitly drawn on (for example, he does not make explicit why it is that they can ‘get rid of’ numbers by subtracting or adding them, or why they need to ‘do the same on both sides’).

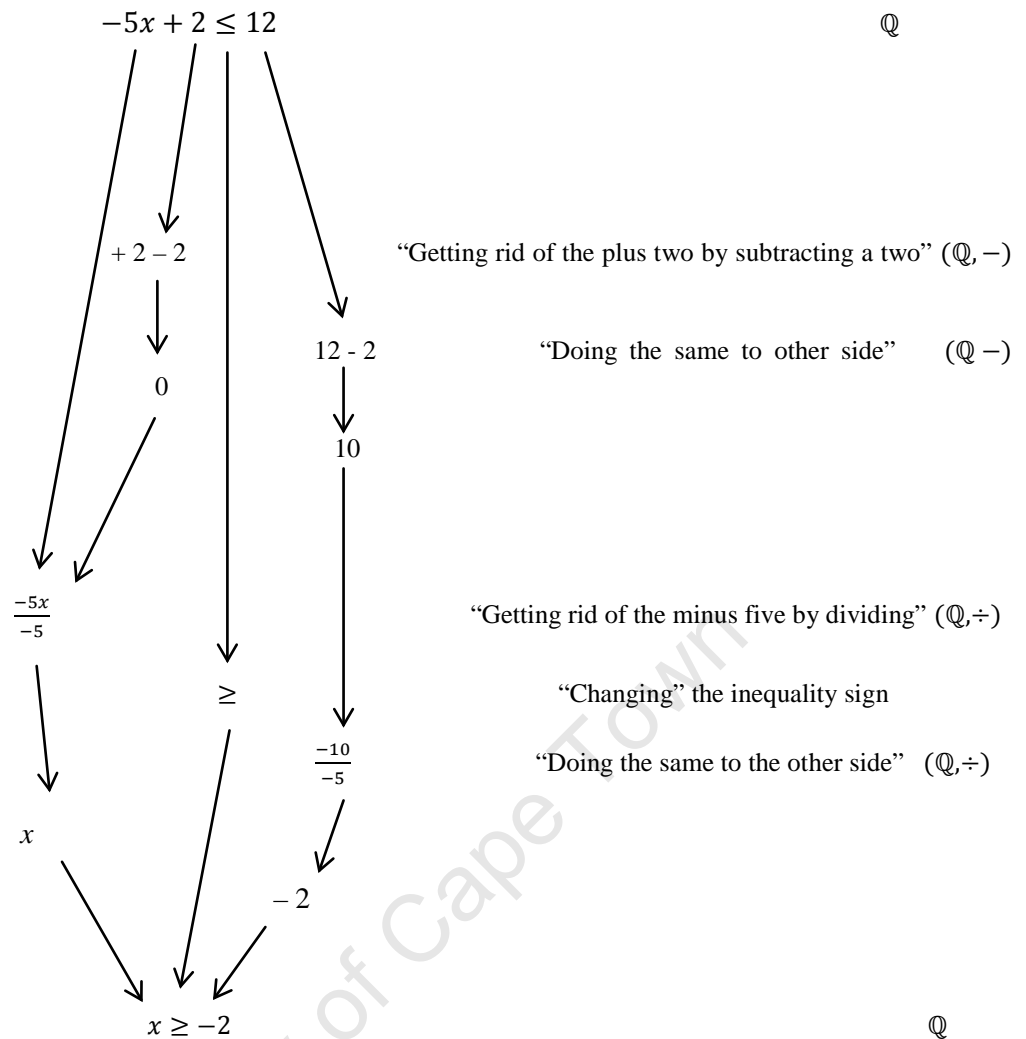
The teacher works through three examples in this event – each one with a slightly different focus. I now describe the operational activity entailed in the procedure using diagrams, maps and tables.



**Figure A1.2** Diagrammatic representation of worked example one

**Table A1.2** Operational activity of worked example one

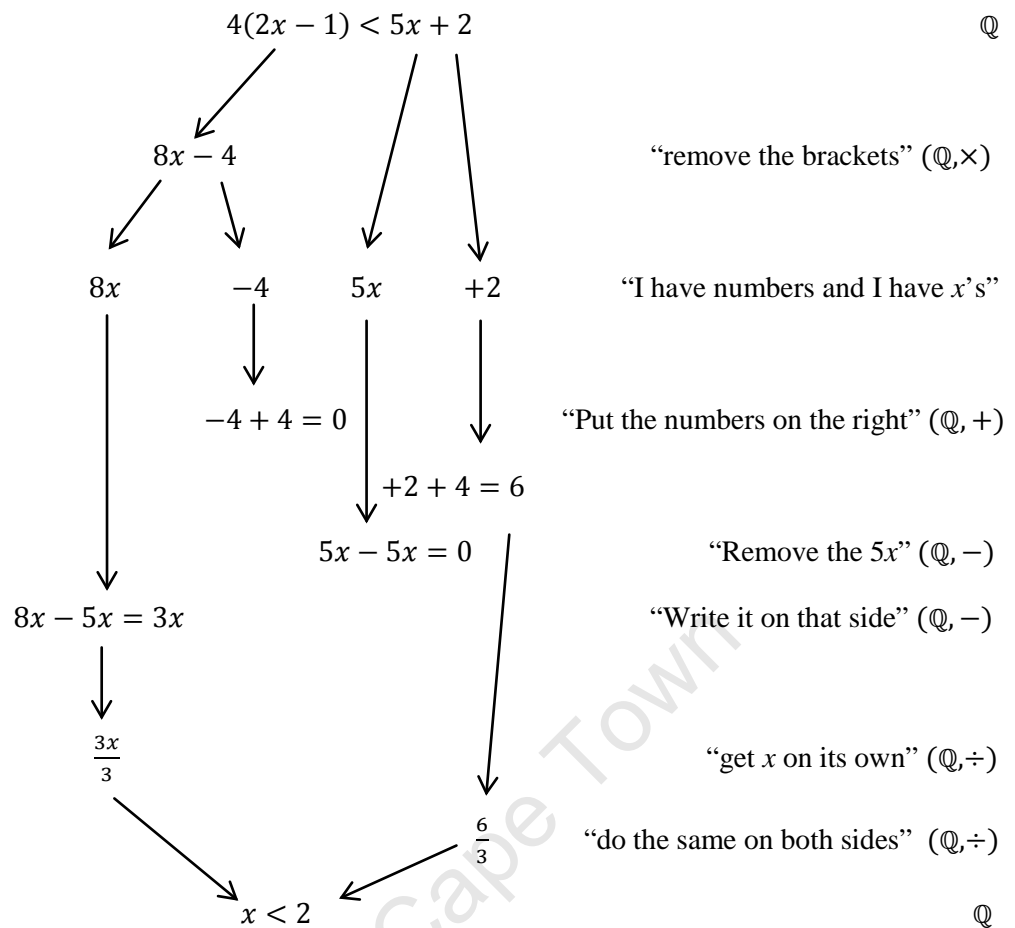
T	Input	Domain	Operation	Output	Codomain
1	$x + 3 < 2$	ℚ	Subtraction	$x + 3 - 3 < 2 - 3$	ℚ
2	$x + 3 - 3 < 2 - 3$	ℚ	Simplification	$x < -1$	ℚ
3	$x < -1$	ℚ	Plot point at $x = -1$	Point at $x = -1$	ℚ
4	‘Less than’ symbol only	ℚ	Represented as	Open circle	ℚ
5	‘Less than’ symbol	ℚ	Represented as	Arrow to the left	ℚ



**Figure A1.3** Diagrammatic representation of worked example two

**Table A1.3** Operational activity of worked example two

T	Input	Domain	Operation	Output	Codomain
1	$-5x + 2 \leq 12$	$\mathbb{Q}$	Subtraction	$-5x + 2 - 2 \leq 12 - 2$	$\mathbb{Q}$
2	$-5x + 2 - 2 \leq 12 - 2$	$\mathbb{Q}$	Simplification	$-5x \leq 10$	$\mathbb{Q}$
3	$-5x \leq 10$	$\mathbb{Q}$	Division and changing the inequality sign	$\frac{-5x}{-5} \geq \frac{10}{-5}$	$\mathbb{Q}$
4	$\frac{-5x}{-5} \geq \frac{10}{-5}$	$\mathbb{Q}$	Simplification	$x \geq -2$	$\mathbb{Q}$
5	'Equal to' symbol	$\mathbb{Q}$	Represent as	Closed circle	$\mathbb{Q}$
6	'Greater than' symbol	$\mathbb{Q}$	Represent as	Arrow to the right	$\mathbb{Q}$



**Figure A1.4** Diagrammatic representation of worked example three

In this example, as in the others, the teacher uses both mathematical and non-mathematical language to describe his steps. Below I show the operational activity for both of these:

**Table A1.4** Operational activity of worked example three based on mathematical language used

T	Input	Domain	Operation	Output	Codomain
1	$4(2x - 1) < 5x + 2$	Q	Multiply	$8x - 4 < 5x + 2$	Q
2	$8x - 4 < 5x + 2$	Q	Addition	$8x - 4 + 4 < 5x + 2 + 4$	Q
3	$8x - 4 + 4 < 5x + 2 + 4$	Q	Simplification	$8x < 5x + 6$	Q
4	$8x < 5x + 6$	Q	Subtraction	$8x - 5x < 5x - 5x + 6$	Q
5	$8x - 5x < 5x - 5x + 6$	Q	Simplification	$3x < 6$	Q
6	$3x < 6$	Q	Division	$\frac{3x}{3} < \frac{6}{3}$	Q
7	$\frac{3x}{3} < \frac{6}{3}$	Q	Simplification	$x < 2$	Q

**Table A1.5** Operational activity of worked example three based on non-mathematical language used

T	Input	Domain	Operation	Output	Codomain
1	$4(2x - 1) < 5x + 2$	$\mathbb{Q}$	Existential shift	$/4(2x - 1) < 5x + 2/$	$\mathbb{X}$
2	$/4(2x - 1)/$	$\mathbb{X}$	Remove the brackets	$/8x - 4/$	$\mathbb{X}$
3	$/8x/, /-4/, /</, /5x + 2/$	$\mathbb{X}$	“Put the numbers on the right” - shift	$/8x - 4 + 4/, /</, /5x + 2 + 4/$	$\mathbb{X}$
4	$/8x - 4 + 4/, /</, /5x + 2 + 4/$	$\mathbb{X}$	Existential shift	$8x - 4 + 4 < 5x + 2 + 4$	$\mathbb{Q}$
5	$8x - 4 + 4 < 5x + 2 + 4$	$\mathbb{Q}$	Addition	$8x < 5x + 6$	$\mathbb{Q}$
6	$8x < 5x + 6$	$\mathbb{Q}$	Existential shift	$/8x < 5x + 6/$	$\mathbb{X}$
7	$/8x < 5x + 6/$	$\mathbb{X}$	Remove the $5x$ , “take” to LHS – shift	$/8x/, /-5x/, /</, /5x - 5x/, /+6/$	$\mathbb{X}$
8	$/8x/, /-5x/, /</, /5x - 5x/, /+6/$	$\mathbb{X}$	Existential shift	$8x - 5x < 5x - 5x + 6$	$\mathbb{Q}$
9	$8x - 5x < 5x - 5x + 6$	$\mathbb{Q}$	Subtraction	$3x < 6$	$\mathbb{Q}$
10	$3x < 6$	$\mathbb{Q}$	Existential shift	$/3x/, /</, /6/$	$\mathbb{X}$
11	$/3x/, /</, /6/$	$\mathbb{X}$	“get $x$ on its own” - shift	$/\frac{3x}{3}/, /</, /6/$	$\mathbb{X}$
12	$/\frac{3x}{3}/, /</, /6/$	$\mathbb{X}$	Existential shift	$\frac{3x}{3} < \frac{6}{3}$	$\mathbb{Q}$
13	$\frac{3x}{3} < \frac{6}{3}$	$\mathbb{Q}$	Division	$x < 2$	$\mathbb{Q}$

Generally, his procedure is as follows:

**Step 1:** ‘Get rid of’ terms which are added to or subtracted from the  $x$  term by subtracting or adding them on both sides (“but what I do on the left hand side I must also do on the right hand side” – the notion of equality is not explicit, it’s recontextualised as this rule).

**Step 2:** Check if the inequality sign changes direction (if it’s negative it changes direction, if it’s positive it stays as it is).

**Step 3:** ‘Get rid of’ the coefficient of  $x$  by dividing. Do the same to the other side.

**Step 4:** Decide whether you need an open or closed dot - if the inequality sign includes ‘equal to’ then the dot is closed, if not it is open.

**Step 5:** Decide whether your arrow should go to the left or the right - if the inequality sign is ‘greater than’, the arrow goes to the right, if ‘less than’, it goes to the left.

So, given an inequality of the form  $ax + b \leq c$  where  $a \neq 0$   $\{x, a, b, c \in \mathbb{R}, a \neq 0 | ax + b \leq c\}$ , the teacher solves it as follows:

$$ax + b \leq c$$

$$ax + b - b \leq c - b$$

$$\frac{ax}{a} \leq \frac{c - b}{a}$$

$$x \leq \frac{c - b}{a} \text{ if } a > 0 \text{ or } x \geq \frac{b - c}{a} \text{ if } a < 0$$

If solution is of the form  $x \leq k$  or  $x \geq k$ , the dot is closed.

If solution is of the form  $x < k$  or  $x > k$ , the dot is open.

If solution is of the form  $x \leq k$  or  $x < k$ , the arrow points to the left.

If solution is of the form  $x \geq k$  or  $x > k$ , the arrow points to the right.

### 3) Activation of the mathematics encyclopaedia

#### From the field of production

A linear inequality is a line with restricted domain. The set of solutions of a real linear inequality makes up a half-space of the n-dimensional real space, one of the two defined by the corresponding linear equation.

In order to solve a linear inequality graphically, we would sketch the two lines represented by the inequality and read off the solution set. So for a linear inequality of the form  $ax + b \leq c$  where  $a \neq 0$   $\{x, a, b, c \in \mathbb{R}, a \neq 0 | ax + b \leq c\}$ , we would sketch  $y = ax + b$  and  $y = c$  and read off the solution for  $ax + b \leq c$  based on the point at which  $ax + b = c$ .

Courant & Robbins (1941: 322) remind us of the elementary rules which govern arithmetical operations with inequalities:

1. If  $a > b$ , then  $a + c > b + c$  (any number may be added to both sides of an inequality).
2. If  $a > b$  and the number  $c$  is positive, then  $ac > bc$  (an inequality may be multiplied by any positive number).
3. If  $a < b$ , then  $-b < -a$  (the sense of an inequality is reversed if both sides are multiplied by  $-1$ ). Thus  $2 < 3$  but  $-3 < -2$ .
4. If  $a$  and  $b$  have the same sign, and if  $a < b$ , then  $\frac{1}{a} > \frac{1}{b}$ .
5.  $|a + b| \leq |a| + |b|$ .

The relations of “less than” and “greater than” are central to the solving of linear inequalities. As explained by Courant & Robbins (1941: 57), “the rational number A is said to be less than the rational number B ( $A < B$ ) and B is said to be greater than A ( $B < A$ ) if  $B - A$  is positive. It then follows that, if  $A < B$ , the points (numbers) between A and B are those which are both  $> A$  and  $< B$ . Any such pair of distinct points, together with the points between them, is called a segment, or interval,  $[A, B]$ .”

In order to solve linear inequality algebraically, the properties of reals are the ground up on which the procedure rests. In order to solve a linear inequality of the form  $ax + b \leq c$  where  $a \neq 0$  and  $\{x, a, b, c \in \mathbb{R}, a \neq 0 | ax + b \leq c\}$ , we can make use of the additive and multiplicative inverses of reals. One way in which this can be done is by rewriting  $c$  as the sum of  $b$  and some other real number,  $d$ :  $ax + b \leq d + b$ .

Based on the existence of additive inverses for all reals, we could then say that  $ax \leq d$ . We would then rewrite  $d$  as a product of  $a$  and another real number,  $e$ :  $ax \leq ae$ , and from there, based on the existence of multiplicative inverses for all reals, conclude that  $x \leq e$  if  $a > 0$  or  $x \geq e$  otherwise if  $a < 0$ .

There are many ways of solving an inequality of this form by exploiting the additive and multiplicative inverses of reals. Because of the associativity of addition and multiplication over the reals, the order in which the operations are carried out does not matter.

## From the field of recontextualisation

### Curriculum

Linear inequalities form part of learning outcome two of the FET Mathematics National Curriculum Statement (2003) – “*The learner is able to investigate, analyse, describe and represent a wide range of functions and solve related problems*” (DoE, 2003: 12). The curriculum specifically refers to linear inequalities by stating that learners should be able to “solve linear inequalities in one variable and illustrate the solution graphically” (DoE, 2003: 26).

The curriculum gives no indication of methods used to solve linear inequalities, so let’s have a look at the textbook which the teacher looks at during the lesson (he does not explicitly refer to it but consults it during the lesson and also chooses one of his examples from the textbook exercise).

### Textbook

The textbook has an activity (pg 184) which links the graphical and algebraic solution of linear inequalities. This activity also explains why it is that the inequality sign changes when dividing by a negative number through using graphical and algebraic examples. It then goes on to give a procedure for solving linear inequalities based on two general rules (given on page 185 of Classroom Mathematics, Grade 10).

“We may add any term (positive or negative) to both sides of an inequality”.

“We may multiply both sides of an inequality by a non-zero number, but when multiplying or dividing by a **negative** number, the **direction** of the inequality sign must be **reversed**”.

## Secondary data production P1 Lesson 1 EE1

### 1) Realisation of content

The intended topic of this lesson is the solving of linear inequalities. The elements of the mathematics encyclopaedia which are activated by this topic are described above.

When this teacher speaks about “getting rid of” terms, and putting the “numbers on the right and the  $x$ ’s on the left” he is operating on character strings, as seen in Table A1.5 – a number is not something which we can ‘get rid of’ or ‘put on the right’. This indicates an existential shift – real numbers are changed into character strings which can be moved around in this way. This suggests that there may be a point of difference between the content associated with the intended topic, which has the field of reals as its domain, and the content which is realised, with its shift in domain from reals to character strings.

But, despite this, the teacher does specify which mathematical operations he needs to do in order to ‘get rid of’ things – subtract, add or divide. Thus although he uses non-mathematical language in the way described above, and does not explicitly draw on the propositional ground (he does not give reasons why they can, for

example, subtract to get rid of something or why they must ‘do the same to both sides’), he is still explicitly carrying out the appropriate mathematical operations.

But despite this the notion of equality and the existence of additive and multiplicative inverses for all reals are implicit in this process – they are recontextualised as ‘doing the same to the other side’ and ‘getting rid of’ or moving terms. In addition to this, the teacher’s emphasis on the order in which things must be ‘gotten rid of’ (first the number added to or subtracted from the variable term, then the coefficient of the variable – he does not entertain any other possibilities) also reveals a difference from the intended topic, where any order is possible due to the associativity of addition and multiplication over the reals. Although this difference does not involve any violation of mathematical principles, it does not explicitly draw on the resources within the encyclopaedia (which functions as an open text), but instead functions as a closed text (which does produce aberrant readings, as we will see later).

The operations of addition, subtraction, multiplication and division over the reals are used in this teacher’s procedure, but are recontextualised as ‘getting rid of’ or ‘putting numbers on the right’. ‘Getting rid of’ is not a function – there is not one unique solution for each input. It is a context-dependent manipulation, as in some cases “getting rid of” is mapped to subtraction, in others division and in still others addition – “getting rid of” takes a very particular form depending on the context. This is clear when learners show confusion over whether to add, subtract or divide in order to ‘get rid of’ numbers. The criteria for ‘getting rid of’ numbers are not clear. But, although the teacher refers to ‘getting rid of’, which, on its own is not a function, his mention and use of the mathematical operations needed in order to do so (albeit without much explanation of why) means that the operations he is carrying out are in fact functions, and his procedure does not violate the propositions, processes and rules of the mathematics encyclopaedia.

In terms of the curriculum – the curriculum does not give any specifications about method, so it is difficult to make any claims about the correspondence of the realised topic with the curriculum. The curriculum states that learners should be able to “solve linear inequalities in one variable and illustrate the solution graphically” (DoE, 2003: 26), which is the stated topic of this event. The textbook gives principles to be used in solving linear inequalities, but does not use the same language used by the teacher. The textbook does not specify the order in which numbers should be added, subtracted, multiplied or divided in order to solve for  $x$  - it seems that this is a criterion put in place by the teacher, with interesting implications, as will be discussed in the next section.

## 2) Regulation of the learner

Generally, this teacher uses non-mathematical language – such as “get rid of”, “remove” brackets and numbers, “put the numbers on the right”, “take” numbers to the “other side”, but he also gives the mathematical equivalents – subtract, add, multiply and divide. Let’s examine the language he uses more closely. He says they need to “get rid of” a particular number by subtracting that number, and in one example by adding it (he does not make it clear why he chooses to add or subtract in order to “get rid of” – additive inverses are implicit). Then he says they should do the same to the right hand side - the notion of equality is implicit – it sounds like a rule learners are familiar (‘do the same to both sides’) with but without explicitly drawing on the notion of equality, or of preserving identity at the level of value. Then they have to “get rid of” something else by dividing – again, no clear explanation of why they now divide (the notion of multiplicative inverses is implicit). So although the notion of equality and the operative properties of the reals are the ground of this procedure, they are implicit and not necessarily available to the learners. The learners could carry out the procedure without explicitly drawing on or understanding these properties – they could just see the procedure as a series of steps in order to “get the  $x$ ’s on one side and the numbers on the other”, by “getting rid of”  $x$ ’s or numbers which are on the wrong side. Even though the teacher clearly understands what “getting rid of” means and is aware of the operations he is carrying out in order to do this, these criteria are not explicitly transmitted to the learners. It thus seems that one of the primary regulative resources in this evaluative event is the steps and rules making up the procedure, rather than the

propositional ground underlying a linear inequality (which is a secondary regulative resource in this evaluative event). Another primary regulative resource is the way in which the solution looks and is set out – as the teacher appeals to the iconic by saying things like “this is the way your solution should look” and referring to the spatial shifting of numbers.

In this evaluative event, 8 appeals are made to iconic or spatial features of solutions, 3 to procedural features of solutions and 7 to mathematical propositions, processes and rules, thus the total number of appeals to iconic and procedural features of solutions exceeds the appeals made to mathematical factors.

This analysis suggests that the topic of linear inequalities is being taught in a procedural manner which forecloses the propositional ground (notion of equality and the operator properties of addition and multiplication over the reals) so that learners rely on the iconic and procedural features of the solutions and do not acquire the criteria for “getting rid of” something which are transmitted by the teacher. Necessity is thus situated within the rules generated by the teacher, which appeal predominantly to iconic and procedural features, and is external to mathematics. There are thus mathematical absences which require insertions by the learner – they need to make decisions about what to “get rid of”, the order in which to “get rid of” numbers, and how to do this, rather than drawing on the propositional ground in order to solve the inequality. The way in which the evaluative criteria are transmitted renders the Symbolic nature of these notions under the aspect of the Imaginary. The image held by the teacher of the learners has a stronger regulative effect on the way in which the procedure is constituted and the language which is used than the propositional ground underlying linear inequalities. The regulation of the learner in this event is under the aspect of the Imaginary.

### Summary

This event falls within quadrant II– the realised topic differs from the intended topic, but still corresponds with the mathematics encyclopaedia. The regulation of the learner is under the aspect of the Imaginary.

### Primary data production P1 Lesson 1 EE2

#### 1) Generating evaluative events

Table A1.6 School P1 Lesson 1 EE2

Time	Evaluative event/sub-event	Activity	Type
21:46 – 40:49	E <sub>2</sub>	Working on exercise number 1	Exercise
32:59 – 34:46	E <sub>2,1</sub>	Explanation of rewriting $\frac{8}{3}$ as $2\frac{2}{3}$	Expository

#### 2) Describing operational activity

In this evaluative event the teacher sets the class an exercise:

- 1)  $x + 30 < 25$
- 2)  $-4x + 3 \leq 11$
- 3)  $7(x - 2) \geq 4x - 20$
- 4)  $-2(x - 3) > 5x - 78$

As he walks around looking at their work he says: “x’s on the one side, numbers on the other side”, and “people I want to see your answer and the number line. Okay? Just don’t do answer then we go for the number line, we don’t do it like that. Right?”

A little while later a student asks how to write down eight over three. The teacher asks the class to simplify eight over three, and they struggle.

Learners: Two. No. Two and ...  
 Teacher: Two and?  
 Learners: A lot of numbers will come.  
 Learner: No, two and two over three.

Some other learners are not happy with that, one asks “what number is that?”

The teacher intervenes:

Teacher: Right, okay, put the pens down and listen to me carefully here. Eight over three so I can rewrite that as a mixed fraction which is now two and two thirds. Right? Okay, and that’s less than so, do we sign, do I shade?

Learners: Left.

Teacher: To the left. So I can start anywhere and say that’s my two and two thirds. Now what’s the number to the left of two and two thirds?

Learner: Three and two.

Teacher: What’s the number left ..

Learner: Two and one.

Learner: Two and a half.

Teacher: Two, that’s right. I don’t have to, and then?

Learners: One.

Teacher: I don’t need to count in thirds. It’s two, one and then?

Learners: Nought.

Transcript School P1 Lesson 2

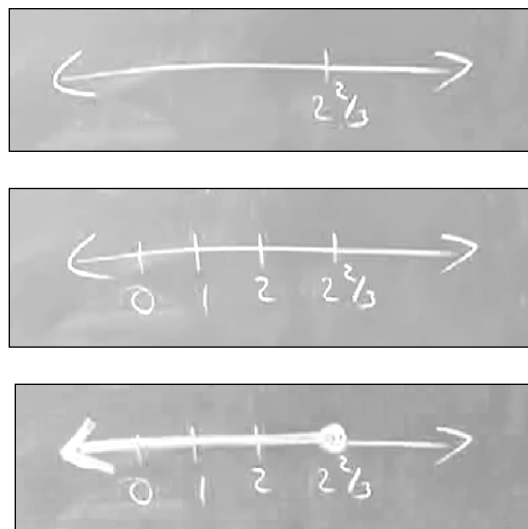
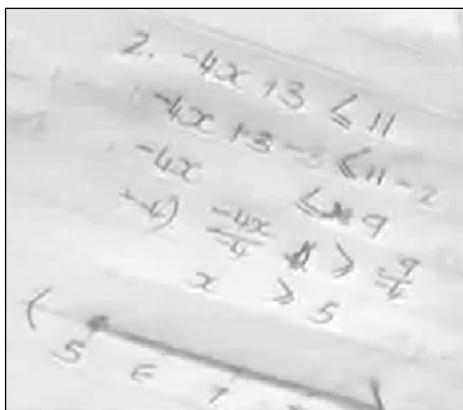


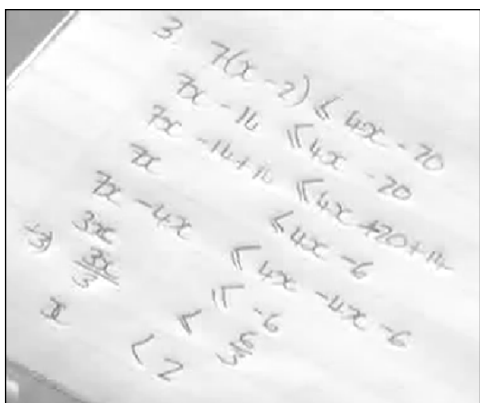
Figure A1.5 Representing  $\frac{8}{3}$  on a number line

Later another learner is not sure whether to draw the arrow to the left or to the right. The teacher tells him to “do it, do it with your hand. No man, not like that. Make your linear line there. Indicate ... around that draw your open circle. Then you need to shade to the left”

As they work, we see two learners’ work:



This learner incorrectly subtracts two on the right hand side in her second line. She also forgets to change the inequality sign when she divides by negative four, but goes back and changes it when she realises. She also gives an incorrect answer in the last step of nine divided by negative four as five (she is subtracting instead of dividing). This reinforces the lack of clarity seen earlier in how learners should ‘get rid of’ terms – there is confusion about whether to subtract or divide.



In this question, the learner changes the inequality sign in the last two lines (she leaves out the equal line). She also writes negative six as positive six from line six to line seven – not sure if this is a slip or if she is intentionally leaving it out.

Figure A1.6 Two learners’ work on the exercise

### 3) Activating the mathematics encyclopaedia

See previous event (P1 L1 EE1)

## Secondary data production P1 Lesson 1 EE2

### 1) Realisation of content

In this event, learners are working on an exercise which involves solving linear inequalities. The first learner’s work shown in Figure A1.6 suggests that she does not acquire the criteria for ‘getting rid of’ terms, despite the teacher’s use of both mathematical and non-mathematical equivalents to explain this process – based on this example, the intended topic is not realised. Based on this learner’s work, it also seems that the way in which the content is realised does not correspond with the mathematics encyclopaedia. It must be noted that the work seen above could represent an exception to the work produced by other learners in the class (which we do not have access to in our records), but based on previous work in the context, which suggests that learners often produce uniform reproductions of the mathematics in these schools, we can make tentative conclusions about the realisation of content based on a few learners’ work.

## 2) Regulation of the learner

The teacher's response to learners' questions in this event reveals once again that learners are regulated through appeals to the iconic and procedural features of the solution set out by the teacher, rather than the propositional ground underlying the topic. For example, the teacher says things like "x's on the one side, numbers on the other side" and "people I want to see your answer and the number line. Okay? Just don't do answer then we go for the number line, we don't do it like that. Right?". Later another learner is not sure whether to draw the arrow to the left or to the right. The teacher tells him to "do it, do it with your hand. No man, not like that. Make your linear line there. Indicate ... around that draw your open circle. Then you need to shade to the left". There are not many appeals made in this event – only four in total (1 to empirical testing, 1 to iconic or spatial features of solutions, 1 to procedural features of solutions and 1 to the mathematics encyclopaedia). Appeals to extra-mathematical factors thus outweigh the one appeal made to the field of mathematics in this event. It seems that again in this event necessity resides with the teacher and his procedure, and is thus external to mathematics. The regulation of the learners is thus under the aspect of the Imaginary once again.

### Summary

In this event, the realised topic does not correspond with the intended topic or the encyclopaedia, placing the event in quadrant IV. The regulation of the learner depends primarily on the Imaginary.

## Primary data production P1 Lesson 2 EE1

### 1) Generating evaluative events

Table A1.7 Evaluative events P1 Lesson 2

Time	Evaluative event/sub-event	Activity	Type
00:00 – 29:00	E <sub>1</sub>	Working through homework questions on solving linear inequalities and representing solutions on a number line	Expository
00:00 – 12:00	E <sub>1,1</sub>	Solving inequality $3x - 1 \leq 7$ and representing solution on number line	Expository
12:00 – 19:28	E <sub>1,2</sub>	Solving inequality $-3x + 14 < 2$ and representing solution on number line	Expository
19:28 – 29:00	E <sub>1,3</sub>	Solving inequality $3(2x + 5) \leq 18$ and representing solution on number line	Expository
29:00 – 37:02	E <sub>2</sub>	Learners writing and working on next question, teacher walking around checking.	Exercise

### 2) Describing operational activity

The teacher starts by marking the homework on linear inequalities, saying that "nobody should write because there's no need to write if you've done your homework yesterday".

He goes around checking who has done the homework, as he says "I had a talk with a few parents yesterday and it's amazing how little they know about you." ... check ... "I didn't choose you to be here, I'm just here to teach you. Get in the right frame of mind and start working."

He tells them not to write while he goes through it, but that he'll give them time afterwards, saying that "two things will happen then" – one, that they will understand the work, two, that they will have it in their books.

I have not described the operational activity of this event in as much detail as the previous event – here I just describe how the teacher approaches each question and his comments to learners, without constructing diagrams or tables because the examples used here are of the same type and form as those in the previous lesson and are approached using the same procedure and steps.

**Number one:**  $3x - 1 \leq 7$

"Ok the first thing I need to do I need to get my x's on one side and I need to get my constants or my numbers on the other side".

In this lesson, for the first time, he tells them that it doesn't matter which side (before he had said left for variables and right for constants).

He adds one on both sides, giving this explanation: "To get rid of this negative one, I need to add a one, okay because a negative one plus one is equal to nought when they cancel." – here for the first time he does explicitly draw on additive inverses and explain why they can add a one to 'get rid of' a negative one, possibly because it is the first time they are 'getting rid of' a negative number in this series of lessons.

He then asks them: "Ok I only want  $x$  so what do I need to get rid of now?"

"The three, so I must divide by three". Once again, he specifies the order in which they must "get rid" of numbers. This time he doesn't explain why he chooses to divide by three in order to 'get rid of' the three.

$$\begin{array}{l} 3x - 1 \leq 7 \\ 3x - 1 + 1 \leq 7 + 1 \\ 3x \leq 8 \\ \frac{3x}{3} \leq \frac{8}{3} \end{array}$$

**Figure A1.7** Solving  $3x - 1 \leq 7$

He then says that because he's dividing "with a positive number", his inequality sign does not change.

Once he has finished the solution and represented it on the number line he tells them "That is how your solution needs to look like".

**Number two:**  $-3x + 14 < 2$

He starts this question by asking "What number on the left hand side should I get rid of?" – most learners answer fourteen (they are now clear about which one he wants them to "get rid of" first, unlike in the previous lesson).

$$\begin{aligned} \text{2) } & -3x + 14 < 2 \\ & -3x + 14 - 14 < 2 - 14 \\ & -3x < -12. \end{aligned}$$

**Figure A1.8** ‘Getting rid of’ the fourteen

His next question is: “I have negative three  $x$ , but I only want  $x$  so what do I do to get rid of negative three?”

When dividing negative twelve by negative three, he gives the learners a hint: “a negative divided by a negative is of course a ...” to which the learners reply “positive”.

**Number 3:**  $3(2x + 5) \leq 18$

As they work through this example, the teacher says:

“Now there’s brackets so what do I need to do?”

“I need to multiply”

“What do I need to get rid of?”

“what I do on the left hand side I must do on the right hand side”

“What do I need to get rid of? I only need the  $x$  so what must I do?”

“is the six negative or positive? Does the sign change?”

It seems that the procedure has become automatic by this point – the teacher runs through a series of questions as he does this example and learners chant out the answers in unison. It seems that learners know the procedure well by now.

### 3) Activating the mathematics encyclopaedia

See P1 L1 EE1

## Secondary data production, P1 Lesson 2 EE1

### 1) Realisation of content

In this event the teacher works through the homework exercise on solving linear inequalities. We see the same issues as P1 Lesson 1 EE1, thus the way in which the content is realised differs from the intended topic from the point of view of the encyclopaedia. Despite this, the realised topic still corresponds with the encyclopaedia (i.e. no mathematical propositions are violated), for the same reasons as in EE1 for lesson 1.

### 2) Regulation of the learner

We see the same things emerging here as in the previous lesson. The teacher works through the examples using the same procedure, and specifies the order in which learners must “get rid of” terms. He emphasises the form of the solution - “that is how your solution needs to look like”.

Analysis of this event indicates that the procedure has become like an automaton by this point – after enough repetition and practice, learners can get it right despite themselves. In terms of appeals made to an authorizing ground, the teacher appeals 4 times to iconic and spatial features of solutions, 5 times to procedural features of solutions and 6 times to mathematical propositions, definitions and processes. Once again we see him appealing to both mathematical and non-mathematical language in his description of each step which needs to be taken, but the appeals to extra-mathematical factors exceeds those made to factors from within the field of mathematics. Necessity is situated external to mathematics in the steps which need to be taken, the rules which need to be remembered and the form of the solution, rather than in the propositional ground underlying the procedure (the notion of inverses and of equality). This suggests that despite the appearance of a Symbolic structure in the teacher’s procedure, his procedure is a re-symbolisation of the content with the purpose of regulating the learners to reach the correct solutions despite themselves. The regulation of the learner is thus under the aspect of the Imaginary.

### Summary

In this event, the intended topic is not realised but the realised topic still corresponds with the encyclopaedia – quadrant II. The regulation of the learner depends primarily on the Imaginary.

### Primary data production P1 Lesson 2 EE2

#### 1) Generating evaluative events

Table A1.8 School P1 Lesson 2 EE2

Time	Evaluative event/sub-event	Activity	Type
29:00 – 37:02	E <sub>2</sub>	Learners writing and working on next question, teacher walking around checking.	Exercise

#### 2) Describing operational activity

While the learners are working on the last two questions, two learners call the teacher over and ask him questions – “so do I move that over?”, “do I take out the six?” Their language is interesting – it suggests a physical shifting of numbers. This is a question with  $x$ ’s on both sides –  $3x + 12$  on one and  $9x - 6$  on the other. The girls are asking whether to get rid of the 6, but the teacher says they need to **first** get  $x$ ’s on one side, and asks them how they get rid of the nine  $x$ .

$$\begin{aligned}
 & 3(x+4) = 9(x+2) \\
 & 3x+12 = 9x+18 \\
 & 3x+12-12 = 9x+18-12 \\
 & 3x = 9x+6 \\
 & 3x-9x = 9x+6-9x \\
 & -6x = 6
 \end{aligned}$$

Figure A1.9 First learner’s work on question four

$$\begin{aligned}
 4 \quad & 3(x+4) \leq 9(x+2) \\
 & 3x+12 \leq 9x+18 \\
 & 3x+12-12 \leq 9x+18-12 \\
 & 3x \leq -15 \\
 & \frac{3x}{3} \leq \frac{-15}{3} \\
 & x \leq -5
 \end{aligned}$$

**Figure A1.10** Second learner's work on question four

This learner just “gets rid of” the nine  $x$  without doing anything to it! I think she is actually adding nine to eighteen (twenty seven) and then subtracting twelve. This should give fifteen but she writes negative fifteen. It is not clear what she is doing in this step. Once she has three  $x$  on the left she is able to divide by three.

At this point the teacher stops the learners and instructs them all to write down number four, some protest and say they've done it, but he insists.

He instructs them to do the first step alone – “remove the brackets”, and he goes around to mark the first step.

One girl says “I got it right now I must write it over!”, but writes the following:

$$\begin{aligned}
 & 3(x+4) \leq 9(x+2) \\
 & 3x+12 \leq 9x+18 \\
 & 3x+12-12 \leq 9x+18-18 \\
 & 3x \leq 9x \\
 & \frac{3x}{3} \leq \frac{9x}{3} \\
 & x \leq 3
 \end{aligned}$$

**Figure A1.11** Third learner's work on question four

Here she “gets rid of” the twelve and the eighteen incorrectly – the notion of equality is suspended, she is just “getting rid of” without preserving identity at the level of value. She then changes her final answer from  $3x$  to  $3$ , even though according to her calculations  $3x$  is correct. But she may have changed her answer because she knows that the solution usually has only a number on the right hand side.

This example is slightly different to the previous ones in that it has variables on both sides. But we do not see how the teacher explains the procedure as the lesson ends before he is able to do so.

### 3) Activating the mathematics encyclopaedia

See P1 L1 EE1

## Secondary data production P1 Lesson 2 EE2

### 1) Realisation of content

In this event, learners work on an exercise on solving linear inequalities. The language of the learners who ask a question, discussed in the primary data production, is interesting as it suggests a physical shifting of symbols (“take out”, “move over”) as part of the process of ‘getting rid of’ something. This physical or spatial shifting of symbols (described in Gripper, 2011a) is an unfamiliar operation from the point of view of the encyclopaedia.

The work of three learners seen in this event once again reveals confusion about “getting rid of” terms and about the notions of equality and additive inverses. The above examples suggest that the realised topic does not correspond with the intended topic or the mathematics encyclopaedia in these learners’ work. This conclusion is once again based on the work of a few learners, but as previously discussed, I make tentative conclusions based on a few learners’ work due to the uniformity of work often produced by learners in this context.

### 2) Regulation of the learner

It is difficult to comment on the regulation of the learner in this event as we do not see many interactions between the teacher and learners. The one interaction we see where the teacher stops all the learners and instructs them to start number four again (he has seen a few of them proceeding incorrectly). The ones who have got it correct complain, but he insists they all start again. He tells them to do the first step only – remove the brackets, then goes around to check what they have done, during which time the lesson ends. This short interaction suggests that the teacher treats all learners the same and does not allow for some learners to work at their own pace or acknowledge that some learners may in fact have reached the correct answer. Instead he insists that they all start again, and prescribes the first “step” they should take, instructing them to wait after completing that step before moving on. There is only one appeal made in this event – to a procedural feature of the solution. But I base my analysis of this event on the previous events’ analysis, as the learners’ work on the exercise is regulated by the explanation they have been given in previous events. This suggests that once again necessity is situated external to mathematics – within the teacher, who determines whether the answer is right or wrong, and thus that the regulation of the learners is under the aspect of the Imaginary.

### Summary

In this event, the intended and realised topics do not correspond. The realised topic does not correspond with the mathematics encyclopaedia either – quadrant IV. The regulation of the learner depends primarily on the Imaginary

## Primary data production P1 Grade 10 Lesson 3 EE1

### 1) Generation of evaluative events

Table A1.9 Evaluative events School P1 Lesson 3

Time	Evaluative event/sub-event	Activity	Type
00:00 – 20:08	E <sub>1</sub>	Using “gradient-intercept” method to plot the lines $y = \frac{1}{2}x + 7$ and $y = -\frac{2}{5}x + 4$	Expository
20:08 – 25:33	E <sub>2</sub>	Students working alone on the exercise (sketching two line graphs), teacher answering questions.	Exercise

## 2) Describing the operational activity

The teacher starts this lesson by handing out graph paper to learners and instructing them to take notes about the “second method”, which he also refers to as “method b” and the “gradient intercept method”. He instructs the learners how to draw their axes on the page (also referred to as “grid”) – “count fifteen from the left and then from the bottom, down we count twenty-five”.

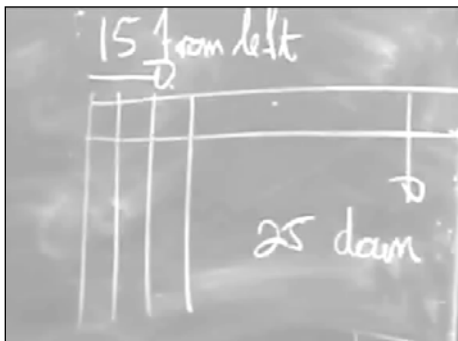


Figure A1.12 Drawing the axes

He explains the position and marking of the axes as follows: “Right, fifteen from the left and I get my  $x$ -axis. Then my  $y$ -axis then on twenty-five from the top I get my  $x$ -axis ... and then you go from nought upward ... on the  $y$ -axis the positive side I go on twelve and then down twelve, seven to the left, seven to the right.” These instructions are focused on direction – up and down, left and right, as well as on counting out the numbers. He initially does not mention the negative numbers, but later says “from nought to twelve and from nought down to negative twelve”. It takes the learners almost 11 minutes to complete the axes. Once the axes are drawn, he gives the instruction to “sketch the following er line graphs” and writes a few equations on the board:

$$1) y = \frac{1}{2}x + 7$$

$$2) y = \frac{2}{3}x - 1$$

$$3) y = -5x + 2$$

$$4) y = -\frac{4}{c}x + 3$$

He then refers to the “first method” from yesterday, which was “with the table”, where he mentions that they used four or five points when in actual fact they only need two points to draw the line – “then you draw it through those two points”. He describes the “table method” as “tedious because you need to know how to substitute correctly, you need to get the correct answer, and then you need to plot the points also in, at, in the same order or correct place”. But he introduces the “gradient intercept method” as easier and quicker. His reason for referring to the gradient intercept method as the easier one with only two points to plot is that “the modern people, they want to do things very quickly” – his explanation of why only two points are needed is related to speed of execution as opposed to the definition of a line.

He starts by referring to the standard form of  $y = mx + c$ , asking a student whether the lines “or the formulas of these lines” he has written on the board are in standard form. She says yes, “because it all ends in numbers”, which he describes as a brilliant answer. When asked, the learners know that the  $m$  represents the gradient, and the  $c$  the  $y$ -intercept. He asks the students what they will need to sketch the line “if the method is the gradient intercept method”, and when they answer the gradient and the  $y$ -intercept, he says “smart thinking”! He then proceeds to give them a note about the method “that you remember every time how to do it”. He instructs them to “take notice, what’s going on so that you will, will can do it very effectively and efficiently and quickly.” This teacher often appeals to efficiency and speed, which appears to be his way of

motivating the learners to use the method he favours – the gradient-intercept method – and to ensure that the learners draw the correct lines, see later discussion on this.

When instructing the students to plot the y-intercept he tells them to make a dot “here” (he points to positive 7 on the y-axis) with no reference to the coordinates of the point. When he asks the students where the gradient is, they say “it’s that line”, another learner says “one”, and another “x”. Some correctly say “it’s a half”. The teacher describes the gradient as “the coefficient of  $x$  or the thing, the number that stands in front of  $x$ ”. Later he refers to it as the “change in  $y$  over the change in  $x$  but that’s for another day”.

Teacher: Okay, now where is the gradient?

Learner: Sir, it’s that line.

Another Learner: One.

Teacher: Okay, where’s the gradient, number one?

Learner: X.

Teacher: Don’t know?

Learners: Half. A half. it’s a half.

Teacher: It’s a half. There’s the gradient  $m$ , so the co-efficient of  $x$  or the thing, the number that stands in front of  $x$  that is my?

Learner: Half.

Teacher: My gradient. Okay. And in this case it is?

Learners: Half.

### Transcript School P1 Lesson 3

In order to plot the second point, he instructs them to use “that number first” (pointing to the number on the numerator, which is one) – “Now what you do for me is the following: you have your y-intercept, now listen carefully, this is the method, okay? Then, we’re always going to use that number first, okay? That’s one, one unit, right?”



**Figure A1.13** “Use that number first”

Again, he specifies the order in which they should proceed – first counting up and then across, which is not mathematically necessary.

Then he tells them to count one unit up from the seven:

Teacher: So I've reached eight now and then I go to the two and I count two units to the?

Learners: Left.

Teacher: Right.

A learner asks how they know why to move to the right:

Learner: Now why, how do you know where to move to the right?

Teacher: Okay, now that's a very good question. It's a very good question. What is the sign in front of this thing? (he points to the gradient,  $\frac{1}{2}$ )

Learner: A plus.

Teacher: A positive, are you with me? Right, so we sometimes will have a positive gradient or a ... ?

Learners & teacher: Negative gradient.

Teacher: Now lines, as you were told last year with positive gradients, they go in that direction (uses his arms to indicate) ... in lines with negative gradients they go in that direction (uses his arm) ... So when I have a positive gradient, I change direction, I count the one unit up and then I change direction to the right with the positive sign ... But just to tell you if I counted my top units then I go to the?

Learner: Right.

Teacher: Left. When it's negative.

Transcript School P1 Grade 10 Lesson 3

**The teacher's procedure for the first example is:**

- 1) Identify the y-intercept from the equation as 7.
- 2) Plot the "y-intercept next to seven, okay? You make a dot there" (plotting the point (0, 7), but he does not state the coordinates).
- 3) Identify the gradient from the equation as  $\frac{1}{2}$ .
- 4) First look at the numerator (which is 1) (Why does he choose the numerator first?)
- 5) If the numerator is positive, count that number of units up from the y-intercept (7), if negative, count down.
- 6) Then look at the denominator (which is 2).
- 7) If the numerator is positive, count that number of units to the right, if negative, count to the left.
- 8) Plot the point.
- 9) Join the two points to get the line.

We can write this more formally:

Step 1: SEL(y-intercept,  $y = \frac{1}{2}x + 7$ ) = 7

Step 2: Plot (0,7)

Step 3: SEL(gradient,  $y = \frac{1}{2}x + 7$ ) =  $\frac{1}{2}$

Step 4: NTR( $\frac{1}{2}$ ) = 1

Step 5: SIG(1,  $\frac{1}{2}$ ) = /+/, ADD(1, 7) (count 1 up from (0,7))

Step 6: DTR( $\frac{1}{2}$ ) = 2,

Step 7: SIG(2,  $\frac{1}{2}$ ) = /+/, ADD(2,0) (count 2 units right from (0,8))

Step 8: Plot (2,8)

Step 9: Join (0,7) and (2,8)

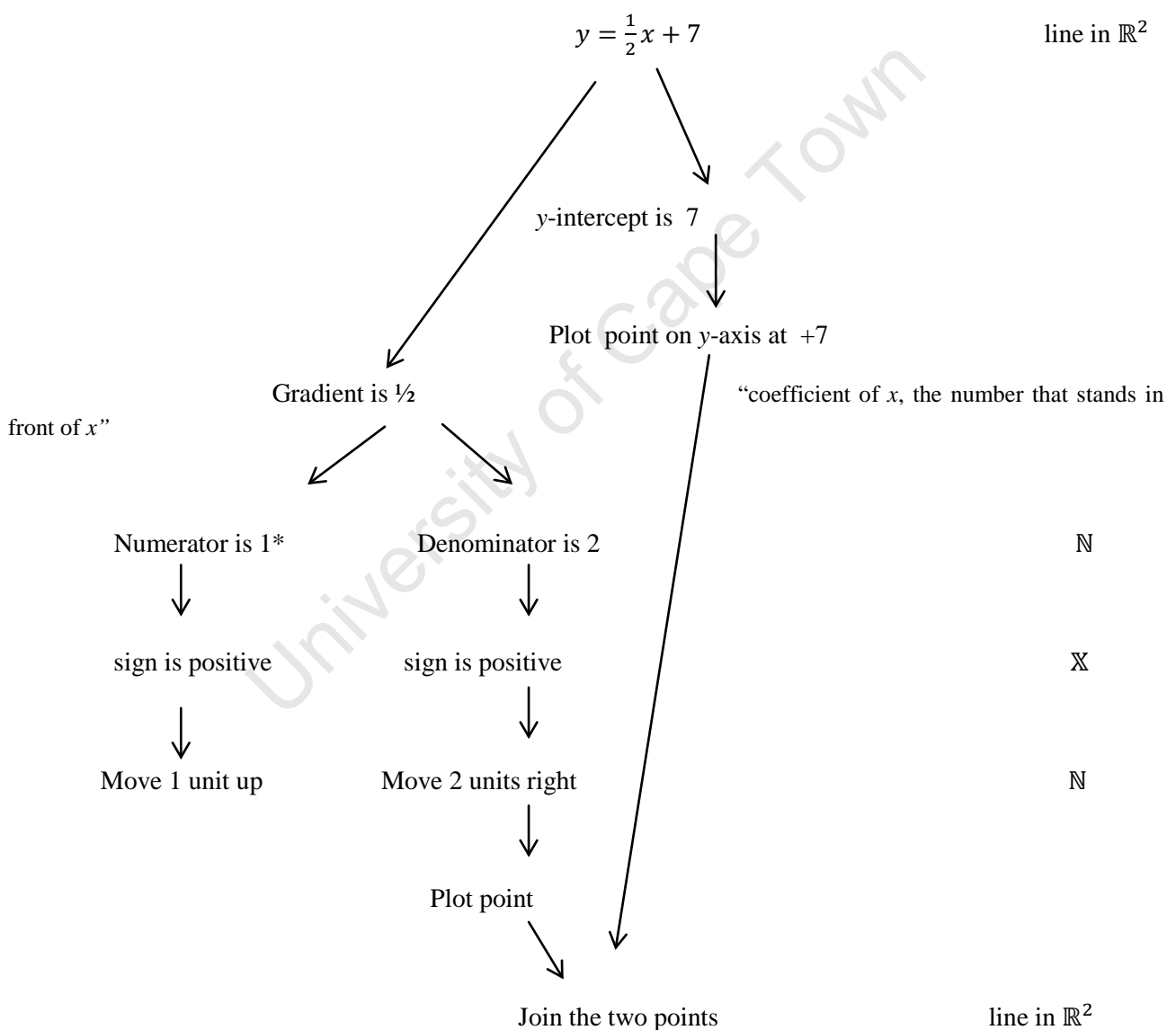


Figure A1.14 Diagrammatic representation of example one

\*NOTE: the teacher starts with the numerator and then moves onto the denominator

**Table A1.10** Operational activity for example one

T	Input	Domain	Operation	Output	Codomain
1	$y = \frac{1}{2}x + 7$	$\{(x, y) \in \mathbb{R}   y = mx + c\}$	Select y-intercept	7	$\{c \in \mathbb{Q}   y = mx + c\}$
2	7	$\{c \in \mathbb{Q}   y = mx + c\}$	Plot	Point on y-axis at 7	coordinate in $\mathbb{R}^2$
3	$y = \frac{1}{2}x + 7$	$\{(x, y) \in \mathbb{R}   y = mx + c\}$	Select gradient	$\frac{1}{2}$	$\{m \in \mathbb{Q}   y = mx + c\}$
4	$\frac{1}{2}$	$\{m \in \mathbb{Q}   y = mx + c\}$	Identify the numerator	1	$\{a \in \mathbb{N}   \frac{a}{b} = m\}$
5	1	$\{a \in \mathbb{N}   \frac{a}{b}\}$	Identify the sign	+	$\mathbb{X}$
6	1, 7	$\mathbb{N}$	Add/count 1 unit <b>up</b> from (0,7)	8	$\mathbb{N}$
7	$\frac{1}{2}$	$\{m \in \mathbb{Q}   y = mx + c\}$	Identify the denominator	2	$\{b \in \mathbb{N}   \frac{a}{b} = m\}$
8	2	$\{b \in \mathbb{N}   \frac{a}{b} = m\}$	Identify the sign	+	$\mathbb{X}$
9	2, 0	$\mathbb{N}$	Add/count 2 units <b>right</b> from (0,8)	2	$\mathbb{N}$
10	2	$\mathbb{N}$	Plot	Point at (2, 8)	coordinate in $\mathbb{R}^2$
11	(0,7) ; (2, 8)	coordinates in $\mathbb{R}^2$	Join	Line through (0, 7) and (2, 8)	Line in $\mathbb{R}^2$

Example two is shown in Figure A1.15

### 3) Activating the Mathematics encyclopaedia

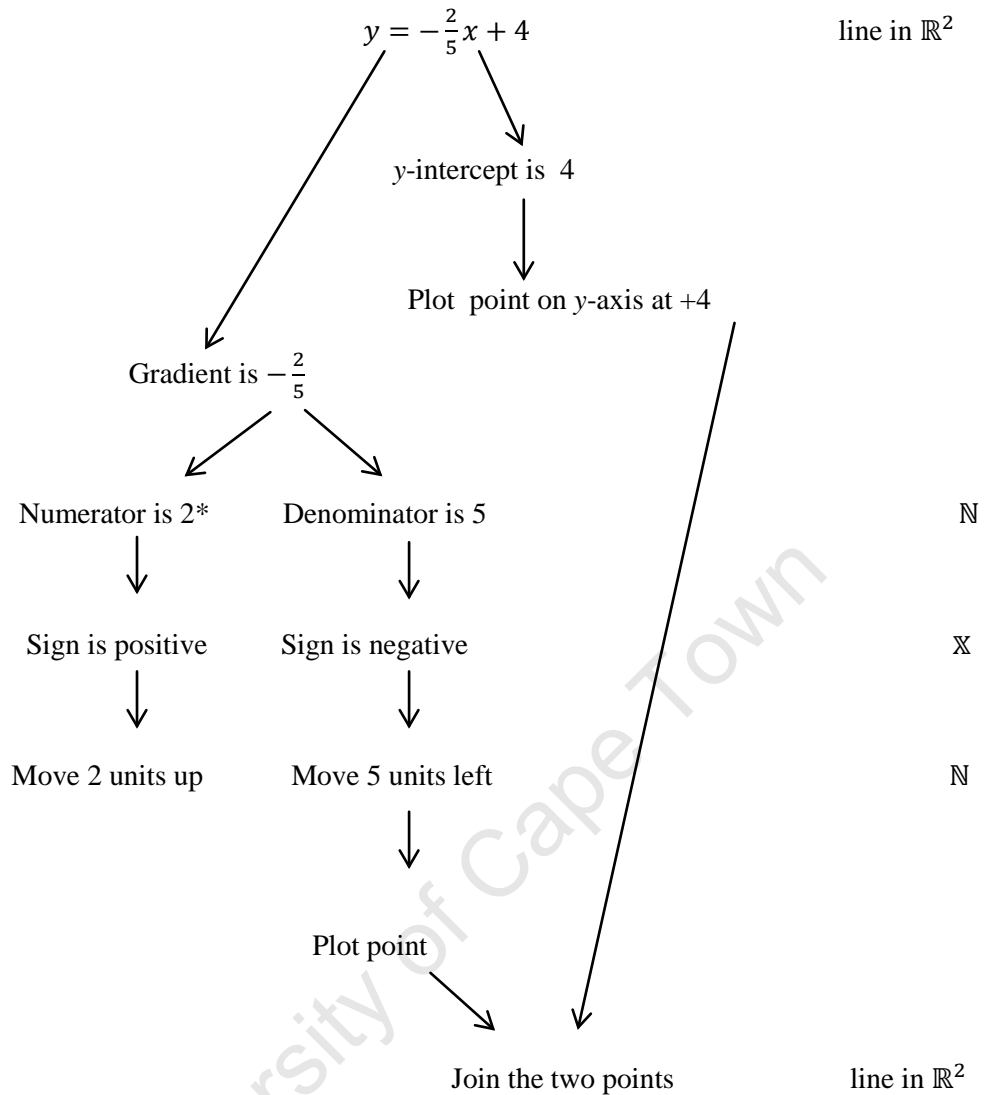
A Euclidean plane with a chosen Cartesian system is called a *Cartesian plane*. The points of a 2-dimensional Cartesian plane can be identified with all possible pairs of real numbers (all points or ordered pairs  $(x,y)$  where  $x$  and  $y$  are real numbers) - that is with the Cartesian product or square,  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$  where  $\mathbb{R}$  is the set of all reals. In  $\mathbb{R}^2$ , every line  $L$  can be described by a linear equation of the form  $L = \{(x, y) | ax + by = c\}$ , with fixed real coefficients  $a$ ,  $b$  and  $c$  such that  $a$  and  $b$  are not both zero. Important properties of these lines are their gradient,  $x$ -intercept and  $y$ -intercept.

A line is a “straight one-dimensional figure having no thickness and extending infinitely in both directions” (Wolfram). In Euclidean geometry, a line is a series of points that extends in two opposite directions without end. It consists of a set of points and is a subset of a plane. In coordinate geometry, lines in a Cartesian plane can be described algebraically by linear equations. In two dimensions, the characteristic equation can also be given by the *gradient--intercept form*:  $y = mx + c$  where:

$m$  is the gradient or slope of the line.

$c$  is the  $y$ -intercept of the line.

$x$  is the independent variable of the function  $y$ .



**Figure A1.15** Diagrammatic representation of second worked example

\*The teacher says: “I go to the top number, how many units must I count” ... “two” – again he starts with the numerator.

The gradient ( $m$ ) of the line through points A  $(x_a, y_a)$  and B  $(x_b, y_b)$  is given by  $m = \frac{y_b - y_a}{x_b - x_a}$  and the equation of this line can also be written:  $y - y_a = m(x - x_a)$ . In order to sketch a line using the gradient and the y-intercept, we would plot and join the two points  $(0; c)$  and  $(0 + (x_b - x_a); c + (y - y_a))$ .

The propositional ground underlying the sketching of a line graph includes the definition of a line, with its properties, as well as the understanding that any point on a line represents an ordered pair of reals in  $\mathbb{R}^2$ . From this description of a line, it follows that one method of sketching a line should not be “easier” or more accurate than another, as for any line only two points are required in order to sketch the line. It is thus interesting that the teacher’s procedure involves privileging of the gradient-intercept method.

## Curriculum

The FET Mathematics NCS learning outcome two states that “*the learner is able to investigate, analyse, describe and represent a wide range of functions and solve related problems*” (DoE, 2003: 12).

The assessment standard which refers to line graphs states that learner should be able to:

“10.2.2 Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence to generalize the effects of the parameters  $a$  and  $q$  on the graphs of functions including:  $y = ax + q$ ” (DoE, 2003: 22).

There is no reference to a textbook during this lesson.

## Secondary data production P1 Lesson 3 EE1

### 1) Realisation of content

The intended topic of this lesson is the sketching of line graphs using the “gradient-intercept” method. Two examples are worked through in this event.

The teacher specifies the order in which to carry out the procedure, which is not mathematically necessary – “we’re always going to use that number first, okay?” – he plots the y-intercept first, then a second point by first counting up or down first and then from left to right. The procedure has a strong focus on counting and moving in certain directions (up/down, right/left).

At one point a learner asks how they know why to move to the right:

Learner: Now why, how do you know where to move to the right?

Teacher: Okay, now that’s a very good question. It’s a very good question. What is the sign in front of this thing? (he points to the gradient,  $\frac{1}{2}$ )

Learner: A plus.

Teacher: A positive, are you with me? Right, so we sometimes will have a positive gradient or a ...?

Learners & teacher: Negative gradient.

Teacher: Now lines, as you were told last year with positive gradients, they go in that direction (uses his arms to indicate) ... in lines with negative gradients they go in that direction (uses his arm) ... So when I have a positive gradient, I change direction, I count the one unit up and then I change direction to the right with the positive sign ... But just to tell you if I counted my top units then I go to the?

Learner: Right.

Teacher: Left. When it’s negative.

## Transcript School P1 Grade 10 Lesson 3

This interaction suggests that the rules for plotting the points are regulating the learners at this point – they are told to look at the sign of the gradient and then go to the left or right based on that, rather than drawing on the definition of gradient. The teacher describes the gradient as “the coefficient of  $x$  or the thing, the number that stands in front of  $x$ ”, and his procedure separates the sign and the value of the gradient in order to enable the generation of a rule for plotting the second point of the line (after the y-intercept has been plotted) – first, if the sign of the numerator is positive, move or count up from the y-intercept the number of

units of the value of the numerator, then if the sign of the denominator is positive, change direction and move to the right the number of units of the value of the denominator. This procedure splits the gradient up into two natural numbers - instead of operating on the gradient as a property of a line, and on Cartesian points, the teacher's procedure operates on the separate signs and values of the gradient. He separates the gradient into two natural numbers, which are used to indicate how many units to 'move' or count, first up and then across, and a sign which is used to indicate which direction to move or count in. There is thus a shift in the domain at this point, suggesting that the intended topic is not realised in this event. The way in which the teacher emphasises the order in which the steps must be followed confirms that the intended topic is not realised – he insists of learners plotting the  $y$ -intercept first, then referring to the numerator of the gradient and moving up, then to the denominator and moving left/right – this order is not mathematically necessary.

Thus the realised topic does not explicitly draw on the propositional ground underlying the intended topic and the intended topic is not realised. But despite this, the procedure is not mathematically incorrect, and mathematical propositions and processes are not violated in this event, placing it in quadrant II.

## 2) Regulation of the learner

Generally in this lesson the relevant mathematical definitions (of a line and of gradient) are not explicitly referred to or used in generating criteria. The features of a line are thus not explicitly regulating the activity of the learners in this lesson. One of the primary regulative resources which the teacher draws on in this procedure is the position of the  $y$ -intercept and the gradient in the equation (it would be interesting to see how the learners dealt with an equation which was not in 'standard form'), as well as the use of negative and positive signs as indicators of which direction to move or count in. The teacher describes the gradient as "the coefficient of  $x$  or the thing, the number that stands in front of  $x$ ", and his procedure separates the sign and the value of the gradient in order to enable the generation of a rule for plotting the second point of the line (after the  $y$ -intercept has been plotted) – if the sign of the numerator is positive, move or count up from the  $y$ -intercept the number of units of the value of the numerator, then if the sign of the denominator is positive, change direction and move to the right the number of units of the value of the denominator.

The concept of a line as a selection of points in the Cartesian plane, each of which is an ordered pair of reals, is not explicitly present in the teacher's procedure due to his focus on gradient-intercept method. Later in the lesson a learner points to the space on her page above the  $x$ -axis and asks "why are we always working there and not here?" as she then points to the space below the  $x$ -axis, which reveals her lack of understanding of the concept of a line or of the Cartesian plane.

In a previous lesson the teacher has taught the learners how to draw line graphs using what he calls the "table method". In this event he describes the "table method" as "tedious because you need to know how to substitute correctly, you need to get the correct answer, and then you need to plot the points also in, at, in the same order or correct place". But he introduces the "gradient intercept method" as easier and quicker. His reason for referring to the gradient intercept method as the easier one with only two points to plot is that "the modern people, they want to do things very quickly" – his explanation of why only two points are needed to plot a line is related to speed of execution as opposed to the definition of a line. The teacher's privileging of the gradient-intercept method, due to it being "easier and quicker" than the other methods could be seen as an attempt to engage and please the learners, and to get them to identify with the gradient-intercept method. As such it can be described as an appeal to the Imaginary. The teacher wants the students to identify with this method so that they will be able to draw the graph despite their lack of knowledge about the definition and properties of a line.

Early in the event he gives them a note about the method "so that you remember every time how to do it". He instructs them to "take notice, what's going on so that you will, will can do it very effectively and efficiently and quickly." This teacher often appeals to efficiency and speed, which appears to be his way of motivating

the learners to use the method he favours – the gradient-intercept method – and to ensure that the learners draw the correct lines.

The teacher does not appear to have high expectations of his learners, as seen in the way that he drills them in the method he chooses, and strongly encourages them to use that method. The questions he asks them also show this. At one point in this event the teacher refers to the standard form of  $y = mx + c$ , asking a student whether the lines “or the formulas of these lines” he has written on the board are in standard form. She says yes, “because it all ends in numbers”, which he describes as a brilliant answer. When asked, the learners know that the  $m$  represents the gradient, and the  $c$  the y-intercept. He asks the students what they will need to sketch the line “if the method is the gradient intercept method”, and when they answer the gradient and the y-intercept, he says “smart thinking”!

Overall in this event, 12 appeals are made to an authorizing ground – four to ease, speed or efficiency; four to iconic or spatial features of solutions; one to procedural features of solutions; and three to mathematical propositions, definitions or processes. This, and the analysis above, suggests that the necessity in this event resides with the teacher and his procedure for sketching a line using the gradient-intercept method, rather than in the field of mathematics (definition of a line). Although the activity being carried out can be described as Symbolic in nature, the regulation of the learner is under the aspect of the Imaginary due to the predominance of appeals to extra-mathematical factors, and the way in which the image held by the teacher of the learners motivates him to replace the content (lines with their definition and propositions which enable sketching) with one specific method (gradient-intercept) in which the order of steps is prescribed.

### Summary

This event falls within quadrant II. The regulation of the learner depends primarily on the Imaginary.

### Primary data production P1 Grade 10 Lesson 3 EE2

#### 1) Generation of evaluative events

Table A1.11 School P1 Lesson 3 EE2

Time	Evaluative event/sub-event	Activity	Type
20:08 – 25:33	E <sub>2</sub>	Students working alone on the exercise (sketching two line graphs), teacher answering questions.	Exercise

#### 2) Description of operational activity

As the learners are working on an exercise, there are quite a few questions for the teacher. First a learner asks about the changing of direction – there seems to be some confusion about how to decide when to go to the left or the right. The teacher reiterates the point that they must go up first and then to the right for a positive gradient and to the left for a negative gradient. He stops them saying “one more thing” in response to a question, writing up the example  $y = 4x - 9$ . He asks them what the y-intercept is, to which they reply “nine”. He says it’s “not nine it’s negative nine. So you go to the y-axis and you make a dot or your plot negative nine”. He then points to the four – “but now, this four, what can I do with this four?” He leads them on:

Teacher: Is it a fraction?

Learners: No.

Teacher: Can I write it as a fraction?

Learners: Yes.

Teacher: How do I write it as a fraction?

Learners: Four over one.

Teacher: Four over one. Exactly. So I do exactly the same thing. I go four units up. It's positive. And one unit to the right.

### Transcript School P1 Lesson 3

Again, he emphasizes the order in which to do things, and separates the gradient into numbers and signs. While the learners are working he goes to help a learner who struggled to understand the first example. She has plotted the point seven on the y-axis, and he then asks her how many units she must count. She says "two". He tells her "no, no, no, you have to start with the top number" to which she replies "oh, one". It seems that he misunderstood her question, because she now says "no, I asked why are we always working there (pointing to the first and second quadrant), what about this? (pointing to the third and fourth quadrant)". He just says: "was that the question. Well do the rest for me please".

#### 3) Activation of mathematics encyclopaedia

See P1 L3 EE1

### Secondary data production P1 Lesson 3 EE2

#### 1) Realisation of content

In this event we once again see the teacher emphasising the order in which steps must be carried out, which is not mathematically necessary. Thus the intended topic is not realised in this event, as in the first event, but the principles of mathematics do not appear to be violated – quadrant II.

#### 2) Regulation of the learner

Once again he emphasises the steps of his method, specifically the order in which points must be plotted, as seen in the exchange described above, where he says: **"no, no, no, you have to start with the top number"**. The way in which the teacher emphasises the order again here suggests that he is placing necessity as external to mathematics and within his procedure with its order of steps. In terms of appeals made to an authorizing ground, he makes one appeal to ease/speed/efficiency, two to iconic or spatial features of solutions and one to procedural features of solutions. He thus does not appeal at all to mathematical propositions, definitions or processes in this event, and the regulation of the learners is thus once again under the aspect of the Imaginary.

### Summary

This event falls in quadrant II. The regulation of the learner depends primarily on the Imaginary.

## Appendix 2: Analysis for School P2

### Primary data production P2 Lesson 1 EE1 and EE3

**Table A2.1** Evaluative events School P2 Lesson 1

Time	Evaluative event/sub-event	Activity	Type
00:00 – 0:40	E <sub>1</sub>	Recap of previous lesson: converting decimal (0,4) to fraction	Expository
0:40 – 05:19	E <sub>2</sub>	Converting recurring decimals to common fractions	Expository
04:00 – 05:19	E <sub>2,1</sub>	Converting 0,7̇ to a fraction	Expository
05:19 – 12:15	E <sub>2,2</sub>	Students working on classwork (converting 0,2̇ and 0,6̇3̇ to fractions)	Exercise
12:15 – 19:38	E <sub>2,3</sub>	Teacher calls student up to do 0,2̇ on board, but she first explaining the “steps”, then learner attempts it, another learner comes up to try.	Expository
19:38 – 24:30	E <sub>2,4</sub>	Learner does conversion of 0,6̇3̇ on board.	Expository
24:30 – 25:24	E <sub>3</sub>	Teacher writes up exercise for homework	Exercise

In this lesson the teacher was 8 minutes late, so the actual length was 33:24

EE1 is very short (40 seconds) the teacher recaps the procedure for converting a decimal to a fraction. She uses 0,4 as an example and writes it as 4 over 10, and then divides the numerator and the denominator by 2.

EE3 is also short (just under 1 minute). In it the teacher writes up a homework exercise.

### Secondary data production P2 Lesson 1 EE1 and EE3

Insufficient data.

### Primary data production P2 Lesson 1 EE2

#### 1) Generation of evaluative events

**Table A2.2** School P2 Lesson 1 EE2

Time	Evaluative event/sub-event	Activity	Type
0:40 – 05:19	E <sub>2</sub>	Converting recurring decimals to common fractions	Expository
04:00 – 05:19	E <sub>2,1</sub>	Converting 0,7̇ to a fraction	Expository
05:19 – 12:15	E <sub>2,2</sub>	Learners working on classwork (converting 0,2̇ and 0,6̇3̇ to fractions)	Exercise
12:15 – 19:38	E <sub>2,3</sub>	Teacher calls learner up to do 0,2̇ on board, first explaining the “steps”. Two learners attempt the solution.	Expository
19:38 – 24:30	E <sub>2,4</sub>	Learner does conversion of 0,6̇3̇ on board.	Expository

In this lesson the teacher was 8 minutes late, so the actual length was 33:24

## 2) Describing operational activity

### Evaluative event 2.1

The stated topic for this event is recurring decimal fractions. In this event, the teacher describes a recurring decimal as “something that is repeating itself”, and the reason she gives with reference to  $0, \dot{7}$  is “because of this dot, it means that this seven is repeating itself”.

The teacher begins an example but she does not state or explain what she will be doing – she just goes straight into the steps for the procedure. She emphasizes the steps throughout the lesson, first introducing them in this evaluative event through the example of converting  $0, \dot{7}$  to a fraction.

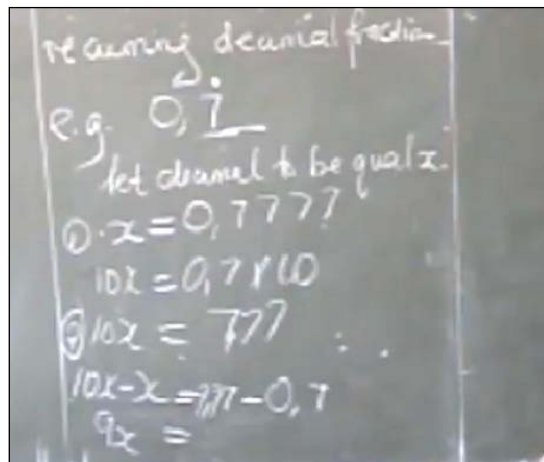
Her first step - “let decimal to be equal  $x$  (she writes  $x = 0, 7$ ), and that seven is repeating itself (she adds more 7’s as she says this)”.

Her second step – “you times by ten. Why are we multiplying by ten? ... Because there is one unit after a comma” (despite the fact that what she has written on the board is  $0,7777$  which clearly has more than one ‘unit’ after the comma – it seems that she uses the word ‘unit’ to mean ‘digit’ which is recurring). She writes  $10x = 0,7 \times 10$  and encourages learners to use their calculators to calculate  $0,7 \times 10$ , also doing so herself. She changes from working with  $x$  as  $0,7777$  to  $0,7$  in this step, although she speaks about  $0,7777$  while pointing at  $0,7$ .

recurring decimal fraction  
e.g.  $0, \dot{7}$   
let decimal to be equal  $x$   
①  $x = 0,7777$   
②  $10x = 0,7 \times 10$   
③  $10x = 7$   
 $10x - x = 7 - 0,7$   
 $9x =$

Figure A2.1 The teacher’s first, second and third steps

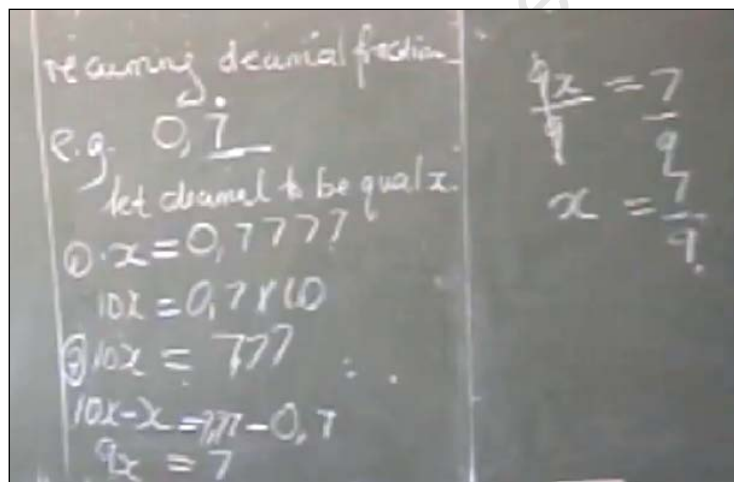
Her third step – “you must subtract equation 1 from equation 2”. She appears to get stuck when subtracting  $0,7$  from  $7$ , and consults the textbook. She then changes the  $7$  in equation 2 and in the subtraction step to  $7,77$ , but she does not change it in the previous step which led to the error, or in the  $0,7$  which she is subtracting:



**Figure A2.2** The teacher changes her second and third steps

The students use their calculators to find the difference between 7,77 and 0,7, and call out that the answer is 7,07. The teacher ignores them and says “the answer is 7”, writing it down.

Her fourth step is “solve for  $x$ ” – in which she divides both sides of the equation  $9x = 7$  by 9, to get an answer of  $x = \frac{7}{9}$



**Figure A2.3** The teacher’s procedure for converting  $0,7̄$  to a fraction

There are a few things which initially stand out from this teacher’s procedure. Firstly, the weak definition offered of a recurring decimal, without any reference to rational numbers. Secondly, the inadequate description of why it is that she multiplies by ten in her “second step”, and the absence of reference to the decimal place value system. She refers to the numbers after the comma as “units”, which is incorrect. Thirdly (which we will see even more clearly in the next evaluative event) the strong emphasis she places on the steps used for converting from a recurring decimal to a common fraction – it seems that the use of these steps regulate the activity, rather than the mathematical operations involved and their objects.

Another interesting feature of this example is her error in writing 0,7 instead of 0,7777... in the second line of her calculation, which carries through and brings confusion to the rest of the problem. She partly corrects it in the subtraction step, but not fully, which leads to an incorrect answer to her subtraction ( $7,77 - 0,7$ ), although a ‘correct’ answer to the original question. Thus local validity is suspended so that the expected answer can be obtained. Let’s examine the operational activity more closely:

Step 1 Let the decimal be equal to  $x$

Step 2 Multiply both sides by 10 because there is one unit after the comma

Step 3 Subtract equation 1 from equation 2

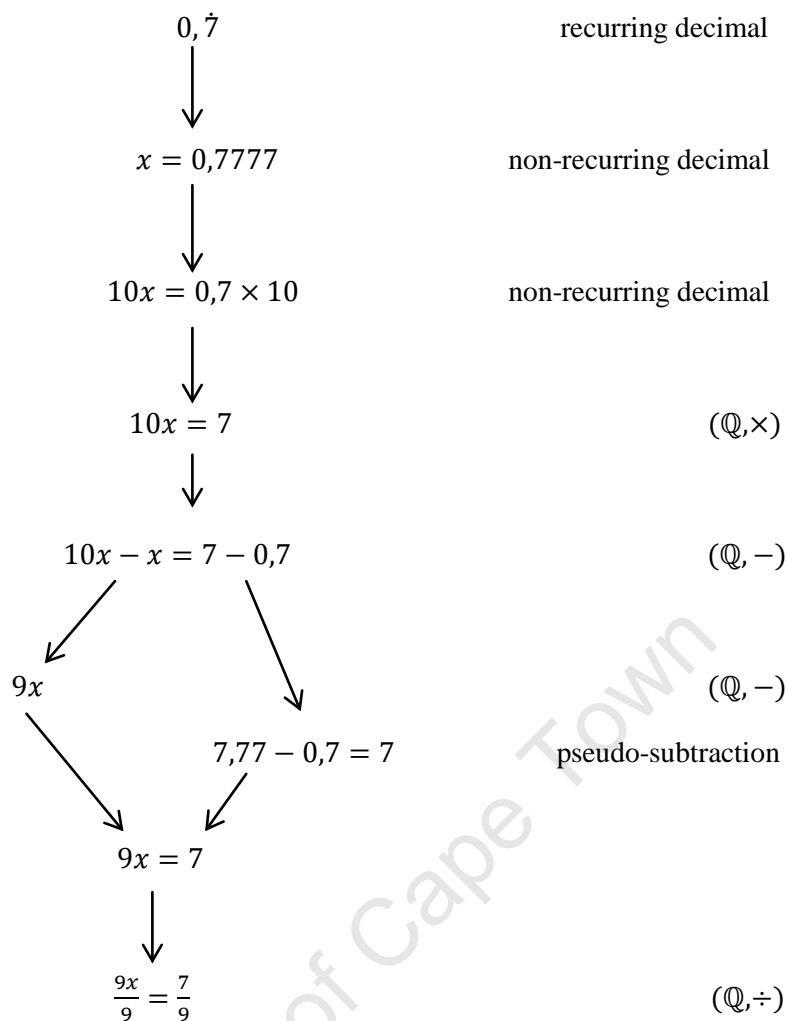
Step 4 Solve for  $x$

The teacher outlines four steps for the procedure, but if we analyse her procedure more closely, there are actually 10 transformations involved:

- 1 Let the decimal equal  $x$ :  $x = 0,7777$  (equation 1)
- 2 Multiply equation 1 by 10:  $10x = 0,7 \times 10$  (equation 2)
- 3 Simplify equation 2:  $10x = 7$
- 4 Subtract equation 1 from equation 2:  $10x - x = 7 - 0,7$  (equation 3)
- 5 Simplify equation 3:  $9x =$
- 6 Change equation 2:  $10x = 7,77$  (new equation 2)
- 7 Change equation 3:  $10x - x = 7,77 - 0,7$  (new equation 3)
- 8 Simplify the new equation 3:  $9x = 7$  (equation 4)
- 9 Divide equation 4 by 9:  $\frac{9x}{9} = \frac{7}{9}$
- 10 Solve equation 4:  $x = \frac{7}{9}$

**Table A2.3** Operational activity of worked example one

T	Input	Domain	Operation	Output	Co-domain
1	$0,7$	Recurring decimals	Existential shift	$x = 0,7777$	Non-recurring decimals
2	$x = 0,7777$	Non-recurring decimals	Multiplication	$10x = 0,7 \times 10$	Non-recurring decimals
3	$10x = 0,7 \times 10$	Non-recurring decimals	Simplification	$10x = 7,77$	Non-recurring decimals
4	$10x - x = 7,77 - 0,7$	Character strings	Pseudo-subtraction	$9x = 7$	Character strings
5	$9x = 7$	Non-recurring decimals	Division	$x = \frac{7}{9}$	Non-recurring decimals



**Figure A2.4** Diagrammatic representation of worked example one

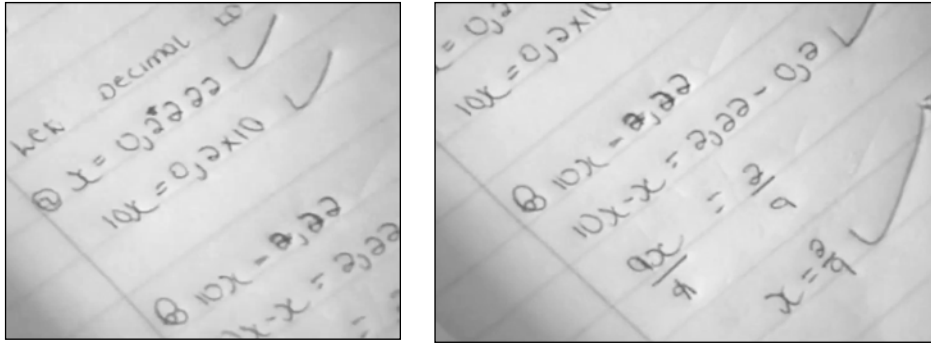
### EE2.2 – 2.4

In this second part of this event, the teacher gives the learners some classwork – two recurring decimals to convert to common fractions:  $0, \dot{2}$  and  $0, \dot{6}\dot{3}$ .

She emphasizes that this is “individual classwork”, but while the students are working on these two questions, the teacher talks a lot and gives instructions about each step. She is busy writing up the steps on the board as she talks, and says “there are four steps in this”:

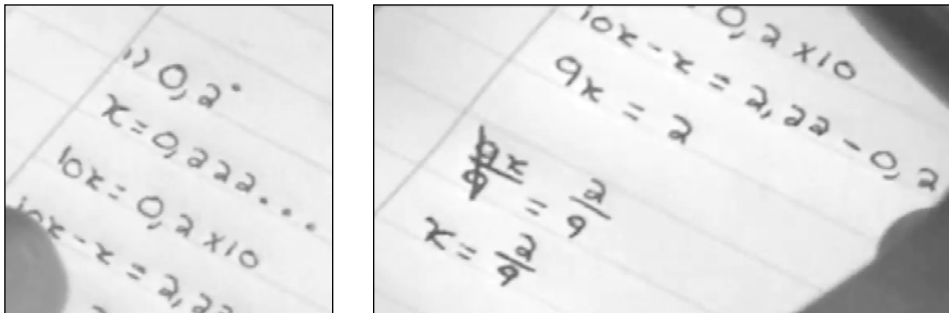
- Step 1: let decimal to be equal  $x$
- Step 2: multiply both equation by power of 10.
- Step 3: subtract equ (1) from equ (2).
- Step 4: solve  $x$

A learner’s work is shown for number one ( $0, \dot{2}$ ). She repeats the teacher’s error from the first example and writes  $0,2 \times 10$  as equal to  $2,22$ . In her subtraction step, the learner incorrectly writes  $2,22 - 0,2$  as equal to  $2$ , and ends up with the correct final answer (exactly as in the teacher’s example). The teacher marks each step of this learner’s work as correct as she moves around the class.



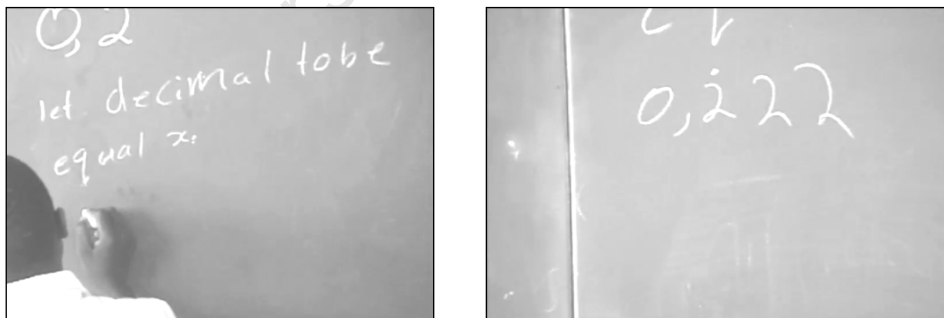
**Figure A2.5** First learner's written work

A second learner's work is shown, his work is also marked correct by the teacher:



**Figure A2.6** Second learner's written work

After 6 minutes, the teacher asks for volunteers to do the problem on the board. The first learner to come to the board starts writing, but doesn't say anything. As he writes, the teacher repeats the steps again, stating that "there are four steps in this activity". The learner is clearly copying steps from the previous example, which is still written up on the board. The teacher instructs him to explain to the class as he goes along, at which point he erases everything that he's done and starts again.



**Figure A2.7** First learner to attempt solution on the board

He tries to explain his procedure but can't make much sense of it, so the teacher calls up another learner.

The second learner does the first three lines below as part of step one, then when prompted by the teacher to do step two, writes down the number 2, even though she has already done what the teacher referred to as step two (which was the multiplication by ten) to yield equation (2). It seems that there is some confusion about whether the numbers refer to the steps or to the equations.

$$\begin{array}{l} \textcircled{1} x = 0,2 \\ 10x = 0,2 \times 10 \\ 10x = 2 \\ \textcircled{2} \end{array}$$

**Figure A2.8** Second learner to attempt solution on the board

She writes  $x = 0,2$  instead of  $0,2222 \dots$ , and gets stuck after she multiplies both sides by 10 (she has  $10x = 2$ ), and sits down.

The third learner keeps the same step 1 as the second learner ( $x = 0,2$ ), but still gets to the correct answer by following the steps used by the teacher, including the incorrect multiplication of  $0,2$  and  $10$  to get  $2,22$  and subtraction of  $0,2$  from  $2,22$  to get  $2$ . She gets a round of applause from the class.

$$\begin{array}{l} 10x = 0,2 \times 10 \\ \textcircled{2} 10x = 2,22 \\ 10x - x = 2,22 - 0,2 \\ 9x \end{array}$$

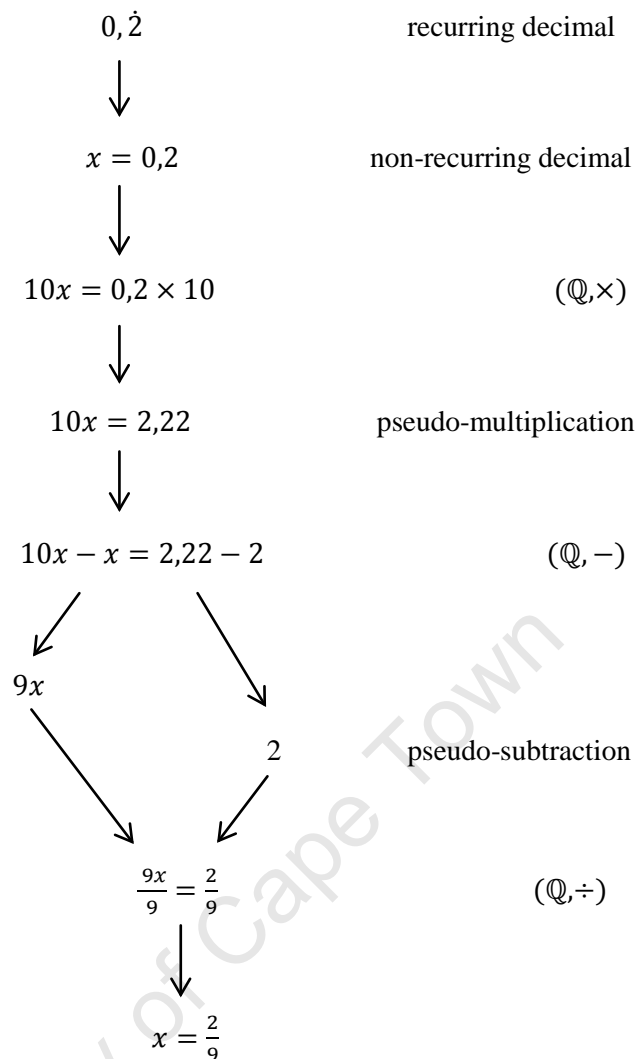
$$\begin{array}{l} 9x = 2 \\ \frac{9x}{9} = \frac{2}{9} \\ x = \frac{2}{9} \end{array}$$

**Figure A2.9** Third learner to attempt solution on the board

The lack of clarity about what a recurring decimal is and about the decimal place value system is clear in the attempts of these learners. The meaning of the notation for a recurring decimal is clearly absent, as seen in the work produced by the learners. A diagrammatic representation of this example is found in Figure A2.10

The teacher's error in the example is seen in all of these learners' work, but the method still gets the two learners whose books were shown and the third learner to write on the board to the correct answer. The learners have regulative resources which would enable them to do the subtraction of  $0,2$  from  $2,22$  correctly in different circumstances, but in this case the criteria of the teacher include an expectation that a natural number will be produced, and so this regulates their computation. But the criteria deployed by the teacher violate the operation of subtraction.

Once again, the teacher's constant reference to the steps is interesting, especially when the learners have already done what a certain step consists of, but she still insists on them stating which step they are busy with – her insistence during this evaluative event further reveals the way in which the procedure is intended as an automaton. But despite her strong emphasis on the steps and her instructions throughout the lesson, it took three learners to get this question right on the board!

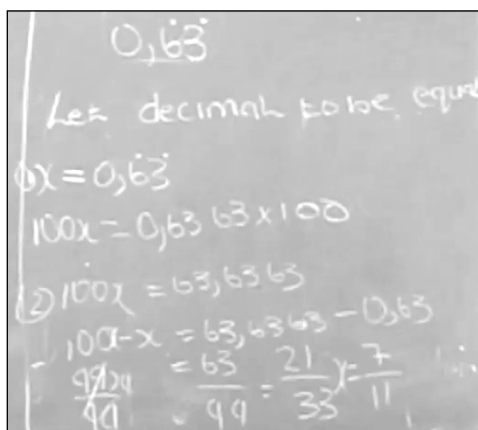


**Figure A2.10** Diagrammatic representation of second example

A learner comes up to the front to do the second question:

The learner writes “let decimal to be equal” and does not say what it is to be equal to!

She multiplies the  $x$  by a hundred, giving the reason that “there are two units after the comma”, but she does not multiply the  $0, \dot{6}\dot{3}$  by 100 until the next line when she puts the number 2 to show that she has moved onto the second step (or the second equation?).



**Figure A2.11** Learner converting  $0, \dot{6}\dot{3}$  to a common fraction

She makes the same error by subtracting 0,63 from 63,6363 and getting 63.

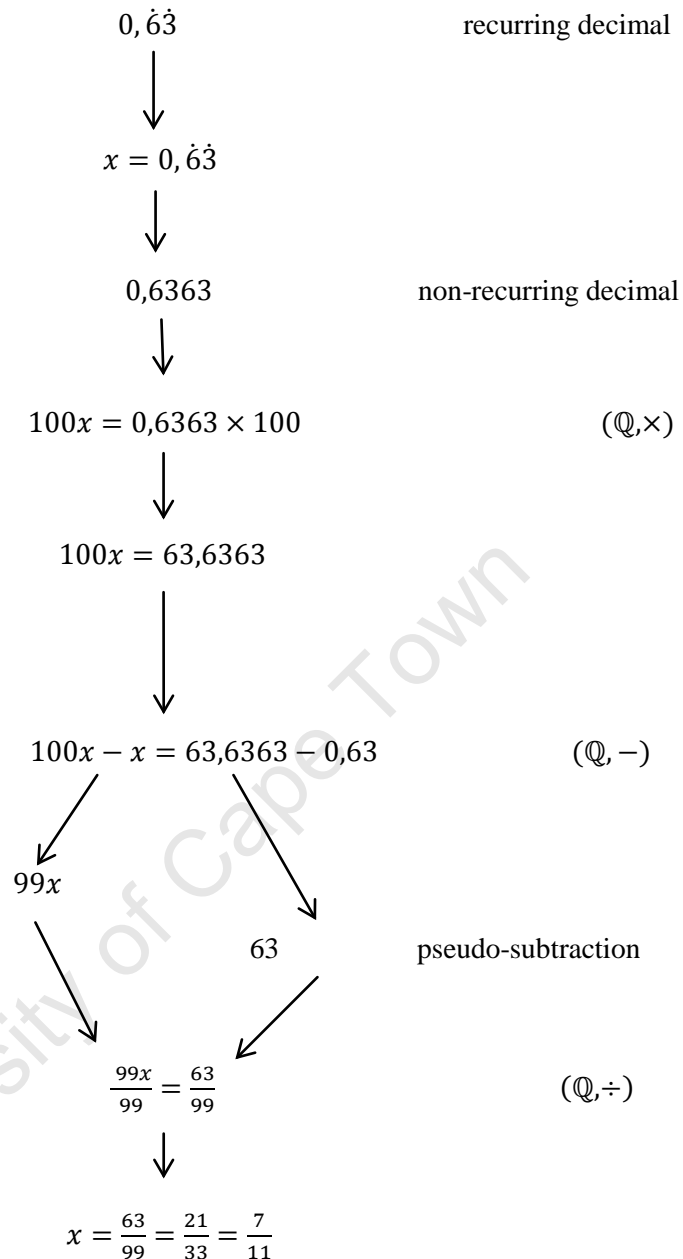


Figure A2.12 Diagrammatic representation of third example

### 3) Activation of the mathematics encyclopaedia

**From the field of production**, the ground on which the algorithm for converting recurring decimals to common fractions rests is the real numbers with their properties. A decimal representation of a real number is called a recurring decimal if at some point it becomes periodic: there is some finite sequence of digits that is repeated indefinitely. A real number has an ultimately periodic decimal representation if and only if it is a rational number. Rational numbers are numbers that can be expressed in the form  $\frac{a}{b}$  where  $a$  and  $b$  are integers and  $b$  is non-zero – in other words, where division is always possible, except by zero. All rational numbers have either finite decimal expansions or recurring decimal expansions.

Reals which have decimal tails consisting of recurring digits can also be presented as the quotient of two integers,  $\frac{a}{b}$  where  $b$  is non-zero, and so are rational. Those reals which have infinite decimal tails not made up of recurring digits can't be written as the quotient of two integers, and so are irrational.

As explained by Stewart & Tall (1977: 23), “the rational numbers may be characterized as those whose decimal expansions repeat at regular intervals”. They remind us that a repeating or recurring decimal is classified as such if from some point on a fixed sequence of digits repeats indefinitely. Courant & Robbins (1941) distinguish between finite and infinite decimals. Finite decimals are those which have a finite number  $n$  of decimal places (any further digits are assumed to be zero) and can be reduced to a fraction with a denominator which is some divisor of  $10^n$ , while infinite decimals are those which do not have a finite number of decimal places, for example  $\frac{1}{3} = 0,3333 \dots$ . An infinite decimal which does not represent a rational number is an irrational number. But despite this distinction, every finite decimal is equal to an infinite recurring decimal, for example 0,4 is equal to 0,399999 ... and 1 is equal to 0,999999 ...

The number of digits in the repeating portion of the decimal expansion of a rational number can be found directly from the multiplicative order of its denominator. The algorithm often used to convert from the decimal expansion to a fraction of the form  $\frac{a}{b}$  where  $a$  and  $b$  are integers and  $b$  is non-zero draws on this. Thus, in order to convert  $0,\dot{7}$  to such a fraction, we would:

Let  $0,\dot{7}$  be equal to  $x$ :  $x = 0,77777 \dots$  (equation 1)

Multiply equation 1 by 10:  $10x = 7,77777 \dots$  (equation 2)

Subtract equation 1 from equation 2:  $9x = 7$  (equation 3)

Solve equation 3:  $x = \frac{7}{9}$

In general, if the recurring decimal has  $r$  as the repetend (the digit which repeats itself), then the fraction that is represented by that repeating decimal is just  $\frac{r}{R}$ , where  $R$  is a number with the same number of digits as  $r$ , but all these digits are 9's.

Thus,

$$0,88888 \dots = \frac{8}{9}$$

$$0,313131 \dots = \frac{31}{99}$$

$$0,567567 \dots = \frac{567}{999} = \frac{21}{37}$$

$$0,42014201 \dots = \frac{4201}{9999} \text{ and so on.}$$

The teacher's procedure rests on the operator properties of multiplication and addition over the field of the rationals (with their inverses, division and subtraction). These properties are shown in Table A2.4.

**Table A2.4** Operator properties of multiplication and addition over the rationals

Axioms	Property
1) $\forall a, b, c \in \mathbb{Q}, a + (b + c) = (a + b) + c$	Associativity of $(\mathbb{Q}, +)$
2) $\forall a, b, c \in \mathbb{Q}, a \times (b \times c) = (a \times b) \times c$	Associativity of $(\mathbb{Q}, \times)$
3) $\forall a, b \in \mathbb{Q}, a + b = b + a$	Commutativity of $(\mathbb{Q}, +)$
4) $\forall a, b \in \mathbb{Q}, a \times b = b \times a$	Commutativity of $(\mathbb{Q}, \times)$
5) $0 \in \mathbb{Q}$ and for $\forall a \in \mathbb{Q}, a + 0 = a = 0 + a$	Additive identity of $(\mathbb{Q}, +)$ is 0
6) $1 \neq 0$ and for $\forall a \in \mathbb{Q}, a \times 1 = a = 1 \times a$	Multiplicative identity of $(\mathbb{Q}, \times)$ is 1
7) $\forall a \in \mathbb{Q}, \exists (-a) \in \mathbb{Q} a + (-a) = 0 = (-a) + a$	Additive inverses
8) $\forall a \in \mathbb{Q}, \exists a^{-1} \in \mathbb{Q} a \times a^{-1} = 1$	Multiplicative inverses

**From the field of recontextualisation**, the National Curriculum Statement for grade ten Mathematics has as its first learning outcome the need for learners to be able to “recognise, describe, represent and work confidently with numbers and their relationships to estimate, calculate and check solutions” (DoE, 2003: 26), and to “calculate confidently and competently with and without calculators, and use rational and irrational numbers

with understanding” (DoE, 2003: 12). As part of this learning outcome, the curriculum states (in Assessment Standard 10.1.2) that learners should be able to “identify rational numbers and convert between terminating or recurring decimals and the form  $\frac{a}{b}$ ;  $a, b, \in \mathbb{Z}; b \neq 0$ ” (DoE, 2003: 26). The curriculum thus associates recurring decimals with the field of rational numbers. But the distinction made here between terminating and recurring decimals does not explicitly include the way in which a terminating decimal can be written with an recurring tail, for example 0,4 can be written as 0,399999 ... and is thus in fact also a repeating or recurring decimal.

The grade ten textbook used for the particular lesson under analysis describes recurring decimals as decimals in which “certain digits are repeated over and over” (Classroom Mathematics Grade 10, pg 14), without explicitly mentioning rational numbers. The textbook contains the following discussion of the procedure for converting recurring decimals to common fractions:

### Converting recurring decimals to common fractions

---

#### General discussion

It is not possible to express recurring decimals as fractions with denominators that are powers of 10. We can however use the following method to convert a recurring decimal to a common fraction.

**Example**  
Convert  $0,\dot{7}$  to a fraction.

*Step 1:* Let the decimal be equal to  $x$ .  $x = 0,77777\dots$

*Step 2:* Multiply both sides of the equation by powers of 10 to get the first of the recurring digits before the decimal comma.  $10x = 7,77777\dots$

*Step 3:* Put the first equation underneath the second.  $x = 0,77777\dots$

*Step 4:* Subtract the first equation from the second.  $9x = 7$

*Step 5:* Solve for  $x$ .  $x = \frac{7}{9}$

**Note:** If two digits recur the equation must be multiplied by powers of 10 so that both recurring digits are before the decimal comma.

**Figure A2.13** Extract from Classroom Mathematics Grade 10 Pg 18

The exercise in the textbook following this example is:

**Exercise 1.12** \_\_\_\_\_ (AS 10.1.2)

1. Convert the following to common fractions in simplest form.
  - a)  $0,\dot{2}$
  - b)  $0,\dot{8}$
  - c)  $0,4\dot{5}$
  - d)  $0,\dot{6}\dot{3}$
  - e)  $0,\dot{1}\dot{3}\dot{5}$
2.
  - a) Convert  $0,\dot{9}$  to a common fraction.
  - b) Do you think that the answer to question 2 a) is correct?
  - c) It is difficult to accept that  $0,\dot{9} = 1$ . Here is another approach to help you to see this.  
 Calculate  $0,\dot{3} \times 3$   
 Convert  $0,\dot{3}$  to a fraction. Now multiply this fraction by 3.  
 Compare the two answers.

**Figure A2.14** Extract from Classroom Mathematics Grade 10 Pg 18

These extracts from the textbook outline the algorithm used to convert recurring decimals to common fractions. The textbook does define rational numbers earlier in the same chapter (“any number that can be written in the form  $\frac{a}{b}$ , where  $a$  and  $b$  are integers and  $b \neq 0$ ” (pg 13), but does not explicitly refer to rational numbers in its treatment of recurring decimals. The textbook also distinguishes between recurring and terminating decimals, but provides an opportunity to discuss the relation between them in Question 2c above.

## Secondary data production P2 Lesson 1 EE2

### 1) Realisation of content

The intended topic of this event is converting recurring decimals to common fractions.

The realised topic does not correspond with the intended topic due to the shifts in domain which take place from recurring to non-recurring decimals. We first see this in the teacher’s first step for example one (making the decimal equal to  $x$ ), where she writes  $x = 0,7777$ , instead of  $0,7777\dots$ , which changes the object she is working on from a recurring decimal to a terminating decimal. This has implications for the rest of the procedure, as she seems to use  $0,7$ ,  $0,77$  and  $0,7777$  interchangeably to represent the recurring decimal  $0,\dot{7}$ . Her final answer is correct despite this interchangeable use of  $0,7$ ,  $0,77$  and  $0,7777$  to represent  $0,\dot{7}$  and her errors related to this. But this does not mean that the realised topic is aligned with the mathematics encyclopaedia, in fact, her procedure violates the principles of mathematics. When she multiplies  $x = 0,7777$  by 10, giving the reason that “there is one unit after the comma” she initially writes down the incorrect answer of  $10x = 7$ , due to her operation on the decimal  $0,7$  instead of  $0,\dot{7}$ . This leads to confusion when she tries to subtract the two equations ( $7 - 0,7$  is  $6,93$ , not  $7$  as the teacher expected it to be), resulting in her addition of two  $7$ ’s after the decimal point ( $10x = 7,77$ ) and an incorrect subtraction of  $0,7$  from  $7,77$  (she writes down  $7$ , although the correct answer is  $7,07$ ). It thus seems that local validity is suspended so that the expected answer can be obtained. The teacher and the learners’ knowledge of the subtraction of decimals is not a regulative resource at this point – what appears to be overdetermining with respect to the validation of the procedure is the question statement and the correctness of the final answer. Thus the initial and terminal points of the procedure have the strongest regulative effect on the ‘steps’ used by the teacher, so much so that they allow suspension of operations that are universally valid and stable.

In this situation, subtraction no longer operates as subtraction over the rationals, but as another form of pseudo-operation. The teacher knows which signifier should appear in the answer ( $7$ ), and whatever the ‘subtraction’ does, the outcome will always be that signifier. An operation such as subtraction has universal validity and is context-independent, but the use of ‘subtraction’ in this situation is radically context-dependent and in fact violates principles of mathematics. This introduces inconsistency, as the operation of subtraction does not function as an operation with a unique, stable output, and thus the realised topic does not correspond with the mathematics encyclopaedia.

Later in the event we see the teacher marking learners’ work as correct, despite their errors with the subtraction ‘step’ – they are following her procedure exactly.

### 2) Regulation of the learner

As mentioned above, the suspension of mathematical operations suggests that the mathematics encyclopaedia is not functioning as the primary regulative resource in this event – necessity does not reside within mathematics, but instead with the teacher and her steps. Nowhere in her explanation does the teacher appeal to the proposition ground underlying the intended topic, but she appeals to the steps (one appeal to procedural features of solutions) as well as how the final answer should look (five appeals to iconic or spatial features of solutions). She appears to consult the textbook a number of times during the lesson in order to see what the solution should look like, and she makes sure that her steps get her there, despite the errors and confusion along the way. It thus appears that a regulative resource used by the teacher in this

example is the way the solution looks in the textbook, but in this case her dependence on the iconic ground of the textbook solution undermines the operation of subtraction of decimals and enables her to accept an incorrect solution to the subtraction of 0,7 from 7,77 in order for her solution to look like the one given in the textbook. The solution in the textbook thus appears to function as a *character distribution matrix*, described by Davis (2010b: 392) as “a resource for the regulation of the presentation of the transformations from one mathematical expression to another as structured by procedures”.

All of this suggests that the regulation of the learner is under the aspect of the Imaginary – in order for the learners to be able to engage with the topic, the teacher reconstitutes it as a series of steps (taken from the textbook) which must be followed religiously, regardless of whether there are mathematical errors or not, to reach a solution which looks a certain way. Necessity is situated as external to the field of mathematics, in the teacher’s steps and the form of the solution.

### Summary

In this event, the realised topic does not correspond with the intended topic or with the encyclopaedia – quadrant IV. The regulation of the learner depends primarily on the Imaginary.

### Primary data production P2 Lesson 2 EE1

#### 1) Generation of evaluative events

**Table A2.5** Evaluative events School P2 Lesson 2

Time	Evaluative event/sub-event	Activity	Type
00:00 – 32:25	E <sub>1</sub>	Working through homework questions on converting recurring decimals to common fractions	Expository
00:00 – 18:05	E <sub>1.1</sub>	Converting $0,1\bar{35}$ to fraction	Expository
18:05 – 32:25	E <sub>1.2</sub>	Converting $0,4\bar{5}$ to fraction	Expository

In this lesson the teacher arrives 13 minutes late and there are 7 minutes at the end in which no work is done.

#### 2) Description of operational activity

The lesson starts with learners going through the homework on the board. The first question is to convert  $0,1\bar{35}$  to a fraction. The first learner who is called up writes the product of one hundred and  $0,1\bar{35}$  as 1,1355, saying something like “three is not recurring”. When she is doing the subtraction step she says “when you minus one comma double one three five five minus zero comma one three five you’re going to get one thirty five.” Her subtraction is completely wrong – she knows what she needs to get and makes sure she gets there. Her completed solution:

Let decimal to be equal  
 $x = 0,135$   
 $1000x = 135,5$   
 $\ominus x = 0,135$   
 $999x = 135 = \frac{45}{333} = \frac{15}{111}$   
 $x = \frac{5}{37}$

**Figure A2.15** First learner’s attempt to convert  $0,1\bar{35}$  to a fraction

The teacher says “I think there is a problem here. Where is the problem, class? There is a problem somewhere, where is that problem?” She calls up another learner to rectify the problem.

The second learner changes the 1,11355 to 135:

Let decimal to be equal  
 $x = 0,135$   
 $1000x = 1,135$   
 $1000x - x = 1,135 - 0,135$   
 $999x = 1,000 = \frac{1000}{999} = \frac{1000}{999}$

**Figure A2.16** Second learner’s correction of first learner’s solution

The original learner asks a question, and says “we don’t see five and one recurring there” – it seems she only sees the one and the five as recurring due to the dots in the question (135), and is confused about why they are not repeated in the solution as she did (she wrote 0,11355 – repeating the one and the five but not the three). There is clearly no understanding of a recurring decimal. The teacher does not seem to understand her question:

Teacher: Where did you get that one comma double one three double five? That one comma ... first you must explain to us, where did you get that one comma? Do it aside. Just do it afresh on that side so we can understand where do you get that one comma...I understand that one and five and three are recurring. What about that one comma ...

Learner: Miss, I thought that because one is recurring ...so ...

Teacher: Wait! Calculate. Show us. Do the calculation.

Learner: I’m going to explain Miss.

Teacher: No. Must show us so we can understand what you are talking about.

Learner: (goes to the board) ... I don’t see what is recurring. I just see the numbers. I know I made a mistake. I didn’t understand, I didn’t see that ...

Teacher: You didn’t understand what?

Learner: The homework you gave us.

Teacher: Now just as a matter, one comma. Where did you get that one comma?

Learner: Miss, ah, one is the number that comes first, so I gonna use that one before the comma because zero is not necessary to us.

Teacher: Therefore you assumed that it must be one?

Learner: Yes miss, and the double one so that I can that is our fraction because if there’s no number that’s a fraction it’s confusing to me.

Teacher: It can be like this, it supposed to be one comma, is one three five then one is recurring and five is recurring but it is this number in between it is three. It means that because these two numbers are recurring, this whole thing is recurring, even that three that is in between. And you're supposed to say the one three five, comma one three five negative zero comma one three five so that you can get one thirty five.

Transcript School P2 Lesson 2

After this exchange the teacher calls up another to do the same question saying she must show the class how the new solution (learner 2) differs from learner 1's solution. She says "you must apply the steps, you can't just calculate without understanding the steps. You must apply all the steps".

Let the decimal be equal  
 $x = 0,135\ 135\ 135\ 135$   
 $1000x = 0,135 \times 1000$   
 $1000x = 135\ 135\ 135 - 0,135$   
 $\frac{999x}{999} = \frac{135}{999} = \frac{43}{333} = \frac{15}{111} \quad x = \frac{5}{37}$

Figure A2.17 Third learner's solution

The learner (3) redoes the solution, consulting her notebook. After a while the teacher says

Teacher: You keep quiet, you don't explain anything to others. You must apply those steps ... which step .. number one, number two, number three, number four, so everybody understands. You can't cut out the steps".

Learner: This is the first step ... this is the second step where I multiply thirty five by a thousand and ...

Teacher: So it means therefore step number two is about what? It is about multiplication.\

Learners: Multiplication.

Teacher: Step number two.

Learner: The third step. It's one thousand  $x$  minus  $x$  and I subtracted ... the one comma one thirty five and it gave me ...

Teacher: Wait it means that's step number three, what step number two says you must subtract equation one from equation two, that's what she did there, you must apply your steps, then the last one ... last step. What is the last step?

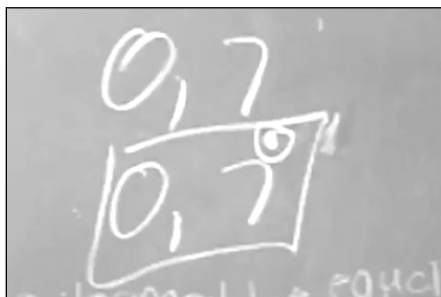
Learners: Division

Teacher: What

Learners: Division.

Teacher: Is it division? ... I gave you four steps on the board. What is it? You must solve for  $x$ .

The teacher then moves onto the second homework question – zero comma forty five recurring, saying “if you don’t apply these steps you are going to have problems”. She also tells them “you must be able to identify if it is recurring decimal fraction. What is the difference between this zero comma seven and this zero comma seven?” (she writes 0,7 and 0,7 with a dot on top of the seven) - “this one is recurring (pointing to the second one) because of what? Because of this dot. Are you with me?”



**Figure A2.18** Which one is recurring?

Her definition is once again based on the dot as the defining feature or property of a recurring decimal fraction, rather than its repetitive nature.

The second question:

A learner comes up to do it, saying that she will write it then explain it when she’s finished.

$$0,4\dot{5}$$

$$100x = 4\dot{5} \times 10$$

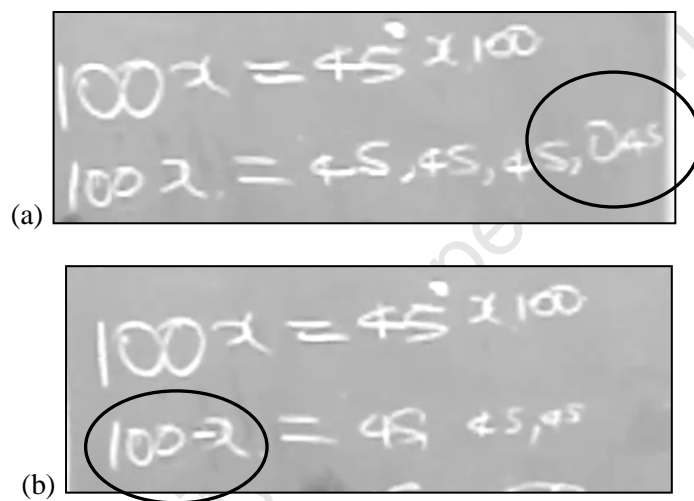
$$100x = 45,45,45$$

$$\frac{99x}{99} = \frac{45}{99} = \frac{5}{11}$$

$$x = \frac{5}{11}$$

**Figure A2.19** First learner’s attempt to convert  $0,4\dot{5}$  to a fraction

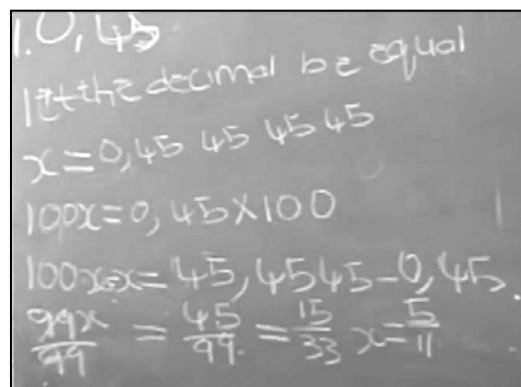
In this example, the learner assumes that both digits (four and five) are recurring, instead of just the five. She incorrectly represents the recurring decimal in the second line as '45', and does not grasp the concept of a decimal fraction as seen in '45,45,45' written in line two. Her answer of  $\frac{45}{99}$  would have been correct for the recurring decimal  $0,4\dot{5}$ , but she goes wrong in her simplification of the fraction. She explains that because there are "two digits" after the comma she uses 100, but it seems that she has written " $\times 10$ " instead of " $\times 100$ " in line two. A learner corrects her and she adds another zero to the '10' in line two. Another learner asks her why she didn't write "let  $x$  equal to" – the learner says "we just see  $x$  on the board, you didn't tell us where that comes from". This learner is quite correct in raising this point, but her question also reveals her dependence on the steps offered by the teacher. Another learner asks why she didn't "put zero comma forty five in the step with one hundred  $x$ " – line two. It seems that the class are able to pick up when things are missing from the procedure – they know it well enough by now. Yet another learner tells her that she "forgot to show at the board how to subtract  $x$  to the hundred  $x$ " – he picked up that she left out the teacher's 'subtraction step'. At this, the learner adds 045 to the end of line two, as seen in Figure A2.20a, at which the class is still not happy so she changes line two again as seen in Figure A2.20b:



**Figure A2.20** First learner makes changes to line two of her solution

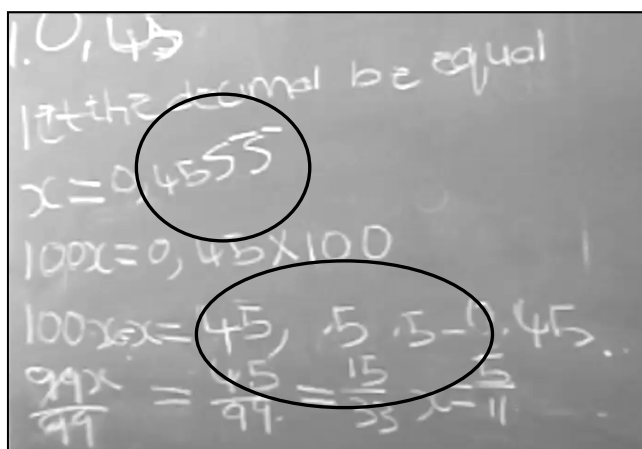
The teacher asks her to explain her solution. She points to the dot and says "that means it's recurring so I showed it, forty five, forty five, forty five, forty five (pointing to the right hand side of line two)". She then moved to the next line and started talking about finding the lowest common denominator (to simplify the fraction  $\frac{45}{99}$ ). She tails off as she points to her final answer and the class applaud her as she sits down. The teacher calls anyone who differs with this learner to come forward and explain themselves.

A second learner comes up and does the following:



**Figure A2.21** Second learner's attempt at converting  $0,45$  to a fraction

The class seems happy with her solution. The teacher says she has a problem “in that recurring, I find that in that that let the decimal be equal to  $x$  and we found that now we saying that forty five is recurring, is not five. ... So it is five now that now is recurring, is supposed to be what?” Between two other learners, they change the solution, but leave the final answer the same:



**Figure A2.22** Changes made by third and fourth learners to second learner’s solution

This is where the lesson ends – presumably the teacher and learners accept the incorrect answer as being the correct fraction, when in fact the correct answer is  $\frac{41}{90}$ . This example does not follow the pattern of the previous ones, and the teacher’s method falls short.

### 3) Activating the Mathematics encyclopaedia

See P2 L1 EE2

## Secondary data production P2 Lesson 2

### 1) Realisation of content

This lesson is taken up with working through a homework exercise on converting recurring decimals to common fractions.

The analysis is the same as the previous lesson EE2:

- Completely wrong subtraction in order to get to ‘right’ answer - “And you’re supposed to say the one three five, comma one three five negative zero comma one three five so that you can get one thirty five.
- No understanding of recurring decimal by learners; teacher’s attempt to define/compare recurring and non-recurring decimals (superficial ... the dot).
- Shift in domain from recurring to non-recurring decimals.

The last example is particularly interesting as it has a slightly different form to the others, and thus the procedure does not work for this example. This illustrates the way in which the teacher’s procedure functions as a closed text and does not give learners access to the supporting propositional ground in order to be able to monitor their use of the procedure and deal with questions which may not be of the same form. Using the initial and terminal points of this procedure as regulative resources will not enable learners to reach the correct answer in this case, so the procedure falls apart. But the teacher does not notice or correct the incorrect solution offered in this example, she is still using the initial and terminal points as a regulative resource and thus the answer appears correct according to her procedure.

This event falls into quadrant IV.

## 2) Regulation of the learner

In this lesson the teacher re-emphasises the steps of her procedure, stating what each one is about. Each time a learner gets stuck or makes a mistake (from the teacher's point of view), she repeats her steps and draws their attention to which step they should be on. She thus appeals to the steps as a strong regulative resource – necessity resides within these steps (set up by her as criteria) rather than within the field of mathematics. The beginning and end points of the procedure (what the solution should look like) as well as the textbook layout (CDM) are also points external to mathematics which are used as regulative resources. The exchange below is interesting:

Teacher: You keep quiet, you don't explain anything to others. You must apply those steps ... which step .. number one, number two, number three, number four, so everybody understands. You can't cut out the steps".\

Learner: This is the first step ... this is the second step where I multiply thirty five by a thousand and ...

Teacher: So it means therefore step number two is about what? It is about multiplication.

Learners: Multiplication.

Teacher: Step number two.

Learner: The third step. It's one thousand  $x$  minus  $x$  and I subtracted ... the one comma one thirty five and it gave me ...

Teacher: Wait it means that's step number three, what step number two says you must subtract equation one from equation two, that's what she did there, you must apply your steps, then the last one ... last step. What is the last step?

Learners: Division.

Teacher: What?

Learners: Division.

Teacher: Is it division? ... I gave you four steps on the board. What is it? You must solve for  $x$ .

Transcript School P2 Lesson 2

Here she is not happy with the learners' description of step four as "division" – she corrects them and says it is "solve for  $x$ ", which is how the textbook describes this step. When a learner asks questions about the procedure the teacher seems to get confused and eventually tells the learner what she's "supposed" to get.

This exchange is interesting because it confirms our suggestion from the previous lesson that the teacher knows what the required solution is, and that regardless of what happens in between the question and the final answer, she makes sure that she gets the expected answer – the beginning and terminal point are thus the regulative resources. The learners clearly pick up on these criteria and treat the questions in a similar way, suspending their knowledge of subtraction and multiplication of decimals in order to reach the expected answer. The teacher's response to learners' questions is interesting, as she does not always answer the question directly, but focuses on what the answer is "supposed" to be. In this event there are 7 appeals to iconic or spatial features of solutions, 4 to procedural features of solutions and only 2 to mathematical propositions, processes or definitions. Necessity is located external to mathematics and within the expected

solution and the ‘steps’ required to get there. This is seen when the learners are unable to convert 0,45 to a fraction as it is of a different form to the others. The teacher’s procedure does not give learners access to the mathematical definitions, processes and objects to enable them to convert a recurring decimal to a fraction, but instead acts as a closed text which forecloses the propositional ground, regulating the learners through the iconic and procedural features of the solution. Generally, the way in which the teacher regulates learners in this event suggests that she is using the steps, the final solution and the textbook layout as a way of ensuring that learners get to the correct solution despite what goes on in between and despite their (and her?) lack of knowledge of the topic. This suggests that regulation of the learners is under the aspect of the Imaginary.

### Summary

This event falls in to quadrant IV. The regulation of the learner depends primarily on the Imaginary.

### Primary data production P2 Lesson 3 EE1

#### 1) Generating evaluative events

Table A2.6 Evaluative events School P2 Lesson 3

Time	Evaluative event/sub-event	Activity	Type
00:00 – 16:00	E <sub>1</sub>	Introduces exponents and explains expanded form with some examples (done on board by learners)	Expository
16:00 -31:35	E <sub>2</sub>	Exercise on simplifying and calculating – teacher calls learners up to do the questions on the board	Exercise/expository

#### 2) Describing operational activity

The teacher announces the topic as “exponents”, and writes the following on the board as a reminder of the ‘laws’ they had learnt:

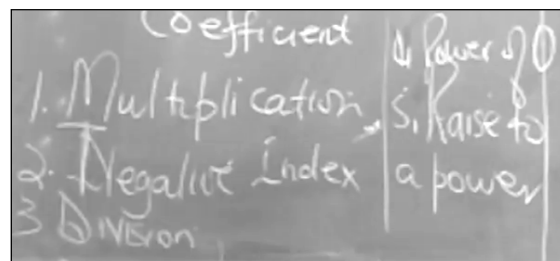
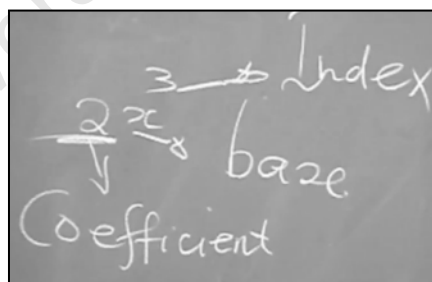


Figure A2.23 Revision of exponents

Her introduction to exponents thus does not draw on the definition of exponentiation. When referring to the ‘laws’ she just lists the headings written below without explaining the actual properties of exponents they represent, saying “you know ... this from grade 9, right?”.

She starts by giving two examples – two to the power three and two to the power two, saying that “whenever you are given ... it means that you must expand ... two times two times two.”

Expanded form  
 $2^3 = 2 \times 2 \times 2 = 8$   
 $2^2 = 2 \times 2 = 4$

Figure A2.24 Expanded form

She then asks the learners to do the following exercise (taken from Classroom Maths 10 pg 151), saying “I want to know whether you understand this”:

Classwork: write out the expanded form.

- 1)  $(ab)^4$
- 2) 4 to the power of 5
- 3)  $ab^4$
- 4)  $1^5$
- 5)  $x^2$

After a few minutes she calls learners up to do the questions on the board. Number one:

1.  $(ab)^4 = a \times b \times a \times b \times a \times b \times a \times b$   
 $= a^4 b^4$

Figure A2.25 First learner’s solution to number one

After some discussion (the teacher asks the class if the answer is right and they reply that it is, but she disagrees and says “no way”), she calls a second learner to come up. This learner erases the answer and writes:

1.  $(ab)^4 = ab \times ab \times ab \times ab$

Figure A2.26 Second learner’s solution to number one

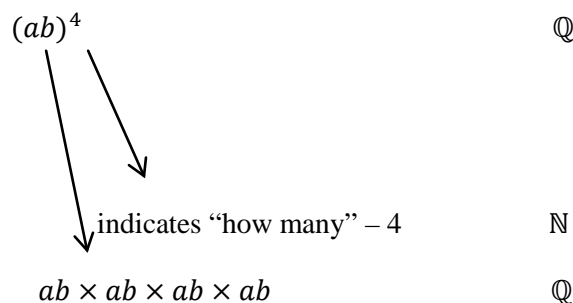
The teacher agrees with this answer. She points out that “times, brackets and dot” means the same thing.

This example reveals the process:

Teacher: 1 2 3 how many..? How many  $ab$ 's?

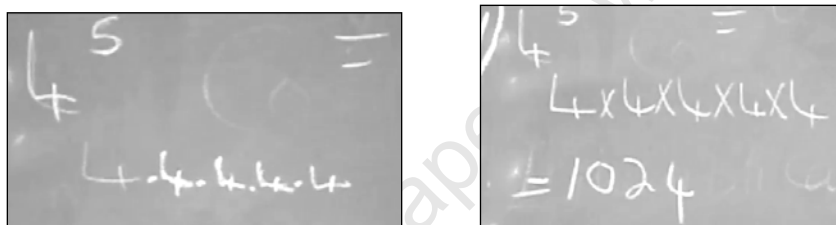
Learner: 4

This suggests that they are using the exponent (4) to show them “how many  $ab$ 's” they should write out in their expansion. The four is thus simply an indication of “how many” – the definition of exponentiation as repeated multiplication is implicit, it is about counting here.



Multiplication is implicit.

Number two:



**Figure A2.27** First learner’s solution and correction of number two

After a comment by the teacher he goes back to the board and replaces the dots with multiplication symbols, again revealing the emphasis by the teacher on what the expected solution looks like.

Numbers three, four and five are done with no discussion.

### 3) Activating the mathematics encyclopaedia

#### From the field of production

The examples covered in this lesson can be located in a class of problems involving arithmetic computations over the set of objects of the form  $a^n$  where  $a, n \in \mathbb{Z}$  and  $a \neq 0$ . The examples thus rely on exponentiation, as they consists of objects of the form  $a^n$ , where a base  $a$  is raised to the power  $n$  -  $a^n$  is defined as  $a \times a \times a \times \dots$  for  $n$  factors of  $a$ . When  $n$  is a positive integer, exponentiation corresponds to repeated multiplication, just as multiplication by a positive integer corresponds to repeated addition:  $a \times n = a + a + a + \dots n$  times

This definition of exponentiation is the foundation of the above lesson on the “expanded form”, but the definition is not explicitly stated in the lesson. The examples in this lesson involve exponentiation and multiplication over the set of rational numbers,  $(\mathbb{Q}, \times)$ . The operatory properties of multiplication over the rationals are listed in Table A2.4.

The teacher mentions ‘laws’, which refer to the following identities satisfied by integer exponentiation: First identity  $a^{m+n} = a^m \cdot a^n$

This identity has the consequence  $a^{m-n} = \frac{a^m}{a^n}$  for  $a \neq 0$  and  $(a^m)^n = a^{mn}$

Another basic identity implicit in this lesson is  $(a \cdot b)^n = a^n \cdot b^n$

Although mentioned by the teacher in the beginning of the lesson, she does not explicitly draw on these identities in her explanation.

The properties of natural number addition and multiplication do not apply to exponentiation. For example, while addition and multiplication over the natural numbers are commutative ( $a + b = b + a$  and  $ab = ba$ ), exponentiation is not commutative ( $a^b \neq b^a$ , for example  $2^3 = 8$ , but  $3^2 = 9$ ). Similarly, while addition and multiplication are associative ( $(a + b) + c = a + (b + c)$  and  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ ), exponentiation is not associative, so  $(a^b)^c \neq a^{b^c}$  for example  $2^3$  to the 4th power is  $8^4$  or 4096, but 2 to the  $3^4$  power is  $2^{81}$ .

While multiplication distributes over addition for natural numbers, distributivity cannot be applied in the same way in exponentiation. So while  $a(b + c) = a \cdot b + a \cdot c$ ,  $(a + b)^c \neq a^c + b^c$  but  $(a \times b)^c = a^c \times b^c$ , as seen in the identity described earlier. So the operation of exponentiation can be distributed over multiplication but not addition (CAN I SAY THIS?). But as discussed by Usiskin (1974), there is correspondence between  $a(b + c) = a \cdot b + a \cdot c$  and  $a^b \cdot a^c = a^{b+c}$ .

## From the field of recontextualisation

### Curriculum

The grade ten curriculum states in LO1, that “when solving problems, the learner is able to recognise, describe, represent and work confidently with numbers and their relationships to estimate, calculate and check solutions” and as an assessment standard, that learners should be able to “10.1.2 (a) Simplify expressions using the laws of exponents for integral exponents”.

### Textbook

All of the questions in this lesson were taken from Classroom Mathematics Grade 10. These questions were selected from an introductory exercise in the textbook (pg 151), which was designed to enable teachers to “ascertain the extent of learners’ knowledge of exponents, evaluate their level of understanding and identify any shortfalls or misunderstandings they may have regarding exponents” (pg 68, teacher’s guide). These specific questions were designed to enable teachers to “assess learners’ ability to do calculations with exponents, both with and without the aid of a calculator” (pg 68).

The instruction of the first part of the exercise is to “write out in expanded form. For example  $2^3 = 2 \times 2 \times 2$ . The questions in this EE are taken from this exercise. The textbook example writes multiplication signs and it is interesting that the teacher, although emphasizing that multiplication signs or dots are acceptable, seems to prefer solutions written with multiplication signs, as opposed to dots, as in the textbook example and answers in the back of the textbook.

## Secondary data production P2 Lesson 3 EE1

### 1) Realisation of content

The announced topic of this event is exponents, which is very broad, but it seems that the intended topic is specifically expanding and simplifying basic expressions containing exponents.

The definition of exponentiation (as repeated multiplication) is implicit and not referred to in the teacher’s “overview” of exponents in this event. The teacher assumes the learners know to multiply, using the word ‘times’ often but not explicitly explaining the link between exponents and multiplication. This exchange is interesting:

- T: “I think you must know that if ever you are given 2 power 3 like .... Or 2 the power 2 it means that you must expand”
- L: “2 times 2 times 2” (*here expand seems synonymous with times or multiply*)
- T: What is the answer?
- L: 6
- T: Why 6? Those who are saying 6 they must raise their hands. You must raise your hands? Those who are saying 6 must be explain to you? I think there are many of you who are saying 6. I think there are not only 2 percent learners who say that. You must raise your hand. All those who are saying the answer is 6 you must raise your hands. Only 2? What are the others are saying, what is the answer is?
- L: 8
- T: Why it's 8?
- L: Because 2 times 2 equals 4.
- T: 2 times 2?
- L: 4 times 2 equals 8.
- T: Times 2 equals?
- L: 8
- T: We are doing expanding, we are expanding here (writes ‘expanded form’ on the board).

### Transcript School P2 Lesson 3

From here she gives them a few examples of ‘expanding’ to practice – it seems that the error made by some learners (two to the power three is six) prompts her to set them these examples – she says “I want to know whether you understand this”. The learners’ error suggests that some of them do not understand the definition of exponentiation as repeated multiplication.

A few minutes later when a learner does the first question on the board (see Figure A2.23), the teacher says that he is not right – “no way”.

Another learner redoes it (see Figure A2.24), and the teacher accepts the second solution although both are correct and in fact the first one is more fully expanded (although the teacher possibly rejects it because the learner simplifies in his final step, although in the next question she calls a learner up to simplify the final step). After this question she emphasises that “times, brackets and dot” means the same thing – multiplication. This is an interesting exchange, as despite her insistence on the different ways multiplication can be represented, the teacher does not accept the first learner’s answer as correct, when in fact the first learner has ‘expanded’ the question more fully than the second one.. In the second example the teacher encourages the learner to change his answer, which contained dots, to contain multiplication signs, confirming that what the solution looks like is important and that she prefers multiplication signs, despite her comment about the different ways to represent multiplication. The learners seem to grasp this and for the next three questions they write multiplication signs.

The process used in this first example shows that they are using the exponent (4) to show them “how many ab’s” they should write out in their expansion – the teacher asks “how many ab’s?”. The four is thus simply

an indication of “how many” – the definition of exponentiation as repeated multiplication is implicit, it has become about counting here. The realised topic thus involves the use of counting as an operation – the exponent is used to indicate “how many” of the base need to be written. Strictly speaking, counting alone in this sense is not a function, as once the number of bases has been decided based on the exponent, these bases could be added, subtracted, multiplied or divided – although the teacher and learners correctly expand the expressions using multiplication.

Based on the above analysis I would say that the intended topic is not realised in this event, but although the realised topic does not explicitly draw on the propositional ground underlying the topic, it does not violate mathematical propositions, definitions or processes.

## 2) Regulation of the learner

As discussed above, despite the first learner correctly expanding an expression, the teacher says her answer is incorrect just because it does not look like the solution in the textbook, and calls another learner up to do the ‘correct’ expansion (both are right). This is interesting as despite her insistence earlier in the event on the different ways multiplication can be represented, the teacher does not accept the first learner’s answer as correct, when in fact the first learner has ‘expanded’ the question more fully than the second one. The answer given in the back of the textbook from which the questions were taken is the same as the second learner, so it appears that the teacher requires that exact answer and is not able to accept any other equivalent answer. This suggests that the evaluative criteria operating here appeal to what the solution is expected to look like, rather than to mathematical necessity, and also that necessity resides with the teacher and what she says is correct. At another point the teacher emphasises what the correct solution should look like and sends a learner back to adjust his (already correct) solution.

There are not many appeals made to an authorising ground in this event (only five). One is made to iconic or spatial features of solutions, two to procedural features of solutions, and two to mathematical propositions, definitions or processes. Thus there is not a big difference between the number of appeals made to mathematical and extra-mathematical factors. But the above analysis suggests that necessity is situated external to mathematics, in the way the solution looks (same as the textbook solutions), and the teacher’s preference for multiplication signs, despite her saying otherwise. This suggests that the regulation of the learner depends primarily on the Imaginary.

## Summary

The event falls into quadrant II. The regulation of the learner depends primarily on the Imaginary.

## Primary data production P2 Lesson 3 EE2

### 1) Generating evaluative events

Table A2.7 School P2 Lesson 3 EE2

Time	Evaluative event/sub-event	Activity	Type
16:00 -31:35	E <sub>2</sub>	Exercise on simplifying and calculating – teacher calls learners up to do the questions on the board	Exercise/expository

### 2) Describing operational activity

Teacher writes up new exercise, also from Classroom Maths pg 151, with the instruction to simplify and calculate, which the learners work on for a few minutes.

Simplify and calculate

a)  $2^3 \times 2$

b)  $3^2 \times 2^3$

c)  $5^2 \times 2^2$

d)  $(5 + 2)^2$

e)  $(5 \times 2)^2$

The teacher calls learners to do the questions on the board. The first two are done without much interaction and both are correct. In the third question, a learner goes straight from five squared times two squared to twenty five times four, and the teacher stops him and tells him to expand first – again, she is focused on the way she expects the solution to look (often based on the textbook solution).

In the fourth question, a learner starts by saying she will “first remove the brackets”. Her solution can be seen in Figure A2.28. The learner is incorrectly applying the distributive law in this question, but the teacher does not notice the error, and calls the next learner up for the fifth question, also seen in Figure A2.28.

④  $(5+2)^2$   
 $5^2 + 2^2 =$   
 $5 \times 5 + 2 \times 2$   
 $25 + 4$   
 $29$

$(5 \times 2)^2$   
 $5 \times 5 \times 2 \times 2$   
 $25 \times 4$   
 $= 100$  →

**Figure A2.28** Learners' solutions to questions four and five

In question five the learner seems to be distributing the power of two to the five and the two inside the brackets, and reaching the correct answer.

The teacher says that they applied two laws today – she states multiplication as the first and does not state the second. She writes up a few more questions which learners spend the rest of the lesson working on.

Generally in this lesson, the operation of raising a number to a specific power is recontextualised to natural number multiplication (or even just counting?), and the main focus of the exercises becomes addition and multiplication of natural numbers. It would be interesting to see how learners handle the last few questions written up, which contain variables.

As the learners work on this exercise in the last few minutes of this event we see a few groups of learners clustered around one learner who is explaining something to the rest.

### 3) Activating the mathematics encyclopaedia

See P2 L3 EE1, with the addition of:

All of the questions in this event were also taken from Classroom Mathematics Grade 10. These questions were selected from an introductory exercise in the textbook (pg 151), which was designed to enable teachers to “ascertain the extent of learners’ knowledge of exponents, evaluate their level of understanding and identify any shortfalls or misunderstandings they may have regarding exponents” (pg 68, teacher’s guide). These specific questions were designed to enable teachers to “assess learners’ ability to do calculations with exponents, both with and without the aid of a calculator” (pg 68).

The instruction of the first part of the exercise is to “write out in expanded form. For example  $2^3 = 2 \times 2 \times 2$ . The second part (which is used in this EE) has the instruction “calculate” and the third part to “simplify and calculate” (also used in this EE). The exercise written on the board during this EE contains questions from the 2<sup>nd</sup> and 3<sup>rd</sup> question in the textbook.

## Secondary data production P2 Lesson 3 EE2

### 1) Realisation of content

The announced topic of this event is “simplifying and calculating” exponential expressions – slightly different to the previous event which was “expanding” exponential expressions. In this event two laws are specifically referred to as ‘what they learnt today’ at the end of the event – the ‘multiplication law’ (when the bases are the same and you are multiplying you add the powers) and another which the teacher does not specify. But it is questionable whether these exponent laws were in fact part of the realised topic.

The recontextualisation of exponentiation to natural number multiplication results in an interesting treatment by the learner and the teacher of questions four and five (figures A2.28 and A 2.29)

In (4), the learner incorrectly distributes the power to each term of the base, while in (5) the learner does the same thing but because the five and two in the base are being multiplied, reaches the correct answer. The teacher does not notice the error in question 4, nor does she draw the learners’ attention to the relationship between these two questions. These questions were selected from an introductory exercise in the textbook, which was designed to enable teachers to “ascertain the extent of learners’ knowledge of exponents, evaluate their level of understanding and identify any shortfalls or misunderstandings they may have regarding exponents” (pg 68, teacher’s guide). These specific questions were designed to enable teachers to “assess learners’ ability to do calculations with exponents, both with and without the aid of a calculator” (pg 68).

Teacher: There are only two laws ke {okay}, you apply i-law {the law} of multiplication and what else? There are 2 laws ke {okay} that you need to apply here. Yi multiplication {its multiplication} and what?

Learners: Addition

Teacher: He? {Hey?}

Learners: Addition

Teacher: Addition and what else?

### Transcript School P2 Lesson 3

In this event the teacher specifically refers to “only two laws” (it is not clear which laws she means), but it does not seem that the learners have used the laws in their simplifications. In fact the exercise she set them does not require use of the exponent laws and was not designed to do so, as seen above. The questions were designed for learners to do calculations with exponents and do not necessarily involve use of exponent laws. The learner who does (5) above uses one of the laws, and the learner in (4) attempts to use the law incorrectly.

It is difficult to say whether the intended topic is realised in this event, but based on the quotes above from the teacher’s guide, I would say that the intended topic from the point of view of the textbook is not realised. As the textbook seems to be a dominant component of the intended topic for the teacher, the intended topic is not realised in this event.

Analysis of the content which is realised shows that it does not correspond with the mathematical encyclopaedia because of the error made in (4) and the lack of correction by the teacher – mathematical propositions and laws are violated in this event without any correction.

## **2) Regulation of the learner**

In this event we see that the teacher still appeals to how the solution should look and the specific procedure that she expects learners to carry out in answering these questions. In the third question, a learner goes straight from five squared times two squared to twenty five times four, and the teacher stops him and tells him to expand first – here she appeals to the way she expects the solution to look (in this case, based on the textbook solution) and the procedure she expects learners to carry out – they must expand the powers before simplifying, which is not mathematically necessary, but is the criteria put in place by the teacher in the previous event. This has implications for the learners' approach to exponential expressions – from this teacher's emphasis on expanding they could approach all exponential expressions with this in mind, which is not necessary to simplifying such expressions and in fact renders the process longer and more cumbersome.

The other questions do not yield any comments from the teacher and just involve learners reading out their answers which the teacher accepts (one incorrectly). From this short interaction I would say that the regulation is under the aspect of the Imaginary based on the same argument as the previous event.

## **Summary**

This event falls into quadrant IV. The regulation of the learner depends primarily on the Imaginary.

## Appendix 3: Analysis for School P3

### Primary data production P3 Lesson 1 EE1

#### 1) Generation of evaluative events

**Table A3.1** Evaluative events School P2 Lesson 1

Time	Evaluative event/sub-event	Activity	Type
00:00 – 14:22	E <sub>1</sub>	Learners write corrections on the board. Teacher comments on their work and makes some changes.	Exercise/expository
14:22 – 32:00	E <sub>2</sub>	Worked examples of simplifying exponential expressions	Expository
14:22 – 24:30	E <sub>2.1</sub>	Simplifying $\frac{5^{2x-1} \cdot 9^{x-2}}{15^{2x-3}}$	Expository
24:30 – 32:00	E <sub>2.2</sub>	Simplifying $\frac{27^{x-1} \cdot 3^{x+1}}{81^{x-1}}$	Expository
32:00 – 37:28	E <sub>3</sub>	Learners work on classwork.	Exercise

#### 2) Description of operational activity

The lesson starts with learners writing the corrections to homework questions on the board with the teacher checking as they go along. The learner doing question three writes the following:

The image shows a student's handwritten attempt at simplifying the expression  $3(x^7 y^8)(x^2 y^2)$ . The student has written:

$$3(x^7 y^8)(x^2 y^2)$$

$$= xy^{7-3} xy^{8-2}$$

$$= 2$$

**Figure A3.1** Learner's attempt at question three

Another learner comes up and redoes it correctly.

A learner gets stuck on question one. She writes:

$$\frac{1}{10^{-2}} = \frac{1}{10^2} = \frac{1}{1} = \frac{10}{1} \times$$

Another learner comes up and writes:

$$\frac{1}{10^{-2}}$$

$$\begin{aligned}
&= \frac{\frac{1}{1}}{\frac{1}{100}} \\
&= \frac{1}{1} \times \frac{1}{100} \\
&= 100
\end{aligned}$$

This is still not correct. The teacher discusses it:

Teacher: Changed ten squared to hundred, can we see what is going on here?

Learners: Yes

Teacher: It means that you have changed here, you have one divided by ten squared. Where does one come from Snazo?

Learner: (inaudible)

Teacher: Oh! Negative and two here. So that one over ten squared, she changed and became hundred; can you see what I am talking about?

Learners: Yes.

Teacher: It means, we have one over one then change the division sign to multiplication which means that she was supposed to have changed her fraction. Snazo is this correct? What answer should be here?

Learners: Hundred over one.

Teacher: Maybe someone else would continue with ten squared here, then one over one ten squared, Right? Then it will be one over one ten squared over one, which is ten squared equals to.....? Hundred

Transcript School P3 Lesson 1

The teacher writes the solution as follows (note the lack of equal signs):

$$\begin{aligned}
&\frac{\frac{1}{1}}{\frac{1}{10^2}} \\
&\frac{1}{1} \times \frac{10^2}{1} \\
&10^2 \\
&= 100
\end{aligned}$$

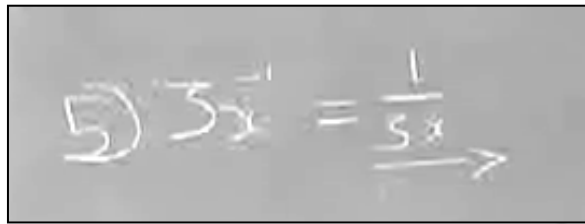
The teacher then also says that the answer to number four is wrong:

$$\frac{a^{-3}}{b^{-4}}$$

$$\begin{aligned}
 &= \frac{1}{\frac{a^3}{\frac{1}{b^4}}} \\
 &= \frac{1}{a^3} \times \frac{b^4}{1} \\
 &= b^4 a^3
 \end{aligned}$$

A learner comes up and changes the answer to the correct one of:  $\frac{b^4}{a^3}$

The learner who first did question five wrote:



**Figure A3.2** First learner's solution to question five

When he is finished, the teacher asks this learner about question five:

Teacher: Is negative one above of  $x$  or three on question five?

Learners: Negative  $x$

Teacher: Heh?

Learners: Negative  $x$

Teacher: Where is  $x$  Unathi?

Learner: Here it is (pointing at the chalkboard).

Teacher:  $x$  goes here (pointing next to the three).

Transcript School P3 Lesson 1

The teacher asks if it is right, and then sends another learner up to redo it. An interesting exchange follows relating to question 5:

Second learner's solution:

$$\begin{aligned}
 &3x^{-1} \\
 &= 3 \cdot \frac{1}{x^1} \\
 &= \frac{3}{1} \times \frac{x}{1} \\
 &= \frac{3}{x}
 \end{aligned}$$

The teacher questions her solution:

Teacher: Is question five correct?

Learners: Yes teacher.

Teacher: How can it be correct? Does three and  $x$  give you three?

Learners: Yes

Teacher: So, how can it be correct? How can it be correct?

Transcript School P3 Lesson 1

Teacher calls the learner back and she changes the last line to  $\frac{3x}{1}$ . A few minutes later the learner comes back again and erases the 1 on the denominator:

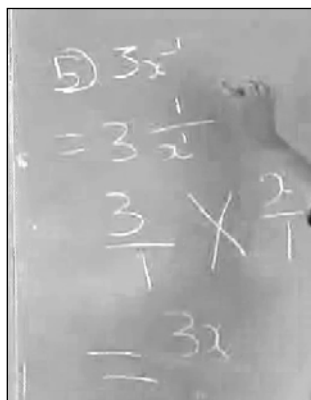


Figure A3.3 The second learner's final solution

Later, the teacher talks through the solution:

Teacher: Is it correct here?

Learners: Yes.

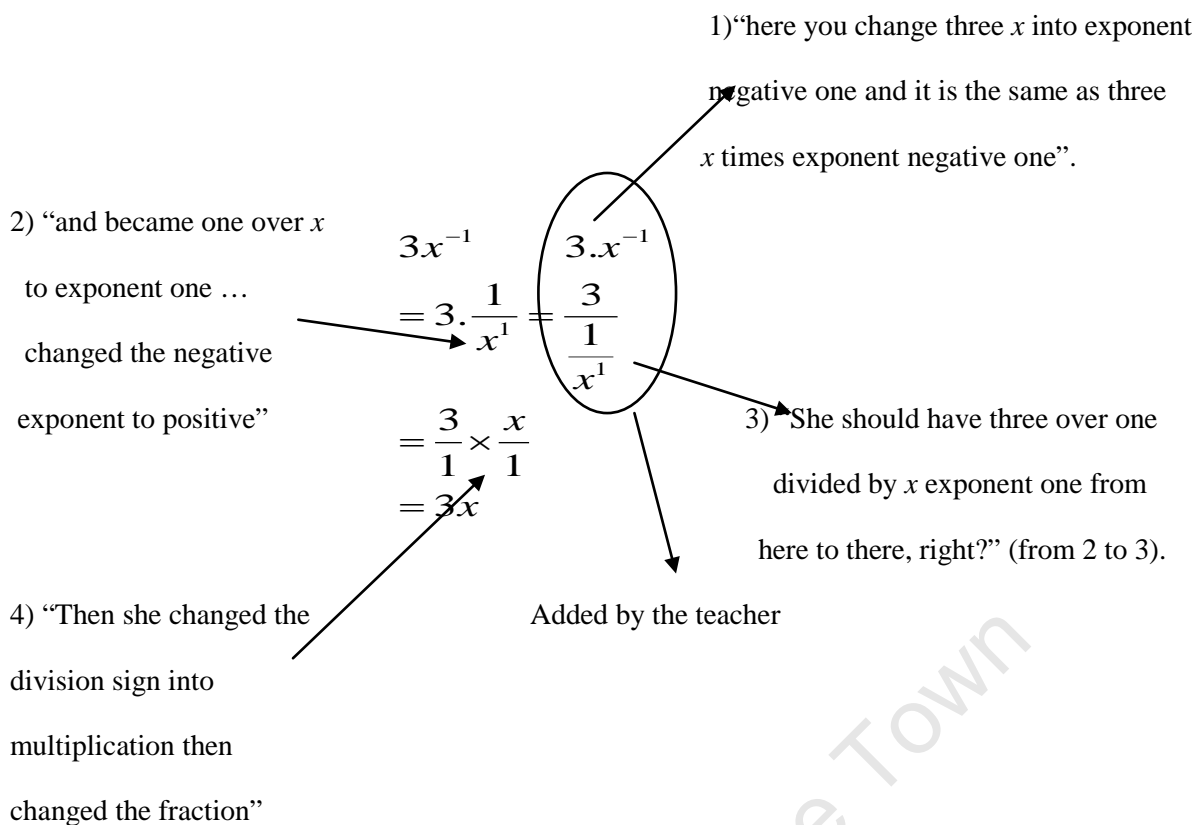
Teacher: Heh?

Learners: Yes.

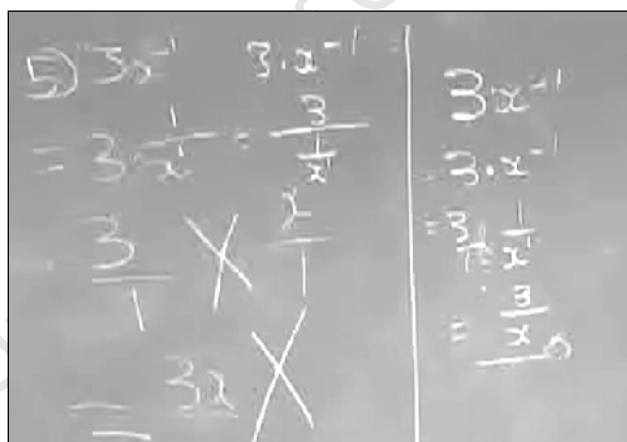
Teacher: Also here you change three  $x$  into exponent negative one and it is the same as three  $x$  times exponent negative one, which is what she did here, then she changed that  $x$  to the exponent negative one, and became one over  $x$  to exponent one ... changed the negative exponent to positive, right? Then three over one times  $x$  over one because she changed ... (inaudible) ... She should have three over one divided by  $x$  exponent one from here to there, right? Then she changed the division sign into multiplication then changed the fraction ...

Transcript School P3 Lesson 1

Teacher edits the learner's solution, but accepts her final answer.



A learner asks a question (it is not clear but sounds as if she asks if the answer should be "three over  $x$ "), and the teacher says "let's start from the beginning", and redoes the solution alongside the original one:



**Figure A3.3** The teacher's first and second solutions

Once she has reached a correct answer of three over  $x$ , she marks the original solution wrong, but doesn't explain why it was wrong. She now moves onto something new. An interesting feature of this teacher's pedagogy which is apparent during this event is the way in which she often refers to changing one thing into something else –

"**Changed** ten squared to hundred, can we see what is going on here?"

"It means that you have **changed** here, you have one divided by ten squared"

"So that one over ten squared, she **changed** and became hundred; can you see what I am talking about?"

"It means, we have one over one then **change** the division sign to multiplication which means that she was supposed to have **changed** her fraction"

“Also here you **change** three  $x$  into exponent negative one and it is the same as three  $x$  times exponent negative one, which is what she did here, then she **changed** that  $x$  to the exponent negative one, and became one over  $x$  to exponent one..... **changed** the negative exponent to positive, right?”

“She should have three over one divided by  $x$  exponent one from here to there, right? Then she **changed** the division sign into multiplication then **changed** the fraction”

### Initial comments

- For each question the first learner got it wrong.
- Generally, learners seem to struggle with these questions.
- The teacher’s treatment of the learner’s solution to question 5.
- The use of “change” as an ‘operation’.

### The teacher’s procedure for number five

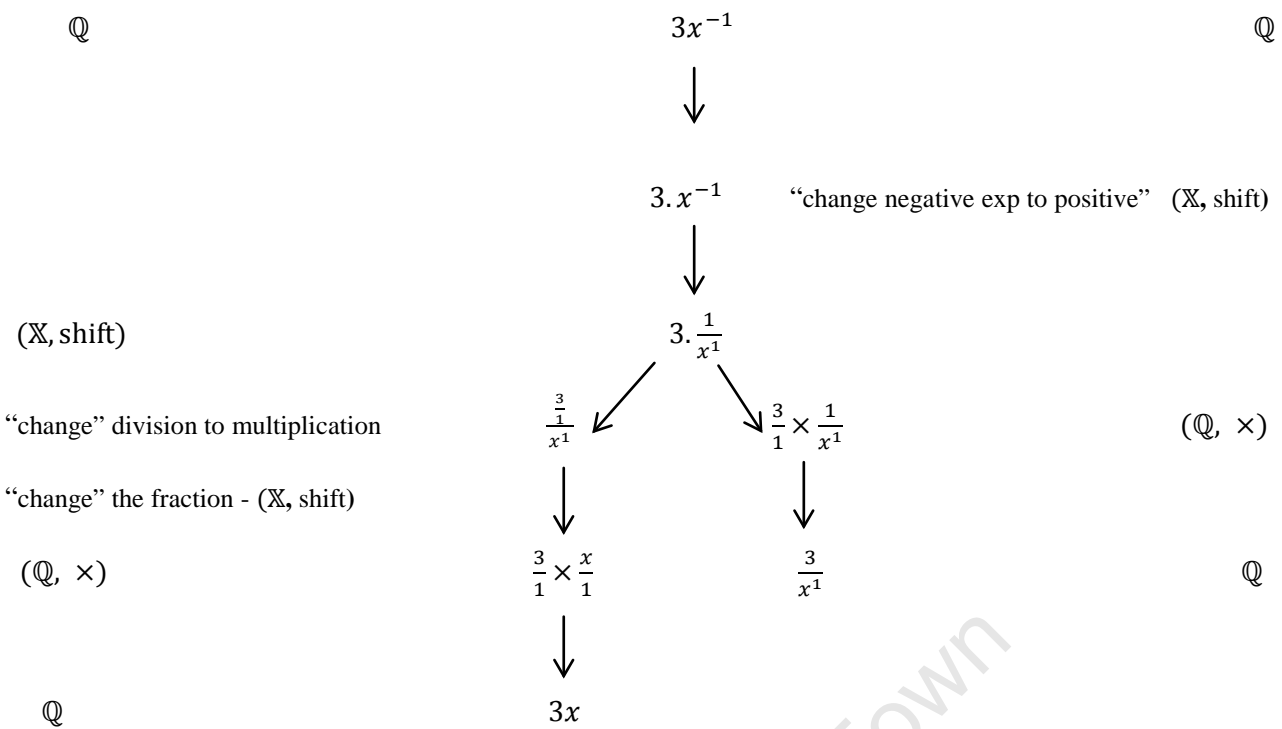
The first time the teacher talks through the solution, she includes the following transformations:

- 1) Rewrite  $3x^{-1}$  as “three  $x$  times exponent negative one” -  $3 \cdot x^{-1}$
- 2)  $3 \cdot x^{-1}$  becomes  $3 \cdot \frac{1}{x^1}$  (“change the negative exponent to positive”)
- 3)  $3 \cdot \frac{1}{x^1}$  becomes  $\frac{3}{x^1}$  (“three over one divided by  $x$  exponent one”)
- 4)  $\frac{3}{x^1}$  becomes  $\frac{3}{1} \times \frac{x}{1}$  (“changed the division sign into multiplication then changed the fraction”)
- 5)  $\frac{3}{1} \times \frac{x}{1}$  becomes  $3x$

It is not clear how the teacher arrives at  $\frac{3}{x^1}$  from  $3 \cdot \frac{1}{x^1}$  in step 3, but it seems as if she looks at the learner’s step 4, which is  $\frac{3}{1} \times \frac{x}{1}$ , and works backwards from there to get her step 3.

When the teacher redoes the solution in response to a learner’s question, the transformations involved are:

- 1) Rewrite  $3x^{-1}$  as  $3 \cdot x^{-1}$
- 2)  $3 \cdot x^{-1}$  becomes  $3 \cdot \frac{1}{x^1}$
- 3)  $3 \cdot \frac{1}{x^1}$  rewritten as  $\frac{3}{1} \cdot \frac{1}{x^1}$   
(she writes the one in as a denominator as she multiplies)
- 4)  $\frac{3}{1} \cdot \frac{1}{x^1}$  becomes  $\frac{3}{x^1}$



**Figure A3.4** Diagrammatic representation of teacher's first (left) and second (right) attempts at question five

**Table A3.2** Operational activity for teacher's first attempt at question five

T	Input	Domain	Operation	Output	Codomain
1	$3 \cdot x^{-1}$	$\mathbb{Q}$	Existential shift	$/3/,/x^{-1}/$	$\mathbb{X}$
2	$/3/,/x^{-1}/$	$\mathbb{X}$	'Change' negative exp to positive	$/3/ \cdot /1/ /x^1/$	$\mathbb{X}$
3	$/3/ \cdot /1/ /x^1/$	$\mathbb{X}$	Spatial shift	$/3/ /1/ x^1/$	$\mathbb{X}$
4	$/3/ /1/ x^1/$	$\mathbb{X}$	'Change' division sign into multiplication and change the fraction	$/\frac{3}{1}/, / \times /, / \frac{x}{1}/$	$\mathbb{X}$
5	$/\frac{3}{1}/, / \times /, / \frac{x}{1}/$	$\mathbb{X}$	Existential shift	$\frac{3}{1} \times \frac{x}{1}$	$\mathbb{Q}$
6	$\frac{3}{1} \times \frac{x}{1}$	$\mathbb{Q}$	Multiply	$3x$	$\mathbb{Q}$

In this example, the teacher does not give the mathematical equivalents for the operation she refers to as 'change', thus the domains and co-domains are often character strings, as seen in the table.

### 3) Activation of Mathematics encyclopaedia

#### From the field of production

Exponential expressions consist of the operation of repeated multiplication over the reals (or the rationals in the case of school mathematics). The domain and codomain of exponential computations are the reals, but at school level the mathematical treatment of the topic first restricts the domain and codomain to positive integers, later extending them to include negative integers and later still, rational exponents. All but one of the questions (number three) in this event has as their main object an exponential expression with a negative exponent. Let's examine number five as an example:

An object such as  $3x^{-1}$  can be written as:

$$\begin{aligned} & \times (3, x^{-1}) \\ & = (3, POW(x, -1)) \end{aligned}$$

It is the product of 3 and  $x$  to the power of negative one, and involves two operations on real numbers. The required simplification of such an object in this example is:

$$3x^{-1} = \frac{3}{x^1}$$

This can be written as:

$$\div (3, POW(x, 1))$$

The 'simplification' has not reduced the number of operations, but has rewritten the expression with a positive exponent instead of a negative one. This example illustrates a procedure for converting an expression with a negative exponent,  $a^{-n}$  (where  $n > 0$ ), to one with a positive exponent by implicitly exploiting the inverse or reciprocal of  $a^{-n}$ , that is  $a^n$  to produce  $\frac{1}{a^n}$ . The notion of an inverse or a reciprocal is central to the simplification of question particularly, with the inverses  $x$  and  $x^{-1}$  as objects in the solution and the initial expression of the above example respectively. The existence of multiplicative inverses for all rationals is the property upon which this 'simplification' is based:  $\forall a \neq 0, a \in \mathbb{Q}, \exists (a^{-1}) \in \mathbb{Q}$  such that  $a \times a^{-1} = 1$ . But in the teacher's procedure, this property is not explicitly drawn on as a regulative resource.

#### From the field of recontextualisation

##### Curriculum

LO1: *When solving problems, the learner is able to recognise, describe, represent and work confidently with numbers and their relationships to estimate, calculate and check solutions.*

10.1.2 (a) Simplify expressions using the laws of exponents for integral exponents.

##### Textbook

The textbook used in this class emphasizes the need to "revise and understand the exponential laws" (Classroom Mathematics Grade 10, pg 153). The definition given in the textbook is: " $a^n = a \times a \times a \dots a$  to  $n$  factors, where  $n$  is a natural number (i.e. the exponents are limited to natural numbers)" (pg 153). The next section in the textbook introduces integral exponents and gives another definition for negative exponents: " $a^{-m} = \frac{1}{a^m}$  ( $a \in \mathbb{R}, m \in \mathbb{N}, a \neq 0$ )" (pg 154). The textbook also lists the laws of exponents, which it states

“hold when  $n, m \in \mathbb{N}$ ” (pg 153). It does not refer to the laws when discussing integral exponents. It seems that textbook focuses on natural number exponents – negative exponents are treated as something which should be changed to positive and then dealt with according to the laws for natural number exponents.

## Secondary data production P3 Lesson 1 EE1

### 1) Realisation of content

The intended topic of this event is the simplification of basic exponential expressions – the learners write up the answers to a homework exercise on the board.

The teacher tells the learners that the questions are wrong (all of them are incorrectly done by the first learner who comes up) and sends up another set of learners to redo the questions. She briefly discusses the others, but the question which she spends time discussing is number five, as discussed in my primary data production. The teacher’s acceptance and perpetuation of the learner’s error in this question is a point at which the realised topic differs from the mathematics encyclopaedia.

Another feature of this event which was highlighted in my primary data production is that the teacher often refers to changing one thing into something else, which suggests that she uses “change” as a general operation to describe a number of transformations, for example “**changed** ten squared to hundred”, and “we have one over one then **change** the division sign to multiplication which means that she was supposed to have **changed** her fraction”

It is not clear what mathematical operations she is performing when she speaks about changing one thing into another – ‘change’ is a very context-dependent operation, which can be mapped onto a number of operations - in the case of ‘changing’ ten squared to a hundred, she is squaring; when she changes the division sign to multiplication and then changes the fraction she is multiplying the inverse etc. These operations are implicit in her explanations – she does not refer to the mathematical equivalents in her explanation. The teacher’s use of “change” as a general operation also has implications for the objects which are being operated on – changing one thing to another is not an operation as there is no one unique output for each input, and so in order to carry out these changes, the objects need to be character strings which can be changed from one thing to another freely. An existential shift thus takes place in order for the ‘operation’ of ‘changing’ to be carried out. This analysis suggests that the realised topic does not correspond with the intended topic, and also that the realised topic does not correspond with the mathematics encyclopaedia.

### 2) Regulation of the learner

The central mathematical object of the questions in this event is an exponential expression with a negative exponent. The propositional ground upon which the procedure for dealing with such expressions is based is the notion of a reciprocal or inverse, as explained in my primary data production. But the teacher does not explicitly appeal to this notion. Analysis of the teacher’s approach to these questions, specifically her use of the word ‘change’ as a context-dependent operation, suggests that mathematical expressions are treated in a procedural and iconic way, so that their components can be physically shifted to effect transformations from one expression to the next. This bends the mathematics in the direction of the learner – the mathematical operations making up this procedure are recontextualised as physical changes or shifts in symbols (change the division to a multiplication, change the fraction etc). In this event, three appeals are made to procedural features of solutions, and none to mathematical propositions, definitions, objects or processes, thus situating necessity as external to the field of mathematics, and rendering the Symbolic under the aspect of the Imaginary.

## Summary

This event falls into quadrant IV. The regulation of the learner depends primarily on the Imaginary.

## Primary data production P3 Lesson 1 EE2

### 1) Generation of evaluative events

**Table A3.3** School P3 Lesson 1 EE2

Time	Evaluative event/sub-event	Activity	Type
14:22 – 32:00	E <sub>2</sub>	Worked examples of simplifying exponential expressions	Expository
14:22 – 24:30	E <sub>2.1</sub>	Simplifying $\frac{5^{2x-1} \cdot 9^{x-2}}{15^{2x-3}}$	Expository
24:30 – 32:00	E <sub>2.2</sub>	Simplifying $\frac{27^{x-1} \cdot 3^{x+1}}{81^{x-1}}$	Expository

### 2) Description of operational activity

The teacher announces the topic of this event as “simplifying expressions”. She does two examples with the class. I have only analysed the first worked example in detail, as the second example is similar and follows the same procedure.

$$\frac{5^{2x-1} \cdot 9^{x-2}}{15^{2x-3}}$$

The following are the ‘steps’ making up the teacher’s procedure for simplifying this expression:

- 1) Identify bases (“look at our bases” ... “what are our bases?”) – five, nine, fifteen
- 2) For each base – decide if it can be written in “exponential form”, if not “find its factors”, or leave it as it is:
  - a. Five – leave it as it is (“We cannot be able to write five exponentially, so five is five to the exponent one, which is you’re finished, it’s done ... it will stay as it is then”)
  - b. Nine –
    - i. Changes to “exponential form” - three squared.
    - ii. Writes three squared in brackets, to the power of  $x$  minus two
  - c. Fifteen – cannot be written in “exponential form” so:
    - i. List its factors (“can fifteen have a base and an exponent? ... no ... so now let us look at the factors of fifteen”) – they list fifteen and one; five and three
    - ii. Select one pair of factors using the other bases as a guide (“from the bases that we already have what factors can we use?”) – five and three.
    - iii. “Put it in brackets (the five and three) then two  $x$  minus three”
- 3) “Get rid of the brackets by using the laws”:
  - a. “Five to the exponent two  $x$  (*minus one*) will remain the same because there is nothing to remove” - rewrites  $5^{2x-1}$

- b. "Then we put our base as it is" (writes down three) and "multiply our exponents" - two times x and two times negative two – writes  $3^{2x-4}$
- c. "Five times three all raised to the exponent two x minus three" – "Five exponent two x minus three times three exponent two x minus three"
- 4) Once you have the same bases ("there is five above and below at the denominator, and also there is three at the top and the bottom"), "take the exponents of the same base and put them together" - writes down one base of five ("our base will be one and that base is five")
- 5) For the exponents: "from the denominator the signs change when replaced on top, the one on top ... remains as it is" – writes down the "top" exponent as is, changes the "signs" of the bottom exponent (this involves separating the exponent (two x minus three) of five in the denominator into two parts and changing the sign in front of each – "two x will be ... negative two x ... and what do we have here? (pointing to the negative three in the exponent) ... positive three").
- 6) Repeats step four and five with the base three
- 7) "Then we are going to look at like terms" – speaks (DOES NOT WRITE THIS STEP) about grouping the "like terms" in the exponent together for each base – "negative one goes with three, negative (x) goes with two x ... which we join them". Adds the "like terms" of base five (two x minus x and minus one plus three) and writes the answer of zero plus two.
- 8) Adds the like terms of base three, writing zero plus negative one.
- 9) Adds zero to two (in the exponent of five) to give two.
- 10) Adds zero to negative one (in the exponent of three) to give negative one.
- 11) "Change five squared" to give twenty five.
- 12) "Change" three to the negative one to give a "positive exponent" and get one third.
- 13) Multiplies twenty five times one on the numerator to give twenty five.
- 14) Multiplies three times one on the denominator to give three.
- 15) Writes and says the final answer - "then we have twenty five over three"

The final solution as written on the board:

$$\begin{aligned}
 & \frac{5^{2x-1} \cdot 9^{x-2}}{15^{2x-3}} \\
 &= \frac{5^{2x-1} \cdot (3^2)^{x-2}}{(5 \cdot 3)^{2x-3}} \\
 &= \frac{5^{2x-1} \cdot 3^{2x-4}}{5^{2x-3} \cdot 3^{2x-3}} \\
 &= 5^{2x-1-2x+3} \cdot 3^{2x-4-2x+3} \\
 &= 5^{0+2} \cdot 3^{0-1} \\
 &= 5^2 \cdot 3^{-1}
 \end{aligned}$$

$$= 25 \cdot \frac{1}{3}$$

$$= \frac{25}{3}$$

Diagram and table for example one:

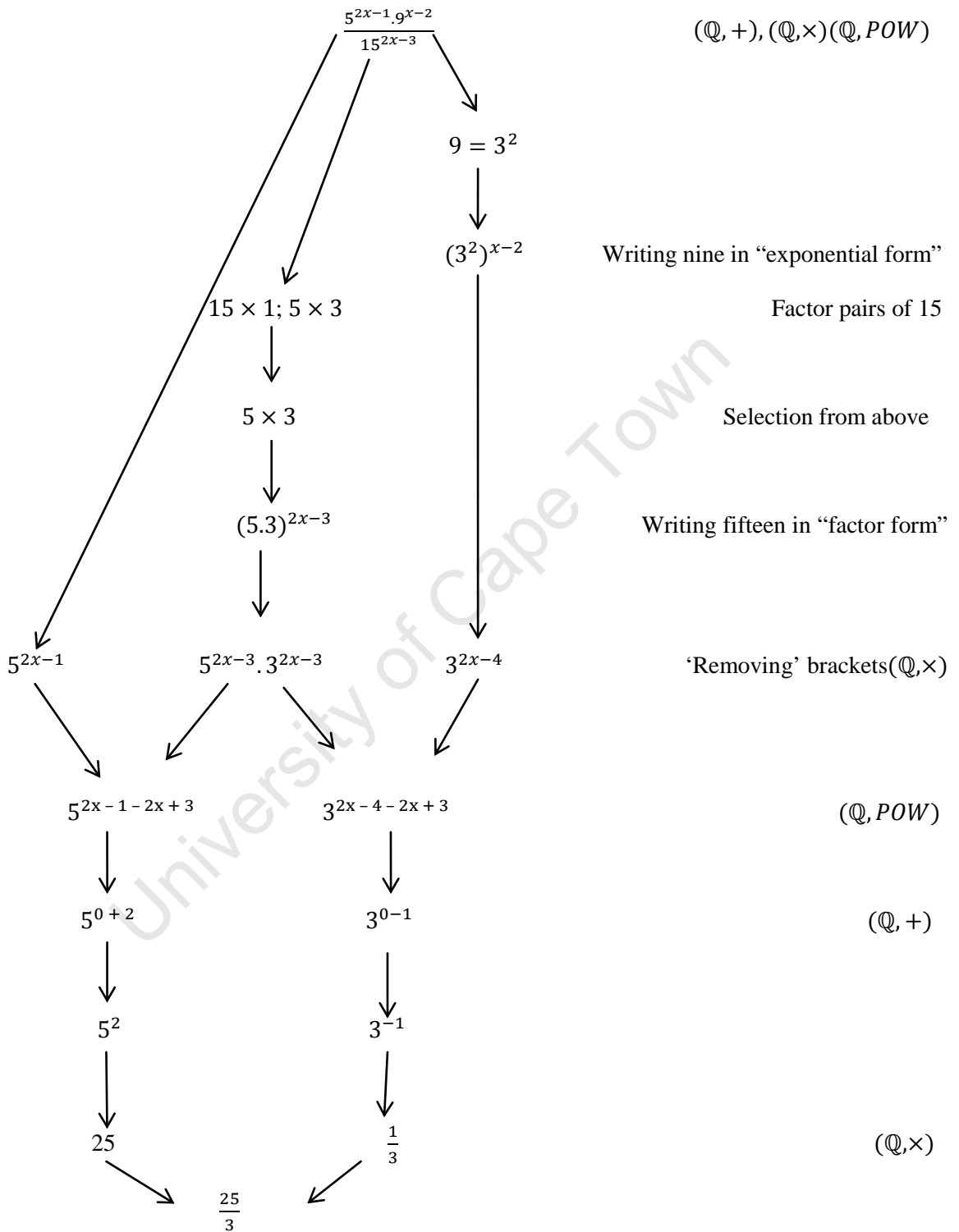


Figure A3.5 Diagrammatic representation of worked example one

**Table A3.4** Operational activity of worked example one

T	Input	Domain	Operation	Output	Codomain
1	$\frac{5^{2x-1} \cdot 9^{x-2}}{15^{2x-3}}$	$\mathbb{Q}$	Identify the bases	5, 9, 15	Bases ( $\mathbb{N}$ )
2b	9	$\mathbb{Q}$	Change nine to “exponential form”	$(3^2)^{x-2}$	$\mathbb{Q}$
2ci	15	$\mathbb{Q}$	List factors of fifteen	15, 1 and 5, 3	Pairs of factors of fifteen
2cii	15, 1 and 5, 3	Pairs of factors of fifteen	Select pair of factors of fifteen	$(5 \cdot 3)^{2x-3}$	$\mathbb{Q}$
3b	$(3^2)^{x-2}$	$\mathbb{Q}$	Multiply the exponents	$3^{2x-4}$	$\mathbb{Q}$
3c	$(5 \cdot 3)^{2x-3}$	$\mathbb{Q}$	Raise five and three to the power of $(2x - 3)$	$5^{2x-3} \cdot 3^{2x-3}$	$\mathbb{Q}$
4	$\frac{5^{2x-1} \cdot 3^{2x-4}}{5^{2x-3} \cdot 3^{2x-3}}$	$\mathbb{Q}$	Select bases which are the same - base five first	$5^{2x-1}, 5^{2x-3}$	$\{a \in \mathbb{Q}   a^x b^y\}$ (a randomly chosen first)
5	$5^{2x-1}, 5^{2x-3}$	$\mathbb{Q}$	Write the base. leave indices from numerator as are, change sign of indices from denominator	$5^{2x-1-2x+3}$	$\mathbb{Q}$
6	$3^{2x-4}, 3^{2x-3}$	$\mathbb{Q}$	Same for base three	$3^{2x-4-2x+3}$	$\{b \in \mathbb{Q}   a^x b^y\}$
7	$5^{2x-1-2x+3}$	$\mathbb{Q}$ (or $\mathbb{Z}$ )	Add the ‘like terms’ of the indices of five	$5^{0+2}$	$\mathbb{Q}$ (or $\mathbb{Z}$ )
8	$3^{2x-4-2x+3}$	$\mathbb{Q}$ (or $\mathbb{Z}$ )	Add the ‘like terms’ of the indices of three	$3^{0-1}$	$\mathbb{Q}$ (or $\mathbb{Z}$ )
9	$5^{0+2}$	$\mathbb{Q}$ (or $\mathbb{Z}$ )	Add	$5^2$	$\mathbb{Q}$ (or $\mathbb{Z}$ )
10	$3^{0-1}$	$\mathbb{Q}$ (or $\mathbb{Z}$ )	Add	$3^{-1}$	$\mathbb{Q}$ (or $\mathbb{Z}$ )
11	$5^2$	$\mathbb{Q}$	Raise five to the power of two	25	$\mathbb{Q}$
12	$3^{-1}$	$\mathbb{Q}$	Change the negative exponent to positive	$\frac{1}{3}$	$\mathbb{Q}$
13	$25 \cdot \frac{1}{3}$	$\mathbb{Q}$	Multiply the numerators	25	$\mathbb{Q}$
14	$\frac{25}{1} \cdot \frac{1}{3}$	$\mathbb{Q}$	Multiply the denominators	3	$\mathbb{Q}$
15	$\frac{25}{1} \cdot \frac{1}{3}$	$\mathbb{Q}$	Put the two together	$\frac{25}{3}$	$\mathbb{Q}$

### 3) Activating the mathematics encyclopaedia

#### From the field of production

The topic of this lesson is simplifying exponential expressions. This specific examples are of a general type of problem involving computations over the set of objects of the form  $a^n$  where  $a, n \in \mathbb{Z}$  and  $a \neq 0$ . The examples thus rely on exponentiation, as they consists of objects of the form  $a^n$ , where a base  $a$  is raised to the power  $n$  -  $a^n$  is defined as  $a \times a \times a \times \dots$  for  $n$  factors of  $a$ .

Simplification of the examples in this event involves computations involving multiplication of expressions of the form  $a^n \times b^m$ , or  $\times (a^n, b^m)$ , and division of expressions of the form  $\frac{a^n}{b^m}$ , or  $\div (a^n, b^m)$ . But in order to carry out these operations, factorization of the bases in this specific example is required so that  $a = b$  in the above expressions, i.e.  $a^n \times a^m$ , or  $\times (a^n, a^m)$ , and  $\frac{a^n}{a^m}$ , or  $\div (a^n, a^m)$ . Both of these involve the operation of multiplication over the set of rational numbers,  $(\mathbb{Q}, \times)$ , the operatory properties of which are listed in Table A2.4. The ground of this procedure thus consists of the axiomatic properties of multiplication over the set of rational numbers as well as the definition of  $a^n$ .

**From the field of recontextualisation**, the curriculum states that in grade ten learners must be able to “simplify expressions using the laws of exponents for integral exponents” (DoE, 2003: 28). The textbook used in this class emphasizes the need to “revise and understand the exponential laws” (Classroom Mathematics Grade 10, pg 153). The definition of an exponent given is: “ $a^n = a \times a \times a \dots a$  to  $n$  factors, where  $n$  is a natural number (i.e. the exponents are limited to natural numbers)” (pg 153). The next section in the textbook introduces integral exponents and gives another definition for negative exponents: “ $a^{-m} = \frac{1}{a^m}$  ( $a \in \mathbb{R}, m \in \mathbb{N}, a \neq 0$ )” (pg 154). The textbook also lists the laws of exponents, which it states “hold when  $n, m \in \mathbb{N}$ ” (pg 153). It does not refer to the laws when discussing integral exponents. It seems that textbook focuses on natural number exponents – negative exponents are treated as something which should be changed to positive and then dealt with according to the laws for natural number exponents.

#### Secondary data production P3 Lesson 1 EE2

##### 1) Realisation of content

As discussed in my primary data production for this event, the intended topic is the simplification of exponential expressions where the terms are multiplied and divided. The examples in this event are of a general type of problem involving computations over the set of objects of the form  $a^n$  where  $a, n \in \mathbb{Z}$  and  $a \neq 0$ . The examples thus rely on exponentiation, as they consist of objects of the form  $a^n$ , where a base  $a$  is raised to the power  $n$  -  $a^n$  is defined as  $a \times a \times a \times \dots$  for  $n$  factors of  $a$ .

The teacher refers to changing bases into “exponential form” a number of times during the lesson. We would expect *exponential form* in the context of grade ten school mathematics to refer to an expression of the form  $a^n$ , where  $a, n \in \mathbb{Z}$  and  $a \neq 0$ . But in this teacher’s procedure, exponential form must mean something different because an object such as  $9^{x-2}$  is already in “exponential form”. The teacher must have something else in mind when she refers to “exponential form”, which she defines as “we having a base and we having an exponent”. It appears that what she has in mind is a simplification of the object  $9^{x-2}$  such that the base is a prime number (i.e. to factorise nine into its prime factors, which yields three squared), although she does not mention primes once in this lesson. The closest she gets is when she refers to the “smallest bases” later in the lesson. It seems that learners are expected to know when they have reached the smallest bases intuitively, without any knowledge of factorizing or of primes.

Similarly, when the teacher refers to looking for the factors of the base, for example when dealing with  $15^{2x-3}$  (“can fifteen have a base and an exponent?”) to which she replies “no, so now let us look at the

factors of fifteen”), it seems that once again she is expecting the students to rewrite the bases as products of their prime factors – “fifteen is going to change into five and three”, despite the differentiation she makes between a base of fifteen and a base of nine. As described in the Princeton Companion to Mathematics (2008), the fundamental theorem of arithmetic is the claim that every positive integer can be expressed in exactly one way as a product of prime numbers. These prime numbers are known as the prime factors of the original number and the product itself is the prime factorization. The teacher differentiates between “exponential form” and finding factors in this lesson depending on the nature of the base, but in both cases she requires learners to rewrite the base as a product of its prime factors. The word she uses to describe this is “change” – “fifteen is going to *change* into five and three”. Thus although she is carrying out the process of prime factorization, it is not explicit as a regulative resource. She recontextualises the process of prime factorization into ‘changing’ a base – either by rewriting it in ‘exponential form’ (when the base can be rewritten as a power of only one prime, for example nine can be written as three squared, eight as two cubed) or ‘finding factors’ (when the base cannot be written as a power of one prime, but as a product of two or more primes, for example fifteen as five times three).

In the case of the power  $5^{2x-1}$ , the teacher says: “we cannot be able to write five exponentially, so five is five to the exponent one, which is you finished, it’s done ... so that means five to two  $x$  minus one will not change it, it will stay as it is ...” As five is a prime base, it cannot be factorised further, but there is no mention of this as the reason to leave it “as it is”. The criteria employed by this teacher’s procedure are thus vague and unnecessarily complicated. Rather than requiring the learners to find the prime factors of each base, the teacher’s procedure involves three cases (exponential form, factor form, no change) possibly in an attempt to simplify things for the learners.

Analysis suggests that the intended topic is not realised in this event – the teacher’s approach to the topic differs from the way in which the topic is constituted in the mathematics encyclopaedia due to the way in which primes, prime factorisation and the definition of exponentiation are rendered implicit in her procedure through her exposition of three options – exponential form, factor form, no change.

But despite the way in which the propositional ground underlying the topic is implicit in the procedure, the realised topic does not violate mathematical principles – the procedure is unnecessarily complicated but still corresponds with the mathematics encyclopaedia.

## 2) Regulation of the learner

Generally, analysis of the teacher’s procedure suggests that her procedure has the specific purpose of regulating the learners to produce the required outcome whether or not they know or draw on the propositional ground underlying the topic. She gives them three separate options to perform on each base, seemingly to simplify matters for them, but instead of simplifying the question her procedure is long and more complicated than is mathematically necessary. It also prevents the learners from engaging with the propositional ground underlying the question. Her procedure thus forecloses the propositional ground underlying the topic and allows for insertions by the learners. Instead of carrying out the process of prime factorisation on each base, the learners need to make decisions about what to do to the base – they have to choose one of the three options presented to them (all of which are in fact recontextualisations of the process of prime factorisation).

In this event the teacher makes seven appeals to iconic or spatial features of solutions, two to procedural features of solutions and two to mathematical propositions, definitions or processes. The appeals to extra-mathematical factors outweigh the appeals to mathematical factors, and thus necessity is situated within the teacher’s criteria and procedure rather than within the field of mathematics in this event due to the way in which the teacher appeals to three separate cases which are not mathematically different. The teacher attempts to simplify things for the learners, probably because she feels that they would not cope with the explicit process of prime factorisation. Because of this, she introduces the three options for ‘changing’ bases,

which is a recontextualisation of the mathematics in the direction of the learner, thus rendering the regulation of the learner under the aspect of the Imaginary.

### Summary

This event falls in quadrant II. The regulation of the learner depends primarily on the Imaginary.

### Primary data production P3 Lesson 1 EE3

#### 1) Generation of evaluative events

**Table A3.5** School P3 lesson 1 EE3

Time	Evaluative event/sub-event	Activity	Type
32:00 – 37:28	E <sub>3</sub>	Learners work on classwork.	Exercise

#### 2) Describing operational activity

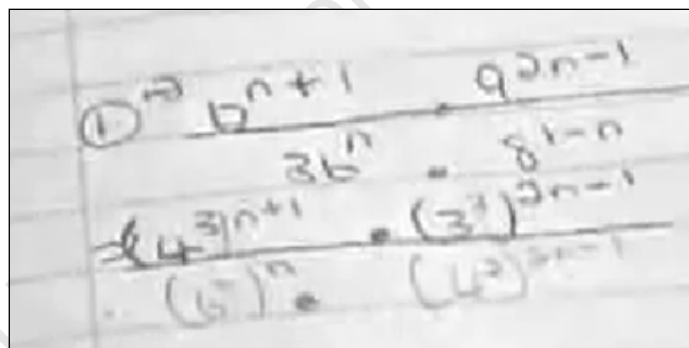
The teacher writes up the classwork exercise:

Simplify the following:

$$1) \frac{12^{n+1} \cdot 9^{2n-1}}{36^n \cdot 8^{1-n}}$$

$$2) \frac{25^n \cdot 5^{n+1} \cdot 25^2}{5^3}$$

Learners work silently with not much interaction with the teacher. While the learners are doing the classwork, we see one of their work:



$$\frac{12^{n+1} \cdot 9^{2n-1}}{36^n \cdot 8^{1-n}}$$

$$= \frac{(4^3)^{n+1} \cdot (3^2)^{2n-1}}{(6^6)^n \cdot (4^2)^{2n-1}}$$

**Figure A3.6** A learner's attempt at question one

#### 3) Activation of the mathematics encyclopaedia

See Lesson 1 EE2

The first question in the exercise set for learners in this event is taken from the textbook (exercise 7.7 page 156, number 8). The teacher has written dots instead of multiplication signs (the textbook uses multiplication signs in this question).

## Secondary data production P3 Lesson 1 EE3

### 1) Realisation of content

In this short event learners work on an exercise set by the teacher. The only material we have to analyse the realisation of content in this event is the example of a learner's work given above (Figure A3.13). In this example, the learner incorrectly rewrites twelve as four to the power three, instead of four times three, nine as three to the power three instead of three squared or three times three, thirty six as six to the power six, and eight as four squared. This learner clearly has not understood the concept of an exponent and is confusing exponentiation with multiplication. She has also not grasped the difference between the teacher's cases of "exponential form" and "finding factors". This confirms our conclusion of the previous event's analysis, that the intended topic is not realised, and in this case neither does the realised topic correspond with the mathematics encyclopaedia. Note that this conclusion is based on the work of only one learner, but due to the general uniformity of work produced by the learners in this context I make a tentative conclusion here and assume that this is not an exception.

### 2) Regulation of the learner

As there is very little interaction between the teacher and learners in this event, besides her instructing them to do the exercise, we do not have much data to make conclusions about the regulation of the learner in this event. No appeals are made to authorizing ground in this short event. But our glimpse of the learner's work above suggests that the teacher's procedure, instead of functioning as an open text, has in fact functioned as a closed text. It seems that this learner does not understand the definition of exponentiation as repeated multiplication, and neither does she understand the process of prime factorisation. The teacher's procedure, although clearly stating which steps should be followed in answering a question such as this one, did not give the learner access to the mathematical principles needed in order to answer this question. In my analysis of the previous event I discussed the way in which the teacher's procedure transformed the mathematical content in the direction of the image of the learner, supposedly in an attempt to simplify things for the learners. But instead of simplifying things, it rendered the mathematical principles implicit. This event consists of learners working on examples based on the teacher's exposition of the procedure in the previous event, so the way in which the teacher regulated learners in the previous event has impacted the mathematics produced by the learners in this event. Thus in this event the regulation of the learner remains under the aspect of the Imaginary.

### Summary

This event falls in quadrant IV. The regulation of the learner is under the aspect of the Imaginary.

## Primary data production P3 Lesson 2 EE1

### 1) Generating evaluative events

Table A3.6 Evaluative events School P3 Lesson 2

Time	Evaluative event/sub-event	Activity	Type
00:00 – 32:41	E <sub>1</sub>	Simplifying $\frac{12^{n+1} \cdot 9^{2n-1}}{36^n \cdot 8^{1-n}}$ on board	Expository
32:41- 36:04	E <sub>2</sub>	Learners work on exercise 7.7 number 7, 9 and 13 on page 58 in their textbook.	Exercise

### 2) Describing operational activity

The lesson starts with learners doing corrections of the previous lesson's classwork on the board. One learner takes over 9 minutes to do the first question:

$$\begin{aligned}
 & \frac{12^{n+1} \cdot 9^{2n-1}}{36^n \cdot 8^{1-n}} \\
 &= \frac{(4 \cdot 3)^{n+1} \cdot (3^2)^{2n-1}}{(9 \cdot 4)^n \cdot (2^3)^{1-n}} \\
 &= \frac{4^{n+1} \cdot 3^{n+1} \cdot 3^{4n-2}}{9^n \cdot 4^n \cdot 2^{3-3n}} \\
 &= \frac{(2^2)^{n+1} \cdot 3^{n+1} \cdot 3^{4n-2}}{(3^3)^{n+1} \cdot (2^2)^{n+1} \cdot 2^{3-3n}} \\
 &= \frac{2^{2n+2} \cdot 3^{n+1} \cdot 3^{4n-2}}{3^{3n+3} \cdot 2^{2n+2} \cdot 2^{3-3n}} \\
 &= 2^{2n+2-2n-2-3+3n} \cdot 3^{n+1+4n-2-3n-3}
 \end{aligned}$$

**Figure A3.7** First learner's attempt at example one

After 7 and a half minutes the teacher asks if he needs help, to which he says no. He does the last line but seems to be struggling with it. He looks back through the question and changes the second-to-last line to:

$$\begin{aligned}
 & 2^{2n+2-2n-2-3+3n} \cdot 3^{n+1+4n-2-3n-3} \\
 &= 2^{-3+3n} \cdot 3^{2n-4}
 \end{aligned}$$

Before writing the negative four he steps away, either to check his book or possibly a calculator to work out  $1 - 2 - 3$ . By the time he has finished the question it took him 9 minutes and 50 seconds. The teacher has come up and is reviewing his solution, asking the class if it is correct. She points out that there is a "problem with this nine" (pointing to the nine on the denominator) – the learner 'changed' it to three cubed instead of three squared. She tells him "when you were changing nine to exponential form, you wrote three cubed instead of three squared". She then erases his solution from the point of that error onwards and asks for another learner to come and finish the question.

Another learner eventually comes up but he starts the question again – he has 'changed' the bases in a different way to the first learner so it seems that he cannot resume from where the teacher wanted him to, so he starts from the beginning and writes the following:

$$\begin{aligned}
 & \frac{12^{n+1} \cdot 9^{2n+1}}{36^n \cdot 8^{1-n}} \\
 & \frac{12^{n+1} \cdot (3^2)^{2n+1}}{(12 \times 3)^n \cdot (2^3)^{2n+1}} \\
 & \frac{12^{n+1} \cdot 3^{4n+2}}{12^n \cdot 3^n \cdot 2^{6n+3}}
 \end{aligned}$$

**Figure A3.8** Second learner's attempt at example one

A third learner comes up and tries to work with the first learner's solution, erasing another part of it before starting. She does not make much head way and after a few minutes the teacher interrupts, and she goes back to her seat.

The second learner completes his attempt, while third learner sits down:

The image shows a piece of paper with handwritten mathematical work. The top line reads  $12^{n+1} \cdot 3^{4n+2} \cdot 2^{6n}$ . Below it, there is a horizontal line, and then the expression is written as  $12 \cdot 3^{n+2} \cdot 2^{6n+2}$ . The work appears to be a student's attempt at simplifying the expression.

**Figure A3.9** Second learner's final solution to example one

This learner has clearly not grasped the criterion about getting to the 'smallest' (i.e. prime) bases. He also makes a slip on the denominator with the power of two cubed.

The teacher now starts the question from the beginning – she does not correct the second learner's attempt. When she writes powers she says "into" for example, "four times three into  $n$  plus one" (which refers to four times three to the power of  $n$  plus one). When breaking down 36, she says "it is 9 times 4 or ... 6 squared" but she writes down 9 times 4. Then she repeats the first step again with the bases of 9 and 4, saying that "since we are still having four and nine it means we have to factorise ... then we put 4 in exponential form and 9 in exponential form" – she interchanges factorizing with 'putting in exponential form'.

"when you're taking this one from the denominator you change the sign" – she first tells them that they must put a base of two and add the exponents, then when they get to the exponent on the denominator she says that they must change the sign – no mention of subtraction here.

- Step 1: Rewrite bases as factors or in exponential form as many times as it takes until there are the same/smallest powers on the numerator and denominator.
- Step 2: Apply law 3 or 4 (to get rid of brackets)
- Step 3: Write down bases and write exponents from the numerator as they are, change the sign of the exponents from the denominator.
- Step 4: Collect like terms.

When a learner asks why she changed the nine and the four, she explains that nine is three times three and four is two times two. She says that they could have changed the 36 into six squared, but that they would have had to change it again to two times three.

A learner asks a question about changing the bases. Another learner starts talking about nine have "three 3s" to which the teacher says "no, there are 2).

The teacher writes on the board that three squared equals three times three, to which the learners reply "ohhhhh"!! The teacher does the same for two squared, saying "you multiply the number by itself". She uses an example of eight, and then says "but there are cases when you "just get the base as exponent", giving the examples of 12 and 15, where you "must find their factors". She is trying to differentiate between numbers which can be written as powers with one base, and numbers which need to be written as powers of more than one prime base. Later she speaks about getting the bases to the "smallest bases".

Let's examine the operational activity of the teacher's procedure (similar to previous lesson's e.g.):

**Table A3.7** Operational activity of example one

T	Input	Domain	Operation	Output	Codomain
1	$\frac{12^{n+1} \cdot 9^{2n-1}}{36^n \cdot 8^{1-n}}$	Q	Identify the bases and then focus on each separately	12, 9, 36, 8	Bases (Q)
2a	12	Base (Q)	Factorise and replace with (4.3)	$(4.3)^{n+1}$	Pair of factors of 12
2b	9	Base (Q)	Change to "exponential form"	$(3^2)^{2n-1}$	Q
2c	36	Base (Q)	Factorise and replace with (9.4)	$(9.4)^n$	Pair of factors of 36
2d	8	Q	Change to "exponential form"	$(2^3)^{1-n}$	Q
3a	$(4.3)^{n+1}$	Q	Raise four and three to the power of $n+1$	$4^{n+1} \cdot 3^{n+1}$	Q
3b	$(3^2)^{2n-1}$	Q	Multiply the exponents	$3^{4n-2}$	Q
3c	$(9.4)^n$	Q	Raise nine and four to the power of n	$9^n \cdot 4^n$	Q
3d	$(2^3)^{1-n}$	Q	Multiply the exponents	$2^{3-3n}$	Q
4	$4^{n+1}$	Q	Change four into 'exponential form'	$(2^2)^{n+1}$	Q
5	$(2^2)^{n+1}$	Q	Multiply the exponents	$2^{2n+2}$	Q
6	$9^n \cdot 4^n$	Q	Change nine and four into 'exponential form'	$(3^2)^n \cdot (2^2)^n$	Q
7	$(3^2)^n \cdot (2^2)^n$	Q	Multiply the exponents	$3^{2n} \cdot 2^{2n}$	Q
8	$\frac{2^{2n+2} \cdot 3^{n+1} \cdot 3^{4n-2}}{3^{2n} \cdot 2^{2n} \cdot 2^{3-3n}}$	Q	Select base 2 – leave exponents on numerator as they are, change sign of exponents on the denominator	$2^{2n+2-2n-3+3n}$	Q
9	$2^{2n+2-2n-3+3n}$	Q	Addition of 'like terms'	$2^{3n-1}$	Q
10	$\frac{2^{2n+2} \cdot 3^{n+1} \cdot 3^{4n-2}}{3^{2n} \cdot 2^{2n} \cdot 2^{3-3n}}$	Q	Select base 3 – leave exponents on the numerator as they are, change sign of exponents on denominator	$3^{n+1+4n-2-2n}$	Q
11	$3^{n+1+4n-2-2n}$	Q	Addition of 'like terms'	$3^{3n-1}$	Q

### 3) Activating the mathematics encyclopaedia

See Lesson 1 EE2

### Secondary data production P3 Lesson 2 EE1

### 1) Realisation of content

In this event the class is doing corrections to the previous lesson's exercise on simplifying exponential expressions. This event consists of just one question, which takes them 32 minutes to work through (four different learners come up to try and tackle the question, eventually the teacher takes over and does it on the board).

In the same way as discussed in P3 L1 EE2, the intended topic is not realised in this event, but the realised topic does not violate the principles of the encyclopaedia, despite them taking a very long route in answering this question!

### 2) Regulation of the learner

Once the teacher has finished going through the example above she asks if the learners have any questions relating to it. Four learners ask questions – I will examine the teacher's response to these questions in order to discuss the regulation of the learners in this event.

#### 1<sup>st</sup> question

Teacher: Who is still having a problem? Who is still struggling with the question? Who still has a problem people? Hmm?..... Are you sharp? I'm asking now, why are you not answering? Where is the problem, boy?

Learner: Here at the bottom, miss.

Teacher: Where about at the bottom?

Learner: There

Teacher: Here (points to the 9)? The thing is here the aim is that we come from this step to get to this side. Our aim is to have a base, the smallest base right? What is that again? Now 3 can help us change this 9 so that it can also have a base of 3 and then when you want to change 9 so it has 3 as a base when you multiply 9 it is 3 and 3, 3 times 3. Like 3 times 3 means when you talk about exponential you talk of the exponent and the base, right? So it means you change 9, you change 3 into 3 squared and then 4 now. 4 is as we know is 2 times 2 then it means writing 2 times 2 as an exponential form it will be? What?

Learners: 2 squared

Teacher: It changes all the time

Teacher: But then ... um ... there were others that were saying 6 squared here. It's okay, 6 squared but then here after that 6 it will be 6 squared which means again I'm going to have to have to change 6 into 2 times 3. Am I correct?

Learners: Yes.

#### Transcript School P3 Lesson 2

In her response to this question she says that two times two is two squared, for the first time in this lesson set associating exponentiation with repeated multiplication. This reference, albeit in passing, recruits appropriate elements of the mathematics encyclopaedia. But despite this, she still does not give the learners access to the idea that they need to rewrite the bases as products of their primes – she refers to smallest bases again. This means that they need to do double the work – they factorise 36 as  $9 \times 4$ , but then still have to factorise 9 and 4. Similarly she shows them that they can use six squared the first time, but will then need to factorise each 6

as  $2 \times 3$ . The process is unnecessarily long – she could have prime factorised 36 initially to yield 2 squared x 3 squared. Her explanation is quite confusing, specifically her use of the word ‘change’ as an operation.

## 2<sup>nd</sup> question

Teacher: Questions?

Learner 1: {inaudible ...} but it was 12 there, it was 12 and n plus 1 then it means there is no suppose to be 4 times 3. I get confused in changing this.

Teacher: P says her problem is changing the bases from the 1<sup>st</sup> step. Anyone can give advice?

Learner 2: (Learner explains but inaudible)

Learner 1: 9 is, there are three 3s. It's three 3s

Teacher: Ha – no, there are 2

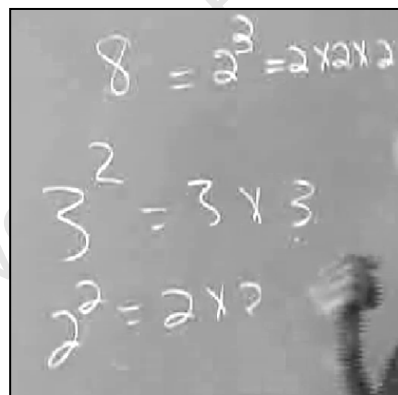
Learner 2: If maybe 13 plus, no man 14

Learner 1: Or I can ask, how many go into 12?

Teacher: Factors for 12

Learner 1: And then if maybe like this because there are two 3s in 9 and then 9 is changed and then when they put 9 I will say 3 squared because you get two 3s from 9.

Teacher: 3 squared means 3 times 3.



**Figure A3.10** “Three squared means three times three”

Learner 1: Ohhhh!

Teacher: 2 squared right? It means it's 2 times 2 - you multiply the same number by itself. Then coming to 8, with 8 if you want to change 8 as ... as ... to exponential form what makes 8?

Learners: It's 2.

Teacher: What is it?

Learner: It's 2 times 2 times 2.

Teacher: 2 times 2 it's 4?

Learner: Yes

Teacher: Then 4 times 2, what is it?

Learners: It's 8

Teacher: But then there are cases whereby you just get the base as exponent, cases like 12 and 15 whereby you must find their factors. The number 15's factor is 15 and 1 plus 5 and 3 which means we must take 5 and 3 because you cannot take 15 because you must still change it. Anyone else with a question?

## Transcript School P3 Lesson 2

The initial response of the teacher here is to ask the other learners if they have any advice for P, whose question seems to be once again about 'changing' the bases. A few learners respond, seemingly trying to explain why we can "change" numbers into other numbers, such as nine into three squared. In their explanation, the learners say that "there are three 3's" in 9 – this is correct if we are adding the three's, but incorrect if we are multiplying them. This highlights the confusion caused by the teacher's use of "change" as an 'operation' during this and the previous lesson, instead of specifying which mathematical operations she is actually carrying out. It also reveals the lack of understanding of the definition of exponentiation. The teacher corrects them saying that "no, there are two" three's in 9, which is not correct for addition, but only correct if multiplication is the operation done to these two three's. Another learner asks the question "how many go into 12?" – it is not clear how many of what she/he is talking about, but the teacher latches onto this and asks for the factors of 12. The previous learner ignores the teacher's reference to factors of 12 and continues discussing nine – "because there are two 3s in 9 and then 9 is changed and then when they put 9 I will say 3 squared because you get two 3s from 9" – again, no reference to multiplication, just to 'changing' nine because "you get two 3's from 9". At this point the teacher finally says "3 squared means 3 times 3" to which the learner(s?) respond "ohhh" – it is almost as if they have heard this before! Encouraged by their response she continues in that vein, discussing two squared and two cubed.

Although the teacher's response to this question did not initially recruit elements of the mathematics encyclopaedia, at the point at which she does so and describes exponentiation as repeated multiplication, it seems that the learners finally understand the process of "changing bases". Her initial explanations drew predominantly on the Imaginary, but at this point her regulation recruits the Symbolic.

### 3<sup>rd</sup> question

Learner: Excuse me miss, I want to ask, when the 5<sup>th</sup> step ...

Teacher: Where? This one or that one?

Learner: On number 1.

Teacher: Yes, yes, 1 2 3 4 5 okay?

Learner: On this one where you put the same exponents or you add all the exponents so you have all exponent for 2 and you add them with all of 3s ... when you are looking for your answer ... (inaudible) ...

Teacher: No you are messing up now, it means you are having a problem with integers. It means you are having a problem with integer addition and subtraction because if for example you get this one, what do you look for there? What is it that you look at?

Learner: Signs

Teacher: Is it not correct

Learners: Yes

Teacher: It's the signs and then what do we do? In solving that 2 minus 3 what do we say? Can I please have one? Yes girl.

Learner: You take the smallest number and use its sign.

Teacher: She says you take the sign of the bigger number right? And then you subtract the smallest to biggest that is how it's done. It means you are having a problem with integers if have a problem here.

### Transcript School P3 Lesson 2

The teacher classifies this learner's question as a "problem with integer addition and subtraction", recruiting the rule about the sign of the bigger number to solve the problem, giving the example of two minus three ("take the sign of the bigger number and then subtract the smallest from the biggest"). Here she responds through reinforcing a rule which they must have learnt previously when dealing with integers – this rule involves unfamiliar operations (see discussion in P6 Lesson 1 EE1.03). She situates necessity within this rule, rather than within mathematical principles related to integers, thus her regulation of the learners at this point is under the aspect of the Imaginary.

#### 4<sup>th</sup> question

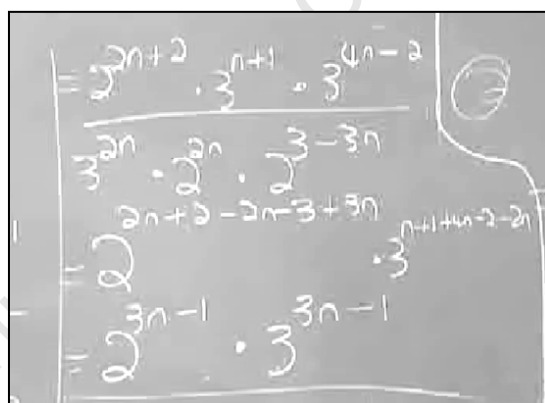


Figure A3.11 Learner asks a question about the second to last line

Learner: (ask question but cannot clearly make out what the question is)

Teacher: (pointing to second to last line). We say here we tried into bases right? And the smallest bases right? Then into what you do now when you come from the step to this step it means you look at the same bases. If you are having 2 there and you having base that is 2 and you have a base that is 2 here it means you gonna have one base that is 2 and then now all those exponents for 2 we write them there with 2 but then coming to those that are in the denominator it means the signs of the exponent will change. If it was positive it means when you write it now it will become negative, right? But now all those are at the top, example there on that base that is 3 we write 3 because 3 is one base. Then 5 and then its exponents of  $n$  plus 1 but then now coming to this 1 what happens to the sign on the denominator? It changes. It depends on whether its negative or positive right? And then if it was negative what will you write?

Her response to this question recruits the criterion of “same bases” – grouping the same bases and ‘writing’ “all those exponents for two”. She reiterates the rule that the signs of the exponents in the denominator will change without drawing on the exponential law upon which this is based – the subtraction of exponents when dividing powers of the same base. Her response here does not recruit the propositional ground underlying the question, and situates necessity within her criteria and rules, rather than within mathematics. This suggests she regulates the learners through recruiting elements of the Imaginary.

Overall in this event the teacher appeals seven times to iconic and spatial features of the solution as authorizing ground, twice to a procedural feature of the solution and only twice to a mathematical proposition, definition, process or rule. Based on this, and the teacher’s response to these four questions analysed above, I would say that necessity is located external to mathematics in this event, and overall her regulation of learners in this event depends on the Imaginary, despite one or two instances where she does recruit the mathematics encyclopaedia (and places necessity within mathematics at those moments). There are more instances where she appeals to and places necessity within criteria and rules which she has generated, some of which do not produce stable outcomes.

### Summary

This event falls into quadrant II. The regulation of the learner depends primarily on the Imaginary.

### Primary data production P3 Lesson 2 EE2

#### 1) Generation of evaluative events

**Table A3.8** School P3 Lesson 2 EE2

Time	Evaluative event/sub-event	Activity	Type
32:41- 36:04	E <sub>2</sub>	Learners work on exercise 7.7 number 7, 9 and 13 on page 58 in their textbook.	Exercise

#### 2) Describing operational activity

In this event learners work on an exercise taken from Classroom Maths grade 10, exercise 7.7 The instruction in the textbook is “Simplify (all variables are positive real numbers), and the teacher selects questions 7, 9 and 13.

One learner’s work on number 7:

$$\begin{aligned} & \frac{5^{2n+1} \cdot 3^{2n-3}}{3^{2n} \cdot 5^{2n}} \\ &= \frac{5^{2n+1} \cdot 3^{2n-3}}{3^{2n} \cdot 5^{2n}} \\ &= 5^{2n-1-2n} \cdot 3^{2n+3-2n} \\ &= 5 \quad .3 \end{aligned}$$

The learner hesitates at this point – he/she does not seem sure about what to do with the exponents. This learner has incorrectly changed the positive one in the exponent of five on the numerator to a negative one,

and similarly the negative three in the exponent of three in the numerator to a positive three – she seems to think that it is necessary to change the sign, probably confusing this with the need to subtract exponents when dividing.

The learner then writes  $4n$  as the exponent of 3 – incorrectly adding instead of subtracting.

$$\frac{5^{2n+1} \cdot 3^{2n-3}}{3^{2n} \cdot 5^{2n}} = \frac{5^{2n+1} \cdot 3^{2n-3}}{3^{2n} \cdot 5^{2n}}$$

$$= \frac{5^{2n-1-2n} \cdot 3^{2n+3-2n}}{5 \cdot 3^{4n}}$$

**Figure A3.12** The learner finishes his solution

He/she then leaves the question as is and moves onto the next one.

We then see another learner’s work – the teacher has marked the first question correctly:

**CLASSWORK**  
 ①  $\frac{5^{2n+1} \cdot 3^{2n-3}}{3^{2n} \cdot 5^{2n}} = \frac{5^{2n+1} \cdot 3^{2n-3}}{3^{2n} \cdot 5^{2n}} = \frac{5^1 \cdot 3^3}{5 \cdot 3^3} = 5 \cdot 3^3$   
 ②  $\frac{10^n \cdot 25^{n-1} \cdot 2}{50^{n+1}} = \frac{(5 \cdot 2)^n \cdot (5^2)^{n-1} \cdot 2}{(5 \cdot 2)^{n+1}} = \frac{5^n \cdot 2^n \cdot 5^{2n-2} \cdot 2}{10^{n+1} \cdot 2^{n+1}} = \frac{5^n \cdot 2^n \cdot 5^{2n-2} \cdot 2}{5^{n+1} \cdot 2^{n+1}} = 5^n \cdot 2^n \cdot 5^n$

**Figure A3.13** Another learner’s attempt at two questions

The second question is not correct – the final answer is incorrect. But he/she may not be finished yet.

And a third learner, also marked by the teacher:

$$\begin{aligned} & \frac{5^{2n+1} \times 3^{n-3}}{5^{2n} \times 3^{2n}} \\ &= \frac{5^{2n+1} \times 3^{n-3}}{3^{2n} \times 5^{2n}} \\ &= 3^{2n+1-3} \times 5^{2n+1-2n} \\ &= 3^{1n-3} \times 5^1 \end{aligned}$$

**Figure A3.14** A third learner's attempt at question seven

The teacher circles the error, but puts a cross on top of the rest of the solution, which is in fact correct.

A fourth learner does the following:

$$\begin{aligned} & \frac{5^{2n+1} \times 3^{n-3}}{3^{2n} \times 5^{2n}} \\ &= 5^{2n+1+2n} \times 3^{n-3+2n} \end{aligned}$$

**Figure A3.15** A fourth learner's attempt at question seven

Here the learner has incorrectly added the exponents of five, and left out the positive one from the numerator. He/she has incorrectly added the  $2n$  from the denominator for both 3 and 5, showing confusion about the 'laws' and when to add or subtract exponents.

### 3) Activating the mathematics encyclopaedia

Same as P1 Lesson 1 EE2

## Secondary data production P3 Lesson 2 EE2

### 1) Realisation of content

Three of the four learners' work which we see on the contain errors. Because learners work is generally uniform in this context, this is enough to make a tentative conclusion. The errors made, mostly related to addition and subtraction of exponents and showing confusion about the 'laws', suggest that the intended topic is not realised and that the realised topic does not correspond with the encyclopaedia.

### 2) Regulation of the learner

In this short event there is very little interaction between the teacher and the learner captured on video. She marks a few learners' work but does not comment. But we can assume that the activity of learners in this event is regulated by the exposition of the procedure through worked examples from the previous events. As discussed, in the previous few events the regulation of the learner is under the aspect of the Imaginary.

## Summary

This event falls into quadrant IV. The regulation of the learner is under the aspect of the Imaginary,

## Primary data production P3 Lesson 3 EE1

### 1) Generation of evaluative events

**Table A3.9** Evaluative events School P3 Lesson 3

Time	Evaluative event/sub-event	Activity	Type
00:00 – 19:27	$E_1$	Worked examples of simplifying exponential expressions.	Expository
00:00 – 05:17	$E_{1.1}$	Simplifying expression $\frac{2^{n+2}-2^{n+1}}{2^n}$	Expository
05:17 – 12:27	$E_{1.2}$	Simplifying expression $\frac{3 \cdot 2^n - 2^{n-2}}{2^{n+1}}$	Expository
08:16 – 12:03	$E_{1.2.1}$	Discussion on adding two fractions	Expository
12:27 – 19:27	$E_{1.3}$	Simplifying expression $\frac{4^n \cdot 2^{n+2} + 8^n}{2^2 \cdot 3^0 \cdot 2^{3n}}$	Expository
19:27 – 21:43	$E_2$	Students working on an exercise.	Exercise

### 2) Describing the operational activity

The stated topic for the lesson is “addition and subtraction” (with reference to exponents). The teacher refers to the previous lesson’s topic as “multiplication”, and then says “but this method is not the same because somewhere somehow we have to take out i-common factor apha (here)”.

She writes two problems on the board, and starts talking about the first one:  $\frac{2^{n+2}-2^{n+1}}{2^n}$

She compares it to the expressions dealt with in the previous lesson, saying that when you first look at this question it seems the same as those expressions, but the difference is “we have this subtraction here right?” – she points to the subtraction sign. She starts by referring to the first term of the numerator  $2^{n+2}$  and explaining to the students that they need to “expand it in a way” using the example of  $(2.3)^{n+1} = 2^{n+1} \cdot 3^{n+1}$  and saying that “what we are going to do is a sort of reverse” of this example. She does not explain why she is able to do so, but just writes the next step:

Her next step is to factorise the numerator. The reason she gives is: “for the mere fact that they are separated by that subtraction sign it means that now we have to take out our common factor”. She goes on to say what they can expect to happen next: “and then if in that common factor there is something that is the same as our denominator we will cancel now right?”.

In order to identify the common factor, she says: “we have two to the exponent n, two to the exponent two, two to the exponent n, two to the exponent one. Which one is common here?” (pointing at each term). The students reply “two to the exponent n”.

She then “takes out” the common factor (there is no explicit reference to the operation of division or the distributivity of multiplication over addition of the reals).

Once she has “taken out” the common factor, she asks:

Teacher: Then where are we going to put what is left?

Learners: In the brackets.

Teacher: In that place of two to the exponent n, one is left because I took it right?

Learners: Yes.

Teacher: So it means that one is left there we are not going to write down because it's multiplication there, okay let me write it down, it will be one times two to the power two, right?

Learner: Yes.

Teacher: Minus, here we have two to the exponent n so what are we writing?

Learners: One.

Teacher: Times two to the exponent?

Learner: One.

Teacher: Divided by?

Learners: Two to the exponent n.

Transcript School P3 Lesson 3

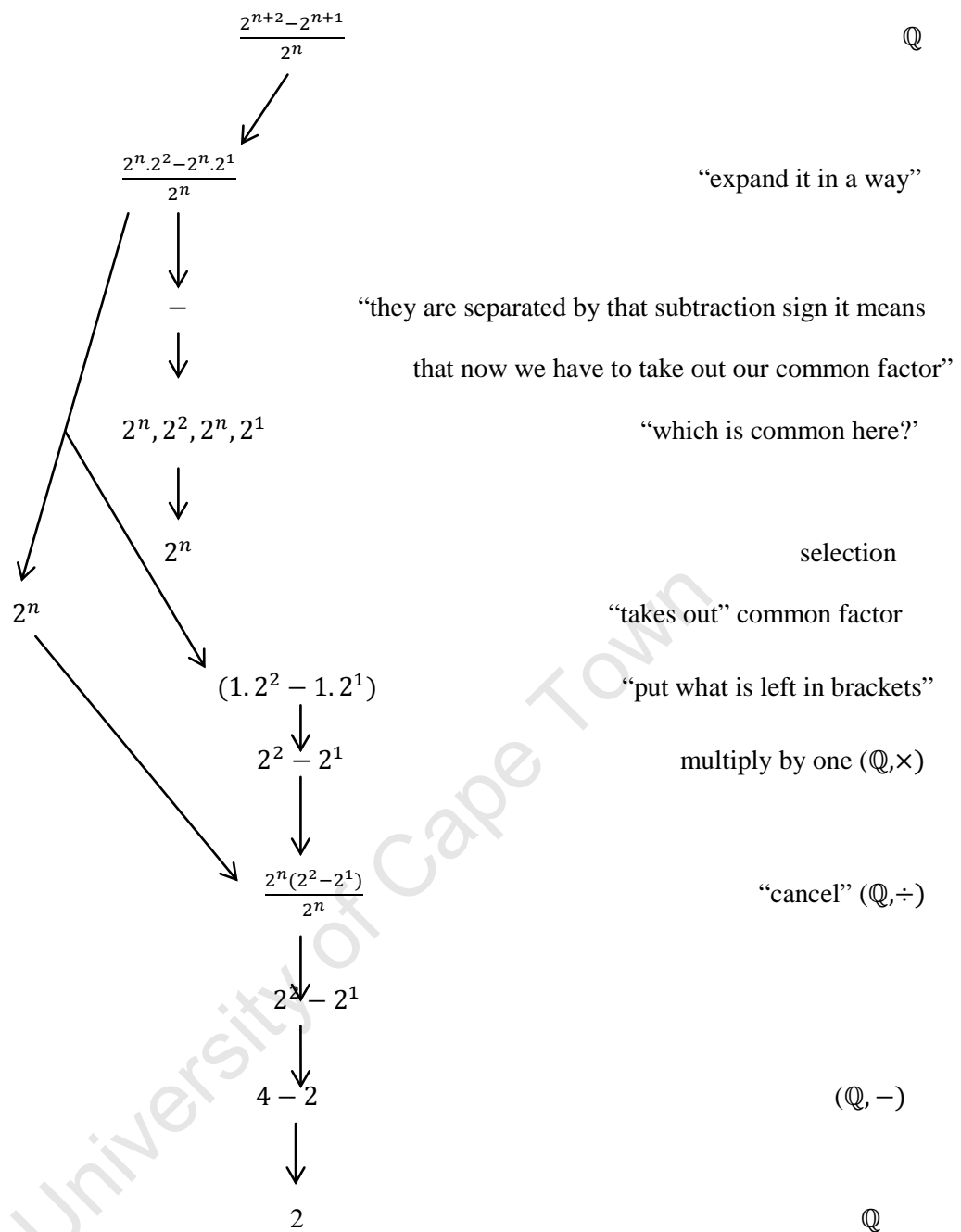
She now rewrites this step without the one's in the brackets- "so this is the same as 2 to the exponent 2 because when we multiply 2 to the exponent 2 with 1 it will be the same thing, nhe? Then minus here when we multiply 1 into 2 to the exponent 1 we will get 2 to the exponent 1 right?"

The next step is to "cancel" the two to the exponent n's on the numerator and denominator – "there we are going to cancel and have a left over of 1 over 1". She then says "we times it by that 2 right?" referring to the one that was "left over" after the cancelling step. She then asks "how many 2s inside right? Then it's we are having 2 to the exponent 2. Minus ..." the learners finish for her "2 to the exponent 1."

She completes the question:

The image shows a chalkboard with handwritten mathematical work. On the left side, the expression  $2^{n+2} - 2^{n+1}$  is written at the top. Below it, the expression is expanded to  $2^n \cdot 2^2 - 2^n \cdot 2^1$ . This is then factored as  $2^n (1 \cdot 2^2 - 1 \cdot 2^1)$ . The final result is  $2^n$ . On the right side, the expression is written as  $2^n (2^2 - 2^1)$ . This is simplified to  $2^2 - 2^1$ , which is  $4 - 2$ , resulting in  $2$ .

Figure A3.16 The completed solution to worked example one



**Figure A3.17** Diagrammatic representation of worked example one

Her procedure for the second example is the same, but she includes a discussion of adding, subtracting and dividing fractions at the end. She refers to the need to look for an LCM when adding or subtracting fractions when they have reached this point:  $3 - \frac{1}{4}$

She tells them that 3’s denominator is one. When looking for the LCM she asks them “What is the number that goes into 1 and as well as into 4?”. They seem unsure, so she asks them if four goes into one, to which they reply no. She asks them how many times one goes into four, to which they say four. She then multiplies three by four. They reach the next step:  $11/4$  divided by 2, so she explains division of fractions. “We have a fraction on top we also have a fraction at the bottom so what must we do?”. “we will change the division into multiplication”. A student calls out “eleven over four times two over one” – the teacher corrects her: “we said when you change the division sign it becomes multiplication, here at the bottom as well we have to change the sign, the numerator becomes the denominator, the denominator becomes the numerator right?”.

In the last example, she starts by saying “we are going back to that step that says we must change our bases” (referring to the previous lesson). When she reaches the ‘common factor step’, she says that “if you look there you can say we do not have common factor because there is nothing that says ... because we have two n, we have n, we have three n so there is nothing common, there is nothing. Is that what you see?” In order to find the common factor she refers to the law which says “when we multiply the same bases we add the exponents”. She applies the law on the numerator, enabling identification of the common factor two to the three n.

**Table A3.10** Operational activity of worked example one

T	Input	Domain	Operation	Output	Codomain
1	$\frac{2^{n+2} - 2^{n+1}}{2^n}$	$\mathbb{Q}$	Expand	$\frac{2^n \cdot 2^2 - 2^n \cdot 2^1}{2^n}$	$\mathbb{Q}$
2	$2^n, 2^2, 2^n, 2^1$	$\mathbb{Q}$	Identifying which is common	$2^n$	$\mathbb{Q}$
3	$\frac{2^n \cdot 2^2 - 2^n \cdot 2^1}{2^n}$	$\mathbb{Q}$	Takes out common factor i.e. divide and ‘put what is left in brackets’	$\frac{2^n(1 \cdot 2^2 - 1 \cdot 2^1)}{2^n}$	$\mathbb{Q}$
4	$\frac{2^n(1 \cdot 2^2 - 1 \cdot 2^1)}{2^n}$	$\mathbb{Q}$	Multiplication by one	$\frac{2^n(2^2 - 2^1)}{2^n}$	$\mathbb{Q}$
5	$\frac{2^n(2^2 - 2^1)}{2^n}$	$\mathbb{Q}$	Cancel i.e. divide $2^n$ by $2^n$	$2^2 - 2^1$	$\mathbb{Q}$
6	$2^2 - 2^1$	$\mathbb{Q}$	Exponentiation	$4 - 2$	$\mathbb{Q}$
7	$4 - 2$	$\mathbb{Q}$	Subtraction	$2$	$\mathbb{Q}$

### 3) Activating the mathematics encyclopaedia

#### From the field of production

The topic of this lesson is simplifying exponential expressions, which involve factorisation. The examples in this event are a general type of problem involving computations over the set of objects of the form  $a^n$  where  $a, n \in \mathbb{Z}$  and  $a \neq 0$ . The examples thus relies on exponentiation, as they consists of objects of the form  $a^n$ , where a base  $a$  is raised to the power  $n$  -  $a^n$  is defined as  $a \times a \times a \times \dots$  for  $n$  factors of  $a$ .

Simplification of these examples involves computations involving multiplication of expressions of the form  $a^n \times b^m$ , or  $\times (a^n, b^m)$ , and division of expressions of the form  $\frac{a^n}{b^m}$ , or  $\div (a^n, b^m)$ . But in order to carry out these operations, factorization is required (‘taking out a common factor’). This involves the operation of multiplication over the set of rational numbers. The ground of this procedure thus consists of the axiomatic properties of multiplication (and its inverse, division) over the set of rational numbers, listed in Table A2.4 for reference, as well as the definition of  $a^n$ .

#### From the field of recontextualisation

#### Curriculum

LO1 – “Simplify expressions using the laws of exponents for integral exponents” (DoE, 2003: 16)

Later in LO2 – “manipulate algebraic expressions by ... factorizing trinomials, factorizing by grouping in pairs, simplifying algebraic fractions with monomial denominators” (DoE, 2003: 24)

Grade 9 RNCS (DoE, 2002: 79) “Uses the laws of exponents to simplify expressions and solve equations”

“Uses factorisation to simplify algebraic expressions and solve equations”.

### **Textbook**

pg 157 – 158, Classroom Maths Grade 10

The textbook starts with an activity which deals with factorizing when the factors are powers. The general discussion below the activity states that “you cannot use these laws (of exponents) to simplify expressions in exponential notation where the terms are separated by + or – ... this is where factorization is useful. It allows you to simplify the expression further. In order to factorise, you need to be able to “reverse” the exponential laws.” (pg 157). This separates out exponent questions into those which deal with multiplication and division, and those in which the terms are added or subtracted, which is a similar approach to the one taken by the teacher.

The third example in this event is adapted from a question on page 158 in the textbook. The teacher changes the  $y$ 's in the textbook to  $n$ 's. She also changes a subtraction sign on the numerator to an addition sign (not sure if this intentional or accidental).

### **Secondary data production P3 Lesson 3 EE1**

#### **1) Realisation of content**

The stated topic for the lesson is “addition and subtraction” (with respect to exponents), so it seems that the intended topic of this event is the simplification of exponential expressions where terms are added or subtracted, which involves factorising by taking out a common factor. The teacher contrasts this topic to that of the previous two lessons, which she describes as multiplication, stating the difference between the two – “but this method is not the same because somewhere somehow we have to take out a common factor here”, thus preparing the learners for the method she is going to use. They work through three examples, but nowhere does she refer to the operatory properties of the rationals, the propositional ground for the procedure.

The teacher uses language such as “taking out” the common factor and what to do with things which are “left” once she has done so – “1 will be left in place of 2 to the exponent  $3n$ ”. She refers to ‘cancelling’, but also says “whatever is left there we divide it with what is common”, “you divide with our common factor”.

In her discussion of adding and subtracting fractions in example two, the teacher’s criteria do not explicitly reference the idea of equivalent fractions. Overall, she does not always explicitly draw on the propositional ground, but she does not violate any mathematical principles in her explanation, nor does she use pseudo-operations (she uses mathematical operations but sometimes describes them in non-mathematical terms, for example “cancel” instead of division). The realised topic thus corresponds with the mathematics encyclopaedia.

#### **2) Regulation of the learner**

When she writes the first example on the board, she compares it to the expressions dealt with in the previous lesson, saying that when you first look at this question it seems the same as those expressions, but the difference is “we have this subtraction here right?” – she points to the subtraction sign between the two terms on the numerator. It seems that she uses the subtraction sign as a trigger or marker for the need to factorise the numerator – when they see a subtraction sign they must take out a common factor. The subtraction sign is thus a signifier or character string which shows the learners what to do next. She tells them that “for the mere

fact that they are separated by that subtraction sign it means that now we have to take out our common factor”. She thus regulates the learner through the use of a signifier or trigger for the next step, appealing to this iconic feature of the question and situating necessity outside of mathematics (she gives no mathematical reason for the need to factorise). This implies that whenever terms are separated by a subtraction sign this is the procedure which must follow, which is not necessarily the case – the procedure thus functions as a closed text. The teacher is also in fact regulating her own activity, as any questions she produces of this type must have common factors, based on her explanation to learners. When she “takes out” the common factor (there is no explicit reference to the operation of division or the distributivity of multiplication over addition of reals).

She also appeals to what the solution should look like as a regulative resource in this event, telling the learners that they can expect the common factor to look like something in the denominator and can then cancel - “and then if in that common factor there is something that is the same as our denominator we will cancel now right?”. She emphasises sameness here as well as in looking for the common factor.

In this event, the teacher appeals once to ease and efficiency, five times to iconic or spatial features of solutions, four times to procedural features of solutions and four times to mathematical propositions, definitions or processes. The appeals to extra-mathematical factors are thus more than double the appeals to mathematical factors, and necessity is situated as external to the field of mathematics. The regulation is predominantly under the aspect of the Imaginary.

### Summary

This event falls into quadrant II. The regulation of the learner depends primarily on the Imaginary.

### Primary data production P3 Lesson 3 EE2

#### 1) Generation of evaluative events

Table A3.11 School P3 Lesson 3 EE2

Time	Evaluative event/sub-event	Activity	Type
19:27 – 21:43	E <sub>2</sub>	Learners working on an exercise.	Exercise

#### 2) Describing operational activity

We do not see any activity of the teacher or learners in this short event. The teacher writes up an exercise for the learners and the lesson ends just over a minute after the teacher has written up this exercise.

#### 3) Activation of the mathematics encyclopaedia

See P3 Lesson 1 EE2 and Lesson 3 EE1

### Secondary data production P3 Lesson 3 EE2

#### 1) Realisation of content

In this short event (just over two minutes) the teacher sets the class an exercise on the simplification of exponential expressions which involve “addition and subtraction”. We do not see any learners’ work.

#### 2) Regulation of the learner

There is no interaction between the teacher and the learners in this event.

**Summary** – insufficient data

## Appendix 4: Analysis for School P6

### Primary data production P6 Lesson 1 EE1

#### 1) Generating evaluative events

Figure A4.1 Evaluative events School P6 Lesson 1

Time	Evaluative event/sub-event	Activity	Type
00:00 – 21:00	E <sub>1</sub>	Finding the difference between successive terms of a number pattern.	Expository
00:00 – 05:30	E <sub>1,1</sub>	Finding the difference between the terms of the number pattern 2; 5; 8.	Expository
05:30 – 21:00	E <sub>1,2</sub>	Finding the difference between the terms of the number pattern -7; -2; 3; 8,	Expository
07:00 – 21:00	E <sub>1,3</sub>	Calculating $-7 + 5$ and $-7 - 5$ .	Expository
21:00 – 25:40	E <sub>2</sub>	Finding term fifty of the number pattern -7; -2; 3; 8.	Expository
25:40 – 48:00	E <sub>3</sub>	Working on a number patterns question from a grade 10 examination.	Exercise

#### 2) Describing operational activity

In this event, learners are asked to “write down the difference between the terms in each of the number patterns” (an activity taken from Classroom Maths Grade 10 Page 19 – 20). When the teacher talks about number patterns in this event, he tells the learners that the “to explore a number pattern the most important thing is to know ... how do you move from” one term to another (he lists a few examples – term one to term two, term two to term three etc). The closest the teacher comes to defining a “pattern” is as “something that continues for this one, the next one, and the next”, and that “we’re not just looking for any number combinations. We’re looking for a pattern that will continue ... for each term and its successor, as well as ... its predecessor.” When discussing the first example, he says: “So now, what is this pattern? How do we know that the next number ... after eight ... Even though we know it, but how do we know? Because, it’s very important because, when it comes to number patterns, we need to develop certain, like ... certain ... We need to actually develop the pattern itself. We need to see what is the relationship. Because the link ... There must be a relationship between .. two, five, eight, eleven, fourteen .. in order for us to .. to .. to be able to pick up the pattern. So, now what is the relationship?”

T: What do you do ... to this two in order for us to become five?

L: You add.

T: So two, from two to five, you add three.

L: Three.

T: What do you do to five in order for it to become eight?

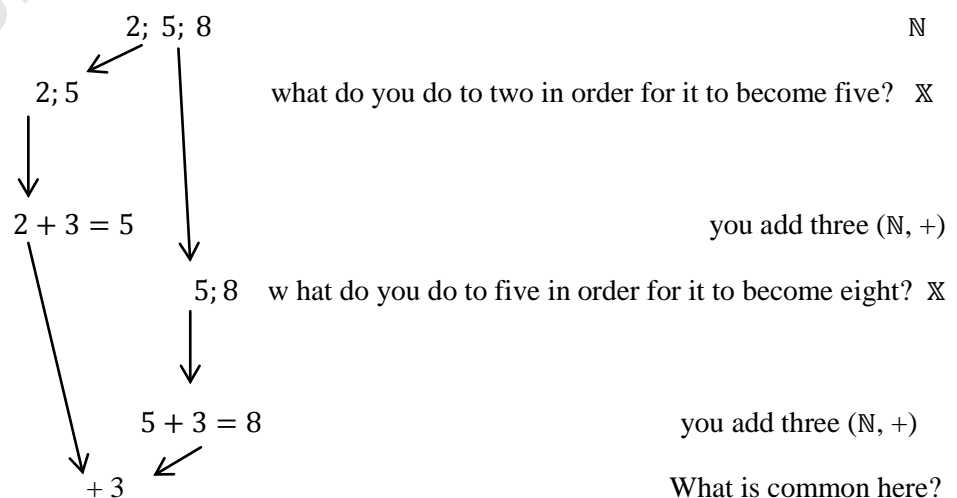
L: You add three.

T: You add three ... So immediately what do you pick up?

- L: (Silent)
- T: We pick up that, okay, there is something common here. And that something common is what?
- L: (Silence)
- T: It's plus three.
- L: Plus three.
- T: Plus three.
- L: Plus three.
- T: Plus three.
- L: Plus three.
- T: Plus three. And we have discovered ... and we've given that a name ... and we've called it the what?
- L: Silence
- T: The common?
- L: (Silence)
- T: What?
- L: (silence)
- T: The common difference ... ne?

Transcript School P6 Lesson 1

The teacher's use of words in this exchange is interesting – he teacher speaks about five 'becoming' eight! This does not make any sense mathematically, but due to the empirical treatment of the notion of "number pattern", and the way in which the subject is positioned as central, it seems that the idea of two becoming five and five becoming eight is from the point of view of the subject who has two, and then five and then eight. The learners' silence suggests that they do not grasp the "relationship" nor the notion of "common difference", even though it seems that they have been introduced to it before.



**Figure A4.1** Diagrammatic representation of procedure for finding the difference between successive terms

**Table A4.1** Operational activity of procedure for finding the difference between successive terms

T	Input	Domain	Operation	Output	Codomain
1	2, 5, 8	$\mathbb{N}$	Existential shift	/2/, /5/, /8/	$\mathbb{X}$
2	/2/	$\mathbb{X}$	What do you <b>do to two</b> in order for it to <b>become five</b> ?	/+3/	$\mathbb{X}$
3	/+3/	$\mathbb{X}$	Existential shift	3	$\mathbb{N}$
4	2+3	$\mathbb{N}$	Addition	5	$\mathbb{N}$
5	5, 8	$\mathbb{N}$	Existential shift	/5/, /8/	$\mathbb{X}$
6	/5/	$\mathbb{X}$	What do you <b>do to five</b> in order for it to <b>become eight</b> ?	/+3/	$\mathbb{X}$
7	/+3/	$\mathbb{X}$	Existential shift	3	$\mathbb{N}$
8	5 + 3	$\mathbb{N}$	Addition	8	$\mathbb{N}$
9	/2 + 3 = 5/ /5 + 3 = 8/	$\mathbb{N}$	Identifying what is <b>common</b>	“plus three”	$\mathbb{N}$

In the next example, the learners are able to produce the next three terms in the sequence easily, but do not know how to answer when the teacher asks them “How do we know that the next number ... after eight ... is going to be thirteen? How do we know that the next number after thirteen ...it’s gonna be eighteen?”

It thus seems that the learners have not grasped the notion of a “common difference” – they are able to add terms to a sequence but are not aware of how they are doing this, it is an inductive process. It seems that the students recognise that they need to extend lists of numbers in a regular way, usually by adding a number.

The teacher now mentions the notion of a “linear pattern” - “If the relationship is the same then we can make the conclusion that it has to be ... It’s what you call a linear pattern, or it’s what you call a linear sequence. [Writes /linear pattern/ on the chalkboard as he speaks.] It’s linear, which means the difference between each and every term is what? It’s the same. Between term one and term two we’ve got the same difference. Between term two and term three is ... it’s the same. Then, once we know that, once we are able to pick that up, then we know we must be dealing with a what? With a linear sequence.”

The notion of a linear pattern seems to be irrelevant to the learners’ realisation criteria. When determining the difference between successive terms of the third sequence, the teacher asks “The question is ... what do you do .. to negative seven ... in order for it to become minus two? ... Do you add five .. or do you subtract five?” – again the notion of “doing” something to a number in an inductive, empirical manner.

It is at this point that the learners’ confusion about whether to add or subtract five (again revealing their lack of understanding of the notion of “pattern” and of the “common difference”) is interpreted by the teacher as a problem with the addition and subtraction of integers. He then launches into an explanation of adding and subtracting integers (sub-event 1.03). The example used by the teacher is interesting – instead of choosing the difference relevant to the context of this question ( $-2 - (-7)$ ), he writes  $-7 - 5$ . This once again suggests that he is using the starting point of the sequence as just that – a point from which to “move” to the next number in an empirical manner. He is trying to show the learners what needs to happen in order for negative seven to “become” negative two.

$$\begin{array}{l}
 7 + 5 = 12 \\
 7 - 5 = 2 \\
 -7 + 5 = -2 \\
 -7 - 5 = -12
 \end{array}$$

**Figure A4.2** Examples used to illustrate the addition and subtraction of integers

Let's examine  $-7 + 5$  as a specific example of the teacher's procedure. The transformations making up the procedure are as follows:

- (1) Separate  $-7$  from  $+5$ .
- (2) Separate the negative sign from  $7$  and the positive sign from  $5$ .
- (3) Decide which of the two natural numbers is bigger,  $7$  or  $5$ .
- (4) Subtract the smaller number,  $5$ , from the larger number  $7$ , to get an answer of  $2$ .
- (5) Add the sign of the larger number, which is a negative sign, to the answer of  $2$  to get a final answer of  $-2$ .

But this list of transformations does not capture what is happening at the level of value. In order to separate negative seven from positive five in the first transformation listed above, an existential shift has taken place – the integer represented by the expression “ $-7 + 5$ ” (i.e.  $-2$ ) is changed at the level of value into something else in order to enable the sundering of negative seven from positive five. Numbers themselves cannot be detached in this way, so implicit in this transformation is an existential shift whereby numbers are changed into characters or symbols which can be freely detached and put back together. This existential shift thus changes the nature of the object at the level of value, but this change is indistinguishable at the level of expression – the signifiers remain constant, but the signified has changed. Existential shifts such as this one are not familiar operations found within the mathematics encyclopaedia but Davis' methodology adopts an extensional stance when analyzing pedagogic situations, which entails accepting whatever emerges operationally (whether familiar or unfamiliar) as participating in the constitution of mathematics in the local pedagogic situation.

In this example, it seems that this existential shift is motivated by a desire to operate on the domain of whole numbers, which is something most learners are familiar with (as discussed by Davis 2010a). In the teacher's procedure, he starts with integers, shifts to whole numbers and operates on these, then returns to the domain of integers.

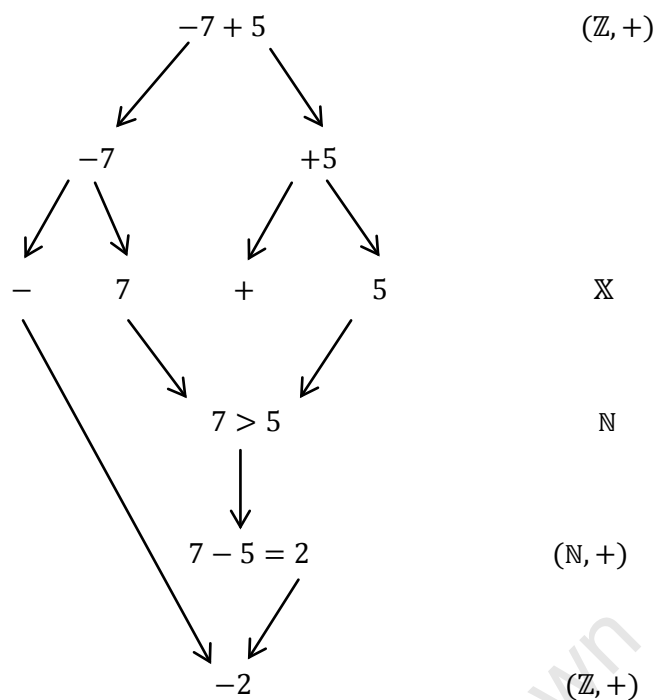


Figure A4.3 Diagrammatic representation of the procedure for calculating  $-7 + 5$

Table A4.3 Operational activity entailed in the procedure for calculating  $-7 + 5$

T	Input	Domain	Operation	Output	Codomain
1	$-7 + 5$	$\mathbb{Z}$	Existential shift	$/-7 + 5/$	$\mathbb{X}$
2	$/-7 + 5/$	$\mathbb{X}$	Sundering	$/-7/,/+5/$	$\mathbb{X}$
3	$/-7/,/+5/$	$\mathbb{X}$	Sundering	$/-/,/7/,/+/,/5/$	$\mathbb{X}$
4	$/7/,/5/$	$\mathbb{X}$	Existential shift	$7, 5$	$\mathbb{N}$
5	$7, 5$	$\mathbb{N}$	Ordering the numbers	$7 > 5$	$\mathbb{N}$
6	$7 - 5$	$\mathbb{N}$	Subtraction over $\mathbb{N}$	$2$	$\mathbb{N}$
7	$2$	$\mathbb{N}$	Existential shift	$/2/$	$\mathbb{X}$
8	$/-/,/2/$	$\mathbb{X}$	Concatenation	$/-2/$	$\mathbb{X}$
9	$/-2/$	$\mathbb{X}$	Existential shift	$-2$	$\mathbb{Z}$

### 3) Activating the mathematics encyclopaedia

#### Field of production

A sequence is a functional relationship. This lesson deals with linear sequences particularly. That linear sequences or patterns are functions is central to the intended topic of this event. The ground of this process is the linear function with its properties. The instruction ‘finding the difference between successive terms’ involves the operation of subtraction over the integers.

From the field of production, the operational resources needed to add and subtract integers are the properties of the integers. Multiplication over the integers,  $(\mathbb{Z}, \times)$ , and addition over the integers,  $(\mathbb{Z}, +)$ , have the following operatory properties:

**Table A4.4** Operator properties of  $(\mathbb{Z}, +)$  and  $(\mathbb{Z}, \times)$ 

Axioms	Properties
$\forall a, b, c \in \mathbb{Z}, a + (b + c) = (a + b) + c$	Associativity of addition
$\forall a, b, c \in \mathbb{Z}, a \times (b \times c) = (a \times b) \times c$	Associativity of multiplication
$\forall a, b \in \mathbb{Z}, a + b = b + a$	Commutativity of addition
$\forall a, b \in \mathbb{Z}, a \times b = b \times a$	Commutativity of multiplication
$\forall a, b, c \in \mathbb{Z}, a \times (b + c) = (a \times b) + (a \times c)$	Distributivity of multiplication over addition
$0 \in \mathbb{Z}$ and for $\forall a \in \mathbb{Z}, a + 0 = a = 0 + a$	Existence of additive identity
$0 \neq 1$ and for $\forall a \in \mathbb{Z}, a \times 1 = a = 1 \times a$	Existence of multiplicative identity
$\forall a \in \mathbb{Z}, \exists (-a) \in \mathbb{Z}$ such that $a + (-a) = 0$	Existence of additive inverses

(adapted from Stewart & Tall, 1977)

These axioms are the operator resources which are available for the addition and subtraction of integers and are thus the ground upon which we would expect any procedure for adding or subtracting integers to rest. But in the case of adding successive terms to a sequence of numbers, this procedure is based on the property of a linear function that the gradient between any two points if the same – as the independent variable increases by a fixed amount, the dependent variable increases uniformly i.e. that a linear function has a constant gradient. The domain of a linear function is the field of reals.

### The curriculum

Number patterns are a central theme in the South African national curriculum statement. The NCS even defines mathematics in terms of patterns – “Mathematics is a human activity that involves observing, representing and investigating patterns and quantitative relationships in physical and social phenomena and between mathematical objects themselves. Through this process, new mathematical ideas and insights are developed” (DoE, 2002: 4).

“Mathematics uses its own specialized language that involves symbols and notations for describing numerical, geometric and graphical relationships” (DoE, 2002: 4).

“Mathematics is based on observing patterns; with rigorous logical thinking, this leads to theories of abstract relations” (DoE, 2003: 9).

“Investigating patterns and relationships allows the learner to develop an appreciation of the aesthetic and creative qualities of Mathematics. These investigations develop mathematical thinking skills such as generalising, explaining, describing, observing, inferring, specialising, creating, justifying, representing, refuting and predicting” (DoE, 2002: 9).

The FET Grade 10 curriculum states that learners should “investigate number patterns (including but not limited to those where there is a constant difference between consecutive terms in a number pattern, and the general term is therefore linear) and hence:

- (a) make conjectures and generalisations;
- (b) provide explanations and justifications and attempt to prove conjectures.” (AS 10.1.3, DoE, 2003:44)

The topic of ‘number patterns’ in school mathematics could thus be described as an attempt to re-insert the functional nature of mathematics into the curriculum, but through the use of inductive logic.

But it seems that in school mathematics the foundational resource when dealing with patterns are the natural numbers (or in some cases integers). Patterns are recontextualised as operations on natural numbers, possibly because teachers know that learners are familiar with natural numbers, and thus the notion of a function,

which is central to the topic of “number patterns” gets reconstituted as operations over natural numbers. This is an example of the image of the learner being inserted, so that the central focus becomes the learner (the pedagogic subject, or the knower), and not the features of mathematics (the knowledge).

In terms of the addition and subtraction of integers, the General Education and Training Revised National Curriculum Statement for Mathematics has as its first learning outcome that “the learner will be able to recognise, describe and represent numbers and their relationships, and to count, estimate, calculate and check with competence and confidence in solving problems” (DoE, 2002: 68). The first mention of integers in the RNCS is in a grade seven assessment standard – “the learner counts forwards and backwards in the following ways: ... in integers for any intervals” (DoE, 2002: 68). For both grades seven and eight, the assessment standards state that the learner “recognises, classifies and represents the following numbers in order to describe and compare them” (DoE, 2002: 68) – integers are listed as part of these assessment standards. Another assessment standard for grade seven states that the learner “estimates and calculates by selecting and using operations appropriate to solving problems that involve ... multiple operations with integers” (DoE, 2002: 70). Integers are thus assumed knowledge by grade ten level, and are referred to as such in FET assessment standards for grade ten, for example, learners should be able to “establish between which two integers any simple surd lies” (DoE, 2003: 16).


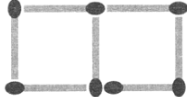

### The textbook

The two activities below make up the section in the textbook used in this class on number patterns. As we see below, there are no definitions or propositions offered in the textbook, and no mention that a linear number sequence is a function.

## Number patterns

**Activity 1.13** (AS 10.1.2)

- Work in pairs.

1. How many matches are in each of the following diagrams?
  - a) (i) 
  - (ii) 
  - (iii) 
  - b) How many matches would you need to make a row of 10 squares?
  - c) How many matches would you need to make a row of  $n$  squares?
2. Sipho is training to run a marathon. He begins by running 30 km per week. Every week, he increases the distance he runs by 4 km.
  - a) How far will he run altogether in the first 6 weeks of training?
  - b) What distance will he run in a week after 12 weeks of training?
3. a) Write down the next three terms in each of the following number patterns.
 

(i) 2; 5; 8	(ii) -7; -2; 3; 8	(iii) 29; 25; 21
(iv) $1\frac{1}{2}$ ; $2\frac{1}{4}$ ; 3	(v) 1,1; 1,5; 1,9	

  - b) Write down the difference between the terms in each of the number patterns in **3 a**).

4. Consider the following number pattern: 4; 7; 10; 13; 16

- What is the difference between successive terms?
- If the difference between successive terms is 3 and  $T_1$  represents term 1,  $T_2$  represents term 2, and so on, complete the following statements.
 

$T_1 = 1 \times 3 + \square$	$T_2 = 2 \times 3 + \square$
$T_3 = 3 \times 3 + \square$	$T_4 = 4 \times 3 + \square$
$T_5 = 5 \times 3 + \square$	
- Calculate the 100th term in the number pattern.
- Write down a formula for the  $n$ th term,  $T_n$ , in the number pattern.

5. Use the same method as in question 4. Write down a formula for the  $n$ th term or  $T_n$  for each number pattern below.

- 1; 3; 5; 7; 9; ...
- 4; 0; 4; 8; ...
- 3; 7; 11; 15; ...
- 7; 13; 19; 25; ...
- $0; \frac{2}{3}; 1\frac{1}{3}; 2; \dots$

Figure A4.4 Activity 1.13 from Classroom Mathematics Grade 10

The teacher’s guide contains the following discussion of activity 1.13, which contains no reference to functions:

**Number patterns**

**Activity 1.13**

- The number patterns in this activity have a constant difference between the terms.
- The general term of these sequences is given by the formula  $T_n = a + (n - 1)d$  where  $a$  is the first term of the sequence and  $d$  is the constant difference.
- These sequences are called **arithmetic sequences**.

Figure A4.5 Extract from teachers’ guide, Classroom Maths Grade 10

In terms of the addition and subtraction of integers, the other component of the topic of this event, the grade ten textbook used in this classroom revises integers – “the set of integers includes the whole numbers and negative numbers” (Laridon et al, 2005: 8). Number lines are central to the textbook’s discussion of integers. A number line is used to show that “positive numbers are to the right of zero and the negative numbers are to the left of zero” (Laridon et al, 2005: 8). The notion of additive inverses is introduced through a discussion of integers and their “opposites”, which are explained as being “the same distance from zero on the number line but in opposite directions” (Laridon et al, 2005: 8). A method for adding integers is given using a number line – “start at the first integer. If you are adding a positive integer, move to the right on the number line. If you are adding a negative integer, move to the left.” (Laridon et al, 2005: 9). One of the examples used is  $-3 + 7$ , which is done using the above “number line” method (pg 9). The commutativity of addition over the integers (the sum of two integers is the same regardless of the order in which they are considered) is not mentioned as a possible operatory resource, where  $-3 + 7 = 7 - 3$ .

The textbook also explains that “subtracting an integer is the same as adding the “opposite” or additive inverse of the number” (Laridon et al, 2005: 9). The example given for this is  $2 - 4 = 2 + (-4) = -2$ . When dealing with multiplication and division, the textbook states that “when you multiply or divide integers that have the same sign, the answer is positive. When you multiply or divide integers with opposite signs, the answer will always be negative” (Laridon et al, 2005: 9).

## Secondary data production P6 Lesson 1 EE1

### 1) Realisation of content

The intended topic of this event is finding the difference between successive terms of a number pattern. The patterns used as examples are all linear patterns. The teacher starts by talking about “exploring the pattern” and based on that gets learners to extend the pattern in order to find the difference between successive terms.

That a sequence is a function is not explicit in this lesson. Instead, sequences are approached empirically – “to explore a pattern”, “how do you move from ...”, “the pattern we have from term one to term two ... should be the same pattern we have from term two to (term three)”, “we need to actually develop the pattern itself”, “what do you do ... to this two ... in order for us to become five?”. These statements place the learner as central and the pattern itself as something to be empirically ‘explored’ and even created by the learners. This empirical emphasis renders the elements of the mathematics encyclopaedia (i.e. a linear function with its properties) implicit in this event.

The teacher speaks about one number becoming another (“What do you do ... to this two .. in order for us to become five?”; “What do you do to five in order for it to become eight?”), it sounds as though the numbers change values, which does not really make sense, but due to the empirical treatment of the notion of “number pattern” discussed above, and the way in which the subject is positioned as central, it seems that the idea of two becoming five and five becoming eight is from the point of view of the subject who has two, and then five and then eight. Later in the event while doing another example the teacher says “the question is ... what do you do to negative seven in order for it to become minus two? ... Do you add five or do you subtract five?” – again the notion of “doing” something to a number in an inductive, empirical manner.

In order to enable learners to find the difference between successive terms of these sequences, the teacher digresses into a discussion of integer addition and subtraction. The operatory properties of addition over the integers are not explicitly drawn on in the procedure for adding integers in this sub-event. In order to add two integers such as negative seven and two using the elements of the encyclopaedia which were outlined earlier, we could exploit the commutativity of addition over the integers. This property enables us to calculate  $2 + (-7)$  instead of  $-7 + 2$  in order to get the answer of  $-5$ . Another way of calculating  $-7 + 2$  would be to draw on the existence of additive inverses for all integers. This would involve replacing 2 with  $7 + (-5)$ , which would yield  $-7 + 7 + (-5)$ , from which the answer of  $-5$  is clearly obtained. The common feature of these two approaches is that they both draw explicitly on the operatory properties of addition over the integers.

When we compare the operational activity in the evaluative event under analysis with these elements within the mathematics encyclopaedia we find some key differences. The objects being operated on are unfamiliar in that they are not objects usually associated with the topic of integer addition. There is a shift from the expected domain of the integers to the domains of the whole or natural numbers. Although whole and natural numbers are part of the set of integers, in this example and others in the context, the learners do not engage with them as integers but as something separate from integers, thus this is considered a shift in the domain (see Basbozkurt, 2010 and Davis, 2010a). In order to enable this shift in the domain, the initial objects (integers) are changed into characters which can be taken apart and then treated as whole numbers. The existential shift which has taken place enables the process of taking numbers apart into characters (symbols and numbers). This breaking apart is an unfamiliar manipulation which is not found in the mathematics encyclopaedia – that of sundering (discussed in detail in Jaffer, 2009). Sundering can be described as a pseudo-operation as it does not necessarily have a stable, unique output for any input, and thus is not a function. For example, the string  $/-7+2/$  could be sundered to yield any of these combinations of character strings:  $/-7/$  and  $/+2/$ ,  $/-/$  and  $/7+2/$ ,  $/-7+ /$  and  $/2/$ , or  $/-/$ ,  $/7/$ ,  $/+ /$  and  $/2/$ . The concatenation which occurs when the negative sign is rejoined to the final answer (referred to in this context as ‘giving the sign of the

bigger number to the answer'), is more stable than sundering, as given two character strings,  $/-/$  and  $/5/$ , there is only one way of concatenating them, yielding  $/-5/$ . Concatenation in this context is thus a function and therefore is considered an operation, although it is not a familiar or expected one when dealing with the topic of integer addition from the perspective of the mathematics encyclopaedia.

In summary, the intended topic of finding the difference between successive terms of a number pattern is/not realised in this event due to the way in which the notion of a linear function is rendered implicit, and the way in which a pattern is recontextualised as something to be empirically explored and created, rather than described. In addition to this, our comparison of the procedure used for adding integers with the elements from the mathematics encyclopaedia has shown a shift at the level of the announced or intended topic (whole numbers are being operated on instead of integers). In addition to this, the realised topic is not aligned with the resources in the mathematics encyclopaedia, as seen by the use of sundering, which is not a function and can thus be described as a pseudo-operation, indicating that the essential property of mathematics that all its operations be functions is not adhered to in this procedure. Thus this evaluative event would fall into quadrant IV in the matrix above.

## 2) Regulation of the learner

Despite the way in which the learners are able to add successive terms to a sequence of numbers, this does not necessarily mean that they grasp the notion of "number pattern". They see a list of numbers and recognize how to add additional numbers to it, but this does not necessarily mean that the notion of "number pattern" is a regulative resource. The fact that the question is framed in such a way as to give them a "pattern" without any description of the type of pattern requires the use of inductive reasoning to answer the question.

The idea of movement between terms suggests the individual physically moving from one term to the next, which renders the notion of common difference as relating to the individual, rather than the sequence. This is another way in which the regulation in this event depends on inductive reasoning. The way in which inductive reasoning is encouraged in this event suggests that the empirical is a strong regulative resource in this event, and thus that necessity is situated within the empirical, instead of the propositional ground of the field of mathematics. The focus on the subject (i.e. the learner) as central in the teacher's procedure, as discussed above, in the role of exploring and even creating the sequence reinforces the strong empirical focus of the regulation in this event.

Through the emphasis on empirical and inductive reasoning in this event, a number pattern is constituted as something sensible which can be discovered inductively by students – the intelligible nature of the mathematics underlying the topic is for the most part ignored. Instead of using the empirical exploration of a sequence as a way of introducing the mathematical principles, the only representation of mathematical ideas in this event are sensible and empirical. Related to the reliance on empirical and inductive reasoning in this event, mathematical definitions are reframed in sensible terms, for example a pattern is defined as "something that continues for this one, the next one, and the next", and a linear pattern as something in which "the difference between each and every term is what? It's the same. Between term one and term two we've got the same difference. Between term two and term three is ... it's the same. Then, once we know that, once we are able to pick that up, then we know we must be dealing with a what? With a linear sequence." Later in the lesson the common difference is defined as "the number we add each time", placing the learner's action of adding as central to the definition. He also says that they have "discovered" the common difference. It seems important to ask why it is that the teacher recruits the empirical in this event. It could be because the teacher does not expect learners to be able to engage with the notion of a linear function and its properties so instead recontextualises a linear number pattern as something which is discovered and explored inductively, rather than presenting a linear function as the mathematical object being discussed and using its properties to make deductions. The teacher is thus bending the mathematical content in the direction of the image of the learner at the point at which the replacement is made.

Another possible reason for attempts at induction is the belief that in order to learn the learners need to be doing practical activities, again suggesting that learners are not able to engage directly with explicit mathematical principles and need to go through a process of discovery in order to grasp these principles. This is exposed by Dowling (1998) as a myth - he argues that while the physical world provides starting points for the regulation of mathematical knowledge, mathematics is principled knowledge, and must be made explicit by the teacher.

Another feature related to the regulation of the learner in this event is the way in which the teacher recontextualises the topic of number patterns into operations over the natural numbers. The way in which the teacher shifts the domain from integers to natural numbers in sub-event 1.3, which the learners are familiar with, suggests that the teacher does not expect the learners to be able to engage with the integers and replaces the content with something that the learners are able to engage with (natural number addition). The topics of integer addition, number patterns and linear functions are recontextualised in this way so that learners can reach the correct answers whether or not they understand the properties of integers and a linear function with its properties. It seems that often in school mathematics the foundational resource when dealing with patterns are the natural numbers. Patterns are recontextualised as operations on natural numbers, and thus the notion of a function, which is central to the topic of “number patterns” gets reconstituted as operations over natural numbers. This is an example of the image of the learner being inserted, so that the central focus becomes the learner (the pedagogic subject, or the knower), and not the features of mathematics (the knowledge), locating necessity outside of mathematics.

In the integer addition sub-event, necessity is also situated outside of the field of mathematics and within the rules generated by the teacher (“so if the signs are the same, what do you do? You take the common sign and then you add. If the signs are not the same what do you do? You subtract ... But first you take the sign of the what? The sign of the bigger number. You look at the bigger number between the two and then you take the sign of the bigger number”) and not within the field of mathematics, as the propositional ground underlying the addition of integers is not referred to, even implicitly, by the teacher. Overall in this event, there are three appeals made to empirical testing (in the process of ‘exploring’ number patterns), three to iconic or spatial features of the procedure, one to the procedural features of the integer addition procedure, and two to mathematical propositions or processes (one involving a description of the four basic operations needed to work with number patterns, the second explaining the relationship between addition and subtraction, multiplication and division as inverses of each other). The appeals made to mathematical factors do not refer to a linear function with its properties, or to the properties of the integers, both of which are central to the intended topic. Thus necessity is located as external to mathematics in this event and the topics are reconstituted in response to the learner. This does result in Symbolic activity, but the re-symbolisation of the content is in the direction of the Imaginary as the image of the learners held by the teacher is regulating the decision made to replace the content.

## Summary

This event falls into quadrant IV. The regulation of the learner depends primarily on the Imaginary.

## Primary data production P6 Lesson 1 EE2

### 1) Generating evaluative events

Table A4.5 School P6 Lesson 1 EE2

Time	Evaluative event/sub-event	Activity	Type
21:00 – 25:40	E <sub>2</sub>	Finding term fifty of the number pattern -7; -2; 3; 8.	Expository

## 2) Describing operational activity

In this short event the teacher shows learners how to find the fiftieth term of the number pattern they have been dealing with - 7; -2; 3; 8.

He asks the learners to calculate term fifty as a way of leading them to the general term for an arithmetic sequence. He has just asked them what the general term is and has not got a response from them, so he asks for term fifty, prompting them to give him the formula  $T_n = a + (n - 1)d$ .

He asks them what the 'a' value is in the sequence they are dealing with to which a number of learners answer "five" (which is the common difference). He tells them the common difference ( $d$ ) is 5, pointing to the sequence he had written up in the previous event, and repeats the question about  $a$ . There is silence for a few seconds and then he asks a learner directly who says "I think it's negative seven, the first term". The teacher agrees and says " $a$  refers to the first term".

When they are simplifying he stops at the point at which  $T_{50} = -7 + 245$  and asks the learners to do this without a calculator. He tells them to "think of what we've just done now" (referring to the integer addition rules). They call out that it is 238. The teacher then tells them that "you have used this principle that we've just talked about because if the signs are not the same you take the ... sign of the bigger number, in this case positive, and then you subtracted the small one from the big one").

He ends the event by telling them that "number patterns are the easiest thing to see. Anyone can see a pattern, right? But what is important for you to be able to continue the pattern ... right?"

His procedure for finding term fifty:

- 1) Write down general term.
- 2) Substitute:  $n = 50$ ,  $a = -7$ ,  $d = 5$
- 3) Simplify (uses BODMAS rule – the brackets first, and integer addition rules).

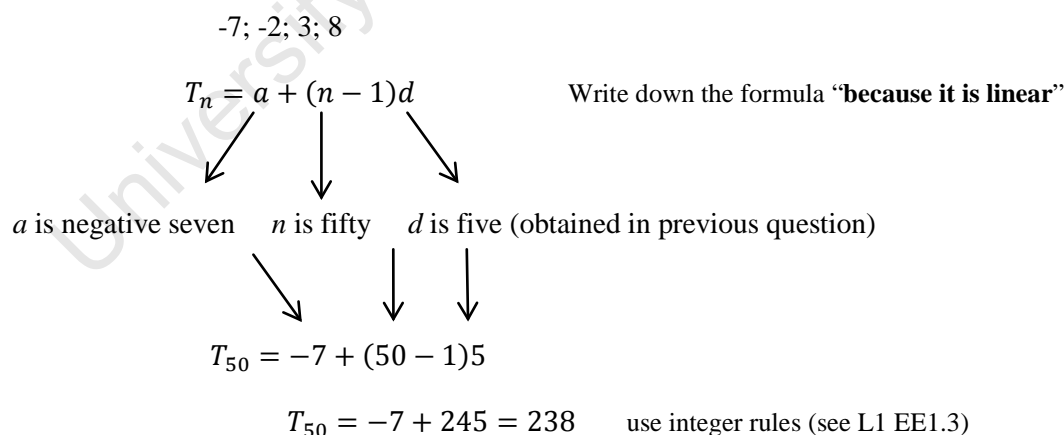


Figure A4.6 Diagrammatic representation of procedure for finding term fifty

## 3) Activating the mathematics encyclopaedia

See P6 Lesson 1 EE1

Finding the 50th term of a number pattern requires knowledge of the  $n$ th term, which is based on the propositional ground of a linear function with its properties – this is discussed in detail in my analysis of the next event where this process is carried out.

## Secondary data production P6 Lesson 1 EE2

### 1) Realisation of content

The intended topic of this short event is to find the fiftieth term of a linear sequence. Once again, the notion of a linear pattern as a function is implicit in the teacher's explanation, which focuses on arithmetic and algebra.

He also briefly refers to the rules for integer addition and subtraction, which were discussed in the previous event, and which depend on a shift in the domain from integers to natural numbers, as well as the use of the unfamiliar operations of concatenation and sundering.

The intended topic is thus not realised and the realised topic does not correspond with the encyclopaedia because of the dependence on the integer rules generated in the previous event – see discussion of L1 EE1.

### 2) Regulation of the learner

This short event does not contain much interaction to analyse, and there is only one appeal made in this event to a procedural feature of a solution. But the teacher's use of the integer rules to regulate the learners in producing the correct solution, as well as the way in which the notion of a linear function is implicit (or absent) from the regulative resources of this event and suggests a rendering of the Symbolic under the aspect of the Imaginary, as discussed in the previous lesson. In this event the teacher also tells them that “number patterns are the easiest thing to see. Anyone can see a pattern, right? But what is important for you to be able to continue the pattern ... right?”, thus again emphasising the empirical, inductive approach to patterns he drew on in the previous event.

## Summary

This event falls into quadrant IV. The regulation of the learner depends primarily on the Imaginary.

## Primary data production P6 Lesson 1 EE3

### 1) Generating evaluative events

Table A4.6 School P6 Lesson 1 EE3

Time	Evaluative event/sub-event	Activity	Type
25:40 – 48:00	E <sub>3</sub>	Learners working on a number patterns question from a grade 10 examination.	Exercise

### 2) Describing operational activity

During this event the learners work on an exercise – a question from a previous grade 10 examination. The teacher tells them that “if you can do this then you did well”, and “if you can do this question then it means you can do number patterns in grade ten” – he thus considers this question as representative of the kinds of questions they should be able to manage, based on their work on number patterns in class:

### QUESTION 3

The science class did experiments on new types of candles which the manufacturers claim will burn very long. A particular candle was lit in a wind free environment and its length was measured every hour. The data regarding the length of the candle was recorded in the following table:

Time in hours	0	1	2	3	4	5
Length in cm	33	31	29	27	25	23



- 3.1 What was the length of the candle after 8 hours? (2)
- 3.2 Use a variable to write an algebraic statement to generalize the relationship between the time in hours and the length of the candle. (3)
- 3.3 After how many hours would the candle have melted completely? (3)
- [8]

Figure A4.7 2008 Grade 10 examination question on “patterns”

When reading through the question, with reference to the table he says to them that “so you see that as time goes on, the length of the candle is ... it’s depreciating ... there is a sort of relationship ... I want you to look at that relationship because it’s ... I mean it’s just a pattern of numbers”.

He tells them they can work in pairs, saying he expects them to be done in ten minutes. The learners work for the rest of the lesson (22 minutes) and a few ask the teacher questions as they work. After five minutes the teacher says “number three almost done by now”, the learners say yes. A few learners call the teacher to answer a question but we do not hear their questions or his answers. At one point he directs an explanation to the whole class:

“In 3.2 they say use a variable ... to write down the relationship between  $n$  and  $T_n$ . ‘ $a$ ’ will fit as ‘ $n$ ’ and then another one will fit as  $T_n$  (pointing to the linear formula written up). Now it goes to function ... we have length, right? Then here we have time (points to table he has drawn) ... the length will be  $T_n$  and then the time will be  $n$ . So you should write it in this form (pointing to the formula again). So which means what do I want to find? The common difference as well as the first term ... this is the general formula for any linear sequence ...  $T_n$  equals  $a$  plus  $n$  minus one times  $d$ . So this is what they want. Generalise ... write down the general formula, right?”

Right near the end of the lesson he says “the problem question seems to be here question number three point two, right? Now let’s read slowly. Use a variable, a letter, to write an algebraic statement to generalise the relationship between the time, in hours, and the length of the candle which is measured in cm (he repeats this twice). Now let me make it easy for you because to write things as a variable you can use  $x$  and  $y$ . You can say ok let the number of hours be  $y$ , I mean hours be  $x$ , and this be  $y$ . But now I’m saying look at the pattern, right? If you happen to see that there is a common difference between the terms then you can use the general formula  $T_n$  equals to  $a$  plus  $n$  minus 1 times  $d$ , ok? You’re going to use that if you see there is a common difference, ok, right? So which means what do you find for me in this formula? You will write down for me the value of  $a$  which is the first term. And you will also write down the value of  $d$  which is the common difference. If you find  $a$  and the common difference this formula, this generalisation, that will be the generalisation that they want ... so basically this is simple ... you want a general formula and if it happens to be linear then this will be your general formula but you’ll have to find the value of  $a$  as well as the value of the common difference, ok?”

$$T_n = a + (n-1)d$$

↓  
1st term
↓  
Common diff

**Figure A4.8** Finding the general term of a linear sequence

He tells them to complete number 3 and 4 for homework.

His procedure for finding the general term:

- 4) “Check the differences” in order to decide if the sequence is linear – if there is a “common difference” between consecutive terms, the sequence is linear, if not it is “another type of thing”.
- 5) **Because the sequence is linear**, write down the formula  $T_n = a + (n - 1)d$ .
- 6) Substitute in the first term ( $a$ ) and the common difference ( $d$ ).

In general terms, given a sequence:  $T_1; T_2; T_3; T_4; T_5$

- 1) If  $T_2 - T_1 = T_3 - T_2$  then the sequence is linear.
- 2) We can thus use the formula  $T_n = a + (n - 1)d$ .
- 3) Substitute  $T_1$  in the place of  $a$  and the difference  $T_2 - T_1$  in the place of  $d$  –

$$T_n = T_1 + (n - 1)(T_2 - T_1).$$

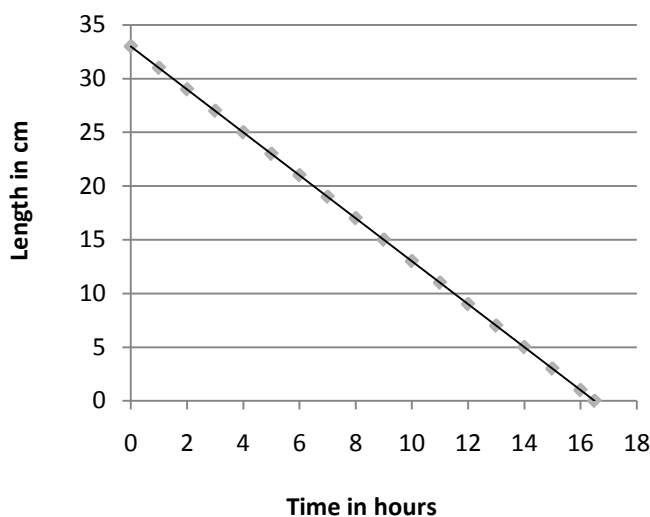
### 3) Activating the mathematics encyclopaedia

The announced topic of the question is “patterns”, and it was taken from a 2008 Grade 10 examination question. The context of the question is a science experiment investigating the relationship between the burning time and the length of the candle. The NCS states that “contexts should be selected in which the learner can use algebraic language and skills to describe patterns and relationships in a way that builds awareness of other Learning Areas” (DoE, 2002: 9). The question is framed in such a way as to appear to be recruiting mathematics into the context of a science class’s experiment – this is reminiscent of Dowling’s (1998) discussion of the “myth of relevance”. The fact that only integral values are obtained for the length in the ‘experiment’, and that there is in fact a regular ‘pattern’, makes the problem seem quite artificial.

Let’s look at the three sub-questions in order to ask what it is that is being examined in this question. Question 3.1 asks for the length of the candle after eight hours. This can be calculated quite simply by extending the table – it is 17cm. In order to extend the table, we needed to recognize that for each hour the length of the candle decreases by 2cm. A more concise way of calculating the answer would be to take the original length of 33cm and subtract two times the number of hours to yield 17cm ( $33 - 2 \times 8 = 17$ ), again recognizing that for each hour the length decreases by 2cm. This leads us to a general formula for the calculating the length of the candle ( $L$ ) in cm, after some time ( $T$ ) in hours:  $L = 33 - 2T$ . Another way of reaching this formula is to calculate a value for the gradient using any pair of values from the table (the

gradient is negative two) and substitute that and the point at which the time is zero (length of 33) or any other point from the table into the equation for a straight line. The table thus represents a line with gradient of negative two and y-intercept of thirty three. This method relies on us identifying the table as representing a linear function, and drawing on the properties of a line, specifically the proposition that a line is the shortest distance between two points – we can use any two points to sketch a line or find its equation.

If we sketch the sequence represented by this equation we yield the following line, with a domain and co-domain of positive reals because of the context of the question (although the table only includes integral values).



**Figure A4.9** The line with equation  $L = 33 - 2T$

It seems that the first part of this question (3.1) requires learners to obtain the answer by extending the table (either actually or mentally – ‘by inspection’), based on the mark allocation of two marks and also the fact that question 3.2 asks for an “algebraic statement” (or a formula), suggesting that this formula is not required in order to complete 3.1.

Question 3.2 asks for an algebraic statement to “generalize the relationship between the time in hours and the length of the candle”. The wording of this question aligns with the National Curriculum Statement, which states that “investigating patterns and relationships allows the learner to develop an appreciation of the aesthetic and creative qualities of Mathematics. These investigations develop mathematical thinking skills such as **generalising**, explaining, describing, observing, inferring, specialising, creating, justifying, representing, refuting and predicting” (DoE, 2002: 9). In the learning outcomes for grade 10, we are told that learner should “investigate number patterns (including but not limited to those where there is a constant difference between consecutive terms in a number pattern, and the general term is therefore linear)” (DoE, 2003: 18). The purpose of the investigation is for learners to “make conjectures and **generalizations**” and “provide explanations and justifications and attempt to prove conjectures” (DoE, 2003: 18). We will later ask whether this question does in fact develop the skills listed above.

In order to find the “algebraic statement”, we can follow the methods outlined above, either using the pattern to generate the formula (33 minus two times T will give us L), or drawing on the properties of a line. Another method is to use the formula for finding the general term of a linear pattern:  $T_n = a + (n - 1)d$ , which is found on the FET formula sheet. This formula would yield a general term of  $T_n = -2n + 35$ , where  $T_n$  represents the length and  $n$  represents the term number (so time zero is term one, time one is term two etc). This differs from the formula for the length which we obtained earlier:  $L = -2T + 33$  - in this

formula  $T$  represents the number of hours, which can be read directly from the table, and not the ‘term number’. This difference will become important when we examine the teacher’s treatment of the question.

In question 3.3, learners are asked after how many hours the candle would have melted completely. This requires learners to realise that the length of a completely melted candle would be zero and to calculate the number of hours this would take, either by substitution into the formula found in question 3.2, or by extending the table until 0 cm is reached. If we do the latter, we see that a length of zero does not appear in the table:

<b>Time</b>	0	1	2		13	14	15	16	17
<b>Length</b>	33	31	29	...	7	5	3	1	-1

**Figure A4.10** Extending the table from question 3.1

This is obviously because the length starts at 33 and decreases by 2cm every hour, which yields only odd numbers. In addition to this, the table contains only integral values, and thus the number of hours taken for a length of zero to be reached cannot be directly read off the table (although we can see that it is 16 and a half hours). The way in which this table is set up is problematic as it gives the impression that only integral values occur (and thus that we are operating on the domain of integers, instead of positive reals – negative reals cannot be included as negative length or time does not make sense in this problem). If the formula is used, we get an answer of 16,5 hours.

Thus the question requires learners to:

- Recognise the pattern shown by the table i.e. that the length of the candle decreases by 2cm each hour.
- Extend the pattern.
- Find a general formula to represent the pattern by identifying the pattern as a linear one.
- Recognising that a length of zero cm is the point at which the candle has burnt out completely.
- Find the time at which this occurs.

## Secondary data production P6 Lesson 1 EE3

### 1) Realisation of content

In this event learners work on a number patterns question taken from a previous grade 10 examination. The intended topic of the question is discussed in detail above, but to summarise, in this question learners need to extend the sequence, find the general formula for the sequence and solve for one of the dependent variable given the independent one. The teacher’s intention was for the learners to finish the question in ten minutes and then to discuss it with them in the remaining five minutes of the lesson, but the learners worked for the full 22 minutes which are left and do not get beyond question 3.2.

It is difficult to say yet whether the intended topic is realised as it is only in the next lesson that the teacher works through this question with the class. But during the time the learners work on the question in this event there are two occasions on which the teacher discusses issues related to question 3.2, which the learners seem to be struggling with. The teacher repeats the procedure for finding the general term of a linear sequence twice. This procedure is outlined in my primary data production. It seems that in this procedure the signifier ‘linear’ is used as a marker or trigger for the procedure, and not as descriptive of the type of relationship represented, and thus the mathematical treatment of such a relationship. The propositional ground of a linear function with its properties is implicit. This will be discussed further when analysing the

next lesson, which is where the teacher works through this question with the class. But at this point it seems that although the intended topic is not realised, the realised topic still corresponds with the mathematics encyclopaedia – the teacher’s procedure for finding the general term, although not violating any mathematical principles, forecloses the propositional ground underlying a linear sequence and consists of algebraic manipulations and arithmetic.

## 2) Regulation of the learner

Interactions in this event:

At one point the teacher says: “if you can do this question then it means you can do number patterns in grade ten”.

He tells them they can work in pairs, saying he expects them to be done in ten minutes. He glances at a learners’ work and tells the whole class that when they do corrections on the board they must mark themselves. He tells them that he will come around to check and sign (referring to the previous exercise).

As the learners work, the teacher explains question 3.2 twice and outlines the procedure for finding the general formula. It seems that the signifier ‘linear’ regulates the procedure without recruiting the notion of a linear function with its properties. It seems that he is regulating the learners to reach the correct answer for 3.2 through repetition of the procedure, which does not explicitly draw on the properties of a linear function but instead on algebraic manipulations. There is only one appeal made in this event to a procedural feature. But based on analysis of the previous and subsequent events, it seems that necessity is situated external to mathematics in the teacher’s constitution of number patterns, and thus that regulation of the learner is under the aspect of the Imaginary.

### Summary

This event falls into quadrant II. The regulation of the learner depends primarily on the Imaginary.

## Primary data production P6 Lesson 2 EE1

### 1) Generating evaluative events

**Table A4.7** Evaluative events P6 Lesson 2

Time	Evaluative event/sub-event	Activity	Type
00:00 – 09:50	E <sub>1</sub>	Discussion of Q3.1 – finding term eight of the pattern	Expository
09:50 – 26:15	E <sub>2</sub>	Discussion of Q3.2 – finding the general term and testing the answer (calculates T <sub>6</sub> )	Expository
11:58 – 13:45	E <sub>2.1</sub>	Discussion about variables	Expository
19:40 – 21:01	E <sub>2.2</sub>	Discussion about independent and dependent variables	Expository
26:15 – 39:00	E <sub>3</sub>	Discussion of Q3.3 – solving for the number of hours	Expository
34:50 – 39:00	E <sub>3.1</sub>	Explanation of like terms	Expository

## 2) Describing operational activity

The teacher starts this event by referring to the homework (question three) and calling learners up to do the questions on the board. The homework question is shown in Figure A4.7

The learner who writes up 3.1 gives the correct answer without any explanation. The learner who writes up 3.2 attempts to calculate a rate, which he gives as 14,4cm/hour. The learner who does 3.3 merely writes 17 hours.

The teacher starts the discussion with 3.1 and says “we have discovered a pattern, we have seen that for every hour that goes on, then ... the candle burns by 2cm”. He uses a table to show how the answer of 17cm is reached:

5	6	7	8
23	21	19	17

Figure A4.11 Table used to answer Question 3.1

Teacher: So wena (you) have to continue the table and then what will happen after six hours, what will happen after seven hours and what will happen after eight hours, so five hours what will be the length, which is twenty-three at five, so what is the length at six?

Learners: Twenty-one

Teacher: Twenty-one. At seven?

Learners: Nineteen

Teacher: At eight?

Learners: Seventeen

Transcript School P6 Lesson 2

They are counting down in two's based on their observation about the lengths in the table they are given (the length decrease by two each time).

## 3) Activating the mathematics encyclopaedia

See P6 Lesson 1 EE2 where each part of this question is discussion in detail.

### Secondary data production, P6 Lesson 2 EE1

#### 1) Realisation of content

The intended topic of this question is discussed in detail in my primary data production for this event when the teacher gives the question to the learners.

Learners can calculate 3.1 without having any notion of the pattern representing a linear function – they can merely extend the table or count mentally to reach the length after 8 hours and the time for the candle to burn

out completely. The teacher uses the table to explain the answer, and prompts the learners to complete it. They do so easily by counting down in two's, as seen in the extract from the transcript above.

But despite the way in which the learners are able to add successive terms to this sequence, it does not necessarily mean that they grasp the notion of “number pattern”. They see a list of numbers and recognize how to add additional numbers to it, but this does not necessarily mean that the notion of “number pattern” is a regulative resource. The fact that the question is framed in such a way as to give them a “pattern” without any description of the type of pattern requires the use of inductive reasoning to answer the question. “Number patterns” is the announced topic of this question, with the pattern dealt with being linear. The topic is thus related to linear functions. But the nature of the question and the way it is dealt with focuses on arithmetic of whole numbers. Are the “mathematical thinking skills such as generalising, explaining, describing, observing, inferring, specialising, creating, justifying, representing, refuting and predicting” (DoE, 2002: 9) listed in the curriculum really being taught and learnt?

Despite the learners’ quick and correct response to this question, it seems that the domain they are operating on is not the field of reals (which is the domain of linear functions), but the natural numbers. Thus the intended topic is not realised in this event.

The realised topic corresponds with the mathematics encyclopaedia as there are no violations of mathematical propositions, processes, definitions or rules.

## 2) Regulation of the learner

The idea of exploration and discovery appears again in this event when the teacher says things like “we have discovered a pattern”. He regulates the learners through appealing to the empirical once again (see discussion for L1 EE1), and the only appeal made in this event to an authorizing ground is one made to empirical testing. Thus the Imaginary is recruited in the regulation of the learners.

### Summary

This event falls into quadrant II. The regulation of the learner depends primarily on the Imaginary.

## Primary data production P6 Lesson 2 EE2

### 1) Generating evaluative events

Table A4.8 School P6 Lesson 2 EE2

Time	Evaluative event/sub-event	Activity	Type
09:50 – 26:15	E <sub>2</sub>	Discussion of Q3.2 – finding the general term and testing the answer (calculates T <sub>6</sub> )	Expository
11:58 – 13:45	E <sub>2,1</sub>	Discussion about variables	Expository
19:40 – 21:01	E <sub>2,2</sub>	Discussion about independent and dependent variables	Expository

### 2) Describing operational activity

The learner who wrote up 3.2 on the board attempted to calculate a rate, which he gives as 14,4cm/hour. The teacher reads through the learner’s answer and then says “let’s go back to the question”. He never actually addresses this learner’s attempt at finding a rate or gradient (which is a valid method of finding the formula,

although the learner makes an error in his calculation of the gradient). Once they have re-read the question (“use a variable to write an algebraic statement to generalize the relationship between the time in hours and the length of the candle”), the teacher asks the class “what is a variable?” He uses the example of  $5a$ , and explains that  $a$  is the variable because “ $a$  can represent anything”. He uses two examples showing different values of  $a$  and then says “a variable is something that changes all the time, it doesn’t have a specific value”.

He asks the class “is there a relationship between the time and the length?”, while pointing to the table he drew in 3.1. He answers his own question - “there is a relationship because we see that every hour that goes by what happens to the candle? ... the candle decreases by 2cm. So every hour the candle decreases by 2cm, ok?” He asks them “is this a linear sort of sequence or is it another type of thing?” He prompts them to check the difference, and asks them what they see (pointing back to the table) ... “by two, by two, all the time. Obviously that means we’ve got a linear sequence, which means we can write it in the form  $T_n = a + (n - 1)d$ .” Here the basis of his identification of the sequence as linear is that it decreases by two each time (i.e. it has a common difference between consecutive terms).

He refers back to the question for 3.2, asking them “why do we want to generalize?” He explains that they can use the generalization to find other terms. He referred to the previous day’s example when it was “easy to find  $T_{50}$ ” once they had the general term (appeal to ease). In order to complete the question, he says “**what will give you that relationship is the common difference as well as the first term (pointing to the formula)**”. He replaces  $a$  in his formula with 33 and writes  $(n - 1)$ , explaining that “ $n$  minus one and  $T_n$  should always be there, they are the variables”. When he asks the learners for the common difference they reply that it is two. He stops and says “there is a difference between two and minus two”. He writes in minus two (without brackets which could be confusing as it looks like subtraction instead of multiplication) as follows:  $T_n = 33 + (n - 1) - 2$ .

The formula he obtains is thus  $T_n = 35 - 2n$ , although he does not simplify it.

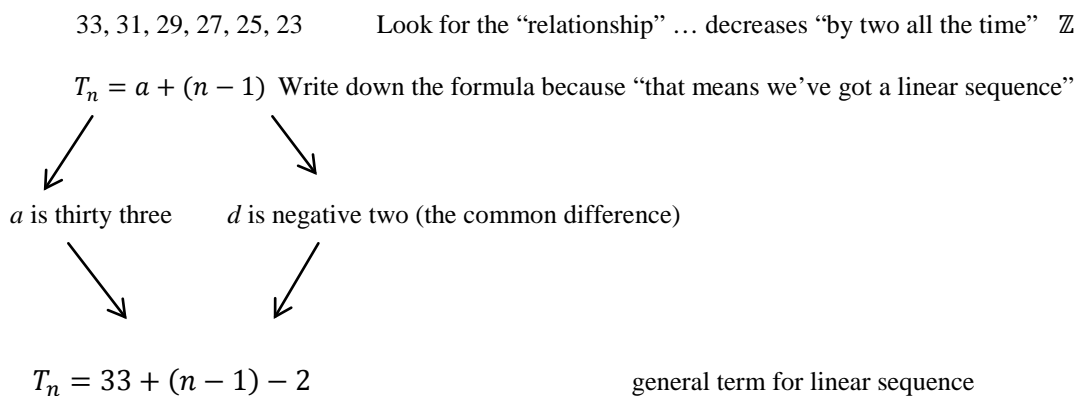
Let’s write out his procedure for obtaining the general term:

- 7) “Check the differences” in order to decide if the sequence is linear – if there is a “common difference” between consecutive terms, the sequence is linear, if not it is “another type of thing”.
- 8) Because the sequence is linear, write down the formula  $T_n = a + (n - 1)d$ .
- 9) Substitute in the first term ( $a$ ) and the common difference ( $d$ ).
- 10) Simplify (although the teacher does not simplify in this case, he does so in another example in the next lesson).

In general terms, given a sequence:  $T_1; T_2; T_3; T_4; T_5$

- 4) If  $T_2 - T_1 = T_3 - T_2$  then the sequence is linear.
- 5) We can thus use the formula  $T_n = a + (n - 1)d$ .
- 6) Substitute  $T_1$  in the place of  $a$  and the difference  $T_2 - T_1$  in the place of  $d$  –

$$T_n = T_1 + (n - 1)(T_2 - T_1).$$



**Figure A4.12** Diagrammatic representation of the procedure for finding the general term in question 3.2

**Table A4.9** Operational activity entailed in the procedure for finding the general term in question 3.2

T	Input	Domain	Operation	Output	Codomain
1	33, 31, 29, 27, 25, 23	$\mathbb{Z}$	Look for the relationship	Length decreases by two	$\mathbb{Z}$
2	Length decreases by two	$\mathbb{Z}$	Decide what pattern it is	Linear pattern	sequence type
3	Linear pattern	sequence type	Use the formula	$T_n = a + (n - 1)d$	general term for linear sequence
4	$T_n = a + (n - 1)d$	general term for linear sequence	Substitute $a = 33$ and $d = -2$	$T_n = 33 + (n - 1) - 2$	general term for linear sequence

He then says “but we before we can be sure that this general term or this general formula is correct, we need to do what? We need to test it. We need to test it with something that we know.” In order to do this, he goes back to the table and redraws it to include all the terms from time zero to 8 hours. He explains that the first term is when the time is zero, not one hour. He calls the time  $n$  (when in fact  $n$  represents the term number according to the formula, and not the number of hours) and length ( $T_n$ ), explaining that “there are two types of variables ...the independent variable and the dependent variable”. He asks the class whether the time depends on the candle or whether the candle depends on the time. He explains why time is independent and length dependent. In order to test he explains that after five hours the term is the sixth term, and substitutes six into his formula. As he substitutes into the formula he says “times negative two” but the way he writes it looks like he is subtracting two (confusing), seen in Figure A4.14.

When adding 33 to negative 10 he uses the “plus and a minus is a minus” story again, doing a few examples and using his diary as an illustration (one diary take away two – which doesn’t make sense in the context of his example).

When they get an answer of 23 for the sixth term (ie. after five hours), he concludes that his generalization is correct.

His procedure for testing the general term:

- 1) Substitute six in the place of  $n$  in the formula obtained.

- 2) Subtract one from six.
- 3) Multiply five and negative two.
- 4) Add thirty three to negative ten.

### 3) Activating the mathematics encyclopaedia

See P6 Lesson 1 EE3

As I mentioned when discussing question 3.1, learners can calculate 3.1. without having any notion of the pattern representing a linear function – they can merely extend the table or count mentally to reach the length after 8 hours and the time for the candle to burn out completely. But in order to do 3.2 learners would need to recognize the pattern as linear, and either draw on the properties of a line to find the equation, or to use the formula for finding  $T_n$  of a linear pattern. The latter method does not require learners to draw on the properties of a line or even to recognize the ‘pattern’ as representing a linear function – it merely requires learners to identify the pattern as linear and to apply the formula, as we will see in the analysis below. So this question, although grounded on the notion of a linear function, with its properties, can be correctly completed without any understanding of this.

## Secondary data production P6 Lesson 2 EE2

### 1) Realisation of content

The intended topic of this event is to find the general formula of a linear pattern. But the nature of the examples and questions, and the way they are dealt with, focuses on arithmetic and algebra ... substituting, simplifying, solving equations, variables, ‘like terms’.

Let’s have a closer look at the procedure for finding the general term of the pattern. Although the method used is valid from the point of view of the encyclopaedia, the notion of a linear function, with its properties, is implicit in the ground of this procedure. The ground seems to be procedural in nature. The announced topic of ‘linear number patterns’ is being taught in a procedural way without any reference to the nature of a linear pattern and the properties of a line. In this procedure the signifier ‘linear’ is used as a marker or trigger for the procedure, and not as descriptive of the type of relationship represented, and thus the mathematical treatment of such a relationship.

The outcome of the procedure is the formula  $T_n = 33 + (n - 1)2$ , which is in fact not correct due to the way in which the negative two is written as if it is being subtracted from the  $(n - 1)$  instead of multiplied. But when the teacher tests the formula, he correctly states that the negative two is being multiplied by the  $(n - 1)$ , and correctly carries out this multiplication. Nevertheless, the way in which the teacher writes the negative two is confusing and could lead to learner misunderstanding and error, although it is probably just a slip and not a conceptual error.

Generally the teacher’s treatment of this question suggests that the intended topic is not realised due to the way in which the notion of a linear function is implicit, but the realised topic still corresponds with the mathematics encyclopaedia.

### 2) Regulation of the learner

It seems that the notion of a linear function with its properties is not regulating this procedure. What is regulating the procedure? The signifier ‘linear’ prompts learners to carry out the procedure without recruiting the notion of a linear function with its properties. The teacher regulates the learners to carry out the procedure and reach the correct answer without relying on any knowledge of linear functions – he thus forecloses the propositional ground underlying the topic and teaches it in a procedural manner. The key to his procedure succeeding is learners recognising that the pattern is linear and responding by using the

formula – from this point, the procedure is like an automaton which leads the learners to the correct answer without any need for the notion of a linear function to be recruited (assuming their algebra is sound, which is not necessarily the case).

Generally the dominant regulative resource appealed to in this lesson is the signifier ‘linear’, which is associated with the formula  $T_n = a + (n - 1)d$ ., instead of with a line and its properties. The procedures which follow consist of arithmetical and algebraic manipulations.

Teacher: Right! We know that. What do you see apha? (Here?). Is this a linear or a sequence okanye (or) is it another type of thing? When you check the difference, right? At the bottom numbers. What happens?

Learners: {Mumbling in the class.....}

Teacher: It decreases by what?

Learners: By two

Teacher: By two, by two, by two, by two all the time, obviously that means you’ve got a linear sequence which means you can write it in the form of  $T_n = a + (n - 1)d$  ...

### Transcript School P6 Lesson 2

Even though the teacher considers it obvious that the sequence is linear, the learners do not necessarily grasp this, or if they do, they do not seem to know why it is linear or what that means. In the previous lesson he explained this procedure twice, and it seemed that the learners were still not happy with it in this lesson. The problem is that the teacher’s procedure, and the learners’ chance of getting the correct general term for this question, depends on them being able to identify the sequence as linear.

In this event, one appeal is made to ease/efficiency as authorizing ground, three to empirical testing, two to procedural features of the solution, and two to mathematical propositions or definitions. Thus the regulation of the learners in this event places necessity as external to mathematics – necessity is situated in the empirical process of identifying a sequence as linear, and in the signifier linear and the formula it is associated with, rather than in the properties of a linear function. The mathematical content (the notion of a linear pattern as a function) is transformed into algebraic and arithmetic manipulations, possibly because the teacher expects the learners to be able to engage with such manipulations. The regulation is thus under the aspect of the Imaginary.

### Summary

This event falls into quadrant II. The regulation of the learner depends primarily on the Imaginary.

### Primary data production P6 Lesson 2 EE3

#### 1) Generating evaluative events

Table A4.10 School P6 Lesson 2 EE3

Time	Evaluative event/sub-event	Activity	Type
26:15 – 39:00	E <sub>3</sub>	Discussion of Q3.3 – solving for the number of hours	Expository
34:50 – 39:00	E <sub>3,1</sub>	Explanation of like terms	Expository

## 2) Describing operational activity

The learner who did 3.3 on the board at the beginning of the lesson just wrote an answer of 17 hours. When discussing 3.3, the teacher refers again to the general term, once again correctly reading the last part as “times negative two” despite the confusing way he has written it (see discussion in Lesson 2 EE2, previous event).

He describes the usefulness of the general term - “any term you want to find you will find using that general term”. He explains that the length will be zero cm when the candle has melted completely. He does not explain that the sequence does not make sense for negative values of length, but can include non-integral positive values. There is no mention that what they are actually doing here is solving a linear equation.

He multiplies out by “removing” the brackets, then refers to getting like terms on one side. He says “like terms are obviously 33 and plus 2”. He asks them what they should do with the minus two  $n$ , explaining that it’s negative so **when he takes it across the equal sign it will be positive**. (‘change sides, change signs’ rule).

3.3.  $T_n = 33 + (n-1) \cdot -2$   
 $0 = 33 + (n-1) \cdot -2$   
 $0 = 33 - 2n + 2$   
 $2n = 33 + 2$   
 $2n = \frac{35}{2}$   
 $n = 17\frac{1}{2}$

Figure A4.13 Solution for question 3.3

This is in fact not the correct answer for the number of hours, as although the equation has been solved correctly, the answer produced is the term number, not the number of hours due to the formula used (as mentioned in my discussion of the question in P6 L1 EE3). So term number seventeen and a half (which in the context of the topic number patterns does not really exist, term numbers are generally whole positive numbers) corresponds with 16 and a half hours, which is the time taken for the candle to reach 0cm in length. This is not noticed by the teacher or learners, who accept 17 and a half as the final answer. The teacher says: “so the candle will not melt after sixteen hours, after how many hours? Seventeen and a half. There is a difference between seventeen and seventeen and a half and when we round it off we can’t round it off seventeen and a half to seventeen we rather round it off to eighteen, so we going to write it as seventeen comma five and the next level is eighteen”.

Let’s work through the teacher’s procedure in this question:

- 1) Replace  $T_n$  with zero in the formula obtained in 3.2.
- 2) Multiply out the brackets.
- 3) Get like terms on one side using the ‘change sides, change signs’ rule.
- 4) Adding the like terms on the right hand side.

- 5) Divide by 2 on both sides.
- 6) Cancel the 2's.

**Table A4.11** Operational activity entailed in solving for the number of hours for the candle to burn out completely

T	Input	Domain	Operation	Output	Codomain
1	$T_n = 33 + (n - 1) - 2$	$\mathbb{Z}$	Substitute	$0 = 33 + (n - 1) - 2$	$\mathbb{Z}$
2	$0 = 33 + (n - 1) - 2$	$\mathbb{Z}$	Multiply	$0 = 33 - 2n + 2$	$\mathbb{Z}$
3	$0 = 33 - 2n + 2$	$\mathbb{Z}$	Existential shift	$/0 = 33 - 2n + 2/$	$\mathbb{X}$
3	$/0 = 33 - 2n + 2/$	$\mathbb{X}$	Spatial Shift	$/2n = 33 + 2/$	$\mathbb{X}$
4	$/2n = 33 + 2/$	$\mathbb{X}$	Existential shift	$2n = 33 + 2$	$\mathbb{Z}$
5	$2n = 33 + 2$	$\mathbb{Z}$	Add	$2n = 35$	$\mathbb{Z}$
6	$2n = 35$	$\mathbb{Z}$	Divide	$\frac{2n}{2} = \frac{35}{2}$	$\mathbb{Z}$
7	$\frac{2n}{2} = \frac{35}{2}$	$\mathbb{Z}$	Simplification	$n = 17,5$	$\mathbb{Z}$

He asks the class why 33 and 2 are like terms – a learner explains: “thirty-three doesn’t have 2n, thirty-three is standing on its own and then two is also standing on its own, it doesn’t have a variable, so thirty-three and two are like terms and they have to be put on one side and the variables must be put on the other side”.

The teacher gives them another example:  $n^2 + 2n - 1 + 3 = 0$  and asks for the like terms. The learners tell him that “it’s minus one and plus three”. The teacher agrees “because they are constants, ok?” His explanation of like terms is that “the variable is the same and the exponent is the same”; “We talk about constants and variables, like constants it’s just a number, it contains no variables, then even though the variable is the same but this is n, but this side is what? It’s n to the power two, when you look at the like terms you need to look at the variable itself as well as the power, do you understand? So n to the power three is not a like term to n to the power two ... but if two n to the power two minus n<sup>2</sup> then your answer will be what? Will be : n<sup>2</sup> because they are like terms because the variable is the same and the exponent is the same. Right!”

### 3) Activating the mathematics encyclopaedia

See L1 EE3

Learners can calculate 3.3 without having any notion of the pattern representing a linear function – they can merely extend the table or count mentally to the time for the candle to burn out completely. So this question, although grounded on the notion of a linear function, with its properties, can be correctly completed without any understanding of this.

## Secondary data production P6 Lesson 2 EE3

### 1) Realisation of content

The intended topic of this event is to find the time taken for the candle to burn out completely, which involves recognising that the length is zero when the candle has burnt out, substituting zero into the linear equation and solving the equation.

The procedure for finding the time taken for the candle to burn out completely – requires further analysis. The procedure used contains the ‘operations’ of SHIFT and CANCEL. From the point of view of the mathematics encyclopaedia there are no operations directly corresponding to ‘changing sides’, ‘changing signs’, ‘shifting over the other side’ or ‘cancelling’. The field of real numbers is not described by any such operations even if the use of these ‘operations’ enables learners to produce statements recognised as solutions to linear equations. A number cannot be ‘shifted to the other side’ - we can only perform such a pseudo-operation on objects amenable to spatial displacement, i.e. character strings. Thus there is a shift in domain in carrying out this procedure from reals to character strings.

In addition to this, the method used does not recruit the notion of a linear function with its properties, and the fact that the point required is the  $x$ -intercept of the function – the point at which the length is zero. As our initial analysis revealed, the incorrect answer is accepted by the teacher and learners, thus the intended topic is not realised, and nor is the realised topic aligned with the encyclopaedia. The announced goal of the question – to find the time taken for the candle to burn out completely – is not achieved. The teacher’s dependence on the linear pattern formula ( $T_n = a + (n - 1)d$ ) results in confusion relating to the number of hours/number of terms – if he had worked through the problem logically and drawn on the notion of a linear function as the ground of the ‘pattern’, this confusion would have been avoided.

Generally, in the teacher’s discussion and mathematical activity, it seems that the object of attention of the topic of “patterns” is simply algebraic manipulation and arithmetic, and that the domain being operated over is the integers (and sometimes the natural numbers as seen in the previous lesson). This suggests that the topic of “patterns” is in fact another way of teaching algebraic and arithmetic skills, and that once again the intended topic is not realised in this event. In addition to this, although the equation has been solved correctly, the answer produced is the term number, not the number of hours due to the formula used (as mentioned previously). So term number seventeen and a half (which in the context of the topic number patterns does not really exist, term numbers are generally whole positive numbers) corresponds with 16 and a half hours, which is the time taken for the candle to reach 0cm in length. This is not noticed by the teacher or learners, who accept 17 and a half as the final answer. Thus the realised topic does not correspond with the encyclopaedia because of the teacher and learners’ acceptance of the answer for the number of terms instead of calculating the number of hours.

## 2) Regulation of the learner

It seems that the notion of a linear function with its properties is not regulating this procedure, as seen in the previous event. Both the teacher and learners incorrectly accept the answer of the number of terms for the number of hours, suggesting that the strongest regulative resource underlying this question is the formula – the answer produced by the formula is not reflected on or questioned in the context of the question. Once again the topic of linear patterns has become about algebraic manipulation, and this has consequences for the mathematics produced in this event – the answer which is accepted is incorrect because of the reliance on the formula to produce the answer, rather than on the notion of a linear function with its properties. In this event three appeals are made to iconic or spatial features of solutions, and three to mathematical features of the solution. But based on the analysis above and on the analysis of the previous events, the regulation in this event is once again under the aspect of the Imaginary.

## Summary

This event falls into quadrant IV. The regulation of the learner depends primarily on the Imaginary.

## Primary data production P6 Lesson 3 EE1

### 1) Generating evaluative events

Table A4.12 Evaluative events School P6 Lesson 3

Time	Evaluative event/sub-event	Activity	Type
00:00 – 3:38	E <sub>1</sub>	Marking homework - finding the general term of the sequence -4; 1; 6; 11; 16; 21	Expository
3:38 – 6:25	E <sub>1.1</sub>	Calculating 200 <sup>th</sup> term of the sequence above	Expository
6:25 – 40:16	E <sub>2</sub>	Activity 1.13 No 4 & 5, 1.14 No 1, 2, 3 from textbook	Exercise
40:16 – 44:15	E <sub>3.1</sub>	Marking Q5: finding general term of the sequence 1, 3, 5, 7	Expository
44:15 - 47:47	E <sub>3.2</sub>	Finding the general term of the sequence -4; 0; 4; 8	Expository

### 2) Describing operational activity

In this evaluative event the procedure from the previous lesson is used. They are working through a homework exercise. As a reminder of the previous lesson, the teacher says “what have we discovered? We have discovered that common difference is what? Five. Five. Five (as he draws lines between the terms)”. The teacher finds the common difference by subtracting first term from the second and the second from the third, but he does not state this explicitly in this event.

He interchanges the use of “arithmetic” and “linear” to describe the sequence. He then substitutes the first term and common difference into  $T_n$  formula  $T_n = a + (n - 1)d$ . By this stage learners are able to recite the formula. He does not simplify once he has substituted in this example. He goes on to check if the general term is correct by substituting in a term number from the sequence.

Figure A4.14 Finding the general term of the sequence - 4; 1; 6; 11

When he has tested, he concludes that “this general term will work no matter ...”

In order to calculate term number 200, he says “the procedure for calculating term number two hundred should be the same as the procedure that we applied for the checking”.

Once has completed this question, he tells the learners that “**those marks that you get for number patterns it means they are already in the bag**”.

### 3) Activating the mathematics encyclopaedia

See P6 Lesson 1 EE1, EE2 and EE3

## Secondary data production P6 Lesson 3 EE1

### 1) Realisation of content

The topic of this event is finding the general term of a given sequence, which is linear. The same procedure from the previous lesson is used. In this event he also checks that the general term is correcting by substituting in a number from the sequence. Same analysis and conclusion as L2 EE2 – the way in which the content is realised does not correspond with the intended topic but does correspond with the encyclopaedia.

### 2) Regulation of the learner

This event is short and consists of the teacher carrying out a procedure which he has worked through a number of times in previous lessons. Once has completed this question, he tells the learners that “those marks that you get for number patterns it means they are already in the bag”- by now the procedure has been practised a number of times and the teacher expects the learners to be very familiar with it and able to get the correct answers in any examination. In this lesson the learners are much more vocal than previous lessons – they chant the formula, next steps and answers, and seem much more confident.

In this event there is one appeal to empirical testing and one to mathematical proposition, process or definition (appealing implicitly to the associativity of addition over the integers to add two integers as part of his simplification). Because of the way in which the procedure has been constituted as algebraic and arithmetic manipulation, as discussed in the previous lesson’s analysis, without drawing on the propositional ground of linear functions, as well as the repetition of the idea of ‘discovering’ the pattern (as previously discussed) and thus an appeal to the empirical, the regulation of the learner is under the aspect of the Imaginary in this event.

### Summary

This event falls into quadrant II. The regulation of the learner depends primarily on the Imaginary.

## Primary data production P6 Lesson 3 EE2

### 1) Generating evaluative events

Table A4.13 School P6 Lesson 3 EE2

Time	Evaluative event/sub-event	Activity	Type
6:25 – 40:16	E <sub>2</sub>	Activity 1.13 No 4 & 5, 1.14 No 1, 2, 3 from textbook	Exercise

### 2) Describing operational activity

He sets the learners a textbook exercise – (Activity 1.13 No 4 & 5, 1.14 No 1, 2, 3) which they work on mostly without interaction with the teacher (a few learners ask him questions which we cannot hear. He also marks a few books) until he notices a common error -  $100 \times 3 + 1 = 103$ . He asks the class to explain “how

does this happen?”, “how do you get 103 from this?”, “initially I thought I was hallucinating or something. But I’ve seen it more than ten times”.

No one volunteers to explain for a while, eventually a boy tries but his answer is not clear. The teacher does not finish the discussion, just says that it “does not exist” without explaining why. He moves on to discuss the next question.

### 3) Activation of the mathematics encyclopaedia

See P6 L1 EE1, 2 and 3

## Secondary data production P6 Lesson 3 EE2

### 1) Realisation of content

In this event the learners work on an exercise without much interaction with the teacher. We do not see the learners’ work, and although the teacher moves around the class marking their books and occasionally answering questions, we do not hear any of these exchanges on the video. So there is insufficient data to discuss the realisation of the topic in this event.

### 2) Regulation of the learner

At one point during this event the teacher stops the class and asks them to explain a common error which he says he has “seen more than ten times”. No one volunteers for a while, so he presses them to explain how this could be true. He offers a voucher to Robben Island for whoever can explain it! A student tries but his reply is not audible and the teacher laughs and dismisses it. He tells the class he should not be seeing this in grade ten and erase what he’s written before moving on – he does not discuss it further.

This short exchange is all we have to work with in this event in terms of the regulation of the learner, as there are no appeals made to any authorizing ground in this event. But although there is not much data with which to make any conclusions about where necessity is situated and thus whether regulation is under the aspect of the Imaginary or the Symbolic, we assume that the learners are regulated by the teacher’s exposition of the procedure being carried out in previous events. The regulation is thus assumed to be under the aspect of the Imaginary.

## Summary

Insufficient data to decide in which quadrant this event lies. The regulation of the learner is under the aspect of the Imaginary.

## Primary data production P6 Lesson 3 EE3

### 1) Generating evaluative events

Table A4.14 School P6 Lesson 3 EE3

Time	Evaluative event/sub-event	Activity	Type
40:16 – 44:15	E <sub>3.1</sub>	Marking Q5: finding general term of the sequence 1, 3, 5, 7	Expository
44:15 - 47:47	E <sub>3.2</sub>	Finding the general term of the sequence -4; 0; 4; 8	Expository

### 2) Describing operational activity

Teacher: Let’s look at number five, will do two of them and then will mark the rest.

Number 5a: 1,3,5,7,9 that is very commonly known nhe? ( right?)

What do we call the set, it is a set of what?

Learner: Number patterns

Teacher: Yes but set of what numbers?

Learners: Even numbers

Teacher: Is it right?!

Learners: No

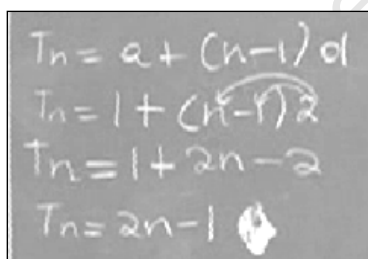
Teacher: Right! What is this set called?

Learners: Odd numbers

Teacher: Now let's look at the pattern behind the set of odd numbers

### Transcript School P6 Lesson 3

In this question the teacher uses the linear formula without discussing that the pattern is arithmetic or linear – he just goes straight to the formula. Here he refers to the common difference as “the number that we add all the time here?”



The image shows a chalkboard with four lines of handwritten algebraic work. The first line is the general formula for an arithmetic sequence:  $T_n = a + (n-1)d$ . The second line shows the first term  $a=1$  and common difference  $d=2$  substituted:  $T_n = 1 + (n-1)2$ . The third line shows the expansion of the brackets:  $T_n = 1 + 2n - 2$ . The fourth line shows the final simplified formula:  $T_n = 2n - 1$ . There is a small drawing of a hand pointing to the final result.

**Figure A4.15** Finding the general term of the set of odd numbers

In this example he multiplies out the brackets and asks learners to simplify – “but now let's write it in simple terms”. He asks them what one minus two is, when they answer minus one he says “WHAT?” He seems to think the answer is one, they argue, and he agrees. “Is there a difference between minus one and negative one? ... You must be confident. When you know something you must be confident about it. If it's right then it's right no matter who says what”.

When finding the common difference of the next question, he asks them “from zero to four what do you get?” “four” “from four to eight” “four” “from negative four to zero” “four”

In this question the class argues over what minus four minus four is, some say negative eight, others say positive eight. Teacher writes negative eight without explaining why. In summary, he says “if it's a linear pattern the most important things is the, is the what? The first term as well as the common difference”. He says they always use the first term and the common difference, but before they know if it's a linear pattern they must first “investigate the difference. If the difference happens to be common then what do you know? That pattern is linear.” Although in the two examples in this event he does not investigate the pattern first but goes straight to the linear formula.

### 3) Activating the mathematics encyclopaedia

See P6 Lesson 1 EE1, 2, 3

## Secondary data production P6 Lesson 3 EE3

### 1) Realisation of content

In this event they mark the exercise on finding the general terms of two sequences, both of which are linear. By now the procedure is very familiar to learners who chant out the steps and the answers confidently. The teacher does not do the first step of his procedure (investigating the sequence to see if it's linear) but goes straight to the formula each time.

He defines the common difference as “the number we add all the time here”, assuming that there is a common difference in both (which there is) – again the common difference is constituted from the point of view of the subject doing the adding, not in terms of the sequence itself. When finding the common difference of the second question, he asks them “from zero to four what do you get?” “four” “from four to eight” “four” “from negative four to zero” “four”, once again making it seem as if learners are starting at one number and moving to another – the idea of movement between terms, as well as the learner being central to the procedure, emerges again here.

Generally, the notion of a linear function is implicit in this event, so the intended topic is not realised but the realised topic does correspond with the encyclopaedia – see previous discussions on this topic.

### 2) Regulation of the learner

He regulates the learners with the formula again and by this point the learners carry out the procedure smoothly. Twice in this event when they are adding integers while simplifying their answers, he argues with them over the correct answer – seemingly to test them, telling them that “you must be confident. When you know something you must be confident about it. If it's right then it's right no matter who says what”. But what is clear is that some of the learners are still not confident with integer addition (they argue over the signs of the answers – for example, some learners said that  $-4 - 4$  is eight, others negative eight. The same happens with  $1 - 2$ , some said one, others negative one.

There are only two appeals to an authorizing ground made in this event – one to empirical testing and one to a procedural feature of a solution, suggesting that necessity is located external to mathematic. As this event is a continuation of the previous event's exercise, and as the learners seem to be regulated by the procedure (which they confidently chant out as the teacher works through these examples), once again the regulation is under the aspect of the Imaginary.

### Summary

This event falls into quadrant II. The regulation of the learner depends primarily on the Imaginary.

## Appendix 5: Analysis for School P7

### Primary data production P7 Lesson 1 EE1

#### 1) Generating evaluative events

Table A5.1 Evaluative events School P7 Lesson 1

Time	Evaluative event/sub-event	Activity	Type
00:00 – 07:00	E <sub>1</sub>	Worked example - simplifying expression $\frac{3^{n-1} \cdot 9^{n+1}}{9^{2n+1}}$	Expository
07:00 – 23:00	E <sub>2</sub>	Learners working on exercise, teacher answering questions and working through answers on board.	Exercise/expository
23:00 – 33:35	E <sub>3</sub>	Exponential equations worked examples	Expository

#### 2) Describing operational activity

The teacher announces the topic they are working on - “we are busy with indices, and we’re busy with the exam-related question”.

He starts going through this question as a worked example:  $\frac{3^{n-1} \cdot 9^{n+1}}{9^{2n+1}}$  which he refers to as “a typical question which you can expect in the exams”, a “typically exam related question”, saying that “if you can master a question like this you’ll be prepared for a question like this at the end of the year” “so I will definitely put a question like this in the exams”. The teacher appeals to the examinations often during this lesson, and to what kinds of questions they can expect in the examinations in grade 10, 11 and 12. He also describes the question as - “This is a definite type of question that you’re supposed to know ... for the final examination, even now for March”.

He starts the worked example by saying “if there’s no prime base you need to convert it to a prime base form” – his first ‘step’ is to convert the bases which are not prime to what he calls “prime base form”. Despite his reference to primes (unlike in a lesson on the same topic in school P3 where primes are not referred to explicitly), his procedure is based on encouraging learners to memorise their prime table, rather than on an understanding of primes. “Now this is a very very important thing – three to the power of two, three to the power of one etcetera you get it from your prime table – it’s in your books, right? You need to memorise the prime table”. The learners seem to either know how to rewrite bases in “prime form”, or could be consulting their books. Once he has completed this ‘step’, he instructs the students to apply the “three index laws” - “as soon as you get to the point where all the bases are written in prime form, in prime base form, you apply the three index laws”.

#### What does the question involve?

- Prime factorization
- Application of exponential identities (‘laws’)
- Simplification to a single power

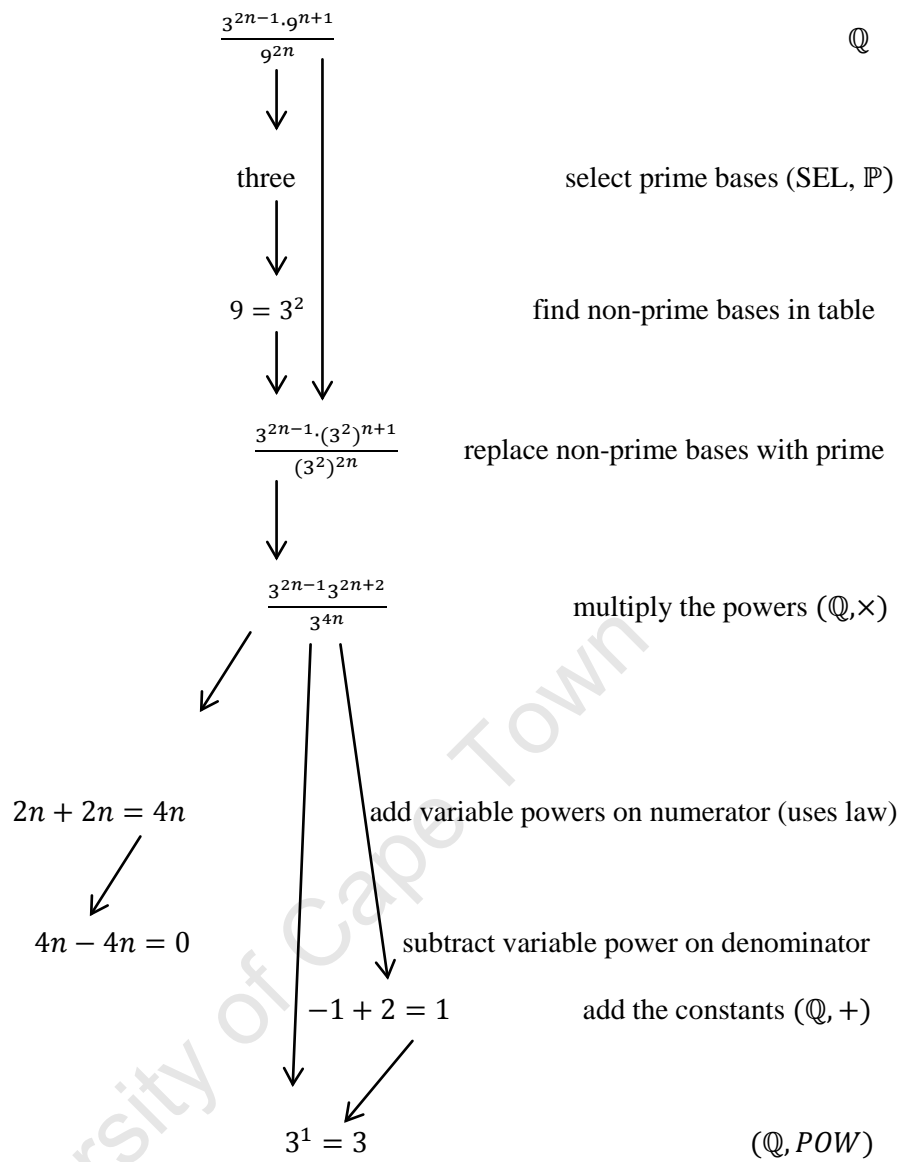
**Teacher's treatment of it – he describes the problem as requiring two steps or processes:**

- 1) Convert bases which are not prime to “prime base form” by using the “prime table” from the learners’ books ... this involves looking at the bases, picking out the composite (non-prime) bases (but this can be done by looking at the table, knowledge of primes is not necessary) and replacing them with their equivalent from the table (**identify** composite bases, **look for them** in the table, **replace** them). “As soon as you get to the point where all the bases are written in prime form, in prime base form, you apply the three index laws” (this point is a trigger for the learners to implement the next ‘step’ ) ...
- 2) Apply the “three index laws” ... He describes one of them as “when you have a double index, right, or a double power, two terms in the power, you multiply the two by the first one and the second one...” He gives the learners a “tip” - “when you get to this type of expressions where you have numerator, denominator, you have bases, indices, etcetera, you will notice that if they should give a variable, a letter,  $m$  or  $x$  for instance. Right, in most cases, in 99% of cases, your  $n$ 's will cancel, your  $m$ 's will cancel, your  $x$ 's will cancel. That's the nature of this type of problems – you'll end up with a numerical value at the end.”

**A closer look at the transformations involved in his actual procedure:**

$$\frac{3^{2n-1} \cdot 9^{n+1}}{9^{2n}}$$

- 1) “how many prime bases do we have in that question” – **identifying and counting** prime bases
- 2) “now if they give you a nine what should you do?” ... “you will get it from your prime table ... you need to memorise that prime table” – **selecting** from prime table
- 3) “three to the power of two ... bracket” – **replacing** nine with three to the power two in brackets
- 4) “okay now you need to apply your laws” “As soon as you get to the point where all the bases are written in prime form .. in prime base form .. you apply the three index laws” “which law applies over here? One, two, three? Three to the power of two bracket  $n$  plus one. What should you do?” ... “you should multiply ... the two by what?” ... “same here, two by the two  $n$ ” (denominator); “when you have a double index, right? Or a double power sign. Two terms in the power, you multiply the two by the first one and by the ... second one” – **multiply powers**
- 5) “so you can do this all in one ... “two  $n$  and two  $n$  gives you what? ... – **adding** variable terms of indices on the numerator (he draws explicitly on the laws for multiplying and dividing powers)
- 6) “four  $n$  subtract four  $n$ . What happens? ... Cancels. Nothing ...  $4n$  disappears” - **subtracting** the variable term on the denominator.
- 7) “look at the numbers. What is minus one add two?” – **add** the ‘numbers’ (i.e. constants)
- 8) “final answer” – **raise** the base (3) to the power (1).



**Figure A5.1** Diagrammatic representation of worked example one

**Table A5.1** Operational activity of worked example one

T	Input	Domain	Operations	Output	Codomain
1	$\frac{3^{2n-1} \cdot 9^{n+1}}{9^{2n}}$	Q	Identify prime bases	3	P (prime)
2	9	C (composite)	Find non-prime bases in prime table and replace	$3^2$	P
3	$(3^2)^{n+1}, (3^2)^{2n}$	Q	Multiply exponents	$3^{2n+2}, 3^{4n}$	Q
4	$2n + 2n$	Q	Add variable terms in numerator	$4n$	Q
5	$4n - 4n$	Q	Subtract variable terms on denominator	0	Q
6	$-1 + 2$	Q	Add constants on numerator	1	Q
7	$3^1$	Q	Exponentiation	3	Q

An extract from the transcript:

Teacher: Two times one. Then you get plus two over here. At the bottom three to the four  $n$ . Okay. Now let's simplify. My answer. Right? Which laws applies now? If I multiply which law do you apply?

Learners: Exponents law.

Teacher: You?

Learners: Add.

Teacher: Yes, you add?

Learners: Exponents.

Teacher: The exponents. . Right?. When you multiply the bases of the same you..

Learners: Add the exponents

Teacher: Add the exponents. What happens when you divide and the bases are the same?

Learners: Subtract .

Teacher: Subtract them. Okay? So you can do this all in one. .. Okay? .. er. Let's check what happens here two  $n$  and two  $n$  gives you what?

Learners: Four  $n$

Teacher: Four  $n$  subtract four  $n$ . What happens? [Learners talking]

Learners:  $n$  [in background]

Teacher: Cancels. Nothing. You see? Two  $n$ , two  $n$  gives you four  $n$  minus four  $n$  gives you?

Learners: Zero [in background]

Teacher: Now you will notice, Right? I'll give you a tip over here. When you get to this type of expressions .. you have numerator .. denominator .. we have bases .. we have indices .. etcetera. You will notice that if they should give a variable .. a letter  $n$  or  $x$  for instance. Right? In most cases .. in ninety nine percent of cases your  $n$ 's will .. cancel. Your  $m$ 's will .. cancel. Your  $x$ 's will .. cancel. That's the nature of this type of problems. You'll end up with a numerical value at the end. Okay? So look at this again. Two  $n$  plus two  $n$  is four  $n$  .. minus four  $n$  .. disappears. Look at the numbers. What is minus one add two?

Learners: Positive one.

Teacher: Positive one. Positive one. Nothing at the bottom. Positive one. Final answer.

Transcript School P7 Lesson 1

The image shows two lines of handwritten mathematical work. The first line is an equation: 
$$= \frac{3^{2n-1} \cdot (3^2)^{n+1}}{(3^2)^{2n}}$$
 The second line is another equation: 
$$= \frac{3^{2n-1} \cdot 3^{2n+2}}{3^{4n}}$$

Figure A5.2 Teacher's procedure for worked example one

## 1) Activating the mathematics encyclopaedia

The topic of the evaluative event in which this example is found is simplifying exponential expressions. This specific example is one of a general type of problem involving arithmetic computations over the set of objects of the form  $a^n$  where  $a, n \in \mathbb{Z}$  and  $a \neq 0$ . The example thus relies on exponentiation, as it consists of objects of the form  $a^n$ , where a base  $a$  is raised to the power  $n$  -  $a^n$  is defined as  $a \times a \times a \times \dots$  for  $n$  factors of  $a$ .

Simplification of this example involves computations involving multiplication of expressions of the form  $a^n \times b^m$ , or  $\times (a^n, b^m)$ , and division of expressions of the form  $\frac{a^n}{b^m}$ , or  $\div (a^n, b^m)$ . But in order to carry out these operations, prime factorization of the bases in this specific example is required so that  $a = b$  in the above expressions, i.e.  $a^n \times a^m$ , or  $\times (a^n, a^m)$ , and  $\frac{a^n}{a^m}$ , or  $\div (a^n, a^m)$ . Both of these involve the operation of multiplication over the set of rational numbers. The ground of this procedure thus consists of the axiomatic properties of multiplication over the set of rational numbers as well as the definition of  $a^n$ .

In order to generate prime factorisations of natural number bases, direct search factorisation (as explained by Davis, 2011a) is the simplest of the procedures available. The procedure is as follows: “for any natural number,  $n$ , test the natural numbers between 1 and  $n$  for proper divisors of  $n$ , starting from 2; every divisor of  $n$  is a factor of  $n$ . Only the divisors between 1 and  $\lfloor \sqrt{n} \rfloor$  need be tested since, if all the natural numbers less than  $\lfloor \sqrt{n} \rfloor$  have been tested, then all possible factors and their cofactors have been tested. As soon as a proper divisor for a given  $n$  is found the process is repeated, until no further proper divisors can be found. The product of the proper divisors is the sought after prime factorization” (Davis, 2011a: 10).

Recall that the smallest proper divisor of a natural number is necessarily prime. This procedure does not depend on learners memorizing primes or rules. Here we simply start from 2 in our search for a divisor and continue until we find one by increasing our potential divisor by one each time, repeating the whole process as many times as required, up to  $\lfloor \sqrt{n} \rfloor$ .

### Curriculum

The curriculum states that in grade ten learners must be able to “simplify expressions using the laws of exponents for integral exponents” (DoE, 2003: 28).

**Textbook** – not used in this event.

## Secondary data production P7 Lesson 1 EE1

### 1) Realisation of content

The intended topic in this event is the simplification of exponential expressions. The teacher’s procedure involves two parts – first, rewrite all bases in what he refers to as “prime base form”, and secondly, apply the index or exponent laws to simplify the expression as far as possible. Although this order is not strictly necessary - the procedure can be carried out by first making as many bases the same as possible, which may not involve prime factorisation, and then once the exponent laws have been applied, rewriting the base as a product of its primes. But despite this emphasis on the order in which the steps must be carried out, the teacher’s procedure still recruits the notion of primes explicitly, unlike in a lesson on the same topic in school P3 where primes are not referred to explicitly. But in this event the procedure, although focusing on the need for bases to be prime, encourages learners to memorise their prime table, rather than on an understanding of primes and the process of prime factorisation. As discussed, direct search factorisation is the simplest of the procedures available for prime factorisation, but the teacher does not reference this or any

other method of prime factorisation, instead insisting that learners memorise their table. But it could also be argued that the examples he uses in this event (he calls a few numbers out asking learners to rewrite them in 'prime base form') are simple enough to warrant this – for example, nine, twenty seven and eight, all of which can be easily written as products of their primes without carrying out a specific procedure.

The teacher also explicitly recruits and refers to the exponent laws, saying things like “when you multiply and the bases are the same you ... add” and “when you divide and the bases are the same ... subtract”. His procedure thus recruits the appropriate elements from the mathematics encyclopaedia, and in does not involve any shifts in the domain being operated over, thus the intended topic is realised in this event, and the realised topic does not violate any mathematical principles and thus corresponds with the mathematics encyclopaedia.

## 2) Regulation of the learner

Features of the regulation in this event:

- Use of 'triggers' in the procedure - “As soon as you get to the point where all the bases are written in prime form, in prime base form, you apply the three index laws” (this point is a trigger for the learners to implement the next 'step' ).
- Emphasis on need to memorise prime table - “Now this is a very very important thing – three to the power of two, three to the power of one etcetera you get it from your prime table – it's in your books, right? You need to memorise the prime table”
- Drawing learners' attention to what the solution will look like and what they should expect to happen - he gives the learners a “tip” - “when you get to this type of expressions where you have numerator, denominator, you have bases, indices, etcetera, you will notice that if they should give a variable, a letter,  $m$  or  $x$  for instance. Right, in most cases, in 99% of cases, your  $n$ 's will cancel, your  $m$ 's will cancel, your  $x$ 's will cancel. That's the nature of this type of problems – you'll end up with a numerical value at the end.” – appealing to the expected solution as a regulative resource.
- The teacher appeals to the examinations often during this lesson, and to what kinds of questions they can expect in the examinations in grade 10, 11 and 12 - “a typical question which you can expect in the exams”, a “typically exam related question”, “if you can master a question like this you'll be prepared for a question like this at the end of the year” “so I will definitely put a question like this in the exams”, “This is a definite type of question that you're supposed to know ... for the final examination, even now for March”.

Overall in this event there is one appeal made to ease/efficiency, three to the impending examinations, one to iconic or spatial features of the solution and three to mathematical features of the solution. I would thus say that necessity is situated external to mathematics and within the teacher's procedure – he appeals to rules, tips, steps and triggers, as well as to the examinations. The regulation of the learners is thus under the aspect of the Imaginary.

## Summary

This event falls into quadrant I. The regulation of the learner depends primarily on the Imaginary.

## Primary data production P7 Lesson 1 EE2

### 1) Generating evaluative events

Table A5.3 School P7 Lesson 1 EE2

Time	Evaluative event/sub-event	Activity	Type
07:00 – 23:00	E <sub>2</sub>	Learners working on exercise, teacher answering questions and working through answers on board.	Exercise/expository

### 2) Describing operational activity

#### Question 1

The first question which the learners work on in class involves a situation where the exponent on the numerator is  $2x$  plus 3 and on the denominator the exponent is  $x$  minus one.

Exponents: Simplify

$$\frac{2^{x+3}}{2^{x-1}} = \frac{2^{2x+3}}{2^{x-1}}$$

Figure A5.3 Question one

Once they have completed the first step above, the teacher explains to the learners that they should subtract the exponents because they are dividing two powers and works through this with them. He then says “now please remember this, whenever you subtract, the bottom from the top, the bottom index from the top index etcetera, then please you need to change the sign of the bottom, it’s much easier. It’s easy to say that negative becomes a positive. So instead of saying three minus minus one, right, it’s easier to say three plus one, that will give us four”. Later he says “just remember when you subtract you change all the bottom signs”.

#### Question 2

2)

$$\frac{3^x}{3^{x-1}} = \frac{3^{x+2}}{3^x}$$

$$= 3$$

Figure A5.4 Question two

In this question, the teacher starts by asking the learners whether there is a need to “break it down to prime base form”. They reply that there is not, and he continues with the question in this way:

Teacher: No. It’s all prime. Right? . Okay? All prime numbers. Now listen carefully. This is shortcut. When you get to Grade Eleven, you’ll have the same type of problems. When you get to Grade Twelve, in your final exams, same type of problems. They test you on this indices. Okay? Now if your bases are all the same what you do is you keep your ..

Learners: Base

Teacher: Base. You see? . Right?. So I’ll show you how to jump from here straight to the answer. Okay? Now let’s see. There’s a  $x$  .. and there’s a  $x$  .. and there’s a two. Can you argue that your desk and your book equals two books?

Learners: No.

Teacher: No. Why not?

Learners: It’s not the same.

Teacher: It’s not the same. Can you see? It’s unlike in mathematics. Now  $x$  plus two is not two  $x$ . Right? It remains  $x$  plus?

Learners: Two.

Teacher: Two. Now let’s go to the  $x$ ’s first. Do you agree this one and that one and that one and that one ... those are like .. (pointing to the  $x$ ’s)

Learners: Yes

Teacher: It’s the same. Okay? Now if you add this  $x$  and that  $x$ , how many  $x$ ’s do we have?

Learners: [in background] Four.

Teacher: We have two  $x$ ’s. Okay. And at the bottom?  $X$  and  $x$ ... if you add it we have?

Learners: Two.

Teacher: Two  $x$ ’s. Now what should we do top to bottom?

Learners: Minus.

Teacher: You minus. Right? You have two  $x$  at the top .. you have two  $x$  at the bottom .. what’s the answer?

Learners: Cancel.

Teacher: The answer will cancel. Can you see? So that  $x$  and that  $x$  and that  $x$  and that  $x$  will cancel. Two  $x$  at the top and two  $x$  at the .. bottom. Let’s go to our numbers. There’s a two and there’s a minus one, are you gonna say two minus one ? Or are you gonna say two minus minus one?

Learners: Two minus minus one

Teacher: Yes, two minus minus one. Now I said it’s easier just to change the sign of the bottom. If it was negative it changes to?

Learners: Positive.

Teacher: To a positive. All Right? Whenever you subtract the value it’s easier just to change the sign of that .. value. ‘Cause if I say minus minus ... what’s a negative times a negative value?

Learners: A positive.

Teacher: A positive value. Whenever you repeat the word negative minus minus change it to a ..

Learners: Positive.

Teacher: Positive. Okay? So I have two at the top and one at the bottom. Your answer therefore will be equal to ...

Learners: Three.

Teacher: It makes sense. And what is three to the power of three?

Learners: Twenty seven.

Transcript School P7

Lesson 1

Below I list the transformations of this procedure:

- 1) Recognise that the bases are all the same and write down the common base.
- 2) “Now let’s go to the  $x$ ’s first” - focus on variables.
- 3) Add  $x$ ’s from the numerator (“We have two  $x$ ’s. Okay”)
- 4) Add  $x$ ’s from the denominator (“And at the bottom?  $x$  and  $x$  ... if you add it we have?”)
- 5) Subtract  $x$ ’s from the denominator from the previous answer (“Now what should we do top to bottom? ... you minus right? You have two  $x$  at the top and two  $x$  at the bottom ... what’s the answer? ... the answer will cancel ... So that  $x$  and that  $x$  and that  $x$  and that  $x$  will cancel”)
- 6) “Let’s go to our numbers. There’s a two and there’s a minus one, are you gonna say two minus one ? Or are you gonna say two minus minus one?” – change the sign of the number on the “bottom” (“Now I said it’s easier just to change the sign of the bottom”) Reason? – “Cause if I say minus minus ... what’s a negative times a negative value?”
- 7) Add the two and one to get three (“your answer therefore will be equal to ... three”)
- 8) Raise three to the power of three to get twenty seven.

Interestingly, the teacher treats the variables and numbers of the indices differently - he explicitly uses the rule “change the sign on the denominator” for the numbers only and not for the variables, which he just subtracts without explicitly ‘changing’ the sign.

**Table A5.4** Operational activity for question two

T	Input	Domain	Operation	Output	Codomain
1	$\frac{3^x \cdot 3^{x+2}}{3^{x-1} \cdot 3^x}$	Q	Identify the common base	3	Base
2	$x + x$	Q	Add variable exponents in the numerator	$2x$	Q
3	$x + x$	Q	Add variable exponents in the denominator	$2x$	Q
4	$2x - 2x$	Q	Subtract the above two answers	0	Q
5	$2 - (-1)$	Q	Subtraction - ‘two minus minus one’ (also refers to ‘change the sign of the bottom’ exponent)	$2 + 1$	Q
6	$2 + 1$	Q	Addition	3	Q
7	$3^3$	Q	Raise three to the power of three	27	Q

### Question 3

The third question he works through, which initially had bases of 25 and 125:

Figure A5.5 Question three

They refer explicitly to prime bases (changing 25 and 125 to five squared and five cubed respectively), and then apply the exponent laws until this point. When dealing with the ‘numbers’ in the exponent, the learners appear to be struggling. The teacher tries to help them: “minus two minus one, mustn’t we change that one to a minus?” using the rule he generated earlier in the event.

He gives another example, where he says that “this sign was positive, I’ve changed it to a negative” while pointing at the exponent three on the denominator, and appealing to the exponent law about division of powers.

Figure A5.6 Using another example to illustrate the division of powers

### 3) Activating the mathematics encyclopaedia

See P7 Lesson 1 EE1

## Secondary data production P7 Lesson 1 EE2

### 1) Realisation of content

In this lesson the teacher works through three of the classwork questions with the learners. In two of the three questions, the bases are all the same and are all prime, thus prime factorisation is not required. These two questions only involve simplification through application of index or exponent laws. The teacher still refers to primes by pointing out to learners that all the bases are prime, thus there is no need to simplify them before applying the exponent laws. In the last question the bases of 25 and 125 are rewritten as five squared and five cubed (without any reference to the method of prime factorisation or the prime table, but explicitly using the notion of a prime base).

The main focus of this event is multiplication and division of exponential expressions, which is recontextualised as adding and subtracting exponents. Specifically, we see a rule generated by the teacher to make things “easier” when dividing powers of the same base – ‘changing the sign of the bottom’. When doing the first question, he tells the learners: “now please remember this, whenever you subtract, the bottom from the top, the bottom index from the top index etcetera, then please you need to change the sign of the bottom, it’s much easier. It’s easy to say that negative becomes a positive. So instead of saying three minus one, right, it’s easier to say three plus one that will give us four”. A few minutes later he says “just remember when you subtract you change all the bottom signs”. This rule which he generates seems to serve as a way of ensuring that learners get to the correct answer, but it does not prevent the learners from

engaging with the addition and subtraction of the integral exponents – they still need to perform integral computations, but these are recontextualised as ‘changing the bottom signs’. The teacher does emphasise that this can only take place when they are dividing powers of the same base and he refers to this property of exponents a few times in this event, giving other simpler examples to illustrate it. So it seems that although he generates a rule about changing the signs, this rule is based on a mathematical property of exponents and he makes this property explicit, suggesting that the intended topic is realised in this event. He also started by using the property of exponents in the previous lesson and then introduced the ‘rule’ in this lesson once learners were familiar with the law upon which it is based.

It is interesting that in this event the teacher treats the variables and constants of the exponents differently – for the variables, he does not draw on his ‘changing the sign of the bottom’ rule, but instead just subtracts the variables of the exponent on the denominator. But when dealing with the constants in the exponents, he explicitly refers to ‘changing the sign’ and then adding, as seen in the following exchange:

Teacher: Two  $x$ 's. Now what should we do top to bottom?

Learners: Minus.

Teacher: You minus. Right? You have two  $x$  at the top .. you have two  $x$  at the bottom .. what's the answer?

Learners: Cancel.

Teacher: The answer will cancel. Can you see? So that  $x$  and that  $x$  and that  $x$  and that  $x$  will cancel. Two  $x$  at the top and two  $x$  at the .. bottom. Let's go to our numbers. There's a two and there's a minus one, are you gonna say two minus one? Or are you gonna say two minus minus one?

Learners: Two minus minus one

Teacher: Yes, two minus minus one. Now I said it's easier just to change the sign of the bottom. If it was negative it changes to?

Learners: Positive.

Teacher: To a positive. All Right? Whenever you subtract the value it's easier just to change the sign of that .. value. 'Cause if I say minus minus ... what's a negative times a negative value?

Learners: A positive.

Teacher: A positive value. Whenever you repeat the word negative minus minus change it to a ..

Learners: Positive.

Teacher: Positive. Okay? So I have two at the top and one at the bottom. Your answer therefore will be equal to ...

Learners: Three.

#### Transcript School P7 Lesson 1

In this exchange we see him refer to the ‘changing signs’ rule as “easier”, but he also gives an explanation – that “cause if I say minus minus ... what's a negative times a negative value? ... a positive value”. We also see his different treatment of the variables and constants – why does he suggesting changing the sign for the constant but not for the variables? Possibly because learners are familiar with the ‘negative times a negative is a positive’ rule from dealing with integers, or possibly in this example (and in the next) it is clear that the variables all “cancel” so there is no need to ‘change the signs of the bottom’.

Overall in this event the content is realised in a way which corresponds with the intended topic and thus the encyclopaedia because of the way in which the exponent laws are explicitly grounding the procedure and as there are no shifts in the domain being operated over (the ‘changing sign’ rule, although potentially involving shifts to the domain of character strings, is motivated by and grounded on explicit mathematical rules, thus I consider this event as falling into quadrant I).

## 2) Regulation of the learner

A feature of the regulation in this event is the way in which the teacher generates a rule which the learners are encouraged to remember and apply - “now please remember this, whenever you subtract, the bottom from the top, the bottom index from the top index etcetera, then please you need to change the sign of the bottom, it’s much easier. It’s easy to say that negative becomes a positive” ... “just remember when you subtract you change all the bottom signs”. Later he says “just remember when you subtract you change all the bottom signs”. This rule which he generates seems to serve as a way of ensuring that learners get to the correct answer, but he still makes explicit the mathematical property upon which this rule is based. At one point learners are arguing about the answer to question three, and he regulates them at this point by referring back to the mathematical property that when dividing powers of the same base we subtract the exponents, using simpler examples to illustrate this:

Teacher: Ja {Yes} ... the law says .. remember .. if I take you back to grade nine ... last year ... can you still remember this? [Writes  $\frac{a^4}{a^3}$  on the board] What’s the answer there?

Learners:  $a$  ... four minus ...

Teacher: It’s  $a$  to the power of four minus ...

Learners: Three.

Teacher: Three. That will be  $a$  to the power of ...

Learners: One.

Teacher: One. Do you agree with that?

Learners: Yes, sir.

Teacher: Okay. Do you agree that I have changed ... this sign was positive .. I’ve changed it to a ...

Learners: Negative.

Teacher: Negative. Do you agree with that? Now if I have two to the power of four over two to the power of three ... Do you agree that the constant remains the same?

Learners: Yes.

Teacher: There you have a base  $a$  .. there you have a base ...

Learners: Two.

Teacher: Two. The base is still the same so therefore isn’t this the same as four minus three?

Learners: Yes sir.

Teacher: Yes; the answer is still two to the power of ..

Learners: One.

Teacher: One. Same thing applies here. Here you have five to the minus two and here you have five to the one. You have to say minus two minus one. Right? Five to the negative two minus one. So that plus one will now change to a negative and a negative two and a negative one gives you a negative three. Okay?

Transcript School P7 Lesson 1

What is also interesting in this event is the teacher’s emphasis on ease – could he be trying to appeal to the learners’ subjective disposition? He gives them a “shortcut” in this event, as an appeal to ease, but he also explains why the shortcut works. In this event there are two appeals to ease or efficiency, one to the exams, one each to iconic and procedural features of solutions and five to mathematical propositions, definitions or rules. There are thus equal appeals made to mathematical and extra-mathematical factors. But because of the teacher’s explanation of why they can “change the bottom signs” and his appeals to the appropriate mathematical rules and processes which make this possible, my analysis suggests that necessity is situated within the field of mathematics and thus that the regulation of the learners, although recruiting elements of the Imaginary in the generation of the rule and the appeal to ease, is predominantly under the aspect of the

Symbolic. Despite attempting to regulate the learners to produce the required answer using fixed steps and rules, the teacher still explains why the rule works mathematically and thus situates necessity within the field of mathematics. He does not transform the mathematics in the direction of the learner, but bases his explanation on mathematical principles.

## Summary

This event falls into quadrant I. The regulation of the learner depends primarily on the Symbolic.

### Primary data production P7 Lesson 1 EE3

#### 1) Generating evaluative events

Table A5.5 School P7 Lesson 1EE3

Time	Evaluative event/sub-event	Activity	Type
23:00 – 33:35	E <sub>3</sub>	Exponential equations worked examples	Expository

#### 2) Describing operational activity

In this evaluative event the teacher introduces a new topic – solving exponential equations, and writes up what he calls a “typical examination question”. Before beginning this example, he does the example  $2x = 16$ , which he refers to as a “typical linear equation”. His method in this example is to “simply divide by the coefficient”. He then moves onto an exponential equation:  $2^x = 16$ , and he tells the learners that “you can do this by going to the table, remember the prime table, it says two to the power four gives you sixteen”. The learners are able to see that the answer is four, but he says “this is mathematics, how do we get to the answer? ... Prove to me that you can get to the answer. Some sort of mathematical reasoning.”

Some learners suggest dividing by two, at which the teacher tells them that “it’s obvious, you can’t divide by two” but doesn’t explain why. He points them back to the previous questions they were working on (simplifying exponential expressions), and prompts them to break the sixteen down to prime bases. They struggle to do that in this context and try to replace the left hand side of the equation with two to the power four instead of replacing sixteen as the teacher intends them to do. A learner then suggests dividing both sides by two and “cancelling”. The teacher says that it is “mathematically incorrect”. He uses the index laws to explain why – two to the power four divided by two is two to the power three. He uses the concept of equality to explain why it is that  $x$  is four, writing “same base same index” on the board:

The image shows a photograph of a whiteboard with handwritten mathematical work. On the left, the equation  $2^x = 2^4$  is written. To the right of this equation, the words "same base" and "same index" are written in two lines, with a box drawn around them. Below the equation, the solution  $x = 4$  is written with a checkmark to its right.

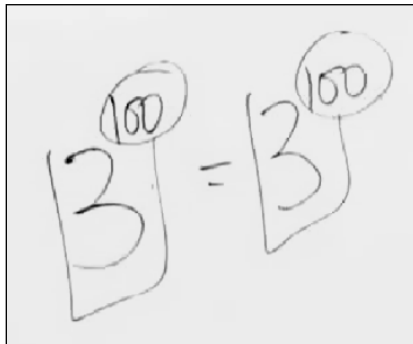
Figure A5.7 Solving an exponential equation

In his explanation, he says the following:

- “what are you going to place in  $x$ ’s position to make the left equal to the right?”

- “the left must be equal to the right”
- “that’s the basis of exponential equations ... same base, same index”
- “if you can make the bases the same, then your indices will be the same”
- “x can only be four”

These suggest that the ground upon which his procedure is based is the notion of equality.



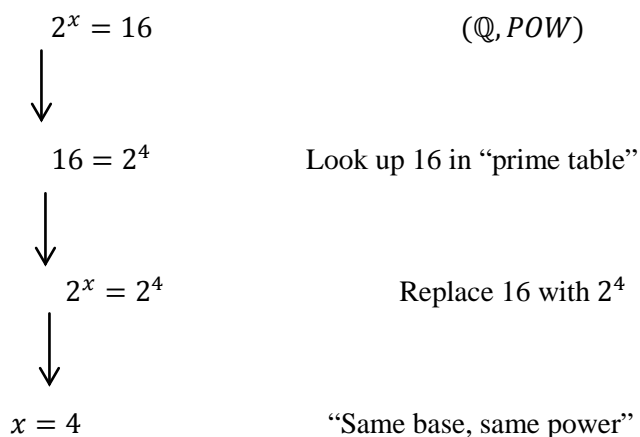
**Figure A5.8** Illustrating the notion of equality

He does three more similar examples involving prime bases,  $2^x = 16$ ,  $3^x = 27$  and  $5^{2x} = 125$ .

In the third example he says - “make the bases the same” therefore “that index,  $2x$ , is equal to three”. What operation is involved in ‘making the bases the same’? In this case, the bases are made the same by looking up numbers in the prime table (which learners are encouraged to memorise earlier in this lesson) and replacing them with what is found there (the prime factorization).

In all of the above examples, they use the “prime table” to “make the bases the same”. In all the examples, the base on the left hand side of the equation is already prime. But if for example, they were given an equation in which both bases were composite and one could be written as a power of the other, the process of prime factorization would not be needed. We can thus see that ‘making’ the bases the same can involve prime factorization of one or both bases, but the bases do not necessarily have to be prime in order to solve the equation, they just need to be the same. But the way in which the teacher describes the procedure, as well as the examples he chooses, suggest that the bases must be prime in order to solve the equation, and that these prime bases must be found in the “prime table”.

Let’s examine the operational activity involved in his first example:



**Figure A5.9** Diagrammatic representation of first worked example

**Table A5.6** Operational activity of first worked example

T	Input	Domain	Operation	Output	Codomain
1	$2^x = 16$	$\mathbb{Q}$	Identifying the base which needs to be “changed”	16	$\mathbb{Q}$
2	16	Numbers not in “prime base form” - $\mathbb{C}$	Looking up in “prime table”	$2^4$	Numbers in “prime base form” - $\mathbb{P}$
3	$16 = 2^4$	$\mathbb{Q}$	Replacing 16 with $2^4$	$2^x = 2^4$	$\mathbb{Q}$
4	$2^x = 2^4$	$\mathbb{Q}$	Applying “same base, same power” principle	$x = 4$	$\mathbb{Q}$

### 3) Activating the mathematics encyclopaedia

#### Field of production

To solve exponential equations without using logarithms, we need to have comparable exponential expressions on either side of the equation so that we can equate the powers and solve. Thus if  $a^x = a^y$ ,  $a \neq 1$ ,  $a > 0$  then  $x = y$ . This principle demonstrates how equations of this type are solved – if the bases are equal then the powers must also be equal, in order for the two sides of the equation to be equal to each other.

Sometimes it is first necessary to convert one side or the other (or both) to some other base before we can solve. So if  $a^x = b^y$ ,  $a, b \neq 1$ ,  $a, b > 0$  we would first need to rewrite either  $a$  as a power of  $b$  or  $b$  as a power of  $a$  in order to solve the equation. So for example, if  $a = b^z$ , we would say  $b^z = b^y$  and thus  $x = y$ . This only applies when both sides of the equation can be written as a power of the same base.

#### Field of recontextualisation

##### Curriculum

In learning outcome two of the FET Mathematics Curriculum it states that “the learner is able to investigate, analyse, describe and represent a wide range of functions and solve related problems” (DoE, 2003: 12). In assessment standard 10.2.5 it states that learners should be able to “solve ... exponential equations of the form  $ka^{x+p} = m$  (including examples solved by trial and error)” (DoE, 2003: 26).

##### Textbook

The textbook used by this teacher (Classroom Maths) defines exponential equations by saying that “the unknown quantity is in the exponent” (pg 181) and gives the form:  $a^x = b$ ,  $a \neq 1$ ,  $a > 0$ .

The principle given in the textbook for solving exponential equations is:

If  $a^x = a^y$ ,  $a \neq 1$ ,  $a > 0$  then  $x = y$ .

The guidelines given for carrying out the procedure (given as part of a worked example) are (pg 181):

- Use the laws of exponents to “express each side as a power of a common base”
- Equate the exponents.

## Secondary data production P7 Lesson 1 EE3

### 1) Realisation of content

The intended topic of this event is the solving of exponential equations. All the exponential equations selected as examples in this event can be rewritten with the same bases on both sides and thus solved using the notion of equality.

Based on my description of the operational activity in this event, the realised topic, although still achieving the same result as the intended topic (the solving of an exponential equation in which both sides can be written with the same base) depends on the process of breaking bases down to prime numbers. The first step of the teacher's procedure involves "making the bases the same" by looking up numbers in the prime table (which learners are encouraged to memorise earlier in this lesson) and replacing them with what is found there (the prime factorization). Generally, 'making' the bases the same in an exponential equation can involve prime factorization of one or both bases, but the bases do not necessarily have to be prime in order to solve the equation, they just need to be the same. The textbook refers to the need to "express each side as a power of a common base" and does not mention primes. But the way in which the teacher describes the procedure, as well as the examples he chooses, suggest that the bases **must** be prime in order to solve the equation, and that these prime bases must be found in the "prime table". The realised topic is thus strongly focused on prime bases, while the intended topic, as found in the mathematics encyclopaedia, does not necessarily include the process of prime factorisation.

But despite this, in terms of the mathematics encyclopaedia, the realised topic does not violate any mathematical principles or definitions. In the teacher's explanation, he says the following:

"what are you going to place in  $x$ 's position to make the left equal to the right?"

"the left must be equal to the right"

"that's the basis of exponential equations ... same base, same index"

"if you can make the bases the same, then your indices will be the same"

" $x$  can only be four"

These suggest that the ground upon which his procedure is based is the notion of equality, and thus that the realised topic is in fact aligned with the mathematics encyclopaedia.

This event can thus be described as one in which there is not correspondence between the intended and realised topics, but in which the realised topic still corresponds with the mathematics encyclopaedia.

### 2) Regulation of the learner

In the teacher's explanation, he appeals eight times to the mathematical propositions, definitions and processes underlying this procedure (specifically, the notion of equality), once to an iconic feature of a solution and once to ease/efficiency. Thus he appeals predominantly to mathematical factors and his explanation situates necessity as internal to mathematics, as can be seen through the centrality of the notion of equality in his procedure, and the way in which he explicitly appeals to this notion. Although the teacher appeals to the need for learners to memorise the prime table, which suggests a reliance on rote learning rather than an understanding of the process of prime factorization, his explanation is explicitly anchored on the mathematical principle of equality, and implicitly on the importance of maintaining identity at the level of value, despite changes at the level of expression. He does not bend the mathematics in the direction of the learner, but repeats the principle of "same base, same power", recruiting the notion of equality, a number of times in this lesson. He also appeals to mathematical principles to explain why it is that you cannot divide

both sides of an exponential equation and “cancel” the base. The regulation of the learner in this event recruits predominantly elements of the Symbolic.

### Summary

This event falls into quadrant II. The regulation of the learner recruits predominantly elements of the Symbolic.

### Primary data production P7 Lesson 2 EE1

#### 1) Generating evaluative events

**Table A5.7** Evaluative events School P7 Lesson 2

Time	Evaluative event/sub-event	Activity	Type
00:00 – 22:52	E <sub>1</sub>	Going through the homework questions on exponential expressions	Expository
18:00 – 22:00	E <sub>1.1</sub>	Fractions discussion	Expository
22:52 – 36:50	E <sub>2</sub>	Exponential equation examples	Expository

#### 2) Describing operational activity

The teacher starts this lesson by asking a learner to go through homework question four:  $\frac{4^{x-1} \cdot 8^{x+1}}{32^{x+1}}$

The learner’s attempt:

$$\begin{aligned}
 &= \frac{(2^2)^{x-1} \cdot (2^3)^{x+1}}{(2^5)^{x+1}} \\
 &= \frac{2^{5x+5}}{2^{5x+5}} \\
 &= 2 \rightarrow
 \end{aligned}$$

**Figure A5.10** Learner’s attempt at homework question four

It seems as if the learner expected to be able to ‘cancel’ the powers in the second to last step, and made sure that she got to that point. The rest of the class discuss whether they agree or not, and seem to disagree with the answer. The teacher adds in the following to the learner’s solution

$$= \frac{2^{2x-2} \cdot 2^{2x+3}}{2^{5x+5}}$$

**Figure A5.11** Teacher’s addition to learner’s solution

And then changes her final answer to two to the power of negative four. The learners don’t seem satisfied so he works through the whole question, saying “the law applies if I multiply .. the bases are the same .. what do I do with the indices? You ... add”; “if you have the same top and bottom you subtract your indices”; “you can write it to a positive index ... just find the reciprocal. Turn it around”. He also explains how he would allocate marks to this question in an exam – one for breaking the powers down to ‘prime base form’, another for applying the exponent law and a third for the final answer.

The teacher does the next question on the board.

$$= \frac{7^{2x-2} \cdot (7^{2x+2})}{7^{3x+2}}$$

$$= \frac{7^{2x-2} \cdot 7^{2x+4}}{7^{3x+2}} = 7^0$$

**Figure A5.12** Teacher’s solution to homework question five

He relies on the exponent laws to show that two minus two is zero. He then goes through a question he had set them from the previous year’s November examination -  $\frac{2^{x+1} \cdot 3^{2x-1}}{18^x}$

When they come across a base of eighteen – which is the most difficult example of prime factorization we see in these examples, the teacher uses the “ladder method” to break it down to a product of prime:

Teacher: What’s the smallest prime number into eighteen?

Learners: Two ... four ...

Teacher: Two. How many times?

Learners: Nine.

Teacher: Nine times ... smallest prime number into nine?

Learners: Three.

Teacher: Three. How many times?

Learners: Two ... Three  
 Teacher: Three times. Smallest prime number into three?  
 Learners: Three ... once.  
 Teacher: Three, once. Still remember this?  
 Learners: Yes sir.

Transcript School P7 Lesson 2

.This method can be describe as: given a natural number  $n$ , divide  $n$  by its smallest prime divisor,  $p_1$ . If the result,  $n_1$ , is greater than 1, then divide  $n_1$  by the smallest prime number by which it is divisible,  $p_2$ . If the result,  $n_2$ , is greater than 1, continue with the procedure, dividing each subsequent  $n_i \neq 1$  by the minimal prime by which it is divisible,  $p_i$  until we get  $n_i = 1$ . We take the prime factorisation of  $n$  to be the product of the powers of the  $p_i$  since the  $p_i \in P$  are the prime factors we seek.

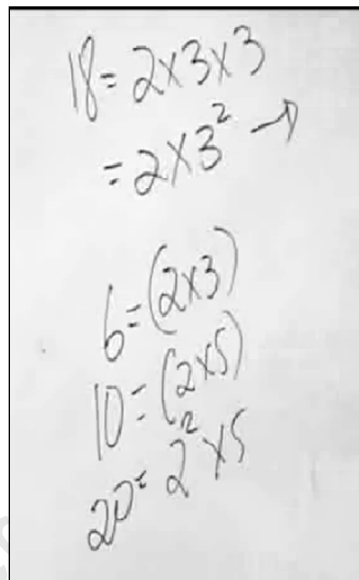


Figure A5.13 Breaking numbers down to prime products

Later in the question he refers to another exponent law: “that’s the rule, the rule is like that, you must multiply by each and every index you have inside”:

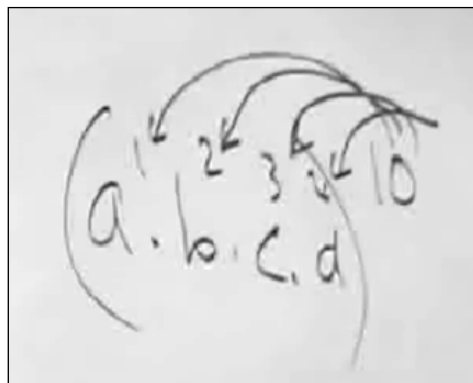
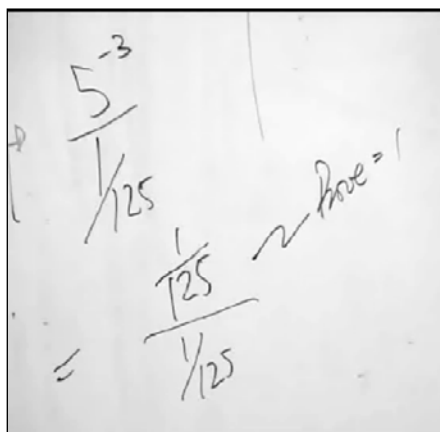


Figure A5.14 An illustration of an exponent law

In order to complete the question he asks them “which law applies here, same top same bottom?”. They struggle to answer so he refers them to their books. He is insistent that they must give the correct law number, they eventually identify “law two” as the correct one and he completes the question referring to this law (relating to division of powers with the same base). The teacher encourages them to repeat the question over and over on their own with their “exam pad, pencil, eraser and table”. He says “you can teach your hand how to do it” through repetition of the method many times.

The learners ask him a question – how to simplify:  $\frac{5^{-3}}{\frac{1}{125}}$

He explains this by referring to division of something by itself – “If your numerator ... your denominator is the same ... you will always get one. It’s actually one over one twenty five divided by the same thing”. Once he’s explained it, he asks them to prove that the answer is one:



**Figure A5.15** Discussion of division of something by itself

In his explanation he refers to division by a fraction - “when you divide by a fraction what happens. You have to turn the thing around. Division is not permissible. Division by a fraction is not permissible. You can’t do that, right? So you don’t divide by a fraction. Instead of divide you change the division to multiplication. You take your fraction and you turn it around ... you call that a reciprocal.”

“Whenever you have a fraction at the bottom, you take your top value as is, don’t change the top value, instead of division you write it as multiplication and then you go to your bottom fraction and then you simply turn it around. The denominator goes to the top and the top goes to the bottom”.

### 3) Activating the mathematics encyclopaedia

See P7 Lesson 1 EE1.

## Secondary data production P7 Lesson 2 EE1

### 1) Realisation of content

The intended topic of this event is the simplification of exponential expressions and the event is spent going through a homework exercise.

The first question they go through is:  $\frac{4^{x-1}.8^{x+1}}{32^{x+1}}$

This is an interesting question for our analysis as there are at least two ways in which it can be done, each involving the same number of ‘steps’ or transformations:

First possible method:  $\frac{4^{x-1}.8^{x+1}}{32^{x+1}} = \frac{4^{x-1}.8^{x+1}}{(4.8)^{x+1}} = \frac{4^{x-1}.8^{x+1}}{4^{x+1}.8^{x+1}} = 4^{-2} = \frac{1}{16}$

Second possible method:  $\frac{4^{x-1} \cdot 8^{x+1}}{32^{x+1}} = \frac{(2^2)^{x-1} \cdot (2^3)^{x+1}}{(2^5)^{x+1}} = \frac{2^{2x-2} \cdot 2^{3x+3}}{2^{5x+5}} = 2^{-4} = \frac{1}{16}$

The teacher has taught the learners in the previous lesson to always change the bases into “prime base form” first and not that this could also be carried out later in the question.

Let’s discuss the learner’s attempt to answer this question, seen in Figure A5.10:

The error made by this learner is line two, the exponent of the numerator. It seems she added two and three to get five, instead of adding negative two and three to get one. It could be that the learner expected to be able to ‘cancel’ the powers in the second to last step, and made sure that she got to that point. The teacher’s emphasis in the previous lesson on the way in which things “cancel” in “these type of questions” and also that the final solution will be a number seemed to have influenced this learner’s treatment of the question.

But in the teacher’s explanation he corrects these errors and refers to the exponent laws quite often. Generally the exponent laws are explicit in the teacher’s criteria - “that’s the rule, the rule is like that, you must multiply by each and every index you have inside”, “which law applies here, same top same bottom?”

Another feature of this event which is interesting is the teacher’s reference to the ladder method to carry out the process of prime factorization – we already noted the lack of a method for this process in previous events, but surmised that it could be due to the small number used (which could be written as products of their primes without any calculation). But eighteen is a different story. In his use of the ladder method, the teacher explicitly refers to prime divisors, and primes are a regulative resource in his explanation. His method, described previously, is a variation of direct search factorization (see P7 Lesson 1 EE1, activation of the mathematics encyclopaedia), and although explicitly referring to prime divisors and the operations of multiplication and division, depends on the operation of “selection” (discussed by Davis, 2011a).

In another example, they need to divide by a fraction. The teacher tells them “when you divide by a fraction what happens. You have to turn the thing around. Division is not permissible. Division by a fraction is not permissible. You can’t do that, right? So you don’t divide by a fraction” – this is not mathematically true, as division by a fraction is permissible. But the teacher then says: “Instead of divide you change the division to multiplication. You take your fraction and you turn it around ... you call that a reciprocal.” Here he draws on the mathematical object of a reciprocal or inverse, but in an iconic way (referring spatial shifting of the numerator and the denominator) - “The denominator goes to the top and the top goes to the bottom”.

Despite this incorrect statement about division by a fraction not being permissible, and the iconic definition of a reciprocal, the objects operated over in this event are mathematical ones and the operations are functions. There are no shifts in the domain and the intended topic is realised.

## 2) Regulation of the learner

In this event, there is a total of 21 appeals made to an authorizing ground - one appeal is made to ease/speed/efficiency, one to the exams, five to iconic or spatial features of solutions and three to procedural features of solutions (making up 10 appeals). The remaining 11 appeals are made to mathematical propositions, definitions, rules and processes. These just outweigh the ten appeals made to extra-mathematical factors, thus suggesting that necessity is situated within the field of mathematics and that regulation of the learner is under the aspect of the Symbolic in this event. Analysis of the interactions between the teacher and learner confirm this – the teacher explicitly situates necessity within the field of mathematics on a number of occasions, both in response to learner questions and in his explanations, appealing to appropriate operational resources from the encyclopaedia at these points, for example the properties of exponents, the process of prime factorisation and the division of something by itself to yield one as the ground of the ‘cancellation’ of terms.

## Summary

This event falls into quadrant I and the regulation of the learner depends primarily on the Symbolic.

### Primary data production P7 Lesson 2 EE2

#### 1) Generating evaluative events

Table A5.8 School P7 Lesson 2 EE2

Time	Evaluative event/sub-event	Activity	Type
22:52 – 36:50	E <sub>2</sub>	Exponential equation worked examples using trial-and-error and logarithms.	Expository

#### 2) Describing operational activity

The teacher begins this event with the question with which he ended the previous lesson:  $2^x = 20$ . The learners call out “two to the power of two times five”, and the teacher writes what we see on the left below, replacing it with the right hand option when learners shout out “no sir, not that times five”:

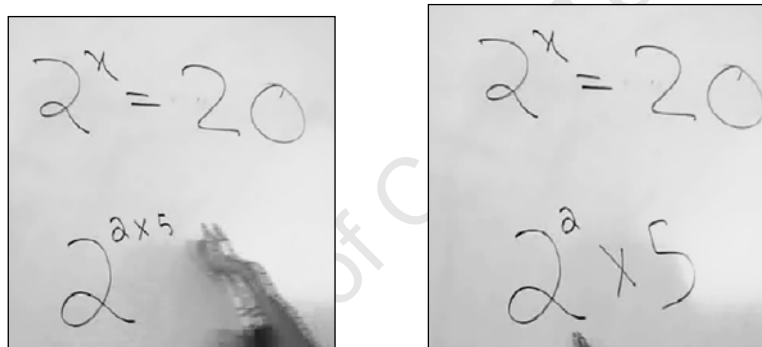


Figure A5.16 Learners suggestion for solving  $2^x = 20$

It seems that the criterion transmitted (and acquired) in the previous lesson is that the bases must be rewritten as a product of primes. The teacher agrees with them that what he has written is the “prime product form” of twenty, but says that “it won’t make any sense in the context of our equation” as they do not have the same bases on the left and the right – thus focusing on the other criterion from the previous lesson – that of “making the bases the same”.

It seems that the learners selected the ‘prime base’ criterion over the ‘same base’ criterion. Now the teacher emphasizes the need for the bases to be the same, and describes the aim of an exponential equation as to rewrite the equation so that the bases are the same on both sides, with no mention of primes this time (unlike in the previous lesson where he emphasized primes) as seen in the extract below:

Teacher: Okay, right. Now look at this. I agree with this. This is the prime product form of ... twenty. Right. But it won’t make any sense in the context of our ... equation. Because on this side I have two to the power of  $x$  ... on this side I have two to the two ... times ... five. Which is correct. Right? But do you have the same bases on the left and the same bases on the right?

Learners: No

Teacher: You don’t have it. Do you see? Right? The aim over here is ... when you deal with an index equation ... let’s assume you have two to the  $x$  equal to sixteen ... the aim over there is to rewrite sixteen as ...

Learners: Two.

Teacher: Two to the power of ... four. Do you have the same bases on the left and on the right?

Learners: Yes.

Teacher: Therefore you can deduce? ... You can say therefore  $x$  will be equal to?

Learners: Four.

Teacher: Does that make sense?

Learners: No.

Teacher: No? if you have two to the  $x$  equal to thirty two ... equal to thirty two, can you break up that?

Learners: Yes.

Teacher: Thirty two is what?

Learners: Two to the power of five.

Teacher: Two to the power of five. So therefore  $x$  will be what?

Learners: Five.

Teacher:  $x$  will be equal to? ... five. Do you agree with that? Okay. Now let's go further. I'll take this off. [Cleans the board]. Understand why that doesn't work, hey? Because the bases aren't the same. Technically two squared times five gives you twenty.

Transcript School P7 Lesson 2

This is a slightly different focus to the previous lesson – there he emphasizes primes and the need to convert bases to prime base form and then to make use of the “same base, same power” principle. In the above explanation he emphasises sameness of bases without any reference to primes.

In order to solve the given example, he refers to the examples from the previous lesson again, asking the learners “do you agree that twenty is the middle between sixteen and thirty two?” The learners then suggest that  $x$  is four comma two or three. The teacher tells them that “the index will lie between the numbers four and five”. He encourages them to use their calculators and “play around”, trying different indices between four and five – he suggests various indices which give answers of 19,6, then 21,1, then 18,37.

Handwritten calculations on a board showing powers of 2 with decimal indices:

$$2^{4.2} = 18.37$$

$$2^{4.3} = 19.6$$

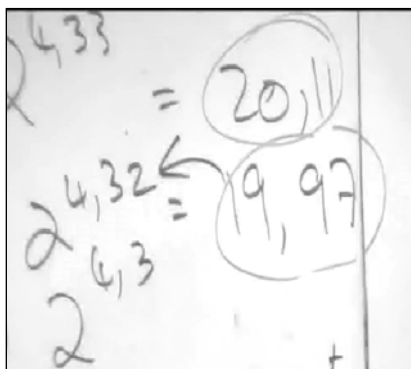
$$2^{4.4} = 21.1$$

$$2^{4.33}$$

Figure A5.17 “Playing around” to solve  $2^x = 20$

In his discussion of the method he tells them “we call this method the method by trial and improvement ... meaning you make an attempt and then you rectify that attempt. Not rectify but you move on”. He tells them that “obviously you won’t get an integral, you won’t get a natural number as index. You need to understand that some numbers will have indices that are not natural, ok, or integers”.

He instructs them to select the one that’s closest to twenty, which they say is nineteen comma nine seven, which he accepts. But then a learner says that he gets nineteen comma nine nine, and the teacher says “that’s close ... what did you use?” The learners says “four comma three two one five”



**Figure A5.18** Selecting the closest answer

Teacher: Did you guess that one?

Learner: No sir I played around.

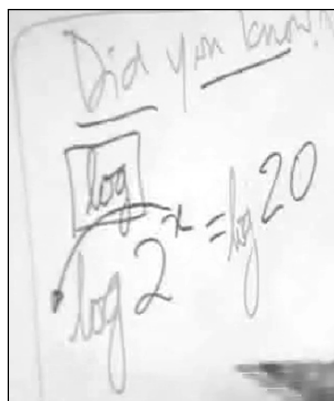
Teacher: You played around. That’s good. You see by means of investigation you can get closer and closer to the answer. Four comma three two one five. Right. Try four comma three two two.

Teacher: You get twenty comma nought one. You know what I did, I took the fifteen and I rounded it off to two.

Transcript School P7 Lesson 2

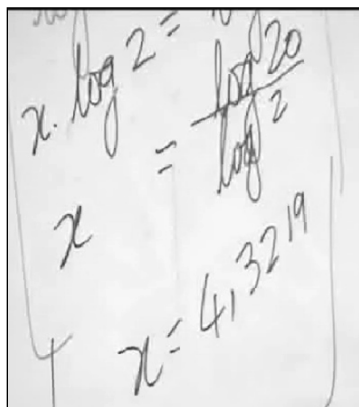
This illustrates the difficulty in arriving at a “final” solution. The teacher does not make it clear at which point learners should accept their solution as the “closest” approximation. “And then you need to select the one that’s closest to twenty. Which one are you going to select?” – the learners reply with different options. The teacher does not write the ‘final’ answer as  $x = \text{something}$ , he just circles the power which they select as closest to twenty.

He then says “I know you’re in grade ten, and I don’t think you’ll mind if I teach you something we use in grade twelve”. He asks them to look for the word “log” on their calculator, saying “now this is nothing to do with wood or something, hey. I’ll tell you what it means. Log is just a different way of explaining indices ... we’re going to play around”. He describes logarithms as “a different way of applying or using exponents”. He introduces what he refers to “log law number three” only – “If you find the log of a value of a base and that base have an index ... the log law, law number three, tells me that if you find the log of a value or an index, you take that index and you place it in ... front, okay? ... so you gonna have x times log two equal to log twenty”. This process (placing the index “in front”) is merely a spatial shift without any reference to the mathematical principles and properties underlying logarithms.



**Figure A5.19** Introducing logarithms

Now he asks them “what process are you going to follow to get rid of log two?” Learners suggest division, resulting in log twenty over log two – the teacher has used logs to get the equation to a familiar form to learners, one in which division is required in order to ‘cancel’ the coefficient of  $x$ .



**Figure A5.20** Solving  $2^x = 20$  using logarithms

They use their calculators to get the answer, and the teacher says “who’s the closest now?” At this point he seems to favour the ‘log method’ as it yields a precise answer, but later he tells them that they do not have to use this method until grade 12. In the next lesson on the same topic, he says that “in grade ten we’re not going to introduce the logs ... logs are a shortcut, you can’t in the examination give us a solution that’s log of ... in the exams when we give you this question we will say by means of trial and improvement ... or we’ll say give the answer to two decimal places”. But he does mention that “nothing stops you using your calculator ... you can use logs” – he seems to encourage the learners to use logs to check their solutions (“so you can use the log to guide you”). He also tells them that in the exams they will be told to solve by means of trial and improvement. “Because this is stated in our document, we get a document from the department, and it’s stated that we have to teach that particular method to you”. He seems to be saying that the only reason he teaches them the trial and improvement method is because it is stated in the curriculum document, but he prefers the “log method”.

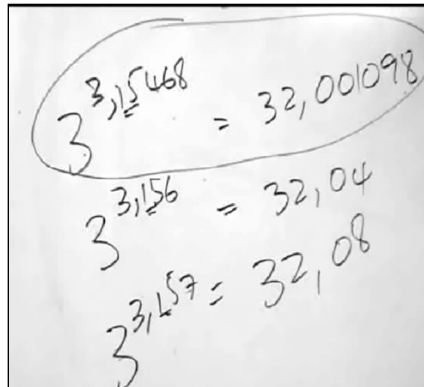
He does another example using trial and improvement:  $3^x = 32$ . A learner asks:

Learner: Sir, why can’t we say two to the power of five?

Teacher: Two to the power of five won’t work because that base is three and that base is gonna be two. The bases won’t be the same. Remember our bases need to be the same on the left and on the right. Okay. Do you agree? There’s your answer. Okay, I’ll give you time. See if you

can figure that one out. First with trial and improvement ... first play around ... play around with the calculator.

## Transcript School P7 Lesson 2



Handwritten calculations showing trial and improvement for solving  $3^x = 32,000$ . The calculations are:

- $3^{15688} = 32,000,98$  (circled)
- $3^{3156} = 32,04$
- $3^{3157} = 32,08$

**Figure A5.21** Another example

Once again the teacher does not write a ‘final’ answer for  $x$  with the trial and improvement method but just circles the one they choose as closest.

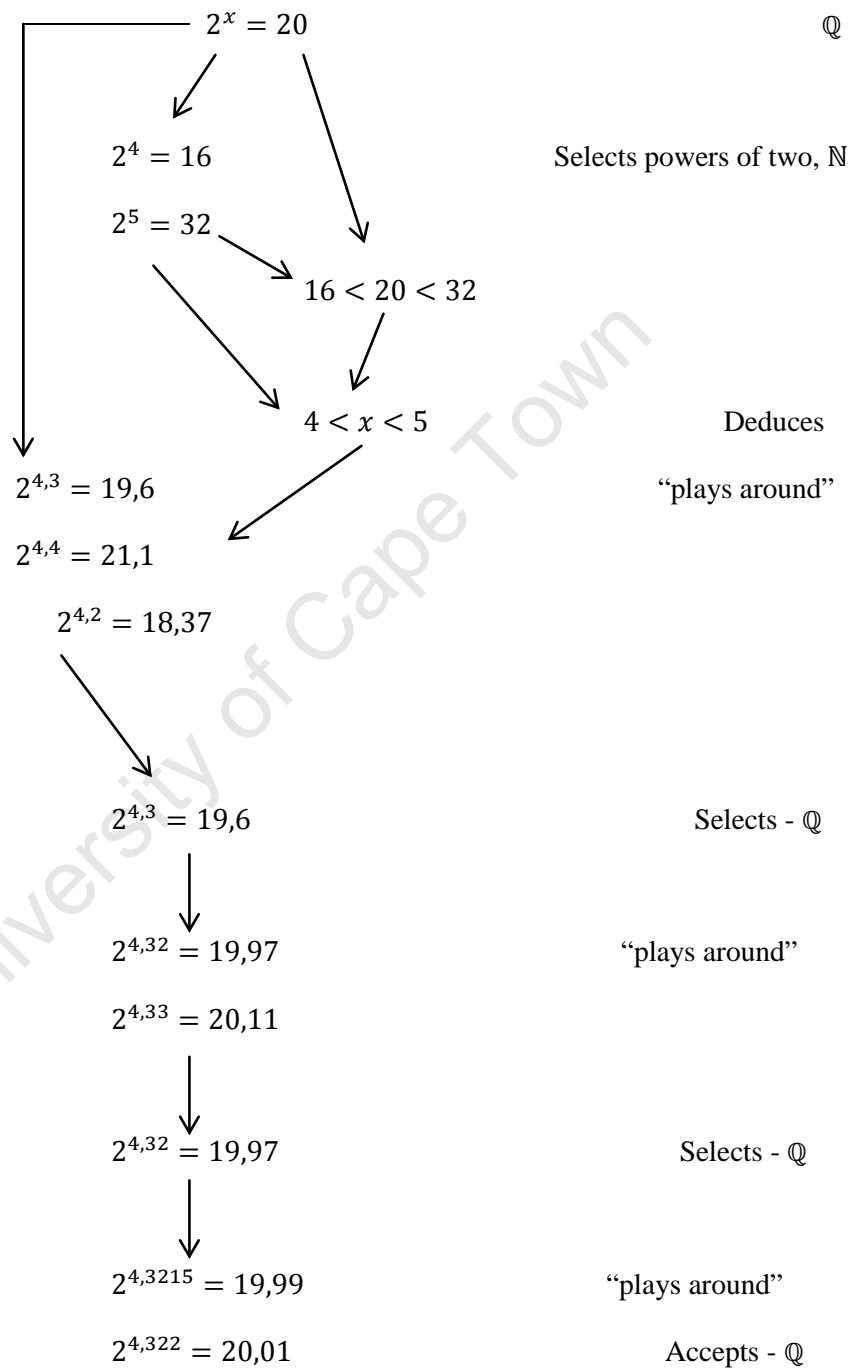
Let’s examine the operational activity of the two methods used more closely – I will just analyse the first example  $2^x = 20$  in detail as all others follow the same process:

**Trial and improvement method for solving  $2^x = 20$ :**

- 1) Find the powers of two on either side of twenty – two to the power of four and two to the power of five as twenty lies between sixteen and thirty two.
- 2) Deduce that the index  $x$  lies between four and five.
- 3) “Play around” with different indices between four and five to find the one which gives the closest answer to twenty.
- 4) “And then you need to select the one that’s closest to twenty. Which one are you going to select?” Select that as the solution (circles it but doesn’t write a final line of  $x =$  solution).

This procedure depends on selection. As explained by Davis (2011a) selection is indicated by  $SEL(c, T) \rightarrow t$ , where  $c$  indicates a specific object-type and  $T$  is the particular set of objects from which a selection of the object,  $t$ , is to be made. In this example,  $c$  is a rational exponent between four and five while  $T$  is the set of all rational numbers between four and five. Selection in this particular example is not a function as the output is not unique for any given input.

Also central to the procedure is what the teacher refers to as “playing around” – trying a succession of exponents (each based on the comparison of the previous selection with the desired answer, in this case twenty) in order to reach the ‘closest’ number to the desired answer.



**Figure A5.22** Diagrammatic representation of solving  $2^x = 20$  using trial and improvement

**Table A5.9** Operational activity involved in solving  $2^x = 20$  using trial and improvement

T	Input	Domain	Operation	Output	Codomain
1	$2^4$	$\mathbb{N}$ (power of 2)	Selection; exp	16	$\mathbb{N}$ (power of 2)
2	$2^5$	$\mathbb{N}$ (power of 2)	Selection; exp	32	$\mathbb{N}$ (power of 2)
3	16, 20, 32	$\mathbb{N}$	Ordering	$2^4 < 20 < 2^5$	$\mathbb{N}$
4	$2^4 < 20 < 2^5$	$\mathbb{N}$	Deduction	$4 < x < 5$	$\mathbb{N}$
5	$2^{4,3}$	$\mathbb{Q}$	“playing around” – trial 1	19,6	$\mathbb{Q}$
6	$2^{4,4}$	$\mathbb{Q}$	“playing around” – trial 2	21,1	$\mathbb{Q}$
7	$2^{4,2}$	$\mathbb{Q}$	“playing around” – trial 3	18,37	$\mathbb{Q}$
8	19,6; 21,1; 18,37	$\mathbb{Q}$	Selecting the closest to 20	19,6	$\mathbb{Q}$
9	$2^{4,32}$	$\mathbb{Q}$	“playing around” – trial 4	19,97	$\mathbb{Q}$
10	$2^{4,33}$	$\mathbb{Q}$	“playing around” – trial 5	20,11	$\mathbb{Q}$
11	19,97; 20,11	$\mathbb{Q}$	Selecting the closest to 20	19,97	$\mathbb{Q}$
12	$2^{4,3215}$	$\mathbb{Q}$	“playing around” – trial 6	19,99	$\mathbb{Q}$
13	19,97; 19,99	$\mathbb{Q}$	Selecting the closest to 20	19,99	$\mathbb{Q}$
14	$2^{4,322}$	$\mathbb{Q}$	“playing around” – trial 7	20,01	$\mathbb{Q}$
15	19,99; 20,01	$\mathbb{Q}$	Selects closest to 20	20,01	$\mathbb{Q}$

### Log method

The teacher presents this method as an alternative to the trial-and-error method, but tells learners they are not expected to know it until grade 12. He also refers to this method as the “shortcut”.

- 1) “Apply the word log on the left and the word log on the right” (i.e. take the logarithm of both sides).
- 2) “Take” the index  $x$  on the left (sunder)
- 3) “place it in front” (apply “log law three”) (shift and concatenate?) – see Figure A5.22
- 4) “Get rid of log two” by dividing in order to “isolate  $x$ ” - see Figure A5.23
- 5) Use the calculator to find the answer of log twenty divided by log two.

This procedure does not depend on selection, but does include the unfamiliar operations of sundering, concatenation and shifting (the latter is similar to Lima & Tall’s (2010) description of procedural embodiment). The ground on which the procedure rests is the definition of a logarithm, with its properties over the reals. But the closest the teacher comes to defining a logarithm is describing it as “a different way of explaining indices ... a different way of applying or using exponents”. He thus mentions the link between the exponential and the logarithmic function, but does not explain it or explicitly draw on this during his

procedure. He refers to the “log law number three” as “if you find the log of a value or an index, you take that index and you place it in front”, suggesting a sundering of the index from its base and spatial shifting of character strings. He also refers to logs as a “word” which should be written on both sides of the equation—the notion of a logarithm as an operation is implicit.

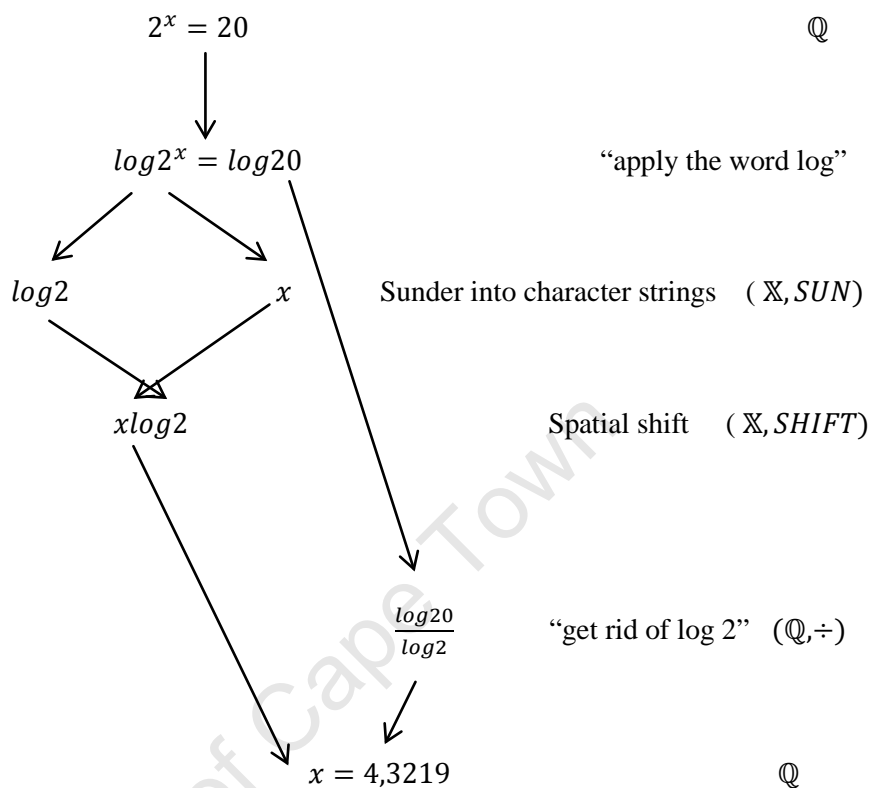


Figure A5.23 Diagrammatic representation of solving  $2^x = 20$  using logarithms

Table A5.10 Operational activity involved in solving  $2^x = 20$  using logarithms

T	Input	Domain	Operation	Output	Codomain
1	$2^x = 20$	Q	“Apply the word log” on both sides	$\log 2^x = \log 20$	Q
2	$\log 2^x = \log 20$	Q	Existential shift	$/\log 2^x/$	X
3	$/\log 2^x/$	X	Sundering	$/\log 2/, /x/$	X
4	$/\log 2/, /x/$	X	Place it in front (shift)	$/x/, /log 2/$	X
5	$/x/, /log 2/, /log 20/$	X	Concatenate	$/x \log 2 = \log 20/$	X
6	$/x \log 2 = \log 20/$	X	“Get rid of $/log 2/$ ”	$\frac{\log 20}{/log 2/}$	X
7	$\frac{\log 20}{/log 2/}$	X	Existential shift	$x = \frac{\log 20}{\log 2}$	Q
8	$x = \frac{\log 20}{\log 2}$	Q	Division	$x = 4,3219$	Q

### 3) Activating the mathematics encyclopaedia

#### Field of production

Axiomatic properties of exponents and logarithms

Exponents	Logarithms
$a^m \times a^n = a^{m+n}$	$\log_a mn = \log_a m + \log_a n$
$a^m \div a^n = a^{m-n}$	$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$
$(a^m)^n = a^{mn}$	$\log_a (x^n) = n \log_a x$
$a^1 = a$	$\log_a a = 1$
$a^0 = 1$	$\log_a 1 = 0$

#### Using trial and improvement to solve exponential equations

Trial and improvement (or trial and error) as a method for solving equations in the field of mathematics is generally an attempt to find a possible solution to an equation – not all possible solutions, and not necessarily the best or closest solution. It is possible to use trial and error to find all solutions or the best solution when a finite number of possible solutions exist. To find all solutions, we would not end the process when one solution is found, but would continue until all solutions have been tried (where possible). To find the best solution, we would compare all the possible solutions found and then, based upon some predefined set of criteria, we would select the best solution. When only one solution can exist, then any solution found is the only solution and so is necessarily the best. Trial and improvement cannot be used to generalize a solution to other problems. It can be used where there is little or no knowledge of the subject.

#### Using logarithms to solve exponential equations

The logarithm of a number to a given *base* is the exponent to which the base must be raised in order to produce that number. The idea of logarithms is to undo the operation of exponentiation. More formally, the logarithm of a number  $b$  with respect to base  $a$  is the exponent to which  $a$  has to be raised in order to yield  $b$ . This way, the logarithm yields the exponent that was used to obtain the power, or the logarithm of  $b$  to base  $a$  is the number  $x$  satisfying the equation  $a^x = b$ . The operation of taking a logarithm is the inverse to the operation of exponentiation. In order to solve exponential equations we can use the fact that the logarithm is the inverse of the exponential function.

An exponential equation can be solved by converting from exponential form to logarithmic form, as the inverse of  $y = a^x$  is  $y = \log_a x$ , which is the same as  $x = a^y$ .

Thus for example, in the equation  $2^x = 20$ , we can rewrite the equation in logarithmic form, drawing on the above definition of a logarithm:  $x = \log_2 20$ . From here we can calculate the value of  $x$  using the natural logarithm, so  $x = \frac{\ln 20}{\ln 2}$ . We could also take the logarithm on both sides of the original equation,  $\log 2^x = \log 20$ , and thus solve for  $x$ .

#### Field of recontextualisation

## Curriculum

In learning outcome two of the FET Mathematics Curriculum it states that “the learner is able to investigate, analyse, describe and represent a wide range of functions and solve related problems” (DoE, 2003: 12). In assessment standard 10.2.5 it states that learners should be able to “solve ... exponential equations of the form  $ka^{x+p} = m$  (including examples solved by trial and error)” (DoE, 2003: 26), as referred to by the teacher in his mention of the “document from the department” which states that particular method. Logarithms are not mentioned in the curriculum at grade 10 level – they are first mentioned in grade 12 where it states that learners should be able to “demonstrate an understanding of the definition of a logarithm and any laws needed to solve real-life problems”.

## Textbook

The textbook (Classroom Maths Grade 10) mentions that not all exponential equations can be expressed as power of equal bases, and also introduces trial and error as a method of finding approximate solutions. They describe the trial and error method more precisely than the teacher does – by referring to it as “interval bisection”. Their method entails finding the greatest power of the base that is less than the required answer, and the smallest power of the base less than the required answer. They then conclude that the exponent lies somewhere between the smallest and greatest exponents. They bisect the interval between those two exponents, calculating the answer with the mid-way exponent. Depending on whether the answer is greater or less than the required answer, they choose an interval half the size (top or bottom half) and bisect that, calculate the answer with that power and continue the process. Pg 183 image.

Written in general terms, this method of “interval bisection” is:

Given  $a^x = b$ ,  $a \neq 1$ ,  $a > 0$  (where  $b$  cannot be expressed as a power of  $a$ )

Find  $x_1$  and  $x_2$  such that  $a^{x_1} < b < a^{x_2}$ , therefore  $a^{x_1} < a^x < a^{x_2}$

Therefore  $x_1 < x < x_2$

Bisect the interval between  $x_1$  and  $x_2$  to get  $x_3$ .

Calculate  $a^{x_3}$ .

If  $a^{x_3} > b$  then  $a^{x_1} < a^x < a^{x_3}$ .

Bisect the interval between  $x_1$  and  $x_3$  to get  $x_4$ .

Calculate  $a^{x_4}$ .

If  $a^{x_4} > b$  then  $a^{x_1} < a^x < a^{x_4}$ .

Bisect the interval between  $x_1$  and  $x_4$  to get  $x_5$ .

Continue the process until  $a^{x_n}$  is within one decimal place of  $b$ .

This is more precise than the teacher’s “playing around” as the only selection involved is the first selection of the greatest power of  $a$  that is less than  $b$  and the smallest power of  $a$  that is greater than  $b$ . Thereafter, the powers are found systematically through bisection of intervals. The teacher’s method involves only selection and guessing and is less systematic. Both methods involve decisions about when to stop i.e. which answer to accept as an approximation. The textbook stops at the closest approximation with only one decimal place. The teacher stops at a different point for each example (in terms of the number of decimal places).

## Secondary data production P7 Lesson 2 EE2

### 1) Realisation of content

The intended topic of this event is the solving of exponential equations in which the two sides of the equation cannot be written as powers of the same base. The curriculum states that trial-and-error should be the method used to solve such equations, and this is the method introduced by the teacher in this event. In my primary data production I describe the elements of the encyclopaedia activated by this topic.

The teacher's procedure for trial-and-improvement depends on selection. As explained by Davis (2011a) selection is indicated by  $SEL(c, T) \rightarrow t$ , where  $c$  indicates a specific object-type and  $T$  is the particular set of objects from which a selection of the object,  $t$ , is to be made. In this example,  $c$  is a rational exponent between four and five while  $T$  is the set of all rational numbers between four and five. Selection in this particular example is not a function as the output is not unique for any given input. Also central to the procedure is what the teacher refers to as "playing around" – trying a succession of exponents (each based on the comparison of the previous selection with the desired answer, in this case twenty) in order to reach the 'closest' number to the desired answer.

The method given in the textbook for trial and improvement is more precise than the teacher's "playing around" as the only selection involved in the textbook method is the first selection of the greatest power of  $a$  that is less than  $b$  and the smallest power of  $a$  that is greater than  $b$ . Thereafter, the powers are found systematically through bisection of intervals. The teacher's method involves only selection and guessing and is less systematic. Both methods involve decisions about when to stop i.e. which answer to accept as an approximation. The textbook stops at the closest approximation with only one decimal place. The teacher stops at a different point for each example (in terms of the number of decimal places).

The teacher introduces the trial-and-improvement method but also shows the learners a procedure using logarithms, seemingly placing more emphasis on this procedure as being the 'shortcut'. In the log procedure the teacher performs existential shifts and operates on strings in order to carry out the manipulations of shifting and sundering and to render the log procedure as familiar to the learners. Thus the log procedure does not depend on selection, but does include the unfamiliar operations of sundering, concatenation and shifting of the exponent and "putting it in front" of the log (the latter is similar to Lima & Tall's (2010) description of procedural embodiment). As explained previously, although concatenation is a function and thus an operation (albeit an unfamiliar one from the point of view of the encyclopaedia), sundering is not a function and thus not an operation. Spatial shifting of symbols (as discussed by Gripper, 2011a) is also a pseudo-operation (i.e. it is not a function – there is not one unique output for each input). The ground on which the procedure rests is the definition of a logarithm, with its properties over the reals. But the closest the teacher comes to defining a logarithm is describing it as "a different way of explaining indices ... a different way of applying or using exponents". He thus mentions the link between the exponential and the logarithmic function, but does not explain it or explicitly draw on this during his procedure. He refers to the "log law number three" as "if you find the log of a value or an index, you take that index and you place it in front", suggesting a sundering of the index from its base and spatial shifting of character strings. He also refers to logs as a "word" which should be written on both sides of the equation– the notion of a logarithm as an operation is implicit.

The dependence of the trial-and-improvement procedure on selection and of the log procedure on sundering and shifting, neither of which are functions in this context, shows us that the intended topic is not realised and neither does the realised topic correspond with the mathematics encyclopaedia.

### 2) Regulation of the learner

Once the teacher has introduced both methods, he seems to favour the 'log method' as it yields a precise answer, but later he tells them that they do not have to use this method until grade 12. In the next lesson on

the same topic, he says that “in grade ten we’re not going to introduce the logs ... logs are a shortcut, you can’t in the examination give us a solution that’s log of ... in the exams when we give you this question we will say by means of trial and improvement ... or we’ll say give the answer to two decimal places”. But he does mention that “nothing stops you using your calculator ... you can use logs” – he seems to encourage the learners to use logs to check their solutions (“so you can use the log to guide you”). He also tells them that in the exams they will be told to solve by means of trial and improvement. “Because this is stated in our document, we get a document from the department, and it’s stated that we have to teach that particular method to you”. He seems to be saying that the only reason he teaches them the trial and improvement method is because it is stated in the curriculum document, but he prefers the “log method”. This could be because he knows that the log method is more likely to get the learners to the correct answer, it is almost a fail-safe method if the learners memorise the steps and carry them out correctly. On the other hand the trial and error method is more open to errors and involves more need for learners to make decisions (due to its dependence on selection). We see a number of examples where the learners select different answers as the closest approximation. This illustrates the difficulty in arriving at a “final” solution. The teacher does not make it clear at which point learners should accept their solution as the “closest” approximation. Thus the teacher seems to regulate the learners by encouraging them to use the log method (despite emphasising that they must show their trial and error method in the exams because “this is stated in our document”). Despite introducing logarithms, the teacher does not make any reference to any reference to the mathematical principles and properties underlying logarithms.

Another key feature of the regulation of the learners in this event is the emphasis on “playing around”, suggesting that the empirical is acting as a regulative resource in this event. The teacher says at one point: “You played around. That’s good. You see by means of investigation you can get closer and closer to the answer”. He uses the idea of “playing around” a lot in this event and encourages the learners to do so. Through this emphasis on the empirical, an exponential equation is constituted as something sensible which can be solved inductively by students – the intelligible nature of the mathematics underlying the topic is not explicit. Instead of using the empirical exploration of an exponential equation as a way of reinforcing the mathematical principles, the only representation of mathematical ideas in this procedure are sensible and empirical. It seems important to ask why it is that the teacher recruits the empirical in this event. The idea of ‘playing around’ seems to be motivated by a desire to engage learners and to make the procedure sound fun and interesting, possibly in an attempt to ensure that learners carry out this procedure which is clearly required by the National Curriculum Statement. The emphasis on “playing around” could be because the teacher does not expect learners to be able to engage with a systematic method for solving an exponential equation by trial-and-improvement (the interval bisection method described previously). The teacher is thus bending the mathematical content in the direction of the image of the learner.

Overall, in this lesson, one appeal is made to ease/efficiency, one to exams, four to empirical testing, two to iconic or spatial features of solutions and five to mathematical propositions, processes and rules. Thus appeals to extra-mathematical factors exceed those made to mathematical factors, and necessity is situated external to the field of mathematics. The regulation of the learner in this event is under the aspect of the Imaginary due to the way in which the teacher re-constitutes the two procedures, and presents them to the learners, in order to regulate the learners to produce the correct answers regardless of their understanding of logarithms or exponents. He is taking the learners into account in his re-constitution and transforming the mathematics in the direction of the image he holds of his learners.

## Summary

This event falls into quadrant IV. The regulation of the learner depends primarily on the Imaginary.

## Primary data production P7 Lesson 3 EE1

### 1) Generation of evaluative events

Table A5.11 Evaluative events School P7 Lesson 3

Time	Evaluative event/sub-event	Activity	Type
00:00 – 20:27	E <sub>1</sub>	Teacher does two examples of solving exponential equations using trial and improvement and checking using logs.	Expository
19:45 – 30:41	E <sub>2</sub>	Learners work on exercise, teacher does a few on the board after a while.	Exercise

### 2) Describing operational activity

In this lesson the teacher reiterates the principle behind equations - “The basic aim of any equation is to equate the left to the right, to have the same balance to both sides” (although he is explicitly drawing on the notion of equality, his explanation seems to suggest that in solving an equation the learner is the one doing the ‘equating’ and ‘balancing’, rather than the notion that an equation by definition involves equality).

Despite his introduction of logs in the previous lesson, he now says that “in grade ten we’re not going to introduce the logs ...logs are a shortcut, you can’t in the examination give us a solution that’s log of ... in the exams when we give you this question we will say by means of trial and improvement ... or we’ll say give the answer to two decimal places”. But he then says “as a method nothing stops you using your calculator ... you can use logs”. He seems to prefer logs but because of the curriculum statement he emphasizes trial and improvement.

When he starts solving the first example ( $5^x = 45$ ), he refers them back to their “list” of prime bases, in order to state that forty five lies between twenty five and one hundred and twenty five. Once again he uses prime bases as a key criterion in the procedure.

He discusses the trial and improvement method again - “... it’s a fancy word for saying you can play around with your calculator. You know, you all know cause you’re supposed to know, that five to the two gives you twenty five, five to the three gives you one twenty five”.

The learners suggest a few possible powers:

Handwritten student work showing trial and improvement for solving  $5^x = 45$ . The work lists several values of  $5^x$  and compares them to 45:

$$\begin{aligned} 5^{2,3} &= 40,5 \\ 5^{2,4} &= 47 \quad ? \\ 5^{2,366} &= 45,05 \\ 5^{2,37} &= 45,34 \\ 5^{2,365} &= 44,98 \\ 5^{2,366} &= 45,02 \end{aligned}$$

Figure A5.24 Solving  $5^x = 45$  using trial and improvement

He asks the learner who suggests two comma three six six why he chose that value, to which the learner replies that it gives forty five comma zero five.

They ‘play around’, first establishing that the number after the decimal place in the power is three: “can you see if you play around with this numbers, it is most certainly three” (pointing to the three in the decimal place immediately after the comma, in the tenths position). From there they try different values, quite randomly – there is no consistent method to their selection of powers. They settle on the bottom one in the figure above (45,02), and the teacher asks them “if you use more decimals will you get closer and closer to the answer?” They keep trying more decimal places to see which answers are closer to forty five. He does not really clarify where they should ‘stop’ i.e. how to decide which answer is most acceptable or closest to forty five.

In response to a learner’s question about the exams - “now in the exams, very good question, how do we mark a question like this? How far do you go? ... we give you about three marks for a question like this, three marks ... let’s assume you get to the exams ... let’s assume I give you two to the power of  $x$  and I give you ... give me a number here ...” Learners suggest eleven.

“first of all this value over here, right, um, you won’t be able, if we give you the value eleven, you won’t be able to break it down to two to the power of something, if it will be sixteen it will be two to the power four, so it’s easy. So it’s impossible for you to break it down to eleven.”

He says that in the exams they will be told to solve by means of trial and improvement - “because this is stated in our document, we get a document from the department, and it’s stated that we have to teach that particular method to you”. He also tells them that when a question says “up to two decimal places” they must use their calculator ...so you can use the log to guide you”

He works through the question saying: “now this is what you do. When you approach this question this is what we want to see”. His procedure is as follows:

- two to the power three gives you eight (no reason for choosing this power)
- two to the power four gives you sixteen (“the one after that”)
- eleven fits between eight and sixteen.
- “that means if I need to find the value for  $x$  it will be three comma something. Now you can play around” (to find the ‘something’) ... “Play around and see what you get”

He describes ‘playing around as’: “Try one answer, see how close it is to eleven. Try again until you get closer. Once you’re happy with the first decimal place, try others for the second etc.” – he gives no indication of where to start and when to stop, nor of why they should choose certain values. He then uses logs to check their answer.

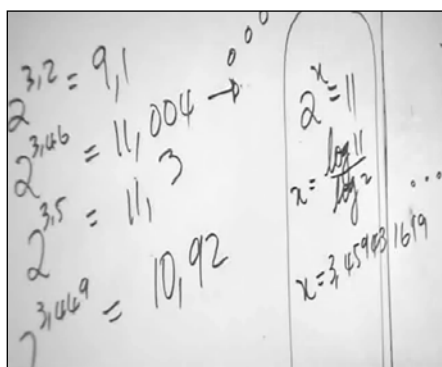


Figure A5.25 Using trial and improvement and logarithms to solve  $2^x = 11$

Once they have found the answer he asks: “now what’s the meaning of that answer ... what does the answer mean to you? ... how would you interpret this answer ... how would you explain this answer to your friend?” – a learner says that the answer is “the number that you use”. He agrees without elaborating or mentioning exponents, but checks their answer, they get ten comma nine nine nine ... and he says “you’re supposed to get eleven”, writing  $2^{3,459431619} = 11$  on the board.

He summarises this topic at the end of the event by saying that there are two ways of solving exponential equations, but that they must do the trial and improvement method for the exams, but can check using the log method.

Trial and improvement – “when you play around with the calculator until you get to a value closer to ...”

Logs – “if you get stuck. But you must write the answer up to two decimal places if you use logs.”

### 3) Activating the mathematics encyclopaedia

See P7 Lesson 2 EE2

## Secondary data production P7 Lesson 3 EE1

### 1) Realisation of content

The teacher uses the same technique as in the previous event – he starts with the trial and error method, focusing on “playing around” and trying different numbers, and then using logs to check his answers. He does two examples – the first takes eight and a half minutes, the second approximately eleven minutes. Using logs straight away would take only one or two minutes, but his process of “playing around” takes much longer. Because of the way in which his trial and error procedure depends on selection, and the way in which he does not equip learners to know when they should stop i.e. when they have reached the “final answer”, as well as the way in which the log procedure does not recruit appropriate elements from the mathematics encyclopaedia but involves the physical shifting of objects, the intended topic is not realised in this event and neither does the realised topic correspond with the mathematics encyclopaedia.

### 2) Regulation of the learner

Once again in this event the teacher does not clarify where they should ‘stop’ i.e. how to decide which answer is most acceptable or closest. In response to a learner’s question about the exams –

“Now. In the exams, very good question. How we mark a question like this. Do you know how far do you go? Do you go up to one two three four five six seven eight nine trials? What do you do in the exams? We give you about three marks for a question like this. Three marks. Okay? Let’s try something quickly. Erm. Let’s assume you get into the exams I’ll mark this for you. Right? If I sit down ... and I set up a question ...let’s assume I give you two to the power of  $x$  and I give you ... let’s see. Give me a number here fifteen ten twelve?”

They settle on eleven as the number. He tells them that “We’ll say by means of trial and improvement. Okay? We’ll set the question like that. Solve for  $x$  by means of trial and improvement. That method. Okay. Because this is stated in our document. We get a document from the department and they say we have to teach that specific method to you. You with me? Right. So it’s clearly stated that you have to use this method. ...Trial and improvement method. Erm. Up to two decimal places [writes /up to 2 dec. places/]. Now if they say this two decimal places what in fact are they telling you? What are they telling you? Are they saying that he carries a calculator, you can’t use a calculator, you have to use a manual? What are they telling you?” The learners respond that they can use a calculator.

“You can. Right. If you see a question and this is accompanying the question. Up to two decimal places, they actually advising you to use a calculator. Okay? Right. They’re telling you, you may use a calculator. And if you understand the calculator correctly? Right? Remember I gave you a shortcut. The shortcut is? Shortcut is

the log. Can you see it. Right. Just do the log; you get a more accurate answer. Okay. So you can use the log to guide you. Are you with me? Right. As a guidance. Now this is what you do. When you approach this question, this is what we want to see. Trial and Improvement”.

The teacher is conveying mixed messages – they must use logs to get a more accurate answer, but only to “guide” them. But the teacher/examiners “want to see” the trial and improvement method – once again he emphasises his preference for the log method, but the need to show the trial and improvement method (which is pointless if one has used the log method – why go through the process of “trial and improvement” when you already know the exact answer? It is merely ‘for show’). What message does this give the learners?

We see the result of this when he starts working through the trial and improvement method with the learners for this question and one learner gives the exact answer. The teacher responds: “You get eleven comma zero zero four. That’s close. Did you make a guess? Guesstimate? Estimate?” To which the learner says “no”, and other learners say “he took a log”. This learner has gone straight to the log, which renders the trial and improvement method artificial and superfluous. Despite this the teacher continues with the trial and improvement for a while longer, trying different options. He then moves on to the log method again – “Okay. .. Now, listen carefully. .. This is a shortcut. Right. On your calculator, if you have two to the power of  $x$  equal to eleven, your  $x$  will be equal to. Right. Your  $x$  will be equal to the following. Listen carefully. All you say is, take log eleven over log two. A quick answer. Just try that for me. Log eleven divided by log two. Just check that answer there”.

After they have found the answer using logs, he summarises: “We have the trial and improvement method so there’s two ways of doing this. Right. Where you can play around with the numbers. Place the banner between two specific knowns ... so you have to play around with the calculator until you get to a band closer to eleven. Right. If you get stuck it doesn’t mean that you can chuck the mark away you can leave the mark. .. Right. If you get stuck, there’s a skill that you can use. You can use the log so they give you two to the power of  $x$  equal to eleven, right,  $x$  will be log eleven over log two.”

Here he describes logs as a “skill” that you can you – no mention of the mathematical definition or properties of a logarithm.

In this lesson there are three appeals to ease or efficiency, three to the examinations, nine to empirical testing, one to an iconic feature of a solution, two to procedural features of solutions and only three to mathematical propositions, definitions, rules or processes. Necessity is thus situated external to the field of mathematics and in the need for empirical testing, as well as in iconic and procedural features of the two methods. Thus regulation is primarily under the aspect of the Imaginary.

### Summary

This event falls into quadrant IV. The regulation of the learner depends primarily on the Imaginary

### Primary data production P7 Lesson 3 EE2

#### 1) Generation of evaluative events

Table A5.12 School P7 Lesson 3 EE2

Time	Evaluative event/sub-event	Activity	Type
19:45 – 30:41	E <sub>2</sub>	Learners work on exercise, teacher does a few on the board after a while.	Exercise

## 2) Describing operational activity

The exercise which learners are working on in this event:

Solve for  $x$  to 2 dec places:

1)  $3^x = 21$

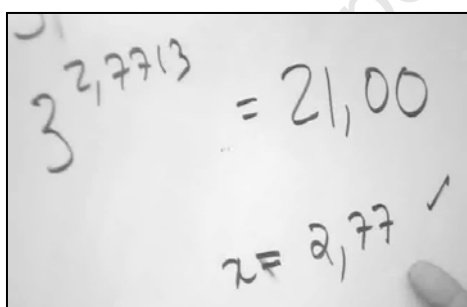
2)  $2^x = 35$

3)  $5^x = 70$

4)  $2^{3x} = 22$

After three minutes he asks if they can mark the exercise. The learners aren't ready so they keep going for a few more minutes. As they work, the teacher points at number three and says "to be very close ... I'll advise you to try to get to the seventy comma nought something or zero zero even ...". After another minute he says "ok can we do it on the board" and starts with number one.

The learners give him the first power of three as "3,7712" and some of them call out more decimal places – they have clearly used logs to get this answer, despite the instruction to use two decimal places and trial and improvement. They get 20,999. He asks if there's someone with a different number. He suggests just using 2,77 – they get 20,97. A learner suggests adding 13 – 2,7713, which a learner calls out as "21,00". The teacher says "that's close hey? I will accept that. But now listen carefully. If they say you must give the answer to two decimal places ... the answer will now be what (writes  $x$  equals) two comma seven" ... a learner says "eight" ... teacher says "no seven one three" Learners – "seven".



The image shows a close-up of a whiteboard with handwritten mathematical work. At the top, the equation  $3^{2,7713} = 21,00$  is written. Below it, the variable  $x$  is set equal to 2,77, with a checkmark next to it, indicating a solution has been found.

**Figure A5.26** Writing the solution to  $3^x = 21$

This is the first time he actually writes the answer as  $x$  equals something, instead of just circling the 'closest' option in their list of trials.

He then goes through numbers two and three, where the learners call out their closest answers. He asks for a volunteer for number four, but the lesson ends.

## 3) Activating mathematics encyclopaedia

See P7 Lesson 2 EE2

## Secondary data production P7 Lesson 3 EE2

### 1) Realisation of content

Learners work on an exercise on solving exponential equations (trial and error/log methods) and they mark the exercise as a class. This method used by the teacher in discussing these questions is the same as the previous two lessons, and the intended topic is thus not realised nor does the realised topic correspond with the encyclopaedia.

### 2) Regulation of the learner

In this event we see an emphasis by the teacher on what “they” want (i.e. the examiners) – “if **they** say you must give the answer to two decimal places”; “but to present your final answer **they** will say that you have to give your answer to two decimal places”; “For **exam purposes**. Okay” – the examinations are once again used to regulate the learners and to motivate them to follow the teacher’s procedure exactly.

Interchanging between the two methods occurs again in this event – the teacher insists on them “playing around” first, but it seems that once again many learners jump straight to the log method and get the exact answer. The teacher continues to “play around” despite that though. Each question takes:

23:10 – 27:00 (4 mins)

27:00 – 28:16 (1 and a bit min)

28:16 – 29:00 (45 seconds)

They are getting quicker at doing the questions by “trial and improvement” although it seems that this may be because learners jump straight to the log method. The teacher keeps emphasising that they must write their answer with two decimal places, seemingly to show that they have used the trial and improvement method.

In this event one appeal is made to the exams, one to empirical testing and one to an iconic feature of a solution – no appeals are made to mathematical propositions or definitions. Once again necessity is situated outside of mathematics and appeals to the method preferred by the curriculum and the examiners, as well as in the need to “play around” but use logs to check the answer. The regulation of the learner is under the aspect of the Imaginary.

### **Summary**

This event falls into quadrant IV. The regulation of the learner depends primarily on the Imaginary.

## Appendix 6: 2008 Grade 12 Mathematics results for the five schools

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**Table A6.1** 2008 Grade 12 Mathematics results for the five schools

School	# Candidates	# Fails < 30%	# Passes at 30%-39%	# Passes at 40%-49%	# Passes at ≥ 50%
P1	66	16 (24,2%)	50 (75,8%)	28 (30,3%)	18 (27,3%)
P2	105	59 (56,2%)	46 (34,8%)	20 (19,1%)	10 (09,5%)
P3	76	11 (14,5%)	65 (85,5%)	47 (61,8%)	34 (44,7%)
P6	126	54 (42,9%)	72 (57,1%)	38 (30,2%)	29 (23,0%)
P7	57	9 (15,8%)	48 (84,2%)	27 (47,4%)	20 (35,1%)
Total	430	149 (34,7%)	281(65,3%)	160 (37,2%)	111 (25,8%)

University of Cape Town

## Appendix 7: Time use in the fifteen lessons

The following table summarises time use in the fifteen lessons:

**Table A7.1** Time usage in the fifteen lessons

School	Lesson	Lesson length	Exposition of mathematical principles	Exposition by worked examples	Marking of worked examples	Working through exercises	Activity unrelated to topic
P1	1	40:49	1:00	14:27	1:47	17:16	6:19
	2	37:02	00:20		21:40	8:02	7:00
	3	25:33	00:10	16:20		5:25	3:38
P2	1	33:24		5:19	13:09	6:56	8:00
	2	52:25			32:25		20:00
	3	39:35	1:00	3:20	22:00	5:15	8:00
P3	1	37:28	00:30	17:30	14:22	5:28	
	2	36:04			32:41	3:23	
	3	21:43	00:30	18:57		2:16	
P6	1	48:00	00:10	25:30		22:20	
	2	39:00	2:43		36:17		
	3	47:47			13:56	33:51	
P7	1	33:35	2:00	15:05	11:30	05:00	1:00
	2	36:50	2:20	13:00	21:30		
	3	30:41	00:10	20:17	7:54	2:20	
Total(%)			2%	27%	41%	21%	9%