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Universe phenomenology as understood from gravitational theories with non-vanishing torsion: cosmology and black holes

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Abstract

In this thesis, we study gravitational theories for which the natural choice of an affine connection is metric compatible while not being symmetric. More specifically, we study gravitational theories constructed on the Riemann-Cartan and Weitzenböck space-times.

Firstly, we outline the mathematical notions needed to construct a definition of space-time. Following this, we introduce the space-time definitions to be made use of throughout this thesis. We then discuss the notions of extremal and auto-parallel curves on the Riemann-Cartan space-time. It is noted that test particles follow extremal curves which are auto-parallel curves of the Levi-Civita connection. Therefore, one must turn to the standard, torsion-free Raychaudhuri equation when studying the focusing conditions that arise in theories constructed on the Riemann-Cartan or Weitzenböck space-times.

Once we have introduced the definitions of the relevant space-times, we move on to review some of the gravitational theories that involve non-vanishing torsion. We first review the Einstein-Cartan theory and two of its modifications. We then review the so-called $f(T)$ theories of gravity before discussing the focusing conditions that arise in this context. By making use of the $f(T)$ field equations together with the torsion-free Raychaudhuri equation, we derive for the first time the $f(T)$ focusing conditions for a one-parameter dependent congruence of timelike auto-parallel curves of the Levi-Civita connection. We then study these focusing conditions for three bi-parametric cosmological models.

Finally, we turn our attention back to the Einstein-Cartan theory and derive the Arnowitt-Deser-Misner formulation of this theory. By making use of this formulation, we derive for the first time the Generalised-Baumgarte-Shapiro-Shibata-Nakamura formulation of the Einstein-Cartan theory. We then consider the case of a vacuum in spherical symmetry and construct a 1-dimensional code to evolve the system numerically. We leave the inclusion of torsion into this code as the subject for future work.

Contents

Declaration

I, Ulrich Karoo Beckering Vinckers, know the meaning of plagiarism and declare that all of the work in the document, save for that which is properly acknowledged, is my own.

It is noted that the results presented in Chapter 5 have given rise to the following article which has been published in the Journal of Cosmology and Astroparticle Physics (JCAP) on 9 December 2020:

U. K. Beckering Vinckers, Á. de la Cruz-Dombriz and F. J. Maldonado Torralba, *Focusing conditions for extended teleparallel gravity theories*, JCAP **12** (2020) 020 [arXiv:2009.04353 [gr-qc]].

It is also noted that the Mathematica, Python and C++ codes constructed by the author and used to produce some of the results in this thesis can be found at `https://www.dropbox.com/sh/cfybc2cysixchlz/AACjKq0NHComUPoXcZz90DH7a?dl=0`.

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To see a World in a Grain of Sand
And a Heaven in a Wild Flower,
Hold Infinity in the palm of your hand
And Eternity in an hour.

William Blake,
Auguries of Innocence

Abbreviations and symbols

Abbreviations

ADM Arnowitt-Deser-Misner

BH Black Hole

BSSN Baumgarte-Shapiro-Shibata-Nakamura

CGR Coincident General Relativity

CSEC Chern-Simons Einstein-Cartan

CSGR Chern-Simons General-Relativity

EC Einstein-Cartan

ECS Einstein-Cartan-Saa

EH Einstein-Hilbert

GBSSN Generalised-Baumgarte-Shapiro-Shibata-Nakamura

GR General Relativity

MCP Minimal Coupling Procedure

PIRK Partially Implicit Runge-Kutta

PDF Probability Density Function

RK Runge-Kutta

TEGR Teleparallel Equivalent of General Relativity

Symbols

$g_{\mu\nu}$ Metric tensor

D Levi-Civita connection

∇ Cartan connection

$\Gamma^{\alpha}{}_{\mu\nu}$	Levi-Civita connection coefficients
$\tilde{\Gamma}^{\alpha}{}_{\mu\nu}$	Cartan connection coefficients
U_4	Riemann-Cartan space-time
V_4	Riemannian space-time
W_4	Weitzenböck space-time
$T^{\alpha}{}_{\mu\nu}$	Torsion tensor
$K^{\alpha}{}_{\mu\nu}$	Contorsion tensor
$W^{\alpha}{}_{\mu\nu}$	Weitzenböck contorsion tensor
$S_{\mu\nu}{}^{\rho}$	Superpotential
$R_{\mu\nu\rho}{}^{\sigma}$	Torsion-free Riemann tensor
$\tilde{R}_{\mu\nu\rho}{}^{\sigma}$	Non-null torsion Riemann tensor
$\tau_{\mu\nu}$	Energy-momentum tensor
$s_{\mu\nu}{}^{\lambda}$	Spin-current tensor
$\mathcal{T}_{\mu\nu}$	Effective energy-momentum tensor
θ	Torsion-free expansion
$\sigma_{\mu\nu}$	Torsion-free shear
$\omega_{\mu\nu}$	Torsion-free twist
$\tilde{\theta}$	Non-null torsion expansion
$\tilde{\sigma}_{\mu\nu}$	non-null torsion shear
$\tilde{\omega}_{\mu\nu}$	Non-null torsion twist
$\overset{\circ}{\theta}$	Geometric expansion
$\overset{\circ}{\sigma}_{\mu\nu}$	Geometric shear
$\overset{\circ}{\omega}_{\mu\nu}$	Geometric twist

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Chapter 1

Introduction

It is well known that General Relativity (GR), which provides us with a description of space, time and gravity, has been usually formulated with symmetric connections only [5]. More specifically, GR is formulated on the so-called Riemannian space-time whose natural choice of an affine connection is the unique, torsion-free and metric compatible connection known as the Levi-Civita connection [6]. In this thesis, we are interested in considering gravitational theories for which the underlying space-times are defined using a natural choice of an affine connection, referred to as the Cartan connection, that is metric compatible while not being symmetric. That is, we consider gravitational theories that include a non-vanishing torsion tensor. The specific space-times that we consider in this thesis are the so-called Riemann-Cartan [7] and Weitzenböck [8] space-times. Below, we briefly discuss these space-times as well as their relevance in the scope of this thesis.

1.1 Gravitation and Riemann-Cartan space-time

The generalisation of the Riemannian space-time to one that includes torsion is known as the Riemann-Cartan space-time [7]. In Chapter 2 of this thesis, we first outline the necessary mathematical notions needed to construct this Riemann-Cartan space-time which we define formally in Chapter 3 following [9]. An example of a gravitational theory that is constructed on the Riemann-Cartan space-time is the so-called Einstein-Cartan (EC) theory which was originally proposed by Élie Cartan in 1922 [10]. In this theory, the torsion tensor is geometrised by the spin-current tensor [7]. As a result of this, in the absence of any fermionic matter, the EC theory of gravity reduces to GR. Therefore, many phenomena that hold in GR, such as Birkhoff's theorem, also hold true in the EC theory. In Chapter 4, we discuss the EC theory as well as two of its modifications. One such modification is the Einstein-Cartan-Saa (ECS) theory of gravity that was first put forward by Alberto Saa in [11] and studied further in [12]. The motivation for such a theory comes from making the following note, given in the next paragraph, on the choice of the volume element used when constructing the gravitational action of the EC theory.

The volume element used in constructing the action of the EC theory is the same as that of GR and does not yield a pure boundary term upon the integration of the Cartan divergence of an arbitrary vector field. In order to obtain a pure boundary term upon such an integration, Saa proposed an alternative volume element which resulted in the action for the ECS theory of gravity. This will be the first modification of the EC theory that we will discuss in this thesis.

The second modification of the EC theory that we discuss is a Chern-Simons modification. Such a modification is carried out here by following [13, 14] in which a Chern-Simons modification to GR is constructed. In the case of spherical symmetry, the Chern-Simons modification to General Relativty (CSGR) vanishes and one is left with the standard GR theory. Therefore spherically symmetric solutions, such as the Schwarzschild solution, are solutions of this modification to GR. However, as mentioned in [14], the Kerr metric violates the so-called Pontryagin constraint and is therefore not a solution of CSGR. This presents motivation for studying the Chern-Simons modification of the Einstein-Cartan (CSEC) theory and the status of the Kerr solution in this theory has been studied in [15].

While we discuss these two aforementioned modifications of the EC theory, we will primarily be interested in the EC theory itself. We make use of the so-called Cartan formalism in reviewing this theory by following what is done in [9, 16–18]. In addition to this, we also discuss the EC theory in the holonomic frame. The latter formulation will be of particular interest when constructing a set of evolution equations to be made use of in carrying out numerical simulations of the theory. To this end, we derive the Gauss, Codazzi and Ricci equations in Riemann-Cartan space-time following what was done in [19, 20] by Baumgarte and Shapiro for the case of Riemannian space-time. In these aforesaid references, the Gauss, Codazzi and Ricci equations of Riemannian space-time are made use of in constructing the Arnowitt-Deser-Misner (ADM) formulation of GR. In Chapter 6 of this thesis, we make use of the Gauss, Codazzi and Ricci equations of Riemann-Cartan space-time in order to derive the ADM formulation of the EC theory. We note that in [21], an ADM-like formulation of the Teleparallel Equivalent of General Relativty (TEGR) is given while an ADM formulation of the so-called Coincident General Relativty (CGR) has been given in [22].

As was shown in [23, 24], one can construct a modification of the ADM formulation based on conformal transformations in the hypersurfaces of Riemannian space-time. Such a formulation is known as the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation of GR. In the construction of this formalism, one makes the assumption that the determinant of the conformal spatial metric is unity. As pointed out by Brown in [25], this presents an issue when working in spherical coordinates as the determinant of the spatial metric in spherical coordinates is not unity. Brown therefore proposed the so-called Generalised-Baumgarte-Shapiro-Shibata-Nakamura (GBSSN) formulation of GR in which one does not make this assumption. By making use of this formulation, Brown carried out the numerical evolution of the system describing a space-time puncture. This was done by assuming a vacuum and taking the metric to be spherically symmetric. The initial data for the metric was taken to be the Schwarzschild solution in isotropic coordinates. Similar numerical simulations using the GBSSN formulation have been carried out in [2, 26, 27]. It is worth noting that the standard BSSN formulation has been used successfully for three-dimensional punture evolution in [28, 29].

Once we have constructed the ADM formulation of the EC theory, we will modify this in order to obtain, for the first time, a GBSSN formulation of the EC theory. In our final statement of the GBSSN evolution equations, we include the torsion tensor by making use of the so-called effective energy-momentum tensor. In order to determine the evolution equation of the torsion tensor, one can therefore decompose the effective energy-momentum tensor and determine the relevant evolution equations for its components. By following the procedure outlined in [30] for evolving the energy-momentum tensor, we apply this method to the effective energy-momentum tensor and obtain the evolution equations for the energy-matter content.

Upon the completion of constructing the GBSSN formulation of the EC theory, we consider the case of spherical symmetry as well as a vanishing energy-matter content. Since the torsion tensor vanishes when the spin-current tensor vanishes in the EC theory, the GBSSN evolution equations will coincide with what is given in [25, 26, 31] for the case of GR. By making use of the evolution equations for this case, we construct a 1-dimensional (1D) code to numerically evolve the system. The modification of this 1D code to include a non-vanishing spin-current tensor is left as the subject for future work.

1.2 Gravitation and Weitzenböck space-time

In this thesis, the most general space-time considered is the Riemann-Cartan space-time. By imposing that the torsion tensor vanishes, the Cartan connection reduces to the Levi-Civita connection which is unique. While one can consider the Riemann-Cartan connection in the case of vanishing torsion, one can also consider the case of vanishing curvature. In such a case, there is no unique choice for the Cartan connection [16]. However, the simplest choice of an affine connection can be found by taking the trivial solution to the second Cartan structure equation. This is referred to as the Weitzenböck gauge with the associated choice of an affine connection being the Weitzenböck connection [8, 32]. We refer to the space-time obtained by taking the natural choice of an affine connection to be the Weitzenböck connection as the Weitzenböck space-time. It is upon this space-time that the so-called

Teleparallel Equivalent of General Relativity (TEGR) is constructed. The reason that this theory is equivalent to GR is that, apart from a surface term that has no contribution, the TEGR action is exactly the Einstein-Hilbert action [33]. The TEGR theory of gravity was first introduced by Albert Einstein himself and the interested reader is directed to the work [34] which contains the first English translations of Einstein's original papers on the subject. The formulation of TEGR through the construction of a gravitational Lagrangian was first done in [35] and later in [36].

In the Lagrangian formulation of TEGR, the Lagrangian density is the so-called torsion scalar T. Theories that are constructed using a Lagrangian density that is a general function $f(T)$ of the torsion scalar are referred to as $f(T)$ theories of gravity. While TEGR is a Lorentz invariant theory, these theories are not necessarily Lorentz invariant [37]. The fact that these theories display this property has led to recent interest [37–40]. In particular, the validity of Birkhoff's Theorem in $f(T)$ theories is discussed in [37, 39, 40]. It was argued in [39] that the first inference of Birkhoff's theorem, which states that a spherically symmetric metric for which a vacuum is considered is static, holds in $f(T)$ theories of gravity. The authors of [39] made this argument by starting with a spherically symmetric metric and then constructing a diagonal tetrad. As pointed out in [37, 40], the ansatz for the tetrad used in [39] is overconstrained and requires that the $f(T)$ theory be TEGR. Therefore, the argument given in [39] is limited to the case of TEGR. As per the terminology used in [37], the diagonal tetrad used in [39] is an example of a "bad tetrad". Further studies conducted in the context of $f(T)$ theories of gravity include the study of junction conditions in these theories [38] as well as constraining specific $f(T)$ theories through the use of Supernovae Type Ia and Baryon Acoustic Oscillation data [41]. The review article [42] discusses a model-independent reconstruction of $f(T)$ theories through the use of the so-called cosmographic method. In Chapter 4, we shall discuss in detail the formulation of $f(T)$ theories following the discussion of the EC theory and its modifications.

We note that for theories that can be written as a function of the torsion scalar, T, a scalar field and its kinetic term, a study of these theories, as well as a comparison with Hubble data, can be found in [43]. Theories resulting from the inclusion of a Gauss-Bonnet term in the $f(T)$ Lagrangian have also been studied [44–49]. In addition, a topic of interest for the case of TEGR is the status of the theory as a gauge theory of the translation group [50–52]. It is also worth mentioning that recently, the set of solutions of $f(T)$ gravity in the Minkowski metric has been studied [53], finding out that a new mode is present at fourth-order perturbation, signaling a strong coupling problem.

1.3 Auto-parallel curves and focusing conditions

In this thesis, we are interested in studying the focusing conditions that arise in the context of the $f(T)$ theories of gravity. More specifically, we wish to consider a congruence of curves and study how a cross-sectional area about this congruence expands or contracts. One can then describe the rate of expansion through the use of the Raychaudhuri equation. In Riemann-Cartan space-time, however, there exist multiple Raychaudhuri equations of which we shall consider three. When one considers the convergence or divergence of neighbouring curves in torsion-free theories of gravity, such as GR, one often turns to the Raychaudhuri equation which was derived in [54]. This Raychaudhuri equation, referred to here as the *torsion-free Raychaudhuri equation*, is known to play a significant role in the proofs of the singularity theorems [55, 56]. This is the first Raychaudhuri equation that we will consider. Due to the fact that the Levi-Civita connection can be considered as an additional structure on Riemann-Cartan space-time [57], the torsion-free Raychaudhuri equation is still valid in this space-time. It is, therefore, possible to still consider the convergence or divergence of curves generated by the Levi-Civita connection in general Riemann-Cartan space-times.

Given the fact that the Levi-Civita connection is not the natural choice of an affine connection in the Riemann-Cartan space-time, the torsion-free Raychaudhuri equation cannot, in general, be used to describe the convergence or divergence of curves generated by the Cartan connection. Consequently, in order to study a congruence of curves generated by the Cartan connection, we aim at determining a Raychaudhuri equation describing the convergence and divergence of these curves. One such equation can be found in [58, 59] and we refer to this result as the *non-null torsion Raychaudhuri equation*. This equation is derived by decomposing the tensor field obtained by acting the Cartan covariant derivative on the tangent vector field of a curve contained in a congruence. While the non-null torsion Raychaudhuri equation is valid in Riemann-Cartan space-time, the associated kinematic quantities do not necessarily describe the physical expansion, shear and twist. In [60,61], a Raychaudhuri equation involving torsion and whose kinematic quantities give the physical expansion, shear and twist is derived. In this thesis, we refer to this result as the *geometric Raychaudhuri equation*. In [62], the geometric Raychaudhuri equation is made use of in studying singularity theorems in affine theories of gravity with torsion. In Chapter 3, we discuss the Raychaudhuri equations mentioned here in greater detail. We note that the more general Raychaudhuri equation for the case of a space-time whose natural choice of an affine connection is not metric compatible in addition to having a non-vanishing torsion tensor is derived in [63].

Once we have derived the non-null torsion and geometric Raychaudhuri equations, we will move on to discuss the implication of such key equations in the context of $f(T)$ theories of gravity. In Chapter 5 of this thesis, we are concerned with studying the convergence or divergence of paths followed by test particles. Since we take test particles to follow extremal curves, which are auto-parallel curves of the Levi-Civita connection, we make use of the torsion-free Raychaudhuri equation when studying the focusing conditions. The $f(T)$ focusing conditions for the case where fundamental congruences are considered have been derived before in [64]. In Chapter 5 of this thesis, we derive for the first time the focusing conditions given a one-parameter dependent congruence of timelike auto-parallel curves of the Levi-Civita connection. Following this, we study these focusing conditions for three paradigmatic $f(T)$ cosmological models. These results have been reported in the article [65].

Chapter 2

Mathematical preliminaries

2.1 Introduction

In this chapter we briefly discuss the mathematical preliminaries needed for the construction of space-times with non-vanishing torsion. We begin by first introducing the definition of a manifold which is used throughout this thesis. This is done in Section 2.2. Following this, in Section 2.3, we define the notion of a tangent space as well as that of a tensor. This is done by following closely the discussions given in [55] by Hawking and Ellis and [5] by Wald. In Section 2.4, we then discuss the notion of orientability of a manifold as well as how integration on a manifold is carried out. We conclude this chapter with a brief dicussion on fibre bundles and principle $\mathcal{G}$-bundles in Section 2.5. Although the discussions given in this chapter are brief, the reader interested in a more in depth study of differential geometry is directed to [6, 57, 66, 67]. For a discussion on the application of differential geometry in the study of physics, the interested reader is directed to [16, 68].

2.2 Definition of a manifold

In this section we introduce the definition of a manifold that is used in this thesis. Although we are predominantly interested in studying 4-dimensional space-times with non-vanishing torsion in this thesis, we will initially discuss the notion of an n-dimensional manifold. Once the notion of an n-dimensional manifold is formalised, we will move on to consider the specific case of a 4-dimensional manifold and we will make use of this notion in order to construct space-times with non-vanishing torsion. It is worth noting that only a brief discussion of the mathematical preliminaries needed to construct a space-time is given here. In particular, many of the statements given here are mentioned without proof and the interested reader is directed to the references mentioned herein for more in-depth discussions on the various concepts that appear in the study of differential geometry.

As mentioned above, the purpose of this section is to provide the reader with the definition of a manifold which is to be used throughout this thesis. Through such a definition, we will be able to construct various metric compatible space-times such as the Riemannian space-time, which has non-vanishing curvature, the Weitzenböck space-time, which has vanishing curvature and non-vanishing torison, as well the Riemann-Cartan space-time which has both non-vanishing curvature and non-vanishing torsion [9].

In what follows, we define a homeomorphism to be a bijective mapping that has a continuous inverse [5]. Intuitively, one can interpret a manifold as being constructed by "pasting" together sets that are homeomorphic to an open subset of a Euclidean space [5, 6]. That is, given an arbitrary point in a manifold, there exists a neighbourhood of that point that is homeomorphic to an open subset of a Euclidean space. To present this notion more formally, we first consider the set

$$\mathbb{R}^n := \left\{ \boldsymbol{x} = \left(x^1, \dots, x^n\right) | x^k \in \mathbb{R} \;\; \forall \, k \in \{1, \dots, n\} \right\} \ , \tag{2.1}$$

where $\mathbb{R}$ is the set of all real numbers [6]. Following what is done in [6], we note the following linear structure on $\mathbb{R}^n$ described by the expressions

$$(\boldsymbol{x}+\boldsymbol{y})^k = x^k + y^k \ , \tag{2.2}$$

and

$$(\alpha\boldsymbol{x})^k = \alpha x^k \ , \tag{2.3}$$

where $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ is a scalar. Equation (2.2) defines addition between elements of $\mathbb{R}^n$ whereas equation (2.3) defines scalar multiplication [6]. It is easily verifiable that the linear structure described by equations (2.2) and (2.3) implies that $\mathbb{R}^n$ is an n-dimensional vector space over the field of real numbers, $\mathbb{R}$ [6]. In addition to this aforementioned linear structure, it is possible to study a standard topological structure on $\mathbb{R}^n$ (see [6,66] and Appendix A of [5]). One can define the function

$$d(\boldsymbol{x},\boldsymbol{y}) := \sqrt{\sum_{k=1}^{n}\left(y^k - x^k\right)^2} \ , \tag{2.4}$$

where $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}$ [6,66]. It is not difficult to show that, from the definition given in equation (2.4), the function $d(\boldsymbol{x},\boldsymbol{y})$ satisfies the following properties [6]:

1. $d(\boldsymbol{x},\boldsymbol{y})$ is positive semi-definite with
$$d(\boldsymbol{x},\boldsymbol{y}) \geq 0 \ , \tag{2.5}$$
for all vectors $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}$;
2. the function is symmetric with
$$d(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{y},\boldsymbol{x}) \ ; \tag{2.6}$$
3. the triangle inequality described by
$$d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z}) \geq d(\boldsymbol{x},\boldsymbol{z}) \ , \tag{2.7}$$
is satisfied for any vectors $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^n$.

Since the above properties are satisfied by the function $d(\boldsymbol{x},\boldsymbol{y})$, it follows that it is a metric on $\mathbb{R}^n$ and we refer to this metric as the *usual metric* on $\mathbb{R}^n$ [6, 66]. In order to see how the standard topological structure on $\mathbb{R}^n$ can be constructed, we first require the definition of a topological space. Following [69] and Appendix A of [5], we have the following definition of a topological space.

Definition 2.2.1. A *topological space* (X,τ) is a set X together with a collection, τ, of subsets of X such that:

1. the set X and the empty set $\varnothing$ are in the collection τ;
2. the union of any elements in τ is in τ;
3. the intersection of any finite number of elements in τ is in τ.

The collection, τ, of subsets of the set X is referred to as a *topology* on X and the subsets that form part of this collection are referred to as *open sets* in the topological space (X,τ).

Where no confusion arises, we will often refer to X simply as the topological space. By following what is done in Appendix A of [5], we wish to introduce a few more definitions. In the above definition, it is stated what is meant by an open set. We note that a subset, say A, of X is said to be a *closed set* if its complement, defined as $X - A := \{x \in X | x \notin A\}$, is an open set [5]. As is well-known, it is possible for a set to be both open as well as closed. Given a set X, one can construct a topological space by simply taking the topology to be the set containing all subsets of X. This is referred to as the *discrete topology*. Given the set X, another simple construction of a topological space is to take the topology to be the so-called *indiscrete topology* which only contains the set X itself as well as the empty set $\varnothing$. Furthermore, if the only elements of the topology that are both open and closed are the set X and the empty set $\varnothing$ then the topological space is said to be *connected*. Consider now the set $A \subset X$ in the topological space (X, τ). The *closure* of the set A, which is denoted as $\bar{A}$, is defined as the intersection of all the closed sets that contain A. On the other hand, the union of all the open sets that contain A is referred to as the *interior* of the set A. One can now define the *boundary* of the set A, which is often denoted as $\dot{A}$, as being the set of all points that are contained in the closure of A but not in the interior of A. Lastly, we turn our attention to the definition of a compact set. An *open cover* of the set $A \subset X$ is defined as being a collection of open sets that is such that A is contained in the union of these sets. If every open cover of the set $A \subset X$ has a finite subcover, then it is said that the set A is *compact*.

Let us now give the so-called natural topological structure of $\mathbb{R}^n$. By making use of the usual metric defined in equation (2.4), we can define the open ball, $B_r(\boldsymbol{x})$, with centre $\boldsymbol{x} \in \mathbb{R}^n$ and radius r as follows:

$$B_r(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}^n | d\left(\boldsymbol{x}, \boldsymbol{y}\right) < r,\ \boldsymbol{x} \in \mathbb{R}^n\} \ , \tag{2.8}$$

where $r > 0$ is a real number [6, 66]. The natural topological structure of $\mathbb{R}^n$ is then constructed by taking the open subsets of $\mathbb{R}^n$, i.e., the sets contained in the topology, to be unions of open balls as defined above through equation (2.8) [5, 6].

Consider a function f defined on an open subset U of $\mathbb{R}^n$. We say that $f \in C^k\left(U\right)$ if all the j-th partial derivatives of f both exist and are continuous for $j \leq k$ where $k \in \mathbb{Z}^+$ [6]. In defining a manifold, we wish to include the condition that the topological space $\mathcal{M}$ be a Hausdorff space. In order to consider the notion of a Hausdorff space, we require the Hausdorff separation axiom which is defined as follows.

Hausdorff separation axiom. Let $\mathcal{M}$ be a topological space. If p and q are two distinct points in $\mathcal{M}$, then there exists disjoint sets $\mathcal{U}, \mathcal{V} \subset \mathcal{M}$ such that $p \in \mathcal{U}$ and $q \in \mathcal{V}$ [55].

A topological space $\mathcal{M}$ that satisfies the Hausdorff separation axiom is referred to as a Hausdorff space [55]. Before moving on to give the definition of a manifold, we first wish to state the notion of a second countable topological space. A topological space is said to be second countable if it has a countable basis for the topology τ [66]. That is, there exists a collection of open sets that is countable and is such that one can write a given open subset of $\mathcal{M}$ as a union of sets contained in the collection [5]. We are now in a position to state the definition of a manifold that is given in [6] and is used in this thesis.

Definition 2.2.2. Let $\mathcal{M}$ be a Hausdorff space that is second countable. If for any $p \in \mathcal{M}$ there exists a neighbourhood $\mathcal{U} \subset \mathcal{M}$ of p such that $\mathcal{U}$ is homeomorphic to some open subset O of $\mathbb{R}^n$ then we refer to $\mathcal{M}$ as an n-dimensional manifold [6].

We note that the neighbourhood in the manifold definition given above is necessarily an open subset [66]. Now, suppose that p is a point in the n-dimensional manifold $\mathcal{M}$. Then, from Definition 2.2.2, there exists an open subset $\mathcal{U}$ of $\mathcal{M}$ such that the map $\psi : \mathcal{U} \to O \subset \mathbb{R}^n$ is a homeomorphism. We shall refer to $(\mathcal{U}, \psi)$ as a coordinate chart [6]. We note that some authors refer to $(\mathcal{U}, \psi)$ as a coordinate system [5]. Now, suppose, in addition to the coordinate chart $(\mathcal{U}, \psi)$, we have the coordinate chart $(\mathcal{V}, \phi)$ defined on the manifold $\mathcal{M}$. In addition, suppose that $\mathcal{U} \cap \mathcal{V} \neq \varnothing$ and consider the following map:

$$\psi \circ \phi^{-1}\Big|_{\phi(\mathcal{U} \cap \mathcal{V})} : \phi\left(\mathcal{U} \cap \mathcal{V}\right) \to \psi\left(\mathcal{U} \cap \mathcal{V}\right) \ . \tag{2.9}$$

Using the fact that the maps ψ and ϕ are homeomorphisms as well as the fact that both the images $\phi\left(\mathcal{U} \cap \mathcal{V}\right)$ and $\psi\left(\mathcal{U} \cap \mathcal{V}\right)$ are non-empty, since we have $\mathcal{U} \cap \mathcal{V} \neq \varnothing$, it is not difficult to see that the map given in equation (2.9) is itself a homeomorphism [6]. Through the consideration of two coordinate charts on a manifold $\mathcal{M}$, we wish to introduce the notion of C^k-compatibility which is defined as follows.

Definition 2.2.3. Let $(\mathcal{U}, \psi)$ and $(\mathcal{V}, \phi)$ be coordinate charts on a manifold $\mathcal{M}$. We say that the coordinate charts are C^k-*compatible* if either $\mathcal{U} \cap \mathcal{V} = \varnothing$ or $\mathcal{U} \cap \mathcal{V} \neq \varnothing$ and the map given in equation (2.9) as well as its inverse are C^k-maps.

We now wish to consider the notion of a differentiable structure on a manifold $\mathcal{M}$. In order to do this, we first wish to introduce the definition of a C^k-atlas.

Definition 2.2.4. Let $(\mathcal{U}_j, \phi_j)$ be a coordinate chart on a manifold $\mathcal{M}$ for $j \in \{1, 2, \ldots, m\}$ where $m \in \mathbb{Z}^+$. Let $\mathcal{A}$ be a collection of coordinate charts defined through the expression

$$\mathcal{A} := \{(\mathcal{U}_1, \psi_1), (\mathcal{U}_2, \psi_2), \ldots, (\mathcal{U}_m, \psi_m)\} \ . \tag{2.10}$$

If $\mathcal{A}$ satisfies the properties:

1. the collection $\{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m\}$ is an open cover of $\mathcal{M}$;
2. any two coordinate charts $(\mathcal{U}_i, \psi_i), (\mathcal{U}_j, \psi_j) \in \mathcal{A}$ satisfy the property of C^k-compatibility;

then $\mathcal{A}$ is said to be a C^k-*atlas* [6, 55].

Using the definition given above, we let $\mathcal{A}$ be a C^k-atlas on the manifold $\mathcal{M}$. If the atlas $\mathcal{A}$ satisfies the condition that if a coordinate chart $(\mathcal{V}, \phi)$ on $\mathcal{M}$ is C^k-compatible with all coordinate charts in $\mathcal{A}$, then the atlas is said to be *maximal* [6]. We refer to a maximal C^k-atlas as a C^k-*differentiable structure* [6]. If a manifold $\mathcal{M}$ has a C^k-differentiable structure given on it we shall refer to it as a C^k-*differentiable manifold* [6].

Let $\mathcal{A}$ be a C^k-atlas as defined in Definition 2.2.4 and let p be any point in $\mathcal{M}$. If there exists an open neighbourhood of p that is such that it only intersects a finite number of the sets $\mathcal{U}_j$ contained in $\mathcal{A}$ then the atlas is said to be *locally finite* [55]. If for any atlas $\mathcal{A}$ of a manifold $\mathcal{M}$ there exists a locally finite atlas

$$\tilde{A} := \{(\mathcal{V}_1, \phi_1), (\mathcal{V}_2, \phi_2), \ldots, (\mathcal{V}_l, \phi_l)\} \ , \tag{2.11}$$

where $l \in \mathbb{Z}^+$ and each $\mathcal{V}_i$ is contained in some $\mathcal{U}_j$, then the manifold $\mathcal{M}$ is said to be *paracompact* [55]. In fact, if we assume that the manifold under consideration is connected, then it satisfies the property of paracompactness since it would be equivalent to the property of being second countable [55, 66]. This is contained in the following theorem which is stated here without proof.

Theorem 2.2.1. *A connected C^k-differentiable manifold satisfies the property of paracompactness if and only if it is second countable [55].*

Following the work [6] we shall refer to a C^∞-manifold as a *smooth manifold*. From now on, when the term manifold is used, we shall be referring to a connected smooth manifold.

2.3 The tangent space, dual space and tensors

Let $\mathcal{M}$ be an n-dimensional manifold and let p be a point in $\mathcal{M}$. We define C_p^∞ to be the set of all C^∞-functions that are defined in a neighbourhood of the point p [6]. We define a tangent vector at the point $p \in \mathcal{M}$ to be a map $\boldsymbol{X}_p : C_p^\infty \to \mathbb{R}$ that is such that, for all $\alpha, \beta \in \mathbb{R}$ and for all $f, g \in C_p^\infty$, we have [5]

1. $\boldsymbol{X}_p(\alpha f + \beta g) = \alpha \boldsymbol{X}_p(f) + \beta \boldsymbol{X}_p(g)$;
2. $\boldsymbol{X}_p(fg) = f(p)\boldsymbol{X}_p(g) + g(p)\boldsymbol{X}_p(f)$.

With this definition in mind, we define $T_p\mathcal{M}$ to be the collection of all tangent vectors at the point $p \in \mathcal{M}$ [5]. It is not difficult to see that the collection $T_p\mathcal{M}$ has the linear structure described by the expressions

$$(\boldsymbol{X}_p + \boldsymbol{Y}_p)(f) = \boldsymbol{X}_p(f) + \boldsymbol{Y}_p(f) \ , \tag{2.12}$$

and

$$\left(\alpha \boldsymbol{X}_p\right)(f) = \alpha \boldsymbol{X}_p(f) \ , \tag{2.13}$$

where $f, g \in C_p^\infty$, $\boldsymbol{X}_p, \boldsymbol{Y}_p \in T_p\mathcal{M}$ and $\alpha \in \mathbb{R}$ [5]. In addition, under this linear structure, the collection $T_p\mathcal{M}$ is a vector space and we refer to this collection as the *tangent space* at p [5]. A proof that the tangent space has the same dimension as that of the associated manifold is given in [5]. Consider an open neighbourhood $\mathcal{U}$ of p which is a point in the n-dimensional manifold $\mathcal{M}$. We let $\psi : \mathcal{U} \to O \subset \mathbb{R}^n$ be a homeomorphism where O is open. For $f \in C_p^\infty$ we introduce the coordinate basis $\{\boldsymbol{e}_\nu\}$ defined as

$$\boldsymbol{e}_\nu(f) := \frac{\partial}{\partial x^\nu}\left(f \circ \psi^{-1}\right)\Big|_{\psi(p)} \ , \tag{2.14}$$

where $\nu \in \{1, 2, \ldots, n\}$ and $\left(x^1, x^2, \ldots, x^n\right)$ are Cartesian coordinates of $\psi(\mathcal{U})$. By making use of the basis constructed above, it is possible to write the tangent vector $\boldsymbol{X}_p \in T_p\mathcal{M}$ as

$$\boldsymbol{X}_p = X_p^\nu \ \boldsymbol{e}_\nu := \sum_{\nu=1}^{n} X_p^\nu \ \boldsymbol{e}_\nu \ , \tag{2.15}$$

where we have implemented the so-called Einstein summation convention, i.e., the summation is implied when an index is repeated. The vector components X_p^ν in equation (2.15) are defined as the value obtained by acting the vector $\boldsymbol{X}_p$ on the C^∞ function $x^\nu \circ \psi : \mathcal{U} \to \mathbb{R}$. That is, we write

$$X_p^\nu := \boldsymbol{X}_p\left(x^\nu \circ \psi\right) \ . \tag{2.16}$$

We denote the collection of linear maps $\boldsymbol{v}_p : T_p\mathcal{M} \to \mathbb{R}$ as $T_p^*\mathcal{M}$ and we write the real value obtained by mapping the vector $\boldsymbol{X}_p$ through the use of $\boldsymbol{v}_p$ as $\langle \boldsymbol{v}_p, \boldsymbol{X}_p \rangle$. Given a coordinate basis, which is defined in equation (2.14), for the tangent space we can define the set of n linear maps $\{\boldsymbol{e}^\nu\}$, which are elements of $T_p^*\mathcal{M}$, through the expression

$$\langle \boldsymbol{e}^\mu, \boldsymbol{e}_\nu \rangle = \delta^\mu_\nu \ . \tag{2.17}$$

We note that, under the linear structure of the operation $\langle \boldsymbol{v}_p, \boldsymbol{X}_p \rangle$, the collection $T_p^*\mathcal{M}$ is a vector space. The set of $\{\boldsymbol{e}^\nu\}$ forms a basis for the elements in $T_p^*\mathcal{M}$ with an arbitrary element $\boldsymbol{v}_p \in T_p^*\mathcal{M}$ written as the following linear combination

$$\boldsymbol{v}_p = (v_p)_\nu \ \boldsymbol{e}^\nu \ , \tag{2.18}$$

where we define the coordinate components through the expression

$$(v_p)_\nu := \langle \boldsymbol{v}_p, \boldsymbol{e}_\nu \rangle \ . \tag{2.19}$$

The collection $T_p^*\mathcal{M}$ is of dimension n and is referred to as the *dual space* of the tangent space with its elements being referred to as *dual vectors* [5, 55]. Now that we have introduced the tangent space and dual space at a point $p \in \mathcal{M}$, we are in a position to introduce the notion of a tensor.

We begin by introducing the Cartesian product Ω_k^l which is defined as follows

$$\Omega_k^l := \underbrace{T_p^*M \times \cdots \times T_p^*M}_{k \text{ factors}} \times \underbrace{T_pM \times \ldots T_pM}_{l \text{ factors}} \ . \tag{2.20}$$

We now define a *tensor of type* (k, l) to be a multilinear map [5]

$$\boldsymbol{T} : \Omega_k^l \to \mathbb{R} \ . \tag{2.21}$$

Given a coordinate chart $(\mathcal{U}, \psi)$ of $p \in \mathcal{M}$ with the basis $\{\boldsymbol{e}_\nu\}$ for the tangent space as well as the basis $\{\boldsymbol{e}^\nu\}$ for the dual space, we can write the type (k, l) tensor as

$$\boldsymbol{T} = T^{\mu_1 \ldots \mu_l}{}_{\nu_1 \ldots \nu_k} \underbrace{\boldsymbol{e}_{\mu_1} \otimes \cdots \otimes \boldsymbol{e}_{\mu_l}}_{l \text{ factors}} \otimes \underbrace{\boldsymbol{e}^{\nu_1} \otimes \cdots \otimes \boldsymbol{e}^{\nu_k}}_{k \text{ factors}} \ , \tag{2.22}$$

where we have once again made use of the Einstein summation convention and $T^{\mu_1 \dots \mu_l}{}_{\nu_1 \dots \nu_k}$ are the coordinate basis components of the tensor associated with $\{\boldsymbol{e}_\nu\}$ [5,55]. We denote the vector space to which a tensor of type (k,l) belongs as $T_p(k,l)\mathcal{M}$. Having now introduced the notion of a tensor, we wish to introduce the notion of a tensor field following [55] by Hawking and Ellis. Let $\mathcal{U} \subset \mathcal{M}$ be an open subset of the manifold $\mathcal{M}$. We define the allocation of an element of the vector space $T_p(k,l)\mathcal{M}$ to a point p in the open set $\mathcal{U}$ as a *tensor field of type* (k,l). Tensor fields will be of particular interest in this thesis when studying aspects of a given space-time. More specifically, we will be interested in describing quantities such as curvature and torsion through the use of tensor fields. Before we move on to discuss specific space-times, we first wish to consider the notion of integration on manifolds.

2.4 Orientability and integration on a manifold

In this section, we wish to discuss, in particular, the notion of integration on a manifold say $\mathcal{M}$. Before introducing this concept, we first wish to define the notion of an orientable manifold as this property of orientability is required when defining integration on a manifold. The definition of orientability is given below following [5,67].

Definition 2.4.1. Let $\mathcal{M}$ be a manifold. We say that $\mathcal{M}$ is an *orientable manifold* if one can select a smooth n-form field that does not vanish anywhere on $\mathcal{M}$. We say that the selected n-form field provides an orientation for $\mathcal{M}$.

For the remainder of this thesis, we will take all manifolds to be orientable as per the definition given above. For such a manifold $\mathcal{M}$ let us denote the selected n-form field, which provides us with the orientation $\mathcal{M}$, as $\boldsymbol{\sigma}$. Let $p \in \mathcal{M}$ and let $(\mathcal{U}, \psi)$ be a coordinate chart of $\mathcal{M}$ with $p \in \mathcal{U} \subset \mathcal{M}$. By making use of the wedge product, we expand the n-form field $\boldsymbol{\sigma}$ in the basis associated with this coordinate chart. That is, we write

$$\boldsymbol{\sigma} = u \; \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n \;, \tag{2.23}$$

where, as per Definition 2.4.1 given above, u is a smooth function that is non-vanishing everywhere. In the case where $u > 0$, we refer to the coordinate chart as *right-handed* whereas, in the case where $u < 0$, we refer to the coordinate chart as *left-handed* [5]. Suppose that $u > 0$ and, therefore, that the coordinate chart is right-handed. Consider an arbitrary n-form field $\boldsymbol{\beta}$ and let us write this n-form field in terms of the basis associated with the coordinate chart

$$\boldsymbol{\beta} = b \; \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n \;. \tag{2.24}$$

We now define the integral of the n-form field over the open neighbourhood $\mathcal{U} \subset \mathcal{M}$ as [5]

$$\int_{\mathcal{U}} \boldsymbol{\beta} := \int_{\psi(\mathcal{U})} b \; \mathrm{d}x^1 \dots \mathrm{d}x^n \;. \tag{2.25}$$

We have thus defined the integral of an arbitrary, smooth n-form field over an open neighbourhood of the manifold $\mathcal{M}$. With this definition in mind, we wish to define the integral of a smooth n-form field over the manifold $\mathcal{M}$. In order to construct such a definition, we first wish to give the definition of a smooth partition of unity on a manifold. Following Willmore [67], we give such a definition below.

Definition 2.4.2. Let $\mathcal{M}$ be a manifold. We define a *smooth partition of unity*, $\{f_i\}$, to be a collection of smooth functions, f_i, on $\mathcal{M}$ that are such that:

1. $f_i \geq 0 \; \forall i$;
2. The collection $\{\mathrm{supp}\,(f_i)\}$ forms a covering of $\mathcal{M}$ that is locally finite;
3. $\sum_i f_i = 1$.

In the second property given above, we have used the notation $\mathrm{supp}\,(f_i)$ to indicate the *support* of the smooth function f_i, i.e., the subset of the domain of f_i whose image does not contain zero. Now, suppose the collection $\{\mathcal{V}_j\}$ of open neighbourhoods of the manifold $\mathcal{M}$ is an open covering of $\mathcal{M}$. If, for each function f_i contained in the partition of unity, there exists an open neighbourhood $\mathcal{V}_j$ contained in the aforementioned open covering such that $\mathrm{supp}\,(f_i) \subset \mathcal{V}_j$, then the partition of unity $\{f_i\}$ is said to be *subordinate* to the open covering $\{\mathcal{V}_j\}$ [67].

Having introduced the notion of a smooth partition of unity on a manifold, we now wish to give the definition of the integral of an arbitrary n-form field over the entirety of the manifold. Since a given manifold, as per the notions given previously, satisfies the property of paracompactness, one can find a countable collection, $\{\mathcal{U}_j\}$, of open neighbourhoods of $\mathcal{M}$ that are locally finite and cover $\mathcal{M}$ [5]. Now, a smooth partition of unity $\{f_i\}$ that is subordinate to the collection $\{\mathcal{U}_j\}$ exists, and we define the integral of an arbitrary n-form $\boldsymbol{\beta}$ field over the entirety of the manifold to be [5]

$$\int_{\mathcal{M}} \boldsymbol{\beta} := \sum_j \int_{\mathcal{U}_j} f_j \boldsymbol{\beta} \ . \tag{2.26}$$

In the above definition we have assumed that, for a coordinate chart $(\mathcal{U}_j, \psi_j)$ where $\mathcal{U}_j$ is part of the aforementioned collection $\{\mathcal{U}_j\}$ and has associated homeomorphisms $\psi_j : \mathcal{U}_j \to O_j \subset \mathbb{R}^n$, the integrability condition

$$\sum_j \int_{\psi_j(\mathcal{U}_j)} f_j |b_j| \mathrm{d}x^1 \dots \mathrm{d}x^n \ , \tag{2.27}$$

is satisfied where b_j is the function used to construct the n-form field $\boldsymbol{\beta}$ using the jth coordinate chart as per equation (2.24) [5]. In order to carry out an integration of a smooth function, say h, over the entirety of the manifold $\mathcal{M}$, we require the notion of a *volume element* which is a continuous and non-vanishing n-form which we shall denote as $\boldsymbol{\zeta}$. For a volume element $\boldsymbol{\zeta}$, we define the integral of the smooth function h over the entirety of the manifold $\mathcal{M}$ as

$$\int_{\mathcal{M}} h := \int_{\mathcal{M}} h\boldsymbol{\zeta} \ . \tag{2.28}$$

We note that while an orientation $\boldsymbol{\sigma}$ is equivalent to an orientation that is obtained by multiplying $\boldsymbol{\sigma}$ by a strictly positive smooth function this is not the case for a volume element. The defining of an integral of a smooth function over $\mathcal{M}$ will play a significant role when we consider specific theories of gravity. This is due to the fact that, when introducing a gravitational theory, we will consider an action which involves such an integral over the manifold included in the definition of the space-time considered. As we shall see in subsequent chapters, although there is a natural choice of a volume element given a space-time, the choice is not unique. This means that it is possible to construct gravitational theories that, while still containing the same smooth function in their respective actions, will differ in their volume elements. An example of such a case is the comparison between the Einstein-Cartan theory and the so-called Einstein-Cartan-Saa theory [11]. In this example, both contain the same smooth function in their action, which is the non-null torsion Ricci scalar and is defined later on, while their actions differ due to having different volume elements.

2.5 Bundles, fibre bundles and principle $\mathcal{G}$-bundles

In this section, we wish to introduce the notion of a fibre bundle and, in particular, discuss the specific type of fibre bundle referred to as a principle $\mathcal{G}$-bundle. The notion of a fibre bundle is essential to the formulation of a gauge theoretic approach to gravitational theories. In particular, it is possible to formulate the Einstein-Cartan theory of gravity as a gauge theory of the Poincaré group [70, 71] and the Teleparallel Equivalent of General Relativity as a gauge theory of the translation group [8]. In later sections of this thesis, we are interested in studying these two aforesaid theories as well as various modifications to these theories. The reader interested in learning about the subject of so-called Poincaré gauge gravity theories in general is directed to [71] by Blagojević as well as [72, 73] by Obukhov. Let us now begin this section by introducing the notion of a bundle. This is given in the following definition from Hawking and Ellis [55].

Definition 2.5.1. Let $\mathcal{E}$ and $\mathcal{M}$ be manifolds and let $\pi : \mathcal{E} \to \mathcal{M}$ be a surjective mapping. We then refer to the tuple $(\mathcal{E}, \mathcal{M}, \pi)$ as a *bundle.*

The manifolds $\mathcal{E}$ and $\mathcal{M}$ in the above definition are often referred to as the *total space manifold* and the *base manifold* respectively [9, 18, 55]. In addition, we refer to the surjective map π as the *projection map* [18]. As an example of a bundle, let us consider the so-called product bundle. Let $\mathcal{M}$ and $\mathcal{A}$ be manifolds and let us introduce the total space manifold as the product manifold $\mathcal{M} \times \mathcal{A}$. Then, the tuple $(\mathcal{M} \times \mathcal{A}, \mathcal{M}, \pi)$ defines a bundle where π is the surjection that is such that $\pi : (p, a) \in \mathcal{M} \times \mathcal{A} \mapsto \pi(p, a) = p \in \mathcal{M}$ for all $a \in \mathcal{A}$ and $p \in \mathcal{M}$ [55]. An example of such a product bundle, which is given in [55], is that of the cylinder C^2. One takes the base manifold to be the circle S^1 and the total space manifold to be the product manifold constructed using S^1 together with the real line $\mathbb{R}^1$, i.e., the total space manifold has the topology $S^1 \times \mathbb{R}^1$. Having discussed the notion of a product bundle, let us now turn our attention to the notion of a fibre bundle. In this thesis, we make use of the following definition used in [9, 18].

Definition 2.5.2. Let $\mathcal{E}, \mathcal{M}$ and $\mathcal{F}$ be manifolds. As before, $\mathcal{E}$ is referred to as the total space manifold and $\mathcal{M}$ is referred to as the base manifold. We refer to $\mathcal{F}$ as the *fibre.* Let $\pi : \mathcal{E} \to \mathcal{M}$ be a surjective mapping. For all $p \in \mathcal{M}$, let there exist a diffeomorphism, say ϕ, that is such that $\phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathcal{F}$ where $\mathcal{U} \subset \mathcal{M}$ is an open neighbourhood of p. In addition, let the following diagram commute

$$\begin{tikzcd}
\pi^{-1}(\mathcal{U}) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \mathcal{U} \times \mathcal{F} \arrow[dl, "\mathrm{proj}_1"] \\
& \mathcal{U} &
\end{tikzcd} ,$$

where $\mathrm{proj}_1 : (p, f) \in \mathcal{U} \times \mathcal{F} \mapsto \mathrm{proj}_1(p, f) = p \in \mathcal{U}$. We then refer to the quadruple $(\mathcal{E}, \mathcal{M}, \pi, \mathcal{F})$ as a *fibre bundle.*

In essence, a fibre bundle is a bundle that is locally a product bundle [55]. That is, the bundle $(\mathcal{E}, \mathcal{U}, \pi)$ is a fibre bundle if it is locally the product manifold $\mathcal{M} \times \mathcal{F}$ where the manifold $\mathcal{F}$ is the fibre. As an example of a fibre bundle, let us consider the tangent bundle. Let $\mathcal{M}$ be a manifold and let us define the total space manifold to be $\mathcal{E} := \bigcup_{p \in \mathcal{M}} \{p\} \times T_p\mathcal{M}$. Let us take the surjection to be the projection map $\mathrm{proj}_1 : (p, v) \in \mathcal{E} \to p \in \mathcal{M}$. In order to conclude that the tangent bundle is indeed a fibre bundle, we require the fibre $\mathcal{F}$. Let us consider an element, say $\boldsymbol{v}$, of the tangent space $T_p\mathcal{M}$. Let the manifold $\mathcal{M}$ be of dimension n and let $E := \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ be a basis for $T_p\mathcal{M}$. Then, we can write the tangent space element $\boldsymbol{v}$ as the following linear combination of elements of the basis E:

$$\boldsymbol{v} = \sum_{k=1}^{n} l^k \boldsymbol{e}_k \ , \tag{2.29}$$

where $l^j \in \mathbb{R} \ \forall j \in \{1, \dots, n\}$. We note that, for an open neighbourhood $\mathcal{U} \subset \mathcal{M}$ of $p \in \mathcal{M}$, the map

$$\phi_{\mathcal{U}} : \left(p, \sum_{k=1}^{n} l^k e_k\right) \in \mathcal{E} \mapsto \phi_{\mathcal{U}}(p, \boldsymbol{v}) = \left(p, \left(l^1, \dots, l^n\right)\right) \in \mathcal{U} \times \mathbb{R}^n \ , \tag{2.30}$$

is a diffeomorphism. Then, the quadruple $(\mathcal{E}, \mathcal{M}, \mathrm{proj}_1, \mathcal{F})$ is a fibre bundle where the fibre is nothing more than $\mathcal{F} := \mathbb{R}^n$.

Before we move on to discuss the notion of a principle fibre bundle, let us first make a note on the diffeomorphism ϕ mentioned in Definition 2.5.2. This choice in a diffeomorphism that maps $\pi^{-1}(\mathcal{U})$ to the product bundle $\mathcal{U} \times \mathcal{F}$ is referred to as a *local trivialisation* [9, 18]. We note that there is a one-to-one correspondence between the notion of a local trivialisation and the notion of a choice of gauge [18]. As a result of such a one-to-one correspondence, it is not uncommon for a local trivialization to simply be referred to as a choice of gauge [9]. In this thesis, we make use of the definition for a $\mathcal{G}$-bundle that is stated in [18] and is given below.

Definition 2.5.3. Let the quadruple $(\mathcal{E}, \mathcal{M}, \pi, \mathcal{F})$ be a fibre bundle as per definition 2.5.2. Let $\mathcal{G}$ be a Lie group that acts as a group of diffeomorphisms on the fibre $\mathcal{F}$ and let $\mathbb{1}$ be the unit element of $\mathcal{G}$. Let the collection $\{\mathcal{U}_1, \dots, \mathcal{U}_m\}$ be an open cover of the base manifold $\mathcal{M}$ and let $\{\phi_1, \dots, \phi_m\}$ be a corresponding collection of local trivialisation maps, i.e., be such that $\phi_j : \pi^{-1}(\mathcal{U}_j) \to \mathcal{U}_j \times \mathcal{F}$ for all $j \in \{1, \dots, m\}$. Consider the collection $\mathcal{A} := \{(\mathcal{U}_1, \phi_1), \dots, (\mathcal{U}_m, \phi_m)\}$. If the collection $\mathcal{A}$ is such that the maps defined as $\phi_{ij} := \phi_i \phi_j^{-1} : (\mathcal{U}_i \cap \mathcal{U}_j) \times \mathcal{F} \to (\mathcal{U}_i \cap \mathcal{U}_j) \times \mathcal{F}$ is such that it has the form

$$\phi_{ij} : (p, f) \in (\mathcal{U}_i \cap \mathcal{U}_j) \times \mathcal{F} \mapsto \phi_{ij}(p, f) = (p, q_{ij}(p) f) \in (\mathcal{U}_i \cap \mathcal{U}_j) \times \mathcal{F} , \tag{2.31}$$

where the maps $q_{ij} : (\mathcal{U}_i \cap \mathcal{U}_j) \to \mathcal{G}$ are such that for all $i, j, k \in \{1, \dots, m\}$ we have

1. $q_{jj}(p) = \mathbb{1}$ for all $p \in \mathcal{U}_j$;
2. $q_{ij}(p) = q_{ji}^{-1}(p)$ for all $p \in \mathcal{U}_i \cap \mathcal{U}_j$;
3. $q_{ij}(p)\, q_{jk}(p)\, q_{ki}(p) = \mathbb{1}$ for all $p \in \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$;

then we refer to $\mathcal{A}$ as a *$\mathcal{G}$-atlas*. By defining a *$\mathcal{G}$-structure* to be an equivalence class of $\mathcal{G}$-atlases, we say that the fibre bundle is a *$\mathcal{G}$-bundle* if a $\mathcal{G}$-structure is specified on the fibre bundle.

The Lie group $\mathcal{G}$ used in the definition given above is often referred to as the *structure group*. In the event that the structure group $\mathcal{G}$ is diffeomorphic to the fibre $\mathcal{F}$, then we say that the $\mathcal{G}$-bundle is a *principle $\mathcal{G}$-bundle* [18]. In many cases, the fibre is taken to be the structure group. In such a case, the diffeomorphism that maps the structure group to the fibre is simply the identity map. Let us consider the so-called linear frame bundle which is an example of a principle $\mathcal{G}$-bundle and is discussed in [9]. Consider the tangent space $T_p\mathcal{M}$ of the n-dimensional base manifold $\mathcal{M}$ of this bundle and let $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ be the standard basis for $\mathbb{R}^n$. Let $\gamma : \mathbb{R}^n \to T_p\mathcal{M}$ be a linear isomorphism and, therefore, the set $\{\gamma(\boldsymbol{e}_1), \dots, \gamma(\boldsymbol{e}_n)\}$ is a basis for $T_p\mathcal{M}$. We now define the set of all linear isomorphisms $\gamma : \mathbb{R}^n \to T_p\mathcal{M}$ by $L_p\mathcal{M}$ and we refer to this set as the *set of linear frames*. We then define $L\mathcal{M} := \bigcup_{p\in\mathcal{M}} \{p\} \times L_p\mathcal{M}$. One can show that the quadruple $(L\mathcal{M}, \mathcal{M}, \pi, GL(\mathbb{R}^n))$, which is referred to as the *linear frame bundle*, is a principle fibre bundle [9]. In this example, the structure group and the fibre coincide and are described by the general linear group $GL(\mathbb{R}^n)$. In the event that the linear isomorphisms are such that the bases $\{\gamma(\boldsymbol{e}_1), \dots, \gamma(\boldsymbol{e}_n)\}$ are orthonormal, then the resulting principle fibre bundle is referred to as the *orthonormal frame bundle* [9]. Following the notation of [9], we denote the total space manifold of the orthonormal frame bundle as $F\mathcal{M}$.

2.6 Chapter conclusions

In this chapter, we introduced the necessary mathematical concepts needed for constructing space-times with torsion. In particular, we introduced the definition of a manifold which we make use of in this thesis as well as the notions of a tangent space and a dual space. We also discussed the construction of tensors through the use of the basis for the tangent space as well the basis for a dual space. Following this, we discussed the notion of integration on manifolds. This will be useful in Chapter 4 when we require a gravitational action constructed on a space-time in order to describe a gravitational theory. Moreover, the notion of a volume element is important when discussing gravitational theories whose volume element is not that of the natural choice. An example of such a theory is the so-called Einstein-Cartan-Saa theory [11] which will be discussed in Section 4.3. We concluded this chapter by discussing the notion of a fibre bundle which is a key concept when describing a gravitational theory as a gauge theory. Examples of these include the Einstein-Cartan theory as a gauge theory of the Poincaré group [74] and the Teleparallel Equivalent of General Relativity as a gauge theory of the translation group [8].

Chapter 3

Space-times with curvature and torsion

3.1 Introduction

In the previous chapter, we introduced the necessary mathematical concepts needed to construct space-times with non-vanishing torsion. In this chapter, we will make use of these mathematical concepts in order to study the so-called Riemann-Cartan space-time whose natural choice of an affine connection is the so-called Cartan connection which is non-symmetric while being metric compatible [74, 75]. In Section 3.2, we outline some notational conventions that we make use of in this thesis. Following this, in Section 3.3 we construct the Riemann-Cartan space-time following [9]. In Section 3.4, we discuss the notions of curvature and torsion in this space-time. In Section 3.5, we discuss the so-called Cartan formalism following [9, 16, 18]. This formalism will allow us to represent some of the notions given in Section 3.4 in a non-holonomic frame. In Section 3.6, we discuss the so-called Riemannian and Weitzenböck space-times which are special cases of the Riemann-Cartan space-time. In Section 3.7, we discuss the notions of auto-parallel and extremal curves in Riemann-Cartan space-time. Lastly, in Sections 3.8, 3.9 and 3.10, we discuss various Raychaudhuri equations that can be constructed on the Riemann-Cartan space-time following the results presented in [54, 58–61].

3.2 Conventions and notation

In the construction of space-times with non-vanishing torsion, we will be concerned with 4-dimensional manifolds where we make use of the definitions given in Chapter 2. When considering a type (k, l) tensor field, say $\boldsymbol{B}$, we will use the notation $B^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l}$ in order to indicate the kind of variables that the tensor field acts on. That is, we will use Greek indices to denote tensor properties. Therefore, any relationships between tensors mentioned in the following sections will be properties of those tensors and will not be specific to any coordinate chart. This notational convention is referred to as the abstract index notation and is made use of[1] in [5]. Furthermore, for a type (k, l) tensor field $\boldsymbol{B}$, we make use of the notational conventions

$$B^{\mu_1\mu_2\cdots\mu_k}{}_{(\nu_1\nu_2\ldots\nu_l)} = \frac{1}{l!}\sum_{\pi} B^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_{\pi(1)}\nu_{\pi(2)}\cdots\nu_{\pi(l)}} \,, \tag{3.1}$$

and

$$B^{\mu_1\mu_2\cdots\mu_k}{}_{[\nu_1\nu_2\ldots\nu_l]} = \frac{1}{l!}\sum_{\pi} \delta_\pi B^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_{\pi(1)}\nu_{\pi(2)}\cdots\nu_{\pi(l)}} \,, \tag{3.2}$$

where π indicates the permutation and δ_π is -1 for odd permutations and 1 for even permutations. In order to illustrate how the notation described in equations (3.1) and (3.2) is used, we consider as

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[1] We note that, while we use Greek indices here for the abstract index notation, Wald makes use of Latin indices. Here, we prefer to reserve the use of Latin indices for dealing with nonholonomic frames. The construction of nonholonomic frames is explained further on in Section 3.5.

an example the case where the tensor field $\boldsymbol{B}$ is of type $(0,3)$. The application of equation (3.1) to the first and third indices of a type $(0,3)$ tensor field $\boldsymbol{B}$ yields

$$B_{(\mu|\alpha|\nu)} = \frac{1}{2}\left(B_{\mu\alpha\nu} + B_{\nu\alpha\mu}\right) \ . \tag{3.3}$$

Through the application of equation (3.2) to the first and third indices of $\boldsymbol{B}$ yields

$$B_{[\mu|\alpha|\nu]} = \frac{1}{2}\left(B_{\mu\alpha\nu} - B_{\nu\alpha\mu}\right) \ . \tag{3.4}$$

In subsequent chapters, when a coordinate basis is introduced, we will then use Greek indices to indicate the coordinate basis components of tensors as was done in the previous chapter. For now however, we will use Greek indices in the capacity of the so-called abstract index notation.

In the following sections, we turn our attention to the construction of a space-time whose natural choice of an affine connection is both metric compatible as well as not necessarily symmetric. We will first introduce the definition of an affine connection following [55] by Hawking and Ellis before moving on to discuss the requirement that the affine connection be metric compatible. The definition of a Riemann-Cartan space-time will then be given by following [9] by Tseng through the use of a 4-dimensional manifold together with the metric tensor as well as the non-symmetric metric compatible affine connection considered as an additional structure on the manifold.

3.3 Construction of Riemann-Cartan space-time

We begin this section by giving the definition of an affine connection which we consider to be an additional structure on a given manifold. Let $\mathcal{M}$ be a 4-dimensional manifold and let p be a point in $\mathcal{M}$. The additional structure referred to as an affine connection, denoted as ∇, is defined to be a description which provides a differential operator $\nabla_{\boldsymbol{X}} = X^{\nu}\nabla_{\nu}$ for all vector fields $\boldsymbol{X}$ at $p \in \mathcal{M}$ and satisfies the properties [55]:

1. for vector fields $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ and $f, g \in C_p^{\infty}$ we have

$$\begin{aligned} \nabla_{f\boldsymbol{Y}+g\boldsymbol{Z}}\boldsymbol{X} &= f\nabla_{\boldsymbol{Y}}\boldsymbol{X} + g\nabla_{\boldsymbol{Z}}\boldsymbol{X} && (3.5)\\ &= fY^{\nu}\nabla_{\nu}\boldsymbol{X} + gZ^{\nu}\nabla_{\nu}\boldsymbol{X} \ ; && (3.6) \end{aligned}$$

2. for vector fields $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ and $\alpha, \beta \in \mathbb{R}$ we have

$$\begin{aligned} \nabla_{\boldsymbol{Z}}\left(\alpha\boldsymbol{X} + \beta\boldsymbol{Y}\right) &= \alpha\nabla_{\boldsymbol{Z}}\boldsymbol{X} + \beta\nabla_{\boldsymbol{Z}}\boldsymbol{Y} && (3.7)\\ &= \alpha Z^{\nu}\nabla_{\nu}\boldsymbol{X} + \beta Z^{\nu}\nabla_{\nu}\boldsymbol{Y} \ ; && (3.8) \end{aligned}$$

3. for vector fields $\boldsymbol{X}, \boldsymbol{Y}$ and $f \in C_p^{\infty}$ we have

$$\begin{aligned} \nabla_{\boldsymbol{Y}}\left(f\boldsymbol{X}\right) &= Y\left(f\right)\boldsymbol{X} + f\nabla_{\boldsymbol{Y}}\boldsymbol{X} && (3.9)\\ &= Y\left(f\right)\boldsymbol{X} + fY^{\nu}\nabla_{\nu}\boldsymbol{X} \ . && (3.10) \end{aligned}$$

In the third property given above, we have used the fact that the description provided by an affine connection is consistent with the idea that a tangent vector field, say $\boldsymbol{Y}$, acting on a function, say $f \in C_p^{\infty}$, yields the directional derivative of f at p, i.e., $\nabla_{\boldsymbol{Y}} f = Y\left(f\right)$ for any affine connection ∇. Furthermore, we note that the affine connection ∇ maps the type (k,l) tensor field $B^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l}$ to the type $(k,l+1)$ tensor field $\nabla_{\sigma}B^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l}$ and we refer to this quantity as the *covariant derivative* of $\boldsymbol{B}$ [5]. It is at this point that we wish to introduce the type $(0,2)$ tensor field $\boldsymbol{g}$ referred to as the metric tensor. Intuitively, the metric tensor provides us with the notion of an inner product on the tangent space $T_p\mathcal{M}$ for all points p in the manifold $\mathcal{M}$. More formally, a *metric tensor* on a manifold $\mathcal{M}$ is a type $(0,2)$ tensor that is nondegenerate and symmetric [5, 55]. The property of nondegeneracy means that $g\left(\boldsymbol{X}, \boldsymbol{Y}\right) = 0$ for all vectors $\boldsymbol{Y} \in T_p\mathcal{M}$ if and only if $\boldsymbol{X}$ is the zero vector. In addition, when we say the metric tensor is symmetric, it is meant that, given two vectors $\boldsymbol{X}, \boldsymbol{Y} \in T_p\mathcal{M}$, we have $g\left(\boldsymbol{X}, \boldsymbol{Y}\right) = g\left(\boldsymbol{Y}, \boldsymbol{X}\right)$. In the abstract index notation, the symmetric property of the metric tensor can be written as $g_{\mu\nu} = g_{\nu\mu}$. In this thesis we take $\boldsymbol{g}$ to be a Lorentz metric with signature $(-, +, +, +)$. Since we are dealing with a Lorentz metric, the manifold $\mathcal{M}$ is taken to be a pseudo-Riemannian manifold since the metric tensor is not necessarily positive semi-definite.

Now that the notions of an affine connection and a metric tensor have been introduced, we wish to introduce the notion of a metric compatible connection for which we have non-vanishing torsion. Let us first begin by introducing the notion of a connection coefficient. Let ∇ and ∇' be two affine connections considered as additional structures on the manifold $\mathcal{M}$. Now, let $\boldsymbol{X}$ be a tangent vector field at $p \in \mathcal{M}$ and let $f \in C_p^\infty$ be a function on $\mathcal{M}$. By making use of the abstract index notation, we consider the map $\left(\nabla_\alpha - \nabla'_\alpha\right)\left(fX^\beta\right)$. The application of the Leibniz rule yields

$$\left(\nabla_\alpha - \nabla'_\alpha\right)\left(fX^\beta\right) = \left[\left(\nabla_\alpha - \nabla'_\alpha\right) f\right] X^\beta + f\left(\nabla_\alpha - \nabla'_\alpha\right) X^\beta \ . \tag{3.11}$$

As already mentioned, the description provided by a given affine connection is consistent with how tangent vector fields provide the notion of a directional derivative of a function on $\mathcal{M}$. That is, for any two affine connections ∇ and ∇' and a function $f \in C_p^\infty$, we have $\nabla_\alpha f = \nabla'_\alpha f$. It therefore follows that equation (3.11) becomes

$$\left(\nabla_\alpha - \nabla'_\alpha\right)\left(fX^\beta\right) = f\left(\nabla_\alpha - \nabla'_\alpha\right) X^\beta \ . \tag{3.12}$$

The above expression implies that the quantity $\left(\nabla_\alpha - \nabla'_\alpha\right) X^\beta$ depends only on the value taken on by the vector field $\boldsymbol{X}$ at $p \in \mathcal{M}$ and it follows that $\left(\nabla_\alpha - \nabla'_\alpha\right)$ is a linear map [5]. Since this is a linear map, we can introduce a type $(1, 2)$ tensor field, say $L^\beta{}_{\alpha\sigma}$, that is such that

$$\left(\nabla_\alpha - \nabla'_\alpha\right) X^\beta = L^\beta{}_{\alpha\sigma} X^\sigma \ . \tag{3.13}$$

We now wish to consider what happens when we act the same map on a dual vector field $\boldsymbol{v}$. We note that $\langle \boldsymbol{v}, \boldsymbol{X} \rangle = v_\alpha X^\alpha \in C_p^\infty$ and thus

$$\left(\nabla_\alpha - \nabla'_\alpha\right)\left(v_\beta X^\beta\right) = 0 \ . \tag{3.14}$$

The application of the Leibniz rule in the above expression yields

$$v_\beta\left(\nabla_\alpha - \nabla'_\alpha\right) X^\beta + X^\beta\left(\nabla_\alpha - \nabla'_\alpha\right) v_\beta = 0 \ . \tag{3.15}$$

By making use of equation (3.13), the above expression becomes

$$X^\beta\left(L^\sigma{}_{\alpha\beta} v_\sigma + \left(\nabla_\alpha - \nabla'_\alpha\right) v_\beta\right) = 0 \ . \tag{3.16}$$

Since the above expression holds for all tangent vector fields $\boldsymbol{X}$, we have

$$\left(\nabla_\alpha - \nabla'_\alpha\right) v_\beta = -L^\sigma{}_{\alpha\beta} v_\sigma \ , \tag{3.17}$$

which describes the action of the map $\left(\nabla_\alpha - \nabla'_\alpha\right)$ on a dual vector field. By extending the notions presented in equations (3.13) and (3.17) it is possible to consider the action of the linear map $\left(\nabla_\alpha - \nabla'_\alpha\right)$ on a type (k, l) tensor field $\boldsymbol{B}$. That is, for a tensor field $\boldsymbol{B}$ of type (k, l), the action of the map $\left(\nabla_\alpha - \nabla'_\alpha\right)$ yields [5]

$$\left(\nabla_\alpha - \nabla'_\alpha\right) B^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} = \sum_{i=1}^{k} L^{\mu_i}{}_{\alpha\sigma} B^{\mu_1 \ldots \sigma \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} - \sum_{i=1}^{l} L^\sigma{}_{\alpha\nu_i} B^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \sigma \ldots \nu_l} \ . \tag{3.18}$$

If we instead consider the map $\left(\nabla_\alpha - \partial_\alpha\right)$ where we have replaced the affine connection ∇'_α with the partial derivative ∂_α, then the map is still in fact linear, however, it does not necessarily define a

$(1,2)$ tensor field but rather a type $(1,2)$ pseudo-tensor. We denote this pseudo-tensor as $\tilde{\Gamma}^{\beta}{}_{\alpha\sigma}$ and we refer to it as a *connection coefficient*. Following equation (3.17), we write [5, 55]

$$(\nabla_\alpha - \partial_\alpha)\, v_\beta = -\tilde{\Gamma}^{\sigma}{}_{\alpha\beta} v_\sigma \implies \nabla_\alpha v_\beta = \partial_\alpha v_\beta - \tilde{\Gamma}^{\sigma}{}_{\alpha\beta} v_\sigma \ . \tag{3.19}$$

Similarly, following equation (3.13), the action of the linear map $(\nabla_\alpha - \partial_\alpha)$ on a vector field $\boldsymbol{X}$ yields

$$(\nabla_\alpha - \partial_\alpha)\, X^\beta = \tilde{\Gamma}^{\beta}{}_{\alpha\sigma} X^\sigma \implies \nabla_\alpha X^\beta = \partial_\alpha X^\beta + \tilde{\Gamma}^{\beta}{}_{\alpha\sigma} X^\sigma \ . \tag{3.20}$$

Through the use of equation (3.18) we conclude that, for a type (k,l) tensor field $\boldsymbol{B}$, we have

$$\nabla_\alpha B^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = \partial_\alpha B^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} + \sum_{i=1}^{k} \tilde{\Gamma}^{\mu_i}{}_{\alpha\sigma} B^{\mu_1 \dots \sigma \dots \mu_k}{}_{\nu_1 \dots \nu_l} - \sum_{i=1}^{l} \tilde{\Gamma}^{\sigma}{}_{\alpha\nu_i} B^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \sigma \dots \nu_l} \ . \tag{3.21}$$

Up until now, we have not assumed that the natural choice of an affine connection is metric compatible. In this thesis, we will only be concerned with affine connections that are metric compatible. Given a metric tensor $\boldsymbol{g}$ on a manifold $\mathcal{M}$, the requirement that the affine connection ∇ is metric compatible implies that [5]

$$\nabla_\alpha g_{\mu\nu} = 0 \ . \tag{3.22}$$

The application of equation (3.21) to the above expression yields

$$0 = \nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \tilde{\Gamma}^{\sigma}{}_{\alpha\mu} g_{\sigma\nu} - \tilde{\Gamma}^{\sigma}{}_{\alpha\nu} g_{\mu\sigma} \ . \tag{3.23}$$

By carrying out cyclic permutations of the indices in the above expression, we obtain [76]

$$0 = \nabla_\mu g_{\nu\alpha} = \partial_\mu g_{\nu\alpha} - \tilde{\Gamma}^{\sigma}{}_{\mu\nu} g_{\sigma\alpha} - \tilde{\Gamma}^{\sigma}{}_{\mu\alpha} g_{\nu\sigma} \ , \tag{3.24}$$

and

$$0 = \nabla_\nu g_{\alpha\mu} = \partial_\nu g_{\alpha\mu} - \tilde{\Gamma}^{\sigma}{}_{\nu\alpha} g_{\sigma\mu} - \tilde{\Gamma}^{\sigma}{}_{\nu\mu} g_{\alpha\sigma} \ . \tag{3.25}$$

By adding equations (3.23) and (3.24) and then subtracting equation (3.25), we obtain

$$0 = \partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} + T_{\alpha\nu\mu} + T_{\mu\nu\alpha} - \tilde{\Gamma}^{\sigma}{}_{\alpha\mu} g_{\sigma\nu} - \tilde{\Gamma}^{\sigma}{}_{\mu\alpha} g_{\sigma\nu} \ , \tag{3.26}$$

where we have defined the *torsion tensor*, $T^{\sigma}{}_{\mu\nu}$, to be [8, 33]

$$T^{\sigma}{}_{\mu\nu} := \tilde{\Gamma}^{\sigma}{}_{\mu\nu} - \tilde{\Gamma}^{\sigma}{}_{\nu\mu} \ . \tag{3.27}$$

By substituting $-\tilde{\Gamma}^{\sigma}{}_{\mu\alpha} = T^{\sigma}{}_{\alpha\mu} - \tilde{\Gamma}^{\sigma}{}_{\alpha\mu}$ into equation (3.26), we obtain the following expression for the connection coefficients associated with a non-symmetric, metric compatible affine connection ∇ [77]

$$\tilde{\Gamma}^{\sigma}{}_{\alpha\mu} = \frac{1}{2} g^{\sigma\nu} \left(\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} + T_{\nu\alpha\mu} - T_{\alpha\mu\nu} + T_{\mu\nu\alpha} \right) \ . \tag{3.28}$$

Given a metric tensor $\boldsymbol{g}$ on a manifold $\mathcal{M}$, the natural choice of an affine connection that is non-symmetric and metric compatible has corresponding connection coefficients given by equation (3.28). For the remainder of this thesis, we will use the symbol ∇ to denote the affine connection which is non-symmetric and metric compatible and therefore has the connection coefficients given by equation (3.28). We are now in a position to give the definition of the Riemann-Cartan space-time which is done following [9] by Tseng.

Definition 3.3.1. Let $\mathcal{M}$ be a manifold and let $\boldsymbol{g}$ be a metric tensor on $\mathcal{M}$. Given this metric tensor, we let ∇ be the non-symmetric and metric compatible affine connection, constructed as an additional structure on the manifold $\mathcal{M}$, with connection coefficients given by equation (3.28). We then define the *Riemann-Cartan space-time* to be the tuple $(\mathcal{M}, \boldsymbol{g}, \nabla)$.

Furthermore, we refer to the connection ∇ used in the above definition as the *Cartan connection* [70, 75]. We have now constructed the Riemann-Cartan space-time as desired. As is done in [75] by Audretsch, we will denote the 4-dimensional Riemann-Cartan space-time defined above as U_4. In the following section we wish to study further the Riemann-Cartan space-time defined in this section by studying the notions of curvature and torsion in this space-time.

3.4 Curvature and torsion

In this section, we turn our attention to studying curvature and torsion on the Riemann-Cartan space-time U_4 as defined in Definition 3.3.1. As already mentioned, the natural choice of an affine connection in Riemann-Cartan space-time is non-symmetric as well as metric compatible. The fact that the natural choice of an affine connection is not necessarily symmetric gives rise to non-vanishing torsion. This can be seen in equation (3.27) where the torsion tensor is defined. In the event that the connection coefficients are symmetric in their second and third indices, the torsion tensor vanishes. However, by assuming that the natural choice of an affine connection, ∇, is not necessarily symmetric, the right-hand side of equation (3.27) is non-vanishing and thus, the Riemann-Cartan space-time possesses non-vanishing torsion. Although we have already provided a definition for the torsion tensor, which is given in equation (3.27), in terms of the connection coefficients $\tilde{\Gamma}^{\sigma}{}_{\mu\nu}$, we wish to provide an alternative definition for the torsion tensor which is given below [9, 55].

Definition 3.4.1. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be arbitrary vector fields at a point p in the manifold $\mathcal{M}$. We define the *torsion tensor*, $\boldsymbol{T}$, at the point p to be the type $(1, 2)$ tensor field that is such that

$$\boldsymbol{T}(\boldsymbol{X}, \boldsymbol{Y}) = \nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} - [\boldsymbol{X}, \boldsymbol{Y}] \ . \tag{3.29}$$

Our goal now is to derive the above relation from the definition given in equation (3.27). This is carried out by following the derivations given in [76, 77]. In order to show how the definition of the torsion tensor given in equation (3.27) leads to equation (3.29), we first note that equation (3.29) can be written as [5]

$$T^{\sigma}{}_{\mu\nu}X^{\mu}Y^{\nu} = X^{\nu}\nabla_{\nu}Y^{\sigma} - Y^{\nu}\nabla_{\nu}X^{\sigma} - [\boldsymbol{X}, \boldsymbol{Y}]^{\sigma} \ . \tag{3.30}$$

Now, let f be a C^{∞} function on the manifold $\mathcal{M}$. We now consider the action of the Lie bracket on f by making use of its definition:

$$\begin{aligned}[\boldsymbol{X}, \boldsymbol{Y}] f &= X^{\nu}\nabla_{\nu}\left(Y^{\mu}\nabla_{\mu}f\right) - Y^{\nu}\nabla_{\nu}\left(X^{\mu}\nabla_{\mu}f\right) \\ &= \left(X^{\nu}\nabla_{\nu}Y^{\sigma} - Y^{\nu}\nabla_{\nu}X^{\sigma}\right)\nabla_{\sigma}f + X^{\mu}Y^{\nu}\left(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}\right) f \ . \end{aligned} \tag{3.31}$$

Let us now consider the action of the covariant derivative ∇_{μ} on the type $(0, 1)$ tensor field $\nabla_{\nu}f$. Through the application of equation (3.19) we obtain the following

$$\nabla_{\mu}\nabla_{\nu}f = \partial_{\mu}\partial_{\nu}f - \tilde{\Gamma}^{\sigma}{}_{\mu\nu}\nabla_{\sigma}f \ . \tag{3.32}$$

By swapping the lower indices in the above expression we obtain the following

$$\nabla_{\nu}\nabla_{\mu}f = \partial_{\nu}\partial_{\mu}f - \tilde{\Gamma}^{\sigma}{}_{\nu\mu}\nabla_{\sigma}f \ . \tag{3.33}$$

By subtracting equation (3.33) from equation (3.32) as well as by making use of the fact that the partial derivatives commute, we obtain

$$\left(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}\right) f = -T^{\sigma}{}_{\mu\nu}\nabla_{\sigma}f \ , \tag{3.34}$$

where we have made use of equation (3.27) in order to introduce the torsion tensor into the above expression. By substituting equation (3.34) into equation (3.31) and by making the argument that the resulting expression must be satisfied for all dual vector fields $\nabla_{\sigma}f$, equation (3.30) is obtained.

Before moving on to discuss the notion of curvature in the Riemann-Cartan space-time, we first wish to make a note on the existence of a metric compatible, symmetric affine connection which we shall denote as D. This metric compatible, symmetric affine connection is referred to as the Levi-Civita connection and is the only metric compatible affine connection in space-times for which only symmetric connections are considered [5, 6]. This result is known as the *Fundamental Theorem of Riemannian Geometry* and the interested reader is directed to [6] for a detailed discussion. We denote the pseudo-tensor used to describe the action of the linear map $(D_\alpha - \partial_\alpha)$ on a dual vector field, say $\boldsymbol{v}$, as $\Gamma^{\sigma}{}_{\alpha\beta}$. That is, we write

$$D_\alpha v_\beta = \partial_\alpha v_\beta - \Gamma^{\sigma}{}_{\alpha\beta} v_\sigma \ . \tag{3.35}$$

By definition, the connection coefficients of the Levi-Civita connection are symmetric in their second and third indices, i.e., $\Gamma^{\sigma}{}_{\alpha\beta} = \Gamma^{\sigma}{}_{\beta\alpha}$. Clearly, the torsion tensor associated with the Levi-Civita connection is zero. Furthermore, the Levi-Civita connection is nothing more than the natural choice of an affine connection in Riemann-Cartan space-time in the absence of torsion. Therefore, by considering equation (3.28) for the case where we are dealing with the torsion-free connection D with symmetric connection coefficients $\Gamma^{\sigma}{}_{\alpha\sigma}$, we have the following expression for the connection coefficients [5, 55, 78]

$$\Gamma^{\sigma}{}_{\alpha\mu} = \frac{1}{2} g^{\sigma\nu} \left(\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} \right) \ . \tag{3.36}$$

These connection coefficients are referred to as *Christoffel symbols* and, as can be seen from the above expression, depend only on the metric tensor [5]. Although the Levi-Civita connection is not the natural choice of an affine connection in the Riemann-Cartan space-time, it can still be considered as an additional structure on the manifold $\mathcal{M}$ included in the definition of U_4 [57]. Therefore, we can still make use of the Levi-Civita connection when considering Riemann-Cartan space-times, however, we must rely on the natural choice of an affine connection, ∇, in order to obtain the relevant geometrical descriptions of the space-time. For example, the curvature and torsion of the space-time U_4 are determined using the Cartan connection ∇.

Since we are interested in considering the Levi-Civita connection as an additional structure on the Riemann-Cartan space-time U_4, we wish to relate it to the natural choice of an affine connection ∇ for such a space-time. In order to do this, we define the so-called *contorsion tensor* which is defined as [8, 33, 77]

$$K_{\mu\nu\alpha} := \frac{1}{2} \left(T_{\mu\nu\alpha} - T_{\nu\alpha\mu} + T_{\alpha\mu\nu} \right) \ . \tag{3.37}$$

By substituting the above expression for the contorsion tensor as well as the expression for the Christoffel symbols given in equation (3.36) into equation (3.28), we obtain the following expression for the connection coefficients of the Cartan connection [77]

$$\tilde{\Gamma}^{\sigma}{}_{\mu\nu} = \Gamma^{\sigma}{}_{\mu\nu} + K^{\sigma}{}_{\mu\nu} \ . \tag{3.38}$$

That is, the contorsion tensor is defined as the difference between the Cartan connection coefficients and the Christoffel symbols. By making use of the above expression, we can use the contorsion tensor to relate the Cartan connection and the Levi-Civita connection [8, 33, 77]. By considering the action of the Cartan connection on a dual vector field, say $\boldsymbol{v}$, we can make use of equation (3.38) in order to obtain

$$\nabla_\alpha v_\beta = \partial_\alpha v_\beta - \Gamma^{\sigma}{}_{\alpha\beta} v_\sigma - K^{\sigma}{}_{\alpha\beta} v_\sigma \ . \tag{3.39}$$

It therefore follows from equation (3.35) that we have

$$\nabla_\alpha v_\beta = D_\alpha v_\beta - K^{\sigma}{}_{\alpha\beta} v_\sigma \implies \left(\nabla_\alpha - D_\alpha \right) v_\beta = -K^{\sigma}{}_{\alpha\beta} v_\sigma \ . \tag{3.40}$$

Therefore, from the above expression, we can interpret the tensor field $K^{\sigma}{}_{\alpha\beta}$ as the specific tensor field that describes the action of the linear map $(\nabla_\alpha - D_\alpha)$ on a dual vector field.

Having found the tensor field that describes the action of the difference between the Cartan connection and the Levi-Civita connection, which we refer to as the contorsion tensor, we wish to now turn our attention to the notion of curvature of the Riemann-Cartan space-time U_4. Here we will be interested in discussing the notion of curvature associated with the Cartan connection as well as the Levi-Civita connection. The reason for this is that the curvature associated with the Levi-Civita connection is well understood in the context of torsion-free space-times and satisfies useful properties that we wish to make use of [5]. In addition, by relating the two aforementioned curvature tensor fields, we will be able to examine how the expressions differ in the presence of torsion. This examination is carried out following [76, 79].

Let f be a smooth function on the manifold $\mathcal{M}$ and let $\boldsymbol{v}$ be a dual vector field. By considering the action of the map $\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha + T^\rho{}_{\alpha\beta} \nabla_\rho\right)$ on $f v_\sigma$, it is not difficult to show that (see [76] for a derivation)

$$\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha + T^\rho{}_{\alpha\beta} \nabla_\rho\right)(f v_\sigma) = f\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha + T^\rho{}_{\alpha\beta} \nabla_\rho\right) v_\sigma \ . \tag{3.41}$$

That is, the action of the map $\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha + T^\rho{}_{\alpha\beta} \nabla_\rho\right)$ on a dual vector field at $p \in \mathcal{M}$ depends only on the value taken on by the dual vector field at the point $p \in \mathcal{M}$. This implies that the map $\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha + T^\rho{}_{\alpha\beta} \nabla_\rho\right)$ is linear and thus, we can introduce a tensor field, say $\tilde{R}_{\alpha\beta\sigma}{}^\rho$, that describes the action of this linear map. That is, we have

$$\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha + T^\rho{}_{\alpha\beta} \nabla_\rho\right) v_\sigma = \tilde{R}_{\alpha\beta\sigma}{}^\rho v_\rho \ . \tag{3.42}$$

We refer to the the tensor field $\tilde{R}_{\alpha\beta\sigma}{}^\rho$ as the *non-null torsion Riemann tensor.* This tensor gives us the notion of curvature in the Riemann-Cartan space-time U_4 since it is the curvature tensor associated with the Cartan connection which is the natural choice of an affine connection in U_4. Furthermore, we define the *non-null torsion Ricci tensor* to be

$$\tilde{R}_{\mu\nu} := \tilde{R}_{\mu\sigma\nu}{}^\sigma \ . \tag{3.43}$$

In addition, we define the *non-null torsion Ricci scalar* to be

$$\tilde{R} := \tilde{R}_\nu{}^\nu \tag{3.44}$$

It is clear from the above discussion that both torsion and curvature are aspects of a given affine connection [33, 68]. It is important to note, however, that since an affine connection is viewed as an additional structure on the space-time manifold $\mathcal{M}$ [55], one can have multiple affine connections defined on $\mathcal{M}$ [57]. It is because of this that we may consider the Levi-Civita connection as an additional structure on U_4. So far, we have discussed the curvature associated with the Cartan connection. We now wish to consider the notion of curvature for which the Levi-Civita connection is considered. Since the Levi-Civita connection is nothing more than the Cartan connection in the absence of torsion, it can be deduced that $(D_\alpha D_\beta - D_\beta D_\alpha)$ is a linear map [5]. It therefore follows that we can introduce a tensor field, say $R_{\alpha\beta\sigma}{}^\rho$, that is such that for a dual vector field $\boldsymbol{v}$ we have [5]

$$(D_\alpha D_\beta - D_\beta D_\alpha) v_\sigma = R_{\alpha\beta\sigma}{}^\rho v_\rho \ . \tag{3.45}$$

We refer to the tensor field $R_{\alpha\beta\sigma}{}^\rho$ as the *torsion-free Riemann tensor.* As was done for the non-null torsion Riemann tensor, we can define the *torsion-free Ricci tensor* as

$$R_{\mu\nu} := R_{\mu\sigma\nu}{}^\sigma \ , \tag{3.46}$$

as well as the *torsion-free Ricci scalar* as

$$R := R_\nu{}^\nu \ . \tag{3.47}$$

Before we move on to discuss the relationship between the non-null torsion Riemann tensor and the torsion-free Riemann tensor, we first wish to discuss the properties satisfied by these tensor fields. We will consider the properties satisfied by the non-null torsion Riemann tensor, following [5, 76], and we will then use these to determine the properties satisfied by the torsion-free Riemann tensor by considering what happens when we have vanishing torsion.

We begin by considering the action of the map given in equation (3.42) on the metric tensor $\boldsymbol{g}$. By carrying out this action and noting that the Cartan connection is metric compatible, we have [76]

$$0 = \left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha + T^\rho{}_{\alpha\beta} \nabla_\rho\right) g_{\mu\nu} = \tilde{R}_{\alpha\beta\mu\nu} + \tilde{R}_{\alpha\beta\nu\mu} \ , \tag{3.48}$$

where we have carried out a contraction of the fourth index in the Riemann tensors through the use of the metric tensor. It can be seen from the above expression that the non-null torsion Riemann tensor is antisymmetric in its third and fourth indices. Clearly the same is true for the torsion-free Riemann tensor since the torsion-free Riemann tensor is nothing more than the non-null torsion Riemann tensor in the case of vanishing torsion. Let us now turn our attention to equation (3.42) explicitly. By carrying out an index substitution in this expression we have

$$\left(\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta + T^\rho{}_{\beta\alpha} \nabla_\rho\right) v_\sigma = \tilde{R}_{\beta\alpha\sigma}{}^\rho v_\rho \ . \tag{3.49}$$

By adding equations (3.42) and (3.49) we obtain [76]

$$\left(T^\rho{}_{\alpha\beta} + T^\rho{}_{\beta\alpha}\right) \nabla_\rho v_\sigma = \left(\tilde{R}_{\alpha\beta\sigma}{}^\rho + \tilde{R}_{\beta\alpha\sigma}{}^\rho\right) v_\rho \ . \tag{3.50}$$

By making use of the fact that the torsion tensor is antisymmetric in its second and third indices and that the above expression holds for an arbitrary dual vector field, we have

$$\tilde{R}_{\alpha\beta\sigma}{}^\rho = -\tilde{R}_{\beta\alpha\sigma}{}^\rho \ . \tag{3.51}$$

That is, the non-null torsion Riemann tensor is antisymmetric in its first and second indices. We again argue that the torsion-free Riemann tensor is also antisymmetric in its first and second indices.

In considering the torsion-free Riemann tensor, it is known [5] that the tensor field obtained by antisymmetrising the first three indices of the torsion-free Riemann tensor vanishes. That is, the torsion-free Riemann tensor satisfies the following property

$$R_{[\alpha\beta\sigma]}{}^\rho = 0 \ , \tag{3.52}$$

where we have made use of the notational convention described in equation (3.2). The above property is referred to as the first Bianchi identity for the torsion-free Riemann tensor. By following the derivation given in the author's previous work [76], we can obtain a corresponding expression for the first Bianchi identity for the non-null torsion Riemann tensor. In order to obtain such an expression, we wish to study the type $(0, 3)$ tensor field $\nabla_{[\alpha} \nabla_\beta v_{\sigma]}$ where v_σ is an arbitrary dual vector field. On the one hand, by making use of equations (3.19) and (3.27), we can write this aforementioned type $(0, 3)$ tensor field as [76]

$$\nabla_{[\alpha} \nabla_\beta v_{\sigma]} = \frac{1}{2} \left(\partial_{[\alpha} T^\rho{}_{\sigma\beta]} v_\rho - T^\rho{}_{[\alpha\beta} \partial_{|\rho|} v_{\sigma]} - T^\rho{}_{[\alpha\beta} T^\lambda{}_{\sigma]\rho} v_\lambda\right) \ . \tag{3.53}$$

That is, we have given an expression for the tensor field $\nabla_{[\alpha} \nabla_\beta v_{\sigma]}$ by writing the Cartan connection in terms of partial derivatives and connection coefficients. On the other hand, through the use of the definition of the non-null torsion Riemann tensor given in equation (3.42), we can write the tensor field $\nabla_{[\alpha} \nabla_\beta v_{\sigma]}$ as [76]

$$\nabla_{[\alpha} \nabla_\beta v_{\sigma]} = \frac{1}{2} \left(\tilde{R}_{[\alpha\beta\sigma]}{}^\rho v_\rho - T^\rho{}_{[\alpha\beta} \partial_{|\rho|} v_{\sigma]} + T^\rho{}_{[\alpha\beta} \tilde{\Gamma}^\lambda{}_{|\rho|\sigma]} v_\lambda\right) \ . \tag{3.54}$$

By equating the right-hand sides of equations (3.53) and (3.54), we obtain [76]

$$\tilde{R}_{[\alpha\beta\sigma]}{}^\rho = \partial_{[\alpha} T^\rho{}_{\sigma\beta]} - T^\lambda{}_{[\alpha\beta} \tilde{\Gamma}^\rho{}_{\sigma]\lambda} \ . \tag{3.55}$$

We have thus obtained the expression for the first Bianchi identity associated with the non-null torsion Riemann tensor. As expected, when the torsion tensor vanishes, equation (3.55) reduces to equation

(3.52). Lastly, we wish to investigate the property referred to as the second Bianchi identity. For the torsion-free Riemann tensor, it is known [5] that it satisfies the second Bianchi identity

$$D_{[\alpha}R_{\beta\sigma]\rho}{}^{\lambda} = 0 \ . \tag{3.56}$$

We wish to derive a similar relation for the case where the non-null torsion Riemann tensor is considered. That is, we wish to consider the type $(1,4)$ tensor field obtained by acting the Cartan covariant derivative on the non-null torsion Riemann tensor and by antisymmetrising the first three indices. Here, we follow the derivation provided in [76] which is carried out following the method given in [5] for the case of a Riemannian space-time. For an arbitrary dual vector field $\boldsymbol{v}$, we consider the type $(0,4)$ tensor field $\nabla_{[\alpha}\nabla_{\beta}\nabla_{\sigma]}v_{\rho}$ and the following two equivalent expressions for the tensor field

$$6\nabla_{[\alpha}\nabla_{\beta}\nabla_{\sigma]}v_{\rho} = \left(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha}\right)\nabla_{\sigma}v_{\rho} + \left(\nabla_{\beta}\nabla_{\sigma} - \nabla_{\sigma}\nabla_{\beta}\right)\nabla_{\alpha}v_{\rho} + \left(\nabla_{\sigma}\nabla_{\alpha} - \nabla_{\alpha}\nabla_{\sigma}\right)\nabla_{\beta}v_{\rho} \tag{3.57}$$

$$= \nabla_{\alpha}\left(\nabla_{\beta}\nabla_{\sigma} - \nabla_{\sigma}\nabla_{\beta}\right)v_{\rho} + \nabla_{\beta}\left(\nabla_{\sigma}\nabla_{\alpha} - \nabla_{\alpha}\nabla_{\sigma}\right)v_{\rho} + \nabla_{\sigma}\left(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha}\right)v_{\rho} \ . \tag{3.58}$$

Let us first turn our attention to equation (3.57). By making use of the definition of the non-null torsion Riemann tensor given in equation (3.42) we have

$$\left(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha}\right)\nabla_{\sigma}v_{\rho} = \tilde{R}_{\alpha\beta\sigma}{}^{\lambda}\nabla_{\lambda}v_{\rho} + \tilde{R}_{\alpha\beta\rho}{}^{\lambda}\nabla_{\sigma}v_{\lambda} - T^{\lambda}_{\alpha\beta}\nabla_{\lambda}\nabla_{\sigma}v_{\rho} \ . \tag{3.59}$$

By substituting equation (3.59) into equation (3.57), one obtains

$$\nabla_{[\alpha}\nabla_{\beta}\nabla_{\sigma]}v_{\rho} = \frac{1}{2}\left(\tilde{R}_{[\alpha\beta\sigma]}{}^{\lambda}\nabla_{\lambda}v_{\rho} + \tilde{R}_{[\alpha\beta|\rho|}{}^{\lambda}\nabla_{\sigma]}v_{\lambda} - T^{\lambda}_{[\alpha\beta}\nabla_{|\lambda|}\nabla_{\sigma]}v_{\rho}\right) \ . \tag{3.60}$$

We now turn our attention to the expression given in equation (3.57). We again make use of the definition of the non-null torsion Riemann tensor given in equation (3.42) and obtain

$$\nabla_{\alpha}\left(\nabla_{\beta}\nabla_{\sigma} - \nabla_{\sigma}\nabla_{\beta}\right)v_{\rho} = \left(\nabla_{\alpha}\tilde{R}_{\beta\sigma\rho}{}^{\lambda}\right)v_{\lambda} + \tilde{R}_{\beta\sigma\rho}{}^{\lambda}\nabla_{\alpha}v_{\lambda} - \nabla_{\alpha}\left(T^{\lambda}_{\beta\sigma}\nabla_{\lambda}v_{\rho}\right) \ . \tag{3.61}$$

By substituting equation (3.61) into the right-hand side of equation (3.58), one obtains

$$\nabla_{[\alpha}\nabla_{\beta}\nabla_{\sigma]}v_{\rho} = \frac{1}{2}\left[\left(\nabla_{[\alpha}\tilde{R}_{\beta\sigma]\rho}{}^{\lambda}\right)v_{\lambda} + \tilde{R}_{[\beta\sigma|\rho|}{}^{\lambda}\nabla_{\alpha]}v_{\lambda} - \nabla_{[\alpha}\left(T^{\lambda}_{\beta\sigma]}\nabla_{\lambda}v_{\rho}\right)\right] \ , \tag{3.62}$$

where we have made use of equation (3.55) in carrying out a simplification. By equating the right-hand sides of equations (3.60) and (3.62), we obtain the following expression for the non-null torsion Riemann tensor

$$\left(\nabla_{[\alpha}\tilde{R}_{\beta\sigma]\rho}{}^{\lambda}\right)v_{\lambda} = 2\tilde{\Gamma}^{\lambda}_{[\alpha\beta}T^{\nu}{}_{\sigma]\lambda}\nabla_{\nu}v_{\rho} + T^{\lambda}_{[\alpha\beta}\nabla_{\sigma]}\nabla_{\lambda}v_{\rho} - T^{\lambda}_{[\alpha\beta}\nabla_{|\lambda|}\nabla_{\sigma]}v_{\rho} \ , \tag{3.63}$$

which is what was obtained in [76] following the method described in [5] where the case of the torsion-free Riemann tensor was considered. Now, by once again making use of the definition of the non-null torsion Riemann tensor given in equation (3.42), we can introduce the non-null torsion Riemann tensor into the right-hand side of equation (3.63). By considering the resulting expression and noting that it should hold for any dual vector field, we obtain the following expression

$$\nabla_{[\alpha}\tilde{R}_{\beta\sigma]\rho}{}^{\lambda} = T^{\nu}{}_{[\alpha\beta}\tilde{R}_{\sigma]\nu\rho}{}^{\lambda} \ . \tag{3.64}$$

The above expression is the second Bianchi identity associated with the non-null torsion Riemann tensor. As expected, when there is vanishing torsion and therefore the Cartan connection reduces to the Levi-Civita connection, the Bianchi identity given in equation (3.64) reduces to equation (3.56) which is the Bianchi identity associated with the torsion-free Riemann tensor. As we shall see in subsequent chapters, the implications of having equations (3.55) and (3.64) being non-vanishing for non-vanishing torsion are significant when considering theories for which one considers non-vanishing torsion [7, 74, 79, 80].

Having now derived the non-null torsion Bianchi identities, we now wish to relate the non-null torsion Riemann tensor to the torsion-free Riemann tensor in order to determine the so-called "non-Riemannian contributions" [74]. The non-Riemannian contributions are the additional terms that contribute to the construction of a tensor field defined using the Cartan connection that have no contribution in the construction of the tensor field when defined using the Levi-Civita connection. That is, the non-Riemannian terms are the terms that contribute to non-null torsion quantities in addition to the torsion-free terms. It is of course expected that these terms vanish in the case of vanishing torsion since, in such a case, the Cartan connection and Levi-Civita connection would coincide. The relating of the non-null torsion Riemann tensor to the torsion-free Riemann tensor allows one to relate the two Ricci scalars, i.e., the non-null torsion and torsion-free Ricci scalars. The reason for studying this is due to the fact that these scalars play an important role when constructing the gravitational action for a given theory [7, 79]. In order to obtain the aforementioned relationship between the non-null torsion Riemann tensor and the torsion-free Riemann tensor, we note that by introducing the connection coefficients of the Cartan connection into the left hand side of equation (3.42) and noting that the resulting expression holds for any dual vector field $\boldsymbol{v}$, we have the following expression for the non-null torsion Riemann tensor

$$\tilde{R}_{\alpha\beta\sigma}{}^{\rho} = \partial_\beta \tilde{\Gamma}^{\rho}{}_{\alpha\sigma} - \partial_\alpha \tilde{\Gamma}^{\rho}{}_{\beta\sigma} + \tilde{\Gamma}^{\delta}{}_{\alpha\sigma}\tilde{\Gamma}^{\rho}{}_{\beta\delta} - \tilde{\Gamma}^{\delta}{}_{\beta\sigma}\tilde{\Gamma}^{\rho}{}_{\alpha\delta} \ . \tag{3.65}$$

It is not difficult to see from the above expression that, when considering the Levi-Civita connection which is simply the Cartan connection in the case of vanishing torsion, the torsion-free Riemann tensor can be written as

$$R_{\alpha\beta\sigma}{}^{\rho} = \partial_\beta \Gamma^{\rho}{}_{\alpha\sigma} - \partial_\alpha \Gamma^{\rho}{}_{\beta\sigma} + \Gamma^{\delta}{}_{\alpha\sigma}\Gamma^{\rho}{}_{\beta\delta} - \Gamma^{\delta}{}_{\beta\sigma}\Gamma^{\rho}{}_{\alpha\delta} \ . \tag{3.66}$$

By introducing the contorsion tensor into the right-hand side of equation (3.65) through the use of equation (3.38) and identifying the Riemannian terms that contribute to the torsion-free Riemann tensor as given in equation (3.66), one obtains the following expression for the non-null torsion Riemann tensor [76, 79]

$$\tilde{R}_{\alpha\beta\sigma}{}^{\rho} = R_{\alpha\beta\sigma}{}^{\rho} + D_\beta K^{\rho}{}_{\alpha\sigma} - D_\alpha K^{\rho}{}_{\beta\sigma} + K^{\rho}{}_{\beta\lambda}K^{\lambda}{}_{\alpha\sigma} - K^{\rho}{}_{\alpha\lambda}K^{\lambda}{}_{\beta\sigma} \ . \tag{3.67}$$

By contracting the first and fourth indices in the above expression, one obtains the following expression which relates the non-null torsion Ricci tensor to the torsion free Ricci tensor

$$\tilde{R}_{\alpha\sigma} = R_{\alpha\sigma} + D_\beta K^{\beta}{}_{\alpha\sigma} - D_\alpha K^{\beta}{}_{\beta\sigma} + K^{\rho}{}_{\alpha\sigma}K^{\beta}{}_{\beta\rho} - K^{\rho}{}_{\beta\sigma}K^{\beta}{}_{\alpha\rho} \ . \tag{3.68}$$

We can now make use of the above expression in order to obtain an expression that relates the non-null torsion Ricci scalar to the torsion-free Ricci scalar. By taking the trace of equation (3.68) one obtains

$$\tilde{R} = R + 2D_\alpha T^\alpha + T \ , \tag{3.69}$$

where we have defined $T^\alpha := T^{\beta\alpha}{}_{\beta}$ to be trace of the torsion tensor and we have defined

$$T := \frac{1}{4}T_{\sigma}{}^{\beta\alpha}T^{\sigma}{}_{\beta\alpha} - \frac{1}{2}T^{\beta\alpha}{}_{\sigma}T^{\sigma}{}_{\beta\alpha} - T^{\beta\alpha}{}_{\beta}T^{\sigma}{}_{\alpha\sigma} \ , \tag{3.70}$$

to be the *torsion scalar* [8, 81]. We note that it is possible to write the torsion scalar as the torsion tensor contracted with a type $(2,1)$ tensor field say $S_{\rho}{}^{\mu\nu}$. In constructing this tensor field, it is useful to introduce the tensor field

$$W_{\nu\mu\alpha} := K_{\nu\mu\alpha} - T_{\nu\mu\alpha} = \frac{1}{2}\left(-T_{\nu\mu\alpha} - T_{\mu\alpha\nu} + T_{\alpha\nu\mu}\right) \ , \tag{3.71}$$

which we shall refer to as the *Weitzenböck contorsion* tensor. By making use of the Weitzenböck contorsion tensor and by defining the tensor field

$$S_{\rho}{}^{\mu\nu} := \frac{1}{2}\left(W^{\mu\nu}{}_{\rho} + \delta^{\mu}_{\rho}T^{\nu} - \delta^{\nu}_{\rho}T^{\mu}\right) \ , \tag{3.72}$$

where δ^{μ}_{ρ} is the Kronecker delta symbol, it is not difficult to show that [8,33,35,36,81]

$$T = T^{\rho}_{\ \mu\nu} S_{\rho}^{\ \mu\nu} \ . \tag{3.73}$$

We note that the tensor field $S_{\rho}^{\ \mu\nu}$ is often referred to as the *superpotential* [8].

The importance of the torsion scalar is apparent from equation (3.69). More specifically, equation (3.69) tells us that the non-Riemannian contribution to the non-null torsion Ricci scalar is composed of the torsion scalar T and a total Levi-Civita covariant derivative of the trace of the torsion scalar. As we shall see in Chapter 4, this plays an important role in gravitational theories for which the associated action contains a Lagrangian density that is linear in either the non-null torsion or torsion-free Ricci scalar. The reason for this is that, for such an action, the total covariant derivative can be integrated out and will therefore have no contribution in the resulting equations of motion. An example of such theories are the so-called Teleparallel Theories of Gravity for which the natural choice of an affine connection yields zero curvature [8,32]. Such theories will be discussed in detail in Chapter 4. For now, however, it will prove to be useful to discuss the decomposition of the torsion tensor as provided in [79,80]. We can then write equation (3.69) in terms of the quantities included in the aforementioned decomposition of the torsion tensor.

The notion of decomposition can be done for any tensor field. Namely, it is possible to decompose a given tensor field into a trace component, a totally symmetric component, a totally antisymmetric component and, lastly, a component that defines the difference between the tensor field and the other decomposed quantities [5,79]. Following what is done in the review article [79] by Shapiro, we consider the following decomposition of the torsion tensor

$$T_{\nu\mu\alpha} = \frac{1}{3}\left(T_{\mu} g_{\nu\alpha} - T_{\alpha} g_{\nu\alpha}\right) + T_{[\nu\mu\alpha]} + Q_{\nu\mu\alpha} \ . \tag{3.74}$$

In the above decomposition, the first term is the trace component while the second term is the totally antisymmetric component [79]. The $Q_{\nu\mu\alpha}$ tensor field defines the difference between the torsion tensor and the sum of the trace component and the totally antisymmetric component. We note that the $Q_{\nu\mu\alpha}$ tensor field is, of course, antisymmetric in its second and third indices and, in addition, it is traceless [79]. Furthermore, it is not difficult to check that this tensor field satisfies the property

$$\epsilon^{\alpha\beta\mu\nu} Q_{\alpha\beta\mu} = 0 \ , \tag{3.75}$$

where $\epsilon^{\alpha\beta\mu\nu}$ is the Levi-Civita tensor. At this point, we wish to introduce the vector field $\boldsymbol{V}$ defined through the following expression [79]

$$V^{\nu} := \epsilon^{\alpha\beta\mu\nu} T_{\alpha\beta\mu} \ . \tag{3.76}$$

We note that, through the use of the vector field defined above, we can write the totally antisymmetric term in the decomposition of the torsion tensor as follows

$$T_{[\nu\mu\alpha]} = -\frac{1}{6}\epsilon_{\nu\mu\alpha\rho} V^{\rho} \ . \tag{3.77}$$

We now wish to write equation (3.69) in terms of the decomposition of the torsion tensor. In order to do this, we turn our attention to the definition of the torsion scalar given in equation (3.70). We have the following expressions for the first two quadratic terms that contribute to the definition of the torsion scalar in equation (3.70)

$$T_{\nu\mu\alpha} T^{\nu\mu\alpha} = \frac{2}{3} T^{\nu} T_{\nu} - \frac{1}{6} V_{\alpha} V^{\alpha} + Q_{\nu\mu\alpha} Q^{\nu\mu\alpha} \ , \tag{3.78}$$

$$T_{\mu\alpha\nu} T^{\nu\mu\alpha} = -\frac{1}{3} T^{\alpha} T_{\alpha} - \frac{1}{6} V_{\alpha} V^{\alpha} + Q_{\mu\alpha\nu} Q^{\nu\mu\alpha} \ . \tag{3.79}$$

Substituting the above two expressions into equation (3.70) yields an expression for the torsion scalar in terms of the quantities included in the decomposition of the torsion tensor. Substituting this resulting expression into equation (3.69) yields[2] the following equation [79, 80]

$$\tilde{R} = R + 2D_\mu T^\mu - \frac{2}{3}T_\alpha T^\alpha + \frac{1}{24}V_\alpha V^\alpha + \frac{1}{2}Q_{\nu\mu\alpha}Q^{\nu\mu\alpha} \ . \tag{3.80}$$

We have thus obtained an expression relating the non-null torsion Ricci scalar to the torsion-free Ricci scalar with the non-Riemannian contributions written in terms of the decomposition of the torsion tensor. This expression has been obtained following [79, 80] by Shapiro and will play an important role when considering specific gravitational theories constructed on the Riemann-Cartan space-time U_4.

When constructing gravitational theories on a Riemann-Cartan space-time, it is often useful to make use of the so-called Cartan formalism which is the subject of the following section.

3.5 Cartan formalism

In this section, we discuss the Cartan formalism in the context of the Riemann-Cartan space-time U_4 defined previously. This discussion is carried out by mainly following what is done in [7] by Trautman, [9] by Tseng and [16] by Nakahara. The reader interested in a more in-depth discussion on the topics mentioned in this section is directed to the aforesaid references. The Cartan formalism involves studying curvature and torsion on a Riemann-Cartan space-time through the use of a non-holonomic basis. Specifically, a non-holonomic basis is a non-coordinate basis composed of smooth vector fields that are orthonormal to each other [5]. We denote these vector fields as $\hat{\boldsymbol{e}}_a$. Let $p \in \mathcal{M}$ where $\mathcal{M}$ is the Riemann-Cartan space-time manifold. Let $\{\partial/\partial x^\nu\}$ be a coordinate basis for the tangent space $T_p\mathcal{M}$ at the point p. The smooth vector fields that make up the non-holonomic basis are then written as the following linear combination of the coordinate basis elements

$$\hat{\boldsymbol{e}}_a = e_a{}^\nu \frac{\partial}{\partial x^\nu} = e_a{}^\nu \partial_\nu \ , \tag{3.81}$$

where the $e_a{}^\nu$ are elements of the general linear group $\mathrm{GL}_4(\mathbb{R})$ [7, 16]. That is, the coefficients $e_a{}^\nu$ are 4×4 invertible matrices with real entries. The above expression implies that the non-holonomic basis elements are linear combinations of rotations of the coordinate basis elements with these rotations being described by the coefficients $e_a{}^\nu$ [16]. As is common for the case where a four-dimensional space-time is considered, we shall refer to $\hat{\boldsymbol{e}}_a$ as a tetrad and refer to its coefficients $e_a{}^\nu$ as the associated *vierbein* [5, 8, 16]. Furthermore, we note that the Greek indices are lowered and raised through the use of the metric tensor and the Latin indices are lowered and raised through the use of the Minkowski metric $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. Consider the dual space $T_p^*\mathcal{M}$ for a point $p \in \mathcal{M}$. We have already introduced the holonomic basis $\{\partial/\partial x^\nu\}$ for the tangent space $T_p\mathcal{M}$. We let $\{\mathrm{d}x^\nu\}$ be the coordinate basis for the dual space. We note that the tangent space basis element $\partial/\partial x^\nu = \partial_\nu$ and the dual space basis element $\mathrm{d}x^\nu$ satisfy the orthogonality relation given in equation (2.17). That is, we have

$$\langle \mathrm{d}x^\mu, \partial_\nu \rangle = \delta^\mu_\nu \ . \tag{3.82}$$

Now, following [16] by Nakahara, we introduce the basis $\left\{\hat{\boldsymbol{\theta}}^a\right\}$ that is dual to the non-holonomic basis $\{\hat{\boldsymbol{e}}_a\}$. That is, we have

$$\langle \hat{\boldsymbol{\theta}}^a, \hat{\boldsymbol{e}}_b \rangle = \delta^a_b \ . \tag{3.83}$$

In terms of a linear combination of elements of the coordinate basis for the dual space, we can write the dual tetrad field $\hat{\boldsymbol{\theta}}^a$ as

$$\hat{\boldsymbol{\theta}}^a = e^a{}_\nu \mathrm{d}x^\nu \ . \tag{3.84}$$

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[2] We note that there is an error in the expression as it is given in [79]. Here, in equation (3.80), we give the corrected equation which is given in [80].

It is important to note that we require the tetrad field to be related to the metric tensor. Given the Riemann-Cartan space-time U_4 with metric tensor $g_{\mu\nu}$, we require that the coefficients of the tetrad field satisfy the following expressions [8, 16]

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \ , \tag{3.85}$$

and

$$\eta_{ab} = g_{\mu\nu} e_a{}^\mu e_b{}^\nu \ . \tag{3.86}$$

Given a space-time with metric tensor $\boldsymbol{g}$, we can construct a tetrad field through the use of equation (3.85). We note, however, that there will not be a unique choice of a tetrad field given a metric tensor. Furthermore, although the metric tensor is Lorentz invariant, the tetrad field does not in general satisfy this property of Lorentz invariance. For theories of gravity where the space-time considered is constructed using an affine connection that is dependent directly on the tetrad field instead of the metric tensor, such as the so-called extended teleparallel theories of gravity, the theories are not necessarily Lorentz invariant [3]. This results from the fact that the connection coefficients will not be Lorentz invariant since the tetrad field itself is not Lorentz invariant. Extended teleparallel theories of gravity will be considered in Chapter 4 where the fact that these theories are not in general Lorentz invariant will be discussed. For now, however, we continue with the discussion of the Cartan formalism using the expressions given above that relate the tetrad fields to the metric tensor.

We note that the metric tensor $\boldsymbol{g}$ can be written as the following tensor product of dual tetrad fields [16]

$$\boldsymbol{g} = g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \eta_{ab} \hat{\boldsymbol{\theta}}^a \otimes \hat{\boldsymbol{\theta}}^b \ , \tag{3.87}$$

where we have made use of equations (3.84) and (3.85) in obtaining the above expression. Furthermore, the conditions of orthonormality given in equations (3.82) and (3.83) imply that $e^a{}_\mu e_a{}^\nu = \delta^\mu_\nu$ and $e^a{}_\mu e_b{}^\mu = \delta^a_b$ respectively. Let us now consider the Cartan connection ∇ given a non-holonomic frame of the form given in equation (3.81). For such a connection, we define the Cartan connection coefficients $\tilde{\Gamma}^k{}_{ab}$ with respect to this non-holonomic basis through the relation [16]

$$\nabla_a \hat{\boldsymbol{e}}_b = e_a{}^\nu \nabla_\nu \hat{\boldsymbol{e}}_b = \tilde{\Gamma}^k{}_{ab} \hat{\boldsymbol{e}}_k \ . \tag{3.88}$$

We can also consider the action of the covariant derivative ∇_ν associated with the Cartan connection on a tetrad field. Through the application of equation (3.20), we have the following

$$\nabla_\nu \hat{\boldsymbol{e}}_b = \left(\partial_\nu e_b^\beta + \tilde{\Gamma}^\beta{}_{\nu\delta} e_b{}^\delta \right) \partial_\beta \ . \tag{3.89}$$

We notice that (3.88) contains the Cartan connection coefficients defined through the use of a non-holonomic basis whereas the connection coefficients in equation (3.89) have been introduced using a coordinate basis $\{\partial/\partial x^\nu\}$. By making use of equations (3.88) and (3.89), we obtain the following relation between the Cartan connection coefficients in a holonomic frame and the Cartan connection coefficients in a non-holonomic frame [16]

$$\tilde{\Gamma}^\beta{}_{\nu\delta} = \tilde{\Gamma}^k{}_{ab} e_k{}^\beta e^a{}_\nu e^b{}_\delta - e^b{}_\delta \partial_\nu e_b{}^\beta = e_k^\beta \left(\delta^k_b \partial_\nu + \tilde{\Gamma}^k{}_{ab} e^a{}_\nu \right) e^b{}_\delta \ . \tag{3.90}$$

Through the use of equation (3.90), one can obtain the following expression for the Cartan connection coefficients in a holonomic frame contracted with a dual space basis element

$$\tilde{\Gamma}^\beta{}_{\nu\delta} \mathrm{d}x^\nu = e_k{}^\beta D^k{}_b e^b{}_\delta \ , \tag{3.91}$$

where

$$D^k{}_b := \delta^k_b \mathrm{d} + \boldsymbol{w}^k{}_b \ , \tag{3.92}$$

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[3] See [37] for a discussion on how the so-called $f(T)$ theories of gravity do not satisfy the property of Lorentz invariance.

is referred to as the *Lorentz covariant derivative* and

$$\boldsymbol{w}^{k}{}_{b} := \tilde{\Gamma}^{k}{}_{ab}\hat{\boldsymbol{\theta}}^{a} \ , \tag{3.93}$$

is known as the *connection one-form* [16]. In terms of the coordinate components of the Cartan connection coefficients, we have the following from equation (3.91)

$$\tilde{\Gamma}^{\beta}{}_{\nu\delta} = e_{k}{}^{\beta} D^{k}{}_{b\nu} e^{b}{}_{\delta} \ , \tag{3.94}$$

where we define $D^{k}{}_{b\nu}$ to be the component of the Lorentz covariant derivative $D^{k}{}_{b}$ with respect to the dual space basis $\{\mathrm{d}x^{\nu}\}$.

With the above expression in mind, we can find an expression for the torsion tensor described in the non-holonomic frame in terms of the Cartan connection coefficients defined in equation (3.88) for a given non-holonomic basis. By making use of equation (3.81), which tells us how we can find the tensor components in a non-holonomic frame given a holonomic frame, as well as by making use of the expression for the torsion tensor given in equation (3.27) one has the following

$$T^{\beta}{}_{\nu\delta} = T^{k}{}_{ab} e_{k}{}^{\beta} e^{a}{}_{\nu} e^{b}{}_{\delta} = \tilde{\Gamma}^{\beta}{}_{\nu\delta} - \tilde{\Gamma}^{\beta}{}_{\delta\nu} \ . \tag{3.95}$$

By making use of equation (3.90) which relates the Cartan connection coefficients in the non-holonomic frame to the Cartan connection coefficients in the holonomic frame, it follows from the above expression that one has

$$T^{k}{}_{ab} = \tilde{\Gamma}^{k}{}_{ab} - \tilde{\Gamma}^{k}{}_{ba} + e_{a}{}^{\nu} e_{b}{}^{\delta} \left(\partial_{\nu} e^{k}{}_{\delta} - \partial_{\delta} e^{k}{}_{\nu} \right) \ . \tag{3.96}$$

The third term in the above expression is nothing more than the Lie bracket of the tetrad fields, i.e., $[\hat{\boldsymbol{e}}_a, \hat{\boldsymbol{e}}_b]^{k}$. Therefore, one has the following expression for the non-holonomic components of the torsion tensor [16]

$$T^{k}{}_{ab} = \tilde{\Gamma}^{k}{}_{ab} - \tilde{\Gamma}^{k}{}_{ba} - [\hat{\boldsymbol{e}}_a, \hat{\boldsymbol{e}}_b]^{k} \ . \tag{3.97}$$

So-far, we have introduced a number of relevant quantities defined on U_4 using the non-holonomic frame. We now wish to show that the connection one-form satisfies the so-called Cartan structure equations [7]. These important equations are geometric statements about the torsion and the curvature in U_4. Before stating and then proving these relations following Nakahara [16], we first wish to introduce the so-called torsion two-form and the curvature two-form. We define the *torsion two-form* and the *curvature two-form* to be

$$\boldsymbol{T}^{a} := \frac{1}{2} T^{a}{}_{bc} \ \hat{\boldsymbol{\theta}}^{b} \wedge \hat{\boldsymbol{\theta}}^{c} \ , \tag{3.98}$$

and

$$\tilde{\boldsymbol{R}}_{b}{}^{a} := \frac{1}{2} \tilde{R}_{dcb}{}^{a} \ \hat{\boldsymbol{\theta}}^{c} \wedge \hat{\boldsymbol{\theta}}^{d} \ , \tag{3.99}$$

respectively. Here, the non-null torsion Riemann tensor components in the non-holonomic frame are written as $\tilde{R}_{dcb}{}^{a}$ and, as before, the torsion tensor components in the non-holonomic frame are written as $T^{a}{}_{bc}$. We now wish to state and prove the following theorem, which describes the satisfaction of the Cartan structure equations, following Nakahara [16].

Theorem 3.5.1. *The equations*

$$\boldsymbol{T}^{a} = \mathrm{d}\hat{\boldsymbol{\theta}}^{a} + \boldsymbol{w}^{a}{}_{b} \wedge \hat{\boldsymbol{\theta}}^{b} \ , \tag{3.100}$$

and

$$\tilde{\boldsymbol{R}}_{b}{}^{a} = \mathrm{d}\boldsymbol{w}^{a}{}_{b} + \boldsymbol{w}^{a}{}_{c} \wedge \boldsymbol{w}^{c}{}_{b} \ , \tag{3.101}$$

which are referred to as the Cartan structure equations, *are satisfied by the connection one-form [9,16].*

Proof. We begin by proving the first Cartan structure equation (3.100) which involves the torsion one-form defined in equation (3.98). Given an arbitrary dual vector field (one-form) $\boldsymbol{v}$, it is not difficult to show that the exterior derivative of this dual vector field satisfies the following expression

$$\mathrm{d}v\left(\boldsymbol{X},\boldsymbol{Y}\right) = X\left(v\left(\boldsymbol{Y}\right)\right) - Y\left(v\left(\boldsymbol{X}\right)\right) - v\left(\left[\boldsymbol{X},\boldsymbol{Y}\right]\right) \ , \tag{3.102}$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are arbitrary vector fields. By making use of the above expression, we can consider the action of $\mathrm{d}\hat{\boldsymbol{\theta}}^a$, which appears on the right-hand side of equation (3.100), on two non-holonomic basis elements. That is, we can consider

$$\mathrm{d}\hat{\boldsymbol{\theta}}^a\left(\hat{\boldsymbol{e}}_b,\hat{\boldsymbol{e}}_c\right) = \hat{\boldsymbol{e}}_b\left(\langle\hat{\boldsymbol{\theta}}^a,\hat{\boldsymbol{e}}_c\rangle\right) - \hat{\boldsymbol{e}}_c\left(\langle\hat{\boldsymbol{\theta}}^a,\hat{\boldsymbol{e}}_b\rangle\right) - \hat{\boldsymbol{\theta}}^a\left(\left[\hat{\boldsymbol{e}}_b,\hat{\boldsymbol{e}}_c\right]\right) \ . \tag{3.103}$$

Now, since the first two brackets in the above expression contain Kronecker deltas as a result of equation (3.83), the first two terms vanish and we are left with

$$\mathrm{d}\hat{\boldsymbol{\theta}}^a\left(\hat{\boldsymbol{e}}_b,\hat{\boldsymbol{e}}_c\right) = -\left[\hat{\boldsymbol{e}}_b,\hat{\boldsymbol{e}}_c\right]^k\langle\hat{\boldsymbol{\theta}}^a,\hat{\boldsymbol{e}}_k\rangle = -\left[\hat{\boldsymbol{e}}_b,\hat{\boldsymbol{e}}_c\right]^a \ . \tag{3.104}$$

We now turn our attention to considering the second term on the right-hand side of equation (3.100). As done above, we consider the action of this term on two non-holonomic basis elements. By writing the wedge product of the connection one-form and a non-holonomic dual basis element $\hat{\boldsymbol{\theta}}^a$ in terms of tensor products, the aforementioned action on two non-holonomic basis elements is

$$\begin{aligned}\left(\boldsymbol{w}^a{}_d\wedge\hat{\boldsymbol{\theta}}^d\right)\left(\hat{\boldsymbol{e}}_b,\hat{\boldsymbol{e}}_c\right) &= \left(\boldsymbol{w}^a{}_d\otimes\hat{\boldsymbol{\theta}}^d - \hat{\boldsymbol{\theta}}^d\otimes\boldsymbol{w}^a{}_d\right)\left(\hat{\boldsymbol{e}}_b,\hat{\boldsymbol{e}}_c\right)\\ &= \langle\boldsymbol{w}^a{}_d,\hat{\boldsymbol{e}}_b\rangle\langle\hat{\boldsymbol{\theta}}^d,\hat{\boldsymbol{e}}_c\rangle - \langle\hat{\boldsymbol{\theta}}^d,\hat{\boldsymbol{e}}_b\rangle\langle\boldsymbol{w}^a{}_d,\hat{\boldsymbol{e}}_c\rangle \ .\end{aligned} \tag{3.105}$$

We note that, from equation (3.93), the action of the connection one-form on a non-holonomic basis element yields the Cartan connection coefficients in a non-holonomic frame, i.e., $\langle\boldsymbol{w}^a{}_d,\hat{\boldsymbol{e}}_b\rangle = \tilde{\Gamma}^a{}_{bd}$. Therefore, it follows that the last expression given above in equation (3.105) becomes

$$\left(\boldsymbol{w}^a{}_d\wedge\hat{\boldsymbol{\theta}}^d\right)\left(\hat{\boldsymbol{e}}_b,\hat{\boldsymbol{e}}_c\right) = \tilde{\Gamma}^a{}_{bd}\delta^d_c - \tilde{\Gamma}^a{}_{cd}\delta^d_b = \tilde{\Gamma}^a{}_{bc} - \tilde{\Gamma}^a{}_{cb} \ . \tag{3.106}$$

By adding equations (3.104) and (3.106) we obtain the following expression for the action of the right-hand side of equation (3.100) on two non-holonomic basis elements

$$\begin{aligned}\left(\mathrm{d}\hat{\boldsymbol{\theta}}^a + \boldsymbol{w}^a{}_d\wedge\hat{\boldsymbol{\theta}}^d\right)\left(\hat{\boldsymbol{e}}_b,\hat{\boldsymbol{e}}_c\right) &= \tilde{\Gamma}^a{}_{bc} - \tilde{\Gamma}^a{}_{cb} - \left[\hat{\boldsymbol{e}}_b,\hat{\boldsymbol{e}}_c\right]^a\\ &= T^a_{\ bc} \ ,\end{aligned} \tag{3.107}$$

where we have made use of equation (3.96) in order to introduce the non-holonomic torsion tensor components. But the right-hand side of equation (3.107) is nothing more than the action of the torsion two-form on two non-holonomic basis elements. Therefore, it follows that

$$\boldsymbol{T}^a\left(\hat{\boldsymbol{e}}_b,\hat{\boldsymbol{e}}_c\right) = \left(\mathrm{d}\hat{\boldsymbol{\theta}}^a + \boldsymbol{w}^a{}_d\wedge\hat{\boldsymbol{\theta}}^d\right)\left(\hat{\boldsymbol{e}}_b,\hat{\boldsymbol{e}}_c\right) \ . \tag{3.108}$$

The above expression implies that the connection one-form satisfies the first Cartan structure equation given in equation (3.100).

We now turn our attention to proving the second Cartan structure equation (3.101). Let us begin by considering the first term on the right-hand side of equation (3.101). We consider the action of this term, which is the exterior derivative of the connection one-form, on two non-holonomic basis elements. By making use of equation (3.102), one has the following expression for the action

$$\mathrm{d}\boldsymbol{w}^a{}_b\left(\hat{\boldsymbol{e}}_c,\hat{\boldsymbol{e}}_d\right) = \hat{\boldsymbol{e}}_c\left(\langle\boldsymbol{w}^a{}_b,\hat{\boldsymbol{e}}_d\rangle\right) - \hat{\boldsymbol{e}}_d\left(\langle\boldsymbol{w}^a{}_b,\hat{\boldsymbol{e}}_c\rangle\right) - \boldsymbol{w}^a{}_b\left(\left[\hat{\boldsymbol{e}}_c,\hat{\boldsymbol{e}}_d\right]\right) \ . \tag{3.109}$$

Since the action of the connection one-form on a non-holonomic basis element yields the Cartan connection coefficients in the non-holonomic basis, the above expression can be written as

$$\mathrm{d}\boldsymbol{w}^a{}_b\left(\hat{\boldsymbol{e}}_c,\hat{\boldsymbol{e}}_d\right) = \hat{\boldsymbol{e}}_c\left(\tilde{\Gamma}^a{}_{db}\right) - \hat{\boldsymbol{e}}_d\left(\tilde{\Gamma}^a{}_{cb}\right) - \tilde{\Gamma}^a{}_{kb}\left[\hat{\boldsymbol{e}}_c,\hat{\boldsymbol{e}}_d\right]^k \ . \tag{3.110}$$

We now wish to consider the second term on the right-hand side of equation (3.101). By writing the wedge product in terms of tensor products, the action of this term on two non-holonomic basis elements is

$$\begin{aligned}\left(\boldsymbol{w}^{a}{}_{e}\wedge w^{e}{}_{b}\right)\left(\hat{\boldsymbol{e}}_{c},\hat{\boldsymbol{e}}_{d}\right) &= \langle\boldsymbol{w}^{a}{}_{e},\hat{\boldsymbol{e}}_{c}\rangle\langle w^{e}{}_{b},\hat{\boldsymbol{e}}_{d}\rangle-\langle w^{e}{}_{b},\hat{\boldsymbol{e}}_{c}\rangle\langle\boldsymbol{w}^{a}{}_{e},\hat{\boldsymbol{e}}_{d}\rangle\\ &= \tilde{\Gamma}^{a}{}_{ce}\tilde{\Gamma}^{e}{}_{db}-\tilde{\Gamma}^{e}{}_{cb}\tilde{\Gamma}^{a}{}_{de}\ ,\end{aligned}\tag{3.111}$$

where we have made use of the fact that the action of the connection one-form on a non-holonomic basis element yields the Cartan connection coefficient in a non-holonomic basis. Therefore, it follows that the action of the right-hand side of equation (3.101) on two non-holonomic basis elements yields

$$\left(\mathrm{d}\boldsymbol{w}^{a}{}_{b}+\boldsymbol{w}^{a}{}_{e}\wedge w^{e}{}_{b}\right)\left(\hat{\boldsymbol{e}}_{c},\hat{\boldsymbol{e}}_{d}\right)=\hat{\boldsymbol{e}}_{c}\left(\tilde{\Gamma}^{a}{}_{db}\right)-\hat{\boldsymbol{e}}_{d}\left(\tilde{\Gamma}^{a}{}_{cb}\right)-\tilde{\Gamma}^{a}{}_{kb}\left[\hat{\boldsymbol{e}}_{c},\hat{\boldsymbol{e}}_{d}\right]^{k}+\tilde{\Gamma}^{a}{}_{ce}\tilde{\Gamma}^{e}{}_{db}-\tilde{\Gamma}^{e}{}_{cb}\tilde{\Gamma}^{a}{}_{de}\ .\tag{3.112}$$

Let us now turn our attention to examining the components of the non-null torsion Riemann tensor in a non-holonomic basis. By making use of equations (3.65) and (3.90) we have the following expression for the non-holonomic components of the non-null torsion Riemann tensor

$$\begin{aligned}\tilde{R}_{dcb}{}^{a}=\tilde{R}_{\mu\nu\sigma}{}^{\rho}e_{c}{}^{\mu}e_{d}{}^{\nu}e_{b}{}^{\sigma}e^{a}{}_{\rho} &= 2\left(\tilde{\Gamma}^{a}{}_{kb}e_{d}{}^{\alpha}e_{c}{}^{\beta}\partial_{[\beta}e^{k}{}_{\alpha]}+e_{d}{}^{\alpha}e_{c}{}^{\beta}e^{k}_{[\alpha}\partial_{\beta]}\tilde{\Gamma}^{a}{}_{kb}+e_{d}{}^{\alpha}e_{c}{}^{\beta}\tilde{\Gamma}^{k}{}_{lb}\tilde{\Gamma}^{a}{}_{mk}e^{l}{}_{[\alpha}e^{m}{}_{\beta]}\right)\\ &= -\tilde{\Gamma}^{a}{}_{kb}\left[\hat{\boldsymbol{e}}_{c},\hat{\boldsymbol{e}}_{d}\right]^{k}+\hat{\boldsymbol{e}}_{c}\left(\tilde{\Gamma}^{a}{}_{db}\right)-\hat{\boldsymbol{e}}_{d}\left(\tilde{\Gamma}^{a}{}_{cb}\right)+\tilde{\Gamma}^{a}{}_{ce}\tilde{\Gamma}^{e}{}_{db}-\tilde{\Gamma}^{e}{}_{cb}\tilde{\Gamma}^{a}{}_{de}\ ,\end{aligned}\tag{3.113}$$

which is nothing more than the right hand side of equation (3.112). We note that the action of the curvature two-form on the non-holonomic basis elements yields the components of the non-null torsion Riemann tensor in this basis which is what is evaluated in the above two expressions. It follows that the connection one-form satisfies equation (3.101). We have thus shown, following [16], that the connection one-form satisfies the Cartan structure equations. □

Theorem 3.5.1 allows one to obtain a number of geometric statements about U_4. In particular, the Cartan structure equations can be used to derive the Bianchi identities in the Cartan formalism [9,16]. We have already obtained the Bianchi identities using space-time indices in Section 3.4. There, the first and second Bianchi identities are given in equations (3.52) and (3.64) respectively. These identities in the Cartan formalism can be obtained by carrying out exterior derivatives of the Cartan structure equations.

Let us first consider the first Cartan structure equation (3.100) and take the exterior derivative of this expression. Carrying out this process yields the following [16]

$$\mathrm{d}\boldsymbol{T}^{a}=\mathrm{d}\left(\mathrm{d}\hat{\boldsymbol{\theta}}^{a}\right)+\mathrm{d}\left(\boldsymbol{w}^{a}{}_{b}\wedge\hat{\boldsymbol{\theta}}^{b}\right)\ .\tag{3.114}$$

As a result of the properties associated with the exterior derivative, the first term on the right-hand side of the above expression vanishes and we are left with

$$\mathrm{d}\boldsymbol{T}^{a}=\mathrm{d}\boldsymbol{w}^{a}{}_{b}\wedge\hat{\boldsymbol{\theta}}^{b}-\boldsymbol{w}^{a}{}_{b}\wedge\mathrm{d}\hat{\boldsymbol{\theta}}^{b}\ .\tag{3.115}$$

Now, through the use of the second Cartan structure equation (3.101), we can substitute the first term on the right-hand side of the above expression for the curvature two-form in order to obtain

$$\mathrm{d}\boldsymbol{T}^{a}=\tilde{\boldsymbol{R}}_{b}{}^{a}\wedge\hat{\boldsymbol{\theta}}^{b}-\boldsymbol{w}^{a}{}_{c}\wedge\left(\boldsymbol{w}^{c}{}_{b}\wedge\hat{\boldsymbol{\theta}}^{b}+\hat{\boldsymbol{\theta}}^{c}\right)\ .\tag{3.116}$$

From the first Cartan structure equation (3.100), it is not difficult to see that the terms in the brackets are nothing more than the torsion two-form and thus, we obtain the first Bianchi identity in the Cartan formalism [9, 16]

$$\tilde{\boldsymbol{R}}_{b}{}^{a}\wedge\hat{\boldsymbol{\theta}}^{b}=\mathrm{d}\boldsymbol{T}^{a}+\boldsymbol{w}^{a}{}_{c}\wedge\boldsymbol{T}^{c}\ .\tag{3.117}$$

It is clear form the above expression that, should the torsion tensor be vanishing, the right-hand side of the above expression vanishes.

In order to obtain the second Bianchi identity in the Cartan formalism, we take the exterior derivative of the second Cartan structure equation (3.101). By taking the exterior derivative of the second Cartan structure equation, one obtains the following

$$\mathrm{d}\tilde{\boldsymbol{R}}_b{}^a = \mathrm{d}\left(\mathrm{d}\boldsymbol{w}^a{}_b\right) + \mathrm{d}\boldsymbol{w}^a{}_c \wedge \boldsymbol{w}^c{}_b - \boldsymbol{w}^a{}_c \wedge \mathrm{d}\boldsymbol{w}^c{}_b \ . \tag{3.118}$$

By substituting the second Cartan structure equation into the above expression, and by noticing that the first term in the above expression vanishes, one obtains

$$\mathrm{d}\tilde{\boldsymbol{R}}_b{}^a = \left(\tilde{\boldsymbol{R}}_c{}^a - \boldsymbol{w}^a{}_d \wedge w^d{}_c\right) \wedge \boldsymbol{w}^c{}_b - \boldsymbol{w}^a{}_c \wedge \left(\tilde{\boldsymbol{R}}_b{}^c - \boldsymbol{w}^c{}_d \wedge w^d{}_b\right) \ . \tag{3.119}$$

By canceling like terms in the above expression, we arrive at the second Bianchi identity which is presented in the Cartan formalism [9, 16]

$$\mathrm{d}\tilde{\boldsymbol{R}}_b{}^a = \tilde{\boldsymbol{R}}_c{}^a \wedge \boldsymbol{w}^c{}_b - \boldsymbol{w}^a{}_c \wedge \tilde{\boldsymbol{R}}_b{}^c \ . \tag{3.120}$$

To conclude this section in which the Cartan formalism is briefly discussed, we wish to note the effect of dealing with a space-time whose natural choice of an affine connection is metric compatible, such as the Cartan connection, on the tetrad field. In order to conduct such a study, we wish to first introduce the so-called *full covariant derivative* which is given as follows

$$\mathcal{D}^k{}_b := \delta^k_b \nabla + \boldsymbol{w}^k{}_b \ . \tag{3.121}$$

While $D^k{}_b$ is only Lorentz covariant, the full covariant derivative given above is both Lorentz covariant as well as space-time covariant. We now wish to consider the so-called *tetrad postulate*[4] which is given by the expression [82]

$$\mathcal{D}^k{}_b e^b{}_\alpha = \nabla_\nu e^k{}_\alpha + \boldsymbol{w}^k{}_{b\nu} e^b{}_\alpha = 0 \ , \tag{3.122}$$

where $\boldsymbol{w}^k{}_b$ is the connection one-form and $\mathcal{D}^k{}_b$ is the full covariant derivative as before. To see how the tetrad postulate implies the metric compatibility of the affine connection, we consider the action of the Cartan covariant derivative on the metric tensor and then make use of equation (3.85) in order to apply equation (3.122). Carrying out this approach, we have

$$\nabla_\nu g_{\alpha\beta} = \nabla_\nu \left(e^a{}_\alpha \eta_{ab} e^b{}_\beta\right) = \eta_{ab} e^b{}_\beta \nabla_\nu e^a{}_\alpha + \eta_{ab} e^a{}_\alpha \nabla_\nu e^b{}_\beta \ . \tag{3.123}$$

We now make use of the tetrad postulate given in equation (3.122) by substituting the action of the Cartan covariant derivative on the vierbein in the above expression for the connection one-form contracted with the vierbein. By taking the connection one-form to be antisymmetric, i.e., $\boldsymbol{w}^a{}_b = -\boldsymbol{w}_b{}^a$, the above expression then yields

$$\nabla_\nu g_{\alpha\beta} = -e^a{}_\beta w_{ac\nu} e^c{}_\alpha - e^a{}_\alpha w_{ac\nu} e^c{}_\beta = 0 \ . \tag{3.124}$$

That is, by taking the connection one-form to be antisymmetric, the associated connection is metric compatible.

This concludes our discussion on the Cartan formalism where a general Riemann-Cartan space-time is considered. In subsequent chapters where the so-called Einstein-Cartan theory of gravity is studied, the formalism discussed in this section will prove to be of great use. In Chapter 4, we discuss specific theories of gravity beginning with the Einstein-Cartan theory. This theory will be discussed in terms of a given holonomic frame as well as in the Cartan formalism outlined in this section. In the following section, we wish to describe some common space-time structures that are specific cases of the Riemann-Cartan space-time U_4. Specifically, we will describe the specific cases of the Riemann-Cartan space-time where either the torsion tensor is required to vanish or the non-null torsion Riemann tensor is required to vanish.

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[4] Given that the Cartan connection is metric compatible, this statement is, in fact, an identity although it is often referred to as the tetrad postulate.

3.6 Riemannian space-time and Weitzenböck space-time

In Section 3.3, we introduced the notion of a Riemann-Cartan space-time and denoted this space-time as U_4. We defined this space-time[5], following [9], as the tuple $(\mathcal{M}, \boldsymbol{g}, \nabla)$ where $\mathcal{M}$ is a manifold, $\boldsymbol{g}$ is the metric tensor and ∇ is the Cartan connection which is both non-symmetric and metric compatible. In this section, we wish to briefly discuss the notions of the so-called Riemannian space-time and the Weitzenböck space-time. Both these space-times are special cases of the Riemann-Cartan space-time and are obtained by imposing certain restrictions on the Cartan connection. That is, additional restrictions are imposed on the natural choice of an affine connection used to construct the space-time.

We first wish to consider the so-called Riemannian space-time which we shall denote as V_4. In this thesis, as was done for the Riemann-Cartan space-time U_4, we construct a definition for the Riemannian space-time V_4 following [9]. The Riemannian space-time is constructed by imposing the condition that the Cartan connection be symmetric. With such a requirement, the Cartan connection and the Levi-Civita connection coincide [74]. That is, the natural choice of an affine connection in the Riemannian space-time V_4 is simply the Levi-Civita connection D whose connection coefficients are the Christoffel symbols which are defined in equation (3.36). We now give the formal definition of a Riemannian space-time below.

Definition 3.6.1. Let $\mathcal{M}$ be a manifold and let $\boldsymbol{g}$ be a metric tensor on $\mathcal{M}$. Given this metric tensor, we let D be the Levi-Civita connection, constructed as an additional structure on the manifold $\mathcal{M}$, with connection coefficients given by equation (3.36). We then define the *Riemannian space-time* to be the tuple $(\mathcal{M}, \boldsymbol{g}, D)$ [9].

Of course, since the Levi-Civita connection is torsion-free, the Riemannian space-time[6] V_4 is torsion-free. We refer to the quantities obtained using the Levi-Civita connection, and which describe Riemannian space-time, as being Riemannian [75]. When we considered Riemann-Cartan space-time previously, we found that quantities, such as the non-null torsion Riemann tensor, that were constructed using the Cartan connection could be written in terms of the associated Riemannian quantities as well as additional terms that arise as a result of having non-vanishing torsion. These additional terms are referred to as *non-Riemannian contributions* to the non-null torsion quantities [74].

As per Definition 3.6.1, the Riemannian space-time V_4 has non-vanishing curvature and vanishing torsion achieved by taking the natural choice of an affine connection to be the Levi-Civita connection. It is in fact possible [32] to construct a space-time whose natural choice of an affine connection leads to the space-time having vanishing curvature and non-vanishing torsion. Such an affine connection is referred to as the Weitzenböck connection [8,32]. The Weitzenböck space-time, which we shall denote as W_4, is constructed by imposing that the Cartan connection must satisfy the additional requirement characterized by the following expression involving the connection coefficients [8,32,33]

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = e_a{}^{\rho} \partial_\mu e^a{}_\nu \ . \tag{3.125}$$

The affine connection with associated connection coefficients given by the above expression is referred to as the *Weitzenböck connection* [8,33]. The expression given above can be derived by first studying the second Cartan structure equation, which is given in equation (3.101), and choosing a particular solution which results in the vanishing of the non-null torsion Riemann tensor. This particular solution is referred to as the *Weitzenböck gauge* and imposes the vanishing of the connection one-form [84]. It is evident from the second Cartan structure equation that the vanishing of the connection one-form implies the vanishing of the non-null torsion Riemann tensor [84]. It can be seen from the definition of the connection one-form, which is given in equation (3.93), that when the connection one-form vanishes then the connection coefficients in the non-holonomic frame vanish. Taking the connection coefficients in the non-holonomic frame to be vanishing in equation (3.90) yields the expression given in equation (3.125). We have thus shown that the choice of the Weitzenböck gauge, which is described

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[5] In the author's Honours thesis [83] this space-time was referred to as the *non-null torsion space-time*. In this work, however, we follow [9] by Tseng and refer to the space-time as the Riemann-Cartan space-time.

[6] In [83], the Riemannian space-time V_4 is referred to as the *torsion-free space-time*.

by the vanishing of the connection one-form, yields the Weitzenböck connection whose connection coefficients in the holonomic frame are given by equation (3.125).

It is also not difficult to show that by substituting the expression for the Weitzenböck connection coefficients in the holonomic frame given in equation (3.125) into equation (3.65), which is used to obtain the components of the non-null torsion Riemann tensor in the holonomic frame, that the non-null torsion Riemann tensor vanishes. For a derivation of this, the reader is directed to [76].

Having now introduced the Weitzenböck connection by taking the particular solution of a vanishing connection one-form known as the Weitzenböck gauge, we are now in a position to define the Weitzenböck space-time W_4. We again follow [9] in presenting a definition for such a space-time.

Definition 3.6.2. Let $\mathcal{M}$ be a manifold and let $\boldsymbol{g}$ be a metric tensor on $\mathcal{M}$. Given this metric tensor, we let ∇ be the Weitzenböck connection, constructed as an additional structure on the manifold $\mathcal{M}$, which has space-time connection coefficients given by equation (3.125). We then define the *Weitzenböck space-time* to be the tuple $(\mathcal{M}, \boldsymbol{g}, \nabla)$ [9].

The fact that this space-time has vanishing curvature and that gravity is a result of non-vanishing torsion has many physical implications [8, 85, 86]. For instance, it is argued in [85, 86] that gravity is, in fact, not a manifestation of curvature but rather a manifestation of torsion. The conclusions found in these aforesaid references are motivated by the so-called Pound-Rebka experiment which is reported in [87].

In this thesis, we are interested in studying a class of theories known as the $f(T)$ theories of gravity which are constructed on this space-time. Before turning our attention to studying specific gravitational theories, however, we will first turn our attention back to the Riemann-Cartan space-time defined in Definition 3.3.1 in order to discuss the notion of an auto-parallel of a given connection. That is, we wish to discuss the notion of a tangent vector field, say ξ^ν, to a curve $\gamma(\lambda)$, where $\lambda \in \mathbb{R}$ is a parameterisation of the curve, that is parallel propagated as described by a given affine connection. This will lead us to the discussion of families of curves, more specifically smooth congruences of curves, generated by the Cartan connection as well as a study of how neighbouring curves in such a congruence diverge or converge with respect to each other. For such a discussion, we require the so-called Raychaudhuri equation for a congruence of curves generated by the Cartan connection.

3.7 Auto-parallels in Riemann-Cartan space-time

In this section, we wish to introduce the notion of an auto-parallel of a given connection. Let us begin by considering a Riemann-Cartan space-time U_4 as per Definition 3.3.1 which is given in Section 3.3. Intuitively, an auto-parallel of a given affine connection is a curve that is "as straight as possible" as described by the affine connection [5]. More specifically, a curve is said to be an auto-parallel of a given affine connection if the parallel propagation, described by the given affine connection, of the tangent vector field of the curve yields a vector field that has the same direction as the tangent vector field. That is, the parallel propagated vector field should have the same direction as the tangent vector field [5]. With this in mind, we write the following definition for an auto-parallel[7] of a given affine connection following [5, 55, 74].

Definition 3.7.1. Let $\gamma(\lambda)$ be a curve in the Riemann-Cartan space-time U_4 with parameterisation $\lambda \in \mathbb{R}$. Let ξ^μ be a tangent vector field to the aforementioned curve $\gamma(\lambda)$ and let $\hat{\nabla}$ be an arbitrary affine connection considered as an additional structure on the manifold $\mathcal{M}$ used in the definition of U_4. We say that the curve $\gamma(\lambda)$ is an *auto-parallel* of the affine connection $\hat{\nabla}$ if

$$\xi^\nu \hat{\nabla}_\nu \xi^\mu = f \xi^\mu \,, \tag{3.126}$$

where $f : \gamma \subset \mathcal{M} \to \mathbb{R}$ is a smooth function on $\gamma(t)$ [5, 55, 74].

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[7] We note that the term "auto-parallel" does not appear in [5, 55] since these books are on GR. Instead the term "geodesic" is used. In the case of a Riemannian space-time, these two notions coincide and so we use the definitions where parallel propogation is concerned. The term "auto-parallel" is used in the review article [74] and we make use of the definition therein.

In the above definition, equation (3.126) implies that the action of $\hat{\nabla}_{\boldsymbol{\xi}}$ on the tangent vector field ξ^μ yields the vector field $f\xi^\mu$ which clearly has a direction that is that of the tangent vector field. Let the connection coefficients that describe the action of the map $\left(\hat{\nabla}_\alpha - \partial_\alpha\right)$ be denoted by $\hat{\Gamma}^\mu{}_{\nu\sigma}$. Now, let $(\mathcal{U}, \psi)$ be a coordinate chart for the manifold $\mathcal{M}$ with $\psi : \mathcal{U} \subset \mathcal{M} \to O \subset \mathbb{R}^4$ and $\mathcal{U}$ such that γ is contained in $\mathcal{U}$. In addition, let the coordinates in the open subset O of $\mathbb{R}^4$ be denoted as $\left(x^0, x^1, x^2, x^3\right)$. For such a coordinate chart, the tangent vector field for the curve $\gamma(\lambda)$ is $\xi^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ and thus, by making use of the aforementioned connection coefficients associated with the affine connection $\hat{\nabla}$, one can write equation (3.126) as [5]

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \hat{\Gamma}^\mu{}_{\nu\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = f \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \, . \tag{3.127}$$

It is not difficult to see that by carrying out a reparameterisation of the curve γ with a parameter τ that satisfies the differential equation

$$\frac{\mathrm{d}^2 \tau}{\mathrm{d}\lambda^2} = f \frac{\mathrm{d}\tau}{\mathrm{d}\lambda} \, , \tag{3.128}$$

one can write equation (3.127) as

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \hat{\Gamma}^\mu{}_{\nu\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = 0 \iff \xi^\nu \hat{\nabla}_\nu \xi^\mu = 0 \, . \tag{3.129}$$

That is, we have introduced a parameterisation, τ, that is such that the action of the operator $\hat{\nabla}_{\boldsymbol{\xi}}$ on the tangent vector field ξ^μ of the curve γ vanishes. Such a parameter is referred to as an *affine parameter* [5, 55]. In this thesis, when considering an auto-parallel of a given affine connection, we will always take the parameterisation of the relevant curve to be the affine parameterisation. Having introduced the notion of an auto-parallel for an arbitrary affine connection, we wish to consider now the notion of extremal curves. For a curve $\gamma(\lambda)$ with original parameterisation $\lambda \in \mathbb{R}$, one writes the length, say l, of this curve as [5, 74]

$$l := \int_p^q \mathrm{d}\lambda \sqrt{g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \, , \tag{3.130}$$

where $p, q \in \mathcal{M}$ are the endpoints of the curve γ. Having introduced the above expression which gives the length of a curve γ we now state the following definition for an extremal curve following [74].

Definition 3.7.2. Let $\gamma(\lambda)$ be a curve in the Riemann-Cartan space-time manifold $\mathcal{M}$. If this curve is such that its length, whose expression is given in equation (3.130), experiences an extremum, then the curve γ is referred to as an *extremal curve*.

The notion of extremal curves is of great importance since the paths followed by test particles[8] are taken to be extremal curves [36, 74, 88]. An argument for this was first introduced in [36]. A brief review of how this argument affects the singularity theorems in the Riemann-Cartan space-time can be found in [88]. It is important to note that the length of a given curve in a Riemann-Cartan space-time does not depend on the torsion tensor [74]. This implies that the presence of torsion has no effect on the paths followed by test particles since the only dynamical quantity used in the construction of the length is the metric tensor [74]. This is an important point to make since this should imply that the paths followed by test particles are then not necessarily auto-parallels of the Cartan connection which is the natural choice of an affine connection in a Riemann-Cartan space-time. Instead, it turns out that the paths followed by test particles are auto-parallels of the Levi-Civita connection[9] [5, 36, 74]. This result is contained in the following theorem which we prove following [5].

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[8] We emphasize that we take test particles to have negligible spin. For the case where a particle in Riemann-Cartan space-time is taken to have non-vanishing spin, the path followed by such a particle will be different from an extremal curve. This is discussed in detail in [75].

[9] We note that we use the term *geodesic* here only when the differential equation describing the auto-parallel curves of the natural choice of an affine connection coincides with the differential equation describing extremal curves.

Theorem 3.7.1. *Let $\gamma(\lambda)$, with original parameterisation λ, be a curve in the manifold $\mathcal{M}$, which forms part of the U_4 tuple as per Definition 3.3.1. If this curve is an extremal curve, then it is an auto-parallel of the Levi-Civita connection D.*

Proof. In order to show that extremal curves are nothing more than auto-parallels of the Levi-Civita connection, we wish to show that the differential equations obtained through extremising the length equation (3.127) are nothing more than the differential equations satisfied by auto-parallels of the Levi-Civita connection. Let us consider the following one parameter family of ordinary variations of the coordinate components $x^\mu \in \mathbb{R}$

$$x^\mu_s := x^\mu + s\delta x^\mu \ , \tag{3.131}$$

where $s \in \mathbb{R}$ and δx^μ is referred to as an ordinary variation and is such that $\delta x^\mu(p) = \delta x^\mu(q) = 0$ [89]. From equation (3.127), the length action for the above family of ordinary variations reads

$$l[s] = \int_p^q \mathrm{d}\lambda \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu_s}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu_s}{\mathrm{d}\lambda}} \ . \tag{3.132}$$

By requiring that the above expression experiences an extremum, one has the following

$$0 \overset{!}{=} \left.\frac{\mathrm{d}l}{\mathrm{d}s}\right|_{s=0} = \frac{1}{2}\int_p^q \mathrm{d}\lambda \left(g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right)^{-1/2} \left(\partial_\sigma g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\delta x^\sigma + 2g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}\delta x^\nu}{\mathrm{d}\lambda}\right) \ . \tag{3.133}$$

Without the loss of generality, we may assume that the original parameterisation of the curve is such that $g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 1$ [5]. In addition, one can perform an integration by parts on the second term in the integrand on the right-hand side of the above expression in order to obtain

$$0 = -\int_p^q \mathrm{d}\lambda \left\{\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \frac{1}{2}g^{\mu\sigma}\left(\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}\right)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right\}\delta x_\sigma \ , \tag{3.134}$$

where we have made use of the fact that the ordinary variation δx^σ vanishes at the points $p, q \in \mathcal{M}$. It is not difficult to see that the integrand contains the Christoffel symbols $\Gamma^\mu{}_{\alpha\beta}$, i.e., the connection coefficients of the Levi-Civita connection, given in equation (3.36). Therefore, since the above equation holds for arbitrary ordinary variations δx^σ, one obtains

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0 \ , \tag{3.135}$$

which are nothing more than the differential equations satisfied by a curve whose tangent vector field is parallel propagated with respect to the Levi-Civita connection, i.e., $\xi^\nu D_\nu \xi^\mu = 0$. It therefore follows that extremal curves are auto-parallels of the Levi-Civita connection. □

The reader may notice that we have refrained from using the term geodesic which is commonly used in the study of GR. The reason for this is due to the theorem given above. In the case of a Riemannian space-time, i.e., the Cartan connection and the Levi-Civita connection coincide, extremal curves are auto-parallels of the natural choice of an affine connection since this connection would be the Levi-Civita connection. In the event that extremal curves and auto-parallels of the natural choice of an affine connection for a given space-time coincide, these curves are referred to as *geodesics* [74]. Of course, this is the case for Riemannian space-times whose natural choice of an affine connection is the Levi-Civita connection. Here, however, we are concerned with Riemann-Cartan space-times whose natural choice of an affine connection, which is the Cartan connection, does not necessarily coincide with extremal curves. We have therefore followed the review article [74] by Hehl in making the distinction and avoiding the term "geodesic". As pointed out in [74], the auto-parallels of the Cartan connection coincide with the auto-parallels of the Levi-Civita connection in the event that the contorsion tensor is totally antisymmetric. In order to see this, consider an auto-parallel curve of the Cartan connection ∇ with tangent vector field $\boldsymbol{\xi}$. By making use of equations (3.20) and (3.38), we have

$$\begin{aligned} 0 = \xi^{\nu}\nabla_{\nu}\xi^{\mu} &= \xi^{\nu}\partial_{\nu}\xi^{\mu} + \tilde{\Gamma}^{\mu}{}_{\nu\beta}\xi^{\nu}\xi^{\beta} \\ &= \xi^{\nu}\partial_{\nu}\xi^{\mu} + \Gamma^{\mu}{}_{\nu\beta}\xi^{\nu}\xi^{\beta} + K^{\mu}{}_{\nu\beta}\xi^{\nu}\xi^{\beta} \ . \end{aligned} \tag{3.136}$$

In the instance where the contorsion tensor is antisymmetric in its second and third indices, and therefore totally antisymmetric, the third term on the right-hand side of equation (3.136) vanishes due to the contraction of symmetric indices with antisymmetric indices. The resulting expression then reduces to the differential equation associated with an auto-parallel curve of the Levi-Civita connection.

Our goal now is to study families of curves in the Riemann-Cartan space-time U_4. In particular, we are interested in studying the rate of expansion of a cross-sectional area enclosing a family of curves. Such a notion is of great use when studying focusing conditions, i.e., the conditions that allow for the convergence or divergence of neighbouring curves. In order to study the rate of expansion of a cross-sectional area enclosing a family of curves contained in U_4, we wish to first specify what we mean by a "family" of curves. Here we will be interested in considering smooth congruences of timelike curves. We define a congruence of curves following Wald [5].

Definition 3.7.3. We define a *congruence* of curves to be a family of curves for which each $p \in \mathcal{M}$ has at most one curve in this family passing through it.

Having now introduced the notion of a congruence of curves, we wish to turn our attention to the consideration of a congruence of timelike curves generated by the Cartan connection. In particular, we will be interested in studying the rate of expansion of a cross-sectional area, say A, that encloses this congruence. Through such a consideration, we follow the derivations given in [5, 58–61, 83] in order to obtain the Raychaudhuri equation for a congruence of timelike curves generated by the Cartan connection. Following the terminology of [83], we refer to the equation derived in [58] as the *non-null torsion Raychaudhuri equation* and we derive this in Section 3.8. We note that the kinematic quantities included in the resulting expression do not describe the relevant physical properties of the congruence. That is, the expansion scalar defined in [58] does not give the physical expansion associated with a curve generated by the Cartan connection [60, 61]. In order to consider the relevant physical kinematic quantities, we turn our attention to the Raychaudhuri equation derived in [60, 61]. We shall derive this equation in Section 3.9 and refer to this as the *geometric Raychaudhuri equation*.

3.8 Non-null torsion Raychaudhuri equation

In this section, we derive[10] the non-null torsion Raychaudhuri equation whose derivation is given in [58] as well as the review article [59]. Consider a cross-sectional area A which encloses a smooth congruence of timelike curves of the Cartan connection. Let us denote the tangent vector field to these curves as ξ^{ν}. We note that we have not assumed that these curves are auto-parallels of the Cartan connection and, therefore, the action of the map $\nabla_{\boldsymbol{\xi}}$ on the tangent vector field ξ^{ν} may be non-vanishing. Our first step is to consider the type $(0,2)$ tensor field $\tilde{B}_{\mu\nu}$ defined through the expression [5]

$$\tilde{B}_{\mu\nu} := \nabla_{\nu}\xi_{\mu} \ , \tag{3.137}$$

where $\xi_{\mu} = g_{\mu\beta}\xi^{\beta}$. We note that, by taking ξ^{ν} to be normalized to unit length, i.e., $g(\boldsymbol{\xi},\boldsymbol{\xi}) = -1$, it is not difficult to show that the tensor field $\tilde{B}_{\mu\nu}$ satisfies the following property

$$\xi^{\mu}\tilde{B}_{\mu\nu} = 0 \ . \tag{3.138}$$

---

[10] The derivation carried out here was conducted by making use of the Mathematica notebook *non_null_torsion_Raychaudhuri.nb* which was constructed by the author and is available at `https://www.dropbox.com/sh/cfybc2cysixchlz/AACjKq0NHComUPoXcZz90DH7a?dl=0`.

Furthermore, through the use of the definition of the non-null torsion Rimeann tensor given in equation (3.42), one has the following expression

$$\left(\nabla_\sigma\nabla_\beta-\nabla_\beta\nabla_\sigma\right)\xi_\alpha=\tilde{R}_{\sigma\beta\alpha}{}^{\rho}\xi_\rho-T^{\rho}{}_{\sigma\beta}\nabla_\rho\xi_\alpha\ . \tag{3.139}$$

By writing the first term on the right-hand side of the above equation in terms of a total Cartan covariant derivative and then taking the trace of the resulting expression, one can obtain [83]

$$\xi^\sigma\nabla_\sigma\tilde{B}=\nabla_\alpha\left(\xi^\sigma\nabla_\sigma\xi^\alpha\right)-\tilde{B}_\alpha{}^{\sigma}\tilde{B}_\sigma{}^{\alpha}-\tilde{R}_{\sigma\rho}\xi^\sigma\xi^\rho-T^{\rho}{}_{\sigma\alpha}\xi^\sigma\tilde{B}^{\alpha}{}_{\rho}\ , \tag{3.140}$$

where $\tilde{B}$ is the trace of the tensor field $\tilde{B}_{\mu\nu}$. We now wish to find a decomposition of the tensor field obtained by projecting $\tilde{B}_{\mu\nu}$ onto the hypersurface. This decomposition should contain symmetric and antisymmetric parts that are both traceless as well as a term which yields the trace of the tensor field $\tilde{B}_{\mu\nu}$. Let us begin by introducing the so-called *spatial metric* which is a metric for the hypersurfaces of constant time and is defined through the expression [5, 59]

$$h_{\mu\nu}:=g_{\mu\nu}+\xi_\mu\xi_\nu\ . \tag{3.141}$$

Having defined the spatial metric, we now wish to introduce the quantities that form part of the decomposition of $h_\mu{}^{\alpha}h_\nu{}^{\beta}\tilde{B}_{\alpha\beta}$. We shall refer to these as the *non-null torsion kinematic quantities*. We first introduce the *non-null torsion expansion scalar* as

$$\tilde{\theta}:=h^{\alpha}{}_{\beta}\nabla_\alpha\xi^\beta=\tilde{B}\ , \tag{3.142}$$

which is simply the trace of $B_{\mu\nu}$. In addition, we define the *non-null torsion shear tensor* as

$$\tilde{\sigma}_{\mu\nu}:=h^{\sigma}{}_{\mu}h^{\rho}{}_{\nu}\nabla_{(\sigma}\xi_{\rho)}-\frac{\tilde{\theta}}{3}h_{\mu\nu}\ , \tag{3.143}$$

which is traceless and is the symmetric term in the decomposition of the tensor field $\tilde{B}_{\mu\nu}$. Lastly, we introduce the tensor field referred to as the *non-null torsion twist tensor* through the following expression

$$\tilde{\omega}_{\mu\nu}:=h^{\sigma}{}_{\mu}h^{\rho}{}_{\nu}\nabla_{[\sigma}\xi_{\rho]}\ , \tag{3.144}$$

which is traceless and is the antisymmetric term in the decomposition of $\tilde{B}_{\mu\nu}$. In terms of the tensor fields defined above, the decomposition of the tensor field $\tilde{B}_{\mu\nu}$ is written as [59]

$$\tilde{B}_{\mu\nu}=\tilde{\omega}_{\mu\nu}+\tilde{\sigma}_{\mu\nu}+\frac{\tilde{\theta}}{3}h_{\mu\nu}-\xi^\sigma\xi_\nu\tilde{B}_{\mu\sigma}\ . \tag{3.145}$$

In terms of the above decomposition, we can write the second term on the right-hand side of equation (3.140) as

$$\tilde{B}_\alpha{}^{\sigma}\tilde{B}_\sigma{}^{\alpha}=\tilde{\sigma}_{\alpha\beta}\tilde{\sigma}^{\beta\alpha}+\tilde{\omega}_{\alpha\beta}\tilde{\omega}^{\beta\alpha}+\frac{\tilde{\theta}^2}{9}h_{\alpha\beta}h^{\beta\alpha}\ , \tag{3.146}$$

and thus, by introducing a suitable parameterisation, say λ, equation (3.140) becomes [59, 83]

$$\begin{aligned}\frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}\lambda}=&\nabla_\alpha\left(\xi^\beta\nabla_\beta\xi^\alpha\right)-\tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu}+\tilde{\omega}_{\mu\nu}\tilde{\omega}^{\mu\nu}-\frac{\tilde{\theta}^2}{3}-\tilde{R}_{\mu\nu}\xi^\mu\xi^\nu\\&-T^{\rho}{}_{\sigma\alpha}\xi^\sigma\left(\tilde{\sigma}^{\alpha}{}_{\rho}+\tilde{\omega}^{\alpha}{}_{\rho}+\frac{\tilde{\theta}}{3}h^{\alpha}{}_{\rho}-\xi^\delta\xi_\rho\nabla_\delta\xi^\alpha\right)\ ,\end{aligned} \tag{3.147}$$

where we have made use of the fact that the non-null torsion shear tensor is symmetric while the non-null torsion twist tensor is antisymmtric. We note that the non-null torsion Raychaudhuri equation is a special case of the Raychaudhuri equation derived in [63] which considers non-metricity in addition to non-vanishing torsion.

As mentioned earlier, for a congruence of curves generated by the Cartan connection, the non-null torsion expansion, shear and twist do not provide us with the relevant physical interpretation. In order to find expansion, shear and twist tensors that give us the desired physical interpretation, we turn our attention to the Raychaudhuri equation derived in [60, 61] and is discussed in the following section

3.9 Geometric Raychaudhuri equation

Here, we derive[11] the Raychaudhuri equation given for the first time in [60, 61]. In this thesis, we refer to this equation as the geometric Raychaudhuri equation.

By making use of the affine connection ∇, we introduce the tensor field $\mathring{\boldsymbol{B}}$ of type $(0,2)$ defined in the following way

$$\mathring{B}_{\alpha\beta} := \nabla_\beta \xi_\alpha + T_{\alpha\nu\beta}\xi^\nu = D_\beta \xi_\alpha + K_{\alpha\nu\beta}\xi^\nu \ . \tag{3.148}$$

By making use of the fact that the tangent vector field is normalised to unit length, it is not difficult to see from the above definition of $\mathring{\boldsymbol{B}}$ that we have

$$\xi^\alpha \mathring{B}_{\alpha\beta} = T_{(\alpha\nu)\beta}\xi^\alpha \xi^\nu \ . \tag{3.149}$$

Furthermore, due to the fact that the torsion tensor is antisymmetric in its second and third indices, we have

$$\xi^\beta \mathring{B}_{\alpha\beta} = \xi^\beta \nabla_\beta \xi_\alpha \ , \tag{3.150}$$

which vanishes when considering auto-parallel curves of the Cartan connection. We now wish to find a physical interpretation of $\mathring{\boldsymbol{B}}$. Through the use of the Lie derivative, we have

$$\pounds_\xi \eta^\beta = \xi^\alpha \nabla_\alpha \eta^\beta - K^\beta{}_{\alpha\nu}\eta^\nu \xi^\alpha - \eta^\alpha \nabla_\alpha \xi^\beta + K^\beta{}_{\alpha\nu}\eta^\alpha \xi^\nu = 0 \ . \tag{3.151}$$

It follows from equations (3.151) and (3.148) that one can write

$$\xi^\alpha \nabla_\alpha \eta^\beta = \left(\nabla_\alpha \xi^\beta + T^\beta{}_{\nu\alpha}\xi^\nu\right)\eta^\alpha = \mathring{B}^\beta{}_\alpha \eta^\alpha \ . \tag{3.152}$$

From the last expression, one can interpret the tensor field $\mathring{\boldsymbol{B}}$ as being a measure of the failure for the deviation vector $\boldsymbol{\eta}$ to be parallel propagated with respect to the Cartan connection. Moreover, $\mathring{\boldsymbol{B}}$ is a linear map that can be used to extract the non-null torsion expansion, shear and twist.

In order to derive the non-null torsion Raychaudhuri equation, we wish to find an expression for $\xi^\rho \nabla_\rho \mathring{B}_{\alpha\beta}$ in terms of the non-null torsion Riemann tensor. From the definition of the non-null torsion Riemann tensor, we have

$$\begin{aligned}\xi^\rho \nabla_\rho \mathring{B}_{\alpha\beta} = {} & \nabla_\beta \left(\xi^\rho \nabla_\rho \xi_\alpha\right) - \left(\nabla_\beta \xi^\rho\right)\left(\nabla_\rho \xi_\alpha\right) - \tilde{R}_{\sigma\beta\rho\alpha}\xi^\sigma \xi^\rho \\ & - T^\rho{}_{\sigma\beta}\xi^\sigma \nabla_\rho \xi_\alpha + \xi^\sigma \nabla_\sigma \left(T_{\alpha\rho\beta}\xi^\rho\right) \ .\end{aligned} \tag{3.153}$$

By taking the trace of equation (3.153), we obtain

$$\xi^\beta \nabla_\beta \mathring{B}_\alpha{}^\alpha = \nabla_\alpha \left(\xi^\beta \nabla_\beta \xi^\alpha\right) - \mathring{B}_\alpha{}^\beta \mathring{B}_\beta{}^\alpha - \tilde{R}_{\alpha\beta}\xi^\alpha \xi^\beta + T^\rho{}_{\beta\alpha}\xi^\beta \mathring{B}^\alpha{}_\rho + \mathring{B}^\rho{}_\nu T_\rho \xi^\nu + \xi^\nu \xi^\rho \nabla_\nu T_\rho \ . \tag{3.154}$$

Let us now define the *geometric expansion* as

$$\mathring{\theta} := h_\alpha{}^\beta \mathring{B}^\alpha{}_\beta \ , \tag{3.155}$$

the *geometric shear* as

$$\mathring{\sigma}_{\alpha\beta} := h^\sigma{}_\alpha h^\rho{}_\beta \mathring{B}_{(\sigma\rho)} - \frac{\theta}{3}h_{\alpha\beta} \ , \tag{3.156}$$

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[11] This derivation was carried out using the Mathematica notebook *Geometric_Raychaudhuri.nb* which was constructed by the author and is available at `https://www.dropbox.com/sh/cfybc2cysixchlz/AACjKq0NHComUPoXcZz90DH7a?dl=0`.

and the *geometric twist*

$$\overset{\circ}{\omega}_{\alpha\beta} := h^{\sigma}{}_{\alpha} h^{\rho}{}_{\beta} \overset{\circ}{B}_{[\sigma\rho]} \ . \tag{3.157}$$

The so-called expansion, shear and twist defined here are the aforementioned physical quantities needed to construct the geometric Raychaudhuri equation using the affine connection ∇. These quantities, referred to as the *geometric kinematic quantities*, provide us with a decomposition of the projected tensor field $h^{\sigma}{}_{\alpha} h^{\rho}{}_{\beta} \overset{\circ}{B}_{\sigma\rho}$. More specifically, we can relate $\overset{\circ}{\boldsymbol{B}}$ to the kinematic quantities through the following expression

$$\overset{\circ}{B}_{\alpha\beta} = \overset{\circ}{\omega}_{\alpha\beta} + \overset{\circ}{\sigma}_{\alpha\beta} + \frac{\overset{\circ}{\theta}}{3} h_{\alpha\beta} - \xi^{\rho}\xi_{\beta} \overset{\circ}{B}_{\alpha\rho} - \xi^{\rho}\xi_{\alpha} \overset{\circ}{B}_{\rho\beta} \ . \tag{3.158}$$

Equation (3.158) is the desired decomposition of $\overset{\circ}{\boldsymbol{B}}$ in terms of the geometric expansion, shear and twist. It is noted, from the definitions given in (3.155), (3.156) and (3.157), that the twist tensor, $\overset{\circ}{\omega}_{\alpha\beta}$, is the antisymmetric part of the projected tensor $h^{\sigma}{}_{\alpha} h^{\rho}{}_{\beta} \overset{\circ}{B}_{\sigma\rho}$, the shear tensor, $\overset{\circ}{\sigma}_{\alpha\beta}$, is the symmetric part and that $\frac{\overset{\circ}{\theta}}{3} h_{\alpha\beta}$ is the trace component in the decomposition. Now that we have obtained this desired decomposition of the projected $\overset{\circ}{B}_{\alpha\beta}$, we wish to turn our attention back to equation (3.154) in order to obtain the Raychaudhuri equation.

By substituting equation (3.158) into equation (3.154) and by choosing a suitable parameterisation, α, of the curves generated by the affine connection ∇, one can obtain the *geometric Raychaudhuri equation*

$$\begin{aligned} \frac{\mathrm{d}\overset{\circ}{\theta}}{\mathrm{d}\alpha} = {} & \nabla_{\alpha}\left(\xi^{\beta}\nabla_{\beta}\xi^{\alpha}\right) - \overset{\circ}{\sigma}_{\alpha\beta}\overset{\circ}{\sigma}^{\alpha\beta} + \overset{\circ}{\omega}_{\alpha\beta}\overset{\circ}{\omega}^{\alpha\beta} - \frac{1}{3}\overset{\circ}{\theta}^{2} - \tilde{R}_{\alpha\beta}\xi^{\alpha}\xi^{\beta} \\ & - T_{\alpha\beta\rho}\xi^{\beta}\left(\frac{1}{3}\overset{\circ}{\theta} h^{\alpha\rho} + \overset{\circ}{\sigma}^{\alpha\rho} - \overset{\circ}{\omega}^{\alpha\rho} - \xi^{\alpha}\xi^{\nu}\nabla_{\nu}\xi^{\rho}\right) + \xi^{\alpha}\nabla_{\alpha}\left(T_{\nu}\xi^{\nu}\right) \ . \end{aligned} \tag{3.159}$$

The above equation is the desired geometric Raychaudhuri equation and was derived in [60, 61] for the first time.

We now wish to give an interpretation for the physical quantities used in the construction of this Raychaudhuri equation. In order to provide such an interpretation, we turn our attention to the cross-sectional area A that encloses the considered congruence of curves generated by the Cartan connection. For an observer traveling along a curve contained in this congruence, the observed expansion or contraction of the cross-sectional area A is measured by the geometric expansion scalar. In addition, the tendency for the cross-sectional area A to become ellipsoidal in shape is described by the geometric shear tensor. Lastly, the geometric twist tensor describes how the cross-sectional A rotates about the congruence as seen by an observer moving along a curve contained in the congruence.

In Chapter 5, we will be interested in considering congruences of timelike curves in order to study conditions relating to the attractiveness of gravity. For such a study, the Raychaudhuri equation will play an important role. In particular, one can study the focusing conditions for test particles in various gravitational theories. It is important to note, however, when studying the focusing conditions for test particles that one considers a congruence of curves followed by test particles. Since we take the paths followed by test particles[12] to be extremal curves, it follows from Theorem 3.7.1 that test particles follow auto-parallel curves of the Levi-Civita connection [74]. This implies that in the case of the Riemann-Cartan space-time U_4, although the natural choice of an affine connection is, by definition, the Cartan connection, one should consider the auto-parallels of the Levi-Civita connection when studying the conditions associated with the requirement that gravity be attractive. Therefore, when studying test particles in the presence of gravity, we will not turn our attention to the non-null torsion or geometric Raychaudhuri equations since these are derived using the Cartan connection

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[12] This is true here since we consider test particles to have negligible spin. The reader interested in the behaviour of spin-particles in the presence of torsion is directed to [75, 90].

whose auto-parallels are not necessarily followed by test particles. Instead, since the Levi-Civita connection is torsion-free, we seek a torsion-free Raychaudhuri equation which describes a congruence of curves generated by the Levi-Civita connection. We derive such an equation in the following section following [5].

3.10 Torsion-free Raychaudhuri equation

Since the Levi-Civita connection is nothing more than the Cartan connection in the absence of torsion, one can derive the torsion-free Raychaudhuri equation directly from either the non-null torsion Raychaudhrui equation (3.140) or the geometric Raychaudhuri equation (3.159). That is, one considers instead a congruence of timelike curves, generated by the Levi-Civita connection, with tangent vector field ξ^{μ}. By considering the case of vanishing torsion, both equations (3.148) and (3.137) reduce to the type $(0,2)$ tensor field

$$B_{\mu\nu} = D_{\nu}\xi_{\mu} \ . \tag{3.160}$$

The above expression implies that this tensor field is constructed through the action of the Levi-Civita covariant derivative on the tangent vector field. In addition, since the above expression is the torsion-free case of equation (3.148), the tensor field $\boldsymbol{B}$ measures the failure of the deviation vector field $\boldsymbol{\eta}$ to be parallel propagated with respect to the Levi-Civita connection. One can introduce the torsion-free expansion scalar θ, shear tensor $\sigma_{\mu\nu}$ and twist tensor $\omega_{\mu\nu}$ by replacing the Cartan connection in equations (3.142), (3.143) and (3.144) with the Levi-Civita connection. More specifically, we write the *torsion-free expansion scalar* as

$$\theta := h^{\alpha}{}_{\beta}B^{\beta}{}_{\alpha} \ , \tag{3.161}$$

the *torsion-free shear tensor* as

$$\sigma_{\alpha\beta} := h^{\sigma}{}_{(\alpha}h^{\rho}{}_{\beta)}B_{\alpha\beta} - \frac{\theta}{3}h_{\alpha\beta} \ , \tag{3.162}$$

and the *torsion-free twist tensor* as

$$\omega_{\alpha\beta} := h^{\sigma}{}_{(\alpha}h^{\rho}{}_{\beta)}B_{\alpha\beta} \ . \tag{3.163}$$

In this thesis, we refer to the last three expressions as the *torsion-free kinematic quantities.*

For the torsion-free case of equation (3.140), the non-null torsion Ricci tensor reduces to the torsion-free Ricci tensor and the terms in brackets have no contribution due to the fact that the Levi-Civita connection yields a vanishing torsion tensor. Through these considerations, one can obtain the torsion-free Raychaudhuri equation from equation (3.140)

$$\frac{\mathrm{d}\theta}{\mathrm{d}\beta} = D_{\alpha}\left(\xi^{\beta}D_{\beta}\xi^{\alpha}\right) - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - \frac{\theta^2}{3} - R_{\mu\nu}\xi^{\mu}\xi^{\nu} \ , \tag{3.164}$$

where β is a suitable parameterisation [5]. Since we would be interested in studying a congruence of timelike auto-parallels of the Levi-Civta connection, we consider the case where the tangent vector field ξ^{ν} is parallel propagated with respect to the Levi-Civita connection. In such a case as well as for a suitable affine parameterisation, say τ which denotes the proper time, the first term on the right-hand side of the above equation vanishes. Lastly, without the loss of generality, we take the tangent vector field ξ^{ν} to hypersurface orthogonal, i.e., the torsion-free twist tensor vanishes with $\omega_{\mu\nu} = 0$. It follows that the Raychaudhuri equation for a congruence of timelike auto-parallels of the Levi-Civita connection is [5]

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{\theta^2}{3} - R_{\mu\nu}\xi^{\mu}\xi^{\nu} \ . \tag{3.165}$$

The above Raychaudhuri equation was derived in [54] by Raychaudhuri. In the study of GR, the above Raychudhuri equation plays a significant role in the investigation of singularity theorems [5, 54–56]. The reader who is interested in the role of the Raychaudhuri equation in the study of singularity theorems is directed to [55] by Hawking and Ellis as well as to [56] by Hawking and Penrose. For a study on singularity theorems for gravitational theories that include torsion, the interested reader is directed to [88]. We will return to the above expression in Chapter 5 where we shall study the focusing conditions in the so-called $f(T)$ theories of gravity.

As a final comment, we wish to compare the torsion-free kinematic quantities defined in equations (3.161), (3.162) and (3.163) to the non-null torsion kinematic quantities as well as the geometric kinematic quantities. By introducing the Cartan covariant derivative into equations (3.161), (3.162) and (3.163) through the use of the contorsion tensor, one can obtain the following which relates the torsion-free, non-null torsion and geometric expansion scalars

$$\theta = \tilde{\theta} + T_\beta \xi^\beta = \overset{\circ}{\theta} \ . \tag{3.166}$$

In addition, one can obtain the following expression that relates the shear tensors

$$\sigma_{\alpha\beta} = \tilde{\sigma}_{\alpha\beta} - \frac{1}{3} T_\rho \xi^\rho h_{\alpha\beta} + h^\sigma{}_\alpha h^\rho{}_\beta K^\mu{}_{(\sigma\rho)} \xi_\mu = \overset{\circ}{\sigma}_{\alpha\beta} \ , \tag{3.167}$$

as well the below expression relating the twist tensors

$$\omega_{\alpha\beta} = \tilde{\omega}_{\alpha\beta} + h^\sigma{}_\alpha h^\rho{}_\beta K^\mu{}_{[\sigma\rho]} \xi_\mu = \overset{\circ}{\omega}_{\alpha\beta} - \xi^\nu h^\sigma{}_\alpha h^\rho{}_\beta K_{\sigma\nu\rho} \ . \tag{3.168}$$

We note that, while the geometric expansion scalar and shear tensor coincide with their torsion-free counterparts, this is not the case for the geometric twist tensor. That is, the difference between the tensor fields $\overset{\circ}{\boldsymbol{B}}$ and $\boldsymbol{B}$ lies entirely in the totally antisymmetric part. This is easily seen by examining the second equality in equation (3.148) since the tensor field $K_{\alpha\nu\beta}\xi^\nu$ is antisymmetric.

3.11 Chapter conclusions

In this chapter, we have discussed the notational conventions used in this thesis as well as the notion of the Riemann-Cartan space-time U_4. By following what was done in [9], we defined the U_4 space-time to be the tuple $(\mathcal{M}, \boldsymbol{g}, \nabla)$ where $\mathcal{M}$ is the space-time manifold, $\boldsymbol{g}$ is the metric tensor and ∇ is the Cartan connection which is non-symmetric and metric compatible. Upon defining U_4, we studied the notions of curvature and torsion in this space-time following [74, 77, 79]. To this end, we introduced the so-called contorsion tensor as the non-Riemannian contribution to the connection coefficients of the Cartan connection. We also made use of the contorsion tensor to relate the non-null torsion Riemann tensor to the torsion-free Riemann tensor. More specifically, we discussed how one can write the non-Riemannian contribution to the non-null torsion Riemann tensor in terms of the contorsion tensor as well as its first order derivatives. We also followed what was done in [5] in order to derive the Bianchi identities associated with the torsion-free and non-null torsion Riemann tensors.

By following what is done in [9,16,18], we discussed the so-called Cartan formalism. This formalism will be made use of in Section 4.2 where we discuss the Einstein-Cartan theory of gravity.

We concluded this chapter with the important topic of the notion of auto-parallel curves in U_4. In the event that the tangent vector field of a curve is parallel propagated in the direction of itself with respect to the Cartan connection, we refer to this curve as an auto-parallel of the Cartan connection. Similarly, if the tangent vector field is parallel propagated in the direction of itself with respect to the Levi-Civita connection, we refer to the curve as an auto-parallel of the Levi-Civita connection. By following [74], we noted that extremal curves are auto-parallel curves of the Levi-Civita connection. This implies that, when considering test particles to follow extremal curves, one should consider a congruence of auto-parallel curves of the Levi-Civita connection when studying the satisfaction or violation of the condition that gravity be attractive. For such a study, one requires the Raychaudhuri equation.

By following [58, 59], we derived the non-null torsion Raychaudhuri equation by decomposing the tensor field obtained by acting the Cartan covariant derivative on the tangent vector field. While this equation is well-defined on U_4, the aforementioned tensor field does not measure the failure for the deviation vector between curves contained in the congruence to be parallel propagated with respect to the Cartan connection. This implies that the kinematic quantities associated with this tensor field do not give the relevant geometrical description. By following what is done in [60, 61], we derived the geometric Raychaudhuri equation by decomposing the tensor field that measures the failure for the deviation vector to be parallel propagated with respect to the Cartan connection. In the case of vanishing torsion, both the non-null torsion and geometric Raychaudhuri equations reduce to the well-known torsion-free Raychaudhuri equation which was derived in the case of Riemannian space-time V_4 in [54]. The tensor field that is decomposed in order to derive the torsion-free Raychaudhuri equation is both the Levi-Civita covariant derivative of the tangent vector field as well as the measure of the failure for the deviation vector field to be parallel propagated with respect to the Levi-Civita connection.

Since test particles follow auto-parallel curves of the Levi-Civita connection, one must turn to the torsion-free Raychaudhuri equation in order to study the satisfaction or violation of the condition that gravity be attractive. In Chapter 5, we will make use of this Raychaudhuri equation in order to study the focusing conditions in $f(T)$ theories of gravity.

For now, however, we turn our attention to discussing various gravitational theories that can be constructed on U_4. This is done in the following chapter.

Chapter 4

Gravitational theories with non-vanishing torsion

4.1 Introduction

In the previous chapter, we discussed space-times for which the natural choice of an affine connection is not necessarily symmetric while still satisfying the property of metric compatibility. In particular, we discussed the so-called Riemann-Cartan space-time whose natural choice of an affine connection provides a description involving non-vanishing torsion in addition to non-vanishing curvature. In this chapter, we study gravitational theories that are constructed using the Riemann-Cartan space-time U_4 as well as theories constructed on the Weitzenböck space-time W_4. In Section 4.2, we discuss the formulation of the so-called Einstein-Cartan (EC) theory. Following this, we discuss two possible modifications to this theory. We first discuss the Einstein-Cartan-Saa (ECS) theory of gravity which results from making a specific modification to the volume element of the action of the EC theory. This is done in Section 4.3. Following this, we discuss the gravitational theory resulting from a Chern-Simons modification to the EC theory in Section 4.4. We conclude this chapter with a discussion on the formulation of $f(T)$ theories of gravity which are constructed on the Weitzenböck space-time. This is done in Section 4.5.

4.2 Einstein-Cartan theory

In this section, we present the so-called Einstein-Cartan (EC) theory of gravity following [91] by Kibble, [92] by Sciama, [7] by Trautman and [9] by Tseng. The EC theory is often viewed as GR with the inclusion of torsion [9]. In this section, we wish to discuss the formulation of the EC theory before moving on to derive the field equations and discuss the phenomenology of the torsion tensor in this theory. We begin this section by first giving a definition for the EC theory. In defining gravitational theories, we require, firstly, a space-time and, secondly, a gravitational action. In constructing the EC theory of gravity, we make use of the Riemann-Cartan space-time defined in the previous chapter. That is, we consider a space-time whose natural choice of an affine connection is the Cartan connection which is metric compatible and not necessarily symmetric which implies that the space-time has non-vanishing torsion. When constructing a gravitational theory for such a space-time, the constructed action will dictate the role of torsion in that theory. In the EC theory, the gravitational action in the Cartan formalism reads

$$S_{\text{EC}} := \frac{1}{2k^2} \int \tilde{\boldsymbol{R}}^{ba} \wedge \boldsymbol{\zeta}_{ab} \ , \tag{4.1}$$

where $k^2 := 8\pi G/c^4$, $\tilde{\boldsymbol{R}}^{ba}$ is the curvature two-form as before and $\boldsymbol{\zeta}_{ab}$ is defined as

$$\boldsymbol{\zeta}_{ab} := * \left(\hat{\boldsymbol{e}}_a \wedge \hat{\boldsymbol{e}}_b\right) = \frac{1}{2}\epsilon_{abqm} \hat{\boldsymbol{\theta}}^q \wedge \hat{\boldsymbol{\theta}}^m \ , \tag{4.2}$$

where ϵ_{abqm} is the Levi-Civita tensor in the non-holonomic frame [9]. We now define the EC theory to be the Riemann-Cartan space-time U_4 together with the action given in equation (4.1) [9]. In addition to the EC action given in equation (4.1), we can introduce the matter action S_m, which has associated matter Lagrangian density $\mathcal{L}_m$, so that the total action considered is

$$S := S_{\text{EC}} + S_{\text{m}} \ . \tag{4.3}$$

Given the total action above, we wish to study the role of curvature and torsion in the EC theory. In order to do this, we require that the total action experiences an extremum under an ordinary variation [89]. That is, we require that

$$\delta S \overset{!}{=} 0 \ . \tag{4.4}$$

By requiring that the total action experiences an extremum, we can follow [7,9] in order to obtain the EC field equations. Let us begin by considering the EC Lagrangian density which has the following form

$$\mathcal{L}_{\text{EC}} := \tilde{\boldsymbol{R}}^{ba} \wedge \boldsymbol{\zeta}_{ab} \ . \tag{4.5}$$

The above expression can be obtained by simply reading off the Lagrangian density from the expression for the EC gravitational action given in equation (4.1). By carrying out an ordinary variation of the Lagrangian density given in equation (4.5) one obtains [9]

$$\delta\left(\mathcal{L}_{\text{EC}}\right) = \delta\left(\tilde{\boldsymbol{R}}^{ba} \wedge \boldsymbol{\zeta}_{ab}\right) = \delta\tilde{\boldsymbol{R}}^{ba} \wedge \boldsymbol{\zeta}_{ab} + \tilde{\boldsymbol{R}}^{ba} \wedge \delta\boldsymbol{\zeta}_{ab} \ . \tag{4.6}$$

Let us first consider the variation of $\boldsymbol{\zeta}_{ab}$. By making use of equation (4.2), one obtains the following for the variation of $\boldsymbol{\zeta}_{ab}$ [9]

$$\begin{aligned}\delta\boldsymbol{\zeta}_{ab} &= \frac{1}{2}\left(\epsilon_{abcd}\delta\hat{\boldsymbol{\theta}}^c \wedge \hat{\boldsymbol{\theta}}^d + \epsilon_{abcd}\hat{\boldsymbol{\theta}}^c \wedge \delta\hat{\boldsymbol{\theta}}^d\right) \\ &= \epsilon_{abcd}\delta\hat{\boldsymbol{\theta}}^c \wedge \hat{\boldsymbol{\theta}}^d \ . \end{aligned} \tag{4.7}$$

In equation (4.2) we defined the two-form field $\boldsymbol{\zeta}_{ab}$ as the Hodge dual of the wedge product between two non-holonomic basis elements. In addition, we can define the four-form field $\boldsymbol{\zeta}$ as

$$\boldsymbol{\zeta} := *1 = \frac{1}{4!}\epsilon_{abcd}\hat{\boldsymbol{\theta}}^a \wedge \hat{\boldsymbol{\theta}}^b \wedge \hat{\boldsymbol{\theta}}^c \wedge \hat{\boldsymbol{\theta}}^d \ , \tag{4.8}$$

the three-form field $\boldsymbol{\zeta}_a$ as

$$\boldsymbol{\zeta}_a := *\hat{\boldsymbol{e}}_a = \frac{1}{3!}\epsilon_{abcd}\hat{\boldsymbol{\theta}}^b \wedge \hat{\boldsymbol{\theta}}^c \wedge \hat{\boldsymbol{\theta}}^d \ , \tag{4.9}$$

the one-form field $\boldsymbol{\zeta}_{abc}$ as

$$\boldsymbol{\zeta}_{abc} := *\left(\hat{\boldsymbol{e}}_a \wedge \hat{\boldsymbol{e}}_b \wedge \hat{\boldsymbol{e}}_c\right) = \epsilon_{abcd}\hat{\boldsymbol{\theta}}^d \ , \tag{4.10}$$

and, lastly, the zero-form field ζ_{abcd} as simply being the Levi-Civita tensor in the non-holonomic frame [9]

$$\zeta_{abcd} := *\left(\hat{\boldsymbol{e}}_a \wedge \hat{\boldsymbol{e}}_b \wedge \hat{\boldsymbol{e}}_c \wedge \hat{\boldsymbol{e}}_c\right) = \epsilon_{abcd} \ . \tag{4.11}$$

Now, by making use of equation (4.10) ,which provides us with the definition of the one-form field $\boldsymbol{\zeta}_{abc}$, it follows from equation (4.7) that we can write the variation of the two-form field $\boldsymbol{\zeta}_{ab}$ as

$$\delta\boldsymbol{\zeta}_{ab} = \delta\hat{\boldsymbol{\theta}}^c \wedge \boldsymbol{\zeta}_{abc} \ . \tag{4.12}$$

Let us now turn our attention to studying the variation of the curvature two-form $\tilde{\boldsymbol{R}}^{ba}$ which appears on the right-hand side of equation (4.6). By making use of the second Cartan structure equation which is given in equation (3.101), we obtain the following expression for the variation of the curvature two-form

$$\delta\tilde{\boldsymbol{R}}^{ba} = \mathrm{d}\delta\boldsymbol{w}^{ab} + \delta\boldsymbol{w}^{a}{}_{c} \wedge \boldsymbol{w}^{cb} + \boldsymbol{w}^{a}{}_{c} \wedge \delta\boldsymbol{w}^{cb} \ . \tag{4.13}$$

Now, we note that the action of the Lorentz covariant derivative, which is defined in equation (3.92), on the variation of the connection one-form yields

$$D\delta\boldsymbol{w}^{ab} = \mathrm{d}\delta\boldsymbol{w}^{ab} + \delta\boldsymbol{w}^{a}{}_{l} \wedge \boldsymbol{w}^{lb} + \boldsymbol{w}^{a}{}_{l} \wedge \delta\boldsymbol{w}^{lb} \ , \tag{4.14}$$

which is nothing more than the right-hand side of equation (4.13). It therefore follows that [9]

$$\delta\tilde{\boldsymbol{R}}^{ba} = D\delta\boldsymbol{w}^{ab} \ . \tag{4.15}$$

Therefore, by substituting equations (4.12) and (4.15) into equation (4.6), one obtains the following expression for the variation of the EC Lagrangian density

$$\delta\mathcal{L}_{\mathrm{EC}} = \left(D\delta\boldsymbol{w}^{ab}\right) \wedge \boldsymbol{\zeta}_{ab} + \tilde{\boldsymbol{R}}^{ba} \wedge \delta\hat{\boldsymbol{\theta}}^{c} \wedge \boldsymbol{\zeta}_{abc} \ . \tag{4.16}$$

By adding and subtracting like terms in the above expression and by grouping terms that contribute to a total Lorentz covariant derivative, one obtains

$$\delta\mathcal{L}_{\mathrm{EC}} = \tilde{\boldsymbol{R}}^{ba} \wedge \delta\hat{\boldsymbol{\theta}}^{c} \wedge \boldsymbol{\zeta}_{abc} + \mathrm{d}\left(\delta\boldsymbol{w}^{ab} \wedge \boldsymbol{\zeta}_{ab}\right) - \delta\boldsymbol{w}^{ab} \wedge D\boldsymbol{\zeta}_{ab} \ , \tag{4.17}$$

where we have made use of the fact that the action of the Lorentz covariant derivative on a smooth function is nothing more than the action of the exterior derivative on that function. We note that the above expression contains the Lorentz covariant derivative of the two-form field $\boldsymbol{\zeta}_{ab}$. By making use of the definition of the two-form field $\boldsymbol{\zeta}_{ab}$ given in equation (4.2), one obtains the following for the action of the Lorentz covariant derivative on this two-form field

$$\begin{aligned} D\boldsymbol{\zeta}_{ab} &= \frac{1}{2} D\left(\epsilon_{abcd}\hat{\boldsymbol{\theta}}^{c} \wedge \hat{\boldsymbol{\theta}}^{d}\right) \\ &= \epsilon_{abcd}\left(D\hat{\boldsymbol{\theta}}^{c}\right) \wedge \boldsymbol{\zeta}_{abc} \ , \end{aligned} \tag{4.18}$$

where we have made use of equation (4.10) in order to introduce the one-form field $\boldsymbol{\zeta}_{abc}$ into the above expression. We now wish to consider the action of the Lorentz covariant derivative on the co-tetrad field $\hat{\boldsymbol{\theta}}^{c}$ which is given in the above expression. It can be seen that the right-hand side of the first Cartan structure equation (3.100) is nothing more than the Lorentz covariant derivative of the co-tetrad field. Thus, the action of the Lorentz covariant derivative on the co-tetrad field $\hat{\boldsymbol{\theta}}^{c}$ is nothing more than the torsion two-form field T^{c}. It therefore follows that equation (4.18) becomes

$$D\boldsymbol{\zeta}_{ab} = \boldsymbol{T}^{c} \wedge \boldsymbol{\zeta}_{abc} \ . \tag{4.19}$$

Substituting the above result into equation (4.17) yields

$$\delta\mathcal{L}_{\mathrm{EC}} = \tilde{\boldsymbol{R}}^{ba} \wedge \delta\hat{\boldsymbol{\theta}}^{c} \wedge \boldsymbol{\zeta}_{abc} - \delta\boldsymbol{w}^{ab} \wedge \boldsymbol{T}^{c} \wedge \boldsymbol{\zeta}_{abc} + \mathrm{d}\left(\delta\boldsymbol{w}^{ab} \wedge \boldsymbol{\zeta}_{ab}\right) \ . \tag{4.20}$$

Through the appropriate modification[1] of the gravitational action, the third term in the above expression has no contribution [5]. We are now in a position to carry out the variation of the EC Lagrangian density with respect to either the co-tetrad field or the connection one-form field in order to obtain two sets of field equations. Let us begin by varying the Lagrangian density with respect to the co-tetrad field $\hat{\boldsymbol{\theta}}^{c}$. By carrying out such a variation, one obtains [9]

$$\frac{\delta\mathcal{L}_{\mathrm{EC}}}{\delta\hat{\boldsymbol{\theta}}^{c}} = -\tilde{\boldsymbol{R}}^{ba} \wedge \boldsymbol{\zeta}_{abc} \ . \tag{4.21}$$

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[1] Such a modification involves the inclusion of the so-called *Gibbons-Hawking-York boundary term* [5, 93, 94]. This term is necessary when considering a space-time manifold that has a non-vanishing boundary since the third term in equation (4.20) will be non-zero.

For the variation of the matter Lagrangian density, we define $\boldsymbol{\tau}_c$ as the three-form field obtained by varying the matter Lagrangian density with repect to the co-tetrad field. That is, we write [7]

$$\boldsymbol{\tau}_c := \frac{\delta \mathcal{L}_{\mathrm{m}}}{\delta \hat{\boldsymbol{\theta}}^c} \ . \tag{4.22}$$

It therefore follows from equations (4.21) and (4.22) that the ordinary variation of the total action with respect to the co-tetrad field yields the following field equation [7, 9]

$$\frac{1}{2k^2} \boldsymbol{\zeta}_{abc} \wedge \tilde{\boldsymbol{R}}^{ba} = -\tau_c \ , \tag{4.23}$$

where we have reintroduced the factor of $1/\left(2k^2\right)$ that was initially omitted when defining the EC Lagrangian density. We now wish to consider the variation of the EC Lagrangian density with respect to the connection one-form. By carrying out such a variation of equation (4.20) one obtains

$$\frac{\delta \mathcal{L}_{\mathrm{EC}}}{\delta \boldsymbol{w}^{ab}} = -\boldsymbol{T}^c \wedge \boldsymbol{\zeta}_{abc} \ . \tag{4.24}$$

As was done in the case of varying the matter Lagrange density with respect to the co-tetrad field, we define $\boldsymbol{s}_{ab}$ to be the three-form field obtained when varying the matter Lagrangian density with respect to the connection one-form field. That is, we write

$$\boldsymbol{s}_{ba} := \frac{1}{2} \frac{\delta \mathcal{L}_{\mathrm{m}}}{\delta \boldsymbol{w}^{ab}} \ . \tag{4.25}$$

It then follows from equations (4.24) and (4.25) that the field equation obtained by varying the total gravitational action with respect to the connection one-form field is [7, 9]

$$\frac{1}{k^2} \boldsymbol{\zeta}_{abc} \wedge \boldsymbol{T}^c = \boldsymbol{s}_{ab} \ , \tag{4.26}$$

where we have once again reintroduced the constant factor of k^2. Equations (4.23) and (4.26) are the desired field equations in the EC theory of gravity and are referred to as the *Einstein-Cartan equations* [7]. Here the EC equations have been written in differential form.

We now wish to write the EC equations in terms of non-holonomic basis components. Such a form of the EC equations is referred to as the Sciama-Kibble form and can be found in [92] by Sciama as well as [91] by Kibble. We shall derive the Sciama-Kibble form of the EC equations given in equations (4.23) and (4.26) by following [9]. Let us begin by considering the EC equation (4.23). Through the use of the definition of the curvature two-form, which is given in equation (3.99), we have

$$\boldsymbol{\zeta}_{abc} \wedge \tilde{\boldsymbol{R}}^{ba} = \frac{1}{2} \tilde{R}_{le}{}^{ba} \hat{\boldsymbol{\theta}}^e \wedge \hat{\boldsymbol{\theta}}^l \wedge \boldsymbol{\zeta}_{abc} \ . \tag{4.27}$$

By substituting the definition for the one-form field $\boldsymbol{\zeta}_{abc}$ given in equation (4.10) into the above expression, one obtains

$$\boldsymbol{\zeta}_{abc} \wedge \tilde{\boldsymbol{R}}^{ba} = \frac{1}{2} \tilde{R}_{le}{}^{ba} \epsilon_{abcd} \hat{\boldsymbol{\theta}}^e \wedge \hat{\boldsymbol{\theta}}^l \wedge \hat{\boldsymbol{\theta}}^d \ . \tag{4.28}$$

By making use of equation (4.9) in order to introduce the two-form field $\boldsymbol{\zeta}_{ab}$ into the right-hand side of the above expression one obtains

$$\boldsymbol{\zeta}_{abc} \wedge \tilde{\boldsymbol{R}}^{ba} = \frac{1}{4} \tilde{R}_{le}{}^{ba} \epsilon_{abcd} \epsilon^{mnld} \hat{\boldsymbol{\theta}}^e \wedge \boldsymbol{\zeta}_{mn} \ . \tag{4.29}$$

Writing the contraction of the Levi-Civita tensors in terms of Kronecker delta symbols yields

$$\begin{aligned} \boldsymbol{\zeta}_{abc} \wedge \tilde{\boldsymbol{R}}^{ba} &= \frac{1}{2} \tilde{R}_{le}{}^{ba} \hat{\boldsymbol{\theta}}^e \wedge \left(\delta^l_c \boldsymbol{\zeta}_{ab} + \delta^l_a \boldsymbol{\zeta}_{bc} + \delta^l_b \boldsymbol{\zeta}_{ca} \right) \\ &= -\tilde{R}_e{}^b \hat{\boldsymbol{\theta}}^e \wedge \boldsymbol{\zeta}_{bc} + \frac{1}{2} \tilde{R}_{ce}{}^{ba} \hat{\boldsymbol{\theta}}^e \wedge \boldsymbol{\zeta}_{ab} \ , \end{aligned} \tag{4.30}$$

where the components of the non-null torsion Ricci tensor in the non-holonomic frame have been introduced into the right-hand side of the above expression. Now, by making use of the following

$$\begin{aligned}\hat{\boldsymbol{\theta}}^a \wedge \boldsymbol{\zeta}_{bc} &= \frac{1}{2}\epsilon_{bclk}\hat{\boldsymbol{\theta}}^a \wedge \hat{\boldsymbol{\theta}}^l \wedge \hat{\boldsymbol{\theta}}^k \\ &= \frac{1}{2}\epsilon_{bclk}\epsilon^{malk}\boldsymbol{\zeta}_m \\ &= \boldsymbol{\zeta}_b\delta^a_c - \boldsymbol{\zeta}_c\delta^a_b \ , \end{aligned} \tag{4.31}$$

we can write equation (4.30) as

$$\boldsymbol{\zeta}_{abc} \wedge \tilde{\boldsymbol{R}}^{ba} = -2\left(\tilde{R}_{cb} - \frac{1}{2}\tilde{R}\eta_{cb}\right)\boldsymbol{\zeta}^b \ . \tag{4.32}$$

By substituting the above expression into the first EC equation (4.23) and by defining the *energy-momentum tensor* through $\boldsymbol{\tau}_c := \tau_{cb}\boldsymbol{\zeta}^b$ [9], one obtains the Sciama-Kibble form of the first EC equation [7]

$$\tilde{R}_{cb} - \frac{1}{2}\tilde{R}\eta_{cb} = k^2\tau_{cb} \ . \tag{4.33}$$

In order to obtain the Sciama-Kibble form of the second EC equation, which is given in equation (4.26), we can carry out a similar process. By carrying out such a process for the left-hand side of equation (4.26), one obtains the following

$$\begin{aligned}\boldsymbol{\zeta}_{abc} \wedge \boldsymbol{T}^c &= \frac{1}{2}T^c{}_{lm}\hat{\boldsymbol{\theta}}^l \wedge (\delta^m_a\boldsymbol{\zeta}_{bc} + \delta^m_b\boldsymbol{\zeta}_{ca} + \delta^m_c\boldsymbol{\zeta}_{ab}) \\ &= -\frac{1}{2}\left[T^c{}_{la}\left(\delta^l_b\boldsymbol{\zeta}_c - \delta^l_c\boldsymbol{\zeta}_b\right) + T^c{}_{lb}\left(\delta^l_c\boldsymbol{\zeta}_a - \delta^l_a\boldsymbol{\zeta}_c\right) + T_l\left(\delta^l_a\boldsymbol{\zeta}_b - \delta^l_b\boldsymbol{\zeta}_a\right)\right] \\ &= -T^c{}_{ba}\boldsymbol{\zeta}_c - T_a\boldsymbol{\zeta}_b + T_b\boldsymbol{\zeta}_a \ , \end{aligned} \tag{4.34}$$

and we remind the reader that $T_l := T^c{}_{lc}$ is the trace of the torsion tensor in the non-holonmic frame. Finally, by substituting equation (4.34) into equation (4.26) and by defining the *spin-current tensor* through $\boldsymbol{s}_{ab} := s_{abc}\boldsymbol{\zeta}^c$ [9], one obtains the Sciama-Kibble form of the second EC equation [7]

$$T^c{}_{ab} + \delta^c_a T_b - \delta^c_b T_a = k^2 s_{ab}{}^c \ . \tag{4.35}$$

We have thus obtained the Sciama-Kibble form of the field equations in the EC theory which have been obtained before in [91, 92]. It is important to note that the second EC equation allows us to describe an important relationship between the torsion tensor and the spin matter content which is contained in the spin-current tensor. Such a relationship can be interpreted from the following theorem which is stated and proved here following [7, 9].

Theorem 4.2.1. *In the event that the spin-current tensor vanishes, the EC theory of gravity reduces to GR [9].*

Proof. We consider the second EC equation in the Sciama-Kibble form given in equation (4.35). By taking the trace of this equation and substituting the resulting expression back into equation (4.35), one obtains the following expression for the components of the torsion tensor in the non-holonomic frame [7]

$$T^c{}_{ab} = \frac{k^2}{2}\left(2s_{ab}{}^c + \delta^c_a s_{bd}{}^d - \delta^c_b s_{ad}{}^d\right) \ . \tag{4.36}$$

It is evident from the above expression that, in the case where the spin-current tensor vanishes, i.e. $s_{ab}{}^c = 0$, then the torsion tensor vanishes. Since the torsion tensor vanishes, the Cartan connection reduces to the Levi-Civita connection and, thus, the non-null torsion Riemann tensor reduces to the torsion-free Riemann tensor. In such a case, the first EC equation, which is given in equation (4.23), reduces to the field equations of GR. We have thus shown that, when the spin-current tensor vanishes, the EC theory of gravity is nothing more than GR. ☐

The theorem proved above implies that GR is simply a degenerate case of the EC theory of gravity [9]. As already mentioned, Theorem 4.2.1 provides us with an important relationship between the torsion tensor and the spin matter content. More specifically, the torsion tensor provides a geometrical interpretation of the spin-current tensor [70]. Such a notion is worth mentioning since such a relationship does not hold between curvature and the energy-momentum tensor. From the first EC equation in the Sciama-Kibble form given in equation (4.33), it can be seen that the vanishing of the energy-momentum tensor implies the vanishing of the non-null torsion Ricci tensor, however, this does not imply the vanishing of the non-null torsion Riemann tensor and therefore does not imply the vanishing of curvature. The fact that the vanishing of the spin-current tensor directly implies the vanishing of torsion implies that the spin-current tensor is geometrised by the torsion tensor in the EC theory of gravity [70].

So far, we have derived the EC field equations in differential form, which are given in equations (4.23) and (4.26) as well as in the Sciama-Kibble form, which are given in equations (4.33) and (4.35). It is not difficult to obtain these field equations in terms of the holonomic basis components by applying the rotations described by the tetrad field to the Sciama-Kibble equations. Nevertheless, here we wish to derive the EC field equations in coordinate components through the direct variation of the gravitational action. The reason for this is that, in subsequent sections, we will follow this method in deriving the field equations for theories that are modifications of the EC theory. Therefore, it will prove to be useful to carry out such a derivation of the coordinate component field equations for the EC theory of gravity which is equivalent to the application of the vierbein to the Sciama-Kibble equations. We begin by considering the EC Lagrangian density given in equation (4.5). By writing the curvature two-form in terms of its non-holonomic components through the use of its definition as well as by making use of the definition of the two-form field $\boldsymbol{\zeta}_{sb}$, we have the following for the EC Lagrangian density

$$\tilde{\boldsymbol{R}}^{ba} \wedge \boldsymbol{\zeta}_{ab} = \frac{1}{4}\tilde{R}_{dc}{}^{ba}\epsilon_{abqm}\hat{\boldsymbol{\theta}}^{c} \wedge \hat{\boldsymbol{\theta}}^{d} \wedge \hat{\boldsymbol{\theta}}^{q} \wedge \hat{\boldsymbol{\theta}}^{m} \ . \tag{4.37}$$

By introducing the four-form field ζ into the above expression through the use of its definition given in equation (4.8) and by contracting indices in the Levi-Civita tensors in order to produce Kronecker delta symbols, one obtains [9]

$$\begin{aligned}\tilde{R}^{ba} \wedge \boldsymbol{\zeta}_{ab} &= \frac{1}{2}\tilde{R}_{dc}{}^{ba}\left(\delta^{c}_{a}\delta^{d}_{b} - \delta^{d}_{a}\delta^{c}_{b}\right)\boldsymbol{\zeta} \\ &= \tilde{R}\boldsymbol{\zeta} \ ,\end{aligned} \tag{4.38}$$

and we remind the reader that $\tilde{R}$ denotes the non-null torsion Ricci scalar. We note that the volume element chosen to construct the action integral in the EC theory is the four-form field $\boldsymbol{\zeta}$. From the definition of this four-form field, we note that it can be written as

$$\boldsymbol{\zeta} = \hat{\boldsymbol{\theta}}^{0} \wedge \hat{\boldsymbol{\theta}}^{1} \wedge \hat{\boldsymbol{\theta}}^{2} \wedge \hat{\boldsymbol{\theta}}^{3} \ . \tag{4.39}$$

The taking of this four-form field to be the volume element in the EC theory, as well as in GR, is referred to as a natural choice of a volume element since it is the simplest construction of a volume element from the co-tetrad fields. In order to write this volume element in terms of the dual basis elements associated with the holonomic frame, we can make use of the definition of the four-form field $\boldsymbol{\zeta}$ as well as the vierbein in order to write it is

$$\begin{aligned}\boldsymbol{\zeta} &= \frac{1}{4!}\epsilon_{abcd}\epsilon^{\mu\nu\delta\rho}e^{a}{}_{\mu}e^{b}{}_{\nu}e^{c}{}_{\delta}e^{d}{}_{\rho}\mathrm{d}x^{0} \wedge \mathrm{d}x^{1} \wedge \mathrm{d}x^{2} \wedge \mathrm{d}x^{3} \\ &= e\ \mathrm{d}x^{0} \wedge \mathrm{d}x^{1} \wedge \mathrm{d}x^{2} \wedge \mathrm{d}x^{3} \ ,\end{aligned} \tag{4.40}$$

where e is the determinant of the vierbein. It is not difficult to see from equation (3.85) that $e = \sqrt{-g}$ and, thus, it follows from the form of the volume element given in equation (4.40) that we can write the EC action as [79]

$$S_{\mathrm{EC}} = \frac{1}{2k^{2}}\int \tilde{R}\sqrt{-g}\ \mathrm{d}^{4}x \ . \tag{4.41}$$

We remind the reader that $k^2 := 8\pi G/c^4$ as before and we now write the Lagrangian density associated with the above action as

$$\mathcal{L}_{\text{EC}} = \tilde{R}\sqrt{-g} \ . \tag{4.42}$$

where we wish to ommit the inclusion of the constant k for simplicity. We now wish to vary the above Lagrangian density with respect to the metric and contorsion tensor components in the holonomic frame. In [95], the Einstein-Hilbert action $S_{\text{EH}} := \int R\sqrt{-g}\ \mathrm{d}^4x$ is considered and the field equations valid for GR are derived by carrying out a variation of this action with respect to the holonomic components of the metric tensor. Here, we shall follow this method in first varying the EC action with respect to the holonomic components of the metric tensor and then varying the EC action with respect to the holonomic components of the contorsion tensor. Let us denote the variation of the EC Lagrangian density, equation (4.42), with respect to the metric and contorsion tensor components in the holonomic frame as $\delta_g\mathcal{L}_{\text{EC}}$ and $\delta_K\mathcal{L}_{\text{EC}}$ respectively. Varying the EC Lagrangian density with respect to the metric yields

$$\delta_g\mathcal{L}_{\text{EC}} = \left(\delta_g\tilde{R}\right)\sqrt{-g} + \tilde{R}\left(\delta_g\sqrt{-g}\right) \ . \tag{4.43}$$

The variation of $\sqrt{-g}$ with respect to the metric yields [95]

$$\delta_g\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\ g_{\mu\nu}\delta g^{\mu\nu} \ . \tag{4.44}$$

Let us now consider the variation of the non-null torsion Ricci scalar. We have the following [95]

$$\delta_g\tilde{R} = \tilde{R}_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta_g\tilde{R}_{\mu\nu} \ . \tag{4.45}$$

Since the metric tensor and the torsion tensor are treated as being independent quantities, the variation of the non-null torsion Ricci tensor with respect to the metric is equivalent to the variation of the torsion-free Ricci tensor with respect to the metric since the non-Riemannian contributions to the construction of the non-null torsion Ricci tensor are independent of the metric tensor. Therefore, by making use of equation (3.66), we obtain the following for the variation of the non-null torsion Ricci tensor with respect to the metric

$$\begin{aligned}\delta_g\tilde{R}_{\mu\nu} = \delta_g R_{\mu\nu} &= \partial_\beta\delta\Gamma^{\beta}{}_{\mu\nu} - \partial_\mu\delta\Gamma^{\beta}{}_{\beta\nu} + \Gamma^{\lambda}{}_{\mu\nu}\delta\Gamma^{\beta}{}_{\beta\delta} + \delta\Gamma^{\lambda}{}_{\mu\nu}\Gamma^{\beta}{}_{\beta\lambda} - \Gamma^{\lambda}{}_{\beta\nu}\delta\Gamma^{\beta}{}_{\mu\lambda} - \delta\Gamma^{\lambda}{}_{\beta\nu}\Gamma^{\beta}{}_{\mu\lambda} \\ &= D_\beta\left(\delta\Gamma^{\beta}{}_{\mu\nu}\right) - D_\mu\left(\delta\Gamma^{\beta}{}_{\beta\nu}\right) \ ,\end{aligned} \tag{4.46}$$

where we have made use of the so-called Palatini identity [95]. It is evident from the last expression that the variation of the non-null torsion Riemann tensor with respect to the metric yields total Levi-Civita covariant derivatives of the Christoffel symbols. Since these are covariant derivatives associated with the Levi-Civita connection and the choice of a volume element in the EC action is compatible with the Levi-Civita connection, the variation of the non-null torsion Ricci tensor yields a pure boundary term. This term can be removed through the appropriate modification of the gravitational action, i.e., by including the so-called Gibbons-Hawking-York boundary term [5, 93, 94]. The variation of the EC Lagrangian density now reads

$$\delta_g\mathcal{L}_{\text{EC}} = \sqrt{-g}\ g^{\mu\nu}D_\beta\left(\delta\Gamma^{\beta}{}_{\mu\nu}\right) - \sqrt{-g}\ D^\nu\left(\delta\Gamma^{\beta}{}_{\beta\nu}\right) + \sqrt{-g}\tilde{R}_{\mu\nu}\delta g^{\mu\nu} - \frac{\sqrt{-g}}{2}\tilde{R}\ g_{\mu\nu}\delta g^{\mu\nu} \ . \tag{4.47}$$

Let us now turn our attention to carrying out the variation of the EC Lagrangian density with respect to the contorsion tensor. In order to carry out such a variation, we turn our attention to equation (3.68) and study the variation of the non-Riemannian terms since these are the only terms that carry dependence on the contorsion tensor. The variation of the EC Lagrangian density with respect to the contorsion tensor yields

$$\begin{aligned}\delta_K\mathcal{L}_{\text{EC}} = \sqrt{-g}\ g^{\mu\nu}\delta_K\tilde{R}_{\mu\nu} &= \sqrt{-g}\ g^{\mu\nu}\left[\frac{\partial\tilde{R}_{\mu\nu}}{\partial K^{\alpha}{}_{\sigma\lambda}} - D_\gamma\left(\frac{\partial\tilde{R}_{\mu\nu}}{\partial\left(D_\gamma K^{\alpha}{}_{\sigma\lambda}\right)}\right)\right]\delta K^{\alpha}{}_{\sigma\lambda} \\ &= \frac{\sqrt{-g}}{2}\left(T^{\nu\lambda}{}_{\mu} + g^{\nu\lambda}T_\mu - \delta^\nu_\mu T^\lambda\right)\delta K^{\mu}{}_{\nu\lambda} \ .\end{aligned} \tag{4.48}$$

It therefore follows from equations (4.47) and (4.48) that the total variation of the EC Lagrangian density is given by the expression

$$\delta\mathcal{L}_{\mathrm{EC}} = \sqrt{-g}\; g^{\mu\nu} D_\beta \left(\delta\Gamma^{\beta}{}_{\mu\nu}\right) - \sqrt{-g}\; D^\nu \left(\delta\Gamma^{\beta}{}_{\beta\nu}\right) + \sqrt{-g}\Big[\left(\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\; g_{\mu\nu}\right)\delta g^{\mu\nu} + \frac{1}{2}\left(T^{\nu\lambda}{}_{\mu} + g^{\nu\lambda}T_\mu - \delta^\nu_\mu T^\lambda\right)\delta K^{\mu}{}_{\nu\lambda}\Big] \;. \tag{4.49}$$

By writing the matter action as $S_{\mathrm{m}} := \int \mathrm{d}^4 x \mathcal{L}_m$ and writing the energy-momentum tensor in holonomic basis components as [95]

$$\tau_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_{\mathrm{m}}}{\delta g^{\mu\nu}} \;, \tag{4.50}$$

it follows from equations (4.49) and (4.50) that the first EC equation in holonomic basis components obtained by varying the total action with respect to the metric is

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R} g_{\mu\nu} = k^2 \tau_{\mu\nu} \;, \tag{4.51}$$

where we have reintroduced the constant factor k^2. Now, by defining the spin-current tensor as

$$s_{\mu}{}^{\lambda\nu} := \frac{4}{\sqrt{-g}}\frac{\delta\mathcal{L}_{\mathrm{m}}}{\delta K^{\mu}{}_{\nu\lambda}} \;, \tag{4.52}$$

it follows from equations (4.49) and (4.52) that the second EC field equation, written in terms of holonomic basis components, obtained by varying the total action with respect to the contorsion tensor is

$$T^{\nu}{}_{\lambda\mu} + \delta^\nu_\lambda T_\mu - \delta^\nu_\mu T_\lambda = k^2 s_{\lambda\mu}{}^{\nu} \;. \tag{4.53}$$

The EC field equations given in equations (4.51) and (4.53) are the holonomic basis component forms of equations (4.33) and (4.33) respectively.

It is not difficult to see that, through the application of the $\mathrm{GL}_4(\mathbb{R})$ matrices $e^{a}{}_{\nu}$ on the Kibble-Sciama equations (4.33) and (4.33), one can obtain the holonomic basis component equations (4.51) and (4.53) respectively. The reason we chose to provide an alternative derivation is due to the fact that we want to use this method when considering gravitational theories that are modifications of the EC theory of gravity. Examples of such theories are considered in subsequent sections. For now, however, we wish to study the energy-momentum tensor and the spin-current tensor. Let us first turn our attention to the EC field equation (4.51) in order to study the energy momentum tensor. As already mentioned, if the spin-current tensor vanishes then, since the torsion tensor geometrises spin in EC theory, the non-null torsion Ricci tensor reduces to the torsion-free Ricci tensor and, therefore, equation (4.51) reduces to the Einstein field equations of GR. Since the first Bianchi identity associated with the torsion-free Riemann tensor vanishes, it follows that the torsion-free Ricci tensor is symmetric and thus, in the case of a vanishing spin-current tensor, the energy-momentum tensor is symmetric. In the event that the spin-current tensor is non-vanishing, however, the torsion tensor is non-vanishing by equation (4.53) and thus, the first Bianchi identity associated with the non-null torsion Riemann tensor is non-vanishing which implies that the non-null torsion Ricci tensor is not necessarily symmetric. In such a case, it follows from equation (4.51) that the energy momentum tensor is not necessarily symmetric. The spin-current tensor, on the other hand, is always antisymmetric in its first and second indices. This can be seen from the fact that the first term on the left-hand side of equation (4.53) is the torsion tensor which is antisymmetric in its second and third indices and that the second and third terms together form an antisymmetric tensor.

Turning our attention to the second Bianchi identity associated with the Cartan connection, which is given in equation (3.64), it is evident that, in the case of a non-vanishing spin-current tensor, the energy-momentum tensor is not necessarily conserved. However, it is possible [7] to construct a conserved and symmetric tensor that is written as a combination of the energy-momentum tensor and the spin-current tensor. In order to do this, we note that the second Bianchi identity associated with the Levi-Civita connection, which is given in equation (3.56), implies the following

$$D_\mu G^\mu{}_\nu = 0 \ , \tag{4.54}$$

where $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor [5]. Let us turn our attention to the first EC field equation given in equation (4.51). By moving the non-Riemannian constributions on the left-hand side of equation (4.51) to the right-hand side, the left-hand side then becomes the Einstein tensor. We can then make use of the second EC equation (4.53) in order to write the non-Riemannian contributions in terms of the spin-current tensor. We can then combine these terms with the energy-momentum tensor in order to write the first EC field equation as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = k^2 \mathcal{T}_{\mu\nu} \ , \tag{4.55}$$

where we define the *effective energy-momentum tensor*, which is both symmetric as well as conserved, to be

$$\begin{aligned}\mathcal{T}_{\mu\nu} := \tau_{\mu\nu} - \frac{1}{2}D_\sigma\left(s_{\mu\nu}{}^\sigma + s^\sigma{}_{\mu\nu} + s^\sigma{}_{\nu\mu}\right) - \pi\Big(4s_\mu{}^\sigma{}_\nu s_\sigma - 4s_\mu{}^{\sigma\rho}s_{\sigma\rho\nu} + 2s_{\sigma\rho\nu}s^{\sigma\rho}{}_\mu \\ - g_{\mu\nu}s_{\sigma\rho\lambda}s^{\sigma\rho\lambda} - 2g_{\mu\nu}s_{\sigma\rho\lambda}s^{\sigma\lambda\rho} + 2g_{\mu\nu}s_\rho s^\rho\Big) \ ,\end{aligned} \tag{4.56}$$

and we have defined $s_\mu := s_{\mu\nu}{}^\nu$ [7]. The energy-momentum tensor combined with the three terms on the right-hand side of the above expression which are covariant derivatives of the spin-current tensor yields the so-called *Belinfante-Rosenfeld tensor* [7]. As noted in [7], the Belinfante-Rosenfeld tensor arises naturally in the EC theory as can be seen in the above expression.

We can make use of the above expression for the effective energy-momentum tensor in order to obtain an expression for the antisymmetric part of the energy-momentum tensor. This antisymmetric part is given below

$$2\tau_{[\alpha\beta]} = D_\sigma s_{\alpha\beta}{}^\sigma - 4\pi\left(s_\alpha{}^{\sigma\rho}s_{\sigma\rho\beta} - s_\beta{}^{\sigma\rho}s_{\sigma\rho\alpha}\right) \ . \tag{4.57}$$

This concludes our discussion of the EC theory of gravity. In the following two sections of this chapter we consider modifications of this theory. The first such modification discussed here, referred to as the Einstein-Cartan-Saa theory of gravity [11,12], involves a different choice of a volume element to the one used in this section. This is the first modification that we wish to consider in this thesis and is discussed in the following section.

4.3 Einstein-Cartan-Saa theory

In this section, we discuss Saa's modification to the EC theory of gravity which is referred to as the Einstein-Cartan-Saa (ECS) theory of gravity. This theory was first presented by Saa in [11]. The approach that is made in Saa's modification to the EC theory is to construct a gravitational action whose integrand consists of the non-null torsion Ricci scalar, as is the case in the standard EC theory, however, the choice of a volume element is different. The motivation for choosing a different volume element lies in an issue surrounding the application of the Minimal Coupling Procedure (MCP) in Riemann-Cartan space-time. Let us first outline this issue which is discussed in detail in [11,12]. In this thesis, we make use of the definition for the MCP that is used in [11].

Consider some action, S, in Minkowski space-time whose Lagrangian density, $\mathcal{L}$, depends on n distinct fields, say ϕ_i for $i \in \{1, \ldots, n\}$, and their first order derivatives, $\partial_\mu\phi_i$. Suppose we were to carry out an ordinary variation of this action and obtain the corresponding equations of motion. In order to determine how these fields behave in the presence of a gravitational field as described by GR, one can apply the MCP to either the action or the equations of motion. Applying the MCP to the Lagrangian density in Minkowski space-time $\mathcal{L}(\phi_i, \partial_\mu\phi_i)$ yields the Lagrangian density $\mathcal{L}(\phi_i, D_\mu\phi_i)$. Since the volume element for GR, $\sqrt{-g}\epsilon_{\mu\nu\sigma\rho}$, is compatible with the Levi-Civita connection, one can comfortably perform an integration by parts in the action. As a result of this, if we were to instead apply the MCP to the equations of motion found in Minkowksi space-time, we would obtain the same result as varying the action. This is not what occurs in the standard EC theory of gravity. Suppose we apply the MCP, as described by the EC theory of gravity, to the action in Minkowski space-time. The application of the MCP to the Lagrangian density $\mathcal{L}\left(\phi_i, \partial_\mu \phi_i\right)$ yields the Lagrangian density $\mathcal{L}\left(\phi_i, \nabla_\mu \phi_i\right)$. In carrying out a variation of this action, one has to first convert the Cartan covariant derivatives of the fields ϕ_i to Levi-Civita covariant derivatives which, upon the application of an integration by parts, will yield pure boundary terms. As a result of this, the equations of motion obtained by carrying out a variation of the action is not the same as the equations obtained by applying the MCP to the equations of motion in Minkowski space-time.

In order to resolve this issue, Saa suggested a modified volume element that will allow for comfortable integration by parts using the Cartan covariant derivative [11]. An issue surrounding the method for constructing such a volume element that is given by Saa in [11] has been disputed by Fiziev in [12]. In [11], Saa stated a particular procedure for calculating the Cartan covariant derivative of the volume element. In [12], Fiziev argued that this aforementioned procedure is actually how one computes the *transposed* Cartan covariant derivative of the volume element. Indeed, as pointed out by Fiziev, the natural choice of a volume element, i.e., the volume element used in the standard EC theory and GR, is compatible with the Cartan covariant derivative. However, it is not compatible with the transposed Cartan covariant derivative. The fact that the natural choice of a volume element is not compatible with the transposed Cartan covariant derivative leads to the MCP issue of standard EC theory.

In order to construct a volume element that allows for comfortable application of integration by parts using the Cartan covariant derivative requires that this volume element in fact be compatible with the transposed Cartan covariant derivative. In this section, we shall show why this is the case and then construct the Einstein-Cartan-Saa (ECS) theory of gravity using this volume element. Let us first begin by studying the natural choice of a volume element and how the integral of the divergence, as described by the Cartan connection, of some arbitrary vector field does not yield a pure boundary term. In holonomic basis components, the Levi-Civita tensor is written as

$$\epsilon_{\mu\nu\sigma\rho} = \sqrt{-g}\ \varepsilon_{\mu\nu\sigma\rho}\ , \tag{4.58}$$

where $\varepsilon_{\mu\nu\sigma\rho}$ is the Levi-Civita symbol as before. The volume element used in the standard EC theory of gravity, which is also the natural choice of a volume element, is written as

$$\zeta = \frac{1}{4!}\epsilon_{\mu\nu\sigma\rho} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\sigma \wedge \mathrm{d}x^\rho\ . \tag{4.59}$$

Let $\boldsymbol{B}$ be a vector field on the Riemann-Cartan space-time manifold $\mathcal{M}$ and let us consider the following volume integral of the Cartan divergence of the vector field $\boldsymbol{B}$

$$I := \int \mathrm{d}^4x \sqrt{-g}\ \nabla_\nu B^\nu\ . \tag{4.60}$$

By converting the Cartan covariant derivative in the integrand of the above expression to a Levi-Civita covariant derivative through the use of the contorsion tensor, we obtain the following

$$I = \int \mathrm{d}^4x \sqrt{-g}\left(D_\nu B^\nu - T_\nu B^\nu\right)\ , \tag{4.61}$$

where we have made use of the fact that the trace of the contorsion tensor is nothing more than the trace of the torsion tensor with a minus sign. Of course, the volume element in the integrand is compatible with the Levi-Civita connection. However, for the sake of completeness, let us show how the first term on the right-hand side of the above expression is a pure boundary term. By making use of the fact that equation (3.36) provides us with the expression

$$\Gamma^{\nu}{}_{\nu\beta} = (-g)^{-1/2}\,\partial_\beta \sqrt{-g}\ , \tag{4.62}$$

it follows that we can write equation (4.61) as [96]

$$I = \int \mathrm{d}^4x \left(\sqrt{-g}\partial_\nu B^\nu + B^\beta \partial_\beta \sqrt{-g}\right) - \int \mathrm{d}^4x T_\nu B^\nu = \int \mathrm{d}^4x \; \partial_\nu \left(\sqrt{-g} B^\nu\right) - \int \mathrm{d}^4x T_\nu B^\nu \; . \tag{4.63}$$

The first term on the right-hand side of the second equality is a boundary term. The second term, however, does not necessarily constitute a boundary term. The goal of [11] was to construct a volume element that would allow for the volume integral of a Cartan divergence $\nabla_\nu B^\nu$ to be a pure boundary term. As pointed out by Fiziev in [12], the natural choice of a volume element is not only compatible with the Levi-Civita connection but also compatible with the Cartan connection. In order to show that this is indeed the case, let us consider the action of the Cartan covariant derivative on the natural choice for a volume element and convert it to a Levi-Civita covariant derivative through the use of the contorsion tensor:

$$\nabla_\mu \epsilon_{0123} = D_\mu \epsilon_{0123} - K^\beta{}_{\nu\beta} \epsilon_{0123} \; . \tag{4.64}$$

The first term on the right-hand side of the above expression vanishes since the natural choice of a volume element is compatible with the Levi-Civita connection and the second term vanishes since the contorsion tensor is antisymmetric in its first and third indices. As desired, and as pointed out by Fiziev in [12], the natural choice for a volume element is compatible with the Cartan covariant derivative, i.e., we have

$$\nabla_\mu \epsilon_{\alpha\beta\sigma\rho} = 0 \; . \tag{4.65}$$

Although the volume element is compatible with the Cartan connection, it is evident from equation (4.63) that the volume integral of a Cartan divergence of an arbitrary vector field does not yield a pure boundary term. As we shall now show, by following what is done in [11], by introducing a new volume element that is compatible with the transposed Cartan covariant derivative, the volume integral of the Cartan divergence of an arbitrary vector field then yields a pure boundary term.

Let us begin by defining the transposed Cartan connection. We define the connection ${}^T\nabla$ whose action on some arbitrary dual vector field, say $\boldsymbol{w}$, yields the type $(0,2)$ tensor field

$${}^T\nabla_\mu w_\nu = \partial_\mu w_\nu - \tilde{\Gamma}^\rho{}_{\nu\mu} w_\rho \; , \tag{4.66}$$

as the *transposed Cartan connection*. In the above, the $\tilde{\Gamma}^\rho{}_{\mu\nu}$ are the Cartan connection coefficients. We note that, while the Levi-Civita and Cartan connections are metric compatible, the transposed Cartan connection is not necessarily metric compatible. This is easily seen by acting the transposed Cartan covariant derivative on the metric tensor as follows

$${}^T\nabla_\alpha g_{\mu\nu} = D_\alpha g_{\mu\nu} - K^\rho{}_{\mu\alpha} g_{\rho\nu} - K^\rho{}_{\nu\alpha} g_{\mu\rho} = -2K_{(\mu\nu)\alpha} \; . \tag{4.67}$$

The above expression implies that the transposed Cartan connection is only metric compatible in the case where the symmetric part $K_{(\mu\nu)\alpha}$ vanishes, i.e., when the contorsion tensor is totally antisymmetric. Before we show the desired result, we first require the notion of Stokes' theorem which is stated here without proof and as given in [66].

Theorem 4.3.1. (Stokes' theorem)*. If $\mathcal{N}$ is an n-dimensional manifold with boundary $\partial\mathcal{N}$ and $\boldsymbol{w}$ is an $(n-1)$-from on $\mathcal{N}$ then*

$$\int_{\mathcal{N}} \mathrm{d}\boldsymbol{w} = \int_{\partial\mathcal{N}} \boldsymbol{w} \; , \tag{4.68}$$

where $\mathrm{d}\boldsymbol{w}$ denotes the exterior derivative of $\boldsymbol{w}$.

For a proof of Stokes' theorem, the reader is directed to [66] by Spivak. We now wish to turn our attention to showing that if a volume element is compatible with the transposed Cartan connection, then the volume integral of the Cartan divergence of an arbitrary vector field is a pure boundary term. Let $\mathcal{M}$ be the Riemann-Cartan space-time manifold and let us consider the following surface integral

$$\int_{\partial\mathcal{M}} i_{\boldsymbol{v}}\bar{\boldsymbol{\zeta}} \ , \tag{4.69}$$

where $\bar{\boldsymbol{\zeta}}$ is a 4-form volume element, $\boldsymbol{v}$ is some arbitrary vector field on $\mathcal{M}$ and $i_{\boldsymbol{v}}$ denotes the interior derivative in the direction of $\boldsymbol{v}$. In holonomic basis components, we write the 4-form volume element as

$$\bar{\boldsymbol{\zeta}} = \frac{1}{4!}\bar{\epsilon}_{\mu\nu\sigma\rho}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\wedge\mathrm{d}x^{\sigma}\wedge\mathrm{d}x^{\rho} \ . \tag{4.70}$$

Substituting the above expression into the surface integral given in equation (4.69) yields

$$\int_{\partial\mathcal{M}} i_{\boldsymbol{v}}\bar{\boldsymbol{\zeta}} = \frac{1}{4!}\int_{\partial\mathcal{M}} \bar{\epsilon}_{\alpha\beta\sigma\rho}v^{\alpha}\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\sigma}\wedge\mathrm{d}x^{\rho} \ . \tag{4.71}$$

We can now make use of Stokes' theorem which is given in equation (4.68) in order to obtain the following from the above expression

$$\int_{\partial\mathcal{M}} i_{\boldsymbol{v}}\bar{\boldsymbol{\zeta}} = \frac{1}{4!}\int_{\mathcal{M}} \partial_{\lambda}\left(\bar{\epsilon}_{\alpha\beta\sigma\rho}v^{\alpha}\right)\mathrm{d}x^{\lambda}\wedge\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\sigma}\wedge\mathrm{d}x^{\rho} \ . \tag{4.72}$$

Since the Christoffel symbols of the Levi-Civita connection are symmetric, we can replace the partial derivative in the above expression with the Levi-Civita covariant derivative in order to obtain

$$\begin{aligned}\int_{\partial\mathcal{M}} i_{\boldsymbol{v}}\bar{\boldsymbol{\zeta}} &= \frac{1}{4!}\int_{\mathcal{M}} D_{\lambda}\left(\bar{\epsilon}_{\alpha\beta\sigma\rho}v^{\alpha}\right)\mathrm{d}x^{\lambda}\wedge\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\sigma}\wedge\mathrm{d}x^{\rho} \\ &= \frac{1}{4!}\int_{\mathcal{M}} \left[\left(D_{\lambda}\bar{\epsilon}_{\alpha\beta\sigma\rho}\right)v^{\alpha} + \bar{\epsilon}_{\alpha\beta\sigma\rho}D_{\lambda}v^{\alpha}\right]\mathrm{d}x^{\lambda}\wedge\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\sigma}\wedge\mathrm{d}x^{\rho} \ . \end{aligned}\tag{4.73}$$

We now wish to consider the case where the volume element $\bar{\boldsymbol{\zeta}}$ is compatible with the transposed Covariant derivative, i.e., the 4-form field $\bar{\epsilon}_{\alpha\beta\sigma\rho}$ satisfies the following

$${}^{T}\nabla_{\lambda}\bar{\epsilon}_{\alpha\beta\sigma} = 0 \ . \tag{4.74}$$

Through the use of the transposed contorsion tensor, we can convert the Levi-Civita covariant derivative in the first term on the right-hand side of equation (4.73) to a transposed Cartan covariant derivative. This gives us

$$\begin{aligned} v^{\rho}D_{\lambda}\bar{\epsilon}_{\alpha\beta\sigma\rho} &= v^{\rho}\,{}^{T}\nabla_{\lambda}\bar{\epsilon}_{\alpha\beta\sigma\rho} + v^{\mu}K^{\nu}{}_{\nu\mu}\bar{\epsilon}_{\alpha\beta\sigma\lambda} \\ &= -T_{\nu}v^{\nu}\bar{\epsilon}_{\alpha\beta\sigma\lambda} \ , \end{aligned}\tag{4.75}$$

where, in the second equality, we have made use of the fact that we assume the volume element to compatible with the transposed Cartan connection as well as the fact that the trace of the contorsion tensor is nothing more than the trace of the torsion tensor with a factor of a minus sign. Let us now turn our attention to the second term on the right-hand side of equation (4.73). By converting the Levi-Civita covariant derivative to a Cartan covariant derivative through the use of the contorsion tensor, we obtain the following

$$\bar{\epsilon}_{\alpha\beta\sigma\rho}D_{\lambda}v^{\rho} = \bar{\epsilon}_{\alpha\beta\sigma\rho}\nabla_{\lambda}v^{\rho} + \bar{\epsilon}_{\alpha\beta\sigma\lambda}T_{\nu}v^{\nu} \ . \tag{4.76}$$

By substituting equations (4.75) and (4.76) into equation (4.73), we can now obtain

$$\int_{\partial\mathcal{M}} i_{\boldsymbol{v}}\bar{\boldsymbol{\zeta}} = \int_{\mathcal{M}} \bar{\zeta}\ \nabla_{\nu}v^{\nu} \ . \tag{4.77}$$

We have thus shown that the volume integral of the Cartan divergence of an arbitrary vector field yields a pure boundary term if the volume element is compatible with the transposed Cartan connection. This is the premise for Saa's modification of the EC theory.

In order to construct the modified action, we require an explicit expression for the volume element. In order to find such a volume element, let us first consider the action of the transposed Cartan covariant derivative on the natural choice for a volume element:

$$ {}^{T}\nabla_{\mu}\epsilon_{01234} = T_{\mu}\epsilon_{0123} \ , \tag{4.78}$$

where we have made use of the fact that the natural choice of a volume element is compatible with the Levi-Civita connection. By introducing a potential function, say Φ, and writing the trace of the torsion tensor as the gradient of this function, i.e., by writing

$$ T_{\mu} = -\partial_{\mu}\Phi \ , \tag{4.79}$$

and substituting this into equation (4.78) one obtains

$$ -{}^{T}\nabla_{\nu}\ln\epsilon_{0123} = \nabla_{\nu}\Phi \ . \tag{4.80}$$

By integrating the above expression, one arrives at the four-form field volume element given in equation (4.70) where we define the following quantity that is obtained by integrating equation (4.80)

$$ \bar{\epsilon}_{\mu\nu\sigma\rho} := \mathrm{e}^{\Phi}\sqrt{-g}\ \varepsilon_{\mu\nu\sigma\rho} \ . \tag{4.81}$$

While the above volume element is compatible with the transposed Cartan connection, it is worth mentioning that it is not compatible with the Levi-Civita connection. We can now construct the following gravitational action by making use of the volume element given above

$$ S_{\mathrm{ECS}} := \frac{1}{2k^{2}}\int \tilde{R}\ \mathrm{e}^{\Phi}\sqrt{-g}\ \mathrm{d}^{4}x \ . \tag{4.82}$$

We now define the *Einstein-Cartan-Saa* (ECS) *theory of gravity* to be the Riemann-Cartan spacetime U_4 together with the gravitational action given above. In the standard EC theory of gravity, the volume element used in the construction of the gravitational action depends only on the metric. In the ECS theory, the associated gravitational action, which is given in equation (4.82), is constructed using a volume element which is dependent on both the metric as well as the potential function Φ for which the trace of the torsion tensor is written as a gradient.

In order to determine which quantities will yield the field equations for the ECS theory of gravity, we turn our attention to equation (3.80) which relates the non-null torsion Ricci scalar to the torsion-free Ricci scalar together with non-Riemannian contributions. We note that the non-Riemannian contributions in equation (3.80) contain a total covariant derivative, associated with the Levi-Civita connection, of the trace of the torsion tensor. Since the ECS gravitational action is constructed using a volume element that is not compatible with the Levi-Civita connection, this total covariant derivative does not contribute as a boundary term and therefore cannot be removed in the gravitational action. It is possible, however, through the use of the contorsion tensor, to introduce a total Cartan covariant derivative of the trace of the torsion tensor into the right-hand side of equation (3.80) which will contribute as a boundary term. By converting the Levi-Civita covariant derivative to a Cartan covariant derivative, we now write the equation that relates the non-null torsion Ricci scalar to the torsion-free Ricci scalar as follows

$$ \tilde{R} = R + 2\nabla_{\mu}T^{\mu} + \frac{4}{3}T_{\alpha}T^{\alpha} + \frac{1}{24}V_{\alpha}V^{\alpha} + \frac{1}{2}Q_{\nu\mu\alpha}Q^{\nu\mu\alpha} \ . \tag{4.83}$$

The covariant derivative given above is associated with the Cartan connection and will therefore contribute as a boundary term since the volume element is compatible with the transposed Cartan connection. We may therefore remove this covariant derivative term in the ECS gravitational action. Let us now turn our attention to the last three terms on the right-hand side of equation (4.83). In order to study the scalar formed by these three terms, we first wish to make a note on the contorsion tensor.

By decomposing the contorsion tensor into a trace component and a traceless component, one can write the contorsion tensor as

$$K_{\alpha\mu\nu} = \frac{1}{3}\left(T_\alpha g_{\mu\nu} - T_\nu g_{\mu\alpha}\right) + P_{\alpha\mu\nu} \ , \tag{4.84}$$

where $P_{\alpha\mu\nu}$ is the traceless part of the contorsion tensor defined through the difference. In constructing the above expression, we have made use of the fact that the trace of the contorsion tensor is nothing more than the trace of the torsion tensor with a factor of a minus sign. Now, by making use of this decomposition together with the decomposition of the torsion tensor, one can obtain the following expression for the scalar produced by contracting the traceless tensor $P_{\alpha\mu\nu}$ with itself

$$P_{\alpha\mu\nu}P^{\alpha\nu\mu} = \frac{1}{24}V_\alpha V^\alpha + \frac{1}{2}Q_{\nu\mu\alpha}Q^{\nu\mu\alpha} \ . \tag{4.85}$$

Therefore, by substituting equation (4.85) into equation (4.83) and substituting the resulting expression as well as equation (4.79) into the ECS action given in equation (4.82), one obtains the following

$$S_{\mathrm{ECS}} = \frac{1}{2k^2}\int \mathrm{d}^4x \ \mathrm{e}^\Phi \sqrt{-g}\left(R + \frac{4}{3}\nabla_\beta\Phi\nabla^\beta\Phi + P_{\alpha\mu\nu}P^{\alpha\nu\mu}\right) \ , \tag{4.86}$$

where we have removed the total Cartan covariant derivative of the trace of the torsion tensor [11]. We now wish to study the field equations associated with the ECS theory of gravity for which the case of a vacuum is considered. That is, the total gravitational action is simply the ECS action given above. We begin by reading off the ECS Lagrangian density from equation (4.86) and write

$$\mathcal{L}_{\mathrm{ECS}} := \mathrm{e}^\Phi\sqrt{-g}\left(R + \frac{4}{3}\nabla_\beta\Phi\nabla^\beta\Phi + P_{\alpha\mu\nu}P^{\alpha\nu\mu}\right) \ , \tag{4.87}$$

where we have omitted the constant k^2 for simplicity. In order to obtain the field equations in the case of a vacuum, we require that the ECS gravitational action experiences an extremum. Let us first consider the case where we vary the action with respect to the traceless tensor $P_{\alpha\mu\nu}$. It is not difficult to see that, by carrying out such a variation, one obtains

$$P_{\alpha\mu\nu} = 0 \ . \tag{4.88}$$

That is, for the case of a vacuum, the ECS theory predicts that the traceless term in the decomposition of the contorsion tensor vanishes. It now remains to vary the ECS action with respect to the metric tensor as well as the potential function Φ. Let us begin by varying the action with respect to the metric tensor. Varying equation (4.86) with respect to the metric tensor yields

$$\delta_g S_{\mathrm{ECS}} = \frac{1}{2k^2}\int \mathrm{d}^4x\sqrt{-g}\left\{\delta_g\left(\mathrm{e}^\Phi R\right) + \mathrm{e}^\Phi\delta g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - \frac{\mathrm{e}^\Phi}{2}g_{\mu\nu}\delta g^{\mu\nu}\left(R + \frac{4}{3}\nabla_\beta\Phi\nabla^\beta\Phi\right)\right\} \ , \tag{4.89}$$

where we use δ_g to denote variation with respect to the metric tensor as before. Let us now turn our attention to the first term in the above expression. By writing the torsion-free Ricci scalar as the contraction of the metric tensor with the torsion-free Ricci tensor, one obtains the following

$$\delta_g\left(e^\Phi R\right) = \mathrm{e}^\Phi R_{\mu\nu}\delta g^{\mu\nu} + \mathrm{e}^\Phi g^{\mu\nu}\delta R_{\mu\nu} \ . \tag{4.90}$$

We have already obtained the variation of the non-null torsion and torsion-free Ricci scalars in equation (4.46). We note that, while the variation of the torsion-free Ricci tensor is a total covariant derivative with respect to the Levi-Civita connection, the second term in the above expression is only a total covariant derivative when the potential function Φ is a constant, i.e., when there is vanishing torsion. This, of course, would then yield the Einstein field equations in a vacuum as one would expect. In order to understand the contribution of this term in the case where the potential function Φ is not a constant, we first note that the variation of the Christoffel symbols is given by the expression [13]

$$\delta\Gamma^{\beta}{}_{\rho\lambda} = \frac{1}{2} g^{\beta\sigma} \left(D_\rho \delta g_{\lambda\sigma} + D_\lambda \delta g_{\rho\sigma} - D_\sigma \delta g_{\rho\lambda} \right) \ , \tag{4.91}$$

By substituting the above expression into equation (4.46) and then the resulting expression into equation (4.90), one obtains the following

$$\delta_g \left(\mathrm{e}^{\Phi} R \right) = \mathrm{e}^{\Phi} R_{\mu\nu} \delta g^{\mu\nu} + \mathrm{e}^{\Phi} D^\mu D^\nu \delta g_{\mu\nu} - \mathrm{e}^{\Phi} g^{\mu\nu} \Box \delta g_{\mu\nu} \ , \tag{4.92}$$

where we define the *torsion-free D'Alembertian* to be $\Box := D_\beta D^\beta$. By substituting the above expression into equation (4.89) and carrying out an integration by parts twice, one obtains the following expression for the variation of the ECS action

$$\begin{aligned} \delta_g S_{\mathrm{ECS}} = \frac{1}{2k^2} \int \mathrm{d}^4 x \sqrt{-g} \Big\{ & \mathrm{e}^{\Phi} R_{\mu\nu} \delta g^{\mu\nu} - \mathrm{e}^{\Phi} \delta g^{\mu\nu} D_\mu \Phi D_\nu \Phi - \mathrm{e}^{\Phi} \delta g^{\mu\nu} D_\mu D_\nu \Phi + \mathrm{e}^{\Phi} g_{\mu\nu} \delta^{\mu\nu} D_\beta \Phi D^\beta \Phi \\ & + \mathrm{e}^{\Phi} g_{\mu\nu} \delta g^{\mu\nu} \Box \Phi + \frac{4}{3} \mathrm{e}^{\Phi} \delta g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - \frac{\mathrm{e}^{\Phi}}{2} g_{\mu\nu} \delta g^{\mu\nu} \left(R + \frac{4}{3} \nabla_\beta \Phi \nabla^\beta \Phi \right) \Big\} \ . \end{aligned} \tag{4.93}$$

It therefore follows that the varying of the ECS gravitational action with respect to the metric tensor leads to the following field equations

$$0 = \frac{\mathrm{e}^{-\Phi}}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\mathrm{ECS}}}{\delta g^{\mu\nu}} = R_{\mu\nu} + \frac{1}{3} \nabla_\mu \Phi \nabla_\nu \Phi - D_\mu D_\nu \Phi + g_{\mu\nu} \left(\Box \Phi + \frac{1}{3} \nabla_\beta \Phi \nabla^\beta \Phi - \frac{1}{2} R \right) \ , \tag{4.94}$$

whose trace is given by the expression

$$0 = -R + \frac{5}{3} \nabla_\beta \Phi \nabla^\beta \Phi + 3 \Box \Phi \ . \tag{4.95}$$

It is not difficult to see that, in the event that the potential function Φ is constant, i.e., the torsion tensor vanishes, equation (4.94) reduces the well-known Einstein field equations as one would expect. Let us now turn our attention to carrying out a variation of the ECS action with respect to the potential function Φ. By varying the ECS Lagrangian density given in equation (4.87) with respect to the potential function, one obtains the following field equations

$$0 = \frac{\mathrm{e}^{-\Phi}}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\mathrm{ECS}}}{\delta \Phi} = R + \frac{4}{3} \nabla_\beta \Phi \nabla^\beta \Phi - \frac{8}{3} \tilde{\Box} \Phi \ , \tag{4.96}$$

where we define the *non-null torsion D'Alembertian* to be $\tilde{\Box} := \nabla_\beta \nabla^\beta$. Now, by adding equation (4.96) to equation (4.95), one obtains the following equation for the potential function

$$\nabla_\nu \Phi \nabla^\nu \Phi + \Box \Phi - \frac{8}{9} \tilde{\Box} \Phi = 0 \ . \tag{4.97}$$

By noting that we can relate the action of the torsion-free D'Alembertian operator on the potential function Φ to that of the non-null torsion D'Alembertian operator through the expression

$$\Box \Phi = \tilde{\Box} \Phi - \nabla_\nu \Phi \nabla^\nu \Phi \ , \tag{4.98}$$

it follows that we can write equation (4.97) as

$$\tilde{\Box} \Phi = 0 \ . \tag{4.99}$$

The above equation implies the propagation of torsion in the ECS theory of gravity [11].

This concludes our discussion of the ECS theory of gravity. We note, however, that this is not the only possible modification of the EC theory that allows for consistent results via the application of the minimal coupling procedure (MCP). For example, Kazmierczak proposed a covariant derivative in [97] that subtracts off the trace of the contorsion tensor from the Cartan connection coefficients. In addition, a covariant derivative was presented in [96] which subtracts off the trace part in the decomposition of the contorsion tensor from the Cartan connection coefficients. These modifications are not discussed in this thesis and the interested reader is directed to the aforesaid references for discussions on these modified versions of the EC theory.

4.4 Chern-Simons modification of the Einstein-Cartan theory

In this section, we consider the Chern-Simons modification of the Einstein-Cartan theory of gravity (CSEC) through the introduction of the Chern-Simons action in addition to the standard Einstein-Cartan action. The Chern-Simons modification of General Relativity (CSGR) is studied in [13] by Jackiw and Pi as well as in the review article [14] by Alexander and Yunes. It turns out that the methods applied in constructing CSGR is easily extended to the case of the EC theory of gravity in order to construct CSEC. In what follows, the approaches found in [13, 14] are made use of in order to consider the case where the torsion tensor is non-vanishing. For a detailed discussion on CSGR, the reader is directed to [13, 14]. It is worth mentioning that the CSEC theory has been studied in [15].

Let us begin our study of CSEC gravity by constructing the total action which is given as follows

$$S := S_{\mathrm{EC}} + S_{\mathrm{CS}} + S_{\theta} + S_{\mathrm{m}} \ , \tag{4.100}$$

where S_{EC} is the EC gravitational action given in equation (4.41);

$$S_{\mathrm{CS}} := \frac{\alpha}{4}\int \sqrt{-g}\ \theta\ {}^*\tilde{R}\tilde{R}\ \mathrm{d}^4x \ , \tag{4.101}$$

is the Chern-Simons action;

$$S_{\theta} := -\frac{\beta}{2}\int \sqrt{-g}\ \left(\nabla_{\nu}\theta\nabla^{\nu}\theta + 2V(\theta)\right)\mathrm{d}^4x \ , \tag{4.102}$$

is the coupling field action and S_{m} is the matter action. Let us first discuss the integrand in the Chern-Simons action given in equation (4.101). The θ factor, which is also used in the construction of the scalar field action, is referred to as the Chern-Simons *coupling field*. Furthermore, we define the *Pontryagin density* [13, 14] to be

$${}^*\tilde{R}\tilde{R} := {}^*\tilde{R}^{\rho\sigma}{}_{\beta}{}^{\alpha}\tilde{R}_{\rho\sigma\alpha}{}^{\beta} \ . \tag{4.103}$$

In the above definition, the Hodge dual of the non-null torsion Riemann tensor is made use of. In order to give an explicit expression for the Hodge dual of the non-null torsion Riemann tensor, we first write

$$\tilde{\boldsymbol{R}} = \tilde{R}_{\mu\nu\rho}{}^{\sigma}\mathrm{d}x^{\mu}\otimes\mathrm{d}x^{\nu}\otimes\mathrm{d}x^{\rho}\otimes\partial_{\sigma} = \frac{1}{2}\tilde{R}_{\mu\nu\rho}{}^{\sigma}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\otimes\mathrm{d}x^{\rho}\otimes\partial_{\sigma} \ , \tag{4.104}$$

where we have made use of the holonomic construction. Through the application of the Hodge dual operator to the above expression, one can obtain the following

$${}^*\tilde{\boldsymbol{R}} = \frac{1}{4}\tilde{R}_{\mu\nu\rho}{}^{\sigma}\epsilon_{\beta\gamma}{}^{\mu\nu}\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\gamma}\otimes\mathrm{d}x^{\rho}\otimes\mathrm{d}x\otimes\partial_{\sigma} \ , \tag{4.105}$$

where $\epsilon_{\mu\nu\sigma\rho}$ is the Levi-Civita tensor as before. Since ${}^*\tilde{\boldsymbol{R}}$ is itself a tensor, we can write it in the form

$${}^*\tilde{\boldsymbol{R}} = \frac{1}{2}{}^*\tilde{R}_{\mu\nu\rho}{}^{\sigma}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\otimes\mathrm{d}x^{\rho}\otimes\partial_{\sigma} \ , \tag{4.106}$$

where ${}^*\tilde{R}_{\mu\nu\rho}{}^{\sigma}$ are the components of this tensor with respect to the holonomic frame. Now, on comparison of the above expression with equation (4.105), we see that one can relate the holonomic tensor components of the Hodge dual of the non-null torsion Riemann tensor to the holonomic tensor components of the non-null torsion Riemann tensor through the expression

$${}^*\tilde{R}_{\beta\gamma\rho}{}^{\sigma} = \frac{1}{2}\epsilon_{\beta\gamma}{}^{\mu\nu}\tilde{R}_{\mu\nu\rho}{}^{\sigma} \ . \tag{4.107}$$

Having now defined the above tensor components, let us now turn our attention to the Pontryagin density given in equation (4.103). As is done in [14], it is possible to write the Pontryagin density as the four-divergence (with respect to the Levi-Civita connection) of a type (1,0) tensor field $\tilde{K}^{\nu}$. This tensor field is referred to as the *topological current* and it is defined through the expression

$$\tilde{K}^{\nu} := \epsilon^{\nu\beta\sigma\rho}\tilde{\Gamma}^{\delta}{}_{\beta\alpha}\left(\partial_{\sigma}\tilde{\Gamma}^{\alpha}{}_{\rho\delta} + \frac{2}{3}\tilde{\Gamma}^{\alpha}{}_{\sigma\lambda}\tilde{\Gamma}^{\lambda}{}_{\rho\delta}\right) . \tag{4.108}$$

It is not difficult to show that the topological current given above satisfies the relation

$$D_{\nu}\tilde{K}^{\nu} = \frac{1}{2}{}^{*}\tilde{R}\tilde{R} . \tag{4.109}$$

Having now defined the topological current, whose divergence yields the Pontryagin density, we can now write the integrand of the Chern-Simons action given in equation (4.101) as a product of the Chern-Simons coupling field and the divergence of the topological current. That is, we can write the Chern-Simons action as

$$S_{\mathrm{CS}} = \frac{\alpha}{4}\int \mathrm{d}^{4}x\sqrt{-g}\;\theta D_{\nu}\tilde{K}^{\nu} . \tag{4.110}$$

It is not difficult to see, from the above expression, that if the Chern-Simons coupling field is a constant, then it follows that the Chern-Simons action vanishes as a result of compact support since the integrand is the divergence of the topological current. In order to express the Chern-Simons action in terms of the Cartan connection coefficients, we can first perform an integration by parts on the above expression in order to obtain

$$S_{\mathrm{CS}} = \frac{\alpha}{2}\left(\int \mathrm{d}^{4}x D_{\nu}\left(\sqrt{-g}\;\theta\tilde{K}^{\nu}\right) - \int \mathrm{d}^{4}x\sqrt{-g}\tilde{K}^{\nu}D_{\nu}\theta\right) . \tag{4.111}$$

The first integral contains the divergence of the topological current and therefore vanishes due to compact support [98, 99]. It follows that, by substituting the definition of the topological current given in equation (4.108) into the above expression, one obtains

$$S_{\mathrm{CS}} = -\frac{\alpha}{2}\int \mathrm{d}^{4}x\; D_{\nu}\theta\;\varepsilon^{\nu\mu\sigma\rho}\tilde{\Gamma}^{\lambda}{}_{\mu\beta}\left(\partial_{\sigma}\tilde{\Gamma}^{\beta}{}_{\rho\lambda} + \frac{2}{3}\tilde{\Gamma}^{\beta}{}_{\sigma\alpha}\tilde{\Gamma}^{\alpha}{}_{\rho\lambda}\right) , \tag{4.112}$$

where $\varepsilon^{\nu\mu\sigma\rho}$ is the Levi-Civita symbol as before. In studying the standard EC theory in Section 4.2, we have already obtained the total variation of the EC action. We now wish to obtain the total variation of the Chern-Simons action by making use of the form given in equation (4.112). In order to do this, we first wish to carry out the variation with respect to the metric and contorsion tensors. That is, we first compute

$$\delta_{g}S_{\mathrm{CS}} + \delta_{K}S_{\mathrm{CS}} = -\frac{\alpha}{2}\int \mathrm{d}^{4}x\; D_{\nu}\theta\;\varepsilon^{\nu\mu\sigma\rho}\tilde{R}_{\sigma\mu\beta}{}^{\lambda}\;\delta\tilde{\Gamma}^{\beta}{}_{\rho\lambda} . \tag{4.113}$$

We now wish to break up the right-hand side into terms that contribute as a result of a variation with respect to the torsion tensor and terms that contribute as a result of a variation with respect to the metric tensor. In order to do this, we write the Cartan connection coefficients as a sum of a Riemannian term, which is the Christoffel symbol, as well as the non-Riemannian term which is the contorsion tensor. That is, through the substitution of equation (3.38) into the above expression, one obtains

$$\delta_{g}S_{\mathrm{CS}} + \delta_{K}S_{\mathrm{CS}} = -\frac{\alpha}{2}\int \mathrm{d}^{4}x\; D_{\nu}\theta\;\varepsilon^{\nu\mu\sigma\rho}\tilde{R}_{\sigma\mu\beta}{}^{\lambda}\;\left(\delta\Gamma^{\beta}{}_{\rho\lambda} + \delta K^{\beta}{}_{\rho\lambda}\right) . \tag{4.114}$$

We now wish to analyse the variation of the Chern-Simons action with respect to the metric tensor. That is, we wish to consider the following integral whose total variation is equivalent to a variation of the metric tensor

$$\delta I := \int \mathrm{d}^{4}x\; D_{\nu}\theta\;\varepsilon^{\nu\mu\sigma\rho}\tilde{R}_{\sigma\mu\beta}{}^{\lambda}\;\delta\Gamma^{\beta}{}_{\rho\lambda} . \tag{4.115}$$

By making use of the variation of the Christoffel symbols given in equation (4.91), equation (4.115) becomes

$$\delta I = \frac{1}{2} \int \mathrm{d}^4 x D_\nu \ \theta \varepsilon^{\nu\mu\sigma\rho} \tilde{R}_{\sigma\mu}^{\ \ \gamma\lambda} \left(D_\lambda \delta g_{\rho\gamma} - D_\gamma \delta g_{\rho\lambda}\right) \ , \tag{4.116}$$

where we have used the fact that the non-null torsion Riemann tensor is antisymmetric in its third and fourth indices whereas the metric tensor is symmetric in its indices. Now, by carrying out an integration by parts in the above expression, and by once again making use of the fact that the non-null torsion Riemann tensor is antisymmetric in its third and fourth indices, one obtains the following

$$\delta I = \int \mathrm{d}^4 x \ \varepsilon^{\nu\mu\sigma\rho} \left(\tilde{R}_{\sigma\mu}{}^{\gamma\lambda} D_\gamma D_\nu \theta + D_\nu \theta \ D_\gamma \tilde{R}_{\sigma\mu}{}^{\gamma\lambda} \right) \delta g_{\rho\lambda} \ . \tag{4.117}$$

We now note that, through the introduction of the type $(2,0)$ tensor field defined as

$$\tilde{C}^{\rho\lambda} := D_\gamma D_\nu \theta \ {}^*\tilde{R}^{\nu(\rho\lambda)\gamma} + \frac{1}{2} \epsilon^{\nu\mu\sigma(\rho|} D_\nu \theta D_\gamma \tilde{R}_{\sigma\mu}{}^{|\gamma|\lambda)} \ , \tag{4.118}$$

we can write equation (4.117) as

$$\delta I = 2 \int \mathrm{d}^4 x \sqrt{-g} \ \tilde{C}^{\rho\lambda} \delta g_{\rho\lambda} \ . \tag{4.119}$$

In this thesis, following the review article [14], we refer to the tensor field defined in equation (4.118) as the *C-tensor*[2]. Now, by making use of equation (4.119), the variation of the Chern-Simons action with respect to the metric tensor and the contorsion tensor reads

$$\delta_g S_{\mathrm{CS}} + \delta_K S_{\mathrm{CS}} = \alpha \int \mathrm{d}^4 x \sqrt{-g} \ \tilde{C}_{\rho\lambda} \delta g^{\rho\lambda} \ - \frac{\alpha}{2} \int \mathrm{d}^4 x \ D_\nu \theta \ \varepsilon^{\nu\mu\sigma\rho} \tilde{R}_{\sigma\mu\beta}{}^{\lambda} \ \delta K^{\beta}{}_{\rho\lambda} \ . \tag{4.120}$$

We note that in order to obtain the total variation of the Chern-Simons action, it is necessary to determine the variation of the Chern-Simons action with respect to the Chern-Simons coupling field. By carrying out such a variation in equation (4.101), it follows that the total variation of the Chern-Simons action is as follows

$$\delta S_{\mathrm{CS}} = \int \mathrm{d}^4 x \left(\alpha \sqrt{-g} \ \tilde{C}_{\rho\lambda} \delta g^{\rho\lambda} \ - \frac{\alpha}{2} D_\nu \theta \ \varepsilon^{\nu\mu\sigma\rho} \tilde{R}_{\sigma\mu\beta}{}^{\lambda} \ \delta K^{\beta}{}_{\rho\lambda} + \frac{\alpha}{4} \sqrt{-g} \ {}^*\tilde{R}\tilde{R} \delta\theta \right) \ . \tag{4.121}$$

In order to obtain the total variation of the total CSEC action given in equation (4.101), we note that the total variation of the coupling field action is

$$\delta S_\theta = -\frac{\beta}{2} \int \mathrm{d}^4 x \sqrt{-g} \left\{ 2 \left(\frac{\mathrm{d}V}{\mathrm{d}\theta} - \Box\theta \right) \delta\theta + \left[\nabla_\mu \theta \nabla_\nu \theta - \frac{1}{2} g_{\mu\nu} \left(\nabla_\beta \theta \nabla^\beta \theta + 2V(\theta) \right) \right] \delta g^{\mu\nu} \right\} \ , \tag{4.122}$$

where we have introduced the *torsion-free d'Alembertian* operator as $\Box := D_\nu D^\nu$. Having now obtained the total variation of the CSEC action, it is now possible to compute the relevant field equations. Through the variation of the CSEC action, it follows from equations (4.49), (4.50), (4.121) and (4.122) that the corresponding field equation is

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R} \ g_{\mu\nu} + 2\alpha k^2 \tilde{C}_{\mu\nu} = k^2 \tau_{\mu\nu} \ . \tag{4.123}$$

As can be seen from its definition, the C-tensor vanishes when the Chern-Simons coupling field is constant which implies that the field equation given above yields the first EC field equation given in

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[2] In three-dimensions, the tensor field defined in equation (4.118) is referred to as the *Cotton-York tensor* [14,100,101]. It is worth mentioning that the Cotton-York tensor gives an alternative representation of the so-called Cotton 2-form through the application of the Hodge star operator [100]. Such a representation is only possible in three-dimensions and the C-tensor does not provide a similar representation in four-dimensions. We therefore follow the terminology used in [14] to distinguish the four dimensional C-tensor from the Cotton-York tensor.

equation (4.51). Let us now turn our attention to the variation of the CSEC action with respect to the contorsion tensor. By making use of equations (4.49), (4.52) and (4.121), one has the following field equation [15]

$$T^{\nu}{}_{\lambda\mu} + \delta^{\nu}_{\lambda} T_{\mu} - \delta^{\nu}_{\mu} T_{\lambda} = k^2 s_{\lambda\mu}{}^{\nu} + 2\alpha k^2 D_{\beta}\theta \; \varepsilon^{\beta\alpha\sigma\nu} \tilde{R}_{\sigma\alpha\mu\lambda} \; . \tag{4.124}$$

We again note that when the coupling field vanishes, the second term on the right-hand side of the above expression vanishes and the result is the second EC equation (4.53).

Before we state the field equations associated with a variation of the coupling field, we first wish to make a note on the implication of the inclusion of the second term on the right-hand side of the above field equation. Suppose that the coupling field is non-vanishing while the spin-current tensor is vanishing. In such a case, it is possible to have non-vanishing torsion in addition to having non-vanishing curvature. Therefore, for the case where the coupling constant is non-vanishing, the vanishing of the spin-current tensor does not necessarily imply the vanishing of the torsion tensor. We now turn our attention to the variation of the CSEC action with respect to the coupling field. From equations (4.121) and (4.122) one has the following field equation

$$\frac{\alpha}{4}\sqrt{-g}\; {}^{*}\tilde{R}\tilde{R} - \beta\left(\frac{\mathrm{d}V}{\mathrm{d}\theta} - \Box\theta\right) = 0 \; . \tag{4.125}$$

The case of a dynamical Chern-Simons modification of the EC theory is where the coupling field action has no contribution, i.e., $\beta = 0$ [14]. In such a case, the above field equation implies that the Pontryagin density must vanish. This is referred to as the *Pontryagin constraint* [14]. Therefore, in the case of a dynamical Chern-Simons modification, a solution of such a theory is such that the Pontryagin density vanishes. In the case of CSGR, i.e., the torsion tensor vanishes, this implies that a solution of GR may not be a solution of CSGR since a metric tensor, while solving the Einstein field equations, may not satisfy the Pontryagin constraint. For example, while there are solutions such as the Schwarzschild solution that satisfy the Pontryagin constraint, the Kerr solution does not satisfy the Pontryagin constraint despite being a solution to the Einstein field equations [14]. One can ask the question of whether the same would be the case for CSEC. That is, does the Kerr solution of the EC theory satisfy the Pontryagin constraint which involves the non-null torsion Riemann tensor? As already mentioned, in considering the case of a vacuum, the EC theory is equivalent to GR and thus, the Kerr solution satisfies the field equations of EC theory. In the case of a vacuum for which the Chern-Simons coupling field is non-vanishing, the torsion tensor is related to the non-null torsion Riemann tensor as per equation (4.124). Therefore, it is possible for the torsion tensor to be non-vanishing and thus for there to be a non-Riemannian contribution in the construction of the non-null torsion Riemann tensor. Clearly, the question of whether the Kerr solution to the EC theory will satisfy the Pontryagin constraint of CSEC is not equivalent to asking the same question for that of CSGR. The authors of [15] present a perturbative approach to solving equation (4.124) and argue that CSEC can accommodate a slow rotating Kerr solution. The interested reader is directed to [15] for more information regarding this question.

This concludes our discussion of gravitational theories for which non-vanishing torsion is considered in addition to non-vanishing curvature. We have considered the standard EC theory of gravity as well as two of its modifications. The first modification considered, which was referred to as the ECS theory of gravity, involved a different choice of a volume element to that used in the original formulation of the EC theory. More specifically, the choice of a volume element was motivated by desiring that the volume element be compatible with the transposed Cartan connection. This would allow for certain physical equations, such as the Dirac equation[3], derived through varying a corresponding action to yield the same result as that which is obtained through applying the MCP to the associated equations valid in Minkowski space-time [11]. The second modification considered, the CSEC theory, involved the consideration of a Chern-Simons action term in addition to the standard EC action term given in equation (4.41).

In the following section, we turn our attention to gravitational theories that are constructed on the Weitzenböck space-time.

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[3] In Appendix A we discuss the construction of the Klein-Gordon and Dirac equations on U_4.

4.5 $f(T)$ theories of gravity

4.5.1 Formulation

In this section we turn our attention to studying the specific teleparallel theories of gravity referred to as $f(T)$ theories of gravity. Such theories are constructed on the Weitzenböck space-time W_4 defined in Chapter 3. As we have done for gravitational theories considered in previous sections, the definition that we wish to use for the so-called $f(T)$ theories of gravity includes a space-time, in this case the Weitzenböck space-time W_4, and a gravitational action. Before we introduce the gravitational action, let us first consider the Einstein-Hilbert (EH) action which is defined as [5,95]

$$S_{\mathrm{EH}} := \int \mathrm{d}^4x\sqrt{-g}\ R\ , \tag{4.126}$$

and we remind the reader that R denotes the torsion-free Ricci scalar. By considering a Weitzenböck space-time, the non-null torsion Ricci scalar vanishes and thus, by substituting equation (3.69) into the EH action, one obtains the following

$$S_{\mathrm{EH}} = \int \mathrm{d}^4x\sqrt{-g}\left(-T - 2D_\nu T^\nu\right)\ . \tag{4.127}$$

The second term in the integrand is a total Levi-Civita covariant derivative and therefore has no contribution in the EH action [33]. Therefore, the EH action used in the definition of GR is equivalent to the gravitational action

$$S_T := \int \mathrm{d}^4x\ e\ T\ , \tag{4.128}$$

on a Weitzenböck space-time and we remind the reader that $e = \sqrt{-g}$ is the determinant of the vierbein [8,33,34,81]. The gravitational action given in equation (4.128) is the action of the Teleparallel Equivalent of General Relativity (TEGR). That is, TEGR is defined to be the Weitzenböck space-time W_4 together with the gravitational action given in equation (4.128). This Lagrangian formulation of TEGR was first introduced in [35] and then later on in [36]. Although this formulation was first discussed in the aforesaid references, the first introduction of TEGR dates back to Einstein and the interested reader is directed to the work [34] for the first English translations of Einstein's original papers on the subject. In considering a Riemannian space-time one can consider, instead of simply the torsion-free Ricci scalar in the gravitational action, a more general function $f(R)$ of the torsion-free Ricci scalar. Such theories of referred to as the $f(R)$ theories of gravity [102]. Similarly, one can consider a Weitzenböck space-time and consider a general function $f(T)$ of the torsion scalar in the gravitational action [33, 40, 103]. Such theories are referred to as the $f(T)$ theories of gravity and are defined to be the Weitzenböck space-time W_4 together with the following gravitational action [33, 40, 103]

$$S_{f(T)} := \int \mathrm{d}^4x\ e\ f(T)\ . \tag{4.129}$$

While TEGR is equivalent to GR, a given $f(R)$ theory of gravity may not in general be equivalent to the $f(T)$ theory of gravity [33]. In this thesis, the $f(T)$ theories are of great interest, however, before discussing these generalised theories, we first wish to make some notes on TEGR. Firstly, we wish to note that it is possible to describe gravitation through the consideration of a gauge theory [8]. In such an interpretation, curvature, which is described by the non-null torsion Riemann tensor, is related to rotations while torsion, which is described by the torsion tensor, is related to translations. In the case of TEGR, for which there is vanishing curvature and non-vanishing torsion, one can interpret the theory as a gauge theory[4] for the group containing only translations [8]. By examining the paths followed by test particles in the presence of gravity as described by TEGR, one can draw

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[4] There have been recent disputes regarding the status of TEGR as a gauge theory of the translation group. Here, we will not discuss this and instead direct the interested reader to [50–52].

an analogy between this theory and Maxwell's theory of electromagnetism. As discussed in Chapter 3, test particles follow auto-parallels of the Levi-Civita connection. That is, the paths followed by test particles are described by the differential equation (3.135). In the study of GR, equation (3.135) is referred to as the *geodesic equation* [5]. In TEGR, torsion, from which gravity results, acts as a force and equation (3.135) is referred to as the *force equation* [8]. It is the force equation that is analogous to the Lorentz force equation in Maxwell's theory of electromagnetism [8]. While Maxwell's theory can be interpreted as a gauge theory of U(1), which is an abelian group, one can also interpret TEGR as an abelian gauge theory [8]. One can study further various gravitational theories with non-vanishing torsion in terms of gauge theories of gravitation and the reader is directed to [9] for a further discussion.

Another note that we wish to make about TEGR, as well as extended Teleparallel Theories of Gravity, is that the dynamical object is the vierbein as opposed to being the metric tensor which is the dynamical object in GR. In order to see why this is the case, we note that the connection coefficients associated with the natural choice of an affine connection for the Weitzenböck space-time, which are given by equation (3.125), depend explicitly on the vierbein instead of the metric tensor. Therefore, the torsion tensor depends explicitly on the vierbein which implies that the dynamical objects for extended Teleparallel Theories of Gravity are the vierbein. The fact that the metric tensor is Lorentz invariant implies that gravitational theories for which the metric tensor is the dynamical object, such as GR, are also Lorentz invariant. On the other hand, gravitational theories for which the vierbein are the dynamical objects are not necessarily invariant under Lorentz transformations since the vierbein are not necessarily invariant under Lorentz transformations [37]. Of course, since TEGR is equivalent to GR, it is invariant under Lorentz transformations. However, the same is not true in general for a given $f(T)$ theory of gravity. We now wish to turn our attention to studying how $f(T)$ theories of gravity are affected by Lorentz transformations of tetrad fields. In order to undergo such a consideration, we first require the field equations associated with an $f(T)$ action.

Let us consider a total gravitational action that consists of the $f(T)$ action given in equation (4.129) together with a matter action, say S_m, with an associated matter Lagrangian density, say $\mathcal{L}_m$. Through the application of the Euler-Lagrange equations, one obtains the following by varying the total gravitational action with respect to the vierbein [83]

$$\frac{\partial\left(ef(T)\right)}{\partial {e^l}_\sigma} - \partial_\nu\left(\frac{\partial\left(ef(T)\right)}{\partial\left(\partial_\nu {e^l}_\sigma\right)}\right) + 2k^2\frac{\delta\mathcal{L}_m}{\delta {e^l}_\sigma} = 0 \; . \tag{4.130}$$

Evaluating the derivatives in the above expression and defining the energy-momentum tensor as the variation of the matter Lagrangian density with respect to the vierbein, i.e.,

$${\tau_\alpha}^\sigma := -\frac{{e^l}_\alpha}{e}\frac{\mathcal{L}_m}{\delta {e^l}_\sigma} \;, \tag{4.131}$$

we have the following [33, 83]

$$fe^{-1}{e^l}_\alpha\frac{\partial e}{\partial {e^l}_\sigma} + f_T\left({e^l}_\alpha\frac{\partial T}{\partial {e^l}_\sigma} - e^{-1}{e^l}_\alpha\partial_\nu\left(e\frac{\partial T}{\partial(\partial_\nu {e^l}_\sigma)}\right)\right) - f_{TT}\left(\partial_\nu T\right){e^l}_\alpha\frac{\partial T}{\partial(\partial_\nu {e^l}_\sigma)} = 2k^2{\tau_\alpha}^\sigma \; . \tag{4.132}$$

The first term on the right-hand side of the above expression involves the derivative of the determinant of the tetrad field with respect to the vierbein and can be evaluated to be the Kronecker delta symbol δ^σ_α. In order to find the derivatives of the torsion scalar, we first note the derivatives of the terms that are quadratic in the torsion tensor with respect to the vierbein [83]

$$\begin{gathered}{e_l}^\sigma\frac{\partial}{\partial {e_l}^\alpha}\left({T_\nu}^{\mu\alpha}{T^\nu}_{\mu\alpha}\right) = 4{T_\lambda}^{\sigma\beta}{T^\lambda}_{\sigma\beta} \;, \quad {e_l}^\sigma\frac{\partial}{\partial {e_l}^\alpha}\left({T^{\mu\alpha}}_\nu{T^\nu}_{\mu\alpha}\right) = 2{T^{\mu\sigma}}_\nu{T^\nu}_{\mu\alpha} - 2{T^{\nu\lambda}}_\alpha{T^\sigma}_{\lambda\nu} \;, \\ {e_l}^\sigma\frac{\partial}{\partial {e_l}^\alpha}\left({T^{\mu\alpha}}_\mu{T^\nu}_{\alpha\nu}\right) = 2{T^{\mu\sigma}}_\mu{T^\nu}_{\alpha\nu} + 2{T^{\nu\lambda}}_\nu{T^\sigma}_{\lambda\alpha} \; .\end{gathered} \tag{4.133}$$

By making use of the above derivatives as well as the definition of the torsion scalar given in equation (3.70), one has the following expression for the derivative of the torsion scalar with respect to the vierbein [83]

$$e^l{}_\alpha \frac{\partial T}{\partial e^l{}_\sigma} = 4T^\nu{}_{\beta\alpha} S_\nu{}^{\sigma\beta} \ , \tag{4.134}$$

where $S_\rho{}^{\mu\nu}$ is the superpotential defined in equation (3.72). In addition, we have the following expressions for the derivatives of the terms that are quadratic in the torsion tensor with respect to the first order derivatives of the vierbein [83]

$$e_l{}^\beta \frac{\partial}{\partial \left(\partial_\nu e_l{}^\alpha\right)} \left(T_\nu{}^{\mu\alpha} T^\nu{}_{\mu\alpha}\right) = 4T_\alpha{}^{\beta\nu} \ , e_l{}^\rho \frac{\partial}{\partial \left(\partial_\nu e_l{}^\alpha\right)} \left(T^{\mu\alpha}{}_\nu T^\nu{}_{\mu\alpha}\right) = 2T^{\rho\nu}{}_\alpha - 2T^{\nu\rho}{}_\alpha \ , \tag{4.135}$$

$$e_l{}^\beta \frac{\partial}{\partial \left(\partial_\nu e_l{}^\alpha\right)} \left(T^{\mu\alpha}{}_\mu T^\nu{}_{\alpha\nu}\right) = 2\delta^\nu_\alpha T^{\mu\beta}{}_\mu - \delta^\beta_\alpha T^{\mu\nu}{}_\mu \ . \tag{4.136}$$

By making use of the above expressions as well as equation (3.70), one can obtain the following expression for the derivative of the torsion scalar with respect to the first order derivatives of the vierbein [83]

$$e^l{}_\alpha \frac{\partial T}{\partial \left(\partial_\nu e^l{}_\sigma\right)} = -4S_\alpha{}^{\sigma\nu} \ . \tag{4.137}$$

It therefore follows from equation (4.132) that the field equations for $f(T)$ theories of gravity are [33, 40]

$$\frac{1}{4} f \delta^\sigma_\alpha + f_T \left(T^\nu{}_{\beta\alpha} S_\nu{}^{\sigma\beta} + e^{-1} e^l{}_\alpha \partial_\nu \left(e e_l{}^\beta S_\beta{}^{\sigma\nu}\right)\right) + f_{TT} S_\alpha{}^{\sigma\nu} \partial_\nu T = \frac{k^2}{2} \tau^\sigma_\alpha \ , \tag{4.138}$$

where f_T and f_{TT} denote the first and second derivatives of the function $f(T)$ with respect to the torsion scalar respectively.

4.5.2 Spherically symmetric space-time

Since the torsion scalar is not necessarily Lorentz invariant due to the fact that it depends explicitly on the vierbein which are not necessarily Lorentz invariant, the left-hand side of the field equations is not necessarily left invariant under Lorentz transformations. In order to elaborate on this notion, let us consider the example of a spherically symmetric space-time in a vacuum. To this end, we consider the following line element

$$\mathrm{d}s^2 = -A^2(t,r)\mathrm{d}t^2 + B^2(t,r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \ , \tag{4.139}$$

where $A(t,r)$ and $B(t,r)$ are arbitrary functions of the coordinate time and the radial distance. In addition, we consider the case of a vacuum, i.e., $\tau^\sigma{}_\alpha = 0$. In GR, and therefore in TEGR, one can show that a metric tensor that is spherically symmetric in a vacuum is in fact static as well asymptotically flat [5, 55]. This result is known as *Birkhoff's theorem* [5, 55, 104]. In this thesis, we shall refer to the notion of a spherically symmetric metric, for which a vacuum is considered, being static as the *first inference* of Birkhoff's theorem. The notion of Birkhoff's theorem in $f(T)$ theories of gravity is an open problem [40], however, an attempt had been made by the authors of [39] to provide a proof for the first inference of Birkhoff's theorem for a general $f(T)$ theory. Following [37] and [40], we will consider an argument for why the authors of [39] were not successful in providing a proof of the first inference of Birkhoff's theorem for a general $f(T)$ theory.

In order to solve the $f(T)$ field equations, one must first provide a tetrad construction according to equation (3.85). A simple choice of a tetrad construction for the line element (4.139) would be the diagonal matrix

$$e^a{}_\mu = \begin{pmatrix} A(t,r) & 0 & 0 & 0 \\ 0 & B(t,r) & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix} \ . \tag{4.140}$$

For the tetrad construction given above, we compute the Weitzenböck connection coefficients as per equation (3.125). Following [76], the non-zero Weitzenböck connection coefficients, in the holonomic frame, are

$$\tilde{\Gamma}^{t}{}_{tt} = \frac{\dot{A}}{A} \ , \qquad \tilde{\Gamma}^{r}{}_{tr} = \frac{\dot{B}}{B} \ , \qquad \tilde{\Gamma}^{t}{}_{rt} = \frac{A'}{A} \ , \qquad \tilde{\Gamma}^{r}{}_{rr} = \frac{B'}{B} \ ,$$
$$\tilde{\Gamma}^{\theta}{}_{r\theta} = \frac{1}{r} \ , \qquad \tilde{\Gamma}^{\phi}{}_{r\phi} = \frac{1}{r} \ , \qquad \tilde{\Gamma}^{\phi}{}_{\theta\phi} = \cot\theta \ . \tag{4.141}$$

It therefore follows from equation (3.27) that the non-zero torsion tensor components, in the holonomic frame, are [76]

$$T^{r}{}_{tr} = -T^{r}{}_{rt} = \frac{\dot{B}}{B} \ , \qquad T^{t}{}_{rt} = -T^{t}{}_{tr} = \frac{A'}{A} \ , \qquad T^{\theta}{}_{r\theta} = -T^{\theta}{}_{\theta r} = \frac{1}{r} \ ,$$
$$T^{\phi}{}_{r\phi} = -T^{\phi}{}_{\phi r} = \frac{1}{r} \ , \qquad T^{\phi}{}_{\theta\phi} = -T^{\phi}{}_{\phi\theta} = \cot\theta \ . \tag{4.142}$$

Therefore, for the diagonal tetrad construction considered, the scalar terms that are quadratic in the torsion tensor are given by the expressions [76]

$$T^{\mu\alpha}{}_{\nu}T^{\nu}{}_{\mu\alpha} = -\frac{(A')^2}{A^2B^2} + \frac{(\dot{B})^2}{A^2B^2} - \frac{2}{r^2B^2} - \frac{\cot^2\theta}{r^2} \ , \tag{4.143}$$

$$T_{\nu}{}^{\mu\alpha}T^{\nu}{}_{\mu\alpha} = 2\left(\frac{(A')^2}{A^2B^2} - \frac{(\dot{B})^2}{A^2B^2} + \frac{2}{r^2B^2} + \frac{\cot^2\theta}{r^2}\right) \ , \tag{4.144}$$

$$T^{\mu\alpha}{}_{\mu}T^{\nu}{}_{\alpha\nu} = -\frac{(\dot{B})^2}{A^2B^2} + \frac{(A')^2}{A^2B^2} + \frac{4A'}{ArB^2} + \frac{4}{B^2r^2} + \frac{\cot^2\theta}{r^2} \ . \tag{4.145}$$

It now follows from equation (3.70) that, for the diagonal tetrad construction given in equation (4.140), the torsion scalar is [39]

$$T = -\frac{2}{B^2r}\left(\frac{1}{r} + \frac{2A'}{A}\right) \ . \tag{4.146}$$

Using the above results, we now turn our attention to the $f(T)$ field equations (4.138) in order to find constraints on the functions $A(t,r)$ and $B(t,r)$ following [39]. The (r,t) field equation is written as

$$\frac{\dot{B}}{A^2Br}f_T = 0 \ . \tag{4.147}$$

Therefore, it follows from the above field equation that the function $B(t,r)$ does not depend on coordinate time, i.e., $\dot{B} = 0$. Therefore, the component g_{00} depends purely on the radius, i.e., $B(t,r) = B(r)$. We now note that the (t,t) field equation is given by the expression

$$\left(\frac{A'}{AB^2r} + \frac{1}{B^2r^2} - \frac{B'}{rB^3} - \frac{1}{2r^2}\right)f_T + \frac{f}{4} = 0 \ . \tag{4.148}$$

In addition, we note that the (r,r) field equation reads

$$\left(-\frac{1}{2r^2} + \frac{2A'}{AB^2r} + \frac{1}{B^2r^2}\right)f_T + \frac{1}{4}f = 0 \ . \tag{4.149}$$

Let us now examine the form of the function $f(T)$. From equation (4.146), which gives the expression for the torsion scalar for the diagonal tetrad construction, one can interpret the torsion scalar as being a function of r and A'/A since we already know that B is purely a function of r. Therefore, one can interpret a function of the torsion scalar, $f(T)$, as a function of r and A'/A. Turning our attention back to the (r,r) field equation (4.149), we note that, for a given $f(T)$ function, one could solve this equation for A'/A in terms of a pure function of r [39]. Denoting such a function as $J(r)$, one then has

$$\frac{A'}{A} = J(r)\ . \tag{4.150}$$

By integrating the above expression and defining the quantity $\bar{A}(r) := e^{\int h(r)\mathrm{d}r}$, one has the following expression for the function A [39]

$$A(r,t) = e^{C(t)}\bar{A}(r)\ , \tag{4.151}$$

where $C(t)$ is a pure function of time obtained through the integration. One can absorb the factor of $e^{C(t)}$ in the construction of the line element by carrying out the coordinate transformation $\mathrm{d}t' = e^{C(t)}\mathrm{d}t$. It follows that the line element can now be written as [39]

$$\mathrm{d}s^2 = -\tilde{A}^2(r)\ \mathrm{d}t'^2 + B^2(r)\ \mathrm{d}r^2 + r^2\ \mathrm{d}\theta^2 + r^2\sin^2\theta\ \mathrm{d}\phi^2\ , \tag{4.152}$$

which is static. It was at this point that the authors of [39] argued that the first inference of Birkhoff's theorem for a general $f(T)$ theory of gravity has been proven. However, as pointed out by Tamanini and Boehmer in [37, 105], there are additional constraints imposed by other field equations. More specifically, the (r,θ) field equation yields [37]

$$\frac{T' f_{TT}\cot\theta}{2r^2} = 0\ , \tag{4.153}$$

where $T' := \partial_r T$. In order to avoid the restrictions imposed by setting T' to zero, one must impose $f_{TT} = 0$ which is the case of TEGR. Therefore, since we have $f_{TT} = 0$, the results presented by Meng and Wang in [39] only hold for the case of TEGR which is already known since TEGR is equivalent to GR. The reader may wonder if the imposing of $T' = 0$ may be a viable restriction. Such a case is studied in depth in [37] where it was noted that such an imposition results in "stringent constraints". Here, however, we simply note that in the case of the Schwarzschild solution, where $A^2 = (1 - 2M/r)$ and $B^2 = (1 - 2M/r)^{-1}$, one has $T' = 4/r^3$. Therefore, the Schwarzschild solution does not satisfy the condition $T' = 0$ and therefore one must resort to imposing $f_{TT} = 0$ which yields TEGR [37]. It follows that one cannot use the diagonal tetrad construction in order to prove Birkhoff's theorem for a general $f(T)$ theory since the tetrad construction will impose that the $f(T)$ theory be TEGR for which it is already known that Birkhoff's theorem holds. Using the terminology of [37], the tetrad constructed in equation (4.140) is referred to as a "bad tetrad".

We now wish to compare how the torsion scalar is affected by making use of a rotated tetrad. For this case, let us consider the following line element which is both spherically symmetric and static [106]

$$\mathrm{d}s^2 = -A^2(r)\mathrm{d}t^2 + B^2(r)\mathrm{d}r^2 + \rho^2(r)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\ , \tag{4.154}$$

where one obtains the static case of equation (4.139) when $\rho(r) = r$. For the choice of a tetrad, we act the SO$(1,3)$ rotation matrix

$$R^a{}_b := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi\sin\theta & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix}\ , \tag{4.155}$$

on the diagonal tetrad $\operatorname{diag}(A, B, \rho, \rho\sin\theta)$. The application of such a rotation yields the following tetrad construction [106]

$$e^a{}_\mu = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B\cos\phi\sin\theta & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ 0 & B\sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ 0 & B\cos\theta & -\rho\sin\theta & 0 \end{pmatrix}\ , \tag{4.156}$$

and we consider the case of a static space-time, i.e., $A = A(r)$ and $B = B(r)$. For such a tetrad construction, the non-zero components of the Weitzenböck connection coefficients are

$$\tilde{\Gamma}^{t}{}_{rt} = \frac{A'}{A} \,, \qquad \tilde{\Gamma}^{r}{}_{\theta\theta} = -\frac{\rho}{B} \,, \qquad \tilde{\Gamma}^{r}{}_{\phi\phi} = -\frac{\rho\sin^2\theta}{B} \,, \qquad \tilde{\Gamma}^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta \,,$$
$$\tilde{\Gamma}^{r}{}_{rr} = \frac{B'}{B} \,, \qquad \tilde{\Gamma}^{\phi}{}_{\phi\theta} = \tilde{\Gamma}^{\phi}{}_{\theta\phi} = \cot\theta \,, \qquad \tilde{\Gamma}^{\phi}{}_{r\phi} = \tilde{\Gamma}^{\theta}{}_{r\theta} = \frac{\rho'}{\rho}, \qquad \tilde{\Gamma}^{\phi}{}_{\phi r} = \tilde{\Gamma}^{\theta}{}_{\theta r} = \frac{B}{\rho} \,. \tag{4.157}$$

It follows from the corresponding torsion tensor components that the scalar terms that are quadratic in the torsion tensor are

$$T^{\mu\alpha}{}_{\nu}T^{\nu}{}_{\mu\alpha} = -\frac{1}{B^2}\left[\frac{A'^2}{A^2} + \frac{2\left(\rho' - B\right)^2}{\rho^2}\right] \,, \tag{4.158}$$

$$T_{\nu}{}^{\mu\alpha}T^{\nu}{}_{\mu\alpha} = \frac{2}{B^2}\left[\frac{A'^2}{A^2} + \frac{2\left(\rho' - B\right)^2}{\rho^2}\right] \,, \tag{4.159}$$

$$T^{\mu\alpha}{}_{\mu}T^{\nu}{}_{\alpha\nu} = \frac{1}{B^2}\left[\frac{4\left(\rho' - B\right)^2}{\rho^2} + \frac{4A'\left(\rho' - B\right)}{A\rho} + \frac{A'^2}{A^2}\right] \,. \tag{4.160}$$

It therefore follows that, for the choice of a rotated tetrad, the torsion scalar takes the form [106]

$$T = -\frac{2\left(\rho' - B\right)}{B^2\rho}\left[\frac{\rho' - B}{\rho} + \frac{2A'}{A}\right] \,. \tag{4.161}$$

We note that by setting $A = B = 1$ and $\rho(r) = r$, one obtains a vanishing torsion scalar which is associated with Minkowski space-time. However, as noted in [106], when the parity transformation $B \to -B$ is performed, the torsion scalar is non-vanishing. It is important to note that equation (4.146) differs to the above expression despite the fact that the quantities are scalars. This is of course due to the fact that the torsion scalar depends explicitly on the vierbein and has been computed with two different tetrad constructions with the second tetrad construction being obtained by carrying out an SO$(1,3)$ rotation on the first. Here, we were interested in discussing how the torsion scalar differs once a Lorentz transformation, in this case a rotation, has been carried out. The reader interested in what restrictions are imposed by the rotated tetrad on a given $f(T)$ theory is directed to [37].

4.5.3 Toroidally symmetric space-time

In this subsection, we consider a space-time which is described by the line element

$$\mathrm{d}s^2 = -A(r)^2\mathrm{d}t^2 + B(r)^2\mathrm{d}r^2 + \rho(r)^2\left[\frac{\mathrm{d}x^2}{1 - kx^2} + \left(1 - kx^2\right)\mathrm{d}y^2\right] \,, \tag{4.162}$$

where $k = -1, 0, 1$ yields hyperbolic, toroidal or spherically symmetric space-times respectively [106]. In particular, we wish to consider how a given $f(T)$ theory is restricted by the choice of a diagonal tetrad. That is, we wish to consider the following tetrad construction [106]

$$e^{a}{}_{\mu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & \frac{\rho(r)}{\sqrt{1-kx^2}} & 0 \\ 0 & 0 & 0 & \rho(r)\sqrt{1-kx^2} \end{pmatrix} \,. \tag{4.163}$$

For the diagonal tetrad construction given above, one has the following non-zero components of the Weitzenböck connection coefficients

$$\tilde{\Gamma}^{t}{}_{rt} = \frac{A'}{A} \,, \qquad \tilde{\Gamma}^{r}{}_{rr} = \frac{B'}{B} \,, \qquad \tilde{\Gamma}^{x}{}_{xx} = \frac{kx}{\sqrt{1 - kx^2}} \,, \qquad \tilde{\Gamma}^{x}{}_{rx} = \tilde{\Gamma}^{y}{}_{ry} = \frac{\rho'}{\rho} \,,$$
$$\tilde{\Gamma}^{y}{}_{xy} = -\frac{kx}{1 - kx^2} \,. \tag{4.164}$$

It follows from the above non-zero components that the scalar terms quadratic in the torsion tensor have the following expressions

$$T^{\rho}{}_{\mu\nu}T^{\mu\nu}{}_{\rho} = -\frac{1}{2}T^{\rho}{}_{\mu\nu}T_{\rho}{}^{\mu\nu} = -\frac{1}{B^2}\left(\frac{A'^2}{A^2}+\frac{2\rho'^2}{\rho^2}\right) - \frac{k^2x^2}{\rho^2\left(1-kx^2\right)} \ , \tag{4.165}$$

$$T_{\nu}T^{\nu} = \frac{1}{B^2}\left(\frac{A'}{A}+\frac{2\rho'}{\rho}\right)^2 + \frac{k^2x^2}{\rho^2\left(1-kx^2\right)} \ , \tag{4.166}$$

and thus, the torsion scalar reads

$$T = -\frac{2}{B^2\rho}\left(\frac{\rho'^2}{\rho}+\frac{2A'\rho'}{A}\right) \ . \tag{4.167}$$

By examining the (r,x) $f(T)$ field equation, one can obtain

$$\frac{kxf_{TT}T'}{2\rho^2} = 0 \ . \tag{4.168}$$

By avoiding the restriction $T'=0$ it is easily seen that one must consider the toroidally symmetric case where $k=0$ or the case of TEGR with a cosmological constant, i.e., $f_{TT}=0$ [106]. It can therefore be concluded that for the cases where $k \neq 0$, the diagonal tetrad is a bad tetrad. However, for the toroidally symmetric case, the field equations do not impose the condition that the second derivative of $f(T)$ with respect to the torsion scalar must vanish and is therefore a good choice of a tetrad for such a case. The reader interested in a more in depth discussion of the toroidally symmetric case for $f(T)$ theories of gravity is directed to [106].

4.5.4 A derivation of the Reissner-Nordström solution in TEGR

In the previous sections, we studied spherically symmetric space-times for a general $f(T)$ theory. In this section, we wish to turn our attention to TEGR, i.e., we consider the case of $f(T)=T$, and study briefly the so-called Reissner-Nordström solution. This solution is well-known and describes the space-time outside an object that is spherically symmetric and carries an electric charge [55]. Up until this point, we have not given a complete example of a derivation of a well-known solution in GR by making use of the TEGR formalism. In this section, we shall derive the Reissner-Nordström solution using the TEGR formalism. Let us start with a spherically symmetric and static metric tensor whose line element is as follows

$$\mathrm{d}s^2 = -A^2(r)\mathrm{d}t^2 + B^2(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right) \ . \tag{4.169}$$

In order to calculate the torsion tensor components, we first require a tetrad construction. Since we are considering the case of TEGR, which is a Lorentz invariant theory, we are not restricted by the choice of a tetrad field. Therefore, in order to make the calculations as simple as possible, we take the tetrad to be of the diagonal form given in equation (4.140). The torsion tensor components are therefore the same as what is given in equation (4.142) except, since we are dealing with a static metric tensor, we have $\dot{B}=0$ which implies that $T^{r}{}_{tr} = -T^{r}{}_{rt} = 0$. We now turn our attention to the consideration of the energy-momentum tensor. The matter action associated with the presence of an electromagnetic field is as follows [8, 89]

$$S_{\mathrm{m}} = -\frac{1}{4}\int \mathrm{d}^4x \ eF_{\mu\nu}F^{\mu\nu} \ , \tag{4.170}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor defined as

$$F_{\mu\nu} := 2D_{[\mu}A_{\nu]} \ . \tag{4.171}$$

In the above definition, D_μ is the space-time covariant derivative associated with the Levi-Civita connection and A_μ is the four vector potential. Let us write down the matter Lagrangian density as follows

$$\mathcal{L}_{\mathrm{m}} = -\frac{1}{4} e F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} e F_{\mu\nu} F_{\alpha\beta} e_a{}^{\alpha} e_b{}^{\mu} e_c{}^{\beta} e_a{}^{\nu} \eta^{ab} \eta^{cd} \ . \tag{4.172}$$

By carrying out a variation with respect to the tetrad field, we obtain

$$\delta\mathcal{L}_{\mathrm{m}} = e F_{\mu\nu} F^{\mu\lambda} e_d{}^{\nu} \delta e^d{}_{\lambda} - \frac{1}{4} e F_{\mu\nu} F^{\mu\nu} e_a{}^{\nu} \delta e^a{}_{\nu} \ . \tag{4.173}$$

It therefore follows from equation (4.131) that the energy-momentum tensor, for the given matter Lagrangian density, is

$$\tau_\nu{}^{\lambda} = -F_{\mu\nu} F^{\mu\lambda} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \delta_\nu^\lambda \ . \tag{4.174}$$

For this derivation, we take the four vector potential to read

$$A_\mu = \left(-q/\left(4\pi r\right), 0, 0, 0\right) \ , \tag{4.175}$$

where q is the charge of the object that is under consideration. We are now in a position to write down the TEGR field equations. For the (t,t) equation, the left-hand side of the field equation is simply the left-hand side of equation (4.148) for the case where $f(T) = T$. The right-hand side can be found by taking the (t,t) component of equation (4.174). This component can be found to be

$$\tau_t{}^t = -\frac{q^2}{8\pi A^2 B^2 r^4} \ . \tag{4.176}$$

It therefore follows that the (t,t) field equation is as follows

$$\frac{1}{2B^2 r^2} - \frac{1}{2r^2} - \frac{B'}{rB^3} = -\frac{q^2}{2A^2 B^2 r^4} \ , \tag{4.177}$$

where we have made use of the expression for the torsion scalar given in equation (4.146). We now wish to obtain the (r,r) field equation. The left-hand side for this equation is nothing more than the left-hand side of equation (4.149) for the case where $f(T) = T$. For the right-hand side, we note that the (r,r) component of the energy-momentum tensor is $\tau_r{}^r = \tau_t{}^t$. We therefore find that, by once again substituting in equation (4.146) which gives the relevant expression for the torsion scalar, the (r,r) field equation is as follows

$$\frac{A'}{AB^2 r} - \frac{1}{2r^2} + \frac{1}{2B^2 r^2} = -\frac{q^2}{2A^2 B^2 r^4} \ . \tag{4.178}$$

We note that the other field equations do not impose any additional constraints to what is described by the (t,t) and (r,r) field equations given above. We now turn our attention to solving these field equations. By subtracting equation (4.177) from equation (4.178), we obtain the following differential equation

$$\frac{A'}{AB^2 r} + \frac{B'}{B^3 r} = 0 \implies \partial_r \left(AB\right) = 0 \ . \tag{4.179}$$

It therefore follows that the vierbein components A and B satisfy

$$AB = l \ , \tag{4.180}$$

where $l \in \mathbb{R}$ is a constant of integration. Since we can rescale the time coordinate through a gauge transformation to $t \to lt$,we can set $l = 1$ [5]. It therefore follows from the above expression that we can write $AB = 1$. We can now write the (t,t) field equation (4.177) as

$$\frac{B'}{rB^3} = \frac{\left(1 - B^2\right)}{2B^2 r^2} + \frac{q^2}{2r^4} \ . \tag{4.181}$$

In order to solve this differential equation, we first consider the case of vacuum, i.e., for now we set $q = 0$. This consideration results in the following differential equation

$$\frac{B'}{B(1-B)(1-B)} = \frac{1}{2r} \ . \tag{4.182}$$

It therefore follows that, by integrating equation (4.182), we obtain

$$B^2 = \frac{1}{1 + C/r} \ , \tag{4.183}$$

where $C \in \mathbb{R}$ is a constant of integration. This gives the Schwarzschild solution with the component B^2 given above and the component $A^2 = 1/B^2$. In order to now find the Reissner-Nordström solution, we let $C = C(r)$ be a function of r and substitute equation (4.183) into equation (4.181) in order to obtain

$$C' = -\frac{q^2}{r^2} \implies C = \frac{q^2}{r} + K \ , \tag{4.184}$$

where $K \in \mathbb{R}$ is a constant of integration. It therefore follows, by substituting the above expression into equation (4.183), that we have the following

$$B^2 = \frac{1}{1 + K/r + q^2/r^2} \ . \tag{4.185}$$

We now require the constant K. In order to find this constant, we consider the vacuum case with $q = 0$ and we study the Newtonian limit, i.e., we compare the solution as $r \to \infty$ to the Newtonian case [5]. In order to do this, we turn our attention to the force equation (geodesic equation in GR) given in equation (3.135). For a test particle with a speed that is much less than the speed of light, we make the approximation $\mathrm{d}x^\mu/\mathrm{d}\tau = (1, 0, 0, 0)$. The force equation yields the following [5]

$$a := \frac{\mathrm{d}^2 x^r}{\mathrm{d}\tau^2} = -\Gamma^r{}_{tt} \ . \tag{4.186}$$

It is not difficult to show that $\Gamma^r{}_{tt} = -K/\left(2B^2 r^2\right)$. In the limit as $r \to \infty$, we have $B \to 1$ and thus, in the Newtonian limit, we have

$$a \approx \frac{K}{2r^2} \ . \tag{4.187}$$

According to Newtonian gravitation, the acceleration reads $a = -GM/r^2$ where M is the mass of the Newtonian gravitational field [5]. It therefore follows that the constant of integration reads $K = -2M$ where we have set $G = 1$. It now follows that the line element reads

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \ . \tag{4.188}$$

The above line element corresponds to the so-called *Reissner-Nordström solution* [55]. Here, we have derived this solution in the TEGR framework as an example as to how one can obtain solutions in this framework.

4.5.5 TEGR solutions that also solve $f(T)$ theories

In Section 4.5.2, we discussed an approach to proving the first inference of Birkhoff's theorem for a general $f(T)$ theory. More specifically, we discussed an attempt by the authors of [39] to prove this by making use of a diagonal tetrad construction. This led to a constraint on the second derivative of the $f(T)$ function, f_{TT}, which required the theory to be TEGR for which it already known that Birkhoff's theorem holds. This example demonstrates that the ansatz for the tetrad, i.e., the imposing of the existence of a particular class of solutions, results in constraints on the $f(T)$ theory. Moreover, this example illustrates the difficulty in extending results that are found to hold for the case of TEGR to other $f(T)$ theories of gravity. The constraint that f_{TT} must vanish is easily avoided by making use of the tetrad construction that is given in equation (4.156) which is obtained by carrying out a rotation of the diagonal tetrad. Of course, TEGR is a Lorentz invariant theory and, therefore, the field equations are unchanged when making use of this rotated tetrad. The fact that the tetrad used in the $f(T)$ field equations gives the same TEGR solution as any other tetrad obtained via a Lorentz transformation provides us with a remarkable result for ultraviolet deformations of TEGR. This result is given in [40] and is the subject of this section. Here, we will follow closely what is done in [40].

Before we discuss this result, we first wish to derive a relation that will be of great use. Let $\boldsymbol{B}$ be a tensor field of type $(2,1)$ and consider the action of the Levi-Civita covariant derivative on this tensor field:

$$D_\nu B_\alpha{}^{\sigma\nu} = \partial_\nu B_\alpha{}^{\sigma\nu} - \Gamma^\sigma{}_{\nu\alpha} B_\beta{}^{\sigma\nu} + \Gamma^\sigma{}_{\nu\beta} B_\alpha{}^{\beta\nu} + \Gamma^\nu{}_{\nu\beta} B_\alpha{}^{\sigma\beta} \ . \tag{4.189}$$

Furthermore, let us assume that the tensor field $\boldsymbol{B}$ is antisymmetric in its second and third indices. The reason for this assumption is that, later on, we would like to apply this result to the torsion tensor and the superpotential tensor both of which are antisymmetric in their second and third indices. For such an assumption, the third term in the above expression vanishes due to the fact that the symmetric indices of the Christoffel symbols contract with the antisymmetric indices of the tensor field $\boldsymbol{B}$. We now substitute the Christoffel symbols in the above expression for the Cartan connection coefficients and the contorsion tensor in order to obtain the following expression

$$D_\nu B_\alpha{}^{\sigma\nu} = \partial_\nu B_\alpha{}^{\sigma\nu} + \tilde{\Gamma}^\nu{}_{\beta\nu} B_\alpha{}^{\sigma\beta} - \tilde{\Gamma}^\beta_{\nu\alpha} B_\beta{}^{\sigma\nu} + K^\beta{}_{\nu\alpha} B_\beta{}^{\sigma\nu} \ . \tag{4.190}$$

We now wish to consider the following expression which appears in the $f(T)$ field equations for the case where the tensor field $\boldsymbol{B}$ is the superpotential [83]

$$e^{-1} e^l{}_\alpha \partial_\nu \left(e e_l{}^\beta B_\beta{}^{\sigma\nu} \right) = \partial_\nu B_\alpha{}^{\sigma\nu} + B_\alpha{}^{\sigma\nu} e_l{}^\beta \partial_\nu e^l{}_\beta - B_\beta{}^{\sigma\nu} e_l{}^\beta \partial_\nu e^l{}_\alpha \ . \tag{4.191}$$

Since we are carrying out these calculations on the Weitzenböck space-time $\mathcal{W}_4$, the second and third terms on the right-hand side of the above expression contain Weitzenböck connection coefficients. It follows that we can write the above expression as

$$e^{-1} e^l{}_\alpha \partial_\nu \left(e e_l{}^\beta B_\beta{}^{\sigma\nu} \right) = \partial_\nu B_\alpha{}^{\sigma\nu} + B_\alpha{}^{\sigma\nu} \tilde{\Gamma}^\beta{}_{\nu\beta} - B_\beta{}^{\sigma\nu} \tilde{\Gamma}^\beta{}_{\nu\alpha} \ . \tag{4.192}$$

By combining equations (4.190) and (4.192), we can obtain the following expression

$$D_\nu B_\alpha{}^{\sigma\nu} = e^{-1} e^l{}_\alpha \partial_\nu \left(e e_l{}^\beta B_\beta{}^{\sigma\nu} \right) + K^\beta{}_{\nu\alpha} B_\beta{}^{\sigma\nu} \ . \tag{4.193}$$

As already mentioned, the above identity will be useful to us when studying relations that involve the divergence of either the superpotential tensor or the torsion tensor. For example, consider equation (3.69) for the case of a Weitzenböck space-time, i.e.,

$$T = -R - 2 D_\nu T^\nu \ . \tag{4.194}$$

Contracting indices in equation (4.193) for the case where the tensor field $\boldsymbol{B}$ is the torsion tensor and substituting the result into the above expression yields

$$T = -R + 2 e^{-1} \partial_\nu \left(e T^\nu \right) \ . \tag{4.195}$$

The purpose of this exercise was to replace the Levi-Civita covariant derivative with a divergence term. Having derived the above expression, we now wish to turn our attention to what is discussed in [40].

Let us begin by considering the case of TEGR with a cosmological constant, i.e., the class of theories that we wish to consider are described by $f(T) = T - 2\sigma$ where σ is a constant. In addition, for the time being, we are only interested in vacuum solutions, i.e., when we have $\tau_\mu{}^\nu = 0$. Now, let the vierbein $e^a{}_\mu$ be a solution to the Einstein field equations. That is, the aforementioned vierbein is a solution to the following equation:

$$\frac{1}{4}\delta^\sigma_\alpha \left(T\left[e^m_\lambda\right] - 2\sigma\right) + T^\nu{}_{\beta\alpha}\left[e^m_\lambda\right] S_\nu{}^{\sigma\beta}\left[e^m_\lambda\right] + e^{-1} e^l{}_\alpha \partial_\nu \left(e e_l{}^\beta S_\beta{}^{\sigma\nu}\left[e^m_\lambda\right]\right) = 0 \; . \tag{4.196}$$

The above expression is obtained by substituting $f(T) = T - 2\sigma$ and $\tau_\alpha{}^\sigma = 0$ into the $f(T)$ field equations (4.138). In addition, we have used the notation $T^\alpha_{\mu\nu}\left[e^m_\lambda\right]$ to indicate explicitly that the torsion tensor is obtained using the tetrad $e^a{}_\alpha$. Taking the trace of the above expression yields the following equation

$$e^l{}_\mu \partial_\nu \left(e e_l{}^\beta S_\beta{}^{\mu\nu}\left[e^m_\lambda\right]\right) = 2e\sigma \; . \tag{4.197}$$

Of course, since the tetrad field solves the Einstein field equations, we have $R[e^a{}_\alpha] = 0$. We now wish to consider a general $f(T)$ function. Let $\bar{e}^a{}_\alpha = \Lambda^a{}_b e^b{}_\alpha$ be a solution to the $f(T)$ field equations for this general function. Following [40], we ask the question as to whether we can find such a tetrad field $\bar{e}^a{}_\alpha$ that allows for the torsion scalar to be a constant, i.e., we have $T = 2\bar{\sigma}$ where $\bar{\sigma}$ is a constant. Since TEGR is a Lorentz invariant theory, the vierbein $\bar{e}^a{}_\alpha$ is a solution to the Einstein field equations. In particular, we note that $R[\bar{e}^a{}_\alpha] = 0$ and

$$\bar{e}^l{}_\mu \partial_\nu \left(e \bar{e}_l{}^\beta S_\beta{}^{\mu\nu}\left[\bar{e}^m_\lambda\right]\right) = 2e\sigma \; , \tag{4.198}$$

where we have made use of the fact that $\bar{e} = e$ is a Lorentz invariant quantity. By making use of the fact that $R[\bar{e}^a{}_\alpha] = 0$, we can make use of equation (4.195) in order to rephrase the question as to whether we can find a tetrad field that allows the torsion scalar to be constant as whether we can find a Lorentz transformation $\Lambda^a{}_b$ that allows for the following equation to be satisfied [40]

$$\bar{\sigma} = e^{-1}\partial_\nu \left(e T^\nu \left[\bar{e}^a{}_\alpha\right]\right) \; . \tag{4.199}$$

Since we have assumed that the tetrad field $\bar{e}^a{}_\mu$ solves the $f(T)$ equations for the case of a vacuum, we can make use of the trace of equation (4.138) in order to obtain the following expression

$$f\left(2\bar{\sigma}\right) - 2\bar{\sigma} f_T\left(2\bar{\sigma}\right) + \bar{e}^l{}_\mu \partial_\nu \left(e \bar{e}_l{}^\beta S_\beta{}^{\mu\nu}\left[\bar{e}^m_\lambda\right]\right) f_T\left(2\bar{\sigma}\right) = 0 \; . \tag{4.200}$$

We can now substitute equation (4.198) into the above expression in order to obtain the following [40]

$$2\left(\bar{\sigma} - \sigma\right) = \frac{f\left(2\bar{\sigma}\right)}{f_T\left(2\bar{\sigma}\right)} \; . \tag{4.201}$$

We have thus shown, following [40], that if we can find a solution $\bar{e}^a{}_\alpha$ that is such that it solves the TEGR field equations and yields a constant torsion scalar with $T[\bar{e}^a{}_\alpha] = 2\bar{\sigma}$, then the tetrad field $\bar{e}^a{}_\alpha$ will be a solution to any $f(T)$ theory satisfying the above expression. As stated in [40], this is a remarkable property of $f(T)$ theories. In particular, we can make specific mention to how one can apply this method to $f(T)$ theories that are ultraviolet deformations of TEGR, i.e., theories that can be written as

$$f(T) = T + O(T^2) \; . \tag{4.202}$$

In order to demonstrate why this is the case, consider a vacuum solution, $\bar{e}^a{}_\alpha$, to the TEGR field equations that is such that $\sigma = \bar{\sigma} = 0$. Turning our attention to the class of theories referred to as ultraviolet deformations of TEGR, we have $f(0) = 0$ and $f_T(0) = 1$ from the above expression. Clearly, for such a consideration, equation (4.201) is satisfied and thus, the tetrad field $\bar{e}^a{}_\alpha$ would solve any theory that is an ultraviolet deformation of TEGR.

4.6 Chapter conclusions

This concludes our discussion of gravitational theories constructed on the Riemann-Cartan and Weitzenböck space-times. The most natural extension of GR to a theory that includes torsion is the EC theory which we discussed first in Section 4.2. We first derived the EC field equations through the use of the Cartan formalism before deriving the field equations in the holonomic frame. Following the discussion on the EC theory, we discussed the ECS theory developed by Saa in [11] and reviewed in [12]. The last modification of the EC theory that we discussed was the CSEC theory which is a Chern-Simons modification that we constructed based on what was done in [13–15].

The last class of theories that we discussed are the $f(T)$ theories of gravity which are constructed on the Weitzenböck space-time. Specifically, we discussed the notion of Birkhoff's theorem for $f(T)$ theories as well as the notions of so-called good and bad tetrads. In the following chapter, we will make use of the formalism discussed here for $f(T)$ theories in order to study the focusing conditions that arise in these theories.

We note that there exists more general modifications of $f(T)$ theories of gravity. One such example is a class of theories that includes a Lagrangian density that can be written as a function of T and the so-called Gauss-Bonnet term [44, 45]. For a study on these class of theories as well as their relevance to cosmology, the interested reader is directed to [44–49].

Chapter 5

Cosmological focusing conditions for $f(T)$ theories

5.1 Introduction

In this chapter, we discuss the focusing conditions that arise in the context of $f(T)$ theories of gravity. In Section 5.2 we outline the tetrad construction to be used in our analysis. Following this, we turn our attention to discussing the focusing conditions in Section 5.3. We will derive the focusing conditions associated with a fundamental congruence in Section 5.3.1 by following what is done in [64]. In Section 5.3.2 we move on to derive, for the first time, the more general focusing conditions associated with a one-parameter dependent congruence of timelike auto-parallel curves of the Levi-Civita connection. We present our conclusions in Section 5.4. We note that the results presented in this chapter have given rise to the article [65].

5.2 Tetrad construction for a Robertson-Walker space-time

We begin this chapter by considering a Robertson-Walker space-time which is described by the line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] , \tag{5.1}$$

where $k = -1, 0, 1$ corresponds to a spherical, flat and hyperbolic spatial geometry respectively [5]. The function $a(t)$ is often referred to as the *scale factor* [107]. In addition, the redshift is written as z and is related to the scale factor through the below expression [107]

$$z = a^{-1} - 1 . \tag{5.2}$$

In the previous chapter, we studied two cases of space-times for which the choice of considering a diagonal tetrad resulted in restricting the $f(T)$ theory to be TEGR with a cosmological constant, i.e., $f_{TT} = 0$. Following the terminology used [37], this led us to classify those tetrads as "bad tetrads". Following [37], we shall see that the choice of a diagonal tetrad given the metric described by the line element in equation (5.1) is a bad tetrad as it too leads to the restriction $f_{TT} = 0$. The diagonal tetrad construction given equation (5.1) is as follows

$$e^a{}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{a(t)}{\sqrt{1-kr^2}} & 0 & 0 \\ 0 & 0 & ra(t) & 0 \\ 0 & 0 & 0 & ra(t)\sin\theta \end{pmatrix} . \tag{5.3}$$

The above tetrad construction yields the following non-vanishing components of the Weitzenböck connection coefficients

$$\tilde{\Gamma}^r{}_{tr} = \tilde{\Gamma}^\theta{}_{t\theta} = \tilde{\Gamma}^\phi{}_{t\phi} = \frac{\dot{a}}{a} , \qquad \tilde{\Gamma}^\theta{}_{r\theta} = \tilde{\Gamma}^\phi{}_{r\phi} = \frac{1}{r} , \qquad \tilde{\Gamma}^r{}_{rr} = \frac{kr}{1-kr^2} , \qquad \tilde{\Gamma}^\phi{}_{\theta\phi} = \cot\theta . \tag{5.4}$$

By making use of the above connection coefficients in order to compute the components of the torsion tensor, one can obtain the following expressions for the scalars that are quadratic in the torsion tensor

$$T_{\nu}{}^{\mu\alpha}T^{\nu}{}_{\mu\alpha} = -6H^2 + \frac{2\left(1+\csc^2\theta\right)}{a^2r^2} - \frac{4k^2}{a^2}\ , \tag{5.5}$$

$$T^{\mu\alpha}{}_{\nu}T^{\nu}{}_{\mu\alpha} = 3H^2 - \frac{\left(1+\csc^2\theta\right)}{a^2r^2} + \frac{2k^2}{a^2}\ , \tag{5.6}$$

$$T^{\nu}T_{\nu} = -9H^2 + \frac{\cot^2\theta}{a^2r^2} - \frac{4k}{a^2}\ , \tag{5.7}$$

where we have defined the *Hubble parameter* to be $H := \dot{a}/a$ as defined in [5]. In addition, we also define the *Hubble constant* to be $H_0 := H(z=0)$ as well as the *dimensionless Hubble parameter* to be $h := H(z)/H_0$. It follows from the above scalar terms that the torsion scalar for the diagonal tetrad construction has the following expression

$$T = 6H^2 - \frac{2}{a^2r^2} + \frac{2k}{a^2}\ . \tag{5.8}$$

Having obtained an expression for the torsion scalar, we turn our attention to the $f(T)$ field equations in order to note any restrictions that might be placed on a given $f(T)$ theory. For the case of a vacuum, we note that the (r,θ) field equation yields

$$-\frac{f_{TT}\cot\theta\ T'}{2a^2r^2} = 0\ . \tag{5.9}$$

By avoiding the restriction $T' = 0$, one arrives at the restriction[1] that $f_{TT} = 0$ which is the case of TEGR with a cosmological constant. Therefore, following the terminology used in [37], we classify the tetrad given in equation (5.3) as a "bad tetrad". We note that the restriction $f_{TT} = 0$ is imposed regardless of the value for k. In [37], after noting that the diagonal tetrad is a bad tetrad, the authors move on to discuss tetrads that are obtained through the application of rotation matrices on the diagonal tetrad. In this thesis, however, we will only be interested in the consideration of the spatially flat Robertson-Walker space-time in Euclidean geometry. That is, we wish to consider a space-time whose metric tensor is described by the line element [5, 107]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)\ . \tag{5.10}$$

Given this metric tensor, one can construct the diagonal tetrad [37]

$$e^{a}{}_{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t) & 0 & 0 \\ 0 & 0 & a(t) & 0 \\ 0 & 0 & 0 & a(t) \end{pmatrix}\ . \tag{5.11}$$

Following what is done in [37,83], we have the following non-vanishing components of the Weitzenböck connection coefficients associated with the above tetrad construction

$$\tilde{\Gamma}^{1}{}_{01} = \tilde{\Gamma}^{2}{}_{02} = \tilde{\Gamma}^{3}{}_{03} = H\ , \tag{5.12}$$

where $H := \dot{a}/a$ is the Hubble parameter as before. By making use of the above non-vanishing connection coefficient components, one has the following expressions for the scalars that depend quadratically on the torsion tensor

$$T^{\nu}{}_{\mu\alpha}T_{\nu}{}^{\mu\alpha} = -6H^2\ , \qquad T^{\mu\alpha}{}_{\nu}T^{\nu}{}_{\mu\alpha} = 3H^2\ , \qquad T_{\nu}T^{\nu} = -9H^2\ . \tag{5.13}$$

It therefore follows that the torsion scalar has the following expression

$$T = 6H^2\ . \tag{5.14}$$

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[1] It is important to emphasize that, in this calculation, we have used $g_{tt} = -1$. We note that, for an arbitrary g_{tt} component, this restriction is not necessarily imposed.

It is important to note that equation (5.8) does not reduce to equation (5.14) in the case of flat spatial geometry, i.e., $k = 0$. The reason for this is as follows. In order to obtain the tetrad construction given in equation (5.11) from equation (5.3), one must first perform a coordinate transformation which results in an off-diagonal tetrad construction. One can then apply Lorentz transformations in order to obtain equation (5.11) for the case of a flat spatial geometry. The fact that the torsion scalar will not remain unchanged is due to the fact that it is not necessarily invariant under Lorentz transformations.

Although the torsion scalar is not necessarily invariant under Lorentz transformations, since it is constructed from the vierbein, the Ricci scalar, which is constructed from the metric tensor, is Lorentz invariant. Therefore, whether calculated using equation (5.14) or (5.8), the expression $-T - 2D_\nu T^\nu$ yields the Lorentz invariant Ricci scalar

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \ , \tag{5.15}$$

as per equation (3.69) [5, 83].

We now turn our attention to studying the $f(T)$ field equations which are given in [37]. In order to derive these field equations, we first wish to specify the type of enery matter content that we wish to consider. In this thesis, we are interested in making use of an energy matter content that is of the general perfect fluid form [5, 107]

$$\tau_{\mu\nu} = \rho\chi_\mu\chi_\nu + P\left(g_{\mu\nu} + \chi_\mu\chi_\nu\right) \ , \tag{5.16}$$

where χ^ν is the four-velocity associated with a comoving observer, i.e., $\chi^\mu = (1, 0, 0, 0)$. Now, from the general expression for the $f(T)$ field equations given in equation (4.138), we note that the $(0, 0)$ field equation is [37]

$$\frac{1}{4}f - 3H^2 f_T = -4\pi\rho \ . \tag{5.17}$$

In addition, by subtracting the above expression from the $(1, 1)$ $f(T)$ field equation, one obtains the following [37]

$$\frac{1}{4}f - 3H^2 f_T - \dot{H}\left(f_T + 12H^2 f_{TT}\right) = 4\pi P \tag{5.18}$$

Let us consider what happens when we consider the case of TEGR with a cosmological constant in equations (5.17) and (5.18). For the case of TEGR with a cosmological constant, we have $f(T) = T + 6H_0^2\,\sigma$ where σ is a constant. By substituting this $f(T)$ function into equation (5.17), one obtains the following expression

$$H^2 = \frac{8\pi\rho}{3} + H_0^2\,\sigma \ . \tag{5.19}$$

The above equation is referred to as the *Friedmann equation* in the case of spatially flat geometry [108]. Now, for the case of TEGR with a cosmological constant, equations (5.18) becomes

$$2\frac{\ddot{a}}{a} = -H^2 + 3H_0^2\,\sigma - 8\pi P \ . \tag{5.20}$$

For the consideration of spatially flat geometry, the above equation is referred to as the *acceleration equation* [108]. Here, we wish to refer to equations (5.17) and (5.18) as the *modified Friedmann equation* and the *modified acceleration equation* respectively. Let us turn our attention to the Friedmann equation (5.19) which holds in the case of TEGR with a cosmological constant. We can divide this equation by H_0^2 and evaluate the resulting expression at the present day value of $z = 0$ in order to obtain [108, 109]

$$\Omega_{m0} + \sigma = 1 \ . \tag{5.21}$$

In this expression, we have taken the present day value of the dimensionless Hubble parameter to be $h(z=0)=1$ and, in addition, we have defined the so-called *energy density parameter* Ω_{m0} as follows [107–109]

$$\Omega_{m0} := \frac{8\pi\rho_0}{3H_0^2} \ . \tag{5.22}$$

Equation (5.21) is often referred to as the *closure relation* for the case of TEGR.

In this section, we have discussed the spatially-flat Robertson-Walker space-time in the context of $f(T)$ theories of gravity. The general $f(T)$ field equations are stated in equations (5.17) and (5.18). We also discussed the ΛCDM model as an $f(T)$ theory. For a further discussion on the cosmological constant, Dark Energy and the ΛCDM model, the reader is directed to [110] by Zeldovich as well as the review article [111]. In this thesis, we will make use of the $f(T)$ field equations mentioned in this section in order to obtain the $f(T)$ focusing conditions in the following section.

5.3 Focusing conditions for $f(T)$ theories

Previously, in Chapter 3, we have already noted that for a theory constructed on the Riemann-Cartan space time U_4, test particles follow the auto-parallels of the Levi-Civita connection. Therefore, for $f(T)$ theories of gravity, which are constructed on the Weitzenböck space-time W_4, test particles follow auto-parallels of the Levi-Civita connection. We have already pointed out, following [8, 35, 74], that these paths are not geodesics since the differential equations describing extremal curves do not necessarily coincide with the differential equations describing the auto-parallels of the Weitzenböck connection which is the natural choice of an affine connection in $\mathcal{W}_4$. In torsion-free theories of gravity such as GR one refers to equation (3.135), which describes the paths followed by test particles, as the geodesic equation. In theories of gravity constructed on W_4, one refers to this equation as the force equation [8]. The fact that test particles follow auto-parallels of the Levi-Civita connection is of great importance when deriving the conditions that arise as a result of imposing that gravity be attractive. In order to obtain such conditions, we turn our attention to the consideration of a smooth congruence of curves which are followed by test particles. More specifically, we wish to consider a smooth congruence of timelike auto-parallel curves of the Levi-Civita connection.

The equation that describes the rate of expansion of a cross-sectional area enclosing this congruence is nothing more than the torsion-free Raychaudhuri equation (3.165). Since it is a cross-sectional area enclosing a congruence of auto-parallel curves of the Levi-Civita connection that gives us an interpration of the affect of gravity on test particles, the requirement that gravity be attractive is described by the inequality

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} \leq 0 \ . \tag{5.23}$$

In order to study the satisfaction or violation of this inequality, one studies the right-hand side of equation (3.165). Clearly the first two terms on the right-hand side of equation (3.165) are not positive. We therefore wish to study the requirement that the third term is not positive. That is, we wish to study the requirement that

$$R_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \ . \tag{5.24}$$

We remind the reader that here $\boldsymbol{\xi}$ is the tangent vector field of a curve contained in the aforementioned congruence. Through the consideration of the above inequality, one can obtain the so-called focusing conditions by introducing the energy matter content into this inequality through the use of the $f(T)$ field equations. In order to study how the Ricci tensor contracted with the four-velocity can be non-negative, we turn our attention to the Einstein tensor defined as follows [5]

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \ . \tag{5.25}$$

By relating the Ricci tensor and the Ricci scalar to the torsion tensor through the use of equations (3.68) and (3.69), one can write the Einstein tensor as [83]

$$G_{\mu\nu} = D_\nu T_\mu - D_\mu T_\nu + 2D_\beta S_{\nu\mu}{}^{\beta} - D_\beta T^{\beta}{}_{\mu\nu} + \frac{1}{2} T g_{\mu\nu} + K^{\alpha}{}_{\beta\nu} K^{\beta}{}_{\mu\alpha} + K^{\alpha}{}_{\mu\nu} T_\alpha \ . \tag{5.26}$$

The above expression is a completely geometric statement in the context of the Weitzenböck space-time W_4 and is therefore theory independent. It is at this point, however, that we wish to introduce the energy matter content into the above expression for the Einstein tensor. This is done by substituting the general expression for the $f(T)$ field equations given in equation (4.138) into the above expression. This yields the following expression for the Einstein tensor which is specific to $f(T)$ theories of gravity [83]

$$\begin{aligned} G_{\mu\nu} = {} & \frac{1}{f_T}\left(k^2 \tau_{\mu\nu} - 2 f_{TT} S_{\nu\mu}{}^{\beta} \partial_\beta T - \frac{1}{2} f g_{\mu\nu}\right) + W^{\beta}{}_{\lambda\nu} S_{\beta\mu}{}^{\lambda} + D_\nu T_\mu \\ & - D_\mu T_\nu - D_\lambda T^{\lambda}{}_{\mu\nu} + \frac{1}{2} T g_{\mu\nu} + K^{\alpha}{}_{\beta\nu} K^{\beta}{}_{\mu\alpha} + K^{\alpha}{}_{\mu\nu} T_\alpha \ . \end{aligned} \tag{5.27}$$

Taking the trace of the above equation yields the following

$$G = \frac{1}{f_T}\left(k^2 \tau - 2 f_{TT} T^{\nu} \partial_\nu T - 2f\right) + 2W^{\beta}{}_{\nu\alpha} S_{\beta}{}^{\alpha\nu} + 2T + K^{\alpha}{}_{\nu\beta} K^{\nu\beta}{}_{\alpha} + T^{\alpha} T_\alpha \ . \tag{5.28}$$

We now turn our attention to studying the conditions that led to the satisfaction of inequality (5.24). We note that the following condition allows for equation (5.24) to be satisfied [5]

$$G_{\mu\nu} \xi^\mu \xi^\nu \geq 0 \ . \tag{5.29}$$

The above condition is referred to here as the *weak focusing condition*. Alternatively, the following inequality allows for equation (5.24) to be satisfied

$$G_{\mu\nu} \xi^\mu \xi^\nu \geq -\frac{1}{2} G \ , \tag{5.30}$$

where G is the determinant of the Einstein tensor [5]. The above expression is referred to as the *strong focusing condition*. We have now stated the weak and strong focusing conditions. For the case of $f(T)$ theories of gravity, the authors of [64] derived the focusing conditions given a fundamental congruence, i.e., the tangent vector field $\boldsymbol{\xi}$ describes comoving observers. In this thesis, we derive more general focusing conditions for $f(T)$ theories of gravity by taking the tangent vector field $\boldsymbol{\xi}$ to describe a one-parameter dependent congruence of timelike auto-parallel curves of the Levi-Civita connection. In the following section, we summarise the results that are obtained by the authors of [64]. These results are specific to comoving observers. Following this discussion, we derive the focusing conditions given a one-parameter dependent congruence of time-like auto-parallel curves of the Levi-Civita connection. We will then move on to discuss the derived focusing conditions for namely three kinds of $f(T)$ cosmological models. These include cosmological models of a polynomial form, an exponential form and a hyperbolic tangent form.

5.3.1 Focusing conditions for a fundamental congruence

Let us consider the case where the tangent vector field of the curves contained in the congruence is that of a comoving observer, i.e., $\xi^\mu = (1, 0, 0, 0)$. By considering equation (5.27) for the case of a spatially-flat Robertson-Walker space-time as well as an energy momentum tensor that is of a general perfect fluid form, one can show that the weak focusing condition holds if and only if we have [64]

$$\frac{1}{f_T}\left(\rho + \frac{1}{16\pi}\left(f - f_T T\right)\right) \geq 0 \ . \tag{5.31}$$

In the case of TEGR, i.e., $f(T) = T$, the above inequality reduces to the weak energy condition $\rho \geq 0$. In addition, the strong focusing condition holds if and only if we have [64]

$$\frac{1}{4\pi f_T}\left(4\pi\left(\rho+3P\right)-\frac{1}{2}\left(f-f_T T\right)+3f_{TT}H\dot{T}\right)\geq 0\ . \tag{5.32}$$

We note that in the case of TEGR, the above inequality reduces to the inequality $\rho + 3P \geq 0$ which is nothing more than the strong energy condition as expected. These are the focusing conditions that were obtained by Liu and Reboucas in [64]. We note that in [64] the conditions that we have referred to here as focusing conditions are referred to as energy conditions. In this thesis, we reserve the term *energy conditions* for the focusing conditions in the case of TEGR which depends solely on the energy-matter content. That is, we reserve the terms weak and strong energy conditions specifically for the inequalities $\rho \geq 0$ and $\rho + 3P \geq 0$ respectively.

In the following section we derive for the first time the $f(T)$ weak and strong focusing conditions for the case of a one-parameter dependent congruence of timelike auto-parallel curves of the Levi-Civita connection. In the special case where the congruence is taken to be a fundamental congruence, the focusing conditions obtained in the following section reduce to the inequalities discussed in this section.

5.3.2 Focusing conditions for a one-parameter dependent congruence

In this section we derive for the first time the $f(T)$ weak and strong focusing conditions for a one-parameter dependent cogruence of timelike auto-parallel curves of the Levi-Civita connection. These results have given rise to the article [65] in which they are discussed. Let us begin by considering a smooth congruence of curves with tangent vector field $\boldsymbol{\xi}$. In this thesis, we are interested in studying the satisfaction/violation of the weak and strong focusing conditions for one-parameter dependent congruences of timelike auto-parallels of the Levi-Civita connection for which the tangent vector field is given by the following expression [112]

$$\xi^\alpha = \gamma\left(1,\beta,0,0\right)\ ,\ \text{with}\ \ \gamma := \frac{1}{\sqrt{1-a^2\beta^2}}\ . \tag{5.33}$$

In order for these curves to be auto-parallels of the Levi-Civita connection, we require that the tangent vector field be parallel propagated with respect to the Levi-Civita connection, i.e., $\xi^\alpha D_\alpha \xi^\nu = 0$. Such a requirement implies that the parameter β be of the form

$$\beta = \frac{C}{a\sqrt{a^2+C^2}}\ , \tag{5.34}$$

where $C \in \mathbb{R}$ is the parameter that, when varied, produces different tangent vector fields that are parallel propagated with respect to the Levi-Civita connection. We note that when $C = 0$ the four-velocity ξ^α given in equation (5.33) reduces to the four-velocity associated with a comoving observer, χ^α, used in the construction of the energy-momentum tensor given in equation (5.16). The weak and strong focusing conditions associated with the case of a fundamental congruence were obtained in [64] and are given in Section 5.3.1.

We now wish to make use of the four-velocity given in equation (5.33) in order to obtain the weak and strong focusing conditions for a one-parameter, C, dependent congruence of timelike auto-parallel curves of the Levi-Civita connection. Let us first obtain the weak focusing condition by making use of equation (5.29). The Einstein tensor for a general $f(T)$ theory is given in equation (5.27). By considering the case of a spatially-flat Robertson-Walker space-time, whose associated vierbein is given in equation (5.11), and contracting the Einstein tensor with the four-velocity given in equation (5.33), we find the $f(T)$ weak focusing condition to be[2]

$$8\pi\rho_{eff} := \frac{1}{f_T}\left[8\pi\rho+\frac{1}{2}\left(f-f_T T\right)+a^2\beta^2\left(2Hf_{TT}\partial_t T-\frac{1}{2}f\right)\right]+3a^2\beta^2H^2\geq 0\ , \tag{5.35}$$

---

[2] We note that, since we are making use of natural units here, i.e., $G = c = 1$, we have $k^2 = 8\pi$ which is what is used in the expressions that follow.

and we refer to ρ_{eff} as the *effective energy-density*. We now wish to turn our attention to the strong focusing condition which is given in equation (5.30). The trace of the Einstein tensor for a general $f(T)$ theory is given in equation (5.28). For the specific case of a spatiall-flat Robertson-Walker space-time, the trace of the Einstein tensor reads [83]

$$G = \frac{1}{f_T}\left(8\pi\tau + 6f_{TT}H\partial_t T - 2\left(f - f_T T\right)\right) \ . \tag{5.36}$$

We now find that, through the use of equation (5.30) the $f(T)$ strong focusing condition is as follows

$$\begin{aligned} 4\pi\left(\rho_{eff} + 3P_{eff}\right) := \frac{1}{f_T}\bigg[&4\pi\left(\rho + 3P\right) - \frac{1}{2}\left(f - f_T T\right) + 3f_{TT}H\partial_t T \\ &+ a^2\beta^2\Big(2Hf_{TT}\partial_t T - \frac{1}{2}f\Big)\bigg] + 3a^2\beta^2H^2 \geq 0 \ , \end{aligned} \tag{5.37}$$

where we have introduced the so-called *effective pressure*, P_{eff}. We have thus obtained the $f(T)$ weak and strong focusing conditions which are given in equations (5.35) and (5.37) respectively.

Consider now the case of a fundamental congruence. In such a case, we have $C = 0$ which implies, from equation (5.34), that we have $\beta = 0$. It is not difficult to see that in the case where $\beta = 0$, equations (5.35) and (5.37) reduce to equations (5.31) and (5.32) respectively. That is, the weak and strong focusing conditions given here for a one-parameter, C, dependent congruence of timelike auto-parallel curves reduce to the weak and strong focusing conditions given in [64] when a fundamental congruence ($C = 0$) is considered.

Let us now return our attention to the more general case of a congruence, i.e., we allow for a general value of the parameter C. We note that the one-parameter C appears in equations (5.35) and (5.37) through the parameter β via equation (5.34). Our goal now is to study these focusing conditions for three specific bi-parametric $f(T)$ theories of gravity. That is, we will consider three $f(T)$ theories whose $f(T)$ functions depend on two parameters. We will then study the satisfaction or violation of the focusing conditions in the parameter space.

5.3.3 Focusing conditions and cosmological models

In this section, we study the focusing conditions for three specific bi-parametric $f(T)$ cosmological models. Here, we will allow the $f(T)$ function describing a specific cosmological model to depend on two values: σ and α. By changing the values of α and σ in the $f(T)$ function, one obtains different $f(T)$ theories. For each choice of cosmological model considered, we first solve the field equation (5.18) for the dimensionless Hubble parameter defined as $h(z) := H(z)/H_0$ [107]. For this study, we take the Hubble parameter H_0 to be that of the ΛCDM solution as it is obtained in [1]. For a given cosmological model, we shall obtain a solution $h(z)$ for each combination of α and σ values. In this note, we are interested in studying the focusing conditions for the case of dust, i.e., the fluid in equations (5.17) and (5.18) would be pressureless. In such a case, the energy-momentum tensor then takes the form

$$\tau_{\mu\nu} = \rho\chi_\mu\chi_\nu \ , \tag{5.38}$$

where $\chi^\mu = (1, 0, 0, 0)$ is the four-velocity associated with a comoving observer. The conservation equation for the energy-momentum tensor gives the following

$$D_\mu {\tau^\mu}_0 = 0 \implies \dot{\rho} = -3H\rho \implies \rho = \rho_0 a^{-3} \ , \tag{5.39}$$

where ρ_0 is a constant. By writing the above expression in terms of the redshift, we obtain the following expression for the energy density [107]

$$\rho(z) = \rho_0\left(1 + z\right)^3 \ . \tag{5.40}$$

Let us now turn our attention to the field equation (5.18) in order to obtain the dimensionless Hubble parameter $h(z)$. We are particularly interested in studying the focusing conditions at the present day value of $z = 0$. In solving such a differential equation, we impose the initial condition: $h(1100) = h_{\Lambda}(1100)$, where

$$h_{\Lambda}(z) := \sqrt{{}^{\Lambda}\Omega_{m_0}\,(1+z)^3 + 1 - {}^{\Lambda}\Omega_{m_0}}\ , \tag{5.41}$$

is the ΛCDM solution [107]. In addition, for illustrative purposes we take the ΛCDM energy density parameter to be ${}^{\Lambda}\Omega_{m_0} = 0.315$ as found in [1]. Once (5.18) has been solved in order to obtain $h(z)$ values for a particular cosmological model, we can then make use of the first field equation (5.17) in order to find the energy density parameter values for Ω_{m_0} which is defined in equation (5.22). Since we expect to obtain different Ω_{m_0} values for different $f(T)$ functions, the parameter Ω_{m_0} will depend on the values of α and σ as described by equation (5.17).

In this thesis, we shall consider three paradigmatic $f(T)$ cosmological models with the first being of a polynomial form [113], the second being of an exponential form [114] and the third being of a hyperbolic tangent form [115]. In constructing these cosmological models, we make use of the dimensionless torsion scalar, $\bar{T}$, defined through the expression

$$\bar{T} := \frac{T}{6H_0^2}\ . \tag{5.42}$$

Before proceeding with the aforementioned $f(T)$ cosmological models, we shall impose two theoretical constraints on the models under consideration. These constraints have been discussed in the context of $f(R)$ theories of gravity in [116–118]. In this note, we make use of these $f(R)$ conditions[3], as given in [117], in order to impose similar conditions on $f(T)$ theories. The first condition we wish to impose on an $f(T)$ theory reads

$$\lim_{T\to\infty} \frac{f(T)}{T} = 1\ , \tag{5.43}$$

and ensures that the $f(T)$ theory of gravity behaves like the TEGR theory of gravity in the limit as $T \to \infty$, i.e., at early stages of the cosmological evolution. In addition, the second condition to be imposed reads [115]

$$f_T > 0\ . \tag{5.44}$$

As can be seen in equations (5.35) and (5.37), this condition ensures that the contribution of the energy matter content in the focusing conditions is always nonnegative, i.e., the *effective Newton's constant* $G_{eff} := G/f_T$ is always positive. By imposing this condition, one also ensures that the gravitons are not ghosts [117].

Finally, recent developments in the field of Gravitational Waves have placed strong constraints on the gravitational wave speed [119]. It is shown in [120] that such a constraint is trivially satisfied for $f(T)$ gravities. Consequently, the cosmological models discussed below are constructed in such a way that the conditions given in equations (5.43) and (5.44) are satisfied. In addition, in the limit where a cosmological model behaves as TEGR plus a cosmological constant, the parameter σ will take on the role of the cosmological constant.

Model 1: $f = 6H_0^2\left(\bar{T} + \sigma\bar{T}^{\alpha}\right)$

The first cosmological model that we wish to consider[4] is of the polynomial form presented in [113,121]. Herein we restrict our consideration of α and σ values to the region $(\sigma, \alpha) \in [0,1] \times [-1,1)$. We note that the conditions (5.43) and (5.44) for the combinations of α and σ values considered are satisfied.

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[3] In the context $f(R)$ theories of gravity, an additional condition states that f_{RR} cannot be negative in order to ensure a stable model against the so-called Dolgov-Kawasaki instability [117, 118]. Nonetheless, such a condition does not have a counterpart on the sign of f_{TT} .

[4] The results given below were obtained using the Mathematica notebook *Polynomial_model.nb* which was constructed by the author and is available at `https://www.dropbox.com/sh/cfybc2cysixchlz/AACjKq0NHComUPoXcZz90DH7a?dl=0`.

We also note that the conditions (5.43) and (5.44) are satisfied for larger values of σ, however, the analysis regarding these larger σ values can be understood from the results given here. The evolution equation (5.18) for the dimensionless Hubble parameter $h(z)$ becomes

$$\frac{\mathrm{d}h}{\mathrm{d}z} = \frac{3}{2}\,\frac{h^2 + \sigma\left(2\alpha - 1\right)h^{2\alpha}}{\left[h + \sigma\alpha\left(2\alpha - 1\right)h^{2\alpha-1}\right]\left(1+z\right)}\ . \tag{5.45}$$

Upon solving (5.45) for the dimensionless Hubble parameter, we turn our attention to studying the weak and strong focusing conditions for this cosmological model. In terms of $h(z)$, the weak focusing condition (5.35) now reads

$$\begin{aligned}\frac{8\pi\rho_{eff}}{3H_0^2} = \left(\frac{1}{1+\sigma\alpha h^{2\alpha-2}}\right)&\left[\Omega_{m_0}(1+z)^3 + \sigma\left(1-\alpha\right)h^{2\alpha}\right.\\ &\left.- \frac{\beta^2}{\left(1+z\right)^2}\left(\frac{4}{3}\sigma\alpha\left(\alpha-1\right)h^{2\alpha-1}h'\left(1+z\right) + h^2 + \sigma h^{2\alpha}\right)\right] + \frac{\beta^2 h^2}{\left(1+z\right)^2} \geq 0\ .\end{aligned} \tag{5.46}$$

where h' denotes the derivative of the dimensionless Hubble parameter with respect to the redshift. In addition, for this cosmological model, the strong focusing condition (5.37) yields

$$\begin{aligned}\frac{4\pi\left(\rho_{eff}+3P_{eff}\right)}{3H_0^2} = \left(\frac{1}{1+\sigma\alpha h^{2\alpha-2}}\right)&\left[\frac{1}{2}\Omega_{m_0}\left(1+z\right)^3 - 2\sigma\alpha\left(\alpha-1\right)h^{2\alpha-1}h'\left(1+z\right)\right.\\ &\left.- \sigma\left(1-\alpha\right)h^{2\alpha} - \frac{\beta^2}{\left(1+z\right)^2}\left(\frac{4}{3}\sigma\alpha\left(\alpha-1\right)h^{2\alpha-1}h'\left(1+z\right) + h^2 + \sigma h^{2\alpha}\right)\right]\\ &+ \frac{\beta^2h^2}{\left(1+z\right)^2} \geq 0\ .\end{aligned} \tag{5.47}$$

As already mentioned, Ω_{m_0} is dependent upon α and σ through (5.17). More specifically for this model, by evaluating (5.17) today, i.e., at $z = 0$, one gets

$$\Omega_{m_0} = h_0^2 + \sigma\left(2\alpha-1\right)h_0^{2\alpha}\ , \tag{5.48}$$

where $h_0 := h(z=0)$. We note that, since we have taken the Hubble parameter H_0 to be the value associated with the ΛCDM solution, we have $h_\Lambda(z=0) = 1$. The first panel in Figure 5.1 shows $\bar{\Omega}_{m0} := \Omega_{m0} - {}^{\Lambda}\Omega_{m0}$ against the cosmological parameters α and σ. In the second and third panels of this figure, we give the 3D plots of equation (5.46) and equation (5.47) respectively for $C = 10^4$ against the parameters α and σ. In Figure 5.2 we show, in the first panel, the dimensionless Hubble parameter at the present day value h_0 for various combinations of α and σ values. In addition, in the second panel we show the 3D plot of f_T. It is clear from this plot that the condition (5.44) is satisfied. For the sake of completeness, we give the 3D plot of f_{TT} in the third panel of Figure 5.2.

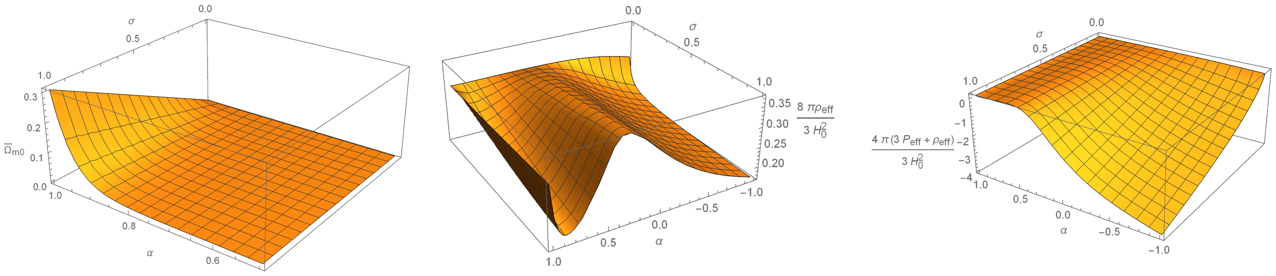

Figure 5.1: 3D plots associated with the polynomial cosmological model $f = 6H_0^2\left(\bar{T} + \sigma\bar{T}^{\alpha}\right)$. In the first panel, we show the 3D plot of $\bar{\Omega}_{m0} := \Omega_{m0} - {}^{\Lambda}\Omega_{m0}$ which is defined as the difference between the energy density parameter Ω_{m0}, given in equation (5.48), and the ΛCDM energy density parameter ${}^{\Lambda}\Omega_{m0}$ which we set to 0.315 [1]. In the second panel, we give the 3D plot of equation (5.46) against the cosmological parameters α and σ. In the third panel, we show the 3D plot of equation (5.47). In the second and third panels, we have set $C = 10^4$.

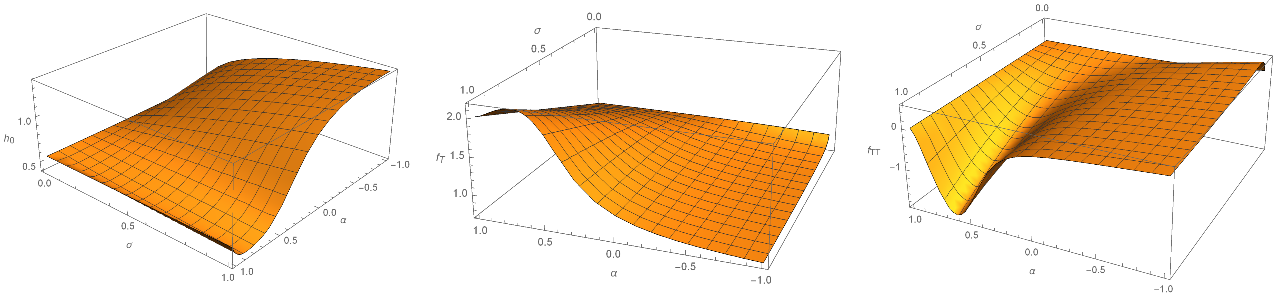

Figure 5.2: 3D plots associated with the polynomial cosmological model $f = 6H_0^2\left(\bar{T} + \sigma\bar{T}^{\alpha}\right)$. In the first panel, we show the 3D plot of the dimensionless Hubble parameter at the present day value h_0 for combinations of α and σ values. In the second panel, we give the 3D plot of the derivative f_T. From this plot, it is clear that the condition (5.44) is satisfied. In the third panel, we show the 3D plot of the second derivative f_{TT}.

Let us now consider the $\{\alpha, \sigma\}$ regions for which the weak and strong focusing conditions, as provided in (5.46) and (5.47) respectively, are satisfied or violated. Figure 5.3 contains region plots showing the satisfaction or violation of the weak and strong focusing conditions in the aforementioned $\{\alpha, \sigma\}$ parameter space at $z = 0$. The blue regions indicate where both the weak and strong focusing conditions are satisfied, whereas the orange regions show where only the weak focusing condition is satisfied. The first panel shows the region plot for the case of a fundamental congruence, i.e., when $C = 0$. The second panel shows the region plot for the case where $C = 1$ and the third panel shows the region plot associated with the case where $C = 10^4$. It is evident from Figure 5.3 that there exist $\{\alpha, \sigma\}$ combinations allowing for the strong focusing condition to be violated. In addition, Figure 5.3 illustrates how the satisfaction of the strong focusing condition changes as the value for C changes, i.e., when considering different four-velocities of the form given in equation (5.33). As the value for C is increased, the $\{\alpha, \sigma\}$ parameter space area violating the strong focusing condition increases. For all cases considered, the weak focusing condition remains satisfied. In order to illustrate where in these contour plots the ΛCDM solution appears, we have plotted it as a single point. Figure 5.4 shows the profile plots of equation (5.46) for constant σ values . The first, second and third panels show such profile plots for the cases $C = 0$, $C = 1$ and $C = 10^4$ respectively. Analogously, the constant σ profile plots of equation (5.47) are shown in Figure 5.5. For the latter scenario, as the value of C increases, the limiting value of α at which the strong focusing condition is violated gets higher.

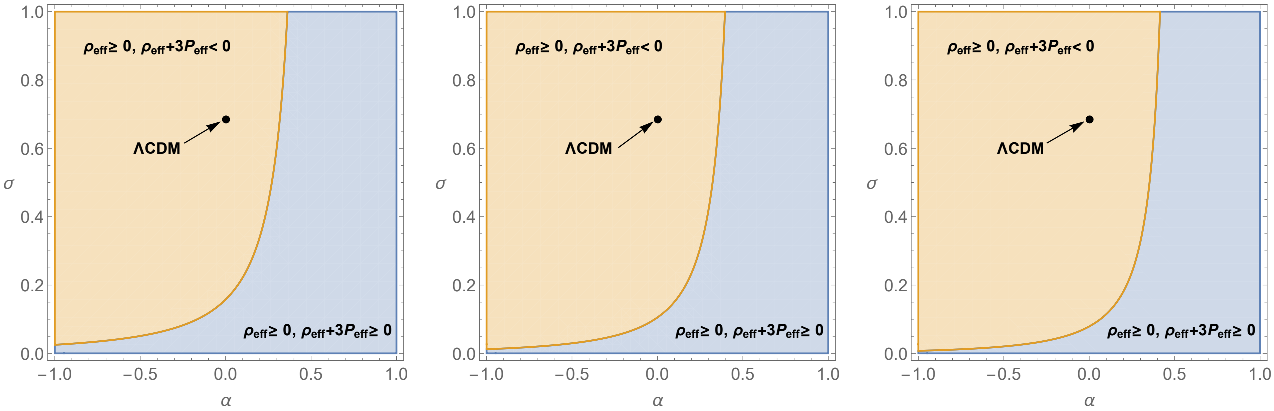

Figure 5.3: Region plots showing the satisfaction or violation of the weak and strong focusing conditions for the polynomial cosmological model $f = 6H_0^2\left(\bar{T} + \sigma\bar{T}^{\alpha}\right)$. The blue regions indicate where both the weak and the strong focusing conditions are satisfied whereas the orange regions indicate where only the weak focusing condition is satisfied. The left, middle and right panels show the region plots for the cases where $C = 0$, $C = 1$ and $C = 10^4$ respectively. We note that the weak focusing condition is satisfied for all values of C considered. In addition, we note that when the value of C is increased, the parameter space area violating the strong focusing condition is bigger. In these plots, we have indicated where the ΛCDM solution appears by plotting it as a point.

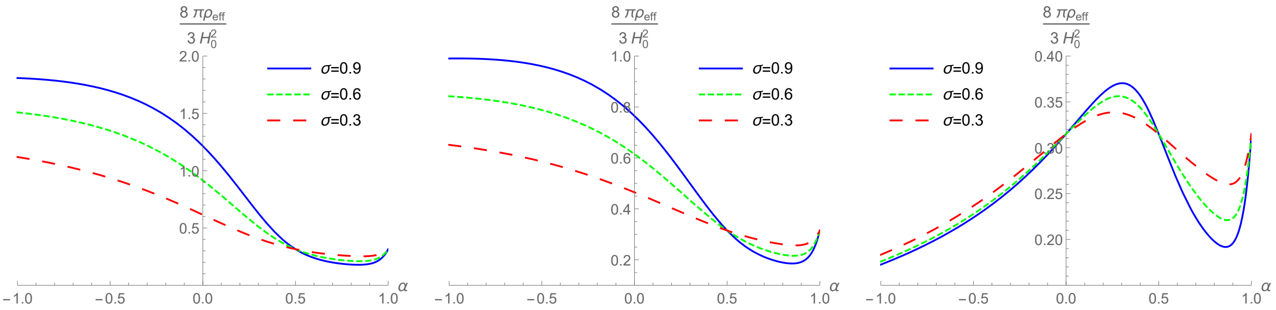

Figure 5.4: σ-constant profile plots of equation (5.46) for the polynomial cosmological model $f = 6H_0^2\left(\bar{T} + \sigma\bar{T}^{\alpha}\right)$. The solid blue curves show the $\sigma = 0.9$ profiles, the dotted green curves show the $\sigma = 0.6$ profiles and the dashed red curves show the $\sigma = 0.3$ profiles. The left, middle and right panels correspond to $C = 0$, $C = 1$ and $C = 10^4$ respectively.

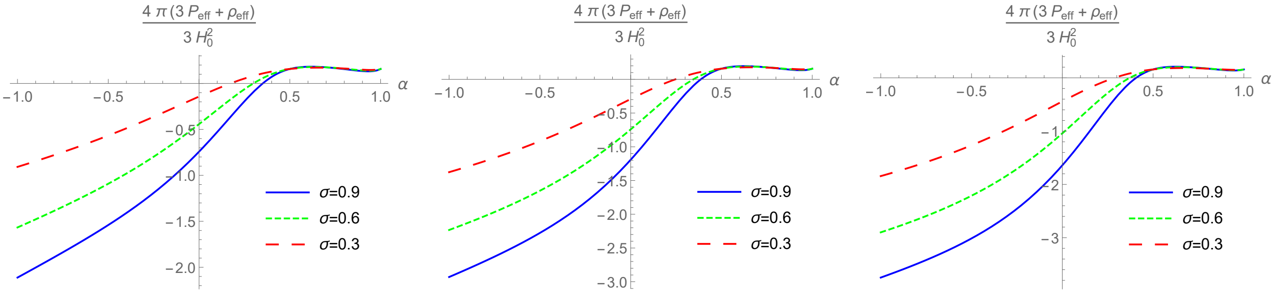

Figure 5.5: σ-constant profiles of equation (5.47) for the polynomial cosmological model $f = 6H_0^2\left(\bar{T} + \sigma\bar{T}^{\alpha}\right)$. The solid blue curves show the $\sigma = 0.9$ profiles, the dotted green curves show the $\sigma = 0.6$ profiles and the dashed red curves show the $\sigma = 0.3$ profiles. The left, middle and right panels correspond to $C = 0$, $C = 1$ and $C = 10^4$ respectively.

Model 2: $f = 6H_0^2\left(\bar{T} + \sigma\, \mathrm{e}^{\alpha\bar{T}}\right)$

The second cosmological model considered[5] is of the exponential form given above [114]. Herein, we shall restrict our consideration to σ and α values such that $(\sigma, \alpha) \in [0, \mathrm{e}] \times [-1, 0]$. We note that the

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[5] The results given below were obtained using the Mathematica notebook *Exponential_model.nb* which was constructed by the author and is available at `https://www.dropbox.com/sh/cfybc2cysixchlz/`

conditions (5.43) and (5.44) are satisfied in this region. For the exponential cosmological model given above, equation (5.18) for the case of vanishing pressure reads

$$\frac{\mathrm{d}h}{\mathrm{d}z} = \frac{3}{2} \frac{h^2 + \sigma \mathrm{e}^{\alpha h^2} \left(2\alpha h^2 - 1\right)}{\left(h + \sigma \alpha h \ \mathrm{e}^{\alpha h^2} + 2\sigma \alpha^2 h^3 \mathrm{e}^{\alpha h^2}\right)(1+z)} \ . \tag{5.49}$$

Now, from equation (5.35), the weak focusing condition for the exponential cosmological model reads

$$\begin{aligned} \frac{8\pi \rho_{eff}}{3H_0^2} = & \frac{1}{\left(1 + \sigma \alpha \ \mathrm{e}^{\alpha h^2}\right)} \Bigg[\Omega_{m_0} (1+z)^3 + \sigma \mathrm{e}^{\alpha h^2} \left(1 - \alpha h^2\right) \\ & - \frac{\beta^2}{(1+z)^2} \left(\frac{4}{3} h^3 (1+z) h' \sigma \alpha^2 \mathrm{e}^{\alpha h^2} + h^2 + \sigma \mathrm{e}^{\alpha h^2} \right) \Bigg] + \frac{\beta^2 h^2}{(1+z)^2} \geq 0 \ , \end{aligned} \tag{5.50}$$

where we obtain the values for $h(z)$ by solving the differential equation (5.49). In addition, from equation (5.37), we find that the strong focusing condition for the exponential cosmological model reads

$$\begin{aligned} \frac{4\pi \left(\rho_{eff} + 3P_{eff}\right)}{3H_0^2} = & \frac{1}{\left(1 + \sigma \alpha \ \mathrm{e}^{\alpha h^2}\right)} \Bigg[\frac{1}{2} \Omega_{m_0} (1+z)^3 - \sigma \mathrm{e}^{\alpha h^2} (1 - \alpha h^2) - 2(1+z) h^3 h' \sigma \alpha^2 \mathrm{e}^{\alpha h^2} \\ & - \frac{\beta^2}{(1+z)^2} \left(\frac{4}{3} h^3 (1+z) h' \sigma \alpha^2 \mathrm{e}^{\alpha h^2} + h^2 + \sigma \mathrm{e}^{\alpha h^2} \right) \Bigg] + \frac{\beta^2 h^2}{(1+z)^2} \geq 0 \ . \end{aligned} \tag{5.51}$$

As done for the case of the polynomial model, the energy density parameter Ω_{m_0} depends on the values of σ and α. More specifically, by evaluating (5.17) at $h(z=0) = h_0$, one can obtain

$$\Omega_{m_0} = h_0^2 + \sigma \mathrm{e}^{\alpha h_0^2} \left(2\alpha h_0^2 - 1\right) \ . \tag{5.52}$$

In Figure 5.6, we give the 3D plot of $\bar{\Omega}_{m0} := \Omega_{m0} - {}^{\Lambda}\Omega_{m0}$. In the second and third panels, we give the 3D plots of equations (5.50) and (5.51) respectively. In Figure 5.7, we plot h_0 in the first panel, f_T in the second panel and f_{TT} in the third panel.

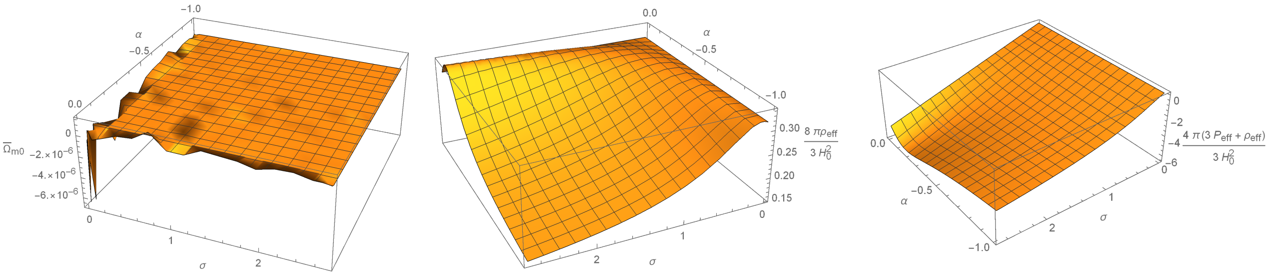

Figure 5.6: 3D plots associated with the exponential cosmological model $f = 6H_0^2\left(\bar{T} + \sigma \mathrm{e}^{\alpha \bar{T}}\right)$. In the first panel, we show the 3D plot of $\bar{\Omega}_{m0} := \Omega_{m0} - {}^{\Lambda}\Omega_{m0}$ which is defined as the difference between the energy density parameter Ω_{m0}, given in equation (5.52), and the ΛCDM energy density parameter ${}^{\Lambda}\Omega_{m0}$ which we set to 0.315 [1]. In the second panel, we give the 3D plot of equation (5.50) against the cosmological parameters α and σ. In the third panel, we show the 3D plot of equation (5.51). In the second and third panels, we have set $C = 10^4$.

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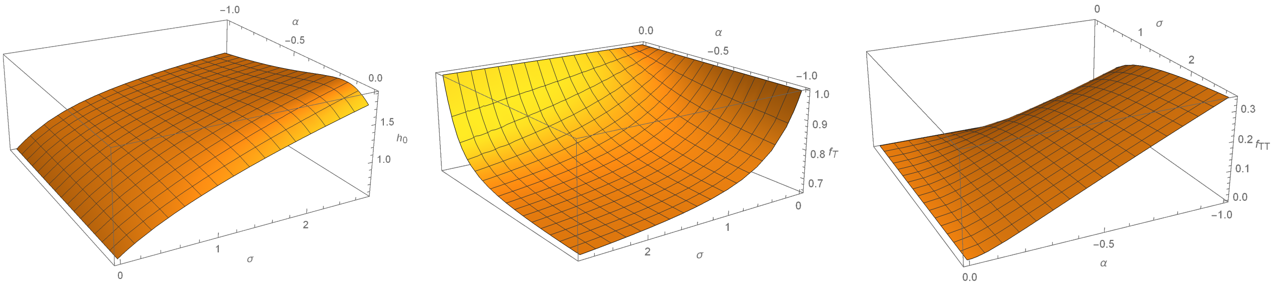

Figure 5.7: 3D plots associated with the exponential cosmological model $f = 6H_0^2\left(\bar{T} + \sigma e^{\alpha\bar{T}}\right)$. In the first panel, we show the 3D plot of the dimensionless Hubble parameter at the present day value h_0 for combinations of α and σ values. In the second panel, we give the 3D plot of the derivative f_T. From this plot, it is clear that the condition (5.44) is satisfied. In the third panel, we show the 3D plot of the second derivative f_{TT}.

As is done for the first model above, the satisfaction or violation of the weak and strong focusing conditions for this model in the $\{\alpha,\, \sigma\}$ parameter space can be studied using (5.50) and (5.51) respectively. Thus, Figure 5.8 shows the region plots of the satisfaction or violation of the weak and strong focusing conditions. The blue regions indicate where both the weak and strong focusing conditions are satisfied. The orange regions indicate where only the strong focusing condition is violated. There, the first, second and third panels are produced by considering the cases where $C = 0$, $C = 1$ and $C = 10^4$ respectively. As observed in the polynomial model above, when the value of C is increased, the $\{\alpha,\, \sigma\}$ parameter space where the strong focusing condition is violated increases. Also, for this cosmological model, the weak focusing condition remains satisfied. Figures 5.9 and 5.10 show constant σ profiles of equations (5.50) and (5.51) respectively. A comment worthwhile to be made concerns the constant σ profiles in Figure 5.10. As the value of C increases (from left to right panels) and for a fixed σ, the left-hand side of equation (5.51) seems to take smaller values which could be eventually negative. Therefore the strong focusing condition may be violated.

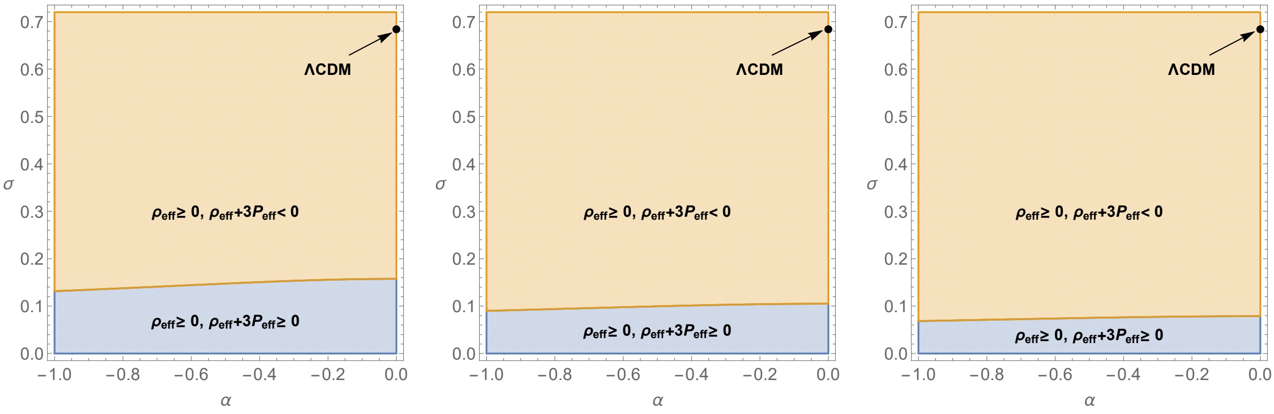

Figure 5.8: Region plots showing the satisfaction or violation of the weak and strong focusing conditions for the exponential cosmological model $f = 6H_0^2\left(\bar{T} + \sigma e^{\alpha\bar{T}}\right)$. The blue regions indicate where both the weak and the strong focusing conditions are satisfied whereas the orange regions indicate where only the weak focusing condition is satisfied. The left, middle and right panels show the region plots for the cases of $C = 0$, $C = 1$ and where $C = 10^4$ respectively. The location of the ΛCDM solution is plotted as a point. We note that the weak focusing condition is satisfied for all values of C considered. In addition, we note that when the value of C is increased, the parameter space area violating the strong focusing condition is bigger.

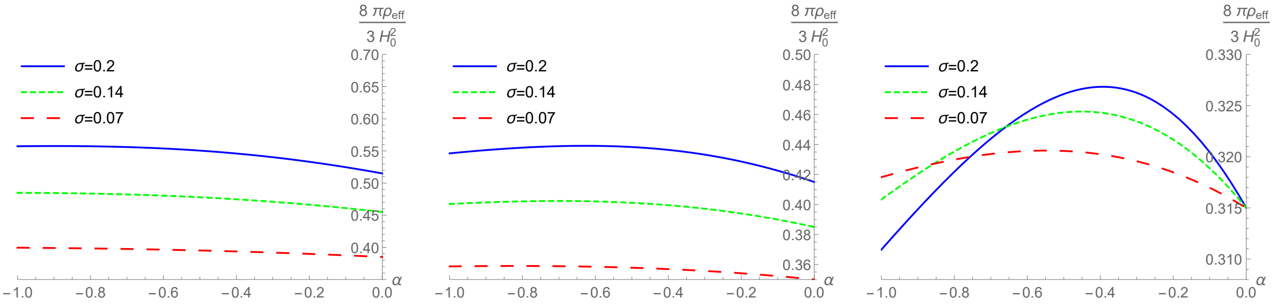

Figure 5.9: σ-constant profiles of equation (5.50) for the exponential cosmological model $f = 6H_0^2\left(\bar{T} + \sigma e^{\alpha\bar{T}}\right)$. The solid blue curves show the $\sigma = 0.2$ profiles, the dotted green curves show the $\sigma = 0.14$ profiles and the dashed red curves show the $\sigma = 0.07$ profiles. The left, middle and right panels correspond to $C = 0$, $C = 1$ and $C = 10^4$ respectively.

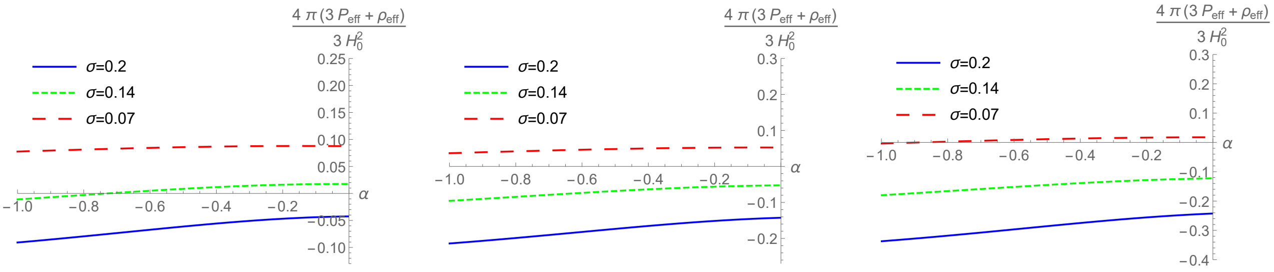

Figure 5.10: σ-constant profiles of equation (5.51) for the exponential cosmological model $f = 6H_0^2\left(\bar{T} + \sigma e^{\alpha\bar{T}}\right)$. The solid blue curves show the $\sigma = 0.2$ profiles, the dotted green curves show the $\sigma = 0.14$ profiles and the dashed red curves show the $\sigma = 0.07$ profiles. The left, middle and right panels correspond to $C = 0$, $C = 1$ and $C = 10^4$ respectively.

Model 3: $f = 6H_0^2\left(\bar{T} + \sigma\bar{T}^\alpha \tanh\left(1/\bar{T}\right)\right)$

The third and last cosmological model under consideration[6] is the hyperbolic tangent model whose $f(T)$ Lagrangian is given above [115]. We have considered the α and σ parameters lying in the $(\sigma, \alpha) \in [0, 1/2] \times [-1, 1]$ interval for which the $f(T)$ conditions (5.43) and (5.44) are satisfied. For this model, equation (5.18) for the case of vanishing pressure reads

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}z} = \frac{3}{2(1+z)}\left[h^2 + \sigma\left(2\alpha - 1\right)h^{2\alpha}\tanh\left(1/h^2\right) - 2\sigma h^{2\alpha-2}\,\mathrm{sech}^2\left(1/h^2\right)\right]\Big[\Big(h + \sigma\alpha\left(2\alpha - 1\right)\\ \times h^{2\alpha-1}\tanh\left(1/h^2\right) - \sigma\left(4\alpha - 3\right)h^{2\alpha-3}\,\mathrm{sech}^2\left(1/h^2\right) - 4\sigma h^{2\alpha-5}\,\mathrm{sech}^2\left(1/h^2\right)\tanh\left(1/h^2\right)\Big)\Big]^{-1}.\end{aligned} \tag{5.53}$$

As done for the previous two cosmological models, we solve the above differential equation and obtain values for $h(z)$ for the various combinations of α and σ values. From equation (5.35), we find that

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[6] The results given below were obtained using the Mathematica notebook *Hyperbolic_tangent_model.nb* which was constructed by the author and is available at `https://www.dropbox.com/sh/cfybc2cysixchlz/AACjKqONHComUPoXcZz90DH7a?dl=0`.

the weak focusing condition for this cosmological model in the case of dust reads

$$\begin{aligned}
&\frac{8\pi\rho_{eff}}{3H_0^2}=\Big[1+\sigma\alpha h^{2\alpha-2}\tanh\left(1/h^2\right)-\sigma h^{2\alpha-4}\operatorname{sech}^2\left(1/h^2\right)\Big]^{-1}\Big[\Omega_{m_0}\left(1+z\right)^3\\
&+\sigma\left(1-\alpha\right)h^{2\alpha}\tanh\left(1/h^2\right)-\sigma h^{2\alpha-2}\operatorname{sech}^2\left(1/h^2\right)-\frac{4h'\beta^2}{3\left(1+z\right)}\Big(\sigma\alpha\left(\alpha-1\right)h^{2\alpha-1}\\
&\times\tanh\left(1/h^2\right)-2\sigma\left(\alpha-1\right)h^{2\alpha-3}\operatorname{sech}^2\left(1/h^2\right)-2\sigma h^{2\alpha-5}\operatorname{sech}^2\left(1/h^2\right)\tanh\left(1/h^2\right)\Big)+h^2\\
&+\sigma h^{2\alpha}\tanh\left(1/h^2\right)\Big]+\frac{\beta^2h^2}{\left(1+z\right)^2}\geq 0\,. \qquad (5.54)
\end{aligned}$$

In addition, we find that, from equation (5.37), the strong focusing condition for the hyperbolic tangent cosmological model in the case of dust reads

$$\begin{aligned}
&\frac{4\pi\left(\rho_{eff}+3P_{eff}\right)}{3H_0^2}=\Big[1+\sigma\alpha h^{2\alpha-2}\tanh\left(1/h^2\right)-\sigma h^{2\alpha-4}\operatorname{sech}^2\left(1/h^2\right)\Big]^{-1}\Big[\frac{1}{2}\Omega_{m_0}\left(1+z\right)^3\\
&-\sigma\left(1-\alpha\right)h^{2\alpha}\tanh\left(1/h^2\right)+\sigma h^{2\alpha-2}\operatorname{sech}^2\left(1/h^2\right)-2\Big(\sigma\alpha\left(\alpha-1\right)h^{2\alpha-1}\tanh\left(1/h^2\right)\\
&-2\sigma\left(\alpha-1\right)h^{2\alpha-3}\operatorname{sech}^2\left(1/h^2\right)-2\sigma h^{2\alpha-5}\operatorname{sech}^2\left(1/h^2\right)\tanh\left(1/h^2\right)\Big)\Big)\left(1+z\right)h'\\
&-\frac{4h'\beta^2}{3\left(1+z\right)}\Big(\sigma\alpha\left(\alpha-1\right)h^{2\alpha-1}\tanh\left(1/h^2\right)-2\sigma\left(\alpha-1\right)h^{2\alpha-3}\operatorname{sech}^2\left(1/h^2\right)\\
&-2\sigma h^{2\alpha-5}\operatorname{sech}^2\left(1/h^2\right)\tanh\left(1/h^2\right)\Big)\Big)+h^2+\sigma h^{2\alpha}\tanh\left(1/h^2\right)\Big]+\frac{\beta^2h^2}{\left(1+z\right)^2}\geq 0\,. \qquad (5.55)
\end{aligned}$$

As done for the previous two cosmological models, the energy density parameter Ω_{m_0} depends upon the cosmological parameters α and σ through equation (5.17) at the present day value of $z=0$. In particular

$$\Omega_{m_0}=h_0^2+\sigma\left(2\alpha-1\right)h_0^{2\alpha}\tanh\left(1/h_0^2\right)-2\sigma h_0^{2\alpha-2}\operatorname{sech}^2\left(1/h_0^2\right)\,. \qquad (5.56)$$

In Figure 5.11, we give the 3D plot of $\bar{\Omega}_{m0}:=\Omega_{m0}-{}^{\Lambda}\Omega_{m0}$. In the second and third panels, we give the 3D plots of equations (5.54) and (5.55) respectively. In Figure 5.12, we plot h_0 in the first panel, f_T in the second panel and f_{TT} in the third panel.

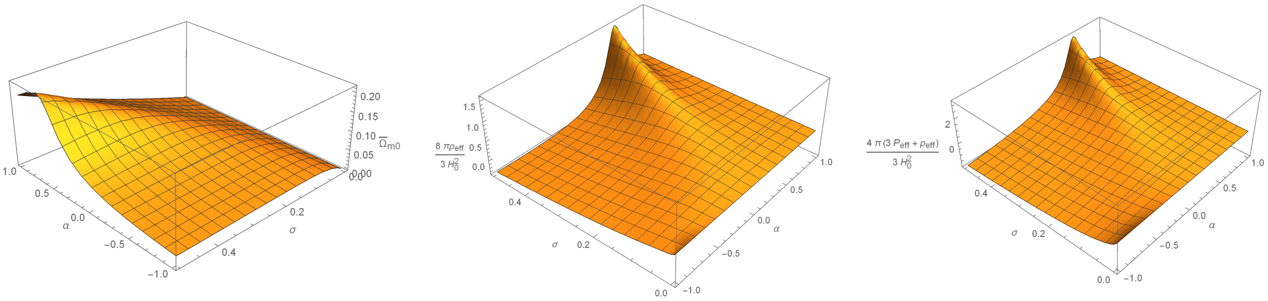

Figure 5.11: 3D plots associated with the hyperbolic tangent cosmological model $f=6H_0^2\left(\bar{T}+\sigma\bar{T}^{\alpha}\tanh\left(1/\bar{T}\right)\right)$. In the first panel, we show the 3D plot of $\bar{\Omega}_{m0}:=\Omega_{m0}-{}^{\Lambda}\Omega_{m0}$ which is defined as the difference between the energy density parameter Ω_{m0}, given in equation (5.56), and the ΛCDM energy density parameter ${}^{\Lambda}\Omega_{m0}$ which we set to 0.315 [1]. In the second panel, we give the 3D plot of equation (5.54) against the cosmological parameters α and σ. In the third panel, we show the 3D plot of equation (5.55). In the second and third panels, we have set $C=10^4$.

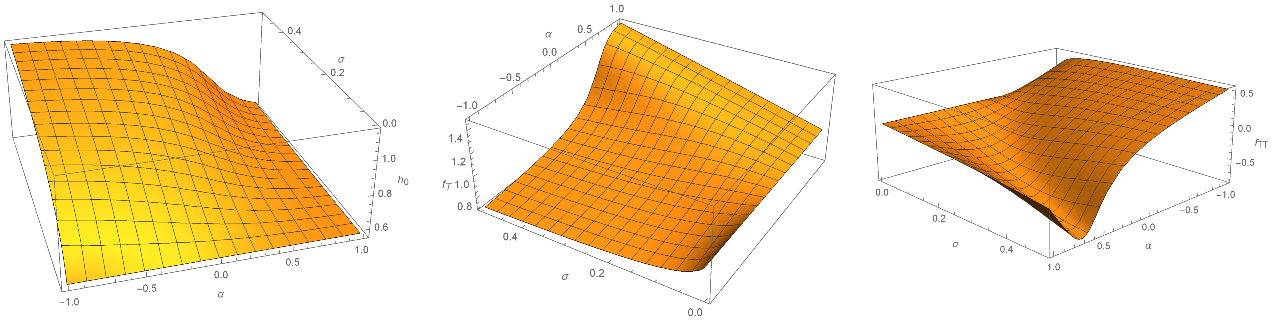

Figure 5.12: 3D plots associated with the hyperbolic tangent cosmological model $f = 6H_0^2\left(\bar{T} + \sigma\bar{T}^{\alpha}\tanh\left(1/\bar{T}\right)\right)$. In the first panel, we show the 3D plot of the dimensionless Hubble parameter at the present day value h_0 for combinations of α and σ values. In the second panel, we give the 3D plot of the derivative f_T. From this plot, it is clear that the condition (5.44) is satisfied. In the third panel, we show the 3D plot of the second derivative f_{TT}.

Figure 5.13 contains the region plots showing the satisfaction or violation of the weak and strong focusing conditions for the hyperbolic tangent cosmological model. The left, middle and right panels correspond to the choices $C = 0$, $C = 1$ and $C = 10^4$ respectively. As was noticed for the previous two cosmological models, the higher the value of C the bigger the area of the $\{\alpha,\, \sigma\}$ region for which the strong focusing condition is violated. For this cosmological model, unlike the two previous models, we found that as the value for C increases regions for which the weak focusing condition is violated arise (see right panel in Figure 5.13). Figure 5.14 shows the constant σ profiles of equation (5.54). In the right panel, which corresponds to $C = 10^4$, there are α and σ combinations resulting in negative values for the effective energy density, ρ_{eff}. This demonstrates that, although the weak focusing condition may be satisfied for the case of a fundamental congruence ($C = 0$), there may be values for C for which the weak focusing condition is not satisfied. Finally, Figure 5.15 shows constant σ profiles of equation (5.55). There we see that, as the value of C increases (from left to right panels) and for a fixed σ, equation (5.55) seems to take smaller values which could be eventually negative. Therefore the strong focusing condition may be violated.

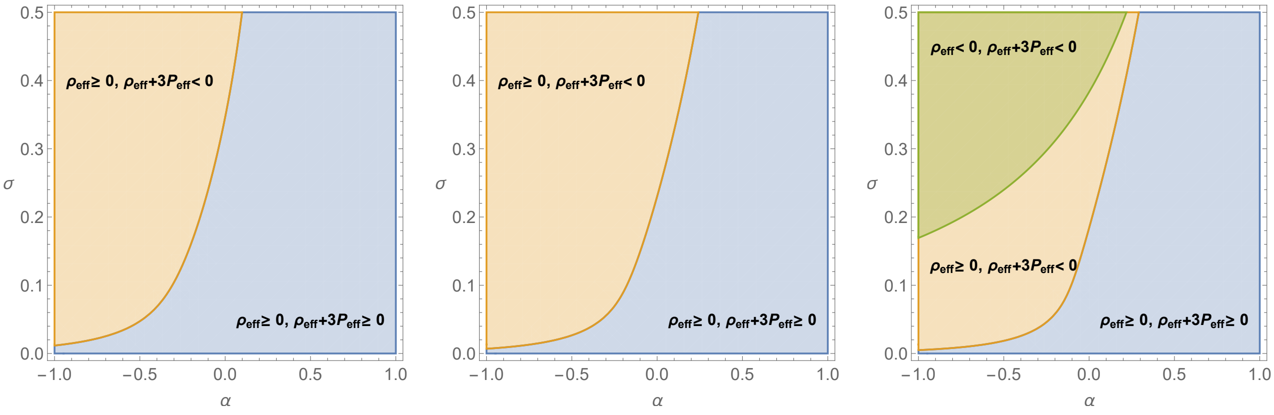

Figure 5.13: Region plots showing the satisfaction or violation of the weak and strong focusing conditions for the hyperbolic tangent cosmological model $f = 6H_0^2\left(\bar{T} + \sigma\bar{T}^{\alpha}\tanh\left(1/\bar{T}\right)\right)$. The blue regions indicate where both the weak and the strong focusing conditions are satisfied whereas the orange regions indicate where only the weak focusing condition is satisfied. The green region indicates where both the weak and the strong focusing conditions are violated. The left, middle and right panels show the region plots for the cases of $C = 0$, $C = 1$ and $C = 10^4$ respectively. We note that more regions for which the strong focusing condition is violated arise when the value for C is increased. In addition, we note that, while the weak focusing condition is satisfied for the case of a fundamental congruence, regions for which the weak focusing condition is violated arise by increasing the value of C.

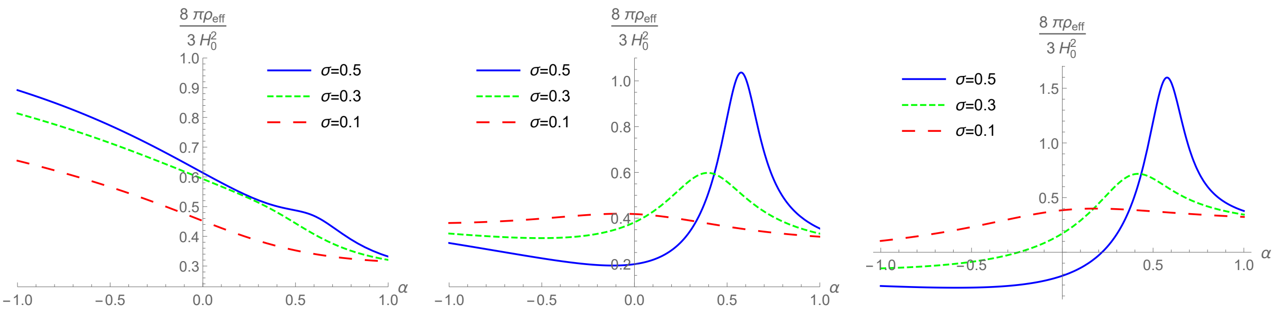

Figure 5.14: Constant σ profile plots of equation (5.54) for the hyperbolic tangent cosmological model $f = 6H_0^2\left(\bar{T} + \sigma\bar{T}^{\alpha}\tanh\left(1/\bar{T}\right)\right)$. The solid blue curves show the $\sigma = 0.5$ profiles, the dotted green curves show the $\sigma = 0.3$ profiles and the dashed red curves show the $\sigma = 0.1$ profiles. The left, middle and right panels correspond to $C = 0$, $C = 1$ and $C = 10^4$ respectively.

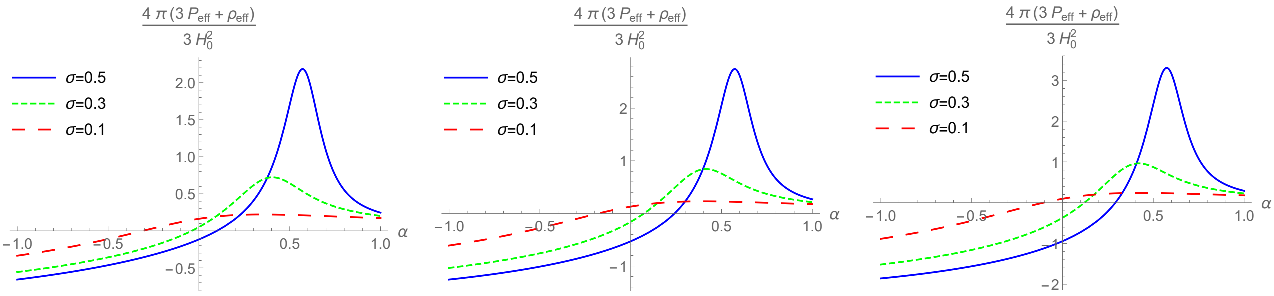

Figure 5.15: Constant σ profile plots of equation (5.55) for the hyperbolic tangent cosmological model $f = 6H_0^2\left(\bar{T} + \sigma\bar{T}^{\alpha}\tanh\left(1/\bar{T}\right)\right)$. The solid blue curves show the $\sigma = 0.5$ profiles, the dotted green curves show the $\sigma = 0.3$ profiles and the dashed red curves show the $\sigma = 0.1$ profiles. The left, middle and right panels correspond to $C = 0$, $C = 1$ and $C = 10^4$ respectively.

5.4 Chapter conclusions

In this chapter, we studied the weak and strong focusing conditions for a one-parameter dependent congruence of timelike auto-parallels of the Levi-Civita connection in the context of $f(T)$ theories. By noting that in the context of $f(T)$ theories of gravity, test particles follow auto-parallels of the Levi-Civita connection we made use of the torsion-free Raychadhuri equation in order to determine the inequalities to be satisfied so such conditions hold. Through the consideration of a spatially-flat Robertson-Walker space-time, we derived the weak and strong focusing conditions for a one-parameter, C, dependent congruence of timelike auto-parallels of the Levi-Civita connection. It was also shown that, in the case of a fundamental congruence, i.e., when $C = 0$, the obtained focusing conditions reduced to those previously found in [64] as expected. Once the general expressions were obtained, we also examined the obtained weak and strong focusing conditions for three viable bi-parametric $f(T)$ cosmological models when the only standard fluid present is dust matter. These cosmological models were of polynomial, exponential and hyperbolic tangent forms. Thus, results presented in this work are expected to cover a wide phenomenology of more convoluted $f(T)$ models. For each cosmological model under consideration, we determined the Hubble parameter evolution in terms of the redshift using the $f(T)$ second Friedman equation, whereas the first one was used to determine the matter density parameter today as a function of the $f(T)$ parameters. Once the Hubble parameter had been solved for, the weak and strong focusing conditions were studied for a redshift value of $z = 0$. For each cosmological model considered, region plots were produced for the cases of $C = 0$, $C = 1$ and $C = 10^4$. It was found that, for all three cosmological models, the increase in the value for C resulted in the parameter-space region where the strong focusing condition is violated to be bigger. It was also found that, for the polynomial and exponential cosmological models, the weak focusing condition remained satisfied when the value for C was increased. However, in the case of the hyperbolic tangent model it was found that a sufficient increase in the value for C eventually resulted in the appearance of parameter-space regions for which the weak focusing condition is violated. Consequently, for this model this result would imply that although the weak focusing condition for a fundamental congruence is satisfied, one cannot conclude that the weak focusing condition would be satisfied for all possible auto-parallel curves of the Levi-Civita connection, i.e., for all possible values of C.

As a final comment, we wish to make a note on the focusing conditions that arise in the so-called $f(R)$ and $f(Q)$ theories of gravity. Analyses of the focusing conditions in the context of $f(R)$ theories can be found in [122,123]. The authors of [122] reported $f(R)$ models for which regions exist where the weak focusing condition is violated in addition to regions that allow for the weak focusing condition to be satisfied. The authors of [123] reported $f(R)$ models for which the weak as well as the strong focusing conditions can be satisfied or violated. The authors of [124] studied the focusing conditions in the context of $f(Q)$ theories of gravity. In [124], the focusing conditions are studied using the present day values for the Hubble and deceleration parameters. There, $f(Q)$ models were reported where the weak as well as the strong focusing conditions could be satisfied or violated. In [64], $f(T)$ models where the weak focusing condition could be satisfied or violated were reported. There, the authors made use of the present day values for the Hubble and deceleration parameters. Here, the $f(T)$ field equations for each considered model were solved and the solution $h(z)$ was substituted into the focusing conditions. In addition to this, we considered a one-parameter depend congruence, i.e., a general value for C. For each of the models considered, regions existed where the weak and strong focusing conditions could be satisfied. For each of these models, regions for which the strong focusing condition could be violated were found. Lastly, we note that for a value of $C = 10^4$, regions exist for the hyperbolic tangent model where the weak focusing condition can be violated. These obtained results are specific to timelike auto-parallel curves of the Levi-Civita connection. For discussions on neighbouring null curves in $f(T)$ theories, the interested reader is directed to [64, 125].

Chapter 6

ADM and GBSSN formulations of Einstein-Cartan theory

6.1 Introduction

In this chapter, we consider the EC theory of gravity and derive the Arnowitt-Deser-Misner (ADM) formulation as well as the Generalised-Baumgarte-Shapiro-Shibata-Nakamura formulation of the EC theory. In Section 6.2, we study foliations of the Riemann-Cartan space-time U_4 and derive the non-null torsion Gauss, Codazzi and Ricci equations. In Section 6.3, we derive the ADM formulation of the EC theory of gravity by following what was done in [19,20] for the case of GR. In Section 6.4, we derive the GBSSN formulation of the EC theory by following what was done in [25,31] for the case of GR. Since the torsion tensor is geometrised by the spin-current tensor in the EC theory, we discuss the evolution of the decomposed effective energy-momentum tensor in 6.5 by following what was done in [30]. Finally, in Section 6.6, we consider the spherically symmetric vacuum GBSSN equations and evolve the system using the well-known Gamma-driver condition. We construct a 1-dimensional (1D) code to carry out this evolution using the fourth-order Runge-Kutta (RK) method. Since such a case has a vanishing energy-matter content, this system is the same as what occurs in GR. Similar systems have been studied before in [2,26,27,31]. Methods that were applied in these aforesaid references are made use of here when constructing our 1D code.

6.2 Foliations of space-time

In this section, we consider the Riemann-Cartan space-time U_4 and the construction of spacelike hypersurfaces which we shall denote as Σ. Here we shall derive the non-null torsion versions of the Gauss, Codazzi and Ricci equations. These derivations are carried out by following the torsion-free derivations that are done in [19,20] by Baumgarte and Shapiro. Once we have derived these equations, we will be able to study how the equations are modified as a result of removing the assumption of vanishing torsion. We note that the results presented in this section are completely geometric and are not specific to the EC theory of gravity. These geometric results, however, are still particular to the Riemann-Cartan space-time.

We take the hypersurfaces Σ to be spacelike surfaces of a function, say f, that is C^∞ on the Riemann-Cartan space-time[1] manifold $\mathcal{M}$. We define the dual vector field $\boldsymbol{\Omega}$ as

$$\Omega_\mu = \nabla_\mu f \ , \tag{6.1}$$

with norm $-\alpha^2$. We now define the vector field $\boldsymbol{n}$ as being such that

$$n^\mu := -\alpha \nabla^\mu f \ . \tag{6.2}$$

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[1] See Definition 3.3.1 for the definition used here for the Riemann-Cartan space-time U_4.

The above definition implies that $\boldsymbol{n}$ is both normalised as well as timelike, i.e., we have $g(\boldsymbol{n},\boldsymbol{n}) = -1$. In addition, we take $\boldsymbol{n}$ to be orthogonal to the hypersurface Σ. We define the *spatial metric*, $\boldsymbol{\gamma}$, through the expression

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \ . \tag{6.3}$$

Our goal now is to perform projections of four-dimensional tensor fields using the spatial metric in order to obtain three-dimensional tensor fields. These three-dimensional tensor fields "live" in the hypersurfaces of $\mathcal{M}$, i.e., we can lower and raise the indices of these tensors using the spatial metric. Let us begin by defining the three-dimensional (projected) covariant derivative. In this note, we are interested in carrying out such a projection for both the covariant derivative with respect to the Cartan connection as well as the covariant derivative with respect to the Levi-Civita connection. Let $\boldsymbol{B}$ be a tensor field of type (k,l) on the space-time manifold $\mathcal{M}$. We define the *three-dimensional Cartan covariant derivative* of the tensor field $\boldsymbol{B}$ as [19, 30]

$${}^3\nabla_\alpha B^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l} = \gamma_\alpha{}^\beta\gamma^{\mu_1}_{\sigma_1}\dots\gamma^{\mu_k}_{\sigma_k}\gamma^{\rho_1}_{\nu_1}\dots\gamma^{\rho_l}_{\nu_l}\nabla_\beta B^{\sigma_1\dots\sigma_k}{}_{\rho_1\dots\rho_l} \ . \tag{6.4}$$

By replacing ∇_μ with D_μ in the above expression, we obtain the definition for the *three-dimensional Levi-Civita covariant derivative* ${}^3D_\mu$. Let us now turn our attention to the vector field $\boldsymbol{n}$ which is used in the definition of the spatial metric and is taken to be orthogonal to the hypersurface Σ. Since we have chosen this normal vector field to be hypersurface orthogonal, $\boldsymbol{\eta}$ satisfies the following expression

$$D_{[\mu}n_{\nu]} = 0 \ . \tag{6.5}$$

We also note that $\boldsymbol{\eta}$ gives the direction of evolution of the proper time [19].

It is at this point that we wish to define the *extrinsic curvature tensor* as

$$K_{\mu\nu} := -\gamma_\mu{}^\alpha\gamma_\nu{}^\beta\nabla_\alpha n_\beta \ . \tag{6.6}$$

In [19], the extrinsic curvature tensor is symmetric, however, we note here that it is not immediately clear whether this extrinsic curvature tensor defined above is symmetric or not. It turns out that the above extrinsic curvature tensor is symmetric as a result of requiring that the projected torsion tensor lies completely in the hypersurface Σ. This will be made clear later on. For now, let us assume that the extrinsic curvature tensor is not necessarily symmetric. Let us now study the various tensor fields that exist within the hypersurface Σ. First, we wish to show that the three-dimensional Cartan covariant derivative, and therefore the three-dimensional Levi-Civita covariant derivative, is compatible with the spatial metric. We consider the following which results from the definition of the three-dimensional Cartan covariant derivative and the definition of the spatial metric:

$${}^3\nabla_\alpha\gamma_{\mu\nu} = \gamma_\alpha{}^\beta\gamma_\mu{}^\sigma\gamma_\nu{}^\rho\nabla_\beta g_{\sigma\rho} + \gamma_\alpha{}^\beta\gamma_\mu{}^\sigma\gamma_\nu{}^\rho\nabla_\beta\left(n_\sigma n_\rho\right) \ . \tag{6.7}$$

The first term on the right-hand side of the above expression vanishes since the Cartan covariant derivative is compatible with the metric tensor. The second term vanishes due to the fact that $n^\mu\gamma_\mu{}^\beta = 0$ as is easily verifiable from the definition of the spatial metric. It follows that the three-dimensional Cartan covariant derivative is compatible with the spatial metric, i.e., we have

$${}^3\nabla_\mu\gamma_{\alpha\beta} = 0 \ . \tag{6.8}$$

The above argument also holds for the three-dimensional Levi-Civita covariant derivative which we denote as ${}^3D_\mu$.

We now wish to study the three-dimensional torsion tensor associated with the three-dimensional Cartan covariant derivative ${}^3\nabla_\mu$. Let $\boldsymbol{w}$ be a vector field in Σ, i.e., we have $\langle\boldsymbol{n},\boldsymbol{w}\rangle = 0$, and consider the action of the three-dimensional Cartan covariant derivative on this dual vector field:

$${}^3\nabla_\mu w_\nu = \partial_\mu w_\nu - {}^3\tilde{\Gamma}^\rho_{\mu\nu}w_\rho \ . \tag{6.9}$$

By making use of the fact that the three-dimensional Cartan covariant derivative is compatible with the spatial metric, one can show that

$$ {}^3\tilde{\Gamma}^{\mu}{}_{\beta\alpha} = \frac{1}{2}\gamma^{\sigma\mu}\left(\partial_\alpha\gamma_{\beta\sigma} + \partial_\beta\gamma_{\sigma\alpha} - \partial_\sigma\gamma_{\alpha\beta} + {}^3T^{\lambda}{}_{\sigma\beta}\gamma_{\alpha\lambda} + {}^3T^{\lambda}{}_{\sigma\alpha}\gamma_{\beta\lambda} - {}^3T^{\lambda}{}_{\alpha\beta}\gamma_{\sigma\lambda}\right) \ , \tag{6.10} $$

where we have defined the *three-dimensional torsion tensor* as

$$ {}^3T^{\lambda}{}_{\sigma\alpha} := 2\ {}^3\tilde{\Gamma}^{\lambda}{}_{[\sigma\alpha]} \ . \tag{6.11} $$

We can write the three-dimensional Christoffel symbols, i.e., the connection coefficients associated with the three-dimensional Levi-Civita covariant derivative, as [19]

$$ {}^3\Gamma^{\mu}{}_{\beta\alpha} = \frac{1}{2}\gamma^{\sigma\mu}\left(\partial_\alpha\gamma_{\beta\sigma} + \partial_\beta\gamma_{\sigma\alpha} - \partial_\sigma\gamma_{\alpha\beta}\right) \ . \tag{6.12} $$

Furthermore, in a similar way to what is done for the case of space-time, we define the *three-dimensional contorsion tensor*[2] as the difference between the two three-dimensional connection coefficients:

$$ {}^3L^{\alpha}{}_{\mu\nu} := {}^3\tilde{\Gamma}^{\alpha}{}_{\mu\nu} - {}^3\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}\left({}^3T^{\alpha}{}_{\mu\nu} - {}^3T_{\mu\nu}{}^{\alpha} + {}^3T_{\nu}{}^{\alpha}{}_{\mu}\right) \ . \tag{6.13} $$

For a C^∞ function, say f, on the hypersurface Σ, we note that the application of the map ${}^3\nabla^3_{[\alpha}\nabla_{\beta]}$ on this function yields the following:

$$ 2\,{}^3\nabla_{[\alpha}{}^3\nabla_{\beta]}f = -{}^3T^{\lambda}{}_{\alpha\beta}{}^3\nabla_\lambda f \ . \tag{6.14} $$

On the other hand, by making use of the definition of the three-dimensional Cartan covariant derivative, ${}^3\nabla_\mu$, we find that

$$ {}^3\nabla_\alpha{}^3\nabla_\beta f = \gamma_\alpha{}^\mu\gamma_\beta{}^\lambda\nabla_\mu\nabla_\lambda f - n^\lambda K_{\alpha\beta}\nabla_\lambda f \ . \tag{6.15} $$

In order to interpret the result of antisymmetrising the above expression, we need to obtain an expression for the antisymmetrised extrinsic curvature tensor. By making use of the fact that the normal vector $\boldsymbol{n}$ is hypersurface orthogonal, it follows from the definition of the extrinsic curvature tensor that

$$ 2K_{[\alpha\beta]} = -{}^3T^{\sigma}{}_{\alpha\beta}n_\sigma \ . \tag{6.16} $$

That is, the antisymmetrised extrinsic curvature tensor is equal to the three-dimensional torsion tensor contracted with the unit normal vector field $\boldsymbol{n}$. We now wish to relate the three-dimensional torsion tensor to the space-time torsion tensor. We note that we have the following from equations (6.14), (6.15) and (6.16)

$$ \gamma_\rho{}^{\lambda}{}^3T^{\rho}{}_{\alpha\beta}\nabla_\lambda f = \gamma_\alpha{}^\mu\gamma_\beta{}^\nu T^{\lambda}{}_{\mu\nu}\nabla_\lambda f + n^\lambda n_\rho{}^3T^{\rho}{}_{\alpha\beta} \ . \tag{6.17} $$

Since f is an arbitrary smooth function on Σ, ${}^3\nabla_\mu f$ is an arbitrary dual vector field in Σ. It now follows from the above equation that

$$ {}^3T^{\lambda}{}_{\alpha\beta} = \gamma_\alpha{}^\mu\gamma_\beta{}^\nu T^{\lambda}{}_{\mu\nu} \ , \tag{6.18} $$

which relates the space-time torsion tensor to the three-dimensional torsion tensor. As it turns out, equation (6.11) is not enough to define entirely the three-dimensional torsion tensor. This is due to the fact that, in order for the three-dimensional torsion tensor to be restricted to the hypersurface Σ, we need the additional requirement that

$$ n_\sigma{}^3T^{\sigma}{}_{\mu\nu} := 0 \ . \tag{6.19} $$

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[2] In Chapters 3 and 4, we denote the space-time contorsion tensor as $\boldsymbol{K}$. In this chapter, we denote the three-dimensional contorsion tensor as $\boldsymbol{L}$ to avoid confusion with the extrinsic curvature tensor.

The above expression forms part of the definition of the three-dimensional torsion tensor and allows for this tensor to lie entirely in the hypersurface Σ. That is, the three-dimensional torsion tensor is now defined through equation (6.11) together with equation (6.19). It now follows from equation (6.18) that

$$ {}^3T^{\lambda}{}_{\alpha\beta} = \gamma_\rho{}^{\lambda}\gamma_\alpha{}^{\mu}\gamma_\beta{}^{\nu}T^{\rho}{}_{\mu\nu} \ . \tag{6.20} $$

Let us now return our attention to the extrinsic curvature tensor. We note that equation (6.16) together with equation (6.19) implies that $K_{[\alpha\beta]} = 0$ and, thus, the extrinsic curvature tensor is symmetric. Consider now the action of the Lie derivative on the spatial metric in the direction of the vector field $\boldsymbol{n}$:

$$ \pounds_{\boldsymbol{n}}\gamma_{\alpha\beta} = 2\gamma_\alpha{}^{\sigma}\gamma_\beta{}^{\rho}\nabla_{(\sigma}n_{\rho)} \tag{6.21} $$

$$ = -2K_{\alpha\beta} \ . \tag{6.22} $$

That is, we can write the extrinsic curvature tensor as the Lie derivative of the spatial metric in the direction of the unit normal vector field.

Up until now, we have only considered the projected torsion tensor and how this relates to the space-time torsion tensor. We now wish to study the curvature projected into the hypersurface Σ. Let us begin by defining the three-dimensional non-null torsion Riemann tensor through the linear map

$$ \left(2{}^3\nabla_{[\mu}{}^3\nabla_{\nu]} + {}^3T^{\rho}{}_{\mu\nu}{}^3\nabla_\rho\right) w_\sigma = {}^3\tilde{R}_{\mu\nu\sigma}{}^{\rho}w_\rho \ , \tag{6.23} $$

together with the requirement that [19]

$$ {}^3\tilde{R}_{\alpha\beta\sigma\rho}n^\rho := 0 \ . \tag{6.24} $$

The last expression ensures that the three-dimensional non-null torsion Riemann tensor lies entirely within the hypersurface Σ. One can think of equation (6.19) as being the torsion analogue of the above expression. Following what is given in [19], one can also define the three-dimensional torsion-free Riemann tensor as

$$ 2{}^3D_{[\mu}{}^3D_{\nu]}w_\sigma = {}^3R_{\mu\nu\sigma}{}^{\rho}w_\rho \ , \tag{6.25} $$

together with the requirement

$$ {}^3R_{\alpha\beta\sigma\rho}n^\rho := 0 \ . \tag{6.26} $$

Since our goal is to ultimately derive the ADM equations associated with the EC theory of gravity, we require the Gauss, Codazzi and Ricci equations. In order to derive these equations for the case of a hypersurface of the U_4 space-time manifold, we follow what is done by the authors of [19] where these equations are derived for the case of vanishing torsion. We first wish to derive the Gauss equation. Following [19], we first consider the following for a spatial dual vector field $\boldsymbol{w}$:

$$ {}^3\nabla_\alpha{}^3\nabla_\beta w_\sigma = \gamma_\alpha{}^{\delta}\gamma_\beta{}^{\mu}\gamma_\sigma{}^{\nu}\nabla_\delta\nabla_\mu w_\nu - \nabla_\mu w_\nu\gamma_\sigma{}^{\nu}n^\mu K_{\alpha\beta} - w^\nu K_{\beta\nu}K_{\alpha\sigma} \ . \tag{6.27} $$

By antisymmetrising the above expression and by making use equations (6.20) and (6.23) together with the fact that the extrinsic curvature tensor is symmetric, we obtain the following

$$ {}^3\tilde{R}_{\alpha\beta\sigma\rho} = \gamma_\alpha{}^{\delta}\gamma_\beta{}^{\mu}\gamma_\sigma{}^{\nu}\tilde{R}_{\delta\mu\nu\rho} + K_{\alpha\rho}K_{\beta\sigma} - K_{\beta\rho}K_{\alpha\sigma} \ . \tag{6.28} $$

Finally , by making use of the requirement that the three-dimensional non-null torsion Riemann tensor lies entirely within the hypersurface Σ, i.e., by invoking equation (6.24), we can contract both sides of the above expression with the spatial metric in order to obtain the *non-null torsion Guass equation*

$$ {}^3\tilde{R}_{\alpha\beta\sigma\rho} = \gamma_\alpha{}^{\delta}\gamma_\beta{}^{\mu}\gamma_\sigma{}^{\nu}\gamma_\rho{}^{\lambda}\tilde{R}_{\delta\mu\nu\lambda} + K_{\alpha\rho}K_{\beta\sigma} - K_{\beta\rho}K_{\alpha\sigma} \ . \tag{6.29} $$

Having derived the Gauss equation for the case of a Riemann-Cartan space-time, we now turn our attention to deriving the Codazzi equation.

We begin by considering the following projection of the non-null torsion Riemann tensor by contracting three times with the spatial metric and once with the unit normal vector field

$$\gamma_\alpha{}^\mu\gamma_\beta{}^\nu\gamma_\sigma{}^\rho n^\lambda \tilde{R}_{\mu\nu\rho\lambda} = \gamma_\alpha{}^\mu\gamma_\beta{}^\nu\gamma_\sigma{}^\rho \left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu + T^\delta{}_{\mu\nu}\nabla_\delta\right) n_\rho \ . \tag{6.30}$$

From the definition of the extrinsic curvature tensor in equation (6.6), we note the following

$$\gamma_\alpha{}^\mu\gamma_\beta{}^\nu\gamma_\sigma{}^\rho\nabla_\mu\nabla_\nu n_\rho = -{}^3\nabla_\alpha K_{\beta\sigma} + K_{\alpha\beta}n^\lambda\nabla_\lambda n_\sigma \ . \tag{6.31}$$

By substituting the above expression into equation (6.30) and by making use of the fact that the extrinsic curvature tensor is symmetric in its two indices, we find the *non-null torsion Codazzi equation* to be

$$\gamma_\alpha{}^\mu\gamma_\beta{}^\nu\gamma_\sigma{}^\lambda \tilde{R}_{\mu\nu\lambda\epsilon}n^\epsilon = {}^3\nabla_\beta K_{\alpha\sigma} - {}^3\nabla_\alpha K_{\beta\sigma} - {}^3T^\lambda{}_{\alpha\beta}K_{\lambda\sigma} \ . \tag{6.32}$$

It now remains to derive the Ricci equation for the case of a Riemann-Cartan space-time. Following what was done in [19], we begin by acting the Lie derivative on the extrinsic curvature tensor as follows

$$\pounds_{\boldsymbol{n}} K_{\alpha\beta} = n^\sigma\nabla_\sigma K_{\alpha\beta} + K_{\alpha\sigma}\nabla_\beta n^\sigma + K_{\beta\sigma}\nabla_\alpha n^\sigma - n^\sigma K_{\alpha\lambda}T^\lambda_{\ \beta\sigma} - n^\sigma K_{\beta\lambda}T^\lambda_{\ \alpha\sigma} \ . \tag{6.33}$$

By making use of the definition of the extrinsic curvature tensor, we can write the first term on the right-hand side of the above expression as

$$\begin{aligned} n^\sigma\nabla_\sigma K_{\alpha\beta} &= -n^\sigma\nabla_\sigma\left(\gamma_\alpha{}^\lambda\nabla_\lambda n_\beta\right) \\ &= -n^\sigma\nabla_\sigma\nabla_\alpha n_\beta - n^\sigma\nabla_\sigma\left(n_\alpha\tilde{a}_\beta\right) \ , \end{aligned} \tag{6.34}$$

where we have defined $\tilde{a}_\alpha := n^\sigma\nabla_\sigma n_\alpha$. We now turn our attention to the second term on the right-hand side of equation (6.33). By making use of the fact that $K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha\tilde{a}_\beta$, which follows from the definition of the extrinsic curvature tensor, we have

$$K_{\sigma\alpha}\nabla_\beta n^\sigma = -K_{\alpha\sigma}K_\beta{}^\sigma - K_{\sigma\alpha}n_\beta\tilde{a}^\sigma \ . \tag{6.35}$$

Consider the following expression

$$n^\sigma\nabla_\alpha\nabla_\sigma n_\beta = \nabla_\alpha\tilde{a}_\beta - K_\alpha{}^\sigma K_{\sigma\beta} - n_\alpha\tilde{a}^\sigma K_{\sigma\beta} \ . \tag{6.36}$$

In the final expression, we wish to introduce the so-called lapse function. The lapse function is introduced through the consideration of an acceleration vector. We have already defined the acceleration vector $\tilde{a}^\mu := n^\beta\nabla_\beta n^\nu$ which is defined using the Cartan covariant derivative. It is also possible to define an acceleration vector using the Levi-Civita covariant derivative. We write such an acceleration vector as $a^\mu := n^\beta D_\beta n^\mu$. Since there are two acceleration vectors, we have two options for the lapse function. When deriving the evolution equations, the lapse function plays the role of measuring how much proper time has elapsed [19]. Since proper time is the affine parameter associated with timelike auto-parallel curves of the Levi-Civita connection, we define the lapse function using the Levi-Civita covariant derivative. That is, we rely on the Levi-Civita connection to provide us with the lapse function. More specifically, the *lapse function* α can be introduced through the expression

$$a_\mu := n^\beta D_\beta n_\mu = {}^3D_\mu \ln\alpha \ . \tag{6.37}$$

By making use of the previous results, we now find the *non-null torsion Ricci equation* to be

$$\begin{aligned} \pounds_{\boldsymbol{n}} K_{\mu\nu} = {}& -\gamma_\mu{}^\sigma\gamma_\nu{}^\rho n^\alpha n^\beta\tilde{R}_{\alpha\sigma\rho\beta} - \frac{1}{\alpha}{}^3D_\mu{}^3D_\nu\alpha + \alpha{}^3D_\mu\left(\frac{1}{\alpha}n^\sigma n^\lambda T_{\lambda\sigma\nu}\right) - K_\mu{}^\beta K_{\nu\beta} - K_\mu{}^\epsilon n^\alpha\gamma_\epsilon{}^\beta\gamma_\nu{}^\delta T_{\beta\delta\alpha} \\ & + {}^3L^\lambda{}_{\mu\nu}\left(\frac{1}{\alpha}{}^3D_\lambda\alpha - n^\sigma n^\delta T_{\sigma\delta\lambda}\right) \ . \end{aligned} \tag{6.38}$$

Since the extrinsic curvature tensor is torsion-free as a result of equation (6.19), we can take the torsion-free part of the right-hand side of the above expression in order to write

$$\pounds_{\boldsymbol{n}} K_{\mu\nu} = -\gamma_{\mu}{}^{\sigma}\gamma_{\nu}{}^{\rho} n^{\alpha} n^{\beta} R_{\alpha\sigma\rho\beta} - \frac{1}{\alpha}{}^{3}D_{\mu}{}^{3}D_{\nu}\alpha - K_{\mu}{}^{\beta}K_{\nu\beta} \ . \tag{6.39}$$

We shall refer to this expression as the *torsion-free Ricci equation*. In this section, we obtained the non-null torsion Gauss, Codazzi and Ricci equations. In the following section we make use of these results in order to derive the ADM formulation of the EC theory of gravity.

6.3 The ADM formulation of the Einstein-Cartan theory of gravity

The non-null torsion Gauss, Codazzi and Ricci equations derived in the previous section are geometric results and are specific to the Riemann-Cartan space-time U_4. In particular, these aforementioned equations are not specific to the EC theory of gravity and the EC field equations have not, as of yet, been invoked. In this section, we make use of the results obtained in the previous section together with the EC field equations in order to obtain the so-called Arnowitt-Deser-Misner (ADM) formulation of the EC theory of gravity. We begin by choosing the coordinate system which is discussed in [19] and leads to the line element being of the form

$$\mathrm{d}s^2 = -\alpha^2\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + \beta^i\mathrm{d}t\right)\left(\mathrm{d}x^j + \beta^j\mathrm{d}t\right) \ . \tag{6.40}$$

Let us turn our attention to the non-null torsion Gauss equation (6.29). By contracting indices once in the non-null torsion Gauss equation and then taking the trace of the resulting expression, we obtain

$$\begin{aligned} {}^{3}\tilde{R} + K^2 - K_{\alpha\beta}K^{\alpha\beta} &= \gamma_{\mu}{}^{\alpha}\gamma_{\nu}{}^{\beta}\tilde{R}^{\mu}{}_{\beta\alpha}{}^{\nu} \\ &= \tilde{R} + 2n^{\mu}n^{\nu}\tilde{R}_{\mu\nu} \ , \end{aligned} \tag{6.41}$$

where K is the trace of the extrinsic curvature tensor and we have used the fact that the non-null torsion Riemann tensor is antisymmetric in its first and second indices. By making use of the first EC field equation (4.51), we can introduce the energy-momentum tensor into equation (6.41) in order to obtain

$$ {}^{3}\tilde{R} + K^2 - K_{\alpha\beta}K^{\alpha\beta} = 16\pi n^{\mu}n^{\nu}\tau_{\mu\nu} \ . \tag{6.42}$$

By defining the *energy density* $\rho := n^{\mu}n^{\nu}\tau_{\mu\nu}$, we can write the above expression as

$$ {}^{3}\tilde{R} + K^2 - K_{ij}K^{ij} = 16\pi\rho \ . \tag{6.43}$$

We shall refer to the above expression as the *non-null torsion Hamiltonian constraint*. In Section 4.2, not only was the energy-momentum tensor $\tau_{\mu\nu}$ defined but also the so-called effective energy-momentum tensor $\mathcal{T}_{\mu\nu}$. This definition of the effective energy-momentum tensor is given in equation (4.56). In the EC theory, the torsion tensor is completely described by the spin-current tensor as per equation (4.36). It therefore makes sense to include the torsion tensor into the energy-matter content through the effective energy-momentum tensor. The evolution of the torsion tensor is then obtained by evolving the energy-matter content. We therefore wish to obtain a Hamiltonian constraint equation whose non-Riemannian contributions are contained entirely in the energy-matter content. In order to do this, we turn our attention to the torsion-free version of equation (6.41) which is given below

$$ {}^{3}R + K^2 - K_{\alpha\beta}K^{\alpha\beta} = R + 2n^{\mu}n^{\nu}R_{\mu\nu} \ . \tag{6.44}$$

We emphasize that the above expression is a completely geometric statement and holds true in U_4. In order to introduce the effective energy-momentum tensor into the above expression, we make use of equation (4.55) in order to obtain the following

$$ {}^{3}R + K^2 - K_{ij}K^{ij} = 16\pi\overset{\circ}{\rho} \ , \tag{6.45}$$

where $\overset{\circ}{\rho} := n^\mu n^\nu \mathcal{T}_{\mu\nu}$ is the *effective energy density*. We refer to the above expression as the *torsion-free Hamiltonian constraint.*

As a result of the choice of coordinates in equation (6.40), the entirety of a spatial tensor is now described by only its spatial components [19]. Because of this, we have switched to Latin indices in the above which run from 1 to 3. That is, the Latin indices indicate the spatial components of a given tensor. We now turn our attention to the Codazzi equation (6.32). Contracting indices once yields

$$ {}^3\nabla_\beta K^\beta{}_\alpha - {}^3\nabla_\alpha K - {}^3T^\lambda{}_{\alpha\beta} K^\beta{}_\lambda = -\gamma_\alpha{}^\mu n^\epsilon \tilde{R}_{\mu\epsilon} \ . \tag{6.46} $$

By making use of equation (4.51) and defining the *momentum density* as

$$ j_\alpha := -\gamma_\alpha{}^\mu n^\lambda \tau_{\mu\lambda} \ , \tag{6.47} $$

equation (6.46) becomes

$$ {}^3\nabla_j K^j{}_i - {}^3\nabla_i K - {}^3T^k_{ij} K^j{}_k = 8\pi j_i \ . \tag{6.48} $$

We refer to the above expression as the *non-null torsion momentum constraint.* In order to write an expression for the momentum constraint equation whose non-Riemannian contributions are contained in the energy-matter content, we turn our attention to the torsion-free version of equation (6.46) which is written below

$$ {}^3D_\beta K^\beta{}_\alpha - {}^3D_\alpha K = -\gamma_\alpha{}^\mu n^\epsilon R_{\mu\epsilon} \ . \tag{6.49} $$

We note that the above expression is a geometric statement that is valid on U_4. We can make use of equation (4.55) and define the *effective momentum density* as

$$ \overset{\circ}{j}_\alpha := -\gamma_\alpha{}^\mu n^\lambda \mathcal{T}_{\mu\lambda} \ , \tag{6.50} $$

in order to write equation (6.49) as

$$ {}^3D_j K^j{}_i - {}^3D_i K = 8\pi \overset{\circ}{j}_i \ . \tag{6.51} $$

We refer to equation (6.51) as the *torsion-free momentum constraint.*

Having obtained the above constraint equations, we now wish to obtain an evolution equation for the spatial metric and the extrinsic curvature tensor. Before doing this, we note that the Lie derivative in the direction of the vector field

$$ t^\mu = \alpha n^\mu + \beta^\mu \ , \tag{6.52} $$

is a natural time derivative [19]. In the above, α is the lapse function, as before, which measures the proper time between hypersurfaces Σ and $\boldsymbol{\beta}$ is the *shift* vector which measures the shifting of spatial coordinates [19]. We note, in particular, that the lapse function describes the proper time measured since we have described the lapse function using the Levi-Civita connection. From equation (6.22), it follows that the ADM evolution equation for the spatial metric is

$$ \partial_t \gamma_{ij} = -2\alpha K_{ij} + {}^3D_i \beta_j + {}^3D_j \beta_i \ . \tag{6.53} $$

In addition, we note that the evolution equation for the trace of the spatial metric is

$$ \partial_t \ln \gamma^{1/2} = -\alpha K + {}^3D_i \beta^i \ . \tag{6.54} $$

We now wish to derive the evolution equations for the extrinsic curvature tensor as well as its trace. Let us first define the stress-energy tensor and effective stress-energy tensor as

$$ S_{\alpha\beta} := \gamma_\alpha{}^\mu \gamma_\beta{}^\nu \tau_{\mu\nu} \ , \qquad \text{and} \qquad \overset{\circ}{S}_{\alpha\beta} := \gamma_\alpha{}^\mu \gamma_\beta{}^\nu \mathcal{T}_{\mu\nu} \ , \tag{6.55} $$

respectively. By making use of the torsion-free Ricci equation (6.39), one can obtain the following evolution equation for the extrinsic curvature tensor

$$\begin{aligned}\partial_t K_{ij} = \alpha\left({}^3R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j\right) + \beta^k{}^3D_k K_{ij} + K_{ik}{}^3D_j\beta^k + K_{kj}{}^3D_i\beta^k \\ - 8\pi\alpha\left(\overset{\circ}{S}_{ij} - \frac{1}{2}\gamma_{ij}\left(\overset{\circ}{S} - \overset{\circ}{\rho}\right)\right) - {}^3D_i{}^3D_j\alpha \ . \end{aligned} \tag{6.56}$$

It follows from the above expression that the evolution equation for the trace of the extrinsic curvature tensor is

$$\partial_t K = 4\pi\alpha\left(\overset{\circ}{S} + \overset{\circ}{\rho}\right) + \alpha K_{ij}K^{ij} + \beta^k\partial_k K - {}^3D^2\alpha \ . \tag{6.57}$$

We have thus obtained the ADM equations for the EC theory of gravity.

6.4 The GBSSN formulation of the Einstein-Cartan theory of gravity

In [19,20], a discussion of the so-called Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation of GR is given. In this formulation, it is assumed that the determinant of the conformal spatial metric is unity. As pointed out in [25,31], the conformal spatial metric is not unity in spherical coordinates. For this reason, a Generalised-BSSN (GBSSN) formulation is presented in [25]. In this section, we wish to apply this method to the EC theory of gravity. That is, we will obtain the GBSSN formulation of the EC theory. As the GBSSN formulation is a modification of the ADM formalism, we will make use of the results derived in the previous section. In Section 6.4.1, we will first discuss conformal transformations on the Riemann-Cartan space-time U_4 which will be required when we derive the GBSSN formulation of the EC theory.

6.4.1 Conformal transformations on U_4

Here, we introduce the conformal spatial metric, the conformal torsion tensor and the subsequent transformations that will allow us to construct the BSSN formulation of the EC theory of gravity. We begin by introducing the conformal spatial metric tensor, which we shall denote as $\bar{\boldsymbol{\gamma}}$, through the expression

$$\bar{\gamma}_{ij} := e^{4\phi}\gamma_{ij} \ , \tag{6.58}$$

where ϕ is a scalar function and is referred to as the *conformal factor* [5,19]. We now wish to consider the action of the Cartan covariant derivative on the conformal spatial metric. Let us first define the type $(1,2)$ tensor field $\boldsymbol{C}$ through the following expression

$$C^k{}_{ij} := {}^3\tilde{\Gamma}^k{}_{ij} - {}^3\bar{\tilde{\Gamma}}^k{}_{ij} \ , \tag{6.59}$$

where the ${}^3\bar{\tilde{\Gamma}}^k{}_{ij}$ denote the connection coefficients of the conformal three-dimensional Cartan covariant derivative ${}^3\bar{\nabla}_i$. Let us now consider the action of the three-dimensional Cartan covariant derivative on the conformal metric tensor:

$${}^3\nabla_k\bar{\gamma}_{ij} = 4e^{4\phi}\gamma_{ij}{}^3\nabla_k\phi \ . \tag{6.60}$$

We notice that the above expression with the three-dimensional Levi-Civita covariant derivative used instead of the three-dimensional Covariant covariant derivative holds. Let $\boldsymbol{w}$ denote a spatial dual vector field. The tensor field $\boldsymbol{C}$ allows us to write

$${}^3\nabla_i w_j = {}^3\bar{\nabla}_i w_j - C^k{}_{ij}w_k \ . \tag{6.61}$$

We can write the action of the three-dimensional Cartan covariant derivative on the conformal metric tensor as

$$ {}^3\nabla_k \bar{\gamma}_{ij} = {}^3\bar{\nabla}_k \bar{\gamma}_{ij} - C^l{}_{ki} \bar{\gamma}_{lj} - C^l{}_{kj} \bar{\gamma}_{il} \ , \tag{6.62} $$

and we note that the first term on the right-hand side of the above expression vanishes. We now wish to obtain the resulting conformal transformation of the Cartan connection coefficients, i.e., we wish to find $C^k{}_{ij}$ in terms of the spatial metric. By making use of the above expression, we have

$$ \begin{aligned} 0 &= -{}^3\bar{\nabla}_k \bar{\gamma}_{ij} + {}^3\bar{\nabla}_i \bar{\gamma}_{jk} + {}^3\bar{\nabla}_j \bar{\gamma}_{ki} \\ &= 4e^{4\phi} \left(-\gamma_{ij} {}^3\nabla_k \phi + \gamma_{jk} {}^3\nabla_i \phi + \gamma_{ki} {}^3\nabla_j \phi \right) + \left(C^l{}_{ik} - C^l{}_{ki} \right) \bar{\gamma}_{lj} + \left(C^l{}_{jk} - C^l{}_{kj} \right) \bar{\gamma}_{li} + \left(C^l{}_{ij} + C^l{}_{ji} \right) \bar{\gamma}_{lk} \ . \end{aligned} \tag{6.63} $$

Having already introduced the conformal transformation of the spatial metric tensor, we now require the conformal transformation of the torsion tensor in order to evaluate the right-hand side of the above expression since

$$ C^k{}_{ij} - C^k{}_{ji} = {}^3T^k{}_{ij} - {}^3\bar{T}^k{}_{ij} \ , \tag{6.64} $$

where ${}^3\bar{T}^k{}_{ij} := {}^3\bar{\tilde{\Gamma}}^k{}_{ij} - {}^3\bar{\tilde{\Gamma}}^k{}_{ji}$ is the *three-dimensional conformal torsion tensor*. Following [79], we note two possibilities for how the three-dimensional torsion tensor transforms[3]. The first such possibility is referred to as *weak conformal symmetry* of the three-dimensional torsion tensor and implies that the torsion tensor does not transform at all, i.e., we have

$$ {}^3\bar{T}^k{}_{ij} = {}^3T^k{}_{ij} \ . \tag{6.65} $$

The second option for a conformal transformation of the three-dimensional torsion tensor that we note here is referred to as *strong conformal symmetry* of the three-dimensional torsion tensor and is such that only the trace component undergoes a transformation, i.e., we have

$$ {}^3\bar{T}^k_{ij} = {}^3T^k_{ij} + h \left(\delta^k_j \partial_i - \delta^k_i \partial_j \right) \sigma \ , \tag{6.66} $$

where h and σ are some C^∞ functions on the hypersurface Σ. In this note, we consider only weak conformal symmetry and therefore the tensor field $\boldsymbol{C}$ is symmetric in its second and third indices. It therefore follows from equation (6.63) that the three-dimensional Cartan connection coefficients transform as

$$ {}^3\bar{\tilde{\Gamma}}^k{}_{ij} = {}^3\tilde{\Gamma}^k{}_{ij} + 2\gamma^{kl} \left(\gamma_{lj} {}^3\nabla_i \phi + \gamma_{li} {}^3\nabla_j \phi - \gamma_{ij} {}^3\nabla_l \phi \right) \ , \tag{6.67} $$

where γ^{ij} denotes the inverse of the spatial metric tensor. Let us now turn our attention to deriving the conformal transformation of the three-dimensional non-null torsion Riemann tensor. For a spatial dual vector field $\boldsymbol{w}$, we have

$$ {}^3\bar{\tilde{R}}_{ijk}{}^l w_l = 2 {}^3\bar{\nabla}_{[i} {}^3\bar{\nabla}_{j]} w_k + {}^3T^l{}_{ij} {}^3\bar{\nabla}_l w_k \ , \tag{6.68} $$

where we have made use of the fact that we are considering weak conformal symmetry of the torsion tensor. Following from the above expression, one can obtain the following expression which describes the conformal transformation of the three-dimensional non-null torsion Riemann tensor:

$$ {}^3\bar{\tilde{R}}_{ijk}{}^l = {}^3\tilde{R}_{ijk}{}^l + 2 {}^3\nabla_{[i} C^l{}_{j]k} + 2C^m{}_{[i|k|} C^l{}_{j]m} + {}^3T^m{}_{ij} C^l{}_{mk} \ . \tag{6.69} $$

Having discussed the conformal transformations of the spatial metric and the three-dimensional torsion tensor, we are now in a position to derive the GBSSN formulation of the EC theory of gravity.

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[3] In fact, there is a third possibility for the conformal transformation which is rather complicated and we do not mention here. The interested reader is directed to [79] for information regarding this.

6.4.2 GBSSN evolution equations

In this section, we discuss the GBSSN formulation of the EC theory. For a discussion on the BSSN formulation of GR, the interested reader is directed to the work [19, 20] by Baumgarte and Shapiro. The BSSN formulation is a modification of the ADM equations. This modification arises by making use of the conformal transformations in the previous section. That is, as an example, instead of evolving explicitly the spatial metric, one evolves the conformal spatial metric as well as the conformal scalar. In the standard BSSN formulation, one makes the requirement that the determinant of the conformal metric be unity [19, 20]. That is, in the standard BSSN formulation, one takes $\bar{\gamma} = 1$. This is particularly useful for 3D numerical simulations where the coordinate system is Cartesian. The reason for this is that the Minkowksi metric in Cartesian coordinates has a determinant of unity. In this thesis, we are interested in obtaining simulations for spherically symmetric solutions. For such simulations, we will be making use of spherical coordinates. In spherical coordinates, the determinant of the Minkowksi metric is not unity. Therefore, one cannot make use of the standard BSSN and instead must consider the so-called Generalised-BSSN (or GBSSN) formulation which was first presented by Brown in [25]. In this section, we give the GBSSN formulation of the EC theory of gravity by following what is done in [19, 20, 25, 31] for the case of GR.

Let us begin by deriving the evolution equation for the conformal factor ϕ. As already mentioned, we do not impose the requirement that the determinant of the conformal spatial metric be unity. Instead, we impose the following condition on the time evolution of the determinant of the conformal spatial metric [25, 31]

$$(\partial_t - \pounds_{\boldsymbol{\beta}}) \ln \bar{\gamma} = -2v \, {}^3D_k \beta^k \ . \tag{6.70}$$

In the above, the parameter v is either 0 or 1. In the case where $v = 0$ the above is referred to as the *Euler condition* whereas in the case where $v = 1$ the above is referred to as the *Lagrange condition*. In order to derive the evolution equation for the conformal scalar, let us first take the determinant of equation (6.58) in order to obtain

$$\phi = -\frac{\ln(\gamma/\bar{\gamma})}{12} \ . \tag{6.71}$$

We can make use of this expression together with equation (6.54), which describes the evolution of the determinant of the spatial metric, in order to obtain the following

$$\partial_t \phi = \frac{\alpha}{6} K - \frac{v}{6} \partial_k \beta^k + v \beta^k \partial_k \phi - \frac{v}{6} \beta^k \partial_k \ln \bar{\gamma}^{1/2} \ . \tag{6.72}$$

In the above we have made use of the expression given below which follows from equation (6.67)

$${}^3\Gamma^k{}_{kj} = {}^3\bar{\Gamma}^k{}_{kj} - 6 \, \partial_j \phi \ , \tag{6.73}$$

where we have denoted the connection coefficients of the three-dimensional conformal Levi-Civita connection as ${}^3\bar{\Gamma}^k{}_{ij}$. Let us now turn our attention to studying the evolution of the trace of the extrinsic curvature tensor. We first decompose the extrinsic curvature tensor, $\boldsymbol{K}$, into a traceless component and a trace component, i.e., we write

$$K_{ij} = A_{ij} + \frac{1}{3} K \gamma_{ij} \ , \tag{6.74}$$

where the tensor $\boldsymbol{A}$ has zero trace. We now allow the tensor field $\boldsymbol{A}$ to undergo the same conformal transformation as the spatial metric, i.e., it is rescaled as

$$\bar{A}_{ij} = e^{4\phi} A_{ij} \ . \tag{6.75}$$

As per the above rescaling, we note that we have $\bar{A}_{ij} \bar{A}^{ij} = A_{ij} A^{ij}$. Turning our attention to the previous evolution equation for the trace of the extrinsic curvature tensor, which is given in equation (6.57), we now write the evolution equation as

$$\partial_t K = 4\pi\alpha \left(\overset{\circ}{S} + \overset{\circ}{\rho} \right) + \alpha \left(\bar{A}_{ij} \bar{A}^{ij} + \frac{1}{3} K^2 \right) + \beta^k \partial_k K - \overset{\circ}{D}{}^2 \alpha \ . \tag{6.76}$$

Let us now turn our attention to obtaining an evolution equation for the conformal spatial metric $\boldsymbol{\gamma}$. We begin by noting that, from the definition of the conformal spatial metric, we have the following expression for its time derivative

$$e^{-4\phi}\partial_t\bar{\gamma}_{ij} = \partial_t\gamma_{ij} + 4\gamma_{ij}\partial_t\phi \ . \tag{6.77}$$

By making use equations (6.53), (6.72) and (6.74) as well as the fact that

$$2\gamma_{k(i}\,{}^3\overset{\circ}{\Gamma}{}^k_{j)l} = \partial_l\gamma_{ij} \ , \tag{6.78}$$

one can obtain the following evolution equation for the conformal spatial metric $\bar{\boldsymbol{\gamma}}$

$$\partial_t\bar{\gamma}_{ij} = -2\alpha\bar{A}_{ij} + 2\bar{\gamma}_{k(i}\partial_{j)}\beta^k + \beta^k\partial_k\bar{\gamma}_{ij} + 4\left(v-1\right)\bar{\gamma}_{ij}\beta^k\partial_k\phi - \frac{2}{3}v\bar{\gamma}_{ij}\partial_k\beta^k - \frac{2}{3}v\bar{\gamma}_{ij}\beta^k\partial_k\ln\bar{\gamma}^{1/2} \ . \tag{6.79}$$

We now turn our attention to finding an evolution equation for the tensor field $\bar{\boldsymbol{A}}$. From equation (6.75), which gives the rescaling of the traceless part of the extrinsic curvature tensor, we have the following expression for its time derivative

$$e^{-4\phi}\partial_t\bar{A}_{ij} = 4A_{ij}\partial_t\phi + \partial_t A_{ij} \ . \tag{6.80}$$

By substituting equation (6.72) and the traceless part of equation (6.56) into the above expression as well as by making use of the fact that

$${}^3D_k A_{ij} + 2A_{k(i}\,{}^3D_{j)}\beta^k = \beta^k\partial_k A_{ij} + 2A_{k(i}\partial_{j)}\beta^k \ , \tag{6.81}$$

one can obtain the following evolution equation

$$\begin{aligned}\partial_t\bar{A}_{ij} = & -2\alpha\bar{A}_{ik}\bar{A}^k{}_j + \alpha K\bar{A}_{ij} - e^{4\phi}\left({}^3D_i{}^3D_j\alpha\right)^{\mathrm{TF}}_{ij} + \alpha e^{4\phi}R^{\mathrm{TF}}_{ij} - 8\pi\alpha e^{4\phi}\overset{\circ}{S}{}^{\mathrm{TF}}_{ij} \\ & + 2\bar{A}_{k(i}\partial_{j)}\beta^k + \beta^k\partial_k\bar{A}_{ij} - \frac{2v}{3}\bar{A}_{ij}\beta^k\partial_k\ln\bar{\gamma}^{1/2} + 4\left(v-1\right)\bar{A}_{ij}\beta^k\partial_k\phi - \frac{2v}{3}\bar{A}_{ij}\partial_k\beta^k \ .\end{aligned} \tag{6.82}$$

In the above, we follow the notational covention that is used by the authors of [19] and use the superscript TF to denote the traceless part of the given object. The last evolution equation that we wish to obtain is that of the so-called *conformal Levi-Civita connection functions* defined as [19]

$${}^3\bar{\Gamma}^i = -\partial_j\bar{\gamma}^{ij} - \bar{\gamma}^{il}\partial_l\ln\bar{\gamma}^{1/2} \ . \tag{6.83}$$

The evolution equation for the Levi-Civita connection functions reads[4]

$$\begin{aligned}\partial_t\ {}^3\bar{\Gamma}^i = & \ 2\alpha\,{}^3\bar{\Gamma}^i{}_{jk}\bar{A}^{jk} - 16\pi\alpha\bar{\gamma}^{ik}\overset{\circ}{j}_k - \frac{4}{3}\alpha\bar{\gamma}^{ik}\partial_k K - 12\alpha\bar{A}^{ij}\partial_j\phi + 4(1-v)\,{}^3\bar{\Gamma}^i\beta^k\partial_k\phi - {}^3\bar{\Gamma}^k\partial_k\beta^i \\ & - 2\bar{A}^{ij}\partial_j\alpha + \beta^k\partial_k{}^3\bar{\Gamma}^i + \bar{\gamma}^{kj}\partial_k\partial_j\beta^i + 2(1-v)\bar{\gamma}^{il}\partial_k\phi\partial_l\beta^k + 2(1-v)\bar{\gamma}^{il}\beta^k\partial_l\partial_k\phi \\ & + \frac{2v}{3}\,{}^3\bar{\Gamma}^i\beta^k\partial_k\ln\bar{\gamma}^{1/2} + \frac{2v}{3}\,{}^3\bar{\Gamma}^i\partial_k\beta^k + \frac{v}{3}\bar{\gamma}^{ij}\left(\partial_j\partial_k\beta^k + \partial_l\ln\bar{\gamma}^{1/2}\partial_j\beta^l + \beta^k\partial_j\partial_k\ln\bar{\gamma}^{1/2}\right) \ .\end{aligned} \tag{6.84}$$

In this section, we have derived the GBSSN formulation of the EC theory of gravity. This was carried out by modifying the ADM formulation of the EC theory of gravity which was constructed in the previous section. More specifically, we replaced the ADM evolution equations for the spatial metric tensor and the extrinsic curvature tensor with evolution equations for the conformal scalar, the trace of the extrinsic curvature tensor, the conformal spatial metric, the rescaled traceless part of the extrinsic curvature tensor and the conformal Levi-Civita connection functions.

Up until now, we have not discussed a $3+1$ decomposition of the energy-matter content. As mentioned in Section 4.2, the EC theory reduces to GR in the case of a vanishing spin-current tensor. This means that, in order to obtain results that may differ from GR, it is necessary to consider a non-vanishing energy-matter content. In the following section, we follow what is done in [30] in order to obtain a $3+1$ decomposition of the energy-matter content for the EC theory of gravity.

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[4] For a derivation, the reader is directed to the Mathematica notebook *ADM_and_GBSSN_EC_theory.nb* which was constructed by the author and is available at https://www.dropbox.com/sh/cfybc2cysixchlz/AACjKq0NHComUPoXcZz90DH7a?dl=0.

6.5 Decomposition of the energy-matter content

Let us turn our attention to constructing a $3+1$ decomposition of the energy-matter content. In doing this, we wish to obtain a set of evolution equations for the energy-matter content. Here, we carry out the decomposition by following what was done by Gourgoulhon in [30] for the case of GR. Since, in the presence of torsion, the energy-momentum tensor is not necessarily conserved, we do not study this tensor directly when carrying out our decomposition. Instead, we study the effective energy-momentum tensor given in equation (4.56) which is both conserved as well as symmetric. Following what is done in [30], let us define the *effective stress-energy tensor* through the contraction of the effective energy-momentum tensor with the spatial metric

$$\overset{\circ}{S}_{\mu\nu} := \gamma_{\mu\alpha}\gamma_{\nu\beta}\mathcal{T}^{\alpha\beta} \ . \tag{6.85}$$

Substituting in the definition of the spatial metric into the above expression yields

$$\overset{\circ}{S}_{\mu\nu} = \mathcal{T}_{\mu\nu} + n_\mu n^\beta \mathcal{T}_{\beta\nu} + n_\nu n^\beta \mathcal{T}_{\beta\mu} + n_\mu n_\nu n^\alpha n^\beta \mathcal{T}_{\alpha\beta} \ . \tag{6.86}$$

By defining the *effective energy density* as

$$\overset{\circ}{\rho} := n^\mu n^\nu \mathcal{T}_{\mu\nu} \ , \tag{6.87}$$

and the *effective momentum density* as

$$\overset{\circ}{j}_\mu = -\gamma_{\mu\nu} n_\beta \mathcal{T}^{\nu\beta} \ , \tag{6.88}$$

we obtain the following expression from equation (6.85) after rearranging [30]

$$\mathcal{T}_{\mu\nu} = \overset{\circ}{S}_{\mu\nu} + 2n_{(\mu}\overset{\circ}{j}_{\nu)} + \overset{\circ}{\rho} n_\mu n_\nu \ . \tag{6.89}$$

We have thus obtained the 3+1 decomposition of the effective energy-momentum tensor which is given above. Let us now turn our attention to obtaining the 3+1 evolution equations for the energy-matter content. From the torsion-free differential Bianchi identity, we know that the effective energy-momentum tensor is conserved and thus

$$0 = D_\mu \mathcal{T}^\mu{}_\nu = D_\mu \overset{\circ}{S}^\mu{}_\nu + \overset{\circ}{j}^\mu D_\mu n_\nu + n_\nu D_\mu \overset{\circ}{j}^\mu + \overset{\circ}{j}_\nu D_\mu n^\mu + D_{\boldsymbol{n}} \overset{\circ}{j}_\nu + n_\nu D_{\boldsymbol{n}} \overset{\circ}{\rho} + \overset{\circ}{\rho} n_\nu D_\mu n^\mu + \overset{\circ}{\rho} D_{\boldsymbol{n}} n_\nu \ . \tag{6.90}$$

By substituting in the extrinsic curvature tensor and making use of the fact that the effective momentum density $\overset{\circ}{\boldsymbol{j}}$ and extrinsic curvature tensor $\boldsymbol{K}$ are orthogonal to the unit normal vector $\boldsymbol{n}$, we obtain

$$0 = D_\mu \mathcal{T}^\mu{}_\nu = D_\mu \overset{\circ}{S}^\mu{}_\nu - \overset{\circ}{j}^\mu K_{\mu\nu} + n_\nu D_\mu \overset{\circ}{j}^\mu - K\overset{\circ}{j}_\nu + D_{\boldsymbol{n}} \overset{\circ}{j}_\nu + n_\nu D_{\boldsymbol{n}} \overset{\circ}{\rho} - \overset{\circ}{\rho} K n_\nu + \overset{\circ}{\rho} a_\nu \ , \tag{6.91}$$

and we remind the reader that K is the trace of the extrinsic curvature tensor. By contracting the above expression with the unit normal vector to the hypersurface Σ, we obtain

$$0 = n^\nu D_\mu \mathcal{T}^\mu{}_\nu = n^\nu D_\mu \overset{\circ}{S}^\mu{}_\nu - D_\mu \overset{\circ}{j}^\mu - \overset{\circ}{j}^\mu a_\mu - D_{\boldsymbol{n}} \overset{\circ}{\rho} + \overset{\circ}{\rho} K \ . \tag{6.92}$$

Let us now study the right-hand side of the above expression. Consider the action of the three-dimensional Levi-Civita covariant derivative on the effective momentum density:

$${}^3D_\mu \overset{\circ}{j}^\mu = \gamma_\mu{}^\nu D_\nu \overset{\circ}{j}^\mu = D_\mu \overset{\circ}{j}^\mu - \overset{\circ}{j}^\nu a_\nu \ . \tag{6.93}$$

In addition, the first term on the right-hand side of equation (6.92) can be written as

$$n^\nu D_\mu \overset{\circ}{S}^\mu{}_\nu = -\overset{\circ}{S}^\mu{}_\nu D_\mu n^\nu = \overset{\circ}{S}^{\mu\nu} K_{\mu\nu} \ , \tag{6.94}$$

where we have made use of the fact that the effective stress-energy tensor $\mathring{\boldsymbol{S}}$ is orthogonal to the unit normal vector field $\boldsymbol{n}$. Lastly, we note that the fourth term on the right-hand side of equation (6.92) is nothing more than $\mathcal{L}_{\boldsymbol{n}}\mathring{\rho}$, i.e., the Lie derivative of the effective energy density with respect to the unit normal vector to the hypersurface Σ. By making use of the above expressions as well as the vector field $\boldsymbol{t}$, we obtain the following from equation (6.92)

$$\partial_t \mathring{\rho} = \alpha \left(K_{ij} \mathring{S}^{ij} - {}^3D \cdot \mathring{j} + 2\mathring{j} \cdot a + \mathring{\rho} K \right) + \beta^i \partial_i \mathring{\rho} \ , \tag{6.95}$$

and we have written the above expression using spatial indices. The above expression describes the evolution of the effective energy density obtained by contracting equation (6.90) with the unit normal vector field. Let us now consider the contraction of equation (6.90) with the spatial metric. Such a contraction yields the following

$$0 = \gamma_\sigma{}^\nu D_\mu \mathcal{T}^\mu_{\ \nu} = \gamma_\sigma{}^\nu D_\mu \mathring{S}^\mu{}_\nu - \mathring{j}^\mu K_{\mu\sigma} - K \mathring{j}_\sigma + \gamma_\sigma{}^\nu D_{\boldsymbol{n}} \mathring{j}_\nu + \mathring{\rho} a_\sigma \ , \tag{6.96}$$

where we have introduced the extrinsic curvature tensor $\boldsymbol{K}$ on the right-hand side of the second equality. Now, we note that the first term on the right-hand side of the second equality can be written as

$$\gamma_\sigma{}^\nu D_\mu \mathring{S}^\mu{}_\nu = \gamma_\sigma{}^\nu \, {}^3D_\mu \mathring{S}^\mu{}_\nu + \mathring{S}^\nu{}_\sigma a_\nu \ . \tag{6.97}$$

Furthermore, we note that the Lie derivative of the effective momentum density with respect to the unit normal vector yields

$$\mathcal{L}_{\boldsymbol{n}} \mathring{j}_\sigma = \gamma_\sigma{}^\nu D_{\boldsymbol{n}} \mathring{j}_\nu - \mathring{j}^\nu K_{\sigma\nu} \ . \tag{6.98}$$

It follows that, after the introduction of the vector field $\boldsymbol{t}$, equation (6.96) becomes

$$\partial_t \mathring{j}_i = \alpha \left({}^3D_j \mathring{S}^j{}_i + \mathring{S}^k{}_i a_k - K \mathring{j}_i + \mathring{\rho} a_i \right) + \mathcal{L}_\beta \mathring{j}_i \ , \tag{6.99}$$

which is the evolution equation for the effective momentum density.

In this section, we have obtained the $3+1$ decomposition of the energy matter content for the EC theory of gravity. In particular, we have derived the energy conservation equation, which is given in (6.95), as well as the momentum conservation equation, which is given in (6.99).

6.6 GBSSN equations in spherical symmetry

In Section 6.4.2 we obtained the GBSSN equations for the EC theory of gravity. In this section, we wish to study these equations for the case where the conformal spatial metric tensor is spherically symmetric. The spherically symmetric, vacuum GBSSN equations for GR have been obtained before in [2, 26, 31]. Here, we shall extend these equations for the inclusion of torsion according to the EC theory of gravity. Here, we present these equations by adding the necessary modifications to what is done in [2, 26, 31].

6.6.1 Evolution equations

Here, we make use of spherical coordinates[5] (r, θ, ψ). Since we take the conformal spatial metric tensor to be spherically symmetric, we can write this tensor $\bar{\boldsymbol{\gamma}}$ as follows

$$\bar{\gamma}_{ij} = \begin{pmatrix} \bar{\gamma}_{rr} & 0 & 0 \\ 0 & \bar{\gamma}_{\theta\theta} & 0 \\ 0 & 0 & \bar{\gamma}_{\theta\theta} \sin^2 \theta \end{pmatrix} \ . \tag{6.100}$$

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[5] We note that we have used ψ to denote the azimuthal angle and reserved ϕ to denote the conformal scalar.

Furthermore, we make the following ansatz for the rescaled traceless part of the extrinsic curvature tensor

$$\bar{A}_{ij} = \bar{A}_{rr} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\bar{\gamma}_{\theta\theta}/\left(2\bar{\gamma}_{rr}\right) & 0 \\ 0 & 0 & -\bar{\gamma}_{\theta\theta}\sin^2\theta/\left(2\bar{\gamma}_{rr}\right) \end{pmatrix} . \tag{6.101}$$

As pointed out by Brown in [31], while in three-dimensional simulations it is necessary to ensure that $\bar{A}_{ij}$ is traceless, this is not necessary here since $\bar{A}_{ij}$ is traceless by construction.

The ansatz for the conformal Levi-Civita connection function is

$${}^3\bar{\Gamma}^i = \begin{pmatrix} {}^3\bar{\Gamma}^r \\ -\cos\theta/\left(\bar{\gamma}_{\theta\theta}\sin\theta\right) \\ 0 \end{pmatrix} . \tag{6.102}$$

With these assumptions in mind, we now turn our attention to stating the evolution equations. We begin with the evolution equation for the lapse which we take to be

$$\partial_t\alpha = \mu_1\beta^r\partial_r\alpha - 2\alpha K \ . \tag{6.103}$$

In addition, we take the evolution equation for the shift vector to be

$$\partial_t\beta^r = B^r + \mu_2\beta^r\partial_r\beta^r \ , \tag{6.104}$$

where B^r is referred to as the *auxiliary field* and has the following evolution equation

$$\partial_t B^r = \mu_3\partial_t {}^3\bar{\Gamma}^r + \mu_4\beta^r\partial_r B - \mu_5\beta^r\partial_r\overset{\circ}{\Gamma}{}^r - \eta B \ . \tag{6.105}$$

In the above, we have introduced the gauge parameters $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \eta \in \mathbb{R}$. These parameters are constant and will be specified later on when carrying out our simulations. We note, however, that it is necessary that μ_3 does not exceed $3/4$ in order to ensure that, at first order, the shift vector does not propagate faster than the speed of light. For an explanation of this fact, the interested reader is directed to [126].

From equation (6.72), one can show that the evolution equation for the conformal scalar function is

$$\partial_t\phi = \frac{\alpha}{6}K - \frac{v}{6}\partial_r\beta^r + v\beta^r\partial_r\phi - \frac{v\beta^r}{12}\left(\frac{\partial_r\bar{\gamma}_{rr}}{\bar{\gamma}_{rr}} + \frac{2\partial_r\bar{\gamma}_{\theta\theta}}{\bar{\gamma}_{\theta\theta}}\right) . \tag{6.106}$$

Equation (6.76) in this coordinate system gives the following evolution equation for the trace of the extrinsic curvature tensor

$$\begin{aligned} \partial_t K = {} & 4\pi\alpha\left(\overset{\circ}{S} + \overset{\circ}{\rho}\right) - \frac{e^{4\phi}}{\bar{\gamma}_{rr}}\left(\partial_r^2\alpha - \frac{\left(\partial_r\bar{\gamma}_{rr}\right)\left(\partial_r\alpha\right)}{2\bar{\gamma}_{rr}} - 2\left(\partial_r\phi\right)\left(\partial_r\alpha\right) + \frac{\left(\partial_r\bar{\gamma}_{\theta\theta}\right)\left(\partial_r\alpha\right)}{\bar{\gamma}_{\theta\theta}}\right) \\ & + \alpha\left(\frac{3\bar{A}_{rr}^2}{2\bar{\gamma}_{rr}^2} + \frac{K^2}{3}\right) + \beta^r\partial_r K \ . \end{aligned} \tag{6.107}$$

For the (r, r) component of the conformal spatial metric, equation (6.79) gives

$$\begin{aligned} \partial_t\bar{\gamma}_{rr} = {} & -2\alpha\bar{A}_{rr} + 2\left(1 - \frac{v}{3}\right)\bar{\gamma}_{rr}\partial_r\beta^r + \beta^r\partial_r\bar{\gamma}_{rr} + 4\left(v-1\right)\bar{\gamma}_{rr}\beta^r\partial_r\phi \\ & - \frac{v}{3}\bar{\gamma}_{rr}\beta^r\left(\frac{\partial_r\bar{\gamma}_{rr}}{\bar{\gamma}_{rr}} + \frac{2\partial_r\bar{\gamma}_{\theta\theta}}{\bar{\gamma}_{\theta\theta}}\right) . \end{aligned} \tag{6.108}$$

Similarly, equation (6.79) yields the following evolution equation for the (θ, θ) component of the conformal spatial metric

$$\begin{aligned} \partial_t\bar{\gamma}_{\theta\theta} = {} & \frac{\alpha\bar{\gamma}_{\theta\theta}A_{rr}}{\bar{\gamma}_{rr}} - \frac{2v}{3}\bar{\gamma}_{\theta\theta}\partial_r\beta^r + \beta^r\partial_r\bar{\gamma}_{\theta\theta} + 4\left(v-1\right)\bar{\gamma}_{\theta\theta}\beta^r\partial_r\phi \\ & - \frac{v}{3}\bar{\gamma}_{\theta\theta}\beta^r\left(\frac{\partial_r\bar{\gamma}_{rr}}{\bar{\gamma}_{rr}} + \frac{2\partial_r\bar{\gamma}_{\theta\theta}}{\bar{\gamma}_{\theta\theta}}\right) . \end{aligned} \tag{6.109}$$

We now turn our attention to obtaining the evolution equation for the rescaled traceless part of the extrinsic curvature tensor given the assumptions in equations (6.100) and (6.101). In order to do this, we first wish to find the components of the three-dimensional torsion-free Ricci tensor

$$ {}^3R_{ac} = \partial_d {}^3\Gamma^d{}_{ac} - \partial_a {}^3\Gamma^d{}_{dc} + {}^3\Gamma^e{}_{ac} {}^3\Gamma^d{}_{de} - {}^3\Gamma^e{}_{dc} {}^3\Gamma^d{}_{ae} \ . \tag{6.110} $$

The non-vanishing components of the conformal three-dimensional Levi-Cevita connection coefficient are

$$ \begin{aligned} &{}^3\bar{\Gamma}^r{}_{rr} = \frac{\partial_r \bar{\gamma}_{rr}}{2\bar{\gamma}_{rr}} \ , \quad {}^3\bar{\Gamma}^\theta{}_{r\theta} = {}^3\bar{\Gamma}^\psi{}_{r\psi} = \frac{\partial_r \bar{\gamma}_{\theta\theta}}{2\bar{\gamma}_{\theta\theta}} \ , \quad {}^3\bar{\Gamma}^r{}_{\theta\theta} = {}^3\bar{\Gamma}^r{}_{\psi\psi} / \sin^2\theta = -\frac{\partial_r \bar{\gamma}_{\theta\theta}}{2\bar{\gamma}_{rr}} \ , \\ &{}^3\bar{\Gamma}^\theta{}_{\psi\psi} = -\sin\theta\cos\theta \ , \quad {}^3\bar{\Gamma}^\psi{}_{\psi\theta} = \frac{\cos\theta}{\sin\theta} \ . \end{aligned} \tag{6.111} $$

In addition, we have

$$ C^r{}_{rr} = C^\theta{}_{r\theta} = C^\psi{}_{r\psi} = 2\partial_r\psi \ , \qquad C^r{}_{\theta\theta} = C^r{}_{\psi\psi} / \sin^2\theta = -\frac{2\bar{\gamma}_{\theta\theta}\partial_r\phi}{\bar{\gamma}_{rr}} \ , \tag{6.112} $$

and thus, we find that the non-vanishing components of the Levi-Civita connection coefficient are

$$ \begin{aligned} &{}^3\Gamma^r{}_{rr} = \frac{\partial_r \bar{\gamma}_{rr}}{2\bar{\gamma}_{rr}} - 2\partial_r\phi \ , \quad {}^3\Gamma^\theta{}_{r\theta} = {}^3\Gamma^\psi{}_{r\psi} = \frac{\partial_r \bar{\gamma}_{\theta\theta}}{2\bar{\gamma}_{\theta\theta}} - 2\partial_r\phi \ , \quad {}^3\Gamma^\psi{}_{\psi\theta} = \frac{\cos\theta}{\sin\theta} \ , \\ &{}^3\Gamma^\theta{}_{\psi\psi} = -\sin\theta\cos\theta \ , \quad {}^3\Gamma^r{}_{\theta\theta} = {}^3\Gamma^r{}_{\psi\psi} / \sin^2\theta = -\frac{\partial_r \bar{\gamma}_{\theta\theta}}{2\bar{\gamma}_{rr}} + \frac{2\bar{\gamma}_{\theta\theta}\partial_r\phi}{\bar{\gamma}_{rr}} \ . \end{aligned} \tag{6.113} $$

In addition, we make note of the following derivatives

$$ \begin{aligned} &\partial_d {}^3\Gamma^d{}_{rr} = \frac{\partial_r^2 \bar{\gamma}_{rr}}{2\bar{\gamma}_{rr}} - \frac{(\partial_r \bar{\gamma}_{rr})^2}{2\bar{\gamma}_{rr}^2} - 2\partial_r^2\phi \ , \qquad \partial_r {}^3\Gamma^d{}_{dr} = \frac{\partial_r^2 \bar{\gamma}_{rr}}{2\bar{\gamma}_{rr}} - \frac{(\partial_r \bar{\gamma}_{rr})^2}{2\bar{\gamma}_{rr}^2} + \frac{\partial_r^2 \bar{\gamma}_{\theta\theta}}{\bar{\gamma}_{\theta\theta}} - \frac{(\partial_r \bar{\gamma}_{\theta\theta})^2}{\bar{\gamma}_{\theta\theta}^2} - 6\partial_r^2\phi \ , \\ &\partial_\theta {}^3\Gamma^\psi{}_{\psi\theta} = -1 - \frac{\cos^2\theta}{\sin^2\theta} \qquad \partial_\theta {}^3\Gamma^\theta{}_{\psi\psi} = -\cos^2\theta + \sin^2\theta \ , \\ &\partial_d {}^3\Gamma^d{}_{\theta\theta} = -\frac{\partial_r^2 \bar{\gamma}_{\theta\theta}}{2\bar{\gamma}_{rr}} + \frac{(\partial_r \bar{\gamma}_{\theta\theta})(\partial_r \bar{\gamma}_{rr})}{2\bar{\gamma}_{rr}^2} + \frac{2\partial_r \bar{\gamma}_{\theta\theta}\partial_r\phi}{\bar{\gamma}_{rr}} - \frac{2\bar{\gamma}_{\theta\theta}(\partial_r \bar{\gamma}_{rr})(\partial_r\phi)}{\bar{\gamma}_{rr}^2} + \frac{2\bar{\gamma}_{\theta\theta}\partial_r^2\phi}{\bar{\gamma}_{rr}} \ . \end{aligned} \tag{6.114} $$

We can now compute the components of the three-dimensional torsion-free Ricci tensor which are given below

$$ {}^3R_{rr} = -\frac{\partial_r^2 \bar{\gamma}_{\theta\theta}}{\bar{\gamma}_{\theta\theta}} + \frac{(\partial_r \bar{\gamma}_{\theta\theta})^2}{\bar{\gamma}_{\theta\theta}^2} + 4\partial_r^2\phi + 2{}^3\Gamma^\theta{}_{r\theta} {}^3\Gamma^r{}_{rr} - 2\left({}^3\Gamma^\theta{}_{r\theta}\right)^2 \ . \tag{6.115} $$

$$ \begin{aligned} {}^3R_{\theta\theta} = {}^3R_{\psi\psi} / \sin^2\theta = &-\frac{\partial_r^2 \bar{\gamma}_{\theta\theta}}{2\bar{\gamma}_{rr}} + \frac{(\partial_r \bar{\gamma}_{\theta\theta})(\partial_r \bar{\gamma}_{rr})}{2\bar{\gamma}_{rr}^2} + \frac{2\partial_r \bar{\gamma}_{\theta\theta}\partial_r\phi}{\bar{\gamma}_{rr}} \\ &- \frac{2\bar{\gamma}_{\theta\theta}(\partial_r \bar{\gamma}_{rr})(\partial_r\phi)}{\bar{\gamma}_{rr}^2} + \frac{2\bar{\gamma}_{\theta\theta}\partial_r^2\phi}{\bar{\gamma}_{rr}} + {}^3\Gamma^r{}_{\theta\theta} {}^3\Gamma^r{}_{rr} + 1 \ . \end{aligned} \tag{6.116} $$

It follows that the three-dimensional Ricci scalar is

$$ {}^3R = {}^3\overset{\circ}{R}_{rr} / \bar{\gamma}_{rr} + 2{}^3R_{\theta\theta} / \bar{\gamma}_{\theta\theta} \ . \tag{6.117} $$

We can now determine the (r, r) component of the traceless part of R_{ij}

$$ {}^3R_{rr}^{\mathrm{TF}} = \frac{2}{3}\left({}^3R_{rr} - \bar{\gamma}_{rr} {}^3R_{\theta\theta} / \bar{\gamma}_{\theta\theta}\right) \ . \tag{6.118} $$

In addition, in order to compute the right-hand side of the evolution equation for $\bar{A}_{rr}$, we require

$$ \left({}^3D_r {}^3D_r\alpha\right)^{\mathrm{TF}} = \frac{2}{3}\left(\partial_r^2\alpha - {}^3\Gamma^r{}_{rr}\partial_r\alpha + \frac{\bar{\gamma}_{rr}}{\bar{\gamma}_{\theta\theta}} {}^3\Gamma^r{}_{\theta\theta}\partial_r\alpha\right) \ . \tag{6.119} $$

By making use of equation (6.82), we can now write the evolution equation for $\bar{A}_{rr}$ as follows

$$\begin{aligned}\partial_t \bar{A}_{rr} = &-\frac{2\alpha \bar{A}_{rr}^2}{\bar{\gamma}_{rr}} + \alpha K \bar{A}_{rr} - e^{4\phi}\left(\overset{\circ}{D}_r \overset{\circ}{D}_r \alpha\right)^{\mathrm{TF}} + \alpha e^{4\phi}\, {}^3\overset{\circ}{R}{}^{\mathrm{TF}}_{rr} + 2\left(1-\frac{v}{3}\right)\bar{A}_{rr}\partial_r\beta^r + \beta^r\partial_r\bar{A}_{rr} \\ &+ 4\left(v-1\right)\bar{A}_{rr}\beta^r\partial_r\phi - \frac{v}{3}\bar{A}_{rr}\beta^r\left(\frac{\partial_r\bar{\gamma}_{rr}}{\bar{\gamma}_{rr}} + \frac{2\partial_r\bar{\gamma}_{\theta\theta}}{\bar{\gamma}_{\theta\theta}}\right) - 8\pi\alpha e^{4\phi}\overset{\circ}{S}{}^{TF}_{rr}\ . \qquad (6.120)\end{aligned}$$

Lastly, through the use of equation (6.84), we can write the evolution equation for the conformal Levi-Civita connection function as

$$\begin{aligned}\partial_t {}^3\bar{\Gamma}^r = &\frac{2\alpha\bar{A}_{rr}}{\bar{\gamma}_{rr}^2}{}_3\bar{\Gamma}^r{}_{rr} - \frac{2\alpha\bar{A}_{rr}}{\bar{\gamma}_{rr}\bar{\gamma}_{\theta\theta}}{}_3\bar{\Gamma}^r{}_{\theta\theta} - \frac{4}{3}\frac{\alpha\partial_r K}{\bar{\gamma}_{rr}} - \frac{12\alpha\bar{A}_{rr}\partial_r\phi}{\bar{\gamma}_{rr}^2} + 4\left(1-v\right){}^3\bar{\Gamma}^r\beta^r\partial_r\phi \\ &- {}^3\bar{\Gamma}^r\partial_r\beta^r - \frac{2\bar{A}_{rr}\partial_r\alpha}{\bar{\gamma}_{rr}^2} + \beta^r\partial_r{}^3\bar{\Gamma}^r + \frac{\partial_r^2\beta^r}{\bar{\gamma}_{rr}} + 2\left(1-v\right)\frac{\partial_r\phi\partial_r\beta^r}{\bar{\gamma}_{rr}} + \frac{2\left(1-v\right)\beta^r\partial_r^2\phi}{\bar{\gamma}_{rr}} \\ &+ \frac{v_3}{3}\bar{\Gamma}^r\beta^r\left(\frac{\partial_r\bar{\gamma}_{rr}}{\bar{\gamma}_{rr}} + \frac{2\partial_r\bar{\gamma}_{\theta\theta}}{\bar{\gamma}_{\theta\theta}}\right) + \frac{2v_3}{3}\bar{\Gamma}^r\partial_r\beta^r - \frac{16\pi\alpha\overset{\circ}{j}_r}{\bar{\gamma}_{rr}} \\ &+ \frac{v}{3\bar{\gamma}_{rr}}\left[\partial_r^2\beta^r + \partial_r\beta^r\left(\frac{\partial_r\bar{\gamma}_{rr}}{2\bar{\gamma}_{rr}} + \frac{\partial_r\bar{\gamma}_{\theta\theta}}{\bar{\gamma}_{\theta\theta}}\right) + \beta^r\left(\frac{\partial_r^2\bar{\gamma}_{rr}}{2\bar{\gamma}_{rr}} - \frac{\left(\partial_r\bar{\gamma}_{rr}\right)^2}{2\bar{\gamma}_{rr}^2} + \frac{\partial_r^2\bar{\gamma}_{\theta\theta}}{\bar{\gamma}_{\theta\theta}} - \frac{\left(\partial_r\bar{\gamma}_{\theta\theta}\right)^2}{\bar{\gamma}_{\theta\theta}^2}\right)\right]\ . \\ &\qquad (6.121)\end{aligned}$$

We have now written the GBSSN evolution equations for the EC theory of gravity given the assumptions in equations (6.100), (6.101) and (6.102).

6.7 Puncture evolution with GBSSN

In this section, we carry out a numerical evolution of a black hole (BH) puncture using the GBSSN formulation of EC theory. Here, we take the conformal spatial metric to be spherically symmetric and consider the case of a vacuum. Since we are considering a vanishing energy-matter content, the EC description coincides with the GR description. We note that such a study has been carried out before in [2, 26, 31]. Here, we follow what is done in these aforesaid references in order to construct our 1D code used for BH puncture evolution. We leave the inclusion of torsion into this 1D as the subject of future work.

6.7.1 System of evolution equations

In carrying out our numerical simulations, we do not evolve the GBSSN equations given in Section 6.6.1 as they are stated. Instead, we have found that by inroducing new quantities, as was done in [26], we obtain a more robust and stable code. Below, we outline the modified system of GBSSN equations used for our simulations.

By following what was done in [26], we define $a := \bar{\gamma}_{rr}$, $b := \bar{\gamma}_{\theta\theta}/r^2$ and $A_a := \bar{A}_r{}^r$. We now wish to introduce these new quantities into the GBSN evolution equations and, in addition, find the evolution equations for these new variables. The evolution equation for the lapse function is unchanged and, for completeness, state the equation below again

$$\partial_t\alpha = \mu_1\beta^r\alpha' - 2\alpha K\ , \qquad (6.122)$$

where we have used a prime to denote partial differentiation with respect to r. The evolution equation for the conformal scalar, equation (6.106), can now be written as

$$\partial_t\phi = \frac{\alpha}{6}K - \frac{v}{6}\beta^{r\prime} + v\beta^r\phi' - \frac{v\beta^r}{12}\left(\frac{a'}{a} + \frac{4}{r} + \frac{2b'}{b}\right)\ . \qquad (6.123)$$

The evolution equation (6.107) for the trace of the extrinsic curvature tensor is now written as

$$\partial_t K = -\frac{e^{4\phi}}{a}\left[\alpha'' - \alpha'\left(\frac{a'}{2a} + 2\phi' - \frac{2}{r} - \frac{b'}{b}\right)\right] + \alpha\left(\frac{3}{2}A_a^2 + \frac{K^2}{3}\right)\ . \qquad (6.124)$$

For the radial component of the conformal spatial metric, equation (6.108) gives us the following evolution equation

$$\partial_t a = -2\alpha a A_a + 2\left(1 - \frac{v}{3}\right) a\beta^{r\prime} + \beta^r a' + 4\,(v-1)\,a\beta^r\phi' - \frac{v}{3} a\beta^r \left(\frac{a'}{a} + \frac{4}{r} + \frac{2b'}{b}\right) . \tag{6.125}$$

By making use of equation (6.109), one can find the following evolution equation for the new variable b

$$\partial_t b = \alpha b A_a - \frac{2v}{3} b\beta^{r\prime} + \frac{2\beta^r b}{r} + \beta^r b' + 4\,(v-1)\,b\beta^r\phi' - \frac{v}{3} b\beta^r \left(\frac{a'}{a} + \frac{4}{r} + \frac{2b'}{b}\right) . \tag{6.126}$$

Before moving on to state the remaining evolution equations, we wish to make a note on the evolution equation for the shift vector given in equation (6.104). In this equation, one makes use of the conformal connection function ${}^3\bar{\Gamma}^r$. In Minkowski space-time, one has ${}^3\bar{\Gamma}^r = -2/r$ which implies that the conformal connection function is singular at the origin. This led Brown to introduce a regularised conformal connection function in [31]. This regularised connection function was of the form ${}^3\bar{\Gamma}^r_{\text{reg}} := \bar{\Gamma}^r + 2/r$. While ${}^3\bar{\Gamma}^r_{\text{reg}}$ is not a true tensor, the regularised quantity

$$\bar{\Delta}^r := {}^3\bar{\Gamma}^r - {}^3\bar{\gamma}^{ij}\,\mathring{\Gamma}^r{}_{ij} = \bar{\Gamma}^r + \frac{2}{rb} , \tag{6.127}$$

where ${}^3\mathring{\Gamma}^k{}_{ij}$ is obtained from the Minkowski metric in spherical coordinates, is in fact a true tensor. The constraint equation for this quantity can be written as

$$\bar{\Delta}^r = \frac{a'}{2a^2} - \frac{b'}{ab} + \frac{2}{rb} - \frac{2}{ra} . \tag{6.128}$$

This regularised quantity was made use of in [26] and is what we introduce into our 1D code. To this end, we introduce the following quantity

$$\Delta^k{}_{ij} = {}^3\Gamma^k{}_{ij} - {}^3\mathring{\Gamma}^k{}_{ij} . \tag{6.129}$$

We will introduce this into the construction of the Ricci tensor. For now, let us write the evolution equation for A_a

$$\partial_t A_a = \alpha K A_a - \frac{e^{4\phi}}{a}\left({}^3D_r{}^3D_r\alpha\right)^{\text{TF}} + \frac{\alpha e^{4\phi}}{a}{}^3R^{\text{TF}}_{rr} + \beta^r A_a' . \tag{6.130}$$

We remind the reader that the superscript TF is used to denoted the trace-free part of the tensor. On comparing the evolution equation for A_a with the evolution equation (6.120) for $\bar{A}_{rr}$, one can notice that for A_a the evolution equation contains only one term that carries a dependence on the shift vector. This is the advection term which appears as the last term on the right-hand side of the above expression.

In order to compute the right-hand side of the above expression, we note that

$$\left({}^3D_r{}^3D_r\alpha\right)^{\text{TF}} = \frac{2}{3}\left[\alpha'' - \alpha'\left(\Gamma^r{}_{rr} + \Gamma^\theta{}_{r\theta}\right)\right] . \tag{6.131}$$

In addition, we also note that the (r,r) component of the trace-free part of the Ricci tensor is as follows

$$\begin{aligned} {}^3R^{\text{TF}}_{rr} = & -\frac{1}{r^2} + \frac{2b'}{rb} + \frac{b''}{2b} - \frac{a''}{2a} + \left(\frac{a'}{2a^2} - \frac{b'}{ab}\right) a' + a\left[\bar{\Delta}^{r\prime} + \frac{2}{rb}\left(\frac{1}{r} + \frac{b'}{b}\right)\right] + \frac{a'^2}{2a^2} + 2\phi'' \\ & - \frac{a'}{2a}\left(\frac{2}{r} + \frac{b'}{b}\right) - 2\phi'\left(\frac{2}{r} + \frac{b'}{b} - \frac{a'}{a}\right) - \frac{a}{r^2 b} + 3\Gamma^r{}_{rr}\Gamma^\theta{}_{r\theta} - 4\mathring{\Gamma}^\theta{}_{\theta r}\Delta^\theta{}_{\theta r} - 2\Delta^\theta{}_{\theta r}\Delta^\theta{}_{\theta r} . \end{aligned} \tag{6.132}$$

An important note to make is that we have introduced $\bar{\Delta}^r$ into the above expression. In addition, we note that our code is more stable when we use the common practice of including $\bar{\Delta}^r$ into the evolution equations only when it appears differentiated. This is also what is done by Brown in [31] when introducing ${}^{3}\bar{\Gamma}^{r}_{\text{reg}}$.

It now remains to find the evolution equation for $\bar{\Delta}^r$. By making use of equations (6.121), (6.128) and (6.126) we obtain

$$\begin{aligned}
\partial_t \bar{\Delta}^r = {} & \frac{2\alpha A_a}{a}\left(\bar{\Gamma}^{r}{}_{rr} + \bar{\Gamma}^{\theta}{}_{r\theta}\right) - \frac{4}{3}\frac{\alpha K'}{a} - \frac{12\alpha A_a \phi'}{a} + 4\left(1-v\right)\beta^r\phi'\left(\frac{a'}{2a^2} - \frac{b'}{ab} - \frac{2}{ar}\right) \\
& - \left(\frac{a'}{2a^2} - \frac{b'}{ab} - \frac{2}{ar}\right)\beta^{r\prime} - \frac{2A_a\alpha'}{a} + \beta^r\left(\bar{\Delta}^{r\prime} + \frac{2}{r^2 b} + \frac{2b'}{b^2 r}\right) + \frac{\beta^{r\prime\prime}}{a} + \frac{2\left(1-v\right)\phi'\beta^{r\prime}}{a} + \frac{2(1-v)\beta^r\phi''}{a} \\
& + \frac{v\beta^r}{3}\left(\frac{a'}{2a^2} - \frac{b'}{ab} - \frac{2}{ar}\right)\left(\frac{a'}{a} + \frac{4}{r} + \frac{2b'}{b}\right) + \frac{2v\beta^{r\prime}}{3}\left(\frac{a'}{2a^2} - \frac{b'}{ab} - \frac{2}{ar}\right) \\
& + \frac{v}{3a}\left[\beta^{r\prime\prime} + \beta^{r\prime}\left(\frac{a'}{2a} + \frac{2}{r} + \frac{b'}{b}\right) + \beta^r\left(\frac{a''}{a} - \frac{a'^2}{a^2} - a'\left(\frac{a'}{2a^2} - \frac{b'}{ab}\right)\right.\right. \\
& \left.\left. - a\left(\frac{a''}{2a^2} - \frac{a'^2}{a^3} - \frac{b''}{ab} + \frac{b'^2}{ab^2} + \frac{a'b'}{a^2 b} + \frac{2a'}{a^2 r} + \frac{2}{ar^2}\right)\right)\right] \\
& - \frac{2}{rb^2}\left[\alpha b A_a - \frac{2v}{3}b\beta^{r\prime} + \frac{2\beta^r b}{r} + \beta^r b' + 4\left(v-1\right)b\beta^r\phi' - \frac{v}{3}b\beta^r\left(\frac{a'}{a} + \frac{4}{r} + \frac{2b'}{b}\right)\right] . \qquad (6.133)
\end{aligned}$$

In the above expression, we have only included $\bar{\Delta}^r$ when it appears differentiated and as part of the advection term $\beta^r\bar{\Delta}^{r\prime}$.

Having now stated the evolution of $\bar{\Delta}^r$, we can now write the evolution equation for the shift. This is given as follows

$$\partial_t\beta^r = B^r + \mu_2\beta^r\beta^{r\prime} \ . \qquad (6.134)$$

For the evolution equation for the auxiliary field B^r, we replace the conformal connection function with the regularised quantity $\bar{\Delta}^r$. That is, we use

$$\partial_t B^r = \mu_3\partial_t\bar{\Delta}^r + \mu_4\beta^r B' - \mu_5\beta^r\bar{\Delta}^{r\prime} - \eta B \ . \qquad (6.135)$$

This choice of gauge is referred to as the *Gamma-driver condition* [26, 31]. In our analysis, we follow [2, 26] and set $\mu_2 = \mu_4 = \mu_5 = 0$ and we set $\mu_3 = 3/4$. In addition, we set the dampening coefficient to $\eta = 3/2$. For the evolution of the lapse function, we set $\mu_1 = 0$ in equation (6.103). Furthermore, since we are conisdering the case of a vacuum, we have already set $\tau_{ij} = 0$ and $s_k{}^{ij} = 0$. As a result of Theorem 4.2.1, it follows that, since the spin-current tensor is vanishing, we have a vanishing torsion tensor. Therefore, the GBSSN equations used here are the same as when considering GR. As noted in [31], the Lagrange condition is more physical than the Euler condition and, therefore, we set $v = 1$ in our analysis.

Lastly, in order to obtain better stability, instead of evolving the conformal scalar function ϕ, we evolve the quantity χ which is defined below

$$\chi := \mathrm{e}^{2\phi} \ . \qquad (6.136)$$

This is also what is done in [2, 26].

6.7.2 Numerics and initial data

In the construction of our 1D code, we make use of the fourth-order Runge-Kutta (RK) method to evolve the system of evolution equations given above. Before stating these methods, we first wish to make a note on the grid construction used. Here, we use a staggered grid approach by placing the origin between the first two grid points. That is, we construct our radial gid points, r_n, as [31]

$$r_n = (-1/2 + n)\,\mathrm{d}r \ , \qquad (6.137)$$

where $\mathrm{d}r$ is our space-step and $n \in \{0, \ldots, N-1\}$. In addition we note that, for our analysis, we set $N = 30000$ in order to place the outer boundary sufficiently far-away. For the space-step, we use $\mathrm{d}r = 0.01$ and for the time-step we use $\mathrm{d}t = \mathrm{d}r/2$. Furthermore, we set the mass of the BH to be $M = 1$.

It is clear from the above construction that the origin lies between the grid points $r_0 = -\mathrm{d}r/2$ and $r_1 = \mathrm{d}r/2$. We now wish to state the initial data to be used. For the lapse function, we make use of the so-called *pre-collapsed* initial data, which is also used in [31], which is of the form

$$\alpha(r,0) = \left(1 + \frac{M}{2r}\right)^{-2} , \tag{6.138}$$

where $M = 1$ is the mass of the BH [26]. For the components of the conformal spatial metric, we use the initial data associated with a unit 2-sphere, i.e., we use

$$a(r,0) = b(r,0) = 1 . \tag{6.139}$$

In order for the spatial metric to be the Schwarzschild BH solution at the initial state $t = 0$, we take the initial data of the conformal scalar function to be

$$\phi(r,0) = -\ln\left(1 + \frac{M}{2r}\right) \implies \chi(r,0) = \left(1 + \frac{M}{2r}\right)^{-2} . \tag{6.140}$$

Lastly, we take the initial data for the trace and rescaled traceless parts of the extrinsic curvature tensor to be vanishing, i.e., we set $K(r,0) = \bar{A}_a(r,0) = 0$.

Having stated the initial data, we now wish to mention the boundary conditions that are used in our analysis. Here, we make use of symmetric boundary conditions for scalars as well as type $(0,2)$ tensors at the inner boundary. For vectors at the inner boundary, we use antisymmetric boundary conditions. More specifically, we set $\alpha(r_0,t_m) = \alpha(r_1,t_m)$, $a(r_0,t_m) = a(r_1,t_m)$, $b(r_0,t_m) = b(r_1,t_m)$, $\chi(r_0,t_m) = \chi(r_1,t_m)$ $K(r_0,t_m) = K(r_1,t_m)$, $A_a(r_0,t_m) = A_a(r_1,t_m)$, $\beta_r(r_0,t_m) = \beta_r(r_1,t_m)$, $B^r(r_0,t_m) = B^r(r_1,t_m)$ and $\bar{\Delta}^r(r_0,t_m) = \bar{\Delta}^r(r_1,t_m)$. For a more in-depth discussion on this choice of inner boundary conditions, the interested reader is directed to [126]. For the outer boundary conditions, we simply freeze the last grid points according to the initial data. That is, we set $\alpha(r_{N-1},t_m) = \alpha(r_{N-1},t_0)$, $a(r_{N-1},t_m) = a(r_{N-1},t_0)$, $b(r_{N-1},t_m) = b(r_{N-1},t_0)$, $\chi(r_{N-1},t_m) = \chi(r_{N-1},t_0)$, $K(r_{N-1},t_m) = K(r_{N-1},t_0)$, $A_a(r_{N-1},t_m) = A_a(r_{N-1},t_0)$, $\beta^r(r_{N-1},t_m) = \beta^r(r_{N-1},t_0)$, $B^r(r_{N-1},t_m) = B^r(r_{N-1},t_0)$ and $\bar{\Delta}^r(r_{N-1},t_m) = \bar{\Delta}^r(r_{N-1},t_0)$.

We now turn our attention to the time evolution of the GBSSN equations. Below, we briefly discuss the fourth-order RK numerical scheme which is used to evolve the system. For this scheme, we set the time step to be $\mathrm{d}t = \mathrm{d}r/2$. In addition, for advection terms such as $\beta^r a'$, we use the forward finite-difference scheme to calculate first-order spatial derivatives. That is, for some quantity X, we write

$$\partial_r X(r_n,t_m) \approx \frac{X(r_{n+1},t_m) - X(r_n,t_m)}{\mathrm{d}r} . \tag{6.141}$$

For all other derivatives, we use a second order finite difference stencil. That is, we calculate the first-order spatial derivatives using [127]

$$\partial_r X(r_n,t_m) \approx \frac{X(r_{n+1},t_m) - X(r_{n-1},t_m)}{2\ \mathrm{d}r} , \tag{6.142}$$

and calculate the second-order spatial derivatives using

$$\partial_r^2 X(r_n,t_m) \approx \frac{X(r_{n+2},t_m) - 2X(r_n,t_m) + X(r_{n-1},t_m)}{\mathrm{d}r^2} . \tag{6.143}$$

We now wish to briefly outline the fourth-oder RK method following [127]. Let $X_i \in \{\alpha, a, b, \chi, K, \bar{A}_a, \beta^r, B^r, \bar{\Delta}^r\}$ and let f_i denote the right-hand side of the GBSSN evolution equation associated with X_i. Given $X_i(r_n, t_m)$, we calculate the quantity X_i at the following time-step through

$$X_i(r_n, t_{m+1}) = X_i(r_n, t_m) + \mathrm{d}r\left(k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4,i}\right)/6 \ , \tag{6.144}$$

where we define

$$k_{1,i} := f_i\left(X_j\right) \ , \tag{6.145}$$

$$k_{2,i} := f_i\left(X_j + \mathrm{d}r\frac{k_{1,j}}{2}\right) \ , \tag{6.146}$$

$$k_{3,i} := f_i\left(X_j + \mathrm{d}r\frac{k_{2,j}}{2}\right) \ , \tag{6.147}$$

$$k_{4,i} := f_i\left(X_j + \mathrm{d}r k_{3,j}\right) \ . \tag{6.148}$$

By making use of the above scheme, we evolve the vacuum GBSSN equations using the Gamma-driver condition. We remind the reader that, for our analysis, we consider the case where $\eta = 3/2$. In addition, we note that we carry out our simulations for up to $t = 100M$ in order to show the behaviour of the solutions for later times as well as to demonstrate the stability of the code.

6.7.3 Results

Here, we present the results obtained using our 1D code[6]. In Figure 6.1, we plot the lapse function against r/M at time values of $t = 0M$, $5M$, $10M$ and $15M$. In the left plot, we show the lapse function for all 30000 grid points. In the right plot, we show the lapse function for the first 3000 grid points. In Figure 6.2 we show the radial component a for $t = 0$, $5M$, $10M$ and $15M$. In the left plot, we show a for all 30000 grid points whereas in the right plot we show a for the first 5000 grid points. Figures 6.3 and 6.4 show the plots of the shift β^r and $\ln(1-\chi)$ respectively.

In Figure 6.5, we plot a against r/M at time $t = 100M$. Similarly, in Figure 6.6, we plot β^r at time $t = 100M$. In Figure 6.7, we plot the maximum value of a in the left plot as well as the maximum value of β^r in the right plot. These maximum values are plotted against time up until $t = 100M$. We also plot the log of the root-mean-square (RMS) values of the Hamiltonian constraint $\mathcal{H}$ up to $t = 100M$ in Figure 6.8. It is evident from Figures 6.7 and 6.8 that the numerical solution starts to converge around $t \sim 20M$ as expected [2–4].

We also plot the apparent horizon position and mass. In order to calculate the position of the apparent horizon, we make use of the so-called *apparent horizon function* given in [128] and stated below

$$\Theta(a, b, \chi, A_a, K) := \frac{\chi}{\sqrt{a}}\left(-\frac{2\chi'}{\chi} + \frac{2}{r} + \frac{b'}{b}\right) + A_a - \frac{2K}{3} \ . \tag{6.149}$$

In order to find the apparent horizon, r_{AH}, we solve the equation $\Theta(a, b, \chi, A_a, K) = 0$ at each time-step. For a detailed discussion on finding apparent horizons in the context of numerical relativity, the reader is directed to [128] by Thornburg. Once we have determined r_{AH} for a given time-step, we calculate the apparent horizon mass. This is done by evaluating the below expression at r_{AH}

$$M_{\mathrm{AH}} := \frac{\sqrt{r^2 b\sqrt{a}}}{2\chi} \ . \tag{6.150}$$

In Figure 6.9 we plot the position of the apparent horizon as well as the apparent horizon mass M_{AH} in the left and right plots respectively. It is evident from these plots that the apparent horizon position and mass begin to stabilise around $t \sim 20M$. In addition, we note that the apparent horizon mass remains within 1.6% of unity for the entire duration of the simulation. We also note that, at $t = 100M$, we have $r_{\mathrm{AH}} \approx 0.977$ and $M_{\mathrm{AH}} \approx 1.005$.

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[6] This 1D code constructed by the author is available at `https://www.dropbox.com/sh/cfybc2cysixchlz/AACjKq0NHComUPoXcZz90DH7a?dl=0` and is entitled *numerical_ simulation.cpp*. In addition, we have also made available the Python codes that were used to plot the results.

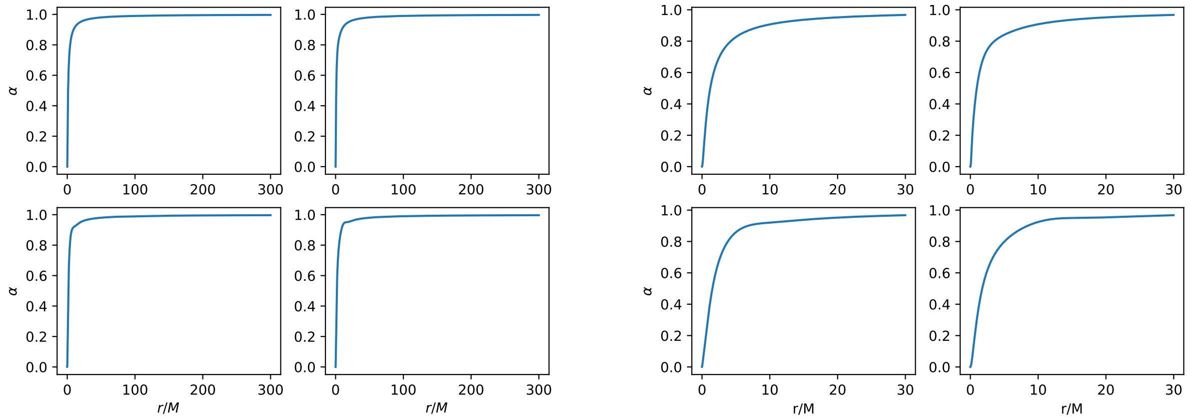

Figure 6.1: Plots of the lapse function α where we have taken the damping coefficient to be $\eta = 3/2$. In the left plot, we have shown the lapse function for all 30000 grid points whereas in the right plot we have shown this quantity for the first 3000 grid points. In both plots, the top-left panel shows the profile at $t = 0M$, the top-right at $t = 5M$, the bottom-left at $t = 10M$ and the bottom-right at $t = 15M$.

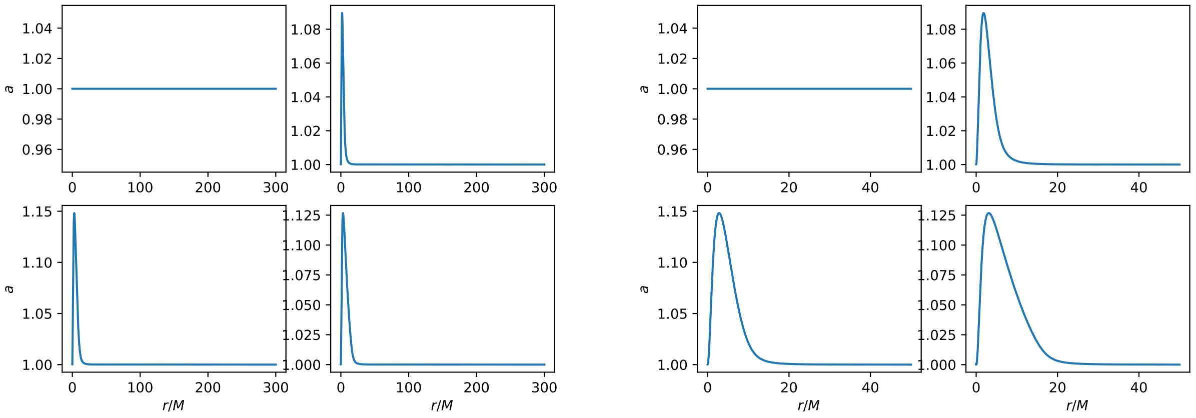

Figure 6.2: Plots of the radial component a where we have taken the damping coefficient to be $\eta = 3/2$. In the left plot, we have shown the radial component for all 30000 grid points whereas in the right plot we have shown this quantity for the first 5000 grid points. In both plots, the top-left panel shows the profile at $t = 0M$, the top-right at $t = 5M$, the bottom-left at $t = 10M$ and the bottom-right at $t = 15M$.

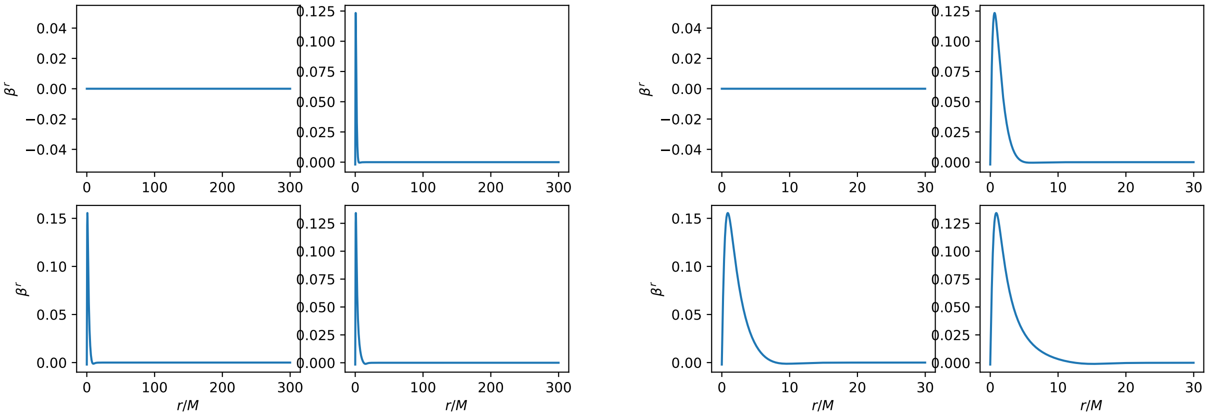

Figure 6.3: Plots of the shift β^r where we have taken the damping coefficient to be $\eta = 3/2$. In the left plot, we have shown the shift for all 30000 grid points whereas in the right plot we have shown this quantity for the first 3000 grid points. In both plots, the top-left panel shows the profile at $t = 0M$, the top-right at $t = 5M$, the bottom-left at $t = 10M$ and the bottom-right at $t = 15M$.

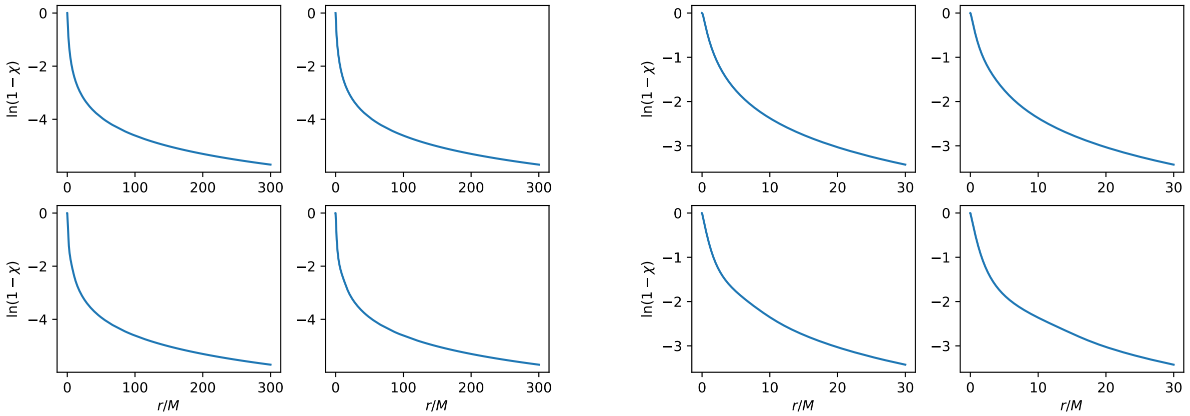

Figure 6.4: Plots of $\ln(1-\chi)$ where we have taken the damping coefficient to be $\eta = 3/2$. In the left plot, we have given the profile for all 30000 grid points whereas in the right plot we have shown this quantity for the first 3000 grid points. In both plots, the top-left panel shows the profile at $t = 0M$, the top-right at $t = 5M$, the bottom-left at $t = 10M$ and the bottom-right at $t = 15M$.

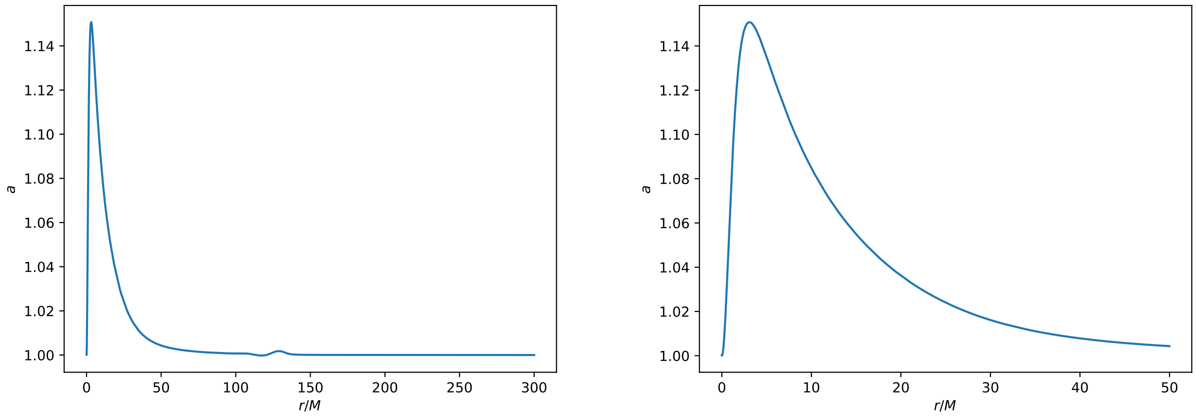

Figure 6.5: Plots of the radial component a for the case where $\eta = 3/2$ at $t = 100M$. In the left plot, we show the radial component a for all 30000 grid points whereas in the right plot we have shown a for the first 5000 grid points.

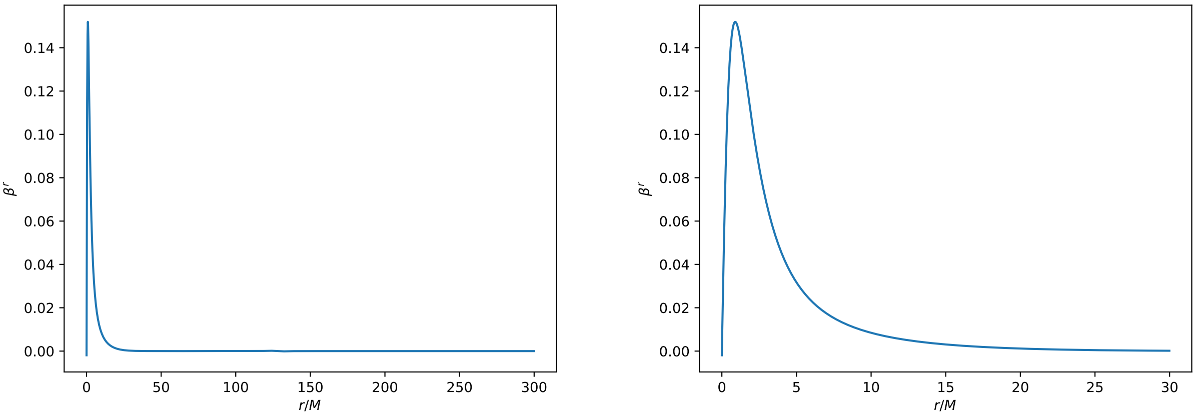

Figure 6.6: Plots of the shift β^r for the case where $\eta = 3/2$ at $t = 100M$. In the left plot, we show the shift for all 30000 grid points whereas in the right plot we have shown the shift for the first 3000 grid points.

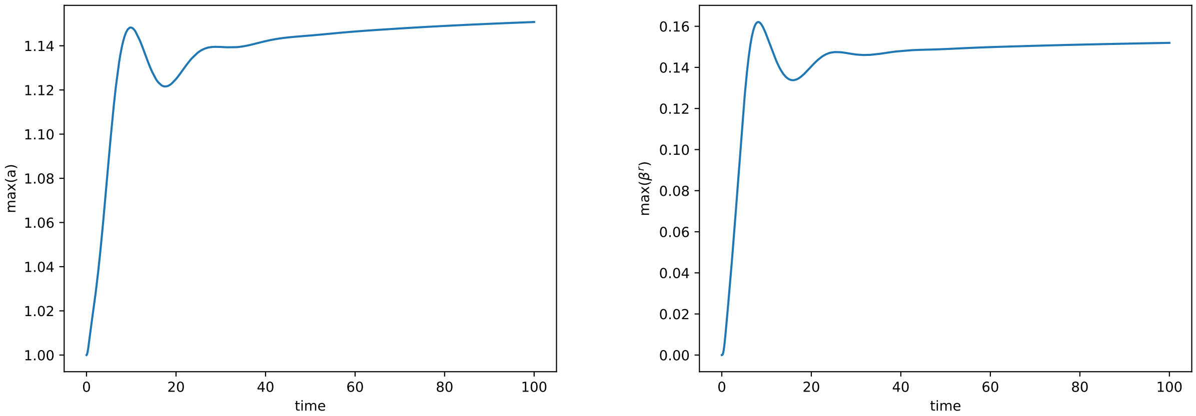

Figure 6.7: Plots of the maximum values of radial component a and the shift β^r for the first $t = 100M$. The left plot shows the evolution of the maximum value of the radial component a while the right plot shows the evolution of the maximum value of the shift β^r. It is clear from these plots that the solution starts to stabilise around $t \sim 20M$ as expected [2–4].

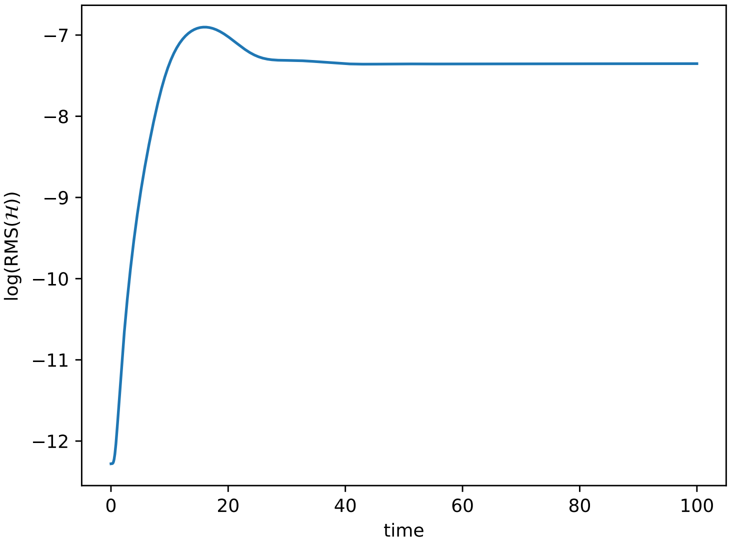

Figure 6.8: Plot of the natural log of the root-mean-square (RMS) of the Hamiltonain constraint $\mathcal{H}$. It is clear from this plot that the numerical solution begins to stabilise around $t \sim 20M$ which is expected.

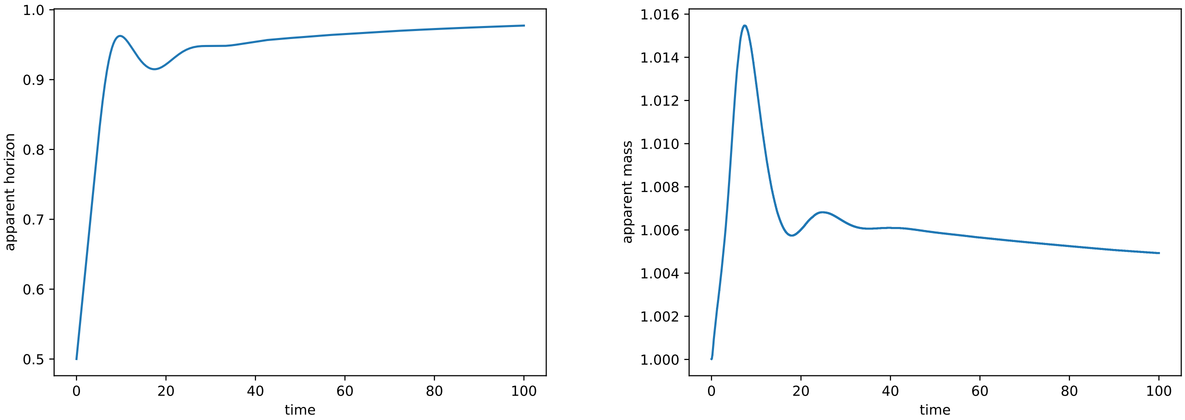

Figure 6.9: The left plot shows the position of the apparent horizon whereas the right plot shows the apparent horizon mass M_{AH}. It is clear from both plots that the solution starts to stabilise around $t \sim 20M$. In addition, the apparent horizon mass does not drift more than 1.6% away from unity.

6.8 Chapter conclusions

In this chapter, we have derived the ADM and BSSN formulations of the EC theory of gravity by following what has been done in [19] with regards to GR. In addition, we presented a $3+1$ decomposition of the energy-matter content for the EC theory of gravity by following what was done in [30] for the case of GR. Before we derived the ADM formulation of the EC theory of gravity, we first studied spacelike hypersurfaces of the Riemann-Cartan space-time which is the underlying space-time of the EC theory of gravity. In doing this, we derived the non-null torsion Gauss, Codazzi and Ricci equations. We noted that the well-known torsion-free Gauss, Codazzi and Ricci equations are well-defined on a space-like hypersurface of U_4 since the three-dimensional Levi-Civita connection is well-defined.

Following this, we turned our attention to determining an ADM formulation of the EC theory of gravity by making use of the EC field equations discussed in Section 4.2. In order to derive the ADM equations, it was necessary to introduce a lapse function. In the ADM formulation of GR, one simply defines the lapse functions through the natural choice of an acceleration vector. In the case of the EC theory, the underlying space-time U_4 admits both the Cartan connection as well as the Levi-Civita connection. Therefore, there are two acceleration vectors: one which measures the acceleration of the unit normal vector as described by the Cartan connection and the other which measures the acceleration as described by the Levi-Civita connection. Since there are two acceleration vectors, there are two choices for the lapse function. In order for the lapse function to describe how much proper time has elapsed, it is necessary to define the lapse function using the acceleration vector that is associated with the Levi-Civita connection. The reason for this is that, since test particles follow auto-parallel curves of the Levi-Civita connection, the acceleration vector associated with this connection will give the proper acceleration of the unit normal vector to the hypersurface.

Since the torsion tensor is determined entirely by the spin-current tensor in the EC theory [7], we derived the ADM equations by making use of the torsion-free Gauss, Codazzi and Ricci equations and introduced the torsion tensor through the use of the effective energy-momentum tensor. This concluded our derivation of the ADM formulation of the EC theory. Following this, we turned our attention to finding a $3+1$ decomposition of the energy-matter content. In order to carry out such a decomposition, we considered the effective energy-momentum tensor which is both symmetric and, by the second torsion-free Bianchi identity (3.56), conserved. By following what was done in [30], we obtained the evolution equations for the effective energy density and the effective momentum density.

Having derived the ADM formulation of the EC theory, we then derived the GBSSN formulation of the EC theory which is a modification of the ADM formulation. This was carried out by again following what was done in [19, 20] for the case of GR. While the BSSN formulation assumes that the determinant of the conformal spatial metric is unity, this is not assumed in the more general GBSSN formulation. As pointed out in [25, 31], the determinant of the conformal spatial metric in spherical coordinates is not unity, implying the need for the GBSSN formulation.

We concluded this chapter by considering the vacuum GBSSN evolution equations in the case of spherical symmetry. Since the EC theory reduces to GR in the case of a vanishing spin-current tensor, in this instance the GBSSN evolution equations reduce to that of GR which are reported in [26, 31]. We constructed a 1D code to numerically evolve the system of GBSSN equations that describes the evolution of a BH puncture. The 1D code that we constructed made use of the fourth-order RK method for time integration. We reported our results as profile plots at time values of $t = 0M, 5M, 10M, 15M$ and $100M$. In addition, we reported the evolution of the maximum values of a and β^r up to $t = 100M$ as well as the evolution of the log of the RMS of the Hamiltonian constraint up to $t = 100M$. We note that the application of GBSSN in the case of spherical symmetry in GR has been done before in [2, 26, 31].

Chapter 7

Conclusions

In this thesis, we have studied gravitational theories for which the natural choice of an affine connection is metric compatible and not necessarily symmetric. That is, we allow for the torsion tensor to be non-vanishing. Below, we mention the main points that have been discussed in this work.

In Chapter 2, we discussed some of the mathematical notions needed to construct a definition of space-time. In particular, following [55, 66], we introduced the definition of a manifold that we made use of throughout this thesis. Following this, we discussed the notions of the tangent space and dual space. We also discussed how integration is carried out on a manifold. Such a discussion proved to be useful since, when constructing gravitational theories, an action integral is required for the definition of the theory. We concluded the chapter by discussing the notion of a fibre bundle and mentioned how the Teleparallel Equivalent of General Relativity (TEGR) can be written as a gauge theory of the translation group [8] while the Einstein-Cartan (EC) theory can be written as a gauge theory of the Poincaré group [70].

In Chapter 3, we introduced the definitions of the various space-times that we made use of throughout this thesis. We defined the Riemann-Cartan space-time to be the tuple consisting of the space-time manifold, the metric tensor and the Cartan connection as the natural choice of an affine connection [9]. Furthermore, in the event that the Cartan connection is the Weitzenböck connection, the space-time was then referred to as the Weitzenböck space-time. In addition to these definitions, we discussed the notions of auto-parallel and extremal curves in these space-times. In particular we noted that since test particles follow extremal curves, test particles follow auto-parallel curves of the Levi-Civita connection and not necessarily auto-parallel curves of the Cartan connection [74]. We discussed three Raychaudhuri equations that can be constructed on the Riemann-Cartan space-time. We argued that, since test particles follow auto-parallel curves of the Levi-Civita connection, one must turn to the torsion-free Raychaudhuri equation to derive the relevant focusing conditions of the theory. This argument is what led us to make use of the torsion-free Raychaudhuri equation in Chapter 5 when determining the focusing conditions for $f(T)$ theories.

In Chapter 4, we reviewed some of the gravitational theories that have non-vanishing torsion. We began the chapter by first discussing the EC theory of gravity by following what was done in [7, 9]. Following this, we reviewed the Einstein-Cartan-Saa (ECS) theory which was first presented in [11]. We noted that, in order to obtain a volume element that yields a pure boundary term upon the integration of the Cartan divergence of an arbitrary vector field, one must ensure that this volume element is parallel with respect to the transpose of the Cartan connection. Another modification to the EC theory that we studied was the Chern-Simons Einstein-Cartan (CSEC) theory which was constructed based on what was done in [13, 14] for the case of General Relativity (GR). We concluded the chapter by reviewing the $f(T)$ theories of gravity.

In Chapter 5, we studied the focusing conditions that arise in the context of the $f(T)$ theories of gravity. In particular we derived, for the first time, the $f(T)$ focusing conditions for a one-parameter dependent congruence of timelike auto-parallel curves of the Levi-Civita connection. We then studied these focusing conditions for three bi-parametric $f(T)$ cosmological models. These results have given rise to the article [65]. Future prospects could involve determining these focusing conditions for the spatially-flat Robertson-Walker space-time in spherical coordinates.

In Chapter 6, we derived the Arnowitt-Deser-Misner (ADM) and Generalised-Baumgarte-Shapiro-Shibata-Nakamura (GBSSN) formulations of the EC theory. We first derived the non-null torsion and torsion-free Gauss, Codazzi and Ricci equations given a Riemann-Cartan space-time. We then made use of the torsion-free Gauss, Codazzi and Ricci equations together with the effective energy-momentum tensor in order to derive the ADM equations. Since the torsion tensor was included entirely in the energy-matter content, we then derived a set of evolution equations for the energy-matter content by decomposing the effective energy-momentum tensor. Following this, we derived for the first time the GBSSN formulation of the EC theory. We then considered the vacuum GBSSN equations for the case of spherical symmetry. Such evolution equations coincide with those of GR and are reported in [25, 31]. We then constructed a 1-dimensional code to evolve this system. Such systems have been numerically studied before in [2, 26, 27, 31]. Modifications of our 1-dimensional code to include non-vanishing torsion is left as the subject for future work. We noted that torsion can be included via the effective energy-momentum tensor which contains the spin-current tensor.

Appendix A

Klein-Gordon and Dirac equations in Riemann-Cartan space-time

Here, we wish to derive the Klein-Gordon and Dirac equations in the presence of curvature and torsion. More specifically, we wish to derive these aforementioned equations given the Riemann-Cartan space-time U_4. In deriving each of these equations, we require their formulation in Minkowski space-time. We discuss these formulations in Sections A.1 and A.2 by following what was done in [89] by Kleinert, [129] by Sakurai, and [130] by Scheck. We then following what is done in [79] by Shapiro and apply the minimal coupling procedure (MCP) in order to derive the corresponding equations in U_4.

A.1 The Klein-Gordon equation in Minkowski space-time

Consider the Minkowski space-time described by the tuple $\left(\mathbb{R}^{1,3}, \boldsymbol{\eta}, \partial\right)$ where $\boldsymbol{\eta}$ is the Minkowski metric with signature $(+,-,-,-)$. Here, we denote such a space-time as M_4. Let us consider the Schrödinger equation, without forces, for a particle:

$$i\partial_t \left|\Psi\right\rangle = \hat{H}\left|\Psi\right\rangle \ , \tag{A.1}$$

where the ket $\left|\Psi\right\rangle$ is a Hilbert space element, $\hat{H}$ is the Hamiltonian operator and we have used natural units to set $\hbar = 1$. The Klein-Gordon equation

$$\left(-\partial^2 - m^2\right)\phi = 0 \ , \tag{A.2}$$

where $\phi(x)$ is a scalar field is the field theoretic analogue of the Schrödinger equation (A.1). While the wavefunction $\Psi(x) := \left\langle x|\Psi(x)\right\rangle$ can be described by a probability density function (PDF), a solution to the Klein-Gordon equation (A.2) cannot, in general, be described by a PDF [130]. Here, we allow $\phi(x)$ to be a complex scalar field and we denote its complex conjugate by $\phi^*(x)$. Let S be a frame which has coordinates x^μ and let S' denote a frame that is obtained through the action of a Lorentz transformation on S, i.e., the coordinates of S' are $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ where $\Lambda^\mu{}_\nu$ is a matrix describing the Lorentz transformation. The scalar field $\phi(x)$ has no spin degrees of freedom and is a *Lorentz scalar*, i.e., in the frame S' we have $\phi'(x') = \phi(x)$. Let us consider the following action constructed on the Minkowski space-time M_4

$$S_{KG} = \int \mathrm{d}^4x \ \phi^*(x)\left(-\partial^2 - m^2\right)\phi(x) \ . \tag{A.3}$$

Varying the above action with respect to $\phi^*(x)$ yields the Klein-Gordon equation (A.2). An equivalent action can be obtained by carrying out an integration by parts on the above expression. This yields the following action

$$S_{KG} = \int \mathrm{d}^4x \left[\left(\partial_\nu\phi^*\right)\left(g^{\mu\nu}\partial_\mu\phi\right) - m^2\phi^*\phi\right] \ , \tag{A.4}$$

after requiring that the scalar field vanishes at the boundaries. It is easily verifiable that varying the above expression with respect to $\phi^*(x)$ yields the Klein-Gordon equation (A.2). When we wish to find the Klein-Gordon equation in the presence of curvature and torsion, we will apply the MCP approach to the above expression. Before doing this, however, we wish to apply Noether's theorem in order to find a quantity that is conserved under a global internal symmetry [98]. First, consider a one-parameter, u, family of $U(1)$ symmetry transformations of the scalar field $\phi(x)$:

$$\phi_u := \mathrm{e}^{-iuG}\phi \ , \tag{A.5}$$

where iG is an element of the associated Lie Algebra $u(1)$. We note that we obtain different ϕ_u variations for different $u \in \mathbb{R}$. In addition, we note that the symmetry transformation of ϕ^* from the above equation is nothing more than

$$\phi_u^* = \mathrm{e}^{iuG}\phi^* \ . \tag{A.6}$$

We now require that the Lagrangian density $\mathcal{L}\left(\phi, \phi^*, \partial_\nu\phi, \partial_\nu\phi^*\right)$ is invariant under a global internal symmetry transformation of the form given above, i.e., we have [98]

$$\mathcal{L}\left(\phi_u, \phi_u^*, \partial_\nu\phi_u, \partial_\nu\phi_u^*\right) = \mathcal{L}\left(\phi, \phi^*, \partial_\nu\phi, \partial_\nu\phi^*\right) \ . \tag{A.7}$$

From the above expression, it follows that we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0}\left[\mathcal{L}\left(\phi_u, \phi_u^*, \partial_\nu\phi_u, \partial_\nu\phi_u^*\right) - \mathcal{L}\left(\phi, \phi^*, \partial_\nu\phi, \partial_\nu\phi^*\right)\right] = 0 \ . \tag{A.8}$$

Carrying out the differentiation with respect to the parameter u yields

$$\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi\right)}\left.\frac{\mathrm{d}\phi_u}{\mathrm{d}u}\right|_{u=0} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^*\right)}\left.\frac{\mathrm{d}\phi_u^*}{\mathrm{d}u}\right|_{u=0}\right) = 0 \ , \tag{A.9}$$

where we have made use of the Euler-Lagrange equations. By reading off the Lagrangian density from equation (A.4) and determining the derivatives of the scalar field with respect to u from equations (A.5) and (A.6), it can be found that

$$j_\nu := \phi^* \overleftrightarrow{\partial_\nu} \phi = \phi^*\partial_\nu\phi - \phi\partial_\nu\phi^* \ , \tag{A.10}$$

is a conserved *Noether current* with $\partial_\nu j^\nu = 0$. This concludes our discussion of the Klein-Gordon equation in Minkowski space-time M_4. In Section A.3, we derive the Klein-Gordon equation in the Riemann-Cartan space-time U_4 through the application of the MCP.

A.2 Dirac equation in Minkowski space-time

In this section, we follow what is done in [89,129] in order to discuss the Dirac equation in Minkowski space-time M_4. Let us first discuss the Lorentz transformations that yield a frame S', with coordinates x'^μ, from a frame S, with coordinates x^μ. As mentioned previously, the coordinates transform as $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. Let us study the Lorentz transformation, Λ, that yields a rotation, by an angle say α, about the $z-$axis. Such a transformation, which we shall denote as $R_3^\mu{}_\nu\left(\alpha\right)$, is written as

$$R_3\left(\alpha\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{A.11}$$

where we have suppressed space-time indices. The fact that the transpose of the above matrix is equal to the inverse of the above matrix together with the fact that the determinant is unity implies that it is an element of the Lie group $SO(1,3)$. We now wish to find a matrix, say $L_3^\mu{}_\nu$, that allows us to obtain the above Lorentz transformation matrix through the use of the exponential map, i.e., $R_3{}^{\mu}{}_{\nu} = \left[e^{-i\alpha L_3}\right]^{\mu}{}_{\nu}$. Calculating the right-hand side through the use of a Taylor expansion and dropping the space-time indices yields

$$\mathrm{e}^{-i\alpha L_3} = \sum_{n=0}^{\infty} \frac{(-i\alpha L_3)^n}{n!} = -i \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1} L_3^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n} L_3^{2n}}{(2n)!} \ . \tag{A.12}$$

Let us consider the following 4×4 matrix as an ansatz:

$$L_3 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{A.13}$$

It is not difficult to show that

$$L_3^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{A.14}$$

By substituting the last two expressions into equation (A.12), we obtain the following

$$e^{-i\alpha L_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \cos\alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \sin\alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \tag{A.15}$$

which is nothing more than the rotation matrix $R^{\mu}_{\ \nu}(\alpha) \in SO(1,3)$. Through the use of the exponential map, one can obtain the $SO(1,3)$ element $R^{\mu}_{\ \nu}(\alpha)$ from the matrix $L^{\mu}_{\ \nu}$. What we have considered so far is simply a rotation about the z-axis. Let us now consider rotations about the x- and y- axes. The matrices

$$L_1 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ , \qquad L_2 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \ , \tag{A.16}$$

produce the rotation matrices $R_1 \in SO(1,3)$ and $R_2 \in SO(1,3)$ respectively. That is, one can obtain the $SO(1,3)$ elements describing rotations about the x- and y-axes through the mappings $R_1 = e^{-i\alpha L_1}$ and $R_2 = e^{-i\alpha L_2}$ respectively.

Up until now, we have only discussed rotations. To find all the generators of the Lorentz group $SO(1,3)$ we must consider, in addition to rotations, boosts. Consider the following Lorentz transformation that describes a boost in the x-direction

$$B_1 = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{A.17}$$

where $\beta := v/c$, v is the velocity of the boosted frame and c is the speed of light. Introducing the *rapidity*, $w := \operatorname{arctanh}\beta$, allows us to write the above Lorentz transformation as

$$B_1 = \begin{pmatrix} \cosh w & \sinh w & 0 & 0 \\ \sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{A.18}$$

We now wish to find a matrix, say N_1, that can produce the above Lorentz transformation through the use of the exponential map. As an ansatz, we consider

$$N_1 = i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{A.19}$$

It is not difficult to show that

$$e^{-iN_1 w} = \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n} N_1^{2n}}{(2n)!} - i \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1} N_1^{2n+1}}{(2n+!)!} \ . \tag{A.20}$$

By making use of the fact that we have

$$N_1^2 = - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \tag{A.21}$$

it follows from equation (A.20) that the exponential map yields

$$e^{-iwN_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \cosh w \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \sinh w \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \tag{A.22}$$

which is nothing more than the Lorentz transformation $B_1 \in SO(1,3)$ which describes a boost in x-direction. In addition, the matrices

$$N_2 = i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \qquad N_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ , \tag{A.23}$$

through the exponential map yield $B_2 = \mathrm{e}^{-iwN_2}$ and $B_3 = \mathrm{e}^{-iwN_3}$ which are the Lorentz transformations that describe boosts in the y- and z-directions respectively [89]. We note that an arbitrary Lorentz transformation, say Λ, can be written as [89]

$$\Lambda = e^{-i\left(\alpha^k L_k + \boldsymbol{w}^k N_k\right)} \ . \tag{A.24}$$

Consider now the matrices given in equations (A.13), (A.16), (A.19) and (A.23). These generators satisfy the following set of commutation relations

$$[L_i, L_j] = i \sum_{k=1}^{3} \varepsilon_{ijk} L_k \ , \tag{A.25}$$

$$[L_i, N_j] = i \sum_{k=1}^{3} \varepsilon_{ijk} N_k \ , \tag{A.26}$$

$$[N_i, N_j] = -i \sum_{k=1}^{3} \varepsilon_{ijk} L_k \ , \tag{A.27}$$

where $i, j, k \in \{1, 2, 3\}$ and we remind the reader that we use ε_{ijk} to denote the Levi-Civita symbol [89]. The above relations form a *Lie algebra* for the Lie group $SO(1,3)$ for which the Lorentz transformations R_i and B_i are elements for $i \in \{1, 2, 3\}$.

In the above, we have briefly discussed the associated Lie algebra of the Lorentz transformations on the Minkowski space-time M_4. We now wish to turn our attention to studying spin-1/2 fields by making use of the Lie algebra construction given in equations (A.25), (A.26) and (A.27). Let us first define the *Pauli spin matrices* as follows [89]

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma^1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{A.28}$$

Following [89], and with the above definitions in mind, we note that the matrices of smallest dimension that satisfy the subalgebra described by equation (A.25) are

$$L_i := \frac{\sigma^i}{2} \ . \tag{A.29}$$

Turning our attention to the entire Lie algebra, which is given by equations (A.25), (A.26) and (A.27), we note that there are two distinct options for ensuring the satisfaction of this Lie algebra. The first such option is to define

$$M_i := -\frac{i\sigma^i}{2} \ . \tag{A.30}$$

The second option is to instead define

$$M_i := \frac{i\sigma^i}{2} \ . \tag{A.31}$$

The first complete set of generators is then equation (A.29) together with equation (A.30) while the second complete set of generators is then equations (A.29) and (A.31). Given the first set of generators, an arbitrary Lorentz transformation can be written as

$$\Lambda = \mathrm{e}^{-i\left(a^k\sigma_k - ib^k\sigma_k\right)} \ , \tag{A.32}$$

where $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. On the other hand, given the second set of generators, an arbitrary Lorentz transformation can be written as [89]

$$\Lambda = \mathrm{e}^{-i\left(a^k\sigma_k + ib^k\sigma_k\right)} \ . \tag{A.33}$$

Given that we have two distinct sets of generators, we introduce two vector fields whose associated transformation law is either that of equation (A.32) or of equation (A.33). That is, we introduce the so-called *Weyl spinors*, φ and ξ, that behave in the following way under a Lorentz transformation [89]:

$$\varphi(x) \longrightarrow \mathrm{e}^{-i\left(a^k\sigma_k - ib^k\sigma_k\right)}\varphi(x) \ , \tag{A.34}$$

$$\chi(x) \longrightarrow \mathrm{e}^{-i\left(a^k\sigma_k + ib^k\sigma_k\right)}\chi^\nu(x) \ . \tag{A.35}$$

By making use of the Weyl spinors introduced above, the goal now is to construct an action integral that is invariant under Lorentz transformations and from which we can extract the equations of motion for a spin-1/2 field. The construction of such an action integral that is quadratic in the Weyl spinors as well as their first order derivatives is given in [89]. Here, we shall simply state this action integral and the interested reader is directed to [89] for the construction. This action integral reads

$$S = \int \mathrm{d}^4x \left(i\varphi^\dagger(x)\bar{\sigma}^\mu\partial_\mu\varphi(x) + i\chi^\dagger(x)\sigma^\mu\partial_\mu\chi(x) - m_1\varphi^\dagger(x)\chi(x) - m_2\chi^\dagger(x)\varphi(x) \right) \ , \tag{A.36}$$

where m_1 and m_2 are constants, the superscript † is used to denote the Hermitian conjugate and we define

$$\sigma^\mu := \left(\mathbb{1}_{2\times 2}, \sigma^i\right) \ , \quad \bar{\sigma}^\mu := \left(\mathbb{1}_{2\times 2}, -\sigma^i\right) \ . \tag{A.37}$$

In equation (A.36) we have omitted the space-time indices following [89]. We now wish to read off the Lagrangian density from equation (A.36) and obtain the equations of motion. By carrying out an ordinary variation of the action with respect to the Weyl spinor Hermitian conjugate $\varphi^\dagger$, one can obtain the following equations of motion:

$$i\bar{\sigma}^\mu\partial_\mu\varphi(x) - m_1\chi(x) = 0 \ . \tag{A.38}$$

On the other hand, the ordinary variation of the action with respect to the Weyl spinor Hermitian conjugate $\chi^\dagger$ yields the following equation of motion:

$$i\sigma^\mu\partial_\mu\chi(x) - m_2\varphi(x) = 0 \ . \tag{A.39}$$

Now, by acting $i\sigma^\mu\partial_\mu$ on the equation of motion (A.38), one obtains

$$-\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu\varphi(x) - m_1 i\sigma^\mu\partial_\mu\chi(x) = 0 \ . \tag{A.40}$$

Now, we not that the anti-commutation relation

$$\sigma^{(\mu}\bar{\sigma}^{\nu)} = \eta^{\mu\nu} \ , \tag{A.41}$$

implies that we have

$$\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu = \mathbb{1}_{2\times2}\partial^2 \ . \tag{A.42}$$

Therefore, by substituting equation (A.39) into equation (A.38) as well as by making use of the above expression we obtain the following equation

$$\left(-\partial^2 - m_1 m_2\right)\varphi(x) = 0 \ . \tag{A.43}$$

We can apply a similar procedure to equation (A.39) in order to obtain the following

$$\left(-\partial^2 - m_1 m_2\right)\chi(x) = 0 \ . \tag{A.44}$$

It is at this point we wish to define a new object which is the so-called *Dirac spinor* and is defined as

$$\psi(x) := \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix} \ . \tag{A.45}$$

For the sake of simplicity, from here on-wards we set $m := m_1 = m_2$ and we define the so-called $\gamma-$*matrices* as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \ . \tag{A.46}$$

In addition, we define $\bar{\psi}(x) := \psi^\dagger(x)\gamma^0$. We now write the following action integral which is referred to as the *Dirac action* [89]

$$S_{1/2} = \int \mathrm{d}^4x \ \bar{\psi}(x)\left(i\gamma^\mu\partial_\mu - m\mathbb{1}_{4\times4}\right)\psi(x) \ . \tag{A.47}$$

By carrying out an ordinary variation of the above Dirac action with respect to $\bar{\psi}(x)$, one can obtain the following equation of motion

$$\left(i\gamma^\mu\partial_\mu - m\mathbb{1}_{4\times4}\right)\psi(x) = 0 \ . \tag{A.48}$$

By writing the above equation in terms of the Weyl spinors $\varphi(x)$ and $\chi(x)$ through the use of the definition of the Dirac spinor, one obtains equations (A.38) and (A.39). Equation (A.48) is referred to as the *Dirac equation* in the Minkowski space-time M_4. Before moving on to discuss this equation for a Riemann-Cartan space-time $\mathcal{U}_4$, we first wish to apply Noether's theorem in order to obtain conserved quantities for a global internal symmetry. Let us consider a one-parameter, $u \in \mathbb{R}$, family of symmetry variations of the Dirac spinor field, $\psi(x)$, described by the Lie group $U(1)$. We write

$$\psi_u := e^{-iuG}\psi \ . \tag{A.49}$$

Of course, the one-parameter family of symmetry variations for the adjoint Dirac spinor is given as follows

$$\bar{\psi}_u := e^{-iuG}\bar{\psi} \ . \tag{A.50}$$

Given a Lagrangian density, $\mathcal{L}$, that depends on the Dirac spinor, its first order derivative as well as the adjoint Dirac spinor, a global internal symmetry of this Lagrangian density implies that

$$\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0}\left(\mathcal{L}\left(\psi_u, \bar{\psi}_u, \partial_\nu\psi_u\right) - \mathcal{L}\left(\psi, \bar{\psi}, \partial_\nu\psi\right)\right) = 0 \ . \tag{A.51}$$

The above expression together with the Euler-Lagrange equations of motion imply that

$$\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\partial_\nu\psi}\left.\frac{\mathrm{d}\psi_u}{\mathrm{d}u}\right|_{u=0}\right) = 0 \ . \tag{A.52}$$

By substituting equation (A.49) into the above expression, as well as by reading off the Dirac Lagrangian density from the Dirac action given in equation (A.47) we obtain the following conserved Noether current [89]

$$j^\nu = \bar{\psi}\gamma^\nu\psi \ . \tag{A.53}$$

In Section A.4, we wish to derive the Dirac equation given the Riemann-Cartan space-time $\mathcal{U}_4$ by applying the MCP to the Dirac action in M_4. The MCP will not be applied directly to the Dirac action as it is given in equation (A.47). By first writing the Dirac action as

$$S_{1/2} = \frac{1}{2}\int \mathrm{d}^4x \left(i\bar{\psi}(x)\gamma^\mu\partial_\mu + i\bar{\psi}(x)\gamma^\mu\partial_\mu - 2m\mathbb{1}_{4\times4}\bar{\psi}(x)\psi(x)\right) \ , \tag{A.54}$$

and then carrying out an integration by parts on the second term in the intgrand, we obtain the following integral for the Dirac action

$$S_{1/2} = \frac{1}{2}\int \mathrm{d}^4x \left(i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - i\left(\partial_\mu\bar{\psi}(x)\right)\gamma^\mu\psi(x) - 2m\mathbb{1}_{4\times4}\bar{\psi}(x)\psi(x)\right) \ . \tag{A.55}$$

In Section A.4, it is this form of the Dirac action that the MCP will be applied to.

A.3 The Klein-Gordon equation in Riemann-Cartan space-time

Here, we consider the Riemann-Cartan space-time U_4. Our goal in this section is to derive the Klein-Gordon equation for U_4 by applying the MCP as described in the review article [79] by Shapiro. Before stating the action integral to be used, we wish to make a note on the volume element that will be used. While it is possible to consider an alternative to the natural choice of a volume element, which is done in the ECS theory of gravity, in this section we will consider action integrals that are constructed using the natural choice of a volume element. The Klein-Gordon equation in Minkowski space-time is studied in Appendix A.1. Applying the MCP procedure to equation (A.4), which is the action in Minkowski space-time, we obtain the following

$$S_{KG} = \int \mathrm{d}^4x\sqrt{-g}\left[\left(D_\nu\phi^*\right)\left(D^\nu\phi\right) - m^2\phi^*\phi\right] \ . \tag{A.56}$$

and we remind the reader that D_ν denotes the Levi-Civita covariant derivative. In constructing the above action from equation (A.4), we have introduced the natural choice for a volume element. Since ϕ is a C^∞ function on the Riemann-Cartan space-time manifold, the dual vector field obtained by carrying out a covariant derivative on the scalar field is independent of the covariant derivative used, i.e., $\nabla_\nu \phi = D_\nu \phi = \partial_\nu \phi$. Since the volume element is parallel with respect to the Levi-Civita connection, we have written the derivatives in the action in terms of Levi-Civita covariant derivatives. Since the volume element is parallel with respect to the Levi-Civita connection, an integration by parts of the above action integral yields

$$S_{KG} = \int \mathrm{d}^4x\sqrt{-g}\phi^* \left(-D^2 - m^2\right)\phi \ . \tag{A.57}$$

Therefore, by requiring an extremum of the action under ordinary variations of the complex conjugate of the scalar field, ϕ^*, we obtain the Klein-Gordon equation in Riemann-Cartan space-time

$$\left(-D^2 - m^2\right)\phi = 0 \ . \tag{A.58}$$

It is worth noting that torsion does not play a role in the above expression. This is not surprising since, as is the case for the Einstein-Cartan theory of gravity, an absence of spin degrees of freedom implies that there is no torsion contribution. Let us now turn our attention to obtaining a conserved quantity, through the use of Noether's theorem, that arises through the occurrence of a global internal symmetry. By applying a similar method to what was done in the previous section for the case of a Minkowski space-time, we have the following Noether current

$$j_\nu := \phi^* \overleftrightarrow{D_\nu} \phi \ , \tag{A.59}$$

with the associated conservation equation $D_\nu j^\nu = 0$. This concludes our discussion of the Klein-Gordon equation in the Riemann-Cartan space-time. We now wish to turn our attention to the Dirac equation. In deriving the Dirac equation for U_4, we follow a similar approach to what was done for the Klein-Gordon equation.

A.4 The Dirac equation in Riemann-Cartan space-time

In this section, we wish to derive the Dirac equation for the Riemann-Cartan space-time by following what is done in [75, 79]. The derivation in this section will be carried out by applying the MCP to the Dirac action given in equation (A.55). Before we apply the MCP to the Dirac action, we first wish to consider how the Dirac spinor behaves as a result of the action of the full covariant derivative, defined in equation (3.121). As suggested in [79], the result of the Dirac spinor being acted on by the full covariant derivative is

$$\mathcal{D}_\mu \psi = \partial_\mu \psi - \frac{i}{2} w^{ab}{}_\mu \sigma_{ab} \psi \ , \tag{A.60}$$

where $\boldsymbol{w}^{ab}$ is the connection one-form defined in equation (3.93) and

$$\sigma_{ab} := i\gamma_{[a}\gamma_{b]} \ . \tag{A.61}$$

On the other hand, the action of the full covariant derivative on the adjoint Dirac spinor field is [79]

$$\mathcal{D}_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \frac{i}{2}\bar{\psi}\ w^{ab}{}_\mu \sigma_{ab} \ . \tag{A.62}$$

We note that the connection one-form is also referred to as the *spinor connection* [8]. In addition, we note that we have defined the spinor connection in a different manner to what is done in [79]. Let us now turn our attention to the antisymmetric object σ_{ab} defined in equation (A.61). Here, the gamma matrices with Latin indices denote the gamma matrices in the non-holonomic frame. That is, the γ^a are given by equation (A.46) and satisfy

$$\gamma^{(a}\gamma^{b)} = \eta^{ab} \ , \tag{A.63}$$

where η^{ab} is the Minkowski metric. We remind the reader that, in this section, we make use of the signature $(+,-,-,-)$. We introduce the space-time gamma matrices, γ^μ, by contracting the tetrad coefficients with the gamma matrices in the non-holonomic frame, i.e., we write

$$\gamma^\mu = e_a{}^\mu \gamma^a \ . \tag{A.64}$$

The anti-commutation relation satisfied by the space-time gamma matrices is as follows

$$\gamma^{(\mu}\gamma^{\nu)} = g^{\mu\nu} \ , \tag{A.65}$$

where $g_{\mu\nu}$ is the space-time metric [75]. Let us now turn our attention to equation (A.60) which describes the action of the full covariant derivative on the Dirac spinor. By assuming the tetrad postulate, i.e., that the action of the full covariant derivative on the vierbein vanishes, we have the following expression for the space-time components of the connection one-form

$$w^k{}_{b\nu} = e^k{}_\alpha \nabla_\nu e_b{}^\alpha \ . \tag{A.66}$$

The above expression can be derived from equation (3.122) which follows directly from the tetrad postulate. By substituting equation (A.61) into equation (A.60), and noting that the connection one-form is antisymmetric, i.e., $\boldsymbol{w}^{ab} = -\boldsymbol{w}^{ba}$, one can obtain the following expression

$$\mathcal{D}_\mu \psi = \partial_\mu \psi + \frac{1}{4} w^{ab}{}_\mu \gamma_a \gamma_b \psi \ . \tag{A.67}$$

Now, by substituting equation (A.66), which relates the spinor connection to the vierbein, into the above expression we obtain

$$\mathcal{D}_\mu \psi = \partial_\mu \psi + \tilde{\Gamma}_\mu \psi \ , \tag{A.68}$$

where we have defined Γ_μ as

$$\tilde{\Gamma}_\mu := \frac{1}{4} e^a{}_\nu \left(\nabla_\mu e_b{}^\nu\right) \gamma_a \gamma^b \ . \tag{A.69}$$

In addition, we also have the following equation for the adjoint Dirac spinor field

$$\mathcal{D}_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \bar{\psi} \tilde{\Gamma}_\mu \ . \tag{A.70}$$

The last three expressions are what is given in [75]. It is also worth noting that the action of the total covariant derivative on the gamma matrices vanishes, i.e., $\mathcal{D}_\nu \gamma^\alpha = 0$ [75]. In order to show this, consider the following which follows from equation (A.70) [75]

$$\mathcal{D}_\mu \gamma^\alpha = \partial_\mu \gamma^\alpha + \tilde{\Gamma}^\alpha{}_{\mu\beta} \gamma^\beta + \tilde{\Gamma}_\mu \gamma^\alpha - \gamma^\alpha \tilde{\Gamma}_\mu \ . \tag{A.71}$$

By substituting equations (A.64) and (A.69) into the above expression yields

$$\mathcal{D}_\mu \gamma^\alpha = \gamma^a \nabla_\mu e_a{}^\alpha + \frac{1}{4} \gamma_\nu \gamma^a \nabla_\mu \gamma^\alpha e_a{}^\nu - \frac{1}{4} \gamma^\alpha \gamma_\nu \gamma^a \nabla_\mu e_a{}^\nu \ . \tag{A.72}$$

Substituting equation (A.65) into the third term on the right-hand side of the above expression yields the following

$$\mathcal{D}_\mu \gamma^\alpha = \frac{1}{2} \gamma^a \nabla_\mu e_a{}^\alpha + \frac{1}{4} \gamma_\nu \gamma^a \gamma^\alpha \nabla_\mu e_a{}^\nu + \frac{1}{4} e^a{}_\beta \gamma_\nu \gamma^\alpha \gamma^\beta \nabla_\mu e_a{}^\nu \ . \tag{A.73}$$

Finally, by substituting equation (A.65) into the third term on the right-hand side of the above expression we obtain

$$\mathcal{D}_\mu \gamma^\alpha = \frac{1}{2} \gamma^a \nabla_\mu e_a{}^\alpha + \frac{1}{4} \gamma_\nu \gamma^a \gamma^\alpha \nabla_\mu e_a{}^\nu + \frac{1}{2} \gamma_\nu e^{a\alpha} \nabla_\mu e_a{}^\nu - \frac{1}{4} \gamma_\nu \gamma^a \gamma^\alpha \nabla_\mu e_a{}^\nu = 0 \ , \tag{A.74}$$

as desired. So far, we have only considered how the Dirac spinor behaves as a result of the action of the full covariant derivative whose spin connection depends on the Cartan covariant derivative of the vierbien. By defining the torsion-free spin connection

$$\overset{\circ}{\boldsymbol{w}}{}^k{}_b := \Gamma^k{}_{ab}\hat{\boldsymbol{\theta}}^a \ , \tag{A.75}$$

we can introduce the *torsion-free full covariant derivative* as

$$\overset{\circ}{\mathcal{D}}{}^k{}_b := \delta^k_b D + \overset{\circ}{\boldsymbol{w}}{}^k{}_b \ . \tag{A.76}$$

We remind the reader that the $\Gamma^k{}_{ab}$ denotes the Christoffel symbols in the non-holonomic frame and $\hat{\boldsymbol{\theta}}$ denotes the dual basis in the non-holonomic frame. The torsion-free full covariant derivative of the spinor field is written as

$$\overset{\circ}{\mathcal{D}}_\mu \psi = \partial_\mu \psi - \Gamma_\mu \psi \ , \tag{A.77}$$

where the term Γ_μ is found by replacing the Cartan covariant derivative in equation (A.69) with the Levi-Civita covariant derivative. In addition, the torsion-free full covariant derivative of the adjoint Dirac spinor field is given by the expression

$$\overset{\circ}{\mathcal{D}}_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \bar{\psi}\Gamma_\mu \ . \tag{A.78}$$

Let us now turn our attention to applying the MCP to the Dirac action of Minkowski space-time M_4 which is given in equation (A.55). Replacing the partial derivatives in this expression with the full covariant derivative $\mathcal{D}$, one obtains the following action integral

$$S_{1/2} = \frac{i}{2}\int \mathrm{d}^4x\ e\left(\bar{\psi}\gamma^\nu \mathcal{D}_\nu \psi - \mathcal{D}_\mu \bar{\psi}\gamma^\mu \psi + 2im\bar{\psi}\psi \mathbb{1}_{4\times 4}\right) \ , \tag{A.79}$$

where e is the determinant of the tetrad field, i.e., $e = \det e^a{}_\mu$. Consider the second term in the integrand on the right-hand side of the above expression. By making use of equations (A.70) and (A.78), we can convert the full covariant derivative to the torsion-free covariant derivative as follows:

$$\mathcal{D}_\mu \bar{\psi} = \overset{\circ}{\mathcal{D}}_\mu \bar{\psi} + \bar{\psi}\left(\Gamma_\mu - \tilde{\Gamma}_\mu\right) \ . \tag{A.80}$$

By substituting the above expression into equation (A.79) we obtain the following

$$S_{1/2} = \frac{i}{2}\int \mathrm{d}^4x\ e\left(\bar{\psi}\gamma^\nu \mathcal{D}_\nu \psi - \overset{\circ}{\mathcal{D}}_\mu \bar{\psi}\gamma^\mu \psi - \bar{\psi}\left(\Gamma_\mu - \tilde{\Gamma}_\mu\right)\gamma^\mu\psi + 2im\bar{\psi}\psi \mathbb{1}_{4\times 4}\right) \ . \tag{A.81}$$

Of course, since the full covariant derivative reduces to the torsion-free covariant derivative in the case of vanishing torsion, it follows that, since the gamma matrices are compatible with the full covariant derivative, the gamma matrices are also compatible with the torsion-free full covariant derivative, i.e., $\overset{\circ}{\mathcal{D}}_\mu \gamma^\nu = 0$. By making use of this result together with the fact that the volume element ed^4x is compatible with the torsion-free full covariant derivative, an integration by parts of the second term in the integrand in the above expression yields the following

$$S_{1/2} = \frac{i}{2}\int \mathrm{d}^4x\ e\left(\bar{\psi}\gamma^\nu \mathcal{D}_\nu \psi + \bar{\psi}\gamma^\mu \overset{\circ}{\mathcal{D}}_\mu \psi - \bar{\psi}\left(\Gamma_\mu - \tilde{\Gamma}_\mu\right)\gamma^\mu\psi + 2im\bar{\psi}\psi \mathbb{1}_{4\times 4}\right) \ . \tag{A.82}$$

We can now convert the torsion-free full covariant derivative $\overset{\circ}{\mathcal{D}}$ to the full covariant derivative $\mathcal{D}$ and obtain the following

$$S_{1/2} = \frac{i}{2}\int \mathrm{d}^4x\ e\left(\bar{\psi}\gamma^\nu \mathcal{D}_\nu \psi + \bar{\psi}\gamma^\mu \mathcal{D}_\mu \psi + \bar{\psi}\gamma^\mu\left(\Gamma_\mu - \tilde{\Gamma}_\mu\right)\psi - \bar{\psi}\left(\Gamma_\mu - \tilde{\Gamma}_\mu\right)\gamma^\mu\psi + 2im\bar{\psi}\psi \mathbb{1}_{4\times 4}\right) \ . \tag{A.83}$$

By writing the difference between the non-null torsion and torsion-free spin connections in terms of the contorsion tensor, we obtain the following

$$\gamma^{\mu}\left(\Gamma_{\mu}-\tilde{\Gamma}_{\mu}\right)-\left(\Gamma_{\mu}-\tilde{\Gamma}_{\mu}\right)\gamma^{\mu}=\frac{1}{4}K_{\epsilon\alpha\beta}\left(\gamma^{\epsilon}\gamma^{\beta}\gamma^{\alpha}-\gamma^{\alpha}\gamma^{\epsilon}\gamma^{\beta}\right) \tag{A.84}$$

$$=T_{\alpha}\gamma^{\alpha}\ . \tag{A.85}$$

In order to obtain the second equality in the above, we applied the anti-commutation relation given in equation (A.65) to the first term in equation (A.84) twice. By substituting the above result into the Dirac action given in equation (A.83), we obtain the following space-time integral for the Dirac action

$$S_{1/2}=i\int \mathrm{d}^4x\ e\ \bar{\psi}\left(\gamma^{\nu}\mathcal{D}_{\nu}\psi+\frac{1}{2}T_{\nu}\gamma^{\nu}\psi+im\psi\mathbb{1}_{4\times4}\right)\ . \tag{A.86}$$

By varying the above action with respect to the adjoint Dirac spinor field, we obtain the Dirac action of Riemann-Cartan space-time as is given in [75, 79]

$$i\gamma^{\mu}\mathcal{D}_{\mu}\psi+\frac{i}{2}T_{\mu}\gamma^{\mu}\psi-m\psi=0\ . \tag{A.87}$$

Clearly, the above equation is different to that which would have been obtained by applying the MCP to the Dirac equation of Minkowski space-time. More specifically, if one were to apply the MCP to the Dirac equation of Minkowski space-time, which is given in equation (A.48), the resulting expression would not contain the second term in the above expression. That is, the term $iT_{\mu}\gamma^{\mu}\psi/2$ does not appear in the Dirac equation obtained through the application of the MCP.

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