FINITE STRIP ANALYSIS OF CURVED PLATE STRUCTURES

by

R.M. Downie

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ABSTRACT

A finite difference based finite strip method of analysis is presented for the solution of curved plate structures subject to normal loading.

By applying variational methods to the principle of minimum potential energy the governing differential equation for a curved finite strip element is formed. The set of simultaneous differential equations resulting from a system of such strips are then solved using finite difference and Gauss reduction techniques. Various boundary conditions including continuity over interior supports may be considered.

DECLARATION OF CANDIDATE

I, Roy Murray Downie, hereby declare that this thesis is my own work and that it has not been submitted for a degree at another University.

Signed by candidate
....
September 1979

DEDICATION

To my parents

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* Professor and Head of Department of Civil Engineering, University of Cape Town.

Senior Lecturer in Structures, Department of Civil Engineering, University of Cape Town

NOTATION

Lower case characters

a	Local coordinate system across strip
b	Width of finite strip
$\underset{\sim}{b}_i$	Unit base vectors in cartesian coordinate system
$\underset{\sim}{d}$	Vector of generalised displacements for plate
$d\theta_i$	Angular nodal spacing along system of finite strips
$\underset{\sim}{e}_i$	Unit base vectors in curvilinear coordinate system
$\underset{\sim}{g}_i$	Base vectors in curvilinear coordinate system
h	Thickness of plate or strip
h_i	Scale factors in curvilinear coordinate system
ℓ	Central length along plate or strip
$\rho(a,y^2)$	Loading function at the point (a,y^2)
q	Load intensity
$\underset{\sim}{r}$	Position vector relative to cartesian coordinate system
r_i, r_j	Nodal coordinates for strip in radial direction
r_m	Mean radial coordinate for strip
r,θ	Plane polar coordinate directions
s_i	Length along local curvilinear y^i axis
$\underset{\sim}{u}$	Vector of generalised displacements at nodal lines i and j
$\underset{\sim}{v}$	In-plane displacement vector
$w(a,y^2)$	Displacement function at the point (a,y^2)
w_3, w_3^*	Vertical displacement in x^3, y^3 direction
$\theta_1, \theta_2, \theta_1^*, \theta_2^*$	Rotational displacements about y^1 and y^2 axes respectively
$\underset{\sim}{\theta}, \underset{\sim}{\theta}^*$	Vector of rotational displacements
x^1, x^2, x^3	Cartesian coordinate directions
y^1, y^2, y^3	Local curvilinear coordinates
y_i^1, y_j^1	Nodal coordinates for strip in y^1 direction
z	Vertical distance from middle plane of plate

Upper case characters

A	Surface area of plate or strip
D_{11}, D_{22}, D_1	
D_r, D_θ	Flexural rigidities for an orthotropic plate
$D_{12}, D_{r\theta}$	Torsional rigidities for an orthotropic plate
E_1, E_2	Modulus of elasticity in y^1 and y^2 directions
E_r, E_θ	Modulus of elasticity in r and θ directions
G	Modulus of elasticity in shear
I	Moment of inertia for beam element
L	Length along boundary of plate
$\underset{\sim}{M}_1, \underset{\sim}{M}_2$	Vector of bending moment components [eqns (3.1), (3.2)]
M_1, M_2	Bending moments acting on boundary of plate [eqns (3.7)]
M_{11}, M_{22}	Bending moments per unit length about the y^2 and y^1 axes respectively [eqns (3.7)]
M_{12}, M_{21}	Twisting moments per unit length about the y^2 and y^1 axes respectively [eqns (3.7)]
$M_{rr}, M_{\theta\theta}$	Bending moments per unit length about the θ and r axes respectively
$M_{r\theta}$	Twisting moment per unit length about the θ axis
P	Point load
$\underset{\sim}{Q}$	Vector of shearing force components
Q	Shearing force acting on boundary of plate [eqns (3.7)]
Q_1, Q_2	Shearing forces per unit length along the y^1 and y^2 axes respectively
U_b	Total potential energy of a finite strip in bending subject to normal forces
X^i	Cartesian coordinate system
Y^i	Orthogonal curvilinear coordinate system

Vectors and Matrices

[A],[B],[C], [D],[F],[J]	Strip stiffness matrices
{A}	Vector of unknown constants
{M}	Vector of bending and twisting moments
{X}	Curvature matrix
{u}	Nodal displacement vector
$[C_a]$	Nodal coordinate matrix
$[C_b]$	Displacement transformation matrix
[E]	Vector of nodal loads and moments
[K]	Finite difference stiffness matrix

Greek characters

$\alpha_1 \ldots \alpha_5$	Fourth derivative central operator pattern values
$\beta_1 \ldots \beta_3$	Second derivative central operator pattern values
γ_1, γ_2	Shearing strains in y^1 and y^2 coordinate directions [eqn (4.7)]
γ_{12}, γ_{21}	Twisting shear strains [eqns (4.21)]
δ_{ij}	Kronecker delta
δU_b	First variation of the total potential energy U_b
$\varepsilon_1, \varepsilon_2$	Direct strains in the y^1 and y^2 coordinate directions [eqns (4.21)]
$\varepsilon_{12}, \varepsilon_{21}$	Components of twisting shear strain [eqns (4.21)]
χ_1, χ_2	Bending curvatures in the y^1 and y^2 coordinate directions [eqns (4.8)]
χ_{12}, χ_{21}	Components of twisting curvature [eqns (4.8)]
σ_1, σ_2	Bending stresses in y^1 and y^2 coordinate directions [eqns (4.22)]
τ_{12}, τ_{21}	Shear stresses in y^1 and y^2 coordinate directions [eqns (4.22)]
μ_1, μ_2	Poisson's ratio in y^1 and y^2 coordinate directions
μ_r, μ_θ	Poisson's ratio in the r and θ coordinate directions
$\underset{\sim}{\nu}$	Unit normal outward vector acting on boundary of plate
ν_1, ν_2	Components of unit normal outward vector in y^1 and y^2 directions respectively
λ	Aspect ratio for rectangular or curved plate defined as ℓ/plate width

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CHAPTER 1

INTRODUCTION

1.1 General

During the past few decades a number of numerical and analytical techniques have been developed for the solution of structural and continuum mechanics problems. Many of these techniques have been based on the two well known and extremely powerful methods of analysis, namely the finite difference and finite element methods.

In the finite difference method the governing differential equation and the set of prescribed boundary differential equations for the structure are replaced by their corresponding finite difference approximations. The method involves placing a mesh of equally or variably spaced grid lines on the structure and applying the governing differential equation at each node. The boundary differential equations are applied at points along the boundary in a similar manner. In this way the set of simultaneous differential equations obtained may be replaced by the corresponding set of simultaneous linear equations. The solution to this set of equations yields the unknown displacements, from which the bending moment and shear force quantities may be subsequently derived. Although the method of finite differences has been used to solve a variety of structural problems, the application has been found to be limited when the boundary conditions and the geometry become more involved.

In the finite element method, the continuum is replaced by an equivalent idealised structure composed of a number of discrete elements connected together along their side and at their common node points. By assuming specific piecewise displacement fields within each element which are continuous across the boundaries, the use of energy theorems make it possible to derive the equilibrium equations for a set of finite elements. In this way the set of simultaneous equations can be formed for the complete structure. Boundary conditions are applied by initialising the appropriate interior and boundary displacement values. From the solution to this set of equations, the nodal displacements are found, which in turn are used to derive the internal stresses and stress resultants. The method has been extended to include a number of different types and shapes of element and a large variety of structural problems may now be solved. For this reason the finite element method has become a very powerful and widely used method of structural analysis.

The finite element method does however, involve the solution to a relatively large set of simultaneous equations, with the corresponding demand on storage capacity and computation involved. In addition, the considerable amount of data preparation required and the excessive volume of results obtained may also be considered a disadvantage to the method. Therefore for certain types of problems it has been found worthwhile to develop methods which result in a reduced amount of computation, storage and input data requirements. This has lead to the development of the finite strip method of analysis, which may be regarded as an extension of the finite element method.

In the finite strip method, the structure is idealised into a number of long strip elements connected along their sides or nodal lines. The displacement in the longitudinal direction is assumed to vary in accordance with harmonic series functions which satisfy the prescribed boundary conditions at the ends of each strip. By using a finite element subdivision in the transverse direction only, it is therefore possible to reduce the dimension of the problem, with the corresponding saving in computation and storage requirements.

The present work represents a further development of the finite strip method, which is extended to the analysis of curved plate structures and where the advantages of both the finite element and finite difference methods are utilized.

1.2 Review

The harmonic finite strip method was first developed by Cheung [1,2,3] and used in the analysis of simply supported and clamped rectangular plate structures. The method has since been extended to the analysis of skew [6,7,8] and curved [10,11,12,13] plates and box girder bridges with similar boundary conditions. Restrictions on the method are mainly the requirement that the structure be prismatic in the longitudinal direction and the limited types of boundary conditions that may be handled at the ends of the strips. In addition, the type and number of interior supports that may reasonably be considered, limits the analysis of continuous structures.

Du Preez [15] developed a general finite strip method which may be applied to straight, skewed and curved slabs of variable cross-section and arbitrary boundary conditions. By applying variational methods to the principle of virtual work and the associated principle of minimum potential energy, the force-displacement relationships for a finite strip element are found in the form of a set of second order differential equations. Different techniques are used to solve the sets of equations obtained by considering a system of finite strips. One of the methods of solution described was that of using the above equations to formulate the stiffness matrix for each strip element and solving the system of finite strips by the direct stiffness method.

Louw [35] adopted a similar approach for the analysis of flat plates, where the set of governing differential equations for the system of finite strips was formed. The method of solution chosen was that of replacing the partial derivatives of the displacement variables by their finite difference approximations and solving the resulting set of simultaneous linear equations. Barker [36] subsequently extended the method to include the analysis of folded plate and translational shell structures.

In the present work, the finite strip method is formulated in terms of orthogonal curvilinear coordinates, enabling plate structures of an arbitrary curved nature to be analysed. The set of governing differential equations is solved using finite difference techniques, where a variable spacing of nodes has been incorporated

1.3 Scope

The scope of the present work on the finite strip method includes the following set of objectives:

i) To develop the equilibrium equations and the strain-displacement relationships for a curved plate in orthogonal curvilinear coordinates.

ii) To derive the expression for the total potential energy of a curved finite strip in orthogonal curvilinear coordinates.

iii) To formulate the governing differential equation for a curved finite strip in plane polar coordinates.

iv) To investigate the use of variable spacing finite difference approximations and their application in the finite strip analysis of curved plates.

v) To analyse a series of typical curved plate structures and to compare the results with those obtained using the finite element and other methods of analysis.

CHAPTER 2

THE CURVILINEAR COORDINATE SYSTEM

Introduction

In the following work it will be necessary to derive the equilibrium equations and the strain-displacement relations for an arbitrary curved plate structure and to develop the governing differential equations for a curved finite strip element. To describe a continuum of this shape it will therefore be necessary to formulate these equations in curvilinear coordinates. Due to the generalised form of the equations obtained it will be possible to specialise at a later stage to any prescribed coordinate system.
The choice of the coordinate system will depend on the geometry of the structure to be analysed, although the present formulation will be restricted to orthogonal systems.

This chapter serves to briefly describe the correspondence between the curvilinear and cartesian coordinate systems and to develop the relationships between the various scalar, vectors and tensor quantities and their derivatives. In the text the curvilinear coordinate system will be referred to as the Y^i system and the cartesian coordinate system as the X^i system.

Useful references in this field include those of Borg [20], Leipholz [24], Flügge [25] and Miller [26].

2.1 Relationship between the X^i and Y^i Systems

Let the cartesian coordinates at any point P be represented by (x^1, x^2, x^3) and the corresponding set of curvilinear coordinates by (y^1, y^2, y^3). Since there is to be a unique relationship between the two systems we may write the transformations as:

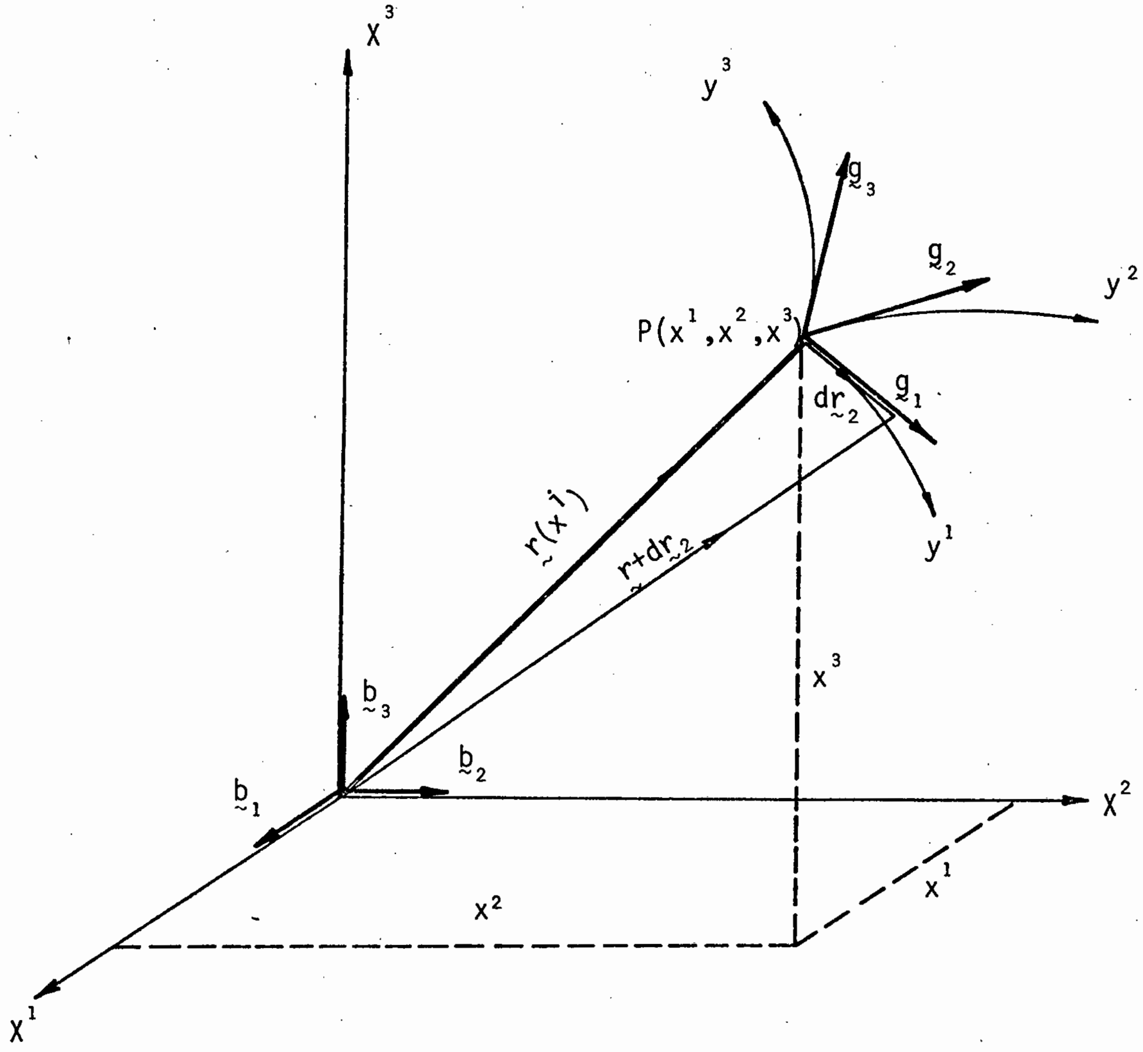


FIGURE 2.1 Curvilinear coordinate system Y^i relative to the cartesian axes X^i showing the position vector $\underset{\sim}{r}(x^i)$ referred to the X^i system

$$x^1 = x^1(y^1, y^2, y^3)$$
$$x^2 = x^2(y^1, y^2, y^3)$$
$$x^3 = x^3(y^1, y^2, y^3)$$

or $$x^i = x^i(y^1, y^2, y^3) \qquad (i = 1,2,3)$$

and $$y^i = y^i(x^1, x^2, x^3) \qquad (i = 1,2,3) \qquad (2.1)$$

Let the unit base vectors in the X^i system be denoted by $\underset{\sim}{b}_i$ and the position vector assumed relative to the X^i system by $\underset{\sim}{r} = \underset{\sim}{r}(x^1, x^2, x^3)$, as shown in Figure 2.1. Then

$$\begin{aligned} \underset{\sim}{r} &= x^1\underset{\sim}{b}_1 + x^2\underset{\sim}{b}_2 + x^3\underset{\sim}{b}_3 \\ &= x^i\underset{\sim}{b}_i \end{aligned} \qquad (2.2)$$

The usual summation convention for repeated indices is made use of here.

Due to the correspondence in Eqn. (2.1) it can be seen that the position vector $\underset{\sim}{r}$ may therefore be described relative to the X^i system in terms of the curvilinear coordinates y^i.

2.2 Base Vectors in Y^i System

In changing from the X^i system to the Y^i system, it is necessary to introduce the base vectors $\underset{\sim}{g}_i$. Using the chain rule for differentiation, we can write

$$\begin{aligned} d\underset{\sim}{r} &= \frac{\partial \underset{\sim}{r}}{\partial y^1} dy^1 + \frac{\partial \underset{\sim}{r}}{\partial y^2} dy^2 + \frac{\partial \underset{\sim}{r}}{\partial y^3} dy^3 \\ &= \frac{\partial \underset{\sim}{r}}{\partial y^i} dy^i \end{aligned} \qquad (2.3)$$

or $$d\underset{\sim}{r} = \underset{\sim}{g}_i dy^i \qquad (2.4)$$

where

$$\underset{\sim}{g}_i = \frac{\partial \underset{\sim}{r}}{\partial y^i} = \frac{\partial x^k}{\partial y^i} \underset{\sim}{b}_k$$

are the base vectors in the curvilinear coordinate system and are as shown in Figure 2.1 and Figure 2.2. Eqn. (2.4) indicates that by using the $\underset{\sim}{g}_i$ vectors as base vectors, the vector $d\underset{\sim}{r}$ has been expressed in terms of its contravariant components dy^i, that is, the components with respect to the basis $\underset{\sim}{g}_i$.

2.3 The Differential of Arc Length

The differential of arc length in the X^i system may be obtained from

$$\begin{aligned} ds^2 = d\underset{\sim}{r}.d\underset{\sim}{r} &= dx^1\underset{\sim}{b}_1.dx^1\underset{\sim}{b}_1 + dx^2\underset{\sim}{b}_2.dx^2\underset{\sim}{b}_2 + dx^3\underset{\sim}{b}_3.dx^3\underset{\sim}{b}_3 \\ &= dx^1dx^1 + dx^2dx^2 + dx^3dx^3 \\ &= dx^idx^i \\ &= \delta_{ij}dx^idx^j \end{aligned} \tag{2.5}$$

where δ_{ij}, referred to as the Kronecker delta, is defined such that

$$\delta_{ij} = 1 \quad \text{for} \quad i = j, \qquad \delta_{ij} = 0 \quad \text{for} \quad i \neq j$$

Using Eqn. (2.3), the length of the line element in the Y^i system may be expressed as follows:

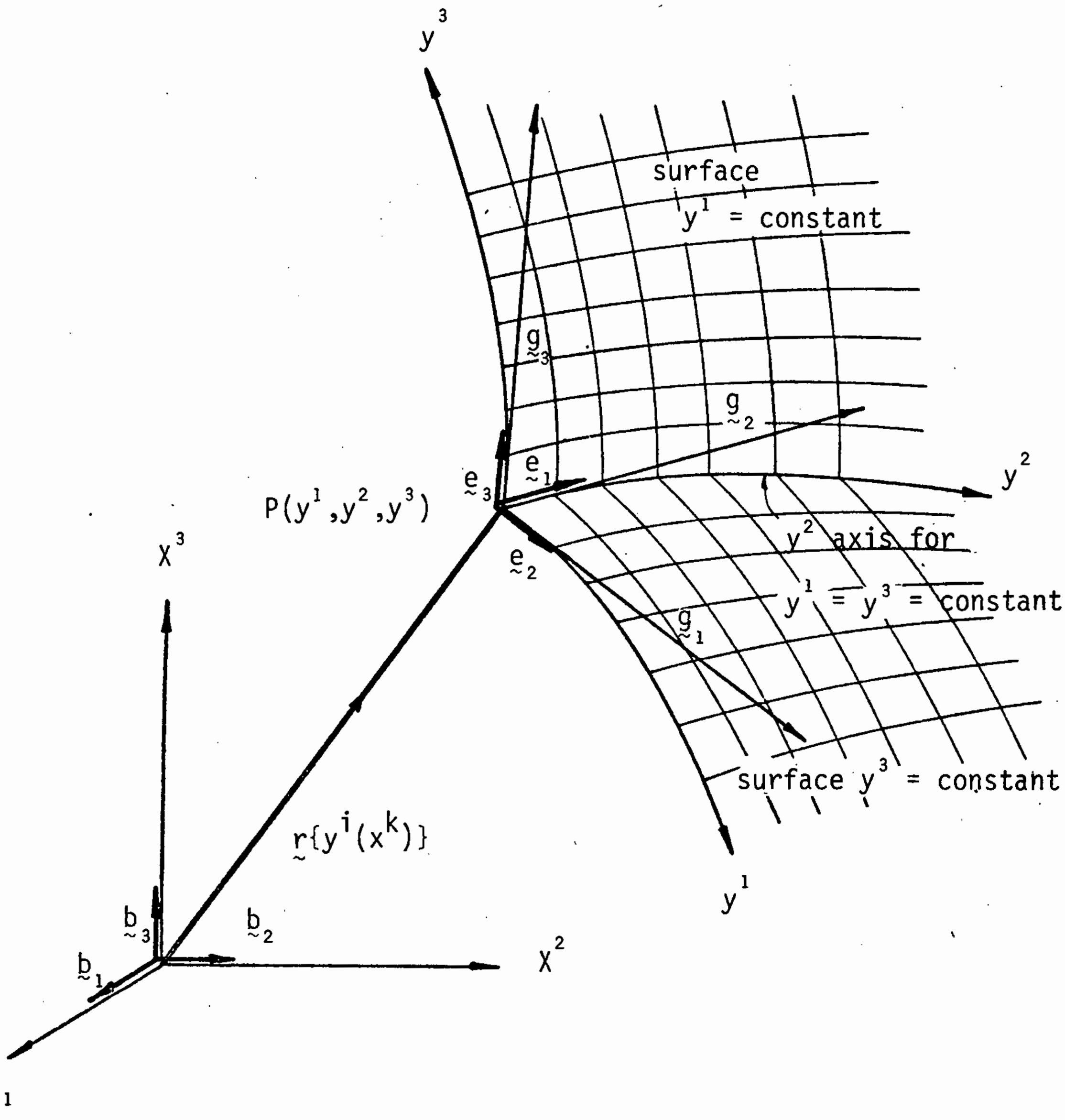


FIGURE 2.2 Local curvilinear axes showing base vectors $\underset{\sim}{g}_i$ and unit base vectors $\underset{\sim}{e}_i$

$$ds^2 = d\underset{\sim}{r}.d\underset{\sim}{r} = \frac{\partial \underset{\sim}{r}}{\partial y^1}.\frac{\partial \underset{\sim}{r}}{\partial y^1} dy^1 dy^1 + \frac{\partial \underset{\sim}{r}}{\partial y^1}.\frac{\partial \underset{\sim}{r}}{\partial y^2} dy^1 dy^2 + \frac{\partial \underset{\sim}{r}}{\partial y^1}.\frac{\partial \underset{\sim}{r}}{\partial y^3} dy^1 dy^3$$

$$+ \frac{\partial \underset{\sim}{r}}{\partial y^2}.\frac{\partial \underset{\sim}{r}}{\partial y^1} dy^2 dy^1 + \frac{\partial \underset{\sim}{r}}{\partial y^2}.\frac{\partial \underset{\sim}{r}}{\partial y^2} dy^2 dy^2 + \frac{\partial \underset{\sim}{r}}{\partial y^2}.\frac{\partial \underset{\sim}{r}}{\partial y^3} dy^2 dy^3$$

$$+ \frac{\partial \underset{\sim}{r}}{\partial y^3}.\frac{\partial \underset{\sim}{r}}{\partial y^1} dy^3 dy^1 + \frac{\partial \underset{\sim}{r}}{\partial y^3}.\frac{\partial \underset{\sim}{r}}{\partial y^2} dy^3 dy^2 + \frac{\partial \underset{\sim}{r}}{\partial y^3}.\frac{\partial \underset{\sim}{r}}{\partial y^3} dy^3 dy^3$$

$$= \frac{\partial \underset{\sim}{r}}{\partial y^i}.\frac{\partial \underset{\sim}{r}}{\partial y^1} dy^i dy^1 + \frac{\partial \underset{\sim}{r}}{\partial y^i}.\frac{\partial \underset{\sim}{r}}{\partial y^2} dy^i dy^2 + \frac{\partial \underset{\sim}{r}}{\partial y^i}.\frac{\partial \underset{\sim}{r}}{\partial y^3} dy^i dy^3$$

$$\therefore \quad ds^2 = \frac{\partial \underset{\sim}{r}}{\partial y^i}.\frac{\partial \underset{\sim}{r}}{\partial y^j} dy^i\, dy^j \qquad (2.6)$$

$$= \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j\, dy^i dy^j$$

$$= \underset{\sim}{g}_{ij} dy^i dy^j \qquad (2.7)$$

where $\underset{\sim}{g}_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j$ (2.8)

2.4 Metric Coefficients and the Metric Tensor

The elements $\underset{\sim}{g}_{ij}$ in Eqn (2.8) above are referred to as the metric coefficients and form the components of the metric tensor, which may be expressed in matrix form as follows:

$$\underset{\sim}{g}_{ij} = \begin{bmatrix} \underset{\sim}{g}_{11} & \underset{\sim}{g}_{12} & \underset{\sim}{g}_{13} \\ \underset{\sim}{g}_{21} & \underset{\sim}{g}_{22} & \underset{\sim}{g}_{23} \\ \underset{\sim}{g}_{31} & \underset{\sim}{g}_{32} & \underset{\sim}{g}_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \underset{\sim}{r}}{\partial y^1}\frac{\partial \underset{\sim}{r}}{\partial y^1} & \frac{\partial \underset{\sim}{r}}{\partial y^1}\frac{\partial \underset{\sim}{r}}{\partial y^2} & \frac{\partial \underset{\sim}{r}}{\partial y^1}\frac{\partial \underset{\sim}{r}}{\partial y^3} \\ \frac{\partial \underset{\sim}{r}}{\partial y^2}\frac{\partial \underset{\sim}{r}}{\partial y^1} & \frac{\partial \underset{\sim}{r}}{\partial y^2}\frac{\partial \underset{\sim}{r}}{\partial y^2} & \frac{\partial \underset{\sim}{r}}{\partial y^2}\frac{\partial \underset{\sim}{r}}{\partial y^3} \\ \frac{\partial \underset{\sim}{r}}{\partial y^3}\frac{\partial \underset{\sim}{r}}{\partial y^1} & \frac{\partial \underset{\sim}{r}}{\partial y^3}\frac{\partial \underset{\sim}{r}}{\partial y^2} & \frac{\partial \underset{\sim}{r}}{\partial y^3}\frac{\partial \underset{\sim}{r}}{\partial y^3} \end{bmatrix}$$

In the case of othogonal curvilinear coordinates, the base vectors $\underset{\sim}{g}_i$ are by definition mutually orthogonal and consequently $\underset{\sim}{g}_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j = 0$ for $i \neq j$. The resulting metric tensor is now a diagonal matrix and the components become

$$\underset{\sim}{g}_{ij} = \begin{bmatrix} \underset{\sim}{g}_{11} & 0 & 0 \\ 0 & \underset{\sim}{g}_{22} & 0 \\ 0 & 0 & \underset{\sim}{g}_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \underset{\sim}{r}}{\partial y^1}\frac{\partial \underset{\sim}{r}}{\partial y^1} & 0 & 0 \\ 0 & \frac{\partial \underset{\sim}{r}}{\partial y^2}\frac{\partial \underset{\sim}{r}}{\partial y^2} & 0 \\ 0 & 0 & \frac{\partial \underset{\sim}{r}}{\partial y^3}\frac{\partial \underset{\sim}{r}}{\partial y^3} \end{bmatrix}$$

or

$$\underset{\sim}{g}_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_i = \frac{\partial \underset{\sim}{r}}{\partial y^i} \cdot \frac{\partial \underset{\sim}{r}}{\partial y^i}$$

$$= \frac{\partial x^k}{\partial y^i} \cdot \frac{\partial x^k}{\partial y^i} \qquad \text{(no sum on i)} \qquad (2.9)$$

2.5 Unit Base Vectors in Y^i System

The base vectors $\underset{\sim}{g}_i$ referred to the curvilinear coordinate axes may be related to the corresponding unit base vectors $\underset{\sim}{e}_i$ as follows:

$$\underset{\sim}{e}_i = \frac{\dfrac{\partial \underset{\sim}{r}}{\partial y^i}}{\left| \dfrac{\partial \underset{\sim}{r}}{\partial y^i} \right|} = \frac{\underset{\sim}{g}_i}{| \underset{\sim}{g}_i |} = \frac{1}{h_i} \underset{\sim}{g}_i \qquad \text{(no sum on i)} \qquad (2.10)$$

where $\quad h_i = | \underset{\sim}{g}_i | = \left| \dfrac{\partial \underset{\sim}{r}}{\partial y^i} \right| \qquad$ (no sum on i)

are the scale factors, relating the dimensions of the relative lengths of the line element between the X^i and Y^i systems.

It can be seen from Eqn (2.10) that the unit vectors $\underset{\sim}{e}_i$ are obtained by dividing the base vectors $\underset{\sim}{g}_i$ by the scale factors h_i. The scale factors are related to the metric coefficients through Eqns (2.8) and (2.10) as follows:

From Eqn (2.10)

$$\underset{\sim}{g}_i = \underset{\sim}{e}_i h_i = \frac{\partial \underset{\sim}{r}}{\partial y^i} \qquad \text{(no sum on i)} \qquad (2.11)$$

From Eqn (2.8)

$$g_{ij} = g_i \cdot g_j = e_i h_i \cdot e_j h_j = h_i h_j \quad \text{(no sum on i)}$$

and in the case where i = j, we obtain

$$\therefore \quad g_{ii} = h_i^2 \quad \text{(no sum on i)} \tag{2.12}$$

The length of the line element in curvilinear coordinates may now be expressed in terms of the scale factors. Restating Eqn (2.7) and using Eqn (2.12) we have

$$ds^2 = g_{ij} dy^i dy^j$$

$$= h_i h_j dy^i dy^j$$

and in the case of orthogonal coordinates

$$h_i h_j = g_{ij} = 0 \quad \text{for } i \neq j$$

$$\therefore \quad ds^2 = \sum_{i=1}^{3} h_i^2 (dy^i)^2 \tag{2.13}$$

Referring to Figure 2.2 at point $P(y^1, y^2, y^3)$ where y^2 and y^3 are held constant the element of arc length along the y^1 axis may be written as:

$$ds_1^2 = h_1^2 dy^1 dy^1$$

$$\therefore \quad ds_1 = h_1 dy^1$$

Similarly $ds_2 = h_2 dy^2$

and $ds_3 = h_3 dy^3$

or $ds_i = h_i dy^i$ (no sum on i) (2.14)

2.6 Derivatives of Base Vectors

For subsequent work in orthogonal curvilinear coordinates it is necessary to obtain the relationships between the derivatives of the unit base vectors $\underset{\sim}{e}_i$ with respect to the curvilinear axes.

Since the set of base vectors and unit base vectors are mutually orthogonal, we may write

$$\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = 1 \quad \text{for} \quad i = j$$

$$\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = 0 \quad \text{for} \quad i \neq j$$

or $$\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$$

Similarly $$\underset{\sim}{g}_i \cdot \underset{\sim}{g}_j = 0 \quad \text{for } i \neq j \tag{2.15}$$

Differentiating Eqns (2.15) in turn with respect to y^k $(k \neq i \neq j)$ we obtain the following set of equations

$$\frac{\partial \underset{\sim}{g}_i}{\partial y^k} \cdot \underset{\sim}{g}_j + \underset{\sim}{g}_i \frac{\partial \underset{\sim}{g}_j}{\partial y^k} = 0 \qquad \text{(i,j,k cyclic permutation)} \tag{2.16}$$

and subtracting cyclic pairs of Eqn (2.16)

$$\underset{\sim}{g}_k \left(\frac{\partial \underset{\sim}{g}_j}{\partial y^i} - \frac{\partial \underset{\sim}{g}_i}{\partial y^j} \right) + \underset{\sim}{g}_j \cdot \frac{\partial \underset{\sim}{g}_k}{\partial y^i} - \frac{\partial \underset{\sim}{g}_k}{\partial y^j} \cdot \underset{\sim}{g}_i = 0 \tag{2.17}$$

and since

$$\frac{\partial \underset{\sim}{g}_i}{\partial y^j} = \frac{\partial}{\partial y^j} \left(\frac{\partial \underset{\sim}{r}}{\partial y^i} \right) = \frac{\partial}{\partial y^i} \left(\frac{\partial \underset{\sim}{r}}{\partial y^j} \right) = \frac{\partial \underset{\sim}{g}_j}{\partial y^i} \tag{2.18}$$

Eqn (2.17) may now be rewritten as

$$\frac{\partial \underset{\sim}{g}_i}{\partial y^k} \cdot \underset{\sim}{g}_j - \underset{\sim}{g}_i \cdot \frac{\partial \underset{\sim}{g}_j}{\partial y^k} = 0 \tag{2.19}$$

By referring to Eqn (2.16) we may conclude that

$$\underset{\sim}{g}_i \cdot \frac{\partial \underset{\sim}{g}_j}{\partial y^k} = 0 \quad \text{(i,j,k cyclic permutation)} \tag{2.20}$$

Differentiating Eqn (2.12) with respect to y^k

$$\underset{\sim}{g}_i \cdot \frac{\partial \underset{\sim}{g}_i}{\partial y^k} = h_i \frac{\partial h_i}{\partial y^k} \tag{2.21}$$

or
$$\underset{\sim}{g}_i \cdot \frac{\partial \underset{\sim}{g}_k}{\partial y^i} = h_i \frac{\partial h_i}{\partial y^k} \tag{2.22}$$

Differentiating Eqns (2.15) in turn with respect to y_k (i,j = all pairs of 1,2,3 for each k) the following set of equations is obtained.

$$\frac{\partial \underset{\sim}{g}_i}{\partial y^k} \cdot \underset{\sim}{g}_j + \underset{\sim}{g}_i \cdot \frac{\partial \underset{\sim}{g}_j}{\partial y^k} = 0 \qquad \text{(i,j = all pairs of 1,2,3)}$$

or
$$\underset{\sim}{g}_j \cdot \frac{\partial \underset{\sim}{g}_i}{\partial y^k} = - \underset{\sim}{g}_i \cdot \frac{\partial \underset{\sim}{g}_j}{\partial y^k} \tag{2.23}$$

Substituting Eqn (2.22) into Eqn (2.23) with k = i we obtain

$$\underset{\sim}{g}_j \cdot \frac{\partial \underset{\sim}{g}_i}{\partial y^i} = - h_i \frac{\partial h_i}{\partial y^j} \quad \text{(i,j = all pairs of 1,2,3)} \tag{2.24}$$

Having established the above sets of relations in Eqns (2.20), (2.21), (2.22) and (2.24), the derivatives of the base vector $\underset{\sim}{g}_i$ may now be obtained. The three components of the vector $\partial \underset{\sim}{g}_i/\partial y^i$ with respect to the local curvilinear axes will be

$$\frac{\partial \underset{\sim}{g}_i}{\partial y^i} \cdot \underset{\sim}{e}_i \; ; \qquad \frac{\partial \underset{\sim}{g}_i}{\partial y^i} \cdot \underset{\sim}{e}_j \; ; \qquad \frac{\partial \underset{\sim}{g}_i}{\partial y^i} \cdot \underset{\sim}{e}_k \qquad (i = 1,2,3)$$

Using Eqn (2.10) these components may be written in terms of the base vectors as follows

$$\frac{\partial \underset{\sim}{g}_i}{\partial y^i} \cdot \frac{\underset{\sim}{g}_i}{h_i} \; ; \qquad \frac{\partial \underset{\sim}{g}_i}{\partial y^i} \cdot \frac{\underset{\sim}{g}_j}{h_j} \; ; \qquad \frac{\partial \underset{\sim}{g}_i}{\partial y^i} \cdot \frac{\underset{\sim}{g}_k}{h_k} \qquad (i = 1,2,3)$$

By referring to Eqns (2.21) and (2.24), these components become

$$\frac{\partial h_i}{\partial y^i} \; ; \qquad - \frac{h_i}{h_j} \cdot \frac{\partial h_i}{\partial y^j} \; ; \qquad - \frac{h_i}{h_k} \cdot \frac{h_i}{\partial y^k}$$

thus we obtain

$$\frac{\partial \underset{\sim}{g}_i}{\partial y^i} = \frac{\partial h_i}{\partial y^i} \frac{\underset{\sim}{g}_i}{h_i} - \frac{h_i}{h_j^2} \frac{\partial h_i}{\partial y^j} \underset{\sim}{g}_j - \frac{h_i}{h_k^2} \frac{\partial h_i}{\partial y^k} \underset{\sim}{g}_k \qquad (i,j,k \text{ cyclic permutation}) \tag{2.25}$$

Similarly the vector $\partial \underset{\sim}{g}_j/\partial y^k$ may be resolved into its three components along the local curvilinear axes as follows:

$$\frac{\partial \underset{\sim}{g}_j}{\partial y^k} \cdot \underset{\sim}{e}_i \; ; \qquad \frac{\partial \underset{\sim}{g}_j}{\partial y^k} \cdot \underset{\sim}{e}_j \; ; \qquad \frac{\partial \underset{\sim}{g}_j}{\partial y^k} \cdot \underset{\sim}{e}_k$$

Referring to Eqns (2.20), (2.21) and (2.22) these components become

$$0 \; ; \qquad \frac{\partial h_j}{\partial y^k} \; ; \qquad \frac{\partial h_k}{\partial y^j}$$

Thus we obtain:

$$\frac{\partial \underset{\sim}{g}_j}{\partial y^k} = \frac{\partial h_j}{\partial y^k}\frac{\underset{\sim}{g}_j}{h_j} + \frac{\partial h_k}{\partial y^j}\frac{\underset{\sim}{g}_k}{h_k} \qquad (j,k = \text{all pairs of } 1,2,3) \qquad (2.26)$$

Eqns (2.25) and (2.26) give the required relationships between the derivatives of the base vectors in orthogonal curvilinear coordinates.

2.7 Derivatives of Unit Base Vectors

The relation between the derivatives of the unit base vectors with respect to the curvilinear axes may be obtained by substituting Eqn (2.10) into Eqns (2.25) and (2.26) as follows:

From Eqn (2.25)

$$\frac{\partial}{\partial y^i}(\underset{\sim}{e}_i h_i) = \frac{\partial h_i}{\partial y^i}\underset{\sim}{e}_i - \frac{h_i}{h_j^2}\frac{\partial h_i}{\partial y^j}(\underset{\sim}{e}_j h_j) - \frac{h_i}{h_k^2}\frac{\partial h_i}{\partial y^k}(\underset{\sim}{e}_k h_k)$$

$$\therefore \quad \frac{\partial \underset{\sim}{e}_i}{\partial y^i} h_i + \underset{\sim}{e}_i \frac{\partial h_i}{\partial y^i} = \frac{\partial h_i}{\partial y^i}\underset{\sim}{e}_i - \frac{h_i}{h_j}\frac{\partial h_i}{\partial y^j}\underset{\sim}{e}_j - \frac{h_i}{h_k}\frac{\partial h_i}{\partial y^k}\underset{\sim}{e}_k$$

$$\therefore \quad \frac{\partial \underset{\sim}{e}_i}{\partial y^i} = -\frac{\partial h_i}{\partial y^j}\frac{\underset{\sim}{e}_j}{h_j} - \frac{\partial h_i}{\partial y^k}\frac{\underset{\sim}{e}_k}{h_k} \qquad (i,j,k \text{ cyclic permutation}) \qquad (2.27)$$

Similarly from Eqn (2.26)

$$\frac{\partial}{\partial y^k}(\underset{\sim}{e}_j h_j) = \frac{\partial h_j}{\partial y^k}\underset{\sim}{e}_j + \frac{\partial h_k}{\partial y^j}\underset{\sim}{e}_k$$

$$\therefore \quad \frac{\partial \underset{\sim}{e}_j}{\partial y^k} h_j + \underset{\sim}{e}_j \frac{\partial h_j}{\partial y^k} = \frac{\partial h_j}{\partial y^k}\underset{\sim}{e}_j + \frac{\partial h_k}{\partial y^j}\underset{\sim}{e}_k$$

$$\therefore \quad \frac{\partial \underset{\sim}{e}_j}{\partial y^k} = \frac{\partial h_k}{\partial y^j}\frac{\underset{\sim}{e}_k}{h_j} \qquad (j,k = \text{all pairs of } 1,2,3) \tag{2.28}$$

Eqns (2.27) and (2.28) are the relationships between the derivatives of the unit base vectors in orthogonal curvilinear coordinates

2.8 Gauss' Divergence Theorem

The Divergence Theorem of Gauss in two dimensions, sometimes referred to as Green's Theorem in the plane, equates the line integral around a boundary to the area integral taken over the enclosed domain. In the subsequent work it will be necessary to make use of Gauss' Theorem as applied to vector quantities and formulated in terms of orthogonal curvilinear coordinates.

Let $\underset{\sim}{M}$ be a vector function defined on the surface A and around the boundary L and let $\underset{\sim}{M}$ be continuously differentiable on A. Then the area integral of the divergence of $\underset{\sim}{M}$ taken over the surface A is equal to the line integral of $\underset{\sim}{M}$ taken over the closed boundary L surrounding the area A, or

$$\int_L (M_1\nu_1 + M_2\nu_2 + M_3\nu_3)dL = \iint_A \left(\frac{\partial M_1}{\partial x^1} + \frac{\partial M_2}{\partial x^2} + \frac{\partial M_3}{\partial x^3} \right) dA$$

$$\therefore \quad \int_L M_i\nu_i \, dL = \iint_A \frac{\partial M_i}{\partial x^i} dA$$

$$\text{or} \quad \int_L \underset{\sim}{M}.\underset{\sim}{\nu} \, dL = \iint_A \nabla.\underset{\sim}{M} \, dA \tag{2.29}$$

where

$\underset{\sim}{\nu} \;=\; \nu_i \underset{\sim}{e}_i$ and ν_i are the components of the unit normal outward vector $\underset{\sim}{\nu}$

$\underset{\sim}{M} \;=\; M_i \underset{\sim}{e}_i$ and M_i are the components of the vector $\underset{\sim}{M}$

$\nabla . \underset{\sim}{M} \;=\;$ Divergence of $\underset{\sim}{M}$ = Div $\underset{\sim}{M}$

The Divergence of the vector $\underset{\sim}{M}$ expressed in orthogonal curvilinear coordinates will be as follows:

$$\text{Div } \underset{\sim}{M} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial y^1} (h_2 h_3 M_1) + \frac{\partial}{\partial y^2} (h_3 h_1 M_2) + \frac{\partial}{\partial y^3} (h_1 h_2 M_3) \right\}$$

where h_i are the scale factors as previously defined. In two dimensions, $h_3 = 1$, $M_3 = 0$ and thus

$$\int_L (M_1 \nu_2 + M_2 \nu_2) dL = \iint_A \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial y^1} (h_2 M_1) + \frac{\partial}{\partial y^2} (h_1 M_2) \right\} dy^1 dy^2 \tag{2.30}$$

Consider specifically the case of a vector function of the form:

$$\int_L (M_{12} \nu_1 + M_{22} \nu_2) \underset{\sim}{e}_1 \; dL$$

Let the unit base vector $\underset{\sim}{e}_1$ be decomposed into its components with respect to the invariant cartesian unit base vectors $\underset{\sim}{b}_i$ such that

$$\underset{\sim}{e}_1 = t_1 \underset{\sim}{b}_1 + t_2 \underset{\sim}{b}_2 + t_3 \underset{\sim}{b}_3 = t_i \underset{\sim}{b}_i$$

$$\therefore \quad \int_L (M_{12} \nu_1 + M_{22} \nu_2)(t_1 \underset{\sim}{b}_1 + t_2 \underset{\sim}{b}_2 + t_3 \underset{\sim}{b}_3) dL$$

$$= \underset{\sim}{b}_1 \int_L (M_{12} t_1 \nu_1 + M_{22} t_1 \nu_2) dL + \underset{\sim}{b}_2 \int_L (M_{12} t_2 \nu_1 + M_{22} t_2 \nu_2) dL$$

$$+ \; \underset{\sim}{b}_3 \int_L (M_{12} t_3 \nu_1 + M_{22} t_3 \nu_2) dL$$

Applying Gauss' theorem as in Eqn (2.30) above

$$\int_L (M_{12}\nu_1 + M_{22}\nu_2)(t_1\underset{\sim}{b}_1 + t_2\underset{\sim}{b}_2 + t_3\underset{\sim}{b}_3)dL$$

$$= \underset{\sim}{b}_1 \iint_A \frac{1}{h_1h_2}\left\{\frac{\partial}{\partial y^1}(M_{12}t_1h_2) + \frac{\partial}{\partial y^2}(M_{22}t_1h_1)\right\} dA$$

$$+ \underset{\sim}{b}_2 \iint_A \frac{1}{h_1h_2}\left\{\frac{\partial}{\partial y^1}(M_{12}t_2h_2) + \frac{\partial}{\partial y^2}(M_{22}t_2h_1)\right\} dA$$

$$+ \underset{\sim}{b}_3 \iint_A \frac{1}{h_1h_2}\left\{\frac{\partial}{\partial y^1}(M_{12}t_3h_2) + \frac{\partial}{\partial y^2}(M_{22}t_3h_1)\right\} dA$$

$$= \iint_A \frac{1}{h_1h_2}\left[\frac{\partial}{\partial y^1}\left\{M_{12}h_2(t_1\underset{\sim}{b}_1 + t_2\underset{\sim}{b}_2 + t_3\underset{\sim}{b}_3)\right\}\right.$$

$$\left. + \frac{\partial}{\partial y^2}\left\{M_{22}h_1(t_1\underset{\sim}{b}_1 + t_2\underset{\sim}{b}_2 + t_3\underset{\sim}{b}_3)\right\}\right] dA$$

$$\therefore \int_L (M_{12}\underset{\sim}{e}_1\nu_1 + M_{22}\underset{\sim}{e}_1\nu_2)\, dL$$

$$= \iint_A \frac{1}{h_1h_2}\left\{\frac{\partial}{\partial y^1}(M_{12}h_2\underset{\sim}{e}_1) + \frac{\partial}{\partial y^2}(M_{22}h_1\underset{\sim}{e}_1)\right\} dA \tag{2.31}$$

The form of Gauss' divergence theorem as expressed in Eqns (2.30) and (2.31) may be applied directly in the subsequent work in curvilinear coordinates.

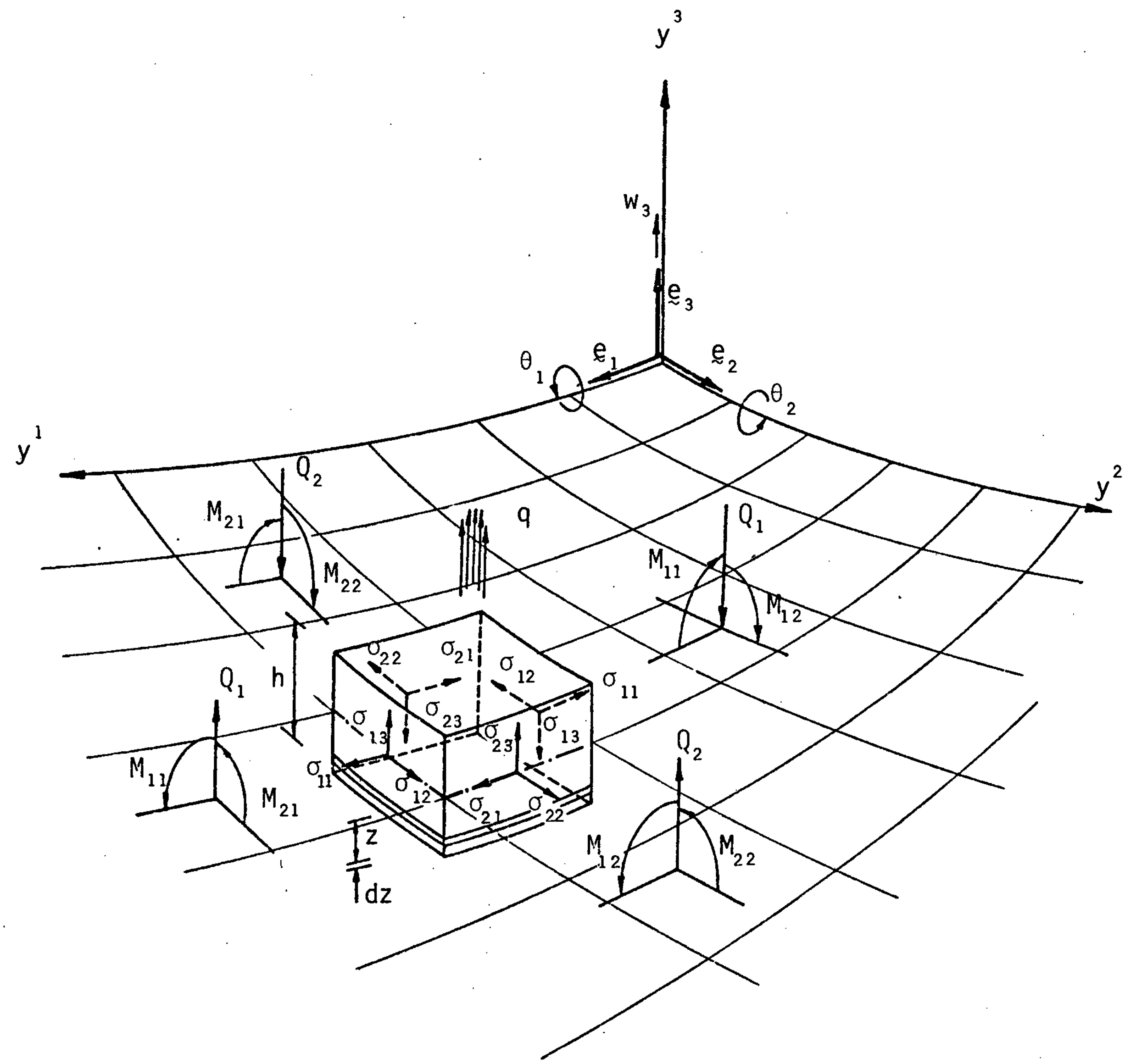


FIGURE 3.1 Element of orthtrophic plate showing positive stresses, stress resultants and displacements

$$\underset{\sim}{M}_1 = M_{12}\underset{\sim}{e}_1 + M_{11}\underset{\sim}{e}_2 \tag{3.1}$$

$$\underset{\sim}{M}_2 = M_{22}\underset{\sim}{e}_1 + M_{21}\underset{\sim}{e}_2 \tag{3.2}$$

and the resultant of the components acting along the boundary of the element will be:

$$\underset{\sim}{M} = M_1\underset{\sim}{e}_1 + M_2\underset{\sim}{e}_2 = M_i\underset{\sim}{e}_i \tag{3.3}$$

An element on the boundary of the plate with components of the unit normal outward vector $\underset{\sim}{\nu}$ is shown in Figure 3.3. The unit normal may be expressed in terms of these components as follows.

$$\underset{\sim}{\nu} = \nu_1\underset{\sim}{e}_1 + \nu_2\underset{\sim}{e}_2 = \nu_i\underset{\sim}{e}_i \tag{3.4}$$

where $\nu_1 = \sin\theta, \qquad \nu_2 = \cos\theta$

Note that the relative lengths of the sides and boundary of the element may be defined in terms of the components of the unit normal vector $\underset{\sim}{\nu}$.

Expressing Eqn. (3.3) in terms of the components of Eqn. (3.4)

$$\underset{\sim}{M} = \underset{\sim}{M}_1\nu_1 + \underset{\sim}{M}_2\nu_2 = \underset{\sim}{M}_i\nu_i \tag{3.5}$$

Substituting Eqns. (3.1), (3.2),(3.3) into Eqn. (3.5) gives

$$M_1\underset{\sim}{e}_1 + M_2\underset{\sim}{e}_2 = (M_{12}\underset{\sim}{e}_1 + M_{11}\underset{\sim}{e}_2)\nu_1 + (M_{22}\underset{\sim}{e}_1 + M_{21}\underset{\sim}{e}_2)\nu_2$$

$$= (M_{12}\nu_1 + M_{22}\nu_2)\underset{\sim}{e}_1 + (M_{11}\nu_1 + M_{21}\nu_2)\underset{\sim}{e}_2 \tag{3.6}$$

and therefore

$$M_1 = M_{12}\nu_1 + M_{22}\nu_2$$

$$M_2 = M_{11}\nu_1 + M_{21}\nu_2$$

also $Q = Q_1\nu_1 + Q_2\nu_2$ (3.7)

By referring to Figure 3.2 and considering the equilibrium of the element, these equations may be obtained directly by summing the components of the stress resultants over the sides of the element.

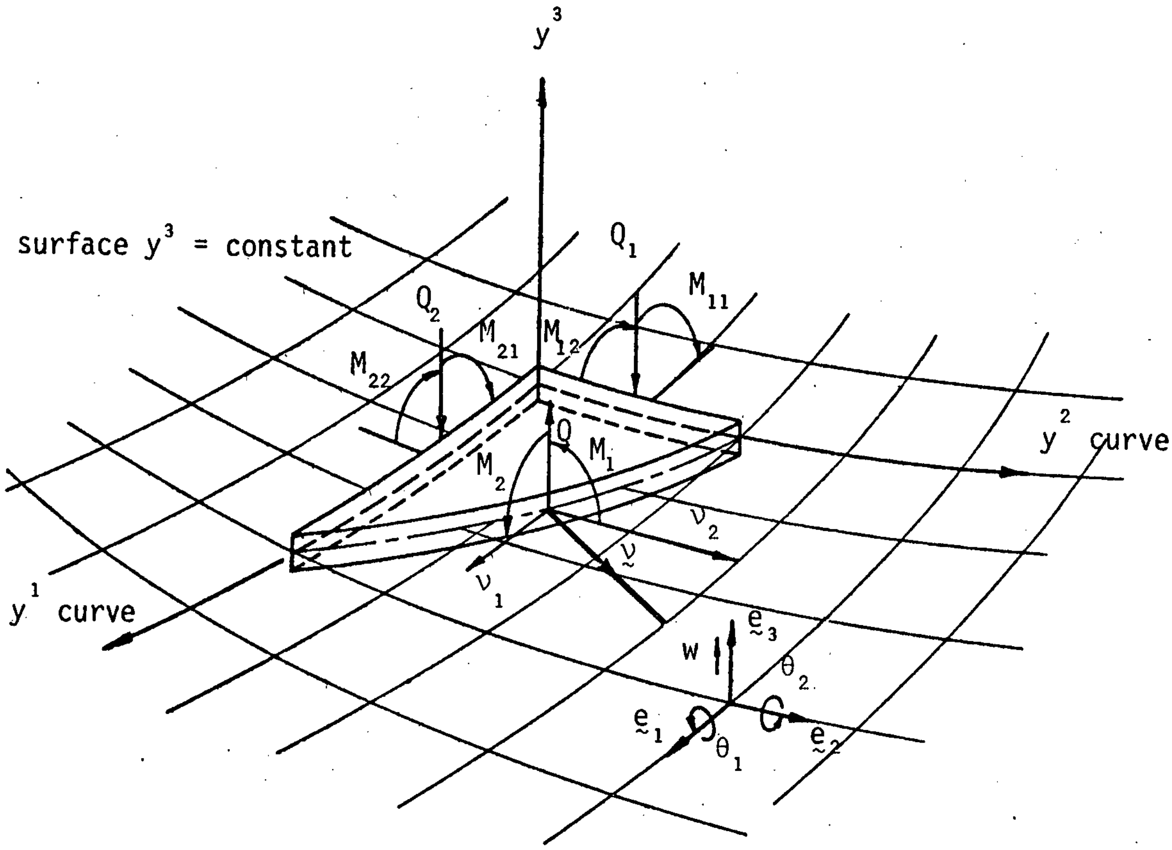


FIGURE 3.2 Element on boundary of plate showing positive stress resultants

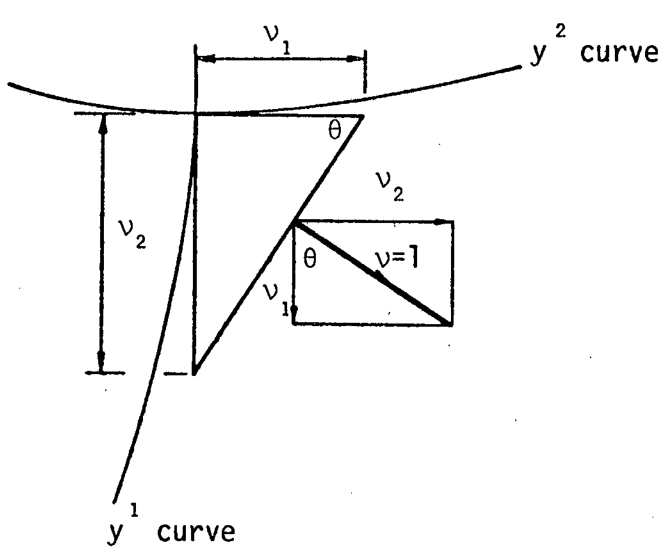


FIGURE 3.3 Element on boundary of plate showing components of unit normal vector $\underset{\sim}{\nu}$.

3.3 Derivation of Equilibrium Equations in Orthogonal Curvilinear Coordinates

Consider an element on the boundary of the plate as shown in Figure 3.4. For equilibrium of the complete surface the sum of all the external forces must be zero, and therefore:

$$\int_L \underset{\sim}{M} dL + \int_L (\underset{\sim}{r} x \underset{\sim}{Q}) dL + \iint_A (\underset{\sim}{r} x \underset{\sim}{\rho}) dA = 0 \qquad (3.8a,b,c)$$

and for vertical equilibrium

$$\int_L \underset{\sim}{Q} dL + \iint_A \underset{\sim}{\rho} dA = 0 \qquad (3.8d)$$

Expand each part of Eqn. (3.8) in turn. For Eqn. (3.8a) substitute Eqn. (3.6) as follows.

$$\int_L \underset{\sim}{M} dL = \int_L (M_1 \underset{\sim}{e}_1 + M_2 \underset{\sim}{e}_2) dL$$

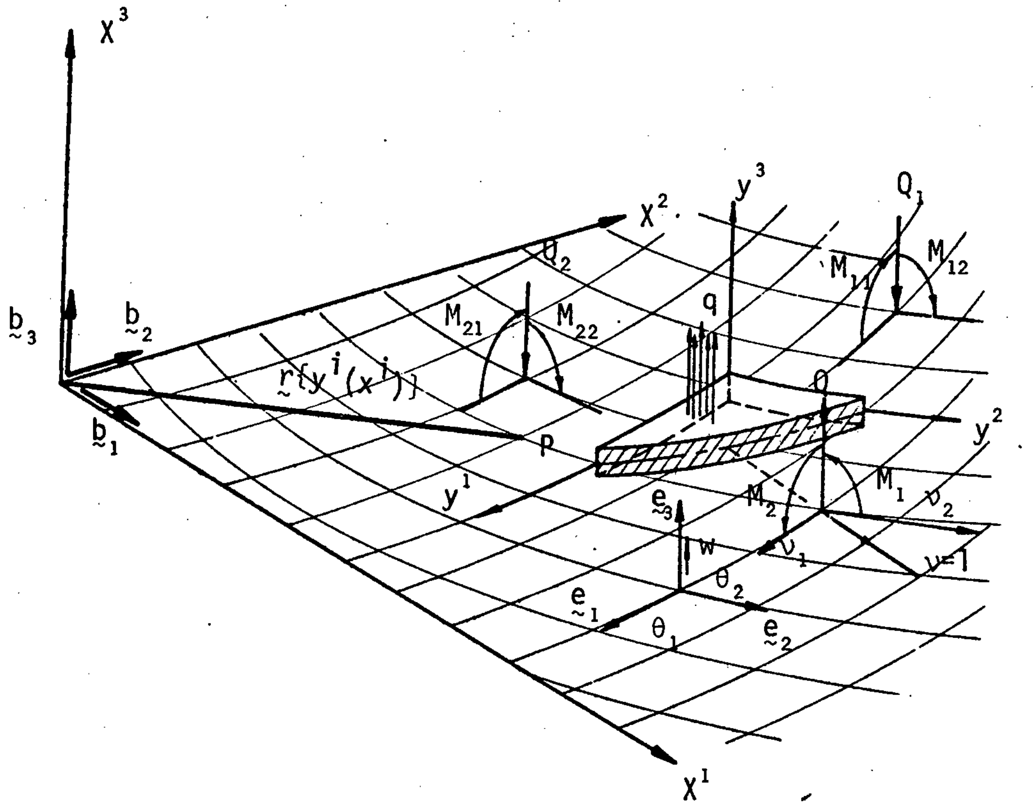
$$= \int_L \left\{ (M_{12}\nu_1 + M_{22}\nu_2) \underset{\sim}{e}_1 + (M_{11}\nu_1 + M_{21}\nu_2) \underset{\sim}{e}_2 \right\} dL$$

$$= \int_L \left\{ M_{12}\nu_1 \underset{\sim}{e}_1 + M_{22}\nu_2 \underset{\sim}{e}_1 + M_{11}\nu_1 \underset{\sim}{e}_2 + M_{21}\nu_2 \underset{\sim}{e}_2 \right\} dL$$

Using Gauss' Divergence theorem to convert the above boundary integral to a surface integral, apply Eqn. (2.31)

$$\int_L \underset{\sim}{M} dL = \iint_A \left[\frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial y^1} (M_{12} h_2 \underset{\sim}{e}_1) + \frac{\partial}{\partial y^2} (M_{22} h_1 \underset{\sim}{e}_1) \right.\right.$$

$$\left.\left. + \frac{\partial}{\partial y^1} (M_{11} h_2 \underset{\sim}{e}_2) + \frac{\partial}{\partial y^2} (M_{21} h_1 \underset{\sim}{e}_2) \right\} \right] dA$$



<u>FIGURE 3.4</u> Element on boundary of plate showing external forces

$$\int_L \underset{\sim}{M} dL = \iint_A \left[\frac{1}{h_1 h_2} \left\{ \frac{\partial M_{12}}{\partial y^1} \underset{\sim}{e}_1 h_2 + M_{12} \frac{\partial \underset{\sim}{e}_1}{\partial y^1} h_2 + M_{12} \underset{\sim}{e}_1 \frac{\partial h_2}{\partial y^1} \right.\right.$$

$$+ \frac{\partial M_{22}}{\partial y^2} \underset{\sim}{e}_1 h_1 + M_{22} \frac{\partial \underset{\sim}{e}_1}{\partial y^2} h_1 + M_{22} \underset{\sim}{e}_1 \frac{\partial h_1}{\partial y^2}$$

$$+ \frac{\partial M_{11}}{\partial y^1} \underset{\sim}{e}_2 h_2 + M_{11} \frac{\partial \underset{\sim}{e}_2}{\partial y^1} h_2 + M_{11} \underset{\sim}{e}_2 \frac{\partial h_2}{\partial y^1}$$

$$\left.\left. + \frac{\partial M_{21}}{\partial y^2} \underset{\sim}{e}_2 h_1 + M_{21} \frac{\partial \underset{\sim}{e}_2}{\partial y^2} h_1 + M_{21} \underset{\sim}{e}_2 \frac{\partial h_1}{\partial y^2} \right\} \right] dA$$

$$= \iint_A \left\{ \frac{1}{h_1} \frac{\partial M_{12}}{\partial y^1} \underset{\sim}{e}_1 + \frac{1}{h_1} M_{12} \frac{\partial \underset{\sim}{e}_1}{\partial y^1} + \frac{1}{h_1 h_2} M_{12} \frac{\partial h_2}{\partial y^1} \underset{\sim}{e}_1 \right.$$

$$+ \frac{1}{h_2} \frac{\partial M_{22}}{\partial y^2} \underset{\sim}{e}_1 + \frac{1}{h_2} M_{22} \frac{\partial \underset{\sim}{e}_1}{\partial y^2} + \frac{1}{h_1 h_2} M_{22} \frac{\partial h_1}{\partial y^2} \underset{\sim}{e}_1$$

$$+ \frac{1}{h_1} \frac{\partial M_{11}}{\partial y^1} \underset{\sim}{e}_2 + \frac{1}{h_1} M_{11} \frac{\partial \underset{\sim}{e}_2}{\partial y^1} + \frac{1}{h_1 h_2} M_{11} \frac{\partial h_2}{\partial y^1} \underset{\sim}{e}_2$$

$$\left. + \frac{1}{h_2} \frac{\partial M_{21}}{\partial y^2} \underset{\sim}{e}_2 + \frac{1}{h_2} M_{21} \frac{\partial \underset{\sim}{e}_2}{\partial y^2} + \frac{1}{h\ h} M_{21} \frac{\partial h_1}{\partial y^2} \underset{\sim}{e}_2 \right\} dA$$

Using Eqns. (2.27) and (2.28) the derivatives of the unit base vector may be replaced as follows

$$\frac{\partial \underset{\sim}{e}_1}{\partial y^1} = -\frac{\partial h_1}{\partial y^2} \frac{\underset{\sim}{e}_2}{h_2}; \qquad \frac{\partial \underset{\sim}{e}_1}{\partial y^2} = \frac{\partial h_2}{\partial y^1} \frac{\underset{\sim}{e}_2}{h_1}$$

$$\frac{\partial \underset{\sim}{e}_2}{\partial y^1} = \frac{\partial h_1}{\partial y^2} \frac{\underset{\sim}{e}_1}{h_2}; \qquad \frac{\partial \underset{\sim}{e}_2}{\partial y^2} = -\frac{\partial h_2}{\partial y^1} \frac{\underset{\sim}{e}_1}{h_1} \qquad (3.9)$$

Using the relationships in Eqns. (3.9) above, Eqn. (3.8a) becomes

$$\int_L \underset{\sim}{M} dL = \iint_A \left\{ \frac{1}{h_1}\frac{\partial M_{12}}{\partial y^1}\underset{\sim}{e}_1 - \frac{1}{h_1 h_2} M_{12}\frac{\partial h_1}{\partial y^2}\underset{\sim}{e}_2 + \frac{1}{h_1 h_2} M_{12}\frac{\partial h_2}{\partial y^1}\underset{\sim}{e}_1 \right.$$

$$+ \frac{1}{h_2}\frac{\partial M_{22}}{\partial y^2}\underset{\sim}{e}_1 + \frac{1}{h_1 h_2} M_{22}\frac{\partial h_2}{\partial y^1}\underset{\sim}{e}_2 + \frac{1}{h_1 h_2} M_{22}\frac{\partial h_1}{\partial y^2}\underset{\sim}{e}_1$$

$$+ \frac{1}{h_1}\frac{\partial M_{11}}{\partial y^1}\underset{\sim}{e}_2 + \frac{1}{h_1 h_2} M_{11}\frac{\partial h_1}{\partial y^2}\underset{\sim}{e}_1 + \frac{1}{h_1 h_2} M_{11}\frac{\partial h_2}{\partial y^1}\underset{\sim}{e}_2$$

$$\left. + \frac{1}{h_2}\frac{\partial M_{21}}{\partial y^2}\underset{\sim}{e}_2 - \frac{1}{h_1 h_2} M_{21}\frac{\partial h_2}{\partial y^1}\underset{\sim}{e}_1 + \frac{1}{h_1 h_2} M_{21}\frac{\partial h_1}{\partial y^2}\underset{\sim}{e}_2 \right\} dA$$

and regrouping with common unit base vectors,

$$\int_L \underset{\sim}{M} dL = \iint_A \left[\left\{ \frac{1}{h_1}\frac{\partial M_{12}}{\partial y^1} + \frac{1}{h_1 h_2} M_{12}\frac{\partial h_2}{\partial y^1} + \frac{1}{h_2}\frac{\partial M_{22}}{\partial y^2} + \frac{1}{h_1 h_2} M_{22}\frac{\partial h_1}{\partial y^2} \right.\right.$$

$$\left. + \frac{1}{h_1 h_2} M_{11}\frac{\partial h_1}{\partial y^2} - \frac{1}{h_1 h_2} M_{21}\frac{\partial h_2}{\partial y^1} \right\} \underset{\sim}{e}_1 \quad + \left\{ - \frac{1}{h_1 h_2} M_{12}\frac{\partial h_1}{\partial y^2} + \frac{1}{h_1 h_2} M_{22}\frac{\partial h_2}{\partial y^1} \right.$$

$$\left.\left. + \frac{1}{h_1}\frac{\partial M_{11}}{\partial y^1} + \frac{1}{h_1 h_2} M_{11}\frac{\partial h_2}{\partial y^1} + \frac{1}{h_2}\frac{\partial M_{21}}{\partial y^2} + \frac{1}{h_1 h_2} M_{21}\frac{\partial h_1}{\partial y^2} \right\} \underset{\sim}{e}_2 \right] dA$$

For Eqn. (3.8b) substitute Eqn (3.7) as follows

$$\int_L (\underset{\sim}{r} x \underset{\sim}{Q}) dL \quad = \quad \int_L (\underset{\sim}{r} x Q \underset{\sim}{e}_3) dL$$

$$= \int_L \{\underset{\sim}{r} x (Q_1 \nu_1 + Q_2 \nu_2)\} \underset{\sim}{e}_3 dL$$

$$= \int_L \{(\underset{\sim}{r} x Q_1 \underset{\sim}{e}_3)\nu_1 + (\underset{\sim}{r} x Q_2 \underset{\sim}{e}_3)\nu_2\} dL$$

and applying Gauss' Divergence theorem to the above cross products, Eqn. (3.8b) becomes

$$\iint_A \left[\frac{1}{h_1h_2}\left\{\frac{\partial}{\partial y^1}\left(\underset{\sim}{r}\times Q_1\underset{\sim}{e}_3h_2\right)+\frac{\partial}{\partial y^2}\left(\underset{\sim}{r}\times Q_2\underset{\sim}{e}_3h_1\right)\right\}\right]dA$$

$$= \int_A \left[\frac{1}{h_1h_2}\left\{\frac{\partial \underset{\sim}{r}}{\partial y^1}\times Q_1\underset{\sim}{e}_3h_2 + \underset{\sim}{r}\times\frac{\partial Q_1}{\partial y^1}\underset{\sim}{e}_3h_2 + \underset{\sim}{r}\times Q_1\frac{\partial \underset{\sim}{e}_3}{\partial y^1}h_2 + \underset{\sim}{r}\times Q_1\underset{\sim}{e}_3\frac{\partial h_2}{\partial y^1}\right.\right.$$

$$\left.\left. + \frac{\partial \underset{\sim}{r}}{\partial y^2}\times Q_2\underset{\sim}{e}_3h_1 + \underset{\sim}{r}\times\frac{\partial Q_2}{\partial y^2}\underset{\sim}{e}_3h_1 + \underset{\sim}{r}\times Q_2\frac{\partial \underset{\sim}{e}_3}{\partial y^2}h_1 + \underset{\sim}{r}\times Q_2\underset{\sim}{e}_3\frac{\partial h_1}{\partial y^2}\right\}\right]dA$$

Using Eqn. (2.11) and since $\partial \underset{\sim}{e}_3/\partial y^i = 0$, the above expression becomes

$$\iint_A \left[\frac{1}{h_1h_2}\left\{\underset{\sim}{e}_1h_1\times Q_1\underset{\sim}{e}_3h_2 + \underset{\sim}{r}\times\frac{\partial Q_1}{\partial y^1}\underset{\sim}{e}_3h_2 + \underset{\sim}{r}\times Q_1\underset{\sim}{e}_3\frac{\partial h_2}{\partial y^1}\right.\right.$$

$$\left.\left. + \underset{\sim}{e}_2h_2\times Q_2\underset{\sim}{e}_3h_1 + \underset{\sim}{r}\times\frac{\partial Q_2}{\partial y^2}\underset{\sim}{e}_3h_1 + \underset{\sim}{r}\times Q_2\underset{\sim}{e}_3\frac{\partial h_1}{\partial y^2}\right\}\right]dA$$

$$= \iint_A \left\{ - Q_1\underset{\sim}{e}_2 + \frac{1}{h_1}\left(\underset{\sim}{r}\times\underset{\sim}{e}_3\right)\frac{\partial Q_1}{\partial y^1} + \frac{1}{h_1h_2}\left(\underset{\sim}{r}\times\underset{\sim}{e}_3\right)Q_1\frac{\partial h_2}{\partial y^1}\right.$$

$$\left. + \underset{\sim}{e}_1Q_2 + \frac{1}{h_2}\left(\underset{\sim}{r}\times\underset{\sim}{e}_3\right)\frac{\partial Q_2}{\partial y^2} + \frac{1}{h_1h_2}\left(\underset{\sim}{r}\times\underset{\sim}{e}_3\right)Q_2\frac{\partial h_2}{\partial y^2}\right\}dA$$

$$= \iint_A \left[\left\{\frac{1}{h_1}\frac{\partial Q_1}{\partial y^1} + \frac{1}{h_1h_2}Q_1\frac{\partial h_2}{\partial y^1} + \frac{1}{h_2}\frac{\partial Q_2}{\partial y^2} + \frac{1}{h_1h_2}Q_2\frac{\partial h_1}{\partial y^2}\right\}\left(\underset{\sim}{r}\times\underset{\sim}{e}_3\right)\right.$$

$$\left. - Q_1\underset{\sim}{e}_2 + Q_2\underset{\sim}{e}_1\right]dA$$

and rewriting Eqn. (3.8c) as

$$\iint_A \left(\underset{\sim}{r}\times\rho\underset{\sim}{e}_3\right)dA = \iint \rho\left(\underset{\sim}{r}\times\underset{\sim}{e}_3\right)dA,$$

the like terms of Eqns. (3.8) may be regrouped as follows:

$$\iint_A \left\{ \frac{1}{h_1} \frac{\partial M_{12}}{\partial y^1} + \frac{1}{h_1 h_2} M_{12} \frac{\partial h_2}{\partial y^1} + \frac{1}{h_2} \frac{\partial M_{22}}{\partial y^2} + \frac{1}{h_1 h_2} M_{22} \frac{\partial h_1}{\partial y^2} \right.$$

$$\left. + \frac{1}{h_1 h_2} M_{11} \frac{\partial h_1}{\partial y^2} - \frac{1}{h_1 h_2} M_{21} \frac{\partial h_2}{\partial y^1} + Q_2 \right\} \underset{\sim}{e}_1 dA \qquad (3.10a)$$

$$+ \iint_A \left\{ \frac{-1}{h_1 h_2} M_{12} \frac{\partial h_1}{\partial y^2} + \frac{1}{h_1 h_2} M_{22} \frac{\partial h_2}{\partial y^1} + \frac{1}{h_1} \frac{\partial M_{11}}{\partial y^1} + \frac{1}{h_1 h_2} M_{11} \frac{\partial h_2}{\partial y^1} \right.$$

$$\left. + \frac{1}{h_2} \frac{\partial M_{21}}{\partial y^2} + \frac{1}{h_1 h_2} M_{21} \frac{\partial h_1}{\partial y^2} - Q_1 \right\} \underset{\sim}{e}_2 dA + \qquad (3.10b)$$

$$\iint_A \left\{ \frac{1}{h_1} \frac{\partial Q_1}{\partial y^1} + \frac{1}{h_1 h_2} Q_1 \frac{\partial h_2}{\partial y^1} + \frac{1}{h_2} \frac{\partial Q_2}{\partial y^2} + \frac{1}{h_1 h_2} Q_2 \frac{\partial h_1}{\partial y^2} + \rho \right\} (\underset{\sim}{r} x \underset{\sim}{e}_3) dA = 0$$

$$(3.10c)$$

For Eqn. (3.8d), rewrite as

$$\int_L \underset{\sim}{Q} dL + \iint_A \underset{\sim}{\rho} dA = \int_L Q \underset{\sim}{e}_3 dL + \iint_A \rho \underset{\sim}{e}_3 dA = 0$$

and substituting Eqn (3.7) as follows we obtain

$$\int_L (Q_1 \nu_1 + Q_2 \nu_2) \underset{\sim}{e}_3 dL + \iint_A \rho \underset{\sim}{e}_3 dA = 0$$

and applying Gauss' Divergence theorem, using Eqn. (2.31)

$$\iint_A \left[\frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial y^1} (Q_1 \underset{\sim}{e}_3 h_2) + \frac{\partial}{\partial y^2} (Q_2 \underset{\sim}{e}_3 h_1) \right\} + \rho \underset{\sim}{e}_3 \right] dA = 0$$

$$\therefore \quad \iint_A \left[\frac{1}{h_1 h_2} \left\{ \frac{\partial Q_1}{\partial y^1} \underset{\sim}{e}_3 h_2 + Q_1 \frac{\partial \underset{\sim}{e}_3}{\partial y^1} h_2 + Q_1 \underset{\sim}{e}_3 \frac{\partial h_2}{\partial y^1} \right. \right.$$

$$\left. \left. + \frac{\partial Q_2}{\partial y^2} \underset{\sim}{e}_3 h_1 + Q_2 \frac{\partial \underset{\sim}{e}_3}{\partial y^2} h_1 + Q_2 \underset{\sim}{e}_3 \frac{\partial h_1}{\partial y^2} \right\} + \rho \underset{\sim}{e}_3 \right] dA = 0$$

$$\therefore \quad \iint_A \left\{ \frac{1}{h_1} \frac{\partial Q_1}{\partial y^1} + \frac{1}{h_1 h_2} Q_1 \frac{\partial h_2}{\partial y^1} + \frac{1}{h_2} \frac{\partial Q_2}{\partial y^2} + \frac{1}{h_1 h_2} Q_2 \frac{\partial h_1}{\partial y^2} + \rho \right\} \underset{\sim}{e}_3 dA = 0 \tag{3.10d}$$

The above Eqn. of (3.10) are the expanded form of Eqns. (3.8), which may be more simply represented as:

$$\iint_A E_1 \underset{\sim}{e}_1 dA + \iint_A E_2 \underset{\sim}{e}_2 dA + \iint_A E_3 (\underset{\sim}{r} x \underset{\sim}{e}_3) dA = 0 \tag{3.11a}$$

and
$$\iint_A E_3 \underset{\sim}{e}_3 dA = 0 \tag{3.11b}$$

where in each case E_i represents the expression contained in parenthesis. Since Eqn. (3.12) applies to an arbitrary area of the plate

$$E_3 \underset{\sim}{e}_3 = 0 \tag{3.12a}$$

Since $\underset{\sim}{e}_3$ is non-zero, it follows that $E_3 = 0$. Eqn. (3.11) therefore becomes

$$\iint_A (E_1 \underset{\sim}{e}_1 + E_2 \underset{\sim}{e}_2) dA = 0$$

and for an arbitrary area

$$E_1 \underset{\sim}{e}_1 + E_2 \underset{\sim}{e}_2 = 0 \tag{3.12b}$$

Therefore $E_1 = E_2 = 0$ since the two orthogonal components of a zero vector are each zero. The Eqns. of (3.8) therefore reduce to $E_i = 0$ or

$$\frac{1}{h_1}\frac{\partial M_{11}}{\partial y^1} + \frac{M_{11}}{h_1 h_2}\frac{\partial h_2}{\partial y^1} + \frac{1}{h_2}\frac{\partial M_{12}}{\partial y^2} - \frac{2M_{12}}{h_1 h_2}\frac{\partial h_1}{\partial y^2} + \frac{M_{22}}{h_1 h_2}\frac{\partial h_2}{\partial y^1} - Q_1 = 0$$

$$\frac{1}{h_2}\frac{\partial M_{22}}{\partial y^2} + \frac{M_{22}}{h_1 h_2}\frac{\partial h_1}{\partial y^2} + \frac{1}{h_1}\frac{\partial M_{12}}{\partial y^1} + \frac{2M_{12}}{h_1 h_2}\frac{\partial h_2}{\partial y^1} + \frac{M_{11}}{h_1 h_2}\frac{\partial h_1}{\partial y^2} + Q_2 = 0$$

$$\frac{1}{h_1}\frac{\partial Q_1}{\partial y^1} + \frac{Q_1}{h_1 h_2}\frac{\partial h_2}{\partial y^1} + \frac{1}{h_2}\frac{\partial Q_2}{\partial y^2} + \frac{Q_2}{h_1 h_2}\frac{\partial h_1}{\partial y^2} + \rho = 0 \qquad (3.13)$$

where in the case of orthogonal curvilinear coordinates $\sigma_{12} = \sigma_{21}$ and therefore $M_{12} = - M_{21}$.

The above Eqns. (3.13) are the three equilibrium equations for the plate expressed in orthogonal curvilinear coordinates.

The expressions for the shearing forces Q_1 and Q_2 may be substituted into the third equation of (3.13) to obtain the governing differential equation of equilibrium.

3.4 Equilibrium Equations in Plane Polar Coordinates

The relationship between the cartesian and the polar coordinate systems may be expressed as follows

$$x^1 = r\cos\theta \qquad \text{or} \qquad x^1 = y^1 \cos y^2$$
$$x^2 = r\sin\theta \qquad\qquad x^2 = y^1 \sin y^2$$
$$x^3 = z \qquad\qquad x^3 = y^3$$

where $y^1 = r, \quad y^2 = \theta, \quad y^3 = z.$

The diagonal components of the metric tensor may be found using Eqn. (2.9) as follows

$$g_{11} = \left(\frac{\partial x^1}{\partial y^1}\right)^2 + \left(\frac{\partial x^2}{\partial y^1}\right)^2 + \left(\frac{\partial x^3}{\partial y^1}\right)^2 = (\cos y^2)^2 + (\sin y^2)^2 = 1$$

$$g_{22} = \left(\frac{\partial x^1}{\partial y^2}\right)^2 + \left(\frac{\partial x^2}{\partial y^2}\right)^2 + \left(\frac{\partial x^3}{\partial y^2}\right)^2 = (y^1 \cos y^2)^2 + (-y^1 \sin y^2)^2$$

$$= (y^1)^2 = r^2$$

$$g_{33} = \left(\frac{\partial x^1}{\partial y^3}\right)^2 + \left(\frac{\partial x^2}{\partial y^3}\right)^2 + \left(\frac{\partial x^3}{\partial y^3}\right)^2 = (1)^2 = 1$$

In matrix form

$$g_{ij} = \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \\ 0 & r^2 & \\ 0 & 0 & 1 \end{bmatrix}$$

The scale factors may be obtained using Eqn. (2.12)

$$\text{i.e. } h_1 = \sqrt{g_{11}} = 1, \qquad h_2 = \sqrt{g_{22}} = r \tag{3.14}$$

The derivatives of the scale factors with respect to the respective axes will be

$$\frac{\partial h_1}{\partial y^1} = \frac{\partial}{\partial r}(1) = 0, \qquad \frac{\partial h_1}{\partial y^2} = \frac{\partial}{\partial \theta}(1) = 0$$

$$\frac{\partial h_2}{\partial y^1} = \frac{\partial r}{\partial r} = 1, \qquad \frac{\partial h_2}{\partial y^2} = \frac{\partial r}{\partial \theta} = 0 \tag{3.15}$$

Substituting Eqns. (3.14) and (3.15) into the Eqns. of (3.18) gives

$$\frac{\partial M_{rr}}{\partial r} + \frac{M_{rr}}{r} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} + \frac{M_{\theta\theta}}{r} - Q_r = 0$$

$$\frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial \theta} + \frac{\partial M_{r\theta}}{\partial r} + \frac{2M_{r\theta}}{r} + Q_\theta = 0$$

$$\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} + \rho = 0 \qquad (3.16)$$

The Eqns. of (3.16) are the equilibrium equations for the plate expressed in plane polar coordinates.

The equilibrium equations in cartesian coordinates may be obtained in a similar manner. By setting $dx^1 = dr$, $dx^2 = rd\theta$ and r large, the above equations of (3.16) will also simplify to those expressed in cartesian coordinates [35].

CHAPTER 4

DERIVATION OF STRAIN-DISPLACEMENT RELATIONS

4.1 Relationship Between Strains and Generalised Displacements in Orthogonal Curvilinear Coordinates

4.1.1 Using the Principle of Virtual Work

The generalised displacements d_i for the plate as shown in Figures 3.1 and 3.4 may be written in vector form as follows

$$\underset{\sim}{d} = d_i \underset{\sim}{e}_i \tag{4.1}$$

or $\underset{\sim}{d} = \theta_1 \underset{\sim}{e}_1 + \theta_2 \underset{\sim}{e}_2 + w_3 \underset{\sim}{e}_3$

where $d_1 = \theta_1$, $d_2 = \theta_2$, and $d_3 = w_3$

The equilibrium equations of (3.12a) and (3.12b) may be expressed and restated as

$$E_i \underset{\sim}{e}_i = 0 \tag{4.2}$$

or $E_1 \underset{\sim}{e}_1 + E_2 \underset{\sim}{e}_2 + E_3 \underset{\sim}{e}_3 = 0$

For the purpose of obtaining the shear strains and the curvatures for the plate, the dot product of Eqns (4.1) and (4.2) is found as follows

$$E_i \underset{\sim}{e}_i \cdot d_j \underset{\sim}{e}_j = 0$$

$\therefore$, $E_i d_j \delta_{ij} = 0$

i.e. $E_i d_i = 0$

Hence

$$\iint_A E_i d_i \, dA = 0$$

Or
$$\iint_A (E_1\theta_1 + E_2\theta_2 + E_3 w_3) dA = 0 \qquad (4.3)$$

Eqn (4.3) can be interpreted as an expression of the principle of virtual work where each component represents the work done by the external forces in the respective curvilinear coordinate directions.

This will become more apparent once the terms are expanded and Gauss' divergence theorem has been applied. Using Eqn (3.10a), the first term of Eqn (4.3) may be expressed as follows:

$$\iint_A E_1\theta_1 dA = \iint_A \left\{ \frac{1}{h_1}\frac{\partial M_{12}}{\partial y^1}\theta_1 + \frac{1}{h_1 h_2} M_{12}\frac{\partial h_2}{\partial y^1}\theta_1 + \frac{1}{h_2}\frac{\partial M_{22}}{\partial y^2}\theta_1 \right.$$
$$\left. + \frac{1}{h_1 h_2} M_{22}\frac{\partial h_1}{\partial y^2}\theta_1 + \frac{1}{h_1 h_2} M_{11}\frac{\partial h_1}{\partial y^2}\theta_1 - \frac{1}{h_1 h_2} M_{21}\frac{\partial h_2}{\partial y^1}\theta_1 + Q_2\theta_1 \right\} dA$$

The above equation may be expanded as follows:

$$\iint_A E_1\theta_1 dA = \iint_A \left\{ \frac{1}{h_1}\frac{\partial}{\partial y^1}(M_{12}\theta_1) - \frac{1}{h_1} M_{12}\frac{\partial\theta_1}{\partial y^1} + \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial y^1}(M_{12}\theta_1) \right.$$
$$+ \frac{1}{h_2}\frac{\partial}{\partial y^2}(M_{22}\theta_1) - \frac{1}{h_2} M_{22}\frac{\partial\theta_1}{\partial y^2} + \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial y^2}(M_{22}\theta_1) + \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial y^2}(M_{11}\theta_1)$$
$$\left. - \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial y^1}(M_{21}\theta_1) + Q_2\theta_1 \right\} dA$$

$$\therefore \quad \iint_A E_1\theta_1 dA = \iint_A \left\{ \frac{1}{h_1h_2} \frac{\partial}{\partial y^1} (M_{12}\theta_1)h_2 + \frac{1}{h_1h_2} \frac{\partial h_2}{\partial y^1} (M_{12}\theta_1) \right.$$

$$\left. + \frac{1}{h_1h_2} \frac{\partial}{\partial y^2} (M_{22}\theta_1)h_1 + \frac{1}{h_1h_2} \frac{\partial h_1}{\partial y^2} (M_{22}\theta_1) \right\} dA \tag{4.4a}$$

$$- \iint_A \left\{ \frac{1}{h_1} M_{12} \frac{\partial \theta_1}{\partial y^1} + \frac{1}{h_2} M_{22} \frac{\partial \theta_1}{\partial y^2} - \frac{1}{h_1h_2} \frac{\partial h_1}{\partial y^2} (M_{11}\theta_1) \right.$$

$$\left. + \frac{1}{h_1h_2} \frac{\partial h_2}{\partial y^1} (M_{21}\theta_1) - Q_2\theta_1 \right\} dA \tag{4.4b}$$

The terms of (4.4a) may be simplified as follows

$$\iint_A \left\{ \frac{1}{h_1h_2} \frac{\partial}{\partial y^1} (M_{12}\theta_1h_2) + \frac{1}{h_1h_2} \frac{\partial}{\partial y^2} (M_{22}\theta_1h_1) \right\} dA$$

Using Gauss' divergence theorem to convert an area integral to a boundary integral, the above expression becomes

$$\int_L \left\{ (M_{12}\theta_1\nu_1) + (M_{22}\theta_1\nu_2) \right\} dL$$

$$= \int_L (M_{12}\nu_1 + M_{22}\nu_2)\theta_1 \; dL$$

Applying Eqn (3.7) to the above result, Eqn (4.4) reduces to

$$\iint_A E_1\theta_1 dA = \int_L M_1\theta_1 dL$$

$$- \iint_A \left\{ \frac{1}{h_1} M_{12} \frac{\partial \theta_1}{\partial y^1} + \frac{1}{h_2} M_{22} \frac{\partial \theta_1}{\partial y^2} + \frac{1}{h_1h_2} \frac{\partial h_2}{\partial y^1} (M_{21}\theta_1) \right.$$

$$\left. - \frac{1}{h_1h_2} \frac{\partial h_1}{\partial y^2} (M_{11}\theta_1) - Q_2\theta_1 \right\} dA \tag{4.5a}$$

Using Eqn (3.10b) the second term of Eqn (4.3) may be expanded in a similar manner. The result is stated in Eqn (4.5b) as follows

$$\iint_A E_2\theta_2 dA = \int_L M_2\theta_2 dL$$

$$- \iint_A \left\{ \frac{1}{h_2} M_{21} \frac{\partial\theta_2}{\partial y^2} + \frac{1}{h_1} M_{11} \frac{\partial\theta_2}{\partial y^1} + \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y^2} (M_{12}\theta_2) \right.$$

$$\left. - \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial y^1} (M_{22}\theta_2) + Q_1\theta_2 \right\} dA \qquad (4.5b)$$

Using Eqn (3.10c), the third term of Eqn (4.3) may be expressed as follows

$$\iint_A E_3 w_3 dA = \iint_A \left\{ \frac{1}{h_1} \frac{\partial Q_1}{\partial y^1} w_3 + \frac{1}{h_1 h_2} Q_1 \frac{\partial h_2}{\partial y^1} w_3 \right.$$

$$\left. + \frac{1}{h_2} \frac{\partial Q_2}{\partial y^2} w_3 + \frac{1}{h_1 h_2} Q_2 \frac{\partial h_1}{\partial y^2} w_3 + \rho w_3 \right\} dA$$

Expanding the above expression, we find that

$$\iint_A E_3 w_3 dA = \iint_A \left\{ \frac{1}{h_1} \frac{\partial}{\partial y^1} (Q_1 w_3) - \frac{1}{h_1} Q_1 \frac{\partial w_3}{\partial y^1} + \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial y^1} (Q_1 w_3) \right.$$

$$\left. + \frac{1}{h_2} \frac{\partial}{\partial y^2} (Q_2 w_3) - \frac{1}{h_2} Q_2 \frac{\partial w_3}{\partial y^2} + \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y^2} (Q_2 w_3) + \rho w_3 \right\} dA$$

$$= \iint_A \left\{ \frac{1}{h_1 h_2} \frac{\partial}{\partial y^1} (Q_1 w_3) h_2 + \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial y^1} (Q_1 w_3) + \frac{1}{h_1 h_2} \frac{\partial}{\partial y^2} (Q_2 w_3) h_1 \right.$$

$$\left. + \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y^2} (Q_2 w_3) - \frac{1}{h_1} Q_1 \frac{\partial w_3}{\partial y^1} - \frac{1}{h_2} Q_2 \frac{\partial w_3}{\partial y^2} + \rho w_3 \right\} dA$$

$$\text{i.e.} \quad \iint_A E_3 w_3 dA = \iint_A \left\{ \frac{1}{h_1 h_2} \frac{\partial}{\partial y^1} (Q_1 w_3 h_2) + \frac{1}{h_1 h_2} \frac{\partial}{\partial y^2} (Q_2 w_3 h_1) \right.$$

$$\left. - \frac{1}{h_1} Q_1 \frac{\partial w_3}{\partial y^1} - \frac{1}{h_2} Q_2 \frac{\partial w_3}{\partial y^2} + \rho w_3 \right\} dA$$

Applying Gauss' divergence theorem the above equation becomes

$$\iint_A E_3 w_3 dA = \int_L \left\{ Q_1 w_3 \nu_1 + Q_2 w_3 \nu_2 \right\} dL$$

$$- \iint_A \left\{ \frac{1}{h_1} Q_1 \frac{\partial w_3}{\partial y^1} + \frac{1}{h_2} Q_2 \frac{\partial w_3}{\partial y^2} - \rho w_3 \right\} dA$$

Using Eqn (3.7), the above result reduces to the following:

$$\iint_A E_3 w_3 dA = \int_L Q\, w_3 dL + \iint_A \rho w_3 dA - \iint_A \left\{ \frac{1}{h_1} Q_1 \frac{\partial w_3}{\partial y^1} \right.$$

$$\left. + \frac{1}{h_2} Q_2 \frac{\partial w_3}{\partial y^2} \right\} dA \tag{4.5c}$$

By combining the terms of Eqn (4.5), Eqn (4.3) may now be expressed as follows

$$\iint_A E_3 w_3 dA = \int_L \{M_1 \theta_1 + M_2 \theta_2 + Q w_3\} dL + \iint_A \{\rho w_3\} dA$$

$$- \iint_A \left\{ \frac{1}{h_1} \frac{\partial \theta_1}{\partial y^1} M_{12} + \frac{1}{h_2} \frac{\partial \theta_1}{\partial y^2} M_{22} + \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial y^1} M_{21} \theta_1 \right.$$

$$- \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y^2} M_{11}\theta_1 - Q_2\theta_1 + \frac{1}{h_2} \frac{\partial \theta_2}{\partial y^2} M_{21} + \frac{1}{h_1} \frac{\partial \theta_2}{\partial y^1} M_{11}$$

$$+ \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y^2} M_{12}\theta_2 - \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial y^1} M_{22}\theta_2 + Q_1\theta_2$$

$$+ \left. \frac{1}{h_1} \frac{\partial w_3}{\partial y^1} Q_1 + \frac{1}{h_2} \frac{\partial w_3}{\partial y^2} Q_2 \right\} dA = 0$$

Regrouping common terms the above expression may now be identified as a statement of the principle of virtual work:

$$\int_L \{ M_1\theta_1 + M_2\theta_2 + Q\, w_3 \} \, dL + \iint_A \{ \rho w_3 \} \, dA$$

$$- \iint_A \left\{ M_{11} \left(\frac{1}{h_1} \frac{\partial \theta_2}{\partial y^1} - \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y^2} \theta_1 \right) + M_{12} \left(\frac{1}{h_1} \frac{\partial \theta_1}{\partial y^1} + \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y^2} \theta_2 \right) \right.$$

$$+ M_{21} \left(\frac{1}{h_2} \frac{\partial \theta_2}{\partial y^2} + \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial y^1} \theta_1 \right) + M_{22} \left(\frac{1}{h_2} \frac{\partial \theta_1}{\partial y^2} - \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial y^1} \theta_2 \right)$$

$$+ \left. Q_1 \left(\frac{1}{h_1} \frac{\partial w_3}{\partial y^1} + \theta_2 \right) + Q_2 \left(\frac{1}{h_2} \frac{\partial w_3}{\partial y^2} - \theta_1 \right) \right\} dA = 0 \qquad (4.6)$$

From Eqns (4.6) the strain quantities may be identified as follows

(i) Shear Strains

$$\gamma_1 = \frac{1}{h_1} \frac{\partial w_3}{\partial y^1} + \theta_2 \qquad (4.7a)$$

$$\gamma_2 = \frac{1}{h_2} \frac{\partial w_3}{\partial y^2} - \theta_1 \qquad (4.7b)$$

(ii) Bending and Twisting Curvatures

$$\chi_1 = \frac{1}{h_1}\frac{\partial\theta_2}{\partial y^1} - \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial y^2}\theta_1 \tag{4.8a}$$

$$\chi_2 = \frac{1}{h_2}\frac{\partial\theta_1}{\partial y^2} - \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial y^1}\theta_2 \tag{4.8b}$$

$$\chi_{12} = \frac{1}{h_1}\frac{\partial\theta_1}{\partial y^1} + \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial y^2}\theta_2 \tag{4.8c}$$

$$\chi_{21} = \frac{1}{h_2}\frac{\partial\theta_2}{\partial y^2} + \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial y^1}\theta_1 \tag{4.8d}$$

If the shear strains as expressed in Eqn (4.7) are assumed to be negligible, then

$$\theta_2 = -\frac{1}{h_1}\frac{\partial w_3}{\partial y^1} \tag{4.9a}$$

and $$\theta_1 = \frac{1}{h_2}\frac{\partial w_3}{\partial y^2} \tag{4.9b}$$

By substituting the above Eqns of (4.9) into those of (4.8), the bending curvatures simplify to the following

$$\chi_1 = -\frac{1}{h_1^2}\frac{\partial^2 w_3}{\partial y^1 \partial y^1} + \frac{1}{h_1^3}\frac{\partial h_1}{\partial y^1}\frac{\partial w_3}{\partial y^1} - \frac{1}{h_1 h_2^2}\frac{\partial h_1}{\partial y^2}\frac{\partial w_3}{\partial y^2}$$

$$\chi_2 = \frac{1}{h_2^2}\frac{\partial^2 w_3}{\partial y^2 \partial y^2} - \frac{1}{h_2^3}\frac{\partial h_2}{\partial y^2}\frac{\partial w_3}{\partial y^2} + \frac{1}{h_1^2 h_2}\frac{\partial h_2}{\partial y^1}\frac{\partial w_3}{\partial y^1}$$

The twisting curvatures are obtained in a similar manner as follows

$$\chi_{12} = \frac{1}{h_1h_2}\frac{\partial^2 w_3}{\partial y^1 \partial y^2} - \frac{1}{h_1h_2^2}\frac{\partial h_2}{\partial y^1}\frac{\partial w_3}{\partial y^2} - \frac{1}{h_1^2h_2}\frac{\partial h_1}{\partial y^2}\frac{\partial w_3}{\partial y^1}$$

$$\chi_{21} = -\frac{1}{h_1h_2}\frac{\partial^2 w_3}{\partial y^2 \partial y^1} + \frac{1}{h_1^2h_2}\frac{\partial h_1}{\partial y^2}\frac{\partial w_3}{\partial y^1} + \frac{1}{h_1h_2^2}\frac{\partial h_2}{\partial y^1}\frac{\partial w_3}{\partial y^2}$$

and since $\chi_{12} = -\chi_{21}$,

$$\chi_{12} - \chi_{21} = \frac{2}{h_1h_2}\frac{\partial^2 w_3}{\partial y^1 \partial y^2} - \frac{2}{h_1h_2^2}\frac{\partial h_2}{\partial y^1}\frac{\partial w_3}{\partial y^2} - \frac{2}{h_1^2h_2}\frac{\partial h_1}{\partial y^2}\frac{\partial w_3}{\partial y^1} \qquad (4.10)$$

The above Eqns of (4.10) are the curvatures for the plate where shear strains have been neglected, expressed in orthogonal curvilinear coordinates.

An alternative method of deriving the strain-displacement relations for the plate will be given in 4.1.2.

4.1.2 Using the Concept of a Rigid Body Movement

Let the surface of the plate undergo a rigid body movement effected by translating and rotating the plane at point $P_1(y_1^1, y_1^2)$. Let these rigid body displacements at point P_1 take place in a positive sense as shown in Figure 4.1 i.e.

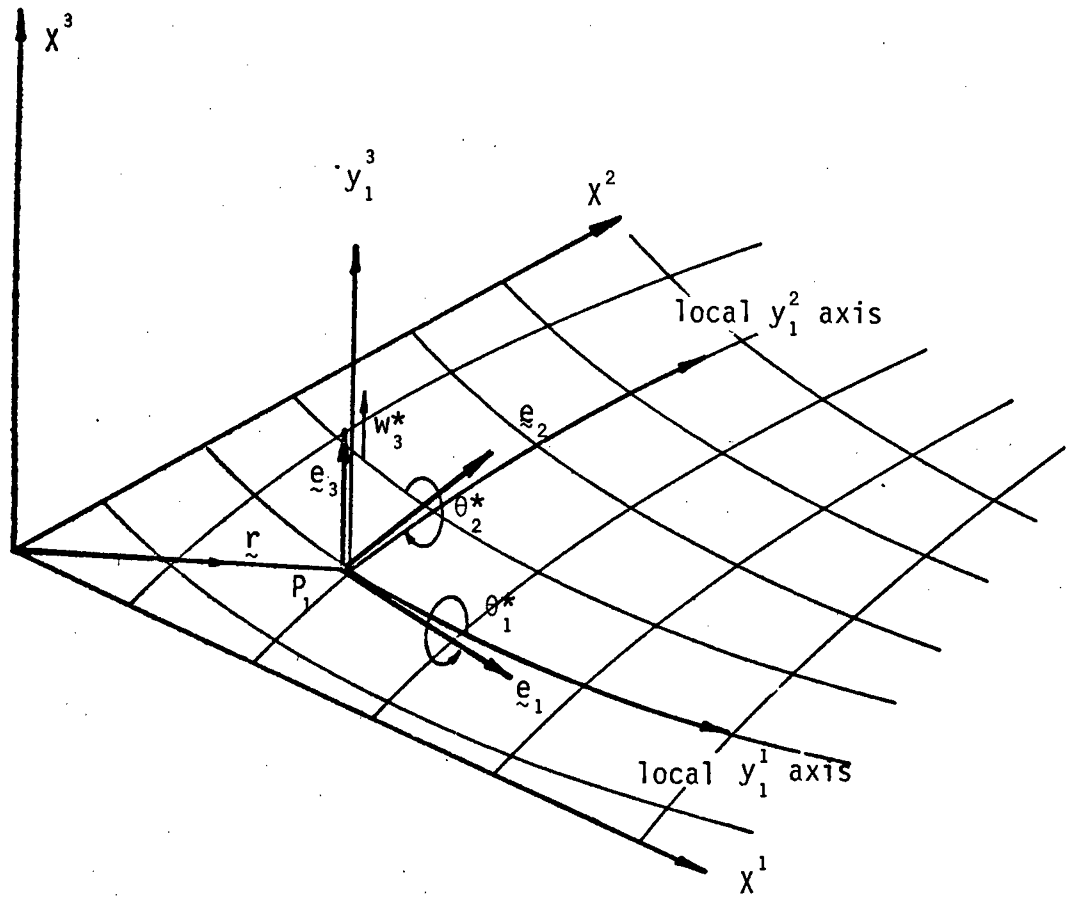
$$\underset{\sim}{d}^* = \theta_1^*\underset{\sim}{e}_1 + \theta_2^*\underset{\sim}{e}_2 + w_3^*\underset{\sim}{e}_3$$

$$= \underset{\sim}{\theta}_1^* + \underset{\sim}{\theta}_2^* + \underset{\sim}{w}_3^*$$

The resulting surface displacements and rotations may be described as follows

$$\underset{\sim}{w}_3 = \underset{\sim}{w}_3^* + \underset{\sim}{\theta}^* x \underset{\sim}{r} = \underset{\sim}{w}_3^* - \underset{\sim}{r} x \underset{\sim}{\theta}(y^i) \qquad (4.11a)$$

and $$\underset{\sim}{\theta}(y^i) = \underset{\sim}{\theta}_1^* + \underset{\sim}{\theta}_2^* = \underset{\sim}{\theta}^* \qquad (4.11b)$$



<u>FIGURE 4.1</u> Plate surface at point P_1 showing applied displacements $\underset{\sim}{d}^*$

For these rigid body movements the strain quantities will be zero. Differentiating Eqns (4.11) with respect to the local curvilinear axes y_1^1 and y_1^2, we obtain,

$$\frac{\partial \underset{\sim}{w}_3}{\partial y_1^1} + \frac{\partial \underset{\sim}{r}}{\partial y_1^1} \times \underset{\sim}{\theta}(y^i) = 0$$

$$\frac{\partial \underset{\sim}{w}_3}{\partial y_1^2} + \frac{\partial \underset{\sim}{r}}{\partial y_1^2} \times \underset{\sim}{\theta}(y^i) = 0 \qquad (4.12)$$

and $$\frac{\partial \underset{\sim}{\theta}(y^i)}{\partial y_1^1} = 0 \; ; \qquad \frac{\partial \underset{\sim}{\theta}(y^i)}{\partial y_1^2} = 0 \qquad (4.13)$$

Using Eqns (2.11) and (2.14), the Eqns of (4.12) and (4.13) may be rewritten as

$$\frac{1}{h_1}\frac{\partial \underset{\sim}{w}_3}{\partial y^1} + \underset{\sim}{e}_1 \times \underset{\sim}{\theta}_1 = 0$$

$$\frac{1}{h_2}\frac{\partial \underset{\sim}{w}_3}{\partial y^2} + \underset{\sim}{e}_2 \times \underset{\sim}{\theta}_1 = 0 \qquad (4.14)$$

and $$\frac{1}{h_1}\frac{\partial \underset{\sim}{\theta}(y^i)}{\partial y^1} = 0 \; ; \qquad \frac{1}{h_2}\frac{\partial \underset{\sim}{\theta}(y^i)}{\partial y^2} = 0 \qquad (4.15)$$

From Eqns (4.14) the shear strain quantities may be identified as follows

$$\gamma_1 = \left[\frac{1}{h_1}\frac{\partial \underset{\sim}{w}_3}{\partial y^1} + \underset{\sim}{e}_1 \times \underset{\sim}{\theta} \right] \cdot \underset{\sim}{e}_3$$

$$= \left[\frac{1}{h_1} \frac{\partial \underset{\sim}{w}_3}{\partial y^1} + \underset{\sim}{e}_1 \times \{\theta_1 \underset{\sim}{e}_1 + \theta_2 \underset{\sim}{e}_2\} \right] \cdot \underset{\sim}{e}_3$$

$$= \left[\frac{1}{h_1} \frac{\partial w_3}{\partial y^1} \underset{\sim}{e}_1 + \frac{1}{h_1} w_3 \frac{\partial \underset{\sim}{e}_3}{\partial y^1} + \underset{\sim}{e}_1 \times \theta_1 \underset{\sim}{e}_1 + \underset{\sim}{e}_1 \times \theta_2 \underset{\sim}{e}_2 \right] \cdot \underset{\sim}{e}_3$$

$$= \frac{1}{h_1} \frac{\partial w_3}{\partial y^1} + \theta_2 \tag{4.16a}$$

$$\gamma_2 = \left[\frac{1}{h_2} \frac{\partial \underset{\sim}{w}_3}{\partial y^2} + \underset{\sim}{e}_2 \times \underset{\sim}{\theta} \right] \cdot \underset{\sim}{e}_3$$

$$= \left[\frac{1}{h_2} \frac{\partial \underset{\sim}{w}_3}{\partial y^2} + \underset{\sim}{e}_2 \times \{\theta_1 \underset{\sim}{e}_1 + \theta_2 \underset{\sim}{e}_2\} \right] \cdot \underset{\sim}{e}_3$$

$$= \left[\frac{1}{h_2} \frac{\partial w_3}{\partial y^2} \underset{\sim}{e}_3 + \frac{1}{h_2} w_3 \frac{\partial \underset{\sim}{e}_3}{\partial y^2} + \underset{\sim}{e}_2 \times \theta_1 \underset{\sim}{e}_1 + \underset{\sim}{e}_2 \times \theta_2 \underset{\sim}{e}_2 \right] \cdot \underset{\sim}{e}_3$$

$$= \frac{1}{h_2} \frac{\partial w_3}{\partial y^2} - \theta_1 \tag{4.16b}$$

From Eqns (4.15) the bending strains may be identified as follows

$$\chi_1 = \left[\frac{1}{h_1} \frac{\partial \underset{\sim}{\theta}}{\partial y^1} \right] \cdot \underset{\sim}{e}_2$$

$$= \left[\frac{1}{h_1} \frac{\partial}{\partial y^1} \{\theta_1 \underset{\sim}{e}_1 + \theta_2 \underset{\sim}{e}_2\} \right] \cdot \underset{\sim}{e}_2$$

$$= \left[\frac{1}{h_1} \left\{ \frac{\partial \theta_1}{\partial y^1} \underset{\sim}{e}_1 + \theta_1 \frac{\partial \underset{\sim}{e}_1}{\partial y^1} + \frac{\partial \theta_2}{\partial y^1} \underset{\sim}{e}_2 + \theta_2 \frac{\partial \underset{\sim}{e}_2}{\partial y^1} \right\} \right] \cdot \underset{\sim}{e}_2$$

$$= \left[\frac{1}{h_1} \left\{ \frac{\partial \theta_1}{\partial y^1} \underset{\sim}{e}_1 + \theta_1 \left(- \frac{\partial h_1}{\partial y^2} \frac{\underset{\sim}{e}_2}{h_2} - \frac{\partial h_2}{\partial y^3} \frac{\underset{\sim}{e}_3}{h_3} \right) + \frac{\partial \theta_2}{\partial y^1} \underset{\sim}{e}_2 \right.\right.$$

$$\left.\left. + \theta_2 \frac{\partial h_1}{\partial y^2} \frac{\underset{\sim}{e}_1}{h_2} \right\} \right] \cdot \underset{\sim}{e}_2$$

$$= \frac{1}{h_1} \frac{\partial \theta_2}{\partial y^1} - \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y^2} \theta_1 \qquad (4.17a)$$

$$\chi_2 = \left[\frac{1}{h} \frac{\partial \underset{\sim}{\theta}}{\partial y^2} \right] \cdot \underset{\sim}{e}_1$$

$$= \left[\frac{1}{h_2} \frac{\partial}{\partial y^2} \{ \theta_1 \underset{\sim}{e}_1 + \theta_2 \underset{\sim}{e}_2 \} \right] \cdot \underset{\sim}{e}_1$$

$$= \left[\frac{1}{h_2} \left\{ \frac{\partial \theta_1}{\partial y^2} \underset{\sim}{e}_1 + \theta_1 \frac{\partial \underset{\sim}{e}_1}{\partial y^2} + \frac{\partial \theta_2}{\partial y^2} \underset{\sim}{e}_2 + \theta_2 \frac{\partial \underset{\sim}{e}_2}{\partial y^2} \right\} \right] \cdot \underset{\sim}{e}_1$$

$$= \left[\frac{1}{h_2} \left\{ \frac{\partial \theta_1}{\partial y^2} \underset{\sim}{e}_1 + \theta_1 \frac{\partial h_2}{\partial y^1} \frac{\underset{\sim}{e}_2}{h_1} + \frac{\partial \theta_2}{\partial y^2} \underset{\sim}{e}_2 + \theta_2 \left(\frac{\partial h_2}{\partial y^3} \frac{\underset{\sim}{e}_3}{h_3} - \frac{\partial h_2}{\partial y^1} \frac{\underset{\sim}{e}_1}{h_1} \right) \right\} \right] \cdot \underset{\sim}{e}_1$$

$$= \frac{1}{h_2} \frac{\partial \theta_1}{\partial y^2} - \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial y^1} \theta_2 \qquad (4.17b)$$

From Eqns (4.15) the twisting shear strains may be obtained as follows:

$$\chi_{12} = \left[\frac{1}{h_1} \frac{\partial \underset{\sim}{\theta}}{\partial y^1} \right] \cdot \underset{\sim}{e}_1$$

$$= \left[\frac{1}{h_1} \frac{\partial}{\partial y^1} \{ \theta_1 \underset{\sim}{e}_1 + \theta_2 \underset{\sim}{e}_2 \} \right] \cdot \underset{\sim}{e}_1$$

$$= \left[\frac{1}{h_1} \left\{ \frac{\partial \theta_1}{\partial y^1} \underset{\sim}{e}_1 + \theta_1 \frac{\partial \underset{\sim}{e}_1}{\partial y^1} + \frac{\partial \theta_2}{\partial y^1} \underset{\sim}{e}_2 + \theta_2 \frac{\partial \underset{\sim}{e}_2}{\partial y^1} \right\} \right] \cdot \underset{\sim}{e}_1$$

$$= \left[\frac{1}{h_1} \left\{ \frac{\partial \theta_1}{\partial y^1} \underset{\sim}{e}_1 - \theta_1 \frac{\partial h_1}{\partial y^2} \frac{\underset{\sim}{e}_2}{h_2} - \theta_1 \frac{\partial h_2}{\partial y^3} \frac{\underset{\sim}{e}_3}{h_3} + \frac{\partial \theta_2}{\partial y^1} \underset{\sim}{e}_2 + \theta_2 \frac{\partial h_1}{\partial y^2} \frac{\underset{\sim}{e}_1}{h_2} \right\} \right] \cdot \underset{\sim}{e}_1$$

$$= \frac{1}{h_1} \frac{\partial \theta_1}{\partial y^1} + \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y^2} \theta_2 \tag{4.17c}$$

$$\chi_{21} = \left[\frac{1}{h_2} \frac{\partial \underset{\sim}{\theta}}{\partial y^2} \right] \cdot \underset{\sim}{e}_2$$

$$= \left[\frac{1}{h_2} \frac{\partial}{\partial y^2} \{ \theta_1 \underset{\sim}{e}_1 + \theta_2 \underset{\sim}{e}_2 \} \right] \cdot \underset{\sim}{e}_2$$

$$= \left[\frac{1}{h_2} \left\{ \frac{\partial \theta_1}{\partial y^2} \underset{\sim}{e}_1 + \theta_1 \frac{\partial \underset{\sim}{e}_1}{\partial y^2} + \frac{\partial \theta_2}{\partial y^2} \underset{\sim}{e}_2 + \theta_2 \frac{\partial \underset{\sim}{e}_2}{\partial y^2} \right\} \right] \cdot \underset{\sim}{e}_2$$

$$= \left[\frac{1}{h_2} \left\{ \frac{\partial \theta_1}{\partial y^2} \underset{\sim}{e}_1 + \theta_1 \frac{\partial h_2}{\partial y^1} \frac{\underset{\sim}{e}_2}{h_1} + \frac{\partial \theta_2}{\partial y^2} \underset{\sim}{e}_2 - \theta_2 \frac{\partial h_2}{\partial y^3} \frac{\underset{\sim}{e}_3}{h_3} - \theta_2 \frac{\partial h_2}{\partial y^1} \frac{\underset{\sim}{e}_1}{h_1} \right\} \right] \cdot \underset{\sim}{e}_2$$

$$= \frac{1}{h_2} \frac{\partial \theta_2}{\partial y^2} + \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial y^1} \theta_1 \tag{4.17d}$$

The Eqns of (4.16) are the shear strains and those of (4.17) the bending and twisting curvatures, as previously derived in 4.1.1 Eqns (4.7) and (4.8) respectively.

4.2 Strain-Displacement Relationships in polar coordinates

Using the Eqns of (3.14) and (3.15) and defining the rotations such that $\theta_r = \theta_2$, $\theta_\theta = \theta_1$, $w = w_3$, the strains as given in Eqns (4.16) and (4.17) simplify to those in plane polar coordinates as follows

i) Shear strains

$$\gamma_r = \frac{\partial w}{\partial r} + \theta_r , \qquad \gamma_\theta = \frac{1}{r} \frac{\partial w}{\partial \theta} - \theta_\theta \tag{4.18}$$

ii) Bending and twisting curvatures

$$\chi_r = \frac{\partial \theta_r}{\partial r}, \qquad \gamma_\theta = \frac{1}{r}\frac{\partial \theta_\theta}{\partial \theta} - \frac{\theta_r}{r}$$

$$\chi_{r\theta} = \frac{\partial \theta_\theta}{\partial r}, \qquad \chi_{\theta r} = \frac{1}{r}\frac{\partial \theta_r}{\partial \theta} + \frac{\theta_\theta}{r} \tag{4.19}$$

If the shear strains as expressed in eqn (4.16) are neglected, then

$$\theta_r = -\frac{\partial w}{\partial r}, \qquad \theta_\theta = \frac{1}{r}\frac{\partial w}{\partial \theta}$$

and the curvatures of eqn (4.17) simplify to the following form:

$$\chi_r = -\frac{\partial^2 w}{\partial r^2} \tag{4.20a}$$

$$\chi_\theta = \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} \tag{4.20b}$$

$$\chi_{r\theta} = \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}$$

$$\chi_{\theta r} = -\frac{1}{r}\frac{\partial^2 w}{\partial \theta \partial r} + \frac{1}{r^2}\frac{\partial w}{\partial \theta}$$

and since $\chi_{r\theta} = -\chi_{\theta r}$

$$\chi_{r\theta} - \chi_{\theta r} = \frac{2}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{2}{r^2}\frac{\partial w}{\partial \theta} \tag{4.20c}$$

The Eqns of (4.20) are the curvatures for the plate in plane polar coordinates where the shear strains have been neglected.

4.3 Relationship Between Moment and Curvature in Orthogonal Curvilinear Coordinates

Consider an element of an orthotrophic plate as shown in Figure 3.1

From Hooke's law and considering the effect of Poisson's ratio in the lateral direction, the total bending and shearing strains may be expressed as follows:

$$\varepsilon_1 = \frac{\sigma_1}{E_1} - \mu_2 \frac{\sigma_2}{E_2}$$

$$\varepsilon_2 = \frac{\sigma_2}{E_2} - \mu_1 \frac{\sigma_1}{E_1}$$

$$\gamma_{12} = \{\varepsilon_{12} + \varepsilon_{21}\} = \frac{\tau_{12}}{G} = \frac{\tau_{21}}{G} \tag{4.21}$$

where E_1, E_2, μ_1, μ_2 are the orthotrophic material constants [34] and the shear modulus G for an orthotrophic plate is given as [30]

$$G \doteq \frac{\sqrt{E_1 E_2}}{2(1 + \sqrt{\mu_1 \mu_2})}$$

Rewriting the eqns of (4.21) in terms of stresses we obtain the following

$$\sigma_{11} = \frac{E_1}{(1 - \mu_1 \mu_2)} \{\varepsilon_1 + \mu_2 \varepsilon_2\}$$

$$\sigma_{22} = \frac{E_2}{(1 - \mu_1 \mu_2)} \{\varepsilon_2 + \mu_1 \varepsilon_1\}$$

$$\tau_{12} = G\gamma_{12} = G\{\varepsilon_{12} + \varepsilon_{21}\} = \tau_{21} \tag{4.22}$$

Let the in-plane displacement vector $\underset{\sim}{v}$ be defined as

$$\underset{\sim}{v} = v_1\underset{\sim}{e}_1 + v_2\underset{\sim}{e}_2 \tag{4.23}$$

where v_1 and v_2 are the components of $\underset{\sim}{v}$ in the y^1 and y^2 directions respectively. At any point P within the plate the in-plane displacements may be expressed in terms of the distance z from the middle plane and the rotations θ_1 and θ_2, as follows:

$$\underset{\sim}{v} = \underset{\sim}{v}_0 + z(\underset{\sim}{e}_3 \times \underset{\sim}{\theta}) \tag{4.24}$$

where $\underset{\sim}{\theta} = \theta_1\underset{\sim}{e}_1 + \theta_2\underset{\sim}{e}_2$

and $\underset{\sim}{v}_0$ is the vector of in-plane displacements at the middle plane of the plate (assumed equal to zero in the case of loading normal to the surface of the plate),

From eqn (4.23) the bending strain and the twisting shear strain quantities may be identified as follows

$$\varepsilon_1 = \left\{ \frac{\partial \underset{\sim}{v}}{\partial s_1} \right\} . \underset{\sim}{e}_1$$

$$= \left\{ \frac{\partial v_1}{\partial s_1} \underset{\sim}{e}_1 + v_1 \frac{\partial \underset{\sim}{e}_1}{\partial s_1} + \frac{\partial v_2}{\partial s_1} \underset{\sim}{e}_2 + v_2 \frac{\partial \underset{\sim}{e}_2}{\partial s_1} \right\} . \underset{\sim}{e}_1$$

$$= \frac{\partial v_1}{\partial s_1} \tag{4.25a}$$

By definition $\partial v_1/\partial s_1$ is the bending strain in the y^1 direction where, from eqn (2.14), $\partial s_1 = h_1 \partial y^1$ is the physical length along a y^1 axis. Similarly

$$\varepsilon_2 = \frac{\partial \underset{\sim}{v}}{\partial s_2} . \underset{\sim}{e}_2 = \frac{\partial v_2}{\partial s_2} \tag{4.25b}$$

and

$$\gamma_{12} = \left\{\frac{\partial \underset{\sim}{v}}{\partial s_1}\right\} \cdot \underset{\sim}{e}_2 + \left\{\frac{\partial \underset{\sim}{v}}{\partial s_2}\right\} \cdot \underset{\sim}{e}_1 = \frac{\partial v_2}{\partial s_1} + \frac{\partial v_1}{\partial s_2} \tag{4.25c}$$

The relationship between the bending strain of eqn (4.25) and the corresponding curvature may be obtained by differentiating eqn (4.24) as follows

$$\varepsilon_1 = \left\{\frac{\partial \underset{\sim}{v}}{\partial s_1}\right\} \cdot \underset{\sim}{e}_1$$

$$= z \frac{\partial}{\partial s_1} \left\{ \underset{\sim}{e}_3 \times \underset{\sim}{\theta} \right\} \cdot \underset{\sim}{e}_1$$

$$= z \frac{1}{h_1} \left\{ \frac{\partial \underset{\sim}{e}_3}{\partial y^1} \times \underset{\sim}{\theta} + \underset{\sim}{e}_3 \times \frac{\partial \underset{\sim}{\theta}}{\partial y^1} \right\} \cdot \underset{\sim}{e}_1$$

$$= z \frac{1}{h_1} \underset{\sim}{e}_3 \times \left\{ \frac{\partial \theta_1}{\partial y^1} \underset{\sim}{e}_1 + \theta_1 \frac{\partial \underset{\sim}{e}_1}{\partial y^1} + \frac{\partial \theta_2}{\partial y^1} \underset{\sim}{e}_2 + \theta_2 \frac{\partial \underset{\sim}{e}_2}{\partial y^1} \right\} \cdot \underset{\sim}{e}_1$$

$$= z \frac{1}{h_1} \underset{\sim}{e}_3 \times \left\{ \frac{\partial \theta_1}{\partial y^1} \underset{\sim}{e}_1 - \theta_1 \frac{\partial h_1}{\partial y^2} \frac{\underset{\sim}{e}_2}{h_2} - \theta_1 \frac{\partial h_1}{\partial y^3} \frac{\underset{\sim}{e}_3}{h_3} + \frac{\partial \theta_2}{\partial y^1} \underset{\sim}{e}_2 \right.$$

$$\left. + \theta_2 \frac{\partial h_1}{\partial y^2} \frac{\underset{\sim}{e}_1}{h_2} \right\} \cdot \underset{\sim}{e}_1$$

$$= z \frac{1}{h_1} \left\{ \frac{\partial \theta_1}{\partial y^1} \underset{\sim}{e}_2 + \theta_1 \frac{\partial h_1}{\partial y^2} \frac{\underset{\sim}{e}_1}{h_2} - \frac{\partial \theta_2}{\partial y^1} \underset{\sim}{e}_1 + \theta_2 \frac{\partial h_1}{\partial y^2} \frac{\underset{\sim}{e}_2}{h_2} \right\} \cdot \underset{\sim}{e}_1$$

$$= - z \left\{ \frac{1}{h_1} \frac{\partial \theta_2}{\partial y^1} - \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y^2} \theta_1 \right\}$$

By referring to eqn (4.8) or (4.17a), the curvature in the y direction is identified and the above expression can be rewritten as

$$\varepsilon_1 = - z \chi_1 \tag{4.26a}$$

Similarly the bending strain of eqn (4.25b) may be related to the curvature as follows

$$
\begin{aligned}
\varepsilon_2 &= \left\{ \frac{\partial \underset{\sim}{v}}{\partial s_2} \right\} \cdot \underset{\sim}{e}_2 \\
&= z \frac{\partial}{\partial s_2} \left\{ \underset{\sim}{e}_3 \times \underset{\sim}{\theta} \right\} \cdot \underset{\sim}{e}_2 \\
&= z \frac{1}{h_2} \left\{ \frac{\partial \underset{\sim}{e}_3}{\partial y^2} \times \underset{\sim}{\theta} + \underset{\sim}{e}_3 \times \frac{\partial \underset{\sim}{\theta}}{\partial y^2} \right\} \cdot \underset{\sim}{e}_2 \\
&= z \frac{1}{h_2} \underset{\sim}{e}_3 \times \left\{ \frac{\partial \theta_1}{\partial y^2} \underset{\sim}{e}_1 + \theta_1 \frac{\partial \underset{\sim}{e}_1}{\partial y^2} + \frac{\partial \theta_2}{\partial y^2} \underset{\sim}{e}_2 + \theta_2 \frac{\partial \underset{\sim}{e}_2}{\partial y^2} \right\} \cdot \underset{\sim}{e}_2 \\
&= z \frac{1}{h_2} \underset{\sim}{e}_3 \times \left\{ \frac{\partial \theta_1}{\partial y^2} \underset{\sim}{e}_1 + \theta_1 \frac{\partial h_2}{\partial y^1} \frac{\underset{\sim}{e}_2}{h_1} + \frac{\partial \theta_2}{\partial y^2} \underset{\sim}{e}_2 - \theta_2 \frac{\partial h_2}{\partial y^3} \frac{\underset{\sim}{e}_3}{h_3} \right. \\
&\qquad \left. - \theta_2 \frac{\partial h_2}{\partial y^1} \frac{\underset{\sim}{e}_1}{h_1} \right\} \cdot \underset{\sim}{e}_2 \\
&= z \frac{1}{h_2} \left\{ \frac{\partial \theta_1}{\partial y^2} \underset{\sim}{e}_2 - \theta_1 \frac{\partial h_2}{\partial y^1} \frac{\underset{\sim}{e}_1}{h_1} - \frac{\partial \theta_2}{\partial y^2} \underset{\sim}{e}_1 - \theta_2 \frac{\partial h_2}{\partial y^1} \frac{\underset{\sim}{e}_2}{h_1} \right\} \cdot \underset{\sim}{e}_2 \\
&= z \left\{ \frac{1}{h_2} \frac{\partial \theta_1}{\partial y^2} - \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial y^1} \theta_2 \right\}
\end{aligned}
$$

By referring to eqn (4.8b) or (4.17b), the above expression becomes

$$\varepsilon_2 = z\chi_2 \tag{4.26b}$$

The relationship between the twisting shear strain of eqn (4.25c) and the corresponding curvatures is obtained by differentiating eqn (4.24) as follows

$$\gamma_{12} = \left\{\frac{\partial \underset{\sim}{v}}{\partial s_1}\right\} \cdot \underset{\sim}{e}_2 + \left\{\frac{\partial \underset{\sim}{v}}{\partial s_2}\right\} \cdot \underset{\sim}{e}_1$$

$$= z\frac{1}{h_1}\left\{\frac{\partial \theta_1}{\partial y^1}\underset{\sim}{e}_1 + \theta_1 \frac{\partial h_1}{\partial y^2}\frac{\underset{\sim}{e}_1}{h_2} - \frac{\partial \theta_2}{\partial y^1}\underset{\sim}{e}_1 + \theta_2 \frac{\partial h_1}{\partial y^2}\frac{\underset{\sim}{e}_2}{h_2}\right\} \cdot \underset{\sim}{e}_2$$

$$+ \ z\frac{1}{h_2}\left\{\frac{\partial \theta_1}{\partial y^2}\underset{\sim}{e}_2 - \theta_1 \frac{\partial h_2}{\partial y^1}\frac{\underset{\sim}{e}_1}{h_1} - \frac{\partial \theta_2}{\partial y^2}\underset{\sim}{e}_1 - \theta_2 \frac{\partial h_2}{\partial y^1}\frac{\underset{\sim}{e}_2}{h_1}\right\} \cdot \underset{\sim}{e}_1$$

$$= z\left[\left\{\frac{1}{h_1}\frac{\partial \theta_1}{\partial y^1} + \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial y^2}\theta_2\right\} - \left\{\frac{1}{h_2}\frac{\partial \theta_2}{\partial y^2} + \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial y^1}\theta_1\right\}\right]$$

Be referring to eqns (4.8c) and (4.8d) or eqns (4.17c) and (4.17d), the above expression becomes

$$\gamma_{12} = z\,\{\chi_{12} - \chi_{21}\} \tag{4.26c}$$

The relationship between the stresses and curvatures may therefore be obtained by substituting eqns (4.26) into those of (4.22) i.e.

$$\sigma_{11} = \frac{E_1 z}{(1 - \mu_1\mu_2)}\{\chi_1 - \mu_2\mu_2\} \tag{4.27a}$$

$$\sigma_{22} = \frac{-E_2 z}{(1 - \mu_1\mu_2)}\{\chi_2 - \mu_1\mu_1\} \tag{4.27b}$$

$$\tau_{12} = G\,z\{\chi_{12} - \chi_{21}\} = \tau_{21} \tag{4.27c}$$

The stress resultants are in turn related to the stresses as follows

$$M_{11}ds_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{11}ds_2 z\,ds_3$$

$$M_{22}ds_1 = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{22}ds_1 z\,ds_3$$

$$M_{12}ds_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{12}ds_2 z\,ds_3$$

$$M_{21}ds_1 = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{21}ds_1 z\,ds_3 \tag{4.28}$$

Substituting the eqns of (4.27) into those of (4.28) and integrating over the limits yield the relationships between the moments per unit length and the curvatures

$$M_{11} = \frac{E_1h^3}{12(1 - \mu_1\mu_2)}\{\chi_1 - \mu_2\chi_2\}$$

$$M_{22} = \frac{E_2h^3}{12(1 - \mu_1\mu_2)}\{\chi_2 - \mu_1\chi_1\}$$

$$M_{12} = \frac{Gh^3}{12}\{\chi_{12} - \chi_{21}\}$$

$$M_{21} = -\frac{Gh^3}{12}\{\chi_{12} - \chi_{21}\} \tag{4.29}$$

or more simply

$$M_{11} = D_{11}\{\chi_1 - \mu_2\chi_2\}$$

$$M_{22} = D_{22}\{\chi_2 - \mu_1\chi_1\}$$

$$M_{12} = D_{12}\{\chi_{12} - \chi_{21}\}$$

$$M_{21} = -D_{12}\{\chi_{12} - \chi_{21}\} \qquad (4.30)$$

The first three eqns of (4.30) may be represented in matrix form as follows.

$$\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \begin{bmatrix} D_{11} & -D_1 & 0 \\ -D_1 & D_{22} & 0 \\ 0 & 0 & D_{12} \end{bmatrix} \cdot \begin{Bmatrix} \chi_1 \\ \chi_2 \\ (\chi_{12} - \chi_{21}) \end{Bmatrix}$$

(4.31)

or

$$\{M\} = [D]\{\chi\} \qquad (4.32)$$

χ_1, χ_2, χ_{12}, χ_{21} are the curvatures as given in eqn (4.8) and D_{11}, D_{22} are the bending rigidities in the mutually orthogonal coordinate directions and D_{12} is the torsional rigidity

where

$$D_{11} = \frac{E_1h^3}{12(1 - \mu_1\mu_2)}, \qquad D_{12} = \frac{Gh^3}{12}$$

$$D_{22} = \frac{E_2h^3}{12(1 - \mu_1\mu_2)}$$

h = plate thickness

and from Betti's reciprocal theorem

$$D_1 = \mu_2 D_{11} = \mu_1 D_{22} \tag{4.33}$$

Eqn (4.32) is the relationship between the moments and the curvatures for the plate in orthogonal curvilinear coordinates.

4.4 Moment Curvature Relationship in Polar Coordinates

Using the expression for the curvatures as given in eqn (4.20), the Moment Curvature relationship in Polar Coordinates may be obtained by substituting these values into eqn (4.31) as follows

$$\begin{Bmatrix} M_r \\ M_\theta \\ M_{r\theta} \end{Bmatrix} = \begin{bmatrix} D_r & -D_1 & 0 \\ -D_1 & D_\theta & 0 \\ 0 & 0 & D_{r\theta} \end{bmatrix} \cdot \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial r^2} \\ \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r} \\ \dfrac{2}{r}\dfrac{\partial^2 w}{\partial r \partial \theta} - \dfrac{2}{r^2}\dfrac{\partial w}{\partial \theta} \end{Bmatrix} \tag{4.34}$$

The above eqn (4.34) is the relationship between the moments and the curvature in plane polar coordinates

where

$$D_r = \frac{E_r h^3}{12(1 - \mu_r \mu_\theta)} , \qquad D_{12} = \frac{Gh^3}{12}$$

$$D_\theta = \frac{E_\theta h^3}{12(1 - \mu_r \mu_\theta)} , \qquad D_1 = \mu_\theta D_r = \mu_r D_\theta$$

h = plate thickness

CHAPTER 5

GOVERNING DIFFERENTIAL EQUATION FOR A CURVED FINITE STRIP

5.1 Derivation of Displacement Function

Let the set of curvilinear axes in the middle plane of the plate be chosen such that the y^1 axes describe the coordinate limits across the width of the strip and the y^2 axes define those coordinate limits along the length of the strip. Figure 5.1 shows a curved finite strip bounded by two coordinate axes y^1_i and y^1_j, referred to as the nodal lines i and j for each strip. For the present formulation based on orthogonal curvilinear coordinates it will be sufficient to define the vertical deflection w between the sides of the strip in terms of the displacement and rotation at each nodal line as shown in Figure 5.2. The generalised displacements $\underset{\sim}{u}$ along the nodal lines of the strip will be defined as follows

$$\underset{\sim}{u} = [w_i(y^2)\ \theta_i(y^2)\ w_j(y^2)\ \theta_j(y^2)]^T \tag{5.1}$$

The position of a point within the strip width may be conveniently defined in terms of a local axis a, such that

$$a = y^1 - y^1_i \qquad (y^1_j \geq y^1 \geq y^1_i) \tag{5.2}$$

From eqn 5.2 it therefore follows that

(i) At $y^1 = y^1_i$, $\quad a = y^1_i - y^1_i = 0$

(ii) At $y^1 = y^1_j$, $\quad a = y^1_j - y^1_i = b$

where b is the width of the strip

For the nodal displacements $\underset{\sim}{u}$ chosen a third order polynomial, defined in terms of 4 unknown constants will therefore be used

i.e.

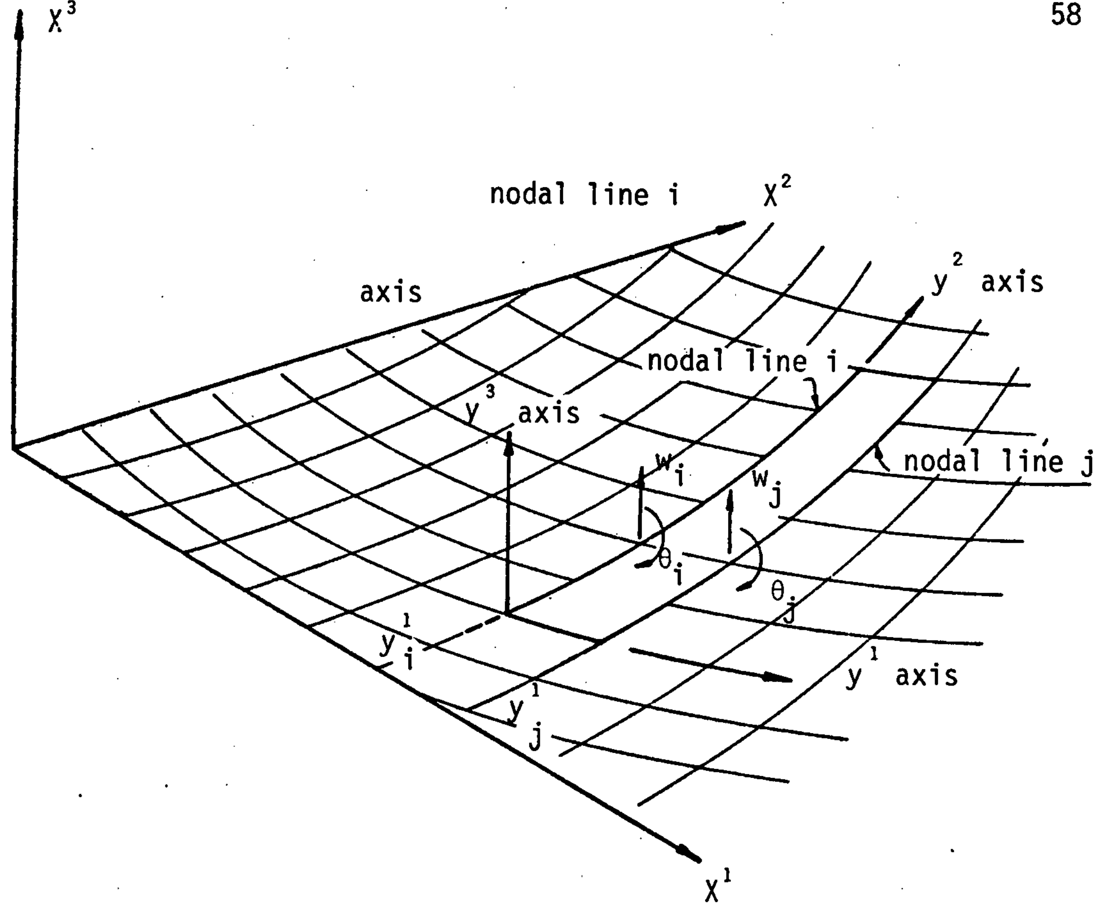


FIGURE 5.1 Single finite strip bounded by nodal lines i and j

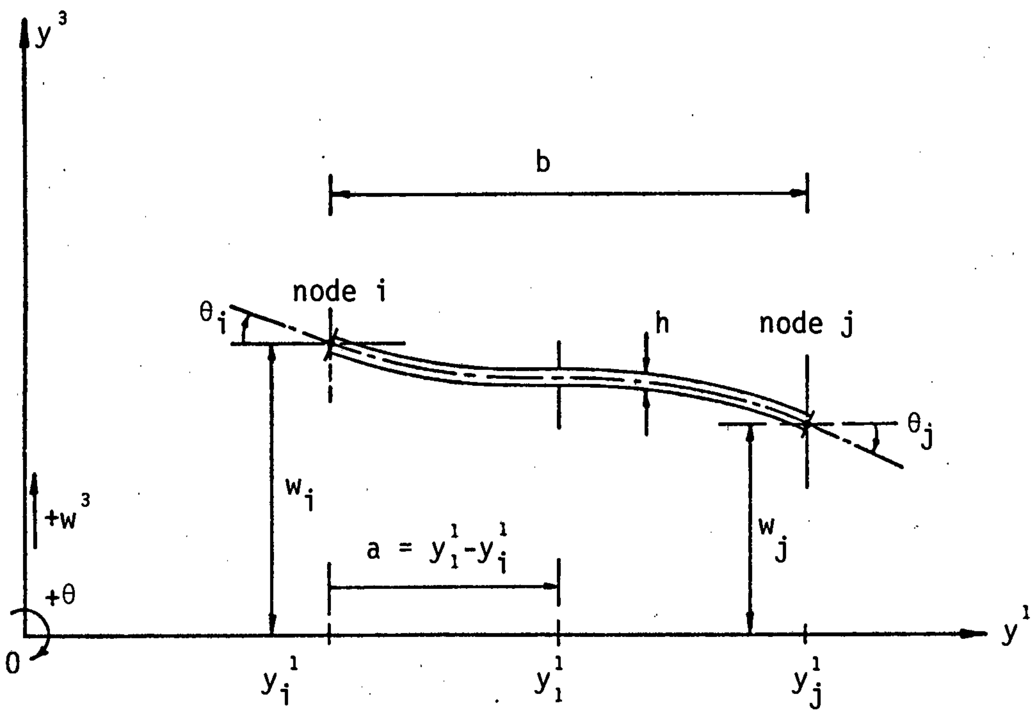


FIGURE 5.2 Transverse flexure of strip showing positive displacements

$$w(a,y^2) \;=\; A_1 + A_2 a + A_3 a^2 + A_4 a^3 \tag{5.3}$$

$$= [1\ a\ a^2\ a^3][A_1\ A_2\ A_3\ A_4]^T$$

$$= [1\ a\ a^2\ a^3]\{A\} \tag{5.4}$$

Substitute eqn (5.2) into (5.3) above and differentiate partially with respect to the y^1 axis

$$w(a,y^2) \;=\; A_1 + A_2(y^1 - y_i^1) + A_3(y^1 - y_i^2) + A_4(y^1 - y_i^3)$$

$$\therefore \frac{\partial w}{\partial y^1}(a,y^2) \;=\; A_2 + 2A_3(y^1 - y_i^1) + 3A_4(y^1 - y_i^1)$$

$$= A_2 + 2A_3 a + 3A_4 a^2$$

$$= [0\ 1\ 2a\ 3a^2]\{A\} \tag{5.5}$$

Referring to Figure 5.2 the boundary conditions at the two ends i and j of the strip are clearly

i) <u>At Node i</u> $\quad (y^1 = y_i^1\ , \quad a = 0)$

$$w(a,y^2) \;=\; w_i(y^2)$$

$$\frac{\partial w}{\partial y^1}(a,y^2) \;=\; -\,\theta_i(y^2) \tag{5.6a}$$

ii) <u>At Node j</u> $\quad (y^1 = y_j^1\ , \quad a = b)$

$$w(a,y^2) \;=\; w_j(y^2)$$

$$\frac{\partial w}{\partial y^1}(a,y^2) \;=\; -\,\theta_j(y^2) \tag{5.6b}$$

Using the above eqns of (5.6) and applying eqns (5.4) and (5.5) to the nodes i and j, we obtain a set of equations relating the nodal displacements {u} to the unknown constant {A} through their nodal coordinates as follows

$$\{u\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & b & b^2 & b^3 \\ 0 & -1 & -2b & -3b^2 \end{bmatrix} \{A\}$$

or $\{u\} = [C_a]\{A\}$ (5.7)

where $[C_a]$ is the nodal coordinate matrix.

The undetermined constants $\{A\}$ may be expressed directly in terms of the nodal displacements $\{u\}$ by inverting the $[C_a]$ matrix. i.e.

$$\{A\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \frac{-3}{b^2} & \frac{2}{b} & \frac{3}{b^2} & \frac{1}{b} \\ \frac{2}{b^3} & \frac{-1}{b^2} & \frac{-2}{b^3} & \frac{-1}{b^2} \end{bmatrix} \{u\}$$

or $\{A\} = [C_a]^{-1}\{u\}$ (5.8)

where $[C_a]^{-1}$ is the displacement transformation matrix.
Eqn (5.4) may now be expressed in terms of eqn (5.8) above as follows

$$w(a,y^2) = [1 \; a \; a^2 \; a^3][C_a]^{-1}\{u\}$$

$$= \left[\left(1 - \frac{3a^2}{b^2} + \frac{2a^3}{b^3}\right)\left(-a + \frac{2a^2}{b} - \frac{a^3}{b^2}\right)\left(\frac{3a^2}{b^2} - \frac{2a^3}{b^3}\right)\left(\frac{a^2}{b} - \frac{a^3}{b^2}\right)\right]\{u\}$$

or $w(a,y^2) = [C_b]\{u\}$ (5.9)

Eqn (5.9) is the required displacement function for the strip, defined in terms of the nodal coordinates and the generalised displacements. The above displacement function will clearly lead to compatibility of displacement and rotation between adjacent strip elements, since the vertical deflection w(a) and the first derivative with respect to y^1 are continuous. However by differentiating eqn (5.5) with respect to y^1 it can be seen that the second derivative varies linearly with a and will therefore be discontinuous at the boundaries between adjacent strips. By referring to eqns (4.8) it is evident that the curvatures are derived from combinations of the second derivative and therefore these will also be discontinuous. When applying eqn (4.32) to determine the stress resultants, it will therefore be necessary to mean the values obtained at common nodal boundaries. To ensure continuity of strain across the strip boundaries it would be necessary to include a second derivative term when defining the generalised displacement $\underset{\sim}{u}$. This would in turn require the choice of a 5th order displacement function which has been used for non-orthogonal systems such as skew plate structures [6,7,8], but does not appear justified in general.

5.2 Potential Energy of a Finite Strip

Considering strains due to bending, the total potential energy U_b of a finite strip as shown in Figure 5.1 is given by

$$U_b = \tfrac{1}{2}\int_{y_i^2}^{y_j^2}\int_{y_i^1}^{y_j^1} \{M_{11}X_1 + M_{22}X_2 + M_{12}X_{12} + M_{21}X_{21}\}h_1 dy^1 h_2 dy^2$$

$$- \int_{y_i^2}^{y_j^2}\int_{y_i^1}^{y_j^1} \{w(a,y^2)\rho(a,y^2)\}\, h_1 dy^1 h_2 dy^2 \qquad (5.10)$$

Substituting eqn (5.9) and noting that $M_{12} = - M_{21}$,

$$U_b = \tfrac{1}{2}\int_{y_i^2}^{y_j}\int_{y_i^1}^{y_j} \{M_{11}X_1 + M_{22}X_2 + M_{12}(X_{12} - X_{21})\}\ h_1dy^1h_2dy^2$$

$$- \int_{y_i^2}^{y_j^2}\int_{y_i^1}^{y_j^1} [C_b]\{u\}\rho(a,y^2)h_1dy^1h_2dy^2$$

$$= \tfrac{1}{2}\int_{y_i^2}^{y_j^2}\int_{y_i^1}^{y_j^1} \{X\}^T\{M\}h_1dy^1h_2dy^2 - \int_{y_i^2}^{y_j^2}\int_{y_i^1}^{y_j^1} \{u\}^T[C_b]^T\rho(a,y^2)h_1dy^1h_2dy^2 \qquad (5.11)$$

Substituting eqn (4.32) into the first term of eqn (5.11) we obtain

$$U_b = \tfrac{1}{2}\int_{y_i^2}^{y_j^2}\int_{y_i^1}^{y_j^1} \{X\}^T[D]\{X\}h_1dy^1h_2dy^2$$

$$- \int_{y_i^2}^{y_j^2}\int_{y_i^1}^{y_j^1} \{u\}^T[C_b]^T\rho(a,y^2)h_1dy^1h_2dy^2 \qquad (5.12)$$

Eqn (5.12) represents total potential energy for a curved finite strip element expressed in orthogonal curvilinear coordinates. The first term is the strain energy due to bending stress resultants and the second terms represent the potential energy of the surface forces.

In the case of plane polar coordinates the expression in eqn (5.12) above reduced to the following

$$U_b = \tfrac{1}{2}\int_{\theta_i}^{\theta_j}\int_{r_i}^{r_j} \{X\}^T[D]\{X\}drrd\theta$$

$$- \int_{\theta_i}^{\theta_j}\int_{r_i}^{r_j} \{u\}^T[C_b]^T\rho(a,\theta)drrd\theta \qquad (5.13)$$

where the curvature matrix {X} and the elasticity matrix [D] are as previously defined in 4.2 and 4.3 respectively for polar coordinates.

5.3 Derivation of Governing Differential Equation for Curved Finite Strip

To evaluate the strain energy contributions in eqn (5.12) it will first be necessary to express the curvatures in terms of the derivatives of the displacement function $w(a,y^2)$ as defined in eqn (5.9).
By referring to the eqns of (4.10) for the curvatures in orthogonal curvilinear coordinates, it can be seen that the following derivatives will be required.

$$\frac{\partial w}{\partial y^1} = \left[\left(\frac{-6a}{b^2} + \frac{6a^2}{b^3}\right)w_i \;\; \left(-1 + \frac{4a}{b} - \frac{3a^2}{b^2}\right)\theta_i \;\; \left(\frac{6a}{b^2} - \frac{6a^2}{b^3}\right)w_j \;\; \left(\frac{2a}{b} - \frac{3a^2}{b^2}\right)\theta_j\right]$$

$$\frac{\partial^2 w}{\partial y^1 \partial y^1} = \left[\left(\frac{-6}{b^2} + \frac{12a}{b^3}\right)w_i \;\; \left(\frac{4}{b} - \frac{6a}{b^2}\right)\theta_i \;\; \left(\frac{6}{b^2} - \frac{12a}{b^3}\right)w_j \;\; \left(\frac{2}{b} - \frac{6a}{b^2}\right)\theta_j\right]$$

$$\frac{\partial w}{\partial y^2} = \left[\left(1 - \frac{3a^2}{b^2} + \frac{2a^3}{b^3}\right)w'_i \;\; \left(-a + \frac{2a^2}{b} - \frac{a^3}{b^2}\right)\theta'_i \;\; \left(\frac{3a^2}{b^2} - \frac{2a^3}{b^3}\right)w'_j \;\; \left(\frac{a^2}{b} - \frac{a^3}{b^2}\right)\theta'_j\right]$$

$$\frac{\partial^2 w}{\partial y^2 \partial y^2} = \left[\left(1 - \frac{3a^2}{b^2} + \frac{2a^3}{b^3}\right)w''_i \;\; \left(-a + \frac{2a^2}{b} - \frac{a^3}{b^2}\right)\theta''_i \;\; \left(\frac{3a^2}{b^2} - \frac{2a^3}{b^3}\right)w''_j \;\; \left(\frac{a^2}{b} - \frac{a^3}{b^2}\right)\theta''_j\right]$$

$$\frac{\partial^2 w}{\partial y^1 \partial y^2} = \left[\left(\frac{-6a}{b^2} + \frac{6a^2}{b^3}\right)w'_i \;\; \left(-1 + \frac{4a}{b} - \frac{3a^2}{b^2}\right)\theta'_i \;\; \left(\frac{6a}{b^2} - \frac{6a^2}{b^3}\right)w'_j \;\; \left(\frac{2a}{b} - \frac{3a^2}{b^2}\right)\theta'_j\right] \tag{5.14}$$

where $w'_i = \dfrac{\partial w_i}{\partial y^2}$, $\theta'_i = \dfrac{\partial \theta_i}{\partial y^2}$

$w'_j = \dfrac{\partial w_j}{\partial y^2}$, $\theta'_j = \dfrac{\partial \theta_j}{\partial y^2}$

and
$$w''_i = \frac{\partial^2 w_i}{\partial y^2 \partial y^2}, \qquad \theta''_i = \frac{\partial^2 \theta_i}{\partial y^2 \partial y^2}$$

$$w''_j = \frac{\partial^2 w_j}{\partial y^2 \partial y^2}, \qquad \theta''_j = \frac{\partial^2 \theta_j}{\partial y^2 \partial y^2}$$

Using the above eqns of (5.14), the curvatures as given in eqn (4.10) may now be expressed in terms of the derivatives of the displacement function and are listed in Appendix A for the case of orthogonal curvilinear and plane polar coordinates. These expressions for the curvatures will be used to evaluate the strain energy term of eqn (5.12) which will require multiplying out and subsequent expanding. The many elements generated by this procedure can now be integrated with respect to y^1 and evaluated. To perform these integrations however, it will be necessary at this stage to specialise to a particular coordinate system. For the purpose of illustration the polar coordinate system will be chosen, where the integrations of eqn (5.13) will be of the following form

$$\int_{r_i}^{r_j} \frac{1}{r^n} \{k_1 + k_2 a + k_3 a^2 + \ldots\} r dr \qquad (5.15)$$

and $n = 0,1,2,3,4$ $\qquad a = r - r_i,$ $\qquad k = \text{constant}$

The above integrations will in general include terms involving $\ell n r$ and when evaluated will therefore result in terms of the form $\ell n \; r_j/r_i$. Since it will be convenient to evaluate the integrals in terms of the nodal coordinate limits associated with each strip, the ℓn term will be replaced by its series sum equivalent as follows:

$$\ell n \frac{r_j}{r_i} = \ell n \left(1 + \frac{b}{r_j} \right)$$

$$= \frac{b}{r_i} - \frac{1}{2}\left(\frac{b}{r_i}\right)^2 + \frac{1}{3}\left(\frac{b}{r_i}\right)^3 - \frac{1}{4}\left(\frac{b}{r_i}\right)^4 + \ldots \qquad (5.16a)$$

$$= \frac{b}{r_j} + \frac{1}{2}\left(\frac{b}{r_j}\right)^2 + \frac{1}{3}\left(\frac{b}{r_j}\right)^3 + \frac{1}{4}\left(\frac{b}{r_j}\right)^4 + \ldots \qquad (5.16b)$$

Using eqns (5.16) above it will now be possible to evaluate the integrals of eqn (5.15) directly in terms of the bounding coordinate radii r_i and r_j and the strip width b. The formulas for these integrations together with the derivation of typical cases are given in Appendix B. The use of numerical integration may however become necessary for more complex curvilinear coordinate systems and would depend on the values of the scale factor and their derivatives obtained.

The integration of the second term of eqn (5.13) representing the potential energy of the surface forces may be expressed as follows

$$[E] = \int_{r_i}^{r_j} [C_b]^T \rho(r,\theta) r dr \tag{5.17}$$

By referring to the $[C_b]$ matrix in eqn (5.9) it can be seen that these integrations do not involve ℓn terms and may therefore be readily evaluated. It will be shown later that eqn (5.17) represents the vector of nodal loads per unit length of strip.

The governing differential equation for a finite strip may be found by applying variational methods to the Principle of Minimum Potential Energy which states that "A system is in a configuration of stable equilibrium if, and only if, the value of the potential energy is a minimum". The terms in the total potential energy expression of eqn (5.13) may now be combined and written as follows

$$U_b = \int_{\theta_i}^{\theta_j} I(\underset{\sim}{u}, \underset{\sim}{u}', \underset{\sim}{u}'') \, d\theta \tag{5.18}$$

where the quantity I is defined in terms of the unknown displacement variables $\underset{\sim}{u}$, $\underset{\sim}{u}'$, and $\underset{\sim}{u}''$, the generalised displacements and their partial derivatives with respect to θ. From the Principle of Minimum Potential energy, U_b will be a minimum when the strip is in a state of stable equilibrium. The variational approach is to seek a stationary value for U_b defined by the appropriate integration of the unknowns over the domain. The integral U_b will be stationary when its first variation with respect to $\underset{\sim}{u}$, $\underset{\sim}{u}'$ and $\underset{\sim}{u}''$ vanishes. This will occur when

$$\delta U_b = 0 \tag{5.19}$$

The first variation of U_b with respect to each component of $\underset{\sim}{u}$, $\underset{\sim}{u}'$ and $\underset{\sim}{u}''$ is found to be

$$\delta U_b = \int_{\theta_i}^{\theta_j} \Big\{ \delta \underset{\sim}{u}''^T \Big[[A]\underset{\sim}{u}'' + [B]\underset{\sim}{u} \Big] + \delta \underset{\sim}{u}'^T \Big[[C]\underset{\sim}{u}' \Big] + \delta \underset{\sim}{u}^T \Big[[B]^T \underset{\sim}{u}'' + [D]\underset{\sim}{u} - [E] \Big] \Big\} d\theta \tag{5.20}$$

where the matrices [A], [B], [C], [D] and [E] are given in Appendix C for the polar coordinate system chosen.

Integrating eqn (5.20) with respect to θ by parts we obtain

$$\begin{aligned} \delta U_b = {} & \delta u'^T \Big[[A]\underset{\sim}{u}'' + [B]\underset{\sim}{u} \Big] \Big|_{\theta_i}^{\theta_j} - \delta \underset{\sim}{u}^T \Big[[A]\underset{\sim}{u}''' + [B]\underset{\sim}{u}' \Big] \Big|_{\theta_i}^{\theta_j} \\ & + \int_{\theta_i}^{\theta_j} \delta \underset{\sim}{u}^T \Big[[A]\underset{\sim}{u}'''' + [B]\underset{\sim}{u}'' \Big] d\theta + \delta \underset{\sim}{u}^T \Big[[C]\underset{\sim}{u}' \Big] \Big|_{\theta_i}^{\theta_j} \\ & - \int_{\theta_i}^{\theta_j} \delta \underset{\sim}{u}^T \Big[[C]\underset{\sim}{u}'' \Big] d\theta + \int_{\theta_i}^{\theta_j} \delta \underset{\sim}{u}^T \Big[[B]^T \underset{\sim}{u}'' + [D]\underset{\sim}{u} - [E] \Big] d\theta \end{aligned} \tag{5.21}$$

Eqn (5.21) may be regrouped with common displacement variations as follows

$$\delta U_b = \delta \underset{\sim}{u}'^T\Big[[A]\underset{\sim}{u}'' + [B]\underset{\sim}{u}\Big]\Big|_{\theta_i}^{\theta_j} - \delta\underset{\sim}{u}^T\Big[[A]\underset{\sim}{u}''' + \big[[B] - [C]\big]\underset{\sim}{u}'\Big]\Big|_{\theta_i}^{\theta_j}$$

$$+ \int_{\theta_i}^{\theta_j}\Big\{\delta\underset{\sim}{u}^T\Big[[A]\underset{\sim}{u}'''' + \big[[B] - [C] + [B]^T\big]\underset{\sim}{u}'' + [D]\underset{\sim}{u} - [E]\Big]\Big\}\,d\theta$$

(5.22)

From eqn (5.19), for δU_b to be zero, each component of eqn (5.22) must be zero and therefore

$$\delta u'^T\Big[[A]\underset{\sim}{u}'' + [B]\underset{\sim}{u}\Big]\Big|_{\theta_i}^{\theta_j} = 0 \qquad (5.23a)$$

$$\delta u^T\Big[[A]\underset{\sim}{u}''' + \big[[B] - [C]\big]\underset{\sim}{u}'\Big]\Big|_{\theta_i}^{\theta_j} = 0 \qquad (5.23b)$$

$$[A]\underset{\sim}{u}'''' + \big[[B] - [C] + [B]^T\big]\underset{\sim}{u}'' + [D]\underset{\sim}{u} - [E] = 0 \qquad (5.24)$$

Eqn (5.24) is the governing differential equation for a curved finite strip and those of (5.23) are the differential equations applicable at the strip boundaries.

5.4 Solution of a System of Finite Strips

To obtain a solution of the curved plate structure the continuum is subdivided transversely into a series of finite strip elements connected along their sides or nodallines, as shown in Figure 5.3. By applying eqn (5.24) to each strip the set of governing differential equations for the system of finite strips may be found.

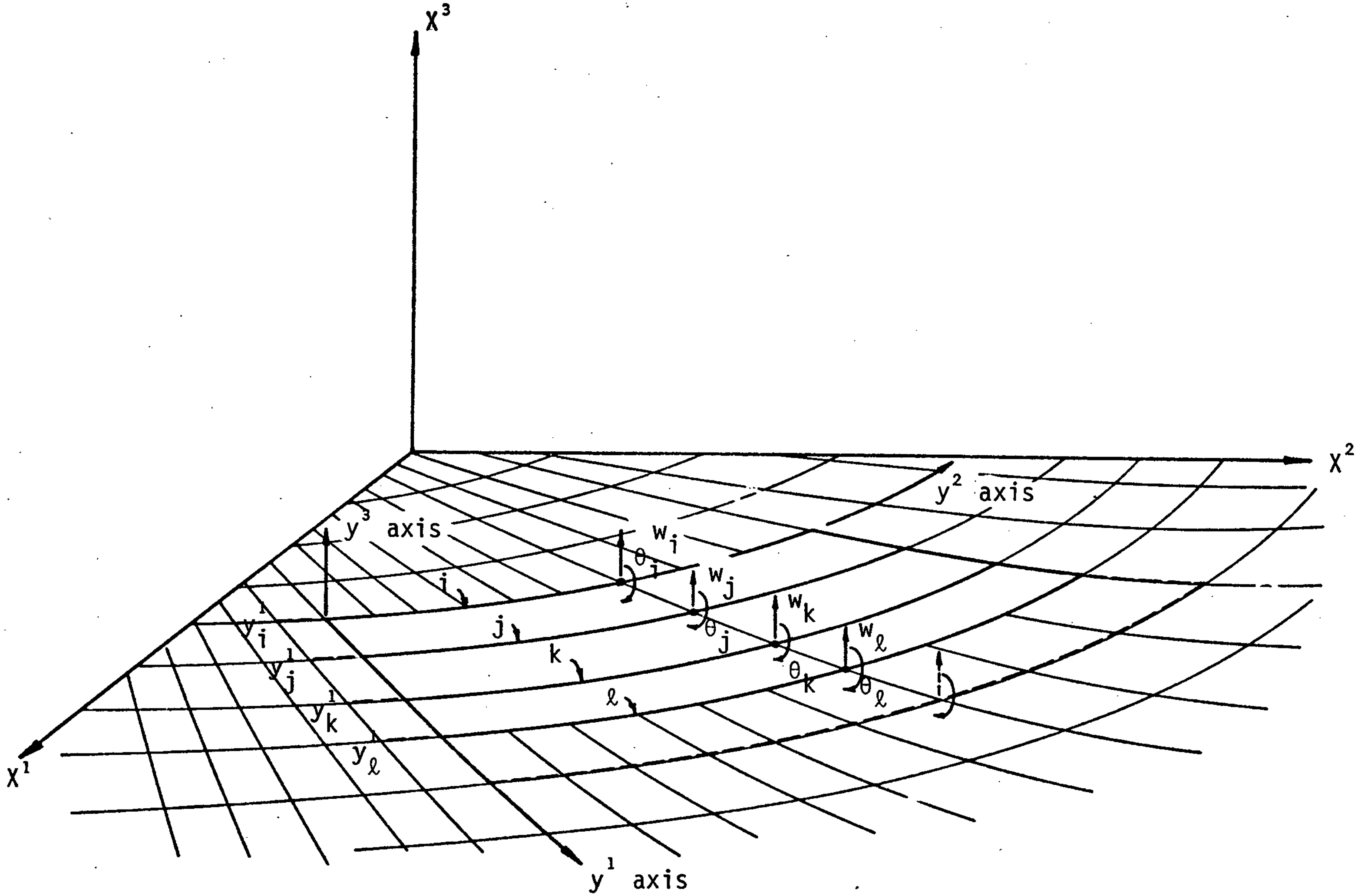


FIGURE 5.3 System of finite strip connected along nodal line boundaries

Since the generalised displacements $\underset{\sim}{u}$ and their partial derivatives will be common along the nodal line boundaries, the equations for adjacent strips may be combined to form the governing differential equation along the nodal lines. In this way the set of simultaneous differential equations for the plate, subdivided into a system of finite strip elements are formulated. By applying the appropriate boundary condition differential equations to the ends of each strip, the above set of simultaneous differential equations may now be uniquely solved to yield the generalised displacements throughout the plate. From the resulting set of displacements and rotations along the strip boundaries the curvatures and corresponding stress resultants may be subsequently determined.

In the present analysis the system of finite strips is subdivided longitudinally into a series of nodal cross-sections, thereby enabling the partial derivative of $\underset{\sim}{u}$ to be replaced by their finite difference equivalent. The finite difference form of the governing differential equation combined for a set of adjacent strips at one common interior nodal cross-section for unequal spacing of nodes is given in Appendix D1. In this way the set of simultaneous differential equations may be conveniently replaced by the corresponding set of simultaneous linear equation formulated directly in terms of the unknown generalised displacements $\underset{\sim}{u}$ throughout the plate. The finite difference form of this set of simultaneous equations compiled for a system of adjoining finite strips and nodal cross-sections is shown in Appendix D2 and may be written as

$$[K]\{d\} = [E] \tag{5.25}$$

where [K] is the finite difference stiffness matrix for the structure
{d} is the vector of nodal displacements and rotations
[E] is the vector of corresponding nodal loads and moments

Eqn (5.25) is subsequently solved using normal Gauss reduction techniques to yield the set of unknown displacements {d}. Using the relationships as given in Appendix A and the corresponding equations of (4.31) or (4.34), the curvatures and stress resultant may be subsequently derived to obtain the complete solution to the curved plate structure.

5.5 Application of Boundary Condition Equations

For a unique solution to a system of curved finite strips it is clear that the governing differential equation must be satisfied at any interior point along the length of each strip and the boundary differential equations must be satisfied at the two ends of each strip.

Since the governing differential equation is of fourth order and given in terms of two basic variables w and θ, a total of eight possible boundary conditions may be identified from eqns (5.23a) and (5.23b) as follows

5.5.1 Simply supported edge

In the case of the simply supported edge the vertical displacement and the curvature in the longitudinal direction are zero. The first variation of $\underset{\sim}{u}$ will therefore be zero and $\delta\underset{\sim}{u}'$ will be non-zero. From the boundary differential eqns of (5.23a) and (5.23b), it follows that:

(i) $\underset{\sim}{u} = 0$

(ii) $[A]\underset{\sim}{u}'' + [B]\underset{\sim}{u} = 0$ (5.26)

5.5.2 Clamped edge

For the clamped edge boundary condition, the vertical displacement an the slope in the longitudinal direction are zero. For these boundary conditions to be satisfied the first variation of $\underset{\sim}{u}$ and $\underset{\sim}{u}'$ must be zero. Eqns (5.23a) and (5.23b) therefore simplify to the following

(i) $\underset{\sim}{u} = 0$

(ii) $\underset{\sim}{u}' = 0$ (5.27)

5.5.3 Guided edge

In the case where, due to symmetry of geometry and loading, the edge may be assumed to be guided, the slope in the longitudinal direction and the vertical shear strain will be zero. The first variation of $\underset{\sim}{u}'$ will therefore be zero and $\delta\underset{\sim}{u}$ will be non-zero. From the boundary differential eqn of (5.23a) and (5.23b) it follows that

$$\text{(i)} \quad \underset{\sim}{u}' = 0$$

$$\text{(ii)} \quad [A]\underset{\sim}{u}''' + \Big[[B] - [C]\Big]\underset{\sim}{u}' = 0 \qquad (5.28)$$

5.5.4 Free edge

For the free edge boundary condition, the curvature in the longitudinal direction and the combined vertical shear strain and twisting curvature will be zero. For these boundary conditions to be satisfied the first variation of $\underset{\sim}{u}$ and $\underset{\sim}{u}'$ will be non-zero and eqn (5.23a) and (5.23b) become

$$\text{(i)} \quad [A]\underset{\sim}{u}'' + [B]\underset{\sim}{u} = 0$$

$$\text{(ii)} \quad [A]\underset{\sim}{u}''' + \Big[[B] - [C]\Big]\underset{\sim}{u}' = 0 \qquad (5.29)$$

In the finite difference application the strips are subdivided into a series of element and the governing differential equation is applied at each node point. At the boundary and the penultimate node however this process generates a set of ficticious displacement and rotations. In each of the above cases, applying the two appropriate boundary condition equations on the boundary enables these ficticious displacements to be uniquely eliminated.

The formulation of the finite difference stiffness matrix for simply supported and clamped boundary conditions is given in Appendix E1 for unequal spacing of nodes. In the case of the free edge boundary condition the elimination process involved in the application of eqns (5.28) becomes more complex and in the present formulation, as given in Appendix E2, a constant nodal spacing has been used.

5.6 Load Vector for Types of Loading Considered

The load vector [E] as defined in eqn (5.17) will be evaluated for the different types of loading to be considered and is restated as follows

$$[E] = \int_{r_i}^{r_j} [C_b]^T \rho(r,\theta) r dr \tag{5.17}$$

By referring to the potential energy term of eqn (5.10) it is to be noted that the basic load intensity is expressed as a force/unit area. By integrating across the strip the nature of loading will therefore transform to that of an intensity / unit length.

In the polar coordinate system the integrations of eqn (5.17) will transform the loading to that of an intensity/unit rotation, since the scale factor $h_2 = r$ in the $y^2 = \theta$ direction is included in the integration. For each type of loading the elements of the resulting load vector will therefore represent the nodal load and fixed end moment quantities/unit rotation along the sides i and j of the strip. Providing the loading across the strip is of a continuous nature, it will be necessary to express the load initially as a force/unit surface area prior to evaluating the load vector. The three basic types of loading to be considered are as follows

5.6.1 Patch loading

The patch loading can be considered to represent the general case of a uniformily distributed load and will be defined as constant across the strip but may vary linearly along the length of the strip. The loading expression $\rho(r,\theta)$ in eqn (5.17) will be of the form

$$\rho(r,\theta) = q_0 + k\,\theta \tag{5.30}$$

where $k = \dfrac{q_1 - q_0}{\Delta\theta}$

and is as shown in Figure 5.5(i)

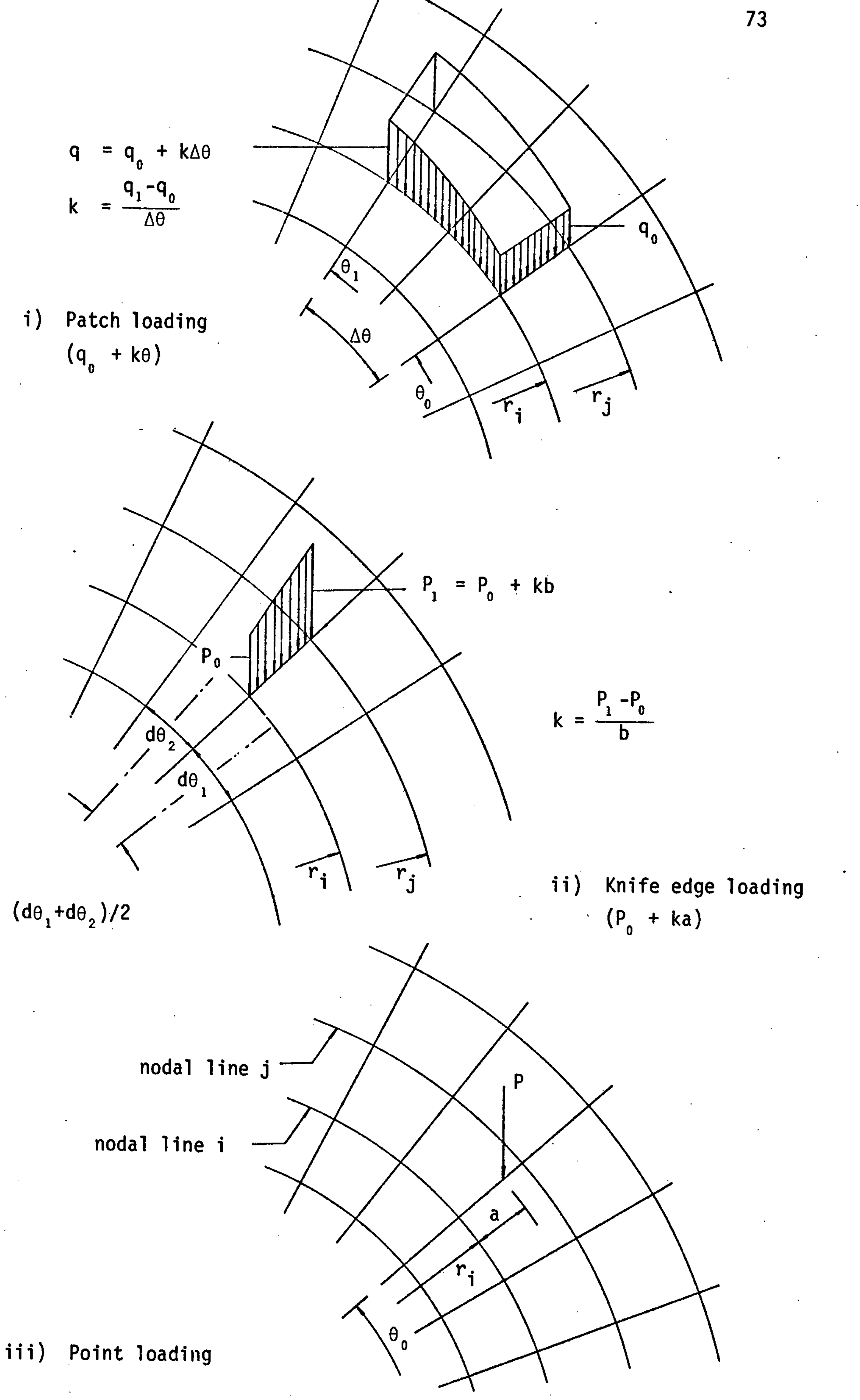


FIGURE 5.4 Different types of loading considered

5.6.2 Knife edge loading

The knife edge load along a radial line at $\theta = \theta_1$ may vary linearly across the strip and will be discontinuous along the length of the strip. In the subsequent finite difference representation however the load may be assumed to be distributed in the θ direction over a nodal spacing of physical length $ds = rd\theta$. The loading expression $\rho(r,\theta)$ in eqn (5.17) will therefore be as follows

$$\rho(r,\theta) = \frac{P_0 + ka}{rd\theta} \tag{5.31}$$

where $k = \dfrac{P_1 - P_0}{b}$

and is as shown in Figure 5.5(ii)

5.6.3 Point loading

The point load acts at a distance x across the strip and at a position $\theta = \theta_1$ along the length of the strip. The load will be assumed to be distributed over a nodal spacing $ds = rd\theta$ as in the case of the line load. However, since the point load is discontinuous in the radial direction as well, the loading expression $\rho(r,\theta)$ in eqn (5.17) will be written as

$$\rho(r,\theta) = \frac{P}{rd\theta} \tag{5.32}$$

where $r = r_i + x$ $\qquad$ x = constant in this case

and is as shown in Figure 5.5(iii)

The load vector is obtained by substituting the appropriate expression for the loading function into eqn (5.17) and evaluating the integral. The load vector [E] is given in Appendix C5 for the above loading types considered. It is to be noted that in each case multiplying by $d\theta$ transforms the elements from an intensity/unit rotation to those of nodal loads. In the case of variable spacing finite difference approximations the spacing $d\theta$ at nodal cross-section i is replaced by $(d\theta_{i-1} + d\theta_i)/2$.

CHAPTER 6

DERIVATION OF VARIABLE SPACING FINITE DIFFERENCE APPROXIMATIONS

Introduction

Since the present work on the finite strip method is based on that of replacing the partial derivatives of $\underset{\sim}{u}$ by the corresponding finite difference approximations, the possibility of formulating the various operator patterns in terms of a variable spacing was considered. A separate investigation [39] was therefore carried out to determine whether the use of variable spacing finite difference approximations was justified in the case of simple and continuous beam problems. The investigation was also initiated to resolve the contradiction noted between a description of the method used to derive the formulas for variable spacing finite difference approximations and the various operator patterns listed [35]. It became apparent that there were in fact two possible formulations for the operator patterns, one set based on a combination of 2nd order polynomial expressions and the other set based on a similar 4th order expression. It was therefore found necessary to examine each set of operator patterns and determine on which to base the present work on the finite strip method. Furthermore the apparent non-symmetry of the resulting finite difference stiffness matrix formulated in terms of a variable spacing also required further investigation. The following sections cover briefly the derivations of the various central and boundary operator patterns and include a discussion and evaluation of the results.

6.1 Central Operator Patterns Based on 2nd Order Expressions

The variable spacing finite difference approximations based on 2nd order polynomial expressions are obtained in the following manner. Consider a smooth function $y = f(x)$ with ordinate values given at a series of 3 unequally spaced points as shown in Figure 6.1 Taking a local axis $x' = x$, $y = 0$, the values of the derivatives of y are found for the point $x = x'$ as follows.

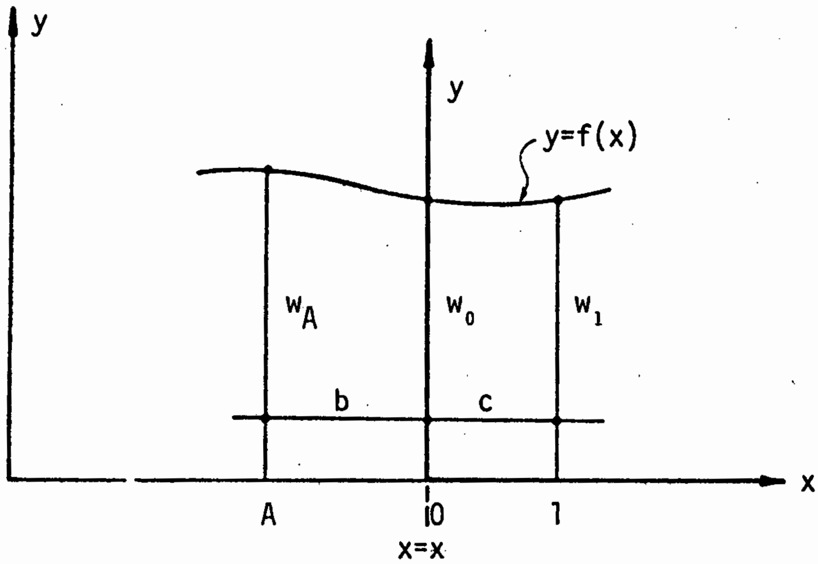


FIGURE 6.1   Function y = f(x) for three unequally spaced nodes

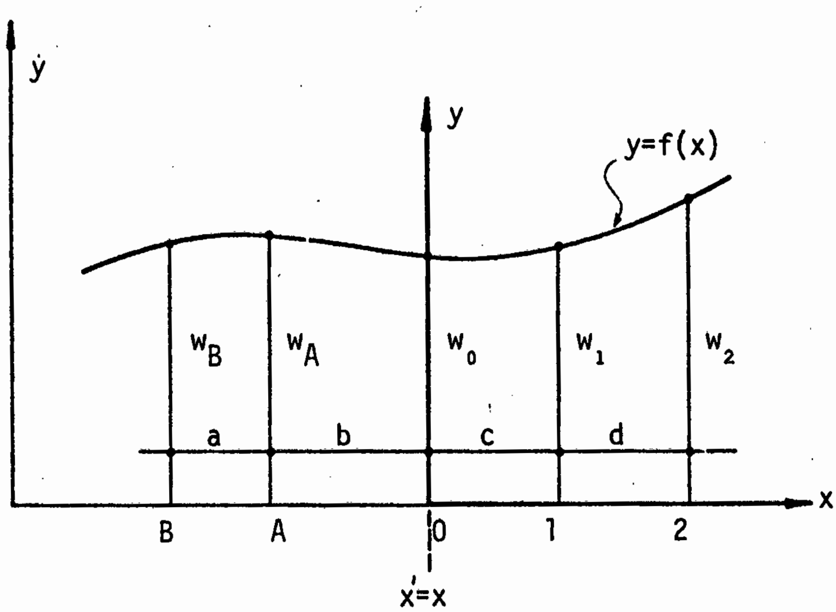


FIGURE 6.2   Function y = f(x) for five unequally spaced nodes

The basic polynomial expression of 2nd order will have the form

$$y = A_0 + A_1 x + A_2 x^2 \tag{6.1}$$

Differentiating successively with respect to x:

$$\frac{dy}{dx} = A_1 + 2A_2 x$$

$$\frac{d^2y}{dx^2} = 2A_2 \tag{6.2}$$

The constants A in the above equations are solved by using the following initial conditions.

i) At $x = 0$, $y = w_0$
ii) At $x = -b$, $y = w_A$
iii) At $x = c$, $y = w_1$

Substituting these initial conditions into eqn (6.1) in turn, the following set of equastions is obtained.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -b & b^2 \\ 1 & c & c^2 \end{bmatrix} \cdot \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} w_0 \\ w_A \\ w_1 \end{Bmatrix}$$

By solving for the values of {A}, the required expressions for the derivatives at x = 0 are obtained as follows

$$y' = \frac{dy}{dx} = A_1 = \frac{-cw_A}{b(b+c)} + \frac{(c-b)w_0}{bc} + \frac{bw_1}{c(b+c)}$$

$$y'' = \frac{d^2y}{dx^2} = 2A_2 = \frac{2w_A}{b(b+c)} - \frac{2w_0}{bc} + \frac{2w_1}{c(b+c)} \tag{6.3}$$

The eqns of (6.3) give the basic operator patterns for the 1st and 2nd derivatives of y, based on a 2nd order polynomial expression.

To obtain higher derivatives of y it is necessary to consider the same basic function with ordinate values given at additional unequally spaced points as shown in Figure 6.2 Applying eqns (6.3) to the interior nodes gives:

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{dy_0}{dx} = \frac{-cw_A}{b(b+c)} + \frac{(c-b)w_0}{bc} + \frac{bw_1}{c(b+c)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=-b} = \frac{d^2y_A}{dx^2} = \frac{2w_B}{a(a+b)} - \frac{2w_A}{ab} + \frac{2w_0}{b(a+b)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = \frac{d^2y_0}{dx^2} = \frac{2w_A}{b(b+c)} - \frac{2w_0}{bc} + \frac{2w_1}{c(b+c)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=c} = \frac{d^2y_1}{dx^2} = \frac{2w_0}{c(c+d)} - \frac{2w_1}{cd} + \frac{2w_2}{d(c+d)} \qquad (6.4)$$

The 3rd and 4th derivatives of y in terms of the basic second order polynomial expression may be obtained from the eqns of (6.4) as follows

$$\frac{d^3y_0}{dx^3} = \frac{d}{dx}\left(\frac{d^2y_0}{dx^2}\right)$$

$$= \frac{-c}{b(b+c)}\left(\frac{d^2y_A}{dx^2}\right) + \frac{(c-b)}{bc}\left(\frac{d^2y_0}{dx^2}\right) + \frac{b}{c(b+c)}\left(\frac{d^2y_1}{dx^2}\right) \qquad (6.5)$$

$$\frac{d^4y_0}{dx^4} = \frac{d^2}{dx^2}\left(\frac{d^2y_0}{dx^2}\right)$$

$$= \frac{2}{b(b+c)}\left(\frac{d^2y_A}{dx^2}\right) - \frac{2}{bc}\left(\frac{d^2y_0}{dx^2}\right) + \frac{2}{c(b+c)}\left(\frac{d^2y_1}{dx^2}\right) \qquad (6.6)$$

By substituting the appropriate eqns of (6.4) into eqns (6.5) and (6.6), the basic operator patterns for the 3rd and 4th derivatives of y are obtained. A summary of the above variable spacing finite difference approximations based on 2nd order polynomials is given in Appendix F1.

6.2 Boundary Operator Patterns Using 2nd Order Expressions

The variable spacing finite difference operator patterns for the derivatives on the edge and at the pen-ultimate node will be derived for the different boundary conditions to be considered.

The procedure will involve initially setting up the appropriate basic operator pattern at the node under consideration. By applying the operator pattern for the known zero derivatives on the boundary, a relationship between the real and fictitious displacements is found. In this way the fictitious displacements are eliminated and the required boundary operator pattern is obtained.

6.2.1 Boundary operator patterns for 4th derivative

6.2.1.1 Free edge: end node

The two boundary conditions on the free edge will be as follows

$$\text{(i)} \quad \text{At} \quad x = 0 , \qquad y'' = 0 \tag{6.7a}$$

$$\text{(ii)} \quad \text{At} \quad x = 0 , \qquad y''' = 0 \tag{6.7b}$$

Applying the basic operator pattern for the 4th derivative at the end node with a = d and b = c as shown in Figure 6.3, the following simplified form is obtained

$$y'''' = \frac{2w_B}{c^2(c+d)} - \frac{2(c+d)w_A}{c^4 d} + \left\{\frac{4}{c^4} + \frac{4}{c^3(c+d)}\right\} w_0 - \frac{2(c+d)w_1}{c^4 d} + \frac{2w_2}{c^2 d(c+d)} \tag{6.8}$$

To eliminate the fictitious displacements w_A and w_B, the basic operator pattern for the 2nd and 3rd derivatives are applied at the end node, and from eqns (6.7) are equated to zero as follows:

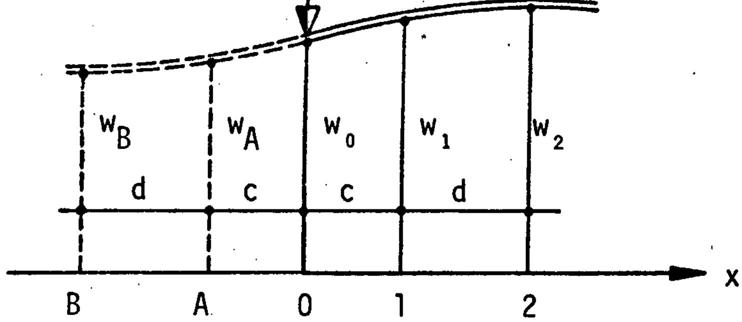


FIGURE 6.3 Free Edge: Operating at end node

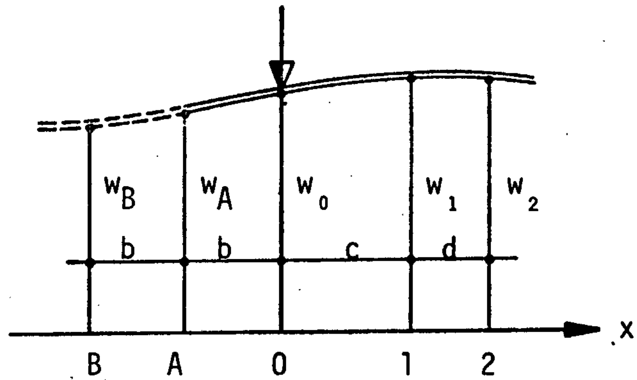


FIGURE 6.4 Free Edge: Operating at pen-ultimate node

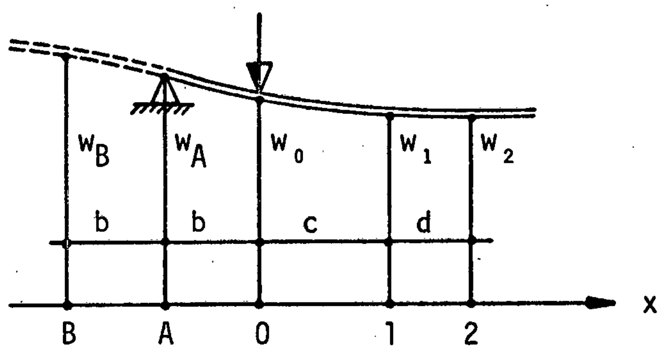


FIGURE 6.5 Simply Supported Edge: Operating at pen-ultimate node

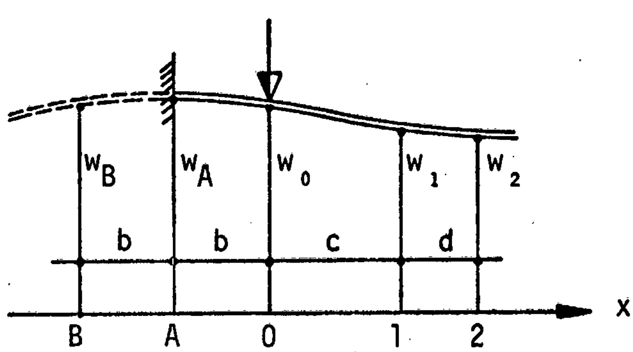


FIGURE 6.6 Clamped Edge: Operating at pen-ultimate node

$$y'' = \frac{2w_A}{2c^2} - \frac{2w_0}{c^2} + \frac{2w_1}{2c^2} = 0$$

$$\therefore \quad w_A = 2w_0 - w_1 \tag{6.9a}$$

$$y''' = \frac{-w_B}{cd(c+d)} + \frac{w_A}{c^2 d} - \frac{w_1}{c^2 d} + \frac{w_2}{cd(c+d)} = 0$$

$$w_B = \frac{(c+d)}{c}\{w_A - w_1\} + w_2$$

from eqn (6.9a), the above equation becomes

$$w_B = \frac{2(c+d)}{c}\{w_0 - w_1\} + w_2 \tag{6.9b}$$

and substituting the eqns of (6.9) into eqn (6.8)

$$y'''' = \frac{4w_0}{c^3(c+d)} - \frac{4w_1}{c^3 d} + \frac{4w_2}{c^2 d(c+d)} \tag{6.10}$$

Eqn (6.10) gives the required operator pattern for the 4th derivative on the free edge. In the case of the simply supported and clamped edge boundary conditions, the operator patterns are not required since the displacement at the ends will be zero.

6.2.1.2 Free edge: pen-ultimate node

The boundary conditions at the free edge are now expressed as follows

(i) At $x = -b$, $y'' = 0$ (6.11a)

(ii) At $x = -b$, $y''' = 0$ (6.11b)

To eliminate the fictitious displacement w_B resulting from the application of the basic equation, the operator pattern for the 2nd deriviative is applied at the end node with a = b and from eqn (6.11a) is equated to zero giving the relationship

$$w_B = 2w_A - w_0 \tag{6.12}$$

Applying the basic operator pattern for the 4th derivative at the pen-ultimate node with a = b, as shown in Figure 6.4, and substituting eqn (6.12) above, the following result is obtained

$$y'''' = \frac{-4w_A}{b^2c(b+c)} + \left\{\frac{4}{b^2c^2} + \frac{4}{c^2(b+c)(c+d)}\right\} w_0 - \frac{4(b+d)w_1}{bc^2d(b+c)} - \frac{4w_2}{cd(b+c)(c+d)}$$

Eqn (6.13) gives the required operator pattern for the 4th deriviative at the pen-ultimate node for the free edge boundary condition.

6.2.1.3 Simply supported edge: pen-ultimate node

In the case of the simply supported edge as shown in Figure 6.5, the following boundary conditions will apply

(i) At $x = -b$, $y = 0$ (6.14a)

(ii) At $x = -b$, $y'' = 0$ (6.14b)

The operator pattern at the pen-ultimate node will therefore be the same as that of the free edge, as given in eqn (6.13), but with w_A equal to zero.

6.2.1.4 Clamped edge: pen-ultimate node

The boundary conditions at a clamped edge, as shown in Figure 6.6, will be as follows

(i) At $x = -b$, $y = 0$ (6.15a)

(ii) At $x = -b$, $y' = 0$ (6.15b)

To eliminate the fictitious displacement w_B resulting from the application of the basic equation, the operator pattern for the 1st derivative is applied at the end node with a = b and from eqn (6.14) is equated to zero giving

$$y' = \frac{-bw_B}{b(2b)} + \frac{bw_0}{b(2b)} = 0$$

$$w_B = w_0 \tag{6.16a}$$

and from eqn (6.15b)

$$w_A = 0 \tag{6.16b}$$

Using eqns (6.16) the basic operator pattern for the 4th derivative therefore simplifies to the following

$$y'''' = \left\{ \frac{4}{b^2c^2} + \frac{4}{b^2(b+c)} + \frac{4}{c^2(b+c)(c+d)} \right\} w_0 - \frac{4(b+d)w_1}{bc^2d(b+c)} + \frac{4w_2}{cd(b+c)(c+d)} \tag{6.17}$$

Eqn (6.17) gives the required operator pattern for the 4th derivative at the pen-ultimate node for the clamped edge boundary condition.

The above set of boundary operator patterns together with the basic operator pattern may now be set up to form the 4th derivative finite difference matrix for the different boundary conditions. For these operator patterns based on second order polynomials, symmetry may be obtained by multiplying each row by the average nodal spacing (b+c)/2. The 4th derivative matrices for the different boundary conditions are given in Appendix F2.

6.2.2 Boundary operator patterns for 2nd derivative

Since the basic operator patterns for the 2nd derivative is given in terms of 3 nodal points only, the boundary operator patterns are obtained directly. The 2nd derivative matrices for the different boundary conditions are as given in Appendix F3.

6.3 Central Operator Patterns Based on a 4th Order Expression

The variable spacing finite difference approximations based on 4th order polynomial expressions are obtained in the following manner. Referring to Figure 6.2, the general expression for a smooth function $y = f(x)$ of 4th order will have the form

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 \tag{6.18}$$

Differentiating successively with respect to x:

$$\frac{dy}{dx} = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 \tag{6.19a}$$

$$\frac{d^2y}{dx^2} = 2A_2 + 6A_3x + 12A_4x^2 \tag{6.19b}$$

$$\frac{d^3y}{dx^3} = 6A_3 + 24A_4x \tag{6.19c}$$

$$\frac{d^4y}{dx^4} = 24A_4 \tag{6.19d}$$

The constants {A} in eqns (6.18) are solved using the following initial displacement condition

(i)	At	$x = -(a+b)$,	$y = w_B$
(ii)	At	$x = -b$,	$y = w_A$
(iii)	At	$x = 0$,	$y = w_0$
(iv)	At	$x = c$,	$y = w_1$
(v)	At	$x = (c+d)$,	$y = w_2$

Substituting these nodal coordinates into eqn (6.18), the following set of simultaneous equations is obtained:

$$\begin{bmatrix} 1 & -(a+b) & (a+b)^2 & -(a+b)^3 & (a+b)^4 \\ 1 & -b & b^2 & -b^3 & b^4 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & c & c^2 & c^3 & c^4 \\ 1 & (c+d) & (c+d)^2 & (c+d)^3 & (c+d)^4 \end{bmatrix} \cdot \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} w_B \\ w_A \\ w_0 \\ w_1 \\ w_2 \end{Bmatrix}$$

By solving for the values of $\{A\}$ the required expressions for the derivatives at $x = 0$ are found from the eqns of (6.19). A summary of the above variable spacing finite difference approximations based on a 4th order polynomial is given in Appendix F4.

6.4 Boundary Operator Patterns Using a 4th Order Expression

The variable spacing finite difference operator patterns required for the 2nd and 4th derivatives on the edge and at the pen-ultimate node will be derived for the different boundary conditions to be considered. In this case the process involves setting up the basic 4th order polynomial expression and its derivatives at the node point under consideration and initially applying the appropriate boundary differential equation. The remaining nodal displacements are then successively equated to the basic function now satisfying the prescribed boundary condition equations. The solution to this system of equations yields the required boundary operator patterns for the various derivatives.

6.4.1 Free edge: End node

Using the relevant boundary condition as given in eqns (6.7), the values of A_2 and A_3 are found from eqns (6.19b) and (6.19c) as follows

(i) $A_2 = 0$

(ii) $A_3 = 0$

and eqn (6.17) therefore reduces to

$$y = A_0 + A_1 x + A_4 x^4 \tag{6.20}$$

Eqn (6.20) is the basic displacement function now satisfying the specified boundary conditions on the free edge. To solve for the remaining constants, the known displacements as shown in Figure 6.3 are used. i.e.

(i) At $x = 0$, $y = w_0$

(ii) At $x = c$, $y = w_1$

(iii) At $x = c+d$, $y = w_2$

These nodal displacements are applied to eqn (6.20) as follows

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & c^4 \\ 0 & (c+d) & (c+d)^4 \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} w_0 \\ w_1 \\ w_2 \end{Bmatrix}$$

Solving the above set of equations yields the values of the unknown constants {A}. Substituting the value of A_4 obtained, eqn (6.18d) becomes

$$y'''' = \frac{24w_0}{cf(f^2+fc+c^2)} - \frac{24w_1}{cd(f^2+fc+c^2)} + \frac{24w_2}{fd(f^2+fc+c^2)} \tag{6.21}$$

where $f = c+d$

Eqn (6.21) gives the required boundary operator pattern for the 4th derivative on the free edge.

6.4.2 Free edge: Pen-ultimate node

Applying the relevant boundary condition eqn of (6.11a) the constant A_2 in eqn (6.19b) may be expressed as

$$\text{(i)} \quad A_2 = 3A_3 b - 6A_4 b^2$$

and eqn (6.18) therefore becomes

$$y = A_0 + A_1 x + A_3(3bx^2 + x^3) + A_4(-6b^2x^2 + x^4) \qquad (6.22)$$

Eqn (6.22) is the basic expression now satisfying the prescribed boundary condition on the free edge. The remaining constants are solved using the known displacements as shown in Figure 6.4. i.e.

(i)	At	$x = 0$,	$y = w_0$
(ii)	At	$x = -b$,	$y = w_A$
(iii)	At	$x = c$,	$y = w_1$
(iv)	At	$x = f$,	$y = w_2$

where $f = c+d$

Applying these nodal coordinates to eqn (6.22), the following set of equations is obtained.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -b & 2b^3 & -5b^4 \\ 0 & c & (3bc^2+c^3) & (-6b^2c^2+c^4) \\ 0 & f & (3bf^2+f^3) & (-6b^2f^2+f^4) \end{bmatrix} \cdot \begin{Bmatrix} A_0 \\ A_1 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} w_0 \\ w_A \\ w_1 \\ w_2 \end{Bmatrix}$$

From the solution to the above set of equations the values of the unknown constants {A} are found. By substituting the values of {A} into the eqns of (6.19), the required free edge boundary operator patterns for the derivatives at the pen-ultimate node may be identified and are given in Appendices F5 and F6 for the 2nd and 4th derivatives respectively.

6.4.3 Simply supported edge: Pen-ultimate node

From the appropriate boundary condition eqns of (6.14), the constants A_0 in eqn (6.18) and A_2 in eqns (6.19) may be expressed as follows.

(i) $A_0 = A_1 b - 2A_3 b^3 + 5A_4 b^4$

(ii) $A_2 = 3A_3 b - 6A_4 b^2$

The operator patterns at the penultimate node will therefore be the same as those of the Free edge, as given in Appendices F5 and F6, but with w_A equal to zero.

6.4.4 Clamped edge: Pen-ultimate node

Applying the relevant boundary condition eqns of (6.15), the constants A_0 and A_1 in eqns (6.19) may be expressed as follows:

(i) $A_0 = A_2 b^2 - 2A_3 b^3 + 3A_4 b^4$
(ii) $A_1 = 2A_2 b - 3A_3 b^2 + 4A_4 b^3$

Substituting the above values into eqn (6.18) the following basic function is obtained.

$$y = A_2(b^2 + 2bx + x^2) + A_3(-2b^3 - 3b^2x - x^3) + A_4(3b^4 + 4b^3x + x^4) \quad (6.22)$$

The remaining constants are solved using the known displacements as shown in Figure 6.6. i.e.

(i) At $x = 0$, $y = w_0$
(ii) At $x = c$, $y \quad w_1$
(iii) At $x = f$, $y = w_2$

where $f = c+d$

The above nodal coordinates are applied to eqn (6.22) to yield the following set of equations.

$$\begin{bmatrix} b^2 & -2b^3 & 3b^4 \\ (b^2+2bc+c^2) & (-2b^3-3b^2c+c^3) & (3b^4+4b^3c+c^4) \\ (b^2+2bf+f^2) & (-2b^3-3b^2f+f^3) & (3b^4+4b^3f+f^4) \end{bmatrix} \cdot \begin{Bmatrix} A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} w_0 \\ w_1 \\ w_2 \end{Bmatrix}$$

From the values of {A} obtained, the required clamped edge boundary operator patterns for the derivatives at the pen-ultimate node are found as previously described, and are given in Appendices F5 and F6 for the 2nd and 4th derivatives respectively.

6.5 Discussion and Evaluation of Operator Patterns

For the purpose of evaluating the two sets of operator patterns, the program STRIPB [39] was developed and used to analyse a series of simple and continuous beam problems. The method of analysis was based on the well known governing differential equation for a beam element

$$EI \frac{d^4y}{dx^4} = w \tag{6.23}$$

The structure was subdivided into a number of elements and corresponding node points incorporating an arbitrary variable spacing. Eqn (6.23) was replaced by its finite difference equivalent and applied at each node point. This process generated a system of simultaneous linear equations, the solution to which yielded the unknown nodal displacements. By a similar approach the differential equation

$$EI \frac{d^2y}{dx^2} = M \tag{6.24}$$

was expressed in terms of finite differences and applied to each node in turn, obtaining the complete solution to the problem. From a comparison of the two sets of results of the series of examples analysed [39], the following relevant points were noted.

(i) In each case where the loading and support conditions were such that all the derivatives were continuous, the 4th order analyses gave exact results for displacements and bending moments, for both variable and constant spacing of nodes. For these conditions, the finite difference representation of the set of governing differential equations was of sufficient accuracy as to yield the exact values of the unknown displacements. Subsequent back-substitution by applying the five point 2nd derivative central operator pattern within the span and the appropriate boundary operator pattern at the ends gave the exact bending moments.

(ii) For a given set of displacements, the operator patterns based on 4th order polynomial expressions gave more accurate results when compared with the more crude patterns based on 2nd order or a combination of 2nd order expressions. However, from the examples analysed it was found that, although in all cases of uniform loading the 4th order analysis gave improved displacements, this was not the case when comparing bending moments, particularly in areas of high curvature, where the 4th order analyses tended to over estimate the correct values. In nearly all cases, and noticably the continuous beam problems, the 2nd order analyses were found to be more consistent and well behaved, particularly when comparing results of the point load cases. It is suspected that this is partially due to the fact that the 2nd degree polynomial can deal more effectively with discontinuities in the bending moment.

(iii) The point load was handled extremely well in the 2nd order analysis and very poorly in the 4th order analysis. In the case of the 2nd order analysis, exact results were obtained for simply supported and cantilever beam problems. In the general continuous case it was found that by concentrating the spacing at the boundaries and at interior supports, the results converged fairly rapidly. In the case of the 4th order analysis, any form of point loading lead to significant errors in the resulting bending moments.

(iv) From the convergence studies it was clear that no advantage was to be gained in concentrating the spacing at a load point or in any region of high but continuous curvature within a span. However, by concentrating the spacing at the boundaries and areas over interior supports, considerable advantage was to be gained. In the case of a continuous beam loaded uniformly, it was found that in the 2nd order analysis, it was possible to use five elements per span of variable spacing compared to eight constantly spaced elements for the same acceptable three per cent error in the support moments.

(v) The finite difference matrices resulting from the operator patterns based on 2nd order expressions were readily made symmetrical. This was achieved by successively multiplying each row of the matrix by the average nodal spacing at the corresponding node. In addition, multiplying the right-hand side of the equations by the above values conveniently transformed the load intensity vector to a vector of nodal loads. However, these facts did not appear to be possible in the case of the matrices based on a 4th order expression.

6.6 Concluding Remarks

(i) The use of variable spacing finite difference approximations would appear to be justified and should prove particularly useful in the analysis of continuous structures.

(ii) In most cases the operator patterns based on a combination of 2nd order polynomial expressions were found to be more suitable and reliable than the corresponding 4th order expressions.

(iii) The above operator patterns were adopted for the present work on the finite strip method, where the partial derivatives of $\underset{\sim}{u}$ in the governing differential equation are replaced by their corresponding finite difference approximations.

(iv) The choice of the 2nd order expressions was based on the superior results obtained in most of the problems analysed, in particular the point load cases and the more general continuous beam problems.

(v) The fact that the resulting derivative matrices became symmetric was a necessary pre-condition for the use of variable spacing finite difference approximations.

CHAPTER 7

RESULTS OF NUMERICAL EXAMPLES

7.1 Structures Analysed

In order to demonstrate the accuracy of the finite strip method, a number of typical curved plate structures have been analysed. For this purpose the general formulation of the method in curvilinear coordinates was specialised to the polar coordinate system. In each case the results obtained are compared with either classical or finite element analyses.

Examples 1 and 2 illustrate the limiting case of a square and rectangular plate respectively, defined in terms of a large radius and a small angle. The rate of convergence of the method is shown using different numbers of finite strip elements and a varying number of angular finite difference subdivisions. Results are compared with those of Cheung [1,2] and Timoshenko [33].

Example 3 gives the results for a simply supported isotropic curved plate with an aspect ratio $\lambda = 2$. The rate of convergence of the method is shown and the distributions of longitudinal and transverse bending moments and displacements are compared with those using an 8 noded iso-parametric finite element analysis.

Example 4 gives the results for a simply supported isotropic curved plate with an aspect ratio $\lambda = 4$. The results are compared with those of Scordelis [12] and with the closed form solution of the plate equation [13], and using the finite element analysis.

In example 5 a three span isotropic curved plate structure is analysed, illustrating the advantage of the finite difference based solution where continuity over any number of interior supports may be included. Distributions of longitudinal bending moments and displacements are compared with those using a variable nodal spacing and with the iso-parametric finite element analysis.

In the case of problems 3 and 5 a comparison of CPU times and total cost between the finite strip and finite element analyses is given. In both cases a three dimensional plot of the displaced shape of the structure has also been included.

7.1.1 Example 1. Simply supported square plate

An isotropic square plate subject to a unit uniform loading throughout was analysed for the ends of the strips simply supported.

The material properties and dimensions were as follows

Elastic Modulus	$E_r = E_\theta = 10,92$
Poissons' ratio	$\mu_r = \mu_\theta = 0,30$
Plate thickness	$h = 1,0$
Torsional rigidity	$D_r = D_\theta = 1,0$
Span length	$\ell = 1,0$

The results showing the rate of convergence of the method are given in Table 7.1, and are compared with the harmonic finite strip solution of Cheung [1] and the classical series solution of the isotropic plate equation of Timoshenko [33].

7.1.2 Example 2. Simply supported rectangular plate

An isotropic rectangular plate subject to a unit uniform loading throughout was analysed for the ends of the strips simply supported. For comparison purposes, an aspect ratio of $\lambda = 2$ was used.

The material properties and dimensions were as follows

Elastic Modulus	$E_r = E_\theta = 10,92$
Poissons' ratio	$\mu_r = \mu_\theta = 0,30$
Plate thickness	$h = 1,0$
Torsional rigidity	$D_r = D_\theta = 1,0$
Span length	$\ell = 1,0$

No of strips*	No of X-sects	Midspan at centre			Midspan on edge	
		w_3	M_1	M_2	w_3	M_2
1	7	0,01342	0,0326	0,1244	0,01534	0,1333
	9	0,01329	0,0327	0,1244	0,01520	0,1333
	11	0,01322	0,0328	0,1244	0,01513	0,1333
	13	0,01319	0,0328	0,1244	0,01509	0,1333
	15	0,01317	0,0328	0,1244	0,01507	0,1333
2	7	0,01340	0,0280	0,1229	0,01534	0,1317
	9	0,01326	0,0281	0,1229	0,01519	0,1317
	11	0,01320	0,0281	0,1229	0,01513	0,1317
	13	0,01317	0,0281	0,1229	0,01509	0,1318
	15	0,01315	0,0281	0,1229	0,01507	0,1318
3	7	0,01339	0,0273	0,1227	0,01534	0,1313
	9	0,01326	0,0274	0,1227	0,01519	0,1314
	11	0,01320	0,0275	0,1227	0,01513	0,1314
	13	0,01317	0,0275	0,1227	0,01509	0,1314
	15	0,01315	0,0275	0,1227	0,01507	0,1314
4	7	0,01339	0,0271	0,1226	0,01534	0,1312
	9	0,01326	0,0272	0,1226	0,01519	0,1313
	11	0,01320	0,0272	0,1226	0,01513	0,1313
	13	0,01317	0,0272	0,1226	0,01509	0,1313
	15	0,01315	0,0273	0,1226	0,01507	0,1313
Timoshenko [33]		0,01309	0,0271	0,1225	0,01509	0,1318
Cheung [1]		0,01309	0,0271	0,1224	0,01500	0,1314
Multiplier		$q\ell^4/D$	$q\ell^2$		$q\ell^4/D$	$q\ell^2$

TABLE 7.1 Example 1. Square plate. Results for unit uniform loading

* Due to symmetry, only half the structure was analysed

No of strips*	No of X-sects	Midspan at centre			Midspan on edge	
		w_3	M_1	M_2	w_3	M_2
1	7	0,01403	0,0144	0,1244	0,01496	0,1285
	9	0,01389	0,0145	0,1244	0,01482	0,1286
	11	0,01383	0,0146	0,1244	0,01476	0,1286
	13	0,01379	0,0146	0,1244	0,01473	0,1286
	15	0,01377	0,0146	0,1244	0,01470	0,1286
2	7	0,01403	0,0125	0,1238	0,01496	0,1280
	9	0,01389	0,0126	0,1238	0,01482	0,1280
	11	0,01383	0,0127	0,1238	0,01476	0,1280
	13	0,01379	0,0127	0,1238	0,01472	0,1280
	15	0,01377	0,0127	0,1238	0,01470	0,1280
3	7	0,01403	0,0122	0,1237	0,01496	0,1278
	9	0,01389	0,0123	0,1237	0,01482	0,1279
	11	0,01383	0,0123	0,1237	0,01476	0,1279
	13	0,01379	0,0124	0,1237	0,01472	0,1279
	15	0,01377	0,0124	0,1237	0,01470	0,1279
4	7	0,01403	0,0121	0,1237	0,01496	0,1278
	9	0,01389	0,0122	0,1237	0,01482	0,1278
	11	0,01383	0,0122	0,1237	0,01476	0,1279
	13	0,01379	0,0122	0,1237	0,01472	0,1278
	15	0,01377	0,0123	0,1237	0,01470	0,1278
Timoshenko [33]		0,01377	0,0102	0,1235	0,01443	0,1259
Cheung [2]		0,01368	0,01236	0,1233	0,01461	0,1274
Multiplier		$q\ell^4/D$	$q\ell^2$		$q\ell^4/D$	$q\ell^2$

TABLE 7.2 Example 2. Rectangular plate. Results for unit uniform loading

* Due to symmetry, only half the structure was analysed

The results showing the rate of convergence of the method are given in Table 7.2, and are compared with those of Cheung [2] and Timoshenko [33]. A typical data element for the case of 1 strip/half structure and 8 finite difference sub-divisions together with the corresponding results is given in Appendix G for the above example.

7.1.3 Example 3. Simply supported curved plate

The simply supported isotropic curved plate shown in Figure 7.2, with an aspect ratio λ = 2 was analysed for the following loading conditions

i) Load Case 1: Unit uniform loading throughout
ii) Load Case 2: Central unit point load

The plate was sub-divided transversely into 6 finite strip elements, consisting of 4 interior strips of 2,0 m width and 2 edge strips of 1,0 m width. The material properties and dimensions were as follows

Elastic Modulus	$E_r = E_\theta = 10{,}92$
Poissons' ratio	$\mu_r = \mu_\theta = 0{,}30$
Plate thickness	$h = 1{,}0$ m
Torsional rigidity	$D_r = D_\theta = 1{,}0$
Total width	$b = 10{,}0$ m
Centre line span	$\ell = 20{,}0$ m
Centre line radius	$R = 50{,}0$ m
Subtended angle	$\theta = 0{,}4$ rad.

The results showing the rate of convergence of the method are given in Table 7.3 and 7.4 for load case 1 and 2 respectively. The distributions of longitudinal and transverse bending moments and displacements are compared with those obtained using the PAFEC finite element program and are given in Figures 7.3 and 7.4 for load case 1 and 2 respectively.

For these comparisons, the finite strip and finite element sub-divisions in the transverse direction were both as described above. In the finite strip analysis the finite difference spacing in the longitudinal direction

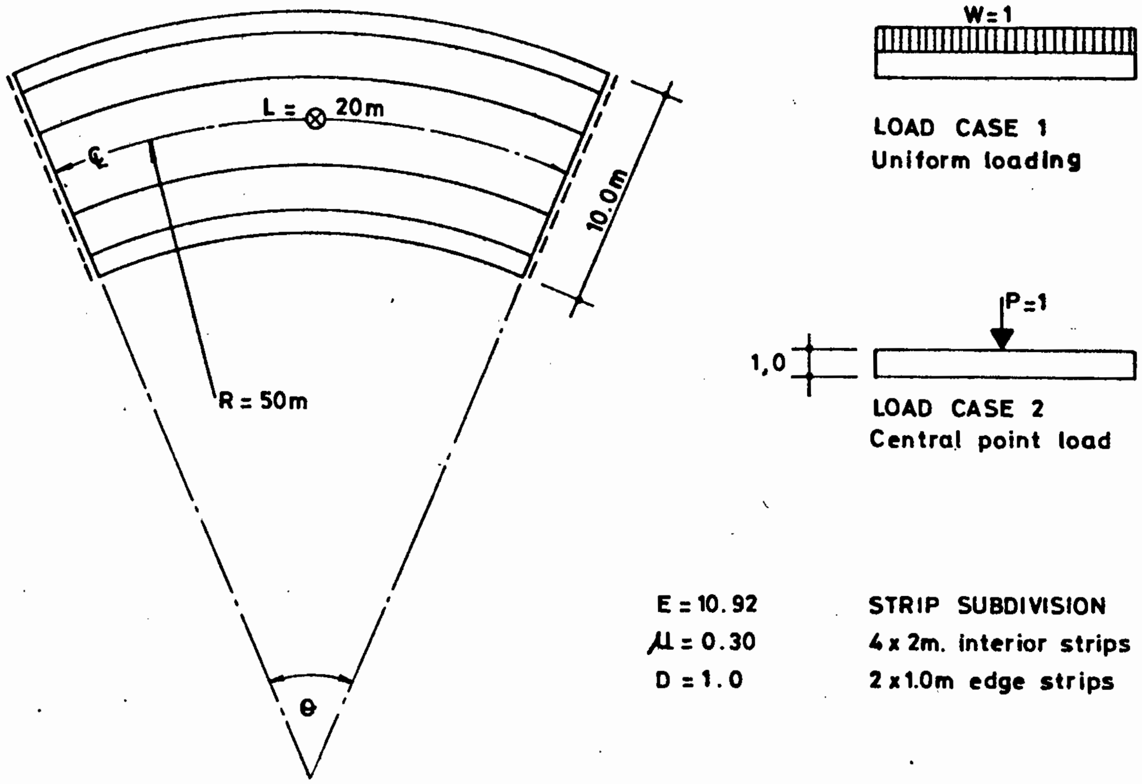


FIGURE 7.1 Example 3. Simply supported curved plate

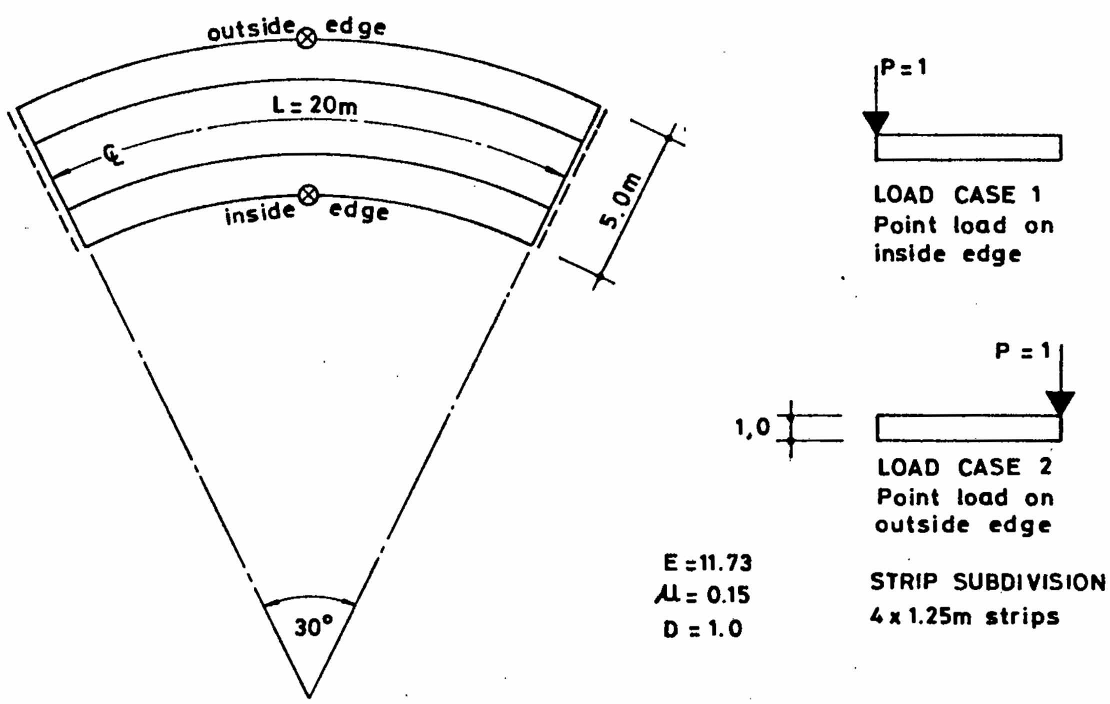


FIGURE 7.2 Example 4. Simply supported curved plate

No of strips	No of X-sect	Midspan at centre		Transverse w_3 at midspan		
		M_1	M_2	Inner edge	Centre line	Outer edge
4	9	4,709	52,08	2092	2426	3087
	11	4,723	52,07	2082	2414	3073
	13	4,731	52,06	2076	2407	3065
	15	4,736	52,06	2073	2403	3061
6	9	4,640	52,04	2091	2424	3085
	11	4,653	52,03	2081	2412	3072
	13	4,663	52,01	2075	2406	3063
	15	4,670	52,00	2071	2401	3058
8	9	4,567	52,00	2090	2423	3084
	11	4,578	51,95	2078	2410	3067
	13	4,588	51,95	2073	2404	3060
	15	4,591	51,95	2070	2400	3056
Finite element		4,97	50,90	2011	2328	2961
Multiplier		-q/D	q/d	-q/D		

TABLE 7.3 Example 3. Curved plate. Unit uniform loading throughout

No of strips	No of X-sect	Midspan at centre		Transverse w_3 at midspan		
		M_1	M_2	Inner edge	Centre line	Outer edge
4	9	26,47	61,06	1635	1953	2367
	11	27,57	62,62	1616	1932	2341
	13	28,25	63,86	1607	1920	2327
	15	28,71	64,87	1601	1913	2319
6	9	27,52	61,40	1634	1952	2366
	11	28,86	63,09	1615	1931	2341
	13	29,75	64,45	1606	1919	2326
	15	30,37	65,56	1599	1911	2317
8	9	28,98	61,86	1633	1951	2365
	11	30,77	63,71	1614	1929	2337
	13	32,07	65,26	1604	1918	2324
	15	33,05	66,56	1599	1911	2316
Finite element		39,0	73,20	1546	1845	2231
Multiplier x 10^{-2}		-P/D	P/D	-P/D		

TABLE 7.4 Example 3. Curved plate. Unit point load at midspan on centreline

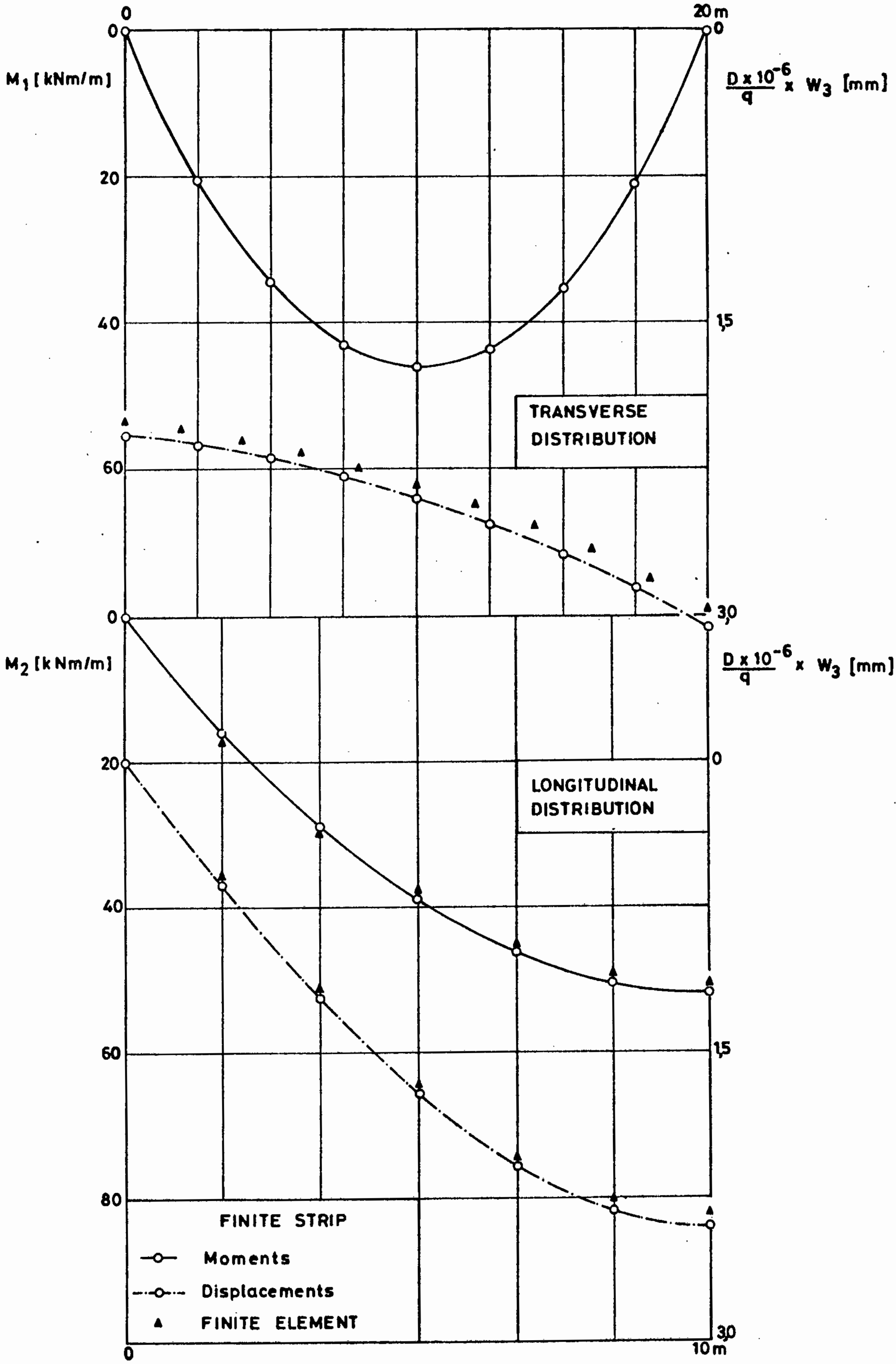


FIGURE 7.3 Example 3. Results for load case 1

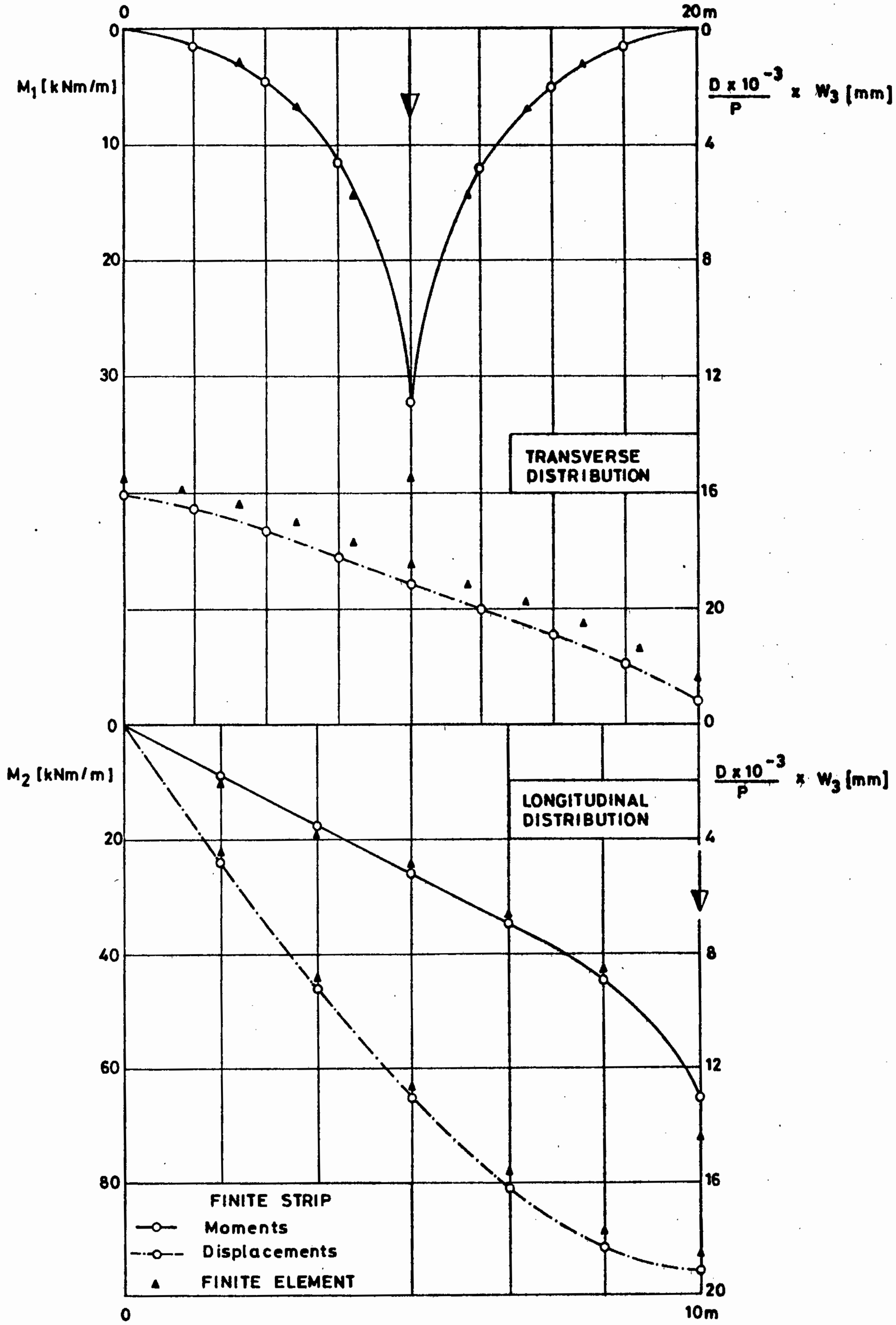
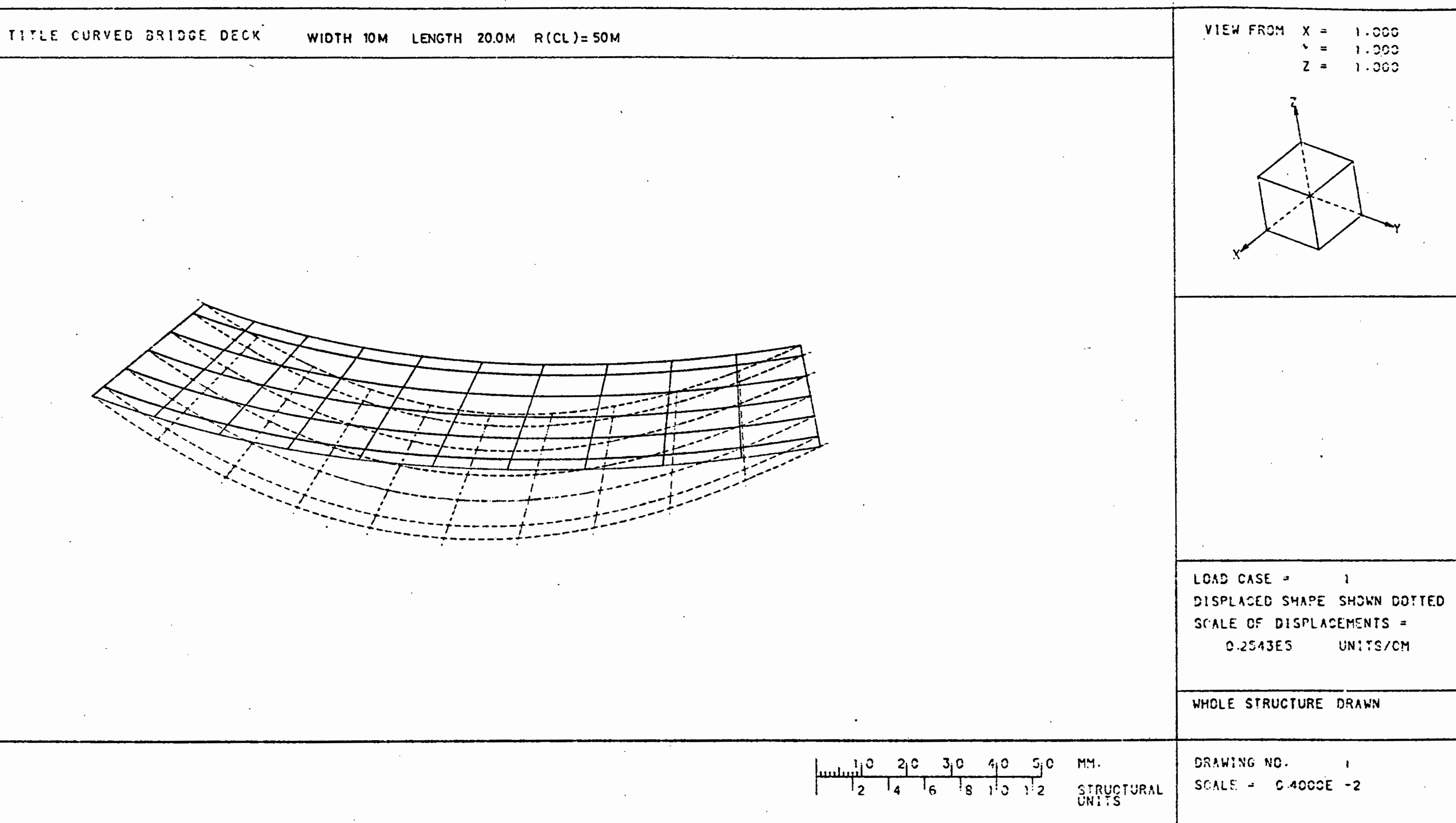


FIGURE 7.4 Example 3. Results for load case 2



<u>FIGURE 7.5</u> Exampel 3. Displaced shape of structure for load case 1

consisted of 10 equally spaced angular sub-divisions along each nodal line, giving a total of 198 degrees of freedom. Due to the superior accuracy obtained in using the thin facet shell type 8 noded iso-parametric finite element, 6 longitudinal sub-divisions were used, giving a total of 36 elements and 665 degrees of freedom.

The time, cost and pages involved in the two analyses for 4 load cases were as follows

	Finite Strip	Finite Element
CPU time	5,37 secs	3 min 8,67 secs
Total cost	R 2,14	R 90,30
Number of pages	18	110

In the case of the PAFEC finite element analysis, the above figures included a three dimensional plot of the displaced shape of the structure for load case 1, which is shown in Figure 7.5.

7.1.4 Example 4. Simply supported curved plate

The simply supported isotropic curved plate shown in Figure 7.2, with an aspect ratio λ = 4 was analysed for the following loading conditions

i) Load Case 1: Unit point load at midspan on inner edge
ii) Load Case 2: Unit point load at midspan on outer edge

The plate was sub-divided transversely into 4 equally spaced finite strips of 1,25 m width. The material properties and dimensions of the plate were as follows

Elastic Modulus	$E_r = E_\theta = 11{,}73$
Poissons' ratio	$\mu_r = \mu_\theta = 0{,}15$
Plate thickness	$h = 1{,}0$ m
Torsional rigidity	$D_r = D_\theta = 1{,}0$
Centre line span	$\ell = 20{,}0$ m
Total width	$b = 5{,}0$ m
Subtended angle	$\theta = 30°$
Centre line radius	$R = 38{,}2$ m

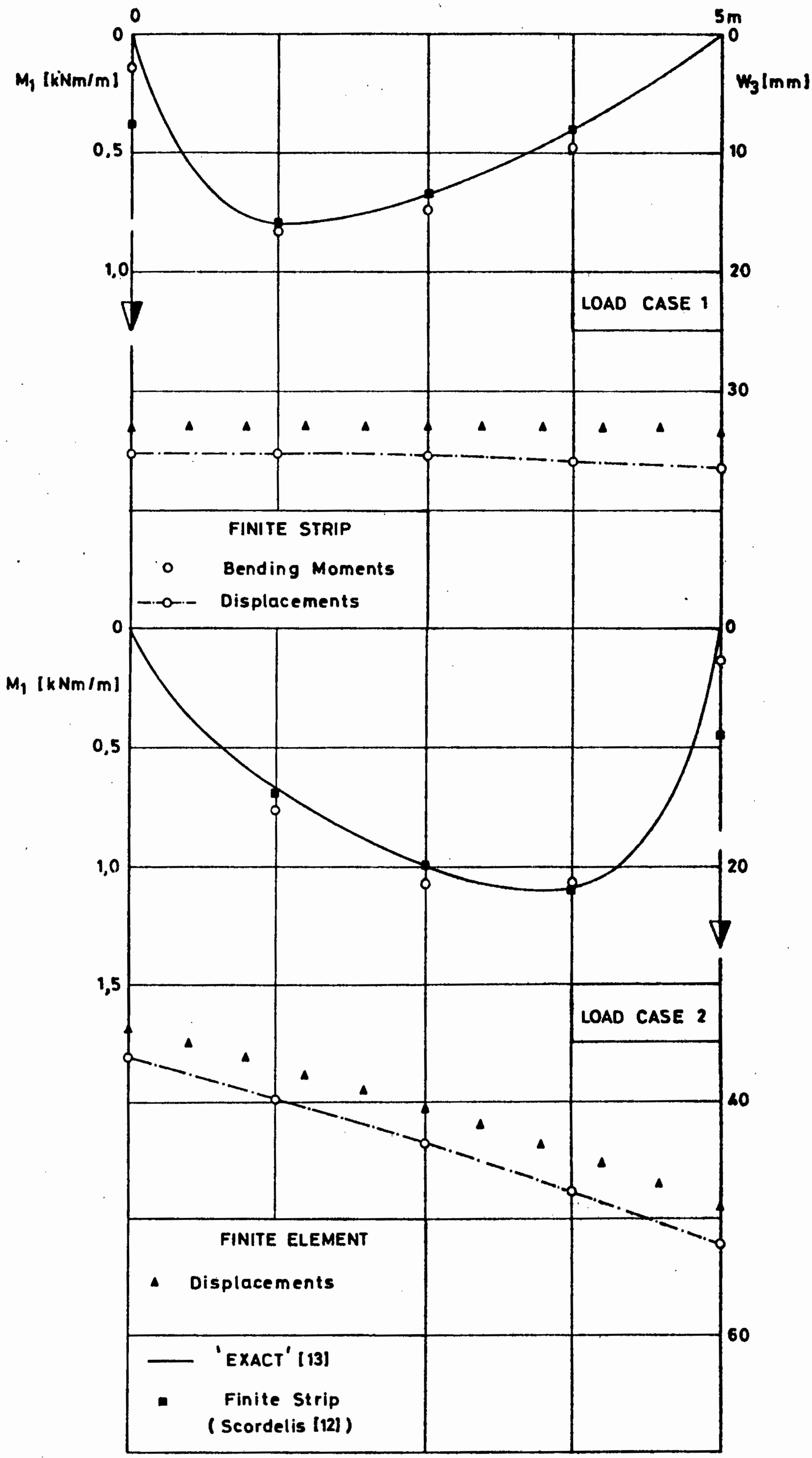


FIGURE 7.6 Example 4. Transverse moments and displacements

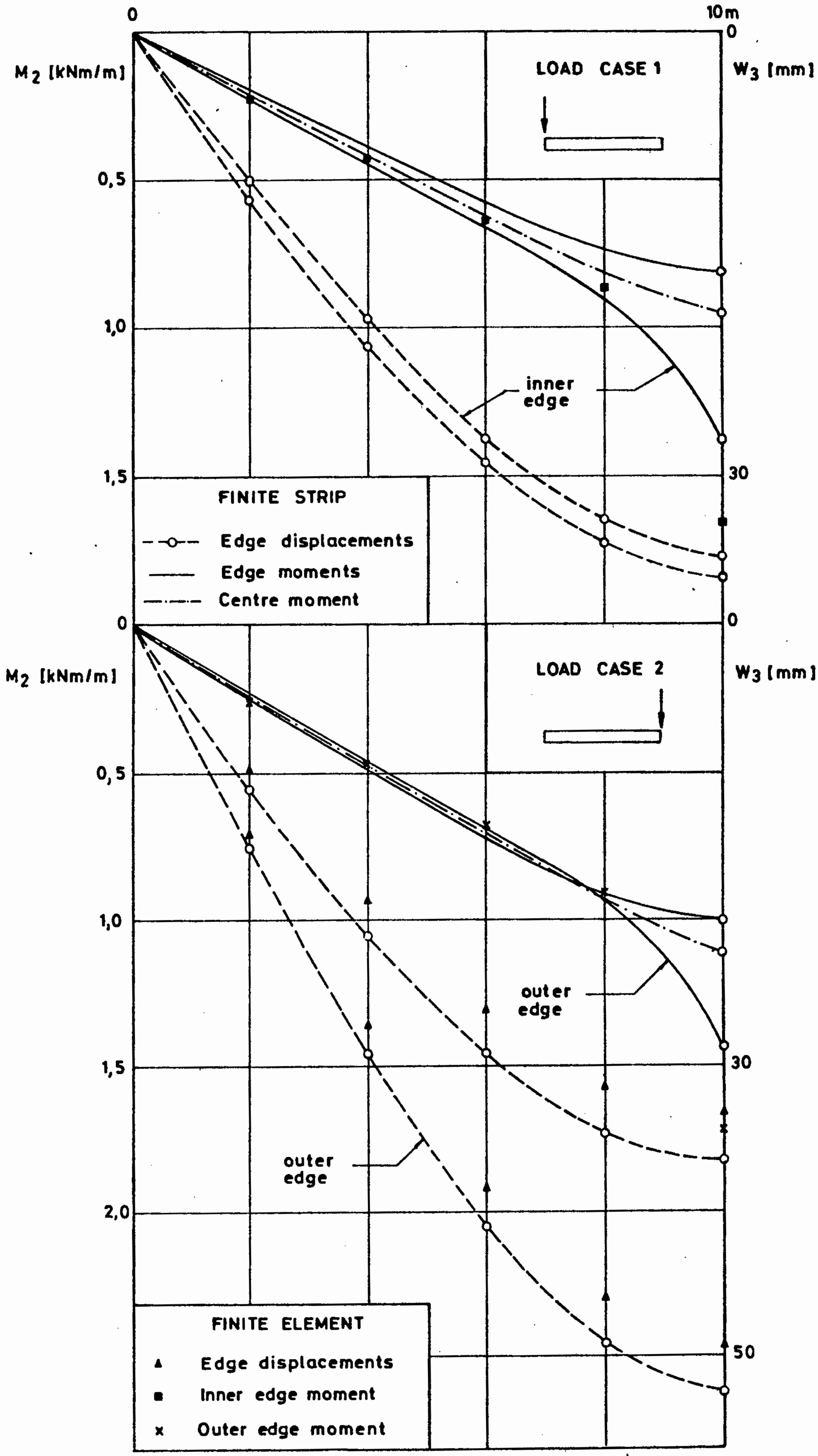


FIGURE 7.7 Example 4. Longitudinal moments and displacements

The transverse bending moment and displacement distributions at midspan are given in Figure 7.6 for the two load cases. The bending moment distributions are compared with those obtained using the harmonic finite strip analysis Scordelis [12] and with the closed form solution of the plate equation [13]. The transverse displacements are compared with those of the finite element analysis. The distribution of longitudinal bending moments and displacements are given in Figure 7.7 for each load case and are also compared with the results obtained using the finite element analysis.

The transverse subdivision of the plate for the harmonic finite strip analysis and the finite element analysis were both as described above. In the longitudinal direction the finite difference and finite element sub-divisions were the same as those in the previous example.

7.1.5 Example 5. Continuous curved bridge deck

Details of the three span continuous isotropic curved bridge deck are given in Figure 7.8. The structure was analysed for the following loading conditions

i) Load Case 1: Dead load only
ii) Load Case 2: Patch load on inside carriageway, Area A
iii) Load Case 3: Patch load on central 3 m width, Area B
iv) Load Case 4: Patch load on outside carriageway, Area C

The plate was sub-divided transversely into 8 finite strip elements, consisting of 6 interior strips of 1,5 m width and two ege strips of 0,5 m width. The material properties and dimensions were as follows

Elastic Modulus	$E_r = E_\theta = 25$ GPa
Poissons' ratio	$\mu_r = \mu_\theta = 0{,}30$
Material density	$\rho = 25$ kN/m
Plate thickness	$h = 1{,}0$ m
End spans	$\ell = 20{,}0$ m
Central span	$\ell = 32{,}0$ m
Total width	$b = 10{,}0$ m
Central radius	$R = 100{,}0$ m

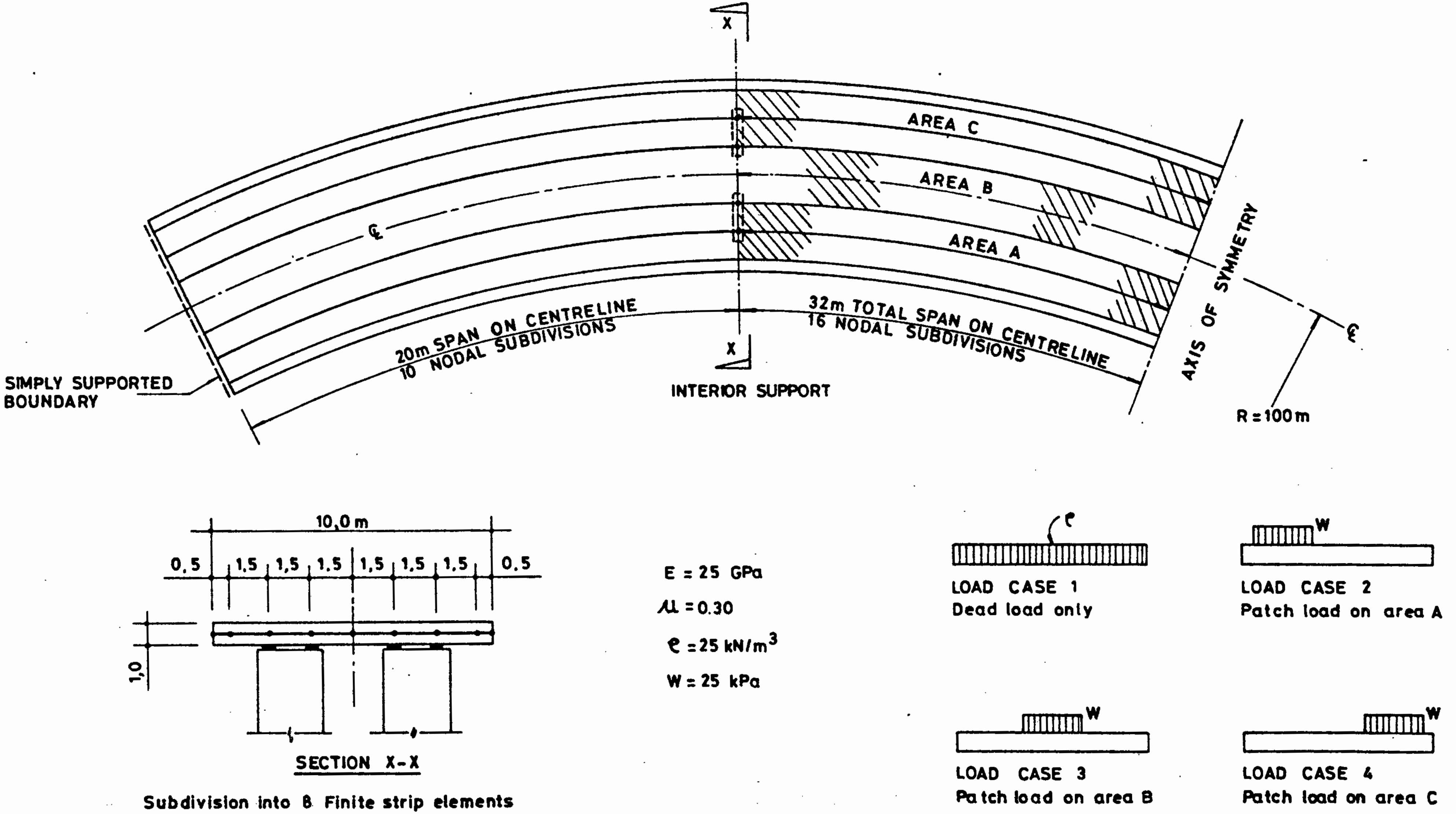


FIGURE 7.8 Example 5. Details of three span continuous isotropic curved bridge deck

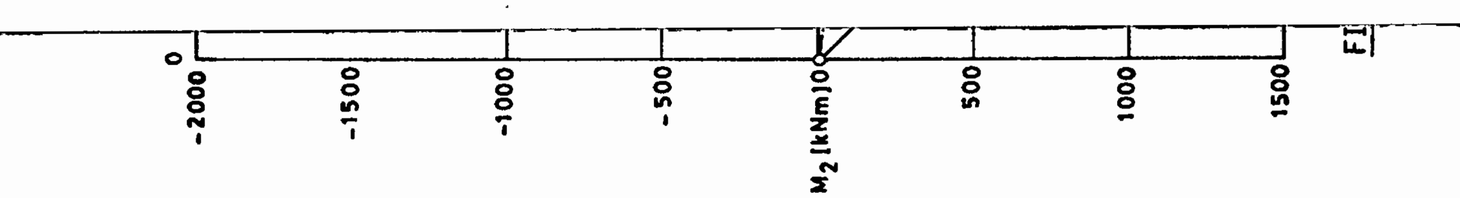
The longitudinal bending moment and displacement distribtuions along the deck centre line for load case 1 are given in Figure 7.9. The results for a constant nodal spacing are compared with those where the spacing has been concentrated at the boundaries and over the interior supports. These distributions are further compared with the results obtained using the finite element analysis. The transverse distribution of bending moments and displacements across the middle of the central span are given in Figure 7.10 for the remaining 3 load cases.

For comparison purposes, the sub-division of the structure in the transverse direction is the same in both analyses. In the finite strip analysis, the finite difference spacing in the longitudinal direction consisted of 10 angular sub-divisions for each of the two end spans and 16 sub-divisions for the central span, giving a total of 666 degrees of freedom for the whole structure. Values of the variable angular nodal spacing used may be found in Appendix G, where the data element and typical results are given for this problem. In the finite element analysis, 5 longitudinal sub-divisions were used for the end spans and 8 sub-divisions for the central span. However, due to the longitudinal symmetry of the loading and geometry, it was only necessary to specify half the structure, giving a total number of 72 elements and 1255 degrees of freedom.

The time and cost involved in the two analyses for the 4 load cases were as follows

	Finite Strip	Finite Element
CPU time	19,36 secs	7 min 7,77 secs
Total cost	R5,13	R144,77

In the finite strip analysis the figures given above are for the analysis of the WHOLE structure (ref. Appendix G - Coding of input). In the case of the finite element analysis, the above figures also included various three dimensional plots of the displaced shape of the structure. A plot of the displacements for load case 1 is shown in Figure 7.11.



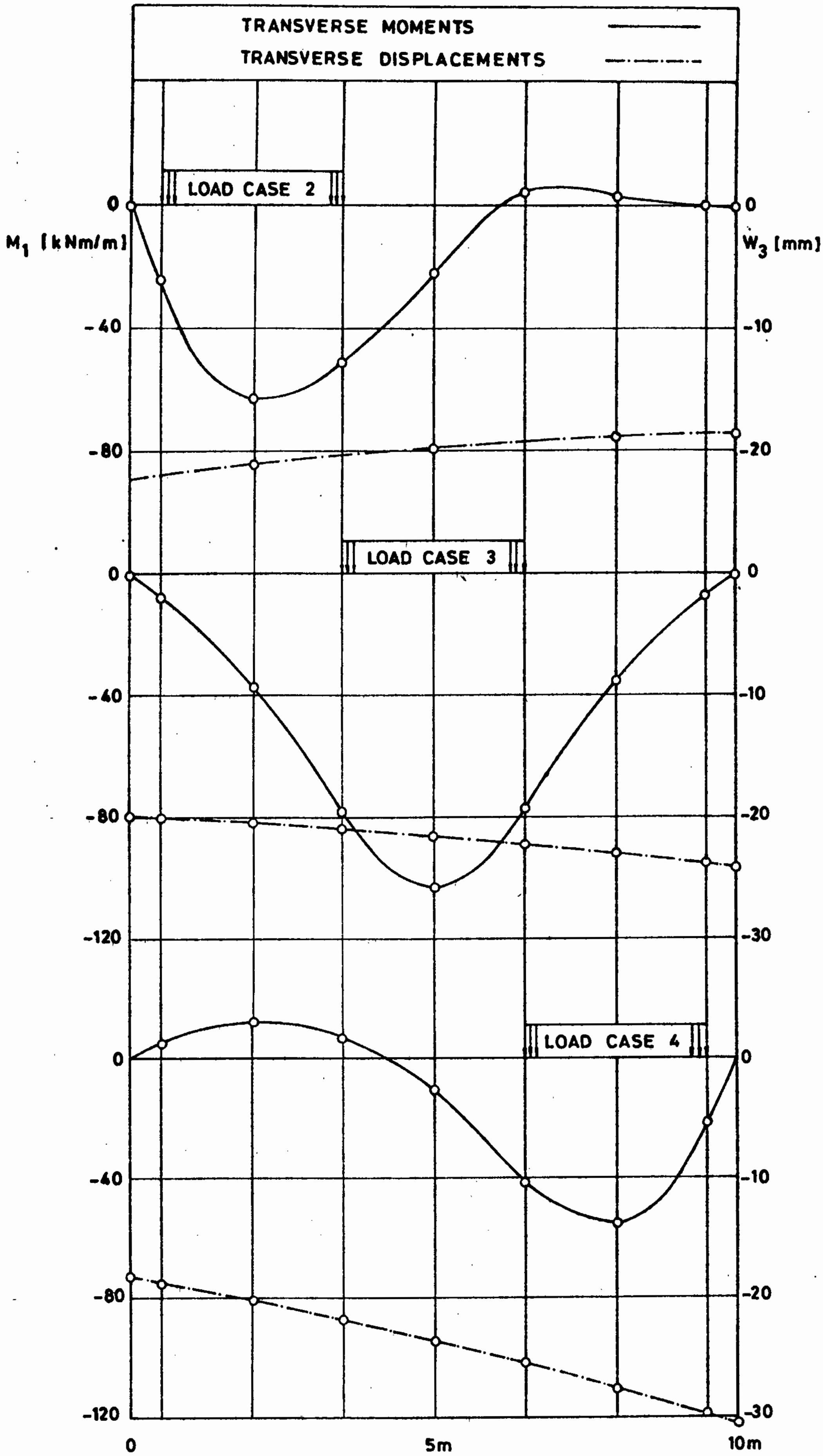


FIGURE 7.10 Example 5. Transverse bending moments and displacements at midspan

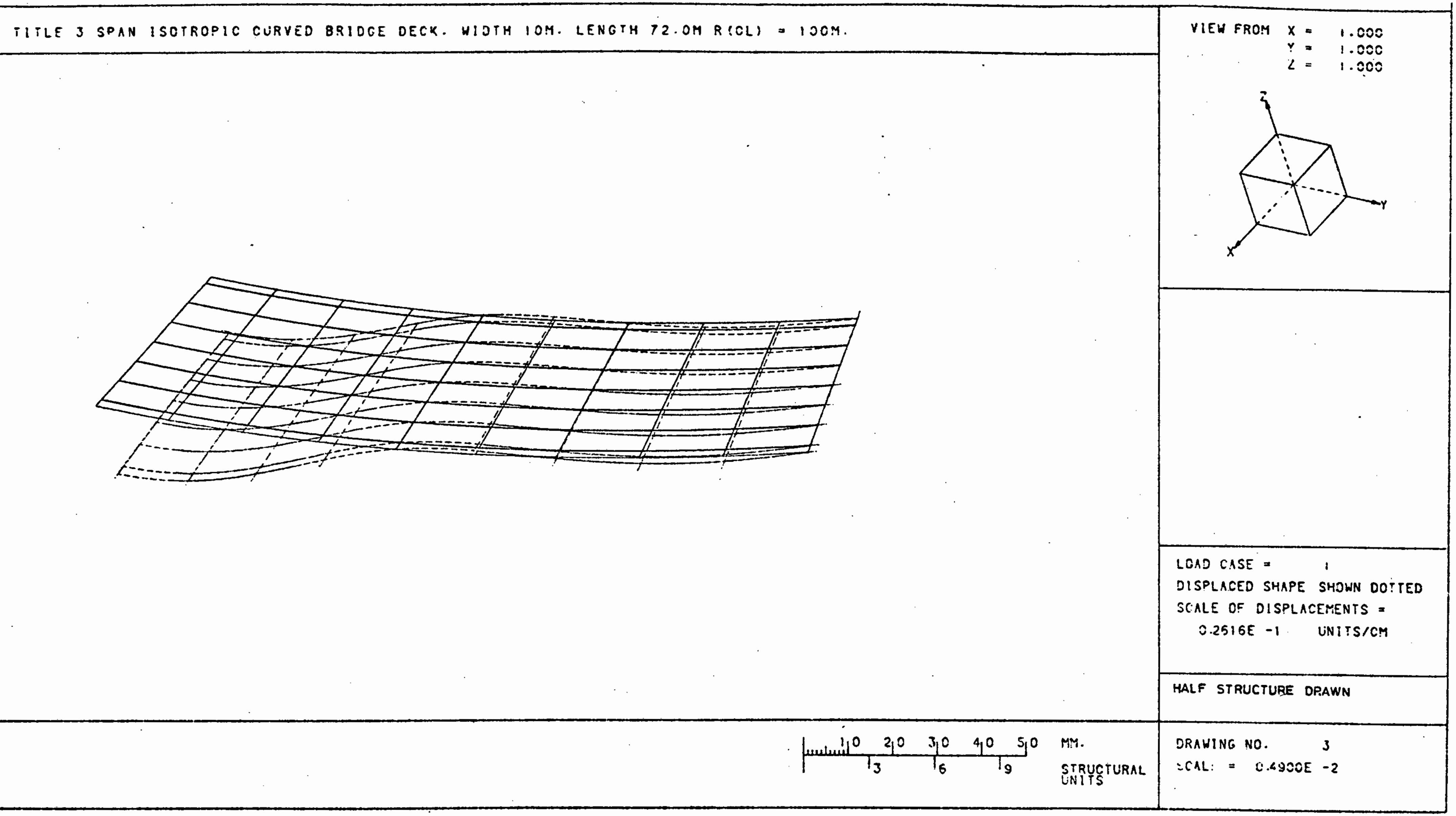


FIGURE 7.11 Example 5. Displaced shape of half structure for load case 1

7.2 Discussion of Results

From the convergence studies it can be seen that in all cases of uniform loading the number of strips used had little effect on the results obtained for a given number of nodal cross-sections. In the point load case however, improved results were obtained by concentrating the spacing of the finite strip elements in the vicinity of the point load. This also had the effect of reducing the lack of curvature compatibility between adjacent strips, particularly at the point of application of the load.

In all cases of loading the number of nodal cross-sections used in the finite difference sub-division had a greater effect on improving the results than did the number of finite strip elements used. This trend was particularly marked in the values of the displacements obtained, which were typically in excess of the correct values.

For a given number of strips and nodal cross-sections, it can be seen that greater accuracy was obtained in all cases of uniform loading compared to the point load cases. This is due to the discontinuity in the derivatives associated with a concentrated loading compared to the continuous nature of the derivatives for a uniform loading. The set of governing differential equations are therefore to some extent misrepresented in the case of a point load and well represented for a uniform loading.

It was found of interest that for the simply supported square and rectangular plates subject to a uniform loading, the maximum displacement did not occur at midspan on the centreline but on the outer edges. This is considered to be due to the increased transverse flexibility of the plate along the free edges and the effect of anti-clastic curvature.

From the results of simply supported and continuous plate structures analysed it can be seen how the outside edge of the curved plate becomes more flexible and the interior edge becomes stiffer as the horizontal curvature increases. This effect occurs due to the increased span at the outer edge and the correspondingly decreased span at the interior edge for a given central arc length. For the simply supported curved plate the effect is well illustrated by the transverse displacement distributions at midspan particularly in the point load cases. It was found interesting that for the point load at midspan on the interior edge, the minimum transverse displacement occurred at the point of application of the load. The effect is also seen in the longitudinal bending moments at midspan on the edges, where the moment at the outer edge for the point load applied on this edge is greater than the moment at the interior edge for the load on this edge.

From the results of the 3 span continuous plate example it was found that by concentrating the spacing at the boundaries and over the interior supports, improved results were obtained in the areas of discontinuous curvature for the same total number of nodal points used. The example also served to illustrate the advantage of using a finite difference subdivision in the longitudinal direction, where continuity over the set of interior supports was included without any difficulty.

Although it is not reasonable to make a direct comparison between the CPU time and total costs for the finite element and finite strip analyses, the relative amounts indicate that considerable advantage may be gained in using the finite strip method presented. By referring the Coding of input and the data elements as given in Appendix G, it can be seen that the specification of the structure and loading require a minimum of data preparation. In addition, it is shown in example 3 that the volume of results generated is greatly reduced.

CHAPTER 8

CONCLUSIONS

8.1 General

By applying variational methods to the principle of minimum potential energy, the governing differential equation for a finite strip was found and the resulting system of simultaneous differential equations was solved using finite difference techniques. By adopting a formulation in terms of orthogonal curvilinear coordinates, the finite strip method was successfully applied to the analysis of plate structures of an arbitrary curved geometry.

The method of solution chosen was that of replacing the partial derivatives of the displacement variables in the governing differential equation by their finite difference equivalent and solving the resulting set of simultaneous linear equations. It was found that by using global coordinate variables, the method of finite differences could be readily applied in the present finite strip analysis formulated in terms of orthogonal coordinate systems.

One of the limitations of the harmonic finite strip method is that the cross-section of the structure is required to be prismatic. By formulating the finite strip method in terms of curvilinear coordinates, it is evident that structures of a variable cross-section in the longitudinal and the transverse directions may now be analysed.

By applying the finite difference form of the governing differential equation at all node points throughout the structure, a set of ficticious displacements was generated at the ends of the strips. The application of the finite difference form of the boundary differential equations at the end and pen-ultimate nodes enabled these displacements to be uniquely eliminated. By following this procedure it was possible to formulate the resulting finite difference stiffness matrix for the structure for all the boundary conditions described.

A further restriction in the harmonic finite strip method is the limited type of boundary condition that may be considered at the ends of the strip and the severe limitation on the number of intermediate supports that can be handled. In the present formulation of the finite strip method, in addition to the four different types of boundary conditions that may be considered, the use of finite differences has made possible the incorporation of any number of interior node supports, thereby enabling structures of a continuous nature to be solved. Since the free edge boundary condition may now be considered at the ends of the strips, the analysis of arbitrary curved continuous plate structures constructed in a stage sequence has also become possible.

From the results of the curved plate structures analysed, it was found that excellent agreement was obtained between the finite strip method presented and the finite element and other alternative analyses. Since the choice of the method of analysis used for a structural problem normally depends on the size of the structure and the total cost involved, the finite strip method of analysis is considered to offer clear advantages for the range of structures that may be analysed.

8.2 Further Work

The formulation of the finite strip method for orthogonal curvilinear systems has enabled a variety of arbitrary shaped curved plate structures to be analysed. Structures such as skew plates and those having a combined curved and skew form are by definition excluded. The possibility of extending the method to general non-orthogonal curvilinear coordinate systems should therefore be considered, since this formulation would place no restriction on the geometry of the structure that may be analysed.

In the finite strip method presented, only the strain energy due to bending forces was considered, thereby limiting the analysis to curved plate structures. By including the strain energy contributions due to in-plane forces, the analysis may therefore be extended to include curved folded plate and shell structures.

The application of the finite difference method required the structure to be subdivided longitudinally into a number of equally or variably spaced nodal cross-sections. By concentrating the spacing of the nodes in the boundary regions and in areas where the derivatives were discontinuous, it was shown that improved results were obtained. An investigation into the optimisation of the finite difference spacing chosen should therefore prove worthwhile.

The solution of the system of differential equations has been based on the finite difference method. Although it would appear that this method holds definite advantages, consideration should nevertheless be given to alternative methods of solution.

BIBLIOGRAPHY

1. CHEUNG, Y.K. — The Finite Strip Method in the Analysis of Elastic Plates with Two Opposite Simply Supported Edges. ICE Proc., Vol 40, May 1968, 1 - 7.

2. CHEUNG, Y.K. — Orthotropic Right Bridges by the Finite Strip Method. Second International Symposium on Concrete Bridge Design. ACI Publication SP 26, Paper SP 26-8, 182 - 205.

3. CHEUNG, Y.K. — Finite Strip Analysis of Elastic Slabs. ASCE., Eng. Mech. Division, Vol 94, December 1968, 1365 - 1379.

4. CHEUNG, M.S.; CHEUNG, Y.K. — Static and Dynamic Behaviour of Rectangular Plates Using Higher Order Finite Strips. Build. Sci., Vol 7, Sept 1972, 151 - 158.

5. GHALI, A.; TADROS, G.S. — Convergence of Semi-analytic Solution of Plates. Publ IABSE, 34-I, 1974, 25 - 33.

6. BROWN, T.G. — Finite Strip Analysis of Skew Structures. Ph.D. Thesis, University of Calgory, Alberta, Canada, 1974.

7. BROWN, T.G.; GHALI, A. — Semi-analytic Solution of Skew Plates in Bending. ICE Proc., Vol 57, Part 2, March 1974, 165 - 175.

   LOO, Y.C.; CUSENS, A.R. — Discussion of above paper, ICE Proc., Vol 57, Part 2, Sept 1974, 567 - 569.

8. BROWN, T.G.; GHALI, A. — Semi-analytic Solution of Skew Box Girder Bridges. ICE Proc., Vol 59, Part 2, Sept 1975, 487 - 500.

9. MUKHOPADHYAY, M. — Finite Strip Method of Analysis of Clamped Skew Plates in Bending. ICE Proc., Vol 61, Part 2, March 1976, 189 - 195.

10. CHEUNG, Y.K. — The Analysis of Cylindrical Orthotrophic Curved Bridge Decks. IABSE Publications, Vol 29 II, 1969, 41 - 52.

11. CHEUNG, M.S.; CHEUNG, Y.K. — Analysis of Curved Box Girder Bridges by Finite Strip Method. IABSE Publications, Vol 31 I, 1971, 1 - 19.

12. SCORDELIS, A.C.
MEYER, C.A.
Analysis of Curved Folded Plate Structures. Report No UC SESM 70-8, University of California, Berkely, 1970.

13. MEYER, C.A.
Analysis and Design of Curved Box Girder Bridges. Ph.D. Thesis, University of California, Berkely, 1970. (awaiting microfiche ordered through University Microfilms International in July 1979).

14. LOO,Y.C.
CUSENS, A.R.
The Finite Strip Method in Bridge Engineering. Viewpoint Publications, 1978.

15. DU PREEZ, R.J.
SAICE, CSIR Symposium on Bridge Deck Analysis. Kyalami Ranch, 1973, paper nos 2, 4, 14.

16. SPARKS, A.D.W.
SAICE Convention on Innovation in Civil Engineering. University of Stellenbosch, 1976, 69 - 77.

17. FUNG, Y.C.
Foundation of Solid Mechanics. Prentice Hall, 1965.

18. FUNG, Y.C.
A First Course in Continuum Mechanics. Prentice Hall, 1969.

19. VINSON, J.R.
Structural Mechanics: The Behaviour of Plates and Shells. John Wiley, 1974.

20. BORG, S.F.
Matrix-tensor Methods in Continuum Mechanics. Van Nostrand, 1963.

21. MASE G.E.
Theory and Problems of Continuum Mechanics. McGraw-Hill, 1970.

22. BOAZ, M.L.
Mathematical Methods in the Physical Sciences. John Wiley, 1966.

23. WASHIZU, K.
Variational Methods in Elasticity and Plasticity. Pergamon Press Ltd., 2nd Edition, 1974.

24. LEIPHOLZ, H.
Theory of Elasticity. Noordhoff International Publishing, 1974.

25. FLÜGGE, W. — Tensor Analysis and Continuum Mechanics. Springer-Verlag, 1972.

26. MILLER, K.S. — A Short Course in Vector Analysis. C.E. Merrill Books Inc., 1962.

27. GREEN, A.E., ZERNA, W. — Theoretical Elasticity. Oxford Press, 2nd Edition, 1968.

28. MURNAGHAN, F.D. — Introduction to Applied Mathematics. John Wiley, 1948.

29. SPIEGAL, M.R. — Vector Analysis and an Introduction to Tensor Analysis. Shaum Publishing Co, 1959.

30. SZILARD, R. — Theory and Analysis of Plates - Classical and Numerical Methods. Civil Eng. and Eng. Mechanics series, Prentice Hall, 1974.

31. HODGE, P.G. — Continuum Mechanics: An Introductory Text for Engineers. McGraw Hill, 1970.

32. SALVADORI, M.G., BARON, M.L. — Numerical Methods in Engineering. Prentice Hall, 1952.

33. TIMOSHENKO, S.P., WOINOWSKY-KRIEGER, S. — Theory of Plates and Shells. McGraw-Hill, 2nd Edition, 1959.

34. GHALI, A., NEVILLE, A.M. — Structural Analysis - A Unified Classical and Matrix Approach. Intext Educational Publications, 1972.

35. VAN H. LOUW, J.G. — Finite Strip Analysis of Flat Plates. M.Sc. (Eng) Thesis, University of Cape Town, 1975.

36. BARKER, R. — A Finite Difference Based Finite Strip Method for the Analysis of Translational Shell Structures. M.Sc.(Eng) Thesis, University of Cape Town, 1976.

37. DE KOCK, M.O. — Skeletal Structures. Postgraduate Course given at the Department of Civil Engineering, UCT, 1975.

38. DOYLE, W.S. Surface Structures. Postgraduate course given at the Department of Civil Engineering, University of Cape Town, 1976.

39. DOWNIE, R.M. A Report on the Use of Variable Spacing Finite Difference Approximations in the Analysis of Simple and Continuous Beam Problems. May 1979, p 1 - 209. Available in the Civil Engineering Department, University of Cape Town.

APPENDIX A

Curvature Matrices

$$
\{\chi\} = \left\{ \begin{array}{c} \chi_1 \\[2ex] \chi_2 \\[2ex] (\chi_{12} - \chi_{21}) \end{array} \right\} = \left\{ \begin{array}{l}
-\dfrac{1}{h_1^2}\dfrac{\partial^2 w}{\partial y^1 \partial y^1} + \dfrac{1}{h_1^3}\dfrac{\partial h_1}{\partial y^1}\dfrac{\partial w}{\partial y^1} - \dfrac{1}{h_1 h_2^2}\dfrac{\partial h_1}{\partial y^2}\dfrac{\partial w}{\partial y^2} \\[2ex]
\dfrac{1}{h_2^2}\dfrac{\partial^2 w}{\partial y^2 \partial y^2} - \dfrac{1}{h_2^3}\dfrac{\partial h_2}{\partial y^2}\dfrac{\partial w}{\partial y^2} + \dfrac{1}{h_1^2 h_2}\dfrac{\partial h_2}{\partial y^1}\dfrac{\partial w}{\partial y^1} \\[2ex]
\dfrac{2}{h_1 h_2}\dfrac{\partial^2 w}{\partial y^1 \partial y^2} - \dfrac{2}{h_1 h_2^2}\dfrac{\partial h_2}{\partial y^1}\dfrac{\partial w}{\partial y^2} - \dfrac{2}{h_1^2 h_2}\dfrac{\partial h_1}{\partial y^2}\dfrac{\partial w}{\partial y^1}
\end{array} \right\}
$$

$$
= \left\{ \begin{array}{l}
\dfrac{-1}{h_1^2} \left\{ \left(-\dfrac{6}{b^2} + \dfrac{12a}{b^3} \right) w_i \left(\dfrac{4}{b} - \dfrac{6a}{b^2} \right) \theta_i \left(\dfrac{6}{b^2} - \dfrac{12a}{b^3} \right) w_j \left(\dfrac{2}{b} - \dfrac{6a}{b^2} \right) \theta_j \right\} \\[3ex]
\dfrac{+1}{h_1^3}\dfrac{\partial h_1}{\partial y^1} \left\{ \left(-\dfrac{6a}{b^2} + \dfrac{6a^2}{b^3} \right) w_i \left(-1 + \dfrac{4a}{b} - \dfrac{3a^2}{b^2} \right) \theta_i \left(\dfrac{6a}{b^2} - \dfrac{6a^2}{b^3} \right) w_j \left(\dfrac{2a}{b} - \dfrac{3a^2}{b^2} \right) \theta_j \right\} \\[3ex]
\dfrac{-1}{h_1 h_2^2}\dfrac{\partial h_1}{\partial y^2} \left\{ \left(1 - \dfrac{3a^2}{b^2} + \dfrac{2a^3}{b^3} \right) w_i' \left(-a + \dfrac{2a^2}{b} - \dfrac{a^3}{b^2} \right) \theta_i' \left(\dfrac{3a^2}{b^2} - \dfrac{2a^3}{b^3} \right) w_j' \left(\dfrac{a^2}{b} - \dfrac{a^3}{b^2} \right) \theta_j' \right\} \\[4ex]
\dfrac{1}{h_2^2} \left\{ \left(1 - \dfrac{3a^2}{b^2} + \dfrac{2a^3}{b^3} \right) w_i'' \left(-a + \dfrac{2a^2}{b} - \dfrac{a^3}{b^2} \right) \theta_i'' \left(\dfrac{3a^2}{b^2} - \dfrac{2a^3}{b^3} \right) w_j'' \left(\dfrac{a^2}{b} - \dfrac{a^2}{b^2} \right) \theta_j'' \right\} \\[3ex]
\dfrac{-1}{h_2^3}\dfrac{\partial h_2}{\partial y^2} \left\{ \left(1 - \dfrac{3a^2}{b^2} + \dfrac{2a^3}{b^3} \right) w_i' \left(-a + \dfrac{2a^2}{b} - \dfrac{a^3}{b^2} \right) \theta_i' \left(\dfrac{3a^2}{b^2} - \dfrac{2a^3}{b^3} \right) w_j' \left(\dfrac{a^2}{b} - \dfrac{a^3}{b^2} \right) \theta_j' \right\} \\[3ex]
\dfrac{+1}{h_1^2 h_2}\dfrac{\partial h_2}{\partial y^1} \left\{ \left(-\dfrac{6a}{b^2} + \dfrac{6a^2}{b^3} \right) w_i \left(-1 + \dfrac{4a}{b} - \dfrac{3a^2}{b^2} \right) \theta_i \left(\dfrac{6a}{b^2} - \dfrac{6a^2}{b^3} \right) w_j \left(\dfrac{2a}{b} - \dfrac{3a^2}{b^2} \right) \theta_j \right\} \\[4ex]
\dfrac{2}{h_1 h_2} \left\{ \left(-\dfrac{6a}{b^2} + \dfrac{6a^2}{b^3} \right) w_i' \left(-1 + \dfrac{4a}{b} - \dfrac{3a^2}{b^2} \right) \theta_i' \left(\dfrac{6a}{b^2} - \dfrac{6a^2}{b^3} \right) w_j' \left(\dfrac{2a}{b} - \dfrac{3a^2}{b^2} \right) \theta_j' \right\} \\[3ex]
\dfrac{-2}{h_1 h_2^2}\dfrac{\partial h_2}{\partial y^1} \left\{ \left(1 - \dfrac{3a^2}{b^2} + \dfrac{2a^3}{b^3} \right) w_i' \left(-a + \dfrac{2a^2}{b} - \dfrac{a^3}{b^2} \right) \theta_i' \left(\dfrac{3a^2}{b^2} - \dfrac{2a^3}{b^3} \right) w_j' \left(\dfrac{a^2}{b} - \dfrac{a^3}{b^2} \right) \theta_j' \right\} \\[3ex]
\dfrac{-2}{h_1^2 h_2}\dfrac{\partial h_1}{\partial y^2} \left\{ \left(-\dfrac{6a}{b^2} + \dfrac{6a^2}{b^3} \right) w_i \left(-1 + \dfrac{4a}{b} - \dfrac{3a^2}{b^2} \right) \theta_i \left(\dfrac{6a}{b^2} - \dfrac{6a^2}{b^3} \right) w_j \left(\dfrac{2a}{b} - \dfrac{3a^2}{b^2} \right) \theta_j \right\}
\end{array} \right\}
$$

APPENDIX A1 Curvature matrix in orthogonal curvilinear coordinates

$$\{\chi\} = \begin{Bmatrix} \chi_r \\ \\ \chi_\theta \\ \\ (\chi_{r\theta} - \chi_{\theta r}) \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial r^2} \\ \\ \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r} \\ \\ \dfrac{2}{r}\dfrac{\partial^2 w}{\partial r \partial \theta} - \dfrac{2}{r^2}\dfrac{\partial w}{\partial \theta} \end{Bmatrix}$$

$$= \begin{Bmatrix} \left(\dfrac{6}{b^2} - \dfrac{12a}{b^3}\right) w_i \quad \left(\dfrac{-4}{b} + \dfrac{6a}{b^2}\right)\theta_i \quad \left(\dfrac{-6}{b^2} + \dfrac{12a}{b^3}\right) w_j \quad \left(\dfrac{-2}{b} + \dfrac{6a}{b^2}\right)\theta_j \\ \\ \dfrac{1}{r^2}\left(1 - \dfrac{3a^2}{b^2} + \dfrac{2a^3}{b^3}\right) w_i'' \quad \dfrac{1}{r^2}\left(-a + \dfrac{2a^2}{b} - \dfrac{a^3}{b^2}\right)\theta_i'' \quad \dfrac{1}{r^2}\left(\dfrac{3a^2}{b^2} - \dfrac{2a^3}{b^3}\right) w_j'' \quad \dfrac{1}{r^2}\left(\dfrac{a^2}{b} - \dfrac{a^3}{b^2}\right)\theta_j'' \\ \dfrac{1}{r}\left(\dfrac{-6a}{b^2} + \dfrac{6a^2}{b^3}\right) w_i \quad \dfrac{1}{r}\left(-1 + \dfrac{4a}{b} - \dfrac{3a^2}{b^2}\right)\theta_i \quad \dfrac{1}{r}\left(\dfrac{6a}{b^2} - \dfrac{6a^2}{b^3}\right) w_j \quad \dfrac{1}{r}\left(\dfrac{2a}{b} - \dfrac{3a^2}{b^2}\right)\theta_j \\ \\ \dfrac{2}{r}\left(\dfrac{-6a}{b^2} + \dfrac{6a^2}{b^3}\right) w_i' \quad \dfrac{2}{r}\left(-1 + \dfrac{4a}{b} - \dfrac{3a^2}{b^2}\right)\theta_i' \quad \dfrac{2}{r}\left(\dfrac{6a}{b^2} - \dfrac{6a^2}{b^3}\right) w_j' \quad \dfrac{2}{r}\left(\dfrac{2a}{b} - \dfrac{3a^2}{b^2}\right)\theta_j' \\ \dfrac{-2}{r^2}\left(1 - \dfrac{3a^2}{b^2} + \dfrac{2a^3}{b^3}\right) w_i' \quad \dfrac{-2}{r^2}\left(-a + \dfrac{2a^2}{b} - \dfrac{a^3}{b^2}\right)\theta_i' \quad \dfrac{-2}{r^2}\left(\dfrac{3a^2}{b^2} - \dfrac{2a^3}{b^3}\right) w_j' \quad \dfrac{-2}{r^2}\left(\dfrac{a^2}{b} - \dfrac{a^3}{b^2}\right)\theta_j' \end{Bmatrix}$$

APPENDIX A2 Curvature matrix in polar coordinates

APPENDIX B

Integration Formulae

$$\int_{r_i}^{r_j} rdr = br_i + \frac{b^2}{2} = br_j - \frac{b^2}{2} \qquad \text{B1.1}$$

$$\int_{r_i}^{r_j} ardr = \frac{b^2}{2} r_i + \frac{b^3}{3} = \frac{b^2}{2} r_j - \frac{b^3}{6} \qquad \text{B1.2}$$

$$\int_{r_i}^{r_j} a^2 rdr = \frac{b^3}{3} r_i + \frac{b^4}{4} = \frac{b^3}{3} r_j - \frac{b^4}{12} \qquad \text{B1.3}$$

$$\int_{r_i}^{r_j} a^3 rdr = \frac{b^4}{4} r_i + \frac{b^5}{5} = \frac{b^4}{4} r_j - \frac{b^5}{20} \qquad \text{B1.4}$$

$$\int_{r_i}^{r_j} a^4 rdr = \frac{b^5}{5} r_i + \frac{b^6}{6} = \frac{b^5}{5} r_j - \frac{b^6}{30} \qquad \text{B1.5}$$

$$\int_{r_i}^{r_j} a^5 rdr = \frac{b^6}{6} r_i + \frac{b^7}{7} = \frac{b^6}{6} r_j - \frac{b^7}{42} \qquad \text{B1.6}$$

$$\int_{r_i}^{r_j} a^6 rdr = \frac{b^7}{7} r_i + \frac{b^8}{8} = \frac{b^7}{7} r_j - \frac{b^8}{56} \qquad \text{B1.7}$$

<u>General Formula</u>

$$\int_{r_i}^{r_j} a^n rdr = \frac{b^{(n+1)}}{(n+1)} r_i + \frac{b^{(n+2)}}{(n+2)} = \frac{b^{(n+1)}}{(n+1)} r_j - \frac{b^{(n+2)}}{(n+1)(n+2)} \qquad \text{B1}$$

<u>APPENDIX B1</u> Integration of $a^n rdr$ terms

$$\int_{r_i}^{r_j} \frac{1}{r}\,dr = \frac{b}{r_i} - \frac{b^2}{2r_i^2} + \frac{b^3}{3r_i^3} - \dots \qquad \text{B2.1}$$

$$\int_{r_i}^{r_j} \frac{a}{r}\,dr = \frac{b^2}{2r_i} - \frac{b^3}{3r_i^2} + \frac{b^4}{4r_i^3} - \dots \qquad \text{B2.2}$$

$$\int_{r_i}^{r_j} \frac{a^2}{r}\,dr = \frac{b^3}{3r_i} - \frac{b^4}{4r_i^2} + \frac{b^5}{5r_i^3} - \dots \qquad \text{B2.3}$$

$$* \int_{r_i}^{r_j} \frac{a^3}{r}\,dr = \frac{b^4}{4r_i} - \frac{b^5}{5r_i^2} + \frac{b^6}{6r_i^3} - \dots \qquad \text{B2.4}$$

$$\int_{r_i}^{r_j} \frac{a^4}{r}\,dr = \frac{b^5}{5r_i} - \frac{b^6}{6r_i^2} + \frac{b^7}{7r_i^3} - \dots \qquad \text{B2.5}$$

$$\int_{r_i}^{r_j} \frac{a^5}{r}\,dr = \frac{b^6}{6r_i} - \frac{b^7}{7r_i^2} + \frac{b^8}{8r_i^3} - \dots \qquad \text{B2.6}$$

$$\int_{r_i}^{r_j} \frac{a^6}{r}\,dr = \frac{b^7}{7r_i} - \frac{b^8}{8r_i^2} + \frac{b^9}{9r_i^3} - \dots \qquad \text{B2.7}$$

<u>General Formula</u>

$$\int_{r_i}^{r_j} \frac{a^n}{r}\,dr = \frac{b^{(n+1)}}{(n+1)r_i} - \frac{b^{(n+2)}}{(n+2)r_i^2} + \frac{b^{(n+3)}}{(n+3)r_i^3} - \dots \qquad \text{B2}$$

* See overleaf for derivation of formula for typical case of n = 3

<u>APPENDIX B2</u> Integration formuli for a^n/r terms

$$\int_{r_i}^{r_j} \frac{a^3}{r}\,dr = \int_{r_i}^{r_j} \frac{1}{r}(r-r_i)^3 dr$$

$$= \int_{r_i}^{r_j} \frac{1}{r}(r^3 - 3r^2 r_i + 3rr_i^2 - r_i^3)dr$$

$$= \int_{r_i}^{r_j} (r^2 - 3rr_i + 3r_i^2 - \frac{r_i^3}{r})dr$$

$$= \left[\frac{1}{3} r^3 - \frac{3}{2} r_i r^2 + 3r_i^2 r - r_i^3 \ell n r \right]_{r_i}^{r_j}$$

$$= \left[\frac{1}{3}(r_j^3 - r_i^3) - \frac{3}{2} r_i(r_j^2 - r_i^2) + 3r_i^2(r_j - r_i) - r_i^3 \ell n \frac{r_j}{r_i}\right]$$

$$= \left[\frac{b}{3}(r_j^2 + r_i r_j + r_i^2) - \frac{3}{2} r_i b(r_i + r_j) + 3r_i^2 b - r_i^3 \ell n \frac{r_j}{r_i}\right]$$

$$= \left[\frac{b}{3}\{(r_i+b)^2 + r_i(r_i+b) + r_i^2\} - \frac{3}{2} r_i b(2r_i+b) + 3r_i^2 b - r_i^3 \ell n \frac{r_j}{r_i}\right]$$

$$= \left[\frac{b}{3}\{3r_i^2 + 3r_i b + b^2\} - 3r_i^2 b - \frac{3}{2} r_i b^2 + 3r_i^2 b - r_i^3 \ell n \frac{r_j}{r_i}\right]$$

$$= \left[br_i^2 + r_i b^2 + \frac{b^3}{3} - \frac{3}{2} r_i b^2 - r_i^3 \ell n \frac{r_j}{r_i}\right]$$

$$= \left[br_i^2 - \frac{1}{2} b^2 r_i + \frac{b^3}{3} - r_i^3 \ell n \frac{r_j}{r_i}\right]$$

and using eqn (5.16a) of Chapter 5;

$$\ell n \frac{r_j}{r_i} = \frac{b}{r_i} - \frac{1}{2}\left(\frac{b}{r_i}\right)^2 + \frac{1}{3}\left(\frac{b}{r_i}\right)^3 - \frac{1}{4}\left(\frac{b}{r_i}\right)^4 + \frac{1}{5}\left(\frac{b}{r_i}\right)^5 - \frac{1}{6}\left(\frac{b}{r_i}\right)^6 + \ldots$$

$$\int_{r_i}^{r_j} \frac{a^3}{r}\,dr = \left[br_i^2 - \frac{1}{2}b^2 r_i + \frac{b^3}{3} - br_i^2 + \frac{1}{2}b^2 r_i - \frac{b^3}{3} + \frac{b^4}{4r_i} - \frac{b^5}{5r_i^2} + \frac{b^6}{6r_i^3} - \ldots\right]$$

$$= \frac{b^4}{4r_i} - \frac{b^5}{5r_i^2} + \frac{b^6}{6r_i^3} - \ldots \qquad \text{B2.4}$$

Derivation of typical case of a^n/r integration formuli

$$\int_{r_i}^{r_j} \frac{1}{r^2}\,dr = \frac{b}{r_i^2} - \frac{2b^2}{2r_i^3} + \frac{3b^3}{3r_i^4} - \ldots \qquad \text{B3.1}$$

$$\int_{r_i}^{r_j} \frac{a}{r^2}\,dr = \frac{b^2}{2r_i^2} - \frac{2b^3}{3r_i^3} + \frac{3b^4}{4r_i^4} - \ldots \qquad \text{B3.2}$$

$$\int_{r_i}^{r_j} \frac{a^2}{r^2}\,dr = \frac{b^3}{3r^2} - \frac{2b^4}{4r^3} + \frac{3b^5}{5r^4} - \ldots \qquad \text{B3.3}$$

$$* \int_{r_i}^{r_j} \frac{a^3}{r^2}\,dr = \frac{b^4}{4r^2} - \frac{2b^5}{5r^3} + \frac{3b^6}{6r^4} - \ldots \qquad \text{B3.4}$$

$$\int_{r_i}^{r_j} \frac{a^4}{r^2}\,dr = \frac{b^5}{5r_i^2} - \frac{2b^6}{6r_i^3} + \frac{3b^7}{7r_i^4} - \ldots \qquad \text{B3.5}$$

$$\int_{r_i}^{r_j} \frac{a^5}{r^2}\,dr = \frac{b^6}{6r_i^2} - \frac{2b^7}{7r_i^3} + \frac{3b^8}{8r_i^4} - \ldots \qquad \text{B3.6}$$

$$\int_{r_i}^{r_j} \frac{a^6}{r^2}\,dr = \frac{b^7}{7r_i^2} - \frac{2b^8}{8r_i^3} + \frac{3b^9}{9r_i^4} - \ldots \qquad \text{B3.7}$$

General Formula

$$\int_{r_i}^{r_j} \frac{a^n}{r^2}\,dr = \frac{b^{(n+1)}}{(n+1)r_i^2} - \frac{2b^{(n+2)}}{(n+2)r_i^3} + \frac{3b^{(n+3)}}{(n+3)r_i^4} - \ldots \qquad \text{B3}$$

* See overleaf for derivation of formula for typical case of $n = 3$

APPENDIX B3 Integration formuli for a^n/r^2 terms

$$\int_{r_i}^{r_j} \frac{a^3}{r^2}\,dr = \int_{r_i}^{r_j} \frac{1}{r^2}\,(r-r_i)^3 dr$$

$$= \int_{r_i}^{r_j} \frac{1}{r^2}\,(r^3 - 3r^2 r_i + 3rr_i^2 - r_i^3)dr$$

$$= \int_{r_i}^{r_j} \left(r - 3r_i + \frac{3r_i^2}{r} \quad \frac{r_i^3}{r^2}\right)dr$$

$$= \left[\frac{r^2}{2} - 3r_i r + 3r_i^2 \ell n r + \frac{r_i^3}{r}\right]_{r_i}^{r_j}$$

$$= \left[\frac{1}{2}(r_j^2 - r_i^2) - 3r_i(r_j - r_i) + 3r_i^2 \ell n\,\frac{r_j}{r_i} + r_i^3\left(\frac{1}{r_j} - \frac{1}{r_i}\right)\right]$$

Subtracting the eqns of B5.1, Appendix B

$$\left(\frac{1}{r_j} - \frac{1}{r_i}\right) = \frac{-b}{r_i^2} + \frac{b^2}{r_i^3} - \frac{b^3}{r_i^4} + \frac{b^4}{r_i^5} - \frac{b^5}{r_i^6} + \frac{b^6}{r_i^7}$$

$$\int_{r_i}^{r_j} \frac{a^3}{r^2}\,dr = \left[\frac{1}{2}b(r_i+r_j) - 3r_i b + 3r_i^2 \ell n\,\frac{r_j}{r_i} + r_i^3\left(\frac{-b}{r_i^2} + \frac{b^2}{r_i^3} - \frac{b^3}{r_i^4} + \frac{b^4}{r_i^5} - \frac{b^5}{r_i^6} + \frac{b^6}{r_i^7}\right)\right]$$

$$= \left[br_i + \frac{1}{2}b^2 - 3r_i b + 3r_i^2 \ell n\,\frac{r_j}{r_i} - br_i + b^2 - \frac{b^3}{r_i} + \frac{b^4}{r_i^2} - \frac{b^5}{r_i^3} + \frac{b^6}{r_i^4}\right]$$

$$= \left[\frac{3}{2}\,b^2 - 3r_i b + 3r_i^2 \ell n\,\frac{r_j}{r_i} - \frac{b^3}{r_i} + \frac{b^4}{r_i^2} - \frac{b^5}{r_i^3} + \frac{b^6}{r_i^4}\right]$$

and using eqn (5.16a)

$$\ell n\,\frac{r_j}{r_i} = \frac{b}{r_i} - \frac{1}{2}\left(\frac{b}{r_i}\right)^2 + \frac{1}{3}\left(\frac{b}{r_i}\right)^3 - \frac{1}{4}\left(\frac{b}{r_i}\right)^4 + \frac{1}{5}\left(\frac{b}{r_i}\right)^5 - \frac{1}{6}\left(\frac{b}{r_i}\right)^6 + \ldots$$

$$\int_{r_i}^{r_j} \frac{a^3}{r^2}\,dr = \left[\frac{3}{2}\,b^2 - 3r_i b - \frac{b^3}{r_i} + \frac{b^4}{r_i^2} - \frac{b^5}{r_i^3} + \frac{b^6}{r_i^4}\right.$$

$$\left. + \; 3r_i b - \frac{3}{2}\,b^2 + \frac{b^3}{r_i} - \frac{3b^4}{4r_i^2} + \frac{3b^5}{5r_i^3} - \frac{3b^6}{6r_i^4}\right]$$

$$\int_{r_i}^{r_j} \frac{a^3}{r^2}\,dr = \frac{b^4}{4r_i^2} - \frac{2b^4}{5r_i^3} + \frac{3b^5}{6r_i^4} - \ldots \qquad \text{B3.4}$$

Derivation of typical case of a^n/r^2 integration formuli

$$\int_{r_i}^{r_j} \frac{1}{r^3}\,dr = \frac{b}{r_i^3} - \frac{3b^2}{2r_i^4} + \frac{6b^3}{3r_i^5} - \ldots \qquad \text{B4.1}$$

$$\int_{r_i}^{r_j} \frac{a}{r^3}\,dr = \frac{b^2}{2r^3} - \frac{3b^3}{3r^4} + \frac{6b^4}{4r^5} - \ldots \qquad \text{B4.2}$$

$$\int_{r_i}^{r_j} \frac{a^2}{r^3}\,dr = \frac{b^3}{3r_i^3} - \frac{3b^4}{4r_i^4} + \frac{6b^5}{5r_i^5} - \ldots \qquad \text{B4.3}$$

$$* \int_{r_i}^{r_j} \frac{a^3}{r^3}\,dr = \frac{b^4}{4r_i^3} - \frac{3b^5}{5r_i^4} + \frac{6b^6}{6r_i^5} - \ldots \qquad \text{B4.4}$$

$$\int_{r_i}^{r_j} \frac{a^4}{r^3}\,dr = \frac{b^5}{5r_i^3} - \frac{3b^6}{6r_i^4} + \frac{6b^7}{7r_i^5} - \ldots \qquad \text{B4.5}$$

$$\int_{r_i}^{r_j} \frac{a^5}{r^3}\,dr = \frac{b^6}{6r_i^3} - \frac{3b^7}{7r_i^4} + \frac{6b^8}{8r_i^5} - \ldots \qquad \text{B4.6}$$

$$\int_{r_i}^{r_j} \frac{a^6}{r^3}\,dr = \frac{b^7}{7r_i^3} - \frac{3b^8}{8r_i^4} + \frac{6b^9}{9r_i^5} - \ldots \qquad \text{B4.7}$$

<u>General Formula</u>

$$\int_{r_i}^{r_j} \frac{a^n}{r^3}\,dr = \frac{b^{(n+1)}}{(n+1)r_i^3} - \frac{3b^{(n+2)}}{(n+2)r_i^4} + \frac{6b^{(n+3)}}{(n+3)r_i^5} - \ldots \qquad \text{B4}$$

* See overleaf for derivation of formula for typical case of $n = 3$

<u>APPENDIX B4</u> Integration formuli for a^n/r^3 terms

$$\int_{r_i}^{r_j} \frac{a^3}{r^3}\,dr = \int_{r_i}^{r_j} \frac{1}{r}\,(r-r_i)^3\,dr$$

$$= \int_{r_i}^{r_j} \frac{1}{r^3}\,(r^3 - 3r^2 r_i + 3rr_i^2 - r_i^3)\,dr$$

$$= \int_{r_i}^{r_j} \left(1 - \frac{3r_i}{r} + \frac{3r_i^2}{r^2} - \frac{r_i^3}{r^3}\right)dr$$

$$= \left[r - 3r_i \ln r - \frac{3r_i^2}{r} + \frac{r_i^3}{2r^2}\right]_{r_i}^{r_j}$$

$$= \left[(r_j - r_i) - 3r_i \ln \frac{r_j}{r_i} - 3r_i^2\left(\frac{1}{r_j} - \frac{1}{r_i}\right) + \frac{r_i^3}{2}\left(\frac{1}{r_j^2} - \frac{1}{r_i^2}\right)\right]$$

Subtracting the eqns of B5.1 and those of B5.2, Appendix B

$$\left(\frac{1}{r_j} - \frac{1}{r_i}\right) = \frac{-b}{r_i^2} + \frac{b^2}{r_i^3} - \frac{b^3}{r_i^4} + \frac{b^4}{r_i^5} - \frac{b^5}{r_i^6} + \frac{b^6}{r_i^7}$$

$$\left(\frac{1}{r_j^2} - \frac{1}{r_i^2}\right) = \frac{-2b}{r_i^3} + \frac{3b^2}{r_i^4} - \frac{4b^3}{r_i^5} + \frac{5b^4}{r_i^6} - \frac{6b^5}{r_i^7} + \frac{7b^6}{r_i^8}$$

and using eqn (5.16a)

$$\ln \frac{r_j}{r_i} = \frac{b}{r_i} - \frac{1}{2}\left(\frac{b}{r_i}\right)^2 + \frac{1}{3}\left(\frac{b}{r_i}\right)^3 - \frac{1}{4}\left(\frac{b}{r_i}\right)^4 + \frac{1}{5}\left(\frac{b}{r_i}\right)^5 + \frac{1}{6}\left(\frac{b}{r_i}\right)^6 - \ldots$$

$$\int_{r_i}^{r_j} \frac{a^3}{r^3}\,dr = \left[b - 3b + \frac{3b^2}{2r_i} - \frac{b^3}{r_i^2} + \frac{3b^4}{4r_i^3} - \frac{3b^5}{5r_i^4} + \frac{3b^6}{6r_i^5} + 3b - \frac{3b^2}{r_i} + \frac{3b^3}{r_i^2}\right.$$

$$\left. - \frac{3b^4}{r_i^3} + \frac{3b^5}{r_i^4} - \frac{3b^6}{r_i^5} - b + \frac{3b^2}{2r_i} - \frac{4b^3}{2r_i^2} + \frac{5b^4}{2r_i^3} - \frac{6b^5}{2r_i^4} + \frac{7b^6}{2r_i^5}\right]$$

$$\int_{r_i}^{r_j} \frac{a^3}{r^3}\,dr = \frac{b^4}{4r_i^3} - \frac{3b^5}{5r_i^4} + \frac{6b^6}{6r_i^5} - \ldots \qquad \text{B4.4}$$

Derivation of typical case of a^n/r^3 integration formulae

Basic Formula: $\frac{1}{r_j} = \frac{1}{r_i} - \frac{b}{r_i r_j}$ (exactly)

Using the above formula the following useful relationships follow:

$$\frac{1}{r_j} = \frac{1}{r_i} - \frac{b}{r_i^2} + \frac{b^2}{r_i^3} - \frac{b^3}{r_i^4} + \dots = \frac{1}{r_m} - \frac{b}{2r_m^2} + \frac{b^2}{4r_m^3} - \frac{b^3}{8r_m^4} + \dots$$

$$\frac{1}{r_i} = \frac{1}{r_j} + \frac{b}{r_j^2} + \frac{b^2}{r_j^3} + \frac{b^3}{r_j^4} + \dots = \frac{1}{r_m} + \frac{b}{2r_m^2} + \frac{b^2}{4r_m^3} + \frac{b^3}{8r_m^4} + \dots \qquad \text{B5.1}$$

$$\frac{1}{r_j^2} = \frac{1}{r_i^2} - \frac{2b}{r_i^3} + \frac{3b^2}{r_i^4} - \frac{4b^3}{r_i^5} + \dots = \frac{1}{r_m^2} - \frac{2b}{2r_m^3} + \frac{3b^2}{4r_m^4} - \frac{4b^3}{8r_m^5} + \dots$$

$$\frac{1}{r_i^2} = \frac{1}{r_j^2} + \frac{2b}{r_j^3} + \frac{3b^2}{r_j^4} + \frac{4b^3}{r_j^5} + \dots = \frac{1}{r_m^2} + \frac{2b}{2r_m^3} + \frac{3b^2}{4r_m^4} + \frac{4b^3}{8r_m^5} + \dots \qquad \text{B5.2}$$

$$\frac{1}{r_j^3} = \frac{1}{r_i^3} - \frac{3b}{r_i^4} + \frac{6b^2}{r_i^5} - \frac{10b^3}{r_i^6} + \dots = \frac{1}{r_m^3} - \frac{3b}{2r_m^4} + \frac{6b^2}{4r_m^5} - \frac{10b^3}{8r_m^6} + \dots$$

$$\frac{1}{r_i^3} = \frac{1}{r_j^3} + \frac{3b}{r_j^4} + \frac{6b^2}{r_j^5} + \frac{10b^3}{r_j^6} + \dots = \frac{1}{r_m^3} + \frac{3b}{2r_m^4} + \frac{6b^2}{4r_m^5} + \frac{10b^3}{8r_m^6} + \dots \qquad \text{B5.3}$$

$$\frac{1}{r_j^4} = \frac{1}{r_i^4} - \frac{4b}{r_i^5} + \frac{10b^2}{r_i^6} - \frac{20b^3}{r_i^7} + \dots = \frac{1}{r_m^4} - \frac{4b}{2r_m^5} + \frac{10b^2}{4r_m^6} - \frac{20b^3}{8r_m^7} + \dots$$

$$\frac{1}{r_i^4} = \frac{1}{r_j^4} + \frac{4b}{r_j^5} + \frac{10b^2}{r_j^6} + \frac{20b^3}{r_j^7} + \dots = \frac{1}{r_m^4} + \frac{4b}{2r_m^5} + \frac{10b^2}{4r_m^6} + \frac{20b^3}{8r_m^7} + \dots \qquad \text{B5.4}$$

Note that $r_j = r_i + b$, $\qquad r_m = r_i + \frac{b}{2} = r_j - \frac{b}{2}$

<u>APPENDIX B5</u> Relationship between $\frac{1}{r_i^n}$, $\frac{1}{r_j^n}$, $\frac{1}{r_m^n}$

APPENDIX C

Strip and Load Matrices

$$[A] = \begin{bmatrix} \frac{13bD\theta}{35r_m^3} + \frac{3b^2D\theta}{10r_m^4} & & & \text{Symmetrical} \\ \frac{-11b^2D\theta}{210r_m^3} - \frac{b^3D\theta}{35r_m^4} & \frac{b^3D\theta}{105r_m^3} + \frac{b^4D\theta}{280r_m^4} & & \\ \frac{9bD\theta}{70r_m^3} & \frac{-13b^2D\theta}{420r_m^3} + \frac{b^3D\theta}{280r_m^4} & \frac{13bD\theta}{35r_m^3} - \frac{3b^2D\theta}{10r_m^4} & \\ \frac{13b^2D\theta}{420r_m^3} + \frac{b^3D\theta}{280r_m^4} & \frac{-b^3D\theta}{140r_m^3} & \frac{11b^2D\theta}{210r_m^3} - \frac{b^3D\theta}{35r_m^4} & \frac{b^3D\theta}{105r_m^3} - \frac{b^4D\theta}{280r_m^4} \end{bmatrix}$$

APPENDIX C1 [A] matrix in polar coordinates for two series terms

$$[B] = \begin{bmatrix} \frac{-6D1}{5br_m} - \frac{D1}{2r_m^2} - \frac{D\theta}{2r_m^2} & \frac{11D1}{10r_m} + \frac{7bD1}{20r_m^2} - \frac{bD\theta}{10r_m^2} & \frac{6D1}{5br_m} + \frac{D1}{2r_m^2} + \frac{D\theta}{2r_m^2} & \frac{D1}{10r_m} + \frac{3bD1}{20r_m^2} + \frac{bD\theta}{10r_m^2} \\ \frac{D1}{10r_m} + \frac{bD1}{20r_m^2} + \frac{bD\theta}{10r_m^2} & \frac{-2bD1}{15r_m} - \frac{b^2D1}{30r_m^2} & \frac{-D1}{10r_m} - \frac{bD1}{20r_m^2} - \frac{bD\theta}{10r_m^2} & \frac{bD1}{30r_m} - \frac{b^2D1}{60r_m^2} - \frac{b^2D\theta}{60r_m^2} \\ \frac{6D1}{5br_m} - \frac{D1}{2r_m^2} - \frac{D\theta}{2r_m^2} & \frac{-D1}{10r_m} + \frac{3bD1}{20r_m^2} + \frac{bD\theta}{10r_m^2} & \frac{-6D1}{5br_m} + \frac{D1}{2r_m^2} + \frac{D\theta}{2r_m^2} & \frac{-11D1}{10r_m} + \frac{7bD1}{20r_m^2} - \frac{bD\theta}{10r_m^2} \\ \frac{D1}{10r_m} - \frac{bD1}{20r_m^2} - \frac{bD\theta}{10r_m^2} & \frac{bD1}{30r_m} + \frac{b^2D1}{60r_m^2} + \frac{b^2D\theta}{60r_m^2} & \frac{-D1}{10r_m} + \frac{bD1}{20r_m^2} + \frac{bD\theta}{10r_m^2} & \frac{-2bD1}{15r_m} + \frac{b^2D1}{30r_m^2} \end{bmatrix}$$

<u>APPENDIX C2</u> [B] matrix in polar coordinates for two series terms

$$[C] = \begin{bmatrix} \frac{24Dr\theta}{5br_m} + \frac{4Dr\theta}{r_m^2} & & & \\ & & \text{Symmetrical} & \\ \frac{-2Dr\theta}{5r_m} + \frac{bDr\theta}{5r_m^2} & \frac{8bDr\theta}{15r_m} + \frac{2b^2Dr\theta}{15r_m^2} & & \\ \frac{-24Dr\theta}{5br_m} & \frac{2Dr\theta}{5r_m} - \frac{bDr\theta}{5r_m^2} & \frac{24Dr\theta}{5br_m} - \frac{4Dr\theta}{r_m^2} & \\ \frac{-2Dr\theta}{5r_m} - \frac{bDr\theta}{5r_m^2} & \frac{-2bDr\theta}{15r_m} & \frac{2Dr\theta}{5r_m} + \frac{bDr\theta}{5r_m^2} & \frac{8bDr\theta}{15r_m} - \frac{2b^2Dr\theta}{15r_m^2} \end{bmatrix}$$

<u>APPENDIX C3</u> [C] matrix in polar coordinates for two series terms

$$
[D] = \begin{bmatrix}
\dfrac{12r_m Dr}{b^3} & & & \text{Symmetrical} \\[2ex]
\dfrac{-6r_m Dr}{b^2} + \dfrac{Dr}{b} & \dfrac{4r_m Dr}{b} - Dr - D1 & & \\[2ex]
\dfrac{-12r_m Dr}{b^3} & \dfrac{6r_m Dr}{b^2} - \dfrac{Dr}{b} & \dfrac{12r_m Dr}{b^3} & \\[2ex]
\dfrac{-6r_m Dr}{b^2} - \dfrac{Dr}{b} & \dfrac{2r_m Dr}{b} & \dfrac{6r_m Dr}{b^2} + \dfrac{Dr}{b} & \dfrac{4r_m Dr}{b} + Dr + D1
\end{bmatrix}
$$

<u>APPENDIX C4</u> [D] matrix in polar coordinates for two series terms

C5.1 Load vector for linear distributed loading $(q_0 + k\theta)$

$$[E] = \begin{bmatrix} \frac{b}{2} r_m - \frac{b^2}{10} \\ \frac{-b^2}{12} r_m + \frac{b^3}{120} \\ \frac{b}{2} r_m + \frac{b^3}{10} \\ \frac{b^2}{12} r_m + \frac{b^3}{120} \end{bmatrix} \{q_0 + k\theta\} \qquad k = \frac{q_1 - q_0}{\Delta\theta}$$

or in terms of r_i and r_j

$$[E] = \begin{bmatrix} \frac{b}{2} r_i + \frac{3b^2}{20} \\ \frac{-b^2}{12} r_i - \frac{b^3}{30} \\ \frac{b}{2} r_j - \frac{3b^2}{20} \\ \frac{b^2}{12} r_j - \frac{b^3}{30} \end{bmatrix} \{q_0 + k\theta\} \qquad k = \frac{q_1 - q_0}{\Delta\theta}$$

<u>APPENDIX C5</u> [E] matrix in polar coordinates

C5.2 Load vector for knife edge loading (P_0 + ka)

$$[E] = \begin{bmatrix} \dfrac{P_0 b}{2d\theta} + \dfrac{3b^2 k}{20d\theta} \\[2ex] \dfrac{-P_0 b^2}{12d\theta} - \dfrac{b^2 k}{30d\theta} \\[2ex] \dfrac{P_0 b}{2d\theta} + \dfrac{7b^2 k}{20d\theta} \\[2ex] \dfrac{P_0 b^2}{12d\theta} + \dfrac{b^3 k}{30d\theta} \end{bmatrix} \qquad k = \frac{P_1 - P_0}{b}$$

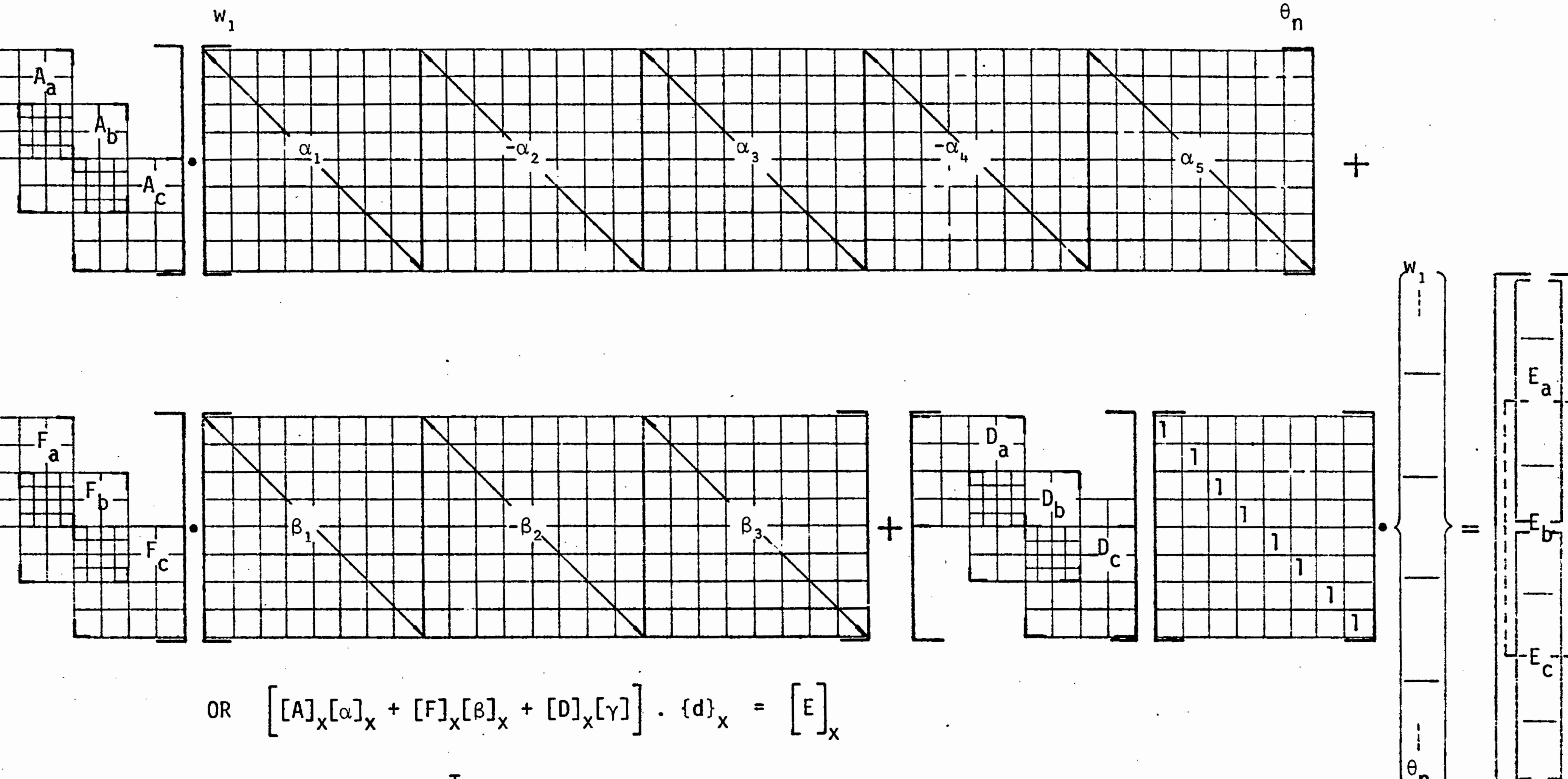
C5.3 Load vector for point loading P_0

$$[E] = \begin{bmatrix} \left(1 - \dfrac{3x^2}{b^2} + \dfrac{2x^3}{b^3}\right) \\[2ex] \left(-x + \dfrac{2x^2}{b} - \dfrac{x^3}{b^2}\right) \\[2ex] \left(\dfrac{3x^2}{b^2} - \dfrac{2x^3}{b^3}\right) \\[2ex] \left(\dfrac{x^2}{b} - \dfrac{x^3}{b^2}\right) \end{bmatrix} \frac{P_0}{d\theta} \qquad b \geq x \geq 0$$

APPENDIX D

Formulation of Stiffness Matrix from Governing Differential Equation

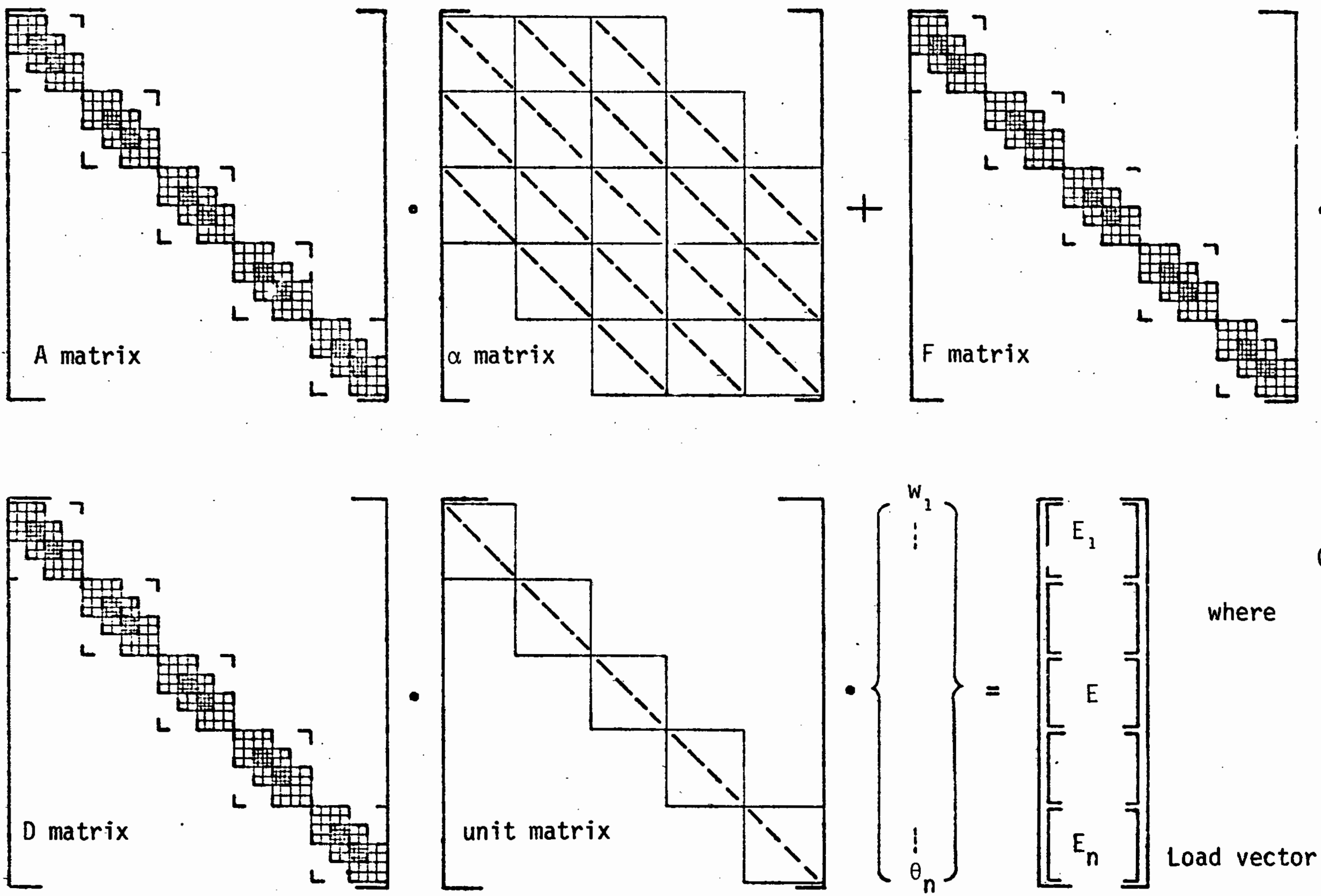
Apply governing differential equation to adjacent strips, a, b, c at interior nodal cross-section x:



OR $\left[[A]_x[\alpha]_x + [F]_x[\beta]_x + [D]_x[\gamma] \right] \cdot \{d\}_x = \left[E \right]_x$

where $[F] = [B] + [B]^T - [C]$ Values of α and β may be found in Appendix F1

APPENDIX D1 Finite difference form of governing differential equation compiled for three adjacent strips at one common interior nodal cross-section for unequal spacing of nodes



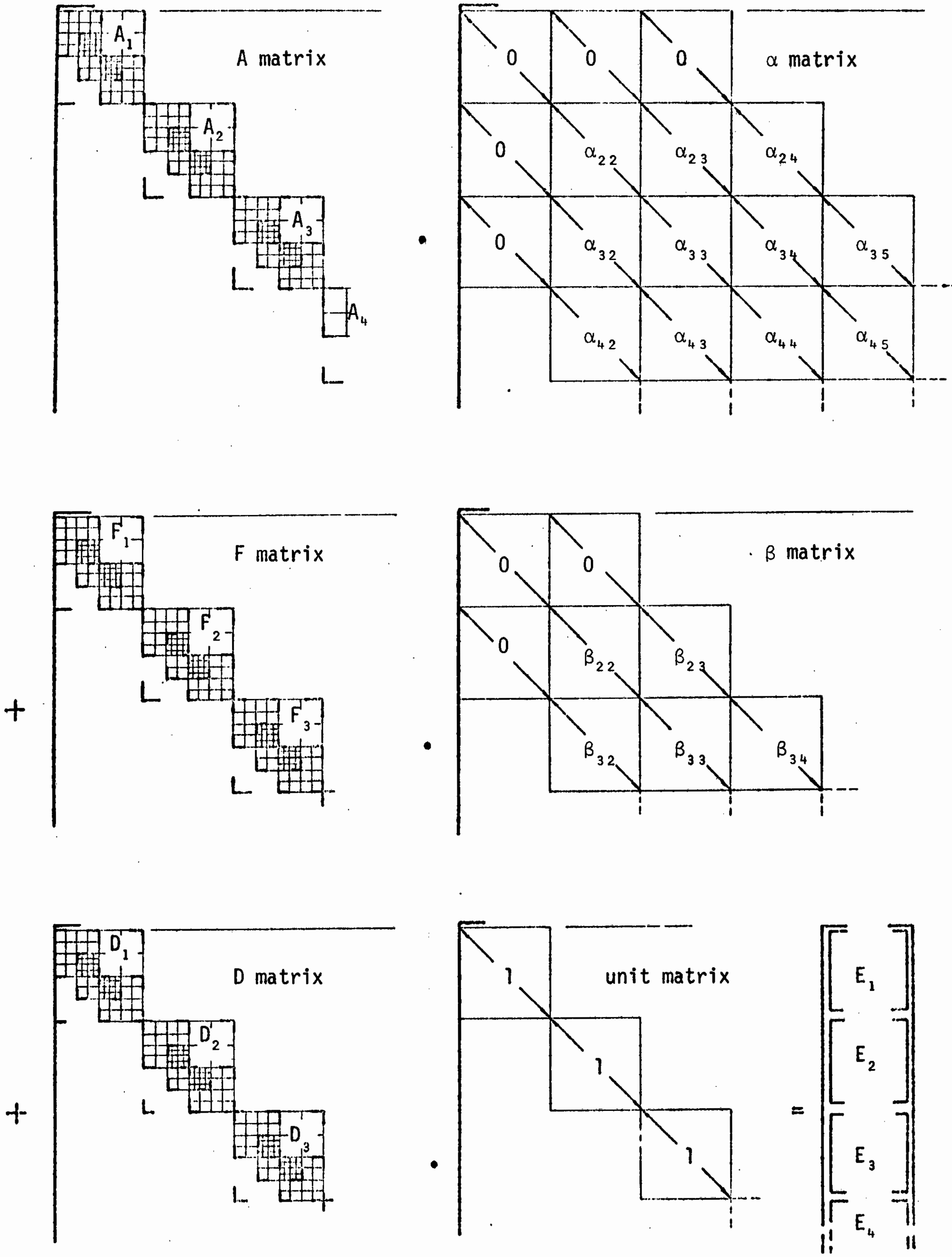
OR $[K]\{d\} = [E]$

where $[K]$ is the finite difference stiffness matrix for the structure

APPENDIX D2 Finite difference form of governing differential equation compiled for 3 strips and 5 nodal cross-sections

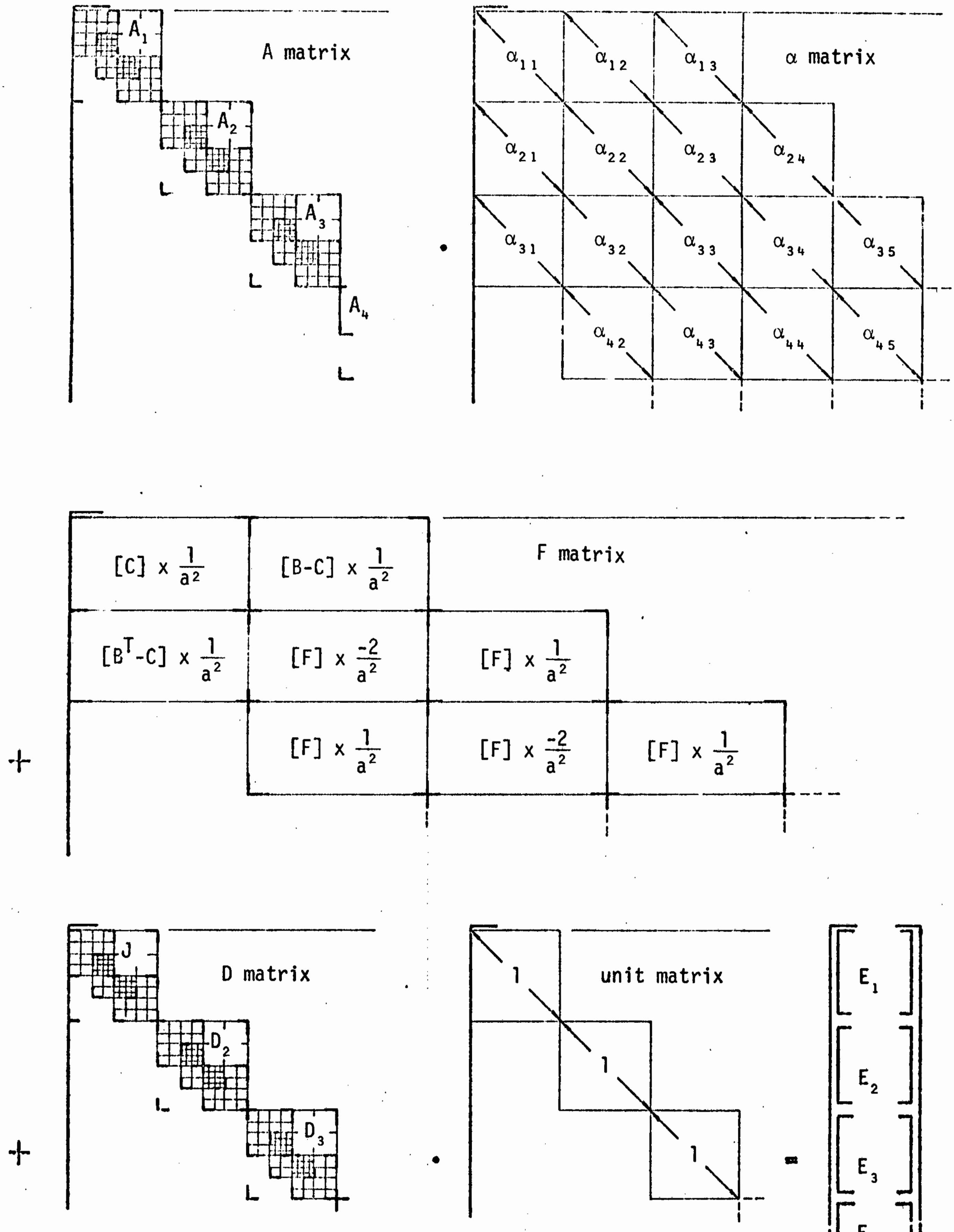
APPENDIX E

Formulation of Stiffness Matrix for Different Boundary Conditions



refer to Appendix F2 and F3 for appropriate values of α and β

APPENDIX E1 Formulation of finite difference stiffness matrix for simply supported and fixed edge boundary conditions



where $[J] = [D] - [B]^T[A]^{-1}[B]$ and $[F] = [B] + [B]^T - [C]$

refer to Appendix F3 for values of α(for constant spacing of nodes = a)

APPENDIX E2 Formulation of finite difference stiffness matrix for free edge boundary condition

APPENDIX F

Finite Difference Approximations

<u>1st Derivative:</u>

$$y' = \frac{-c}{b(b+c)} w_A + \frac{(c-b)}{bc} w_0 + \frac{b}{c(b+c)} w_1 \qquad \text{f 1.1}$$

<u>2nd Derivative:</u>

$$y'' = \frac{2}{b(b+c)} w_A - \frac{2}{bc} w_0 + \frac{2}{c(b+c)} w_1 = \beta_1 w_A - \beta_2 w_0 + \beta_3 w_1 \qquad \text{f 1.2}$$

<u>3rd Derivative:</u>

$$y''' = \frac{-2c\ w_B}{ab(a+b)(b+c)} + \frac{2(c^2+ac-ab)\ w_A}{ab^2c(b+c)} + \left\{\frac{2(b-c)}{b^2c^2} - \frac{2c}{b^2(a+b)(b+c)} + \frac{2b}{c^2(b+c)(c+d)}\right\} w_0 - \frac{2(b^2+bd-cd)\ w_1}{bc^2d(b+c)} + \frac{2b\ w_2}{c^2(b+c)(c+d)} \qquad \text{f 1.3}$$

<u>4th Derivative:</u>

$$y'''' = \frac{4\ w_B}{ab(a+b)(b+c)} - \frac{4(a+c)\ w_A}{ab^2c(b+c)} + \left\{\frac{4}{b^2c^2} + \frac{4}{b^2(a+b)(b+c)} + \frac{4}{c^2(b+c)(c+d)}\right\} w_0 - \frac{4(b+d)\ w_1}{bc^2d(b+c)} + \frac{4\ w_2}{c^2(b+c)(c+d)} \qquad \text{f 1.4}$$

$$= \alpha_1 w_B - \alpha_2 w_A + \alpha_3 w_0 - \alpha_4 w_1 + \alpha_5 w_2$$

<u>Angular spacing at node i</u> $\quad a = d\theta_{i-2} \quad b = d\theta_{i-1} \quad c = d\theta_i \quad d = d\theta_{i+1}$

<u>APPENDIX F1:</u> Summary of variable spacing finite difference operator patterns based on 2nd order polynomials

1. Free Edge Boundary

$$\alpha = \left[\begin{array}{cccccc}
\frac{4}{a^3(a+b)} & \frac{-4}{a^3b} & \frac{4}{a^2b(a+b)} & 0 & 0 \\[2ex]
\frac{-4}{a^2b(a+b)} & \frac{4}{a^2b^2}+\frac{4}{b^2(a+b)(b+c)} & \frac{-4(a+c)}{ab^2c(a+b)} & \frac{4}{bc(a+b)(b+c)} & 0 \\[2ex]
\frac{4}{ab(a+b)(b+c)} & \frac{-4(a+c)}{ab^2c(b+c)} & \frac{4}{b^2c^2}+\frac{4}{b^2(a+b)(b+c)}+\frac{4}{c^2(b+c)(c+d)} & \frac{-4(b+d)}{bc^2d(b+c)} & \frac{4}{cd(b+c)(c+d)} \\[2ex]
0 & \frac{4}{bc(b+c)(c+d)} & \frac{-4(b+d)}{bc^2d(c+d)} & \frac{4}{c^2d^2}+\frac{4}{c^2(b+c)(c+d)}+\frac{4}{d^2(c+d)(d+e)} & \frac{-4(c+e)}{cd^2e(c+d)} \\[2ex]
0 & 0 & \frac{4}{cd(c+d)(d+e)} & \frac{-4(c+e)}{cd^2e(d+e)} & \frac{4}{d^2e^2}+\cdots \\[2ex]
0 & 0 & 0 & \cdots\cdots & \cdots\cdots
\end{array}\right.$$

Note: To obtain symmetry multiply row i by average nodal spacing at node i (refer overleaf)

2. Simply Supported Edge As for Free Edge, with row 1 and column 1 equal to zero.

APPENDIX F2: Boundary operator pattern for 4th derivative based on 2nd order polynomials

1. <u>Free Edge Boundary</u>

$$
\begin{bmatrix}
\frac{2}{a^2(a+b)} & \frac{-2}{a^2b} & \frac{2}{ab(a+b)} & 0 & 0 & 0 \\
\frac{-2}{a^2b} & \frac{2(a+b)}{a^2b^2}+\frac{2}{b^2(b+c)} & \frac{-2(a+c)}{ab^2c} & \frac{2}{bc(b+c)} & 0 & 0 \\
\frac{2}{ab(a+b)} & \frac{-2(a+c)}{ab^2c} & \frac{2(b+c)}{b^2c^2}+\frac{2}{b^2(a+b)}+\frac{2}{c^2(c+d)} & \frac{-2(b+d)}{bc^2d} & \frac{2}{cd(c+d)} & 0 \\
0 & \frac{2}{bc(b+c)} & \frac{-2(b+d)}{bc^2d} & \frac{2(c+d)}{c^2d^2}+\frac{2}{c^2(b+c)}+\frac{2}{d^2(d+e)} & \frac{-2(c+e)}{cd^2e} & \cdots\cdots \\
0 & 0 & \frac{2}{cd(c+d)} & \frac{-2(c+e)}{cd^2e} & \frac{2(d+e)}{d^2e^2}+\frac{2}{d^2(c+d)}+\frac{2}{e^2(e+f)} & \cdots\cdots \\
0 & 0 & 0 & \cdots\cdots & \cdots\cdots & \cdots\cdots
\end{bmatrix}
$$

<u>Note:</u> Symmetry has been obtained by multiplying row i by the average nodal spacing at node i

2. <u>Simply Supported Edge</u> As for Free Edge, with row 1 and column 1 equal to zero.

<u>APPENDIX F2</u>: Symmetrical boundary operator pattern for 4th derivative based on 2nd order polynomials

3. Fixed Edge Boundary

$$\alpha = \left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 \\
0 & \frac{4}{a^2b^2} + \frac{4}{a^3(a+b)} + \frac{4}{b^2(a+b)(b+c)} & \frac{-4(a+c)}{ab^2c(a+b)} & \frac{4}{bc(a+b)(b+c)} & 0 \\
0 & \frac{-4(a+c)}{ab^2c(b+c)} & \frac{4}{b^2c^2} + \frac{4}{b^2(a+b)(b+c)} + \frac{4}{c^2(b+c)(c+d)} & \frac{-4(b+d)}{bc^2d(b+c)} & \frac{4}{cd(b+c)(c+d)} \\
0 & \frac{4}{bc(b+c)(c+d)} & \frac{-4(b+d)}{bc^2d(c+d)} & \frac{4}{c^2d^2} + \frac{4}{c^2(b+c)(c+d)} + \frac{4}{d^2(c+d)(d+e)} & \frac{-4(c+e)}{cd^2e(c+d)} \\
0 & 0 & \frac{4}{cd(c+d)(d+e)} & \frac{-4(c+e)}{cd^2e(d+e)} & \frac{4}{d^2e^2} + \cdots \\
0 & 0 & 0 & \cdots & \cdots
\end{array}\right.$$

<u>Note:</u> To obtain symmetry, multiply row i by average nodal spacing at node i (refer overleaf)

APPENDIX F2: Boundary operator pattern for 4th derivative based on 2nd order polynomials

3. Fixed Edge Boundary

$$
\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2(a+b)}{a^2b^2} + \frac{2}{a^3} + \frac{2}{b^2(b+c)} & \frac{-2(a+c)}{ab^2c} & \frac{2}{bc(b+c)} & 0 & 0 \\
0 & \frac{-2(a+c)}{ab^2c} & \frac{2(b+c)}{b^2c^2} + \frac{2}{b^2(a+b)} + \frac{2}{c^2(c+d)} & \frac{-2(b+d)}{bc^2d} & \frac{2}{cd(c+d)} & 0 \\
0 & \frac{2}{bc(b+c)} & \frac{-2(b+d)}{bc^2d} & \frac{2(c+d)}{c^2d^2} + \frac{2}{c^2(b+c)} + \frac{2}{d^2(d+e)} & \frac{-2(c+e)}{cd^2e} & \cdots \\
0 & 0 & \frac{2}{cd(c+d)} & \frac{-2(c+e)}{cd^2e} & \frac{2(d+e)}{d^2e^2} + \frac{2}{d^2(c+d)} + \frac{2}{e^2(e+f)} & \cdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots
\end{array}\right.
$$

Note: Symmetry has been obtained by multiplying row i by the average nodal spacing at node i

APPENDIX F2: Symmetrical boundary operator pattern for 4th derivative based on 2nd order polynomials

1. <u>Free Edge</u>

$$\beta = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{2}{a(a+b)} & \frac{-2}{ab} & \frac{2}{b(a+b)} & 0 & 0 \\ 0 & \frac{2}{b(b+c)} & \frac{-2}{bc} & \frac{2}{c(b+c)} & 0 \\ 0 & 0 & \frac{2}{c(c+d)} & \frac{-2}{cd} & \frac{2}{d(c+d)} \\ & & & \cdots & \cdots \end{bmatrix}$$

2. <u>Clamped Edge</u>

$$\beta = \begin{bmatrix} 0 & \frac{2}{a^2} & 0 & 0 & 0 \\ 0 & \frac{-2}{ab} & \frac{2}{b(a+b)} & 0 & 0 \\ 0 & \frac{2}{b(b+c)} & \frac{-2}{bc} & \frac{2}{c(b+c)} & 0 \\ 0 & 0 & \frac{2}{c(c+d)} & \frac{-2}{cd} & \frac{2}{d(c+d)} \\ & & \cdots & \cdots & \cdots \end{bmatrix}$$

3. <u>Simply Supported Edge</u> As for Free Edge, with row 1 and column 1 = 0

<u>APPENDIX F3:</u> Boundary operator patterns for 2nd derivative based on 2nd order polynomials

1st Derivative: Not required

2nd Derivative:

$$y'' = \frac{-2(bf-cf+bc)\ w_B}{ae(f+e)(c+e)} + \frac{2(ce-cf+ef)\ w_A}{ab(b+f)(b+c)} - \frac{2(bf-cf+bc-be+ce+ef)\ w_0}{bcef} + \frac{2(bf-be+ef)\ w_1}{cd(c+e)(b+c)} - \frac{2(ce-be+bc)\ w_2}{df(f+e)(b+f)} \qquad \text{f 4.2}$$

3rd Derivative:

$$y''' = \frac{-6(c+f-b)w_B}{ae(f+e)(c+e)} + \frac{6(c+f-e)\ w_A}{ab(b+f)(b+c)} + \frac{6(b-c+e-f)\ w_0}{bcef} - \frac{6(b+e-f)\ w_1}{cd(c+e)(b+c)} + \frac{6(b+e-c)\ w_2}{df(f+e)(b+f)} \qquad \text{f 4.3}$$

4th Derivative:

$$y'''' = \frac{24\ w_B}{ae(f+e)(c+e)} - \frac{24\ w_A}{ab(b+f)(b+c)} + \frac{24\ w_0}{bcef} - \frac{24\ w_1}{cd(c+e)(b+c)} + \frac{24\ w_2}{df(f+e)(b+f)} \qquad \text{f 4.4}$$

Angular spacing at node i $\quad a = d\theta_{i-2} \quad b = d\theta_{i-1} \quad c = d\theta_i \quad d = d\theta_{i+1} \quad e = a+b \quad f = c+d$

APPENDIX F4: Summary of variable spacing finite difference operation patterns based on 4th order polynomials

1. <u>Free Edge Boundary</u>   Let $x = a(3a+4b+2c) + b(b+c)$

$$
\left[\begin{array}{ccccc}
\dfrac{12}{a(a+b)\{(a+b)^2+a(a+b)+a^2\}} & \dfrac{-12}{ab\{(a+b)^2+a(a+b)+a^2\}} & \dfrac{12}{b(a+b)\{(a+b)^2+a(a+b)+a^2\}} & 0 & 0 \\[2ex]
\dfrac{-24(3a+2b+c)}{a(a+b)(a+b+c).x} & \dfrac{24(2a+2b+c)}{ab(a+b).x} & \dfrac{-24(2a+b+c)}{bc(a+b).x} & \dfrac{24(2a+b)}{c(b+c)(a+b+c).x} & 0 \\[2ex]
\dfrac{24}{a(a+b)(a+b+c)(a+b+c+d)} & \dfrac{-24}{ab(b+c)(b+c+d)} & \dfrac{24}{bc(a+b)(c+d)} & \dfrac{-24}{cd(b+c)(a+b+c)} & \dfrac{24}{d(c+d)(b+c+d)} \; \ldots \\[2ex]
0 & \dfrac{24}{b(b+c)(b+c+d)(b+c+d+e)} & \dfrac{-24}{bc(b+c)(c+d+e)} & \dfrac{24}{cd(b+c)(d+e)} & \dfrac{-24}{de(c+d)(b+c+d)} \\[2ex]
0 & 0 & \dfrac{24}{c(c+d)(c+d+e)(c+d+e+f)} & \dfrac{-24}{cd(d+e)(d+e+f)} & \dfrac{24}{de(c+d)(e+f)}
\end{array}\right.
$$

2. <u>Simply Supported Edge</u>   As for Free Edge with row 1 and column 1 = 0

<u>APPENDIX F5:</u> Boundary operator pattern for 4th derivative based on 4th order polynomials

3. <u>Clamped Edge Boundary</u>

$$\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{24}{a^2b(b+c)} & \frac{-24}{bc(a+b)^2} & \frac{24}{c(b+c)(a+b+c)^2} & 0 & 0 \\
0 & \frac{-24}{ab(b+c)(b+c+d)} & \frac{24}{bc(a+b)(c+d)} & \frac{-24}{cd(b+c)(a+b+c)} & \frac{24}{d(c+d)(b+c+d)(a+b+c+d)} & \\
0 & \frac{24}{b(b+c)(b+c+d)(b+c+d+e)} & \frac{-24}{bc(c+d)(c+d+e)} & \frac{24}{cd(b+c)(d+e)} & \frac{-24}{de(c+d)(b+c+d)} & \cdots \\
0 & 0 & \frac{24}{c(c+d)(c+d+e)(c+d+e+f)} & \frac{-24}{cd(d+e)(d+e+f)} & \frac{24}{de(c+d)(e+f)} & \cdots
\end{array}\right.$$

<u>APPENDIX F5</u>: Boundary operator pattern for 4th derivative based on 4th order polynomials

1. <u>Clamped Edge</u> Let $e = a+b$, $f = c+d$, $g = b+c$

$$
\begin{array}{cccccc}
w_1 & w_2 & w_3 & w_4 & w_5 & \\
0 & \dfrac{2(a+b)(a+g)}{a^2bg} & \dfrac{-2a(a+g)}{bc(a+b)^2} & \dfrac{2a(a+b)}{cg(a+g)^2} & 0 & 0 \\
0 & \dfrac{-2\{(b+c)(2a-b)+a(2b-a)\}}{a^2bg} & \dfrac{2a(2g-a)}{bc(a+b)^2} & \dfrac{-2a(2b-a)}{c(b+c)(a+g)^2} & 0 & 0 \\
0 & \dfrac{2(ce-cf+ef)}{ab(b+f)(b+c)} & \dfrac{-2(bf-cf+bc-be+ce+ef)}{bcef} & \dfrac{2(bf-be+ef)}{cd(c+e)(b+c)} & \dfrac{-2(ce-be+bc)}{df(f+e)(b+f)} & \cdots\cdots \\
0 & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots
\end{array}
$$

<u>APPENDIX F6</u>: Boundary operator patterns for 2nd derivative based on 4th order polynomials

2. <u>Free Edge</u> Let $x = a(3a+4b+2c) + bg$

$$
\begin{array}{cccccc}
 & w_1 & w_2 & w_3 & w_4 & \\
\left[\right. & 0 & 0 & 0 & 0 & 0 \\
 & \dfrac{6a\{(3a+b+g)(b^2+ab-a^2)+(a+g)x\}}{a(a+b)(2a+b)(a+g)x} & \dfrac{-6a\{(2a+b+g)(b^2+ab-a^2)+gx\}}{abg(2a+b)x} & \dfrac{6a\{(2a+g)(b^2+ab-a^2)+cx\}}{bc(a+b)(2a+b)x} & \dfrac{-6a(b^2+ab-a^2)}{cg(a+g)x} & 0 \\
 & \dfrac{-2(bf-cf+bc)}{ae(f+e)(c+e)} & \dfrac{2(ce-cf+ef)}{ab(b+f)(b+c)} & \dfrac{-2(bf-cf+bc-be+ce+ef)}{bcef} & \dfrac{2(bf-be+ef)}{cd(c+e)(b+c)} & \cdots \\
 & 0 & \cdots & \cdots & \cdots &
\end{array}
$$

3. <u>Simply Supported Edge</u> As for Free Edge, with column 1 equal to zero

<u>APPENDIX F6</u>: Boundary operator patterns for 2nd derivative based on 4th order polynomials

APPENDIX G

Coding of Input for Program STRIP

Typical Results for Examples 2 and 5

CODING OF INPUT FOR PROGRAM STRIP

The coding of input data required to define the curved plate structure to be analysed and to specify the different boundary conditions and loading types will be as follows:

1. TITLE (2 cards)

Column	Format	Description
1-80	40A2	ITLE: Title cards for structure

(Comment: A total of two cards must be used)

2. PARAMETERS (1 card)

Column	Format	Description
1- 5	I5	NST: Total number of strips in structure
6-10	I5	NXS: Total number of nodal x-sections used in finite difference approximations
11-15	I5	NLC: Total number of load cases
16-27	F12.2	RD(1): Radius of first nodal line

(Comment: Integer values must be rightly adjusted)

3. MATERIAL PROPERTIES (1 card)

Column	Format	Description
1-15	F15.2	ES: Modulus of elasticity in longitudinal direction
16-30	F15.2	ER: Modulus of elasticity in transverse direction
31-45	F15.2	GRS: Shear modulus
46-53	F8.4	VS: Poissons' ratio in longitudinal direction
54-61	F8.4	VR: Poissons' ratio in transverse direction

(Note: Only two Moduli and one Poissons ratio need be given - the remaining section properties are calculated by default)

4. STRIP DIMENSIONS (NST cards)

Column	Format	Description
1-12	F12.4	W(I): Width of strip I
13-24	F12.4	DT(I): Depth of strip I

(Note: One card is used for each strip)

5. NODAL SPACING (NXS-1 cards)

Column	Format	Description
1-80	8F10.6	DA(I): Angular nodal finite difference spacing in radians. May be constant or variable

(Comment: One card required for each set of eight consecutive values)

6. END BOUNDARY CONDITIONS (1 card)

Column	Format	Description
1-16	8A2	IB: Boundary condition at start of structure
20-36	8A2	IE: Boundary condition at end of structure

where: IB, IE may be either SIMPLY SUPPORTED
or CLAMPED EDGE
or GUIDED EDGE
or FREE EDGE

(Comment: The present version of the program does not include the guided and free edge boundary conditions)

7. LOAD CASE (1 card)

Column	Format	Description
1- 2	I2	LCN: Load case number
4-64	30A2	LABL: Heading for description of loading type

(Comment: LCN should start with 1 and increase consecutively for each additional load case)

8. LOADING TYPE (Repeated for as many similar or different loading types required to specify the above load case)

Column	Format	Description
1- 8	4A2	IL: Identity of loading type
9-13	I5	NLB: Nodal line number at start of loading
14-18	I5	NLE: Nodal line number at end of loading
19-23	I5	NXB: x-section number at start of loading
24-28	I5	NXE: x-section number at end of loading
28-40	F12.4	WMAG: Magnitude of load or load intensity

where: IL may be either POINT
or UNIFORM
or LINE

and: WMAG should have a negative value if acting downwards

(Comment: In the case of a point load NLB and NLE will have the same integer value. Similarly NXB and NXE will be equal. In the case of a line load only NXB and NXE will be equal)

9. CONTINUATION (1 card)

Column	Format	Description
1- 8	4A2	IL: Continuation control card

where: IL may be either NEXT ... (i)
or LAST ... (ii)

For (i) Steps 7, 8, 9 are repeated (ii) End of last loading case

10. INTERIOR BOUNDARY CONTROL (1 card)

Column	Format	Description
1- 5	I5	NDR: Number of interior displacements or rotations to be initialised

(Comment: If there are no interior boundary conditions, then NDR must be set equal to zero)

11. INTERIOR BOUNDARY CONDITIONS (Repeated for as many displacements or rotations as required)

Column	Format	Description
1-12	6A2	IZ: Identity of displacement type
13-22	F10.4	DMAG: Initial value to be assigned to the displacement type
23-27	I5	NLN: Nodal line number at which the interior displacement occurs
28-32	I5	NXN: x-section number at which the interior displacement occurs

where: IZ may be either DISPLACEMENT
or ROTATION

PROGRAM RESTRICTIONS

Maximum number of strips NST = 8

Maximum number of x-sections NXS = 40

Maximum number of load cases NLC = 14

Dimensioning will therefore be as follows:

Strip Matrices

Compiled for one nodal x-section MSM = 2*NST+2 = 18

Stiffness Matrix

Half band width	NBW = 2*MSM+4 = 40
Augmented band width	NBL = NBW + NLC = 54
Matrix size	NSM = NXS*MSM = 720

The program is fully operational on the UNIVAC 1106 at the University of Cape Town at the present time. Either card input may be used or data may be contained in a datafile or in an element of a program file. The runstream via card input will be as follows:

```
@RUN RUNID,ACCT.NO/USERID,PROJECTID,TIME,PAGES
@PASSWD Password
@ASG,A STRIP.
@XQT STRIP.ABS

   data for problem 1

@XQT STRIP.ABS

   data for problem 2

@FIN
```

Data elements for examples 2 and 5, together with typical results may be found overleaf.

```
FINITE*STRIP(1).DATA19
        EX 2. ISOTROPIC RECTANGULAR PLATE  1 STRIP  9 NODAL X-SECTS. R 125000.0H
        ************************************************************************
            1     9     1    124999.50
                 10.92             10.92                      .3
              .25           1.0
           .000001    .000001    .000001    .000001    .000001    .000001    .000001    .000001
        SIMPLY SUPPORTED     SIMPLY SUPPORTED
         1  UNIT UNIFORM LOADING THROUGHOUT
        UNIFORM        1     2     1     9      -1.0000
        LAST
            7
        ROTATION         .000000     2     2
        ROTATION         .000000     2     3
        ROTATION         .000000     2     4
        ROTATION         .000000     2     5
        ROTATION         .000000     2     6
        ROTATION         .000000     2     7
        ROTATION         .000000     2     8
```

EX 2. ISOTROPIC RECTANGULAR PLATE 1 STRIP 9 NODAL X-SECTS. R 125000.0M
****

MATERIAL AND SECTION PROPERTIES *** UNITS ARE IN N , M , S

ELASTIC MODULUS	ES =	.10920+02	ER =	.10920+02
SHEAR MODULUS	GRS =	.42000+01		
POISSONS RATIO	VS =	.3000	VR =	.3000

STRIP NO	LINE NO	RADIUS	WIDTH	DEPTH	BENDING AND TORSIONAL RIGIDITY DS	DR	DRS
	1	124999.50					
1			.25	1.00	.1000+01	.1000+01	.3500+00
	2	124999.75					

ANGULAR NODAL SPACING (RAD) USED IN FINITE DIFFERENCE APPROXIMATIONS

.000001 .000001 .000001 .000001 .000001 .000001
.000001 .000001

LONGITUDINAL NODAL SPACING (M) ALONG CENTRAL ARC , RADIUS 124999.62

.125000 .125000 .125000 .125000 .125000 .125000
.125000 .125000

LOADING CASE NO 1 UNIT UNIFORM LOADING THROUGHOUT

LOADING TYPE	NODAL LINE NO FROM	AT	TO	CROSSECTION NO FROM	AT	TO	MAGNITUDE OF LOAD
UNIFORM	1		2	1		9	-1.0000

INITIAL CONDITIONS OF DISPLACEMENT OR ROTATION ALONG STRUCTURE

NO	TYPE	NODAL LINE	X-SECTION	MAGNITUDE
1	ROTATION	2	2	.000000
2	ROTATION	2	3	.000000
3	ROTATION	2	4	.000000
4	ROTATION	2	5	.000000
5	ROTATION	2	6	.000000
6	ROTATION	2	7	.000000
7	ROTATION	2	8	.000000

BOUNDARY CONDITION AT START OF STRUCTURE SIMPLY SUPPORTED
BOUNDARY CONDITION AT END OF STRUCTURE SIMPLY SUPPORTED

LOAD CASE NO 1. RESULTS OF GENERALISED DISPLACEMENTS

X-SECTION	NODAL LINE	DISPLACEMENT	ROTATION
1	1	.000000	.000000
	2	.000000	.000000
2	1	-.576853-02	-.327406-02
	2	-.539780-02	.000000
3	1	-.105728-01	-.587237-02
	2	-.989997-02	.000000
4	1	-.137281-01	-.751251-02
	2	-.128613-01	.000000
5	1	-.148252-01	-.807109-02
	2	-.138919-01	.000000
6	1	-.137281-01	-.751252-02
	2	-.128613-01	.000000
7	1	-.105728-01	-.587238-02
	2	-.989998-02	.000000
8	1	-.576853-02	-.327407-02
	2	-.539781-02	.000000
9	1	.000000	.000000
	2	.000000	.000000

LOAD CASE NO 1. RESULTS OF BENDING AND TWISTING MOMENTS

X-SECTION	NODAL LINE	M1	M2	M12
1	1	.000000	.000000	.183351-01
	2	.000000	.000000	.241823-06
2	1	-.171915-02	.566757-01	.164429-01
	2	-.779880-02	.545014-01	.221760-06
3	1	-.229220-02	.967238-01	.118679-01
	2	-.119736-01	.933318-01	.167183-06
4	1	-.252840-02	.120626+00	.615652-02
	2	-.139585-01	.116634+00	.894189-07
5	1	-.259580-02	.128576+00	.158937-07
	2	-.145408-01	.124405+00	.410158-13
6	1	-.252840-02	.120626+00	-.615649-02
	2	-.139585-01	.116634+00	-.894188-07
7	1	-.229217-02	.967239-01	-.118679-01
	2	-.119736-01	.933319-01	-.167183-06
8	1	-.171914-02	.566758-01	-.164430-01
	2	-.779881-02	.545015-01	-.221760-06
9	1	.000000	.000000	-.183351-01
	2	.000000	.000000	-.241823-06

```
FINITE*STRIP(1).DATAEX5V
        EX.5 THREE SPAN ISOTRIPIC CURVED BRIDGE DECK. 8 STRIPS 37 Y-SECT R(CL)=100M
        ************************************************************************
           8    37     4         95.00
           25000000.       25000000.                      .3
              .5          1.0
             1.5          1.0
             1.5          1.0
             1.5          1.0
             1.5          1.0
             1.5          1.0
             1.5          1.0
              .5          1.0
         .010      .015      .025      .025      .025      .025      .025      .025
         .015      .010      .010      .015      .0225     .0225     .0225     .022
         .0225     .0225     .0225     .0225     .0225     .0225     .0225     .022
         .015      .010      .010      .015      .025      .025      .025      .025
         .025      .025      .015      .010
        SIMPLY SUPPORTED    SIMPLY SUPPORTED
         1  DEAD LOAD ONLY
        UNIFORM      1    9    1    37     -25.0000
        NEXT
         2  PATCH LOAD  CENTRAL SPAN  INSIDE CARRIAGEWAY
        UNIFORM      2    4   11    27     -25.0000
        NEXT
         3  PATCH LOAD  CENTRAL SPAN  CENTRAL 3M WIDTH
        UNIFORM      4    6   11    27     -25.0000
        NEXT
         4  PATCH LOAD  CENTRAL SPAN  OUTSIDE CARRIAGEWAY
        UNIFORM      6    8   11    27     -25.0000
        LAST
           8
        DISPLACEMENT    .000000    3   11
        DISPLACEMENT    .000000    4   11
        DISPLACEMENT    .000000    6   11
        DISPLACEMENT    .000000    7   11
        DISPLACEMENT    .000000    3   27
        DISPLACEMENT    .000000    4   27
        DISPLACEMENT    .000000    6   27
        DISPLACEMENT    .000000    7   27
```

EX.5 THREE SPAN ISOTROPIC CURVED BRIDGE DECK, 8 STRIPS 37 X-SECT R(CL)=100M
****

MATERIAL AND SECTION PROPERTIES *** UNITS ARE IN N , M , S

ELASTIC MODULUS	ES	=	.25000+08	ER	=	.25000+08
SHEAR MODULUS	GRS	=	.96154+07			
POISSONS RATIO	VS	=	.3000	VR	=	.3000

STRIP NO	LINE NO	RADIUS	WIDTH	DEPTH	BENDING AND TORSIONAL RIGIDITY DS	DR	DRS
	1	95.00					
1			.50	1.00	.2289+07	.2289+07	.8013+06
	2	95.50					
2			1.50	1.00	.2289+07	.2289+07	.8013+06
	3	97.00					
3			1.50	1.00	.2289+07	.2289+07	.8013+06
	4	98.50					
4			1.50	1.00	.2289+07	.2289+07	.8013+06
	5	100.00					
5			1.50	1.00	.2289+07	.2289+07	.8013+06
	6	101.50					
6			1.50	1.00	.2289+07	.2289+07	.8013+06
	7	103.00					
7			1.50	1.00	.2289+07	.2289+07	.8013+06
	8	104.50					
8			.50	1.00	.2289+07	.2289+07	.8013+06
	9	105.00					

ANGULAR NODAL SPACING (RAD) USED IN FINITE DIFFERENCE APPROXIMATIONS

.010000	.015000	.025000	.025000	.025000	.025000
.025000	.025000	.015000	.010000	.010000	.015000
.022500	.022500	.022500	.022500	.022500	.022500
.022500	.022500	.022500	.022500	.022500	.022500
.015000	.010000	.010000	.015000	.025000	.025000
.025000	.025000	.025000	.025000	.015000	.010000

LONGITUDINAL NODAL SPACING (M) ALONG CENTRAL ARC , RADIUS 100.00

1.000000	1.500000	2.500000	2.500000	2.500000	2.500000
2.500000	2.500000	1.500000	1.000000	1.000000	1.500000
2.250000	2.250000	2.250000	2.250000	2.250000	2.250000
2.250000	2.250000	2.250000	2.250000	2.250000	2.250000
1.500000	1.000000	1.000000	1.500000	2.500000	2.500000
2.500000	2.500000	2.500000	2.500000	1.500000	1.000000

LOADING CASE NO 1 DEAD LOAD ONLY

LOADING TYPE	NODAL LINE NO FROM	AT	TO	CROSSECTION NO FROM	AT	TO	MAGNITUDE OF LOAD
UNIFORM	1		9	1		37	-25.0000

LOADING CASE NO 2 PATCH LOAD CENTRAL SPAN INSIDE CARRIAGEWAY

LOADING TYPE	NODAL LINE NO FROM	AT	TO	CROSSECTION NO FROM	AT	TO	MAGNITUDE OF LOAD
UNIFORM	2		4	11		27	-25.0000

LOADING CASE NO 3 PATCH LOAD CENTRAL SPAN CENTRAL 3M WIDTH

LOADING TYPE	NODAL LINE NO FROM	AT	TO	CROSSECTION NO FROM	AT	TO	MAGNITUDE OF LOAD
UNIFORM	4		6	11		27	-25.0000

LOADING CASE NO 4 PATCH LOAD CENTRAL SPAN OUTSIDE CARRIAGEWAY

LOADING TYPE	NODAL LINE NO FROM	AT	TO	CROSSECTION NO FROM	AT	TO	MAGNITUDE OF LOAD
UNIFORM	6		8	11		27	-25.0000

INITIAL CONDITIONS OF DISPLACEMENT OR ROTATION ALONG STRUCTURE

NO	TYPE	NODAL LINE	X-SECTION	MAGNITUDE
1	DISPLACEMENT	3	11	.000000
2	DISPLACEMENT	4	11	.000000
3	DISPLACEMENT	6	11	.000000
4	DISPLACEMENT	7	11	.000000
5	DISPLACEMENT	3	27	.000000
6	DISPLACEMENT	4	27	.000000
7	DISPLACEMENT	6	27	.000000
8	DISPLACEMENT	7	27	.000000

BOUNDARY CONDITION AT START OF STRUCTURE SIMPLY SUPPORTED
BOUNDARY CONDITION AT END OF STRUCTURE SIMPLY SUPPORTED

LOAD CASE NO 1. RESULTS OF GENERALISED DISPLACEMENTS

X-SECTION	NODAL LINE	DISPLACEMENT	ROTATION
1	1	.000000	.000000
	2	.000000	.000000
	3	.000000	.000000
	4	.000000	.000000
	5	.000000	.000000
	6	.000000	.000000
	7	.000000	.000000
	8	.000000	.000000
	9	.000000	.000000
2	1	-.842748-03	-.270521-04
	2	-.831690-03	-.174947-04
	3	-.821895-03	.277432-05
	4	-.836554-03	.159970-04
	5	-.868422-03	.262598-04
	6	-.915332-03	.365234-04
	7	-.979496-03	.498326-04
	8	-.106846-02	.705343-04
	9	-.110610-02	.803775-04
3	1	-.198825-02	-.609954-04
	2	-.196319-02	-.398074-04
	3	-.194119-02	.693635-05
	4	-.197629-02	.380876-04
	5	-.205205-02	.623853-04
	6	-.216344-02	.866965-04
	7	-.231563-02	.118057-03
	8	-.252563-02	.165594-03
	9	-.261368-02	.187216-03
4	1	-.320852-02	-.893408-04
	2	-.317205-02	-.573324-04
	3	-.314438-02	.154045-04
	4	-.320684-02	.651652-04
	5	-.333461-02	.104341-03
	6	-.351987-02	.143557-03
	7	-.377067-02	.193631-03
	8	-.411249-02	.267270-03
	9	-.425404-02	.299710-03
5	1	-.314017-02	-.745592-04
	2	-.311098-02	-.430048-04
	3	-.310363-02	.281220-04
	4	-.318378-02	.759903-04
	5	-.332627-02	.113121-03
	6	-.352313-02	.150245-03
	7	-.378247-02	.198374-03
	8	-.413023-02	.270561-03
	9	-.427334-02	.302685-03
6	1	-.179860-02	-.158437-04
	2	-.179576-02	.389461-05
	3	-.183547-02	.453127-04
	4	-.192362-02	.704017-04
	5	-.204318-02	.884372-04
	6	-.218883-02	.106280-03
	7	-.236558-02	.131351-03
	8	-.259167-02	.174171-03
	9	-.268377-02	.194872-03
7	1	.355993-03	.815873-04
	2	.315967-03	.783013-04

	3	.208625-03	.643721-04
	4	.122436-03	.509639-04
	5	.546901-04	.395129-04
	6	.420552-05	.274814-04
	7	-.266206-04	.133654-04
	8	-.366443-04	.106071-05
	9	-.366485-04	-.735800-06
8	1	.241823-02	.202342-03
	2	.232643-02	.165366-03
	3	.214950-02	.764985-04
	4	.207965-02	.214657-04
	5	.207562-02	-.147023-04
	6	.212432-02	-.519566-04
	7	.224134-02	-.108720-03
	8	.246623-02	-.196261-03
	9	.257307-02	-.231499-03
9	1	.304552-02	.312967-03
	2	.290930-02	.233519-03
	3	.270104-02	.621314-04
	4	.267222-02	-.107323-04
	5	.271435-02	-.446082-04
	6	.280656-02	-.795259-04
	7	.297200-02	-.153672-03
	8	.331688-02	-.323775-03
	9	.349803-02	-.402434-03
10	1	.191236-02	.351022-03
	2	.176349-02	.246239-03
	3	.157739-02	.321371-04
	4	.157824-02	-.887413-05
	5	.159622-02	-.303493-04
	6	.166955-02	-.526296-04
	7	.175980-02	-.899693-04
	8	.202803-02	-.299744-03
	9	.220393-02	-.405893-03
11	1	.335626-03	.369122-03
	2	.179682-03	.256102-03
	3	.000000	.287965-04
	4	.000000	.182166-04
	5	-.561541-04	.194429-06
	6	.000000	-.196551-04
	7	.000000	-.208244-04
	8	.158248-03	-.241582-03
	9	.307734-03	-.358063-03
12	1	-.193994-02	.389805-03
	2	-.210932-02	.289506-03
	3	-.237039-02	.888418-04
	4	-.246471-02	.613003-04
	5	-.256199-02	.531142-04
	6	-.262374-02	.440250-04
	7	-.268837-02	.198528-04
	8	-.259459-02	-.177010-03
	9	-.248112-02	-.278899-03
13	1	-.653380-02	.409057-03
	2	-.672088-02	.340833-03
	3	-.711527-02	.203018-03
	4	-.738073-02	.163902-03
	5	-.762547-02	.163052-03
	6	-.786926-02	.160922-03
	7	-.808933-02	.119812-03
	8	-.817900-02	-.179804-04
	9	-.815341-02	-.859735-04
14	1	-.152575-01	.393603-03

	2	-.154492-01	.373847-03
	3	-.159793-01	.339716-03
	4	-.164877-01	.343421-03
	5	-.170199-01	.367622-03
	6	-.175897-01	.390239-03
	7	-.181794-01	.390738-03
	8	-.187436-01	.355485-03
	9	-.189168-01	.336494-03
15	1	-.246167-01	.355204-03
	2	-.247998-01	.376974-03
	3	-.254132-01	.441239-03
	4	-.261257-01	.509667-03
	5	-.269441-01	.581690-03
	6	-.278701-01	.652397-03
	7	-.288986-01	.717968-03
	8	-.300223-01	.780428-03
	9	-.304180-01	.802413-03
16	1	-.332819-01	.315960-03
	2	-.334537-01	.370603-03
	3	-.341227-01	.518030-03
	4	-.350003-01	.650479-03
	5	-.360708-01	.776209-03
	6	-.373284-01	.900860-03
	7	-.387759-01	.103065-02
	8	-.404277-01	.117517-02
	9	-.410286-01	.122914-02
17	1	-.402125-01	.285588-03
	2	-.403751-01	.363916-03
	3	-.410824-01	.573687-03
	4	-.420834-01	.757847-03
	5	-.433498-01	.929552-03
	6	-.448716-01	.110025-02
	7	-.466558-01	.128161-02
	8	-.487285-01	.148719-02
	9	-.494910-01	.156397-02
18	1	-.446652-01	.267126-03
	2	-.448222-01	.359700-03
	3	-.455527-01	.607812-03
	4	-.466302-01	.825025-03
	5	-.480201-01	.102689-02
	6	-.497103-01	.122770-02
	7	-.517099-01	.144178-02
	8	-.540500-01	.168462-02
	9	-.549147-01	.177501-02
19	1	-.461948-01	.260678-03
	2	-.463497-01	.358016-03
	3	-.470876-01	.619027-03
	4	-.481905-01	.847489-03
	5	-.496220-01	.105971-02
	6	-.513691-01	.127087-02
	7	-.534417-01	.149616-02
	8	-.558727-01	.175161-02
	9	-.567720-01	.184655-02

LOAD CASE NO 1. RESULTS OF BENDING AND TWISTING MOMENTS

X-SECTION	NODAL LINE	M1	M2	M12
1	1	.000000	.000000	.605991+02
	2	.000000	.000000	.439715+02
	2	.000000	.000000	.439715+02
	3	.000000	.000000	.941519+01
	3	.000000	.000000	.941519+01
	4	.000000	.000000	-.122088+02
	4	.000000	.000000	-.122088+02
	5	.000000	.000000	-.281660+02
	5	.000000	.000000	-.281660+02
	6	.000000	.000000	-.434277+02
	6	.000000	.000000	-.434277+02
	7	.000000	.000000	-.627379+02
	7	.000000	.000000	-.627379+02
	8	.000000	.000000	-.924884+02
	8	.000000	.000000	-.924884+02
	9	.000000	.000000	-.106598+03
2	1	-.235889+00	.146703+03	.570528+02
	2	-.730993+01	.143934+03	.416695+02
	2	-.818863+01	.144197+03	.416695+02
	3	-.208550+02	.140291+03	.889920+01
	3	-.204641+02	.140174+03	.889920+01
	4	-.267042+02	.139486+03	-.118894+02
	4	-.266060+02	.139456+03	-.118894+02
	5	-.288214+02	.140328+03	-.272796+02
	5	-.288324+02	.140331+03	-.272796+02
	6	-.274056+02	.142163+03	-.420039+02
	6	-.275275+02	.142199+03	-.420039+02
	7	-.216295+02	.144890+03	-.605670+02
	7	-.220469+02	.145015+03	-.605670+02
	8	-.873572+01	.149276+03	-.886649+02
	8	-.781072+01	.148999+03	-.886649+02
	9	-.233953+00	.151238+03	-.101599+03
3	1	-.255567+00	.319464+03	.427560+02
	2	-.132560+02	.314132+03	.314827+02
	2	-.147614+02	.314584+03	.314827+02
	3	-.414416+02	.305572+03	.605366+01
	3	-.409651+02	.305429+03	.605366+01
	4	-.550354+02	.303010+03	-.106907+02

	4	-.548613+02	.302958+03	-.106907+02
	5	-.597937+02	.304204+03	-.232211+02
	5	-.598173+02	.304211+03	-.232211+02
	6	-.562595+02	.307816+03	-.352191+02
	6	-.564718+02	.307880+03	-.352191+02
	7	-.428644+02	.313612+03	-.501598+02
	7	-.433347+02	.313754+03	-.501598+02
	8	-.155004+02	.322520+03	-.717267+02
	8	-.139823+02	.322065+03	-.717267+02
	9	-.253883+00	.326015+03	-.809896+02
4	1	-.284792+00	.477991+03	.866710+01
	2	-.175746+02	.470665+03	.510676+01
	2	-.193744+02	.471205+03	.510676+01
	3	-.571954+02	.457517+03	-.304045+01
	3	-.568176+02	.457404+03	-.304045+01
	4	-.779937+02	.452714+03	-.834435+01
	4	-.778261+02	.452664+03	-.834435+01
	5	-.852643+02	.453705+03	-.121776+02
	5	-.852937+02	.453714+03	-.121776+02
	6	-.795037+02	.458729+03	-.158369+02
	6	-.797079+02	.458791+03	-.158369+02
	7	-.590139+02	.467256+03	-.205613+02
	7	-.593564+02	.467359+03	-.205613+02
	8	-.202092+02	.479703+03	-.274851+02
	8	-.183883+02	.479156+03	-.274851+02
	9	-.300233+00	.484262+03	-.304219+02
5	1	-.286647+00	.471980+03	-.298038+02
	2	-.177452+02	.464635+03	-.253854+02
	2	-.195527+02	.465177+03	-.253854+02
	3	-.579089+02	.451610+03	-.143412+02
	3	-.575304+02	.451496+03	-.143412+02
	4	-.790930+02	.447140+03	-.594301+01
	4	-.789207+02	.447089+03	-.594301+01
	5	-.866269+02	.448544+03	.958046+00
	5	-.866679+02	.448556+03	.958046+00
	6	-.809728+02	.453981+03	.763044+01
	6	-.811955+02	.454048+03	.763044+01
	7	-.602092+02	.462938+03	.151350+02
	7	-.605498+02	.463040+03	.151350+02
	8	-.205900+02	.475829+03	.240907+02
	8	-.187317+02	.475271+03	.240907+02

	9	-.306276+00	.480515+03	.274368+02
6	1	-.265800+00	.300709+03	-.650972+02
	2	-.140336+02	.295241+03	-.527555+02
	2	-.156484+02	.295725+03	-.527555+02
	3	-.441684+02	.287164+03	-.232610+02
	3	-.436423+02	.287007+03	-.232610+02
	4	-.585975+02	.286098+03	-.277862+01
	4	-.583573+02	.286026+03	-.277862+01
	5	-.640420+02	.288968+03	.127561+02
	5	-.641099+02	.288988+03	.127561+02
	6	-.612347+02	.294034+03	.277919+02
	6	-.615657+02	.294133+03	.277919+02
	7	-.474168+02	.301285+03	.462234+02
	7	-.479160+02	.301434+03	.462234+02
	8	-.171538+02	.311946+03	.706439+02
	8	-.154225+02	.311427+03	.706439+02
	9	-.285526+00	.315962+03	.803026+02
7	1	-.251369+00	-.358230+02	-.885875+02
	2	-.524078+01	-.371460+02	-.686788+02
	2	-.644082+01	-.367860+02	-.686788+02
	3	-.108034+02	-.347081+02	-.238792+02
	3	-.986152+01	-.349907+02	-.238792+02
	4	-.813797+01	-.291604+02	.269869+01
	4	-.774099+01	-.292795+02	.269869+01
	5	-.806799+01	-.240520+02	.198562+02
	5	-.817779+01	-.240190+02	.198562+02
	6	-.118577+02	-.205990+02	.365486+02
	6	-.124143+02	-.204320+02	.365486+02
	7	-.152670+02	-.180001+02	.607867+02
	7	-.161954+02	-.177216+02	.607867+02
	8	-.866567+01	-.133408+02	.987702+02
	8	-.720449+01	-.137792+02	.987702+02
	9	-.269172+00	-.112117+02	.114867+03
8	1	-.267949+00	-.534347+03	-.876150+02
	2	.110439+02	-.528691+03	-.612071+02
	2	.103127+02	-.528471+03	-.612071+02
	3	.544274+02	-.510173+03	-.774991+01
	3	.567007+02	-.510855+03	-.774991+01
	4	.928295+02	-.497142+03	.116524+02
	4	.931634+02	-.497243+03	.116524+02
	5	.102934+03	-.491304+03	.184373+02

	5	.102799+03	-.491264+03	.184373+02
	6	.882917+02	-.490651+03	.250719+02
	6	.877108+02	-.490477+03	.250719+02
	7	.505494+02	-.495983+03	.429192+02
	7	.483566+02	-.495325+03	.429192+02
	8	.791179+01	-.502792+03	.897880+02
	8	.898567+01	-.503115+03	.897880+02
	9	-.292086+00	-.504695+03	.112343+03
9	1	-.594124+00	-.116822+04	-.480272+02
	2	.353210+02	-.115443+04	-.191964+02
	2	.332403+02	-.115381+04	-.191964+02
	3	.175051+03	-.112741+04	.307733+02
	3	.183724+03	-.113002+04	.307733+02
	4	.290182+03	-.112434+04	.126590+02
	4	.286452+03	-.112322+04	.126590+02
	5	.295516+03	-.113035+04	.359846+01
	5	.295547+03	-.113035+04	.359846+01
	6	.284376+03	-.112603+04	-.538725+01
	6	.287732+03	-.112704+04	-.538725+01
	7	.184196+03	-.113260+04	-.248328+02
	7	.176153+03	-.113018+04	-.248328+02
	8	.348458+02	-.114817+04	.199857+02
	8	.366748+02	-.114872+04	.199857+02
	9	-.589735+00	-.115654+04	.471543+02
10	1	-.171226+01	-.152389+04	-.132736+02
	2	.404519+02	-.151589+04	.644425+01
	2	.304730+02	-.151289+04	.644425+01
	3	.295643+03	-.155659+04	.377494+02
	3	.309639+03	-.156079+04	.377494+02
	4	.529788+03	-.161704+04	-.679186+01
	4	.517738+03	-.161342+04	-.679186+01
	5	.395520+03	-.162962+04	-.147960+02
	5	.396366+03	-.162987+04	-.147960+02
	6	.519541+03	-.162945+04	-.222631+02
	6	.530782+03	-.163283+04	-.222631+02
	7	.333317+03	-.159324+04	-.701468+02
	7	.318936+03	-.158893+04	-.701468+02
	8	.394933+02	-.154820+04	-.418367+02
	8	.484894+02	-.155090+04	-.418367+02
	9	-.166458+01	-.155376+04	-.208386+02
11	1	-.289033+01	-.162040+04	.149139+01
	2	.352901+02	-.162704+04	-.227719+01

	2	-.215533+02	-.160998+04	-.227719+01
	3	.439651+03	-.188836+04	-.132220+02
	3	.347644+03	-.186075+04	-.132220+02
	4	.826761+03	-.215189+04	-.236962+02
	4	.858662+03	-.216146+04	-.236962+02
	5	.300138+03	-.186808+04	-.335588+02
	5	.302155+03	-.186868+04	-.335588+02
	6	.859406+03	-.218698+04	-.429110+02
	6	.819547+03	-.217502+04	-.429110+02
	7	.413874+03	-.194721+04	-.518390+02
	7	.499462+03	-.197289+04	-.518390+02
	8	-.185295+02	-.167428+04	-.601911+02
	8	.438692+02	-.169300+04	-.601911+02
	9	-.300758+01	-.167879+04	-.628622+02
12	1	-.172919+01	-.146143+04	.164024+02
	2	.404396+02	-.145367+04	-.108597+02
	2	.304434+02	-.145067+04	-.108597+02
	3	.295894+03	-.149513+04	-.640492+02
	3	.309943+03	-.149934+04	-.640492+02
	4	.530260+03	-.155635+04	-.404924+02
	4	.518162+03	-.155272+04	-.404924+02
	5	.395780+03	-.156955+04	-.521330+02
	5	.396629+03	-.156980+04	-.521330+02
	6	.519960+03	-.156965+04	-.632963+02
	6	.531249+03	-.157304+04	-.632963+02
	7	.333616+03	-.153366+04	-.333250+02
	7	.319185+03	-.152933+04	-.333250+02
	8	.394439+02	-.148890+04	-.783571+02
	8	.484799+02	-.149161+04	-.783571+02
	9	-.168932+01	-.149460+04	-.104706+03
13	1	-.640815+00	-.101168+04	.518117+02
	2	.351718+02	-.998551+03	.153774+02
	2	.330030+02	-.997900+03	.153774+02
	3	.175693+03	-.973814+03	-.564369+02
	3	.184562+03	-.976475+03	-.564369+02
	4	.291610+03	-.972951+03	-.594740+02
	4	.287681+03	-.971772+03	-.594740+02
	5	.295659+03	-.980567+03	-.695319+02
	5	.295698+03	-.980578+03	-.695319+02
	6	.285597+03	-.976797+03	-.786737+02
	6	.289153+03	-.977864+03	-.786737+02

	7	.185104+03	-.983636+03	-.774183+02
	7	.176868+03	-.981165+03	-.774183+02
	8	.345628+02	-.999460+03	-.139027+03
	8	.365059+02	-.100004+04	-.139027+03
	9	-.669740+00	-.100810+04	-.171617+03
14	1	-.352689+00	-.298335+03	.915429+02
	2	.130846+02	-.292875+03	.571167+02
	2	.123488+02	-.292654+03	.571167+02
	3	.633006+02	-.275545+03	-.182037+02
	3	.658087+02	-.276297+03	-.182037+02
	4	.106750+03	-.264588+03	-.562065+02
	4	.107012+03	-.264667+03	-.562065+02
	5	.117938+03	-.261088+03	-.802890+02
	5	.117810+03	-.261050+03	-.802890+02
	6	.102023+03	-.262303+03	-.103302+03
	6	.101515+03	-.262151+03	-.103302+03
	7	.594446+02	-.269761+03	-.136961+03
	7	.570419+02	-.269040+03	-.136961+03
	8	.944159+01	-.278994+03	-.200855+03
	8	.105565+02	-.279329+03	-.200855+03
	9	-.422156+00	-.281964+03	-.229393+03
15	1	-.300175+00	.308778+03	.100230+03
	2	-.303728+01	.307059+03	.715132+02
	2	-.410859+01	.307380+03	.715132+02
	3	-.198095+01	.307967+03	.320555+01
	3	-.922976+00	.307650+03	.320555+01
	4	.526226+01	.311477+03	-.430647+02
	4	.569357+01	.311348+03	-.430647+02
	5	.674965+01	.314049+03	-.776631+02
	5	.663656+01	.314083+03	-.776631+02
	6	.114819+01	.314671+03	-.110925+03
	6	.550893+00	.314850+03	-.110925+03
	7	-.711874+01	.314183+03	-.152114+03
	7	-.815496+01	.314493+03	-.152114+03
	8	-.742571+01	.315777+03	-.208626+03
	8	-.595846+01	.315337+03	-.208626+03
	9	-.422126+00	.316633+03	-.231332+03
16	1	-.365303+00	.784091+03	.876376+02
	2	-.127380+02	.777443+03	.656877+02
	2	-.142767+02	.777904+03	.656877+02
	3	-.393678+02	.765973+03	.106803+02

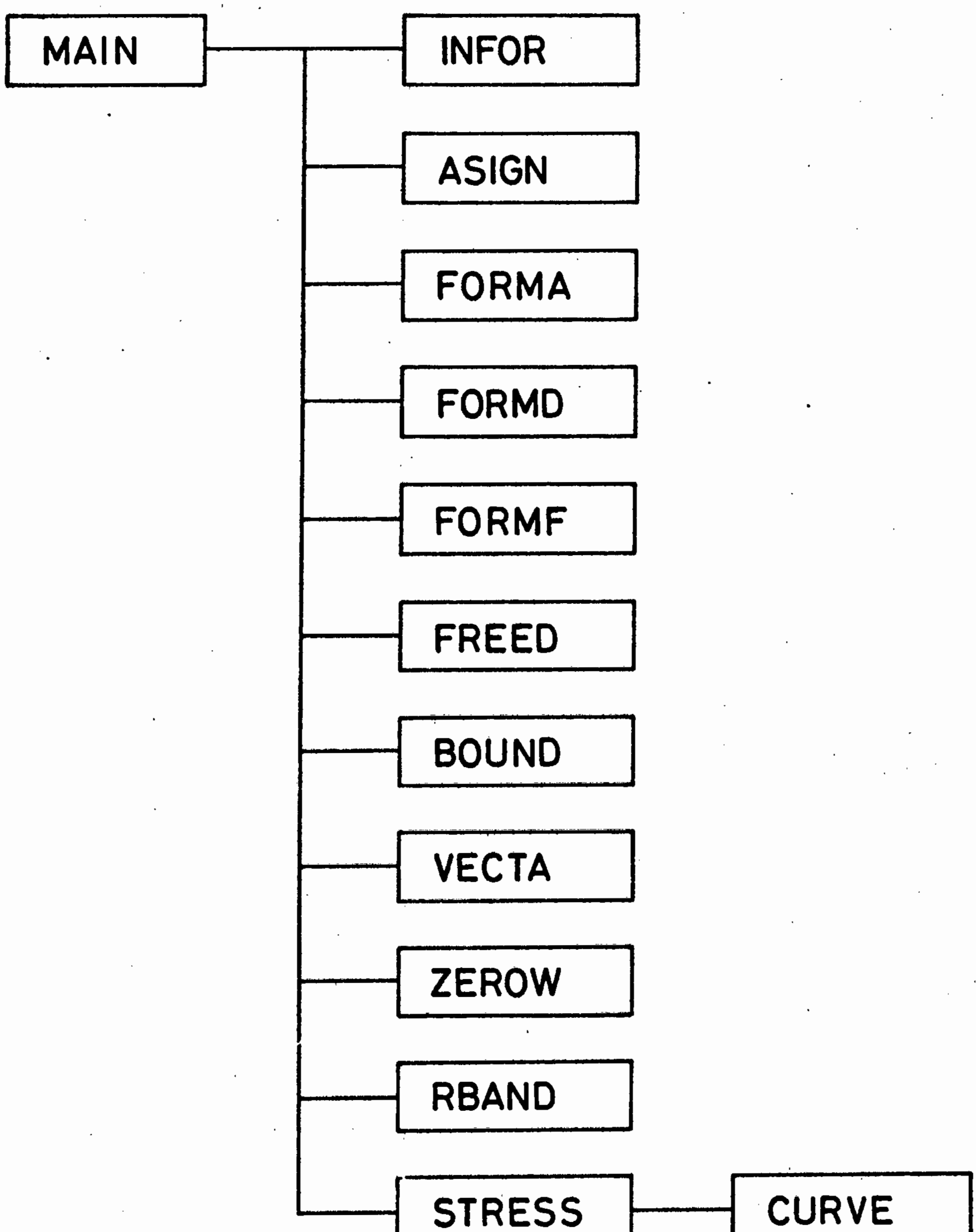
	3	-.387988+02	.765803+03	.106803+02
	4	-.514991+02	.761575+03	-.311557+02
	4	-.512451+02	.761499+03	-.311557+02
	5	-.562893+02	.761118+03	-.654576+02
	5	-.563590+02	.761139+03	-.654576+02
	6	-.545991+02	.762895+03	-.983641+02
	6	-.549490+02	.763000+03	-.983641+02
	7	-.435124+02	.766995+03	-.135274+03
	7	-.440399+02	.767153+03	-.135274+03
	8	-.171057+02	.774787+03	-.179855+03
	8	-.153823+02	.774270+03	-.179855+03
	9	-.409000+00	.777560+03	-.196687+03
17	1	-.316978+00	.112372+04	.632247+02
	2	-.182762+02	.111403+04	.484576+02
	2	-.200034+02	.111454+04	.484576+02
	3	-.600102+02	.109450+04	.102996+02
	3	-.596619+02	.109439+04	.102996+02
	4	-.824399+02	.108447+04	-.204187+02
	4	-.822820+02	.108442+04	-.204187+02
	5	-.904680+02	.108127+04	-.467187+02
	5	-.905088+02	.108128+04	-.467187+02
	6	-.846354+02	.108306+04	-.718739+02
	6	-.848432+02	.108312+04	-.718739+02
	7	-.630516+02	.108915+04	-.987307+02
	7	-.633388+02	.108924+04	-.987307+02
	8	-.222998+02	.109974+04	-.129192+03
	8	-.204747+02	.109920+04	-.129192+03
	9	-.438519+00	.110337+04	-.140287+03
18	1	-.352808+00	.132716+04	.329440+02
	2	-.210988+02	.131584+04	.255297+02
	2	-.230023+02	.131641+04	.255297+02
	3	-.702865+02	.129188+04	.608337+01
	3	-.700475+02	.129181+04	.608337+01
	4	-.976972+02	.127857+04	-.999360+01
	4	-.975931+02	.127854+04	-.999360+01
	5	-.107305+03	.127356+04	-.240142+02
	5	-.107319+03	.127356+04	-.240142+02
	6	-.994046+02	.127487+04	-.374032+02
	6	-.995372+02	.127491+04	-.374032+02
	7	-.726697+02	.128133+04	-.514014+02
	7	-.728842+02	.128139+04	-.514014+02
	8	-.246709+02	.129237+04	-.668126+02

	8	-.227752+02	.129180+04	-.668126+02
	9	-.367063+00	.129610+04	-.723222+02
19	1	-.326398+00	.139496+04	.136018+00
	2	-.219103+02	.138301+04	.169069+00
	2	-.238928+02	.138360+04	.169069+00
	3	-.733860+02	.135743+04	.279063+00
	3	-.731794+02	.135737+04	.279063+00
	4	-.102311+03	.134298+04	.414692+00
	4	-.102220+03	.134296+04	.414692+00
	5	-.112387+03	.133736+04	.557976+00
	5	-.112404+03	.133736+04	.557976+00
	6	-.103807+03	.133854+04	.691106+00
	6	-.103906+03	.133857+04	.691106+00
	7	-.754551+02	.134532+04	.795584+00
	7	-.755913+02	.134536+04	.795584+00
	8	-.255614+02	.135711+04	.869058+00
	8	-.235901+02	.135652+04	.869058+00
	9	-.512424+00	.136111+04	.895549+00

APPENDIX G

Sequence of Subroutines
Listing of Program STRIP

SEQUENCE OF SUBROUTINES FOR PROGRAM STRIP



FINITE*STRIP(1).MAIN

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      CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

C     A FINITE DIFFERENCE BASED FINITE STRIP METHOD OF ANALYSIS HAS BEEN
C     DEVELOPED FOR THE SOLUTION OF CURVED PLATE STRUCTURES SUBJECT TO
C     NORMAL LOADING. SIMPLY SUPPORTED AND CONTINUOUS STRUCTURES MAY BE
C     ANALYSED. A FINITE ELEMENT SUBDIVISION IS USED ACROSS THE PLATE ,
C     WHERE A THIRD ORDER POLYNOMIAL DISPLACEMENT FUNCTION DEFINES THE
C     VARIATION IN DISPLACEMENT ACROSS EACH STRIP ELEMENT. USING THE
C     PRINCIPLE OF MINIMUM POTENTIAL ENERGY THE GOVERING DIFFERENTIAL
C     EQUATION IS FORMED. THE SOLUTION INVOLVES SUBSTITUTING FINITE
C     DIFFERENCE APPROXIMATIONS TO THE DIFFERENITAL EQUATION ALONG THE
C     LENGTH OF THE STRIPS, WHERE IN THE PRESENT FORMULATION, THE NODAL
C     SPACING MAY BE VARIED IN ANY REGION. THE RESULTING SET OF
C     SIMULTANEOUS LINEAR EQUATIONS IS THEN SOLVED. THE DISPLACEMENTS
C     AND ROTATIONS OBTAINED AT EACH NODAL CROSS SECTION ARE THEN USED
C     TO SOLVE FOR THE INTERNAL FORCES. THE UNITS ARE IN  N , M , S.

C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

C     DIMENSIONING IS FOR A MAX OF 8 STRIPS 40 X-SECTIONS 14 LOAD CASES

      COMMON ITLE(80),RD(9),W(8),DT(8),DS(8),DR(8),DRS(8),D1(8)
      COMMON DL(39),DA(39),AX(4,4),BX(4,4),CX(4,4),DX(4,4),FX(4,4)
      COMMON FDSM(720,54),AA(18,18),DD(18,18),FF(18,18),IB(8),IE(8)
      COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,ER,GRS,VS,VR,G(3)
      COMMON WI1,WI2,WI3,TI1,TI2,TI3,WJ1,WJ2,WJ3,TJ1,TJ2,TJ3
      COMMON LABL(30),IL(4),IZ(6)
      INTEGER BC(6)/'CL','SI','FR','GU','DI','RO'/
      INTEGER WL(5)/'PO','LI','UN','NE','LA'/

C     READ AND PRINT TITLE CARDS FOR PLATE

      READ(8,100) ITLE
  100 FORMAT (40A2)
      WRITE(5,200)ITLE
  200 FORMAT(1H1,7X,40A2,/,8X,40A2)

C     PROGRAM INFOR - READ MATERIAL CONSTANTS AND PLATE GEOMETRY
C                        CALCULATE STRIP PROPERTIES , NODAL SPACING
      CALL INFOR

      WRITE(5,201) ES , ER , GRS , VS , VR
  201 FORMAT(///,8X,'MATERIAL AND SECTION PROPERTIES  ***  UNITS ARE IN
     1 N , M , S',/,8X,61('-'),//,8X,'ELASTIC MODULUS    ES  = ',E12.5,
     24X,'ER  = ',E12.5,//,8X,'SHEAR MODULUS     GRS  = ',E12.5,//,8X,
     3'POISSONS RATIO     VS  =',F9.4,8X,'VR  =',F9.4,/)
      WRITE(5,202)
  202 FORMAT(/,8X,'STRIP   LINE   RADIUS   WIDTH   DEPTH   BENDING AND T
     1ORSIONAL RIGIDITY',/,10X,'NO',5X,'NO',/,51X,'DS',9X,'DR',9X,'DRS')
      DO 600 I=1,NST
      WRITE(5,203) I , RD(I)
  600 WRITE(5,204) I , W(I) , DT(I) , DS(I) , DR(I) , DRS(I)
  204 FORMAT(1H ,8X,I2,20X,F6.2,F7.2,3E11.4)
  203 FORMAT(1H ,15X,I2,F12.2)
      WRITE(5,203) NNL , RD(NNL)
      WRITE(5,205)
  205 FORMAT(//,8X,'ANGULAR NODAL SPACING (RAD) USED IN FINITE DIFFERENC
     1E APPROXIMATIONS',/)
      WRITE(5,206)  (DA(I),I=1,NDA)
  206 FORMAT(1H ,3X,6F12.6)
```

```
      RAVE = (RD(1) + RD(NNL))/2.
      WRITE(5,207) RAVE
  207 FORMAT(/,8X,'LONGITUDINAL NODAL SPACING (M) ALONG CENTRAL ARC',
     1' , RADIUS',F11.2,/)
      WRITE(5,206)  (DL(I),I=1,NDA)

C     SUBROUTINE ASIGN - INITIALIZE FINITE DIFFERENCE STIFFNESS MATRIX

      CALL ASIGN

C     SUBROUTINE FORMA - COMPILE 'A' MATRIX FOR COMPLETE CROSS-SECTION

      CALL FORMA

C     SUBROUTINE FORMD - COMPILE 'D' MATRIX FOR COMPLETE CROSS-SECTION

      CALL FORMD

C     SUBROUTINE FORMF - COMPILE 'F' MATRIX FOR COMPLETE CROSS-SECTION

      CALL FORMF

C     SUBROUTINE FREED - COMPILE ELTS OF FDSM FREE OF BDY CONDITIONS

      CALL FREED

C     SUBROUTINE BOUND - COMPILE ELTS OF FDSM SUBJECT TO BDY CONDITIONS

      CALL BOUND

C     SUBROUTINE VECTA - COMPILE ELTS OF LOAD VECTOR FOR NLC LOAD CASES

      CALL VECTA

C     SUBROUTINE ZEROW - SET BDY CONDS AT ENDS AND ALONG NODAL LINES

      CALL ZEROW

C     SUBROUTINE RBAND - INVERT AUGMENTED STIFFNESS MATRIX AS COMPILED

      CALL RBAND

      DO 400  LCN = 1,NLC
      WRITE(5,15) LCN
   15 FORMAT(1H1,/,8X,'LOAD CASE NO',I3,'.    RESULTS OF GENERALISED DIS
     1PLACEMENTS',/,8X,56('-'),//,8X,'X-SECTION',4X,'NODAL LINE',4X,
     2'DISPLACEMENT',7X,'ROTATION',/)
      JC = NBW + LCN
      DO 500 I = 1,NXS
      I1 = 2*(NNL*(I-1)) + 1
      I2 = I1 + 1
      WRITE(5,16) I , FDSM(I1,JC) , FDSM(I2,JC)
   16 FORMAT(1H ,10X,I2,12X,'1',8X,E12.6,6X,E12.6)
      DO 500 J = 2,NNL
      I1 = 2*(NNL*(I-1) + J) - 1
      I2 = I1 + 1
  500 WRITE(5,17) J , FDSM(I1,JC) , FDSM(I2,JC)
   17 FORMAT(1H ,23X,I2,8X,E12.6,6X,E12.6)
      WRITE(5,18) LCN
   18 FORMAT(1H1,/,8X,'LOAD CASE NO',I3,'.    RESULTS OF BENDING AND TWI
     1STING MOMENTS',/,8X,59('-'),//,8X,'X-SECTION',4X,'NODAL LINE',9X,
     2'M1',14X,'M2',14X,'M12',/)

C     SUBROUTINE STRES - CALCULATE STRESS RESULTANTS FROM DISPLACEMENTS

  400 CALL STRES(JC)
      CALL EXIT
      END
```

INITE*STRIP(1).INFOR

```
      SUBROUTINE INFOR
      COMMON ITLE(80),RD(9),W(8),DT(8),DS(8),DR(8),DRS(8),D1(8)
      COMMON DL(39),DA(39),AY(4,4),BY(4,4),CX(4,4),DX(4,4),FX(4,4)
      COMMON FDSM(720,54),AA(18,18),DD(18,18),FF(18,18),TB(8),TE(8)
      COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,ER,GRS,VS,VR,G(3)
      COMMON WT1,WT2,WT3,TT1,TT2,TT3,WJ1,WJ2,WJ3,TJ1,TJ2,TJ3
      COMMON LABL(30),TL(4),TZ(6)
      INTEGER BC(6)/'CL','SI','FR','GU','DI','RO'/

C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

C     READ IN ALL INFORMATION REQUIRED TO DEFINE CURVED PLATE STRUCTURE

C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

C     READ IN NO OF STRIPS, NO OF X-SECT, RADIUS OF 1ST NODAL LINE

      READ(8,101) NST , NXS , NLC , RD(1)
101   FORMAT(3I5,F12.2)

C     NNL = NO OF NODAL LINES              NDA = NO OF DA'S (ANGULAR
C     MSM = SIZE OF AA,FF MATRIX           FINITE DIFFERENCE SPACING)
C     NBL = AUGMENTED WIDTH OF FDSM        NSM = SIZE OF MATRIX FDSM
C     NLC = NO OF LOAD CASES               NBW = BAND WIDTH OF FDSM

      NNL = NST + 1
      NDA = NXS - 1
      MSM = 2*NST + 2
      NSM = MSM*NXS
      NBW = 2*MSM + 4
      NBL = 3*MSM
      IF(MSM-4-NLC)1,2,2
    1 NBL = NBW + NLC

C     READ IN MATERIAL PROPERTIES  ES , ER , GRS , VR , VS

    2 READ(8,102) ES , ER , GRS , VS , VR
102   FORMAT(3F15.2,2F8.4)

C     CALC REMAINING SECTION PROPERTIES IF UNSPECIFIED

      IF(ES)10,11,10
11    ES = ER*VS/VR
10    IF(ER)12,13,12
13    ER = ES*VR/VS
12    IF(VS)14,15,14
15    VS = VR*ES/ER
14    IF(VR)16,17,16
17    VR = VS*ER/ES
16    IF(GRS)18,19,18
19    GRS = 0.5*(ES*ER)**.5/(1 + (VS*VR)**.5)

C     READ IN  STRIP WIDTHS AND STRIP THICKNESS

18    READ(8,103) (W(I),DT(I),I=1,NST)
103   FORMAT(2F12.4)

C     READ IN ANGULAR FINITE DIFFERENCE SPACING  DA(I)

      READ(8,104) (DA(I),I=1,NDA)
104   FORMAT(8F10.6)
```

```
C      CALC NODAL RADII AND RIGIDITY CONSTANTS  DS , DR , D1 , DRS

       DO 500 I=1,NST
       RD(I+1) = RD(I) + W(I)
       DS(I)   = ES*DT(I)**3/(12.*(1 - VS*VR))
       DR(I)   = ER*DT(I)**3/(12.*(1 - VS*VR))
       D1(I)   = DR(I)*VS
  500  DRS(I)  = GRS*DT(I)**3/12.

C      CALC FINITE DIFFERENCE SPACING ALONG CENTRAL ARC

       DO 501 I=1,NDA
  501  DL(I)   = DA(I)*(RD(1) + RD(NNL))/2.

C      READ IN BEGIN AND END BOUNDARY CONDITIONS  IB , IE

       READ(8,105) IB , IE
   105 FORMAT(8A2,4X,8A2)
       RETURN
       END
```

FINITE*STRIP(1).ASIGN

```
      SUBROUTINE ASIGN
      COMMON ITLE(80),PD(9),W(8),DT(8),DS(8),DR(8),DRS(8),D1(8)
      COMMON DL(39),DA(39),AX(4,4),BX(4,4),CX(4,4),DX(4,4),FX(4,4)
      COMMON FDSM(720,54),AA(18,18),DD(18,18),FF(18,18),TB(8),TE(8)
      COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,FR,GRS,VS,VR,G(3)

C     INITIALISE F D STIFFNESS MATRIX EQUAL TO ZERO

      DO 1  I = 1,NSM
      DO 1  J = 1,NBL
    1 FDSM(I,J) = 0.0
      RETURN
      END
```

```
ITE*STRIP(1).FORMA
            SUBROUTINE FORMA
            COMMON ITLE(80),RD(9),W(8),DT(8),DS(8),DR(8),DRS(8),D1(8)
            COMMON DL(39),DA(39),AX(4,4),BX(4,4),CX(4,4),DX(4,4),FX(4,4)
            COMMON FDSM(720,54),AA(18,18),DD(18,18),FF(18,18),IB(8),IF(8)
            COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,ER,GRS,VS,VR,G(3)
            COMMON WI1,WI2,WI3,TI1,TI2,TI3,WJ1,WJ2,WJ3,TJ1,TJ2,TJ3
            COMMON LABL(30),TL(4),IZ(6)

      C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      C     SET UP 'A' MATRIX FOR EACH STRIP AND COMPILE FOR COMPLETE X-SECT

      C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

            DO 1 I  = 1,MSM
            DO 1 J  = 1,MSM
          1 AA(I,J) = 0.0

            DO 50 II = 1,NST
            B  =  W(II)
            RI = RD(II)
            RJ = RD(II+1)
            RM = (RI + RJ)/2.

            DO 2 I  = 1,4
            DO 2 J  = 1,4
          2 AX(I,J) = 0.0

      C     SET UP ELEMENTS OF  MATRIX AX FOR STRIP II

            AX(1,1) =      13.*B/(35.*RM**3)   + 3.*B**2/(10.*RM**4)
            AX(2,1) = -11.*B**2/(210.*RM**3) -      B**3/(35.*RM**4)
            AX(3,1) =       9.*B/(70.*RM**3)
            AX(4,1) =  13.*B**2/(420.*RM**3) +      B**3/(280.*RM**4)
            AX(2,2) =       B**3/(105.*RM**3) +     B**4/(280.*RM**4)
            AX(3,2) = -13.*B**2/(420.*RM**3) +      B**3/(280.*RM**4)
            AX(4,2) =      -B**3/(140.*RM**3)
            AX(3,3) =      13.*B/(35.*RM**3)   - 3.*B**2/(10.*RM**4)
            AX(4,3) =  11.*B**2/(210.*RM**3) -      B**3/(35.*RM**4)
            AX(4,4) =       B**3/(105.*RM**3) -     B**4/(280.*RM**4)

      C     MULTIPLY AX MATRIX BY BENDING RIGIDITY DS FOR STRIP II

            DO 3  I = 1,4
            DO 3  J = 1,I
          3 AX(I,J) = AX(I,J) * DS(II)

      C     ASSIGN VALUES TO SYMMETRICAL ELEMENTS OF MATRIX AX

            AX(1,2) =  AX(2,1)
            AX(1,3) =  AX(3,1)
            AX(1,4) =  AX(4,1)
            AX(2,3) =  AX(3,2)
            AX(2,4) =  AX(4,2)
            AX(3,4) =  AX(4,3)

      C     ADD AX MATRIX FOR STRIP II TO MATRIX AA

            DO 4 I = 1,4
            K = 2*II-2+I
            DO 4 J = 1,4
            L = 2*II-2+J
          4 AA(K,L) = AA(K,L) + AX(I,J)

         50 CONTINUE
            RETURN
            END
```

FINITE*STRIP(1).FORMD

```
      SUBROUTINE FORMD
      COMMON ITLE(80),RD(9),W(8),DT(8),DS(8),DR(8),DRS(8),D1(8)
      COMMON DL(39),DA(39),AX(4,4),BX(4,4),CX(4,4),DX(4,4),FX(4,4)
      COMMON FDSM(720,54),AA(18,18),DD(18,18),FF(18,18),IB(8),IE(8)
      COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,ER,GRS,VS,VR,G(3)
      COMMON WT1,WT2,WT3,TT1,TT2,TT3,WJ1,WJ2,WJ3,TJ1,TJ2,TJ3
      COMMON LABL(30),TL(4),TZ(6)

C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

C     SET UP 'D' MATRIX FOR EACH STRIP AND COMPILE FOR COMPLETE X-SECT

C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      DO 1 I  = 1,MSM
      DO 1 J  = 1,MSM
    1 DD(I,J) = 0.0

      DO 50 II = 1,NST
      B  =  W(II)
      RI =  RD(II)
      RJ =  RD(II+1)
      RM =  (RI + RJ)/2.

C     . SET UP ELEMENTS OF MATRIX DX FOR STRIP II

      DO 2 I  = 1,4
      DO 2 J  = 1,4
    2 DX(I,J) = 0.0

      DX(1,1) = 12.*RM/B**3
      DX(2,1) = -6.*RM/B**2 + 1./B
      DX(4,1) = -6.*RM/B**2 - 1./B
      DX(2,2) =  4.*RM/B    - (1. + VS)
      DX(4,2) =  2.*RM/B
      DX(4,4) =  4.*RM/B    + (1. + VS)

      DX(3,1) = -DX(1,1)
      DX(3,2) = -DX(2,1)
      DX(3,3) =  DX(1,1)
      DX(4,3) = -DX(4,1)

      DO 4  I = 1,4
      DO 4  J = 1,I
    4 DX(I,J) = DX(I,J)*DR(II)

C     ASSIGN VALUES TO SYMMETRICAL ELEMENTS OF MATRIX 'DX'

      DX(1,2) = DX(2,1)
      DX(1,3) = DX(3,1)
      DX(1,4) = DX(4,1)
      DX(2,3) = DX(3,2)
      DX(2,4) = DX(4,2)
      DX(3,4) = DX(4,3)

C     ADD DX MATRIX FOR STRIP II TO MATRIX DD

      DO 5 I = 1,4
      K = 2*II-2+I
      DO 5 J = 1,4
      L = 2*II-2+J
    5 DD(K,L) = DD(K,L) + DX(I,J)

   50 CONTINUE
      RETURN
      END
```

```
FINITE*STRIP(1).FORMF
           SUBROUTINE FORMF
           COMMON ITLF(80),RD(9),W(8),DT(8),DS(8),DR(8),DRS(8),D1(8)
           COMMON DL(39),DA(39),AX(4,4),BX(4,4),CX(4,4),DX(4,4),FX(4,4)
           COMMON FDSM(720,54),AA(18,18),DD(18,18),FF(18,18),TB(8),TE(8)
           COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,ER,GRS,VS,VR,G(3)
           COMMON WT1,WT2,WT3,TT1,TT2,TT3,WJ1,WJ2,WJ3,TJ1,TJ2,TJ3
           COMMON LABL(30),IL(4),IZ(6)

     C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

     C     SET UP 'F' MATRIX FOR EACH STRIP AND COMPILE FOR COMPLETE X-SECT

     C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

           DO 1 I  = 1,MSM
           DO 1 J  = 1,MSM
         1 FF(I,J) = 0.0

           DO 50 II = 1,NST
           B  =  W(II)
           RI =  RD(II)
           RJ =  RD(II+1)
           RM =  (RI + RJ)/2.

     C     SET UP 'INITIAL' COMPONENTS OF MATRIX BX FOR STRIP IT

           DO 2 I  = 1,4
           DO 2 J  = 1,4
         2 BX(I,J) = 0.0

           BX(1,1) =   -6.*D1(II)/(5.*B*RM)
           BX(2,1) =        D1(II)/(10.*RM)
           BX(1,2) =   11.*D1(II)/(10.*RM)
           BX(2,2) = -2.*B*D1(II)/(15.*RM)
           BX(4,2) =      B*D1(II)/(30.*RM)

           BX(3,1) = -BX(1,1)
           BX(4,1) =  BX(2,1)
           BX(3,2) = -BX(2,1)

           BX(1,3) = -BX(1,1)
           BX(2,3) = -BX(2,1)
           BX(3,3) = -BX(3,1)
           BX(4,3) = -BX(4,1)

           BX(1,4) =  BX(4,1)
           BX(2,4) =  BX(4,2)
           BX(3,4) = -BX(1,2)
           BX(4,4) =  BX(2,2)

     C     SET UP 'RADIAL' COMPONENTS OF MATRIX BX IN MATRIX AX

           DO 3 I  = 1,4
           DO 3 J  = 1,4
         3 AX(I,J) = 0.0

           AX(1,1) =       -D1(II)/(2.*RM**2)  -      DS(II)/(2.*RM**2)
           AX(2,1) =      B*D1(II)/(20.*RM**2) +    B*DS(II)/(10.*RM**2)
           AX(1,2) =   7.*B*D1(II)/(20.*RM**2) -    B*DS(II)/(10.*RM**2)
           AX(2,2) = -B**2*D1(II)/(30.*RM**2)
           AX(3,2) =   3.*B*D1(II)/(20.*RM**2) +    B*DS(II)/(10.*RM**2)
```

```
      AX(4,2) =  B**2*D1(II)/(60.*RM**2) + B**2*DS(II)/(60.*RM**2)
      AX(3,1) =  AX(1,1)
      AX(4,1) = -AX(2,1)

      AX(1,3) = -AX(1,1)
      AX(2,3) = -AX(2,1)
      AX(3,3) = -AX(3,1)
      AX(4,3) = -AX(4,1)

      AX(1,4) =  AX(3,2)
      AX(2,4) = -AX(4,2)
      AX(3,4) =  AX(1,2)
      AX(4,4) = -AX(2,2)

C     SUM 'INITIAL' AND 'RADIAL' COMPONENTS INTO MATRIX BX

      DO 4  I = 1,4
      DO 4  J = 1,4
    4 BX(I,J) = BX(I,J) + AX(I,J)

C     SET UP 'INITIAL' COMPONENTS OF MATRIX CX FOR STRIP II

      DO 5 I  = 1,4
      DO 5 J  = 1,4
    5 CX(I,J) = 0.0

      CX(1,1) =   24./(5.*B*RM)
      CX(2,1) =   -2./(5.*RM)
      CX(2,2) =  8.*B/(15.*RM)
      CX(4,2) = -2.*B/(15.*RM)

      CX(3,1) = -CX(1,1)
      CX(4,1) =  CX(2,1)
      CX(3,2) = -CX(2,1)
      CX(3,3) =  CX(1,1)
      CX(4,3) = -CX(2,1)
      CX(4,4) =  CX(2,2)

C     SET UP 'RADIAL' COMPONENTS OF MATRIX CX IN MATRIX AX

      DO 6 I  = 1,4
      DO 6 J  = 1,4
    6 AX(I,J) = 0.0

      AX(1,1) =       4./RM**2
      AX(2,1) =        B/(5.*RM**2)
      AX(2,2) = 2.*B**2/(15.*RM**2)

      AX(4,1) = -AX(2,1)
      AX(3,2) = -AX(2,1)
      AX(3,3) = -AX(1,1)
      AX(4,3) =  AX(2,1)
      AX(4,4) = -AX(2,2)

C     SUM 'INITIAL' AND 'RADIAL' COMPONENTS INTO MATRIX CX

      DO 7  I = 1,4
      DO 7  J = 1,I
    7 CX(I,J) = CX(I,J) + AX(I,J)

C     NOW MULTIPLY MATRIX CX BY TORSIONAL RIGIDITY DRS(II)

```

```
      DO 8  I = 1,4
      DO 8  J = 1,I
    8 CX(I,J) = CX(I,J) * DRS(II)

C     ASSIGN VALUES TO SYMMETRICAL ELEMENTS OF MATRIX CX

      CX(1,2) = CX(2,1)
      CX(1,3) = CX(3,1)
      CX(1,4) = CX(4,1)
      CX(2,3) = CX(3,2)
      CX(2,4) = CX(4,2)
      CX(3,4) = CX(4,3)

C     CALCULATE FX MATRIX FROM ELEMENTS OF BX AND CX MATRICES

      DO 9  I = 1,4
      DO 9  J = 1,4
    9 FX(I,J) = BX(I,J) + BX(J,I) - CX(I,J)

C     ADD FX MATRIX FOR STRIP II TO MATRIX FF

      DO 10 I = 1,4
      K = 2*II-2+I
      DO 10 J = 1,4
      L = 2*II-2+J
   10 FF(K,L) = FF(K,L) + FX(I,J)

   50 CONTINUE
      RETURN
      END
```

```
INITE*STRIP(1).FREED
              SUBROUTINE FREED
              COMMON ITLE(80),RD(9),W(8),DT(8),DS(8),DR(8),DRS(8),D1(8)
              COMMON DL(39),DA(39),AX(4,4),BX(4,4),CX(4,4),DX(4,4),FX(4,4)
              COMMON FDSM(720,54),AA(18,18),DD(18,18),FF(18,18),IB(8),IE(8)
              COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,ER,GPS,VS,VR,G(3)
              COMMON WT1,WT2,WT3,TT1,TT2,TT3,WJ1,WJ2,WJ3,TJ1,TJ2,TJ3
              COMMON LABL(30),IL(4),IZ(6)

        C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

        C     COMPILE ELEMENTS OF STIFFNESS MATRIX FREF FROM BOUNDARY CONDITIONS

        C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

              NX2 = NXS - 2

        C     COMPONENTS OF A MATRICES COMMON TO ALL BOUNDARY CONDITIONS

              DO  1 IA = 3,NX2

              A = DA(IA-2)
              B = DA(IA-1)
              C = DA(IA)
              D = DA(IA+1)

        C     NOTE. OPERATOR PATTERN HAS BEEN MULTIPLIED BY (A + B)/2.

              G(1) =  2.*(B+C)/(B*B*C*C) + 2./(B*B*(A+B)) + 2./(C*C*(C+D))
              G(2) = -2.*(B+D)/(B*C*C*D)
              G(3) =  2./(C*D*(C+D))

              K  = 1
              LC = MSM
              IR = (IA-1)*MSM + 1
              LR = IR + MSM - 1

              DO  2 I = IR,LR
              L = MSM - LC + 1
              DO  3 J = 1,LC
              FDSM(I,J) = FDSM(I,J) + AA(K,L)*G(1)
            3 L  =  L + 1
              K  =  K + 1
            2 LC = LC - 1
              DO 1 ID = 2,3
              K  = 1
              IC = (ID-1)*MSM + 1
              LC = IC + MSM - 1
              DO 1 I = IR,LR
              L  = 1
              DO 5 J = IC,LC
              FDSM(I,J) = FDSM(I,J) + AA(K,L)*G(ID)
            5 L  =  L + 1
              K  =  K + 1
              IC = IC - 1
            1 LC = LC - 1

        C     COMPONENTS OF F AND D MATRICES COMMON TO ALL BOUNDARY COND.

              NX1 = NXS - 1

              DO 11 IA = 2,NX1
```

```

      A = DA(IA-1)
      B = DA(IA)

C     NOTE. OPERATOR PATTERN HAS BEEN MULTIPLIED BY (A + B)/2.

      G(1) = -(A+B)/(A*B)
      G(2) =  1./B
      G(3) =  (A+B)/2.

      K  = 1
      LC = MSM
      IR = (IA-1)*MSM + 1
      LR = IR + MSM - 1

      DO 12 I=IR,LR
      L  = MSM - LC + 1
      DO 13 J = 1,LC

C     MULTIPLY ELEMENTS OF STRIP MATRIX DD BY (A + B)/2.

      FDSM(I,J) = FDSM(I,J) + FF(K,L)*G(1) + DD(K,L)*G(3)
   13 L  =  L + 1
      K  =  K + 1
   12 LC = LC - 1

C     COMPONENTS OF F MATRIX COMMON TO ALL BOUNDARY COND.

      K  = 1
      IC = MSM + 1
      LC = IC + MSM - 1
      DO 11 I =IR,LR
      L  = 1
      DO 15 J = IC,LC
      FDSM(I,J) = FDSM(I,J) + FF(K,L)*G(2)
   15 L  =  L + 1
      K  =  K + 1
      IC = IC - 1
   11 LC = LC - 1
      RETURN
      END
```

INITE*STRIP(1).BOUND

```
            SUBROUTINE BOUND
            COMMON ITLF(80),RD(9),W(8),DT(8),DS(8),DR(8),DRS(8),D1(8)
            COMMON DL(39),DA(39),AX(4,4),BX(4,4),CX(4,4),DX(4,4),FX(4,4)
            COMMON FDSM(720,54),AA(18,18),DD(18,18),FF(18,18),IB(8),IE(8)
            COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,ER,GRS,VS,VR,G(3)
            COMMON WI1,WI2,WI3,TI1,TI2,TI3,WJ1,WJ2,WJ3,TJ1,TJ2,TJ3
            COMMON LABL(30),TL(4),TZ(6)
            INTEGER BC(6)/'CL','SI','FR','GU','DI','RO'/

      C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      C     COMPILE ELTS OF STIFFNESS MATRIX SUBJECT TO BOUNDARY CONDITIONS

      C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      C     PENULT X-SECT. BOUNDARY CONDITIONS AT BEGINNING OF STRUCTURE

            A = DA(1)
            B = DA(2)
            C = DA(3)

            DO  1 I = 1,4
            IF(IB.EQ.BC(I)) GO TO (10,15,15,10),I
          1 CONTINUE

      C     BOUNDARY CONDITION HERE IS A CLAMPED EDGE

         10 G(1) =  2.*(A+B)/(A*A*B*B) + 2./(B*B*(B+C)) + 2./(A**3)
            G(2) = -2.*(A+C)/(A*B*B*C)
            G(3) =  2./(B*C*(B+C))
            GO TO 99

      C     BOUNDARY CONDITION HERE IS SIMPLY SUPPORTED

         15 G(1) =  2.*(A+B)/(A*A*B*B) + 2./(B*B*(B+C))
            G(2) = -2.*(A+C)/(A*B*B*C)
            G(3) =  2./(B*C*(B+C))

      C     ASSIGN AA MATRIX COMPONENTS TO STIFFNESS MATRIX

         99 IR = MSM + 1
            LR = IR + MSM - 1
            K  =  1
            LC = MSM

            DO  2 I = IR,LR
            L = MSM - LC + 1
            DO  3 J = 1,LC
            FDSM(I,J) = FDSM(I,J) + AA(K,L)*G(1)
          3 L  =  L + 1
            K  =  K + 1
          2 LC = LC - 1

            DO  4 ID = 2,3
            K  =  1
            IC =  (ID-1)*MSM + 1
            LC =  IC + MSM - 1
            DO  4 I = IR,LR
            L  =  1
            DO  5 J = IC,LC
            FDSM(I,J) = FDSM(I,J) + AA(K,L)*G(ID)
```

```
    5 L  =  L + 1
      K  =  K + 1
      IC = IC - 1
    4 LC = LC - 1

C     PENULT X-SECTION. BOUNDARY CONDITIONS AT END OF STRUCTURE

      A = DA(NDA)
      B = DA(NDA-1)
      C = DA(NDA-2)

      DO  6 I = 1,4
      IF(IE.EQ.BC(I)) GO TO (30,35,35,30),I
    6 CONTINUE

C     BOUNDARY CONDITION HERE IS A CLAMPED EDGE

   30 G(1) =  2.*(A+B)/(A*A*B*B) + 2./(B*B*(B+C)) + 2./(A**3)
      GO TO 199

C     BOUNDARY CONDITION HERE IS SIMPLY SUPPORTED

   35 G(1) =  2.*(A+B)/(A*A*B*B) + 2./(B*B*(B+C))

C     ASSIGN AA MATRIX COMPONENTS TO STIFFNESS MATRIX

  199 NX2 = NXS - 2
      IR  = NX2*MSM + 1
      LR  = IR + MSM - 1
      K   =  1
      LC  = MSM

      DO  7 I = IR,LR
      L = MSM - LC + 1
      DO  8 J = 1,LC
      FDSM(I,J) = FDSM(I,J) + AA(K,L)*G(1)
    8 L  =  L + 1
      K  =  K + 1
    7 LC = LC - 1
      END
```

```
INITE*STRIP(1).VECTA
            SUBROUTINE VECTA
            COMMON ITLE(80),RD(9),W(8),DT(8),DS(8),DR(8),DRS(8),DI(8)
            COMMON DL(39),DA(39),AX(4,4),BX(4,4),CX(4,4),DX(4,4),FX(4,4)
            COMMON FDSM(720,54),AA(18,18),DD(18,18),FF(18,18),IB(8),IE(8)
            COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,ER,GRS,VS,VR,G(3)
            COMMON WI1,WI2,WI3,TI1,TI2,TI3,WJ1,WJ2,WJ3,TJ1,TJ2,TJ3
            COMMON LABL(30),TL(4),IZ(6)
            INTEGER BC(6)/'CL','SI','FR','GU','DT','RO'/
            INTEGER WL(5)/'PO','LI','UN','NE','LA'/

      C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      C     COMPILE LOAD VECTOR FOR POINT , LINE AND UNIFORMILY DIST LOADING

      C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      C     READ IN LOAD CASE NUMBER  LCN AND DESCRIPTION OF LOADING

            WRITE(5,40)
         40 FORMAT(1H1)
         30 READ(8,80) LCN , LABL
         80 FORMAT(I2,2X,30A2)
            IF(LCN.GT.NLC) GO TO 35
            WRITE(5,45) LCN , LABL
         45 FORMAT(//,8X,'LOADING CASE NO',I2,2X,30A2,//,9X,'LOADING',6X,
           1'NODAL LINE NO',7X,'CROSSECTION NO',5X,'MAGNITUDE',/,10X,'TYPE',7X
           2,'FROM',4X,'AT',4X,'TO',4X,'FROM',4X,'AT',4X,'TO',5X,'OF LOAD',/)
            JC = NBW + LCN

      C     READ IN LOADING TYPE , NODAL LINE LIMITS , X-SECT LIMITS , MAG

         55 READ(8,81) IL , NLB , NLE , NXB , NXE , WMAG
         81 FORMAT(4A2,4I5,F12.4)
            DO 1  I = 1,5
            IF(IL.EQ.WL(I)) GO TO (15,20,25,30,35),I
          1 CONTINUE

      C     POINT LOADING - ASSIGN NODAL LOAD TO LOAD VECTOR

         15 IR = 2*(NNL*(NXB - 1) + NLB) - 1
            WRITE(5,50) IL , NLB , NXB , WMAG
         50 FORMAT(1H ,8X,4A2,11X,I2,18X,I2,7X,F12.4)
            FDSM(IR,JC) = FDSM(IR,JC) + WMAG
            GO TO 55

      C     LINE LOADING - CALC NODAL LOAD AND ASSIGN TO LOAD VECTOR

         20 NS = NLE - NLB
            IR = 2*(NNL*(NXB - 1) + NLB) - 1
            WRITE(5,54) IL , NLB , NLE , NXB , WMAG
         54 FORMAT(1H ,8X,4A2,5X,I2,10X,I2,12X,I2,7X,F12.4)
            DO 21   K = 1,NS
            IN = NLB + K - 1
            B  = W(IN)
            FDSM(IR,JC)   = FDSM(IR,JC)   + WMAG*B/2.
            FDSM(IR+1,JC) = FDSM(IR+1,JC) - WMAG*B**2/12.
            FDSM(IR+2,JC) = FDSM(IR+2,JC) + WMAG*B/2.
            FDSM(IR+3,JC) = FDSM(IR+3,JC) + WMAG*B**2/12.
         21 IR = IR + 2
            GO TO 55
```

```
C     UNIFORM LOADING - CALC NODAL LOAD AND ASSIGN TO LOAD VECTOR

   25 NS = NLE - NLB
      NX = NXE - NXB - 1
      WRITE(5,60) IL , NLB , NLE , NXB , NXE , WMAG
   60 FORMAT(1H ,8X,4A2,5X,I2,10X,I2,6X,I2,10X,I2,1X,F12.4)

C     X-SECT AT START - ASSIGN HALF NODAL LOADING TO THESE NODES

      IR = 2*(NNL*(NXB - 1) + NLB) - 1
      DO 26  K = 1,NS
      IS = NLB + K - 1
      B  = W(IS)
      RI = RD(IS)
      RJ = RD(IS+1)
      AD = DA(NXB)/2.
      FDSM(IR,JC)   = FDSM(IR,JC)   + (B*RI/2. +  3.*B**2/20.)*AD*WMAG
      FDSM(IR+1,JC) = FDSM(IR+1,JC) - (B**2*RI/12. + B**3/30.)*AD*WMAG
      FDSM(IR+2,JC) = FDSM(IR+2,JC) + (B*RJ/2. -  3.*B**2/20.)*AD*WMAG
      FDSM(IR+3,JC) = FDSM(IR+3,JC) + (B**2*RJ/12. - B**3/30.)*AD*WMAG
   26 IR = IR + 2
      IF(NX.EQ.0) GO TO 75

C     INTERMEDIATE X-SECT - ASSIGN FULL NODAL LOADING TO THESE NODES

      DO 27  L = 1,NX
      IX = NXB + L
      IR = 2*(NNL*(IX - 1) + NLB) - 1
      DO 27  K = 1,NS
      IS = NLB + K - 1
      B  = W(IS)
      RI = RD(IS)
      RJ = RD(IS+1)
      AD = (DA(IX-1) + DA(IX))/2.
      FDSM(IR,JC)   = FDSM(IR,JC)   + (B*RI/2. +  3.*B**2/20.)*AD*WMAG
      FDSM(IR+1,JC) = FDSM(IR+1,JC) - (B**2*RI/12. + B**3/30.)*AD*WMAG
      FDSM(IR+2,JC) = FDSM(IR+2,JC) + (B*RJ/2. -  3.*B**2/20.)*AD*WMAG
      FDSM(IR+3,JC) = FDSM(IR+3,JC) + (B**2*RJ/12. - B**3/30.)*AD*WMAG
   27 IR = IR + 2

C     X-SECT AT END - ASSIGN HALF NODAL LOADING TO THESE NODES

   75 IR = 2*(NNL*(NXE - 1) + NLB) - 1
      DO 28  K = 1,NS
      IS = NLB + K - 1
      B  = W(IS)
      RI = RD(IS)
      RJ = RD(IS+1)
      AD = DA(NXE-1)/2.
      FDSM(IR,JC)   = FDSM(IR,JC)   + (B*RI/2. +  3.*B**2/20.)*AD*WMAG
      FDSM(IR+1,JC) = FDSM(IR+1,JC) - (B**2*RI/12. + B**3/30.)*AD*WMAG
      FDSM(IR+2,JC) = FDSM(IR+2,JC) + (B*RJ/2. -  3.*B**2/20.)*AD*WMAG
      FDSM(IR+3,JC) = FDSM(IR+3,JC) + (B**2*RJ/12. - B**3/30.)*AD*WMAG
   28 IR = IR + 2
      GO TO 55
   35 RETURN
      END
```

FINITE*STRIP(1).ZFROW

```
      SUBROUTINE ZFROW
      COMMON ITLE(80),RD(9),W(8),DT(8),DS(8),DR(8),DRS(8),D1(8)
      COMMON DL(39),DA(39),AX(4,4),BX(4,4),CX(4,4),DX(4,4),FX(4,4)
      COMMON FDSM(720,54),AA(18,18),DD(18,18),FF(18,18),IB(8),IE(8)
      COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,ER,GRS,VS,VR,G(3)
      COMMON WI1,WI2,WI3,TI1,TI2,TI3,WJ1,WJ2,WJ3,TJ1,TJ2,TJ3
      COMMON LABL(30),IL(4),IZ(6)
      INTEGER BC(6)/'CL','SI','FR','GU','DI','RO'/

C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

C     SET BOUNDARY CONDITIONS. INITIALISE DISPLACEMENTS AND ROTATIONS

C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      DO  1 I = 1,4
      IF(IB.EQ.BC(I)) GO TO (15,15,20,20),I
    1 CONTINUE

C     START BOUNDARY. SET DISPLACEMENTS AND ROTATIONS TO ZERO

   15 DO  100 I = 1,MSM
      FDSM(I,1) = 1.0
      DO  100 J = 2,NBL
  100 FDSM(I,J) = 0.0

   20 DO  2 I = 1,4
      IF(IE.EQ.BC(I)) GO TO (25,25,30,30),I
    2 CONTINUE

C     END  BOUNDARY. SET DISPLACEMENTS AND ROTATIONS TO ZERO

   25 DO  200 L = 1,MSM
      I = NSM - MSM + L
      FDSM(I,1) = 1.0
      DO  250 K = 1,NLC
      J = NBW + K
  250 FDSM(I,J) = 0.0
      DO  200 J = 2,NBW
      I = I - 1
  200 FDSM(I,J) = 0.0

C     READ IN NUMBER OF DISPLACEMENTS OR ROTATIONS TO BE ASSIGNED

   30 READ(8,61) NDR
   61 FORMAT(I5)
      IF(NDR)600,600,70
   70 WRITE(5,62)
   62 FORMAT(//,8X,'INITIAL CONDITIONS OF DISPLACEMENT OR ROTATION ALONG
     1 STRUCTURE',//,9X,'NO',6X,'TYPE',8X,'NODAL LINE',6X,'X-SECTION',
     26X,'MAGNITUDE',/)
      DO 500 NBC = 1,NDR
      READ(8,63) IZ , DMAG , NLN , NXN
   63 FORMAT(6A2,F10.6,2I5)
      WRITE(5,64) NBC , IZ , NLN , NXN , DMAG
   64 FORMAT(1H ,7X,I2,4X,6A2,6X,I2,12X,I2,10X,F10.6)

C     DETERMINE ROW NUMBER OF DISPLACEMENT/ROTATION TO BE ASSIGNED

      IC = 1
      IF(IZ.EQ.BC(6)) IC = 0
```

```
      I   = 2*(NNL*(NXN - 1) + NLN) - IC

C     SET ALL R/C = 0.0  DIAG = 1.0  ASSIGN DIS/ROT IN ROW I = DMAG

      DO  300 J = 1,NBL
  300 FDSM(I,J) = 0.0
      FDSM(I,1) = 1.0
      DO  350 K = 1,NLC
      J = NBW + K
  350 FDSM(I,J) = DMAG
      MC = NBW
      IF(I.LT.MC) MC = I
      DO  400 J = 2,MC
      I = I - 1
  400 FDSM(I,J) = 0.0
  500 CONTINUE
  600 CONTINUE

      WRITE(5,208) IB , IE
  208 FORMAT(/,8X,'BOUNDARY CONDITION AT START OF STRUCTURE',8X,
     18A2,/,8X,'BOUNDARY CONDITION AT  END  OF STRUCTURE',8X,8A2)

      RETURN
      END
```

```
INITE*STRIP(1).RBAND
      SUBROUTINE RBAND
      COMMON ITLE(80),RD(9),W(8),DT(8),DS(8),DR(8),DRS(8),D1(8)
      COMMON DL(39),DA(39),AX(4,4),BX(4,4),CX(4,4),DX(4,4),FX(4,4)
      COMMON A(720,54),AA(18,18),DD(18,18),FF(18,18),IB(8),IF(8)
      COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,FR,GPS,VS,VR,G(3)
      COMMON WI1,WI2,WI3,TI1,TI2,TI3,WJ1,WJ2,WJ3,TJ1,TJ2,TJ3
      COMMON LABL(30),IL(4),IZ(6)
      DOUBLE PRECISION FACT
      INTEGER P,Q,R,S,T

C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C     THE SET OF SIMULTANEOUS EQUATIONS CONTAINED IN MATRIX A IS SOLVED
C     USING NORMAL GAUSS REDUCTION TECHNIQUES. THE NUMBER OF EQUATIONS
C     TO BE SOLVED IS  N , AND THE HALF BAND WIDTH IS  M . MATRIX A IS
C     AUGMENTED WITH  NLC  COLUMN VECTORS, EACH A SEPERATE LOADING CASE.
C
C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      N   = NSM
      M   = NBW
      NN  = N - 1
      MM  = M - 1
      NNN = M + 1
      MMM = M + NLC

C     MATRIX A HAS N ROWS , AND IS AUGMENTED FROM COLS NNN TO MMM

      P = M
      DO 1  I = 1,NN
      III = I + 1
      Q = MM
      K = 2
      IF(P.GT.N) P=N
      DO 2  L = III,P
      FACT = A(I,K)/A(I,1)
      IF(P.EQ.N) Q=N-L+1
      T = K
      DO 3  J = 1,Q
      A(L,J)  = A(L,J) - FACT*A(I,T)
    3 T = T + 1
      DO 4  R = NNN,MMM
    4 A(L,R)  = A(L,R) - FACT*A(I,R)
      Q = Q - 1
    2 K = K + 1
    1 P = P + 1
      DO 5  I = N,2,-1
      II  = I - 1
      DO 6  R = NNN,MMM
    6 A(I,R)  = A(I,R)/A(I,1)
      S = I - M + 1
      IF(I.LT.M) S=1
      T = 2
      DO 5  J = II,S,-1
      DO 7  R = NNN,MMM
    7 A(J,R)  = A(J,R) - A(I,R)*A(J,T)
    5 T = T + 1
      DO 8  R = NNN,MMM
    8 A(1,R)  = A(1,R)/A(1,1)
      RETURN
      END
```

```
INITE*STRIP(1).STRES
      SUBROUTINE STRES(JC)
      COMMON ITLE(80),RD(9),W(8),DT(8),DS(8),DR(8),DRS(8),D1(8)
      COMMON DL(39),DA(39),AX(4,4),BX(4,4),CX(4,4),DX(4,4),FX(4,4)
      COMMON FDSM(720,54),AA(18,18),DD(18,18),FF(18,18),IB(8),IE(8)
      COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,ER,GRS,VS,VR,G(3)
      COMMON WI1,WI2,WI3,TI1,TI2,TI3,WJ1,WJ2,WJ3,TJ1,TJ2,TJ3
      COMMON LABL(30),TL(4),TZ(6)
      INTEGER BC(6)/'CL','SI','FR','GU','DI','RO'/

C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

C     CALCULATE STRESS RESULTANTS FROM DISPLACEMENTS AND ROTATIONS

C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

C     FIRST X-SECTION - SET UP DISPLACEMENTS AND ROTATIONS

      A = DA(1)
      B = DA(1)

      DO 100  I = 1,NST
      I2 = 2*I - 1
      I3 = 2*(NNL+I) - 1
      G  = 1.0
      IF(IB.EQ.BC(2)) G = -1.0

      WI1 = FDSM(I3,JC)*G
      WI2 = FDSM(I2,JC)
      WI3 = FDSM(I3,JC)

      TI1 = FDSM(I3+1,JC)*G
      TI2 = FDSM(I2+1,JC)
      TI3 = FDSM(I3+1,JC)

      WJ1 = FDSM(I3+2,JC)*G
      WJ2 = FDSM(I2+2,JC)
      WJ3 = FDSM(I3+2,JC)

      TJ1 = FDSM(I3+3,JC)*G
      TJ2 = FDSM(I2+3,JC)
      TJ3 = FDSM(I3+3,JC)

  100 CALL CURVE(I,1,A,B)

C     INTERIOR X-SECTIONS - SET UP DISPLACEMENTS AND ROTATIONS

      NX1 = NXS - 1
      DO 200 J = 2,NX1
      A = DA(J-1)
      B = DA(J)

      DO 200 I = 1,NST
      I1 = 2*(NNL*(J-2) + I) - 1
      I2 = 2*(NNL*(J-1) + I) - 1
      I3 = 2*(NNL*J + I) - 1

      WI1 = FDSM(I1,JC)
      WI2 = FDSM(I2,JC)
      WI3 = FDSM(I3,JC)

      TI1 = FDSM(I1+1,JC)
```

```
      TI2 = FDSM(I2+1,JC)
      TI3 = FDSM(I3+1,JC)

      WJ1 = FDSM(I1+2,JC)
      WJ2 = FDSM(I2+2,JC)
      WJ3 = FDSM(I3+2,JC)

      TJ1 = FDSM(I1+3,JC)
      TJ2 = FDSM(I2+3,JC)
      TJ3 = FDSM(I3+3,JC)

  200 CALL CURVE(I,J,A,B)

C     LAST X-SECTION - SET UP DISPLACEMENTS AND ROTATIONS

      A = DA(NX1)
      B = DA(NX1)

      DO 300  I = 1,NST
      I1 = 2*(NNL*(NXS-2) + I) - 1
      I2 = 2*(NNL*(NXS-1) + I) - 1
      G = 1.0
      IF(IE.EQ.BC(2)) G = -1.0

      WI1 = FDSM(I1,JC)
      WI2 = FDSM(I2,JC)
      WI3 = FDSM(I1,JC)*G

      TI1 = FDSM(I1+1,JC)
      TI2 = FDSM(I2+1,JC)
      TI3 = FDSM(I1+1,JC)*G

      WJ1 = FDSM(I1+2,JC)
      WJ2 = FDSM(I2+2,JC)
      WJ3 = FDSM(I1+2,JC)*G

      TJ1 = FDSM(I1+3,JC)
      TJ2 = FDSM(I2+3,JC)
      TJ3 = FDSM(I1+3,JC)*G

  300 CALL CURVE(I,NXS,A,B)
      RETURN
      END
```

```
INITE*STRIP(1).CURVE
      SUBROUTINE CURVE(IS,JX,B,C)
      COMMON ITLF(80),RD(9),W(8),DT(8),DS(8),DR(8),DRS(8),D1(8)
      COMMON DL(39),DA(39),AX(4,4),BX(4,4),CX(4,4),DX(4,4),FX(4,4)
      COMMON FDSM(720,54),AA(18,18),DD(18,18),FF(18,18),IB(8),IE(8)
      COMMON NST,NXS,NNL,NDA,MSM,NSM,NBL,NBW,NLC,ES,ER,GRS,VS,VR,G(3)
      COMMON WI1,WI2,WI3,TI1,TI2,TI3,WJ1,WJ2,WJ3,TJ1,TJ2,TJ3

C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

C     CALCULATE CURVATURES AND BENDING AND TWISTING MOMENTS FOR STRIP

C     CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

C     CALCULATE 1ST AND 2ND DERIVATIVES OF W AND THETA W R T Y2 AXIS

      D1WI = -C*WI1/(B*(B+C)) + (C-B)*WI2/(B*C) +  B*WI3/(C*(B+C))
      D1TI = -C*TI1/(B*(B+C)) + (C-B)*TI2/(B*C) +  B*TI3/(C*(B+C))
      D1WJ = -C*WJ1/(B*(B+C)) + (C-B)*WJ2/(B*C) +  B*WJ3/(C*(B+C))
      D1TJ = -C*TJ1/(B*(B+C)) + (C-B)*TJ2/(B*C) +  B*TJ3/(C*(B+C))

      D2WI = 2.*WI1/(B*(B+C)) -     2.*WI2/(B*C) + 2.*WI3/(C*(B+C))
      D2WJ = 2.*WJ1/(B*(B+C)) -     2.*WJ2/(B*C) + 2.*WJ3/(C*(B+C))

C     CALCULATE THE 3 CURVATURES K1  K2  K12 AT NODAL LINES I AND J

      JS   = IS + 1
      BB   = W(IS)
      RI   = RD(IS)
      RJ   = RD(JS)

      CRI  =  (6./BB**2)*WI2  - (4./BB)*TI2
     .       -(6./BB**2)*WJ2  - (2./BB)*TJ2
      CSI  =  (1./RI**2)*D2WI - (1./RI)*TI2
      CRSI = -(2./RI**2)*D1WI - (2./RI)*D1TI

      CRJ  = -(6./BB**2)*WI2  + (2./BB)*TI2
     .       +(6./BB**2)*WJ2  + (4./BB)*TJ2
      CSJ  =  (1./RJ**2)*D2WJ - (1./RJ)*TJ2
      CRSJ = -(2./RJ**2)*D1WJ - (2./RJ)*D1TJ

C     CALCULATE MOMENTS AT NODAL LINES IS AND JS STRIP IS X-SECT JX

      BRI  = DR(IS)*(CRI - VS*CSI)
      BSI  = DS(IS)*(CSI - VR*CRI)
      BRSI = DRS(IS)*CRSI

      BRJ  = DR(IS)*(CRJ - VS*CSJ)
      BSJ  = DS(IS)*(CSJ - VR*CRJ)
      BRSJ = DRS(IS)*CRSJ

      IF(IS.GT.1)GO TO 50
      WRITE(5,20) JX , BRI , BSI , BRSI , BRJ , BSJ , BRSJ
   20 FORMAT(1H ,12X,I2,11X,'1',4X,3(4X,E12.6),/,26X,'2',4X,
     13(4X,E12.6),/)
      RETURN
   50 WRITE(5,25) IS , BRI , BSI , BRSI , JS , BRJ , BSJ , BRSJ
   25 FORMAT(1H ,24X,I2,4X,3(4X,E12.6),/,25X,I2,4X,3(4X,E12.6),/)
      RETURN
      END
```

APPENDIX H

List of Postgraduate Courses Taken

LIST OF POSTGRADUATE COURSES* TAKEN TO COMPLETE THE REQUIREMENTS OF THE DEGREE

Year	Course	Credit Rating
1975	CE 508 Skeletal Structures	5
1975	CE 516 Prestressed Concrete	5
1975	CE 524 Structural Dynamics	3
1975	CE 525 Coastal Engineering	5
1976	CE 515 Surface Structures	5
1976	CE 519 Steel Structures	3
1976	CE 506 Properties of Concrete	4
1977	CE 533 Bridge Engineering	4
Thesis	Finite Strip Analysis of Curved Plates	10
	Total	44

Number of credits required for the degree 40

* A brief summary of the content of each course is contained overleaf

BRIEF SUMMARY OF POSTGRADUATE COURSES TAKEN

CE 508 Skeletal Structures

Matrix formulation, solution of large sets of linear simultaneous equations, stiffness and flexibility matrices for elastic analysis, the force and displacement methods of linear analysis, elements of stability and vibration.

CE 516 Prestressed Concrete

Limit state design, partial prestressing, bending, shear, torsion, continuous structures, composite construction and recent development.

CE 524 Structural Dynamics

Principles of dynamics, natural modes of vibration. Energy methods. Forced vibrations, differential equations, normal mode and frequency response methods. Damping. Introduction to random excitation.

CE 525 Coastal Engineering

Establishment of design criteria for coastal structures. Prediction from meteorological data of swell characteristics approaching a coast line. The dynamics of water and sediment motion in the coastal zone. Estimates of water surface elevations and forces acting on coastal structures. Stability of armour units in marine construction.

CE 515 Surface Structures

Basic equations of elasticity of two and three dimensional stress problems, finite differences, finite element methods and experimental methods of analysing flat plates, deep beams, folded plates and shells.

CE 519 Steel Structures

Modern aspects of steel design and construction. Theorems of limit analysis for plane frames. Plastic analysis of steel frames.

CE 506 Properties of Concrete

The properties and behaviour of fresh concrete, of hardened concrete and of the constituent materials, testing and control.

CE 533 Bridge Engineering

Historical development, structural action and materials; bridge types and construction methods, aesthetics, economics, loading, analysis and design methods for superstructures and substructures including foundations, bearings and expansion points, failures, modifications and repairs.

Total number of credits for the above courses: 34

APPENDIX I

Examination Papers

UNIVERSITY OF CAPE TOWN

DEPARTMENT OF CIVIL ENGINEERING

UNIVERSITY EXAMINATION: JUNE, 1975

COURSE CE 508 - SKELETAL STRUCTURES

Time allowed: 4 hours   Notes are allowed

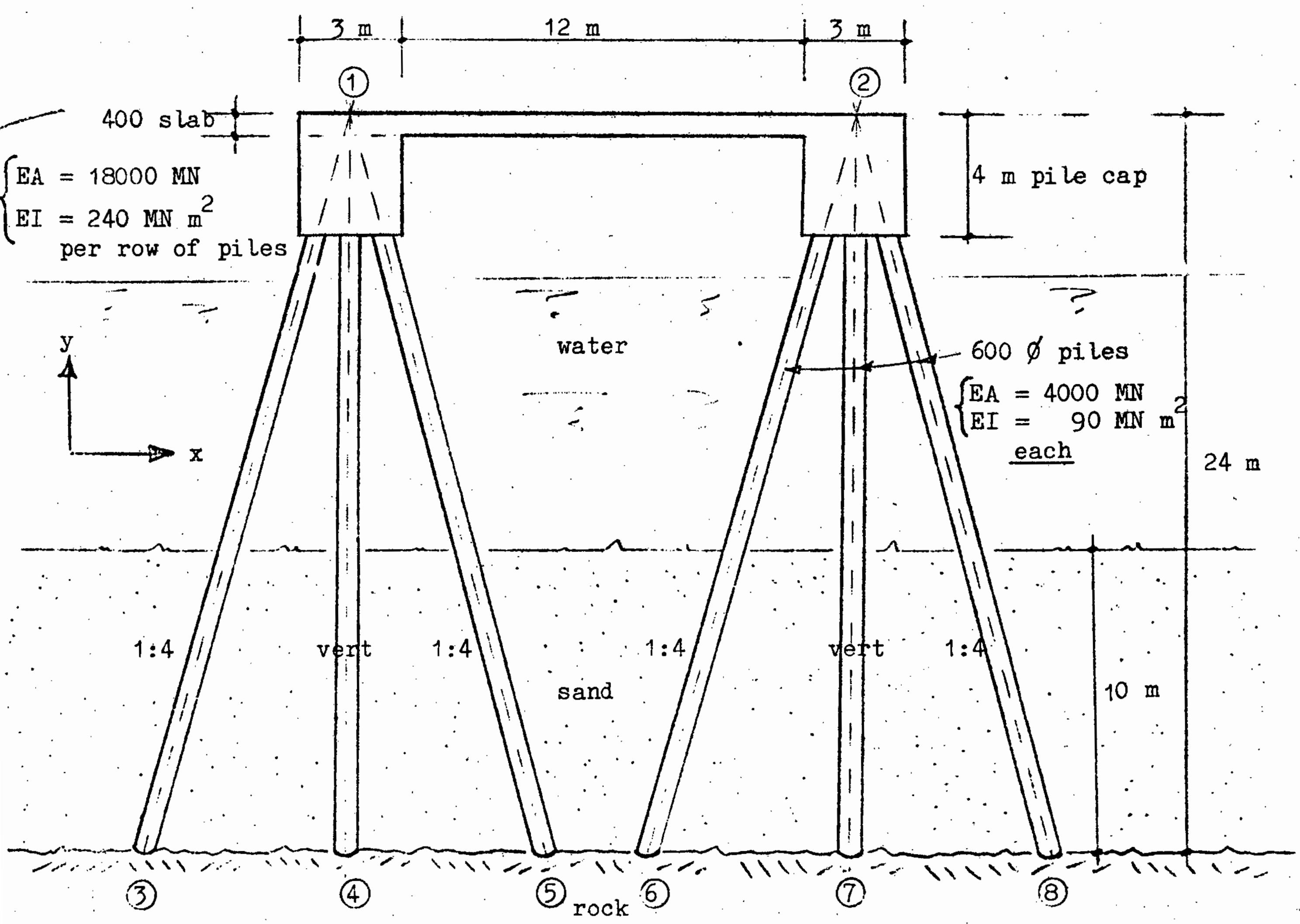
Part A: For each of the five structures shown below determine the degree of static and of effective kinematic indeterminateness, select the most suitable method of analysis, give the order of all the relevant matrices required for solution by the chosen method. State clearly what assumptions are made.

[40 marks]

Part B: Compile the matrices for any two of these structures; one analysed by the FORCE method and one analysed by the DISPLACEMENT method. Do not attempt to complete all the arithmetic processes, but give sufficient detail to show clearly the principles and operations involved.

[60 marks]

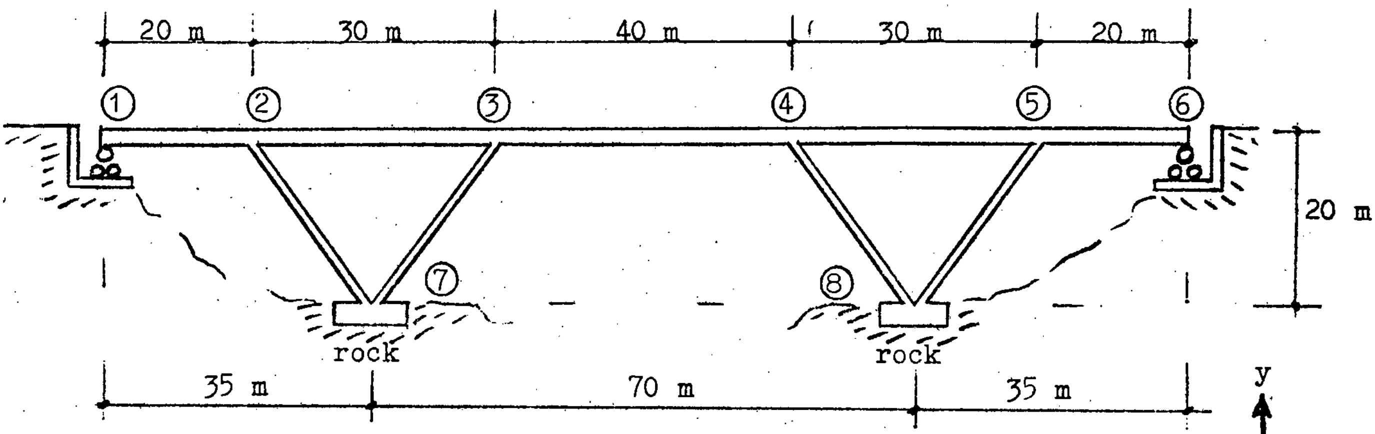
1. Jetty with vertical and horizontal loads applied to the top surface in the xy plane. The pile rows are at 3 m spacing along the jetty.



CROSS-SECTION

/2. ....

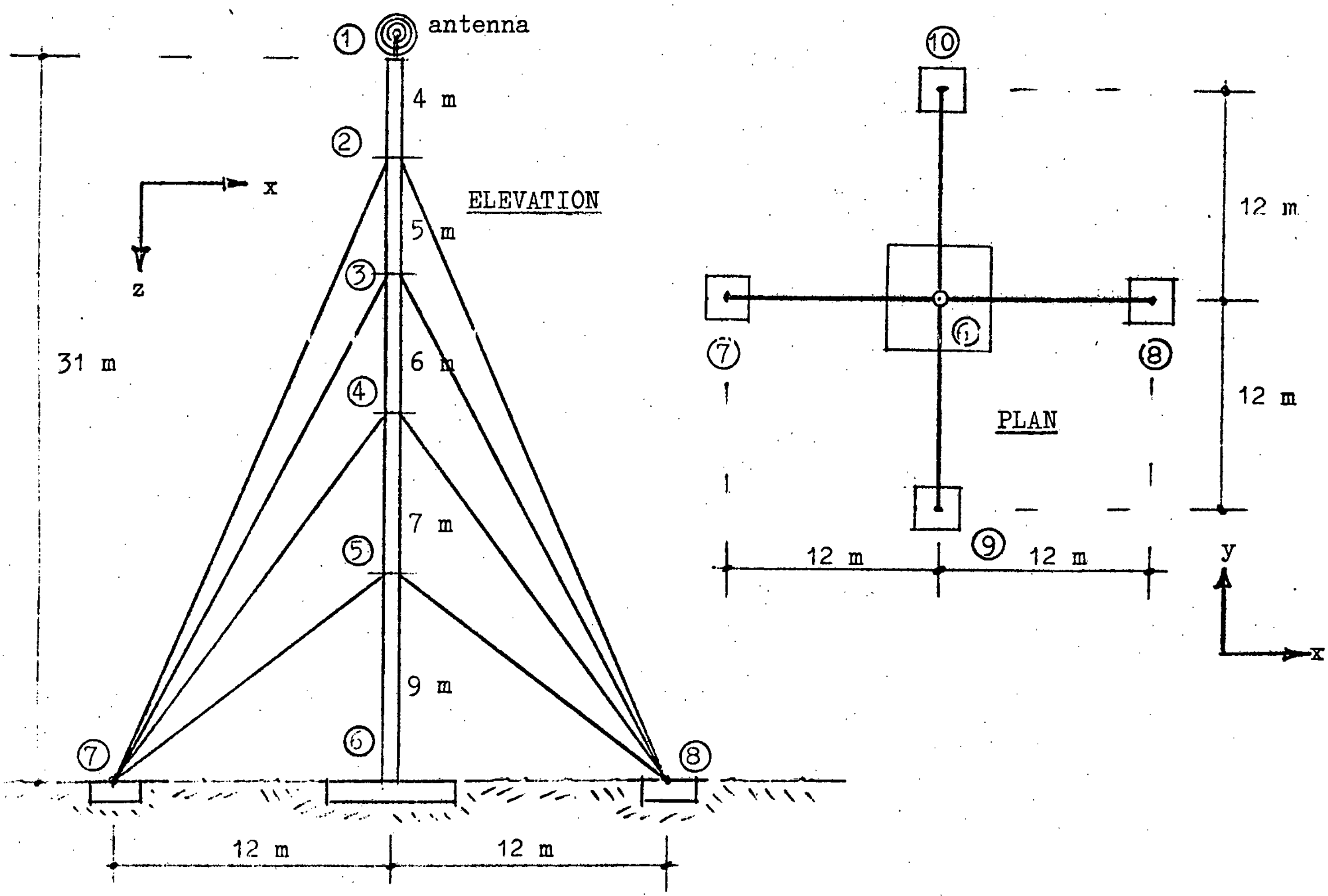
2. Bridge, monolithic concrete beam-slab deck and inclined columns, with vertical and horizontal loading applied to the deck.



deck : EA = 140 000 MN; EI = 40 000 MN m^2

columns: EA = 36 000 MN; EI = 3 000 MN m^2

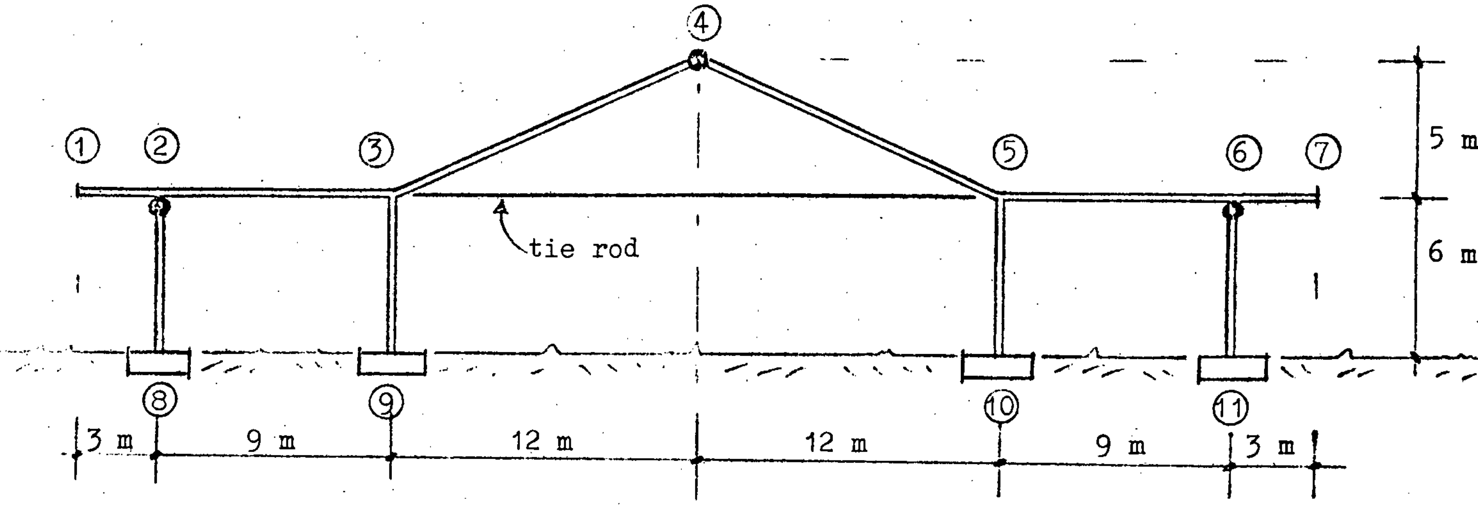
3. Tower, consisting of a single vertical tubular column fixed at the base and stayed at right angles on four levels with steel wire guy ropes, which are sufficiently pretensioned not to go slack. Loading is applied at the top only.



column : EA = 1300 MN; EI = 7 MN m^2(200 ∅)

wire rope: EA = 36 MN each: (15 mm ∅.)

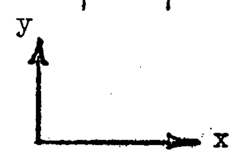
4. Building, tied steel portal frame with two side bays. Wind, dead and imposed roof loading.



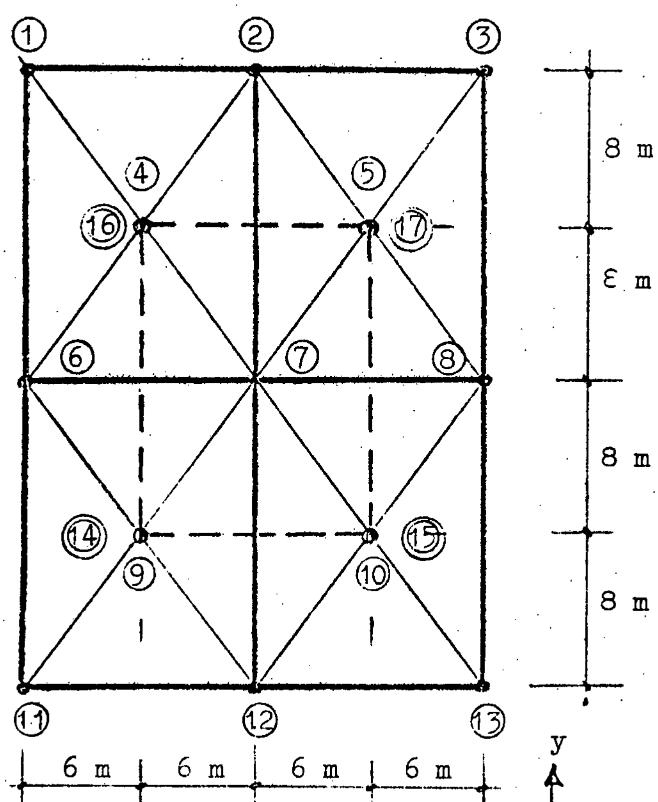
Rafters: EA = 1000 MN; EI = 20 MN m^2

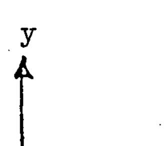
Columns: EA = 1200 MN; EI = 9 MN m^2

Tie rod: EA = 60 MN (20 ⌀).

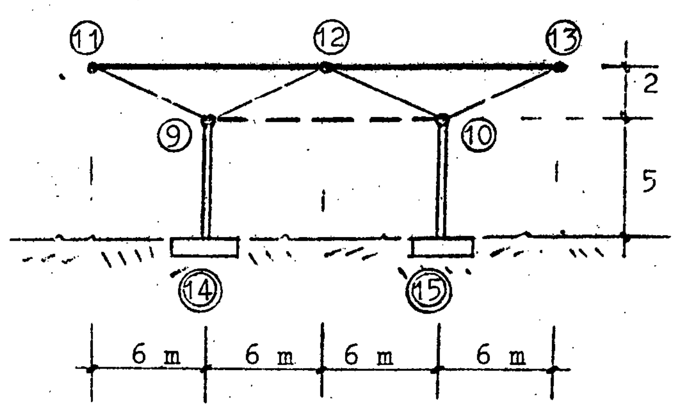


5. Roof, ball-jointed, double-layer, three-way grid on four columns fixed at their bases, all of tubular steel construction, with vertical and horizontal loading applied to the top joints.

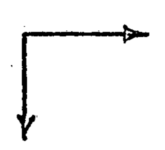




PLAN



ELEVATION



top layer	:	EA = 600 MN, (200 mm ⌀)
inclined	:	EA = 900 MN, (300 mm ⌀)
bottom layer	:	EA = 1200 MN, (400 mm ⌀)
columns	:	EA = 3000 MN, (500 mm ⌀); EI = 100 MN m^2

UNIVERSITY OF CAPE TOWN

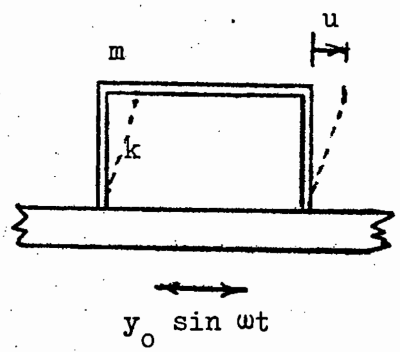
DEPARTMENT OF CIVIL ENGINEERING

UNIVERSITY EXAMINATION: 16th AUGUST 1975

COURSE CE 524 - STRUCTURAL DYNAMICS

Time allowed: 3 hours

1. A model frame is to be tested on a vibration table which executes sinusoidal vibrations of chosen amplitude y_o and frequency w rad/sec. Equipment is available for measuring the amplitude u_m of the displacement of the frame relative to the table, and the phase angle ϕ between the sinusoidal motion of the frame and that of the table.



(a) Treating the frame as a one degree of freedom system with linear viscous damping, derive the relations for the steady state response

$$u_m = y_o \frac{\omega^2}{p^2} \frac{1}{\sqrt{(1 - \frac{\omega^2}{p^2})^2 + 4\zeta^2 \frac{\omega^2}{p^2}}} \; ; \quad u_m(p^2 - \omega^2) = y_o\omega^2 \cos \phi$$

where $p = \sqrt{k/m}$ is the natural frequency and ζ is the damping ratio.

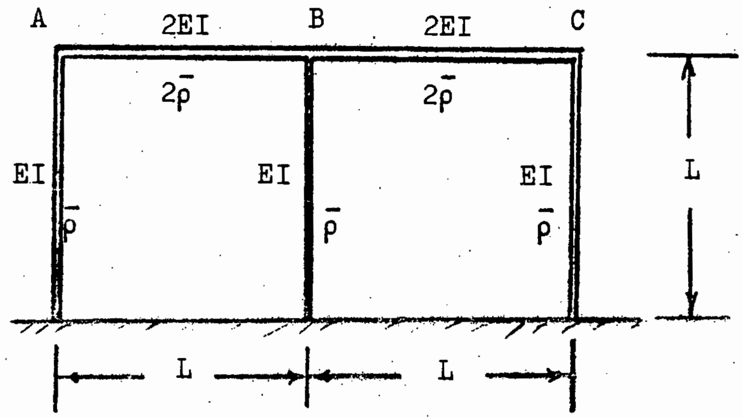
(b) If ζ is small, and the forcing frequency ω can be varied over a sufficient range, suggest a simple way of determining p and ζ by means of two measurements.

(c) If the energy dissipated per cycle W_d is taken to be proportional to the maximum elastic strain energy in the cycle ("structural damping") explain the concept of "equivalent viscous damping" and derive the relation

$$\zeta_e = \frac{\gamma}{4\pi} \frac{p}{\omega}$$

where $\gamma = W_d/(\frac{1}{2} k u_m^2)$.

2.

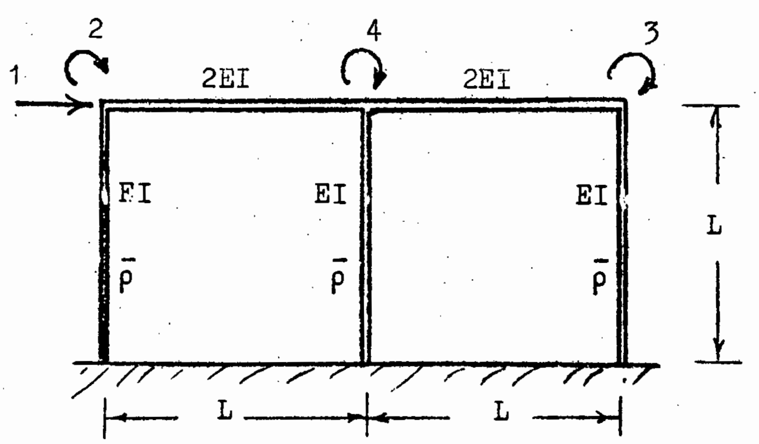


Estimate the fundamental natural frequency (associated with side-sway) of the two-bay frame shown, using Rayleigh's method and accounting for the flexibility of the beam members AB, BC and for the distributed mass of the three columns.

To simplify the calculation, you may assume the rotation at B to be the same as those at A and C. What relation will the result obtained using this simplification have to the frequency obtained when the three rotations are not all assumed equal?

(Note that the deflection functions (cubic polynomials) used to derive the standard stiffness and mass matrices may be combined to obtain reasonable deflection for the frame members.)

3.



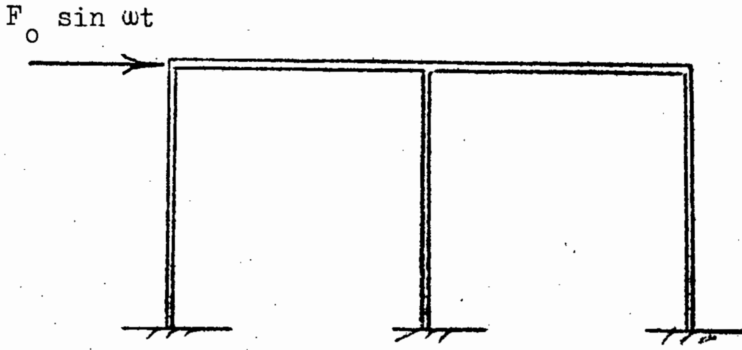
For the same two-bay frame considered in question 2, number the generalized displacements and forces as shown.

(a) Derive the (4 × 4) stiffness matrix $\underset{\sim}{k}$ and the consistent mass matrix $\underset{\sim}{m}$ for this frame.

(b) Derive also a lumped mass matrix for this frame. If this matrix rather than the consistent mass matrix is used, derive the corresponding stiffness matrix that must be used with it.

4. Suppose the load on the same frame consists of a sinusoidal force at the beam level, i.e. the load vector is

$$\underline{F} = F_o \sin \omega t \{1\ 0\ 0\ 0\}$$



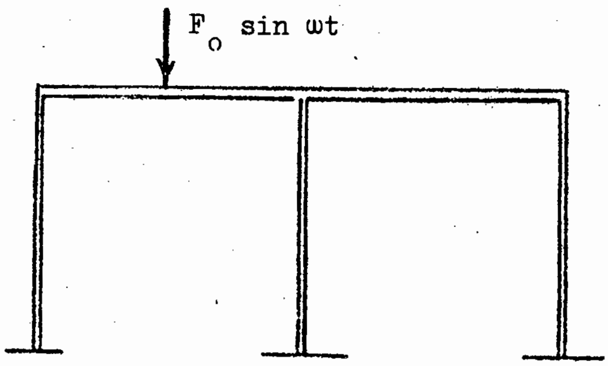
(a) Find the steady state response of the frame, as a sum of contributions from the four normal modes of the frame.

(The attached sheet shows the four mode vectors of the frame, normalized with respect to the mass matrix, with the corresponding natural frequencies).

(b) If the forcing frequency ω can be varied over a wide range, discuss briefly how the response of the frame would depend upon ω.

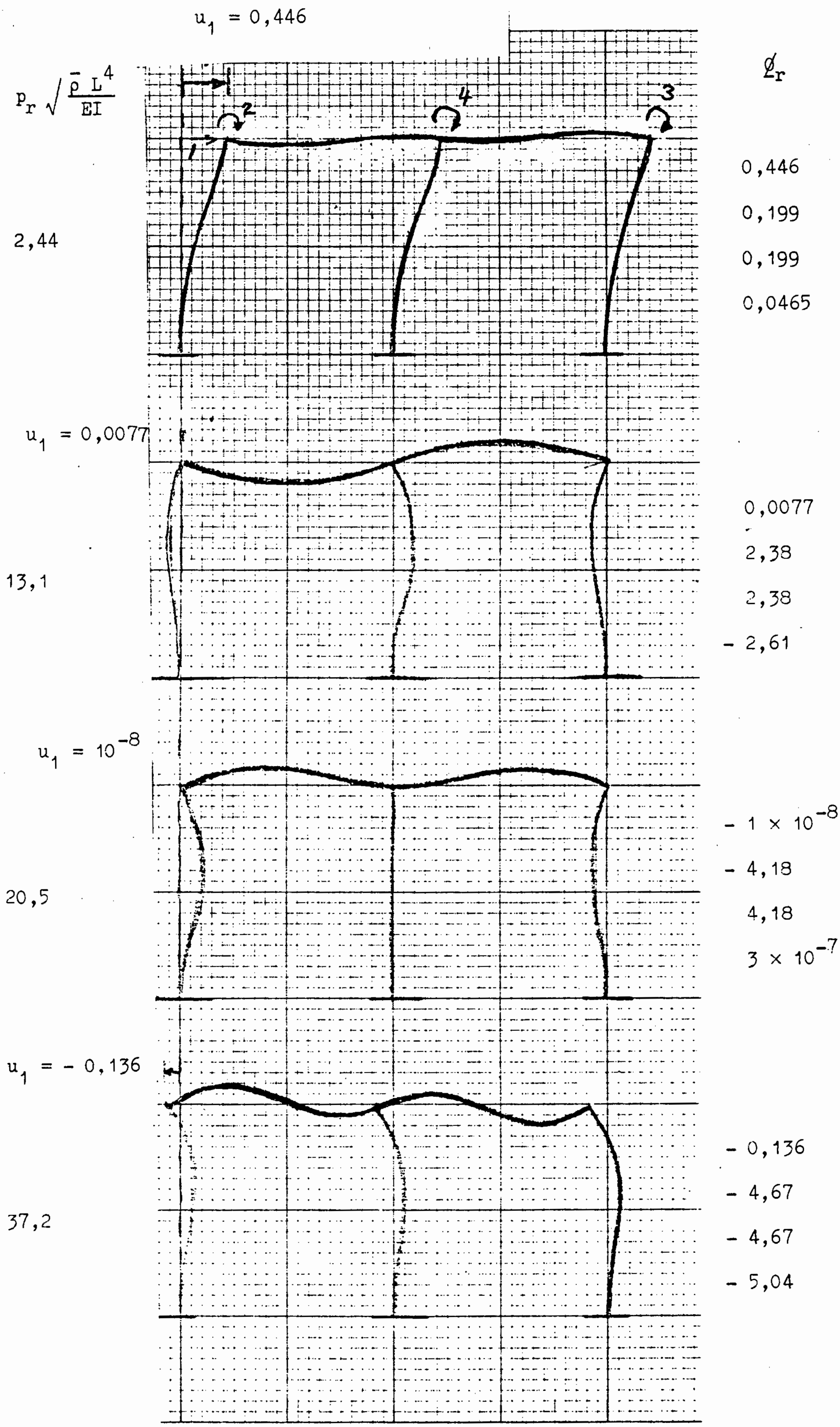
5. [Extra credit; do this only when you have finished the first four questions].

Suppose the load on the frame consists of a sinusoidal force at the midpoint of one of the beams.



(a) Find the appropriate load vector to use with the stiffness matrix and consistent mass matrix.

(b) Discuss briefly the qualitative differences, if any, that you would expect in the response of the frame to this load as compared to the horizontal load specified in Problem 4. (Assume again that the exciting frequency ω may vary over a wide range).



$u_1 = 0,446$
$p_r \sqrt{\frac{\bar{\rho} L^4}{EI}}$
$\varnothing_r$
2,44
0,446
0,199
0,199
0,0465
$u_1 = 0,0077$
13,1
0,0077
2,38
2,38
- 2,61
$u_1 = 10^{-8}$
20,5
$- 1 \times 10^{-8}$
- 4,18
4,18
3×10^{-7}
$u_1 = - 0,136$
37,2
- 0,136
- 4,67
- 4,67
- 5,04

UNIVERSITY OF CAPE TOWN

DEPARTMENT OF CIVIL ENGINEERING

UNIVERSITY EXAMINATION: DECEMBER, 1975

M.Sc. IN CIVIL ENGINEERING

COURSE CE 516: PRESTRESSED CONCRETE

Time allowed: 3 hours — Attempt ALL questions

Any books and notes may be used

1. (a) State as concisely as possible the reason why it is essential in prestressed concrete design to check both the serviceability and and ultimate limit states.

[3 marks]

(b) In a partially prestressed member a certain amount of untensioned reinforcement is normally required. Show, with the aid of appropriate diagrams, how you would determine, in the design of a particular member according to CP 110, the relative costs of using prestressing steel, high yield reinforcing steel or mild steel to provide the additional reinforcement required. Assume that unit rates for the three types of steel are available.

[12 marks]

2. Write brief notes (about half a page) on each of the following:-

(a) The main factors affecting creep in prestressed concrete listed in order of importance with respect to a typical structure such as a highway bridge.

(b) The reasons for avoiding either too small or too large a percentage of steel in a prestressed concrete member.

(c) Compare and contrast flexural cracking and shear cracking.

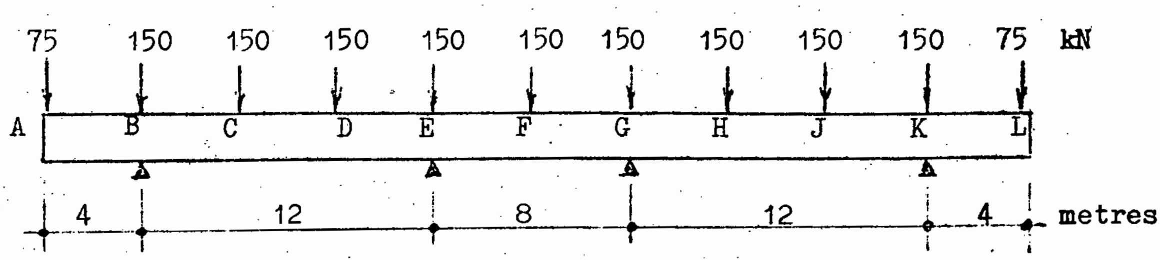
(d) Linear transformation of a cable profile.

(e) The mechanism of failure of a concrete beam in torsion, with comments on the effect of shear and bending moment in combination with torsion.

[25 marks]

/3. ....

3.



The rectangular, post-tensioned concrete beam shown above is simply supported at B, E, G and K. The beam carries eleven equally spaced columns supporting an upper floor. The sustained loads transmitted by these columns to the beam are as shown (viz. 75 kN at A and L; 150 kN at all the other points). In addition to its self-weight, the beam has to carry a sustained load of 20 kN/m throughout its length.

Use the load balancing method, balancing all sustained loads, to give a suitable preliminary design (i.e. beam size, cable profile and prestress force).

The design should comply with the following constraints:-

(a) Minimum width of beam : 250 mm

(b) Maximum depth of beam : 1200 mm

(c) Minimum cover distance to centroid of tendons: 100 mm

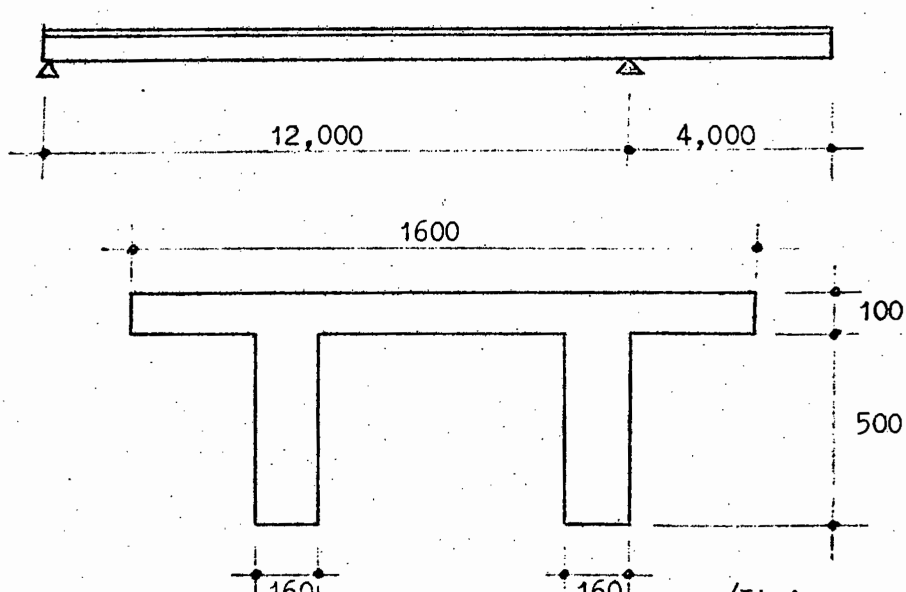
(d) Maximum average concrete stress : 6 MPa.

Assume furthermore that prestressing losses are negligible so that the prestressing force is constant throughout.

Assume Weight of Concrete = 25 kN/m^3

[20 marks]

4.



/It is proposed ...

4. (Continued)

It is proposed to build a prestressed concrete floor spanning 12 metres with an additional 4 metre cantilever. Pretensioned double-T units of the uniform cross-section shown above are to be used. These are to be designed according to CP 110 as Class 3 members, using the minimum prestressing force and providing additional untensioned reinforcement where necessary.

The floor is required to carry an imposed loading of 10 kN/m^2 over any part.

Prestressing is to be accomplished using straight horizontal tendons only.

Determine the following:-

(a) The minimum prestressing force required and its eccentricity. (Consider only the support points and midspan, neglecting the slightly higher moments just to the left of midspan).

(b) The area of additional untensioned reinforcement required at midspan using either prestressing wire or high yield reinforcing steel.

Comment on the merits and demerits of the resulting design.

Necessary data:

Use Concrete Grade 50 (i.e. f_{cu} = 50 MPa)

Permissible compressive stress for serviceability limit state: 16,7 MPa
Permissible hypothetical tensile stress for serviceability limit state (including depth factor): 5,8 MPa.

Weight of concrete = 25 kN/m^3

Minimum cover distance to centroid of prestressing tendons: 80 mm

Factors of Safety for Ultimate Limit State: Dead Load: 1,4
Imposed Load: 1,6

Characteristic strength of prestressing wire: f_{pu} = 1550 MPa

Residual prestress (after losses): 0,6 f_{pu}

Characteristic strength of high yield steel: f_y = 460 MPa

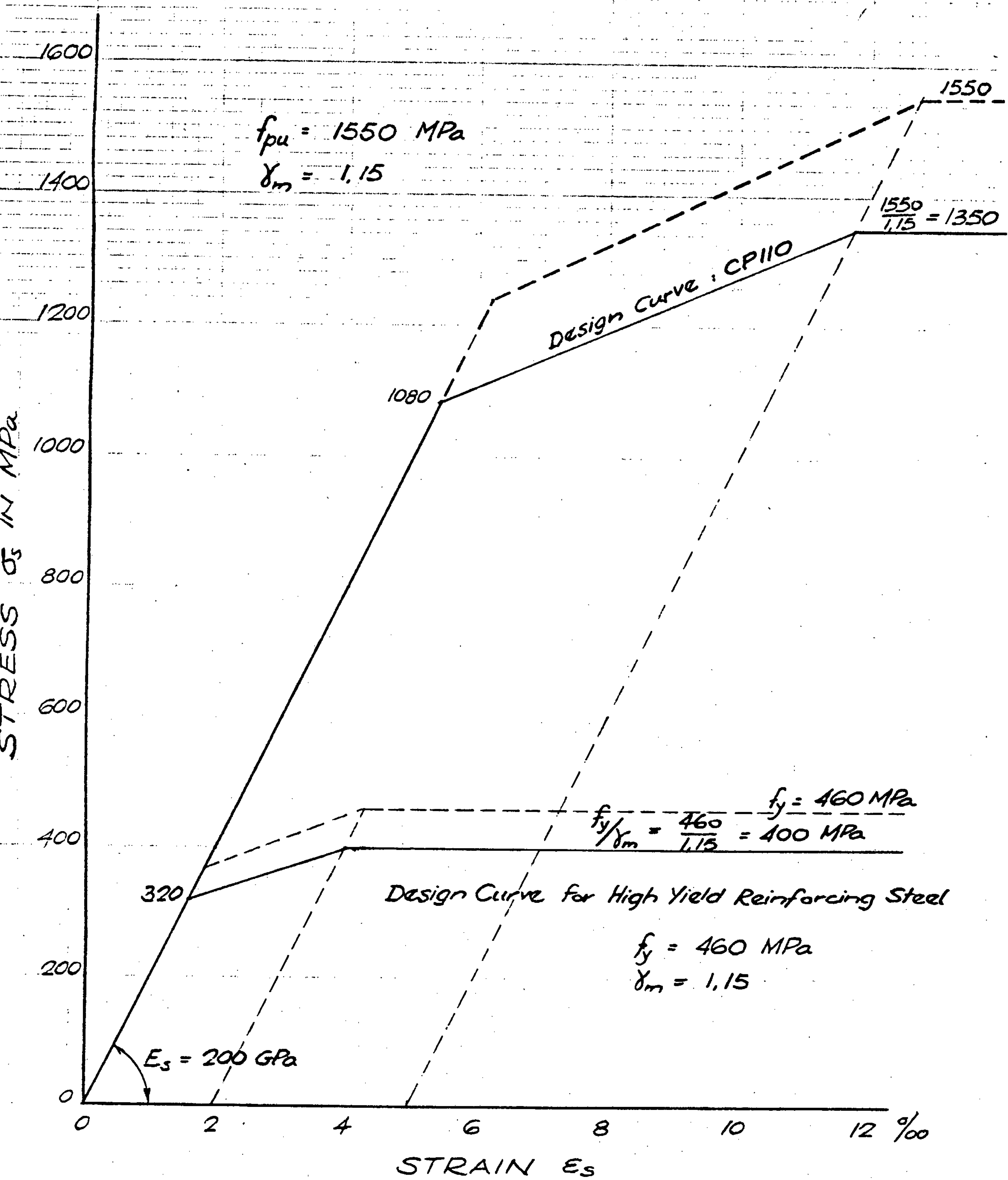
Cost of untensioned prestressing wire: R700/ton

Cost of high yield reinforcing steel: R350/ton

Modulus of Elasticity for steel: 200 GPa

Design curves for prestressing wire and high yield steel are attached.

[40 marks]



1600
1400
1200
1000
800
600
400
200
0
STRESS σ_s IN MPa
f_{pu} = 1550 MPa
γ_m = 1,15
1550
$\frac{1550}{1,15}$ = 1350
Design Curve : CP110
1080
f_y = 460 MPa
$f_y/\gamma_m = \frac{460}{1,15}$ = 400 MPa
320
Design Curve for High Yield Reinforcing Steel
f_y = 460 MPa
γ_m = 1,15
E_s = 200 GPa
0
2
4
6
8
10
12 ‰
STRAIN ε_s

UNIVERSITY OF CAPE TOWN

DEPARTMENT OF CIVIL ENGINEERING

M.Sc IN CIVIL ENGINEERING

UNIVERSITY EXAMINATION : FEBRUARY 1976

CE 525 : Coastal Engineering

All Questions may be attempted — Time : 3 hours

Constants

Sea water density = 1025 kg/m^3

Sea water weight = 10 kN/m^3

1. A swell of 10 second period with a deep water wave height of 3 m approaches a beach with the wave crests parallel to the shore. Trace the progress of this wave in shoaling water through to the breaker point including the following calculations :-
   - (a) the wave length and wave celerity in deep water
   - (b) the water depth at which the wave begins to be affected by the presence of the sea bed.
   - (c) the wave length and wave celerity for water depths at 10 m intervals between d=80 m and d=10 m, and at 1 m intervals between d=10 m and d=1 m.
   - (d) the depth of water in which the wave breaks, the type of breaker and the wave height at breaking. Ignore the effect of wave set up or down.

     d slope in 50
   - (e) sketch the effect of wave set up and down including an estimate of depths.
   - (f) estimate the wave heights in the surf zone.
   - (g) calculate the energy flow in W/m in water depths of 10 m, 5 m, and 2 m.

2. A cylindrical pipe is laid on the sea bed across a harbour entrance in 10 m of water, the pipe diameter being 1 m and the axis of the pipe is parallel to the local wave crests. If the local wave length is 50 m, estimate the wave period, and find the peak magnitudes of the velocity and acceleration force components per metre length of pipe. Estimate the peak resultant force in the inshore direction, and the timing of this in relation to the passage of a wave crest

   H=2m C_D =1,2 C_M =2,5

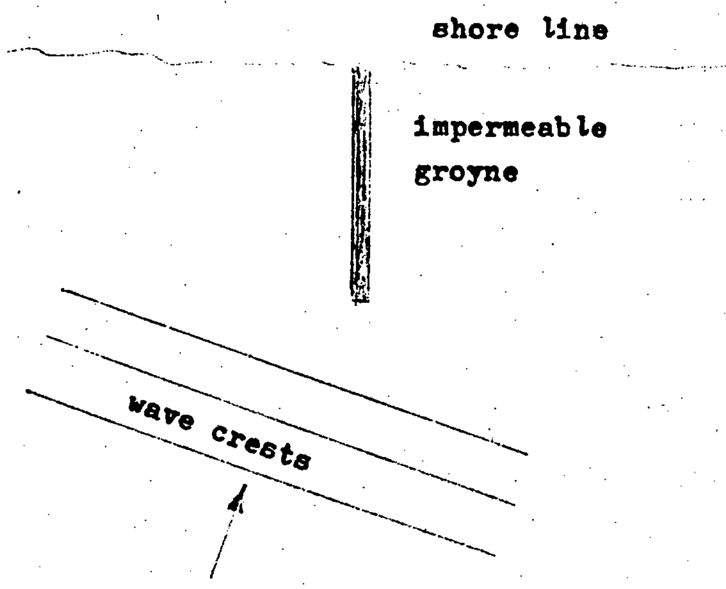
3.(a). A storm at sea generates waves with a period range of 6 to 12 seconds. The resulting swell travels towards a harbour 400 km away. Estimate the time required for the longest waves to cover the intervening distance, assuming deep water throughout. Also estimate how much later the shortest waves will begin to arrive.

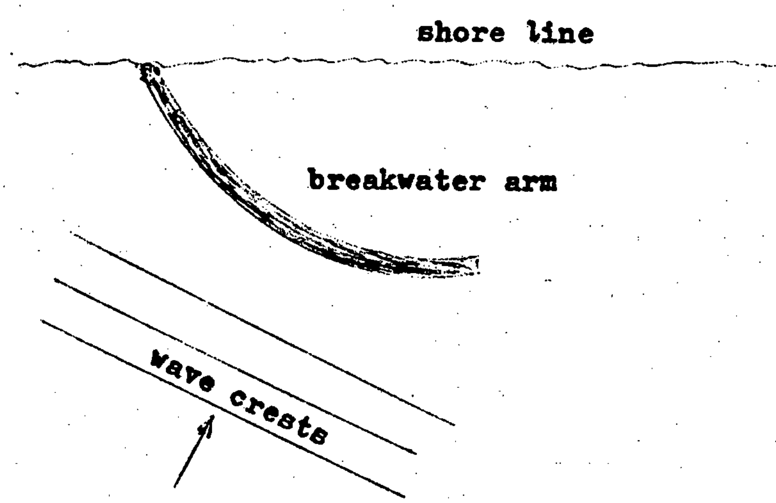
(b). A refraction diagram is constructed for a bay and the spacing between a particular pair of adjacent orthogonals doubles in travelling from deep water to the 10 m depth, the wave period being 7 seconds. Estimate the percentage change in wave height ocurring between these zones on the assumption that no breaking waves are present between the zones.

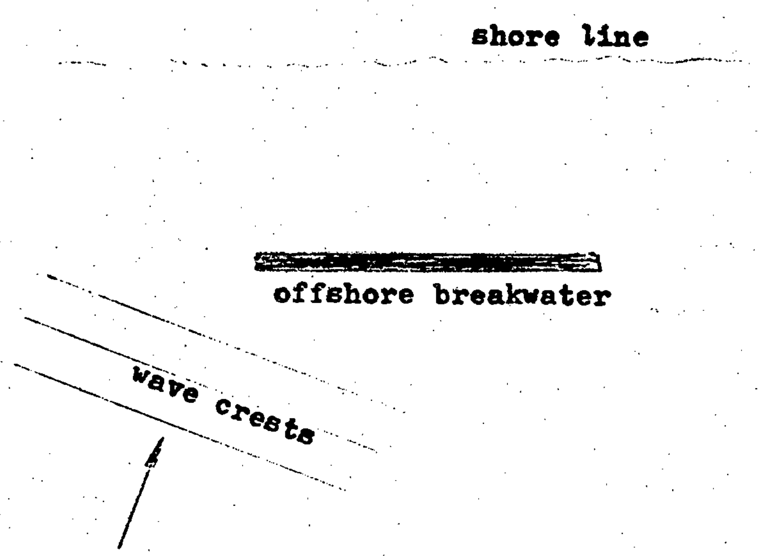
(c). Suggest some of the requirements you would incorporate into a specification for armour blocks.

4. The overleaf page shows the plan views of three separate coastal structures on which oblique waves impinge. In each case indicate areas where you consider deposition or erosion will occur, and also estimate the shape of the breaker line once stable conditions are established

5. There is a continuous dissipation of energy due to tidal movements of water over the earths surface, and in some instances useful power is abstracted from the sea in tidal power schemes. Suggest what effect this may have on the dynamics of the earth-moon system over very long periods of time.


shore line
impermeable
groyne
wave crests


shore line
breakwater arm
wave crests


shore line
offshore breakwater
wave crests

UNIVERSITY OF CAPE TOWN

DEPARTMENT OF CIVIL ENGINEERING

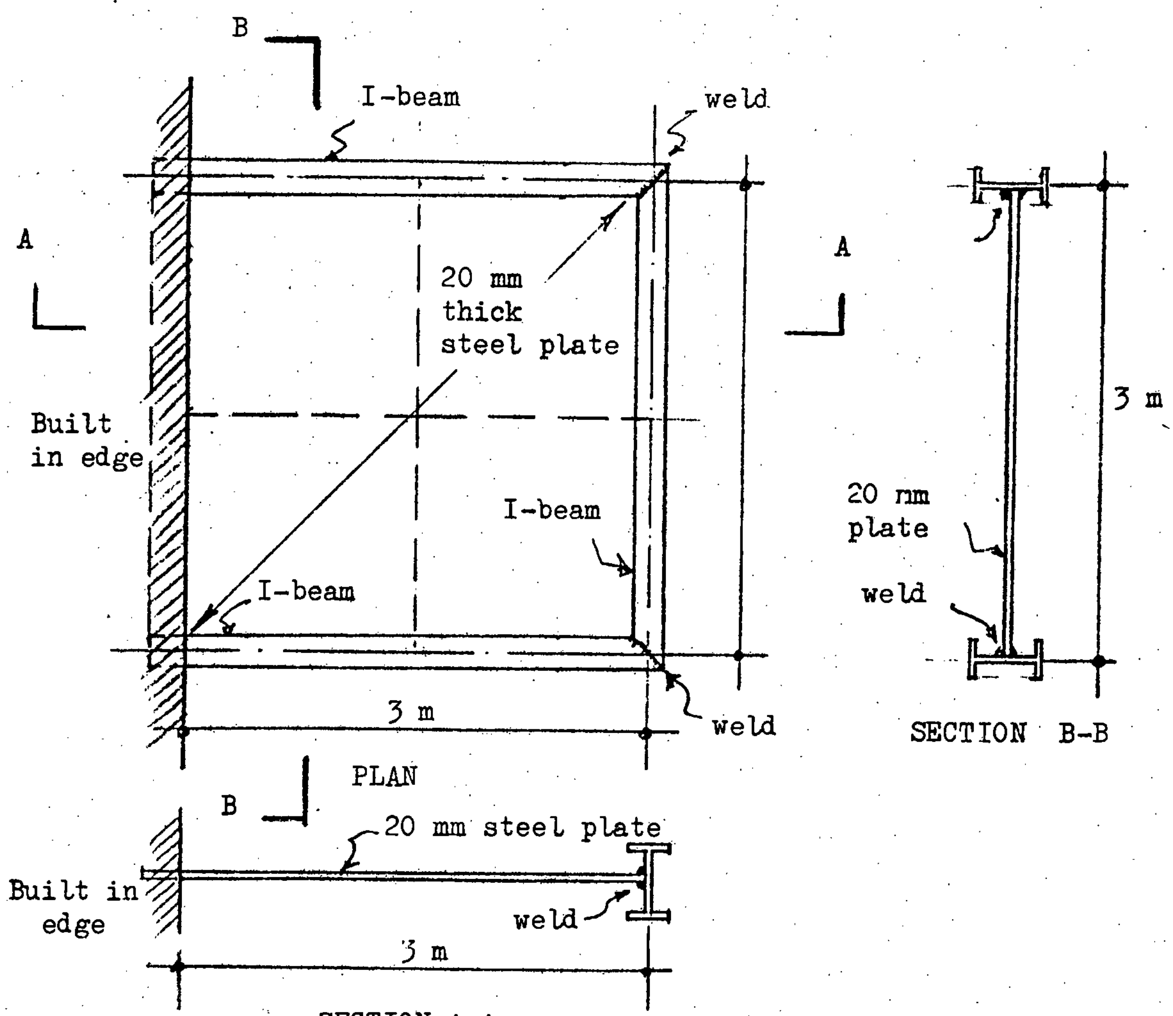
UNIVERSITY EXAMINATION - 21st AUGUST, 1976

COURSE CE 515: SURFACE STRUCTURES

Time allowed: 3 hours

Answer both questions

1.



SECTION A-A

A 20 mm thick plate is welded to a rigid framework on three sides and the plate and beams are built into a substantial concrete wall on the other side as shown.

The properties of the steel beams and the plate are given below.

The plate is subjected to a uniformly distributed load of 10 kN/m^2.

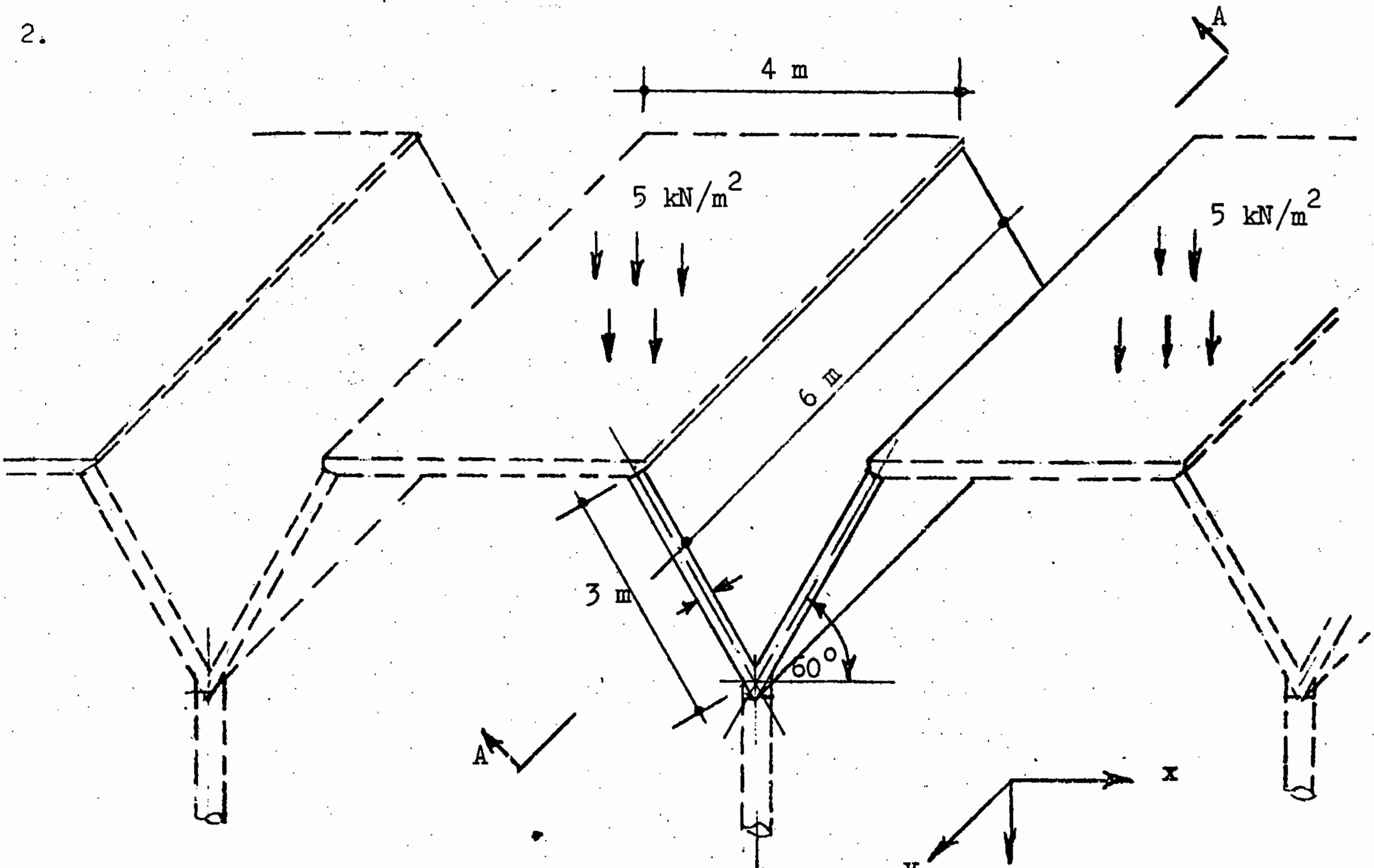
Show all the steps necessary to analyse this structure for displacements and bending moments.

(Hint: Demonstrate the method with a coarse grid as shown).

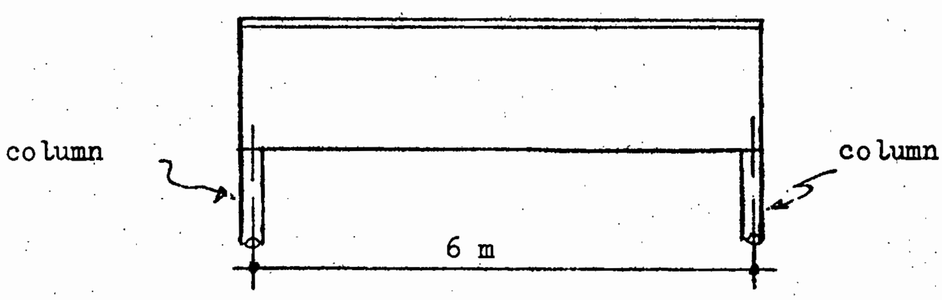
Section Properties:

Beams:	E = 200 GPa	$I = 1{,}7 \times 10^{-3}$ m^4	Plate:	E = 200 GPa
	G = 80 GPa	$J = 0{,}05 \times 10^{-3}$ m^4		$\nu = 0{,}3$
		$A = 22 \times 10^{-3}$ m^2		h = 20 mm

2.



Oblique View



Longitudinal Section A-A

Show what steps are required to determine the displacements, stresses and bending moments in the V-shaped portion only of the roof structure shown.

The horizontal slabs are subjected to a uniformly distributed load of 5 kN/m^2.

Note: (1) There is no moment connection between the horizontal slabs and the V-shaped sections.

(2) All slabs are 100 mm thick.

$$[H] = \frac{E}{ab(1-\nu^2)} \begin{bmatrix} -b & -\nu a & 0 & \nu a & b & 0 & 0 & 0 \\ -\nu b & -a & 0 & a & \nu b & 0 & 0 & 0 \\ (-ca-db) & (-cb-da) & (ca+db) & (-cb+da) & (-ca+db) & (cb+da) & (ca-db) & (cb-da) \\ 0 & -\nu a & -b & \nu a & 0 & 0 & b & 0 \\ 0 & -a & -\nu b & a & 0 & 0 & \nu b & 0 \\ (-ca+db) & (-cb-da) & (ca-db) & (-cb+da) & (-ca-db) & (cb+da) & (ca+db) & (cb-da) \\ -b & 0 & 0 & 0 & b & -\nu a & 0 & \nu a \\ -\nu b & 0 & 0 & 0 & \nu b & -a & 0 & a \\ (-ca-db) & (-cb+da) & (ca+db) & (-cb-da) & (-ca+db) & (cb-da) & (ca-db) & (cb+da) \\ 0 & 0 & -b & 0 & 0 & -\nu a & b & \nu a \\ 0 & 0 & -\nu b & 0 & 0 & -a & \nu b & a \\ (-ca+db) & (-cb+da) & (ca-db) & (-cb-da) & (-ca-db) & (cb-da) & (ca+db) & (cb+da) \end{bmatrix}$$

Where $c = \frac{1-\nu}{4}$ and $d = \frac{\nu}{2}$

Stress/Displacement Matrix for Plane Stress Rectangular Elements

Finally the element stiffness matrix for in-plane forces becomes:-

(See over)

$$
\left[K_e^p\right] = \frac{Eh}{180(1-\nu^2)}
\begin{bmatrix}
p\cdot\frac{b}{a} + q\cdot\frac{a}{b} & & & & & & & \\
s & p\cdot\frac{a}{b} + q\cdot\frac{b}{a} & & & & \text{S Y M M E T R I C A L} & & \\
r\cdot\frac{b}{a} - q\cdot\frac{a}{b} & -t & p\cdot\frac{b}{a} + q\cdot\frac{a}{b} & & & \text{V A L U E S} & & \\
t & -p\cdot\frac{a}{b} + q\cdot\frac{b}{a} & -s & p\cdot\frac{a}{b} + q\cdot\frac{b}{a} & & & & \\
-p\cdot\frac{b}{a} + q\cdot\frac{a}{b} & t & -r\cdot\frac{b}{a} - q\cdot\frac{a}{b} & s & p\cdot\frac{b}{a} + q\cdot\frac{a}{b} & & & \\
-t & r\cdot\frac{a}{b} - q\cdot\frac{b}{a} & s & -r\cdot\frac{a}{b} - q\cdot\frac{b}{a} & -s & p\cdot\frac{a}{b} + q\cdot\frac{b}{a} & & \\
-r\cdot\frac{b}{a} - q\cdot\frac{a}{b} & -s & -p\cdot\frac{b}{a} + q\cdot\frac{a}{b} & -t & r\cdot\frac{b}{a} - q\cdot\frac{a}{b} & t & p\cdot\frac{b}{a} + q\cdot\frac{a}{b} & \\
-s & -r\cdot\frac{a}{b} - q\cdot\frac{b}{a} & t & r\cdot\frac{a}{b} - q\cdot\frac{b}{a} & -t & -p\cdot\frac{a}{b} + q\cdot\frac{b}{a} & s & p\cdot\frac{a}{b} + q\cdot\frac{b}{a}
\end{bmatrix}
$$

where:- $p = 60 + \frac{30\nu^2}{(1-\nu)}$; $q = 22.5\,(1-\nu)$; $r = 30 - \frac{30\nu^2}{(1-\nu)}$; $s = 22.5(1+\nu)$; $t = 22.5(1-3\nu)$

Plane Stress Rectangular Element Stiffness Matrix

UNIVERSITY OF CAPE TOWN

DEPARTMENT OF CIVIL ENGINEERING

UNIVERSITY EXAMINATION: JUNE 1976

COURSE CE 506 - PROPERTIES OF CONCRETE

Time allowed: 3 hours   5th June, 1976

Part A consists of fifteen multiple-choice questions. Each question is followed by five suggested answers; select the one which is best in each case and circle one of (a), (b), (c), (d) or (e) for each question. This portion of the examination paper must NOT be removed from the Examination Room and must be handed in for marking.
Part B consists of five questions. Answer all questions.

PART A - Multiple-Choice Section (All questions of equal value)

Question A1: In controlling the quality of concrete produced for a project, a test is needed which:

(a) gives the true strength of the material;

(b) gives, for variations in testing procedures, the least variation in results;

(c) gives the true strength of the specimen;

(d) gives a clearly defined stress pattern;

(e) is easy to carry out.

Question A2: In design of concrete mixes according to CP 110 Concrete Structures Code, the target strength chosen is directly related to:

(a) the design strength f_{cu};

(b) the design strength f_{cu} plus 1,65 times the standard deviation 'σ'.

(c) the design strength f_{cu} plus the standard deviation 'σ';

(d) the design strength f_{cu} plus the coefficient of variation 'v';

(e) the design strength f_{cu} plus 1,65 times the coefficient of variation 'v'.

Question A3: The most important aspect of sampling from a pre-mixed concrete truck is to:

(a) protect the sample from wind and sun;

(b) obtain a representative sample in order to carry out further tests;

(c) ensure that the concrete is properly mixed;

(d) check the workability and slump;

(e) obtain a sufficient quantity of concrete to carry out further tests.

/Question A4: ....

Question A4: For a water/cement ratio of 0,6 by weight the use of rounded river gravel in place of crushed aggregate of cubic shape and rough texture will:

(a) show little difference in compressive strength but increase flexural strength;

(b) increase compressive strength by about 10% and also increase flexural strength;

(c) decrease compressive strength by about 10% but increase flexural strength;

(d) increase compressive strength slightly but lower flexural strength;

(e) decrease slightly, both compressive and flexural strengths.

Question A5: The Unit Water Method of Mix Design, described in lectures, suggests that the grading of the combined aggregate be made finer than the recommended grading when:

(a) the maximum aggregate size is larger;

(b) the maximum aggregate size is smaller;

(c) the coarse aggregate is crushed material;

(d) the cement content is higher;

(e) the cement content is lower.

Question A6: An increase in the proportion of aggregate material in the sieve range 2,00 mm to 9,5 mm (No. 8 to 3/8") will tend to:

(a) make the concrete harsh and liable to honeycomb;

(b) make the finishability of the concrete better;

(c) improve the economy of the mix;

(d) increase the amount of water required;

(e) reduce the amount of water required.

Question A7: The addition of an air entraining agent to a concrete mix usually leads to:

(a) a more economical mix;

(b) a stronger concrete;

(c) a decrease in the required sand percentage;

(d) a decrease in cement content;

(e) a denser concrete because of improved workability.

/Question A8: .....

Question A8: In the Unit Water Method of Mix Design, described in lectures, the estimated water content for a particular slump is fixed by:

(a) the maximum size of the aggregate;

(b) the grading of the aggregate;

(c) the shape of the aggregate;

(d) (a) and (b) above;

(e) (a) and (c) above.

Question A9: Capillary water in hydrated cement paste is:

(a) water held in areas of restricted adsorption of the gel structure;

(b) water occupying space beyond the range of surface forces of the solid phase of the gel structure.

(c) water existing in cavities and channels up to 100 times greater than the size of gel pores;

(d) both (b) and (c) above;

(e) water chemically combined such that it is part of the solid matter in the hardened paste.

Question A10: Plastic shrinkage of concrete is caused by:

(a) removal of capillary and gel pore water;

(b) the absorbtion of mixing water by porous or dry aggregates;

(c) sedimentation and settling of solids in the concrete mix;

(d) bleeding of free water to the top surface of the concrete where it is often lost by evaporation or drainage;

(e) all of (b), (c) and (d) above.

Question A11: The secant elastic modulus of concrete is increased by:

(a) increased water:cement ratio and increased paste content;

(b) constant water:cement ratio and increased paste content;

(c) increased water:cement ratio and decreased water content;

(d) constant water:cement ratio and air entrainment;

(e) decreased water:cement ratio and decreased paste content;

Question A13: Decreasing the water/cement ratio influences the ultrasonic pulse velocity because:

(a) poor compaction leads to voids;

(b) a decrease in the density causes the pulse velocity to increase;

(c) an increase in strength (due to a lowering of the water cement ratio) causes the pulse velocity to increase;

(d) an increase in the density causes the pulse velocity to increase;

(e) an excess of paste causes the pulse velocity to decrease.

/Question A14: ....

Question A14: Rapid Hardening Portland cement can be manufactured by:

(a) more finely grinding the Portland cement;

(b) changing the ratio of $C_2S:C_3S$;

(c) intergrinding some high alumina cement with the Portland cement;

(d) both (a) and (b) above;

(e) all of (a), (b) and (c) above.

Question A15: Excessive bleeding of concrete can be corrected by:

(a) adding more cement;

(b) adding crusher dust or other fine material;

(c) by air entrainment;

(d) both of (a) and (b) above;

(e) all of (a), (b) and (c) above

[Total 20 marks]

PART B

Question B1:

(a) A laboratory trial mix of concrete with 30 kg of water, 50 kg of cement, 130 kg of sand and 180 kg of stone gave a 28-day strength which was too low, a slump of 110 mm and real mortar excess of 8%. It is decided that a reduction in water/cement ratio to 0,56 will probably correct the strength requirement. What mix would you suggest for a second trial to give a slump of 60 mm and a real mortar excess of 2% given that the densities of the water, cement, sand and stone are 1000, 3150, 2600 and 2750 kg/m^3 respectively.

(b) The compressive strength of the second trial mix after 28 days' storage at 18°C is 33 MPa. Using Plowman's method, determine how long it would take to reach the same strength at 25°C. What will be the compressive strength after 3 days at 25°C?

[20 marks]

Question B2:

Consider an average structural grade concrete made with 20 mm river gravel aggregate (irregular gravel), normal Portland cement, water/cement ratio (by weight) 0,60 aggregate/cement ratio 6,0, and slump of 75 mm.

(i) Calculate the effect on strength of adding water so as to increase the slump to 150 mm.

(ii) How does this strength change compare with that expected to result from changing from gravel to crushed coarse aggregate but maintaining the aggregate/cement ratio at 6,0 and slump at 75 mm?

(iii) If a graded river gravel with maximum size 80 mm was used in place of the 20 mm gravel, comment on the expected water demand, water/cement ratio and resulting compressive strength of the concrete.

Clearly state the assumptions made in each case.

[15 marks]

Question B3:

(i) Explain briefly how the progressive hydration of cement may lead to self-desiccation of concrete.

(ii) Calculate the gel/space ratio for a concrete with a water/cement ratio of 0,60 at an age of 14 days at which time 60 per cent of the cement had hydrated. Comment on the expected compressive strength corresponding to this gel/space ratio.

(iii) 100 g of cement and 20 g of water are placed in one sealed container and 100 g of cement and 60 g of water are placed in another sealed container. Calculate in both instances the maximum degree of hydration possible, the volume of gel formed, the weight of chemically combined water and the weight of free water in the capillary pores.

[20 marks]

/Question B4: ....

Question B4: "When concrete specimens are loaded axially in compression they always fail in tension". Briefly discuss this statement and go on to discuss the effect of specimen size and shape, and also the effectiveness of capping materials on the apparent ultimate compressive strength of concrete test specimens.

[10 marks]

Question B5: A considerable number of different types of test procedures have been devised to measure "workability" of concrete. Discuss the reasons for the multiplicity of methods used. List ways in which the workability of concrete can be increased without increasing the water content.

[20 marks]

UNIVERSITY OF CAPE TOWN

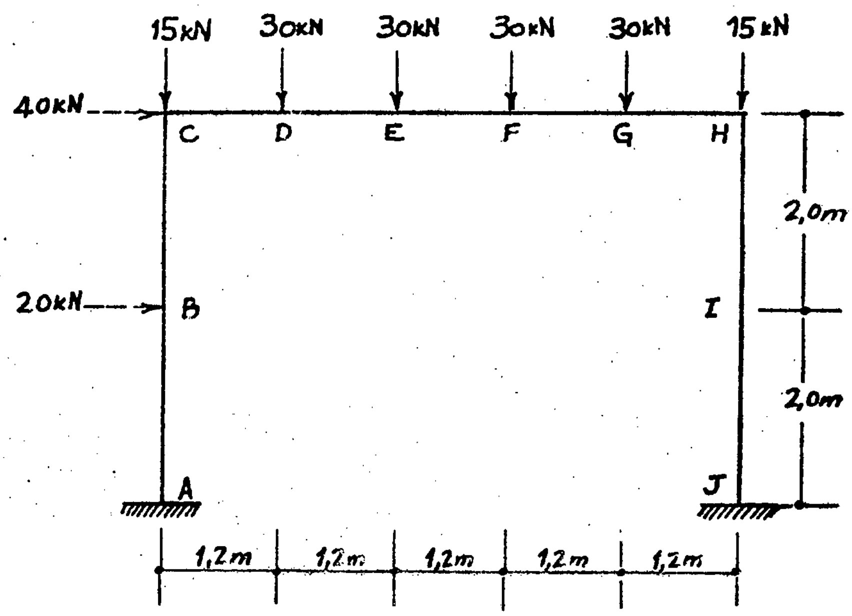
DEPARTMENT OF CIVIL ENGINEERING

UNIVERSITY EXAMINATION: 13th NOVEMBER, 1976

CE 519 - STEEL STRUCTURES

Time allowed: 3 hours

1.



The rectangular frame shown above is to be designed by plastic methods. The loads shown are working loads: the vertical loads represent dead plus superimposed loads and the horizontal loads represent wind loads. The wind loads may act from left to right (at B and C as shown) or from right to left (at H and I).

1. Use limit analysis to determine the least value of M_p for which the frame can equilibrate all factored load combinations using the following assumptions:

   (a) the frame is designed with a uniform section,

   (b) the load factor for dead plus superimposed load alone is 1,75, and for dead plus superimposed load plus wind load 1,4.

   Draw the bending moment and shear force diagrams, and determine the axial loads in the members, for the collapse conditions.

[40 marks]

2. Using the Abridged Version of the Handbook on Hot Rolled Structural Steel Sections, and the Design Recommendations issued, select an appropriate parallel flange I-section for this design. The yield stress is to be taken as 250 MPa.

   Choose your section/sections with respect to the collapse bending moments, shear forces and axial loads. Consider

   (a) whether the section or sections chosen is/are compact,

   (b) whether shear stiffeners are required,

   (c) lateral stability and the points at which lateral bracing is required,

   (d) in-plane buckling.

[35 marks]

3. The moments computed from an elastic analysis with uniform E.I. over the entire frame are given below. Tension on the inside of the frame is taken as positive.

Section	Moment due to vertical loads only (kNm)	Moment due to wind only acting from left to right (kNm)
A	+ 27,00	- 70,91
B	- 13,50	+ 1,93
C	- 54,00	+ 34,77
D	+ 18,00	+ 20,36
E	+ 54,00	+ 5,95
F	+ 54,00	- 8,45
G	+ 18,00	- 22,86
H	- 54,00	- 37,27
I	- 13,50	+ 9,95
J	+ 27,00	+ 57,16

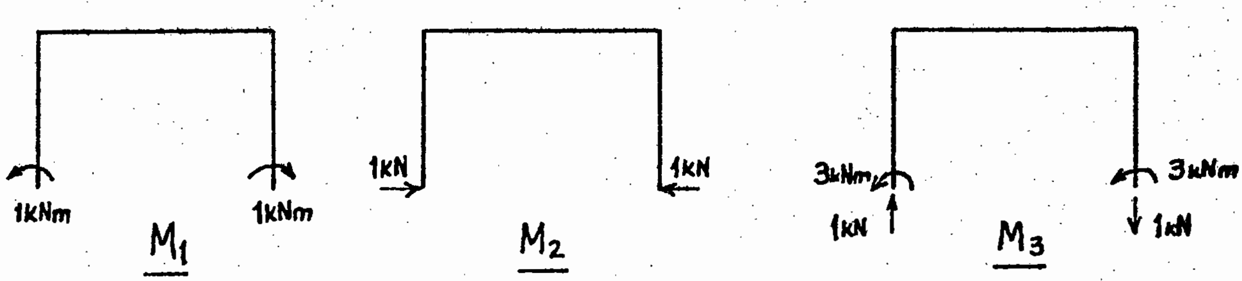
Assume for simplicity that the dead and superimposed load together may or may not act. The wind may act from left to right or from right to left.

For the M_p value calculated in part (a), determine the load factor against failure by alternating plastic deformation at any section.

Do you consider this result to be significant in determining member sizes?

[10 marks]

4.



Using the three independent self-stress systems associated with the force systems shown above, write down the compatibility equations for the structure analysed in 1. above at the point of collapse. Assume plastic hinge rotations at each of the hinges in the mechanism.

You may take the following values for the integrals below:

$$\int M_1 \frac{M}{EI}\, ds = -0{,}0076 \text{ kNm}$$

4. (Continued)

$$\int M_2 \frac{M}{EI}\, ds = -\ 0{,}0491 \text{ kNm}$$

$$\int M_3 \frac{M}{EI}\, ds = +\ 0{,}0240 \text{ kNm}$$

M is the collapse bending moment diagram, and moments causing tension on the inside are positive. The integrals extend over the whole structure.

Hence determine which is the last hinge to form.

[15 marks]

UNIVERSITY OF CAPE TOWN

DEPARTMENT OF CIVIL ENGINEERING

UNIVERSITY EXAMINATION   JULY 1977

COURSE CE 533 - BRIDGE ENGINEERING

Time allowed: THREE HOURS   Notes are allowed

1. Write a brief critical review of the main ideas contained in the road traffic bridge loading specifications covered in this course and discuss in particular the effects of using simplified or equivalent loading systems.

[20 marks]

2. Give a critical evaluation of the methods of analysis for hollow concrete slab bridges, considering different deck plan shapes and different void configurations.

[20 marks]

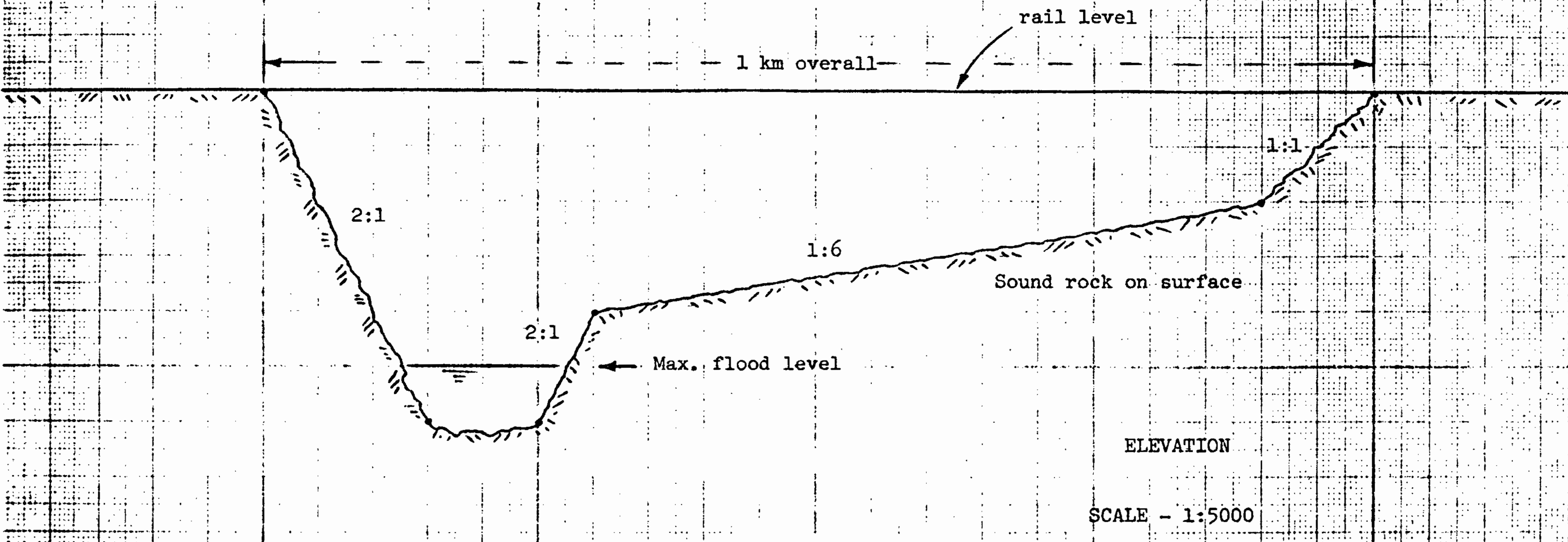
3. For each of the four bridge sites on the attached sheets, select the most suitable type of bridge structure and construction method.

Draw adequate sketches of the superstructure, substructure and foundations directly on these sheets and hand them in. State all assumptions clearly and describe the construction method adequately. List brief reasons for all the major decisions.

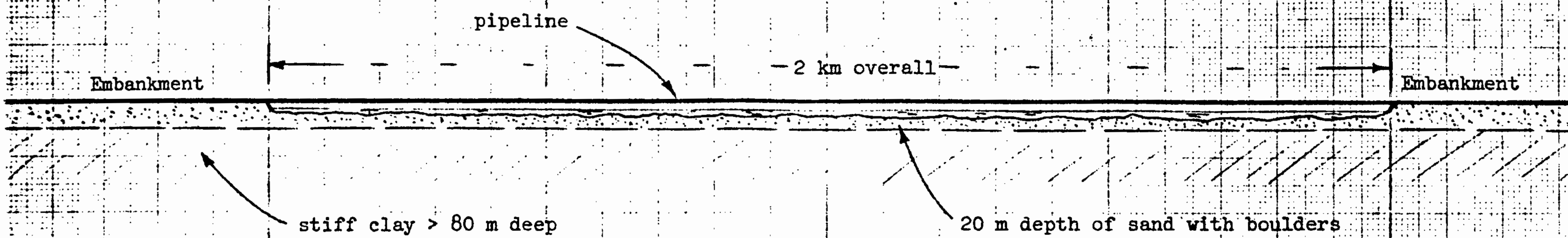
[15 marks each]

3a/ . . . . .

3(a) A straight and level single railway line crossing a gorge of great scenic beauty 100 km inland from the coast:



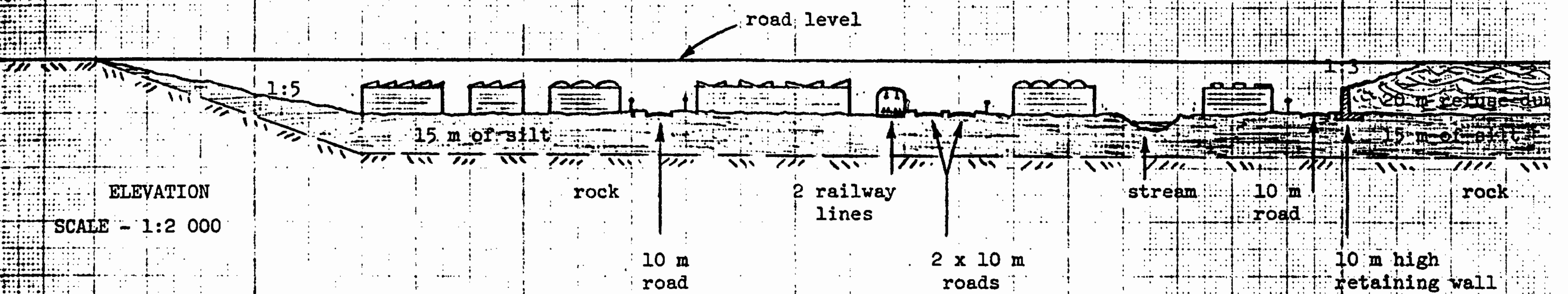
3(b) A straight and level 2 m diameter steel water pipeline crossing an estuary on the coast. The water depth in the estuary is between 2 m and 6 m, with ± 1 m tidal fluctuation. A minimum headroom of 10 m is required for yachting.



ELEVATION

SCALE - 1:10 000

3(c) A straight and level single carriageway two-lane road with sidewalks, 12,0 m overall width, crossing an industrial estate situated in a 20 m deep valley.

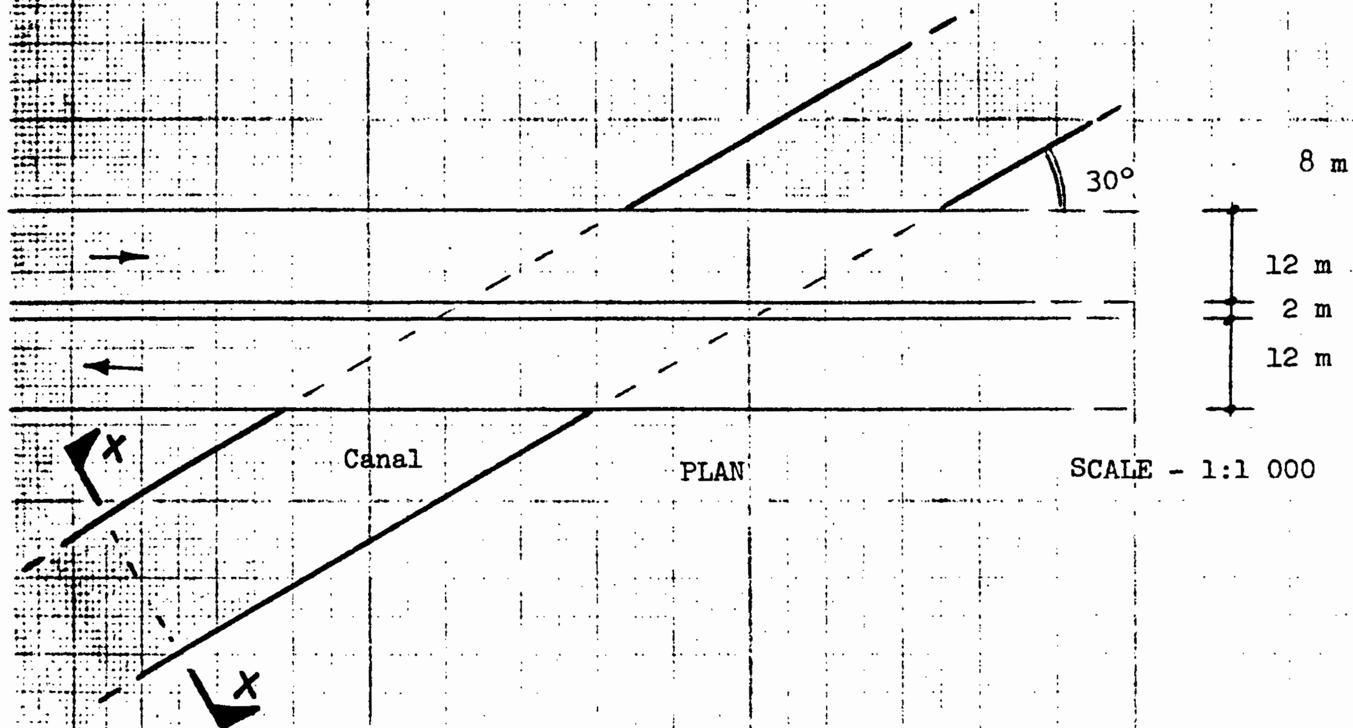


ELEVATION

SCALE - 1:2 000

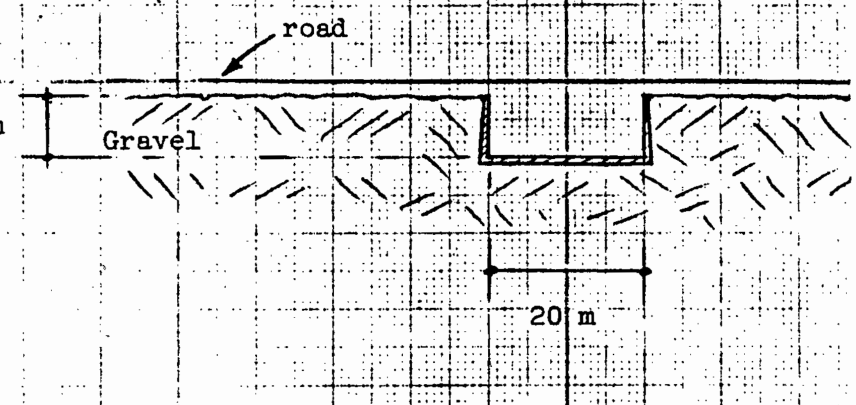
3(d) A dual carriageway rural freeway crossing an existing RC irrigation canal on level ground in the Karroo.

No piers are allowed in the canal and the canal may not be subjected to additional loads.



PLAN

SCALE - 1:1 000



SECTION XX