Reinsurance and Dividend Management.

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

Signed by candidate

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November 27, 2014

Abstract

In this dissertation we set to find the dual optimal policy of a dividend payout scheme for shareholders with a risk-averse utility function and the retention level of received premiums for an insurance company with the option of reinsurance. We set the problem as a stochastic control problem. We then solve the resulting second-order partial differential equation known as Hamilton-Jacobi-Bellman equation. We find out that the optimal retention level is linear with the current reserve up to a point whereupon it is optimal for the insurance company to retain all business. As for the optimal dividend payout scheme, we find out that it is optimal for the company not to declare dividends and we make further explorations of this result.

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Notation

$\mathbb{R}$	the real numbers
$\mathbb{P}$	basic probability measure
Ω	sample or event space on which probabilities are defined
$\mathcal{F}$	σ-algebra of the probability space
$\{\mathcal{F}_t\}_{t\geq 0}$	filtration
$W(t)$, W_t	standard Brownian motion
$\mathbb{E}\{X\}$	the expectation of the random variable X
S^n	the set of all $(n \times n)$ symmetric matrices
$C([0,\infty);\mathbb{R}^n)$	the set of all continuous functions $\varphi : [0,\infty) \to \mathbb{R}^n$
$\mathcal{B}(U)$	the Borel σ-field generated by all the open sets of U
$:=$	equals by definition
$\mathbf{W}^n$	$:= C([0,\infty);\mathbb{R}^n)$
$\mathbf{W}^n_t$	$:= \{\zeta(\cdot \wedge t) \mid \zeta(\cdot) \in \mathbf{W}^n\}, \quad t \geq 0$
$\mathcal{B}_t(\mathbf{W}^n)$	$:= \mathcal{B}(\mathbf{W}^n_t)$
$\mathcal{B}_{t+}(\mathbf{W}^n)$	$:= \cap_{s>t}\mathcal{B}(\mathbf{W}^n_t), \quad t \geq 0$
$\mathcal{A}^n(U)$	the set of all $\{\mathcal{B}_{t+}(\mathbf{W}^n)\}_{t\geq 0}$-progressively measurable processes $\eta : [0,\infty) \times \mathbf{W}^n \to U$
$L^p_{\mathcal{G}}(\Omega;\mathbb{R}^n)$	the set of $\mathbb{R}^n$ - valued $\mathcal{G}$-measurable random variables X such that $\mathbb{E}\vert X(t)\vert^p < \infty$, $p \in [1,\infty)$
D_x	$:= \frac{\partial}{\partial x}$
D_{xx}	$:= \frac{\partial^2}{\partial x^2}$
$\vert \cdot \vert$	absolute value or distance function where appropriate

Chapter 1

Introduction

In any financial firm the goal is to optimise net income (dividends) for the shareholders. It is no different for an insurance firm. The insurance business aims to generate dividend income for the insurance company's shareholders, and naturally we question how best the company can maximise the expected present value of the shareholders' income. The study of optimal dividends problems was pioneered by de Finetti around 1957 [4]. It has been shown empirically that it is possible to increase the expected present value of net income (dividends) to shareholders of an insurance firm by effecting reinsurance.

Lately there has been increasing attention towards insurance-related problems using stochastic control theory. This is so because a company, an insurance company in this instance, has to devise a way of controlling investment strategies all the while having to pay dividends in a way that maximises or minimises a particular objective function observing certain constraints. As such we have a stochastic control problem [17].

Stochastic control is the study in control theory that deals with the existence of uncertainty either in observations of the data or in the drivers of the evolution of the data. A control system can be viewed informally as a dynamical system containing a parameter (control or input) which can be manipulated to influence the behaviour of the system so as to achieve a desired goal.

In this dissertation we aim to find the reinsurance and dividend policy that maximise the expected discounted utility of dividend payouts over a time period. This dissertation focusses only on one form of reinsurance which is proportional reinsurance. We begin by explicitly defining the problem in Chapter 2.

In Chapter 3 we look at stochastic control theory, reinsurance basics, characteristics of utility functions and dividend distribution policies within companies. In Section 3.1 we look at tools which aid us in calculations carried out in Chapter 5. We identify assumptions and properties to be satisfied by the results. We then take a look at the general framework of reinsurance in terms of the forms and types of contracts and its role in risk reduction. We examine the properties of utility functions of Hyperbolic Absolute Risk Averse type in which the utility function we examine falls under. We finally take a look at considerations taken into account when companies declare dividends.

In Chapter 4 we look at academic papers which have examined similar problems. These papers provide a theoretical framework on which we base this dissertation. We explore the methods used in these papers noting conditions under which the methods are applied. We look at the results obtained and further analysis carried out.

In Chapter 5 we formulate the Hamilton-Jacobi-Bellman equation so as to solve for the value function. With this value function we then proceed to calculate the optimal retention level and dividend strategy. We do a further analysis of the effect of exogenous parameters on the optimal strategies obtained. We make final remarks in Chapter 6.

Chapter 2

Problem Statement

The following is a detailed description of the problem statement adapted from Yong and Zhou [16].

We consider a model of an insurance company, which can choose a reinsurance policy to manage risks. In addition there is a choice of the amount of dividends to be paid out to shareholders. The objective is then to find the reinsurance and dividend policy that maximise the discounted expected total utility of dividend payouts over a time period.

Here a is the proportion of all premiums that the company individually insures so reinsurance allows the company to divert a proportion $(1-a)$ of all premiums to another company, and consequently, the fraction $(1-a)$ of each claim is paid by the reinsurer.

We describe the model as follows: Denote the value of the liquid assets of the company at time s by $X(s)$. Let $c(s)$ be the dividend rate paid out to the shareholders at time s. Then the dynamics of $X(s)$ are given by

$$\begin{cases} \mathrm{d}X(s) = [a(s)\mu - c(s)]\mathrm{d}s + a(s)\sigma \mathrm{d}W(s), & s \in [t, T], \\ X(t) = x_t. \end{cases} \tag{2.0.1}$$

Here μ is the difference between the premium rate and the expected payments on claims per unit time (called the safety loading), $(1-a(s))$ is the reinsurance fraction at time s, and x_t is the initial value of the liquid assets of the company. W denotes the Weiner process (or standard Brownian motion). By the nature of the problem, the following constraints must be satisfied:

$$X(s) \geq 0, \quad 0 \leq a(s) \leq 1, \quad \forall s \in [t, T]. \tag{2.0.2}$$

The objective of the company is to choose the dividend payout scheme $c(\cdot)$ and the risk management policy $a(\cdot)$, both being nonanticipative, such that (2.0.2) is satisfied and the following discounted expected total utility of dividend payouts over $[t, T]$ is maximised:

$$\mathbb{E}\left[\int_t^T e^{-rs} U(c) \, \mathrm{d}s\right], \tag{2.0.3}$$

where $r > 0$ is the discount rate, s is the dummy time variable and $(\Omega, \mathcal{F}, \mathbb{P})$ is the applicable probability space.

Chapter 3

Preliminaries

3.1 Stochastic control theory

The central idea in solving the problem presented in Chapter 2 is that we formulate it in terms of a second order partial differential equation known as the Hamilton-Jacobi-Bellman equation. Upon solving this HJB equation for the value function, explained shortly, we are able to attain the optimal reinsurance strategy, a and the optimal dividend distribution strategy, c. Theorems and discussions in this section provide a mathematical framework on which we can apply the method of solution.

Stochastic control is the 'study of dynamical systems[1] subject to random perturbations and which can be controlled in order to optimise some performance criterion' [11].

We begin by giving a mathematical formulation to the optimisation problem; firstly we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a standard Brownian motion process $\{W_t\}_{t\geq 0}$, defined on a filtered probability space, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The filtration $\mathcal{F}_t$ represents the information available to the observer of the process at time t and any decision is completely based upon this information.

[1] A dynamical system is one which changes in time (in some well defined way); what changes is the state of the system. For such systems, the basic problem is to predict the future behaviour. For this purpose stochastic differential equations are well suited.

We then consider the real valued controlled state process X which satisfies;

$$\begin{cases} \mathrm{d}X(t) = b(t, X(t), u)\mathrm{d}t + \sigma(t, X(t), u)\mathrm{d}W(t), & t \geq 0, \\ X(0) = R. \end{cases} \tag{3.1.1}$$

where the control $u = \{u(s)\}_{0 \leq s \leq T}$ is a progressively measurable[2] process valued in the control set U, a subset of $\mathbb{R}^2$, since we are considering only two control variables; the dividend payout rate and the retention level. We impose that the initial reserve $X(0)$ is $\mathcal{F}_0$-measurable and without loss of generality we assumed that it is equal to a deterministic value, R.

Associated with the controlled state process X is the cost functional or performance index defined as

$$\mathcal{J}(t, x, u(\cdot)) = \mathbb{E}\{\int_0^T f(t, x(t), u(t))\,\mathrm{d}t + h(x(T))\} \tag{3.1.2}$$

where $f(t, x(t), u(t))$ is the integrand in the value function to be defined in the next subsection. In this dissertation, the performance index is given by (2.0.3) and there is no explicit boundary condition thus we will always consider $h(x(T))$ to be equal to zero in the calculations.

We now introduce some assumptions for the problem setting:

1. $(\mathcal{A}, d)$ is a Polish space [3] and $T > 0$.

2. The maps $b : [0, T] \times \mathbb{R} \times U \to \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \times U \to \mathbb{R}$, $f : [0, T] \times \mathbb{R} \times U \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous, and there exists a constant $L > 0$ such that for $\varphi(t, x, u) = b(t, x, u), \sigma(t, x, u), f(t, x, u), h(x)$;

$$\begin{cases} |\varphi(t, x, u) - \varphi(t, \hat{x}, u)| \leq L|x - \hat{x}|, & \forall t \in [0, T],\ x, \hat{x} \in \mathbb{R},\ u \in U, \\ |\varphi(t, 0, u)| \leq L, & \forall (t, u) \in [0, T] \times U. \end{cases} \tag{3.1.3}$$

The above assumptions state that b, σ, f and h are bounded uniformly over x and are globally Lipschitz functions of x. Under the above assumptions, for

[2] Progressive measurability implies that the stopped process is also measurable.
[3] A Polish space is a metric space that is complete and separable.

any $(t,x) \in [0,T) \times \mathbb{R}$ and $u \in U$, (3.1.1) admits a unique strong solution $x(\cdot) \equiv x(\cdot; t, x, u(\cdot))$.

The Value function

An admissible policy is one such that both $a(\cdot)$ and $c(\cdot)$ are nonanticipative[4] where $0 \le a(\cdot) \le 1$ and $c \le x$. For a given admissible policy u we define the return or gain function V_u by

$$V_u(t,x) = \mathbb{E} \int_t^T -e^{-rs} c^{\eta} \, \mathrm{d}s$$

where r is the discount rate, s is the time dummy variable and $\eta < 0$. Here we have already substituted into the performance index given by (2.0.3), the utility function, $U(c) = -c^{\eta}$, which we are going to do the analysis with. The initial objective is then to find optimal return function otherwise known as the value function, which is defined as

$$V(t,x) = \sup_{u \in U} V_u(t,x) \tag{3.1.4}$$

and finally an optimal policy u^* that satisfies $V(t,x) = V_u(t,x)$ for all (t,x).

Proposition 1. Let the assumptions in (3.1.3) hold. The value function $V(t,x)$ satisfies the following:

$$|V(t,x)| \le K(1+|x|), \qquad \forall (t,x) \in [0,T) \times \mathbb{R}, \tag{3.1.5}$$

$$|V(t,x) - V(\hat{t},\hat{x})| \le K\{|x-\hat{x}| + (1+|x| \vee |\hat{x}|)|t-\hat{t}|^{\frac{1}{2}}\}, \quad \forall t,\hat{t} \in [0,T),\ x,\hat{x} \in \mathbb{R}. \tag{3.1.6}$$

The above proposition due to Yong and Zhou [16] says that the value function is bounded and Lipschitz continuous. From the Picard-Lindelőf theorem in the study of differential equations, Lipschitz continuity guarantees the existence and uniqueness of the solution to an initial value problem.

[4] Nonanticipative refers to a process that is $\mathcal{F}_{t^-}$ −measurable.

Dynamic Programming Principle

A fundamental principle in control theory is Bellman's optimality principle. It is also known as the dynamic programming principle (DPP). Formally it means that 'if one has followed an optimal control decision until some arbitrary observation time say τ, then, given this information, it remains optimal to use it after τ' [11]. It speaks more to the global nature of the optimal decision at any point in time.

More formally; for all $\tau \in \mathcal{T}_{t,T}$, the set of stopping times valued in [t,T]:

$$V(t,x) = \sup_{u \in U} \mathbb{E}\left[\int_t^\tau f(s, X(s), u(s))\,\mathrm{d}s + V(\tau, X(\tau))\right]. \tag{3.1.7}$$

The Dynamic Programming Principle allows us to study the optimal control problem by looking at the value function, $V(t,x) := \sup_{u \in U} \mathcal{J}(t,x,u(\cdot))$.
On the time interval $[t,T]$ we can be split the optimisation problem into two subproblems:

- Firstly, the optimisation problem is solved on the sub-interval $[\tau, T]$, with f as the running cost and h as the terminal cost as presented in (3.1.2), thus determining the value function $V(\tau, \cdot)$, at time τ.

- Secondly, the optimisation problem is solved on the sub-interval $[t, \tau]$ with f as the running cost and $V(\tau, \cdot)$ from the first step as the terminal cost.

The essence of the dynamic programming principle is that: globally optimal implies locally optimal. The dynamic programming principle gives a necessary condition for the control, u, to be optimal.

The Hamilton-Jacobi-Bellman Equation

Solving the dynamic programming equation (3.1.7) directly is not easy. The Hamilton-Jacobi-Bellman equation is a partial differential equation that the value function should satisfy based on the dynamic programming equation. It is the 'infinitesimal version of the dynamic programming principle' [11]. Generally, the HJB equation is a fundamental partial differential equation for stochastic control and stochastic differential games.

Generally, the partial differential equation (PDE) is derived as follows:

$$-\frac{\partial}{\partial t}V(t,x) - \sup_{u\in U}\mathbb{H}(t,x,D_xV(t,x),D_{xx}V(t,x)) = 0, \qquad \forall (t,x)\in[0,T)\times\mathbb{R}^n, \tag{3.1.8}$$

where for $(t,x,p,M)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times S_n$:

$$\mathbb{H}(t,x,p,M) = \left[b(t,x,u).p + \frac{1}{2}\mathrm{tr}(\sigma(t,x,u)\sigma'(t,x,u)M) + f(t,x,u)\right].$$

The function $\mathbb{H}$ is the generalised Hamiltonian operator of the associated control problem.

The following proposition, in the one-dimensional setting, from Yong and Zhou [16] states that the value function is a solution to the Hamilton-Jacobi-Bellman partial differential equation.

Proposition 2. Suppose the assumptions in (3.1.3) hold and the value function $V\in C^{1,2}([0,T]\times\mathbb{R})$. Then V is a solution of the following terminal value problem (of a possibly degenerate) second-order partial differential equation (the Hamilton-Jacobi-Bellman equation):

$$\begin{cases} -\frac{\partial}{\partial t}V(t,x) - \sup_{u\in U}\mathbb{H}(t,x,D_xV(t,x),D_{xx}V(t,x)) = 0, & \forall (t,x)\in[0,T)\times\mathbb{R}, \\ V(T,x) = g(x), & x\in\mathbb{R}. \end{cases} \tag{3.1.9}$$

where $\mathbb{H}(\cdot,\cdot,\cdot,\cdot)$ is defined as in (3.1.8).

Proof. See Pham [12]. □

The Verification Theorem

The verification technique allows one to test whether an admissible policy is optimal or not. It also shows one how to construct an optimal control.

Similarly to Pham [11], using the classical verification approach one finds a smooth solution to the Hamilton-Jacobi-Bellman equation, and checks that this candidate solution coincides with the value function under suitable sufficient conditions. From this we get a verification theorem as a consequence of an optimal control.

We present without proof an adapted form of the verification theorem from Chighoub and Mezerdi [1].

The Verification Theorem. Let V be a classical solution of (3.1.9) , such that for some constants $c_1 \geq 1$, $c_2 \in (0, \infty)$, $|V(t,x)| \leq c_2(1 + |x|^{c_1})$. Then, for all $(t,x) \in [0,T] \times \mathbb{R}$, and $u \in U$,

$$V(t,x) \geq \mathcal{J}(t,x;u) \text{ given by (3.1.2).} \tag{3.1.10}$$

Furthermore, if there exists $u^* \in U$ such that with probability 1,

$$\begin{aligned} &(t,x^*) \in [0,T] \times \mathbb{R}, \text{ Lebesgue almost every } t \leq T, \\ \text{and} \quad &u^* \in \arg\max_u \mathbb{H}(t,x,D_x V(t,x^*), D_{xx} V(t,x^*)), \end{aligned}$$

then it follows that $V(t,x) = \mathcal{J}(t,x;u^*)$.

In a sense the verification theorem provides sufficient conditions for optimality.

Remark 1. In this section we outlined the framework underpinning the method of solution. We gave the assumptions which we will take into account when solving the problem described in Chapter 2. We introduced the value function as the supremum over all admissible control strategies of (2.0.3). The dynamic programming principle allows one to the construct an optimal control strategy through the value function . This is done by calculating the value function as the unique solution to the constructed HJB equation. Thereafter the optimal control strategy is constructed.

3.2 Reinsurance

3.2.1 What is Reinsurance?

It is insurance for insurance companies. Basically an insurance company shares part of its liability with another. The former is known as the primary or ceding company while the latter is known as the reinsurer. Like primary insurance, reinsurance is a mechanism for spreading risk. However it is a transaction between insurance companies only. As such the reinsurance market forms a secondary market for insurance risks. The only parties to the reinsurance agreement are the insurer and the reinsurer; all rights and obligations run only between them. The reinsurance contract does not change the direct, or original, insurer's responsibility to its policyholder and the insurer must fulfill the terms of its policy whether or not it has reinsurance or whether or not the reinsurer is rightly or wrongly refusing to perform [14]. It is important to note that reinsurance is not co-insurance, where separate insurers assume shares of the same insurance risk. Nor is it substitution of one insurer for another [14].

Similarities between insurance and reinsurance include

1. protection against uncertain, future events
2. transfer of risk in both cases
3. requirement of a payment of premium
4. and naturally, underwriting (selecting, analysing, pricing) skills.

One of the main differences between insurance and reinsurance is that a reinsurance contract is customised to the buyer as opposed to insurance where an average premium exists. Each reinsurance contract must be individually priced to meet the particular needs and risk level of the reinsured [2]. Also in insurance buyers have varying knowledge levels while in reinsurance buyers are assumed to be knowledgeable - no *contra proferentem*[5], and as such insurance is highly regulated while reinsurance is unregulated.

'Reinsurance is both a risk management and a financing decision' [13]. As a risk management tool; it is optimal risk sharing among risk-averse agents. An insurer's decision to purchase reinsurance is similar to that of any non-financial firm to purchase insurance in the first place. As a financing tool; purchasing reinsurance explicitly reduces the minimum capital requirement. For a firm in search of an optimal financial structure, reinsurance is seen as one of the levers. This is supported by the findings of Garven and Lamm-Tennant [6].

3.2.2 How Reinsurance works?

Risk of loss is spread via the following measures

- risks are shifted from one insurer to another
- single insurer's burden is reduced by sharing of risks
- sharing risk increases capacity for an insurer thus allowing larger books of business to be covered
- reinsurance often employs a "subscription" business model.

3.2.3 Types and Characteristics of contracts

There are two types of reinsurance:

1. "Facultative" Reinsurance
   Mainly provides additional capacity by reinsuring the primary policies individually. As such the reinsurer has full knowledge of the each policy's risk and reserves the right to accept or reject additional cover.

[5] *Contra proferentem* refers to the principle where the drafter of a standard contract is liable for any ambiguities in the contract.

2. "Treaty" Reinsurance

   A transaction covering specified losses for an aggregated set of policies according to the terms of the reinsurance contract.

3.2.4 Forms of Reinsurance

Pro Rata Reinsurance (Proportional)

This is the first of the two forms. This is characterised by a proportional sharing of liability and premium between the primary insurer and the reinsurer according to a pre-agreed percentage.

The two most common types of proportional reinsurance are "quota share" and "surplus share" reinsurance.

- Under quota share reinsurance, the reinsurer assumes an agreed percentage of each risk from the first Rand, up to any limit assigned. For example, if there is a $R100$ loss under a 40% quota share reinsurance contract, the cedent would bear $R60$ of that loss and the reinsurer concurrently would bear $R40$ of that loss. The percentage always reflects the percentage of loss borne by the reinsurer. The portion of the risk that the reinsurer assumes is called the "ceded risk", and the portion that the cedent keeps is referred to as the reinsurance "retention".

- Surplus share is similar to quota share reinsurance in that premiums and losses are shared on a proportional basis, but differs in that the portion of the reinsured policy the direct insurer retains is expressed as a fixed monetary amount, and the reinsurance may or may not apply from the first Rand. Premium is shared based on the ratio of retained liability, and the reinsurer agrees to pay the same *pro rata* portion of any loss and expense incurred by the cedent.

Statistical considerations of Proportional Reinsurance

If the claim is for an amount X, then the insurer or ceding party will pay Y where

$$Y = aX, \qquad 0 < a < 1$$

with a being the retention level.

As the amount paid by the insurer on a claim X is $Y = aX$ and the amount paid by the reinsurer is $Z = (1-a)X$, the distribution of both of these amounts can be found by a simple change of variable.

$$f_y(y) = \frac{1}{a} f_x\left(\frac{y}{a}\right)$$
$$f_z(z) = \frac{1}{1-a} f_x\left(\frac{z}{1-a}\right)$$

Example 1

This example serves to illustrate the benefit of effecting proportional reinsurance for both the insurer and reinsurer. If X represents the size of an individual claim and the retention level, a, is 0.6, then:

- The amount paid by the insurer is $Y = 0.6X$.
- The mean amount paid by the insurer is $\mathbb{E}(Y) = 0.6\mathbb{E}(X)$.
- The variance of the amount paid by the insurer is $Var(Y) = 0.36Var(X)$.

Under the same scenario,

- The amount paid by the reinsurer is $Z = 0.4X$.
- The mean amount paid by the reinsurer is $\mathbb{E}(Z) = 0.4\mathbb{E}(X)$.
- The variance of the amount paid by the reinsurer is $Var(Z) = 0.16Var(X)$.

Thus the variance of payable claims is reduced for both the insurer and reinsurer.

For completeness

$$\begin{aligned} Var(X) &= Var(Y+Z) \\ &= Var(Y) + Var(Z) + 2Cov(Y,Z) \\ &= Var(aX) + Var((1-a)X) + 2Cov(aX,(1-a)X) \\ &= a^2Var(X) + (1-a)^2Var(X) + 2a(1-a)Var(X) \\ &= Var(X) \end{aligned}$$

Where $a = 0.6$ the covariance term amounts to $0.48Var(X)$ hence the reduction in the variance of the amounts paid by the insurer and reinsurer is due to the covariance between $Y$and Z.

3.2.5 Non-Proportional Reinsurance

This is the second form of reinsurance. Under non-proportional, the reinsurer's liability is not triggered until the cedent's losses exceed a specified monetary amount, called the "retention". If losses to the ceding company are less than the retention, the reinsurer owes nothing. The reinsurance agreement will include a limit of liability for each claim above which the reinsurer is not obligated to pay. Under non-proportional reinsurance there is excess of loss and stop loss reinsurance forms. With individual excess of loss (XOL) reinsurance, the reinsurer will be required to make a payment when the claim amount for any individual claim exceeds a specified excess point or retention. With stop loss reinsurance, the reinsurer will be required to make payments if the total claim amount for a specified group of policies exceeds a specified amount (usually expressed as a percentage of the gross premium).

Non-proportional reinsurance tends to cost less than does quota share reinsurance because the reinsurer does not participate in every loss.

In this dissertation we only consider the case where a company effects quota share proportional reinsurance to be precise. The goal is to find the optimal retention level which is the amount of business they keep from the initial set of contracts.

3.3 Utility Functions

A utility function represents the investor's preferences. An investor who is risk-averse tries to avoid risk. A risk-seeking investor takes more chances and a risk-neutral one is indifferent between choices with equal expected payoffs regardless of differing risk levels. In this dissertation we assume that all shareholders have the same utility function;

$$U(c) = -(c+\delta)^{\eta}, \quad \eta < 0,\, \delta \geq 0$$

In this dissertation $\delta = 0$. The utility function represents the risk averse nature of the shareholders. To be more specific the utility function exhibits Hyperbolic Absolute Risk Aversion. Other examples of HARA type utility functions are

$$U(c) = \log(c+\delta), \quad \delta \geq 0 \qquad \text{and} \qquad U(c) = c^{\eta}, \quad 0 \leq \eta \leq 1.$$

Characterisation of utility functions

$U(c)$ denotes the utility function for money. At all times we assume that every individual prefers more money to less money so,

$$U'(c) > 0.$$

It is important to note the linear transformation invariance of the utility function so that

$$V(c) = a + bU(c), \quad b > 0,$$

is an equivalent utility function to $U(c)$.

If

- $U''(c) < 0$, then U(c) is strictly concave $\implies$ Risk Averse
- $U''(c) = 0$, then U(c) is neither concave nor convex $\implies$ Risk Neutral
- $U''(c) > 0$, then U(c) is strictly convex $\implies$ Risk Seeking.

Measuring Risk Aversion

It is useful to have a measure of the degree to which investors are risk averse. Pratt developed the most commonly used risk aversion measure,

$$r_A(c) = -\frac{U''(c)}{U'(c)}$$

where $r_A(c)$ is the absolute risk aversion coefficient. For risk averse individuals $U''(c) < 0$, therefore $r_A(c)$ will be positive for risk averse individuals. $r_A(c)$ is the same for any equivalent of $U(c)$.

Risk aversion coefficients of HARA type utility functions

- For $U(c) = c^\eta$, $0 \leq \eta \leq 1$

$$r_A(c) = -\frac{U''(c)}{U'(c)} = -\frac{\eta(\eta-1)c^{\eta-2}}{\eta c^{\eta-1}} = \frac{1-\eta}{c}.$$

- For $U(c) = \log(c)$

$$r_A(c) = -\frac{U''(c)}{U'(c)} = -\frac{-c^{-2}}{c^{-1}} = \frac{1}{c}.$$

- For $U(c) = -c^\eta$, $\eta < 0$

$$r_A(c) = -\frac{U''(c)}{U'(c)} = -\frac{-\eta(\eta-1)c^{\eta-2}}{-\eta c^{\eta-1}} = \frac{1-\eta}{c}.$$

For HARA type utility functions risk aversion decreases as wealth increases.

Risk tolerance is the extent to which an investor is willing to accept more risk in exchange for the possibility of a higher return. The risk tolerance coefficient is given by the reciprocal of the absolute risk aversion coefficient;

$$\text{Risk Tolerance} = \frac{1}{r_A(c)}.$$

By taking reciprocal of the risk aversion coefficients we, as expected, see that risk tolerance increases as wealth increases.

3.4 Dividend Payout Strategies

Lintner [9] noted three consistent patterns of how firms set dividends;

1. Firstly, target dividend payout ratios are set based on the proportion of earnings firms are comfortable with distributing as dividends in the long run.

2. Secondly, dividends are varied according to long-term variation of earnings, but can be increased only to levels which the firms feel they can maintain. Generally, dividends lag earnings because firms avoid cutting dividends.

3. Finally, the main concern is the change in dividends rather than level of dividends.

In support of the second point above, Fama and Babiak [5] confirmed empirically that dividend changes tend to follow earnings changes.

In support of Lintner [9], Cyert and Marsh [3] have a model which predicts how a manager approaches dividend policy. They say a manager will;

1. Set dividend payout ratios that are informed by industry norms.

2. Change dividends in line with changes in earnings.

3. Employ a simple rule of thumb for example raising dividends only after 30% or more increase in earnings.

4. Avoid knee-jerk reactions to short-term changes such as stockholders attitudes.

In this dissertation the only restriction is that dividends be less than or equal to the current reserve when they are declared. We do not set out to find the frequency of declaration of dividends rather the optimal strategy of declaration. It will be shown in Chapter 5 that the optimal dividend policy follows a given functional form.

Chapter 4

Literature Review

In this chapter we look at previous studies on optimising shareholder value by effecting proportional reinsurance while optimally paying out dividends for the example of an insurance company. We take an in-depth look at four academic papers, outlining their specific research questions, methodology and the results obtained. We lastly show their similarities and differences to this dissertation.

4.1 Controlling Risk Exposure and Dividend Pay-Out Schemes: Insurance Company Example by B. Højgaard and M. Taksar [7]

Here an insurance company is considered. Its liquid assets evolve as an arithmetic Brownian motion with a constant positive drift and a constant diffusion coefficient in the absence of a control. The liquid assets of the company at time t, are described by a stochastic process $\{R(t)\}_{t\geq 0}$. This was referred to as the risk process. In general

$$\begin{cases} \mathrm{d}R(t) = \mu \mathrm{d}t + \sigma \mathrm{d}W(t), & t \geq 0, \\ R(0) = x, \end{cases} \tag{4.1.1}$$

with x as the initial reserve and μ and σ both being positive. The diffusion coefficient is interpreted as the risk exposure and the drift term is understood as potential profit. The objective was to find a policy of paying out dividends and effecting reinsurance to reduce risk exposure[1], that maximises the expected total discounted dividends paid until time of bankruptcy or in the utility sense maximised the expected total discounted utility of consumption for the shareholders. These shareholders are in essence assumed to have the same utility function of dividends; $U(x) = x$ where x is the wealth. This is representative of their risk-neutrality. Here the expected present value of the net dividends provides insight into the company's value.

Control policy π was defined by a 2-dimensional stochastic process $\{a_\pi(t), L_\pi(t)\}$ where $0 \leq a_\pi(t) \leq 1$ is the risk exposure and $L_\pi(t) \geq 0$ is the non-decreasing process whose value corresponds the cumulative amount of dividends distributed up to time t. More specifically the risk process is given as

$$\begin{cases} \mathrm{d}R_\pi(t) = \mu a_\pi(t)\mathrm{d}t + \sigma a_\pi(t)\mathrm{d}W(t) - \mathrm{d}L_\pi(t), \quad t \geq 0, \\ R(0) = x - L_0^\pi, \end{cases}$$

where x is the initial reserve. Another assumption is that cheap reinsurance[2] is in effect. Mathematically, the objective was to find π^* such that

$$V(x) = \sup_{\pi \in \Pi} \mathbb{E} \int_0^{\tau_\pi} e^{-rt} \mathrm{d}L_\pi(t)$$

where r is the discount factor and $\tau_\pi = \inf\{t : R_\pi(t) = 0\}$ is the bankruptcy time.

This problem was treated as a singular/regular stochastic control problem. The method of solution used was the standard Hamilton-Jacobi-Bellman equation formulation approach under the assumption that the optimal value function given above is sufficiently smooth.

[1] By reducing risk exposure the company in turn reduces its potential profit as it has ceded responsibility of a proportion of the premiums

[2] Cheap reinsurance is when the insurer and reinsurer both have the same safety loading i.e. they both add an equal amount to the basic premium to cover the expense of securing and maintaining the business.

Two cases were considered

1. rate of dividend payout is bounded by some positive constant, M.

2. no restriction on the rate of dividend payout.

In the first case the HJB equation was as follows

$$\max_{\{a\in[0,1],\, l\in[0,M]\}} \left[\frac{1}{2}\sigma^2 a^2 V''(x) + (\mu a - l)V'(x) - rV(x) + l\right] = 0$$

and in the second case was as follows

$$\max\left(\max_{a\in[0,1]} \left[\frac{1}{2}\sigma^2 a^2 V''(x) + \mu a V'(x) - rV(x)\right], 1 - V'(x)\right) = 0$$

with $V(0) = 0$ in both cases.

They showed that there are two levels of the reserve $u_0 < u_1$ which act as policy changing points. The risk exposure which represents the retention level monotonically increases between 0 and u_0 as a function of current reserve from 0 to maximum possible. In the first case, dividends are paid out at the maximal rate when the reserve exceeds u_1. However in the second case, every excess above u_1 is distributed as dividends. They also showed that for M small enough there was only one switching point $u_0 = u_1$ in which case the optimal risk exposure is always less than optimal. They further went on to do a numerical comparison which supported that there is a gain[3] when proportional reinsurance is effected.

4.2 Optimal Risk and Dividend Control for a Company with a debt liability by M.I.Taksar and X.Y.Zhou [15]

They considered an example of a corporation which can choose to effect a business policy from several available control policies based on expected profits

[3] Here the gain refers to increased risk reduction which in turn increases time of bankruptcy.

and respective risks. It can also choose the amount of dividends to be paid out to the shareholders whilst servicing a constant debt obligation. The objective was to find the policy which maximises the expected total discounted dividend payouts until the time of bankruptcy.

The value of the liquid assets of the company at time t, was denoted by $x(t)$. The liquid assets follow a diffusion process with the diffusion and drift coefficients which are affine functions of the risk control variable.

$$\begin{cases} \mathrm{d}x(t) = (a(t)\mu - \delta)\mathrm{d}t + a(t)\sigma \mathrm{d}W(t) - \mathrm{d}c(t), & t \geq 0, \\ x(0) = 0, \end{cases}$$

where

- μ is profit rate, the difference between the premium rate and the expected payments on claims per unit time.
- $a(t)$ is the risk process and $(1 - a(t))$ is the reinsurance fraction at time t.
- $c(t)$ is the total amount in dividends paid out to the shareholders up to time t.
- δ is the liability rate, the rate at which the debt is repaid.

The management of the company chooses the control 'variables', $a(t)$ and $c(t)$. The objective was to maximise

$$\mathbb{E} \int_0^{\tau} e^{-rt} \mathrm{d}c(t)$$

where τ is the time of bankruptcy; $\tau = \inf\{t \geq 0 : x(t) = 0\}$. Mathematically the model is a mixed regular/singular control problem. The bounded functional $a(t)$ represents the regular control while $c(t)$ represents the singular control. The optimal value of the functional $c(t)$ will cause the process $x(t)$ to be reflected at the boundary b_1 defined hereafter in their findings. They also employed the standard HJB equation approach to find the value function i.e. the maximum of the expectation above. The associated HJB equation was

$$\max\left(\max_{a\in[0,1]}\left[\frac{1}{2}\sigma^2a^2V''(x)+(a\mu-\delta)V'(x)-rV(x)\right],1-V'(x)\right)=0$$

which they separated into

$$\begin{aligned} &\max_{a\in[0,1]}\left[\frac{1}{2}\sigma^2a^2V''(x)+(a\mu-\delta)V'(x)-rV(x)\right]\leq 0,\\ &V'(x)\geq 1,\\ \text{and}\quad &(1-V'(x))\max_{a\in[0,1]}\left[\frac{1}{2}\sigma^2a^2V''(x)+(a\mu-\delta)V'(x)-rV(x)\right]=0. \end{aligned}$$

The results showed that the qualitative decision of the corporation's management depends on the ordering of the profit rate μ and liability rate δ.

1. When $\mu<\delta$ then all available reserve is declared as dividends and the company shuts down.

   The next 2 non-trivial scenarios were the crux of the study where $\mu>\delta$.

2. When $\mu\leq 2\delta$ the company assumes all the risk since it is optimal to not have reinsurance at all.

3. if $\mu\geq 2\delta$ then there are two levels of the current reserve to note; b_0 and b_1 with $b_0\leq b_1$. In the interval $[0,b_0]$ the optimal risk level $a(x)$ is strictly an increasing function of the reserve. For all $x>b_0$ then it is optimal for the corporation to assume all the risk i.e. $a(x)=1$. b_1 provides somewhat a desirable upper bound for the corporation's wealth and hence any amount in excess of b_1 is paid out as dividends.

4.3 Enhancing insurer value through reinsurance, dividends and capital optimisation: An expected utility approach by Y. Krvavych [8]

Here similarities between maximisation of shareholders value under a solvency constraint imposed by a regulatory authority and maximisation of shareholder value using utility approach with a specific isoelastic[4] utility function are investigated. In order to find the corresponding utility function, a discrete-time ruin model of the surplus (Crammer-Lundberg)[5] is used:

$$S(t) = S(t-1) + c - (L(t) - L(t-1)), \quad t = 1, 2, \ldots$$

where

- $S(0)$ is initial surplus, S_0.
- c is the annual premium.
- $\Delta L(t)$ is annual total claims which are independent identically compound Poisson distributed.

The author argued that any preferred utility function of wealth by the insurer is related to the fixed level of insolvency risk. As such a concave utility function was proposed and it was derived to be

$$U(x) = \frac{x^{1-m}}{1-m}$$

with $m = |\ln \epsilon| > 0$ and $m \neq 1$ for all $\epsilon < 10\%$ where ϵ is the maximally acceptable ruin probability. The derived utility function exhibits constant

[4] Isoelastic refers to a class of utility functions with constant relative risk aversion. By extension the utility can be expressed in terms of constant consumption or some other economic variable that a decision-maker is concerned with.

[5] Cramer-Lundberg model is the classical compound-Poisson risk model. The model describes an insurance company with incoming cash premiums at a constant rate, c, and outgoing claims which arrive according to a Poisson process with intensity λ and are independent and identically distributed non-negative random variables with F distribution and mean parameter, μ.

relative risk aversion and is therefore isoelastic. Using this isoelastic function the three controls are dividend rate, leverage ratio and retention level of quota-share proportional reinsurance. Here the surplus follows geometric Brownian motion,

$$\begin{cases} \mathrm{d}S(t) = (\mu - \delta)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t), & t \in \mathbb{Z}^+, \\ S(0) = S_0, \end{cases}$$

where μ and σ are composed of the underwriting profit, leverage ratio, retention level and administrative expenses. δ represents dividend payment rate. What we do not explicitly show in the above stochastic differential equation is that the surplus is composed of investment income and underwriting profit each following a different Brownian motion independently. The value function which corresponds to the shareholders' value was defined as the present value of future dividend payments up to bankruptcy (or ruin);

$$V(x) = V(x; a, \lambda, \delta) = \mathbb{E}\left[\int_0^{\tau} e^{-rs}U(d_s)\,\mathrm{d}s + e^{-r\tau}U(B)\right]$$

subject to the given boundary condition $V(x^*) = U(B)$ where $U(B) = \frac{B^{1-m}}{1-m}$. The other coefficients are as follows;

- B is an insolvency cost.
- r is the force of interest.
- τ is the time of insolvency; $\tau = \inf\{t : S_t = x^*\}$.
- x^* denotes the minimal capitalisation level at which the insurance regulator considers the insurer financially solvent.
- d_t is the amount of dividends paid at time t.

The corresponding HJB equation which the value function must satisfy was as follows;

$$\frac{1}{2}\sigma^2 x^2 V''(x) + (\mu - \delta)xV'(x) - rV(x) + U(\delta x) = 0, \quad x \geq x^*,$$

subject to the boundary condition $V(x^*) = U(B)$. By inspection the HJB equation is in the form of a second order Cauchy-Euler ordinary differential equation and was solved as such. The author left it to the reader to work out the optimal policy as

$$(a^*, \lambda^*, \delta^*) = \arg\max_{\{a\in(0,1),\, \lambda>0,\, \delta\in(0,1)\}} V(x; a, \lambda, \delta).$$

As such all three controls are dependent on the insolvency cost.

4.4 On maximising dividends with investments and reinsurance by G.S. Ongkeko, R.C.H. Del Rosario and M.T. Castillo [10]

They consider an insurance company whose uncontrolled surplus follows an arithmetic Brownian motion. Within its business activities the company invests in risky and non-risky assets, has the option to reduce its risk by effecting proportional reinsurance and pays out dividends every now and then. The goal of the company is to maximise expected discounted future dividends by effecting proportional reinsurance while conducting investments in risky assets and paying out dividends. The company surplus is therefore controlled by the retained proportion of premiums, $a(t)$, by the proportion of the surplus invested in risky assets denoted by $b(t)$ and by the rate at which dividends are paid out denoted by $l(t)$.

The controlled surplus process: $\{X_u(t)\}_{t\geq 0}$ is described by the following stochastic differential equation,

$$\begin{cases} \mathrm{d}X_u(s) = a_u(s)(\mu \mathrm{d}s + \sigma \mathrm{d}W_1(s)) + b_u(s)X_u(s)(r_p \mathrm{d}s + \sigma_p \mathrm{d}W_2(s)) \\ \qquad + (1 - b_u(s))X_u(s)r_F \mathrm{d}s - l_u(s)\mathrm{d}s, \qquad s \in [t, T], \\ X_u(t) = x. \end{cases}$$

An admissible control is u such that $\{a_u(t), b_u(t), l_u(t)\}$ is adapted to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$, with $0 \leq a_u(t) \leq 1$, $0 \leq b_u(t) \leq 1$ and $0 \leq l_u(t) \leq X_u(t)$.

The two Brownian motions $\{W_1(t)\}_{t\geq 0}$ and $\{W_2(t)\}_{t\geq 0}$ are independent with respect to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$. Initially, the uncontrolled surplus $\{R_t\}_{t\geq 0}$ evolves according to

$$\mathrm{d}R(t) = \mu \mathrm{d}t + \sigma \mathrm{d}W_1(t),$$

where μ, $\sigma > 0$. The risky asset price process $\{P(t)\}_{t\geq 0}$, follows a geometric Brownian motion according to

$$\mathrm{d}P(t) = P(t)(\mu_p \mathrm{d}t + \sigma_p \mathrm{d}W_2(t)).$$

The non-risky price process $\{A(t)\}_{t\geq 0}$, is as follows;

$$\mathrm{d}A(t) = A(t) r_F \mathrm{d}t.$$

The shareholders have a power utility function, $G(x) = x^{\eta}, 0 < \eta < 1$, which represents their risk averse nature. The performance index for each admissible policy, u, was given by

$$V^u(t,x) = \mathbb{E}_{t,x} \int_t^{T \wedge \tau} e^{-\delta s}[l_u(s)]^{\eta}\, \mathrm{d}s$$

and the value function is given by

$$V(t,x) = \sup_{u\in U} V^u(t,x).$$

They used the HJB equation approach to find the optimal policy that maximised the expected value of the discounted future dividends. The corresponding HJB equation which the value function must satisfy was

$$\sup_{u\in U} \left\{ V_t + \frac{1}{2}(a^2\sigma^2 + b^2x^2\sigma_p^2)V_{xx} + (a\mu - l + x[r_F + b(r_p - r_F)])V_x + e^{-\delta t}[l(t)]^{\eta} \right\}$$
$$= 0.$$

Upon solving the HJB equation the optimal reinsurance was one which increased monotonically with the surplus up to a point where it was optimal to retain all business i.e. $a_u^*(t) = 1$. Also it was optimal to invest a constant proportion of the surplus in the risky asset and this proportion depended on the risk premium, $r_P - r_F$. They also showed cases where it was optimal to invest the whole proportion of the surplus in risky assets and none at all, implying it was optimal to invest all the surplus in the non-risky assets. The optimal dividend payout strategy was to declare dividends as less than or equal to the current surplus depending on the derived functional form of the optimal payout strategy.

4.5 In conclusion

Højgaard and Taksar [7] and Taksar and Zhou [15] examine the case where the shareholders are assumed to have a risk neutral utility function. Treatment of the results in these two papers is very similar. They both invoke the theory of reflecting boundary stochastic differential equations. The main difference is that while Højgaard and Taksar [7] consider cases of bounded and unbounded dividend payout rates Taksar and Zhou [15] present their results under different cases of how the profit rate and liability rate interact. Krvavych [8] and Ongkeko *et al.* [10] look at risk averse utility functions the difference being that in the former a constant relative risk averse utility function is examined while in the latter a hyperbolic absolute risk averse utility function is looked at. In this dissertation we look at a hyperbolic risk averse utility function defined in Chapter 3, Section 3.3.

Common in all four papers examined and also this dissertation are the control variables namely the proportion of premiums retained and the dividend payout rate. Krvavych [8] and Ongkeko *et al.* [10] have in addition as controls the leverage ratio and investment proportion into risky assets, respectively. This dissertation looks at the time horizon $[t, T]$ with an arbitrary t which can also be zero. However the first three papers consider the performance index over the time interval $[0, \tau]$ where τ represents the time of bankruptcy. Ongkeko *et* *al.* [10] look at the interval $[t, T \wedge \tau]$ the upper bound being a fixed final time or time of ruin, whichever happens first.

Looking at these papers cements that we ought to follow the Hamilton-Jacobi-Bellmnan partial differential equation approach. Because of the different constraints and considerations in each of these papers we see that different HJB equations are obtained. From there, there are also different methods of solution. It is important to note the effect of the lower limit of the time intervals used in these papers. When the starting time is zero then the value function, as in the first three papers, is a function of one variable, the surplus only. When the starting time is an arbitrary time t, then the value function is explicitly a function of two variables, the time and the company surplus. This is the case in [10] and this dissertation.

The problem we examine in this dissertation is more of a starting point and as such has no explicit constraints such as boundary conditions on the value function, constant debt obligation, insolvency constraint and so on. Presence of more components in the controlled surplus process can give rise to more spatial dimensions to the surplus process. In Krvavych [8] and Ongkeko *et al.* [10] we see that it is possible that the stochastic differential equation for the company surplus follows a multi-dimensional diffusion process if components of the controlled surplus also evolve as some diffusion process.

Thus we have now established a literature based framework for this dissertation.

4.6 Objective and Significance of this study

The general objective of this study is to maximise the expected discounted utility of total dividend payouts over a time period by the appropriate dividend policy and reinsurance strategy.

The specific objectives of this dissertation are to:

1. determine the optimal reinsurance strategy i.e. the retention level $a \in (0, 1]$.
2. find the optimal dividend payout scheme $c(\cdot)$ with a view to maximising expected discounted utility of total dividend payouts over a finite time horizon.

In this dissertation the surplus, X, evolves as in (2.0.1) and the performance index to be maximised is given by (2.0.3). We follow the same method of formulating the appropriate Hamilton-Jacobi-Bellman equation to obtain the optimal value function. From there we calculate the optimal retention level and dividend pay-out scheme.

The significance of this dissertation is as follows:

1. It will assist decision-makers in the insurance industry (such as managers of insurance companies) in choosing appropriate retention percentages. The optimal nature of the retention percentage overtly minimises the probability of ultimate ruin although we do not look at this in great detail in this study.
2. It will add to the existing body of knowledge on mathematical applications in the insurance industry.

3. Optimal reinsurance reduces the impact of losses to the primary insurer. It can also increase the capacity of the primary insurer to issue more policies.

4. Generally reinsurance enables a client to get coverage that would be too great for any one company to assume.

5. Beyond income generation an optimal dividend payout scheme provides a way for investors to assess a company as an investment prospect.

Chapter 5

Results

5.1 Solution to the Hamilton-Jacobi-Bellman partial differential equation

Generally there is no universal solution to the Hamilton-Jacobi-Bellman equation in stochastic control problems, so firstly we make an assumption about the qualitative nature of the solution. We proceed to find the solution and finally verify that the obtained solution satisfies any assumptions made along the way.

The HJB equation is obtained by applying (3.1.8) to (2.0.1) and (2.0.3) with $U(c) = -c^\eta$ already substituted into (2.0.3) so that

$$V_t + \sup_{\{a \in (0,1],\, c \leq x\}} \{\frac{1}{2}a^2\sigma^2 V_{xx} + (a\mu - c)V_x - c^\eta\} = 0, \quad \eta \leq 0. \tag{5.1.1}$$

We present the solution of the HJB equation given by (5.1.1) as a proposition and its corresponding proof.

Proposition 3. The solution to the HJB equation (5.1.1) is given by $V(t,x) \in C^{1,2}([0,T] \times \mathbb{R})$ where $V(t,x)$ is a separable function of the form $g(t)x^\eta$ and is explicitly given by

$$V(t,x) = -\left[\frac{2\sigma^2(\eta-1)^2}{\mu^2\eta}(e^{-\frac{\mu^2\eta}{2\sigma^2(\eta-1)^2}(T-t)} - 1)\right]^{1-\eta} x^\eta. \tag{5.1.2}$$

Proof. We begin by computing the maximisers of (5.1.1) with a view to substituting them into (5.1.1) so as to proceed without the supremum function.

We firstly combine all the terms in a, under a function f such that

$$f(a) = \frac{1}{2}a^2\sigma^2 V_{xx} + a\mu V_x.$$

Differentiating once with respect to a to find the critical point, a^*,

$$f'(a) = a\sigma^2 V_{xx} + \mu V_x = 0.$$

The optimal reinsurance proportion is then given by

$$a^* = \frac{-\mu V_x}{\sigma^2 V_{xx}}. \tag{5.1.3}$$

Secondly we combine all the terms in c, under a function h such that

$$h(c) = -cV_x - c^\eta.$$

Differentiating once with respect to c to find the critical point, c^*,

$$h'(c) = -V_x - \eta c^{\eta-1} = 0.$$

The optimal dividend payout rate is then given by

$$c^* = \left(-\frac{V_x}{\eta}\right)^{\frac{1}{\eta-1}}. \tag{5.1.4}$$

Substituting the maximisers a^* and c^* into (5.2.5) we get

$$V_t - \frac{1}{2}\frac{\mu^2 V_x^2}{\sigma^2 V_{xx}} - \left(-\frac{V_x}{\eta}\right)^{\frac{1}{\eta-1}} V_x - \left(-\frac{V_x}{\eta}\right)^{\frac{\eta}{\eta-1}} = 0. \tag{5.1.5}$$

We **conjecture** that the solution to (5.1.5) and in turn (5.1.1) is of the form $V(t,x) = g(t)x^\eta$.

Given $V(t,x) = g(t)x^\eta$,

then $V_t = g'(t)x^\eta$, $V_x = g(t)\eta x^{\eta-1}$, and $V_{xx} = g(t)\eta(\eta-1)x^{\eta-2}$.

Substituting into (5.1.5) we have

$$x^\eta\left[g'(t) - \frac{1}{2}\frac{\mu^2\eta}{\sigma^2(\eta-1)}g(t) + (\eta-1)\left[-g(t)\right]^{\frac{\eta}{\eta-1}}\right] = 0.$$

Let $A = \frac{1}{2}\frac{\mu^2\eta}{\sigma^2(\eta-1)}$ and $B = (\eta - 1)$.

Then $g(t)$ satisfies the differential equation

$$g'(t) - Ag(t) + B(-g(t))^{\frac{\eta}{\eta-1}} = 0 \tag{5.1.6}$$

which is a Bernoulli type differential equation.

We let $\kappa = \frac{\eta}{\eta-1}$ and $y(t) = [-g(t)]^{1-\kappa}$, then

$$\frac{\mathrm{d}}{\mathrm{dt}}y(t) = (1-\kappa)[-g(t)]^{-\kappa} - \frac{\mathrm{d}}{\mathrm{dt}}g(t).$$

Substituting the above transformation into (5.1.6) we have

$$-\frac{1}{1-\kappa}[-g(t)]^{\kappa}\frac{\mathrm{d}}{\mathrm{dt}}y(t) + Ay(t)^{\frac{1}{1-\kappa}} + By(t)^{\frac{\kappa}{1-\kappa}} = 0,$$

then

$$\frac{\mathrm{d}}{\mathrm{dt}}y(t) - (1-\kappa)Ay(t) = (1-\kappa)B$$

which is a linear first-order differential equation.

The integrating factor is then given by

$$I = e^{\int_t^T -(1-\kappa)A\,ds} = e^{-(1-\kappa)A(T-t)},$$

then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}Iy(t) &= I(1-\kappa)B \\
Iy(t) &= \int_t^T I(1-\kappa)B\,ds \\
y(t) &= e^{(1-\kappa)A(T-t)}(1-\kappa)B\int_t^T e^{-(1-\kappa)A(T-s)\,ds} \\
&= e^{(1-\kappa)A(T-t)}\frac{B}{A}[1 - e^{-(1-\kappa)A(T-t)}].
\end{aligned}$$

In terms of $g(t)$,

$$\begin{aligned}
g(t) &= -y(t)^{\frac{1}{1-\kappa}} \\
&= -\{\frac{B}{A}[e^{(1-\kappa)A(T-t)} - 1]\}^{\frac{1}{1-\kappa}} \\
&= -\left[\frac{2\sigma^2(\eta-1)^2}{\mu^2\eta}(e^{-\frac{\mu^2\eta}{2\sigma^2(\eta-1)^2}(T-t)} - 1)\right]^{1-\eta}.
\end{aligned}$$

Therefore,

$$V(t,x) = -\left[\frac{2\sigma^2(\eta-1)^2}{\mu^2\eta}(e^{-\frac{\mu^2\eta}{2\sigma^2(\eta-1)^2}(T-t)} - 1)\right]^{1-\eta} x^\eta. \tag{5.1.7}$$

It is evident that $V(t,x) \in C^{1,2}([0,T] \times \mathbb{R})$. By application of the Verification Theorem, we see that $V(t,x) = g(t)x^\eta$ is the solution we set to find. □

5.2 Analysis of Results

The Reinsurance proportion

Substituting the expressions for $V_x(t,x)$ and $V_{xx}(t,x)$ into (5.1.3) we get the following retention level;

$$a^*(t,x) = \frac{\mu}{\sigma^2(1-\eta)}x. \tag{5.2.1}$$

$a^*(t,x)$ is directly proportional to the surplus, x. $a^*(t,x)$ is an increasing function of the surplus. The increase of the retention level with the surplus is consistent with characteristics of HARA type utility functions explored earlier; Risk tolerance increases with wealth.

The reinsurance proportion increases as the risk premium $\frac{\mu}{\sigma^2}$ of the risk process increases. We first analyse this by looking at the effect of μ. If we view, for now only, $a^*(t,x)$ as a function of μ only, mathematically

$$\frac{\mathrm{d}}{\mathrm{d}\mu}a^*(t,x) = \frac{x}{\sigma^2(1-\eta)}.$$

Since $\frac{\mathrm{d}}{\mathrm{d}\mu}$ is positive, $a^*(t,x)$ is an increasing function of μ and, as asserted, of the risk premium, $\frac{\mu}{\sigma^2}$. We can also illustrate this by increasing the risk premium through an increase in μ, graphically.

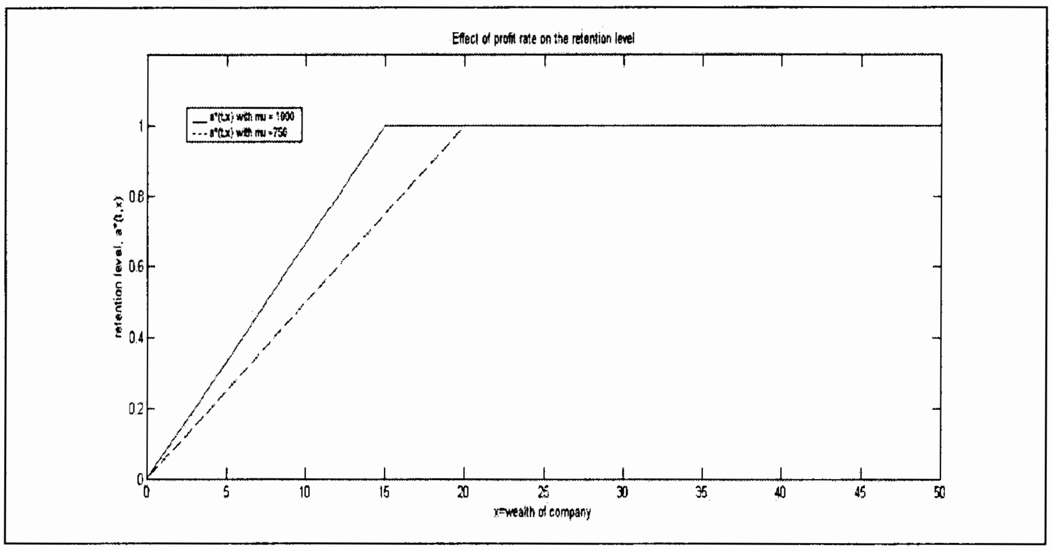

Fig. 5.1: Simulations of $a^*(t,x)$ for which $\sigma = 100$ and $\eta = -0.5$.

If we, in this instance, consider $a^*(t,x)$ as a function of σ^2 only, mathematically

$$\frac{\mathrm{d}}{\mathrm{d}\sigma^2}a^*(t,x) = -\frac{\mu x}{(\sigma^2)^2(1-\eta)}.$$

Thus, $a^*(t,x)$ is a decreasing function of σ^2 but since σ^2 is in the denominator we reach the earlier assertion that $a^*(t,x)$ is an increasing function of the risk premium, $\frac{\mu}{\sigma^2}$. We also show the effect of the increase in the risk premium by reducing σ, graphically.

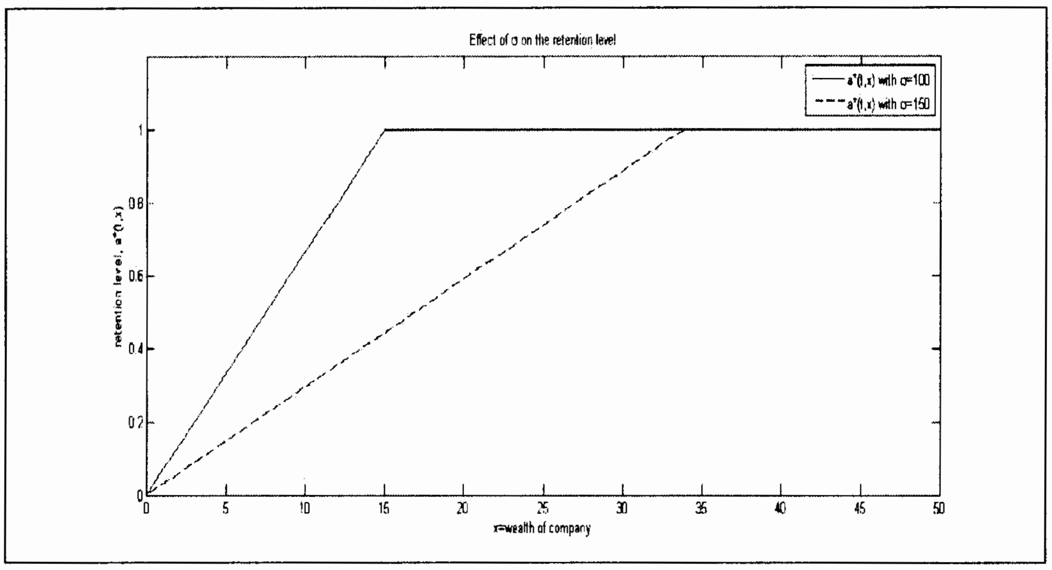

Fig. 5.2: Simulations for $a^*(t,x)$ for which $\mu = 1000$ and $\eta = -0.5$.

In both cases of increasing the risk premium $\frac{\mu}{\sigma}$ either via an increase in μ or a reduction in σ, the effect is that maximum retention level of 1 is reached faster.

The insurance company buys proportional reinsurance if the current surplus x is below a surplus threshold $\hat{x} = \frac{\sigma^2(1-\eta)}{\mu}$ otherwise they retain all the business i.e $a^*(t,x) = 1$.

$$a^*(t,x) = \begin{cases} \frac{\mu}{\sigma^2(1-\eta)}x, & x < \frac{\sigma^2(1-\eta)}{\mu} \\ 1, & \text{otherwise.} \end{cases} \tag{5.2.2}$$

Here $\hat{x} = \frac{\sigma^2(1-\eta)}{\mu}$ determines the switching point for the policy makers. As such the company wealth is split into two regions for the reinsurance decision as shown next.

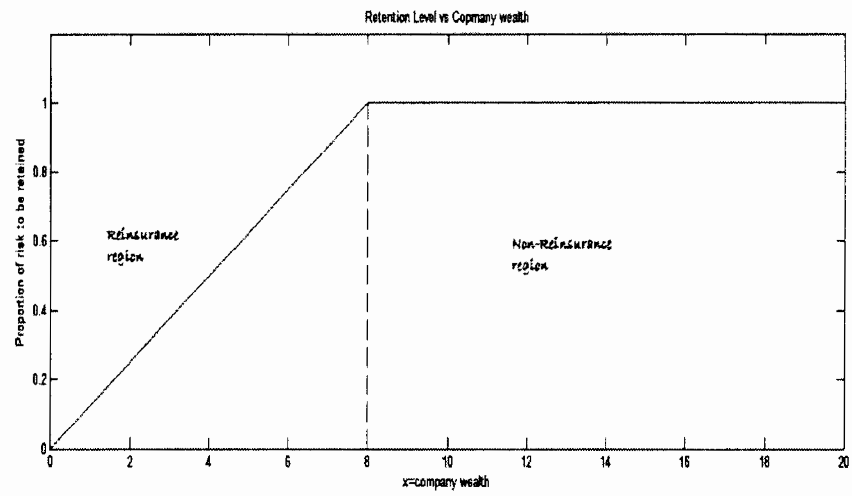

Fig. 5.3: Reinsurance decision regions

The threshold amount x^* decreases as η approaches 0 and increases as η approaches $-\infty$. This is consistent with risk averse behaviour. If the shareholders are more risk averse i.e. η is lower, then the threshold is higher and the opposite is true. As such the parameter η represents the shareholders' risk appetite. This is illustrated on the next page.

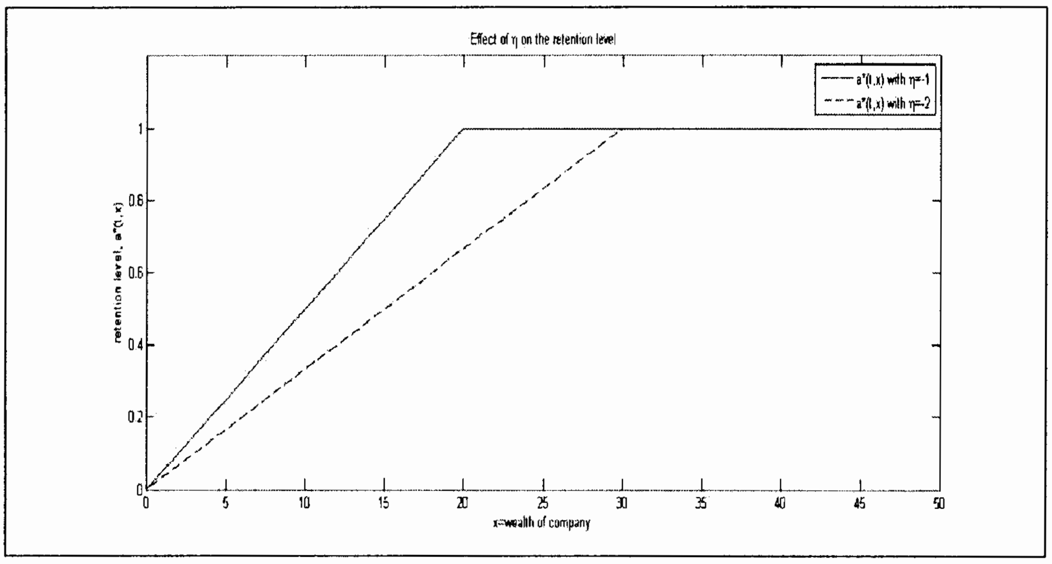

Fig. 5.4: Simulations for $a^*(t,x)$ for which $\mu = 1000$ and $\sigma = 100$

The Dividend payout

By substituting the expression for $V_x(t,x)$ in (5.1.4) we obtain the optimal dividend payout function in terms of t and x.

$$\begin{aligned} c^*(t,x) &= \left(-\frac{V_x}{\eta}\right)^{\frac{1}{\eta-1}} \\ &= (-g(t)x^{\eta-1})^{\frac{1}{\eta-1}} \\ &= \left(\left[\frac{2\sigma^2(\eta-1)^2}{\mu^2\eta}(e^{-\frac{\mu^2\eta}{2\sigma^2(\eta-1)^2}(T-t)} - 1)\right]^{1-\eta} x^{\eta-1}\right)^{\frac{1}{\eta-1}} \\ &= \frac{x}{\{\frac{2\sigma^2(\eta-1)^2}{\mu^2\eta}(e^{-\frac{\mu^2\eta}{2\sigma^2(\eta-1)^2}(T-t)} - 1)\}}. \end{aligned} \tag{5.2.3}$$

On examination of the denominator, for arbitrary values of σ, μ, η and time, we see that the denominator is always negative, mainly due to the presence of the parameter η which is negative, and in turn the optimal dividend to be declared too. In light of this, the optimal policy is that the company not declare dividends to its shareholders. In such a case the shareholders may only obtain gains through share price increase.

Remark 2. It is somewhat peculiar that the optimal dividend to be declared is negative implying that no dividends should be declared by the company. To further examine the no dividends policy we try an alternate form of the value function; $V(t,x) = -g(t)x^\eta$, as a solution to (5.1.5) see if we observe any different policy for dividends for the company and present the results in the following proposition.

Proposition 4. The solution to the HJB equation (5.1.1) is given by $V(t,x) \in C^{1,2}([0,T] \times \mathbb{R})$ where $V(t,x)$ is a separable function of the form $-g(t)x^\eta$ and is explicitly given by

$$V(t,x) = -\left[\frac{2\sigma^2(\eta-1)^2}{\mu^2\eta}(e^{-\frac{\mu^2\eta}{2\sigma^2(\eta-1)^2}(T-t)} - 1)\right]^{1-\eta} x^\eta. \tag{5.2.4}$$

Moreover, when we conjecture both $g(t)x^\eta$ and $-g(t)x^\eta$ as the forms of the value function, we see that the value function is unique. By extension the optimal dividend policy is the same.

Proof. Again the HJB equation is the following;

$$V_t + \sup_{\{a\in(0,1],\, c\leq x\}} \{\frac{1}{2}a^2\sigma^2 V_{xx} + (a\mu - c)V_x - c^\eta\} = 0, \quad \eta \leq 0. \tag{5.2.5}$$

We carry out the same treatment of the maximisers of (5.2.5) as in section 5.1 up to the point of the conjecture. We then substitute $V(t,x) = -g(t)x^\eta$ and its corresponding partial derivatives into (5.1.5) to get

$$-g'(t)x^\eta - \frac{1}{2}\frac{\mu^2\eta}{\sigma^2(1-\eta)}g(t)x^\eta - (g(t)x^{\eta-1})^{\frac{1}{\eta-1}}. - \eta g(t)x^{\eta-1} - (g(t)x^{\eta-1})^{\frac{\eta}{\eta-1}} = 0$$

such that

$$x^\eta[-g'(t) + \frac{1}{2}\frac{\mu^2\eta}{\sigma^2(\eta-1)}g(t) + (\eta-1)(g(t))^{\frac{\eta}{\eta-1}}] = 0$$

or

$$x^\eta[g'(t) - Ag(t) - Bg(t)^\kappa] = 0$$

by letting $A = \frac{1}{2}\frac{\mu^2\eta}{\sigma^2(\eta-1)}$, $B = (\eta-1)$ and $\kappa = \frac{\eta}{\eta-1}$.

Then $g(t)$ as before satisfies the differential equation

$$g'(t) - Ag(t) - B(g(t))^{\kappa} = 0 \tag{5.2.6}$$

which is a Bernoulli type differential equation.

By letting $y(t) = g(t)^{1-\kappa}$ and obtaining $\frac{\mathrm{d}}{\mathrm{dt}}y(t) = (1-\kappa)[g(t)]^{-\kappa}\frac{\mathrm{d}}{\mathrm{dt}}g(t)$ and substituting both expressions in (5.2.6), we get

$$\frac{\mathrm{d}}{\mathrm{dt}}y(t) - A(1-\kappa)y - B(1-\kappa) = 0$$

which is a linear first-order differential equation.

The integrating factor is then given by

$$I = e^{\int_t^T -(1-\kappa)A\,ds} = e^{-(1-\kappa)A(T-t)},$$

then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}Iy(t) &= I(1-\kappa)B \\
Iy(t) &= \int_t^T I(1-\kappa)B\,ds \\
&= \frac{B}{A}[1 - e^{-A(1-\kappa)(T-t)}] \\
y(t) &= \frac{B}{A}[e^{A(1-\kappa)(T-t)} - 1] \\
&= \frac{2\sigma^2(\eta-1)^2}{\mu^2\eta}[e^{-\frac{1}{2}\frac{\mu^2\eta}{\sigma^2(\eta-1)^2}(T-t)} - 1] \\
&= g(t)^{-\frac{1}{\eta-1}}.
\end{aligned}$$

Now

$$V(t,x) = -\left[\frac{2\sigma^2(\eta-1)^2}{\mu^2\eta}[e^{-\frac{1}{2}\frac{\mu^2\eta}{\sigma^2(\eta-1)^2}(T-t)} - 1]\right]^{1-\eta} x^{\eta}. \tag{5.2.7}$$

Exactly the same as $V(t,x)$ obtained in (5.1.7).

In this case

$$\begin{aligned} c^*(t,x) &= \left(-\frac{V_x}{\eta}\right)^{\frac{1}{\eta-1}} \\ &= (g(t)x^{\eta-1})^{\frac{1}{\eta-1}} \\ &= \left(\left[\frac{2\sigma^2(\eta-1)^2}{\mu^2\eta}(e^{-\frac{\mu^2\eta}{2\sigma^2(\eta-1)^2}(T-t)}-1)\right]^{1-\eta} x^{\eta-1}\right)^{\frac{1}{\eta-1}} \\ &= \frac{x}{\{\frac{2\sigma^2(\eta-1)^2}{\mu^2\eta}(e^{-\frac{\mu^2\eta}{2\sigma^2(\eta-1)^2}(T-t)}-1)\}}, \end{aligned} \tag{5.2.8}$$

which is negative and the same as we initially found out in 5.2.3. □

Remark 3. We notice that tweaks to the form of the value function do not change the final optimal dividend policy. Perhaps this is because the moment we elect to use the utility function; $U(c) = -c^\eta$, $\eta < 0$, the performance index given by (2.0.3) is always negative. However, not shown explicitly in this dissertation is that for the choice of the utility function; $U(c) = c^\eta$, $0 \leq \eta \leq 1$, both the value function and the optimal dividend policy are always negative. We could not go on further with the log utility function because the corresponding HJB partial differential equation was difficult to solve for a smooth and closed form value function, therefore we could not discern the form of the optimal policy in terms of both dividends and the retention policy.

A negative payout ratio generally implies that the firm or company is generating negative earnings or net loss. This could be so because the wealth function model chosen in this dissertation (2.0.1), an arithmetic Brownian motion with drift, allows negative values. So from the onset we expected problems of negativity affecting results. Choosing a surplus/wealth process which does not allow negative values could be the way out but then it also comes with complexity of solutions.

Chapter 6

Conclusion

We saw that for proportional reinsurance, the retention level which is akin to the proportion of risk taken by the company is directly proportional to the surplus. This draws parallels with the risk tolerance properties of the HARA type utility functions. We further saw that the risk premium $\frac{\mu}{\sigma^2}$ affects the surplus at which a company assumes all risk. The larger the risk premium the smaller the company reserve at which the company assumes all the risk. On examining the effect of the parameter η on the size of the surplus at which the company assumes all the risk; the larger η is, the smaller the surplus at which all the risk is assumed. η represents the risk appetite of the shareholders. Within this framework we established the existence of a switching point at which the company assumes all the risk thereby not effecting any reinsurance at all.

We found out that it is optimal to not declare dividends for the company. This is surprising because at least initially shareholders would expect income in the form of dividends every once in a while. The possible reasons for the no dividends policy could be the hyperbolic absolute risk averse nature of the utility function and the evolution of the surplus based on the drift and diffusion components; the surplus follows an arithmetic Brownian motion. In this case shareholders would only obtain value increase through share price appreciation either through share buy-backs or good company performance.

The no dividends policy limits the analysis of how the time horizon interacts with the surplus when dividends are declared. Therefore we cannot determine the point beyond which declaring dividends may lead to company ruin or if before that a reflecting boundary for the surplus exists. It would be interesting and important to involve an insolvency constraint likely in the form of a boundary condition. Other extensions of this dissertation would be consideration of corporate debt repayment and investments into assets (risky and non-risky). All this would affect the initial formulation of the stochastic differential equation which describes the surplus of the company.

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