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**ANALYSIS OF CDO TRANCHE
VALUATION AND THE 2008 CREDIT
CRISIS**

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Abstract

The causes of the 2008 financial crisis were wide ranging. Some financial commentators have suggested there were significant inadequacies in the models used to price complex derivatives such as *synthetic Collateralised Debt Obligations (CDOs)*. We discuss the technical properties of CDOs and the modeling approaches used by CDO traders and the watchdog credit rating agencies. We look at how the pricing models fared before and during the financial crisis. Comparing our model prices to market synthetic CDO prices, we investigate how well these pricing models captured the underlying financial risks of trading in CDOs.

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0.1 Introduction

Synthetic Collateralised Debt Obligations (CDOs) are CDOs that are synthesized through credit default swaps¹ hence the name synthetic CDOs. Cash CDOs on the other hand are securities whose coupon and capital repayments are paid from a pool of secured cashflows e.g. mortgage payments, credit card receivables. Here we look only at synthetic CDOs to investigate CDO pricing models. The securitisation of sub class bonds contributed to the worsening of Cash CDO risk profiles but are not analysed here. The majority of traditional CDOs are fully cash funded. The modeling implications of the cash CDOs are however very similar to synthetic CDOs and the possible modeling pitfalls encountered there can be highlighted in synthetic CDOs pricing [7].

In the years leading up to the 2008 financial crisis, the credit market had witnessed huge growth both in types credit derivatives, and the notional amount outstanding. The International Swaps and Derivatives Association reported in April 2007 that total notional amount on outstanding credit derivatives was \$35.1 trillion with a gross market value of \$948 billion. A report in *The Times* on September 15, 2008, reported that the "Worldwide credit derivatives market was valued at \$62 trillion". The reference pools backing the tranching products are usually pure corporate default risk, more recently mixed reference pools of corporate and structured finance risk have become popular through synthetic CDOs vehicles.

Bluhm,C, 2003 analyses the different factors which have contributed to the success of CDO trading:(i) spread arbitrage opportunities, (ii) regulatory capital relief, (iii) funding and (iv) economic risk transfer. Having exchange traded standardized single tranche CDO products have also helped stimulate the growth of synthetic CDOs, e.g., the Dow Jones CDX and ITRAXX series, these single tranche CDOs have enjoyed rapid expansion.

The main market participants in CDOs have been banks, hedge funds, insurance companies, pension funds, and other large corporates [3]. These products were sold by market makers (usually investment banks) and many pension funds took the role of insurer betting that huge defaults would not occur. There has subsequently been discussions as to whether this was a suitable asset class for pension funds and whether there was adequate appreciation of the CDOs' risk.

Synthetic CDOs can be used to provide protection against possible losses due to default of the underlying firms. Like cash CDOs the investor can break up

¹see Section 1.1

the capital structure into different tranches and buy protection for the tranche they are interested in. For example, a tranche with attachment points A to detachment point B% will bear the portfolio losses in excess of A% of the initial value of the portfolio, up to B%. The tranche absorbing the first losses, called equity tranche, is characterized by $A = 0$ and $B > 0$. The holders of a tranche characterized by attachment points 3 to 7% would not suffer any loss as long as the total portfolio loss is lower than 3% of its initial value.

Tranche number	Tranche name	Attachment points (%)	
		Lower A	Upper B
1	Equity	0	3
2	Mezzanine 1	3	7
3	Mezzanine 2	7	10
4	Mezzanine 3	10	15
5	Senior	15	30

Figure 1: Example of a CDO tranche structure.

If we have a group of defaultable instruments (e.g. bonds, loans, credit default swaps) from different firms put together. The losses on each reference portfolio depend on the **default probability** of each firm and the losses when a default occurs. The loss on the multi-name portfolio additionally depend on the degree of dependence between the reference firms known as **default correlation**. Dependency is commonly called correlation an abuse of language as correlation is a complete description of the dependency of jointly Gaussian random variables. The default correlation plays an important role in the timing of a firm's default and as a consequence the distribution of the portfolio losses. For a tranching investor, default correlation also determines what share of the portfolio credit risk stays within each tranche.

Using the industry standard Gaussian Copula model², we demonstrate the importance of default correlation. Figure E.2 show two excess loss distributions (i.e., the vertical axis shows probability of loss exceeding values on horizontal axis) for the same average default probability but for two different default correlation assumptions (10% and 30%) are plotted [3]. From the graphs we can see the importance of correlation, if you are a holder of a tranche with 8% subordination your expected losses are lower when correlation is 30% than when it is 10%. The opposite is true for the more senior tranches. At 30% correlation,

²see Chapter 1 for explanation of Gaussian Copula

there is a much higher expected loss for the protection seller, and therefore they would require a higher premium to be paid in, by you, in compensation. Higher correlations imply higher losses for senior tranches and lower losses for equity tranches. Given the importance of this parameter, investors are justifiably concerned about how the default correlation should be quantified.

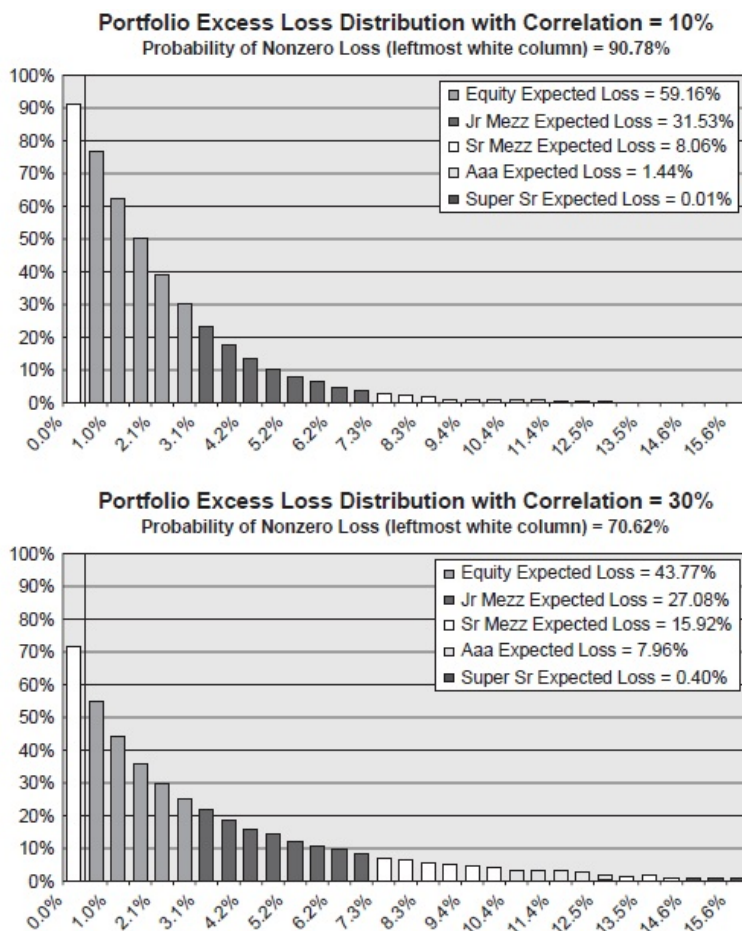


Figure 2: Tranche Implied Correlation *Source: Citigroup*

In this paper we aim to show how the Gaussian Copula model used for pricing coped, in the fairly sophisticated CDO market . In Chapter 1 Literature review, we give a brief summary of the ideas and terminology used in the synthetic CDOs markets. Section 1.1 explores the mechanics of CDO transactions and we describe the different types of default correlation in section 1.1.3. Section 1.3 explores the one factor Copula model, the following sections then proceed to work through the math and give detailed algorithms for the calculation of portfolio loss distributions and synthetic CDOs prices. The model implementation

is explained in Chapter 2. In the results chapter we present the model prices before during and after the 2008 recession period. The model prices from two different approaches the one factor copula method and the large homogeneous portfolio method are both presented. Finally, we end with concluding remarks on how well these models coped in assisting investors price and manage their portfolio risks.

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Chapter 1

Literature Review

1.1 Mechanics

CDOs are a structured asset backed security with multiple tranches. These tranches are constructed by grouping together securities (bonds, loans and for synthetic CDOs indices) that are of similar credit quality. The senior tranches are considered the safest securities. Interest and principal payments are made according to seniority so that junior tranches are only paid after all senior tranches have been settled i.e those in the lower tranches suffer losses first. In synthetic CDOs payments are made to the buyer when there is a default in the indices tracked otherwise the buyer has to make regular payments to the CDO seller for the duration of the synthetic CDO see Figure 1.1.

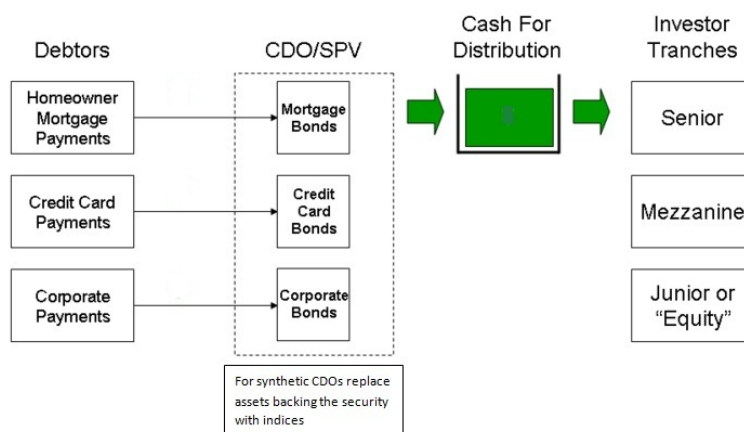


Figure 1.1: CDO securitisation

In a sense, synthetic CDOs are like selling (or buying insurance) on portions of the loss on a portfolio. The valuation problem is then trying to determine the fair price of this insurance [1]. In order to price CDOs it is crucial to observe that tranching is not a linear operation (i.e., the expected loss of the tranche is not the expected loss of the whole portfolio multiplied by the tranche size). When computing the price (marking to market) of a tranche at a point in time, one has to take into account the expectation of the future tranche losses under the pricing measure.

Since the tranche is a non linear function of the loss, the expectation will depend on all moments of the loss and not just the expected loss. Alternative to working with all the moments of the loss we can specify the loss distribution for each tranche explicitly. The loss distribution of the portfolio is characterised by the marginal distributions of the single name defaults and by the dependency among the defaults of different names. A complete description can be provided by either the whole multivariate distribution or the market default model, a Copula function.

The Copula is the multivariate distribution once the marginal distributions have been standardised to uniform distributions¹. To model dependence the market assumes (arbitrarily) that there is a Gaussian Copula connecting the defaults. If the CDO has 125 reference names, the copula would require a parameterization matrix with 7750 entries of pairwise correlation parameters. When looking at a tranche market practice is to assume that all these 7750 parameters are all equal to each other, a very drastic assumption [2]. The model is used to find the prices of related products by finding tranches that are liquid for which prices are known, finding the Implied correlation and reverse engineering this to value related products.

1.1.1 Credit Indices

In order to find market implied default rates for single names we can use credit default swaps which have a very liquid market. For multiname instruments we look at credit indices. A credit index is an index whose value is derived from a basket of credit default swaps with the same term on different names. Unlike other multi-name credit derivatives, such as first-to-default baskets, synthetic CDOs provide unleveraged exposure to the names in the basket of credit default swaps. The indeces are rolled out at end of a polling of dealers views (on the expected defaults) contribute to the ranking of the most liquid CDSs [2]. This exercise is managed by Markit. The indeces have fixed term and every six

¹see section 1.2 for more details

months a new series is rolled out, usually in March and September, to reflect the names in the credit derivative market that fit the rules for each index at that time. The details of the various credit indices can be found on the markit web page *www.markit.com*.

The index is given by a pool of names $1, 2, \dots, M$, typically $M = 125$, each with a notional of $1/M$ so that the total pool notional is 1. The CDO cashflow has a default leg and a premium leg. The default leg consists of once off protection payments made by the protection seller to the protection buyer each time one or more names default. The payments correspond to the loss increment and protection is valid until final maturity T or until all the names in the pool have defaulted T_b if this is earlier.

In exchange for loss increase payments, a periodic premium with rate S is paid from the protection buyer to the protection seller, until final maturity T_b . This premium is computed on a notional that decreases each time a name in the pool defaults and decreases by an amount corresponding to the notional of that name (without taking account of any recovery) see Figure 1.2.

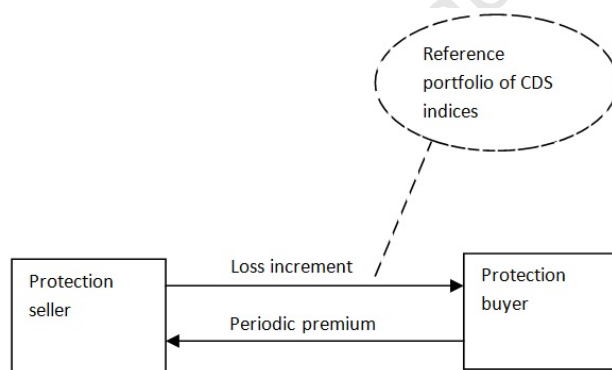


Figure 1.2: CDO payments structure

Example If we have a CDO with tranches 0%-5%, 6%-30% and 31%-100% a 100 reference companies and a portfolio notional of 100. Then the buyer of the protection makes periodic fixed payments (called the spread) to the seller of protection semiannually or quarterly in arrears for a specific tranche. The cash settlement is the face value of the reference entity (here notional 1) less the post-default market price (i.e. less recovery R). All the reference entities that default during the year are removed and do not receive any further spread (or protection) payments. If three reference entities default only the 0%-5% tranche is affected a payment of $3(1-R)$ is paid to the protection buyer. If more reference entities default and we end up with after recovery losses greater than

5% of the portfolio then only then do we start suffering losses for the 6%-30% tranche and so on.

1.1.2 Product's Payoff and Prices

We denote

- M - The number of single names in the reference portfolio. For the main DJ-iTraxx and CDX indices, M is equal to 125.
- \bar{L}_t - The portfolio cumulated loss up to time t divided by M
- \bar{C}_t - The number of defaulted names up to time t divided by M
- $D(s, t)$ - The discount factor between time s and time t
- Δ_i - Is the year fraction between times T_{i-1} and T_i
- S_0 - Is the fair market spread

Since at each default part of the defaulted loss is recovered, we have $0 \leq \bar{L}_t \leq \bar{C}_t \leq 1$. We also note that $L_t = \sum_{i=1}^M 1_{\{\tau_i \leq t\}} LGD_i$. LGD_i is the the loss given default has occurred. $LGD_i = 1 - R_i$ were R_i is the recovery rate at time i . The pool normalised loss is $\bar{L}_{T_i} = \frac{1}{M} L_t$.

We can define the default leg $DefLeg(0)$ (present value of expected future losses) and the premium leg $PremLeg(0)$ (the present value of expected future premium payments) two legs of the index is given as follows:

$$DefLeg(0) = \int_0^T D(0, t) d\bar{L}_t$$

$$PremLeg(0) = S_0 \sum_{i=1}^b \Delta_i D(0, T_i) (1 - \bar{C}_{T_i})$$

In market quotes of the $PremLeg$, the actual notional used would be an average of the notional between times $[T_{i-1}, T_i]$ (since the defaults do not all occur at T_i), but we we replace it with $1 - \bar{C}_{T_i}$ in our model as is commonly done.

When we calculate $PremLeg$ for tranches we do not only consider the number of defaults, we have to account for amount of recovery as well, thus we replace \bar{C}_{T_i} with \bar{L}_{T_i} . The amount of recovery will influence whether or not we cross attachment points see equation(1.4). The fair value market quotes are the values of S_0 that result in the two legs being equal $PremLeg = DefLeg$. If we can find a model for the loss and the number of defaults, we can use that loss

distribution together with the number of defaults plug them inside the two legs and find the risk-neutral expectation (and thus price).

$$S_0 = \frac{E_0 \left[\int_0^T D(0, t) d\bar{L}_t \right]}{E_0 \left[\sum_{i=1}^b \delta_i D(0, T_i) (1 - \bar{C}_t) \right]}$$

As is market practice we can assume a flat term structure for interest rates and rewrite the above equation as follows

$$S_0 = \frac{\int_0^T D(0, t) dE_0[\bar{L}_t]}{\sum_{i=1}^b \delta_i D(0, T_i) (1 - E_0[\bar{C}_t])}$$

The same assumption of a flat default-free interest rate is used again later to obtain *analytical pricing* formulas². If we further assume independent defaults then the $D(s, t)$ terms would become $P(s, T) = E[D(s, T)]$, the zero coupon bond prices.

CDO Tranches. Synthetic CDOs with a pool of Credit Default Swaps on different names, $1, 2, \dots, M$, typically $M = 125$ and maturity T are tranced i.e. the loss of the resulting pool between the points A and B with $0 \leq A \leq B \leq 1$. We can define this tranche loss $\bar{L}_t^{A,B}$ as

$$\bar{L}_t^{A,B} := \frac{1}{B-A} [(\bar{L}_t - A) 1_{\{A < \bar{L}_t \leq B\}} + (B - A) 1_{\{\bar{L}_t > B\}}] \quad (1.1)$$

An alternative expression that is also useful (depending on whether we are looking for implied compound or base correlation³) is

$$\bar{L}_t^{A,B} := \frac{1}{B-A} [B\bar{L}_t^{0,B} - A\bar{L}_t^{0,A}] \quad (1.2)$$

Suppose $0 < A \leq B \leq 1$. In calculations $\bar{L}_t^{A,B}$ is deduced from the total loss \bar{L}_t . Once enough names have defaulted and the loss \bar{L}_t has reached A , the count towards $\bar{L}_t^{A,B}$ starts. The loss is rescaled by the thickness $B - A$, of the tranche which implies $0 < \bar{L}_t^{A,B} \leq 1$. Once the loss reaches B the count starts for the next tranche.

When pricing CDO tranche $[A, B]$ we are interested in the tranche premium $S_0^{A,B}$ also called tranche spread. The spread is paid quarterly on the survived average notional. For the equity tranche (the tranche with $A=0$) an extra premium called the upfront premium could also be charged $U_0^{A,B}$, this is meant to reduce counterparty credit risk to the protection seller. In times of great

²see section 1.3.2

³see section 1.1.3 for definition of Base and Compound Correlation

uncertainty this upfront premium could also be required on all other tranches. The premiums are paid periodically say at times $T_1, T_2, \dots, T_b = T$. If we make the assumption that the $S_0^{A,B}$ payments are made on the notional remaining at each date T_i rather than the average over $[T_{i-1}, T_i]$.

We can then define the default and premium legs for each tranche as follows.

$$DefLeg(0) = \int_0^T D(0, t) d\bar{L}_t^{A,B} \quad (1.3)$$

$$PremLeg(0) = U_0^{A,B} + S_0^{A,B} D_{v01_{A,B}} \quad (1.4)$$

$$D_{v01_{A,B}} = \sum_{i=1}^b \Delta_i D(0, T_i) (1 - \bar{L}_{T_i}^{A,B}) \quad (1.5)$$

The tranche value is calculated by taking the risk neutral expectation of the discounted payoff. The tranche spread is the spread that equates the $PremLeg$ to the $DefLeg$. We obtain

$$S_0^{A,B} = \frac{E_0 \left[\int_0^T D(0, t) d\bar{L}_t^{A,B} \right] - U_0^{A,B}}{E_0 \left[\sum_{i=1}^b \delta_i D(0, T_i) (1 - \bar{L}_{T_i}^{A,B}) \right]} \quad (1.6)$$

If we assume interest rates are deterministic and default-free, we can then rewrite (1.6)

$$S_0^{A,B} = \frac{\int_0^T D(0, t) dE_0[\bar{L}_t] - U_0^{A,B}}{\sum_{i=1}^b \Delta_i D(0, T_i) (1 - E_0[\bar{L}_{T_i}^{A,B}])} \quad (1.7)$$

It is important to note that even though the equations above have dL_t in practice the term refers to discrete jumps in the loss function.

1.1.3 Implied Correlation Models

The implied correlation is the correlation that is obtained by reverse engineering the correlation from a given tranche market spread using our assumed loss model and linking the defaults across single names. The implied correlation obtained through a Gaussian Copula is collapsed into one parameter (all the single name pairs are assumed to have the same correlation). The implied correlation can take the form of either **Base** or **Compound** correlation. A Compound correlation models the correlation by breaking the portfolio into a series of increasingly thick equity tranches and treating mezzanine tranches as analogous to a spread of two equity tranches [3]. A Base correlation model looks to find the implied correlation for each tranche independent of the other tranches' correlation parameter. We shall look only at the Compound correlation. The Base correlation

was introduced as an improvement to the Compound correlation and is now the market norm despite the body of research criticising its use (see list below) . A comparison of the two forms of implied correlation is given below [1].

- Compound correlation is more consistent at single tranche level. For Base correlation there is inconsistency at the single tranche level as two components of the same trade are valued with models having two different parameter values.
- The Compound correlation is less robust because for some market CDO spreads compound correlation cannot be implied or can be implied with multiple solutions.
- This correlation allows us to quote spread using a single correlation value with the market accepted Gaussian Copula framework.
- The implied Base correlation is more easily extended to pricing bespoke tranche prices via interpolation. Base correlation is more easily interpolated and leads to the possibility of pricing non-standard detachments.
- Base correlation may lead to negative expected tranche losses thus violating basic no-arbitrage conditions. Depending on the interpolation technique being used, tranche spreads could end up being not be arbitrage free.

The *correlation skew (or smile)* refers to the uneven implied correlation that one gets for the different tranches by the reverse engineering process described above. Since all the tranches are created by the same underlying single names we would expect the resulting implied correlations to be the same across tranches. Market practitioners can still make use of these results. The resultant tranche correlations can be used for example to calculate the prices of similarly credit rated portfolios, to make comparisons of relative price levels or portfolio risk levels etc. The Implied Copula model can be used to calibrate consistently across the capital structure, but not across maturities, as it is an inherently static model. Dynamic loss models have been proposed [7].

1.1.4 Credit Modeling

Credit modeling uses structural hazard rate models. For a tranching (single-tranche) collateralized debt obligation investor, default correlation determines what share of the portfolio risk stays within a tranche, that is, the fair premium relative to the total portfolio spread [3].

In the end what we want to deduce from the various approaches is the default rate distribution (loss distribution), and this will allow us to price the different tranches. When we use the Gaussian single factor Copula⁴ model we have little flexibility, in that one can only play around with the single copula parameter ρ , with scenario probabilities being fixed by the Gaussian assumption. If we are to price a set of instruments (e.g. CDO tranches) with a single model specification (just one parameter) can be unrealistic. Other models have been proposed like the Implied Copula [7] in this approach we can play around with the scenario probabilities so as to obtain a rich variety of possible default rate distributions, which can help in pricing a set of instruments with a single model specification[2].

Other models like the Generalised Poisson Loss model [7] (which can give consistent results over maturities and capital structure) have been proposed which aim to address various weaknesses of the Gaussian copula model. However the relative, ease of implementation and flexibility of the Gaussian Copula model have made it the favored choice for practical purposes. Deeper stochastic models (e.g. GPL), involving large numbers of parameters, are harder to implement and risk-management is more difficult. Thus, copula models, and especially the one-factor Gaussian Copula model, are often chosen for practical reasons.

1.2 Copula

Copulas were introduced by Sklar (1959): important developments in the theory are due to Schweizer and Sklar (1974, 1983), who used them in the context of probabilistic metric spaces, and to Schweizer and Wolff (1981). The Copula was first used in the context of CDOs by David X Li in the year 2000 [6]. The ease with which Copulas allow for the full description of the joint probability density of many random variables from their marginal densities makes them a good candidate to model CDO tranches.

1.2.1 Sklar's Theorem

Definition of a Subcopula. Let n be a fixed positive integer $n \geq 2$ let S_1, S_2, \dots, S_n be subsets of \mathbf{R} , and let H be a mapping from $S_1 \times S_2 \times \dots \times S_n$ into the unit interval $I = [0, 1]$. For $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in the domain of H , with $x_k \leq y_k$ for $k = 1, \dots, n$ the H -volume of the n -box $B = [x_1, y_1] \times \dots \times [x_n, y_n]$ is given by

⁴see section 1.3.1

$$V_H(B) = \Delta_{x_n}^{y_n} \Delta_{x_{n-1}}^{y_{n-1}} \dots \Delta_{x_1}^{y_1} H(t)$$

Where for any $t = (t_1, \dots, t_n)$ in domain of H ,

$$\Delta_{x_k}^{y_k} H(t) = H(t_1, \dots, t_{k-1}, y_k, t_{k+1}, \dots, t_n) - H(t_1, \dots, t_{k-1}, x_k, t_{k+1}, \dots, t_n)$$

The function H is n -increasing if $V_H(B) \geq 0$ for any such n -box B . An n -subcopula C' is a function which satisfies the following [13]:

- 1 $C' : S_1 \times S_2 \times \dots \times S_n \rightarrow I$ where $S_i, i = 1, 2, \dots, n$ are subsets of I which contain $\{0, 1\}$
- 2 $C'(x_1, x_2, \dots, x_n) = 0$ if $x_i = 0$ for some i
- 3 C' is n -increasing
- 4 C' has uniform margins, i.e. $C'(1, 1, \dots, 1, x_i, 1, \dots, 1) = x_i$ for all $x_i \in S_i$

Consider a probability space $(\Omega, \mathfrak{S}, P)$, with Ω a non-empty set, \mathfrak{S} a sigma algebra on Ω and P a probability measure on \mathfrak{S} . Let X and Y be two (Borel-measurable) r.v.s on $(\Omega, \mathfrak{S}, P)$ with values in \mathfrak{R}^* , the extended real line. Let also F, F_1 and F_2 be their joint and marginal distribution functions. As usual, the r.v.s are said to be continuous when their d.f.s are.

(Sklar, 1959)[14] Let $F_1(x), F_2(y)$ be (given) marginal distribution functions. Then, for every $(x, y) \in \mathfrak{R}^{*2}$ (i) if C' is any subcopula whose domain contains $\text{Range } F_1 \times \text{Range } F_2$,

$$C'(F_1(x), F_2(y))$$

is a joint distribution function with margins $F_1(x), F_2(y)$ (ii) conversely, if $F(x, y)$ is a joint distribution function with margins $F_1(x), F_2(y)$ there exists a unique subcopula C' with domain $\text{Range } F_1 \times \text{Range } F_2$, such that [10]

$$F(x, y) = C'(F_1(x), F_2(y))$$

if $F_1(x), F_2(y)$ are continuous, the subcopula is a copula; if not, there exists a copula C such that

$$C(v, z) = C'(v, z)$$

for every $(v, z) \in \text{Range } F_1 \times \text{Range } F_2$

First of all, let us notice that, from the definition, copulas are joint distribution functions of standard uniform random variates. If we take the example of a bivariate joint distribution function we have

$$C(v, z) = Pr(U_1 \leq v, U_2 \leq z)$$

Since copulas are joint distribution functions of standard uniforms, a copula computed at $F_1(x), F_2(y)$ gives a joint distribution function at (x, y) :

$$\begin{aligned} C(F_1(x), F_2(y)) &= Pr(U_1 \leq F_1(x), U_2 \leq F_2(y)) \\ &= Pr(F_1^{-1}(U_1) \leq x, F_2^{-1}(U_2) \leq y) \\ &= Pr(X \leq x, Y \leq y) \\ &= F(x, y) \end{aligned} \tag{1.8}$$

This formulation generalises to cases where there are more than two variables to obtain $M(> 2)$ variable joint distribution. Copulas allow for a separation between the marginal CDF and the dependence structure. **Sklar's theorem** guarantees that if the marginals are continuous, every n -dimensional joint distribution function can be represented as a (unique) n -dimensional Copula. Under Sklar's theorem there exists a unique Copula C such that

$$F(x_1, x_2, \dots, x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n))$$

and conversely

$$C(u_1, u_2, \dots, u_n) = F(F_1^{-1}(u_1), F_2^{-1}(u_2), \dots, F_n^{-1}(u_n))$$

1.3 Gaussian Copula

The default event of a credit reference is a random binary variable (either it has occurred or not). We need to know when the default event occurs and the amount recoverable. In order to model the dependence of the random defaults we can use the Gaussian Copula to model default times. The correlation of default times are a more intuitive random variable to handle than trying to model correlation between default events.

We denote by τ_i the default time of name i in a pool of M names. The Copula formalism allows us to connect the default times of the different names in the most general way. If $p_i(t) = F(\tau_i \leq t)$ is the default probability of name i by time t , we know that the random variable $p_i(\tau_i) = U_i$ is a uniform random variable. As discussed above Copulas are multivariate distributions

on uniform random variables. If U_1, U_2, \dots, U_M is a multivariate uniform then following equation (1.8) we can say a possible multivariate distribution of the default times is $C(u_1, u_2, \dots, u_M)$.

Since the U_1, U_2, \dots, U_M variables are connected through a multivariate distribution C we have a dependence structure on the default times. We can get the marginal distributions since

$$\tau_1 = p_1^{-1}(U_1), \dots, \tau_M = p_M^{-1}(U_M) \quad (1.9)$$

assuming that the p_i are invertible. Once we can simulate these default times we can run different possible paths that have each name defaulting at different times and thus calculate our expected net present values⁵.

An arbitrary assumption is made to introduce the Gaussian Copula into the formulation. The assumption is made that, given standard Gaussian random variables X_i s we have $[U_1, U_2, \dots, U_M] = [\Phi(X_1), \dots, \Phi(X_M)]$ ⁶ [2] i.e the multivariate uniform can be mapped by a multivariate Gaussian random variable $[X_1, X_2, \dots, X_M]$ with a given correlation matrix. We thus have instead of (1.9)

$$\tau_1 = p_1^{-1}(\Phi(X_1)), \dots, \tau_M = p_M^{-1}(\Phi(X_M)) \quad (1.10)$$

The normally distributed latent factors X are the ones used to link the default times together. Equation (1.6) defined the market quotes in terms of expectation of the tranced loss $\bar{L}^{A,B}$. The loss that would be tranced \bar{L}_t is defined in terms of single default times as

$$\bar{L}_t = \sum_{i=1}^M \frac{1}{M} (1 - R_i) 1_{\{\tau_i \leq t\}} = \sum_{i=1}^M \frac{1}{M} (1 - R_i) 1_{\{\Phi(X_i) \leq p_i(t)\}} \quad (1.11)$$

We have to assume a particular structure for the default probabilities p_i . For single names the default probabilities are supposed to be related to the hazard rate λ . The relation is

$$p_i(t) = 1 - \exp\left(-\int_0^t \lambda_i(s) ds\right) \quad (1.12)$$

and for $\tau_i \leq t$

$$p_i(\tau_i) = 1 - \exp(-\Lambda_i(t) * \tau_i) \quad (1.13)$$

Where $\Lambda_i(t) = -\int_0^t \lambda_i(s) ds$. We can construct our model with each of the λ_i and R_i (the recovery rate) being distinct for $i = 1, \dots, M$. This is called a

⁵see equations 1.3 and 1.4

⁶ Φ is the cumulative distribution function of the one-dimensional standard Gaussian distribution.

heterogeneous pool model. Alternatively we can have these values being equal and use a *homogeneous* pool model. The usual way that we would simulate the random values for τ_i the default times is, we would generate a uniform U_i and equate it to (1.13) invert and find τ_i . In our Gaussian Copula model the U_i are generated from $\Phi(X_i)$ so that $\tau_i = \Lambda_i(t)(1 - \Phi(X_i))$. We introduce the clean pool hazard rate⁷ $h = \frac{s}{1-R}$. We can use this hazard rate to calculate the p_i for each obligor from the CDS spread s . We are then able to calculate the implied cumulative probability of default

$$Imp.PD(t, T) = 1 - \exp\left(-\frac{s}{1-R}(T-t)\right) \quad (1.14)$$

1.3.1 Single Factor Copula Model

The model we use here is a one-factor model whereby the defaults are driven by one factor which we take to represent a common economic driver of credit events. A one factor Copula structure is a special case of the Gaussian Copula above where

$$X_i = \sqrt{\rho_i}Z + \sqrt{1 - \rho_i}Y_i \quad (1.15)$$

with Y_i, Z being standard Gaussian Variables. Z is the systematic factor affecting the default times of all names, and Y_i is the idiosyncratic factor affecting just the i th name [2]. The correlation between Gaussian factors X_i and X_j is given by $\sqrt{\rho_i\rho_j}$. If we assume homogeneous dynamics then $\rho_i = \rho_j$ and all pairwise correlation parameters have a single common value i.e $\rho_i = \rho$ and the correlation matrix is then given by

$$\begin{bmatrix} 1 & \rho & \rho & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho & \rho & \dots & 1 & \rho \\ \rho & \rho & \dots & \rho & 1 \end{bmatrix}$$

Conditional on the state of the common economic factor, credits will default when their asset values fall below a pre-specified threshold. This default threshold usually represents the level of debt of a company. Following on from equation

⁷see appendix A for formula derivation

(1.11) we note that the pool's normalised loss

$$\begin{aligned}
\bar{L}_t &= \sum_{i=1}^M \frac{1}{M} (1 - R_i) 1_{\{\tau_i \leq t\}} \\
&= \sum_{i=1}^M \frac{1}{M} (1 - R_i) 1_{\{\Phi(X_i) \leq p_i(t)\}} \\
&= \sum_{i=1}^M \frac{1}{M} (1 - R_i) 1_{\{X_i \leq \Phi^{-1}(p_i(t))\}}
\end{aligned} \tag{1.16}$$

A credit, i , is assumed to default if its asset return, X_i , falls below a pre-specified level or default threshold given by the $\Phi^{-1}(p_i(t))$. There are two possible approaches that we can then follow. We can either use Monte Carlo simulations or use an analytical approach (Large pool homogeneous one-factor Gaussian Copula model LHP) to determine the default losses. In Monte Carlo simulations the term structure of default probabilities for each credit can be calibrated to market spreads or implied from the credit ratings. In the analytical approach discussed in Hull & White we take advantage of the assumption that the obligors are assumed independent conditional on the common market factor. The Monte Carlo method although slower allows more flexibility in allowing heterogeneous correlation and default loss calculation.

1.3.2 Large pool homogeneous one-factor Gaussian Copula model

We denote

- PD - The probability of default.
- LGD - The loss given default.

This can provide a useful direct, analytical method for determining the risk of CDO tranches. Conditional on the systematic factor, Z defaults are independent and the probability of one default is,

$$PD(T; Z; \rho) = \Phi\left(\frac{\Phi^{-1}(PD(T)) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \tag{1.17}$$

The first assumption that we make is that the underlying portfolio is homogeneous in the sense that the PDs, LGDs and factor sensitivities are uniform across debt instruments. The second assumption is that the portfolio is really large. We assume that it contains loans from an infinite number of obligors. By

the Law of large numbers the proportion of defaults $\bar{C} \rightarrow PD(T; Z; \rho)$. The conditional percentage portfolio loss, denoted by $Loss(Z)$ can be directly obtained as LGD times conditional default probability [9].

$$Loss(Z) = LGD \cdot \Phi\left(\frac{\Phi^{-1}(PD(T)) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \quad (1.18)$$

The probability that the loss is larger than some value B can be expressed as the probability that the factor Z is smaller than some critical value $d(B)$. To obtain $d(B)$ we set $Loss(Z) = B$ in equation (1.18) and solve for Z .

$$\begin{aligned} Prob(Loss \geq B) &= Prob(Z \leq d(B)) = \Phi[d(B)] \\ d(B) &= \frac{\Phi^{-1}(p) - \sqrt{1-\rho}\Phi^{-1}(B/LGD)}{\sqrt{\rho}} \end{aligned} \quad (1.19)$$

We express the expected portfolio loss as a percentage of the portfolio notional. For an equity tranche with attachment point 0 and detachment point B , the expected loss as a percentage of the portfolio notional can be written as

$$E(Loss_{(0,B)}) = LGD \cdot E\left[\Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) I\{Z > d(B)\}\right] + B\Phi(d(B)) \quad (1.20)$$

The second term on the right-hand side captures the factor scenarios where the portfolio loss is larger than the detachment point B in this case the entire tranche principal is lost. If the loss is not greater than B the indicator function $I\{Z > d(B)\}$ takes the value 1 ie Z is above $d(B)$. As shown in the appendix there is a closed form solution for the expectation in the first term which leads to the solution.

$$E(Loss_{(0,B)}) = LGD \cdot \Phi_2(\Phi^{-1}(p), -d(B), -\sqrt{\rho}) + B\Phi(d(B)) \quad (1.21)$$

Where $\Phi_2(x, y, \rho)$ is the cumulative standard bivariate normal distribution function with correlation ρ . The formula can be used to determine the expected loss of a tranche with nonzero attachment point A and detachment point B . Recalling that the expected loss formula from above are not expressed as a percentage of the tranche principle, we divide by the difference between a tranche's attachment and detachment points.

$$E(Loss_{(A,B)}) = \frac{1}{A-B} \cdot E(Loss_{(0,A)}) - E(Loss_{(0,B)}) \quad (1.22)$$

For the senior tranche with detachment point $B = 1$, we can set $E(Loss_{(0,1)}) = LGD \times PD$. As the number of obligors increases, the quality of the LHP approximation improves. Also the approximation is affected by the choice of the

correlation parameter ρ and the level of heterogeneity in the portfolio. The usual procedure for implementing a one-factor Copula model involves integrating the value of the underlying instrument over the probability distribution of the factor using Gaussian quadrature.

The problem with that is once you change the tranche you have to change the implied correlation. This implies inconsistency in the loss distribution because we are saying that the dependency of default changes depending on the amount of loss and this should not be the case. As a result it is very difficult to determine the appropriate correlation when the Gaussian Copula model is used to value non-standard credit derivatives such as bespoke CDOs and CDO-squareds. This has led a number of researchers to look for Copulas that fit market prices better than the Gaussian Copula. Among the Copulas that have been considered are the Student-t, double-t, Clayton, Archimedean, and Marshall Ohkin [5]

The implementation of the Monte Carlo approach is discussed further in the next chapter.

Chapter 2

Model Implementation

2.1 Model Setup

We use CDX S9 index CDO market quotes. The price of a CDO tranche is a function of the tranche's notional, the expected default losses and the single name CDS spreads. At each payment date the protection buyer needs to estimate the credit losses expected in the portfolio and distribute these losses to each tranche based on the attachment and detachment levels of each tranche (the default leg). The protection seller then receives a premium for each tranche based on the remaining notional (the premium leg).

The fair tranche spread is that spread that equates the premium leg to the default leg. To determine the joint distribution of defaults we can use the marginal distributions of the loss $L(t)$ up to maturity. The computation of the default leg involves finding the expected losses in each tranche i.e. $E(L(t) - A)^+$ where A are the attachment points of the tranches. In order to calculate this expectation we make the following assumptions

- We ignore the counterparty risk between the parties to the CDO transaction.
- The expected recovery (required for calculating the loss given default LGD) is fixed for all the single names. The recovery rate is 40% for all the obligors.
- A flat term structure for Interest rates. The interest rates are also assumed to be independent of the recovery rate¹. The risk free interest rate is set at 2%.

¹see section 1.1.2

- All the obligors are assumed to contribute an equal amount to the total notional $1/M$.

Unfortunately, it is not possible to relax these assumptions and gain extra realism without a considerably increasing the model complexity. We are however able to reach fairly realistic general conclusions on CDO valuations. If premiums are paid after time fractions Δ_t (for quarterly payments $\Delta_t = 0.25$) have passed then denoting CDO spread by s risk free rate by r the notional at time t as N_t and the loss by L_t . Assuming defaults occur only between coupon payment dates we have

$$\text{Coupon leg} = s \sum_{t=1}^T \frac{\Delta_t E_0[N_t]}{(1+r)^t} \quad (2.1)$$

$$\text{Accrued leg} = s \sum_{t=1}^T \frac{\Delta_t E_0[N_t - N_{t-1}]}{2(1+r)^t} \quad (2.2)$$

$$\text{Default leg} = \sum_{t=1}^T \frac{E_0[L_t - L_{t-1}]}{(1+r)^t} \quad (2.3)$$

The premium leg would be the sum of the Coupon and Accrued coupon legs. We can use a Monte Carlo simulation to find the expected loss function. To carry out a this simulation, we first need to find the implied hazard rate² curve from CDS spreads (if we have CDSs of different terms then instead of one hazard rate value we would need to construct the hazard rate term structure). Once we have the hazard rate curve we can calculate $\Lambda(t)$ ³ were t takes on values t_1, t_2, t_3, \dots, T . We start with shortest dated CDS to longer dated CDSs. The reset dates can be chosen to coincide with with the quarterly coupon dates. Here we have single CDSs over the term of the CDO. The hazard rate curve is used to get the probability of default $p_i(t)$ term structure which we compare with the simulated random values $\Phi(X_i)$ i.e. if $\Phi(X_i) \leq p_i(t)$ then that i^{th} name is said to have defaulted by time t see section 1.1. These default times are determined for each obligor depending hazard rate term structure together with their dependence as determined by the correlation structure imposed by the Copula model i.e. the systematic risk is incorporated through the correlation factor chosen. After running the model you get tranche spreads for each tranche based on you chosen correlation see figure 2.1. We can run the model for several correlation values obtain a range of spreads and use the market spreads to interpolate back to the implied tranche correlation.

²see appendix A for the calculation of the implied hazard rate CDS spread

³see equation 1.3

Deal Parameters		Capital Structure			Running			
Risk-free(%)	2%	Tranche	Attachment	Tranche Spread	Spread	Accrued Leg	Coupon Leg	Default Leg	
Coupons pa	4	Equity	0%	38.79%		0.182851179	1.46639433	0.720524215	
Value Date	19-May-09	Mezz J	3%	13.91%		0.114442702	2.43429959	0.451001708	
Maturity Date	20-Dec-12	Mezz S	7%	7.79%		0.080473264	2.78968227	0.317366512	
Spread of index (bps)	179	Senior J	10%	4.40%		0.058003891	3.02218221	0.22836149	
		Senior	15%	1.51%		0.02176464	3.30158207	0.08573329	
		Super Seni	30%	0.02%		0.03699463	3.38910207	0.00112566	
Copula Parameters		Index	0%	100%					
Simulation Trials	100								
Correlation	50%								
Position of Correlation Matrix									
WatchType	Timer	2.632813							
# Loans	125				Calculate				
# Periods	15								
Notional (\$)	125000								
Avg. LGD	0.60								
Avg. Spread (bps)	233.24								

Figure 2.1: Excel results example

2.2 Generating draws from Copula

The Copulas are constructed using the univariate normal margins. Generating draws from the Copula mentioned in section 1.3 can be done by following these steps found in [10].

- 1 Generate a column vector \mathbf{x} of n independent draws (x_1, x_2, \dots, x_n) from the standard normal distribution.
- 2 Find the Cholesky decomposition⁴ \mathbf{L} of the symmetric and positive definite matrix Σ .
- 3 We can then use equation 1.13 to find the default probabilities $p_i(t)$.
- 4 Using these default probabilities and the X_i s determined in \mathbf{Lx} we can work out the loss function in equation 1.16.

The Cholesky decomposition remains the same for all the simulations and for efficiency is determined outside the MC loop. To help increase random number generation efficiency we use the polar algorithm. This generates two values at a time. For each point accepted, the polar transformation produces a vector with two independent normally distributed elements. This algorithm does not involve any approximations, so it has the proper behavior in the tails of the distribution.

The legs are calculated separately for each tranche. It is crucial to notice that tranching is a non-linear operation. We obtain the cumulated loss at each

⁴see Appendix B

coupon date t $\text{Loss}_{cum}(t)$ for each simulation run j . Using an initial assumption of the correlation ρ we can generate random gaussian Copula values X_i for each obligor i , using equation 1.16 we can determine the dates of default and thus the cumulated loss. We can then go back to each coupon date and apportion the loss and notional level to each tranche, by looking at the marginal tranche loss $L(t)_{A,B}^j = \min(B - A, \max(0, \text{Loss}_{cum}(t) - A))$. The tranche loss L_t in equation (2.3) are then a sum of these tranche marginal losses $L(t)_{A,B}^j$.

2.3 Systematic Risk of CDO tranches

The risk of CDOs is often analysed by looking at either the PD or the expected loss. Indeed Rating agents used this approach to rate the CDO tranches. A CDO tranche with a default probability of 0.1% however does not carry the same risk as a corporate bond with a default probability of 0.1%. This is because the CDO tranches carry significantly higher systematic risk. To measure systematic risk we can look at PDs when the world is in a very bad state. In the one factor model that we used this could be the PD conditional on the value of the factor $Z = -3.09$ for example, a scenario that is 99.9% worse than all possible scenarios. If we take the homogeneous portfolio assumptions and consider the conditional default probability

$$p(Z) = \Phi \left[\frac{\Phi^{-1}(PD(T)) - \sqrt{\rho}Z}{\sqrt{1-\rho}} \right] \quad (2.4)$$

We can determine the probability of a homogeneous tranche with attachment A being hit. If in our portfolio we have N obligors and D defaults in the portfolio then for the tranche with attachment point A suffers losses if the percentage portfolio loss is greater than A

$$\frac{D \times LGD \times EAD}{\sqrt{1-\rho}} > A \quad (2.5)$$

Simplifying and rearranging we have

$$D > A \times N / LGD \quad (2.6)$$

Using the assumption of independent defaults, conditional on the factor Z the number of defaults D follows a binomial distribution. The probability of successfully hitting the tranche with attachment A conditional on Z is therefore given by

$$1 - \text{Binom}(A \times N / LGD, N, P(Z)) \quad (2.7)$$

Where $\text{Binom}(x, N, q)$ is the cumulative probability of observing x or fewer successes out of N trials with success probability q [9].

Chapter 3

Results

3.1 Parameter Estimation Risk of CDO tranches

Parameter Estimation Risk We note the asymmetry in the results in Table 3.1. A decrease from 0.2 to 0.3 has a smaller impact than an increase from 0.3 to 0.4. If we assume that the three scenarios are equally likely then the correct estimate is obtained as a simple average. For example the correct estimate for the 3-7 tranche would be 0.51% almost double 0.30% which is the one that is obtained using the average factor 0.3. This illustrates the importance of parameter estimation risk for a reliable analysis of CDOs. This suggests an area that could have contributed to the financial crisis. There was possibly a neglect by market practitioners of parameter uncertainty [9].

Table 3.1: Expected Loss for different $\sqrt{\rho}$ LHP Model

Tranche	$\sqrt{\rho} = 0.2(\%)$	$\sqrt{\rho} = 0.3(\%)$	$\sqrt{\rho} = 0.4(\%)$
0,3%	25.99	25.6028	24.2850
3,7%	0.0114	0.3008	1.1887
7,100%	0.000	0.0002	0.0045

3.2 Compound Correlation Smile and Invertibility

After collapsing the Gaussian Copula correlation to one parameter ρ , we can using the Monte Carlo simulation, observe the correlation smile. Results are shown in figure 3.1. We notice that there is no bar for the 3-7% tranche this

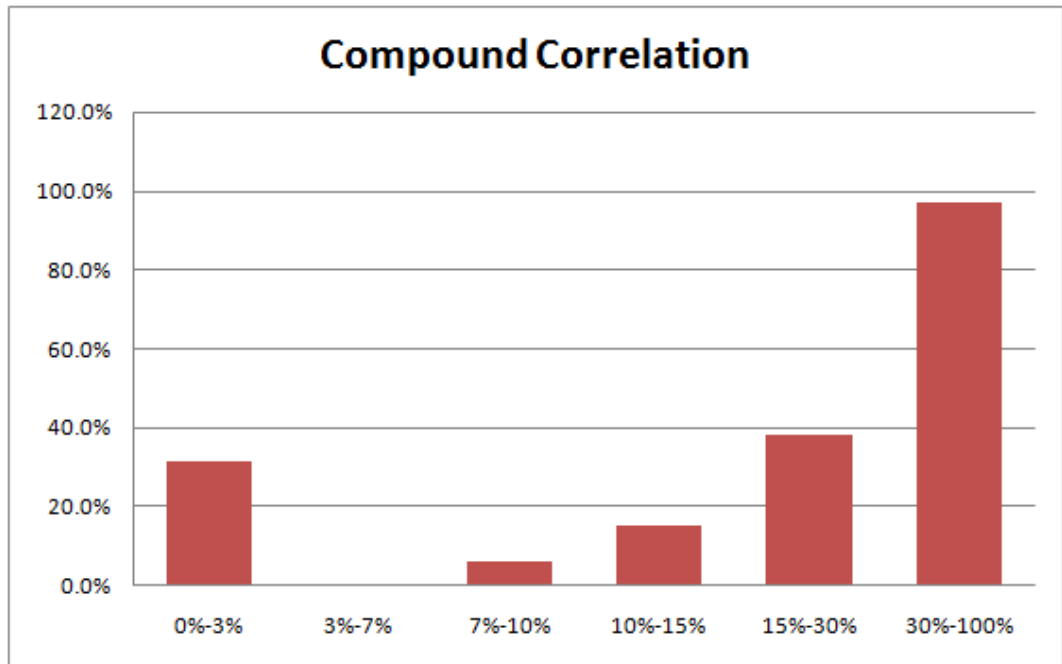


Figure 3.1: Compound Correlation

problems is not atypical in that we often face market spreads where we cannot imply the compound correlation [7].

In the figure 3.2 below we show how we get the implied correlation, by plotting the fair market spread (the horizontal line) and the model fair spread at different correlation levels. The point where the fair market spread intersects with the correlation plot gives the tranche implied correlation. We note that we can get more than one implied tranche compound correlation [7] although this does not happen in our example. Given a market spread we are not always guaranteed that we can imply a compound correlation as noticed for the 3-7% tranche. There is also no implied correlation for the 30-100% tranche. We have used 5000 simulation runs for these results. The Monte Carlo approach is computationally time-consuming and requires a large number of simulations in order to produce enough defaults that can impact the most senior tranches of the CDO.

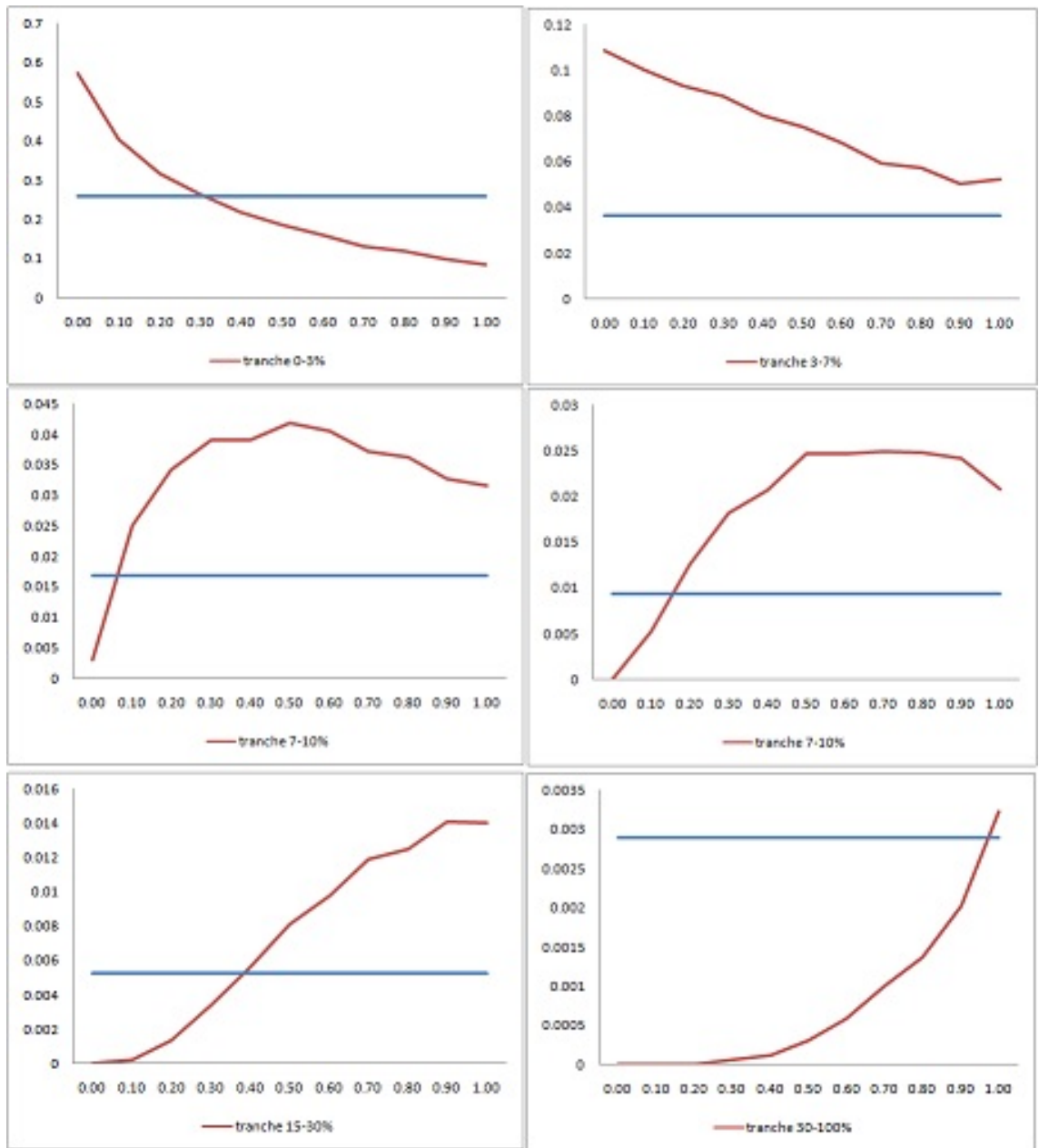


Figure 3.2: Tranche Implied Correlation

Figure 3.3 Shows how the implied correlation varied for CDOs priced on three different dates. The implied correlation is highest for the 31-October 2008 date across tranches. This suggests some sensitivity of the model to market sentiment, in a market crash the stocks correlation approaches 1. The correlation smile exits for all dates across tranches.

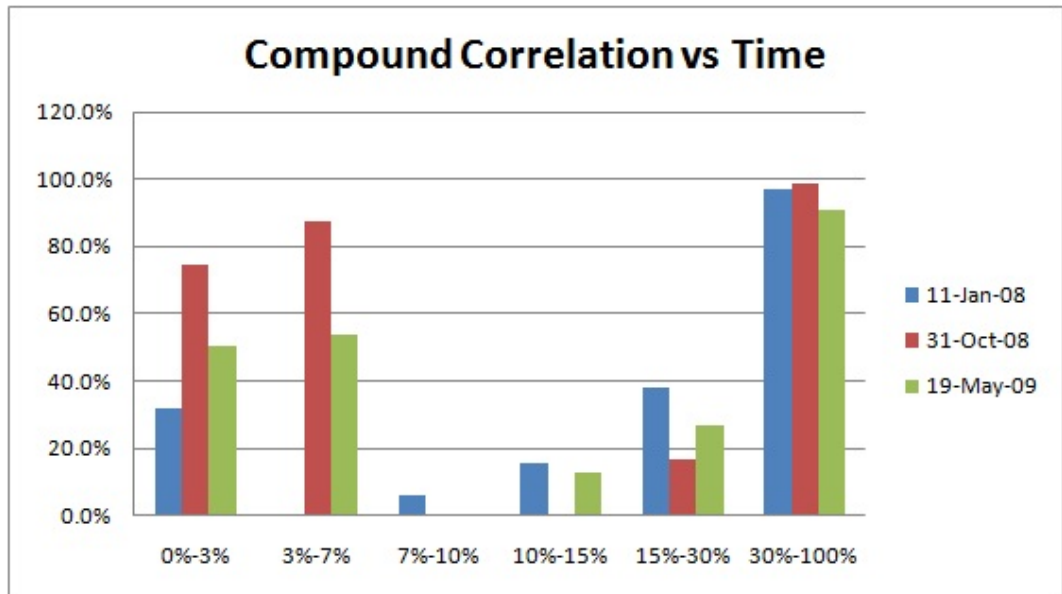


Figure 3.3: Tranche Implied Correlation

3.3 LHP Model vs Copula Simulation Model Results

In Figure 3.4 we show a comparison of the tranche prices that are derived from the LHP Model and the Copula Simulation (CS) for a given correlation.

Date	Source	Tranche					
		0-3%	3-7%	7-10%	10-15%	15-30%	30-100%
11-Jan-08	LHP	0.2071	0.0659	0.0323	0.0167	0.0042	0.0001
	Copula Simulation	0.2576	0.0876	0.0394	0.0185	0.0038	0.0000
	Market	0.2587	0.0363	0.0168	0.0093	0.0052	0.0029
31-Oct-08	LHP	0.1374	0.0815	0.0623	0.0496	0.0315	0.0047
	Copula Simulation	0.3939	0.1768	0.1153	0.0787	0.0377	0.0032
	Market	0.4033	0.1510	0.0700	0.0373	0.0120	0.0061
19-May-09	LHP	0.2461	0.1078	0.0681	0.0455	0.0198	0.0012
	Copula Simulation	0.3831	0.1599	0.0903	0.0546	0.0193	0.0007
	Market	0.3893	0.1586	0.0653	0.0293	0.0095	0.0038

Figure 3.4: Market spreads Vs LHP and Copula Simulation spreads

We set the correlation parameter to the CS equity tranche implied correlation at each valuation date for both models. The CS implied correlation increased

from 31.7% on 11- Jan- 08 to 74,75% on 31- Oct- 08 and then fell to 50.46% on 19- May- 09. The LHP and CS model give roughly the same values on the 11- Jan- 08- date. The difference between the LHP and CS model is significant on the 31- Oct- 08 and 19- May- 09 dates. This suggest that the LHP is a good approximation of the Copula in stable market conditions but is not a reliable approximation in extreme market conditions.

3.4 Systematic Risk of CDO tranches

We compute the conditional PDs for the mezzanine tranche and for an individual bond with the same default probability as the mezzanine tranche with attachment point 3%.

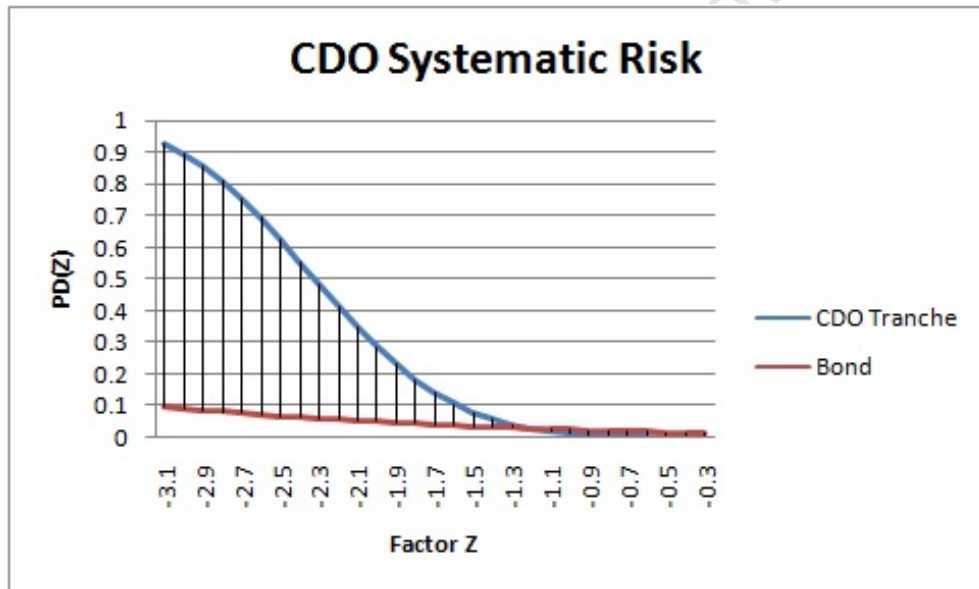


Figure 3.5: Bond Vs CDO Systematic Risk

Results shown in Figure 3.5 below. CDO systematic risk is shown to be significantly higher than Bond systematic risk as the economic environment worsens. If a shock hits the economy, an individual tranche will react very strongly due to its high systematic risk.

Chapter 4

Conclusions

In the results section 3.1 we showed using an illustration the high parameter risk associated with CDO pricing models. Our aim was to examine how well the LHP and Copula pricing (using compound correlation) models were suited for this task. Using CDX S9 index market quotes we compared the CDO tranche model (LHP and Copula Simulation) predicted price against the actual market tranche spread quotes figure 3.4. We observed the compound correlation smile (figure 3.1) and observed how the implied correlation changed over time leading up to the 2008 financial crisis on three different dates (figure 3.3). In section 3.4 we examined deficiencies of using just the probability of default estimates to credit rate CDO tranches. We did so by demonstrating the differences in sensitivity of probability default of ordinary bonds against CDO tranches under various scenarios.

- As expected the implied correlation showed the correlation smile. This is a serious practical draw back of Copula models in general, and the Gaussian Copula model in particular [5]. The one-factor Gaussian Copula model has a single parameter, the correlation, and when matched to market data the correlation displays a clear skew across different tranches of the same portfolio instead of being constant. Other Copula models like the Doublet Copula [5] have been developed in an attempt to flatten the correlation skew. The Gaussian Copula model is therefore largely foundational model, which displays qualitative characteristics observed in practice and through simulations in other models [11]. The implied correlation showed a rise as expected as we moved deeper into the credit crisis.

Market data for CDOs is not readily accessible and this limited our analysis. The data is proprietary and thus costly to acquire. This has limited

the amount of academic research on similar instruments and also analyses of the more complex hedging strategies.

- The reverse engineering to obtain tranche spreads assumes a very liquid market. The CDO market is however very illiquid. The lack of liquidity was even worse leading up to and during the credit crunch. This all means that the models become less robust. There are also huge costs for getting data which limits independent academic studies.
- Obtaining hedging strategies using Compound correlation is difficult to achieve, not least because of the sometimes missing implied tranche correlation but also the possibility of more than one tranche correlation being implied. Hedging strategies have been explored using Base correlation, in its heterogeneous version provided easy to calculate hedge ratios [7].
- Credit rating agencies produce ratings based on default probability or expected loss. Many market participants were not aware of the difference in risk presented by for example a single tranche rated AAA and a bond rated AAA by rating agencies. Many investors might have been caught unaware before 2008 credit crisis. Setting aside the ignorance of investors to the ratings process the credit ratings have been roundly criticised for not taking account of the true CDO risk and thus misleading. Recently the United States prosecutors have moved to bring criminal proceedings against credit rating agency Standard & Poor's for giving AAA ratings to sub-prime mortgages. There is a clear conflict of interest presented because banks are the ones that pay the rating agencies to rate bank assets. The banks then sell the assets to third parties who rely on these ratings.
- The Gaussian Copula model cannot be used to imply correlations at other times other than the valuation date because of its stationarity. Practitioners have moved to devise models like the dynamic Generalised Poisson Model GPM [7] to try and have more dynamic models. The model is therefore not very useful for making long term investment strategies. The Gaussian Copula is difficult to justify, implement and interpret. The added consistency of such models as the GPM should be balanced against their added complexity.
- The choice of parameter values is important and if done incorrectly can lead to significantly different results. In order to correctly interpret and use the model results one needs to consider parameter uncertainty involved.

Appendix A

Terminology

Definition: CDS spread = Premium paid by protection buyer to the seller CDS spread

$$s = \frac{\sum_{\tau} (1 - R - A(\tau)) \frac{PD_{\tau}^0}{(1+r_{\tau})^{\tau}}}{\frac{1}{freq} \sum_{\tau} \frac{1 - \sum_{t=0}^{\tau-1} PD_{\tau}^0}{(1+r_{\tau})^{\tau}}} \quad (\text{A.1})$$

$A(\tau)$ is the accrued interest as a percentage of the notional principle. If we are looking at a one period setup and we arrive at

$$\frac{s}{freq} = (1 - R) \left(\exp\left(\frac{h}{freq}\right) - 1 \right) \quad (\text{A.2})$$

Solving for the hazard rate

$$h = freq \ln \left(1 + \frac{s}{freq(1 - R)} \right) \quad (\text{A.3})$$

Now note that we can approximate $\ln(1 + x)$ by x for small x giving us

$$h = \frac{s}{(1 - R)} \quad (\text{A.4})$$

We can then calculate the implied cumulative probability

$$Imp.PD(t, T) = 1 - \exp \left(- \frac{s}{1 - R} (T - t) \right) \quad (\text{A.5})$$

Quotation: In basis points per annum of the contract's notional amount

Payment: Quarterly Example: A CDS spread of 339 bp for five-year Italian debt means that default insurance for a notional amount of EUR 1 m costs EUR 33,900 per annum; this premium is paid quarterly (i.e. EUR 8,475 per quarter)

Appendix B

Cholesky decomposition

The implementation of Cholesky decomposition $\text{CHOLESKY}(\mathbf{A})$ takes the matrix \mathbf{A} and estimates the decomposition by finding the elements of the matrix \mathbf{C} in $\mathbf{A} = \mathbf{C}\mathbf{C}^T$. Since \mathbf{A} is a lower triangular matrix, each element on the main diagonal of \mathbf{C} is found as $c_{ii} = \sqrt{\left(a_{ii} - \sum_{k=1}^{i-1} c_{ik}^2\right)}$ where x_{ij} is the i th row and j th column entry of the matrix \mathbf{X} . Off-diagonal elements are given by

$$c_{ji} = \frac{\left(a_{ji} - \sum_{k=1}^{i-1} c_{jk}c_{ik}\right)}{c_{ii}}$$

```

'XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
'XX                               Cholesky
'XX -Implements cholesky Decomposition
'XX -Finds elements of matrix C in A = CC' since A is lower triangular.
'XX
'XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

Private Sub choleskym(a() As Double)

Dim j As Integer, S As Double, k As Integer, _
    i As Integer, C() As Double
ReDim C(1 To N, 1 To N)
'Cholesky Decomposition
For j = 1 To N
S = 0
For k = 1 To j - 1
    S = S + C(j, k) ^ 2
Next k
C(j, j) = a(j, j) - S
If C(j, j) <= 0 Then Exit For
C(j, j) = (C(j, j)) ^ 0.5

For i = j + 1 To N
S = 0
For k = 1 To j - 1
    S = S + C(i, k) * C(j, k)
Next k
C(i, j) = (a(i, j) - S) / C(j, j)
Next i
Next j

For i = 1 To N
For j = 1 To N

cholesky(i, j) = C(i, j)

Next j
Next i
End Sub

```

Figure B.1: Cholesky Implementation code

Appendix C

Polar algorithm

```
'XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
'XX
'XX Polar method
'XX
'XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

Public Function NRND() As Double
Dim w As Double, z As Double
Static NRND2 As Double, take2 As Boolean

'Check whether a non-used variable is available
If take2 = True Then
    NRND = NRND2
    take2 = False
Else
'Polar method
Do

NRND = 2 * RND - 1
NRND2 = 2 * RND - 1
w = NRND * NRND + NRND2 * NRND2

Loop Until w < 1
    z = Sqr(-2 * Log(w) / w) 'polar transformation
    NRND = NRND * z
    NRND2 = NRND2 * z
    take2 = True
End If
End Function
```

Figure C.1: Polar algorithm


```

+ lambda(tr + 1) * Application.WorksheetFunction.NormSDist(d)
Else
ELO(tr, R) = LGD * pd(R, 1)
End If

loss(tr, R) = (ELO(tr, R) - ELO(tr - 1, R)) / _
(lambda(tr + 1) - lambda(tr))
mloss(tr, R) = loss(tr, R) - loss(tr, R - 1)
output(R, tr) = mloss(tr, R)
Next tr

For tr = k To 1 Step -1 'recovery calculation
If 1 - lambda(tr) < 1 - LGD Then
d = Application.WorksheetFunction.NormSInv(pd(R, 1)) / w - ((1 - w ^ 2) ^ 0.5) / _
w * Application.WorksheetFunction.NormsInv((1 - lambda(tr)) / (1 - LGD))

ERO(tr, R) = (1 - LGD) * Bivnor(Application.WorksheetFunction.NormSInv(pd(R, 1)), -d, -w) _
+ (1 - lambda(tr)) * Application.WorksheetFunction.NormSDist(d)

Else
ERO(tr, R) = (1 - LGD) * pd(R, 1)
End If

rec(tr, R) = (ERO(tr, R) - ERO(tr + 1, R)) / _
(lambda(tr + 1) - lambda(tr))
mrec(tr, R) = rec(tr, R) - rec(tr, R - 1)
output(R, tr + k + 1) = mrec(tr, R)
Next tr
Next R

LHP_LR = output

End Function

```

Figure D.2: LHP VBA code

Appendix E

Probability of Default matrix generation VBA code

```
'XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
'XX          Probability of Default Matrix (PDMatriX)
'XX -Uses the recovery rate(reco) the CDS spread (spread), the coupon dates
'XX -(cdates) and startdate (startd) to estimate a PDMatriX for each coupon date.
'XX -Assume a constant CDS spread over different Maturities
'XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Private rec() As Double, CDS() As Double, cdates() As Double, writePd As Boolean, N As Integer, Q As Integer, _
startd As Date

Public Property Let SetValues(d As IReadData)

'Read in the data required
Q = d.Q      'Number of periods
N = d.N      'Number of Obligors
writePd = d.drwl
ReDim CDS(1 To N)
ReDim rec(1 To N)
ReDim cdates(1 To Q)

Dim j As Integer
For j = 1 To N
CDS(j) = d.CDS(j) 'Record the CDS spreads
rec(j) = d.RR(j)
Next j

For j = 1 To Q
cdates(j) = d.cdates(j) 'Record the actual coupon dates
Next j

startd = d.strdate
End Property

Public Sub PDMatX(pd() As Double, Optional ninv As Boolean)
Dim j As Integer, i As Integer

For j = 1 To Q
For i = 1 To N
pd(i, j) = 1 - Exp(-(CDS(i) / 10 ^ 4) / (1 - rec(i)) * _
((cdates(j) - startd) / 360))
```

Figure E.1: Probability of default matrix

Finding the inverse of the probabilities of default will allow us to make the comparison $X_i \leq \Phi^{-1}(p_i(t))$ discussed in equation 1.16

```
'Transformation of PD into inverse Normal
If ninv = True Then pd(i, j) = Application.WorksheetFunction.NormSInv(pd(i, j))
If writePd Then
Cells(23 + i, 5 + j) = pd(i, j)
End If
Next i
Next j

End Sub
```

Figure E.2: Probability of default matrix

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