Cosmological Magnetogenesis

by

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Abstract

In this thesis, second-order gauge-invariant perturbation theory is used to develop a self-consistent framework in which the non-linear coupling between inflationary gravitational waves and a large-scale magnetic field is investigated, as means of amplifying the field to within the strength limits required to support the galactic dynamo. For the purpose of performing an entirely gauge-invariant analysis, the seed field and the gravitational wave spectrum are treated as small perturbations of a Friedmann-Lemaître-Robertson-Walker (FLRW) background. The magnetic-gravito interaction is shown to produce a sufficient amplification of both a homogeneous and inhomogeneous magnetic field that is proportional to the magnitude of the gravitational wave induced shear anisotropy and the square of the field's initial comoving scale. Contrasting the results obtained using the weak field approximation to those emerging from our gauge-invariant approach, indicates that the methods agree in the limit of high conductivity, however their corresponding solutions are otherwise only compatible in the limit of infinitely long-wavelength gravitational waves.

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Contents

List of Figures

Conventions and Abbreviations

Conventions

Signature: $(- + ++)$.
Units: $c = h = \kappa = 8\pi G = 1$.
The sign conventions used in this thesis are the same as those employed in Ellis (1971) [35].
The covariant analogue to the Newtonian curl of a vector V_a and tensor T_{ab} are defined as:

$$\mathrm{curl}\, V_a \equiv \epsilon_{abc}\, D^b\, V^c, \quad \mathrm{curl}\, T_{ab} \equiv \epsilon_{cd(a}\mathrm{D}^c V_{b)}{}^d.$$

The curl of a second-rank PSTF tensor S_{ab} is defined by [66, 99]:

$$\mathrm{curl}\, S_{ab} \equiv \epsilon_{cd<a}\, \mathrm{D}^c\, S_{b>}{}^d.$$

Round brackets denote symmetrization and square brackets denote anti-symmetrization.

Abbreviations

GR:	General Relativity.
GI:	gauge-invariant.
GW:	gravitational wave.
EMT:	energy-momentum tensor.
FLRW:	Friedmann-Lemaître-Robertson-Walker.
WFA:	weak field approximation.
CMB:	cosmic microwave background.
MHD:	magnetohydrodynamics.

Chapter 1

Introduction

Astrophysical observations indicate that almost all environments in the Universe are magnetized, and the more we search for extragalactic fields, the more pervading they are revealed to be. The next logical step is to ask why these large-scale magnetic fields are ubiquitous in a Universe as diverse as ours and how did these fields originate? The current properties of these magnetic fields should in principle, reflect their past and give clues to their origins, so we rely on observations to provide the necessary information to answer these questions. Measurements of fields in and around galactic clusters are tentative due to difficulties in determining the electron density of the intergalactic medium. Faraday rotation and Zeeman splitting measurements indicate however, that galactic magnetic fields at high redshifts exist with the roughly the same strength as those found permeating the Milky way. This common property of large scale fields in different galaxies indicates that their origins may be intrinsically connected to cosmological repercussions of the interplay between gravitational and gauge interactions [24]. This suggests that their origins may indeed be primordial, in which case their presence in the very early Universe would have impacted strongly on the geometry of the Universe, as well as the various fundamental processes that sparked their generation [49]. For this reason, the determination of the origin and properties of cosmic magnetic fields is of extreme importance in Cosmology.

This makes magnetogenesis one of the 'hot' topics in current Cosmology, however we have yet to arrive at a well-established theory for their generation with the strengths and scales measured today. The most popular theories include the amplification of a small field by the galactic dynamo and the adiabatic protogalactic collapse at the start of structure formation [49]. Although these mechanisms are shown to yield substantial enhancement, they are not self-sufficient as they presuppose the existence of seed fields. In addition, these seed fields must satisfy certain stringent strength and size criteria in order for the generated fields to agree with the magnitudes observed today. It is therefore, a pertinent problem to provide a mechanism that induces a large enough amplification of a realistically weak *pre-existing* seed field, such that the proposed mechanisms are physically viable.

In this thesis, we investigate the coupling between a large-scale magnetic field and the gravitational wave (GW) spectrum which accompanies most inflationary scenarios. We aim to show that this interaction leads an amplification of the original field, that is sufficient to bring its strength within the limits required for the dynamo mechanism to work. This builds on earlier work by Tsagas *et al.* [98] in which this idea was first introduced within the weak field approximation.

The highly non-linear nature of the Einstein's field equations makes finding exact solutions as well as applying numerical techniques, commonly very complex. In order to solve them analytically, severe symmetry assumptions are often required to simplify the physical models, which then restricts their application [71]. For this reason, we choose to tackle this problem using a perturbative approach, which entails decomposing the real physical Universe into a family of space-times. The properties of the large-scale structure we see today are perturbatively expanded about a fictitious idealized background model, such as a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. Before we can implement perturbation theory in a self-consistent way, we must give due attention to the issue of gauge invariance which has historically plagued such studies [70]. If we consider a large-scale magnetic field residing in the background model as done by Tsagas *et al.* [98], the interaction with linearized gravitational waves manifests as a first-order perturbation of this background. Linear perturbation theory is far simpler because it involves fewer variables and is devoid of mode-mixing. We will see later that magnetic fields do not however satisfy the mathematical requirements for gauge-invariance. This is problematic, as gauge-dependant variables are in general not observable and the physical quantities to which they correspond is unclear [13]. For this reason, we introduce the field as a small vector perturbation on the background spacetime and invoke non-linear perturbation theory in order to study the interaction of interest. Within the context of second-order perturbation theory, we aim to find and solve a closed system of evolution equations for completely gauge-invariant variables which fully characterize the modes induced by the interaction.

In the first part of the analysis, we consider a magnetic field that is on average homogeneous over its coherence scale, as in the work described in [98]. This allows us to directly compare the two studies, which is essential in determining whether the gauge-dependence of their formalism leads to different results. We subsequently consider the more general case where the original magnetic field is inhomogeneous over the same scales in a matter-dominated Universe. The introduction of magnetic spatial gradients at linear order requires a more subtle treatment of the associated spatial currents, necessitating the use of the magnetohydrodynamic approximation (MHD) to provide a framework in order to obtain a tractable solution. By comparing the results of our treatments of homogeneous and inhomogeneous fields, we hope to determine the implications of placing restrictions on the properties of the magnetic field.

We use the 1+3 covariant approach [33, 35, 41, 51] because as it allows Maxwell's equations and Einstein's equations to be written in an intuitive and simple fashion [70]. The outline of this thesis is as follows:

- In chapter (2), Einstein's equations and related identities are discussed.
- Chapter (3) entails a look at the facets of perturbation theory with an indepth study of the gauge problem described in terms of co-ordinates. The gauge problem provides

an incentive to develop the formalism that we adopt in the application. The history of nonlinear perturbation theory is reviewed and finally, a procedure for tackling second-order perturbation problems is outlined.

- Chapter (4) covers the 1+3 covariant approach to Cosmology as well as the treatment of the matter and electromagnetic fields within this approach. The equations governing the dynamics are derived in terms of the defined covariant quantities and FLRW models are introduced.

- We discuss the importance of magnetic fields in Cosmology in chapter (5) together with the current observations and the their accompanying strength constraints. The proposed mechanisms for magnetogenesis are reviewed. We investigate various approximation schemes commonly employed to reconcile the isotropy and homogeneity of the FLRW background and the directional nature of electromagnetic fields.

- In chapter (6) the interaction of gravitational waves and a homogeneous magnetic field, defined as linear perturbations on an FLRW background, is studied for both dust- and radiation-dominated Universes using second-order gauge-invariant perturbation theory. The generated magnetic field solutions are compared with those in Tsagas *et al.* [98].

- In chapter (7), the result for a dust Universe in chapter (6) is recalculated for the case of an inhomogeneous magnetic field, using the MHD approximation.

- In chapter (8) current values of the concomitant quantities are substituted into the magnetic field solutions from chapter (6) and (7) in order to obtain order of magnitude estimates of the resulting amplification.

- In chapter (9) the results from chapter (6) are compared with those from chapter (7) and with the solutions given in Tsagas *et al.* [98].

Chapter 2

General Relativity:

Einstein's incentive for developing the field of Relativity was the inconsistency between Maxwell's theory and Galilean Relativity. Einstein attempted to resolve this discrepancy by generalizing the principles of Galilean relativity to all physics and in so doing, Special Relativity emerged. One of the defining postulates of this theory was the principle of *general covariance*, which states that the general laws of physics should remain the same in all co-ordinate systems. Mathematically, this suggests that the laws of physics should be tensor equations, as tensors are independent of the chosen frame of reference. Relativity also introduced the idea that the spatial and time coordinates are inter-related. The smooth continuous domain which they constitute, will be referred to from here on as the spacetime manifold. A point on the spacetime will be called an event, and the 'distance' (spacelike or timelike) between two events will be referred to as the interval. This first theory is referred to as 'special' because it is limited to bodies moving in the absence of a gravitational field. In 1916 Einstein successfully expanded Special Relativity to include the effect of the gravitational field of a source (mass or energy) on the shape of space and the flow of time.

According to General Relativity (GR), gravity may be interpreted as a manifestation of the curvature of spacetime due to the presence of a mass or energy source. This curvature is quantified by the Riemann tensor, R_{abcd} which describes how the orientation of a vector changes when transported parallel to itself along a curve.

2.1 Ricci identities

The Riemann tensor, being a measure of the spacetime curvature, represents the non-commutivity of the second covariant derivatives. This definition gives rise to the *Ricci identities*. For any 4-vector field X_a, the following holds;

$$R_{abcd}X^d = 2\nabla_{[a}\nabla_{b]}X_c. \tag{2.1}$$

This is loosely analogous to the second derivative of any vector field characterizing the curvature at the point. ¿From (2.1) we see that if the manifold is flat ($R_{abcd} = 0$), any vector that is transported parallel to itself, will return to its starting position unchanged.

We can decompose the Riemann tensor into its trace ($R^a{}_{bac} = R_{bc}$) and trace-free part, referred to as the Weyl tensor C_{abcd};

$$R_{abcd} = C_{abcd} + \frac{1}{2}(R_{ad}g_{bc} - R_{ac}g_{bd} + R_{bc}g_{ad} - R_{bd}g_{ac}) - \frac{1}{6}R(g_{ac}g_{bd} - g_{bc}g_{ad}). \tag{2.2}$$

The long-range curvature

The Weyl tensor C_{abcd} represents the part of the curvature at a point that is not determined locally by a source but by the conditions of the spacetime elsewhere. It equivalently describes the free non-local part of the source's gravitational field that acts over a long range.
It can be decomposed into 'electric' and 'magnetic' trace-free symmetric parts;

$$E_{ab} = C_{acbd}u^c u^d \qquad H_{ab} = \tfrac{1}{2}C_{acst}\epsilon^{st}{}_{bd}u^c u^d. \tag{2.3}$$

E_{ab} represents tidal forces which is a secondary effect of gravity. It arises from a relative acceleration sourced by differences in the gravitational field at different points. The magnetic part H_{ab} has no Newtonian analogue and describes among other things the propagation of oscillations in spacetime (gravitational waves) that are predicted by GR. These two tensors contribute to the Weyl tensor in the following way;

$$C_{abcd} = \left(\epsilon_{abq}\epsilon_{cds} + g_{abpq}g_{cdrs}u^p u^r\right)E^{qs} + \left(\epsilon_{abq}g_{cdrs}u^r + g_{abpq}\epsilon_{cds}u^p\right)H^{qs}, \tag{2.4}$$

where

$$g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}. \tag{2.5}$$

The various symmetries of the curvature tensor

$$R_{abcd} = R_{[ab][cd]} = R_{cdab}, \quad R_{a[bcd]} = 0, \tag{2.6}$$

reduce the number of independent components of this tensor to 20 in 4-dimensions. The tensor C_{abcd} characterizing the conformal properties of spacetime contains 10 of these components, while the remaining 10 are represented by its trace, the Ricci tensor R_{ab}.

Local curvature

The Ricci tensor R_{ab} obtained by contracting the Riemann tensor on the first and third indices ($R^a{}_{cab}$), controls the local gravitational field of a nearby source [95]. The 10 degrees of freedom encompassed by R_{ab} can be expressed in a single dynamical equation describing the properties of the local gravitational field surrounding a source, namely the Einstein equation

$$G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab} = \Lambda g_{ab} + T_{ab}. \tag{2.7}$$

Here Λ is the cosmological constant[1], G_{ab} is the Einstein tensor and T_{ab} is the energy-momentum tensor describing the properties of the source. The contraction of the Ricci tensor is the Ricci scalar $R = R^a{}_a$, which assigns a single real number to each point on

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[1] The cosmological constant is the physical equivalent of the introduction of energy density into a vacuum.

the manifold, characterizing the intrinsic curvature at that point. g_{ab} is referred to as the metric. It is essentially a field which defines the spacetime interval and maps two 4-vectors to reals such that the geometrical properties of spacetime: 'distance' (in space or time), angle and area etc., become measurable quantities. By requiring these quantities to remain unchanged, the metric is forced to be covariantly constant and unique. If we now reconsider the principle of general covariance, we understand that it implies that the metric is the only spacetime quantity on which the general laws of physics can depend.

The Einstein equation (2.7) illustrates the intrinsic relationship between the geometry of spacetime (g_{ab} on LHS) and the matter/energy content (T_{ab} on RHS). The remaining degrees of freedom manifest as the 10 unknown components of the metric g_{ab} and the energy-momentum tensor T_{ab}[2]. If an analytic solution to the Einstein equation is sought, the degrees of freedom contained in (2.7) are expressed as a set of 10 non-linear equations.

2.2 Bianchi identities

We reiterate that coordinate systems are of no significance, in accordance with the invariance of the laws of physics under general coordinate transformations. For this reason, the physics governed by the Einstein equation should only give rise to 6 *independent* equations for determining the 10 elements of the metric tensor [4]. Four degrees of freedom (3 space, 1 time) must remain unspecified so that a co-ordinate system (gauge) can be chosen, from which arbitrary constraints[3] can be extracted and imposed to close the system. In order for the 4 metric coefficients to remain arbitrary, supplementary constraints on the Einstein equation are required. These are the *Bianchi identities* which are derived by taking the covariant derivative of the Riemann tensor

$$\nabla_{[a}R_{bd]de} = 0 \quad \Leftrightarrow \quad \nabla_e R_{abcd} + \nabla_d R_{abec} + \nabla_c R_{abde} = 0. \tag{2.8}$$

Contracting (2.8) twice gives equivalent relations

$$\nabla_a R^a_c = \tfrac{1}{2}\nabla_c R \Rightarrow \nabla^a G_{ac} = 0, \tag{2.9}$$

referred to as the *contracted Bianchi identities.* These relations ensure that only 6 of the 10 equations are independent. We see that the Einstein equation (2.7) is only consistent provided the following holds

$$\nabla^a T_{ac} = 0. \tag{2.10}$$

The Einstein equation (2.7) mathematically encapsulates the equivalence between spacetime and matter and demonstrates how a mass or energy source dictates how space curves, while the geometry of the Universe manipulates matter in 3 directions. The fact that the source affects the very geometry of the space in which it dwells, gives rise to the inherent nonlinearity of this equation. As a result of the inter-dependence of the equations, the Einstein equation cannot always be readily solved and the solutions to (2.7) that are obtained, may not be combined to generate new solutions.

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[2] The symmetric nature of these tensors $g_{(ab)}$, $T_{(ab)}$ removes 6 of their 16 elements.

[3] For example, an FLRW Universe is specified by the imposition of the conditions $g_{00} = 1$ and $g_{0b} = 0$ where $a, b = 0$ is the time dimension and $a, b = 1, 2, 3$ are the spatial dimensions.

Chapter 3

Perturbation Theory and the Gauge problem

The inherent non-linearity of the Einstein equations means that analytical studies are only possible if certain simplifying assumptions about the physical models they describe, are made. However, the exact solutions obtained for these models are commonly too idealized to sufficiently represent the realm of natural phenomena [58]. This is where the relativistic theory of perturbations comes into play. Perturbation theory essentially entails the reformulation of a problem by the addition of a 'small' term to the mathematical description of an exactly solvable problem. In cosmological perturbation theory, we essentially extract approximate solutions to the Einstein equation by treating them as perturbations of some known *exact* solution, referred to as the background [13]. There are various models that function as zeroth-order approximations of the real Universe on large scales (for example, FLRW and Bianchi models) for which the exact Einstein solutions are known. The observed Universe can then be modelled as a spacetime deviating to a small degree from this model.

Mathematically, we perform a perturbative expansion of the matter and kinematic fields in some small parameter quantifying the deviation from the fields in the idealized spacetime. The higher order terms of this expansion become successively smaller and more unimportant. If this expansion is performed to linear order, it can be viewed as splitting the spacetime into two; a 'smooth' fictitious background spacetime $(\mathcal{M}_0, T_0)$ without perturbations (usually FLRW) and a 'lumpy' physical spacetime $(\mathcal{M}, T)^1$ that represents the real inhomogeneous Universe as we experience it ('almost-FLRW'). The perturbations are described in terms of the unique relationship between the two. The perturbative value of a tensor δT is then taken as the difference between the values of a quantity in the background T_0 and its value in the perturbed spacetime T;

$$\delta T \equiv T - T_0. \tag{3.1}$$

However, differential geometry tells us that the comparison of the quantities T and T_0 is only physically meaningful when their values are compared at the same point [72]. In order

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[1] T is the tensor field representing the physical (matter, kinematic) quantities and the geometrical quantities (such as the metric itself).

to extract the perturbation value, we need to specify a map that provides a pairwise identification of points in the two manifolds. This is referred to as a *gauge* Φ and it maps each point in one spacetime to a single corresponding 'image' point in the other spacetime. This mapping characterizes how each point in the idealized background manifold has been disturbed in order to generate the perturbed physical Universe. This gauge-fixing, involves imposing co-ordinate conditions in the two space-times where the gauge choice conventionally refers to a particular choice of co-ordinate system in $\mathcal{M}$. Since co-ordinates in $\mathcal{M}$ correspond to specific chosen co-ordinates in $\mathcal{M}_0$, the co-ordinate choice in $\mathcal{M}$ immediately creates the desired mapping between the two manifolds [37]. We bear in mind that there is nothing intrinsic to the two manifolds that allows us to establish a one-to-one correspondence of the points between them [14]. Since the only imposed restriction is that the 'distance' between the two space-times must be small in some suitable sense [37], there is no reason to choose one mapping over another. This *gauge freedom*[2] is the source of the gauge-problem. We will now look more closely at the process of creating this map.

3.1 Specifying the Gauge Φ

The only measurements that can be made are of the real physical Universe. In reality, we therefore observe the lumpy spacetime $\mathcal{M}$ and choose how to relate the points therein to those in the fictitious spacetime $\mathcal{M}_0$ (i.e. we specify the gauge). This immediately defines the perturbations and their evolutions. Finding the best way to make this correspondence is referred to as the 'fitting problem' for Cosmology [39, 77]. Although the choice of Φ is not unique, there are 4 steps which are generally taken to fully specify the gauge:

(A) We define a family of worldlines $\tilde{\gamma}_0$ in $\mathcal{M}_0$ and a corresponding set $\tilde{\gamma}$ in the spacetime $\mathcal{M}$. It is along these paths that we compare the fluctuations in the quantities in the two manifolds and determine their relative evolutions. The fundamental fluid flow lines are an obvious choice in $\mathcal{M}_0$, and are often the best choice in $\mathcal{M}$. However, in some cases, it may be convenient to consider other choices such as the paths designated by the normals to a chosen set of surfaces.

(B) We define a correspondence between the individual worldlines $\tilde{\gamma}_0^i$ in $\mathcal{M}_0$ and those $\tilde{\gamma}^i$ in $\mathcal{M}$. This assigns each observer in $\mathcal{M}$ to a specific observer in $\mathcal{M}_0$ whose observations we will then compare. In the case where the background model is FLRW or Bianchi, the designation of observers in $\mathcal{M}_0$ becomes insignificant as the spatial homogeneity characteristic of these space-times ensures that the observations of the observers $\tilde{\gamma}_0^i$ will be identical.

(C) We define a family of spatial surfaces $\tilde{\Sigma}_0$ in $\mathcal{M}_0$ and a family of spatial surfaces $\tilde{\Sigma}$ in $\mathcal{M}$. These are 3-spaces that are constant in time in the two manifolds. The logical choice for $\tilde{\Sigma}_0$ in $\mathcal{M}_0$ is the set of surfaces of homogeneity (i.e. where t_0 = constant), which forces the corresponding surfaces $\tilde{\Sigma}$ (t_0 = constant) in $\mathcal{M}$ to be the *idealized* surfaces of constant density (i.e. μ_0 = constant). Essentially, the spatial surfaces where (t = constant) are actually taken to be those 3-spaces in $\mathcal{M}$ where (t_0 = constant). Each spacelike surface

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[2] Here 'gauge-freedom' refers to the freedom available in the choice of mapping.

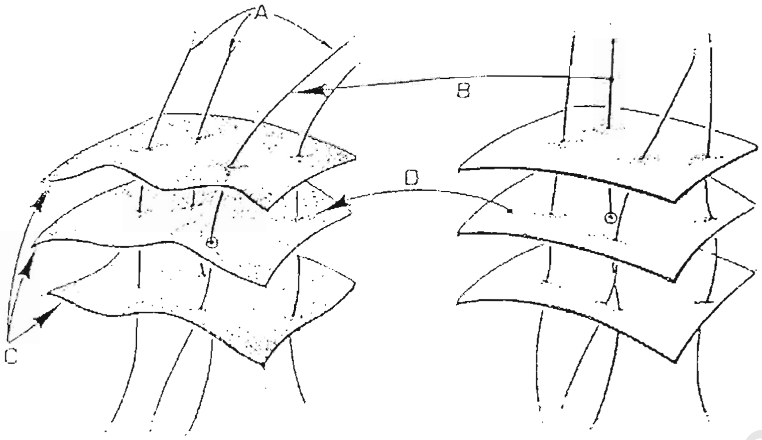

Figure 3.1: The map Φ may be created by making 4 specifications: **(A)** we choose a family of worldlines in each spacetime; **(B)** we specify a unique correspondence between the worldlines in the family in $\mathcal{M}$ and the time lines in the family in $\mathcal{M}_0$; **(C)** we then choose a family of 3-surfaces in each spacetime; **(D)** we specify a particular correspondence between the spacelike surfaces in the family in $\mathcal{M}$ and the spacelike surfaces in the family in $\mathcal{M}_0$ [37].

in the family then corresponds to a particular time value t_0.

(D) We define a correspondence between each particular spatial surface $\tilde{\Sigma}_0^i$ that constitutes the family $\tilde{\Sigma}_0$ in $\mathcal{M}_0$ and each individual spatial surface $\tilde{\Sigma}^i$ in the family $\tilde{\Sigma}$ in $\mathcal{M}$. In doing this, we are effectively assigning a time value t_0 in the background to each event q in the spacetime $\mathcal{M}$. This is critical as this assigns a particular point q in $\mathcal{M}$ to a point in spacetime q_0 in $\mathcal{M}_0$ and completes the specification of the gauge.

In choosing the families of worldlines and 3-surfaces in each spacetime in **(A)** and **(C)**, the intersections of $\tilde{\gamma}$ and $\tilde{\Sigma}$ isolate points in $\mathcal{M}$ and $\mathcal{M}_0$ which are then paired up in a unique manner by our choices of **(B)** and **(D)**.

In this way, we identity the inherent freedom in selecting a gauge that arises from the choice of surfaces in perturbed spacetime and assigning values of quantities to these 3-spaces. This freedom leads to an arbitrariness of certain perturbations such as $\delta\mu$. With $t = t_0$, if we then exercise the gauge freedom in (C) and select spacelike surfaces of constant density in conjunction with choosing ($\mu_0 = \mu$) as allowed by **(D)**, we can specify a gauge where the perturbations in energy density are $\delta\mu = 0$. Similarly, the perturbative values of the energy density and the other quantities can be tailored to any desired values by the choice of a suitable gauge, as illustrated in figure (3.2).

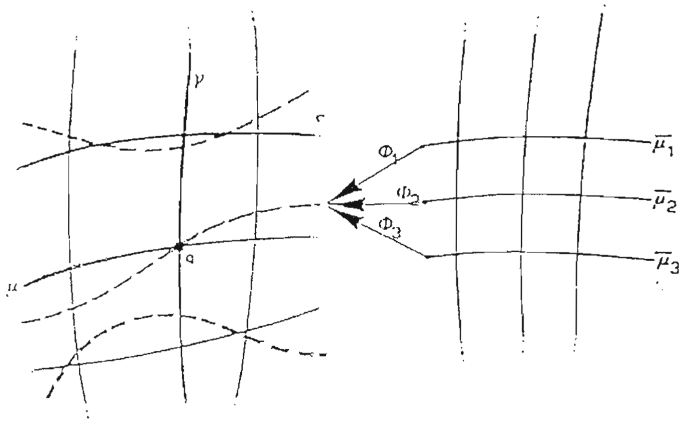

Figure 3.2: Any desired value can be designated to the density perturbation ($\delta\mu = \mu - \mu_0$) at a point q by merely varying the correspondence between surfaces in $\mathcal{M}_0$ and $\mathcal{M}$ in **(D)** [37].

3.2 The Gauge Transformation

It has been shown that definitions of certain perturbations depend critically on the choice of mapping and will therefore, be affected when this correspondence is altered. We now consider the operation that is used to change the mapping which is referred to as a *gauge transformation*. It is represented by an infinitesimal co-ordinate transformation, given by

$$x_0^a \rightarrow x_0^{a\prime} = x_0^a - \epsilon^a(x); \tag{3.2}$$

where $\epsilon^a(x)$ is some arbitrary infinitesimal vector field. This operation can be thought of as merely relabelling the co-ordinates in $\mathcal{M}_0$. The key point is that such a transformation induces a translation of the co-ordinates in the background manifold while the physical spacetime is kept fixed. The operation effectively changes the point in the background $\mathcal{M}_0$ that corresponds to a point in the physical spacetime $\mathcal{M}$. The points in $\mathcal{M}_0$ now correspond to co-ordinates that may have different characteristics in the perturbed spacetime. We use the standard transformation rules to see how an arbitrary tensor on the background spacetime is changed to first-order in ϵ [27];

$$T_0'(x) = T_0(x) + \mathcal{L}_\epsilon T_0(x)\ . \tag{3.3}$$

Here $\mathcal{L}_\epsilon T(x)$ is the Lie derivative of the background tensor field T_0 along the vector field ϵ. For scalars, vectors and tensors, this is

$$\mathcal{L}_\epsilon f = \nabla_a f \epsilon^a, \tag{3.4}$$

$$\mathcal{L}_\epsilon V_a = \nabla_b V_a \epsilon^b + V^b \nabla_b \epsilon_a, \tag{3.5}$$

$$\mathcal{L}_\epsilon T_{ab} = \nabla_c T_{ab} \epsilon^c + T_{ac} \nabla_b \epsilon^c + T_{cb} \nabla_a \epsilon^c. \tag{3.6}$$

We can infer that the value of the perturbation in a quantity which has a non-zero background value and is a function of its co-ordinate position in the background, is affected by this same operation. Using (3.3), we can calculate how the perturbation changes under a gauge transformation

$$\delta T'(x) = T(x) - T_0'(x) = (T(x) - T_0(x)) - \mathcal{L}_\epsilon T_0(x) = \delta T(x) - \mathcal{L}_\epsilon T_0(x)\ . \qquad (3.7)$$

Perturbations which are affected in this manner are termed gauge-dependant. This dependance is the source of the arbitrariness of certain perturbations (such as $\delta\mu$ described in (3.1)) which is central to the gauge problem.

Gauge-dependant variables become problematic when switching between gauges, because perturbations defined in one gauge Φ_1 which are affected by gauge transformations, will not necessarily correspond to variables in another Φ_2. If the gauge Φ is only partially specified (i.e. **(A)**–**(D)** given in section (3.1) are not specified), the residual freedom prohibits the reconstruction of the map solely from observations of the real Universe $\mathcal{M}$. The gauge-dependant perturbations will then not be observable, following from the principle of general covariance [20]. Leaving some residual freedom may also allow physically different points in the background $\mathcal{M}_0$ to be mapped to each other as the limit ($\mathcal{M} = \mathcal{M}_0$) is approached [18]. This gives rise to spurious evolutionary modes of the linearized perturbations which are inherently unphysical. There are 3 ways of approaching the problem in order to side-step such unsatisfactory situations [96]:

- We can initially choose an appropriate gauge, specify it completely (i.e. specify **(A)**-**(D)** completely) and work within it throughout the treatment. The gauge-dependant perturbations, such as $\delta\mu$, pose no problem here. This may however be impractical as different situations can be greatly simplified by a suitable choice of gauge.
- The second approach was first explored by Lifshitz who acknowledged the gauge freedom present in the synchronous gauge[3], kept tabs on it and removed the spurious gauge modes that arose from it. The results must then be transformed into gauge-invariant quantities in order to expose their physical meaning. Although this approach is fundamentally valid, it can become complicated and has in the past been littered with errors [82].
- The third means of avoiding the problematic gauge modes is to establish a complete description of the perturbations in terms of variables that are independent of the mapping and thus do not change under such transformations. These objects are referred to as *gauge-invariant*. Taking equation (3.7) and the definitions of Lie derivatives given in (3.4)-(3.6) into account, Stewart and Walker deduced a clear-cut definition of such quantities [88]:
  Lemma: The linear perturbation δT of a field T_0 on the background spacetime $\mathcal{M}_0$ is said to be gauge-invariant (GI) if and only if it fulfills at least one of the following criteria:

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[3] The synchronous gauge is the co-ordinate system where the proper time is the same as the co-ordinate time and constant spatial co-ordinates are orthogonal to constant time hypersurfaces. This is the natural frame of free-falling observers.

> i.) T_0 vanishes;
> ii.) T_0 is a constant scalar;
> iii.) T_0 is a linear combination of Kronecker deltas with constant co-efficients.
>
> This lemma makes sense if we consider how a gauge transformation alters a field T, given in equation (3.7). When the co-ordinate in the background $\mathcal{M}_0$ corresponding to a point in $\mathcal{M}$ is changed under a gauge transformation, the value of a perturbation of a quantity remains invariant because the background value of that quantity is the same everywhere (i.e. zero as in (i.) or constant as in (ii.)).
> Mathematically the above stated conditions ensure that the Lie derivative of the field T_0 vanishes for all vector fields ϵ on $\mathcal{M}$. From (3.3) it is evident that if these conditions are satisfied, the field T on $\mathcal{M}$ and hence the perturbation remains unchanged by the gauge transformation.

The refinement of an approach to tackle the GI dilemma was achieved over a long period of time and had various contributors. Hawking (1966) [51] was the first to develop an entirely covariant formalism by extending the previous work by Ehlers (1961) [33], that looked at perturbations of the curvature rather than of the metric. It was fully GI in terms of the tensor perturbations, however, it suffered from the same problems in that the variable $\frac{\delta\mu}{\mu}$ that he used in his analysis of density perturbations depended on the chosen gauge. Olson (1976) [77] managed to identify and remove most of these spurious gauge modes but could not eliminate *all* interpretation ambiguities born out of the gauge-dependant definition of the density perturbations. Bardeen (1980) [2] recognized the need to develop an approach where gauge-invariant variables are employed throughout as dynamical degrees of freedom [18] and made the first presentation of a complete set of GI variables that fully described the linear perturbed spacetime. These metric and matter variables comprised of appropriate linear combinations of gauge-dependant perturbations such that the Stewart and Walker GI condition is satisfied. The aspect of this approach that is deemed unsatisfactory is that, by their construction, the variables are defined with respect to a particular co-ordinate system and their physical and geometrical meanings become clear only once a specific hypersurface condition (i.e a time gauge) is specified [11]. Within his approach Bardeen also employs a non-local decomposition of variables into scalar, vector and tensor parts. The variables are thus, only gauge-invariant under a limited set of gauge transformations which acknowledge this splitting. His formalism is also a linear theory and therefore, only applicable to small deviations from the FLRW symmetry. Ellis and Bruni (1989) [37] recognized the potential for a significantly improved approach that is based on the fully covariant formalism presented by Hawking and Olson but where the choice of variable representing the density perturbations is guided by the Stewart and Walker lemma. The covariant definition of the variables ensures that their correlation to physically and geometrically significant quantities is immediately transparent and their exact presentation gives them meaning in any spacetime. Most importantly, they are chosen such that their background values vanish, and are thus GI by definition. The advantage of this approach is that the scalar, vector and tensor perturbations are treated in a unified manner; no decomposition is made, but it can be performed at a later stage. Since no linearization is performed when defining their covariant quantities, the exact evolution equations can be found and then linearized with respect to a chosen background later on.

We choose to move away from the standard approach, which follows the evolution of perturbations defined between observers in $\mathcal{M}_0$ and $\mathcal{M}$ Universes, related by a specific gauge. Instead of fixing the gauge freedom, we will adopt the Ellis and Bruni approach (which is generic to all reference frames) and extract only the GI part of the perturbations [96]. In this approach the fluctuations are defined with respect to two adjacent observers within the same perturbed Universe $\mathcal{M}$.

3.3 Non-linear perturbation theory

As described in the first part of chapter (3), perturbation theory is one method that can used to solve the Einstein equations and involves perturbatively expanding them around a background model. The extent to which the expansion is performed is determined by the phenomena one wishes to probe. We estimate the order of the terms that will yield physically relevant measures of inhomogeneity and anisotropy of the real Universe [76] and eliminate quantities of higher order. In linear perturbation theory, the expansion of the kinematic, gravitational and matter quantities typically stops at first-order, however the fluctuations in the observables are then limited to small deviations from those in the background model. In order to determine when higher order perturbations become significant and must be dropped, it is common practice to compare the magnitudes of linearized variables to background quantities such as the energy density μ. For example, in the Eulerian approach [65, 81] to density perturbations, the concomitant equations are expressed in terms of the density contrast

$$\delta\mu(x,t) = \left(\mu(x,t) - \bar{\mu}\right)/\bar{\mu}. \tag{3.8}$$

This variable can be perturbatively expanded, with solutions that are assumed to be suitably handled by linear theory provided ($\delta\mu(x,t) \ll 1$). However this linear approximation only probes the early stages of gravitational collapse; the density contrast of most structures in the Universe significantly exceeds unity [40].

Alternatively, a Lagrangian approach may be adopted in which the variable that is chosen to follow the evolution of the density fluctuations is the displacement of fluid element from its starting position [90]. Zel'dovich introduced the idea that a universal time function $F(t)$ can be used to translate the final Eulerian position x to the initial Lagrangian position q [22];

$$x(q,t) = q + F(t)\Psi(q), \tag{3.9}$$

where $\Psi(q)$ is the time-independent function of the Lagrangian density field. He also proposed a linear Lagrangian approximation for dust called the Zeldovich approximation [87, 105] in which the time function is approximated to first-order by the growth factor $D(t)$ of the linear modes. The key aspect of the Zeldovich approximation is that a small perturbation in Lagrangian fluid element paths contains nonlinear information concerning the corresponding Eulerian quantities [85]. The displacements grow with the same rate $D(t)$ everywhere, however the growth rate of the density perturbations which they represent, vary with position. This means that displacements remains finite for all time, including those where $\delta\mu(x,t) \to \infty$. This linear theory can therefore be used to probe non-linear structure formation, however it is limited to the quasi-linear regime. Unfortunately there is no general rigorous procedure to establish when non-linearities *need* to be taken into account to avoid skipping over important physical effects [20]. To improve the accuracy of our models, we need to retain the higher order terms in the perturbative expansion of the solutions to the Lagrangian displacement.

The pioneer in this regard was Buchert who extended previous work using the Lagrangian approximation within the context of Newtonian theory, to second [15] and third [16] order. Approaches which are based on Newtonian dynamics are only valid on small scales; a relativistic approach is necessary to accurately model phenomena occurring on super-horizon scales. Lifshitz (see above) was the first to present a linear theory of perturbations with its basis in general relativity. Tomita (1967) [92] built on this work and calculated the second-order terms in the evolution of the scalar perturbations in an Einstein-de Sitte Universe within the synchronous gauge, one of the most widely used gauges in cosmological perturbation theory. Since the degrees of freedom in this gauge are not all fixed, care must be taken to monitor any fictitious modes that emerge.

Materesse, Pantano and Saez [74, 75] recalculated this result using synchronous and comoving coordinates within a relativistic framework. The dynamics are monitored via the observable fluid variables (density, shear and vorticity) and tensors describing the curvature, with the fluid flow equations expressed in Lagrangian form. The problem is simplified by considering an irrotational fluid where signal exchange by the gravitational field is neglected (i.e. terms containing H_{ab} are dropped). Russ *et. al.* [83] extended the above using a tetrad formalism, which is necessary for the implementation of the relativistic Zeldovich approximation, and included the second-order terms that arose from the mixing of growing and decaying linear scalar modes.

Salopek, Stewart and Croudace [84] derived a relativistic non-linear Hamilton-Jacobi[4] equation to simplify the calculation of the higher order corrections to the Zeldovich approximation. Its solution was sought in terms of a spatial gradient expansion, which is advantageous as it allows the concomitant equations to be constructed such that they are invariant under transformations of the spatial co-ordinates. In doing so, they calculated the second-order metric perturbations in the comoving synchronous gauge.

Tomita [93] investigated the inclusion of tensor and vector modes at linear order as seeds for all 3 types of perturbations at second-order.

In [72], Matarrese and co-authors followed the evolution of the perturbations in an Einstein-de Sitte Universe, up to second-order within the synchronous gauge. Generalizing the condition of GI at first-order to second-order, GI metric variables were defined, enabling a second-order gauge transformation to then be performed on the results to obtain the metric perturbations in the Poisson gauge. Within both co-ordinate settings, mode-mixing was identified.

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[4] Hamiltonian-Jacobi theory is widely used in the separation of variables.

Bruni and co-authors have worked extensively on shaping the definition of gauge invariance to an arbitrary order n, as well as generating formulae for the gauge transformations at higher order. In [13], the explicit rules for second- and third-order are presented, using the transformation of tensors between the synchronous and Poisson gauges as an example. The gauge dependence of non-linear perturbations based on two parameters was explored in [12, 89], leading to the derivation of the explicit higher-order gauge transformation rules and a rigorous definition of gauge invariance.

It is evident that the literature in non-linear perturbation theory is limited and the calculations in the bulk of the work are performed within a specific gauge, chosen to simplify the interpretation and evolution of the variables. Since the gauge is completely specified, the perturbations become real geometrical objects and the issue of gauge invariance does not feature. In recent work by Clarkson [20], a semi-rigorous methodology for the treatment of perturbations up to second-order within a 1+3 covariant frame work was presented. Using the procedure to find second-order (two parameter) GI matter quantities set up in [76], Clarkson considered the sourcing of tensorial gravitational waves by scalar density perturbations and visa-versa. This is performed under the assumption that an analogous method for linear perturbations exists. For our application, we choose to employ this methodology as a basis to construct a set of second-order covariant GI quantities describing the magnetic field-GW interaction in an 'almost-almost' FLRW Universe.

Recipe for second-order covariant perturbation theory

In what follows, we outline the general procedure for second-order perturbation theory using our own perturbation scheme to illustrate the method.

- We start by choosing an appropriate background spacetime (FLRW in our case). From the properties of this spacetime we extract those quantities which are non-zero and thus zeroth order.
- We perform the standard linearization procedure about this chosen background spacetime. This is relatively simple and entails merely choosing which types of perturbations to excite at this level. The point is that one can choose the perturbation scheme according to the effects one wishes to examine. In our case we distinguish between two linear space-times to describe small deviations from FLRW background that are sourced by magnetic field and GWs separately[5]. In the first 'almost FLRW' spacetime, only the magnetic field exists. It obeys Maxwell's equations and is therefore divergence-free at linear order. This spacetime may thus be regarded as a class of vector perturbations on the FLRW background. In the second 'almost FLRW' spacetime, the perturbations describing the gravitational waves are limited to purely tensorial ones. This choice of first-order variables is appropriate as the isotropic nature and conformal flatness of FLRW spacetime forces these variables to immediately vanish in the background and in so doing, makes them manifestly GI. Our treatment is sim-

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[5] The choice of the perturbative scheme described here, will be motivated at a later stage in section (5.4)

plified by this limited choice of perturbations but the more generic the first-order perturbations are chosen to be, the more complex the treatment becomes.

- The inherent problem with using the covariant approach to non-linear order is that the gauge transformations at second-order and higher become extremely convoluted and performing these operations throughout the wide range of existing gauges would be far too cumbersome. The alternative is to establish a set of second-order gauge-invariant variables that fully encompass the effects one wishes to study. Although simpler, this is also a non-trivial task. By definition, a second-order gauge-invariant variable is one that vanishes in *both* the background and the first-order space-times [13]. Again basing our choice on the physical effects we wish to study, we identify a set of these second-order GI (SOGI) variables (being either objects entering only at $\mathcal{O}(2)$ or being constructed from zeroth and first-order quantities) and restate the relevant equations entirely in terms of these SOGIs. The appeal of using linear perturbation theory is that the separate scalar, vector and tensor modes evolve independently of each other. However at higher order we encounter mode-coupling, where the different perturbations may affect each others' evolutions. We choose to ignore one of these couplings, namely the back-reaction of the magnetic field on the GWs.
- The equations governing the evolution of the main SOGIs are finally derived. In order to arrive at a closed set of equations, it may be necessary to find the evolution equations of auxiliary SOGIs that arise in the main equations. This is performed in an iterative manner. By converting all multiples of first-order quantities into new SOGIs, the system appears in the form of a set of *linear* differential equations which makes finding their solutions simpler [20].

Chapter 4

The Covariant approach

The ambiguities born out of the interpretation of gauge-dependant quantities led Hawking [51] to develop a perturbative approach based on the work presented in Ehlers' seminal paper [33], in which *covariant* variables which take the same form in all co-ordinate systems, were introduced. Ellis extended and developed Ehlers' approach and performed the unique splitting given in [33] on the exact covariant fluid equations given in [51], into their components in 3-dimensional space and time. This formalism was systematically presented in the famous *Cargèse lectures* [35] and is referred to as the '1+3' covariant approach. In this approach to GR, the metric itself can not provide a covariant description and therefore, does not constitute the fundamental variables. Instead, the kinematic quantities of the fluid: the energy density, pressure, gravito-electric and magnetic quantities are the observables whose perturbations fully describe the inhomogeneity and anisotropy of the real Universe [68]. We will now review the details of the 1+3 covariant formalism that was used to formulate the GI approach to perturbations by Ellis and Bruni [37].

In order to perform this 1+3 decomposition, we need to introduce a universal reference velocity field u^a. This chosen field will form the basis of a reference frame relative to which all motion is defined and quantified. In accordance with the observed average recession of the galaxies, we assume that the matter in the Universe has a locally well-defined preferred motion that can be represented by a unique 4-velocity vector field u^a, satisfying $u_a u^a = -1$. Based on the Copernican principle, we can assume that this holds at each point in the Universe. We then introduce a family of observers, called the *fundamental observers*, travelling such that this field represents the congruence of their worldlines[1]. In so doing, any observations made are those relative to this preferred velocity. Since this timelike unit vector is tangent to the fundamental worldline curve, it is defined in terms of local co-ordinates by

$$u^a = \frac{dx^a}{d\tau}, \tag{4.1}$$

where τ is the *proper* time measured by the fundamental clock. In choosing this field to coincide with the average velocity of the matter in the Universe, it acquires an 'invariant significant' such that the covariant quantities at every point which are defined with respect

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[1] A worldline is a timelike curve which represents the history of a fixed point y^α in space through time.

to u^a, can be decomposed uniquely [35]. The 1+3 split is suitable in any spacetime that has a preferred timelike flow such as cosmological models.

Frame choice:

When adopting the viewpoint of these observers, our frame choice is referred to as the fundamental frame[2]. We are at liberty to adopt any frame through the re-definition of a velocity field u^a. Gauge invariance however, requires us to choose a velocity that reduces to the 4-velocity of the fundamental observers in the FLRW limit [18]. This velocity is then used in the definition of the covariant variables, which are picked out to describe the perturbations as seen by observers comoving with this velocity. Given that the values of the variables depend on our choice of u^a, the reference can be chosen so as to eliminate certain parts of the inhomogeneity and anisotropy. For example, we can choose u^a such that the perturbation describing the acceleration $\dot{u}^a$ vanishes, referred to as the geodesic frame. The above stated restriction on u^a however, ensures that this frame-dependance does not introduce similar ambiguities to those encountered from the freedom in the gauge choice. Since the velocity used in their definition reduces to the same in the background, the variables are entirely GI.

Comoving Co-ordinates

In certain instances, it is convenient to fix the spatial co-ordinates of the events in the spacetime We attach the spatial reference frame to the average positions of the galaxies such that the fundamental observers are at rest with respect to the local matter distribution. In terms of the manifold, we are in effect intersecting the fluid flow lines with local surfaces (τ = constant) and labelling the worldline with the spatial co-ordinates of the point of intersection. This intersection point will then evolve through time but will suffer no change in its co-ordinates (y^1, y^2, y^3) relative to the reference frame. The 3-subspaces with (τ = constant) are dragged along the worldlines onto surfaces with (τ' = constant). The comoving co-ordinates are given by $x^a = (\tau, y^\alpha)$ where τ is the proper time measured along the fluid flow lines from the initial surface S.

4.1 Projection tensors:

Given that this unique timelike velocity exists at each point, the other component of the spacetime at each point must be a preferred instantaneous rest-space that is constant in time. This is *locally* perpendicular to u^a. We can then define a metric g_{ab} for the spacetime that decomposes into parts describing the projection of the vector-space tangent to each point, onto these 3D hypersurfaces and the 1D timelike space itself $U^{ab} = -u^a u^b$ [33]:

$$g_{ab} = h_{ab} + U_{ab} \tag{4.2}$$

where

$$\nabla^c g_{ab} = 0, \quad g^a{}_a = 4. \tag{4.3}$$

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[2] The fundamental frame must be distinguished from the fluid frame, which refers to our frame of reference other than the fundamental one.

The projection tensor h_{ab} has the following properties

$$h^a{}_a = g^a{}_a + u^a u_a = 3 \quad h^a{}_b h^b{}_c = h^a{}_c \quad h_{ab}u^a = 0. \tag{4.4}$$

It is important to note the use of the term *hypersurfaces*. These sub-surfaces divide manifolds into disconnected bulk regions and are only locally orthogonal to u^a. Only in a non-rotating Universe, do these rest-spaces merge to form a surface lying perpendicular to u^a. In this case h_{ab} becomes the metric of the spatial sections [36]. The operation of h_{ab} on any vector X^a extracts the spatial component orthogonal to u^a;

$$X^a_\perp = h^a{}_b X^b \quad X^a_\perp u_a = 0. \tag{4.5}$$

We define the pseudotensor ϵ_{abcd} with the properties

$$\epsilon_{abcd} = \epsilon_{[abcd]}, \quad \epsilon_{0123} = (-det g_{ab})^{-1/2}, \tag{4.6}$$

and the corresponding projected tensor by

$$\epsilon_{abcd}u^d = \epsilon_{abc}. \tag{4.7}$$

Since the contraction of a skew tensor with a symmetric one is zero, it is evident that all contractions of ϵ_{abc} with the 4-velocity are identically zero, implying that ϵ_{abc} is purely spatial. It is therefore interpreted as the effective volume element in the 3-space of the comoving observers.

We can use this universally defined velocity field u^a to define the covariant derivative of a quantity $\nabla_a T_{bc}$. This can then be split in the same manner into a comoving spatial derivative acting on projected quantities

$$D_a T_{bc} = h^s{}_a h^t{}_b h^u{}_c \nabla_s T_{tu} \tag{4.8}$$

and the comoving timelike derivative along the flow lines parallel to u^a

$$\dot{T}_{a...b} = \nabla_c T_{a...b} u^c. \tag{4.9}$$

The projection of a vector into the 3-space of the observer retains only the spatial part

$$V_{<a>} = h_{ab}V^b, \tag{4.10}$$

while the projection of a second rank tensor T_{ab} decomposes it into its irreducible trace, antisymmetric and projected symmetric trace-free (PSTF) components

$$T_{ab} = \frac{1}{3}T h_{ab} + \epsilon_{abc}T^c + T_{<ab>}, \tag{4.11}$$

where

$$T = h_{ab}T^{ab}; \quad T_a = \frac{1}{2}\epsilon_{abc}T^{bc}; \quad T_{<ab>} = \left[h_{(ac}h_{b)d} - \tfrac{1}{3}h_{ab}h^{cd}\right]T^{cd}. \tag{4.12}$$

The decomposition of the variables into their irreducible parts in this manner forms the underlying basis of the 1+3 covariant approach.

4.2 Kinematics

This section closely follows the discussion in [35].

The kinematics are extracted by decomposing the covariant derivative of 4-velocity vector into its components parallel and orthogonal to u^a

$$\nabla_b u_a = v_{ab} - u_a \dot{u}_b, \tag{4.13}$$

where $\dot{u}^b$ is the acceleration vector. This vector vanishes provided the flow lines describe the motion of the particles when they are unaffected by any force other than gravity or inertia (geodesics). The spacelike field v_{ab} defines the change in the preferred 4-velocity on the 3-space of the observer and effectively describes the relative velocities of the adjacent observers.

This quantity can be further split into its symmetric and anti-symmetric components:

$$v_{ab} = \Omega_{ab} + \omega_{ab}, \tag{4.14}$$

where Ω_{ab} is referred to as the expansion tensor and $\omega_{ab} \equiv \omega_{[ab]} = D_{[a}u_{b]}$ is the vorticity tensor. The decomposition of the symmetric part into its trace and PSTF parts gives

$$\Omega_{ab} = \sigma_{ab} + \tfrac{1}{3}h_{ab}\Theta, \tag{4.15}$$

where $\Theta = \Theta^a{}_a = \nabla^a \dot{u}_a$ is the expansion scalar. The quantity $\sigma_{ab} \equiv \mathrm{D}_{<a}u_{b>}$ is referred to as the shear. The magnitudes of the shear and vorticity are given by

$$\sigma^2 = \frac{1}{2}\sigma^{ab}\sigma_{ab}, \quad \omega^2 = \frac{1}{2}\omega^{ab}\omega_{ab}. \tag{4.16}$$

Definitions of additional variables are needed to expose the physical meanings of above kinematic quantities characterizing the dynamics of the Universe. We start with the definition of a vector which relates the motion of two neighbouring particles along worldlines throughout time. Firstly, we choose a curve y^α (i.e. τ = constant) that connects two distinct worldlines. In order to ensure that this curve connects the same two points throughout its progression with the worldlines, we invoke the use of comoving co-ordinates defined in (4). If the spatial positions of the two particles are described by y^v and $(y^v + \delta y^v)$, then the vector joining them will have a time co-ordinate of 0 and position δy^v at all times. This is called the connecting vector and is generally defined by the relation

$$X^a = \frac{\partial x^a}{\partial y^v}\delta y^v. \tag{4.17}$$

Using the identity

$$(\dot{X^a}) = (\nabla_b X^a) u^b = \nabla_b u^a X^b, \tag{4.18}$$

its time derivative can be expressed in terms of the relative velocity vector defined above

$$(\dot{X^a}) = (v^a{}_b - \dot{u}^a u_b) X^b. \tag{4.19}$$

Although the temporal component of the connecting vector X^a is 0, it still has a time part and does not in general lie perpendicular to the preferred velocity u^a (i.e. in some sense it may be tilted). In order to define an entirely spatial vector, we must project X^a into the rest space of the observer by multiplying by $h^a{}_b$. The resulting vector

$$X_\perp^a = h^a{}_b X^b \tag{4.20}$$

is referred to as the *relative position vector* and gives the displacement of the two observers in space only. The time derivative of the connecting vector then represents the rate at which the spatial displacement varies and thus yields a spacelike vector describing the observers' differential velocity

$$V_b = h^a{}_b \dot{(X_\perp^a)}. \tag{4.21}$$

If we then apply (4.18) to the relative position vector

$$\dot{(X_\perp^a)} = \nabla_b u^a X_\perp^b, \tag{4.22}$$

and multiply by $h_a{}^b$, we obtain

$$h_a{}^c \dot{(X_\perp^a)} = V^c = v_b{}^c X_\perp^b, \tag{4.23}$$

since v_{ab} is the spatial part of the $\nabla_b u_a$. Being a vector, $X_\perp^a$ can be expressed in terms of a magnitude and a unit direction within the 3-space

$$X_\perp^a = \delta L \; e^a \tag{4.24}$$

where $e^a e_a = 0$, $e^a u_a = 0$ and $\delta L = (h_{ab} X^a X^b)^{\frac{1}{2}}$.

Taking the time derivative of the relative distance gives

$$\dot{(\delta L)} = \left((X_\perp^b X_b)^{\frac{1}{2}} \right)^{\cdot} = \frac{1}{2} \frac{\dot{(X_\perp^b)}(X_b)}{(\delta L)} + \frac{1}{2} \frac{\dot{(X_b)}(X_\perp^b)}{(\delta L)}. \tag{4.25}$$

Using (4.19), (4.21) and (4.23), we obtain

$$\begin{aligned} \dot{(\delta L)} &= \frac{1}{2} \frac{V_a (X_b)}{(\delta L)} + \frac{1}{2} \frac{((v_{ab} - \dot{u}_a u_b) X^a X_\perp^b)}{(\delta L)} && (4.26) \\ &= \frac{1}{2} \frac{v_{ab} X^b (X_\perp^a)}{(\delta L)} + \frac{1}{2} \frac{v_{ab} X^a X_\perp^b}{(\delta L)}. && (4.27) \end{aligned}$$

If we substitute for the definition of $X_\perp^a$, we arrive at

$$\dot{(\delta L)} = \frac{v_{ab} (\delta L)^2 e^a e^b}{(\delta L)}. \tag{4.28}$$

Since $e^a e^b$ is symmetric, we retain only the symmetric variables constituting v_{ab} and arrive at

$$\frac{\dot{(\delta L)}}{(\delta L)} = v_{ab} e^a e^b = \Theta_{ab} e^a e^b = \frac{1}{3} \Theta + \sigma_{ab} e^a e^b, \tag{4.29}$$

which is the *generalized Hubble law.* This indicates that the rate at which the distance between points changes $\dot{(\delta L)}$ is proportional to their relative distance, where the ratio of proportionality depends on direction (since it contains the term $\sigma_{ab}e^a e^b$) [36]. Using the same definitions, it can be shown that the rate of direction change is

$$h_a{}^b \dot{(e_b)} = \left(\omega_a{}^b + \sigma_a{}^b - h_a{}^b \sigma_{fg}\, e^f\, e^g\right) e_b. \tag{4.30}$$

This reveals the effects of the above defined kinematical quantities on the direction and magnitude of the relative motion.

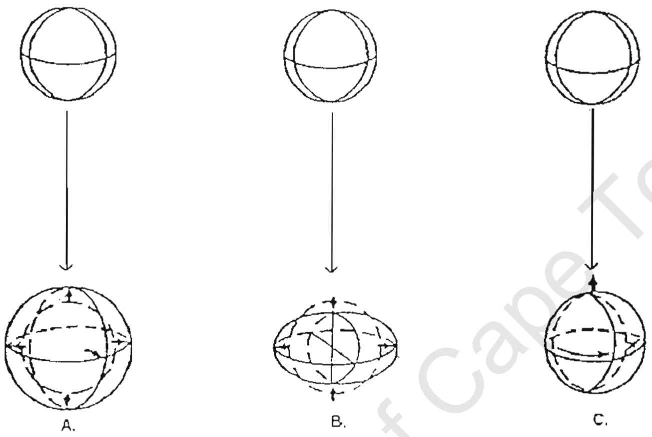

Figure 4.1: **(A.)** The action of Θ alone on a sphere is shown to produce another sphere with a different volume while preserving its orientation. **(B.)** The action of σ_{ab} alone is shown to distort a sphere into an ellipsoid with the same volume, without changing the direction of the principle axes of the shear. **(C.)** The action of ω_a alone is shown to give rise to a rotation of the sphere about a fixed axis [27].

The Expansion scalar

If we consider only the contribution of Θ in (4.29), we see that a positive expansion will lead to an increase in the relative distances of particles by

$$\dot{(\delta L)} = \frac{1}{3}\Theta \delta L. \tag{4.31}$$

Since

$$\dot{e}_a = 0, \tag{4.32}$$

we infer that this variable does not affect the orientation of the particles in relation to each other. Since it is the trace of v_{ab}, it represents the rate of change of the relative velocities of particles in the 3-space in the same direction as the change and quantifies the local rate of isotropic expansion or contraction. It is convenient to define a length scale $a(t)$ by the relation

$$\frac{\dot{a}}{a} = \frac{1}{3}\Theta. \tag{4.33}$$

We can now describe the change in volume V along the flow line as $\frac{\dot{V}}{V} = \frac{3\dot{a}}{a}$.

The Shear

Eliminating all non-shear induced contributions to equations (4.29) and (4.30), we obtain

$$\dot{(\delta L)} = \sigma_{ab}e^a e^b \delta L, \quad \dot{e}_a = \sigma_a{}^b e_b - \sigma_{cd}e^c e^d e_a. \tag{4.34}$$

If we select an orthonormal basis of its eigenvectors, the 3 non-zero components of the shear are its diagonal elements. Its trace-free character requires

$$\sigma_1 + \sigma_2 + \sigma_3 = 0. \tag{4.35}$$

If one component of σ_{ab} is positive (in accordance with an expansion in that direction $\sigma_1 > 0$), then the constraint (4.35) requires at least one to be negative ($\sigma_2 < 0$), yielding contraction in that perpendicular direction. The action of the shear can then be viewed as distorting a spherical space into an ellipsoid without a change of volume (being trace-free $\sigma^a{}_a = 0$). From the equations, we see that shear induces a change in the distance between particles along its principle axis $\sigma_v = \sigma_{ab}e^a e^b$. Only the particles positioned in an eigendirection of the shear will remain unaffected, although the cumulative change in direction will be zero on average.

The Vorticity tensor

Setting $\sigma_{ab} = \Theta_{ab} = 0$ in equations (4.29) and (4.30), we obtain

$$\dot{(\delta L)} = 0, \quad \dot{e}_a = \omega_a{}^b e_b. \tag{4.36}$$

We can infer that particles will remain at fixed positions relative to each other over time but suffer a change in their orientation for non-zero ω_{ab}. To illustrate its action, we define the corresponding vorticity vector ω_a which represents the 3 components of ω_{ab} (as implied by its antisymmetric nature [74])

$$\omega_a = \frac{1}{2}\epsilon_{abcd}u^b\omega^{cd}. \tag{4.37}$$

Since $\omega_{ab}u^b = 0$ we see from (4.37) that $\omega^a\omega_{ab} = 0$. This indicates that ω_a is an eigenvector of the tensor with a zero eigenvalue. This implies that if e^b in the above equation is aligned with the vector ω_a, then no change in direction is registered. Since its action preserves all distances and changes the relative particle orientations in all directions except one, the vorticity induces a pure rotation in the particle positions with respect to a *locally* inertial frame. ω_a represents the axis of this rotation. If $\omega_{ab} = 0$, there is no local rotation and the rest spaces of the particles mesh together to form a spacelike surface normal to u^a, as mentioned earlier.

4.3 Matter

It is standard practice to treat the entire cosmic substratum as a fluid and look to relativistic fluid dynamics for an appropriate description. All information concerning the densities that characterize the matter field is contained within the symmetric tensor T^{ab}, referred to as the energy-momentum tensor (EMT). We will define the EMT in terms of its components in some arbitrary frame.

The momentum density in the a direction has a flux (time rate of flow) across a surface with normals pointing in the b direction which we denote T^{ab}, where $(a, b = 0,\ 1,\ 2,\ 3)$ [28]. Given that the 0 index conventionally denotes the time co-ordinate, the elements T^{a0} are interpreted as the flux per unit volume of the a^{th} component of the 4-momentum across a surface of constant time. T^{0b} represents the rate of flow of the 0-momentum component (energy density) across the spatial surface of constant x^b. By definition the T^{00} component is entirely non-dynamic and thus characterizes the relativistic energy density of the matter field. Since the energy flux density and the momentum density are both essentially the products of mass density and net velocity, T^{a0} is required to be symmetric (i.e. $T^{a0} = T^{0b}$). This illustrates the interchangibility of mass and relativistic energy.

The remaining diagonal elements can be interpreted as the flux per unit volume moving in the same direction as the force (momentum density) inducing it. By definition, they represent the pressure. In the same regard, the off-diagonal elements encode the flux density of the momentum that flows in a direction perpendicular to the momentum and contain the stress of the matter field. The conservation of angular momentum requires $T^{ab} = T^{ba}$. This is necessary to prevent volume elements from experiencing infinite angular acceleration.

The choice of the 4-velocity is however, not unique and can be chosen such that when the EMT is split up in the usual manner into its spatial and temporal parts, it takes on the following form

$$T_{ab} = \mu u_a u_b + 2q_{(a}u_{b)} + \Pi_{ab}, \tag{4.38}$$

where

$$q_a u^a = 0, \quad \Pi_{ab} = \Pi_{ba}, \quad \Pi_{ab}u^b = 0. \tag{4.39}$$

The component $\mu = T_{ab}u^a u^b$ gives the energy density of the fluid. The component $q_a = -h_a{}^b T_{bc}u^c$ represents the elements $T_{0b} \equiv T_{a0}$ and is thus, the relativistic momentum density or the energy flux relative to u^a. The spatial part Π_{ab} can then be decomposed further into its trace and trace-free components;

$$\Pi_{ab} = \pi_{ab} + p h_{ab} \tag{4.40}$$

where

$$\pi_{ab} = \pi_{ba}, \quad \pi^a{}_a = 0 \quad \pi_{ab}u^b = 0. \tag{4.41}$$

The diagonal elements (the trace)

$$p = \frac{1}{3}h^a{}_b T^b{}_a \tag{4.42}$$

describes the isotropic pressure and

$$\pi_{ab} = h_a{}^c h_b{}^d T_{cd} - \frac{1}{3} h_{ab}(h_{cd}T^{cd}) \tag{4.43}$$

represents the anisotropic part of this stress arising from viscosity and elasticity within the fluid.

4.4 Perfect Fluids

The standard definition of a perfect fluid is a comoving fluid with no particle intersections or interactions. The exclusion of any interactions means that the fluid can be characterized by zero viscosity and heat conduction. This forces the components of the EMT which are sourced by these properties, namely the energy-flux and anisotropic stress, to vanish

$$q_a = \pi_{ab} = 0 \tag{4.44}$$

and in so doing, reduces the energy momentum tensor to the form

$$T_{ab} = \mu u_a u_b + p h_{ab}. \tag{4.45}$$

We identify these elements in the EMT as the directional quantities and can therefore describe a perfect fluid as one that appears isotropic in its own rest frame (since it is comoving). Hydrodynamically speaking, the fluid is in an equilibrium state where all fluid velocities are aligned so as to produce a new preferred hydrodynamical velocity and a corresponding projection tensor. The 1+3 splitting of the EMT using these new tensors yields the form of the fluid in equation (4.45) above.

4.5 Energy-momentum Conservation

If we want to solve Einstein's equations, we require auxiliary equations concerning the field of the matter source. We recall the contracted Bianchi identities introduced in section (2.2);

$$\nabla^b T_{ab} = 0.$$

The equation where $a = 0$ describes the conservation of energy, while the equations where ($a = b = 1,\ 2,\ 3$) describes the conservation of each component of momentum. These parts are isolated using the standard space-time splitting. The component of the above identity in the direction of u^a is the *energy-conservation equation* which gives the evolution of the relativistic energy density of the matter along the flow lines,

$$\dot{\mu} + \Theta(\mu + p) + \pi^{ab}\sigma_{ab} + q^a\dot{u}_a + \nabla_a q^a = 0. \tag{4.46}$$

The spatial components that are isolated by projecting orthogonal to u^a is the *momentum-conservation equation* and describes the various pressure contributions to the total acceleration

$$(\mu + p)\dot{u}_a + h_a{}^c(\nabla_c p + \nabla^b \pi_{cb} + \dot{q}_c) + (\omega_a^b + \sigma_a{}^b + \tfrac{4}{3}\Theta h_a{}^b)q_b = 0. \tag{4.47}$$

We identify the term $(\mu + p)$ in (4.47) as the inertial mass density. As previously mentioned, the consistency of the Einstein equation requires that all matter and energy fields combined obey these conservation equations. This illustrates the intrinsic connection between the invariance of physics under co-ordinate transformations and energy conservation which is embedded in GR [4].

4.6 Equation of state

To close the system of fluid equations and complete its description, the conservation equations need to be supplemented by an equation relating the pressure and energy density; the equation of state. The cosmic substratum typically takes on the form of a barotropic fluid, for which the pressure can be represented by a function of the energy density $p = p(\mu)$ only. The fluid is effectively assumed to be in a state of local thermodynamic equilibrium (perfect hydrostatic balance) so that the equation of state becomes

$$p = w\mu, \tag{4.48}$$

where the barotropic constant w takes on various values for different types of fluids. To preserve causality the pressure p must be less than the energy density (i.e. $0 < w < 1$). In our application in chapter (6) and (7), we consider the following two types of barotropic fluids:

- A non-relativistic perfect fluid is referred to as *dust*. Since the energy density dominates over the pressure, we set $w = 0$. This equation of state is typically used to describe the Universe at late times where the fluid particles have been suitably dispersed by the expansion such that the kinetic pressures are negligible.
- *Radiation* refers photons or massive particles travelling at relativistic speeds. Here $w = \frac{1}{3}$ suitably describes the faster fall-off of the energy density than the pressure with the expansion. This occurs because the radiation particles have momentum and consequently suffer additional loss in energy with redshift (not only a decrease in the number density of the particles). Radiation is commonly taken to be a perfect fluid.

4.7 Electromagnetism in GR

GR provides a complete description of the magnetic field via the antisymmetric and traceless Faraday tensor

$$F_{ab} = F_{[ab]}. \tag{4.49}$$

When moving with the 4-velocity u^a, the observer perceives the electromagnetic field as an electric field

$$F_{ab}u^b = E_a, \tag{4.50}$$

and a magnetic field

$$F^*_{ab}u^b = \left(\frac{1}{2}\epsilon_{abcd}F^{cd}\right)u^b = B_a, \tag{4.51}$$

where $F^*_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}$ is the dual of the electromagnetic tensor. As a result of the total skewness of F_{ab} and the alternating tensor ϵ_{abcd} in definition (4.51), the projection of these vectors along u^a are identically zero

$$E_a u^a = 0, \qquad B_a u^a = 0, \tag{4.52}$$

since $u_a u_b$ is symmetric. This implies that they are purely spatial with magnitudes

$$E^a E_a = E^2, \quad B^a B_a = B^2. \tag{4.53}$$

The equation for F_{ab} has been formulated such that it describes the properties of the electromagnetic field under a frame transformation

$$F_{ab} = u_{[a}E_{b]} + \epsilon_{abcd}B^c u^d. \tag{4.54}$$

We infer that the electric and magnetic fields are merely different manifestations of the same unified electromagnetic field. In our application in chapters (6) and (7), we will look at the physically measurable magnetic and electric components of the electromagnetic field rather than the Faraday tensor itself, as it gives rise to mathematical compactness and the physical meaning of the results is more clear when stated in terms of E_a and B_a [95].

Maxwell's equations

Maxwell's equations are given in terms of operations on the Faraday tensor; the first has the form

$$\nabla^b F_{ab} = J_a, \tag{4.55}$$

where J_a is the 4-current giving rise to the electromagnetic field. This is made up of the charge density $q = -J^a u_a$ and the spatial current measured in the rest frame of the fundamental observer $J_a = h_{ab}J^b$. The second equation is a direct consequence of the existence of a 4-potential

$$\nabla_{[c}F_{ab]} = \nabla_c F_{ab} + \nabla_a F_{bc} + \nabla_b F_{ca} = 0. \tag{4.56}$$

Isolating the temporal parts of Maxwell's equations yields the propagation equations

$$\dot{E}_a + \tfrac{2}{3}\Theta E_a = \mathsf{curl}\, B_a + \sigma_{ab}B^b + \epsilon_{abc}\dot{u}^b B^c + \epsilon_{abc}\omega^b E^c + \frac{1}{\epsilon_0} j_a, \tag{4.57}$$

$$\dot{B}_{<a>} + \tfrac{2}{3}\Theta B_a = \sigma_{ab}\tilde{B}^b - \mathsf{curl}\, E_a - \epsilon_{abc}\omega^b B^c, \tag{4.58}$$

which illustrate the conservation of momentum-density for a magnetized medium. The constraint equations are derived by projecting with h_{ab}

$$D^a E_a = \frac{1}{\epsilon_0}\rho_c - 2\omega^a B_a, \tag{4.59}$$

$$D^a B_a = 2\omega^a E_a, \tag{4.60}$$

where ρ_c must be zero for the divergence-free condition. The above indicate that in an irrotational Universe $\omega^a = 0$, the electric and magnetic fields are divergence-free and thus have no scalar contributions.

The Faraday tensor can be used to construct an energy-momentum tensor encompassing the properties of the EM field

$$T^{em}_{ab} = \frac{1}{2}\left(E^2 + B^2\right)\left(\frac{1}{3}h_{ab} + u_a u_b\right) + 2u_{(a}\epsilon_{b)cd}E^c B^d + \Pi_{ab}, \tag{4.61}$$

where $\Pi_{<ab>} = \frac{1}{3}\left(E^2 + B^2\right)h_{ab} - E_a E_b - B_a B_b$. We can easily identify the EM analogues to the components of the EMT of an imperfect fluid in (4.38) as seen by the fundamental observer

$$\mu_{em} = \frac{1}{2}\left(E^2 + B^2\right)u_a u_b, \tag{4.62}$$

$$p_{em} = \frac{1}{6}\left(E^2 + B^2\right)h_{ab}, \tag{4.63}$$

$$q^a_{em} = \epsilon^{abc}E_b B_c, \tag{4.64}$$

$$\pi^{ab}_{em} = \Pi^{ab}. \tag{4.65}$$

Since

$$p_{em} = \frac{1}{3}\mu_{em}, \tag{4.66}$$

the EM field behaves like a radiation fluid with anisotropic stress and energy flux. The fluid description of the field however, does not account for its vector nature as well as its intrinsic coupling to the curvature [99].

Case of Infinite Conductivity

An important assumption which commonly accompanies treatments of electromagnetic fields in Cosmology, is that of infinite conductivity of the cosmic medium. This supposition is valid for the majority of the Universe's evolution and is most applicable during very early stages in its evolution (cf. [7], for an example). Consider the following form of Ohm's law

$$J_a + J^b u_b u_a = \sigma E_a, \tag{4.67}$$

where u^a is the velocity of the fluid and σ is the conductivity of the medium. If we project into the rest-frame of the observer $h^a{}_c$, we find

$$h^a{}_c J_a = \sigma E_c. \tag{4.68}$$

This shows that non-zero spatial currents $h^a{}_c J_a$ can exist in the absence of an electric field provided the medium is perfectly conducting. This assumption allows us to set $E_a = 0$ which greatly simplifies the manipulation of Maxwell's equations.

The alternative means of discounting any electric fields while retaining coupling between the fluid and the magnetic field, is to assume the presence of a pure magnetic field with no associated electric field and currents. When the electric field propagation equation with $E_a = J_a = 0$ is inserted into the exact momentum-conservation equation, the magnetic field terms therein are removed, effectively eliminating the momentum exchange between the field and the fluid. We see that this assumption disconnects the fluctuations in energy density with those in the field [96]. When implementing the weak field approximation (WFA) [96, 97, 99] (which will be discussed in depth at a later stage), the choice of infinite conductivity over setting $E_a = J_a = 0$ is even more important because the key assumption $B^2 \ll \mu$ additionally removes this coupling in the evolution equation of the spatial gradient of the expansion [96].

4.8 Dynamics

Returning to (2.7), we employ the decomposition of the energy-momentum tensor given in (4.38) and split the field equations into the spatial and timelike parts by performing a full contraction over h_{ab} and u^a. This produces 3 equations which are equivalent to (2.7)

$$\begin{aligned} R_{ab}u^a u^b &= \tfrac{1}{2}(\mu + 3p) - \Lambda, && (4.69)\\ R_{ab}u^a h^b{}_c &= -q_c, && (4.70)\\ R_{ab}h^a{}_c h^b{}_d &= \left(\tfrac{1}{2}(\mu - p) + \Lambda\right) h_{cd}. && (4.71) \end{aligned}$$

These equations are the timelike $(0,0)$, space-time $(0,\mu)$ and spacelike (α,β) parts of the Einstein equation. These relations are equivalent to 10 equations of which all need to be satisfied. We will use Einstein equation in the form of (4.69)-(4.71) to derive equations governing the dynamics of the Universe.

4.8.1 The Propagation equations

The Ricci identity given in (2.1) must hold for any vector field and can therefore, be applied to the chosen 4-velocity field

$$R_{abcd}u^d = 2\nabla_{[a}\nabla_{b]}u_c. \qquad (4.72)$$

Isolating the part of this identity along u^a gives an equation describing the evolution of the covariant derivative of the preferred velocity along the fluid flow lines

$$\dot{(\nabla_d u_a)} - \nabla_d \dot{u}_a + \nabla_d u^c \nabla_c u_a = R_{abcd}u^b u^c. \qquad (4.73)$$

We are now equipped to find the evolution equations of the individual kinematic quantities introduced in section (4.2). We can employ the 1+3 decomposition of $(\nabla_d u_a)$ in equation (4.13) and the field equation (4.69) and separate the resultant (4.73) into its trace, anti-symmetric and PSTF parts to find 3 equations. They are referred to as the *propagation equations* because they involve the time derivatives of the kinematic quantities.

The Raychaudhuri equation

The contraction of the propagation equation (4.73) on the indices a and d isolates the trace

$$\dot{(\nabla^a u_a)} - \nabla^a \dot{u}_a + \nabla^a u^c \nabla_c u_a = R^a{}_{bca}u^b u^c = R_{bc}u^b u^c. \qquad (4.74)$$

We now substitute for the RHS from the equation (4.69) and replace $\nabla^a u_a = \Theta$ and find the evolution of the expansion, or the Raychaudari equation

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 = -\tfrac{1}{2}(\mu + 3p) + \dot{u}^a\dot{u}_a + \Lambda - 2(\sigma^2 + \omega^2) + \nabla^a\dot{u}_a. \tag{4.75}$$

We now introduce the scale factor $a(t)$ defined in section (4.2) which relates physical coordinates to comoving coordinates. It shows how the physical separation between comoving fluid elements changes with time and is therefore related to the relative expansion of the Universe. If we replace the LHS of (4.75) using equation (4.33);

$$\Theta = \tfrac{3\dot{a}}{a},$$

we see that it is equivalent to the second derivative of the scale length and therefore gives the curvature.

The contribution of the term $-\frac{1}{2}(\mu + 3p)$ depends critically on the relationship between pressure and density as dictated by the equation of state. For *normal* matter $(\mu + 3p > 0)$ this term is negative and thus slows down the expansion and is so identified as the active gravitational mass density of the fluid. The vorticity term favours the expansion of scale factor, as the local rotation along the fluid flow tends to pull matter apart due to conservation of angular momentum. The shear term reduces the expansion rate because the positive spatial change that is represented by σ_{ab} is perpendicular to the direction of the average motion of matter. The outcome of this competition between these terms determines the fate of the Universe.

The Vorticity propagation equation

The antisymmetric part of the propagation equation (4.73) is found by multiplying it by ϵ^{adk} and represents the evolution of the vorticity

$$\dot{\omega}_{<a>} = \tfrac{1}{2}\mathrm{curl}\,\dot{u}_c - \tfrac{2}{3}\Theta\omega^a + \sigma^a{}_b\omega^b. \tag{4.76}$$

Evidently, the global vorticity is diluted by the expansion.

The Shear propagation equation

If we project (4.73) on the indices a and c and substitute only the trace-free symmetric part of v_{ab} defined in equation (4.14), we arrive at the shear evolution equation

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} + \dot{u}_{<a}\dot{u}_{b>} + \mathrm{D}_{<a}\dot{u}_{b>} - \sigma_{<a}{}^c\sigma_{b>c} - \omega_{<ac}\omega_{b>}{}^c + \tfrac{1}{2}\pi_{ab} - E_{ab}. \tag{4.77}$$

We note the appearance of the electric part of the Weyl tensor E_{ab} and the anisotropic stress π_{ab} in the equation above. This illustrates how the tidal gravitational field sources distortion in the fluid flow.

4.8.2 The Constraint equations

The propagation equations are complemented with 3 additional relations, which are the components of the Ricci identity (4.72) orthogonal to u^a on the index d. They are referred to as the *constraint equations* as they contain no time derivatives of the kinematic quantities, only spatial derivatives.

The $(0,\nu)$ component of the field equations

The first equation is found by contracting on a and c and substituting the RHS for the 'space-time' component of the Einstein equation (4.70). This gives

$$\tfrac{2}{3}\mathrm{D}_a\Theta - \mathrm{D}^b\sigma_{ab} - \mathsf{curl}\,\omega_a - 2\epsilon_{abc}\dot{u}^b\omega^c = q_a, \tag{4.78}$$

which illustrates the direct relationship between the spatial inhomogeneities in the matter and the energy flux q_a.

The divergence of the vorticity equation

To find the constraint equation involving the vorticity we exploit the symmetric nature of the Riemann tensor. The antisymmetric part of the Ricci identity,

$$R_{a[bcd]}u^a = 0 \tag{4.79}$$

directly implies that

$$\nabla_{[b}\nabla_{d]}u_c = 0. \tag{4.80}$$

Multiplying by ϵ^{bcd} gives the identity

$$\mathrm{D}^a\omega_a = \omega^a\dot{u}_a. \tag{4.81}$$

The magnetic part of the Weyl tensor

Finally, we multiply the Ricci identity by ϵ^{cdk} and symmetrize on a and k to obtain an expression for the magnetic part of the Weyl tensor

$$H_{ab} = (\mathsf{curl}\,\sigma)_{ab} - \mathrm{D}_{<a}\omega_{b>} - 2\dot{u}_{<a}\omega_{b>}. \tag{4.82}$$

Evidently, the tensor H_{ab} characterizing gravitational waves is composed partly of the rotation of the shear and the distortion of the vorticity. The action of the shear has been shown to induce distortion without a change in the direction of its principle axis. The magnetic part of Weyl tensor thus describes how this axis rotates.

4.8.3 The other Bianchi identities

The evolution of the shear is given in equation (4.77) is controlled by the electric Weyl component E_{ab} (and thus H_{ab}) whose propagation equations we have yet to find. To determine the evolution of the gravitational field, we use the *other Bianchi identities*. These are the field equations that essentially govern the Weyl tensor, isolating the part of the spacetime curvature that is affected by the matter distribution elsewhere [95]. In 4 dimensions, the identity (2.8) is equivalent to

$$\mathrm{D}^dC_{abcd} = \mathrm{D}_{[b}R_{ca]} - \frac{1}{6}g_{c[a}\mathrm{D}_{b]}R. \tag{4.83}$$

By replacing the Weyl tensor by its form given in equation (2.4) and substituting equation (2.7), we obtain equations for H_{ab} and E_{ab}. Projecting them along u^a gives Maxwell-like field equations that govern the evolution of the gravitational field

$$\begin{aligned} \dot{E}_{ab} + \Theta E_{ab} - \mathrm{curl}\, H_{<ab>} + \tfrac{1}{2}(\mu + p)\sigma_{ab} + \tfrac{1}{6}\Theta\pi_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} - 2\epsilon_{cd<a}\dot{u}^c H_{b>}{}^d \\ + \tfrac{1}{2}\mathrm{D}_{<a}q_{b>} - 3\sigma_{<ac}\left(E^c{}_{b>} - \tfrac{1}{6}\pi_{b>}{}^c\right) - \epsilon_{cd<a}\omega^c\left(E_{b>}{}^d + \tfrac{1}{2}\pi_{b>}{}^d\right) = 0 \end{aligned} \tag{4.84}$$

and

$$\begin{aligned} \dot{H}_{ab} + \Theta H_{ab} + \mathrm{curl}\, E_{ab} - \tfrac{1}{2}\mathrm{curl}\, \pi_{ab} + \tfrac{3}{2}\omega_{<a}q_{b>} - 3\sigma_{<ac}H^c_{b>} \\ = -\epsilon_{cd<a}\left(2\dot{u}^c E_{b>}{}^d + \omega^c H_{b>}{}^d + \tfrac{1}{2}\sigma_{b>}{}^c q^d\right). \end{aligned} \tag{4.85}$$

Contracting with h_{ab} gives the two complementary constraints

$$\mathrm{D}^b E_{ab} - \tfrac{1}{3}\mathrm{D}_a\mu - 3\omega^b H_{ab} + \tfrac{1}{2}\mathrm{D}^b\pi_{ab} + \Theta\tfrac{1}{3}q_a - \tfrac{1}{2}\sigma_{ab}q^b + \epsilon_{abc}\left(\tfrac{3}{2}\omega^b q^c - \sigma^{bd}H^c{}_d\right) = 0 \tag{4.86}$$

and

$$\mathrm{D}^b H_{ab} + (\mu + p)\omega_a + 3\omega^b\left(E_{ab} - \tfrac{1}{6}\pi_{ab}\right) + \epsilon_{abc}\sigma^{bd}\left(E_d{}^c + \tfrac{1}{2}\pi_d{}^c\right) + \tfrac{1}{2}\mathrm{curl}\, q_a = 0. \tag{4.87}$$

We have defined the fundamental variables, their governing equations and the reference frame with which they are identified. We are now equipped to follow the evolution of the perturbations of these quantities as seen by the fundamental observers.

To summarize, we have derived a system of propagation and constraint equations which closes once the equation of state given in section (4.6) has been specified. The constraint equations are necessary when dealing with an initial value problem; they essentially give the initial conditions which must be satisfied by the evolution equations. An initial data set that satisfies these equations can then be sought. It is necessary to check that this set of equations is consistent, which involves confirming that the constraints are satisfied through time as a result of the propagation equations holding. This will be elaborated on in section (6.5).

4.8.4 The vorticity-free case

In section (4.2) the vorticity ω_a was shown to give rise a rotation of a local frame relative to the rest-frame of the galaxies. If $\omega = 0$, the 4-velocity u^a satisfies the relation[3]

$$u_{[a}\nabla_{c]}u_b = 0 \rightarrow u_{[a}u_{b,\,c]} = 0, \tag{4.88}$$

which leads to

$$\omega = 0 \rightarrow \ \textit{locally} \ \exists \ \textit{functions} \ r,\ t : u^a = -rt_{,a}. \tag{4.89}$$

In other words, $w = 0$ gives the condition for which u^a is proportional to a gradient $t_{,a}$ which is the tangent vector that lies orthogonal to the local surfaces $\Sigma_\perp$ (i.e. $t =$ constant). We infer that $\omega = 0$ is then the condition for u^a to be orthogonal to these individual local surfaces. Since all tangent 3-spaces $\Sigma_\perp$ lie normal to u^a, they mesh together to form a spacelike surface orthogonal everywhere to u^a.

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[3] The ',' in the following equation denotes a partial derivative, defined by $f_{,a} \equiv \frac{\partial f}{\partial x^a}$.

The Gauss equation

When $\omega = 0$, it is possible to define the intrinsic curvature of a family of these surfaces $\Sigma_{\perp}$. We use the Ricci identity in 3-space applied to an arbitrary vector field X_a in the 3-space (i.e. $X^a u_a = 0$)

$$\mathrm{D}_c\left(\mathrm{D}_d X_a\right) - \mathrm{D}_d\left(\mathrm{D}_c X_a\right) = {}^3R_{abcd}X^b, \tag{4.90}$$

where ${}^3R_{abcd}$ is the curvature tensor of the 3-spaces. In 3 dimensions, the Ricci tensor has the same number of independent components as the full Riemann tensor which means that the 3-Riemann tensor can be written in terms of the 3-Ricci tensor as follows;

$${}^3R_{abcd} = {}^3R_{ac}h_{bd} - {}^3R_{ad}h_{bc} - \frac{1}{2}\,{}^3R\left(h_{ac}h_{bd} - h_{ad}h_{bc}\right). \tag{4.91}$$

Substituting the definition of the covariant derivative of a vector field on these surfaces $D_b X_a \equiv h_b{}^d h_a{}^c \nabla_d X_c$ as well as h_{ab} and $\nabla_a u_d$ in (4.13), we obtain an expression for the curvature tensor in 3-space in terms of the 4-dimensional curvature;

$${}^3R_{abcd} = h^t{}_a h^f{}_b s h^e{}_c h^s{}_d R_{tfes} - \Theta_{ca}\Theta_{bd} + \Theta_{bc}\Theta_{ad}. \tag{4.92}$$

Contracting (4.92) over 2 indices and substituting for (2.7), (4.38) and (4.78), we obtain

$${}^3R_{ab} = -\dot{\sigma}_{<ab>} - \Theta\sigma_{ab} + \mathrm{D}_{<a}\dot{u}_{b>} + \dot{u}_{<a}\dot{u}_{b>} + \pi_{ab} + \tfrac{1}{3}h_{ab}\left[2\mu + \tfrac{2}{2}\Theta^2 + 2\sigma^2 + 2\Lambda\right]. \tag{4.93}$$

Contracting again ${}^3R = h^{ab3}R_{ab}$ gives the Ricci scalar of these 3-spaces;

$${}^3R = 2\sigma^2 - \tfrac{2}{3}\Theta^2 + 2\Lambda + 2\mu. \tag{4.94}$$

This illustrates how the matter content influences the curvature of the 3D space.

4.9 FLRW Models

Since the process of solving the exact Einstein equations is highly non-trivial, the logical first step towards achieving this is to find a solution to a more simplified matter distribution [57, 78]. Given that the Universe is observed with a high degree of symmetry on large scales (larger than 100 Mpc [62]), it seemed reasonable to begin with a model in which the matter distribution has the same properties in all directions – a characteristic referred to as *isotropy*[4]. In an isotropic spacetime as shown in figure (4.2), points lying along a mapped-out circle centered at a point **A**, must have the same density. This is however only consistent with 2 other overlapping spheres of fixed density centered at points **B** and **C** if the density throughout the spacetime is constant [36]. This illustrates how the condition of isotropy immediately demands that the spacetime must the same on spatial surfaces orthogonal to the fluid flow and is said to exhibit *homogeneity*. In fact, the assumption of isotropy between two distinct worldlines is all that we require to guarantee that the properties throughout the spacetime are independent of location. We note that the condition of homogeneity

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[4] 'Isotropy' refers to spatial isotropy and will continue to be observed only by a comoving observer which moves with a *uniform* u^a. A different observer that travels with a constant velocity *relative* to the fundamental observers will see an anisotropic Universe [78].

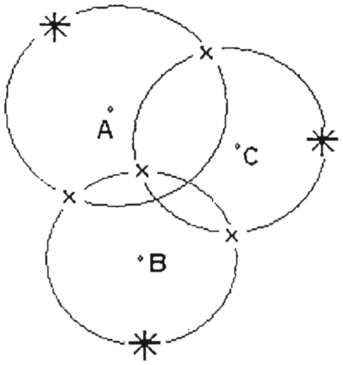

Figure 4.2: The assumption of isotropy requires spatial homogeneity to hold simultaneously [36].

does not however require isotropy to hold simultaneously. Bianchi models are examples of spacetimes that are homogeneous but not isotropic (the Universe expands anisotropically). These models are known as the Friedmann-Lemaître-Robertson-Walker (FLRW) models and are essentially the solutions to the Einstein equation which are homogeneous and isotropic about each point. The assumption of homogeneity and isotropy, initially made for purely aesthetic reasons, turned out to be an extremely good approximation of the real Universe provided we average over scales larger than clusters of galaxies [36]. Besides the high level of isotropy now detected in the Cosmic microwave background, other observations such as galaxy counts as a function of magnitude, and number counts of galaxies lying inside cells spread throughout the Universe (for redshift surveys), support this description on large scales. Provided we assume that there is nothing special about our position in the Universe (i.e. the Copernican principle), this evidence for isotropy lays the basis for the belief in homogeneity and hence our assertion that the Universe is well-described by the FLRW model. FLRW Universes have become standard models of Cosmology and as the limits on anisotropy became stronger, the number of models based on anything but the FLRW metric become limited. Galaxies are however receding from us and for this reason we construct models that are homogeneous and isotropic but not static in time.

Kinematics

Relative to the congruence of fundamental observers with 4-velocity u^a, the kinematics are assumed to be locally isotropic. To eliminate preferred direction in the homogeneous spatial section, all vector quantities must vanish including the 4-acceleration $\dot{u}_a = (\nabla_b u_a)u^b$;

$$\omega_a = \sigma_{ab} = \dot{u}_a = 0. \tag{4.95}$$

Since both the vorticity and the aceleration vanish, then

$$\omega^a = 0 = \dot{u}^a \quad \longrightarrow \quad u_{[c;d]} = 0 \quad \Leftrightarrow \quad u_{[c,d]} = 0, \tag{4.96}$$

which implies that u^a is a gradiant;

$$locally \ \exists \ a \ function \ \ t : u^a = t_{,a}. \tag{4.97}$$

Since u^a is unique, the above implies that the proper time t must be unique up to an additive constant. The surfaces of spatial homogeneity are thus the surfaces of simultaneity (t = constant) which are *orthogonal* to the flow lines [36].

Matter

The condition of isotropy also demands the following constraints

$$\pi_{ab} = q_a = 0, \tag{4.98}$$

to ensure that the EMT and thus the Ricci tensor are isotropic. A perfect fluid is thus a necessary requirement of this model.

Gravitational field

If the constraints (4.95) and (4.98) are substituted into equations (4.77) and (4.82), we see that the electric and magnetic components of the Weyl tensor and hence the Weyl tensor itself, vanish identically

$$E_{ab} = H_{ab} = 0 \Rightarrow C_{abcd} = 0 \ . \tag{4.99}$$

We can then infer that these models are conformally flat. In addition, the spatial uniformity forces the spatial gradients of the energy density μ, the pressure p and the expansion Θ to vanish

$$0 = \mathrm{D}_a \, \mu = \mathrm{D}_a \, \Theta = \mathrm{D}_a \, p. \tag{4.100}$$

Geometry

To eliminate preferred spatial directions (i.e. the eigendirections of the 3-Ricci tensor) the 3D Ricci tensor must also be isotropic and therefore proportional to the isotropic metric h_{ab}

$${}^3R_{ab} = \frac{1}{3} R h_{ab}. \tag{4.101}$$

Replacing the expression for the 3-Riemann tensor by (4.101) in equation (4.91) gives

$${}^3R_{abcd} = K \left(h_{ac} h_{bd} - h_{ad} h_{bc} \right), \tag{4.102}$$

which implies that the 3-spaces have constant curvature

$$C = \frac{{}^3R}{6}. \tag{4.103}$$

The contracted Bianchi identities given in section (2.9) imply that the Ricci scalar is a function of time only and

$$ {}^3\dot{R} = \frac{2}{3}\Theta \; {}^3R. \tag{4.104} $$

The scale factor $a(t)$ here is the length scale in the homogeneous and isotropic FLRW spacetime. We can then write the constant C as

$$ C = \frac{K}{a^2(t)}, \tag{4.105} $$

where K is a constant. From equation (4.103) we see that ${}^3R = \frac{6K}{a^2(t)}$, where K is now referred to as the curvature constant. If we now substitute the restrictions on the kinematics stated in equation (4.95) above into (4.94) and replace ${}^3R$, we obtain the Friedmann equation

$$ \mu + \Lambda = \frac{1}{3}\Theta^2 + \frac{3K}{a^2}, \tag{4.106} $$

describing the intrinsic curvature of the isotropic and homogeneous 3-spaces. The curvature constant K indicates the geometry of the Universe and can be normalized to $K = +1, -1$ for open and closed models if it is not zero as in the case of a flat Universe.

Despite its high symmetry, this model is employed to deal with non-uniformity; we treat any observed large scale structure as small perturbations in density to an otherwise homogeneous FLRW Universe. In linear perturbation theory where the FLRW model forms the background spacetime, the remaining quantities that do not vanish in this spacetime are μ, p, Θ and are termed zeroth-order. We note that these variables appear in the exact equations as coefficients of the GI first-order variables and therefore need their background values only.

Chapter 5

Magnetic fields in Cosmology

This section is based primarily on the detailed reviews [42, 45, 46, 49, 104].

With the exception of the baryon density and the spectrum of energy density fluctuations, there are no remnants of the fundamental processes of the very early Universe that can be detected today. The thermalized pre-recombination period smoothed out any observable effects [42]. We know from observations that large-scale magnetic fields are present in most environments today and if these fields are truly primordial, we speculate that they would have left their mark on many events in the thermodynamical history of the Universe. An example is Big-bang nucleosynthesis (BBN), where the interaction of a magnetic field with the magnetic moment of a neutrino may have given rise to a spin-flip and change of its handedness, introducing an additional neutrino degree of freedom [42]. Depending on their spectrum, magnetic fields existing in protogalactic clouds with strengths of 10^{-12} – 10^{-9}G may have played a significant role in structure formation [102]. The Lorentz force that acts on charges in an inhomogeneous (i.e. $\mathrm{curl}\, B \neq 0$) magnetic field has been shown to induce perculiar velocities [54] which seed density perturbations, and in so doing alter the gravitational perturbations. The presence of primordial magnetic fields may thus provide the solution to the problem of insufficient luminous baryonic matter in the Universe. Hypermagnetic fields, hypothesized to emerge during the electroweak phase, have also been identified as the possible source of the observed baryon asymmetry of the Universe (BAU) [46].

To fully comprehend the part primordial magnetic fields played in the history of the Universe, we must understand their origins. Firstly, we need to gauge the strength and size with which the fields had to be generated in order to agree with observations today. We review the observational tools used to measure the different types of magnetic fields and give some current constraints on their strengths. These values obtained observationally are supplemented with estimates of how primordial fields would have affected the relative elemental abundances during BBN. These limits greatly impinge on the validity of the proposed generating mechanisms.

5.1 Observational tools

Current observations reveal that magnetic fields are prevalent throughout galaxy clusters, disk and spiral galaxies as well as high-redshift condensations. We detect most of these fields indirectly by examining their effects on their immediate environments.

The interaction between a magnetic field and the magnetic dipole moment associated with the orbital angular momentum of atoms, removes the degeneracy of certain energy levels. This manifests as a splitting of single spectral and recombination lines into multiple closely spaced lines, a phenomenon known as the **Zeeman effect**. The measurement of this splitting gives an estimate of the strength of the homogeneous component along the line of site

$$\Delta v_z = \frac{e\overline{B}_{\parallel}}{2\pi m_e}. \tag{5.1}$$

The induced shifts are small and limit the use of this observational tracer to local measurements within our own galaxy. It does however, provide reliable information regarding the local direction of other magnetic fields, which is useful in advancing our understanding of their origins.

Synchrotron emission arises due to the motion of free relativistic electrons in a magnetic field. As an energetic electron encounters a magnetic field, the crossed magnetic field and its electric field trap it along the field lines in a helical trajectory. Since the direction in which the electron moves is continuously changing, the electron is effectively accelerating and consequently emits this signature radiation. The frequency of this radio emission depends on the radius of the trajectory and the strength of the magnetic field. In addition it has an intrinsic polarization which is proportional to

$$\mid \frac{\overline{B}_{\perp}}{B_{tot\perp}} \mid^2 . \tag{5.2}$$

Given that $\overline{B}$ is the homogeneous component, the amount of polarization provides a measure of the uniformity of the magnetic energy and gives an indication of the field's orientation. The electromagnetic radiation is proportional to $n_e B^2$ where n_e is the local electron density. An independent measure of n_e is thus needed and can in some cases be extracted from the X-ray emission from the hot electron gas typically found within galaxy clusters. Because this is not always possible it is standard to assume the equipartition of the field and fluid energy densities to estimate B.

High-redshift magnetic fields and those spanning the intergalactic medium (IGM) are primarily studied using the **Faraday rotation** of polarized electromagnetic radiation. The magnetic field induces a resonance effect, which causes polarized radiation passing through it to separate into two differently polarized rays. Due to the presence of free electrons the two propagation modes experience different refractive indices and consequently travel at different speeds. The mode that is polarized in the same direction as the current induced by the magnetic field, propagates with greater velocity causing the plane of the polarized light to rotate. The angle of rotation $\Delta\chi$ probes both the line-of-sight magnetic field strength B_ℓ

and the size of the field. The faraday rotation measure (RM) out to a maximum redshift z_m is given by [42]

$$RM(z_m) = \frac{\Delta\chi}{\Delta\lambda^2} = 8.1 \times 10^5 \int n_e B_\ell(z)(1+z)^{-2} d\ell(z) \quad rad\ \mathrm{m}^{-2}, \tag{5.3}$$

where $d\ell(z) = 10^{-6} H_0^{-1}(1+z)(1+\Omega_z)^{-\frac{1}{2}} dz$ Mpc. As in the case of synchrotron emission, the drawback of this method is its requirement of an independent measure of n_e along the line of sight, as well as information regarding possible field reversals which would impact on the integral in (5.3). A value for n_e can be calculated from the relative pulse delay and frequency of pulsars. The radio pulses from these objects propagate slower in an ionized interstellar medium than in vacuous space. The comparison of the delay in the arrival of different radio pulses at different wavelengths gives an indication of the column density of electrons. This is made difficult by the low density of the IGM and the faintness of these objects in very distant galaxies. RMs are thus used primarily to place upper limits on the field strengths.

5.2 Current limits on primordial magnetic fields

5.2.1 Observations

Magnetic Fields in Galaxies

Various methods have been used to measure the magnetic field permeating the interstellar medium of the Milky Way and give a mean strength of $3-4$ μG. This value is in agreement with equipartition between the magnetic field, the cosmic radiation and the small-scale turbulence [59]. However, the equipartition of energy between a magnetic field and its environment is purely theoretical and this may not be regarded as clear-cut experimental evidence. Even though other galaxies such as $M33$ with a measured strength of 4 μG appear to share this property, the fields inhabiting others such as $M82$ and the Magellanic Clouds are considerably stronger than their predicted equipartition strengths.

Magnetic Fields in Clusters

Measurements of the Faraday rotation of radiation emitted by radio sources in and behind galactic clusters, allude to the presence of large-scale magnetic fields within[1] many of them. By comparing the RMs of sources through roughly 50 Abell clusters, a phenomenological equation describing the strength of a field within the inter-cluster medium was derived [49]

$$B_{ICM} \sim 2\ \mu\mathrm{G} \left(\frac{L_{frs}}{10\mathrm{k}pc}\right)^{-\frac{1}{2}} (h_{50})^{-1}, \tag{5.4}$$

where L_{frs} is the reversal field length and h_{50} is the reduced Hubble constant. The typical values of L_{frs} of $10-100$ kpc correspond to strengths of $1-10$ μG. The Hydra A cluster

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[1] The magnetic fields 'within' clusters actually refer those in the inter-cluster medium and therefore, cannot be assigned to individual galaxies.

provides an example of a strongly magnetized cluster. The RMs taken continuously from the radio core out to a distance of 50 *kpc*, reveal strong asymmetry between its north and south lobes. This has been attributed to the presence of a field coherent over 100 *kpc* with a strength of 6 μG, with a superimposed tangled field of strength 30μG [91]. High resolution images of radio sources [34] within other clusters indicate that their central regions are permeated by extremely strong fields with strengths in the range of $10 - 30$ μG, with some as large as 70 μG. In these cases the magnetic pressure exceeds the gas pressure indicated by X-ray data, suggesting that these strong fields may well influence the cluster dynamics. The reason for this drop in field strength from the central to the outer parts has not yet been established and may be related to the magnetic field structure, or merely an artifact of the decreasing gas pressure. The strong agreement between the strengths of fields within clusters and those residing in the considerably denser interstellar regions of galaxies, is surprising but entirely reasonable if their origins are indeed primordial [42].

Magnetic Fields in High-redshift objects

The Faraday measurements of high-redshift quasars allow us to gauge the properties of fields in the very distant past. In [59] the RM of the high-redshift quasar PKS 1229-021 ($z = 1.038$) situated behind a spiral galaxy at $z = 0.395$ indicated the presence of a young galactic field with an amplitude in the range $1 - 4$ μG. An intriguing observation was that the field reversals occur over distances roughly equal to the spiral arm separation, in much the same way as the magnetic field in the Milky way.

Magnetic Fields in the IGM

Faraday measurements of magnetic fields located outside galaxies are more difficult owing to the low-density of the IGM and the limited number of sources that can give an independent measure of n_e. RMs of intergalactic magnetic fields have been recently performed on a sample of 309 galaxies and quasars. An upper limit of $B \leq 10^{-9}\mathrm{G} \quad (\Omega_{IG} h_{100}/0.01)^{-1}$ [100] was derived where Ω_{IG} is a measure of the ratio of the ionized gas density to the critical density of the intergalactic plasma. The field was assumed to be homogeneous on horizon scales.

CMB Observations

Magnetic fields present at last scattering, would have imprinted on the Cosmic Microwave background (CMB) leaving their mark on the large-angle anisotropies of its power spectrum. They can cause the Faraday rotation of the direction of linear E-type polarization to produce B-modes. Primordial gravitational waves predicted by inflation, also induce this curl-like polarization in the CMB [86]. Since their source (gravity) is a tensor field they also radiate in the weaker quadrupole mode and their signals therefore, need to be distinguished from those produced by magnetic fields.

The anisotropic stress that is naturally induced by a magnetic field will also impact on the CMB. Barrow, Ferreira and Silk [3] found that the shear anisotropy σ_Λ in a general anisotropic Universe permeated by a large-scale magnetic field, is sourced by the magnetic energy density

$$\frac{\sigma_\Lambda}{H} = \frac{4}{2-\gamma}\left(\frac{\mu_B}{\mu} + \frac{\mu_{gas}}{\mu}\right) + \delta t^{\frac{(\gamma-2)}{\gamma}}, \tag{5.5}$$

where δ is a constant and $0 \leq \gamma \leq 4/3$. Given that the angular anisotropy is proportional to σ_Λ, they were able to extract an upper limit on a homogeneous magnetic field of $B \leq 3.4 \times 10^{-9}\text{G} \quad (\Omega_0 h_{50}^2)^{\frac{1}{2}}$ using the 4-year COBE data. This value is calculated under the assumption that the observed anisotropy is generated only by the field.

5.2.2 Primordial Nucleosynthesis

There are various ways that the presence of a magnetic field in the very early Universe could have influenced BBN. It has been argued that the contribution of a magnetic field to the energy density would lead to an increase in the rate of the expansion and subsequent cooling of the Universe. A temperature that is low enough for the binding energy of deuterium to exceed the mean energy per particle would have been reached more rapidly, thereby increasing the predicted abundance of ^{4}He. In [48] the authors calculate that a magnetic field with strength $B = 1.2 \times 10^{-10}$ G is needed to increase the relative abundance of ^{4}He above 0.24. The presence of a field during this epoch increases the $n \leftrightarrow p$ reaction rates [19, 48]. In response, the ^{4}He abundance drops and counteracts the above-mentioned increase in the expansion rate. This effect is however likely to be negligible. The presence of a magnetic field alters the statistical distribution of the electrons and the positrons. The wave character of the particles force an integral number of de Broglie wavelengths to fit the circumferences of their spiral pathways and in so doing, cause the Lamour radii of their orbits to become quantized. In this case the electrons and positrons occupy Landau energy states. In the more natural case where $eB \ll T^2$, the magnetic field alters the degeneracy of each Landau level and they become bunched with discrete energy values. This causes a fraction of the $e - p^+$ pairs to enter the ground Landau state, resulting in an increase in their number and energy densities which impacts on the Hubble expansion rate. Grasso and Rubinstein combined all these effects and calculated an upper limit on a homogeneous magnetic field of

$$B \leq 1 \times 10^{-11}\ \text{G}, \tag{5.6}$$

for $T = 10^9$K and assuming the ^{4}He abundance does not exceed 0.245 [48]. This limit weakens to roughly 10^{-12}G if the field is assumed to contain inhomogeneities on scales smaller than the current horizon.

We infer that primordial fields are likely to have existed with strengths of roughly 10^{-10}G. The field generation mechanisms that are proposed in the following section, must give rise to fields in this range in order to be viable.

5.3 Magnetogenesis

The origin of cosmological magnetic fields has generated much debate in recent years, with the majority of this work being focused on providing mechanisms that generate these galactic fields on large scales (see [49, 104] and references therein). The candidate mechanisms are diverse, often depending on the required seed field strengths given in section (5.2).

5.3.1 Primordial field amplification

It has been suggested that the fields observed today could be a result of the amplification of relatively large seed fields through protogalactic collapse at the onset of structure formation [67]. The highly conductive nature of the cosmic plasma during this epoch suppresses the diffusion of the magnetic field and we can therefore assume that the magnetic flux is almost conserved. As a patch of matter gravitationally collapses to current measured densities, the flux lines of the frozen-in field are compressed, inducing adiabatic amplification[2]. It can be shown that the magnetic field strength increases by the square of the decrease in the physical galaxy size L during the collapse. Due to flux conservation this can be related to the appropriate energy densities

$$\left(\frac{B_f}{B_i}\right) = \left(\frac{L_i}{L_f}\right)^2 = \left(\frac{\mu_f}{\mu_i}\right)^{\frac{2}{3}}. \tag{5.7}$$

μ_i and μ_f are the energy densities directly before and after the gravitational collapse respectively. It is reasonable to assume that the mean energy density of the patch μ_i is roughly the critical value of the Universe. Using the current value of μ_f we obtain a factor of $\left(\frac{\mu_f}{\mu_i}\right) \sim 10^6$. We conclude that this mechanism induces an amplification of roughly 4 orders of magnitude. Using the current strength measurements of the *galactic fields* of 10^{-6}G given in section (5.2.1), a field present before structure formation needs to be

$$B_i \sim 10^{-10}\ \mathrm{G}. \tag{5.8}$$

Although this mechanism is physically feasible, it requires reasonably strong seed fields to be consistent with current observations.

5.3.2 Galactic Dynamo

Another popular mechanism which requires a relatively weaker pre-existing field, is amplification via the galactic dynamo by means of parametric resonance [79, 59]. The combined effect of differential rotation across the disk and the cyclonic turbulent motions of the ionized gas, is believed to lead to an exponential amplification of a smaller primordial field, by the conversion of the kinetic energy (associated with the electrically conducting fluid) into magnetic energy. If we consider the time evolution of the magnetic field

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \frac{1}{4\pi\sigma}\nabla^2 B, \tag{5.9}$$

we see that the dynamo mechanism is advocated when the second term in (5.9) (representing the diffusion of the magnetic field lines into the plasma) is dominated by the term describing the freezing-in of the field. If the field is in a state of sufficiently low resistivity, the combined differential rotation across the disk and the hydrodynamical turbulence arising from the cyclic motions cause the magnetic field lines to distort. According to [1], the ratio of the magnetic field strength to the fluid energy density $\frac{B}{\mu_m}$ behaves as the distance between the

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[2] The term adiabatic implies no loss of energy to the surroundings. As the area enclosed by field lines decreases, the energy of the magnetic field must correspondingly increase to conserve flux.

fluid elements. The field is therefore amplified as it stretches in this manner. The theory predicts that the amplification continues until the dynamo 'saturate' [45]. This occurs when the back-reaction of the plasma is no longer negligible and the higher order terms $\mathcal{O}\left(B^2\right)$ in (5.9) must be retained. This contribution opposes further growth and stabilizes the amplification. It is hypothesized to initiate once equipartition is achieved and the magnetic energy density is comparable with the kinetic energy of the plasma. This corresponds to an energy density of roughly $\mu_B \approx 2-8\mu\text{G}$. Equipartition is estimated to be achieved in about 10^8 – 10^9 years; on these timescales this mechanism can potentially amplify fields as weak as 10^{-20}G in a CDM dominated Universe with $\Lambda = 0$. The dynamo will operate for longer in a Universe with a non-vanishing cosmological constant and can therefore, amplify seeds as weak as 10^{-30}G to the current measured values [46]. Although the dynamo mechanism is strongly supported by the close correlation between the observed structure of the galactic fields and the spiral pattern of galaxies, there is some argument over its efficiency and the amount of amplification that can be achieved.

5.3.3 Origins of primordial seed magnetic fields

The major problem with these two mechanisms is that they assume the presence of a pre-existing seed field whose origin is still to be established. A further idea relies on turbulence (disrupted flow) and shocks, which occur during the stages of structure formation. This induces weaker magnetic fields via battery-type mechanisms, which operate as a result of large-scale misalignments of gradients in the electron number density and pressure (or temperature) [23, 60, 63].

While the generation of intergalactic magnetic fields in the ejected magnetoplasma surrounding galaxies can be argued within the boundaries of astrophysics as we understand them, the origins of primordial fields in regions devoid of baryonic matter requires a larger stretch of the imagination. There have been numerous attempts to generate early pre-recombination magnetic fields with strengths suitable to support and maintain the dynamo, by exploiting the different out-of-equilibrium epochs that are believed to have taken place between the end of the inflationary era and decoupling [54]. These fields are facilitated by currents that arise from local charge separation generated by vortical velocity fields prevalent in the early plasma (cf. also [9, 71]). One problem with the above mechanisms is that they are casual in nature so the scales over which the fields are coherent cannot exceed the particle horizon during that epoch. Given that such phase transitions took place at very early times where the comoving horizon size was small, tight constraints must be placed on the coherence length of these fields. However, pre-big bang models based on string theory [43], in which vacuum fluctuations of the magnetic field are amplified by the dilaton field, predict super-horizon fields. Inflation has long been suggested as a solution to the causality problem since it naturally achieves correlations on superhorizon scales. However adjustments to the standard inflationary models need to be made since magnetic fields surviving this epoch are small, on account of the inability of vector fields to couple gravitationally to the conformally flat metric resulting from the exponentially fast expansion. A way around this obstacle is by breaking the conformal invariance of electromagnetism since this alters the way the underlying gauge fields couple to gravity. There are many ways of doing this which explains the variety of the proposed mechanisms in the literature [26]. Though such inflationary scenarios have not been without critique [47].

To prevent the dilution of the strength of the field by the inflationary expansion, we need to break the gauge invariance of electromagnetism which demands the conservation of magnetic flux. In an attempt to destroy conformal invariance, alterations to the current gravitational couplings have been made such as the introduction of additional high-energy couplings of photons directly to gravity or a scalar field.

It has also been proposed that inflation is followed by a period of preheating in which the parametric resonance of the causal oscillations of the inflaton field and the accompanying perturbations can lead to amplification on super-horizon scales [5]. Other authors have advocated the breakdown of Lorentz invariance either in the context of string theory and non-commutative varying speed of light theories, or due to the dynamics of large extra dimensions [44, 8]. The success of these proposals however, is usually achieved at the expense of simplicity.

In order for these proposed mechanisms to be viable, they must in addition produce seed fields that satisfy the criteria for the subsequent amplification processes to work. To be a candidate seed field for the galactic dynamo, the induced field must exceed a minimum coherence scale in order to prevent the destabilization of the dynamo action. The time scale over which the amplification takes place also dictates a minimum field strength. For example, in the case of a dark-energy dominated Universe we obtain $B \sim 10^{-34}$ G on a coherence scale of 10 kpc. Davis *et al.* [24] proposed an inflationary mechanism that exploits the natural coupling between the Z-boson and the gravitational background. Unfortunately the produced fields only just fall within dynamo limits in the case of a dark energy dominated Universe. Recently the production of a magnetic seed field due to the rotational velocity of ions and electrons, caused by the nonlinear evolution of primordial density perturbations in the cosmic plasma during pre-recombination radiation and matter eras, was investigated in [73] and a rms amplitude $B \approx 10^{-23}(\lambda/\mathrm{Mpc})^{-2}$ G at recombination on comoving scales $\lambda \geq 1\,\mathrm{Mpc}$ was reported.

In this thesis an alternative mechanism is proposed that looks at the interaction of a *pre-existing* field, such as the one proposed by Davis *et al.*, with a gravitational wave (GW) spectrum. We aim to show using perturbation theory that this interaction can produce a sufficiently large amplification of a seed field present at the end of inflation, to easily meet the above mentioned requirements for the dynamo to work.

5.4 Approximation Schemes

The FLRW model introduced in section (4.9) describes a smooth unperturbed Universe and is therefore chosen as the background spacetime in most perturbation problems. Its high symmetry is however incompatible with the directional nature of EM fields and for this reason, perturbative treatments with EM fields existing in an FLRW background must be well argued. Because EM fields introduce additional variables, perturbation problems involving such fields are generally more complicated and for this reason are usually accompanied by simplifying assumptions. We review some common approximation schemes highlighting their treatment of magnetic fields.

When dealing with EM fields in an FLRW context the logical approach is to introduce them as perturbations of this background. This resolves the inconsistency between their vector nature and the properties of the background and ensures that the first-order EM variables are gauge-invariant. When the linearization procedure is carried out in the usual manner, we see that the energy density which enters the treatment at $\mathcal{O}(B^2)$ is removed from the gravitational propagation equations; Maxwell's and Einstein's equations become decoupled [20]. This perturbation scheme is useful in approaches such as a multi-fluid treatment of the cosmic plasma. In [71] Marklund and co-authors view the globally neutral plasma as two imperfect fluids relative to the fundamental frame. In each fluids' own rest frame the energy-momentum tensor given in terms of its rest frame velocity $u_a^{(i)}$ is that of a perfect fluid. The expression for the anisotropic stress of each component $\pi_{ab}^{(i)}$ is of order $\left(v^a_{(i)}\right)^2$ where $v^a_{(i)}$ is the relative velocity of the fluid and the observer. Since the isotropy of the background prohibits preferred direction, any perculiar velocities only enter at $\mathcal{O}(1)$. As a result the anisotropic stress contributions from the fluids are neglected to first-order. This approach is limiting as the effect of the EM field is only felt in the propagation equation of the relative velocity of the two fluids $v^a = \frac{1}{2}(v^a{}_{(1)} + v^a{}_{(2)})$. In this equation the electric field sources linear velocity perturbations.

However when studying effects occurring over much larger time scales and spanning longer length scales than those characteristic of plasma effects, a 2-species plasma can be accurately described using a single-component fluid model [70], often dubbed the magnetohydrodynamic (MHD) approximation. This handles low-frequency phenomena in a magnetized plasma from the perspective of fluid dynamics. A reduced description is achieved by defining appropriate one-fluid variables representing the bulk quantities. Ohm's law is often brought into MHD applications in order to determine the correct treatment of the associated electric field. It is common practise to take advantage of the high conductivity of the young cosmic plasma and employ the ideal MHD limit ($\sigma \to \infty$). In this limit the flux lines are effectively glued to the plasma elements which means that the associated electric fields become negligible. In the non-ideal MHD limit [70] where σ is assumed to large but not infinitely so, the electric field enters at linear order in the case of a first-order magnetic field. If the mean velocities are much smaller than the speed of light in a vacuum, then the Newtonian MHD approximation is implemented [46] where the fluid dynamics are described by Newtonian dynamics and the background expansion is given by the Friedmann equation. Given that the MHD approximation provides an accurate description of the cosmic medium on large scales where the geometry of the spacetime comes into play, it is commonly employed in the study of the evolution of large-scale magnetic fields in curved space-times. It is also suitable when studying phenomena stemming from turbulence [10]. The description of the MHD approximation is further developed in section (7.2) when it is employed in our treatment of an inhomogeneous magnetic field.

In other work [55, 64] where authors exclude the field from the background, the linear perturbations in the fluid energy density are assumed to be sourced *only* by the magnetic field[3]. Since $\mu_{em} \sim \mathcal{O}\left(B^2\right)$ is first-order, the field B^a is regarded as 'half-order'. The magnetic anisotropic stress can not be cancelled by that of the fluid as it is unperturbed. More importantly, $\pi_{ab} \sim \mathcal{O}\left(B^2\right)$ and is therefore regarded as first-order and retained upon linearization about the FLRW background. This approximation is thus advantageous when one wishes to examine the role of magnetic anisotropies in phenomena such as in the sourcing of gravitational waves. Even though the magnetic field is not $\mathcal{O}(1)$, it is still a perturbation of the background (although smaller) and thus vanishes in the background as required by gauge-invariant condition.

We now review treatments in which a primordial field is allowed to enter the background cosmology. Since the FLRW spacetime can not accommodate the anisotropies naturally induced by a magnetic field, Battaner and co-workers [6] fully randomize the field so that there is effectively no average field on sufficiently large scales. Although the magnetic energy density is still finite $< B^2 > \neq 0$, there is no preferred direction on cosmological scales.

Alternatively we can resolve this inconsistency by constraining its contribution to the total energy-momentum tensor to a sufficiently small fraction, so that it does not disturb the isotropy (i.e. $\pi_{ab} \equiv -\tilde{B}_{<a}\tilde{B}_{b>} = 0.$). This is referred to as the *weak field approximation* [96, 97, 99]. This is achieved if we assume that the energy density of the magnetic field is negligible compared with that of the cosmic fluid at recombination;

$$\tilde{B}^2 \ll \mu. \tag{5.10}$$

Although the upper limits on primordial magnetic fields resulting from measurements of the CMB anisotropy (stated in section (5.2.1)) support such weak initial conditions, there are various subtleties that demand attention. The weakness of the field may lead to negligible stress in background, however π_{ab} can not be set identically to zero; it is merely perturbatively smaller than the energy density of the fluid and can thus only be eliminated in comparison to μ. The treatment of the various magnetic terms becomes unclear; variables of the form $B_a B_b$ are regarded as being directional and assigned as first-order, whilst terms such as B^2 are viewed as energy density-like objects and thus zeroth-order despite them having the same dimensions.

Given that the magnetic field is so weak, second-order matter contributions that are dropped during the linearization process may be of comparable magnitude to the first-order magnetic field terms that are retained. It has been argued [99] that these higher-order terms have negligible influence on the evolution of the density fluctuations to $\mathcal{O}(1)$. One should thus take care when using this approximation in the analysis of other perturbations. The most problematic aspect of this approximation scheme is that the magnetic field resides in the background.

Since it is not a scalar, it does not therefore satisfy the conditions for gauge invariance as defined by the Stewart and Walker lemma.

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[3] In these papers, electric field is neglected on the basis of the assumption of high conductivity of the cosmic medium for the majority of its evolution.

The gauge dependance of B^a means that one can not guarantee that when calculating the part of the magnetic field produced by the coupling of the background field to linear perturbations of the FLRW background (such as gravitational waves), it will lead to physically meaningful results.

A suitable approach that deals with a general magnetic field and does not encounter the GI problem stems from recent work of Clarkson *et al.* [21, 20]. The seed magnetic field is treated as an *on average* homogeneous linear perturbation of the background FLRW model and thus satisfies the Stewart and Walker definition of a GI variable. The couplings to gravitational degrees of freedom that arise when perturbing the background, are taken to be second-order in the perturbative scheme. We use this methodology to develop a self-consistent framework based on second-order perturbation theory, that facilitates rewriting Maxwell's equations in a manner that makes them manifestly gauge-invariant to second-order, with interaction terms that clearly describe the modes induced by the gravity wave-magnetic field interaction. We compare our results with the solutions recovered using the WFA in order to identify discrepancies stemming from the differences in the approaches.

Chapter 6

The homogeneous magnetic field case

The initial restriction to a homogeneous seed field leads to simplification on the technical level, but still encapsulates the main features of the gravito-magnetic interaction. We investigate this coupling with an inhomogeneous seed in chapter (7).

6.1 Perturbation scheme

If we wish to study the interaction between gravitational waves and a magnetic field in a cosmological setting using the approach described in section (5.4), we immediately face a second-order problem in perturbation theory because both the magnetic field as well as GW are absent in the exact FLRW background, and may thus be individually regarded as first-order perturbations. It is convenient to introduce infinitesimal smallness parameters to expose the physical meaning of the perturbations [76]. Using the 1+3 covariant approach [33, 35, 41, 51], we develop a two parameter expansion in two smallness parameters: ϵ_B represents the magnitude of a homogeneous magnetic field and ϵ_g represents the magnitude of the GW. The magnitude of the interaction GW × magnetic field is of order $\mathcal{O}(\epsilon_B\epsilon_g)$ as is the magnitude of the electromagnetic fields generated in such a manner. However, at second-order only terms of order $\mathcal{O}(\epsilon_B\epsilon_g)$ are kept while terms of order $\mathcal{O}(\epsilon_g^2)$ and $\mathcal{O}(\epsilon_B^2)$ are discarded. The latter enter primarily via the Einstein equations and appear as products with first-order variables. These terms are then third order and are dropped accordingly.

The perturbation space-times are divided up and denoted in the following way:

- $\mathcal{B}$ = Exact FLRW as background spacetime, $\mathcal{O}(\epsilon^0)$;
- $\mathcal{F}_1$ = FLRW perturbed by a homogeneous magnetic field whose energy density and curvature are neglected, $\mathcal{O}(\epsilon_B)$;
- $\mathcal{F}_2$ = FLRW with gravitational perturbations $\mathcal{O}(\epsilon_g)$;
- $\mathcal{S} = \mathcal{F}_1 + \mathcal{F}_2$ allows for inclusion of interactions terms of order $\mathcal{O}(\epsilon_B\epsilon_g)$.

We will generally refer to terms of order $\mathcal{O}(\epsilon_B)$ and $\mathcal{O}(\epsilon_g)$ appearing in $\mathcal{F}$ as 'first-order' and to variables of mixed order $\mathcal{O}(\epsilon_B\epsilon_g)$ appearing in $\mathcal{S}$ as 'second-order'.

We will now work in the above defined spacetime $\mathcal{S}$. It must be noted that the absence of an electric field in $\mathcal{F}_1$ and $\mathcal{S}$ does not necessarily imply that there is no electric field at all, but rather that the electric field is perturbatively smaller than the magnetic field (i.e. E^a is of the order $\mathcal{O}(\epsilon_B\epsilon_B)$). This is congruent with the standard assumption that the very early Universe was a good conductor (again see, for example, [7] for a calculation). The inclusion of an electric field in $\mathcal{F}_1$ is possible in principle but would require us to alter the perturbation scheme because the interactions between gravitational waves and the electric field would also then need to be taken into account[1].

6.2 FLRW background

The FLRW models are characterized by a perfect fluid matter tensor and the condition of everywhere-isotropy. Applying the restrictions given in section (4.9) which characterize this spacetime, the key background equations are the energy conservation equation (4.46)

$$\dot{\mu} + \Theta(\mu + p) = 0, \tag{6.1}$$

the Raychaudhuri equation (4.75)

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 + \tfrac{1}{2}(\mu + 3p) + \Lambda \tag{6.2}$$

and the Friedmann equation (4.106)

$$\mu + \Lambda = \frac{1}{3}\Theta^2 + \frac{3K}{a^2}, \tag{6.3}$$

where the constant $K = 0,\ 1,\ -1$ indicates the geometry of the spatial sections.

6.3 First-order perturbations

6.3.1 The homogeneous magnetic field $\tilde{B}_a$

We assume the magnetic field $\tilde{B}^a$ to be spatially homogeneous at first-order ($\mathrm{D}_a\tilde{B}_b = 0$) and regard the gradient of $\tilde{B}^a$ as well as the magnetic anisotropy $\pi_{ab} = -\tilde{B}_{<a}\tilde{B}_{b>}$ as being of second-order. We presuppose that such a field was produced by some primordial process, which left a relic field on average homogeneous over a typical coherence length. Since there are no electric fields or charges in the $\mathcal{F}_1$ spacetime, the magnetic induction equation takes the form

$$\dot{\tilde{B}}_{<a>} + \tfrac{2}{3}\Theta\tilde{B}_a = 0. \tag{6.4}$$

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[1] A more realistic way of describing the interaction between gravitational waves and electromagnetic fields would be to employ a multi-fluid description [9, 71] as discussed earlier, which allows for modelling the currents. This is beyond the scope of this investigation.

As a result, the magnetic field scales as

$$\tilde{B}_a = \tilde{B}_a^0 \left(\frac{a_0}{a}\right)^2 , \tag{6.5}$$

where a denotes the scale factor, e.g., $\Theta = 3\,\dot{a}/a = 3H$ and where H denotes the inverse Hubble length. The adiabatic decay evident in equation (6.5) arises from the expansion of the Universe which conformally dilutes the field lines due to flux conservation [55].

6.3.2 Gravitational waves

In the covariant approach to cosmology, linearized gravitational waves are purely tensorial and are monitored via the electric (E_{ab}) and magnetic (H_{ab}) Weyl constituents, which are not sourced by rotational (vector) and density (scalar) perturbations [17, 29]. The transverse nature of these PSTF tensors means that we only need to eliminate their vector parts in order for them to characterize frame-invariant GW. We isolate the linear tensorial modes by imposing the constraints

$$0 = \mathrm{D}_a\mu = \mathrm{D}_a p = \mathrm{D}_a\Theta = \omega_a = \dot{u}_a . \tag{6.6}$$

These restrictions ensure that the sources of vector modes (spatial gradients and vector perturbations themselves) vanish and lead to the constraints

$$0 = \mathrm{D}^a\sigma_{ab} = \mathrm{D}^a E_{ab} = \mathrm{D}^a H_{ab} = H_{ab} - \mathsf{curl}\,\sigma_{ab} . \tag{6.7}$$

Since the shear tensor is coupled to H_{ab} and E_{ab}, it can also be used as a measure of gravitational waves. Since the magnetic part of the Weyl tensor can be expressed in terms of the shear as shown in equation (4.82), the propagation of these tensor modes are completely governed by the following equations

$$\dot{\sigma}_{<ab>} + \tfrac{2}{3}\Theta\sigma_{ab} = -E_{ab}, \tag{6.8}$$

$$\dot{E}_{<ab>} + \Theta E_{ab} = \mathsf{curl}\,(\mathsf{curl}\,\sigma_{ab}) - \tfrac{1}{2}(\mu + p)\,\sigma_{ab}, \tag{6.9}$$

together with the background equations for Θ and μ. Since every FOGI tensor satisfies the linearized identity[2]

$$\mathsf{curl}\,(\mathsf{curl}\,T_{ab}) = -\mathrm{D}^2 T_{ab} + \tfrac{3}{2}\mathrm{D}_{<a}\mathrm{D}^c T_{b>c} + \left(\mu + \Lambda - \tfrac{1}{3}\Theta^2\right) T_{ab}, \tag{6.10}$$

we note that the wave equation for the shear is closed

$$\ddot{\sigma}_{ab} - \mathrm{D}^2\sigma_{ab} + \tfrac{5}{3}\Theta\dot{\sigma}_{ab} + \left(\tfrac{1}{9}\Theta^2 + \tfrac{1}{6}\mu - \tfrac{3}{2}p + \tfrac{5}{3}\Lambda\right)\sigma_{ab} = 0 \ . \tag{6.11}$$

We see that the gravitational waves are completely determined by shear evolution and the $\sigma - B$ terms will thus provide a suitable description of the interaction.

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[2] The covariant spatial Laplacian is given by $\mathrm{D}^2 = \mathrm{D}^a\mathrm{D}_a$.

6.4 Second-order perturbations: The interaction

The evolution of the magnetic field is governed by Maxwell's equations. If we require charge neutrality and neglect currents as well as the back-reaction of the induced second-order magnetic fields with the shear, Maxwell's equations take on the form:

$$\begin{aligned} \dot{E}_{<a>} + \tfrac{2}{3}\Theta E_a &= \mathsf{curl}\, B_a, & (6.12)\\ \dot{B}_{<a>} + \tfrac{2}{3}\Theta B_a &= \sigma_{ab}\tilde{B}^b - \mathsf{curl}\, E_a, & (6.13)\\ D^a E_a &= 0, & (6.14)\\ D^a B_a &= 0. & (6.15) \end{aligned}$$

Observe that the EM fields have to be divergence-free at all orders (since we disregard vorticity effects) and are thus purely vector perturbations on the background. In equation (6.13) the magnetic field-shear term ($\sigma_{ab}\tilde{B}^b$) is identified as the variable describing their interaction.

The current form of Maxwell's equations is problematic because it comprises of first-order (σ_{ab}) and second-order (E_a, $\mathsf{curl}\, E_a$, $\mathsf{curl}\, B_a$) variables. Furthermore, these equations are not second-order in the usual perturbative sense; the magnetic fields on the LHS and RHS of Maxwell's equations are not the same. In the $\mathcal{S}$ spacetime, the magnetic field that appears on the LHS is the combined first- and second-order magnetic fields and has a non-zero contribution from the linear field in $\mathcal{F}_1$. There are derivative operators of $\mathcal{O}(1)$ acting on the first-order field that are additionally retained but the solution to the linear equation (6.4) does not indicate which modes they generate. As a result equation (6.13) cannot be integrated [20]. Unlike metric-based approaches the solutions for perturbative operators are never sought and they must therefore act on quantities of the same perturbative order [21], which is clearly not the case in equation (6.13).

In order to isolate the purely second-order equations, we need to somehow eliminate the contribution at the $\mathcal{F}_1$ level. In Special Relativity the standard procedure to disentangle the different magnetic field perturbations in a consistent way, involves using a power series expansion of the magnetic field

$$B^a = \epsilon_B B_1^a + \epsilon_g \epsilon_B B_2^a + \mathcal{O}\left(\epsilon_g^2, \epsilon_B^2\right). \tag{6.16}$$

If this expansion is then substituted for B_a in Maxwell's equations in $\mathcal{S}$, the B_1^a part which satisfies the magnetic induction equation (6.4) in $\mathcal{F}_1$, cancels out leaving only the B_2^a component. Moreover, when the full field B^a is multiplied by an $\mathcal{F}_2$ variable only the B_1^a contribution is retained. This expansion works provided the derivatives are partial derivatives, however in this case where frame derivatives are involved, the commutator relations for the various differential operators (cf. the appendix) cannot be consistently satisfied. To illustrate this important point more clearly we consider the commutation relation between the (proper) time derivative and the spatial gradient applied to the magnetic field. If the commutator relation is introduced after the expansion of B^a we obtain

$$\left(D^b B^a\right)^{\cdot}_{\perp} = (\epsilon_B D^b B_1^a)^{\cdot} + (\epsilon_g \epsilon_B D^b B_2^a)^{\cdot}_{\perp}. \tag{6.17}$$

Since the first-order part of the field B^a_1 is free from spatial gradients, we eliminate $D^bB^a_1$ and arrive at

$$\begin{aligned} &= \epsilon_g\epsilon_B\left(D^bB^a_2\right)^{\cdot}_{\perp} \\ &= \epsilon_g\epsilon_B\left[D^b\dot{B}^a_2 - \tfrac{1}{3}\Theta D^bB^a_2\right], \qquad (6.18) \end{aligned}$$

where only the second-order field contributions are kept. This does not agree with the case where the linearized identity for $(D^aB^b)^{\cdot}$ is substituted before using the power series expansion (6.16)

$$\begin{aligned} \left(D^bB^a\right)^{\cdot}_{\perp} &= D^b\dot{B}^a - \tfrac{1}{3}\Theta D^bB^a + H^{bd}\epsilon_{dac}B^c + \sigma^d{}_cD^cB_a \\ &= \epsilon_BD^b\left[\dot{B}^a_1 - \tfrac{1}{3}\Theta B^a_1\right] + \epsilon_g\epsilon_B\left[D^b\dot{B}^a_2 - \tfrac{1}{3}\Theta D^bB^a_2\right] + H^{bd}\epsilon^a{}_{cd}B^c_1 + \sigma^d{}_cD^cB^a_1 \\ &= \epsilon_g\epsilon_B\left[D^b\dot{B}^a_2 - \tfrac{1}{3}\Theta D^bB^a_2\right] + \epsilon_BH^b{}_d\epsilon^{dac}B^1_c, \qquad (6.19) \end{aligned}$$

as the first-order contribution B^a_1 appears in a term that is retained. Here $\perp$ denotes projection onto the fundamental observer's rest space. The discrepancy arises from the interaction between the magnetic Weyl tensor and the magnetic field and illustrates how the commutator relations facilitate the coupling of Maxwell's equations and the curvature of spacetime via frame derivatives [20]. The inconsistency can only be resolved if the interaction terms are eliminated which will render the exercise pointless. For this reason we do not use this power series expansion.

The problem stems from the fact that B^a is not gauge-invariant. We recall the definition of second-order gauge-invariant (SOGI) variables given in section (3.3). We see that B^a has a non-zero contribution in $\mathcal{F}_1$ (B^a_1 still contributes at this perturbative level) and does not therefore vanish at first *and* second-order as required for GI.

So the need arises to define a new variable for the magnetic field that vanishes at all lower orders and satisfactorily describes the effects that we wish to investigate. Examining Maxwell's equation (6.13) reveals that $\beta_a \equiv \dot{B}_{<a>} + \frac{2}{3}\Theta B_a$ is the sought SOGI variable which must be used at second-order instead of the magnetic field B^a. When the magnetic induction equation (6.4) is applied to β^a, this new variable comprises solely of the second-order magnetic field which vanishes in both the background $\mathcal{S}$ *and* $\mathcal{F}_1$. The new variable is truly second-order and satisfies the criterion for gauge invariance. We choose to describe the interaction in terms of the variable $I_a \equiv \sigma_{ab}\tilde{B}^b$. Maxwell's equations can now be rewritten in gauge-invariant terms at second-order, namely

$$\begin{aligned} \dot{E}_{<a>} + \tfrac{2}{3}\Theta E_a &= \mathsf{curl}\, B_a, \qquad (6.20) \\ \beta_a + \mathsf{curl}\, E_a &= I_a. \qquad (6.21) \end{aligned}$$

Note that the term all contributions to the $\mathsf{curl}\, B_a$ term are at least second-order and thus satisfy the GI condition. As previously mentioned the re-definition of all multiples of first-order quantities into new SOGIs means that the system appears in the form of a set of linear differential equations. The usual zeroth-order harmonic functions can then be used to remove the tensorial nature of the SOGI equations as well as the spatial gradients, thereby converting the system into *ordinary* differential equations which are easily integrated.

Since the SOGI β^a is constructed from pure vector modes, we expect the divergence of β^a to be 0. Applying the standard constraints $0 = \mathrm{D}^a B_a = \mathrm{D}^a E_a$ to $D^a\beta_a$ yields

$$\mathrm{D}^a\beta_a = \mathrm{D}^a I_a. \tag{6.22}$$

The latter is equivalent to the expression $\sigma_{ab}\mathrm{D}^a\tilde{B}^b$ which automatically vanishes since spatial gradients are regarded as second-order. This illustrates consistency and we can infer that the interaction variable has no scalar modes.

If the medium is assumed to be infinitely conducting, all electric fields dissipate (or manifest only at third order level or higher) and Maxwell's equation (6.21) reduce to $\beta^a = I^a$. In this case, once the solution for I^a is known, the gauge-dependant generated magnetic field measured by the fundamental observer can be obtained via a standard integration of β^a. It is important to stress that β^a is the fundamental variable whose deviation from zero quantifies the evolution of the magnetic field at second-order in a truly gauge-invariant manner.

6.5 Consistency of the approximation schemes

When implementing a linearization scheme, it is necessary to check that the constraint equations defined in section (4.8.2) are preserved throughout the evolution along the fluid flow lines. This involves finding the time derivative of each constraint and checking that it is identically satisfied as a result of the accompanying propagation equations defined in section (4.8.1) being true. If this demonstrated for all constraints, we can then infer that provided a set of initial data is found that satisfies the initial conditions (given by the set of constraint equations) this set of equations will hold at a later time [11]. The procedure is as follows; we define a variable whose elements are the constraint equations for the system

$$C^A = (C^1, C^2, ...) = 0. \tag{6.23}$$

The next step is to check that

$$\dot{C}^A = F^A(C^B), \tag{6.24}$$

where F^A does not contain any time derivatives and C^B is a linear combination of the elements of C^A. If the constraints are satisfied on the initial surface (i.e. $C^A = 0$ at $t = t_0$) then (6.24) implies that they are satisfied for all time. If an auxiliary constraint not constituting part of the initial set, such as C^i, is contained in C^B, then this process must be repeated for this new equation C^i (or new set C^B) until we arrive at an identically satisfied set. If too many non-trivial constraints C^j arise the scheme is said to be inconsistent [11].

Since our perturbative scheme places additional restrictions on these equations, it is crucial to check that the set is still consistent up to second-order under these conditions. We will also check the evolution of the linear constraints in Tsagas *et. al* [98]. In both cases the calculations will be performed for a flat irrotational Universe with zero cosmological constant given that these assumptions are made when calculating the final results in both treatments. Note that the absence of an electric field at first-order in both investigations removes the energy-flux q_a to linear order. At second-order $q_a \sim \mathcal{O}(\epsilon_B^2)$ and is neglected in accordance with our perturbative scheme. The above mentioned conditions, in conjunction with the restrictions (6.6) required to isolate the GW modes, reduce the constraint equations in both investigations to

$$\begin{aligned} C_1 &= \mathrm{D}^b\sigma_{ab} = 0, \\ C_2 &= H_{ab} - (\mathrm{curl}\,\sigma)_{ab} = 0, \\ C_3 &= \mathrm{D}^b E_{ab} = 0, \\ C_4 &= D^b H_{ab} = 0. \end{aligned}$$

The equations for the divergence and propagation of the vorticity (4.81) and (4.76) are trivially satisfied (i.e. $0 = 0$) and do not need to be checked. Maxwell's equations give rise to 2 constraints:

$$\begin{aligned} C_5 &= \mathrm{D}^a B_a = 0, \\ C_6 &= \mathrm{D}^a E_a = 0. \end{aligned} \tag{6.25}$$

An additional restriction

$$\mathrm{D}_a B^2 = 0 = \epsilon_{abc} B^b \mathrm{curl}\, B^c \tag{6.26}$$

is required to remove the magnetized scalar and vector modes in the WFA [98]. This removes the rotational parts of the magnetic anisotropic stress [69];

$$C_7 = \mathrm{D}^b \pi_{ab} = 0. \tag{6.27}$$

This constraint is unnecessary in our treatment as it enters at $\mathcal{O}(\epsilon_B^2)$ level and is consequently dropped.

6.5.1 Consistency of the Weak field approximation

The key assumption of the weak field approximation [96, 97, 99] is

$$\frac{B^2}{\mu} \ll 1. \tag{6.28}$$

The energy density of the magnetic field is $\mu_B = \frac{B^2}{2}$ and assuming a barotropic equation of state, the magnetic pressure is $p_B = w\mu_B$. Using (6.28)

$$h = \mu_T + p_T = \left(\mu + \tfrac{B^2}{2}\right)(1 + w) = \mu(1 + w) \tag{6.29}$$

and

$$\mu_T + 3p_T = \left(\mu + \tfrac{B^2}{2}\right)(1 + 3w) = \mu(1 + 3w), \tag{6.30}$$

where μ is the energy density of the cosmic fluid. The field is sufficiently weak such that we can write

$$\mu_T = \mu \tag{6.31}$$

and

$$\dot{\mu}_T = -\Theta h = -\Theta\mu(1+w). \tag{6.32}$$

Constraints $(C_1 - C_6)$ are found to be consistent:

$$\begin{aligned}
\dot{C}_1 &= -\Theta\left(C_1 - C_3 + \tfrac{1}{2}C_7\right) = 0,\\
\dot{C}_2 &= -\Theta C_2 = 0,\\
\dot{C}_3 &= -\Theta C_3 = 0,\\
\dot{C}_4 &= -\tfrac{4}{3}\Theta C_4 = 0,\\
\dot{C}_5 &= -\Theta C_5 = 0,\\
\dot{C}_6 &= -\Theta C_6 = 0.
\end{aligned} \tag{6.33}$$

Invoking the linear evolution equation for the anisotropic stress in [99]

$$\dot{\pi}_{ab} = \tfrac{4}{3}\Theta\pi_{ab} - \tfrac{2}{3}B^2\sigma_{ab}, \tag{6.34}$$

we obtain

$$\dot{C}_7 = -\frac{5}{3}\Theta C_7 = 0, \tag{6.35}$$

which indicates that constraint C_7 holds through time.

6.5.2 Consistency of our scheme

All constraints $(C_1) - (C_6)$ are found to be consistent to second-order and evolve with time via the same equations as shown in (6.33).

6.6 Wave equations for the main variables

Having written the key Maxwell's equations as a system of differential equations of purely SOGI variables, we now turn to the derivation of wave equations for the electric and magnetic fields. We again make no assumptions about the spatial geometry or the equation of state and also retain the cosmological constant. This has the advantage of allowing us to draw some conclusions about how these parameters influence the interaction between GW and magnetic fields. In particular, it will turn out that neglecting the current in Maxwell's equations and simultaneously requiring a homogeneous magnetic field at first-order level, leads to consistent equations only in spatially flat models.

6.6.1 Wave equation for the interaction variable

Let us first derive the wave equation for the interaction variable $I_a = \sigma_{ab}\tilde{B}^b$. Even though the shear σ_{ab} belongs to $\mathcal{F}_2$ and the magnetic field $\tilde{B}^a$ to $\mathcal{F}_1$, the commutator relations do not lead to ambiguities for I^a since they manifest themselves only at third-order in this case. Using equations (6.4) and (6.8), we obtain

$$\begin{aligned} \dot{I}_{<a>} &= \dot{\sigma}_{ab}\tilde{B}^b + \sigma_{ab}\dot{\tilde{B}}^b, \\ &= -\tfrac{4}{3}\Theta\sigma_{ab}\tilde{B}^b - E_{ab}\tilde{B}^b. \end{aligned} \tag{6.36}$$

We identify the auxiliary quantity $J_a \equiv E_{ab}\tilde{B}^b$. To find the wave equation for I^a, we need the propagation equation of this new SOGI. Bringing in equations (6.4) and (6.9) we obtain

$$\begin{aligned} \dot{J}_{<a>} &= \dot{\tilde{B}}^b E_{ab} + \tilde{B}^b \dot{E}_{ab} \\ &= \left(-\tfrac{2}{3}\Theta\tilde{B}^b\right) E_{ab} + \tilde{B}^b\left(-\Theta E_{ab} + \mathsf{curl}\,(\mathsf{curl}\,\sigma_{ab}) - \tfrac{1}{2}(\mu+p)\sigma_{ab}\right) \\ &= -\tfrac{5}{3}\Theta E_{ab}\tilde{B}^b + \mathsf{curl}\,(\mathsf{curl}\,\sigma_{ab})\tilde{B}^b - \tfrac{1}{2}\,(\mu+p)\,\sigma_{ab}\tilde{B}^b. \end{aligned} \tag{6.37}$$

Since $\mathsf{curl}\,\tilde{B}^b$ is at least second-order, the above term simplifies to

$$\mathsf{curl}\,(\mathsf{curl}\,\sigma_{ab})\tilde{B}^b = \mathsf{curl}\,\mathsf{curl}\,(I_a) - \sigma_{ab}\mathsf{curl}\,(\mathsf{curl}\,\tilde{B}^b) = \mathsf{curl}\,\mathsf{curl}\,(I_a). \tag{6.38}$$

Using equation (6.10) we obtain

$$\dot{J}_{<a>} + \tfrac{5}{3}\Theta J_a = -\mathrm{D}^2 I_a + \left[\tfrac{1}{2}\,(\mu-p) + \Lambda - \tfrac{1}{3}\Theta^2\right] I_a. \tag{6.39}$$

Eliminating the auxiliary variable J^a, the general closed wave equation for I^a is found to be

$$\ddot{I}_{<a>} - \mathrm{D}^2 I_a + 3\Theta\dot{I}_{<a>} + \left[\tfrac{13}{9}\Theta^2 - \tfrac{1}{6}\mu - \tfrac{5}{2}p + \tfrac{7}{3}\Lambda\right] I_a = 0. \tag{6.40}$$

In the case of infinite conductivity, the solution to equation (6.40) instantly yields the solution of β^a from which the induced magnetic field measured by the fundamental observer may be obtained by integration.

The remaining wave equations are found in the same manner shown above, however their explicit derivations will not be shown.

6.6.2 Wave equation for the electric field

To derive the wave equation for the induced electric field, we first differentiate equation (6.20) and equate the result with the second-order identity

$$(\mathsf{curl}\,B_a)^{\cdot}_{\perp} = -\Theta\,\mathsf{curl}\,B_a + \mathsf{curl}\,\beta_a - H_{ab}\tilde{B}^b \tag{6.41}$$

to obtain

$$\ddot{E}_{<a>} + \tfrac{5}{3}\Theta\dot{E}_{<a>} + \left[\tfrac{4}{9}\Theta^2 - \tfrac{1}{3}\,(\mu+3p) + \tfrac{2}{3}\Lambda\right] E_a = \mathsf{curl}\,\beta_a - H_{ab}\tilde{B}^b. \tag{6.42}$$

Secondly, using equation (6.21) to substitute for $\mathsf{curl}\,\beta_a$ above and the expansion

$$\mathsf{curl}\,\,(\mathsf{curl}\,E_a) = -\mathrm{D}^2 E_a - \left[\tfrac{2}{9}\Theta^2 - \tfrac{2}{3}\,(\mu+\Lambda)\right] E_a, \tag{6.43}$$

we find a forced wave equation for the induced electric field, namely

$$\ddot{E}_{<a>} - \mathrm{D}^2 E_a + \tfrac{5}{3}\Theta \dot{E}_{<a>} + \left[\tfrac{2}{9}\Theta^2 + \tfrac{1}{3}(\mu - 3p) + \tfrac{4}{3}\Lambda\right] E_a = K_a, \tag{6.44}$$

where the forcing term $K_a \equiv \mathrm{curl}\, I_a - H_{ab}\tilde{B}^b = \epsilon_{cd[a}\mathrm{D}\,\sigma_{b]}{}^c B^b$ has no divergence. It is possible to show that the forcing term K^a as well as $\mathrm{curl}\, I_a$ and $H_{ab}\tilde{B}^b$ respectively, can be found from the wave equation

$$\ddot{K}_{<a>} - \mathrm{D}^2 K_a + \tfrac{11}{3}\Theta \dot{K}_{<a>} + \left[\tfrac{22}{9}\Theta^2 - \tfrac{1}{3}(\mu + 9p) + \tfrac{8}{3}\Lambda\right] K_a = 0. \tag{6.45}$$

For example, the wave equation for $\mathrm{curl}\, I_a$ follows by taking the curl of equation (6.40) and using the expansion (6.43), while the case $H_{ab}\tilde{B}^b$ is similar to the derivation of the wave equation for the interaction term I^a.

It will be useful for later purposes to consider the electric field's rotation. By taking the curl of equation (6.44), we immediately arrive at

$$\begin{aligned} \ddot{(\mathrm{curl}\, E_a)}_\perp \quad &- \quad \mathrm{D}^2 (\mathrm{curl}\, E_a) + \tfrac{7}{3}\Theta\, \dot{(\mathrm{curl}\, E_a)}_\perp \\ &+ \quad \left[\tfrac{7}{9}\Theta^2 + \tfrac{1}{6}(\mu - 9p) + \tfrac{5}{3}\Lambda\right] \mathrm{curl}\, E_a = \mathrm{curl}\, K_a. \end{aligned} \tag{6.46}$$

Because $\mathrm{curl}\left(H_{ab}\tilde{B}^b\right) = -\mathrm{D}^2 I_a + \left[-\tfrac{5}{18}\Theta^2 + \tfrac{5}{6}(\mu + \Lambda)\right] I_a$ holds we note the interesting result

$$\mathrm{curl}\, K_a = \left[\tfrac{1}{18}\Theta^2 - \tfrac{1}{6}(\mu + \Lambda)\right] I_a. \tag{6.47}$$

From the above we see that $\mathrm{curl}\, K_a = 0$ for a cosmological model with flat spatial sections. We can infer that the electric field's rotation is not induced by the interaction between magnetic fields and GW at second-order level; the generated electric field is curl-free. Consequently the interaction between magnetic fields and GWs in a spatially flat Universe produces the same magnetic field as in the limit of high conductivity.

Upon closer inspection of the forcing term K^a in equation (6.44) one discovers that this term is actually identically zero as a result of the identity [101]

$$0 = \epsilon^{abc}\, V_b \left(\mathrm{D}_d\, A_c{}^d\right) - 2\, V_b\, \epsilon^{cd[a} \left(\mathrm{D}_c\, A^{b]}{}_d\right), \tag{6.48}$$

which holds for any vector V_a and tensor $A_{ab} = A_{<ab>}$ perpendicular to the congruence u^a. Equation (6.47) thus implies that our chosen perturbative scheme is only consistent if the cosmological model is spatially flat[3]. We therefore see that the requirement of having a spatially homogeneous and thus curl-free magnetic field at first-order can be achieved only when the Universe is spatially flat.

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[3] In light of the commutator relation (6.43), which holds for $\tilde{B}^a$ in $\mathcal{F}_1$, the simultaneous requirement of $\mathrm{D}_a\tilde{B}_b = 0$ and $\mathrm{curl}\, \tilde{B}_a = 0$ is only consistent for a spatially flat Universe – in an open or closed Universe, a current is needed to uphold the magnetic field's homogeneity.

6.6.3 The generated magnetic field

We found that the wave equations for the electric field and its rotation are unforced. Since they are introduced only at second-order, we can infer that they will remain perturbatively smaller than the magnetic field and can therefore be neglected. The generated magnetic field now follows directly from the interaction variable since in this case we have $\beta^a = I^a$. For closed or open models however, a wave equation for β^a is needed to determine the induced magnetic field. The sought-after equation may be obtained by adopting the constraint equation (6.21) to equation (6.46) and substituting for $\mathrm{curl}\, K_a$ via equation (6.47), which leads to

$$\begin{aligned}\ddot{\beta}_{<a>} - \mathrm{D}^2\beta_a + \tfrac{7}{3}\Theta\dot{\beta}_{<a>} + \left[\tfrac{7}{9}\Theta^2 + \tfrac{1}{6}(\mu - 9p) + \tfrac{5}{3}\Lambda\right]\beta_a &= \\ \ddot{I}_{<a>} - \mathrm{D}^2 I_a + \tfrac{7}{3}\Theta\dot{I}_{<a>} + \left[\tfrac{13}{18}\Theta^2 + \tfrac{1}{3}\mu - \tfrac{3}{2}p + \tfrac{11}{6}\Lambda\right] I_a. \qquad (6.49)\end{aligned}$$

Observe that for models with flat spatial sections, the LHS and RHS of the above equation become identical, in agreement with the comment following equation (6.46). A slight simplification is achieved by employing equation (6.40) yielding a forced wave equation for β_a:

$$\begin{aligned}\ddot{\beta}_{<a>} - \mathrm{D}^2\beta_a + \tfrac{7}{3}\Theta\dot{\beta}_{<a>} &+ \left[\tfrac{7}{9}\Theta^2 + \tfrac{1}{6}(\mu - 9p) + \tfrac{5}{3}\Lambda\right]\beta_a \\ &= -\tfrac{2}{3}\Theta\dot{I}_{<a>} - \left[\tfrac{13}{18}\Theta^2 - \tfrac{1}{2}(\mu + 2p - \Lambda)\right] I_a. \qquad (6.50)\end{aligned}$$

It is evident that the variable I^a and hence gravitational waves source fluctuations in the magnetic field variable β^a. Another way to derive equation (6.50) is by differentiating Maxwell's equation (6.21) twice, using equation (6.20) to remove the $\mathrm{curl}\, E_a$-term and applying the corresponding commutation relations. This demonstrates the consistency of our approximation scheme.

6.7 Solutions for flat Universes

After having derived the fundamental equations governing the interaction between GWs and magnetic fields as well as the generated electromagnetic fields, we turn to the task of solving them. For the sake of simplicity we investigate the solutions for spatially flat models with zero cosmological constant Λ only. We assume the matter to obey a barotropic equation of state, $p = w\mu$, with constant barotropic index w.

A unified time variable

The background equations (6.1)-(6.3), together with the assumptions stated above, imply the following evolution equation for the scale factor

$$\frac{\ddot{a}}{a} + \frac{1}{2}(1 + 3w)\left(\frac{\dot{a}}{a}\right)^2 = 0. \qquad (6.51)$$

By choosing initial conditions such that $\Theta_0 \equiv \Theta(t_0) = 3H_0$ for some arbitrary initial time t_0 with the Hubble radius $H = \dot{a}/a$ and intergrating once, we obtain the following relation for the expansion

$$\frac{1}{3}\Theta = \frac{\dot{a}}{a} = \frac{2}{3(1+w)(t-t_0)+2/H_0}. \tag{6.52}$$

Integrating once more we find for the scale factor the solution

$$a(t) = a_0\left[\tfrac{3}{2}H_0\,(1+w)\,(t-t_0)+1\right]^{\frac{2}{3(1+w)}}. \tag{6.53}$$

The introduction of a dimensionless time variable τ, defined as

$$\tau \equiv \tfrac{3}{2}H_0\,(1+w)\,(t-t_0)+1, \tag{6.54}$$

turns out to be useful. The new time variable allows for a convenient integration of almost all equations to be considered later, irrespective of the barotropic index and taking the initial conditions *explicitly* into account as well. For example, the scale factor evolves simply as $a = a_0\,\tau^{2/(3(1+w))}$ and the Hubble radius as $H = H_0/\tau$ where the arbitrary initial time t_0 is now $\tau = 1$.

Harmonic Decomposition

Since we consider Universes with *flat* spatial geometries only, the induced magnetic field can be found by integrating β^a. To this end it suffices to solve for the interaction variable I^a whose wave equation is given in equation (6.40).

A harmonic decomposition [11, 31] is used to take care of the Laplacian operator. It is standard procedure to assume that the time and spatial dependance of each variable is separable such that the variable can be expressed as the product of the time part and the spatial part. This operation effectively decomposes the differential equation for the time variation of an entire perturbation into separate equations describing the time variation of each component of the spatial part of the perturbation each characterized by a comoving wavenumber. Since any disturbance in a quantity can be expressed as the superposition of normal modes, we can decompose the spatial part into the summation over a series of harmonics Q which are covariantly constant $\dot{Q}^{(k)} = 0$ and are conveniently the eigenfunctions of the Laplace-Beltrami operator

$$\mathrm{D}^2 Q = -\frac{k^2}{a^2}Q. \tag{6.55}$$

Using these harmonics we expand arbitrary scalars, vectors and tensors in the following way

$$S = \sum_k \left(S_{(k)}Q_{(k)}\right), \tag{6.56}$$

$$V_a = \sum_k \left(\frac{1}{k}V_{(k)}Q^s_a + V_{(k)}Q^v_a\right), \tag{6.57}$$

$$T_{ab} = \sum_k \left(\frac{1}{k^2}T_{(k)}Q^s_{ab} + \frac{1}{k}T_{(k)}Q^v_{ab} + T_{(k)}Q^T_{ab}\right), \tag{6.58}$$

where $\mathrm{D}_a S_{(k)} = \mathrm{D}_a V_{(k)} = \mathrm{D}_a T_{(k)} = 0$ (i.e. timelike).

The use of harmonics is advantageous because the comoving wavenumber k characterizing each particular spatial variation encompasses the idea of comoving wavenumber k for the inhomogeneities [52]. We can then define a comoving scale $\lambda = \frac{2\pi a}{k}$ for each perturbation. In our application, the harmonic decomposition is particularly useful as it allows us to distinguish the specific situation where the wavelengths of the perturbations are much larger than the Hubble scale ($\frac{2\pi a}{k} \gg \mathrm{H}^{-1}$), in which case the Laplacian operator in equation (6.55) that is proportional to k, can be eliminated in the wave equations of certain perturbations, yielding easily-solved differential equations. Although the use of a plane wave description is mathematically incorrect in curved space, it turns out that the only difference that arises is the allowable values of the wavenumbers. For a flat geometry ($K = 0$) the eigenvalues form a continous spectrum where $k^2 \geq 0$. The spectrum for an open model ($K = +1$) is discrete with $k^2 = \alpha(\alpha + 2)$ where $\alpha = 1,\ 2,\ 3\ldots$ A spacetime with negative curvature ($K = -1$) can accomodate the eigenvalues $k^2 = 1 + \alpha^2$ where $\alpha^2 \geq 0$ [50].

As mentioned previously, GWs are purely tensorial and so we expand the representative shear variable as the tensor harmonics given in (6.58)

$$\sigma_{ab} = \sum_k \sigma^{(k)} Q^{(k)}_{ab}, \tag{6.59}$$

where as usual $\dot{Q}^{(k)}_{<ab>} = 0$ and $\mathrm{D}^2 Q^{(k)}_{ab} = -(k^2/a^2) Q^{(k)}_{ab}$ hold. Each gravitational wave mode is associated with the physical wavelength

$$\lambda_{\mathrm{GW}} = 2\pi a/k. \tag{6.60}$$

The expansion of the magnetic field in pure vector (solenoidal) harmonics is

$$\tilde{B}_a = \sum_n \tilde{B}^{(n)} Q^{(n)}_a, \tag{6.61}$$

and these obey the relations $\dot{Q}^{(n)}_{<a>} = 0$ and $\mathrm{D}^2 Q^{(n)}_a = -(n^2/a^2) Q^{(k)}_a$. Since the magnetic field in $\mathcal{F}_1$ obeys $\mathrm{curl}\, \tilde{B}_a = 0$, it follows that $\mathrm{D}^2 \tilde{B}_a = -\mathrm{curl}\,(\mathrm{curl}\, \tilde{B}_a) = 0$. For this condition to be satisfied, we see from equation (6.55) that n^2 must be 0. The harmonic decomposition of the magnetic field therefore reduces to

$$\tilde{B}_a = \tilde{B}^{(0)} Q^{(0)}_a, \tag{6.62}$$

where $\tilde{B}^{(0)} = \tilde{B}^0 (a_0/a)^2$. This means that the magnetic field $\tilde{B}^a$ is spatially constant (i.e. in agreement with the assumption of homogeneity). Perturbations in $\mathcal{S}$ are conveniently decomposed with the vector harmonics[4]

$$V^{(\ell)}_a \equiv Q^{(k)}_{ab} Q^b_{(n)}, \tag{6.63}$$

which are readily verified to fulfill the standard requirements $\dot{V}^{(\ell)}_{<a>} = 0$ and $\mathrm{D}^2 V^{(\ell)}_a = -(\ell^2/a^2) V^{(\ell)}_a$, where the wavenumber ℓ satisfies

$$\ell^2 = (k_a + n_a)(k^a + n^a). \tag{6.64}$$

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[4] It should be kept in mind that all above introduced harmonics are exclusively defined on the background FLRW spacetime.

Here, k refers to the GW and n to the magnetic field. Generally there is a range of k and n combinations that correspond to a given ℓ mode. This is accounted for by a summation over all permutations of k and n. However because the magnetic field in $\mathcal{F}_1$ only has a zero mode ($n = 0$) in our investigation, the wavenumber ℓ coincides with the wavenumber k of the shear and only a single comoving gravitational wavenumber corresponds to a given interaction mode.

Using the above defined unified time variable τ and the harmonic decomposition above, we transform the wave equation (6.40) for the interaction variable I^a into an ordinary differential equation

$$\begin{aligned}\frac{9}{4}(1+w)^2 I''_{(\ell)} &+ \frac{27(1+w)}{2\tau} I'_{(\ell)} \\ &+ \left[\left(\frac{\ell}{a_0 H_0}\right)^2 \tau^{-\frac{4}{3(1+w)}} + \frac{25-15w}{2\tau^2}\right] I_{(\ell)} = 0,\end{aligned} \tag{6.65}$$

where a prime means differentiation with respect to τ. Initial conditions are chosen as follows:

$$I_{(\ell)}(t_0) = \sigma_{(k)}(t_0)\tilde{B}_0, \tag{6.66}$$

$$I'_{(\ell)}(\tau = 1) = \tilde{B}_0\left[\sigma'_{(k)}(1) - \frac{4}{3(1+w)}\sigma_{(k)}(1)\right]. \tag{6.67}$$

$\tilde{B}_0$ is the initial amplitude of the first-order magnetic field and $\dot{\sigma}_{(k)}(t_0) = 3/2H_0\,(1+w)\,\sigma'_{(k)}(1)$ was used. For every mode k we have initially $\sigma(t_0) = \sigma_0$ and $\sigma'(1) = \sigma'_0$.

6.8 Limiting case where $\ell \to 0$

In the case ($\ell \to 0$), the solution of equation (6.65) is found to be

$$I^{(0)}(\tau) = C_1\,\tau^{-\frac{10}{3(1+w)}} + C_2\,\tau^{\frac{-5+3w}{3(1+w)}}, \tag{6.68}$$

where C_1 and C_2 are constants of integration. If the initial conditions (6.66)–(6.67) are chosen, the corresponding integration constants are

$$C_1 = \frac{(-5+3w)\,I_{(\ell)}(1) - 3\,(1+w)\,I'_{(\ell)}(1)}{5+3w} \tag{6.69}$$

and

$$C_2 = \frac{10 I_{(\ell)}(1) + 3\,(1+w)\,I'_{(\ell)}(1)}{5+3w}. \tag{6.70}$$

This solution is in agreement with the result obtained by multiplying the first-order magnetic field (6.5) with the ($\ell \to 0$) solution of the shear equation (6.11). The total magnetic field in the limit ($k \to 0$) is then

$$\begin{aligned}B^{(0)}(\tau) = \tilde{B}_0\,\tau^{-\frac{4}{3(1+w)}}\Bigg[1 &- \frac{C_1}{\tilde{B}_0 H_0}\frac{2}{3\,(1-w)}\left(\tau^{\frac{-1+w}{3(1+w)}} - 1\right) \\ &+ \frac{C_2}{\tilde{B}_0 H_0}\frac{1}{1+3w}\left(\tau^{\frac{2+6w}{3(1+w)}} - 1\right)\Bigg],\end{aligned} \tag{6.71}$$

where $\tilde{B}_0$ is the magnitude of the first-order magnetic field interacting with the GWs at initial time t_0. For physical reasons, the induced magnetic field must vanish initially. The growing contribution in the second line of equation (6.71) means that the interaction always leads to an amplification of the magnetic field for any physically acceptable choice of equation of state.

Some important special cases are now considered. For the sake of simplicity, we eliminate any growth or decay of the interaction mode initially and set $I'_{(\ell)}(1) = 0$. In the matter-dominated era where the matter is accurately described as dust (i.e. $w = 0$ and $a = a_0\tau^{2/3}$) the magnetic field mode is

$$B^{(0)}_{\text{Dust}}(a) = \tilde{B}_0 \left(\frac{a_0}{a}\right)^2 \left[1 + \frac{2}{3}\frac{\sigma_0}{H_0}\left\{\left(\frac{a_0}{a}\right)^{3/2} - 1\right\} + \frac{2\sigma_0}{H_0}\left\{\frac{a}{a_0} - 1\right\}\right]. \tag{6.72}$$

For a radiation-dominated era where $w = 1/3$ and $a = a_0\tau^{1/2}$, the total magnetic field is

$$B^{(0)}_{\text{Rad}}(a) = \tilde{B}_0 \left(\frac{a_0}{a}\right)^2 \left[1 + \frac{2}{3}\frac{\sigma_0}{H_0}\left\{\frac{a_0}{a} - 1\right\} + \frac{5}{6}\frac{\sigma_0}{H_0}\left\{\left(\frac{a}{a_0}\right)^2 - 1\right\}\right]. \tag{6.73}$$

In the limit ($k \to 0$) the amplification depends mainly on the scale factor and the magnitude of the initial GW distortion relative to the Hubble parameter $(\sigma/H)_0$.

6.9 General case with $\ell \neq 0$

The general solution to the interaction equation (6.65) is

$$\begin{aligned} I_{(\ell)}(\tau) = \tau^{\frac{-5+w}{2(1+w)}} \Bigg[& D_1\, J_1\left(\frac{3w+5}{2\,(1+3w)}, \frac{\ell}{a_0 H_0}\frac{2}{1+3w}\tau^{\frac{1+3w}{3(1+w)}}\right) \\ & + D_2\, J_2\left(\frac{3w+5}{2\,(1+3w)}, \frac{\ell}{a_0 H_0}\frac{2}{1+3w}\tau^{\frac{1+3w}{3(1+w)}}\right)\Bigg], \end{aligned} \tag{6.74}$$

where D_1, D_2 are integration constants and J_1, J_2 denote Bessel functions of the first and second kind, respectively. Observe that in the limit of $\ell \to 0$ the solution (6.68) is recovered. The generated magnetic field relative to the observer moving with 4-velocity u^a, can be calculated from the solution (6.74) analytically for every barotropic parameter w. We will state here only the total magnetic field solution in the case of dust and radiation respectively. For dust where $w = 0$ and $a = a_0\tau^{2/3}$, the full magnetic field is

$$B^{(\ell)}_{\text{Dust}}(a) = \tilde{B}_0 \left(\frac{a_0}{a}\right)^2 \left[1 + \frac{3}{4\pi^2}\left(\frac{\lambda_{\text{GW}}}{\lambda_{\text{H}}}\right)^2_0 \left(\frac{\sigma_0}{H_0} + \frac{\sigma'_0}{2H_0}\right) + \mathcal{O}(a^{-1})\right]. \tag{6.75}$$

For radiation where $w = 1/3$ and $a = a_0\tau^{1/2}$, the total magnetic field modes obey

$$B^{(\ell)}_{\text{Rad}}(a) = \tilde{B}_0 \left(\frac{a_0}{a}\right)^2 \left[1 + \frac{3}{4\pi^2}\left(\frac{\lambda_{\text{GW}}}{\lambda_{\text{H}}}\right)^2_0 \left(\frac{\sigma_0}{H_0} + \frac{2\sigma'_0}{3H_0}\right) + \mathcal{O}(a^{-1})\right]. \tag{6.76}$$

We have introduced the gravitational wavelength $\lambda_{\mathrm{GW}} = 2\pi a/k$ and the Hubble length $\lambda_{\mathrm{H}} = 1/H$. The un-displayed remainders $\mathcal{O}(a^{-1})$ in the expressions above contain functions which oscillate with amplitudes decaying at least as fast as the inverse scale factor a^{-1}. When the limit $(k \rightarrow 0)$ of the full solutions above is taken, the findings (6.72) and (6.73) are rediscovered. The results (6.75)–(6.76) clearly show how the generated magnetic field depends on the initial conditions. The late time behaviour is almost identical for both dust and radiation. Physically, the interaction can only be effective if gravitational wavelength matches the size of the magnetic field region $\lambda_{\mathrm{GW}} \sim \lambda_{\tilde{B}}$.

- In the case of $\lambda_{\mathrm{GW}} \gg \lambda_{\tilde{B}}$ the magnetic field cannot be physically affected by the GW.
- For $\lambda_{\mathrm{GW}} \ll \lambda_{\tilde{B}}$ the effect becomes negligible due to its quadratic dependence on λ_{GW}.

If we divide the findings (6.75)–(6.76) through by the energy density of the background radiation (which decays in the same manner as the original magnetic field) the dominant contribution can be summarized as follows

$$\frac{B}{\mu_\gamma^{1/2}} \simeq \left[1 + \frac{1}{10}\left(\frac{\lambda_{\tilde{B}}}{\lambda_{\mathrm{H}}}\right)_0^2 \left(\frac{\sigma}{H}\right)_0\right] \left(\frac{\tilde{B}}{\mu_\gamma^{1/2}}\right)_0, \tag{6.77}$$

where the wavenumber indices have been suppressed and $\sigma_0' = 0$ was assumed. At late times, a significant amplification of the original magnetic field can be achieved for super-horizon gravitational waves. Note that a result almost identical to (6.77) was obtained in [98], wherein the factor 1/10 is instead replaced by 10. Our result however, holds for *any* finite gravitational wavelength $\lambda_{\mathrm{GW}} \sim \lambda_{\tilde{B}}$ while the result in [98] assumes $\lambda_H \ll \lambda_{\mathrm{GW}}$. Somewhat contrived initial conditions leading to an abrupt amplification of the field were also used in [98] whereas we choose initial conditions such that there is no generated field when the interaction sets in. Furthermore, the seed field $\tilde{B}$ is thought to emerge from the electroweak phase transition after which it starts to interact with the inflationary gravitational wave spectrum.

6.10 Comparison with the weak field approximation

We now compare our results with those given in [98] in which this interaction has recently been investigated using the so-called weak field approximation [96, 97, 99]. We remind the reader that this scheme allows for a weak magnetic test field $\tilde{B}_a$ in the background whose energy density, anisotropic stress and spatial dependence have negligible impact on the background dynamics: $\tilde{B}^2 \ll \mu$ and $\pi_{ab} = -\tilde{B}_{<a}\tilde{B}_{b>} \simeq 0 \simeq \mathrm{D}_a\tilde{B}_b$ to zeroth-order. In order to isolate linear tensor perturbations it is necessary to impose $\mathrm{D}_a\tilde{B}^2 = 0 = \epsilon_{abc}\tilde{B}^b \operatorname{curl}\tilde{B}^c$ in addition to the standard constraints $\omega_a = 0 = \mathrm{D}_a\mu = \mathrm{D}_a p$ associated with pure perfect fluid cosmologies. The main equations governing the induced magnetic field were derived in [98] for the case of a spatially flat Universe with vanishing cosmological constant Λ and a barotropic equation of state $p = w\mu$:

$$\begin{aligned} \ddot{B}_{(\ell)} + \frac{5}{3}\Theta\dot{B}_{(\ell)} + \left[\frac{1}{3}(1-w)\Theta^2 + \frac{\ell^2}{a^2}\right] B_{(\ell)} \\ = 2\left(\dot{\sigma}_{(k)} + \frac{2}{3}\Theta\sigma_{(k)}\right)\tilde{B}_0^{(n)}\left(\frac{a_0}{a}\right)^2, \end{aligned} \tag{6.78}$$

where the GWs are determined by the shear wave equation

$$\ddot{\sigma}_{(k)} + \frac{5}{3}\Theta\dot{\sigma}_{(k)} + \left[\frac{1}{6}\left(1-3w\right)\Theta^2 + \frac{k^2}{a^2}\right]\sigma_{(k)} = 0. \tag{6.79}$$

Here, the shear is harmonically decomposed as $\sigma_{ab} = \sigma_{(k)}Q^{(k)}_{ab}$, while for the induced magnetic field $B^{(\ell)}_a = B_{(\ell)}V^{(\ell)}_a$ with $V^{(\ell)}_a = Q^{(k)}_{ab}Q^b_{(n)}$ was adopted. The background magnetic field evolves as $\tilde{B}_a = \tilde{B}^0_a(a_0/a)^2$ and $\tilde{B}^0_a = \tilde{B}^0_{(n)}Q^{(n)}_a$ is assumed. We now compare our results with the corresponding solutions in [98] to see whether the two formalisms lead to different results. For simplicity, we restrict ourselves here to the case of dust ($w = 0$).

As pointed out above, the only allowed magnetic wavenumber for the interacting magnetic field is $n = 0$ when $\mathrm{D}_a\tilde{B}_b = 0$, which leads to $\ell = k$. The published solution for the generated magnetic field in the weak field approximation (equation (21) in [98]) is not however applicable in the limit $n \to 0$. This can be traced back to the choice for the initial conditions for the generated magnetic field made by the authors of [98] when solving equations (6.78)–(6.79) (see equation (19) in [98]).

In what follows we solve equations (6.78)–(6.79) again, including the full solution for the shear instead of merely keeping the dominant part as done in [98]. We specify the initial conditions by choosing for every mode k of the shear $\sigma_{(k)}(a_0) = \sigma_0$, $\dot{\sigma}_{(k)}(a_0) = 0$ and for every mode $\ell = k$ of the generated magnetic field $B_{(\ell)}(a_0) = 0 = \dot{B}_{(\ell)}(a_0)$. Note that this choice of initial conditions differs from that in [98] but agrees with our choice made in section (6.9). The solution including the background field, for an arbitrary wavenumber k of the shear, has the structure

$$B^{(\ell)}_{\text{Dust}}(a) = \tilde{B}_0\left(\frac{a_0}{a}\right)^2\left[1 + \frac{\sigma_0}{H_0}f\left(\sqrt{a};k\right) + \mathcal{O}\left(a^{-\frac{1}{2}}\right)\right], \tag{6.80}$$

where the function $f\left(\sqrt{a};k\right)$ is built of several oscillatory terms with amplitude $(\lambda_{\mathrm{GW}}/\lambda_{\mathrm{H}})^2_0$ at most and the un-displayed part falls of at least as fast as $a^{-1/2}$. If this is compared with our result (6.75), one observes that it differs by having another time behaviour. More striking however, is that the term $f\left(\sqrt{a};k\right)$ now not only amplifies the seed field but also grows like $\sqrt{a}$ in the long wavelength limit ($k/a_0H_0 \ll 1$). This is in clear contrast to the gauge-invariant result (6.75), where the seed undergoes amplification but then still decays adiabatically as a^{-2}. On the other hand, in the limit ($k \to 0$), the exact full solution is now

$$B^{(0)}_{\text{Dust}}(a) = \tilde{B}_0\left(\frac{a_0}{a}\right)^2\left[1 + \frac{\sigma_0}{H_0}\left\{\frac{20}{3} - 14\left(\frac{a}{a_0}\right)^{1/2} + \frac{36}{5}\left(\frac{a}{a_0}\right) + \frac{2}{15}\left(\frac{a_0}{a}\right)^{3/2}\right\}\right]. \tag{6.81}$$

Again, we obtain a solution whose time behaviour differs from that found in (6.72). However, the weak field solutions agree with our presented solutions in the case ($k \to 0$) when only the dominant part of the solutions is considered (at least in the examples considered above).

A further important remark concerns the issue of conductivity. We have seen earlier that within our assumptions and for spatially flat Universes, the gravito-magnetic interaction leads to an induced magnetic field which is independent of the conductivity of the cosmic medium. This is due to the fact that the interaction does not generate rotational electric field modes which might affect the magnetic field. In the weak field approximation however, the situation is completely different. If one assumes that the conductivity of the cosmic medium is sufficiently high to cause the rapid dissapation of the electric fields to yield a curl-free induced magnetic field, then equation (6.78) no longer applies and one simply has to use

$$\dot{B}_{(\ell)} + \frac{2}{3}\Theta B_{(\ell)} = \sigma_{(k)}\tilde{B}_0^{(n)}\left(\frac{a_0}{a}\right)^2 \tag{6.82}$$

instead, while the equation for the shear (6.79) is unaltered. This means that the weak field approximation produces the same result as our gauge-invariant perturbation approach in the high conductivity limit, and for that case only. It is therefore evident that in the weak field approximation, the conductivity of the cosmic medium has a crucial bearing on the generated magnetic field, in stark contrast to the result of our gauge-invariant approach (see also [20]).

Chapter 7

The inhomogeneous magnetic field case

The purpose of this next chapter is to provide a detailed comparison between the outcome of the magnetic-gravito interaction with a homogeneous and inhomogeneous magnetic field, with the intention of highlighting the repercussions of restricting the nature of the field – namely spatial homogeneity.

7.1 Formalism

In order to investigate the more general case of an inhomogeneous magnetic field residing in the $\mathcal{F}_1$ spacetime, we adopt the same perturbative scheme given in section (5.4) but include spatial gradients of the field (D^aB^b) at this perturbative level:

- $\mathcal{F}_1$ = FLRW perturbed by an inhomogeneous magnetic field whose energy density and anisotropic stress are neglected, $\mathcal{O}(\epsilon_B)$.

The additional $\mathcal{F}_1$ variable $(\mathrm{D}_aB_b) \sim \mathcal{O}(\epsilon_B)$ complicates matters. The linearized commutator for $\mathsf{curl}\,\mathsf{curl}\,B_a$ (c.f, appendix)

$$\mathsf{curl}\,\mathsf{curl}\,B_a = -\mathrm{D}^2B_a + \tfrac{2}{3}\left(\mu + \Lambda - \tfrac{1}{3}\Theta^2\right)B_a. \tag{7.1}$$

reduces to

$$\mathsf{curl}\,\mathsf{curl}\,B_a = -\mathrm{D}^2B_a \tag{7.2}$$

in the flat case. This means that if we wish to consider a curl-free, first-order magnetic field residing in a flat Universe, then (D_bB_a) must vanish at linear-order for consistency. Vanishing spatial gradients however, correspond to a homogeneous first-order field. For non-redundant results we can then only consider open and closed models. This is restrictive as only a limited set of eigenvalues k can be accommodated by these geometries and we will not be able to compare our results with those in chapter (6). To treat the general case encompassing all geometries we must therefore retain $\mathsf{curl}\,B_a$ at the $\mathcal{F}_1$ perturbative level. Since the rotation of a magnetic field naturally induces a current $j_a = \mathsf{curl}\,B_a$, we require a model that provides a suitable treatment of currents at this level. For this we prescribe to magnetohydrodynamics (MHD).

Since the mean-free paths between the electron-ion collisions in a typical plasma are macroscopically long, it is not always obvious that a fluid description is indeed valid. On small scales these interactions are frequent and cause the two species to move relative to each other, generating charge separation effects referred to as plasma oscillations. If we consider much larger scales on which the individual collisions are not explicitly seen, the different species are observed to move together with a common average velocity, allowing the single fluid description of this 2-component system, introduced in section (5.4) as the MHD approximation. Since the young cosmic plasma being studied in this problem is assumed to be highly conducting, it may be described as a continuum without collisional effects and can thus be accurately modelled using the MHD approach. We now pay attention to how the currents, which are established due to the net motion of the fluid by induction, modify the field and in so doing couple the hydrodynamical equations to Maxwell's equations via Ohm's law.

We perform the calculations in an irrotational Universe with non-zero Λ and assuming $p = w\mu$, as done in chapter (6). It has been argued that the reduced MHD description of a plasma as a fluid is only valid in the case of a cold plasma [56], where the fluid pressures are negligible compared with the magnetic pressure (i.e. $p = 0$). Furthermore, this extension of the analysis is being performed primarily for the sake of comparison with the homogeneous case. It is therefore sufficient to calculate the results for the case of *dust* only. We also adopt the geodesic frame in which the acceleration of the fluid frame $\dot{u}^a$ vanishes to all orders.

7.2 Basic equations of MHD

We now define the magnetohydrodynamic variables assuming global charge neutrality (i.e. the number densities of the electrons and ions are roughly equal such that $\rho_c = -e(n_e - n_i) \sim 0$). We assume that the interactions of the ions and electrons collectively isotropize their motions such that in a chosen frame, the properties of the fluid on macroscopic scales can be described in terms of an average velocity v^a. We take this mean motion to be the center of mass velocity of the electron-ion system, defined by

$$v^a = \frac{\mu_e v^a_{(e)} + \mu_i v^a_{(i)}}{\mu_e + \mu_i}, \tag{7.3}$$

where v^a coincides with the velocity of the fundamental observer at zeroth-order. The behaviour of the electromagnetic fields are now examined in terms of this bulk velocity.

Ohm's law is generally formulated in the local rest frame of the conducting fluid and is assumed to hold to all orders

$$j_{<a>} = \sigma(E_a + \epsilon_{abc} v^b B^c), \tag{7.4}$$

where the conductivity σ is regarded as a constant for simplicity. The 3-vector E^a represents the field as observed from the rest-space of the fluid and the second term in (7.4) is the apparent electric field associated with the fluid flow. We make the standard assumption that the cosmic medium is infinitely conducting, referred to as the ideal MHD limit. This is valid considering the early epoch in which the interaction takes place and is also consistent with the treatment of a homogeneous magnetic field in a *flat* Universe in chapter (6); the magnetic-gravito interaction was shown in section (6.6.2) to generate the same magnetic field on flat spatial sections as in the limit of infinite conductivity. By assuming $(\sigma \rightarrow \infty)$, equation (7.4) indicates that the *effective* electric field $(E_a + \epsilon_{abc}v^bB^c)$ must be identically zero in order for the spatial current $j_{<a>}$ to remain finite. The electric field is now determined jointly by the fluid velocity and the magnetic field

$$E_a = -\epsilon_{abc}v^bB^c. \tag{7.5}$$

We note that the electric field is of the order $(\epsilon_B\epsilon_g)$ and thus enters the $\mathcal{S}$ spacetime.

7.3 First-order perturbations

7.3.1 The inhomogeneous magnetic field $\tilde{B}_a$

We assume that the relic magnetic field residing in the $\mathcal{F}_1$ spacetime contains inhomogeneities on scales smaller than the typical coherence scales. The spatial gradients $D_b\tilde{B}_a$ are thus of order $\mathcal{O}(\epsilon_B)$. Given that the magnetic field is a first-order perturbation on the background, the magnetic anisotropy $\pi_{ab} = -\tilde{B}_{<a}\tilde{B}_{b>} \sim \mathcal{O}(\epsilon_B^2)$ is still neglected. Since the associated electric field is perturbatively smaller than the magnetic field and enters only at $\mathcal{S}$, the magnetic induction equation has the same form as in equation (6.4)

$$\dot{\tilde{B}}_{<a>} + \tfrac{2}{3}\Theta\tilde{B}_a = 0.$$

7.3.2 Gravitational waves

The tensorial gravitational waves are again governed completely by the same shear evolution equation (6.8)

$$\ddot{\sigma}_{ab} - \mathrm{D}^2\sigma_{ab} + \tfrac{5}{3}\Theta\dot{\sigma}_{ab} + \left(\tfrac{1}{9}\Theta^2 + \tfrac{1}{6}\mu - \tfrac{3}{2}p + \tfrac{5}{3}\Lambda\right)\sigma_{ab} = 0.$$

We note that the time dependance of the first-order magnetic field and the shear, remain the same as in the homogeneous case. The evolution of the interaction variable is therefore unaffected by the spatial variation of the magnetic field.

7.4 Second-order perturbations: The interaction

We look to Maxwell's equation to determine the nature of the interaction between GWs and the magnetic field. If the back-reaction of the induced field with the shear is ignored, the induction equation now has the form

$$\dot{B}_{<a>} + \tfrac{2}{3}\Theta B_a = \sigma_{ab}\tilde{B}^b + 2D^b\left(v_{[a}B_{b]}\right), \tag{7.6}$$

where the second term (curl E_a) describes the dragging of the field lines by the fluid. Again we are faced with the problem of removing the primary magnetic field component of B^a

from the LHS of (7.6) to ensure that it is truly second-order. Given that the magnetic spatial gradients are now retained in $\mathcal{F}_1$, the commutator relation used as an example in the homogeneous case in section (6.4), is now consistently satisfied when the power series expansion of B^a

$$B^a = \epsilon_B B_1^a + \epsilon_g \epsilon_B B_2^a + \mathcal{O}(\epsilon_g^2, \epsilon_B^2)$$

is applied. Although this expansion does not immediately appear to be invalid, in the event that an inconsistency does exist we choose to represent the magnetic field using the same SOGI variable

$$\beta_a \equiv \dot{B}_a + \frac{2}{3}\Theta B_a, \tag{7.7}$$

identified in chapter (6). We select the same interaction variable $I_a = \sigma_a^b B_b$ and denote the variable $F_a = \mathsf{curl}\, E_a \equiv -2D^b \left(v_{[a} B_{b]}\right)$. We can now restate Maxwell's equations in $\mathcal{S}$ as a system of differential equations in terms of these SOGI variables

$$\begin{aligned} \beta_a + F_a &= I_a, & (7.8)\\ D^a E_a &= 0\,, & (7.9)\\ D^a B_a &= 0\,. & (7.10) \end{aligned}$$

To close the system, the $\dot{E}_a$ equation is replaced by the velocity propagation equation in the cold plasma limit, which comes out of the momentum-conservation equation as

$$\dot{v}_a + \tfrac{1}{3}\Theta v_a = -v^b \mathrm{D}_b v_a - \sigma_{ab} v^b - \epsilon_{abc} j^b B^c. \tag{7.11}$$

This equation indicates that the average fluid velocity which now constitutes the electric field, affects the evolution of the magnetic field which then modifies v^a in a back-reaction. Given that both v^a and B^a are individually regarded as first-order, only the linear part of this equation is needed

$$\dot{v}_{<a>} + \frac{\Theta}{3} v_a = 0. \tag{7.12}$$

The generated magnetic field B^a can be found directly by integrating a linear combination of the I^a and F^a solutions.

If we take the divergence of β^a we obtain the relation

$$\mathrm{D}^a \beta_a = \mathrm{D}^a I_a. \tag{7.13}$$

Substituting for β^a in terms of the magnetic field $\tilde{B}^a$ and imposing the standard constraints $0 = \mathrm{D}^a B_a = \mathrm{D}^a E_a$, consistently yields the same relation (7.13). Since spatial gradients now are regarded as first-order, the latter in equation (7.13) is equivalent to $\sigma_{ab} \mathrm{D}^a \tilde{B}^b$ which can not immediately be set to zero. Since we can not place any further constraints on relation (7.13), the variables are not necessarily divergence-free.

7.5 Consistency of MHD approximation

The constraints from Einstein's gravitational equations $(C_1) - (C_6)$ take the same form as that given in chapter (6) which have been shown to be consistent. We need to check the consistency of Maxwell's equations (7.9) and (7.10) within the MHD approximation;

$$\begin{aligned} C_6 &= \mathrm{D}^a E_a = 0, & (7.14)\\ C_7 &= \mathrm{D}^a B_a = 0. & (7.15) \end{aligned}$$

Replacing E_a with equation (7.5) in (7.14) above, gives

$$C_6 = \mathsf{curl}\, B_a\, v^a - B_a \mathsf{curl}\, v^a = 0. \tag{7.16}$$

Using equations (7.6) and (7.12), we arrive at

$$\begin{aligned} \dot{C}_6 &= -\tfrac{4}{3}\Theta C_6 = 0, & (7.17)\\ \dot{C}_7 &= -\Theta C_7 + 2(\mathrm{D}^b v_a)(\mathrm{D}^a B_b) = 0. & (7.18) \end{aligned}$$

We must now check whether the additional constraint $C_8 = (\mathrm{D}^b v_a)(\mathrm{D}^a B_b) = 0$ is satisfied throughout its evolution along the fluid flow as a consequence of the propagation equations for v^a and B^a being true. Substituting equations (7.12) and (7.6), we find

$$\dot{C}_8 = -\Theta C_8 = 0, \tag{7.19}$$

indicating that C_8 is conserved in time and yields a consistent set of equations.

7.6 Re-scaled variables

It is convenient to re-scale the primary magnetic field variable. By replacing $\mathcal{B}_a = \tilde{B}_a \left(\frac{a}{a_0}\right)^2$ the homogeneous magnetic mode is removed and the quantity $\mathcal{B}^a$ now represents the evolution of the field over and above its adiabatic decay. The part of the new variable representing the first-order magetic field is independent of time and remains equal to its initial amplitude of $\mathcal{B}_0^a = \tilde{B}_0^a$ throughout its evolution. The time dependence of $\mathcal{B}^a$ found in the final solutions then describes the evolution of the *generated* field only. The main variables become

$$\beta_a = \dot{\mathcal{B}}_a = 0, \qquad \mathcal{I}_a = \sigma_{ab}\mathcal{B}^b, \qquad \mathcal{F}_a = -2D^b\left(v_{[a}\mathcal{B}_{b]}\right). \tag{7.20}$$

Using $H = \frac{\Theta}{3} = \dot{a}/a$ we can restate equation (7.8) in terms of these variables

$$\dot{\mathcal{B}}_a + \mathcal{F}_a = \mathcal{I}_a. \tag{7.21}$$

Considering equation (7.21) at some arbitrary initial time t_0, we see that in order for the generated magnetic field to be zero initially, the following must hold;

$$\dot{\mathcal{B}}(t_0) = \sigma_0 \tilde{B}_0 - \mathcal{F}_0 = 0. \tag{7.22}$$

This initial condition requires the magnitude of the electric field rotation to equal the initial interaction

$$\mathcal{F}_0 \equiv \sigma_0 \tilde{B}_0. \tag{7.23}$$

7.7 Evolution equations for the main variables

It has been shown in equation (7.21) that the magnetic field SOGI $\mathcal{B}^a$ can be extracted directly by integrating a linear combination of the solutions to the interaction $\mathcal{I}^a$ and the electric field rotation $\mathcal{F}^a$. We turn to find the evolution equations for these variables, maintaining generality.

7.7.1 Governing equation for the interaction variable

In section (6.8) the solution for I^a obtained by solving its wave equation, was found to agree with the result calculated from the multiplication of the time dependencies of the shear and background magnetic field determined individually. Solving the wave equation for the interaction variable however, requires a harmonic decomposition. As demonstrated in section (7.4), the interaction variable is not necessarily divergence-free and may have a non-zero scalar contribution. Decomposing $\mathcal{I}^a$ as a pure vector (as in the homogeneous case) may thus be incorrect, so we choose to find the solution for $\mathcal{I}^a$ by solving for σ_{ab} and the first-order part of $\mathcal{B}^b$ separately.

We employ the standard harmonic decomposition described in (6.7) to deal with the Laplacian operator present in the wave equation of the shear. Using the unified time variable τ, the shear wave equation (6.11) describing the evolution of the GWs becomes

$$9\left(1+w\right)^2\sigma_{(k)}^{''}+\frac{15}{2\tau}\left(1+w\right)\sigma_{(k)}^{'}+\left[\frac{1}{2}\frac{(3-9w)}{\tau^2}+\frac{k^2}{\left(H_0a_0\right)^2}\tau^{\frac{-4}{3(1+w)}}\right]\sigma_{(k)}=0. \qquad (7.24)$$

The first-order magnetic field component of $\mathcal{B}$ remains constant over time with a magnitude $\tilde{\mathcal{B}}_0$ for all eigenvalues k. The solution for $\mathcal{I}_{(k)}=\sigma_{(k)}\mathcal{B}$ will therefore only contain the gravitational wavenumber k.

7.7.2 Propogation equation for F^a

In the ideal MHD limit, the electric field is expressed as the product of the primary magnetic field and the velocity. Using the linear velocity propagation equation (7.12), the equation describing the evolution of the forcing term $\mathcal{F}^a$ is found to be

$$\mathcal{F}_a^{'}+\frac{2}{3\tau}\mathcal{F}_a=0, \qquad (7.25)$$

where a prime means differentiation with respect to the unified time variable τ. As shown in section (7.6) the magnitude of the vector $\mathcal{F}^a$ at some initial time ($\tau=1$) is required to be $\mathcal{F}_0(1)=\sigma_0\mathcal{B}_0$. The use of the unified time variable removes the barotropic index w. However, given that the velocity equation (7.12) used in its derivation was found assuming $w=0$, equation (7.25) is therefore only valid for dust.

7.8 Solutions for flat Universes

For the sake of comparison, we investigate the dust solutions only for spatially flat models with zero cosmological constant.

7.8.1 Limiting case where $k \to 0$

In the limit ($k \to 0$) the solution to the shear wave equation for a dust Universe is

$$\sigma_{(0)}(\tau) = C_1\tau^{-2} + C_2\tau^{-\frac{1}{3}}, \tag{7.26}$$

where C_1 and C_2 are integration constants. If we specify the initial conditions as

$$\sigma_{(0)}(\tau = 1) = \sigma_0 \tag{7.27}$$

and

$$\sigma'_{(0)}(1) = \sigma'_0, \tag{7.28}$$

the corresponding integration constants are

$$C_1 = \frac{-5\sigma_0(1) - 3\sigma'_0}{5} \tag{7.29}$$

and

$$C_2 = \frac{10\sigma_0(1) + 3\sigma'_0}{5}. \tag{7.30}$$

Dropping indices, the first-order magnetic field represented by $\mathcal{B} = \tilde{B}_0$ has no time dependence and is absorbed into the integration constants when calculating the solution to the interaction variable.

The general equation (7.25) for the forcing term $\mathcal{F}^a$ is independent of a comoving wavenumber and can thus be employed in this special case. Its solution is found by imposing the initial conditions (7.23) and integrating (7.25)

$$\mathcal{F}(\tau) = \tilde{B}_0\sigma_0\tau^{-\frac{4}{3}}. \tag{7.31}$$

We are now equipped with the solutions of both forcing terms and can integrate to solve for the magnetic field in the limit of ($k \to 0$). To simplify matters, we eliminate any initial growth or decay of the interaction mode and set $\sigma'_0 = 0$. In the matter-dominated era where $w = 0$ and $a = a_0\tau^{2/3}$, the magnetic field mode is

$$\mathcal{B}^{(0)}_{\mathsf{Dust}}(a) = \tilde{B}_0\left[1 + \frac{2}{15}\frac{\sigma_0}{H_0}\left\{\left(\frac{a_0}{a}\right)^{3/2} - 1\right\} + \frac{6\sigma_0}{5H_0}\left\{\frac{a}{a_0} - 1\right\} + \frac{2\sigma_0}{H_0}\left\{\left(\frac{a_0}{a}\right)^{1/2} - 1\right\}\right]. \tag{7.32}$$

When comparing the above to the result (6.72) in the homogeneous case, the last term in (7.32) is identified as the contribution from the $\mathcal{F}^a = \mathsf{curl}\, E^a$ term.

7.8.2 General case with $k \neq 0$

The general solution to the shear wave equation is found to be

$$\sigma_{(k)}(\tau) = \tau^{-\frac{7}{6}}\Big[C_1\, J_1\Big(\frac{5}{2\,(1)}, \frac{k}{a_0H_0}2\tau^{\frac{1}{3}} + C_2\, J_2\Big(\frac{5}{2}, \frac{k}{a_0H_0}2\tau^{\frac{1}{3}}\Big)\Big)\Big)\Big], \tag{7.33}$$

where C_1, C_2 are integration constants and J_1, J_2 denote Bessel functions of the first and second kind, respectively. To find the solution for the interaction variable $\mathcal{I}_{(k)}$, we multiply (7.33) by the first-order magnetic field $\mathcal{B} = \tilde{B}_0$.

The magnetic field as seen by an observer moving with 4-velocity u^a is found by adding solutions (7.33) and (7.31) and integrating. For every mode k we adopt the same initial conditions $\sigma(1) = \sigma_0$ and $\sigma'(1) = \sigma'_0$ for convenience. The total magnetic field in the case of *dust* is

$$\mathcal{B}^{(k)}_{\mathrm{Dust}}(\tau) = \tilde{B}_0\left[1 + 3a_0^2\left(\frac{\sigma_0 H_0}{k^2}\right) + \frac{3a_0^2}{2}\left(\frac{\sigma'_0 H_0}{k^2}\right) - \frac{2\sigma_0}{H_0} + \mathcal{O}(\tau^{-1/3})\right]. \tag{7.34}$$

To allow for direct comparison with the corresponding solution in chapter (6), we convert equation (7.34) into a function of the scale factor and introduce the initial gravitational wavelength $(\lambda_{\mathrm{GW}})_0 = 2\pi a_0/k$ and Hubble length $(\lambda_{\mathrm{H}})_0 = 1/H_0$. For *dust* the scale factor varies as $a = a_0\tau^{2/3}$ and the solution becomes

$$\mathcal{B}^{(k)}_{\mathrm{Dust}}(a) = \tilde{B}_0\left[1 + \frac{3}{4\pi^2}\left(\frac{\lambda_{\mathrm{GW}}}{\lambda_{\mathrm{H}}}\right)^2_0\left(\frac{\sigma_0}{H_0} + \frac{\sigma'_0}{2H_0}\right) - \left(\frac{2\sigma_0}{H_0}\right) + \mathcal{O}(a^{-1/2})\right]. \tag{7.35}$$

We do not give the remainders of the solution incorporated in $\mathcal{O}(a^{-\frac{1}{2}})$ as these decay rapidly with time and can be neglected. The result (7.35) indicates that the magnitude of the amplification depends critically on the initial conditions, as shown by the dust solution (6.75) in the homogeneous field case. The solution (7.32) is recovered when taking the long-wavelength limit of the full solution stated above.

We now present equation (7.34) in terms of the magnetic field variable $B_{(k)} = \mathcal{B}_{(k)}\left(\frac{a_0}{a}\right)^2$ to enable direct comparison with the result in chapter (6);

$$B^{(k)}_{\mathrm{Dust}}(a) = \tilde{B}_0\left(\frac{a_0}{a}\right)^2\left[1 + \frac{3}{4\pi^2}\left(\frac{\lambda_{\mathrm{GW}}}{\lambda_{\mathrm{H}}}\right)^2_0\left(\frac{\sigma_0}{H_0} + \frac{\sigma'_0}{2H_0}\right) - \left(\frac{2\sigma_0}{H_0}\right) + \mathcal{O}(a^{-1/2})\right]. \tag{7.36}$$

To simplify matters, we again take $\sigma'_{(k)}(1) = 0$ for granted. If we divide the result (7.36) through by the energy density of the background radiation, the dominant mode is then given by

$$\frac{B}{\mu_\gamma^{1/2}} \simeq \left[1 + \frac{1}{10}\left(\frac{\lambda_{\tilde{B}}}{\lambda_{\mathrm{H}}}\right)^2_0\left(\frac{\sigma}{H}\right)_0 - \left(\frac{\sigma_0}{H_0}\right)\right]\left(\frac{\tilde{B}}{\mu_\gamma^{1/2}}\right)_0, \tag{7.37}$$

where the wavenumber indices have been suppressed. We identify an additional term in the amplification factor $\left(-\frac{\sigma_0}{H_0}\right)$ that is sourced by curl E = - curl $(v \times B)$.

Chapter 8

Application

In order to estimate the amplification of the seed field due to the interaction with GWs, we reproduce the analysis presented in [98] using the same parameter values.

Given that the evolution of the spatially flat Universe is dominated by a dark energy component such as a cosmological constant or quintessence, the minimum seed required for the dynamo mechanism to work is of the order of 10^{-30} G at the time of completed galaxy formation, and coherent on a scale at least as large as the largest turbulent eddy, roughly $\sim$ 100 pc [25]. Using equation (5.7), we find that such a collapsed magnetic field with coherence length $\lambda_{\tilde{B}} \sim 10\,kpc$ on a comoving scale, corresponds to a field of strength

$$B_i = B_f \left(\frac{L_i}{L_f}\right)^2 = 10^{-30}\mathrm{G}\left(\frac{10^2\ \mathrm{pc}}{10^4\ \mathrm{pc}}\right)^2 = 10^{-34}\mathrm{G}, \tag{8.1}$$

provided the field remains frozen into the cosmic plasma from the epoch of radiation decoupling to galaxy formation [25]. Its field strength compared to the energy density of the background radiation prior to structure formation, μ_γ, gives rise to the ratio $\tilde{B}/\mu_\gamma^{1/2} \sim 10^{-29}$, which stays constant as long as the magnetic flux is conserved and the magnetic field is frozen into the cosmic medium.

The current models of inflation are based on the idea that a phase transition took place in the early Universe that generated a vacuum energy density which then naturally induced the exponential expansion. The Hubble parameter H remains constant during this phase and is related to the vacuum energy density μ_{infl} driving it by

$$H = \frac{\sqrt{(\mu_{infl}/3)}}{m_{Pl}} \sim \frac{\lambda M^2}{m_{Pl}}, \tag{8.2}$$

[24, 103] where the Planck mass m_{Pl} is the value for which the Schwarzschild radius and the Compton length are equal (i.e. the Planck length) (see [57, 78] for example). A general prediction of all inflationary scenarios is the production of large scale gravitational waves. Using the observed quadrupole anisotropy in the COBE CMB data, White extracted a value for the inflationary scale of H $= \frac{1}{\lambda_H} \sim 10^{-13}$G [103], assuming that the anisotropy

is sourced only by the inflationary GWs. The scale of the magnetic field therefore implies $\lambda_{\tilde{B}}/\lambda_H \sim 10^{20}$ at the end of inflation. The energy density of these GWs is roughly [98]

$$\mu_{\rm GW} \simeq \left(\frac{1}{\lambda_{\rm GW}}\right)^2 \left(\frac{H}{m_{\rm Pl}}\right)^2 . \tag{8.3}$$

Here $\lambda_{\rm GW}$ denotes the wavelength of the GW. This implies an induced shear anisotropy [98]

$$\left(\frac{\sigma}{H}\right)_0 \simeq \left(\frac{\lambda_H}{\lambda_{\rm GW}}\right)_0 \left(\frac{H}{m_{\rm Pl}}\right) , \tag{8.4}$$

where the zero suffix indicates the end of the inflationary epoch. Typical inflationary models predict $H/m_{\rm Pl} \sim 10^{-6}$, which lies comfortably within the bound $H/m_{\rm Pl} \lesssim 10^{-5}$ stemming from the quadrupole anisotropy of the CMB.

The interaction of such a primordial magnetic field with GWs produced by inflation leads to a substantial amplification of the former. Returning to our result (6.77) for the homogeneous magnetic field case and applying (8.4), we find for the total magnetic field [98]

$$\frac{B}{\mu_\gamma^{1/2}} \simeq \left[1 + \frac{1}{10}\left(\frac{\lambda_{\tilde{B}}}{\lambda_{\rm H}}\right)_0^2 \left(\frac{\lambda_{\rm H}}{\lambda_{GW}}\right) \left(\frac{H}{m_{\rm Pl}}\right)\right] \left(\frac{\tilde{B}}{\mu_\gamma^{1/2}}\right)_0 . \tag{8.5}$$

Since the magnetic field and gravitational wavelengths need to be of comparable size for the interaction to be physically effective, we can write

$$\frac{B}{\mu_\gamma^{1/2}} \simeq \left[1 + \frac{1}{10}\left(\frac{\lambda_{\tilde{B}}}{\lambda_{\rm H}}\right)_0 \left(\frac{H}{m_{\rm Pl}}\right)\right] \left(\frac{\tilde{B}}{\mu_\gamma^{1/2}}\right)_0 . \tag{8.6}$$

Substituting $(\lambda_{\tilde{B}}/\lambda_H)_0 \sim 10^{20}$ and $H/m_{\rm Pl} \sim 10^{-6}$ into the above expression, we find that the gravito-magnetic interaction amplifies the original magnetic field as much as 13 orders of magnitude. This mechanism thus brings an inflationary seed such as in [24] up to $\sim 10^{-21}$ G, which is comfortably within the requirements of the galactic dynamo mechanism [25]. In Universes with zero cosmological constant, the minimum seed for the dynamo has to be raised from $\sim 10^{-30}$ G to $\sim 10^{-23}$ G [61].
If we now consider the final solution (7.37) to the inhomogeneous case, we identify an additional term $\left(-\frac{\sigma}{H}\right)_0$ stemming from the rotation of the electric field. Using equation (8.4) we calculate the magnitude of this contribution;

$$\left(\frac{\sigma}{H}\right)_0 \simeq \left(\frac{\lambda_H}{\lambda_{\rm GW}}\right)_0 \left(\frac{H}{m_{\rm Pl}}\right) = \left(\frac{1}{10^{20}}\right)(10^{-6}) = 10^{-26}. \tag{8.7}$$

This is negligible when compared to the amplification factor of $\sim 10^{13}$. This makes sense given that the ratio in (8.7) is independent of a comoving wavenmber; the time-dependence of the forcing term F_a is sought from a first-order propagation equation in which a Laplacian does not appear. This term does not therefore contain the ratio of the coherence scales of the perturbation and the horizon, which is responsible for the substantial amplification by the interaction. We can infer that the rotation of the associated second-order electric field does not give rise to a significant suppression of the induced magnetic field.

We stress that the efficiency of the mechanism is highly dependent on the ratio between the coherence length $\lambda_{\tilde{B}}$ of the initial magnetic field and the initial size of the horizon λ_H. This ratio however, disappears when the limit ($k \to 0$) is taken (see section (6.8) and (7.8.1)). Even though the solutions (6.72), (6.73) and (7.32) show a growth proportional (quadratic) to the scale factor, the factor of proportionality $(\sigma/H)_0$ ($\sim 10^{-26}$ from (8.7) in our example) is far too small in order to achieve an effective amplification. It follows that the interaction between GWs and on average homogeneous and inhomogeneous magnetic fields is completely negligible in the limit of ($k \to 0$).

Chapter 9

Comparison and Discussion

We recalculated the induced magnetic field employing the weak-field approximation, extending previous results in [98], and compared the solutions with ours derived in a gauge-invariant manner using SOGI variables. It was found that there is a significant difference in the growth behaviour of the magnetic field when SOGI variables are used, as compared to the case of a weak-field approximation scheme. While the two methods agree in the limit of high conductivity, they are otherwise compatible only in the limit of ($k \to 0$) when the dominant part of the solution is considered. The reason why the solutions obtained within the weak field approximation are in general not equivalent to our solutions, comes from the non-gauge-invariance of the WFA; the magnetic field $\tilde{B}_a$ interacting with the GWs is treated as a weak background field and does not vanish exactly in the FLRW background as required by the Stewart and Walker lemma. We remind the reader once more that our procedure solves firstly for the gauge-invariant variable $\beta_a = \dot{B}_{<a>} + \frac{2}{3}\Theta B_a$ from which the magnetic field B_a measured in the frame of reference of u^a, can then subsequently be found. On integrating β, generated magnetic field is found, which is gauge-invariant relative to an observer moving with velocity u^a. In [98], where a non-zero magnetic field exists in the background FLRW model, it is a non-trivial process to extract the part of the field that is produced by the interaction.

The comparison of the homogeneous and inhomogeneous magnetic field cases indicates that the dominant parts of both solutions agree. The additional term in the amplification factor that stems from the rotation of the electric field in the inhomogeneous field case, is found to be negligible, the reason being that this forcing term is an artifact of the MHD approximation: it is introduced so that the spatial currents can be retained. Given that approximation to curl $E_a \simeq -2D^b \left(v_{[a}B_{b]}\right)$ is assumed to hold on each comoving scale, its solution is independent of wavenumber. Since it is the wavenumber k in the boost factor that introduces the gravitational wavelength which in turn gives rise to the resultant amplification, the electric field rotation does not produce a significant amplification. Furthermore, the interaction between GWs and a homogeneous magnetic field generates, in this particular case, no electric fields (at least to second-order in the perturbative scheme).

The presence of a spatially homogeneous field is only consistent in the flat Universe, whereas no restrictions on the spatial geometry arise at any stage in the derivation of the evolution equations with $(\mathrm{D}^b B^a) \neq 0$. This makes sense when we consider the spectrum of allowed wavenumbers for different geometries. For open models ($K = +1$), the lowest wavenumber is $n = 3$. For closed models ($K = -1$), we find that $n \geq 1$. However, the spectrum of wavenumbers in a flat spacetime ($K = 0$) is continuous, with $n \geq 0$. Given that the spatial homogeneity of the background field restricts its associated wavenumber to $n = 0$, we see that this eigenvalue can only be accommodated in a flat Universe. More interesting is the relationship between curvature and magnetic fields. Einstein's theory is geometrical which implies that vectors are directly coupled to the spacetime curvature via the Ricci identity [94]. In [96], Tsagas and Maartens find that the evolution equation of the spatial gradient of the magnetic field contains a term $\epsilon_{acd} B^c H_b^d \sim \mathcal{O}(\epsilon_B \epsilon_g)$, which alludes to a non-local coupling to curvature. This result is confirmed in [99] with the appearance of a term containing the Riemann curvature tensor in the propagation equation of $(D_a B_b)$ and indicates that the curvature sources magnetic inhomogeneities. In the case of a homogeneous magnetic field in chapter (6), the spatial gradients of the magnetic field are at least second-order. In order to preserve the spatial uniformity of the first-order field through time, the Riemann tensor may have to vanish to prevent it from sourcing $(D_a B_b)$ so that the spatial gradients remain small and continue to contribute at higher-order only. In the analysis of the inhomogeneous magnetic field in chapter (7), magnetic spatial eddies exist at first-order and for this reason, the boost from the coupling between the field and the curvature need not necessarily be eliminated.

Finally, our results show that in the presence of gravitational radiation, the magnitude of the magnetic field is amplified proportionally to the shear distortion caused by the propagating waves. Once the amplification is saturated, the magnetic field then dissipates adiabatically as usual. Crucially however, the gravitational boost is also proportional to the square of the field's original scale. This immediately suggests that the mechanism presented here could lead to significant amplification when dealing with *large* scale magnetic fields. Indeed, when applied to fields of roughly 10^{-34} G spanning a comoving scale of roughly 10 *kpc* today, like those produced in [24], our mechanism leads to an amplification of up to 13 orders of magnitude. The size of the boost can easily bring these magnetic fields well within the galactic dynamo requirements, without the need for extra amplification during reheating. In fact, the enhancement is so effective that it can bring the field within the dynamo limits even within conventional cosmological models which are not dark-energy dominated. This is more easily achieved when the extra strengthening of the field due to the adiabatic collapse of the protogalaxy, is also taken into account.

Chapter 10

Conclusion

Although the focus of magnetogenesis in recent years has been the generation of *large-scale* magnetic fields, a self-sufficient mechanism still evades us. The galactic dynamo is indeed physically feasible and has been shown to generate fields with strengths matching current observations, but requires a reasonably strong seed field to work. To make this theory more robust, we need to find a way of producing seed fields that are suitable for subsequent amplification by the dynamo. In this thesis we aimed to provide one such mechanism by investigating the properties of magnetic fields in the presence of cosmological gravitational waves using perturbation theory, building on the work using the weak field approximation in Tsagas *et. al* [98]. We regard the gauge-dependance of the magnetic field variable in [98] as problematic and for this reason, set off in search of an alternative gauge-independent perturbative approach with which to investigate this magnetic-gravito coupling.

After considering the problems stemming from the inherent freedom in the gauge choice, we concluded that the 1+3 covariant gauge-invariant formalism developed by Ellis and Bruni [37] allows for the most clear interpretation of the effects. Aftering reviewing the approximation schemes implemented in the literature, we selected a formalism based upon work presented in Clarkson *et. al.* [20, 21] as best suited to our purposes. The study of the interaction of a magnetic field with GWs using this perturbative scheme requires second-order perturbation theory.

The full set of equations determining the evolution of the gravitational waves and the generated electromagnetic fields was presented initially for the case of a homogeneous magnetic field. The integration showed an amplification of the induced magnetic field arising from the original field's interaction with the GWs. We found that the treatment of a homogeneous field using our chosen perturbative scheme was only consistent in a spatially flat Universe. The results were discussed in different fluid regimes, in particular dust and radiation, and it was established that the dominant contribution to the magnetic field is the same in both of these regimes. The interaction of GWs with a spatially inhomogeneous magnetic field in the dust regime was then studied using the magnetohydrodynamic approximation and was shown to yield an amplification of the same order of magnitude in all spatial geometries. The time behavior of the magnetic field solutions differed appreciably from the results calculated using the weak field approximation. In summary, the magneto-GW interaction was found to boost both a homogeneous and inhomogeneous magnetic field by more than 10 orders of magnitude, which is sufficient to bring primordial seed fields as predicted in [24] within the required limits for amplification by the dynamo mechanism.

Over and above the presentation of a physically viable mechanism for primordial magnetogenesis, the more important achievement is the establishment of a formalism which guides the choice of proper second-order gauge-invariant variables (SOGI). Using this methodology, one is able to obtain results in terms of clearly defined quantities, with no ambiguity concerning the physical validity of the variables. The discrepancies between the results obtained using the weak field approximation and our approach, indicates that a strictly gauge-invariant procedure is as necessary as formerly thought.

Chapter 11

Appendix

11.1 Identities

Below we give some identities used in the calculations, where the variables used are defined in the text.

$$
\begin{aligned}
\epsilon^{abc}\epsilon_{def} &= 3\,!\; h^{[a}{}_{d}h^{b}{}_{e}h^{c]}{}_{f} && (11.1)\\
\epsilon^{abc}\epsilon_{aef} &= 2\; h^{[b}{}_{e}h^{c]}{}_{f}, && (11.2)\\
\epsilon^{abc}\epsilon_{abf} &= 2\; h^{c}{}_{f}, && (11.3)\\
\epsilon^{abc}\epsilon_{abc} &= 3\,!. && (11.4)
\end{aligned}
$$

11.2 Commutation relations

Here we present various commutator relations which have been used in the text. The relations are given up to second-order in our perturbation scheme. The vanishing of vorticity, $\omega_{ab} = 0$, is assumed throughout in conjunction with the constraints $\mathrm{D}_a\mu = \mathrm{D}_a p = 0$ which isolate the pure tensor modes. All appearing tensors are PSTF, $S_{ab} = S_{<ab>}$, and all vectors V_a, W_a are purely spatial.

Commutators for scalars f:

$$
\begin{aligned}
(\mathrm{D}_a f)^{\cdot} &= \mathrm{D}_a \dot{f} - \frac{1}{3}\Theta D_a f - \sigma_a^b \mathrm{D}_b f, && (11.5)\\
\mathrm{curl}\, D_a f &= 0. && (11.6)
\end{aligned}
$$

Commutators for first-order vectors V_a:

$$
\begin{aligned}
(\mathrm{D}_a V_b)^{\cdot}_{\perp} &= \mathrm{D}_a \dot{V}_b - \tfrac{1}{3}\Theta \mathrm{D}_a V_b - \sigma_a{}^c \mathrm{D}_c V_b + H_a{}^d\, \epsilon_{dbc}\, V^c && (11.7)\\
(\mathrm{curl}\, V_a)^{\cdot}_{\perp} &= \mathrm{curl}\, \dot{V}_a - \tfrac{1}{3}\Theta\, \mathrm{curl}\, V_a - \epsilon_{abc}\, \sigma^{bd}\, \mathrm{D}_d\, V^c - H_{ab}\, V^b && (11.8)\\
\mathrm{D}_{[a}\, \mathrm{D}_{b]} V_c &= \left[\tfrac{1}{9}\Theta^2 - \tfrac{1}{3}(\mu + \Lambda)\right] V_{[a}\, h_{b]c} + \left(\tfrac{1}{3}\Theta\, \sigma_{c[a} - E_{c[a}\right) V_{b]} \\
&\quad + h_{c[a}\left(E_{b]d} - \tfrac{1}{3}\Theta\, \sigma_{b]d}\right) V^d && (11.9)
\end{aligned}
$$

Commutators for first-order tensors T_{ab}:

$$\begin{aligned}
\left(\mathrm{D}_a T_{bc}\right)^{\cdot}_{\perp} &= \mathrm{D}_a \dot{T}_{bc} - \tfrac{1}{3}\Theta \mathrm{D}_a T_{bc} - \sigma_a{}^d \mathrm{D}_d T_{bc} + 2H_a{}^d\, \epsilon_{de(b}\, T_{c)}{}^e && (11.10)\\
\left(\mathrm{D}^b T_{ab}\right)^{\cdot}_{\perp} &= \mathrm{D}^b \dot{T}_{ab} - \tfrac{1}{3}\Theta \mathrm{D}^b T_{ab} - \sigma^{bc} \mathrm{D}_c T_{ab} + \epsilon_{abc}\, H^b{}_d\, T^{cd} && (11.11)\\
\left(\mathsf{curl}\, T_{ab}\right)^{\cdot}_{\perp} &= \mathsf{curl}\, \dot{T}_{ab} - \tfrac{1}{3}\Theta\, \mathsf{curl}\, T_{ab} - \sigma_e{}^c\, \epsilon_{cd(a}\, D^e T_{b)}{}^d + 3H_{c<a} T_{b>}{}^c && (11.12)\\
\mathsf{curl}\,\mathsf{curl}\, T_{ab} &= -D^2 T_{ab} + \left(\mu + \Lambda - \tfrac{1}{3}\Theta^2\right) T_{ab} + \tfrac{3}{2} D_{<a} D^c T_{b>c} \\
&\quad +3T_{c<a}\left(E_{b>}{}^c - \tfrac{1}{3}\Theta\sigma_{b>}{}^c\right) && (11.13)
\end{aligned}$$

Commutators for second-order vectors W_a:

$$\begin{aligned}
\left(\mathrm{D}_a W_b\right)^{\cdot}_{\perp} &= \mathrm{D}_a \dot{W}_b - \tfrac{1}{3}\Theta\, \mathrm{D}_a W_b && (11.14)\\
\mathrm{D}_{[a}\, \mathrm{D}_{b]} W_c &= \left[\tfrac{1}{9}\Theta^2 - \tfrac{1}{3}\left(\mu + \Lambda\right)\right] W_{[a}\, h_{b]c} && (11.15)\\
\mathsf{curl}\,\mathsf{curl}\, W_a &= -\mathrm{D}^2 W_a + \mathrm{D}_a\left(\mathsf{div}\, W\right) + \tfrac{2}{3}\left(\mu + \Lambda - \tfrac{1}{3}\Theta^2\right) W_a. && (11.16)
\end{aligned}$$

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