STATIC HEDGING OF BARRIER OPTIONS: A REVIEW OF FOUR METHODS

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ABSTRACT

This paper examines the static hedging of a European up-and-out call option. Four different static hedging models are examined in detail and are implemented. Their hedging performance is examined in a framework that aims to simulate real market conditions. This is done to determine the practical usefulness of the static hedging schemes in comparison with dynamic delta hedging. Only one of the four models, by Derman, Ergener and Kani (1995) seems to show promise when transaction costs and stochastic volatility are taken into account.

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1 INTRODUCTION

Static hedging is one of the most actively researched areas in Financial Mathematics. This paper examines various models proposed for hedging barrier options statically. Previous work on the evaluation of hedges of these options by Tompkins (2002), concluded that while various exotic options can effectively be hedged by either dynamic delta or static hedges, neither methods produced effective hedges for a European up-and-out call option. Tompkins concluded that further research into how this option can be hedged is warranted.

This paper extends the work done by Tompkins (2002) by examining four models for the hedging of a European up-and-out call. In section two of this paper, an overview of the rationale behind barrier options, the various types of barrier options and the pricing thereof is discussed. Section three discusses the preference for static hedging and its main advantages. This is followed by a section that provides a brief overview of the existing literature on the static hedging of barrier options. The sections thereafter examine the four hedging models in detail. Section nine evaluates the performance of the proposed static hedging methodologies and casts some light on their effectiveness. Finally, relevant conclusions are drawn

All the modelling experiments were implemented in Microsoft Excel 2000 and Visual Basic for Applications and were run on a personal computer with 1.2GHz processor with 640MB RAM. The CD-ROM accompanying this paper includes all the programming code of the numerical implementations of the four models reviewed in this paper.

2 BARRIER OPTIONS

Barrier options have been traded in financial markets in larger volumes than vanilla options as far back as the 1950's. Snyder (1969) discusses the use and market for 'special options', which are now known as 'barrier options'. They allow the user to tailor option strategies to their specific market views, generally at a lower cost compared to vanilla options. This section discusses the various types of single European barrier options, their use and pricing in a Black-Scholes framework.

2.1 Types of barrier options

Single barrier options can be grouped into one of two groups: Regular or Reverse barrier options. Reverse barrier options are options that are either born or die in-the-money. This results in a discontinuity in the position needed when dynamically hedging (see discussion in Section 3) which increases trading difficulty and risks. Table 1 below lists the various types of single European barrier options and their payoffs. As soon as the payoff conditions are not met, the barrier option ceases to exist and pays out nothing.

Table 1: Categories of Single barrier options

Option	Type	Payoff at maturity
Down-and-out call	Regular	$\max\{S_T - K, 0\}$ provided $S_t > L,\ \forall t \in [0,T]$, where $L < K$
Down-and-out put	Reverse	$\max\{K - S_T, 0\}$ provided $S_t > L,\ \forall t \in [0,T]$, where $L < K$
Up-and-out call	Reverse	$\max\{S_T - K, 0\}$ provided $S_t < L,\ \forall t \in [0,T]$, where $L > K$
Up-and-out put	Regular	$\max\{K - S_T, 0\}$ provided $S_t < L,\ \forall t \in [0,T]$, where $L > K$
Down-and-in call	Regular	$\max\{S_T - K, 0\}$ provided $\exists t$ s.t. $S_t \leq L$ for $t \in [0,T]$, where $L < K$
Down-and-in put	Reverse	$\max\{K - S_T, 0\}$ provided $\exists t$ s.t. $S_t \leq L$ for $t \in [0,T]$, where $L < K$
Up-and-in call	Reverse	$\max\{S_T - K, 0\}$ provided $\exists t$ s.t. $S_t \geq L$ for $t \in [0,T]$, where $L > K$
Up-and-in put	Regular	$\max\{K - S_T, 0\}$ provided $\exists t$ s.t. $S_t \geq L$ for $t \in [0,T]$, where $L > K$

2.2 Demand for barrier options

The following factors affect the demand for barrier options:

- Cheaper than vanilla options: A major reason for the use of barrier options is that they are cheaper than vanilla options. Consider a speculator who thinks that a share (or index) will go up, but by no more than 20% over some period of time (say 6 months). Let the current price level be 100, volatility 20%, the risk-free rate 10% and assume no dividends. An at-the-money European vanilla call option (strike equals 100) will cost the speculator 8.28. However, a European up-and-out call option with barrier level 120 will cost the speculator only 2.35. Taleb (1997) notes that fund managers use barrier options (particularly down-and-out puts) to hedge their exposure, at reduced cost, but with residual risk.
- Chartist views: Taleb (1997) also explains that the demand for barrier options is affected by market participants believing in serial correlation and trends. Fund managers may prefer to be long provided that the market does not fall below a certain point. These market participants therefore create demand for down-and-out call options.
- Mean-reverting mentality: As opposed to chartist views, some market participants believe in mean-reverting price processes. They believe that if the market rallies too high, it will revert back (fall again) or vice versa. They therefore prefer to hold options that come into existence against the general market direction. This mentality lends itself to knock-in options. For example, those believing in mean reversion may want to hold up-and-in put options in a rallying bull market.

2.3 Pricing barrier options in the Black-Scholes framework

The focus of this paper from this section onwards will be on the European up-and-out barrier option. The derivation in this section is based on the pricing of barrier options as set out by Björk (1998). We work in a space $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}\right)$ where $\mathcal{F}_t$ is generated by a standard Brownian motion B_t, augmented to satisfy the usual conditions. Consider the following arithmetic Brownian motion process X_t:

$$\begin{aligned} dX_t &= \mu dt + \sigma dB_t \\ X_0 &= \alpha \end{aligned} \tag{1}$$

where μ is the constant drift, σ the volatility and B_t a standard Brownian motion. Let τ_β be a stopping time such that,

$$\tau_\beta = \inf\{t : X_t = \beta\}$$

where $\beta > \alpha$

Define the absorbed process $X_t^{\tau_\beta}$ as follows:

$$X_t^{\tau_\beta} = \begin{cases} X_t & \text{if } t < \tau_\beta \\ \beta & \text{if } t \geq \tau_\beta \end{cases} \tag{2}$$

Theorem 2.3.1:

The density $f_\beta(x;t,\alpha)$ of the above absorbed process $X_t^{\tau_\beta}$ where X_t is as defined in (1) above, is given by

$$f_\beta(x;t,\alpha) = \varphi(x;\mu t + \alpha, \sigma^2 t) - \exp\left\{-\frac{2\mu(\alpha-\beta)}{\sigma^2}\right\}\varphi(x;\mu t - \alpha + 2\beta, \sigma^2 t)$$

where $\varphi(x;\mu,\sigma^2)$ denotes the density of a normal distribution with mean μ and variance σ^2. The density $f_\beta(x;t,\alpha)$ holds for the interval $(-\infty,\beta)$ if $\alpha < \beta$.

Proof:

The proof of the above theorem will be broken up into four sub-lemmas.

Lemma 1:

First note that Brownian motion satisfies the Strong Markov Property:

If τ is a stopping time, then $\hat{B}_t \equiv B_{\tau+t} - B_\tau$ is a standard B.M. indep. of the σ-algebra $\mathcal{F}_\tau$.

Let $\tau = \inf\{t : B_t = y\}$ and let $\hat{B}_t = B_{\tau+t} - B_\tau$, where $B_t =$ standard Brownian motion, $B_t^* = \sup\limits_{s \le t} B_s$, $0 \le y$ and $x \le y$. Then,

$$\mathbb{P}(B_t \le x, B_t^* > y) = \mathbb{P}(B_{\tau+(t-\tau)} \le x, \tau < t)$$

$$\begin{aligned}
&= \mathbb{P}(\hat{B}_{t-\tau} \le x - y, \tau < t) \text{ since } \hat{B}_{t-\tau} = B_t - B_\tau \text{ and } B_\tau = y \\
&= \mathbb{P}(\hat{B}_{t-\tau} \le x - y)\mathbb{P}(\tau < t) \text{ since } \hat{B}_{t-\tau} \text{ independent of } \mathcal{F}_\tau \text{ and } \{\tau < t\} \in \mathcal{F}_\tau \\
&= \mathbb{P}(\hat{B}_{t-\tau} \ge y - x)\mathbb{P}(\tau < t) \text{ by symmetry} \\
&= \mathbb{P}(\hat{B}_{t-\tau} \ge y - x, \tau < t) \\
&= \mathbb{P}(B_t - B_\tau \ge y - x, \tau < t) \\
&= \mathbb{P}(B_t \ge 2y - x) \text{ since } B_\tau = y \text{ and } \{B_t \ge 2y - x\} \subseteq \{\tau < t\} \\
&\qquad \text{because } B_t \ge 2y - x \Rightarrow B_t \ge y \Rightarrow \tau < t
\end{aligned}$$

Therefore,

$$\mathbb{P}(B_t \le x, B_t^* > y) = \mathbb{P}(B_t \ge 2y - x) = \mathbb{P}\left(\frac{B_t}{\sqrt{t}} \ge \frac{2y - x}{\sqrt{t}}\right) = 1 - \mathbb{N}\left(\frac{2y - x}{\sqrt{t}}\right) = \mathbb{N}\left(\frac{x - 2y}{\sqrt{t}}\right)$$

since $\frac{B_t}{\sqrt{t}} \sim \mathbb{N}(0,1)$ and where $\mathbb{N}$ is the cumulative distribution function of a standard normal random variable.

□

Lemma 2:

We now investigate $F_t(x, y) = \mathbb{P}(B_t \le x, B_t^* \le y)$, the joint distribution of B_t and B_t^*. We derive an expression for F_t. Now,

$$\begin{aligned} F_t(x,y) &= \mathbb{P}(B_t \le x, B_t^* \le y) \\ &= \mathbb{P}(B_t \le x) - \mathbb{P}(B_t \le x, B_t^* > y) \end{aligned}$$

Also, $\mathbb{P}(B_t \le x, B_t^* > y) = \mathbb{P}(B_t \ge 2y - x)$

It therefore follows from *Lemma 1* that the joint distribution $F_t(x,y)$ of (B_t, B_t^*) is given by

$$F_t(x,y) = \mathbb{N}\left(\frac{x}{\sqrt{t}}\right) - \mathbb{N}\left(\frac{x-2y}{\sqrt{t}}\right)$$

and the marginal density function with respect to x,

$$f_t^x(x,y) = \frac{1}{\sqrt{t}}\left(\varphi\left(\frac{x}{\sqrt{t}}\right) - \varphi\left(\frac{x-2y}{\sqrt{t}}\right)\right)$$

where $\varphi(x)$ is the standard normal density function. This can be seen by differentiating $F_t(x,y)$ with respect to x.

□

Lemma 3:

Next, we aim to extend the result of *Lemma 2* to the case where X_t is an arithmetic Brownian motion starting at 0 with drift μ and volatility σ. Start with a $\mathbb{P}$-Brownian motion B_t, and define $d\mathbb{Q} = e^{\frac{\mu}{\sigma}B_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t} d\mathbb{P}$. Then define $\hat{B}_t = B_t - \frac{\mu}{\sigma}t$ and define $X_t = \sigma\hat{B}_t = \mu t + \sigma B_t$. X_t is then, under $\mathbb{Q}$, an arithmetic Brownian motion with drift μ and volatility σ. Now,

$$\begin{aligned}
&\mathbb{Q}(X_t \le x, X_t^* \le y) \\
&= \mathbb{Q}\left(B_t \le \frac{x}{\sigma}, B_t^* \le \frac{y}{\sigma}\right) \\
&= \int I_{\left\{B_t \le \frac{x}{\sigma}, B_t^* \le \frac{y}{\sigma}\right\}} \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{P} \\
&= \int e^{\frac{\mu}{\sigma}B_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}t} I_{\left\{B_t \le \frac{x}{\sigma}, B_t^* \le \frac{y}{\sigma}\right\}} d\mathbb{P} \\
&= \int_{-\infty}^{\mu/\sigma} e^{\frac{\mu}{\sigma}z - \frac{1}{2}\frac{\mu^2}{\sigma^2}t} f_t^x\left(z, \frac{y}{\sigma}\right) dz \\
&= \int_{-\infty}^{\mu/\sigma} e^{\frac{\mu}{\sigma}z - \frac{1}{2}\frac{\mu^2}{\sigma^2}t} \frac{1}{\sqrt{t}}\left[\varphi\left(\frac{z}{\sqrt{t}}\right) - \varphi\left(\frac{z - 2y/\sigma}{\sqrt{t}}\right)\right] dz \\
&= \mathbb{N}\left(\frac{x - \mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu y}{\sigma^2}} \mathbb{N}\left(\frac{x - 2y - \mu t}{\sigma\sqrt{t}}\right)
\end{aligned}$$

where the last line follows by completing the square. We have now shown that if X_t is an arithmetic $\mathbb{Q}$-Brownian motion, with $X_0 = 0$, then

$$\mathbb{Q}(X_t \le x, X_t^* \le y) = \mathbb{N}\left(\frac{x - \mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu y}{\sigma^2}} \mathbb{N}\left(\frac{x - 2y - \mu t}{\sigma\sqrt{t}}\right)$$

Now, if $X_0 = \alpha$, then we simply replace x by $x - \alpha$ and y by $y - \alpha$ to get

$$\begin{aligned}
F_t(x, y) &= \mathbb{Q}(X_t \le x, X_t^* \le y) \\
&= \mathbb{N}\left(\frac{x - \alpha - \mu t}{\sigma\sqrt{t}}\right) - \exp\left\{\frac{2\mu(y - \alpha)}{\sigma^2}\right\} \mathbb{N}\left(\frac{x + \alpha - 2y - \mu t}{\sigma\sqrt{t}}\right)
\end{aligned}$$

and thus

$$f_t^x(x, y) = \varphi(x, \mu t + \alpha, \sigma^2 t) - \exp\left\{\frac{2\mu(y - \alpha)}{\sigma^2}\right\} \varphi(x, \mu t - \alpha + 2y, \sigma^2 t)$$

where $\varphi(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ is the density of a $\mathbb{N}(\mu, \sigma^2)$ variable.

$\square$

Lemma 4:

Now suppose that

$$\begin{aligned} dX_t &= \mu dt + \sigma dB_t \\ X_0 &= \alpha \end{aligned}$$

under a measure $\mathbb{P}$.

Consider the absorbed process $X_t^{\tau_\beta}$ as defined in (2) and (1) above. It is obvious that $\mathbb{P}\left(X_t^{\tau_\beta} \leq x\right) = 1$ if $x \geq \beta$. Now,

$$\begin{aligned} \mathbb{P}\left(X_t^{\tau_\beta} \leq x\right) &= \mathbb{P}\left(X_t \leq x, X_t^* < \beta\right) \\ &= \mathbb{N}\left(\frac{x-\alpha-\mu t}{\sigma\sqrt{t}}\right) - \exp\left\{\frac{2\mu(\beta-\alpha)}{\sigma^2}\right\}\mathbb{N}\left(\frac{x+\alpha-2\beta-\mu t}{\sigma\sqrt{t}}\right) \end{aligned}$$

Therefore, by *Lemma 3* the density $f_\beta(x;t,\alpha)$ of $X_t^{\tau_\beta}$ is given by

$$f_\beta(x;t,\alpha) = \begin{cases} \varphi(x;\mu t+\alpha,\sigma^2 t) - \exp\left\{-\dfrac{2\mu(\alpha-\beta)}{\sigma^2}\right\}\varphi(x;\mu t-\alpha+2\beta,\sigma^2 t) \ \text{ if } x < \beta \\ 0 \ \text{ if } \ x \geq \beta \end{cases}$$

for $\beta > \alpha$ where $\varphi(x,\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$. *Lemma 4* thus completes the proof of *Theorem 2.3.1.*

□

Theorem 2.3.1 above can now be used to prove the following theorem for up-and-out contracts (call or put).

Theorem 2.3.2:

Consider the following dynamics for the share price process and the bank account,

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t \\ dA_t &= rA_t dt \end{aligned}$$

where A_t now refers to the bank-account at time t and W_t the standard Brownian motion. Let $F(t,s,\Phi)$ be the value at time t of a contingent claim with payoff $\Phi(S(T))$

at time T, given $S_t = s$. Also, let F^{LO} denote the pricing function of an up-and-out contract with maturity T, payoff function $\Phi(S(T))$ and barrier level L. Furthermore, let $F(t,s,\Phi^L)$ refer to the pricing functional of a claim at share price level s at time t and payoff function Φ^L. Then,

$$F^{LO}(t,s,\Phi) = F(t,s,\Phi^L) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t,\frac{L^2}{s},\Phi^L\right)$$

where $S < L$ and $\tilde{r} = r - \frac{1}{2}\sigma^2$. Here $\Phi^L(x) = \Phi(x).I_{\{x<L\}}$ where I refers to the usual indicator function.

Proof:

Let $\mathcal{Z}^{LO} = \begin{cases} \Phi(S(T)) & \text{if } S(t) < L \ \ \forall t \in [0,T] \\ 0 & \text{else} \end{cases}$ (i.e. claim only pays off if barrier has not been hit). Now,

$$\begin{aligned} F^{LO}(0,s,\Phi) &= e^{-rT}\mathbb{E}^{\mathbf{Q}}_{0,s}\left[\mathcal{Z}^{LO}\right] \\ &= e^{-rT}\mathbb{E}^{\mathbf{Q}}_{0,s}\left[\Phi(S(T)).I_{\left\{\sup\limits_{0\le t\le T} S(t)<L\right\}}\right] \\ &= e^{-rT}\mathbb{E}^{\mathbf{Q}}_{0,s}\left[\Phi^L(S^L(T)).I_{\left\{\sup\limits_{0\le t\le T} S(t)<L\right\}}\right] \quad \text{where } S^L \text{ is the absorption of } S \text{ at level } L \\ &= e^{-rT}\mathbb{E}^{\mathbf{Q}}_{0,s}\left[\Phi^L(S^L(T))\right] \end{aligned}$$

where $\mathbb{E}^{\mathbf{Q}}_{0,s}$ refers to the conditional expectation under the risk-neutral measure, conditional on the share price being s at time 0.

$$\mathbb{E}^{\mathbf{Q}}_{0,s}\left[\Phi^L(S^L(T))\right] = \int_0^L \Phi^L(x)h(x)dx$$

where $h(x)$ refers to the density of the stochastic variable $S_L(T)$. From standard Black-Scholes theory and price dynamics,

$$S_T = se^{\tilde{r}T+\sigma W_T} = e^{X(T)}$$

where,

$$\begin{aligned} dX(t) &= \tilde{r}dt + \sigma dW(t) \\ X(0) &= \ln s \end{aligned}$$

and thus $S^L(t) = e^{\left(X^{\ln L}(t)\right)}$. Therefore,

$$\mathbb{E}^{\mathbf{Q}}_{0,s}\left[\Phi^L(S^L(T))\right] = \int_{-\infty}^{\ln L} \Phi^L(e^y) f(y)dy$$

where $f(y)$ is the probability density function of the stochastic variable $X^{\ln L}(T)$ as given by *Theorem 2.3.1* above. So,

$$\begin{aligned} f(y) &= \varphi(y;\tilde{r}T + \ln s, \sigma\sqrt{T}) - \exp\left\{\frac{-2\tilde{r}(\ln s - \ln L)}{\sigma^2}\right\}\varphi(y;\tilde{r}T - \ln s + 2\ln L, \sigma\sqrt{T}) \\ &= \varphi(y;\tilde{r}T + \ln s, \sigma\sqrt{T}) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \varphi\left(y;\tilde{r}T + \ln\left(\frac{L^2}{s}\right), \sigma\sqrt{T}\right) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}^{\mathbf{Q}}_{0,s}\left[\Phi^L(S^L(T))\right] &= \int_{-\infty}^{\ln L} \Phi^L(e^y) f(y)dy \\ &= \int_{-\infty}^{\ln L} \Phi^L(e^y)\varphi(y;\tilde{r}T + \ln s, \sigma\sqrt{T})dy - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \int_{-\infty}^{\ln L} \Phi^L(e^y)\varphi\left(y;\tilde{r}T + \ln\left(\frac{L^2}{s}\right), \sigma\sqrt{T}\right)dy \\ &= \int_{-\infty}^{\infty} \Phi^L(e^y)\varphi(y;\tilde{r}T + \ln s, \sigma\sqrt{T})dy - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \int_{-\infty}^{\infty} \Phi^L(e^y)\varphi\left(y;\tilde{r}T + \ln\left(\frac{L^2}{s}\right), \sigma\sqrt{T}\right)dy \end{aligned}$$

This implies that,

$$\mathbb{E}^{\mathbf{Q}}_{0,s}\left[\Phi^L(S^L(T))\right] = \mathbb{E}^{\mathbf{Q}}_{0,s}\left[\Phi^L(S(T))\right] - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \mathbb{E}^{\mathbf{Q}}_{0,\frac{L^2}{s}}\left[\Phi^L(S(T))\right]$$

and therefore that,

$$F^{LO}(t,s,\Phi) = F(t,s,\Phi^L) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t,\frac{L^2}{s},\Phi^L\right) \qquad \square$$

2.3.1 The European Up-and-out call option formula

Consider a European up-and-out call option with maturity T, strike level K and barrier level L on a share with price S_t at time t. Also, let the share's continuous dividend rate be q, the risk-free rate r and the volatility σ. Therefore, $\tilde{r} = r - q - \frac{1}{2}\sigma^2$. By *Theorem 2.3.2,* we have that

$$F^{LO}(t,s,C(K)) = F(t,s,C^L(K)) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t,\frac{L^2}{s},C^L(K)\right)$$

where $C(K)$ denotes the payoff of a call struck at K and $C^L(K)$ the payoff of a contingent claim that pays

$$\begin{cases} S_T - K & \text{if } S_T < L \\ 0 & \text{otherwise} \end{cases}$$

In other words, the out feature here depends on S_T, not the path of $(S_t)_{t \le T}$.

Consider $H(x;L) = \begin{cases} 1 & \text{if } x > L \\ 0 & \text{if } x \le L \end{cases}$. This contract gives the owner one unit currency if the value of the underlying share exceeds L at maturity T, otherwise nothing. Using risk-neutral valuation (in a Black-Scholes framework), because H is just a binary call with strike L, it is well-known that

$$H(t,s;L) = e^{-r(T-t)} \mathrm{N}\left[\frac{\ln\left(\frac{s}{L}\right) + \tilde{r}(T-t)}{\sigma\sqrt{T-t}}\right]$$

where $H(t,s;L)$ is the price of this contract at time t (maturity T) with share price s at time t.

Now, for $L < K$ the up-and-out call option has no chance of any payoff, so we can ignore this case. If $L > K$, then

$$C^L(K) = C(K) - C(L) - (L-K)H(L)$$

See Figure 1 below for an explanation.

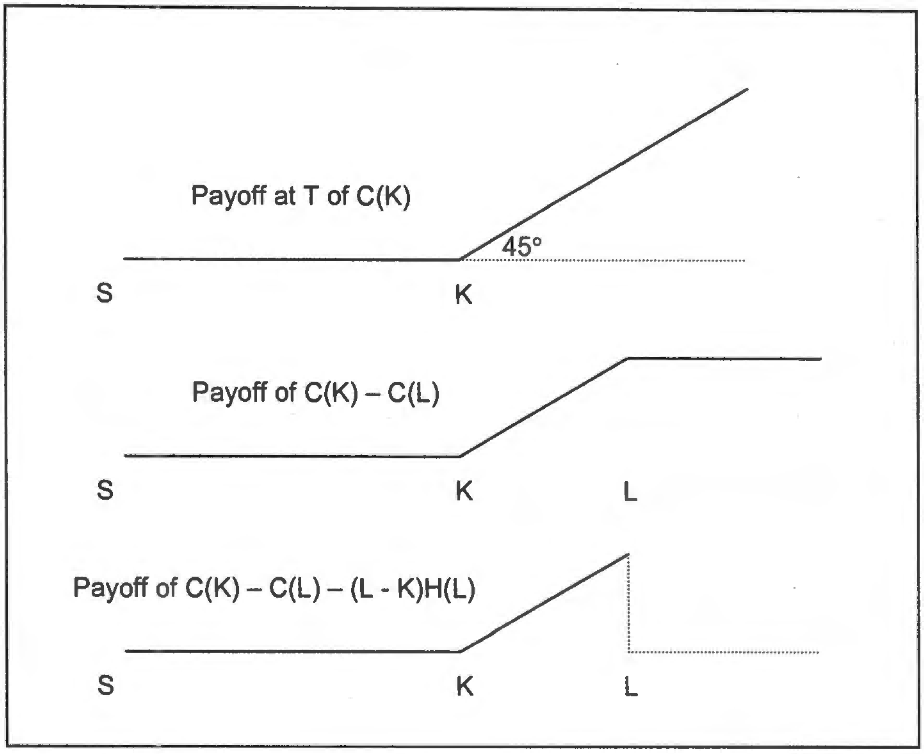


Figure 1: The payoff at maturity of an up-and-out call option synthesized using other securities

Therefore, the price of an up-and-out call option is as follows:

$$\begin{aligned} C^{LO}(t,s,K) &= C(t,s,K) - C(t,s,L) - (L-K)H(t,s,L) \\ &\quad - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \left[C(t,\frac{L^2}{s},K) - C(t,\frac{L^2}{s},L) - (L-K)H(t,\frac{L^2}{s},L) \right] \end{aligned} \tag{3}$$

where $C(t,s,K)$ denotes the price of a vanilla call option with strike K with a share price of s at time t. This follows from the linearity of the pricing functional F, which in turn follows from the linearity of the expectation operator.

2.3.2 *Delta of a European Up-and-out call option*

The delta, or first derivative with respect to the share price, of the European up-and-out call (equation (3)) can thus be computed as

$$\begin{aligned}\frac{\partial C^{LO}(t,s,K)}{\partial s} &= \frac{\partial C(t,s,K)}{\partial s} - \frac{\partial C(t,s,L)}{\partial s} - (L-K)\frac{\partial H(t,s,L)}{\partial s} \\ &\quad -\left(L^{2\tilde{r}/\sigma^2}\right)\left(\frac{-2\tilde{r}}{\sigma^2}\right)\left(s^{\frac{-2\tilde{r}}{\sigma^2}-1}\right)\left[C(t,\frac{L^2}{s},K) - C(t,\frac{L^2}{s},L) - (L-K)H(t,\frac{L^2}{s},L)\right] \\ &\quad -\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}\left(-\frac{L^2}{s^2}\right)\left[\frac{\partial C(t,\frac{L^2}{s},K)}{\partial s} - \frac{\partial C(t,\frac{L^2}{s},L)}{\partial s} - (L-K)\frac{\partial H(t,\frac{L^2}{s},L)}{\partial s}\right]\end{aligned}$$

for $S < L$, where $\frac{\partial C(t,s,K)}{\partial s}$ is the delta of a call with strike K and a share price of s at time t, while $\frac{\partial H(t,s,L)}{\partial s} = \frac{e^{-r(T-t)}}{s\sigma\sqrt{T-t}}\varphi\left[\frac{\ln\left(\frac{s}{L}\right)+\tilde{r}(T-t)}{\sigma\sqrt{T-t}}\right]$.

Section 3.3 contains a discussion on the use of delta for dynamic hedging, its continuity and some graphical depictions.

2.3.3 *Other Barrier Option Formulae*

Formulae for pricing down-and-out contracts ($F_{LO}(t,s,\Phi)$) can be derived in a similar way to up-and-out contracts as discussed in the previous sections.

$$F_{LO}(t,s,\Phi) = F(t,s,\Phi_L) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t, \frac{L^2}{s}, \Phi_L\right)$$

See Björk (1998), but note that the formula for an up-and-out call stated there is incorrect.

The price of down-and-in contracts ($F_{LI}(t,s;\Phi)$) can easily be found through in-out parity,

$$F_{LI}(t,s;\Phi) = F(t,s;\Phi) - F_{LO}(t,s;\Phi), \quad \forall s$$

because the down-and-in and down-and-out contracts are perfectly complementary. When one disappears, the other appears and as a result, they must add up to a vanilla contract ($F(t,s;\Phi)$).

Similarly, the price of up-and-in securities ($F^{LI}(t,s;\Phi)$) can be found, again using in-out-parity:

$$F^{LI}(t,s;\Phi) = F(t,s;\Phi) - F^{LO}(t,s;\Phi), \quad \forall s$$

Both put and call prices for these contracts can thus easily be found in a Black-Scholes framework.

2.3.4 *Black-Scholes prices versus real prices*

In real market conditions, the crossing of the barrier is typically checked discretely (for example once a day). This contrasts with the Black-Scholes framework, where the share price is monitored continuously. Figure 2 below depicts a heuristic argument as to why the price of an up-and-out barrier option would be less under continuous than discrete monitoring of the barrier. Assume that the price of a share is S_t at time t and S_{t+1} at time t, with both being below the barrier level L. In the case of continuous monitoring, there are an infinite number of price paths between time points t and $t+1$. A fair proportion of these paths may cross the barrier level, resulting in the up-and-out option knocking out. However, in the case of discrete monitoring at time points t and $t+1$, none of the share price paths will result in the option knocking out (because only the values at times t and $t+1$ are considered).

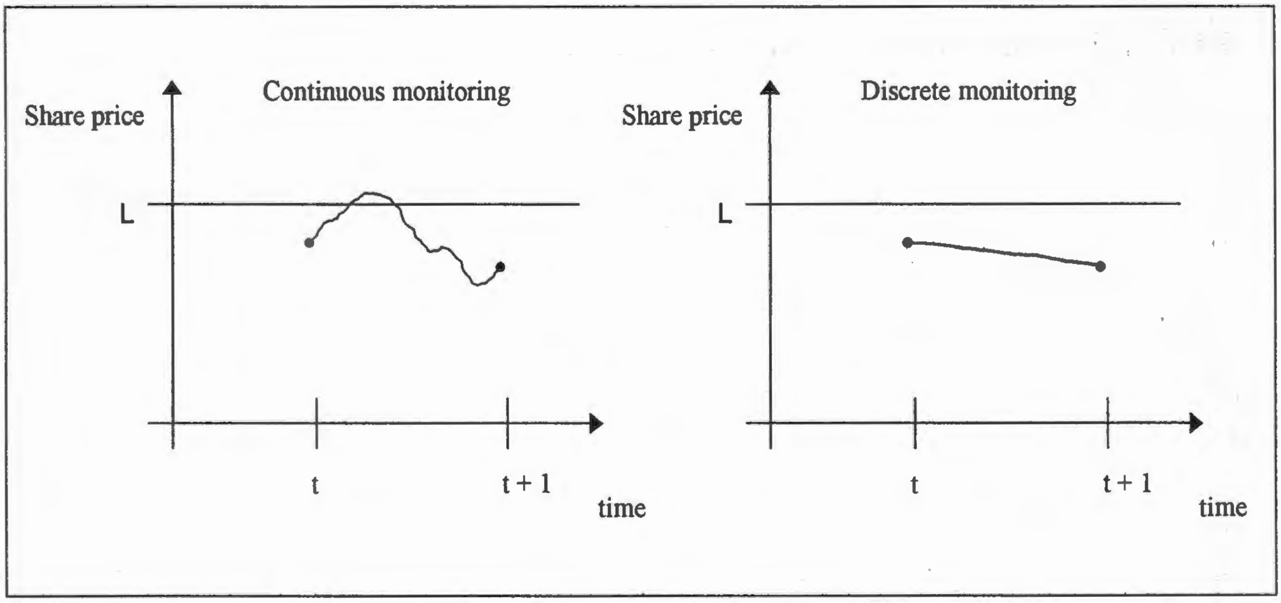


Figure 2: Continuous versus discrete monitoring

The probability of crossing L is therefore greater under continuous monitoring. As a result, the price of an up-and-out option is therefore lower in the Black-Scholes framework than that observed in the market. Monte Carlo simulations confirm the above argument.

3 A CASE FOR STATIC HEDGING

This section motivates why market participants (and especially market makers in options) may prefer static hedging to dynamic hedging in the case of barrier options.

3.1 Dynamic hedging

According to the Black-Scholes (Black and Scholes (1973)) theory, a share option at any instant behaves like a weighted portfolio of risky shares and riskless zero-coupon bonds. Instead of buying an option, you can buy a portfolio of shares and bonds, with weights as given by the Black-Scholes formula, and earn exactly the same return. To accomplish this you must *continuously* adjust the weights according to the formula though. This portfolio with continuous adjustment is called the dynamic replicating portfolio.

3.2 Static hedging

Under static hedging, a portfolio consisting of various securities like standard options is constructed so that it will exactly replicate the value of the target option (option needing to be hedged). This portfolio can consist of securities with various maturities and strike levels, but must have *fixed* weights that will require no further adjustment.

3.3 Dynamic Hedging: Problematic for Barrier Options

In this paper, market makers in options are assumed to be those market participants that have no business reason other than trading profit to take on option positions. They don't have some business or project risk that that needs to be hedged nor are they speculating or placing a directional bet. Their business aim is simply to determine the cost of the option and charge the client a price higher than this (i.e. cost of option + profit = option premium). Dynamic hedging, like delta hedging (see Hull (2003) for a discussion), is not used to determine the cost of the option to the market maker. It is rather a technique used by market makers to protect themselves from losses. Static hedging will prove to be a better tool for determining the cost of the option, especially in the case of a barrier option. Let's consider dynamic hedging of a European up-and-out call option on a share in comparison with a European vanilla call option. Parameters are given in Table 2 below.

Table 2: Option Parameters

Strike Price (X)	R100
Barrier Level (L)	R150
Volatility	20%
Risk-free rate (r)	10%
Continuous dividend yield (q)	5%
Time to maturity in years (T)	1/12

Note that the rates in Table 2 above are given in NACC (nominal annual compounded continuously) form. Figure 3 below shows the Black-Scholes values for the up-and-out call option (UOC) and the vanilla call option at time 0 for various spot prices of the underlying share.

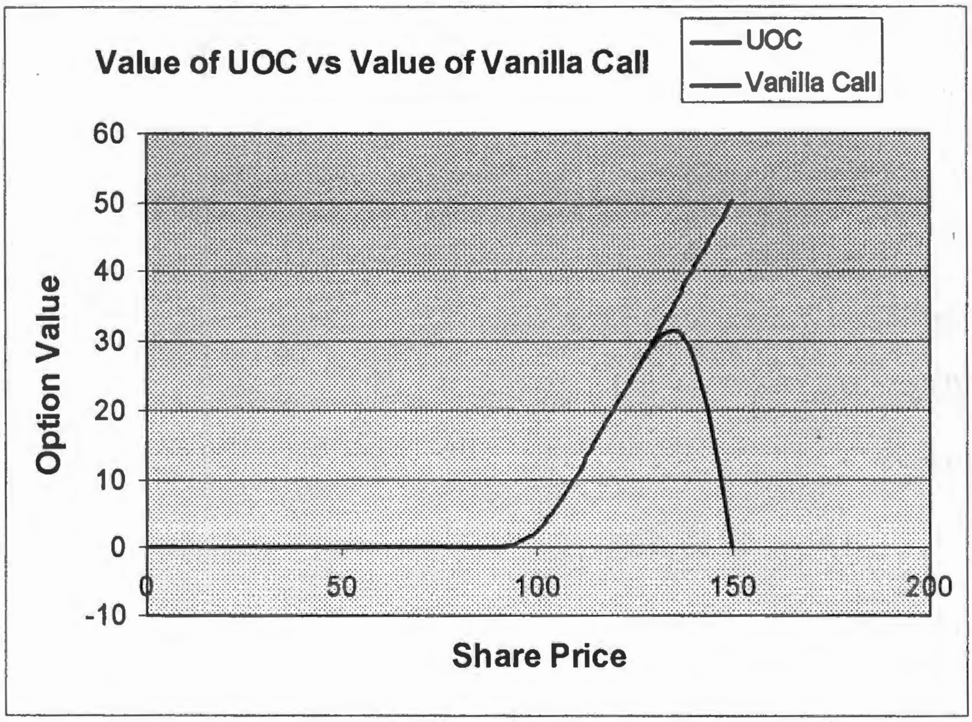


Figure 3: Value of up-and-out call option vs. value of vanilla call option

As discussed in Hull (2003), anyone interested in delta hedging (making a portfolio delta-neutral) will be interested in the portfolio's (or option's) delta and gamma. Figure 4 below depicts the delta for the up-and-out call and the vanilla call.

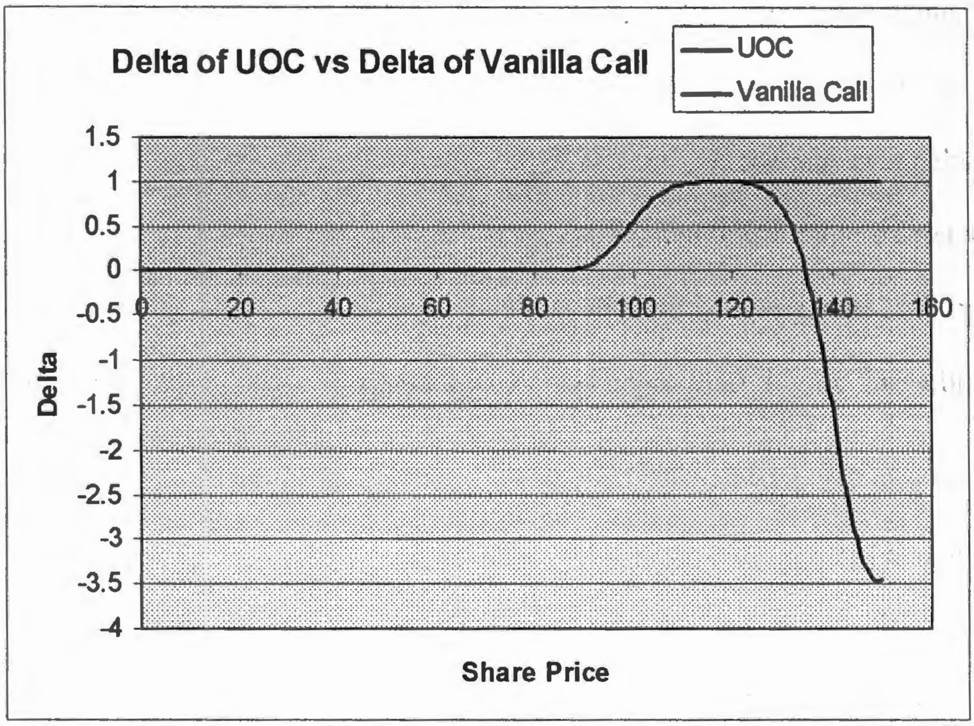


Figure 4: Delta of up-and-out call option vs. delta of vanilla call option

Figure 4 above shows that while the delta for the barrier option almost coincides with that of the vanilla call for share prices less than *R120*, it differs significantly for values close to the barrier option's barrier level *(R150)*. The fact that the barrier's delta changes so much as the spot level tends to *R150* makes delta hedging for the barrier option more difficult. A small price change in the spot level of the underlying can result in large losses when delta hedging, as continuous adjustment is impractical. Figure 5 reinforces this.

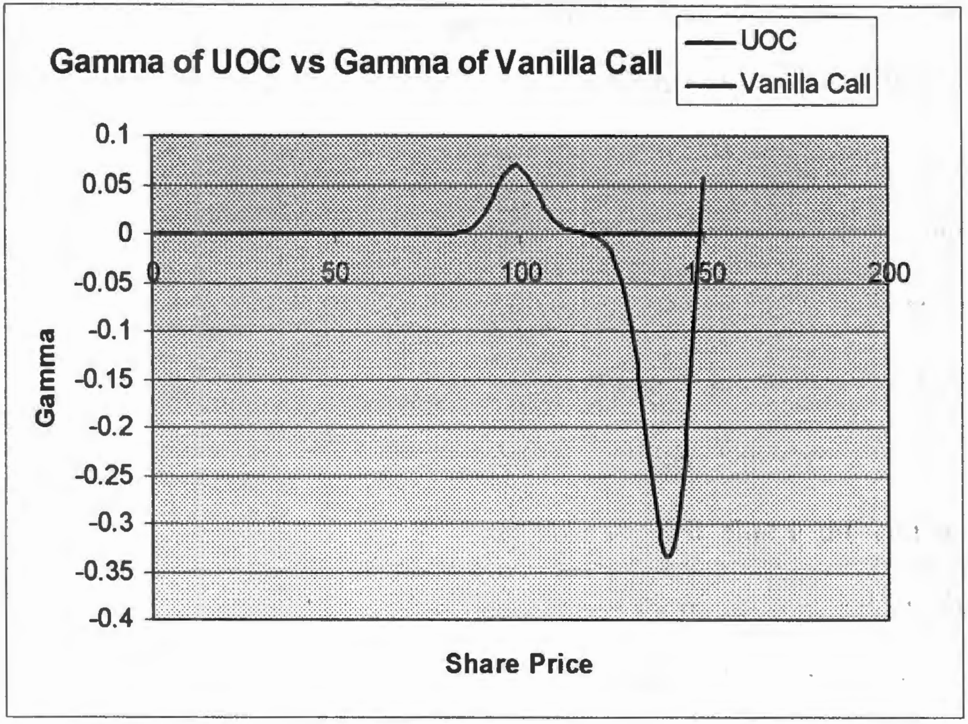


Figure 5: Gamma of up-and-out call vs. gamma of vanilla call option

Figure 5 shows that when the spot price approaches the barrier level (*R150*), the gamma takes on large negative values. These large absolute gamma values imply that delta is highly sensitive to the price of the underlying share. It is then quite risky to leave a delta-neutral portfolio unchanged for any length of time. Frequent rebalancing is therefore imperative. The reason that deltas and gammas for the up-and-out call become such large negative values is that the payoff of the barrier option is discontinuous at the barrier level.

Figure 6 plots the Vega for the two options in question. This shows that a market maker (or anyone who wants to hedge a barrier position) will find it difficult to dynamically vega-hedge an up-and-out call option. The vega of the up-and-out call simply changes too much when the spot level approaches the barrier level. This can be attributed to the fact that a volatility pickup near the barrier increases the likelihood of the price of the underlying passing through the barrier.

In contrast to the up-and-out call, the Greeks of the vanilla call are always positive well-behaved functions making dynamic hedging more practical.

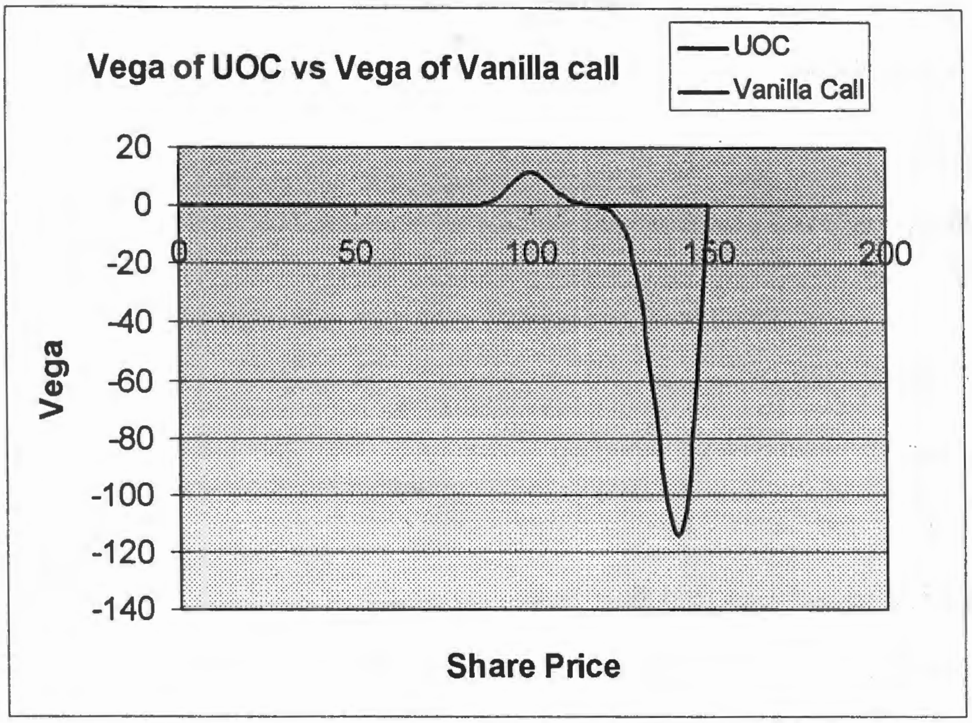


Figure 6: Vega of up-and-out call vs. vega of vanilla call

3.4 Advantages of static hedging over dynamic hedging

The previous subsection showed that dynamic hedging for barrier options (especially the up-and-out call option) could be problematic. This section notes the main advantages of static hedging over dynamic hedging.

- Dynamic hedging requires continuous trading (rebalancing) that will generate infinite transaction costs. One may argue that it is acceptable to trade periodically. This may lead to a low approximation error for securities with low gammas. However, barrier options like up-and-out call options often have regions of large negative gamma, which can lead to substantial losses under dynamic hedging. Static hedging doesn't have this problem, as long as the hedger can trade at the first passage time to the barrier.
- To dynamically hedge barrier options, the hedger needs to continuously estimate the future carrying costs (r), dividend yield (q) and volatility of the barrier option. An error in estimating these parameters will lead to erroneous dynamic hedging. This can be substantial especially in the case of large absolute values for the Greeks, as

observed for the up-and-out call option (but also for other barrier options). Static hedging only requires knowledge about the implied volatility of the vanilla options at inception of the hedge and sometimes when the underlying asset is at the barrier level.

- Static hedging comprises the construction of a portfolio consisting of traded vanilla options (in the case of a barrier option). The market values of these vanilla options provide us with a good estimate of the cost of the target option (the barrier option to be hedged). Dynamic hedging relies on a theoretical framework of option valuation (like Black-Scholes) that ignores transaction costs and assumes market conditions that are unrealistic.

Static hedging is not without its problems though. An illiquid market in the required hedging instruments poses a real problem. An investigation into static hedging does seem warranted though if one takes its advantages into account.

4 AN OVERVIEW OF STATIC HEDGING OF BARRIER OPTIONS

This section contains a brief overview of the literature surrounding static hedging of barrier options to date:

- Bowie and Carr (1994) show how to statically hedge single barrier options and look-back options, based on the assumptions of the Black (1976) model. These results were later extended by Carr and Chou (1996) to include assets with a non-zero underlying carry rate. They show that all down-and-in securities can be decomposed into a static portfolio of Arrow-Debreu securities, which in turn can be statically hedged by vanilla options. In-out parity implies that the same result holds true for out options. Carr, Ellis and Gupta (1998) further extended this research to include assets with non-lognormal distributions. The research done by these authors can be grouped together as they are all based on the underlying principle of 'vertical' replication. This involves static hedging by taking a position in securities with the same maturity as that of the security being hedged, but with various strike levels. Their results are favourable as they provide closed-form analytical solutions for the hedge positions and values.

- Derman, Ergener and Kani (1995) introduced a different approach to static hedging known as 'horizontal' replication. This involves taking positions in various securities with maturities before and equal to that of the hedged security, all with the same strike level. Their model can be considered a discrete-time model as it only provides exact replication at a finite number of time points. The model generates numerical solutions for the hedges and values of barrier options using the available hedging instruments.

- Chou and Georgiev (1998) established an analytical connection between the above two approaches. They also furthered the 'vertical' replication method by showing how to hedge when the drift is a piecewise-constant function of time.

- Tompkins (2002) simulated dynamic hedging using Monte Carlo methods in comparison with various proposed static hedging strategies for various types of options. He concludes that:

  > *"The results varied among the types of exotic securities. Neither dynamic nor static hedging approaches were found to be universally superior." (Tompkins (2002), p.31)*

  Tompkins further notes that the Carr, Ellis and Gupta (1998) method for hedging an up-and-out call is ineffective. Specifically he notes that:

  > *"It was found that both approaches [delta hedging and the Carr, Ellis and Gupta approach of static hedging] failed to adequately address the risks of up-and-out calls, and further research into how these products are hedged is warranted." (ibid, p.32)*

- Thomsen (1998) extended Tompkins's research. Like Tompkins, Thomsen based his research on the assumptions of discrete trading opportunities, positive transaction costs and a stochastic volatility structure. However, Thomsen considers underlying assets with non-zero constant carrying rates, and compares dynamic hedging with the Carr and Chou (1996) static hedge of down-and-in calls. He concludes that not only under a perfect market setting (Black-Scholes framework), but also under conditions of transactions costs and stochastic volatility, the static hedge performs better than dynamic hedging.

- Toft and Xuan (1998) tested the Derman, Ergener and Kani (1995) model for up-and-out calls with rebates by assuming that Heston's stochastic volatility model describes market option prices. They concluded that the static hedge is effective if volatility is moderate or if the rebate is set equal to the intrinsic value of the corresponding standard call.

- Brown, Hobson and Rogers (1998) took a different approach to that of Carr, Ellis and Gupta (1998) and Derman, Ergener and Kani (1995). They use traded call options to establish lower and upper bounds for barrier options. The benefit is that market beliefs and preferences are incorporated into the hedge and that these prices are model independent. They also provide simple arbitrage strategies when barrier options are trading outside of these limits.

- Carr and Picron (1999) use a mixture of 'vertical' and 'horizontal' replication to statically hedge American binary options. They show that the ability to hedge these fundamental path-dependent securities implies that all other single-barrier options can be statically hedged. They conclude that:

  > *"Our simulation of dynamic and static hedges of American binary calls shows that static hedging dominates dynamic hedging in both a mean-variance framework and in a more general utility maximization setting" (Carr and Picron (1999), p.67)*

  They also note that it would be relatively easy to extend the analysis to more complicated barrier options (e.g. lookback and multiple–barrier options), but that it would be challenging to extend the analysis to include time and spatially dependent variance rates, interest rates and dividend yields and still retain analytical solutions.

- Liljefors (2001) examines the efficiency of the Derman, Ergener and Kani (1995) model under dynamic market conditions. He first builds the Derman, Ergener and Kani (1995) portfolio and then optimises it over the number of options to buy/sell (with the flexibility to choose different strikes). This dramatically diminishes the risk compared to the original Derman, Ergener and Kani (1995) portfolio when dynamic market conditions are applied.

- Andersen, Andreasen and Eliezer (2002) present extensions of the results of some of the above-mentioned articles. They derive exact, explicit expressions for the composition of statically replicating portfolios that allow the underlying asset price process to have both jumps and a time- and state-dependent diffusion volatility structure. These results are all based on the assumption that European options are traded for all maturities and strikes, with inelastic supply.

- Dupont (2002) outlines the application of the technique known as mean-square-hedging to static hedging of barrier options. This technique minimizes the hedging error and an extension is derived to make this technique consistent with any prior pricing model or with any linear constraints on the hedging residual. The main benefits of this method of replication are that it does not require any strong assumptions on the availability of traded options with certain strikes or maturities, or on the distribution of the underlying asset.

- Poulsen (2003), much like Thomsen (1998) and Tompkins (2002), compares dynamic hedging with static hedging. Poulsen, however, focuses on model risk (the effect of model misspecification on prices). He finds that,

> *"Static hedging of barrier options is more sensitive to model risk than Δ-hedging. Still, under realistic conditions wrong static hedges may very well outperform correct Δ-hedges. Especially after some natural adjustments."(Poulsen (2003))*

5 THE CARR, ELLIS AND GUPTA MODEL

Carr, Ellis and Gupta (1998) (CEG) introduced static hedges for various European path-dependent exotic options that rely on the *put-call symmetry* (PCS) result developed by Bates (1988). The CEG method of static hedging is also known as 'vertical' replication as it uses options with variable strikes, but fixed maturity (the same maturity as that of the exotic option being hedged) to construct the hedge.

5.1 Model Assumptions

The model relies on the following assumptions:

- Frictionless markets and no arbitrage opportunities.
- Liquid vanilla options market. Vanilla options with the required strike levels are freely available in the market place.
- Underlying price process with zero drift. An example of such a price process is the familiar geometric Brownian motion under the risk neutral measure:

  $$\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t \qquad (4)$$

  where S_t is the price of the underlying asset at time *t*, *r* the nominal risk-free rate, q the continuous dividend yield of the underlying asset, σ the standard deviation of the underlying asset's relative price changes and W_t the familiar standard Brownian motion. The zero drift condition therefore implies that $r = q$, so that S_t is a martingale.
- A symmetric volatility structure (described below)

The above assumptions show that the CEG model has some obvious limitations, particularly the zero-drift condition. Section 5.4 discusses the main limitations of the CEG model.

5.2 Put-call symmetry and a symmetric volatility structure

Denote T as the expiration time of options under consideration and K as the strike price.

Note that under the zero drift assumption, forward prices equal spot prices, since

$$F_t = S_t e^{(r-q)(T-t)} \tag{5}$$

where F_t denotes the forward price for delivery T at time t and thus when $r = q$, $F_t = S_t$ $\forall t$. Therefore, the zero drift condition implies that options written on the spot price behave the same as those written on the forward price.

CEG make the further assumptions that:

- The volatility of the forward price is a known function $\sigma(F_t, t)$ of the forward price F_t and time t.
- The following symmetry condition holds:

$$\sigma(F_t, t) = \sigma(F^2 / F_t, t) \tag{6}$$

for all $F_t \geq 0$ and $t \in [0, T]$ where F is the current forward price F_0.

CEG notes that the above symmetry condition is satisfied in the Black (1976) model where volatility is deterministic ($\sigma(F_t, t) = \sigma(t)$). The symmetry can be seen when the volatility is graphed as a function of $Y_t \equiv \ln(F_t / F)$. If $v(Y_t, t) \equiv \sigma(F_t, t)$, the equivalent condition is:

$$v(y, t) = v(-y, t) \tag{7}$$

for all $y \in \mathbf{R}$ and $t \in [0,T]$, where $Y_t = y$ for ease of notation. This equivalent condition can be derived as follows:

$$\begin{aligned}
v(y,t) &\equiv \sigma(F_t,t) && \text{by definition} \\
&= \sigma(F^2 / F_t,t) && \text{by the symmetry condition} \\
&= \sigma(F.F / F_t,t) && \\
&= \sigma(Fe^{\ln(F/F_t)},t) && \\
&= \sigma(Fe^{-y},t) && \text{since } y \equiv \ln(F_t / F) \\
&= v(-y,t) && \text{since } Fe^y = F_t
\end{aligned}$$

CEG notes that the symmetry condition is satisfied in models with a symmetric volatility smile in the log of K/F. The same holds true for volatility frowns and even more complex volatility structures. Note that volatility here is local volatility ($\sigma(F_t,t)$) and not implied volatility. Refer to Derman and Kani (1994), Dupire (1994) or Kani, Derman and Kamal (1996) for a discussion on local volatility.

Theorem 5.2.1: European Put-call symmetry:

Given frictionless markets, no-arbitrage, zero drift and the symmetry condition, the following relationship holds:

$$C(K_c)K_c^{-1/2} = P(K_p)K_p^{-1/2} \tag{8}$$

where $C(K_c)$ and $P(K_p)$ denotes the price of a European call struck at K_c and a European put struck at K_p respectively and where the geometric mean of the call strike K_c and the put strike K_p is the forward price $F(=S)$ (the price of the asset):

$$(K_c K_p)^{1/2} = F \tag{9}$$

This relation holds at any time before and including expiration.

Proof:

Denote F_t as the forward price at time $t \in [0,T]$ of the underlying for delivery in T years time. Also, let $\sigma(F_t,t)$ be the local volatility rate of the forward price as a function of the forward price F_t and time t. Under the risk-neutral measure, the forward price process is given by

$$\frac{dF_t}{F_t} = \sigma(F_t,t)dW_t \tag{10}$$

where W_t is a standard Brownian motion. Let B_0 denote the price at time 0 of a bond that pays one currency unit at time T and let $C_0(K,T)$ and $P_0(K,T)$ denote the initial value of a European call and put struck at K and maturing at time T. Let $G_c(K,T) \equiv \frac{C_0(K,T)}{B_0}$ and $G_p(K,T) \equiv \frac{P_0(K,T)}{B_0}$ be the forward prices at time 0 of these options for delivery in T years.

Lemma 1:

CEG then shows that both these forward prices satisfy the following partial differential equation (PDE):

$$\frac{\sigma^2(K,T)K^2}{2}\frac{\partial^2 G}{\partial K^2}(K,T) = \frac{\partial G}{\partial T}(K,T) \tag{11}$$

where $K > 0, T > 0$

This result and its proof can be accredited to Dupire (1994). It shows how forward option values change with the strike and maturity, holding the initial time and underlying forward price fixed. The proof is as follows:

Consider the standard result that the forward price of a call is given by its expected payoff under the risk-neutral measure:

$$G_c(K,T) = \int_K^{\infty} (F_T - K)p(F_T,T;F_0,0)dF_T \tag{12}$$

where $p(F_T,T;F_0,0)$ is the transition density function of the forward price showing the probability density function of the forward price at F_T at time T, given that it is at F_0 at time 0. The Kolmogorov forward equation governing this density is:

$$\frac{1}{2}\frac{\partial^2}{\partial K^2}\left[\sigma^2(K,T)K^2 p(K,T;F_0,0)\right]=\frac{\partial}{\partial T}p(K,T;F_0,0) \tag{13}$$

where $K>0, T>0$.

Differentiating (12) twice with respect to K yields:

$$\frac{\partial^2 G_c(K,T)}{\partial K^2}=p(K,T;F_0,0) \tag{14}$$

This result (14) can be attributed to Breeden and Litzenberger (1978).

Substituting (14) into (13) yields:

$$\frac{1}{2}\frac{\partial^2}{\partial K^2}\left[\sigma^2(K,T)K^2\frac{\partial^2 G_c(K,T)}{\partial K^2}\right]=\frac{\partial}{\partial T}\frac{\partial^2 G_c(K,T)}{\partial K^2} \tag{15}$$

where $K>0, T>0$. Integrating twice with respect to K yields (13), the required result. By put-call parity, the same result holds for European puts.

It can easily be verified that Black's formulae for calls and puts satisfy the above equation (15) with $\sigma^2(K,T)=\sigma^2$. The forward call value $G_c(K,T)$ is the unique solution of (11) subject to the boundary conditions:

$$G_c(K,0)=\max\left[F_0-K,0\right],\ K>0$$

$$\lim_{K\to\infty^-}G_c(K,T)=0,\ T>0$$

$$\lim_{K\to 0^+}G_c(K,T)=F_0,\ T>0$$

Similarly, the forward put value $C_p(K,T)$ is the unique solution of (11) subject to the following boundary conditions:

$$G_p(K,0) = \max\left[K - F_0, 0\right],\ K > 0$$

$$\lim_{K \to \infty^-} G_p(K,T):\ K\ ,\ T > 0$$

$$\lim_{K \to 0^+} G_p(K,T) = 0\ ,\ T > 0$$

Consider the normalized call and put values: $u_c(y,T) \equiv G_c(K,T)(KF_0)^{1/2}$ and

$u_p(y,T) \equiv G_p(K,T)(KF_0)^{1/2}$ where $y \equiv \ln(K / F_0)$ and T the maturity as usual.

Lemma 2:

It can be shown by a change of variables that the normalized values both solve the following PDE:

$$\frac{v^2(y,T)}{2}\frac{\partial^2 u}{\partial y^2}(y,T) - \frac{v^2(y,T)}{8}u(y,T) = \frac{\partial u}{\partial T}(y,T) \tag{16}$$

where $y \in (-\infty, \infty), T > 0$ and $v(y,T) \equiv \sigma(F_0 e^y, T)$ is the volatility expressed as a function of y and T. In the case of $u_c(y,T)$, this can be done as follows:

$$\begin{aligned}
\frac{\partial u_c(y,T)}{\partial y} &= \frac{\partial(G_c(K,T)(KF_0)^{-1/2})}{\partial y} \\
&= \frac{\partial(G_c(KF_0)^{-1/2})}{\partial K}\frac{\partial K}{\partial y} \quad \text{dropping the } (K,T) \text{ for ease of use} \\
&= \frac{\partial(G_c(KF_0)^{-1/2})}{\partial K}F_0 e^y
\end{aligned}$$

Therefore,

$$\frac{\partial^2 u_c(y,T)}{\partial y^2} = \frac{\partial\left(\dfrac{\partial(G_c(KF_0)^{-1/2})}{\partial K}F_0 e^y\right)}{\partial y}$$

$$= \frac{\partial\left(\dfrac{\partial(G_c(KF_0)^{-1/2})}{\partial K}K\right)}{\partial K}\frac{\partial K}{\partial y} \quad \text{since } F_0 e^y = K$$

$$= \frac{\partial\left(\left(\dfrac{\partial G_c}{\partial K}\right)K^{1/2}F_0^{-1/2} - \dfrac{1}{2}K^{-1/2}F_0^{-1/2}G_c\right)}{\partial K}K$$

$$= \left(\frac{\partial^2 G_c}{\partial K^2}K^{1/2}F^{-1/2} + \frac{1}{4}K^{-3/2}F_0^{-1/2}G_c\right)K \quad \text{note that the middle terms cancel}$$

$$= K^{3/2}F_0^{-1/2}\frac{\partial^2 G_c}{\partial K^2} + \frac{1}{4}K^{-1/2}F_0^{-1/2}G_c$$

Making $\frac{\partial^2 G_c}{\partial K^2}$ the subject of the above equation and substituting it into (11) yields,

$$K^{-3/2}F_0^{1/2}\frac{\sigma^2(K,T)K^2}{2}\left(-\frac{1}{4}K^{-1/2}F_0^{-1/2}G_c + \frac{\partial^2 u_c(y,T)}{\partial y^2}\right) = \frac{\partial u_c(y,T)}{\partial T}(KF_0)^{1/2}$$

since $\frac{\partial G_c}{\partial T} = \frac{\partial u_c(y,T)}{\partial T}(KF_0)^{1/2}$. Simplifying and noting that $v(y,T) = \sigma(F_0 e^y, T)$ yields the required result (i.e. (16)). A similar argument holds for $u_p(-y,T)$.

□

The normalized forward call value $u_c(y,T)$ is the unique solution of (16) subject to the following boundary conditions:

$$u_c(y,0) = \max\left[e^{-y/2} - e^{y/2}, 0\right],\ y \in \mathbb{R}$$

$$\lim_{y\to\infty^-} u_c(y,T) = 0,\ T > 0$$

$$\lim_{y\to-\infty^+} u_c(y,T) = +\infty,\ T > 0$$

While the normalized forward put value $u_p(y,T)$ is the unique solution of (16) subject to the following boundary conditions:

$$u_p(y,0) = \max\left[e^{y/2} - e^{-y/2}, 0\right],\ y \in \mathbb{R}$$

$$\lim_{y \to \infty^-} u_p(y,T) = +\infty,\ T > 0$$

$$\lim_{y \to -\infty^+} u_p(y,T) = 0,\ T > 0$$

Recall from (7) our symmetry condition, which implies that:

$$v^2(y,T) = v^2(-y,T) \text{ for all } y \in \mathbb{R}, T > 0$$

This together with an analysis of the previous two sets of boundary conditions leads one to conclude that $u_c(y,T)$ and $u_p(-y,T)$ satisfy the same boundary value problem and are thus equal. Therefore,

$$u_c(y,T) = u_p(-y,T) \text{ for all } y \in \mathbb{R}, T > 0$$

Recalling our definitions of the normalized put and call values, we can rewrite this equality as,

$$G_c(K_c,T)(K_cF_0)^{-1/2} = G_p(K_p,T)(K_pF_0)^{-1/2}$$

where $\sqrt{K_cK_p} = F_0$. This can be seen as follows: First, denote K by K_c instead, so that

$$\begin{gathered} u_c(y,T) \equiv G_c(K_c,T)(K_cF_0)^{-1/2} \\ u_p(y,T) \equiv G_p(K_c,T)(K_cF_0)^{-1/2} \\ y \equiv \ln(K_c / F_0) \end{gathered}$$

by our previous definitions. So that $F_0 = K_ce^{-y}$. Also, note that

$$\begin{aligned} \sqrt{K_pK_c} &= F_0 \\ (K_pK_c)^{1/2} &= K_ce^{-y} \\ K_p^{\ 1/2}K_c^{\ -1/2} &= e^{-y} \\ F_0K_p^{\ 1/2}K_c^{\ -1/2} &= F_0e^{-y} \text{ multiplying both sides by } F_0 \\ \sqrt{K_pK_c}K_p^{\ 1/2}K_c^{\ -1/2} &= F_0e^{-y} \\ K_p &= F_0e^{-y} \end{aligned}$$

Now, $u_c(y,T) = u_p(-y,T)$, so that

$$\begin{aligned} G_c(K_c,T)(K_cF_0)^{-1/2} &= G_p(F_0e^{-y},T)(F_0e^{-y}F_0)^{-1/2} \\ &= G_p(K_p,T)(K_pF_0)^{-1/2} \end{aligned}$$

Multiplying both sides by $\sqrt{F_0}B_0$ yields the required result (*Theorem 5.2.1*):

$$C_0(K_c,T)K_c^{-1/2} = P_0(K_p,T)K_p^{-1/2}$$

5.3 Hedging an Up-and-Out Call using CEG

Although the CEG model can be used to hedge various exotic European path-dependent options, the focus in this paper will be on hedging a European up-and-out call option (UOC). Once a hedging strategy for the UOC option is derived, the in-out parity result can be used to determine the hedge or value of an up-and-in call option (see section 2.3.3).

Put-call symmetry will be the fundamental result used to establish the hedge for the UOC in this section. Note that all the instruments considered in this section, have the same maturity as that of the UOC ('vertical' replication).

By definition, an UOC has a knockout barrier set above the current market price (or forward price, since $r = q$, and thus $F_t = S_t \ \forall t$). Note that only barriers set above the strike *(L>K)* is considered, because if $(L \leq K)$ the option will be worthless, as it will knock-out before it can have a positive payoff. Consider the following naïve replicating portfolio:

- A long European call struck at K
- sell (write) $\frac{K}{L}$ European puts struck at $\frac{L^2}{K}$

The reasoning is as follows: The call will match the required payoff at maturity if the barrier has not been hit. The puts will provide a portfolio of zero value along the barrier:

When $F(=S)=L$, the put-call symmetry condition is:

$$C(K)K^{-1/2}=P(K_p)K_p^{-1/2}\text{, where }(KK_p)^{1/2}=L$$

Therefore substituting $K_p=\frac{L^2}{K}$ into the above put-call symmetry equation and solving for $C(K)$ yields:

$$C(K)=\frac{K}{L}P\left(\frac{L^2}{K}\right)$$

In other words, when $F(=S)=L$, the call will exactly be offset by the position in the written puts, leaving us with a portfolio with zero value. However, our hedge is incorrect since when $F(=S)<L$, our naïve replicating portfolio does not match the barrier as our position in puts still has value.

Instead CEG proposes the following replicating portfolio for the UOC using up-and-in securities:

- A long European call struck at *K*
- A European up-and-in put (*UIP*) struck at *K*, with barrier level *L*
- A short position of *(L-K)* up-and-in bonds (*UIB*), with barrier level *L*. By definition, an up-and-in bond *UIB(L)* pays one currency unit at expiration, as long as the barrier *L* has been hit before expiration.

Our portfolio in summary then,

$$UOC(K,L)=C(K)-UIP(K,L)-(L-K)UIB(L) \tag{17}$$

for $L>K,F$. This replication can be seen as follows. Consider the UOC if the barrier is never touched: The required payoff is exactly equal to that of the call, while the up-and-in securities expire worthless as the in-barrier as not been crossed. If the barrier is crossed before expiration (and the UOC knocks out), the in-securities knock in and become vanilla options. Put-call parity tells us that our replicating portfolio can then be liquidated at zero cost: By put-call parity,

$$C(K) = [L - K]B(T) + P(K)$$

since $F(=S) = L$ and where $B(T)$ denotes a zero coupon bond maturing at time T (equivalent to our up-and-in bond).

Note that this replicating portfolio is model independent (since put-call parity is model independent) and holds for any underlying price process. CEG, however, note that the up-and-in securities may not trade or may only trade with heavy friction. The aim is therefore to try and replicate these up-and-in securities using vanilla European options.

Put-call symmetry can be used to show that $\frac{K}{L}$ calls struck at $\frac{L^2}{K}$ will replicate the $UIP(K,L)$ option. Here, $\frac{L^2}{K} = L.\frac{L}{K} > L > K$ since $\frac{L}{K} > 1$ and therefore the call never expires in the money. If the barrier is never hit before maturity, the $UIP(K,L)$ expires worthless as do the calls. Mathematically, if $S_T \leq L$, then $\frac{K}{L}C\left(\frac{L^2}{K}\right) = \frac{K}{L}\left(S_T - \frac{L^2}{K}\right)^+ = 0$ because $S_T - \frac{L^2}{K} \leq L\left(1 - \frac{L}{K}\right) < 0$. Hence, if S_t does not hit the barrier, then $\frac{K}{L}C\left(\frac{L^2}{K}\right) = UIP(K,L)$. If the barrier is hit before expiration though, the above position in calls exactly equal the $P(K)$ which in turn equals the $UIP(K,L)$ since the put has knocked in. Mathematically, if S_t does hit the barrier, however, then

$$\begin{aligned}\frac{K}{L}C\left(\frac{L^2}{K}\right) &= \frac{K}{L}\left(\frac{L^2}{K}\right)^{1/2}\left[\left(\frac{L^2}{K}\right)^{-1/2}C\left(\frac{L^2}{K}\right)\right] \\ &= \frac{K}{L}\left(\frac{L^2}{K}\right)^{1/2}\left(K^{-1/2}P(K)\right) \\ &= P(K)\end{aligned}$$

CEG then show that the $UIB(L)$ can be replicated by buying 2 binary calls (BC) struck at L and $\frac{1}{L}$ European calls struck at L. In other words,

$$UIB(L) = 2BC(L) + \frac{1}{L}C(L) \tag{18}$$

for $L > F$. Note that by definition, the binary calls pay one unit currency at expiry if the underlying asset's price finishes above the barrier L at expiry.

Proof of Up-and-In bond replicating portfolio:

First rewrite the up-and-in bond as a combination of an up-and-in binary call and an up-and-in binary put,

$$UIB(L) = UIBC(L) + UIBP(L) \tag{19}$$

The $UIBC(L)$ will provide the identical payoff as that of the $UIB(L)$ when the underlying asset's price crosses the barrier and ends up above the barrier at expiry (the $UIBP(L)$ will then expiry worthless) If the barrier is crossed before expiry, but the underlying asset's price ends up below the barrier at expiry, the $UIBP(L)$ will identically match the $UIB(L)$ payoff (while the $UIBC(L)$ will expiry worthless)

Observe that an $UIBC(L)$ is identical to a standard $BC(L)$ since it has to knock in to have positive value. Thus $UIB(L) = BC(L) + UIBP(L)$.

CEG then expand the $UIBP(L)$ into the following components using the Chriss and Ong (1995) method of synthesizing using vertical spreads:

$$UIB(L) = BC(L) + \lim_{n \to \infty^-} n\left[UIP(L, L) - UIP(L - n^{-1}, L)\right] \tag{20}$$

and apply put-call symmetry to get

$$UIB(L) = BC(L) + \lim_{n \to \infty^-} n\left[C(L) - (L - n^{-1})L^{-1}C(L^2(L - n^{-1})^{-1})\right] \tag{21}$$

Note that as explained on the previous page,

$UIP(K, L) = \frac{K}{L}C\left(\frac{L^2}{K}\right)$ and thus $UIP(L, L) = \frac{L}{L}C\left(\frac{L^2}{L}\right) = C(L)$ and also that

$UIP(L - n^{-1}, L) = (L - n^{-1})L^{-1}C(L^2(L - n^{-1})^{-1})$.

A good approximation to (21) is as follows:

$$UIB(L) \approx BC(L) + L^{-1} \lim_{n\to\infty^-} C(L^2(L-n^{-1})^{-1}) + \lim_{n\to\infty^-} n\left[C(L) - C(L+n^{-1})\right] \qquad (22)$$

with approximation error of $O(n^{-2})$. Again using Chriss and Ong (1995) the final term can be rewritten as a binary call while the second term tends to $L^{-1}C(L)$ as $n \to \infty$. Therefore,

$$UIB(L) = 2BC(L) + L^{-1}C(L) \qquad (23)$$

Rewriting (17) in terms of (23) and the replicating portfolio of calls for the $UIP(K,L)$ discussed on the previous page, yields the following:

$$UOC(K,L) = C(K) - \frac{K}{L}C\left(\frac{L^2}{K}\right) - \left[2(L-K)BC(L) + \frac{L-K}{L}C(L)\right] \qquad (24)$$

for $L > K, F$. Equation (24) above still contains a binary call. Again, the Chriss and Ong (1995) method of synthesizing can be used to reduce it to an infinite number of vertical spreads of standard calls:

$$BC(L) = \lim_{n\to\infty^-} n\left[C(L) - C(L + \frac{1}{n})\right] \qquad (25)$$

This technique of binary call replication is impractical though. A numerical technique called Richardson extrapolation (see Dahlquist, Björck and Anderson (1974) or Marchuk and Shaidurov (1983) for a derivation) can be used approximate the limit in (25). This works as follows:

Consider three known points for (25). Let these be $n=1$, $n=2$ and $n=3$, so that the values of $C(L)$, $C(L+1)$, $C\left(L+\frac{1}{2}\right)$ and $C\left(L+\frac{1}{3}\right)$ are known. Richardson extrapolation fits a polynomial of the form $f(x) = ax^2 + bx + c$ through these three points where $x_1 = 1$, $x_2 = \frac{1}{2}$ and $x_3 = \frac{1}{3}$ so that $f(x_1) = C(L) - C\left(L + \frac{1}{1}\right)$, $f(x_2) = n\left(C(L) - C\left(L + \frac{1}{2}\right)\right)$ and $f(x_3) = 3\left(C(L) - C\left(L + \frac{1}{3}\right)\right)$ respectively. In other words, $n = 1$ correspond to $x_1 = 1$, $n = 2$ to $x_2 = \frac{1}{2}$ and $n = 3$ to $x_3 = \frac{1}{3}$. Substituting into the polynomial then,

$$\begin{aligned} f(x_1) &= a + b + c \\ f(x_2) &= \frac{a}{4} + \frac{b}{2} + c \\ f(x_3) &= \frac{a}{9} + \frac{b}{3} + c \end{aligned}$$

and therefore

$$\begin{aligned} C(L) - C(L+1) &= a + b + c \\ 2\left(C(L) - C\left(L + \frac{1}{2}\right)\right) &= \frac{a}{4} + \frac{b}{2} + c \\ 3\left(C(L) - C\left(L + \frac{1}{3}\right)\right) &= \frac{a}{9} + \frac{b}{3} + c \end{aligned}$$

In matrix notation,

$$\begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{4} & \frac{1}{2} & 1 \\ \frac{1}{9} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} C(L) - C(L+1) \\ 2\left(C(L) - C\left(L + \frac{1}{2}\right)\right) \\ 3\left(C(L) - C\left(L + \frac{1}{3}\right)\right) \end{pmatrix}$$

inverting,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 & -12 & 9 \\ -2\frac{1}{2} & 16 & -13\frac{1}{2} \\ \frac{1}{2} & -4 & 4\frac{1}{2} \end{pmatrix} \begin{pmatrix} C(L) - C(L+1) \\ 2\left(C(L) - C\left(L + \frac{1}{2}\right)\right) \\ 3\left(C(L) - C\left(L + \frac{1}{3}\right)\right) \end{pmatrix} \qquad (26)$$

Now, $\lim\limits_{n\to\infty^-}$ corresponds to $x=0$, but $f(0)=c$. Therefore,

$$BC(L)\approx\frac{1}{2}\big(C(L)-C(L+1)\big)-4\left(2\left(C(L)-C\left(L+\frac{1}{2}\right)\right)\right)+4\frac{1}{2}\left(3\left(C(L)-C\left(L+\frac{1}{3}\right)\right)\right)$$

and thus

$$BC(L)\approx 6C(L)-0.5C(L+1)+8C\left(L+\frac{1}{2}\right)-\frac{27}{2}C\left(L+\frac{1}{3}\right) \tag{27}$$

CEG notes that the above portfolio of vanilla calls approximates the binary call well (accurate to five decimal places), but the approximation deteriorates near expiration when prices are near the strike. Substituting equation (27) into (24) produces a static hedge consisting of plain vanilla call options of varying strikes, but fixed maturity *T*. The static hedge is as follows,

$$\begin{aligned}UOC(K,L)\approx{}& C(K)-\frac{K}{L}C\left(\frac{L^2}{K}\right)\\&-\left[2(L-K)\left(6C(L)-0.5C(L+1)+8C\left(L+\frac{1}{2}\right)-\frac{27}{2}C\left(L+\frac{1}{3}\right)\right)+\frac{L-K}{L}C(L)\right]\end{aligned}$$

An even better approximation for the binary call (and thus up-and-out call) can be found by using four known points. Refer to Marchuk and Shaidurov (1983) for a derivation of the required positions for four or more points.

5.4 Limitations and restrictions of the CEG model

The limitations and restrictions of the CEG model can be grouped into the following categories:

- Zero drift price process: While this is acceptable for options on futures, it will is unlikely that the price process of any share or index will exhibit a zero drift. Carr and Bowie (1994) and CEG did relax the assumption of zero drift, but they were only able to derive tight bounds for the required static hedges.
- Symmetric volatility structure: Volatility structures do sometimes exhibit a symmetric structure, especially smiles in the case of foreign currency options (see Hull (2003), pp.330-345). However, volatility skews are far more prevalent in the market for equity options.
- Fractional positions: The static hedge derived above requires the hedger to take on a fractional position in certain options. This is fairly impractical.
- Broad, deep market of vanilla options: The hedger requires a market of plain vanilla options with various strike levels. These options may not trade or may only trade infrequently.
- Not a 'pure' static hedge. The hedger is required to liquidate the portfolio as soon as the barrier level is hit. To an extent, continuous monitoring of the position is therefore still required.
- Transaction costs: The CEG model ignores transaction costs. When the replicating portfolio is unwound, transaction costs will be incurred that might not produce the required payoff.

6 THE DERMAN, ERGENER AND KANI MODEL

The Derman, Ergener and Kani (1995) (DEK) model is based on 'horizontal' replication. This involves the building of a static hedging portfolio using securities with a wide range of maturities (i.e. not only securities with the same maturity as that of the target option, as in the case of 'vertical' replication). The DEK model specifies step-by-step how to find a replicating portfolio for any path-independent or 'weakly' path dependent options at all future share price levels and times. A barrier option is an example of a 'weakly' path dependent option where the payoff depends only on whether or not the share price path has crossed a certain level. The DEK model is not applicable to 'fully' path dependent options (e.g. Asian options) though. In general, the DEK replicating portfolio would require an infinite number of hedging instruments. Using a finite number of instruments, the target option can be hedged precisely at a finite number of time points. An increase in the number of instruments will increase the hedging accuracy.

6.1 Model Assumptions

The model relies on the following main assumption(s):

- Black-Scholes assumptions, including most notably:
    - The share price follows the familiar geometric Brownian motion with constant volatility under the risk-neutral measure.
    - Short selling is allowed.
    - Securities are perfectly divisible. In other words, fractional positions in options can be taken in the market.
- Option positions can be liquidated at model implied prices.
- A wide and deep market for vanilla options with a wide range of strikes.

6.2 Mathematical derivation of the DEK model

Assume that the share price process follows a geometric Brownian motion process with constant volatility under the risk-neutral measure. In other words,

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t \tag{28}$$

where S_t denotes the share price at time *t*, r the constant risk-free rate, σ the constant volatility and W_t a standard Brownian motion.

Let $V(S_t, T-t)$ denote the value of a European derivative dependent on S_t at time *t*. The DEK models aims to find a portfolio of standard options that replicates the value of the derivative $V(S_t, T-t)$ for all times *t*, except possibly at expiration *T*. Mathematically,

$$V(S_t, T-t) = \int_t^T w(t,u) Option(S_t, K(u), u-t) du \tag{29}$$

where $Option(S_t, K(u), u-t)$ denotes the value of standard call or put option with strike $K(u)$ and maturity $u-t$ and $w(t,u)$ the number of options (i.e. weight) with maturity $u-t$. The DEK model is interested in a static hedge (i.e. fixed weights). Therefore the weights must be independent of the initial time *t*,

$$\frac{\partial}{\partial t} w(t,u) = 0$$

The first parameter *t* is therefore redundant and can therefore be eliminated to make way for a simpler static weight function $w(u)$. The integral representation in (29) therefore represents the value of the derivative $V(S_t, T-t)$ as a changing portfolio since some of the options mature as $t \to T$, with static weights corresponding to options with changing maturities (the same option keeps the same, constant weight although the option's maturity changes as one moves through time from *t* to *T*). For simplicity, we will assume that the strike prices $K(u)$ coincide with the boundary levels. In other words,

$$K(u) = B(u)$$

This will ensure that there are no cash flows generated by expiring options. Consider for example a European up-and-out call option ($V(S_t, T-t)$ = UOC) with barrier level L. If call options with strike levels less than the barrier level are used to hedge the UOC, cash flows will possibly be generated during the life of the option. This can invalidate the hedge. Using strike levels that coincide only with the barrier level, this will be avoided (A similar argument holds for exotic derivatives with down barriers and double barriers).

We can therefore rewrite equation (29) as

$$V(S_t, T-t) = \int_t^T w(u) Option(S_t, B(u), u-t) du \tag{30}$$

Equation (29) above was defined for all times t, except possibly at expiration T. This was done for the following reason. Take the case of the up-and-out call with $t = T$ and $S_t = L$. The payoff of the up-and-out call is discontinuous at $S_t = L$. This means that the weight $w(T)$ has the form of a Dirac delta function. To avoid the problems that this will pose when determining the integral (the integral will be improper), DEK separates the terminal weights from the integral, so that

$$V(S_t, T-t) = \int_t^T w(u) Option(S_t, B(u), u-t) du + w_T Option_T(S_T, B(T), T-t) \tag{31}$$

where w_T represents the collective weights necessary to hedge the payoff of the target option at expiration while $Option_T(S_T, B(T), T-t)$ represents collectively the standard options necessary for the hedge at expiration. The problem now reduces to finding the weights (of the appropriate standard options) $w(u)$. This can be done as follows. Let $Boundary\_payoff(t)$ denote the known payoff at time t on the boundary. When $S_t = B(t)$,

$$Boundary\_payoff(t) = \int_t^T w(u) Option(B(t), B(u), u-t) du + w_T Option_T(B(t), B(T), T-t) \tag{32}$$

Equation (32) can be used recursively to determine the weights $w(u)$ as follows. Consider $t = T$. First identify an appropriate collection of standard options with expiration T. Both $Boundary\_payoff(T)$ and $Option_T(B(T), B(T), T-T)$ are known and

$$Boundary\_payoff(T) = w_T Option_T(B(T), B(T), T-T)$$

Therefore the weights w_T can be determined. (In certain cases such as the up-and-out call option with a discontinuous payoff at the barrier level, $Option_T(B(T), B(T), T-T)$ will be zero for options. This makes the weights w_T indeterminable. As we will see in the next subsection, modifying our approach so that options expiring at time T hedges the target option at time $T-\varepsilon$, for small $\varepsilon > 0$, will produce a good approximation.)

Equation (32) can then be used recursively to determine the weights for the replicating portfolio of standard options. Doing this in a continuous time space is impractical though (since there are an infinite number of time points and weights). Instead DEK uses discretization of time to approximate the hedge and weights at a finite number of time points.

Consider dividing the time interval (t, T) into a set of equally spaced time points $t = t_0, t_1, t_2, \ldots t_N = T$. Denote the weights at these time points as $w_1, w_2, \ldots w_N$ respectively, where w_N now denotes the collective weights at termination of the target option. The integral representation (31) then becomes,

$$V(S_i, T-i) = \sum_{u=i}^{t_N} w(u) Option(S_i, B(u), u-i) + w_N Option_N(S_i, B(T), T-i) \tag{33}$$

where $i = t_0, \ldots t_N$. Equation (32) becomes,

$$Boundary\_payoff(i) = \sum_{u=i}^{t_N} w(u) Option(B(i), B(u), u-i) + w_N Option_N(B(i), B(T), T-i) \tag{34}$$

Consider $i = t_N$. Equation (34) is then,

$$Boundary\_payoff(T) = w_N Option_N(B(T), B(T), T-T) \tag{35}$$

with $Boundary\_payoff(T)$ and $Option_T(B(T), B(T), T-T)$ known. The weights w_N can therefore be determined. We can then find w_{N-1} from equation (34) as follows,

$$Boundary\_payoff(t_{N-1}) = w_{N-1}Option(B(t_{N-1}), B(t_{N-1}), t_{N-1} - t_{N-1})$$
$$+ w_N Option_N(B(t_{N-1}), B(T), T - t_{N-1})$$

since all quantities in the above equation are known, except w_{N-1} (the option values are simply Black-Scholes prices). This recursive method can be applied to find all the other weights $w_1, w_2, \ldots w_{N-2}$.

6.3 Hedging an Up-and-Out Call using DEK

This section demonstrates the hedging of a European up-and-out call option on a share with parameters as given in Table 3 below.

Table 3: Parameters of the up-and-out call option

Spot price of share (S_0)	R80
Strike price (X)	R100
Barrier Level (L)	R120
Volatility (σ)	20%
Risk-free interest rate (r)	5%
Continuous dividend yield (q)	3%
Option's time to maturity in years (T)	1

Note that the rates in Table 3 above are given in NACC (nominal annual compounded continuously) form. Computing the theoretical price implied by the Black-Scholes model for this up-and-out call option, yields a value of R0.541246.

We now try and find an approximate hedge for the above up-and-out call using the DEK model. We first use only 6 positions in call options with different maturities to construct a perfect hedge at five time points only. The discretization approach as outlined in the previous subsection is used. Also, the modification (also discussed in the previous subsection) is applied to avoid the problem of indeterminable hedging instruments at expiration. Table 4 below displays the calls used to construct the hedge.

Table 4: Hedging the up-and-out call using six vanilla call options

Option	**Strike Price**	**Expiration (Maturity)**	**Time of hedge**
Call 1 (Obvious position)	R100	1	-
Call 2	R120	1	0.8
Call 3	R120	0.8	0.6
Call 4	R120	0.6	0.4
Call 5	R120	0.4	0.2
Call 6	R120	0.2	0

The obvious hedging instrument for the up-and-out call struck at *R100* is a European call option struck at *R100* (Call 1) with the same maturity as that of the up-and-out call option. This option will perfectly hedge the up-and-out call if the out barrier *(R120)* is never hit.

A position in Call 2 (with maturity at time 1) is bought to ensure that if the barrier *(R120)* is hit at time *0.8* (i.e. in 9.6 months time), the payoff (value) of Call 1 will be perfectly offset by Call 2 (i.e. so that the portfolio consisting of both will have zero value once liquidated).

Similarly, a position in Call 3 (with maturity at time 0.8) is purchased to ensure that if the barrier is hit at time 0.6, Call 3 will exactly offset the values of the positions in Calls 1 and 2.

Table 5 below shows the calculated positions in the various options needed for the hedge, as well as the Black-Scholes prices of these options at time 0.

Table 5: Using six vanilla call options to hedge the up-and-out call options

Option Number	Weight	Price at time 0	Option Type	Strike	Expiration
1	1.000000000	R1.385173	Call	100	1
2	-4.532032157	R0.193056	Call	120	1
3	1.836064878	R0.088219	Call	120	0.8
4	0.578953780	R0.026166	Call	120	0.6
5	0.269934045	R0.002788	Call	120	0.4
6	0.154544332	R0.000006	Call	120	0.2

Multiplying the values in the column labelled "Weight" in Table 5 with the corresponding values in the column labelled "Price at time 0" yields the cost of this static hedge (R0.688113).

Figure 7 below plots the value of the hedging portfolio against various times to maturity of the up-and-out call when the price of the underlying share is equal to the barrier level *(R120)*. A 'perfect' hedging portfolio should give us a portfolio of value zero in this situation. The graph in Figure 7 can therefore be seen as an indication of the error inherent in using the hedging portfolio as described above.

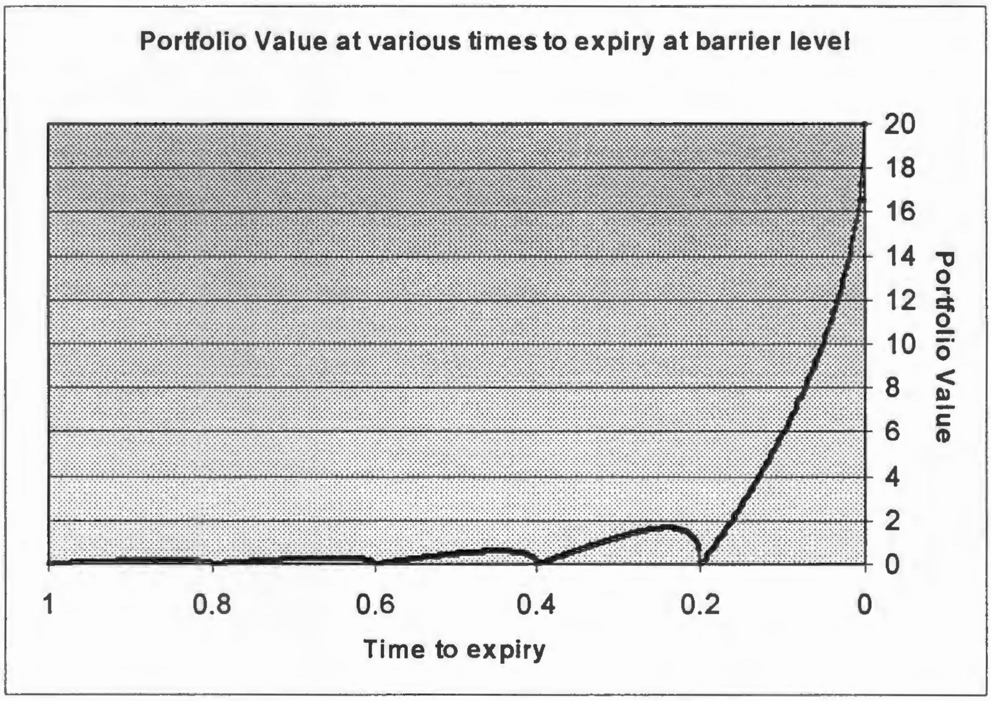


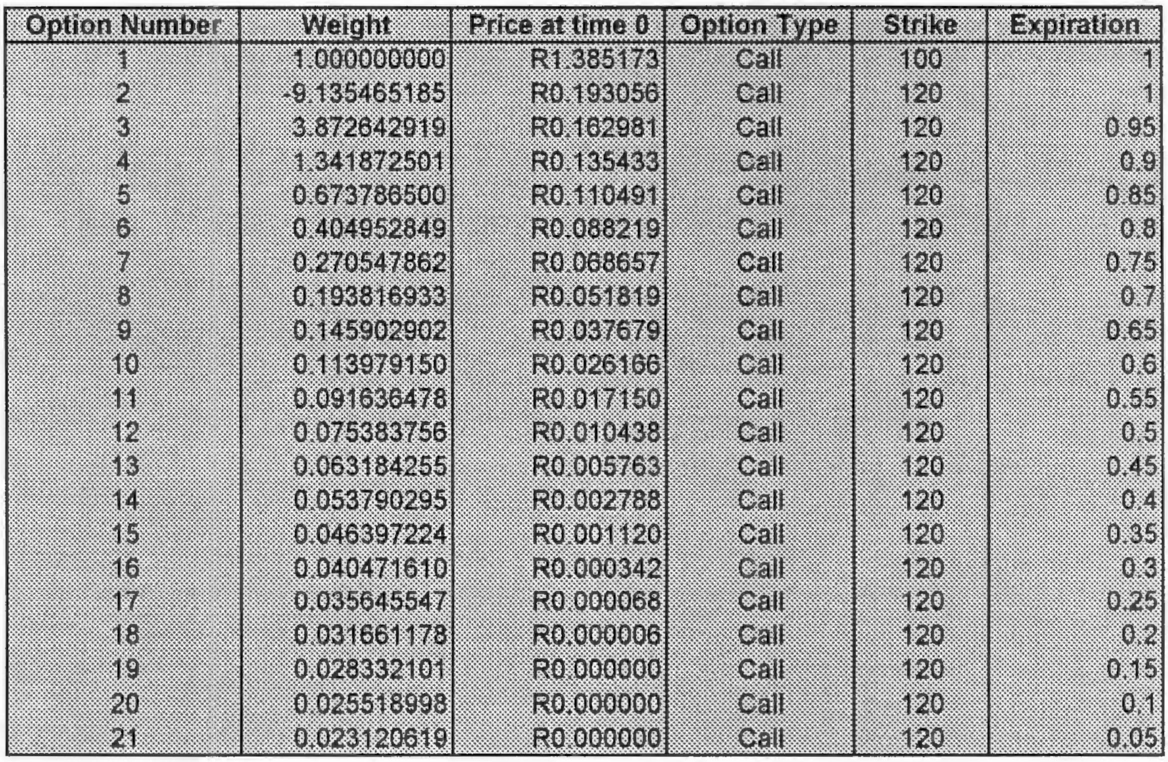
Figure 7: Hedging error of portfolio with six call positions

The graph shows that the hedging portfolio is zero for the hedging times of the five non-obvious call options (i.e. times 0; 0.2; 0.4; 0.6 and 0.8). Note that these times correspond to times 1; 0.8; 0.6; 0.4 and 0.2 to expiry for the up-and-out call. It is evident from Figure 7 that the error is greatest very close to expiry of the up-and-out call. However, this is as expected.

In an identical fashion a second, more accurate hedging portfolio can be constructed using 21 call option positions with different maturities. These options allow the hedge to be 'perfect' at 20 different time points.

Table 6 below shows the detail of this replicating portfolio.

Table 6: Using 21 vanilla call options as a replicating portfolio



Option Number	Weight	Price at time 0	Option Type	Strike	Expiration
1	1.000000000	R1.385173	Call	100	1
2	-9.135465185	R0.193056	Call	120	1
3	3.872642919	R0.162981	Call	120	0.95
4	1.341872501	R0.135433	Call	120	0.9
5	0.673786500	R0.110491	Call	120	0.85
6	0.404952849	R0.088219	Call	120	0.8
7	0.270547862	R0.068657	Call	120	0.75
8	0.193816933	R0.051819	Call	120	0.7
9	0.145902902	R0.037679	Call	120	0.65
10	0.113979150	R0.026166	Call	120	0.6
11	0.091636478	R0.017150	Call	120	0.55
12	0.075383756	R0.010438	Call	120	0.5
13	0.063184255	R0.005763	Call	120	0.45
14	0.053790295	R0.002788	Call	120	0.4
15	0.046397224	R0.001120	Call	120	0.35
16	0.040471610	R0.000342	Call	120	0.3
17	0.035645547	R0.000068	Call	120	0.25
18	0.031661178	R0.000006	Call	120	0.2
19	0.028332101	R0.000000	Call	120	0.15
20	0.025518998	R0.000000	Call	120	0.1
21	0.023120619	R0.000000	Call	120	0.05

The cost of this replicating portfolio is therefore R0.584626. This value can be seen to converge to the Black-Scholes price of R0.541246 as the number of hedging instruments and time points increase. A hedging portfolio consisting of 101 call options with equally spaced time points for hedging can be shown to cost R0.550569.

Figure 8, similar to Figure 7, plots the value of the hedging portfolio of 21 positions in different call options against various times to maturity of the up-and-out call when the price of the underlying share is equal to the barrier level *(R120)*.

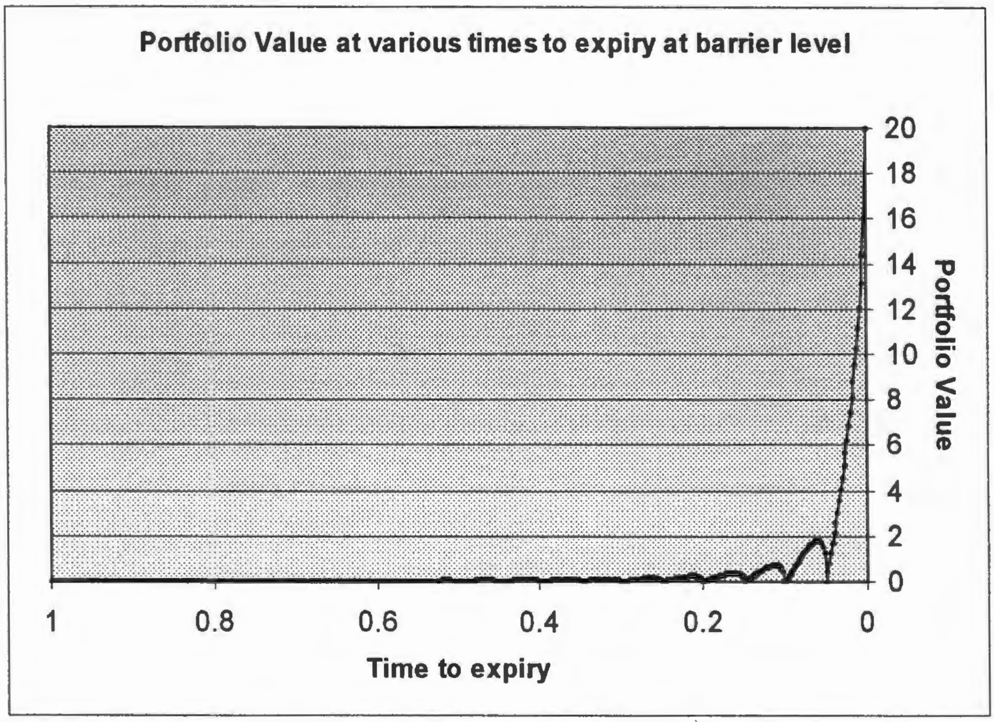


Figure 8: Hedging error of replicating portfolio consisting of 21 call positions with different maturities

As expected, the hedging error shown in Figure 8 is smaller than that of Figure 7. Again, the error is, not surprisingly, greatest close to expiration of the target up-and-out call option.

6.4 Limitations and restrictions of the DEK model

A few of the major limitations and restrictions of the DEK model are as follows:

- Model implied prices. The DEK model assumes that model implied prices will be realized on liquidation of the option positions, either at maturity or when a barrier will be hit. Changing parameters like volatilities and other market conditions and frictions make this unlikely. This also then leads us to our second point:
- Only a 'semi' static hedge. The hedger still needs to monitor the position and liquidate the portfolio as soon as a barrier level is hit. It is therefore not really a 'hedge and forget' pure static hedge. Furthermore, the continuous monitoring of the portfolio implies costs, unaccounted for in the model.
- Notional positions in vanilla options. As evident from the previous section (see Figures 7 and 8), fractional notional positions in vanilla options are required. This is not realistically achievable in the market place.
- Strike levels. Clarke (1998) notes that barrier options frequently have barrier levels far removed from strike levels observed in liquid vanilla option markets. Finding vanilla options with the appropriate strike levels can therefore be problematic.
- Only an approximation. The previous section shows that the DEK model only produces an approximate replicating portfolio in practice (the hedge is only perfect at a few discrete points in time).

7 MEAN-SQUARE HEDGING IN THE DUPONT FRAMEWORK

The DEK and CEG methods of static hedging introduced above were noted to make strong assumptions either on the availability of vanilla options with given strikes and maturities or the distribution of the share price process. Dupont (2002) introduces an alternative method called mean-square hedging. This method closely resembles the general linear model in regression analysis. The writer of a barrier option may have liquid vanilla options with only a few maturities and strikes available to hedge the barrier option. These vanilla options may not have the required strikes and maturities to hedge the barrier in either the DEK or CEG framework. Mean-square hedging provides the hedger with a method that uses only these vanilla options to construct the best possible semi-static hedging portfolio. In addition, any share price process and option valuation method can be used in conjunction with mean-square hedging method.

7.1 The Mean-square Hedging method

Consider a barrier option with price Y_t and underlying share price S_t at time t, strike level K, barrier level H and maturity T. Also, consider a world with n possible states at time T. These n states correspond to n possible states for S_t and Y_t at time T. Denote these by $S_T(\omega_1),\ldots,S_T(\omega_n)$ and $Y_T(\omega_1),\ldots,Y_T(\omega_n)$ respectively. Assume the availability of k hedging instruments (these can be any assets, e.g. vanilla options, bonds etc.) with prices $S_T^1,\ldots,S_T^k$ respectively. However, with n possible states of the world at time T, these assets will each also have n possible states. Group the asset prices of these hedging instruments at maturity T together in a matrix $\mathbf{X}$, where

$$\mathbf{X} = \begin{pmatrix} S_T^1(\omega_1) & .. & .. & S_T^k(\omega_1) \\ S_T^1(\omega_2) & & & S_T^k(\omega_2) \\ \vdots & & . & \vdots \\ S_T^1(\omega_n) & .. & .. & S_T^k(\omega_n) \end{pmatrix}$$

Matrix $\mathbf{X}$ therefore has dimensions $(n \times k)$. Similarly group the barrier option values at maturity T into a matrix $\mathbf{Y}$, where

$$\mathbf{Y} = \begin{pmatrix} Y_T(\omega_1) \\ . \\ . \\ . \\ Y_T(\omega_n) \end{pmatrix}$$

is a $(n \times 1)$ matrix. Denote the position in asset S^i by β^i. Again group these positions together in a matrix $\boldsymbol{\beta}$ such that,

$$\boldsymbol{\beta} = \begin{pmatrix} \beta^1 \\ . \\ . \\ . \\ \beta^k \end{pmatrix}$$

is a $(k \times 1)$ matrix. Consider the equation $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$ where $\mathbf{e}$ is the $(n \times 1)$ matrix,

$$\mathbf{e} = \begin{pmatrix} e_1 \\ . \\ . \\ . \\ e_n \end{pmatrix}$$

In other words, matrix $\mathbf{e}$ is the matrix of errors from regression. Mean-square hedging is concerned with finding the matrix $\boldsymbol{\beta}$ such that $\mathbf{e}'\mathbf{e} = \sum_{i=1}^{n} e_i^2$ is minimized ($\mathbf{e}'$ denotes the transpose of matrix $\mathbf{e}$). In other words, mean-square hedging minimizes the hedging error at maturity, given a set of hedging instruments k and asset prices at maturity for the n possible states of the world.

7.2 Mean-square hedging of an Up-and-out call option

The method of mean-square hedging can be best illustrated by considering an example. Consider the hedging of a European up-and-out call option with asset dynamics given by equation (36) below, and parameters as given in Table 7.

Table 7: Parameters of Up-and-out call options

Spot price of share (S_0)	R80
Strike price (X)	R100
Barrier Level (L)	R120
Volatility (σ)	20%
Drift rate of share price (μ)	20%
Risk-free interest rate (r)	5%
Continuous dividend yield (q)	3%
Option's time to maturity in years (T)	1

Note that the rates in Table 7 above are given in NACC (nominal annual compounded continuously) form. Assume that the hedging instruments available are as given in Table 8.

Table 8: Details of hedging instruments/assets

Instrument	**Type**	**Strike**	**Volatility**	**Maturity in years**
Bank account / zero-coupon bond	-	-	-	1
Option 1	Call	R70	20%	1
Option 2	Call	R75	20%	1
Option 3	Call	R80	20%	1
Option 4	Call	R85	20%	1
Option 5	Call	R90	20%	1
Option 6	Call	R95	20%	1
Option 7	Call	R100	20%	1
Option 8	Call	R105	20%	1
Option 9	Call	R110	20%	1
Option 10	Call	R115	20%	1

Note that all the options listed in Table 8 above are options on the underlying of the barrier option to be hedged. Since the maturity of these options is identical to that of the barrier option, the risk-free rate and dividend yield will be the same as that of the barrier option.

Although an implied volatility skew is common in the marketplace, the assumption here will be that all the options listed in Table 8 and the barrier option will have a constant implied volatility of 20% until maturity.

In this example, it follows that $k = 11$. Next, the asset prices at maturity T for the various states of the world n need to be calculated. As all the assets are options on the same underlying (except for the bank account / risk-free zero coupon bond, in which case $S_T^1(\omega_i) = 1$ for $i = 1,\ldots,n$), the problem reduces to calculating only the share price of the underlying asset for the various states of the world n.

We make the following assumptions about the share price process and option valuation model:

- Assume a geometric Brownian motion share price process, so that

$$\begin{aligned} \frac{dS_t}{S_t} &= (0.2-0.03)dt + 0.2dB_t \\ S_0 &= 80 \end{aligned} \tag{36}$$

  (recalling the values in Table 7), where B_t is a standard Brownian motion under the real-world measure.

- Also, assume that the above vanilla options and barrier can be valued using the Black-Scholes model.

The share prices at maturity T, for the various states can therefore be simulated using Euler's approach:

$$\begin{aligned} S_{t+\Delta t} &= S_t e^{\left(\mu - q - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\left(B_{t+\Delta t} - B_t\right)} \\ &= S_t e^{\left(\mu - q - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon} \\ &= S_t e^{\left(0.2 - 0.03 - \frac{1}{2}(0.2)^2\right)\Delta t + 0.2\sqrt{\Delta t}\varepsilon} \end{aligned} \tag{37}$$

where ε is a sample from the standard normal distribution. In addition, choose:

- $n = 10000$ i.e. there are 10000 states of the world and therefore 10000 simulations of S_T
- $\Delta t = \frac{T}{N} = \frac{1}{365}$. In other words, discrete increments once per day to the share price (365 time steps).

Note that instead of using Monte Carlo simulation, as we do here, one can also use a binomial tree or trinomial tree for the share price evolution.

One further thing must be kept in mind: the up-and-out call option can at any stage (time step) knock out (cross or hit the barrier of *R120*). This is bound to happen for a fair proportion of the 10000 simulations of the share price at maturity. When this happens, the portfolio of assets (options in this example) is liquidated (which implies that this method is semi-static) and respective values accumulated (rolled forward) to maturity using the bank account.

One Monte Carlo simulation with antithetic variate (see Clewlow and Strickland (1998) for a discussion) was run (with the parameters $n = 10000$ (paths) and $N = 365$). Microsoft Excel's Solver function was then used to find the matrix $\boldsymbol{\beta}$ that minimizes $\mathbf{e}'\mathbf{e} = \sum_{i=1}^{n} e_i^2$.

Table 9 below lists the optimal asset positions (weights) and the cost of the hedge at time 0. "Weights" in Table 9 refers to the required position in the particular option. A negative weight indicates that the option needs to be written, while a positive number indicates a purchase. The bank account position indicates the required amount of money that needs to be deposited at time 0.

Table 9: The required asset positions (weights) and the associated costs

Cost of Hedge at time 0	**Weights**	**Price per unit**	**Cost**
Bank Account position	0.000102678	R0.951229	R0.000098
Option 1	0.000210951	R12.873548	R0.002716
Option 2	0.000394170	R9.600374	R0.003784
Option 3	0.000888377	R6.922021	R0.006149
Option 4	0.002590006	R4.832201	R0.012515
Option 5	0.002315505	R3.272927	R0.007578
Option 6	0.004093859	R2.156131	R0.008827
Option 7	0.974863147	R1.385173	R1.350354
Option 8	0.047595411	R0.870090	R0.041412
Option 9	-0.470322139	R0.535732	-R0.251967
Option 10	-1.837051108	R0.324097	-R0.595383
Total Cost of Hedge at time 0			**R0.586085**

To test the quality of the hedging position, the regression measure *R-squared* can be used. Recall that *R-squared* is defined as follows:

$$R^2 = \frac{\sum_{i=1}^{n} \left(\hat{Y}_T(\omega_i) - \bar{Y} \right)^2}{\sum_{i=1}^{n} \left(Y_T(\omega_i) - \bar{Y} \right)^2}$$

where $\overline{Y}$ refers to the arithmetic mean of the elements of the column vector $\mathbf{Y}$ and $\hat{Y}_T(\omega_i)$ the value produced by our hedging portfolio when the share price is in state i. In our example above, the *R-squared* equals 80.61%. This seems to suggest that the hedge is fairly good. However, closer inspection of the hedging errors under the various states indicates that this is not the case. A fair number of the observations have large errors in absolute terms. Figure 9 below shows the errors

$\hat{e}_i = Y_T(\omega_i) - \hat{Y}_T(\omega_i)$ for $i = 1,\ldots,n$ where $\hat{\mathbf{Y}}=\mathbf{X}\hat{\boldsymbol{\beta}}$ with $\hat{\boldsymbol{\beta}}$ being the $\boldsymbol{\beta}$ that minimizes $\mathbf{e}'\mathbf{e} = \sum_{i=1}^{n} e_i^2$.

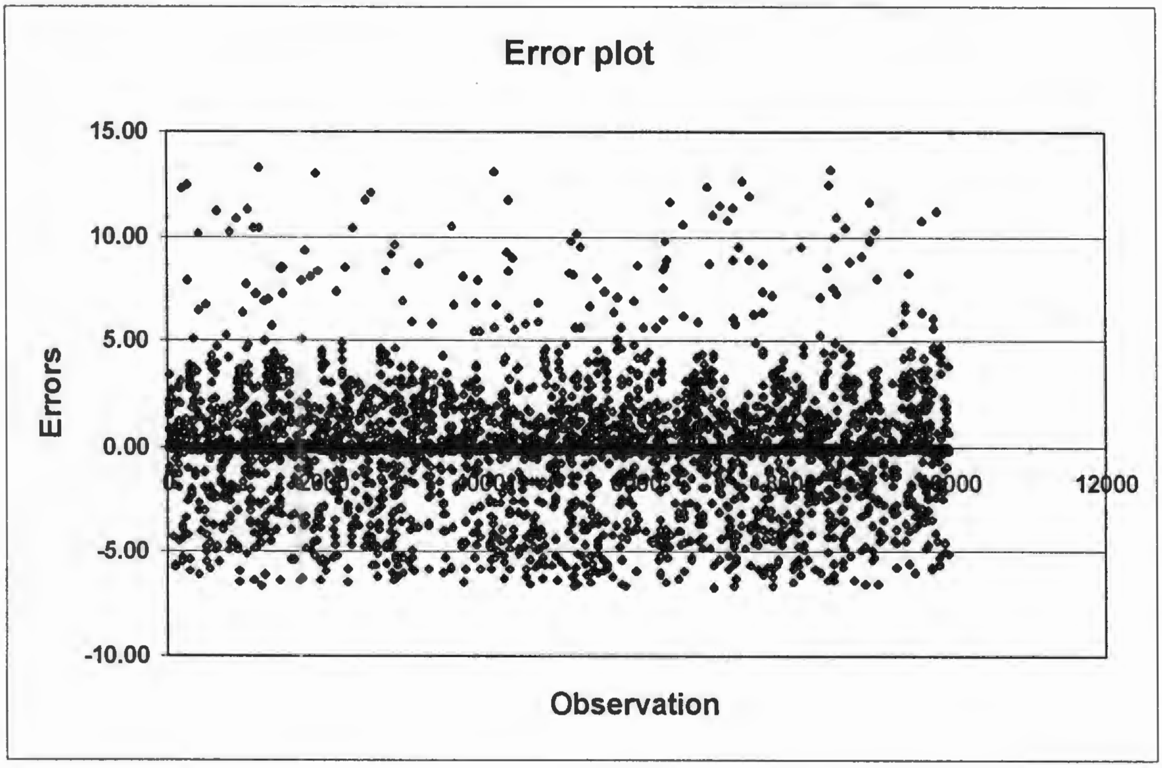


Figure 9: A plot of the observed regression errors under the various states

In other words, an error of *R10* means a loss of *R10* for the writer of the barrier option under that specific state. This loss is large relative to the cost of the hedging strategy.

Consider an alternative hedging portfolio, consisting of the instruments as given in Table 10.

Table 10: Details of hedging instruments/assets

Instrument	Type	Strike	Volatility	Maturity in years
Bank account / zero-coupon bond	-	-	-	1
Option 1	Call	R90	20%	1
Option 2	Call	R95	20%	1
Option 3	Call	R100	20%	1
Option 4	Call	R105	20%	1
Option 5	Call	R110	20%	1
Option 6	Call	R115	20%	1
Option 7	Call	R120	20%	1
Option 8	Call	R130	20%	1
Option 9	Call	R140	20%	1
Option 10	Call	R145	20%	1

A Monte Carlo simulation (with the same up-and-out call option parameters as the previous hedging portfolio) yields a *R-squared* value of 96.16% and positions (weights) as given in Table 11 below.

Table11:The required asset positions (weights) and the associated costs

Cost of Hedge at time 0	Weights	Price per unit	Cost
Bank Account position	-0.000204943	R0.951229	-R0.000195
Option 1	0.000569440	R3.272927	R0.001864
Option 2	-0.005011785	R2.156131	-R0.010806
Option 3	1.017056930	R1.385173	R1.408800
Option 4	-0.068163932	R0.870090	-R0.059309
Option 5	0.306527721	R0.535732	R0.164217
Option 6	-1.763393145	R0.324097	-R0.571511
Option 7	-6.451152969	R0.193056	-R1.245437
Option 8	24.790362913	R0.065899	R1.633661
Option 9	-68.108766272	R0.021597	-R1.470957
Option 10	59.070982796	R0.012219	R0.721795
Total Cost of Hedge at time 0			**R0.572123**

An error plot (see Figure 10) of this portfolio confirms that this semi-static hedge performs much better than our previous portfolio.

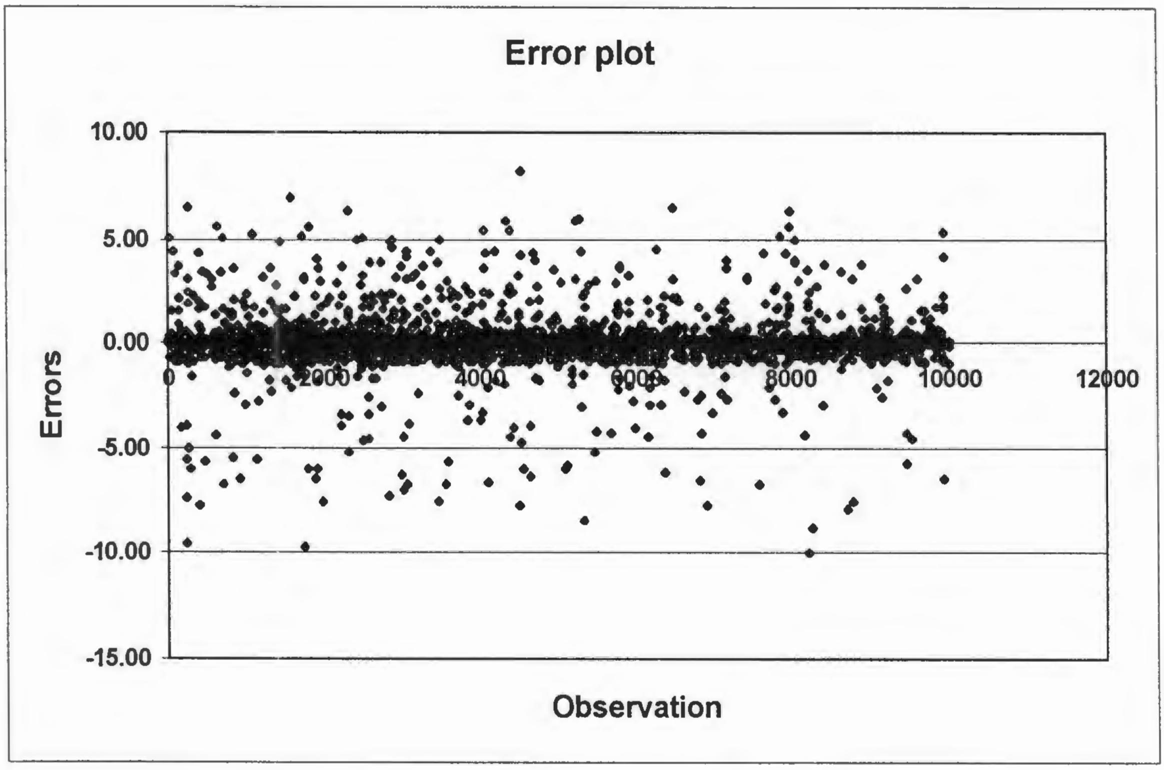


Figure 10: A plot of the observed regression errors under the various states

Although some of the hedging errors are still big in absolute terms, the errors are in general much smaller than those in the previous hedging portfolio. In particular, there are fewer errors with large positive values (these correspond to those states implying a loss to the writer of the barrier option).

An analysis of the latter hedging portfolio's instruments [in Table 10] indicates that the availability of vanilla options (hedging instruments) with high strike levels is crucial for a reasonable hedge to be constructed.

7.3 Advantages of Mean-square hedging

The main advantages of Mean-square hedging are as follows:

- As opposed to the CEG and DEK models discussed earlier, mean-square hedging places no restrictions or assumptions on the share price process. The user is allowed to specify any process he/she sees fit. This further translates into allowing an option pricing model other than the Black-Scholes model.
- The user of this semi-static hedging scheme is allowed to specify the available, liquid vanilla options that can be used to construct a hedge. No assumptions about the availability of options with specific strike levels or maturities are made.
- Although not discussed in this paper, mean-square hedging can incorporate constraints on the hedging error as well as the allowable option positions.
- Mean-square hedging also allows a volatility structure to be specified i.e. a volatility skew or smile can be incorporated. This may be a better representation of reality.

7.4 Limitations of Mean-square hedging

Mean-square hedging does have the following limitations though.

- Computationally the method is more complicated than the CEG and DEK models. Computation time is therefore also greater.
- Mean-square hedging is still only a semi-static hedge. If liquidation is removed (to make it a full static hedge), the hedge deteriorates and hedging errors grow very large.
- Even with a portfolio of hedging instruments with a wide range of strike levels, hedging errors are fairly large in some states.

8 THE ANDERSEN, ANDREASEN AND ELIEZER MODEL

Andersen, Andreasen and Eliezer (2002) (from here on referred to as AAE) extended the results of previous literature on static hedging. They derived exact, explicit expressions for the static hedging of barrier options with almost any complicated knock-out region. Rebates were also included in their analysis. In the case of an out option, a rebate is an amount received by the option holder if the barrier option knocks out before maturity. AAE demonstrated that their technique allows for a local volatility structure, which depends deterministically on time and on the underlying asset as in Dupire (1994). When a stochastic volatility structure is considered, their technique breaks down though (see AAE Appendix B for a discussion).

8.1 Static hedging under the AAE model

The AAE method is based on the following assumptions and results:

- First assume that all interest rates and dividend yields are zero. Also assume that the underlying share has a local volatility structure. Mathematically,

  $$\frac{dS_t}{S_t} = \sigma(t,S_t)dW_t \tag{38}$$

  where S_t denotes the share price at time t and W_t the standard Brownian motion under the risk-neutral measure.

- Let $C(T,K)$ and $P(T,K)$ denote the time-0 prices of European call and put options, with maturity T and strike level K. Denote the marginal density of S_t by $f(t,S)$ so that $f(T,S)$ corresponds to the marginal density of S_T. Now,

  $$\begin{aligned} C(T,K) &= \int_K^{\infty}(S-K)f(T,S)dS \\ &= -\int_{\infty}^{K}(S-K)f(T,S)dS \end{aligned}$$

Hence,

$$C_K(T,K) = -(K-K)f(T,K) - \int_{\infty}^{K} -f(T,S)dS$$
$$= \int_{\infty}^{K} f(T,S)dS \quad \text{...........}(*)$$

and so

$$C_{KK}(T,K) = f(T,K)$$

where subscripts denote partial derivatives. Therefore,

$$f(T,S) = C_{KK}(T,S) \tag{39}$$

This result was first proved by Breeden and Litzenberger (1978) and is model independent.

By put-call parity, $f(T,S) = P_{KK}(T,S)$. Also, from $(*)$ above,

$$\begin{aligned} \int_{B}^{\infty} f(T,S)dS &= -C_K(T,B) \\ \int_{0}^{B} f(T,S)dS &= 1 + C_K(T,B) \end{aligned} \tag{40}$$

AAE prove the following theorem (see pp.17-21 in AAE):

Theorem 8.1.1:

Suppose that the underlying share evolves according to (38) and consider an option that has the value $g(S_T)$ at time T and knocks out on a set $B \subset \Omega$, $\Omega = [0,T] \times (0,\infty)$, with a once differentiable rebate function, R, that depends only on time. Assuming that $\Omega \setminus B$ is an open submanifold in Ω, a static hedge for the option value is defined by

$$
\begin{aligned}
BO(0,S_0) = {} & \int_0^\infty g(S) C_{KK}(T,S)\,dS \\
& -\frac{1}{2}\int_0^T \sum_{S\in\left(\overline{\partial B}\right)(t,\cdot)} \left[BO_S(t,S+) - BO_S(t,S-)\right]\sigma(t,S)^2 S^2 C_{KK}(t,S)\,dt \\
& -\int_0^\infty \sum_{t\in\left(|\partial B\right)(\cdot,S)} \left[BO(t+,S) - BO(t,S)\right] C_{KK}(t,S)\,dS \\
& -\int_{\operatorname{int} B} R'(t) C_{KK}(t,S)\,dt\,dS
\end{aligned}
$$

where ∂B and $\operatorname{int} B$ denote, respectively, the boundary and interior of B, $R' = \frac{dR}{dt}$, $BO_S(t,S+)$ is the limit of $BO_S(t,S+\varepsilon)$ for $\varepsilon \downarrow 0$ and $BO(t,S)$ is the value of a barrier option at time t with share price S at that time. Further, define

$$
\begin{aligned}
|\partial B &= \left\{(t,S)\in\partial B \middle| \exists\varepsilon>0 : (t,S+h)\in\partial B, \forall |h|<\varepsilon\right\} \\
\overline{\partial B} &= \partial B \setminus |\partial B
\end{aligned}
$$

and if $A \subset \Omega$, let

$$
\begin{aligned}
A(t,\cdot) &= \left\{S\in(0,\infty) \middle| (t,S)\in A\right\} \\
A(\cdot,S) &= \left\{t\in[0,T] \middle| (t,S)\in A\right\}
\end{aligned}
$$

The first term in Theorem 8.1.1 above is that part of the barrier price that can be attributed to the payoff at maturity. The second term represents the non-vertical parts of the knock-out barrier, the third the vertical parts and the fourth the rebate.

8.2 Hedging an up-and-out call option

Consider Theorem 8.1.1 in the case of an up-and-out call option with a constant barrier B, no rebate, maturity T and strike level $K < B$ (recall that Theorem 8.1.1 assumes that all interest rates and dividend yields are zero). Note that here there is no rebate, so the fourth term in Theorem 8.1.1 falls away. Also, $|\partial B = \{\ \}$, the empty set, so the third term also falls away. In addition, $g(S) = (S-K)^+ I_{\{S<B\}}$, so that

$$
\begin{aligned}
&\int_0^\infty g(S)C_{KK}(T,S)dS \\
&= \int_0^\infty (S-K)^+ I_{\{S<B\}} C_{KK}(T,S)dS \\
&= \int_K^B (S-K)C_{KK}(T,S)dS
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_K^B (S-K)C_{KK}(T,S)dS &= \int_K^\infty (S-K)C_{KK}(T,S)dS - \int_B^\infty (S-K)C_{KK}(T,S)dS \\
&= C(T,K) - \int_B^\infty (S-K)C_{KK}(T,S)dS \\
&= C(T,K) - \int_B^\infty (S-B+B-K)C_{KK}(T,S)dS \\
&= C(T,K) - \int_B^\infty (S-B)C_{KK}(T,S)dS - \int_B^\infty (B-K)C_{KK}(T,S)dS \\
&= C(T,K) - C(T,B) - (B-K)\int_B^\infty C_{KK}(T,S)dS \\
&= C(T,K) - C(T,B) + (B-K)C_K(T,B)
\end{aligned}
$$

since $C_{KK}(T,S) = f(T,S)$. Lastly, consider the second term in Theorem 8.1.1. Note that for a constant barrier level, the boundary $\overline{\partial B}$ consists only of one piece, so that the summation sign is redundant. Also, for an up-and-out option, $UOC(t,B+) = 0$, so that $UOC_S(t,B+) = 0$. Therefore, the third term reduces to

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T -UOC_S(t,B-)\sigma(t,B)^2 B^2 C_{KK}(t,B)dt \\
&= \frac{1}{2}\int_0^T UOC_S(t,B-)\sigma(t,B)^2 B^2 C_{KK}(t,B)dt \\
&= \int_0^T UOC_S(t,B-)C_T(t,B)dt
\end{aligned}
$$

recalling from Dupire (1994) that $C_T(t,B) = \frac{1}{2}\sigma(t,B)^2 B^2 C_{KK}(t,B)$. The hedging equation for the up-and-out call option under consideration is therefore,

$$UOC(0,S_0)=C(T,K)-C(T,B)+(B-K)C_K(T,B)+\int_0^T UOC_S(t,B-)C_T(t,B)dt \quad (41)$$

8.3 A numerical example

Consider the hedging of a European up-and-out call with parameters as given in Table 12 below. Note that here the volatility is assumed constant, so that $\sigma=\sigma(t,S_t)$.

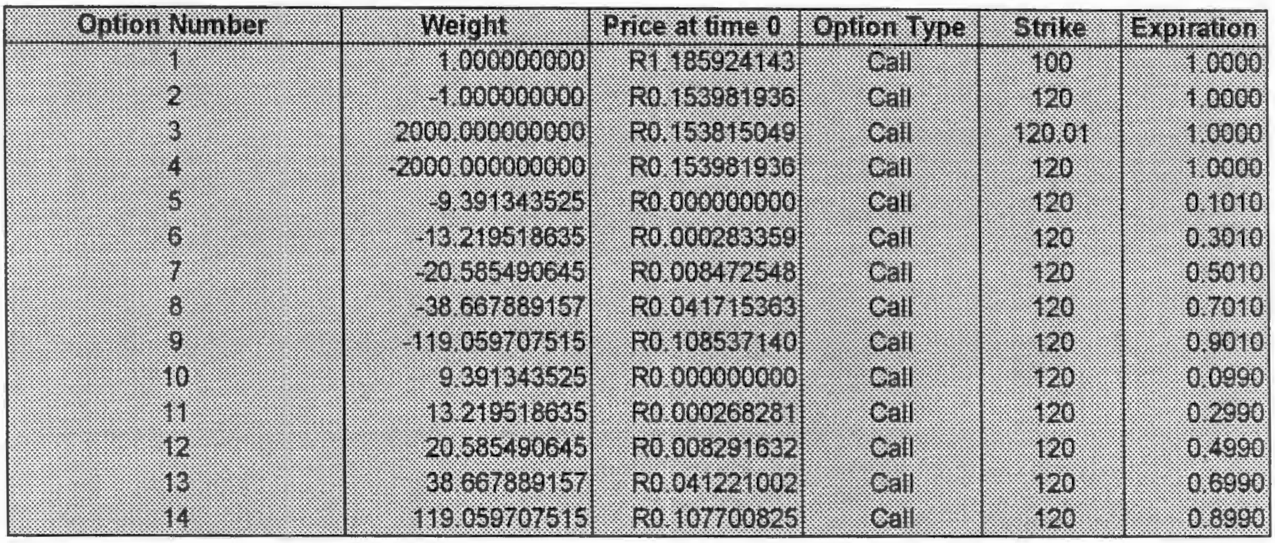
Table 12: Parameters of the European up-and-out call

Spot price of share (S_0)	R80
Strike price (*X*)	R100
Barrier Level (*L*)	R120
Volatility (σ)	20%
Risk-free interest rate (*r*)	0%
Continuous dividend yield (*q*)	0%
Option's time to maturity in years (*T*)	1

Consider an implementation of the AAE hedging equation (41) where the integral term is approximated by a midpoint Riemann sum. An implementation with 5 subintervals for the Riemann sum yields the hedging portfolio indicated by Table 13.

The first two option positions represent the required position for $C(T,K)$ and $C(T,B)$ respectively, while the third and fourth positions are those required for the strike spread $(B-K)C_K(T,B)=(B-K)\left[\frac{C(T,B+\varepsilon)-C(T,B)}{\varepsilon}\right]$. Note that the strike spread involves a first derivative approximation in B to the right, to ensure that there are no unwanted cash flows ($C(T,B-\varepsilon)$ will produce a unwanted cash flow). The value of ε used for the above-mentioned strike spread was 0.01, while the ε value used for the maturity spread in the integral approximation was 0.0001. Note that the choice of ε values will influence the maximum allowable subintervals for the integral approximation.

Table 13: Hedging position for the up-and-out call given in Table 12 above



Option Number	Weight	Price at time 0	Option Type	Strike	Expiration
1	1.000000000	R1.185924143	Call	100	1.0000
2	-1.000000000	R0.153981936	Call	120	1.0000
3	2000.000000000	R0.153815049	Call	120.01	1.0000
4	-2000.000000000	R0.153981936	Call	120	1.0000
5	-9.391343525	R0.000000000	Call	120	0.1010
6	-13.219518635	R0.000283359	Call	120	0.3010
7	-20.585490645	R0.008472548	Call	120	0.5010
8	-38.667889157	R0.041715363	Call	120	0.7010
9	-119.059707515	R0.108537140	Call	120	0.9010
10	9.391343525	R0.000000000	Call	120	0.0990
11	13.219518635	R0.000268281	Call	120	0.2990
12	20.585490645	R0.008291632	Call	120	0.4990
13	38.667889157	R0.041221002	Call	120	0.6990
14	119.059707515	R0.107700825	Call	120	0.8990

Table 13 shows that the required hedge has a total cost of *R0.575558* compared to the Black-Scholes value of *R0.483974*. Increasing the number of subintervals produces a total hedging cost that converges to the Black-Scholes value.

8.4 Non-zero interest rates and dividend yields

In this section we aim to extend the static hedge of the up-and-out call to the case of non-zero, constant risk-free rates and dividend yields. In other words,

$$dS_t = (r - q)S_t dt + \sigma(t, S_t)dW_t \tag{42}$$

where r and q denote the continuously compounded risk-free rate and dividend yield respectively and W_t a standard Brownian motion under the risk-neutral measure. First, consider the case of constant volatility $\sigma(t, S_t) = \sigma$.

AAE suggests non-zero interest rates and dividend yields can easily be incorporated into their model by modelling the forward share price F_t, instead of the share price (equation (38)), so that

$$\frac{dF_t}{F_t} = \sigma dW_t \tag{43}$$

where $F_t = S_t e^{(r-q)(T-t)}$ and T is the option's maturity. By Itô's Lemma, this is equivalent to $dS_t = (r-q)S_t dt + \sigma(t, S_t)dW_t$. AAE notes that in this case, the barrier levels must be represented in terms of forward share price levels, $B(t) = Be^{(r-q)(T-t)}$, and terminal payments must be represented in terms of their discounted values. In other words, the barrier level becomes time-dependent. The resulting hedging equation is then

$$\begin{aligned}
UOC(0,S_0) &= \hat{C}(T,K) - \hat{C}(T,B(T)) + (B-K)\hat{C}_K(T,B(T)) \\
&\qquad + \int_0^T UOC_F(t,B(t)-)\hat{C}_T(t,B(t))dt \\
&= C(T,K) - C(T,B) + (B-K)C_K(T,B) \\
&\qquad + \int_0^T UOC_S(t,B(t)-)\frac{\partial S_t}{\partial F_t}C_T(t,B(t))dt \\
&= C(T,K) - C(T,B) + (B-K)C_K(T,B) \\
&\qquad + \int_0^T e^{-(r-q)(T-t)}UOC_S(t,B(t)-)C_T(t,B(t))dt
\end{aligned} \tag{44}$$

where $\hat{C}(T,B(T))$ refers to the value of a call option on the forward share price with maturity T and strike $B(T)$. However, a call option on the forward share price is equivalent to a call option on the share price with interest rate r and dividend yield q. Therefore $\hat{C}(T,B(T)) \equiv C(T,B(T))$. Futhermore, $C(T,B(T)) = C(T,B)$ since $B(T) = B$.

The only other complexity remaining in constructing the static hedge is then the computation of $UOC(t,B(t)-)$, the value of an up-and-out call option on a share, with underlying process (42) and with moving barrier $B(t) = Be^{(r-q)(T-t)}$. Kunitomo and Ikeda (1992) provide a closed-form solution for such an option. Once $UOC(t,B(t)-)$ is computed, its delta $UOC_S(t,B(t)-)$ can easily be computed numerically.

An implementation of this static hedging scheme (44) however fails to work. The hedge does not seem to converge to the Black-Scholes value[1,2].

8.5 Advantages of the AAE model

The AAE model has the following main advantages:

- The model allows for a price process with jumps. Although not discussed in this paper, the AAE model extends Theorem 8.1.1 to the case of an underlying share price process with jumps. The model therefore provides a static hedging scheme for barrier options with a discontinuous share price process.
- Allows for arbitrary barrier structures. The AAE model allows for the hedging of options with arbitrary barrier structures.

8.6 Limitations of the AAE model

The limitations of the AAE model are similar to those of the other three methods discussed in this paper. These include:

- fractional notional positions unattainable in the market place
- semi-static hedging that requires liquidation of the hedging portfolio upon knockout
- only an approximation (because the integral needs to be approximated when implemented)

---

[1] In a personal communication, Dr. David Eliezer says: "The first thing I am checking is whether the Carr-type formula that you are using can be extended to arbitrarily shaped barriers without additional terms. It isn't clear to me that it can."

[2] At the time of writing, Dr. Peter Carr was working on a paper titled "Semi-Static Hedging and Conservative Fields". This paper aims to extend the formulation of the AAE paper to include deterministic interest rates and dividend yields and to include jumps.

- no transaction costs, and
- a liquid vanilla options market with a continuum of strike levels and maturities remains a problem.

9 EVALUATION OF HEDGING PERFORMANCE

This section aims to evaluate the hedging performance of the four hedging methodologies discussed in this paper in a similar way to that of Tompkins (2002). This is done in a framework that tries to simulate real market conditions by considering models of non-constant volatility, transaction costs and bid-offer spreads. Throughout, the hedging performance of the static hedging schemes will be compared to that of dynamic delta hedging.

9.1 Evaluation Methodology

9.1.1 Main assumptions

- Throughout this section, we will assume that the static hedging schemes replicate the payoff of the up-and-out call exactly, as the dynamic hedging scheme does. Note that none of the four hedging schemes exactly replicate the payoff of the up-and-out call. All four provide only an approximation to the exact payoff, with the approximation improving as the number of different option hedging positions are increased. Loosely, the general order of accuracy of the four methods, in increasing order is: Dupont, DEK, AAE and CEG. This order varies depending on the hedging instruments and number of different positions available. Since this section incorrectly assumes that the static hedges are perfect, its results are biased towards the static hedging schemes when compared to the dynamic delta hedging scheme. This should be borne in mind.
- A constant bid-offer spread of 0.125 $\left(\frac{1}{8}\right)$ is assumed throughout. This implies that when the share price is S_t, a share is bought at a cost of

$(S_t + 0.0625)$ and sold at $(S_t - 0.0625)$, with the simulated share price path providing the mid value of S_t

- Proportional transaction costs of 0.5% are assumed. This implies that when any security (option and/or share) is bought or sold, a cost of 0.5% is incurred. Note that an option transaction will only incur the proportional transaction cost while it will be unaffected by the bid-offer spread. Share transactions will be affected by both (both a bid-offer spread and transaction costs will be incurred).

9.1.2 Modelling assumptions

This section aims to simulate real market conditions. This is done in two ways:

- Stochastic volatility. To simulate the fact that the hedging portfolio's options mostly probably won't be liquidated at model-based prices, Heston's stochastic volatility model will be employed. Fouque, Papanicolaou and Sircar (2000) briefly discuss the Heston model, together with some other stochastic volatility models. Mathematically,

$$\begin{aligned} dY_t &= \alpha(m - Y_t)dt + \beta\sqrt{Y_t}d\hat{Z}_t \\ \sigma_t &= \sqrt{Y_t} \\ d\langle W, \hat{Z}\rangle_t &= \rho dt \\ \hat{Z}_t &= \rho W_t + \sqrt{1-\rho^2}Z_t \end{aligned} \tag{45}$$

where W_t is the standard Brownian motion driving the share price process, σ_t the volatility at time t, α the mean reversion rate, m the mean reversion level and ρ the correlation between the volatility and share price processes. This volatility process was implemented using the familiar Euler method (see Clewlow and Strickland (1998) for a discussion) to get a realization of σ_t for use as an estimate of the market implied volatility at time t.

- Secondly, the Euler approach will be employed, as was done in equation (37) with antithetic variance reduction, to produce simulated share price paths in the real world.

9.1.3 Comparing dynamic delta hedging with static hedging

The performance evaluation approach in the next few sections will then be employed as follows. First, share price and volatility paths will be generated as discussed in the section above. Taking transaction costs into account, the delta hedging costs under the various paths will be calculated and discounted to the inception date of the option under consideration. Refer to Hull (2003), pp. 302-309 for a discussion on delta hedging. Two performance statistics will be calculated:

- Firstly, a statistic called average percentage difference (AD) will be calculated as follows:

$$AD = \frac{HC}{UOC(T,L,K)}$$

where HC is the average hedging cost and $UOC(T,L,K)$ is the Black-Scholes price of a European up-and-out call with strike level K, barrier L and maturity T. Under zero transaction costs and constant volatility, the expectation is that this measure will be equal to one. A value greater than one will indicate that hedging costs are greater than that given by Black-Scholes. In the sections that follow, this statistic is also referred to as Avg. Hedging Cost / Theoretical Value, but where Theoretical value includes the transactions cost at inception of the option.

- Secondly, a statistic called hedging performance will be calculated, where

$$HP = \frac{\sigma_{HC}}{UOC(T,L,K)}$$

and σ_{HC} is the standard deviation of the hedging costs over the various paths. The statistic HP is therefore a measure of the variability of the hedging result

HC. Note that this is the same performance measure as used in Hull (2003) and will be referred to as Std. Dev. HC / Theoretical Value in the sections that follow.

The same will be done for the static hedging method under consideration. In other words, the costs of putting the static hedge in place at time 0 will be computed and added to the discounted value of transaction costs incurred if the up-and-out call knocks out before or at expiration. This will produce a value for HC under each path. Statistics AD and HP will be computed in a similar manner.

9.2 Evaluating the different models

Throughout this section, the stochastic volatility parameterisation for the Heston model, as given in Table 14, will be used.

Table 14: Parameterisation of Heston's stochastic volatility model

Rate of mean reversion (α)	16
Mean reversion level (m)	4.00%
Correlation (ρ)	-0.5
Volatility of volatility (β)	0.1

The mean reversion level of 4% translates into a mean reverting volatility level of 20%. A mean reversion rate of 16 ensures that the stochastic volatility paths will stay within a narrow band around the reverting level of 20%, while the negative correlation of –0.5 ensures a stochastic volatility process that resembles the real world observation of a negative correlation between volatility and price.

9.2.1 Evaluating dynamic hedging under no transaction costs and constant volatility

Before this paper examines the four static hedging models, the dynamic delta hedging method is examined under the conditions of no transaction costs. Table 15 below lists the details of the up-and-out call under consideration.

Table 15: Details of up-and-out call to be hedged

Spot Price of underlying (S_0)	R80.00
Strike price (X)	R100.00
Barrier (L)	R120.00
Volatility (σ) / starting volatility	20.00%
Risk-free interest rate (r) in NACC form	5.00%
Drift rate (μ) of share in real world	20.00%
Continuous dividend yield (q)	0.00%
Option's time to maturity in years (T)	1
Bid-Offer Spread	0
Commission on transaction	0.00%
Notional position	1
Number of path simulations	1000
Number of times the portfolio is rebalanced	365

As shown in Table 15, one thousand (1000) daily share price and volatility paths were simulated. The dynamic delta hedging portfolio was furthermore assumed to be rebalanced on a daily basis (365 times over a one year period).

Table 16: Statistics of dynamic delta hedging under zero transaction costs and constant volatility

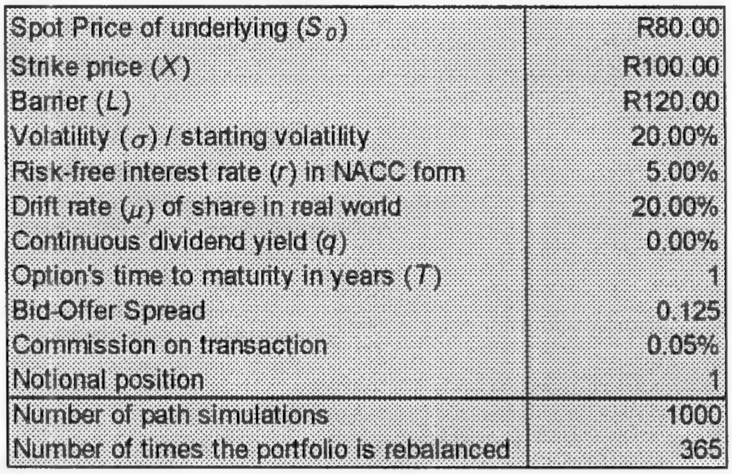
B-S Price of Euro up-and-out call with commission	R0.677679
Avg. Hedging Cost / Theoretical Value	125.74%
Std. Dev. HC / Theoretical Value	127.60%

Dynamic delta hedging does not seem to work well in the case of the up-and-out call. The average hedging cost under the dynamic scheme is approximately 25% greater than that indicated by the Black-Scholes price. As explained in section 3.3, the poor dynamic delta hedging results can be attributed to the fact that the delta of an up-and-out call changes dramatically as the share price approaches the barrier level. Furthermore, a hedging performance statistic of 127.60% is observed. This indicates that the hedging costs under a dynamic hedging scheme are highly variable. Dynamic delta hedging is therefore undesirable and impractical even under zero transaction costs and constant volatility.

9.2.2 *Evaluating the DEK model*

Table 17 below details the parameters of the up-and-out call to be hedged.

Table 17: Details of the up-and-out call to be hedged

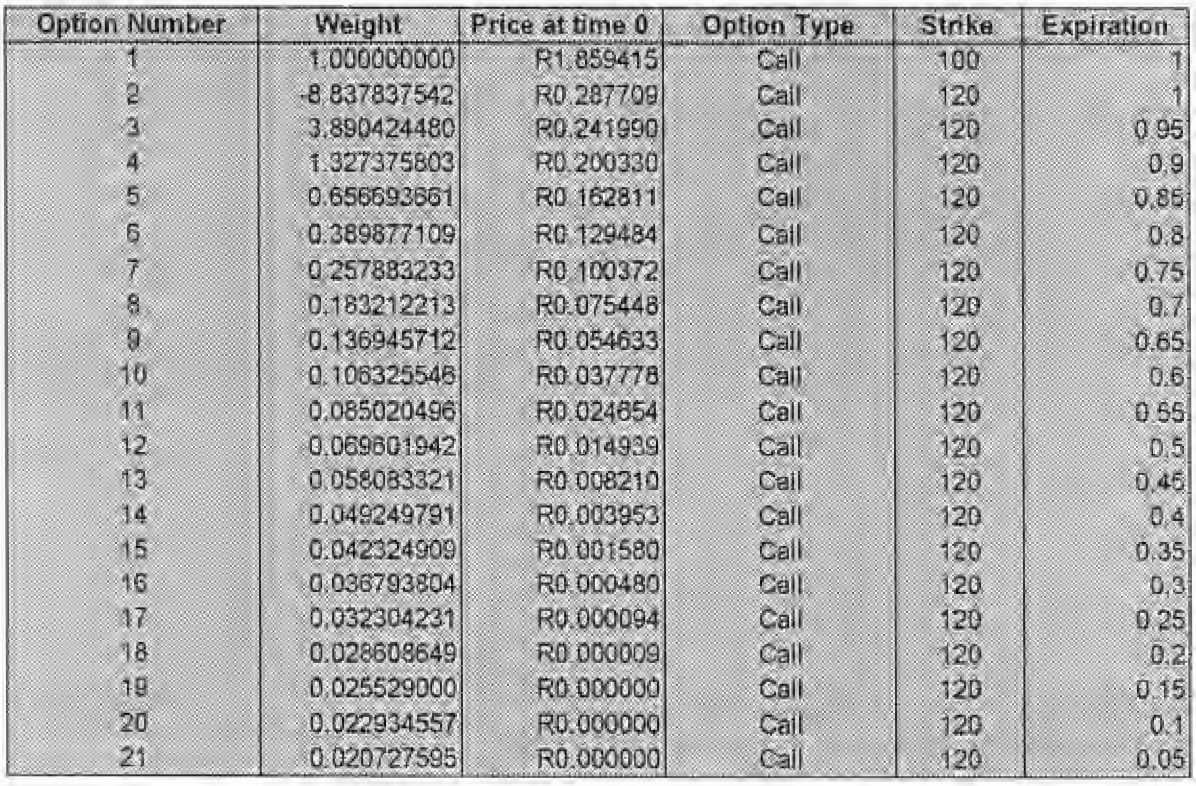


Spot Price of underlying (S_0)	R80.00
Strike price (X)	R100.00
Barrier (L)	R120.00
Volatility (σ) / starting volatility	20.00%
Risk-free interest rate (r) in NACC form	5.00%
Drift rate (μ) of share in real world	20.00%
Continuous dividend yield (q)	0.00%
Option's time to maturity in years (T)	1
Bid-Offer Spread	0.125
Commission on transaction	0.05%
Notional position	1
Number of path simulations	1000
Number of times the portfolio is rebalanced	365

Again, one thousand (1000) daily share price and volatility paths were simulated. The dynamic delta hedging portfolio was also assumed to be rebalanced on a daily basis (365 times over a one year period).

The portfolio of hedging options is displayed in Table 18 below.

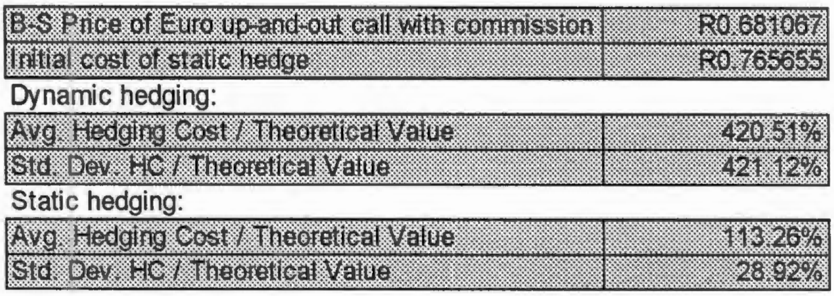
Table 18: Details of the hedging portfolio under the DEK model



Option Number	Weight	Price at time 0	Option Type	Strike	Expiration
1	1.000000000	R1.859415	Call	100	1
2	-8.837837542	R0.287709	Call	120	1
3	3.890424480	R0.241990	Call	120	0.95
4	1.327375803	R0.200330	Call	120	0.9
5	0.656693661	R0.162811	Call	120	0.85
6	0.389877109	R0.129484	Call	120	0.8
7	0.257883233	R0.100372	Call	120	0.75
8	0.183212213	R0.075448	Call	120	0.7
9	0.136945712	R0.054633	Call	120	0.65
10	0.106325546	R0.037778	Call	120	0.6
11	0.085020496	R0.024654	Call	120	0.55
12	0.069601942	R0.014939	Call	120	0.5
13	0.058083321	R0.008210	Call	120	0.45
14	0.049249791	R0.003953	Call	120	0.4
15	0.042324909	R0.001580	Call	120	0.35
16	0.036793804	R0.000480	Call	120	0.3
17	0.032304231	R0.000094	Call	120	0.25
18	0.028608649	R0.000009	Call	120	0.2
19	0.025529000	R0.000000	Call	120	0.15
20	0.022934557	R0.000000	Call	120	0.1
21	0.020727595	R0.000000	Call	120	0.05

Under constant volatility of 20%, the results are as displayed in Table 19.

Table 19: Statistics of the DEK model under constant volatility

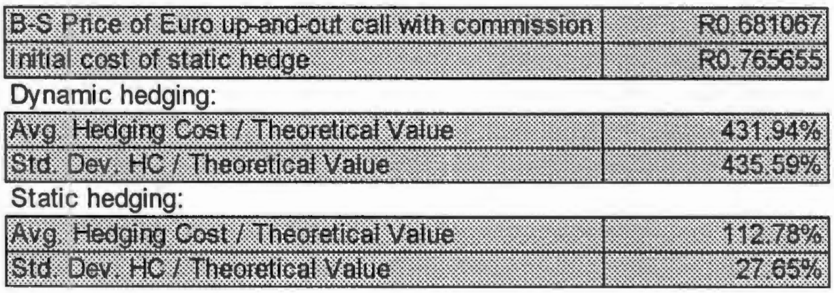


B-S Price of Euro up-and-out call with commission	R0.681067
Initial cost of static hedge	R0.765655
Dynamic hedging:	
Avg. Hedging Cost / Theoretical Value	420.51%
Std. Dev. HC / Theoretical Value	421.12%
Static hedging:	
Avg. Hedging Cost / Theoretical Value	113.26%
Std. Dev. HC / Theoretical Value	28.92%

Note that the initial cost of the static hedge includes commission and that the "Theoretical Value" in the dynamic and static hedge statistics corresponds to 0.681067 and 0.765655 respectively.

If stochastic volatility is instead considered, the resulting hedging statistics are as given in Table 20.

Table 20: Statistics of the DEK model under stochastic volatility



B-S Price of Euro up-and-out call with commission	R0.681067
Initial cost of static hedge	R0.765655
Dynamic hedging:	
Avg. Hedging Cost / Theoretical Value	431.94%
Std. Dev. HC / Theoretical Value	435.59%
Static hedging:	
Avg. Hedging Cost / Theoretical Value	112.78%
Std. Dev. HC / Theoretical Value	27.65%

Tables 19 and 20 indicate that dynamic delta hedging performs poorly under both constant and stochastic volatility when transaction costs are considered. Not only is the hedging cost far greater than the Black-Scholes price, but the standard deviation is also very large, indicating large variability in the hedging costs. The dynamic delta hedging results can partly be explained by noting that the delta of an up-and-out call changes dramatically as the share price approaches the knock-out barrier level. This results not only in a large change in the required number of shares held, but as a consequence, also results in large transaction costs being incurred.

The same cannot be said for the DEK static hedging scheme. Hedging costs are only slightly greater than that indicated by the theoretical DEK price, while the standard deviation is only of moderate size. One would be inclined to conclude that the DEK is preferable for the up-and-out call under consideration. This optimism is dampened though if the reader recalls the possible hedging errors incurred under such a static hedge (especially close to maturity, see Figure 8). However, it is the writer's opinion that the static hedge is still preferable and performs better than dynamic delta hedging.

It is interesting to see that the static hedge seems to perform better under stochastic volatility than constant volatility. This can be explained by first noting that the transaction costs at inception will be identical under constant and stochastic volatility (because the volatility is assumed to be 20% at inception under both). Transaction costs under these two approaches only differ when the up-and-out call option knocks out, at which stage the underlying share price is at its highest. The stochastic volatility will on average then be lower than 20% (the value of volatility at inception) due to the negative correlation between the share price and stochastic volatility process. The value of the individual hedging options (and by implication the hedging portfolio as a whole) at knock-out will therefore be of smaller magnitude than under constant volatility. Transaction costs will as a result also be smaller.

9.2.3 *Evaluating the CEG model*

The same up-and-out call option was considered with the same parameters as those given in Table 17, with the only difference being that the risk-free interest rate and dividend yields were set to 0%. Tables 22 and 23 display the hedging statistics in the case of the CEG model, while Table 21 displays the details of the portfolio of vanilla options used in the static hedge.

Table 21: Details of the hedging portfolio under the CEG model

Option Number	Weight	Price at time 0	Option Type	Strike	Expiration
1	1.00000	R1.18592	Call	100	1
2	-0.83333	R0.01011	Call	144	1
3	-240.00000	R0.15398	Call	120	1
4	20.00000	R0.13812	Call	121	1
5	-320.00000	R0.14585	Call	120.5	1
6	540.00000	R0.14851	Call	120.333333	1
7	-0.16667	R0.15398	Call	120	1

Note that the absolute sizes of the option hedging positions under the CEG model are much larger than those under the DEK model.

Table 22: Statistics of the CEG model under constant volatility

B-S Price of Euro up-and-out call with commission	R0.486394
Initial cost of static hedge	R1.322970
Dynamic hedging:	
Avg. Hedging Cost / Theoretical Value	536.06%
Std. Dev. HC / Theoretical Value	595.98%
Static hedging:	
Avg. Hedging Cost / Theoretical Value	542.19%
Std. Dev. HC / Theoretical Value	937.49%

Table 23: Statistics of the CEG model under stochastic volatility

B-S Price of Euro up-and-out call with commission	R0.486394
Initial cost of static hedge	R1.322970
Dynamic hedging:	
Avg. Hedging Cost / Theoretical Value	594.82%
Std. Dev. HC / Theoretical Value	740.62%
Static hedging:	
Avg. Hedging Cost / Theoretical Value	512.34%
Std. Dev. HC / Theoretical Value	894.17%

Tables 22 and 23 show that while dynamic delta hedging performs poorly, the CEG static hedging scheme does worse. Not only is the average hedging cost much greater than the theoretical price or value, but the standard deviation of the hedging costs is also extremely large. While the static hedge may on average perform to a similar extent to that of dynamic hedging, greater variability of the static hedging costs makes it less attractive.

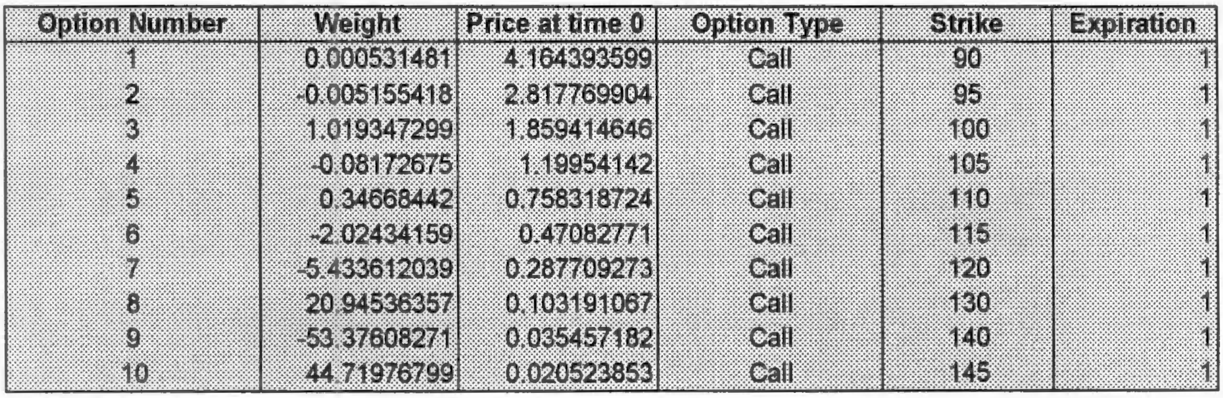
The proportional commission structure seems to make the CEG static hedging scheme impractical. This is not only due to the absolute size of the weights of the option positions held – it is also due to the nominal values of these positions when the portfolio is liquidated (at knock-out when the call options are in the money).

9.2.4 *Evaluating the Dupont model*

Again, the exact up-and-out call detailed in Table 17 was considered under the Dupont (mean-square) hedging methodology.

Table 24 below lists the details of the 10 different option positions used in the static hedge.

Table 24: Details of the hedging portfolio under the Dupont model



Option Number	Weight	Price at time 0	Option Type	Strike	Expiration
1	0.000531481	4.164393599	Call	90	1
2	-0.005155418	2.817769904	Call	95	1
3	1.019347299	1.859414646	Call	100	1
4	-0.08172675	1.19954142	Call	105	1
5	0.34668442	0.758318724	Call	110	1
6	-2.02434159	0.47082771	Call	115	1
7	-5.433612039	0.287709273	Call	120	1
8	20.94536357	0.103191067	Call	130	1
9	-53.37608271	0.035457182	Call	140	1
10	44.71976799	0.020523853	Call	145	1

Compared to the CEG model, the notional positions of the hedging portfolio are far smaller in absolute terms. They are, however, greater than those of the DEK model. In addition, the option positions with large weights are most likely to be out of the money at knockout, reducing the effect of transaction costs when compared with the CEG model. It is therefore not unexpected to observe that Dupont static hedging model performs better than the CEG model, but worse than the DEK static hedging model.

Table 25: Statistics of the Dupont model under constant volatility

B-S Price of Euro up-and-out call with commission	R0.681067
Initial cost of static hedge	R0.766959
Dynamic hedging:	
Avg. Hedging Cost / Theoretical Value	415.49%
Std. Dev. HC / Theoretical Value	435.61%
Static hedging:	
Avg. Hedging Cost / Theoretical Value	126.30%
Std. Dev. HC / Theoretical Value	61.48%

Table 26: Statistics of the Dupont model under stochastic volatility

B-S Price of Euro up-and-out call with commission	R0.681067
Initial cost of static hedge	R0.766959
Dynamic hedging:	
Avg. Hedging Cost / Theoretical Value	428.23%
Std. Dev. HC / Theoretical Value	440.20%
Static hedging:	
Avg. Hedging Cost / Theoretical Value	123.82%
Std. Dev. HC / Theoretical Value	57.45%

While the average hedging cost is reasonably close to the theoretical value indicating that the hedging strategy may be adequate, a fairly large standard deviation measure negates this.

9.2.5 Evaluating the AAE model

The same up-and-out call option was considered with identical parameters as those given in Table 17, but with the risk-free interest rate and dividend yields set to 0%. Tables 28 and 29 display the hedging statistics in the case of the AAE model.

Table 27: Details of the hedging portfolio under the Dupont model

Option Number	Weight	Price at time 0	Option Type	Strike	Expiration
1	1.000000000	R1.185924143	Call	100	1.0000
2	-1.000000000	R0.153981936	Call	120	1.0000
3	2000.000000000	R0.153815049	Call	120.01	1.0000
4	-2000.000000000	R0.153981936	Call	120	1.0000
5	-4.358067862	R0.000000000	Call	120	0.0510
6	-5.079093945	R0.000000123	Call	120	0.1510
7	-6.022711396	R0.000056781	Call	120	0.2510
8	-7.298528715	R0.000921114	Call	120	0.3510
9	-9.096829423	R0.004693259	Call	120	0.4510
10	-11.775273201	R0.013883853	Call	120	0.5510
11	-16.082742071	R0.030380187	Call	120	0.6510
12	-23.853585951	R0.055195453	Call	120	0.7510
13	-40.917727873	R0.088629669	Call	120	0.8510
14	-102.883448960	R0.130508606	Call	120	0.9510
15	4.358067862	R0.000000000	Call	120	0.0490
16	5.079093945	R0.000000101	Call	120	0.1490
17	6.022711396	R0.000052587	Call	120	0.2490
18	7.298528715	R0.000883960	Call	120	0.3490
19	9.096829423	R0.004572454	Call	120	0.4490
20	11.775273201	R0.013633255	Call	120	0.5490
21	16.082742071	R0.029970772	Call	120	0.6490
22	23.853585951	R0.054614747	Call	120	0.7490
23	40.917727873	R0.087877078	Call	120	0.8490
24	102.883448960	R0.129590884	Call	120	0.9490

Note that from Table 27 above, hedging options 3, 4, 14 and 24 all have large weights. In the case of options 3 and 4, the values are far greater than any of the portfolios in the previous hedging methodologies.

Table 28: Statistics of the AAE model under constant volatility

B-S Price of Euro up-and-out call with commission	R0.486394
Initial cost of static hedge	R3.822815

Dynamic hedging:

Avg. Hedging Cost / Theoretical Value	586.32%
Std. Dev. HC / Theoretical Value	714.91%

Static hedging:

Avg. Hedging Cost / Theoretical Value	597.90%
Std. Dev. HC / Theoretical Value	1138.69%

Table 29: Statistics of the AAE model under stochastic volatility

B-S Price of Euro up-and-out call with commission	R0.486394
Initial cost of static hedge	R3.822815
Dynamic hedging:	
Avg. Hedging Cost / Theoretical Value	548.93%
Std. Dev. HC / Theoretical Value	606.39%
Static hedging:	
Avg. Hedging Cost / Theoretical Value	580.98%
Std. Dev. HC / Theoretical Value	1142.78%

These large weights translate into large hedging costs as observed in Tables 28 and 29 above. The static hedging performs worse than the delta hedging scheme and is impractical to use.

10 SUMMARY AND CONCLUSION

This paper reviewed the static hedging of barrier options. In particular, the static hedging of the European up-and-out call option under four models was examined. These four models were: Carr, Ellis and Gupta (1998) (CEG); Derman, Ergener and Kani (1995) (DEK); Dupont (2002) (mean-square hedging) and Andersen, Andreasen and Eliezer (2002) (AAE). All four methods derive static hedging portfolios of vanilla options that replicate the payoff of the up-and-out call option. While none of these models is able to replicate the payoff of the call exactly under all conditions, their approximations converge (under the assumptions made) to a perfect replicating strategy as the number of different vanilla hedging options available increases.

However, none of these models attempts to include transaction costs in the modelling assumptions. When the performance of these hedging strategies is evaluated under conditions similar to those found in financial markets (realised volatility different to that used to construct the hedge, and transaction costs), only one of the four methods performs adequately. Since the DEK model uses a hedging portfolio with reasonably small notional positions in vanilla options, it is not *that* affected by the transaction costs structure. This implies that the model is the most appropriate of the four for a static hedging scheme. Not only does it perform much better than dynamic delta hedging in terms of average hedging cost, but its variability is also within reasonable bounds. However, the suitability of the DEK model is limited by certain factors: the availability of the required range of notional positions in vanilla options and the potentially large hedging errors close to maturity. The CEG, Dupont and AAE models, on the other hand, are impractical to implement because they rely on portfolios containing large notional positions.

Based on these limitations of the four models discussed in this paper, further research into static hedging (taking transaction costs and share price processes with jumps into account) is warranted.

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