Power domination in graphs

by

Dean Reagon

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Supervisor: Dr Imran Allie
Co-supervisor: Prof David Erwin

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Declaration

I hereby declare that this thesis is my own work, that it has not been submitted for any degree or examination at any other academic institution, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Dean Reagon March 2025

Signed: Signed by candidate

Abstract

Domination in graphs has been studied since the 1800s. Many parameters related to domination have been defined and studied since then. Power domination was first defined and studied in the early 2000s. It is an abstraction of how an electrical power system is monitored. In this thesis, we focus on power domination and its natural extension k-power domination. We also look at the propagation radius, which is essentially the number of steps it takes a power dominating set of vertices to monitor a graph. We show how ideas from domination can be extended to power domination and k-power domination. We show how domination differs from k-power domination. We demonstrate how making small changes to a graph affects the domination and power domination number of the graph. The small graph changes we will present are: vertex removal, edge removal and edge contraction. We present upper and lower bounds for how much the power domination number can change and present examples that reach all of these bounds. We present a general bound on the power domination and k-power domination number of connected graphs. We also present a general bound on the power domination number of connected, claw-free, cubic graphs. In all cases we present examples that reach these bounds. We show that finding a power dominating set of a tree is equivalent to finding a spider partition of that tree. We also present a lower bound on the power domination number of a tree with respect to its number of branching vertices. We present an upper bound on the power domination radius with regards to the smallest degree of a vertex in a graph. We also present infinitely many graphs that reach this bound.

Contents

Chapter 1

Introduction

In its essence, domination theory has been studied since the 1800s, specifically in relation to problems of how to arrange pieces on a chess board [6]. However, domination was only properly defined as a graph theoretical concept in 1958 [6]. Domination in graphs has grown into its own field within graph theory since then, with thousands of papers having been published on the subject to date.

A graph is a collection of vertices and edges connecting vertices. A vertex is said to dominate itself and its neighbouring vertices. A set of vertices is said to dominate itself and its neighbouring vertices. A dominating set is a set of vertices which dominates all the vertices in a graph. The domination number of a graph is the minimum cardinality of a dominating set of the graph. A core theme of domination theory is finding the domination number exactly or else finding upper and lower bounds on the domination number for certain classes of graphs.

Domination theory has since branched out into many different sub fields, with many new variants of domination having been defined and investigated. Some examples include total domination, roman domination and eternal domination [13]. All of these new variants of domination change in some way how a set of vertices dominate vertices in a graph. One of the newer types of domination variants is called power domination. Power domination was originally conceived to formalise the problem of how to optimally monitor an electrical network by placing a minimum number of measurement units in the network at various substations. It was first rigorously defined as a graph theoretical concept by Haynes et al. in [15].

Given an electrical network, we can define an underlying graph by letting the substations be vertices and letting there be an edge between vertices if there are power lines connecting the corresponding substations. The placement of phasor measurement units (PMUs) at substations allow for the monitoring of the network. The PMU can monitor the substation at which it is placed as well as any adjacent substations. Then using the Ohm and Kirchhoff current laws, it is then possible to use knowledge of the currently monitored stations to monitor more substations in the network if certain conditions are met. In the language of graphs, we say a set of vertices dominates its closed neighbourhood just as in normal domination. However, unlike normal domination, the dominated or *monitored* vertices can then monitor their neighbours in a process called propagation. The process repeats until the rules of propagation do not allow for more vertices to be monitored. If the whole graph is monitored at the end of this process, then the original set of vertices is a power dominating set of the graph. The power domination number of a graph is the minimum cardinality of a power dominating set of the graph.

In general there will be multiple propagation steps before the entire graph is monitored. A natural question to ask is how many steps it takes a power dominating set to monitor the graph, and what is the smallest number of steps it takes any power dominating set to monitor the graph. The answer is trivially one if we take our power dominating set to be all the vertices in the graph. We therefore restrict our set to be a minimum power dominating set. To solve this we need the notion of the propagation radius. The propagation radius for a given graph is the smallest number of steps it takes for a minimum power dominating set to monitor the whole graph.

In this thesis, we will begin by exploring some fundamental properties of minimal and minimum dominating sets, as well as show some upper and lower bounds on the domination number. We will show how these ideas can be extended to power domination and also reveal how power domination differs from normal domination. We will also look at k-power domination, which is a natural extension of power domination. We will reproduce proofs of theorems from various sources and will provide some original proofs of some lemmas and theorems. This thesis will be split into several chapters which will look at various aspects of domination and power domination.

In Chapter 2, we present some preliminary definitions and results in graph theory that will be used in this thesis. This includes all the basic graph theory necessary for the definitions and results in domination and power domination. We also introduce some basic results from domination in graphs. We define some useful properties of minimal and minimum dominating sets and present some upper and lower bounds on the domination number in terms of various graph parameters.

In Chapter 3, we introduce power domination and look at how it is both similar and different to domination. We will also see what upper and lower bounds there are for the power domination number and will extend these ideas to k-power domination using our own proofs.

In Chapter 4, we introduce a general bound on the power domination and k-power domination number of connected graphs. We also show a general bound on the power domination number of connected, claw-free, cubic graphs. In all cases we characterise the family of graphs that reach these bounds.

In Chapter 5, we demonstrate how making small changes to a graph affects the domination and power domination number of the graph. The small graph changes we will present are: vertex removal, edge removal and edge contraction. We give upper and lower bounds for how much the power domination number can change and present examples that reach all of these bounds.

In Chapter 6, we do an in depth exploration of power domination in trees and unicyclic graphs. We show that finding a power dominating set of a tree is equivalent to finding a spider partition of that tree. We present a lower bound on the power domination number in trees and present an infinite family that shows this bound is sharp. We then present some original results on the power domination number in unicyclic graphs.

In Chapter 7, we present some properties and novel results regarding the power domination radius. We determine an upper bound on the power domination radius in terms of the minimum degree of a vertex in a graph. We present some infinite families of graphs that show that this bound is sharp.

In Chapter 8, we conclude with a brief discussion.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter we will introduce some basic concepts in graph theory and domination theory. We will define and explain the definitions and operations that are used throughout the rest of this thesis. We start by defining a graph and the basic properties that all graphs have. Next we will formerly define domination as well as a dominating set and the domination number. We then present some results on minimal dominating sets and some bounds on the domination number in terms of various graph parameters.

2.2 Basic concepts in graph theory

Definition 2.2.1 (Graph)**.** A *graph* G is a pair $G = (V(G), E(G))$, where $V(G)$ is a non-empty set of elements called *vertices* and $E(G)$ is a set of 2-element subsets of $V(G)$, called *edges*.

If there is no chance of confusion, then we may write V and E instead of $V(G)$ and $E(G)$. An edge $\{u, v\} \in E$ is denoted by uv, where $u, v \in V$. The *order* of G is the number of vertices in G. The *size* of G is the number of edges in G.

Definition 2.2.2. Let $G = (V, E)$ be a graph. Let $u, v \in V$ and $uv \in E$.

(i) We say that uv *joins* u and v, the edge uv is *incident* with u and v, u and v are *adjacent*, and u is a *neighbour* of v (and vice versa).

(ii) The *neighbourhood* of $u \in V$, denoted as $N(u)$, is the set of all vertices in V that are adjacent to u. If u has no neighbours in V, then u is *isolated* in G. that is,
$$N(u) = \{v \in V : uv \in E\}.$$

(iii) The *closed neighbourhood* of u, denoted $N[u]$, is the union of the neighbourhood of u and u itself. That is,
$$N[u] = N(u) \cup \{u\}.$$

(iv) Let $S \subseteq V$. Then the *neighbourhood of* S is the union of the neighbourhood of every vertex in S and the *closed neighbourhood* of S is the union of the closed neighbourhood of every vertex in S. That is,
$$N(S) = \bigcup_{v \in S} N(v), \text{ and}$$
$$N[S] = \bigcup_{v \in S} N[v].$$

Let G be a graph and $v \in V(G)$. We note that, by definition, $v \notin N(v)$ and $v \in N[v]$. Then $N(v) \neq N[v]$ for any $v \in V(G)$. However, the same is not always true for $S \subseteq V$. Indeed if every vertex $v \in S$ has a neighbour $u \in S$ then, $N(S) = N[S]$.

Definition 2.2.3 (Subgraph)**.** Let G and H be graphs. We say that H is a *subgraph* of G, denoted as $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 2.2.4 (Induced subgraph)**.** Let G be a graph and $S \subseteq V(G)$. The subgraph of G *induced* by S, denoted by $G[S]$, is defined as follows: $V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G) : u \in S \text{ and } v \in S\}$.

Similarly, we can define subgraphs induced by subsets of edges. Let G be a graph and $S \subseteq E(G)$. The subgraph of G induced by S, denoted by $G[S]$, is defined as follows: $V(G[S]) = \{v \in V(G) : \text{there exists some edge } uv \in S\}$ and $E(G[S]) = S$.

Definition 2.2.5 (Degree)**.** Let G be a graph. The *degree* of a vertex $v \in V$, denoted as $deg_G(v)$, is the number of vertices adjacent to v. That is, $deg_G(v) = |N(v)|$. When there is no chance of confusion we write $deg(v)$ instead of $deg_G(v)$. We say G is *regular* if every vertex has the same degree. We say G is *d-regular* if every vertex has degree d.

Definition 2.2.6. Let G be a graph. The *minimum degree* of G, denoted $\delta(G)$, is defined as $\delta(G) = \min\{deg(v) : v \in V\}$. The *maximum degree* of G, denoted $\Delta(G)$, is defined as $\Delta(G) = \max\{deg(v) : v \in V\}$. When there is no chance of confusion we write δ and Δ instead of $\delta(G)$ and $\Delta(G)$.

Definition 2.2.7. Let G be a graph.

(i) A *walk* is a sequence of vertices $v_0, ..., v_k$ such that $v_i v_{i+1} \in E$ for $i \in \{0, ..., k-1\}$, and is referred to as a $v_0 - v_k$ walk. The length of the walk $v_0, ..., v_k$ is k. If $v_0 = v_k$, then we say the walk is *closed*.

(ii) A *trail* is a walk in which every edge is distinct.

(iii) A *path* is a trail in which every vertex is distinct.

(iv) A *cycle* is a walk $v_0, ..., v_k$ in which every vertex in $\{v_0, ..., v_{k-1}\}$ is distinct and $v_0 = v_k$. We denote by C_n the graph containing only a cycle of length n.

Remark 2.2.8. Let G be a graph with a $u - v$ walk. Then it is well known that the $u - v$ walk contains a $u - v$ path.

We consider three of the most common ways to make a small change to a graph, namely: edge removal, vertex removal and edge contraction. These are the smallest amount of change you can do to a graph while removing an edge, removing a vertex or identifying two vertices respectively. When studying these it will also provide insight on how domination and power domination behave differently with respect to these changes.

Definition 2.2.9 (Vertex removal)**.** Let G be a graph with $v \in V$. Then $G - v$ is the subgraph of G obtained by removing vertex v and edges incident to v from G.

Definition 2.2.10 (Edge removal)**.** Let G be a graph with $uv \in E$. Then the graph $G - uv$ is defined as: $V(G - uv) = V(G)$ and $E(G - uv) = E(G) - \{uv\}$.

Definition 2.2.11 (Edge contraction)**.** Let G be a graph with $uv \in E$. Then $G \sim uv$

is the graph obtained by removing the edge uv and identifying u and v into a single vertex that we denote w. Let $G' = G - uv$. Then, $G \sim uv$ is defined as,

$$V(G \sim uv) = (V(G) \cup \{w\}) - \{u, v\}, \text{ and}$$

$$E(G \sim uv) = E(G - \{u, v\}) \cup \{xw : x \in N_G(\{u, v\}) - \{u, v\}\}.$$

An example of the various small graph changes described is presented in Figure 2.1.

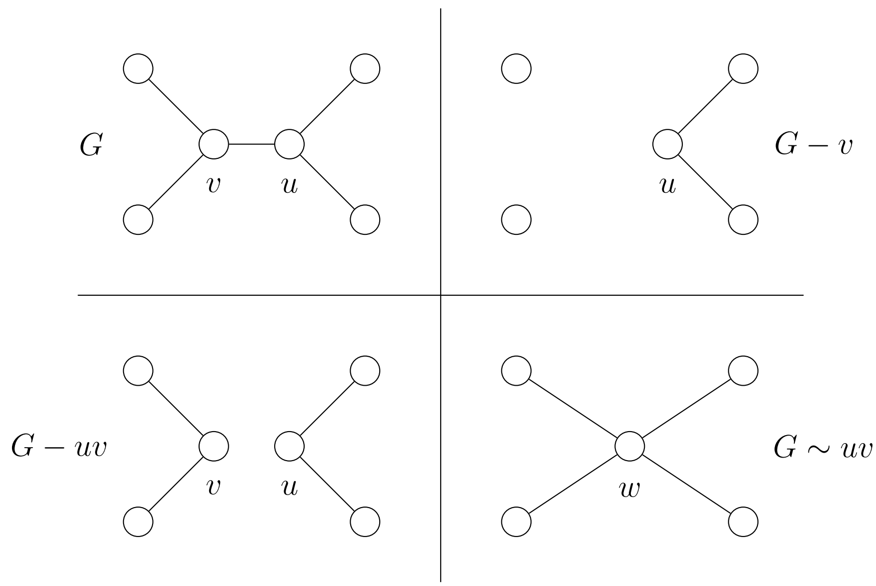

Figure 2.1: from left to right and top to bottom we have: G, $G - v$, $G - uv$, and $G \sim uv$.

Definition 2.2.12. Let G and H be graphs. The *union* of G and H, denoted as $G \cup H$, is the graph defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

Definition 2.2.13 (Connected graph)**.** Let G be a graph. Then G is *connected* if there exists a path from u to v for all $u, v \in V$. If G is not connected, then it is *disconnected*. A *component* of G is a maximal connected subgraph of G. We denote the number of components of G as $\omega(G)$.

Note that if $V(G) \cap V(H) = \emptyset$ then G and H are said to be disjoint, and the union would be a disconnected graph. We also note that every graph is the union of its components. Thus every graph is the disjoint union of connected graphs. Consequently, connected graphs are fundamental, as their study often provides a straightforward path to generalising results to disconnected graphs.

Definition 2.2.14 (complement of a graph)**.** Let G be a graph. The *complement* of G, denoted as $\overline{G}$, is the graph defined by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : u, v \in V(G) \text{ and } uv \notin E(G)\}$.

An example of a graph and its complement are shown in figure 2.2.

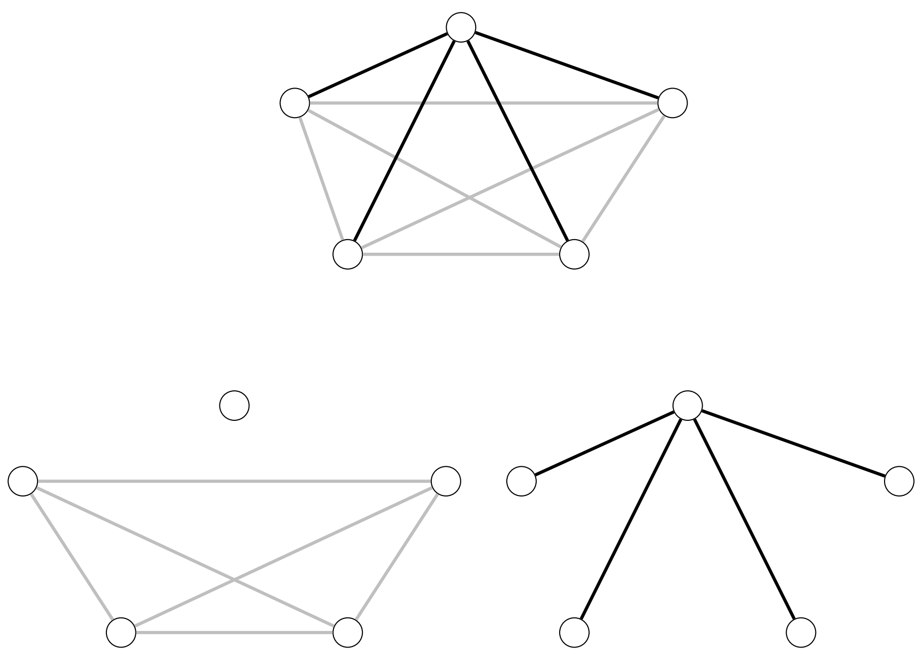

Figure 2.2: An example of complements. The top graph is the union of the two graphs below. The graphs on the bottom left and right are complements of each other.

Definition 2.2.15 (Graph isomorphism)**.** Let G and H be graphs. Let f be a mapping $f : V(G) \to V(H)$. Then f is an *isomorphism* from G to H if it has the following properties:

(i) The mapping f is bijective.

(ii) The edge $f(u)f(v) \in E(H)$ for every $uv \in E(G)$.

(iii) The edge $f(u)f(v) \notin E(H)$ for every $uv \notin E(G)$.

We say G and H are *isomorphic* if there exists an isomorphism from G to H.

Distance in graphs is a well studied subject and has many applications in all areas of graph theory. The distance between any two vertices is the length of a shortest path between them. Distance is useful because it gives us a structured way of talking about how close two vertices are to each other. This is necessary because often it is helpful to know what is the minimum or maximum distance between two vertices with a certain property. In many instances, our goal is to optimize the distance between vertices, either by maximizing or minimizing it.

Definition 2.2.16. Let G be a connected graph and let $u, v \in V(G)$. The *distance* from u to v, denoted by $d(u, v)$, is defined to be the minimum length of a $u-v$ path. A $u-v$ path of minimum length is called a shortest $u-v$ path or a $u-v$ *geodesic*.

Definition 2.2.17. Let G be a connected graph and let $u, v \in V(G)$.

(i) The *eccentricity* of u is defined as $e(u) = \max\{d(u, v) : v \in V(G)\}$. That is, the furthest distance from any vertex in the graph to u.

(ii) The *radius* of G is defined as $rad(G) = \min\{e(u) : u \in V(G)\}$.

(iii) The *diameter* of G is defined as $diam(G) = \max\{e(u) : u \in V(G)\}$.

Definition 2.2.18. Let G be a connected graph. A *central vertex* of G is a vertex u such that $e(u) = rad(G)$. The *centre* of G is the subgraph of G induced by the set of central vertices $\{v \in V(G) : e(v) = rad(G)\}$.

Note that a graph which contains no cycles is called *acyclic*.

Definition 2.2.19. A *forest* is a graph that is acyclic. A *tree* is a connected forest.

Note that every component of a forest is a tree.

2.3 General classes of graphs

Given the diverse structural variations among graphs, generalizing theorems to encompass all graphs within graph theory can be challenging. To achieve more robust results, it is often beneficial to focus on specific graph classes. This approach underscores the significance of classifying graphs into distinct categories. Furthermore, many graph classes exhibit unique properties that prove advantageous when studying domination and power domination.

Definition 2.3.1 (Complete graph)**.** Let G be a graph. We say that G is *complete* if every pair of vertices in $V(G)$ are adjacent. We denote by K_n, the *complete graph* with n vertices.

Definition 2.3.2 (Empty graph)**.** Let G be a graph. We say that G is *empty* if it has no edges. The *empty graph* with n vertices is the complement of the complete graph with n vertices. We therefore denote by $\overline{K_n}$, the empty graph with n vertices.

Definition 2.3.3 (Bipartite graph)**.** Let G be a graph. Then G is *bipartite* if there exists a partition of its vertices into two sets V_1 and V_2, such that the graphs $G[V_1]$ and $G[V_2]$ are both empty. If V_1 and V_2 are such that $|V_1| = i$, $|V_2| = j$ and every vertex in V_1 is adjacent to every vertex in V_2, then G is a *complete bipartite graph* denoted as $K_{i,j}$.

The following class of graphs, spiders, will be considered in particular in the study of power domination in trees and unicyclic graphs.

Definition 2.3.4 (Spider graph)**.** A graph G is a *spider* if it is a tree and it has no more than one vertex with degree higher than 2. This high degree vertex is called the *root*. If there is no vertex of degree 3 or more then any vertex can be the root.

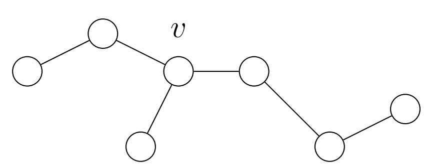

Figure 2.3: An example of a spider. The root is v.

Definition 2.3.5 (Wheel graph)**.** Let G be a graph with order $n+1$. Then G is a *wheel graph* if there exists a vertex $v \in V(G)$ such that $deg(v) = n$ and $G[V(G) - \{v\}]$

is isomorphic to C_n. The wheel graph with $n+1$ vertices is denoted as W_n. That is,

$$V(W_n) = V(C_n) \cup \{v\}, \text{ and}$$

$$E(W_n) = E(C_n) \cup \{uv : u \in V(C_n)\}.$$

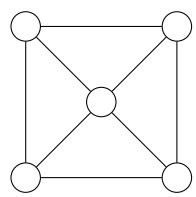

Figure 2.4: The wheel W_4.

2.4 Domination in graphs

In this section, we utilize fundamental graph theoretic concepts to define and explore domination theory. This serves as a foundational framework for power domination theory, as numerous results from domination theory can be extended to power domination theory and subsequently to k-power domination theory. We start by defining dominating sets and domination.

Definition 2.4.1 (Dominating set)**.** Let G be a graph and let $S \subseteq V(G)$. We say that S *dominates* $N[S]$. We say that S is a *dominating set* of G if $N[S] = V(G)$.

Definition 2.4.2 (Domination number)**.** Let G be a graph. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. A minimum dominating set of G is a dominating set of G with minimum cardinality. If S is a minimum dominating set of G, we will call it a $\gamma(G)$-set.

Definition 2.4.3 (Minimal dominating set)**.** Let G be a graph and S a dominating set of G. We say that S is a *minimal dominating set* of G if there is no proper subset of S that is a dominating set of G.

We note that all $\gamma(G)$-sets are minimal dominating sets, but not all minimal dominating sets are $\gamma(G)$-sets.

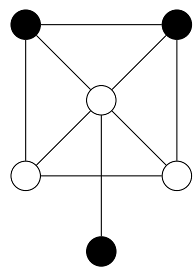

Figure 2.5: In the graph, the black vertices form a minimal dominating set which is not a minimum dominating set.

When discussing minimal dominating sets, it is beneficial to precisely characterise the properties necessary for a set to be a minimally dominating set. One approach is to identify the unique contribution of each vertex within a dominating set. As every vertex in the dominating set dominates its closed neighbourhood, we refer to its unique contribution as its private neighbourhood. This concept enables us to characterise all minimal dominating sets.

Definition 2.4.4 (Private neighbour)**.** Let G be a graph, $S \subseteq V(G)$ and $v \in S$. The *private neighbourhood of* v *with respect to* S, denoted $pn(v, S)$, is the set $N[v] \cap N[S - v]$. A vertex $w \in pn(v, S)$ is a *private neighbour of* v *with respect to* S. To be concise, we write that w is a S-*pn* of v. We say that w is an *external* S-*pn* of v if $w \in pn(v, S)$ and $w \neq v$.

The following are well known essential results in domination theory.

Theorem 2.4.5 ([6])**.** *Let G be a graph and S a dominating set of G. Then S is a minimal dominating set of G if and only if every $v \in S$ has at least one of the following two properties:*

1. *The vertex v has an external S-pn in G.*

2. *The vertex v is an isolated vertex in $G[S]$.*

Proof. Let S be a minimal dominating set of G. Let $v \in S$ not have an external S-pn. Let $S' = S - \{v\}$ then, we know that $N[S - v] \supseteq N[S] - N[v] = V - N[v]$. So S' dominates at least $V - N[v]$.

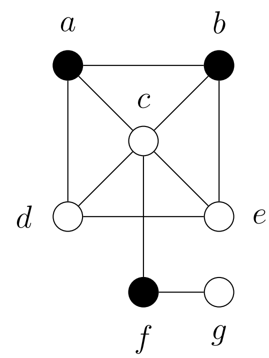

Figure 2.6: An example of a private neighbour. The subset $S = \{a, b, f\}$ is a minimal dominating set. We see that $pn(a, S) = N[a] - N[S - a] = N[a] - N[b, f] = \{d\}$, $pn(b, S) = \{e\}$, and $pn(f, S) = N[f] - N[S - \{f\}] = N[f] - N[a, b] = \{f, g\}$. We see from f that it is possible for a vertex to be its own private neighbour. We also see that the external private neighbour of f is g.

Since v has no external S-pn's we have that $N[v] - N[S - \{v\}] \subseteq \{v\}$. This implies that $N(v) \subseteq N[S - v]$. Thus, S' dominates at least $(V - N[v]) \cup N(v) = V - \{v\}$.

Since S is a minimal dominating set, S' does not dominate V. Therefore, S' does not dominate v. Thus, $v \notin N[S']$ and so v is isolated in $G[S]$. It follows that for any $v \in S$, either v is isolated in $G[S]$ or v has an external S-pn.

Now, conversely, let S be a dominating set of G such that for all $v \in S$, v has at least one of the two properties. Then there are two cases.

1. *Let $v \in S$ have an external S-pn, called w.* Then $S' = S - \{v\}$ does not dominate w and so S' is not a dominating set.
2. *Let $v \in S$ be an isolated vertex in $G[S]$.* Then $S' = S - \{v\}$ does not dominate v and so S' is not a dominating set.

Since every vertex in S has at least one of these properties, $S - \{v\}$ is not a dominating set for any $v \in S$. Therefore, S must be a minimal dominating set of G. □

Theorem 2.4.6 ([6]). *Let G be a graph without isolated vertices and let S be a minimal dominating set of G. Then $V(G) - S$ is a dominating set of G.*

Proof. By Theorem 2.4.5, every vertex $v \in S$ either has an external S-pn or v is isolated in $G[S]$. Let $v \in S$ have an external S-pn. Then v is adjacent to $V(G) - S$. Therefore, v is dominated by $V(G) - S$. Now instead let v be isolated in $G[S]$. Since G has no isolated vertices, v must be adjacent to some vertex in $V(G) - S$. Thus, every vertex in S is adjacent to a vertex in $V(G) - S$. Therefore, $N[V(G) - S] = (V(G) - S) \cup S = V$, so $V(G) - S$ is a dominating set of G. □

Corollary 2.4.7 ([6])**.** *Let G be a graph with order n and no isolated vertices. Then $\gamma(G) \leq \frac{n}{2}$.*

Proof. Let S be a minimal dominating set of G. Then by Theorem 2.4.6, $V(G) - S$ is a dominating set of G. Then $\gamma(G) \leq \min\{|S|, |V(G) - S|\}$. At least one of S or $V(G) - S$ has cardinality less than or equal to $\frac{n}{2}$. Therefore, $\gamma(G) \leq \frac{n}{2}$. □

Theorem 2.4.8 ([6])**.** *Let G be a graph without isolated vertices. Then, G contains a minimum dominating set S such that every vertex $v \in S$ has at least one external S-pn.*

Proof. Assume to the contrary that there is no such minimum dominating set of G. Let S be a minimum dominating set of G such that the size of $G[S]$ is maximised. Then, by our assumption, there exists a vertex $v \in S$ such that v has no external S-pn. Then, by Theorem 2.4.5, v must be isolated in $G[S]$. Since G has no isolated vertices, there must be a vertex $u \in V(G) - S$ that is adjacent to v. Since v has no external S-pn, we can say that $S - \{v\}$ dominates $G - v$. Let $S' = (S - \{v\}) \cup \{u\}$. Since u and v are adjacent, S' dominates $(G - v) \cup v = G$. Therefore, S' is a dominating set and since $|S'| = |S|$, S' is a minimum dominating set. Since v has no external S-pns, u must be adjacent to at least one other vertex in S. Therefore, the size of $G[S']$ is at least one more than the size of $G[S]$. But this contradicts that $G[S]$ is maximised. Therefore, our assumption is incorrect and there exists a minimum dominating set S such that every $v \in S$ has at least one external S-pn. □

We now present a result of our own, which extends Theorem 2.4.8.

Corollary 2.4.9. *Let G be a connected graph with order $n > 2$. Then, there exists a $\gamma(G)$-set, S, such that every vertex $v \in S$ has degree at least two and at least one external S-pn.*

Proof. From Theorem 2.4.8 we know there exists a $\gamma(G)$-set, S, such that every $v \in S$ has at least 1 external S-pn. Let $\{v_1, v_2, ...v_k\}$ be the set of all vertices in S with degree 1 and let $\{u_1, ..., u_k\}$ be the set of vertices such that $u_i v_i \in E(G)$ for all $i \in \{1, ..., k\}$.

For all $i \in \{1, ..., k\}$, v_i has degree one and one external S-pn, so u_i must be the external S-pn of v_i and therefore $u_i \notin S$. Therefore, $S \cap \{u_1, ..., u_k\} = \emptyset$.

Since G is connected with order $n > 2$ and each v_i has degree 1, we must have that each u_i has degree at least 2 or else $G[\{u_i, v_i\}]$ would be a component of G. Let $S' = (S - \{v_1, ..., v_k\}) \cup \{u_1, ..., u_k\}$, then S' dominates G and $|S'| \leq |S|$ and so S' is a $\gamma(G)$-set. Now each $u_i \in S'$ has v_i as its external S'-pn, since v_i has degree 1.

Let $\{w_1, w_2, ..., w_z\}$ be the set of all vertices in S' that do not have a private external neighbour. Then by our definition of S and S', we have $\{w_1, w_2, ..., w_z\} \subseteq S \cap S'$. By our choice of S, w_i has an external S-pn which we will call x_j. Now $x_j \notin \{v_i, ..., v_k\}$ since $x \notin S$. Also, $x_j \notin \{u_1, ..., u_k\}$ since u_i is an external S-pn of v_i for all $i \in \{1, ..., k\}$. Therefore, since x_j is not an external S'-pn of w, we have that x_j is adjacent to some u_i. Then x_j has degree at least 2. Let $S'' = (S' - \{w_1, w_2, ..., w_z\}) \cup \{x_1, x_2, ..., x_z\}$. Then S'' dominates G and $|S''| \leq |S'|$, and so S'' is a $\gamma(G)$-set such that every vertex in S'' has degree at least 2. Assume to the contrary that there exists a $y \in S''$ that has no external S''-pn. Since all v_i have degree 1, $y \notin \{u_1, ..., u_k\}$. Then $y \in \{x_1, .., x_z\}$. Therefore, since every x_j is adjacent to some u_i, $S'' - \{y\}$ dominates G. This contradicts that S'' is a $\gamma(G)$-set, so our assumption is wrong.

Thus, S'' is a $\gamma(G)$-set such that any vertex in S'' has degree at least 2 and at least one external S''-pn. $\square$

A central problem in domination theory is finding the domination number of a given graph. It has been proven that the problem of finding the domination or power domination number of a graph is NP-hard. This means that the most efficient algorithm is not significantly better than checking every single subset of vertices to see if it is a dominating or power dominating set. These problems remain NP-hard even when restricted to bipartite or chordal graphs [15]. Therefore, since it is so hard to compute $\gamma(G)$ directly it becomes interesting to know if there are any upper and lower bounds on $\gamma(G)$. We now take a look at some interesting bounds on the domination number.

Theorem 2.4.10 ([6])**.** *Let G be a graph of order n. Then,* $\lceil \frac{n}{1+\Delta(G)} \rceil \leq \gamma(G) \leq$

$n - \Delta(G)$.

Proof. To prove the lower bound, we note that every vertex v dominates itself and all of its neighbours. Therefore, the maximum number of vertices that a vertex v can dominate is $1 + \Delta(G)$, which occurs if $deg(v) = \Delta(G)$. Therefore, any set $S \subseteq V$ dominates at most $|S|(1 + \Delta(G))$ vertices, so that at least $\lceil \frac{n}{1+\Delta(G)} \rceil$ vertices are needed to dominate the n vertices of G. Therefore, any dominating set of G needs at least $\lceil \frac{n}{1+\Delta(G)} \rceil$ vertices and thus, $\lceil \frac{n}{1+\Delta(G)} \rceil \leq \gamma(G)$.

To prove the upper bound we let v be a vertex in G with degree $\Delta(G)$. Now let $S \subseteq V$ be the vertex set containing v and all vertices not adjacent to v. Then $|S| = n - \Delta(G)$ and every vertex in G is either in S or adjacent to v. So S is a dominating set which means that $\gamma(G) \leq n - \Delta(G)$. □

Let G be a graph with $v \in V$ and let $deg(v) = 1$. Then it is easy to see that either v or its neighbour must be in every dominating set of G. This simple observation can help to find a graph that attains the bound in Corollary 2.4.7.

Definition 2.4.11 (Pendant vertex)**.** Let G be a graph and let $v \in V(G)$. A vertex v is a *pendant vertex* if v has degree one.

Definition 2.4.12 (Pendant edge)**.** Let G be a graph and let $e \in E(G)$. An edge e is a *pendant edge* if e is incident to a pendant vertex.

Definition 2.4.13 (Corona)**.** Let G be a graph of order n with $V = \{v_1, v_2, ..., v_n\}$. Then the corona of G, denoted as $cor(G)$, is defined as follows:

- The vertex set is defined as $V(cor(G)) = \{v_1, ..., v_n, u_1, ..., u_n\}$.
- The edge set is defined as $E(cor(G)) = E(G) \cup \{v_i u_i : i \in \{1, 2, ..., n\}\}$.

That is, to construct $cor(G)$ we take G and then we join a pendant vertex to each vertex of G.

We now show that the upper bound in Theorem 2.4.7 is sharp. We can also characterise all connected graphs that attain this upper bound. The following theorem was stated in [6], with the authors referencing a proof in [19]. We instead provide our own proof of this theorem.

Theorem 2.4.14 ([6]). *Let G be a connected graph of order n with $\gamma(G) = \frac{n}{2}$. Then G is either $K_{2,2}$ or G is the corona of some connected graph.*

Proof. Let $\mathcal{T}$ be the family of coronas of connected graphs. It is easy to verify that all graphs G in $\mathcal{T} \cup \{K_{2,2}\}$ satisfy $\gamma(G) = \frac{n}{2}$. Conversely, let G be a connected graph with $\gamma(G) = \frac{n}{2}$. By Corollary 2.4.9, G contains a $\gamma(G)$-set in which every vertex has degree at least 2 and at least one external private neighbour. Let S be a $\gamma(G)$-set such that all vertices in S have degree at least 2, at least 1 external-Spn and $\omega(G[S])$ is minimised. Therefore, $|N[S]| \geq 2|S|$ with equality if and only if $|pn(v, S) - S| = 1$ for all $v \in S$. Since $|S| = \gamma(G) = \frac{n}{2}$, we have that $2|S| = |V|$. Thus, $|N[S]| = 2|S|$ which implies that, $|pn(v, S)| = 1$ for all $v \in S$.

This implies that $G[S]$ has no isolated vertices, since any vertex in S has degree at least 2. Let $S = \{v_1, v_2, ..., v_{\frac{n}{2}}\}$, and let $N(v_i) - S = \{u_i\}$ for $i = 1, 2, ..., \frac{n}{2}$. If u_i is not adjacent to a vertex u_j for all $i \neq j$, then $G \in \mathcal{T}$. We may now assume that $|S| > 1$ and $u_i u_j \in E$ for some $i \neq j$ and consequently that $G \notin \mathcal{T}$.

Claim (1). *The graph $G[\{v_i, v_j\}]$ is a component in $G[S]$.*

Proof of claim. Assume to the contrary that $G[\{v_i, v_j\}]$ is not a component in $G[S]$. Then without loss of generality, we may let v_i be adjacent to some $v_k \in S - \{v_j\}$. This is shown in Figure 2.7. Let $S' = (S - \{v_i, v_j\}) \cup \{u_j\}$. Then S' is a dominating set, since $S - \{v_i, v_j\}$ dominates $V - \{u_i, v_j, u_j\}$ and u_j dominates $\{u_i, v_j, u_j\}$. We have that $|S'| < |S|$, which contradicts S being a $\gamma(G)$-set. Therefore, $G[\{v_i, v_j\}]$ is a component in $G[S]$.

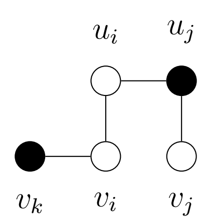

Figure 2.7: Diagram showing that if $G[\{v_i, v_j\}]$ is not a component of $G[S]$ then $S' = (S - \{v_i, v_j\}) \cup \{u_j\}$ is a dominating set of G. The vertices in S' are shown in black.

Claim (2). *The cardinality of S is at most 2.*

Proof of claim. Assume to the contrary that $|S| > 2$. This is shown in Figure 2.8. Then since G is connected, we can say without loss of generality that $u_i u_k \in E$ for some $u_k \in V$ and $u_k \neq u_j$. So $S' = (S - \{v_i, v_k\}) \cup \{u_k\}$ is a dominating set since $S - \{v_i, v_k\}$ dominates $V - \{v_k, u_k, u_i\}$ and $\{u_k\}$ dominates $\{v_k, u_k, u_i\}$. Then, S' is a smaller dominating set than S which contradicts our choice of S. Therefore, $|S| \leq 2$.

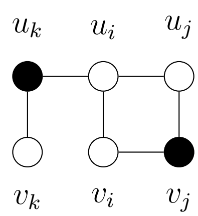

Figure 2.8: Diagram showing that if $|S| > 2$ then $S' = (S - \{v_i, v_k\}) \cup \{u_k\}$ is a dominating set of G. The vertices in S' are shown in black.

We know from Claim 2 and our choice of S, that $|S| = 2$. Then we may assume that $S = \{v_i, v_j\}$, and so $V = \{v_i, v_j, u_i, u_j\}$. We know from Claim 1 that v_i is adjacent to v_j. This is shown in Figure 2.9. Assume that $v_i u_j \in E$, then $\{u_j\}$ is a smaller dominating set than S, which contradicts our choice of S. So $v_i u_j \notin E$. Then by symmetry, we must have $v_j u_i \notin E$. Therefore, either $G \cong K_{2,2}$ or $G \cong P_4$. We note that P_4 is the corona of P_2, and so by our assumption that $G \notin \mathcal{T}$, we must have $G \cong K_{2,2}$.

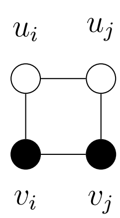

Figure 2.9: Diagram showing a spanning subgraph of G. The vertices of S are shown in black.

□

We now relate the domination number to some other well known parameters in graph theory. The following 4 parameters are all well known and well studied in graph theory. The covering number and edge covering number are both related to covering a graph with as small a set as possible. In essence this is similiar to domination theory. The independence number and edge independence number are also well studied graph parameters.

Definition 2.4.15 (Covering number $\alpha(G)$)**.** Let G be a graph. A *vertex cover* of G is a set $A \subseteq V$ such that every edge in G is incident to some vertex in A. The *covering number* of G, denoted $\alpha(G)$, is the minimum cardinality of a vertex cover of G.

Definition 2.4.16 (Edge covering number $\alpha'(G)$)**.** Let G be a graph without isolated vertices. An *edge cover* of G is a set $A' \subseteq E$ such that every vertex in G is incident to some edge in A'. The *edge covering number* of G, denoted $\alpha'(G)$, is the minimum cardinality of an edge cover of G.

Definition 2.4.17 (Independence number $\beta(G)$)**.** Let G be a graph. An *independent set* of G is a set $B \subseteq V$ such that no two vertices in B are adjacent. The *independence number* of G, denoted $\beta(G)$, is the maximum cardinality of an independent set of G.

Definition 2.4.18 (Edge independence number $\beta'(G)$)**.** Let G be a graph without isolated vertices. An *independent edge set* of G is a set $B' \subseteq E$ such that no two edges in B are adjacent. The *edge independence number* of G, denoted $\beta'(G)$, is the maximum cardinality of an independent edge set of G.

Theorem 2.4.19 ([6])**.** *Let G be a graph without isolated vertices. Then*

$$\gamma(G) \leq min\{\alpha(G), \alpha'(G), \beta(G), \beta'(G)\}$$

Proof. Let S be a maximum independent set of G. Then for any vertex $v \in V(G)-S$, v must be adjacent to at least one vertex in S, otherwise $S \cup \{v\}$ would be a bigger independent set than S which is a contradiction. Therefore, S is a dominating set and $\gamma(G) \leq \beta(G)$.

Let C be a vertex cover of G. Then C is incident to every edge of G. Therefore, every vertex $v \in (V - C)$, not isolated in G, is adjacent to C. But G has no isolated vertices, so every vertex $v \in (V - C)$ is adjacent to C. Then C is a dominating set and therefore $\gamma(G) \leq \alpha(G)$.

We now let X be a minimum edge cover of G. for each edge in X, select an incident vertex and let this collection of vertices be S. Then $|S| \leq |X| = \alpha'(G)$. Then S

dominates every vertex incident to X. Since X is an edge cover, S dominates $V(G)$, so S is a dominating set. Therefore, $\gamma(G) \leq |S| \leq |X| = \alpha'(G)$. Thus, we have the inequality $\gamma(G) \leq \alpha'(G)$.

Let M be a maximum independent edge set of G. Then $|M| = \beta'(G)$. Let $M' = V(G) - V(G[M])$.

Claim (1)**.** *Let $uv \in M$, then there does not exist distinct vertices $x, y \in M'$ such that $ux, vy \in E(G)$.*

Proof of claim. Assume to the contrary that there exists $ux, vy \in E(G)$ such that $x \neq y$. Then $(M - \{uv\}) \cup \{ux, vy\}$ is a bigger independent edge set that M, which is a contradiction.

We now construct a dominating set S such that $|S| = \beta'(G)$. For each edge $ab \in M$ do the following: If $N(M') \cap \{a, b\} = \{a\}$, add a to S. If $N(M') \cap \{a, b\} = \{a, b\}$ or $N(M') \cap \{a, b\} = \emptyset$, add either a or b to S but not both.

Claim (2)**.** *The set S is a dominating set.*

Proof of claim. By construction, S is incident to all edges in M and therefore $N[S]$ is adjacent to all vertices in $G[M]$. Assume to the contrary that there exists a vertex $x \in V(G) - S$, not adjacent to S. Then $x \notin V(G[M])$, so $x \in M'$. Since G has no isolated vertices, x is adjacent to some vertex $u \in V - S$. If $u \in M'$ then $M \cup \{xu\}$ is an independent edge set with greater size than M, which contradicts M being maximised. Therefore, $u \in V(G[M])$, which means there exists a $uv \in M$. Then, by construction of S, $v \in S$ and v is adjacent to some some vertex y in M'. Then by our choice of x and v, $x \neq y$. Therefore, $uv \in M$, $x, y \in M'$ and $xu, vy \in E(G)$ where $x \neq y$. This contradicts Claim 1, which means that the assumption is wrong and S is a dominating set.

Therefore, $\gamma(G) \leq |S| \leq |M| = \beta'(G)$, which implies $\gamma(G) \leq \beta'(G)$.

Thus, we have that $\gamma(G) \leq \min\{\alpha(G), \alpha'(G), \beta(G), \beta'(G)\}$. □

Chapter 3

Introduction to Power Domination

3.1 Introduction

Recall that power domination was originally constructed to formalise the problem of efficiently monitoring an electrical network. Given an electrical network we define a corresponding graph by letting the substations be vertices and joining two vertices with an edge if there are power lines directly connecting the corresponding substations. We then place phasor measurement units (PMUs) at substations to monitor the network. The PMU can monitor the substation it is placed at and any adjacent stations. Then using the Ohm and Kirchhoff current laws, it is then possible to use knowledge of the currently monitored stations to monitor more substations in the network if certain conditions are met. In the language of graphs, we say that our initial set of vertices monitors its closed neighbourhood. Then, the monitored vertices can then monitor their neighbours if certain conditions are met. This process is called propagation. This repeats until the conditions of propagation are not met. In this chapter we will rigorously define power domination and concepts that are related to power domination, such as: power dominating sets, power domination number, and propagation radius. We then show some basic results in power domination. Some of these results will be simple extensions from domination and some will be unique to power domination, such as k-power domination.

3.2 Definitions

We start by properly defining power domination. In order to do this, we use the definition of power domination found in [13].

Definition 3.2.1 (Power domination)**.** Let G be a graph and $S \subseteq V(G)$. We denote by $P_G^i(S)$ the set of vertices *monitored by S at step i*. We define $P_G^i(S)$ recursively as follows:

- If $i = 0$ then $P_G^i(S) = N[S]$. We call step 0 the **domination step**.
- For $i \geq 0$, let $V^i = \{v \in P_G^i(S) : |N[v] - P_G^i(S)| \leq 1\}$. Then

$$P_G^{i+1}(S) = \bigcup_{v \in V^i} N[v]$$

  For $i \geq 1$, we call step i a **propagation step**.

Where there is no chance of confusion we say $P^i(S)$ or P^i instead of $P_G^i(S)$. For $v \in P_G^i(S)$ such that v has exactly one unmonitored neighbour u, we have that $u \in P_G^{i+1}(S)$ and we say that v *propagates* to u in step $i+1$.

We can see that the sequence $(P^i(S))_{i\geq 0}$ is a non-decreasing sequence and that $P^i(S) \subseteq P^{i+1}(S)$. It is also clear that if $P^i(S) = P^{i+1}(S)$, then $P^i(S)$ is a fixed point of the sequence. If G has n vertices then a fixed point must be reached for some $i < n$. If i is the smallest integer such that $P_G^i(S) = P_G^{i+1}(S)$, then we denote by $P_G^\infty(S)$ the set $P_G^i(S)$. That is, $P_G^\infty(S) = P_G^j(S)$ for all $j \geq i$. Therefore, $P^n(S) = P^\infty(S)$. We say that S *power dominates* the set $P_G^\infty(S)$. With the above we can now define a power dominating set.

Definition 3.2.2 (Power dominating set)**.** Let G be a graph and $S \subseteq V$. Then S is a *power dominating set of* G if $P^\infty(S) = V(G)$. For brevity, we say that S is a PDS of G.

Definition 3.2.3 (Power domination number)**.** Let G be a graph. The *power domination number* of G is the minimum cardinality of a PDS of G, denoted $\gamma_p(G)$. If a PDS has cardinality $\gamma_p(G)$, we call it a *minimum-PDS* or a *$\gamma_p(G)$-set*.

Definition 3.2.4 (Propagation radius)**.** Let G be a graph and let $S \subseteq V$ be a PDS of G. Let i be the smallest integer such that $P_G^i(S) = P_G^{i+1}(S)$. Then, the *propagation radius of* S *in* G is $i+1$. We denote the *propagation radius of* S *in* G **as** $rad_p(G, S)$. *The propagation radius of* G is the minimum propagation radius of any $\gamma_p(G)$-set. The propagation radius of G is denoted as $rad_p(G)$. That is,

$$rad_p(G) = \min\{rad_p(G, S) \ : \ S \text{ a } \gamma_p(G)\text{-set}\}$$

Remark 3.2.5. Let G be a graph. We note that $V(G)$ is a PDS of G and that $rad_p(G, V) = 1$. This motivates why the propagation radius of a graph is the minimum of all $\gamma_p(G)$-sets and not all power dominating sets.

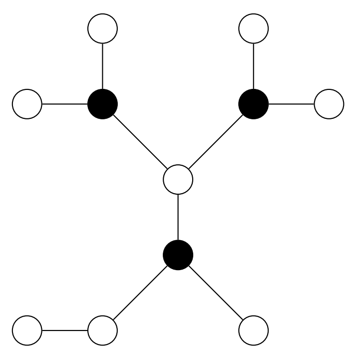

Figure 3.1: Example of a graph G with power domination number 3. The black vertices form a $\gamma_p(G)$-set. The propagation radius of G is 2.

From the definition it is apparent that any dominating set is also a power dominating set. Therefore, for any graph G, $\gamma_p(G) \leq \gamma(G)$.

It is also possible to generalise the idea of power domination to propagate more than one unmonitored neighbour at a time. With this in mind we define k-power domination.

Definition 3.2.6 (k-power domination)**.** Let G be a graph, k a non-negative integer and $S \subseteq V(G)$. We denote by $P_{k,G}^i(S)$ the set of vertices *monitored by* S *at step* i. We define $P_{k,G}^i(S)$ recursively as follows:

- If $i = 0$ then $P_{k,G}^0(S) = N[S]$. We call step 0 the **domination step**.

- For $i \geq 0$, let $V^i = \{v \in P^i_{k,G}(S) : \ |N[v] - P^i_{k,G}(S)| \leq k\}$. Then

$$P^{i+1}_{k,G}(S) = \bigcup_{v \in V^i} N[v]$$

  For $i \geq 1$, we call the step i a **propagation step**.

Similarly to power domination, we can see that the sequence $(P^i_k(S))_{i\geq 0}$ is a non-decreasing sequence. It is also clear that if $P^i_k(S) = P^{i+1}_k(S)$ then $P^i_k(S)$ is a fixed point of the sequence. If G has n vertices then a fixed point must be reached for some $i < n$. If i is the smallest integer such that $P^i_{k,G}(S) = P^{i+1}_{k,G}(S)$, then we denote by $P^\infty_{k,G}(S)$ the set $P^i_{k,G}(S)$. That is, $P^\infty_{k,G}(S) = P^j_{k,G}(S)$ for all $j \geq i$. Therefore, $P^n_k(S) = P^\infty_k(S)$. We say that S *k-power dominates* the set $P^\infty_{k,G}(S)$. We can now define a k-power dominating set.

Definition 3.2.7 (k-power dominating set)**.** Let G be a graph, k a non-negative integer and $S \subseteq V$. Then S is a *k-power dominating set of* G if $P^\infty_k(S) = V(G)$. For brevity, we say that S is a k-PDS of G.

Definition 3.2.8 (k-power domination number)**.** Let G be a graph. The *k-power domination number* of G is the minimum cardinality of a *k-PDS* of G, denoted $\gamma_{p,k}(G)$. If a k-PDS has cardinality $\gamma_{p,k}(G)$, we call it a minimum k-PDS or a $\gamma_{p,k}(G)$-set.

From this definition, we can see that if $k = 1$, then we get the definition for power domination. If $k = 0$ then there is never a propagation step and so we get the definition of domination. In this sense, we can think of domination and power domination as special cases of k-power domination.

Remark 3.2.9. It is clear that for any graph G, we have the inequality:

$$\gamma(G) \geq \gamma_p(G) \geq \gamma_{p,i}(G) \geq \gamma_{p,j}(G)$$

for all $1 \leq i \leq j$.

Definition 3.2.10 (k-propagation radius)**.** Let G be a graph and let $S \subseteq V$ be a k-PDS of G. Let i be the smallest integer such that $P^i_{k,G}(S) = P^{i+1}_{k,G}(S)$. Then, the

k-propagation radius of S, in G, is $i+1$. We denote the *k-propagation radius of S in G as $rad_{p,k}(G,S)$*. The k-propagation radius of G is the minimum k-propagation radius of a $\gamma_{p,k}(G)$-set. The propagation radius of G is denoted as $rad_{p,k}(G)$. That is,

$$rad_{p,k}(G) = \min\{rad_{p,k}(G,S) \ : \ S \text{ a } \gamma_{p,k}(G)\text{-set}\}$$

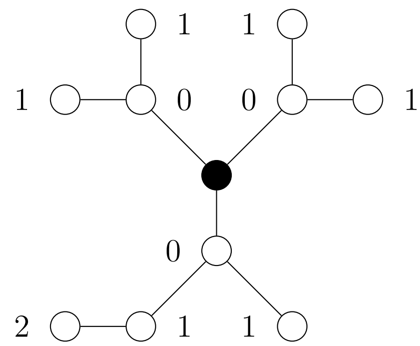

Figure 3.2: Example of a graph G with $k \geq 2$ power domination number 1. The black vertex forms a $\gamma_{p,k}(G)$-set. The k-propagation radius of G is 3. The vertices are labelled according to which step they are monitored at.

3.3 Basic results

We now demonstrate some basic results about power domination and k-power domination.

The following result that shows that the k-power domination number will be one and that any vertex forms a $\gamma_{p,k}$-set so long as every vertex has sufficiently small degree.

Proposition 3.3.1 ([9]). *Let G be a connected graph. If $\Delta(G) \leq k+1$ then $\gamma_{p,k}(G) = 1$ and any vertex of G forms a $\gamma_{p,k}$-set.*

Proof. Let $v \in V(G)$ and let $u \in P^i_{k,G}(\{v\})$ for some $i \geq 0$. Then $N[u] \subseteq P^i_{k,G}(\{v\})$ or $|N[u] - P^i_{k,G}(\{v\})| \leq \Delta(G) - 1 = k + 1 - 1 = k$. Therefore, u propagates to all of its unmonitored vertices in step $i+1$. Therefore, every monitored vertex propagates to all its unmonitored vertices at every step. It follows that any $v \in V(G)$ is a k-PDS and that $\gamma_{p,k}(G) = 1$. $\square$

The following is our own result which shows that the k-power domination number of a connected graph will be one if there is a vertex with sufficiently large degree.

Proposition 3.3.2. *Let G be a connected graph of order n. If $\Delta(G) \geq n-k-1$, then $\gamma_{p,k}(G) = 1$.*

Proof. Let $v \in V(G)$ be such that $deg(v) \geq n-k-1$. Let $S = \{v\}$, then $P^0_{k,G}(S) = N[v]$. Since $|N[v]| \geq n-k$, we have that $|V(G) - P^0_{k,G}(S)| \leq n-(n-k) = k$. Therefore, at every step, every monitored vertex will be adjacent to less than k unmonitored vertices, and will therefore propagate. Therefore, S is a minimum k-PDS of G. □

Corollary 3.3.3. *Let G be a graph, k be a non-negative integer and $\gamma_{p,k}(G) > 1$. Then $k+2 < \Delta(G) < n-k-1$.*

Proof. Let G be a graph, k be a non-negative integer and $\gamma_{p,k}(G) > 1$. Then, by Theorem 3.3.1, $k+2 < \Delta(G)$ and by Theorem 3.3.2, $\Delta(G) < n-k-1$. Therefore, $k+2 < \Delta(G) < n-k-1$. □

The following results were stated without proof in [15]. We now provide our own original proofs here.

Corollary 3.3.4 ([15])**.** *Let $\mathcal{C}$ be the set of all cycle graphs and let $\mathcal{P}$ be the set of all path graphs. Then for any $G \in \mathcal{C} \cup \mathcal{P}$, $\gamma_p(G) = 1$.*

Proof. For any $G \in \mathcal{C} \cup \mathcal{P}$, $\Delta(G) \leq 2$. Therefore, if we take $k = 1$, we have that $\Delta(G) \leq 2 = k+1$. Then by Proposition 3.3.1, $\gamma_p(G) = 1$. □

Corollary 3.3.5 ([15])**.** *Let $\mathcal{K}$ be the set of all complete graphs and let $\mathcal{K}_{2,x}$ be the set of all complete bipartite graphs of the form $K_{2,x}$, where x is a positive integer. Then for any $G \in \mathcal{K} \cup \mathcal{K}_{2,x}$, $\gamma_p(G) = 1$.*

Proof. Let be $G \in \mathcal{K} \cup \mathcal{K}_{2,x}$ and G have order n. Then $\Delta(G) \geq n-2$. If we let $k = 1$, then $\Delta(G) \geq n-2 = n-k-1$. Thus, by Lemma 3.3.2 $\gamma_p(G) = 1$. □

We now present a theorem that extends the idea of Theorem 2.4.9 to k-power domination. We prove that we can always find a $\gamma_{p,k}(G)$-set such that none of its vertices have "small" degree as long as G has at least one vertex that does not have "small" degree.

Proposition 3.3.6 ([9]). *Let G be a connected graph such that $\Delta(G) \geq k+2$. Then G has a $\gamma_{p,k}(G)$-set in which every vertex has degree at least $k+2$.*

Proof. Let S be a $\gamma_{p,k}(G)$-set. Suppose that S has l vertices, $\{v_1, ..., v_l\}$, with degree less than $k+2$.

For each i, $1 \leq i \leq l$, let $u_i \in V$ such that $deg(u_i) \geq k+2$ and $d(u_i, v_i)$ is minimised. Let T_i be a $u_i - v_i$ geodesic for each i, $1 \leq 1 \leq l$. We note that u_i's may not all be distinct. Also, by construction u_i is the only vertex with degree $\geq k+2$ in T_i.

Let $S' = (S - \{v_1, ..., v_l\}) \cup \{u_1, ..., u_l\}$. Label the vertices of T_i as $u_i, t_0, ..., t_r = v_i$. Then $t_0 \in P^0(S')$ since $t_0 \in N(u_i)$. We have that $deg(t_0) < k+2$ and $u_i, t_0 \in P^0(S')$. Therefore, $|N[t_0] - P^0(S')| \leq k+2-2 = k$. Therefore, t_0 propagates to t_1. Similarly t_1 propagates to t_2 and so on until $t_r = v_i$ propagates to $N(v)$. Thus, for all $i \in \{1, ..., l\}$, S' will monitor $N[v_i]$.

Therefore, S' will eventually monitor $N[S - \{v_1, ..., v_l\}] \cup N[v_1, ..., v_l] = N[S]$ and thus S' is a k-PDS of G. Since $|S'| \leq |S|$, S' is $\gamma_{p,k}(G)$-set and S' has no vertex with degree less than $k+2$. $\square$

Chapter 4

Upper bounds in connected graphs

4.1 Introduction

Bounds on the domination number of a graph have been extensively studied, both in general and in specific families of graphs [18, 3]. For a more in-depth list of bounds on the domination number, we refer the reader to [16]. While power domination was first defined 40 years after domination, consequently there has been less research into power domination. Research has been done on the bounds on the power domination number and k-power domination number [9, 8]. Since $\gamma_{p,k}(G) = \sum_{i=1}^{n} \gamma_{p,k}(C_i)$ and $rad_{p,k}(G) = \max\{rad_{p,k}(C_i)|i \in \{1, ..., n\}\}$ where $C_1, ..., C_n$ are the components of G, we will restrict our consideration to connected graphs for this chapter.

4.2 Bounds on the power domination number

We present a general bound on the power domination number for connected graphs. The lower bound is trivially $\gamma_p(G) \geq 1$. A trivial upper bound would be $\gamma_p(G) \leq \gamma(G) \leq \frac{n}{2}$. We present a proof that $\frac{n}{3}$ is the best upper bound for the power domination number of a connected graph. We also characterise all graphs that achieve this bound. This proof was first shown in [5]. We then generalise this proof to find that $\frac{n}{k+2}$ is the best upper bound for k-power domination. Again, we characterise all graphs that achieve this bound. We will present our own proof of this result.

We start be defining a special family of connected graphs:

Definition 4.2.1 ($\mathcal{T}$)**.** Let $\mathcal{T}$ be the family of all graphs such that each graph $G \in \mathcal{T}$ is obtainable from some connected graph H by adding for each $v \in V(H)$: two vertices v' and v''; two edges vv' and vv''; and possibly the edge $v'v''$.

The following lemma was stated to be a simple exercise in [5], where no proof was given. We present our own proof.

Lemma 4.2.2 ([5])**.** *Let $G \in \mathcal{T} \cup \{K_{3,3}\}$ be a graph with order n. Then $\gamma_p(G) = \frac{n}{3}$.*

Proof. Let H be a connected graph such that $G \in \mathcal{T}$ is obtainable from H. Then, G has order $n = 3|V(H)|$.

First, we show that $\gamma_p(G) \geq \frac{n}{3}$. Assume to the contrary that $\gamma_p(G) < \frac{n}{3}$ and that S' is a $\gamma_p(G)$-set. By the pigeonhole principle there is at least one $v \in V(H)$ such that $S' \cap \{v, v', v''\} = \emptyset$. Then v will always have v' and v'' as unmonitored neighbours and so G is unmonitored by S', which is a contradiction. Therefore, $\gamma_p(G) \geq \frac{n}{3}$. If we let $S = V(H) \subseteq V(G)$, then $N[S] = V(G)$ and so S is a PDS of G and $|S| = \frac{n}{3}$. Therefore, $\gamma_p(G) = \frac{n}{3}$.

Now, let $G = K_{3,3}$. Let $\{X, Y\}$ be a bipartition of G with $X = \{v_1, v_2, v_3\}$ and $Y = \{v_4, v_5, v_6\}$. Let $S = \{v_1, v_4\}$. Then, $N[S] = V$ and so S is a PDS of G. Therefore, $\gamma_p(G) \leq \frac{n}{3}$. Assume that $\gamma_p(G) < \frac{n}{3}$. Then, since $\gamma_p(G) < 2$, we must have $\gamma_p(G) = 1$. Without loss of generality we can let $S' = \{v_1\}$ be a PDS of G. Then, $P^0(S) = \{v_1, v_4, v_5, v_6\}$. Now $|N[v_i] - P^0(S)| = 2$ for $i \in \{4, 5, 6\}$, so $\{v_4, v_5, v_6\}$ will not propagate. Therefore, $P^0 = P^1 \neq V$, which means that S' is not a PDS. Therefore, $\gamma_p(K_{3,3}) = 2 = \frac{n}{3}$. □

Theorem 4.2.3 ([5])**.** *Let G be a connected graph of order $n \geq 3$. Then $\gamma_p(G) \leq \frac{n}{3}$ with equality if and only if $G \in (\mathcal{T} \cup \{K_{3,3}\})$.*

Proof. If $\Delta(G) \leq 2$ then by Lemma 3.3.1 $\gamma_p(G) = 1$ and we are done. Let $\Delta(G) \geq 3$. By Lemma 3.3.6, G contains a $\gamma_p(G)$-set in which every vertex has degree at least 3. Let S be a $\gamma_p(G)$-set such that all vertices in S have degree at least 3 and $\omega(G[S])$ is minimised.

Claim (1)**.** *Every vertex $v \in S$ has at least two external S-pns. That is, $|pn(v, S) - S| \geq 2$.*

Proof of claim. Assume to the contrary that there exists a vertex $v \in S$ such that $|pn(v,S) - S| \leq 1$. If v is adjacent to some vertex in S, let $S' = S - \{v\}$. Note that v and its neighbours, except for possibly one, are all monitored by S'. This vertex, if it exists, is then monitored by v. Hence, S' is a PDS of G with $|S'| < |S|$, which is a contradiction. Therefore, v is an isolated vertex in $G[S]$. That is, $N(v) \subseteq V - S$ and $|N(v) - S| \geq 3$. Choose a vertex $u \in N(v) - pn(v,S)$ and let $S'' = (S - \{v\}) \cup \{u\}$. Then, v and its neighbours except for possibly one are all monitored by S''. This vertex, if it exists, is then monitored by v. Thus, S'' is a $\gamma_p(G)$-set of G with $\omega(G[S'']) < \omega(G[S])$, which contradicts the choice of S. Therefore, each vertex in S has at least two external S-pns.

By Claim 1, $|N[S]| \geq 3|S|$, so G has at least $3|S|$ vertices which implies that $\gamma_p(G) \leq \frac{n}{3}$.

By Lemma 4.2.2 we know all graphs G in $\mathcal{T} \cup \{K_{3,3}\}$ with order n, satisfy $\gamma_p(G) = \frac{n}{3}$. Conversely, assume that G is a connected graph with $\gamma_p(G) = \frac{n}{3}$. By Lemma 3.3.6, G contains a $\gamma_p(G)$-set in which every vertex has degree at least 3. Let S be a $\gamma_p(G)$-set such that all vertices in S have degree at least 3 and $\omega(G[S])$ is minimised. By Claim (1), $|pn(v,S) - S| \geq 2$ for all $v \in S$ and therefore $|N[S]| \geq 3|S|$. Since $3|S| = |V|$ we must have that $|N[S]| = |V|$, so $N(v) - S = pn(v,S)$ and $|pn(v,S) - S| = 2$ for all $v \in S$. This implies that $G[S]$ has no isolated vertices, since every vertex in S has degree at least 3. Let $S = \{v_1, v_2, ..., v_{\frac{n}{3}}\}$, and let $N(v_i) - S = \{u_{i1}, u_{i2}\}$ for $i \in \{1, 2, ..., \frac{n}{3}\}$. If no vertex in $\{u_{i1}, u_{i2}\}$ is adjacent to a vertex in $\{u_{j1}, u_{j2}\}$ for all $i \neq j$, then $G \in \mathcal{T}$ and we are done.

Now, without loss of generality, let $|S| > 1$ and let $u_{i1}u_{j1} \in E(G)$ for some $i \neq j$.

Claim (2)**.** *The graph $G[\{v_i, v_j\}]$ is a component in $G[S]$.*

Proof of claim. Assume to the contrary that $G[\{v_i, v_j\}]$ is not a component in $G[S]$. Since $G[S]$ has no isolated vertices, without loss of generality, we can let v_i be adjacent to some $v_k \in S - \{v_j\}$. This is shown in Figure 4.1. let $S' = (S - \{v_i, v_j\}) \cup \{u_{j1}\}$. Then, $P^0(S') = N[S'] \cup N[u_{j1}] \supseteq V - \{u_{i2}, u_{j2}\}$ since, $v_i \in N[v_k]$ and $v_j, u_{i1}, u_{j1} \in N[u_{j1}]$. Then, $V - P^0(S') \subseteq \{u_{i2}, u_{j2}\}$. These two vertices can be monitored in turn by v_i and v_j. Thus, S' is a PDS. But, $|S'| < |S|$, which contradicts our choice of S. Therefore $G[\{v_i, v_j\}]$ is a component in $G[S]$.

Claim (3)**.** *The cardinality of S is at most 2.*

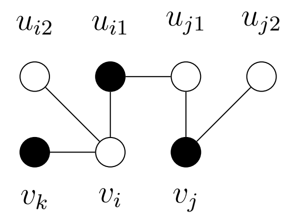

Figure 4.1: Diagram showing that if $G[\{v_i, v_j\}]$ is not a component of $G[S]$ then $S' = (S - \{v_i, v_j\}) \cup \{u_{j1}\}$ is a PDS of G. The vertices in S' are shown in black.

Proof of claim. Assume to the contrary that $|S| > 2$. Then, $\{u_{i1}, u_{i2}\}$ or $\{u_{j1}, u_{j2}\}$ has a vertex adjacent to a vertex in $\{u_{k1}, u_{k2}\}$ for some $v_k \in S - \{v_i, v_j\}$, say $u_{i1}u_{k1} \in E(G)$. Thus, $(S - \{v_i, v_k\}) \cup \{u_{k1}\}$ is a PDS of size smaller than S, which contradicts our choice of S. Therefore $|S| \leq 2$. This is shown in Figure 4.2

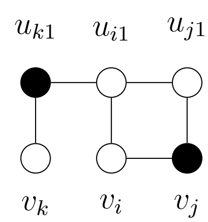

Figure 4.2: Diagram showing that if $|S| > 2$ then $S' = (S - \{v_i, v_k\}) \cup \{u_{k1}\}$ is a PDS of G. The vertices in S' are shown in black.

Let $S = \{v_i, v_j\}$ and $V(G) = \{v_i, v_j, u_{i1}, u_{i2}, u_{j1}, u_{j2}\}$. Since $G[S]$ has no isolated vertices, G must have at least the edges shown in Figure 4.3. Now v_i and v_j must have degree 3, since each has 2 external S-pns. Assume that $u_{j1}u_{j2} \in E(G)$. Then, we can show that $\{u_{j1}\}$ is a PDS of G. We note that $P^0(\{u_{j1}\}) \subseteq \{y_{j1}, v_j, u_{i1}, u_{j2}\}$. Then, v_j will propagate to v_i, which will propagate to u_{i2}. Thus, $\{u_{j1}\}$ is a PDS with smaller cardinality than S. This contradicts S being a minimum PDS. So, $u_{j1}u_{j2} \notin E(G)$. By symmetry, we have $u_{i1}u_{i2} \notin E(G)$. Now, assume that $u_{j1}u_{i2} \notin E(G)$. Then, we can show that $\{v_j\}$ is a PDS. We note that $P^0(\{v_j\}) \subseteq \{v_j, v_i, u_{j1}, u_{j2}\}$, therefore u_{j1} will propagate to v_{i1}, which causes v_i to propagate to u_{i2}. Consequently $\{v_j\}$ is a PDS of size smaller than S, which contradicts our choice of S. Hence, $u_{j1}u_{i2} \in E(G)$ and by symmetry, we have $u_{i1}u_{j2} \in E(G)$. Assume that $u_{i2}u_{j2} \notin E(G)$. Then, we can prove that $\{v_j\}$ is a PDS. We see that $P^0(\{v_j\} \subseteq \{v_j, v_i, u_{j1}, u_{j2}\})$, then u_{j2} propagates to u_{i1}, which causes v_i to propagate to u_{i2}. So $\{v_j\}$ is a PDS of size smaller than S, which contradicts our choice of S. Thus, $u_{i2}u_{j2} \in E(G)$ and therefore, $G \cong K_{3,3}$.

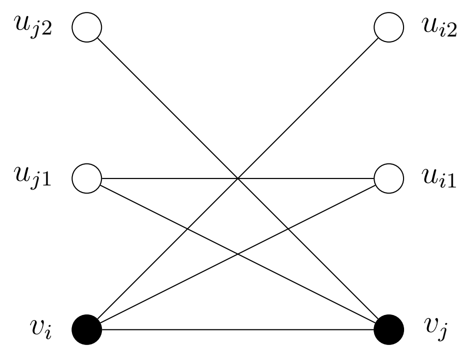

Figure 4.3: Diagram showing a spanning subgraph of G. The vertices of S are shown in black.

□

4.3 Bounds on the k-power domination number

In this section, we generalise the ideas of Theorem 4.2.3 to k-power domination. Notice that the key insight of Theorem 4.2.3 is that we can guarantee a $\gamma_p(G)$-set such that every vertex has at least 2 external private neighbours. This gives us an upper bound on the minimum size of the neighbourhood of a $\gamma_p(G)$-set which gives us an upper bound on $\gamma_p(G)$. We can extend this argument to show that there is a $\gamma_{p,k}(G)$-set where every vertex has at least k+1 external private neighbours.

We first start by defining a family of connected graphs that extend the idea of Definition 4.2.1 to work for k-power domination. This family was first defined in a different way in [4].

Definition 4.3.1 ($\mathcal{T}_k$)**.** Let $\mathcal{T}_k$ be the family of all possible graphs such that, each graph $G \in \mathcal{T}_k$, is obtainable from some connected graph H, by adding for each $v \in V(H)$: $k+1$ vertices $v_0, v_1, ..., v_k$; k+1 edges $vv_0, vv_1, ..., vv_k$; and possibly any subset of edges from $\{v_iv_j | i, j \in \{0, ..., k\}, i \neq j\}$.

Chang et al proved in [4], that for $G \in \mathcal{T}_k$ with order n, $\gamma_{p,k}(G) = \frac{n}{k+2}$. We extend this slightly to show that for every $K_{k+2,k+2}$, with order n, $\gamma_{p,k}(K_{k+2,k+2}) = \frac{n}{k+2}$.

Lemma 4.3.2. *Let $G \in \mathcal{T}_k \cup \{K_{k+2,k+2}\}$ be a graph with order n, then $\gamma_{p,k}(G) = \frac{n}{k+2}$.*

Proof. Let H be a connected graph such that $G \in T_k$ is obtainable from H. Then G has order $n = (k+2)|V(H)|$. Let $S = V(H) \subseteq V(G)$, then $N[S] = V(G)$, so S is a k-PDS of G and $|S| = \frac{n}{k+2}$. Now assume to the contrary that $\gamma_{p,k}(G) < \frac{n}{k+2}$ and that S' is a $\gamma_{p,k}(G)$-set. By the pigeonhole principle there is at least one $v \in V(H)$ such that $S' \cap \{v, v_0, .., v_k\} = \emptyset$. Then, v will always have $v_0, v_1, ..., v_k$ as unmonitored neighbours and so G is unmonitored by S' which is a contradiction. Therefore, $\gamma_{p,k}(G) = \frac{n}{k+2}$.

Now let $G = K_{k+2,k+2}$. Let $X = \{v_1, v_2, .., v_{k+2}\}$ and $Y = \{u_1, u_2, ..., u_{k+2}\}$ be a bipartition of G. Let $S = \{v_1, u_1\}$, then $N[S] = V$ and so S is a k-PDS of G. Now assume that there is a singleton set S', that power dominates G. Without loss of generality let $S' = \{v_1\}$. Then, $P_k^0(S) = \{v_1, u_1, u_2, ..., u_k\}$. Now $|N[u_i] - P_k^0(S)| = k+1$ for $1 \leq i \leq k+2$, so $\{u_1, u_2, ..., u_k\}$ will not propagate. Therefore, $P_k^0 = P_k^1 \neq V$, which means that S' is not a k-PDS. Therefore, $\gamma_{p,k}(K_{k+2,k+2}) = 2 = \frac{n}{k+2}$. Thus, for all graphs $G \in \mathcal{T}_k \cup \{K_{k+2,k+2}\}$ with order n, we have $\gamma_{p,k}(G) = \frac{n}{k+2}$. □

Chang et al proved in [4], that if G is a connected graph of order $n \geq k+2$ then, $\gamma_{p,k}(G) \leq \frac{n}{k+2}$ and that the family of graphs $\mathcal{T}_k$ achieves this bound. Here we extend this theorem slightly to show that equality is reached if and only if $G \in \mathcal{T}_k \cup \{K_{k+2,k+2}\}$.

Theorem 4.3.3. *Let G be a connected graph of order $n \geq k+2$. Then, $\gamma_{p,k}(G) \leq \frac{n}{k+2}$ with equality if and only if $G \in \mathcal{T}_k \cup \{K_{k+2,k+2}\}$.*

Proof. If $\Delta(G) \leq k+1$, then by Lemma 3.3.1, $\gamma_{p,k}(G) = 1$ and we are done. Now, let $\Delta(G) \geq k+2$. By Lemma 3.3.6, G contains a $\gamma_{p,k}(G)$-set in which every vertex has degree at least $k+2$. Let S be a $\gamma_{p,k}(G)$-set such that all vertices in S have degree at least $k+2$ and $\omega(G[S])$ is minimised.

Claim (1). *Every $v \in S$ has at least $k+1$ external S-pn's, that is, $|pn(v, S) - S| \geq k+1$.*

Proof of claim. Assume to the contrary that there exists a vertex $v \in S$ such that $|pn(v, S) - S| \leq k$. If v is adjacent to some vertex in S, let $S' = S - \{v\}$. We note that v is unmonitored and has at most k unmonitored neighbours. These unmonitored vertices, if they exist, are then monitored by v. Thus, S' is a k-PDS of

G with $|S'| < |S|$, which contradicts our choice of S. Hence v is an isolated vertex in $G[S]$, that is, $N(v) \subseteq V - S$ and $|N(v) - S| \geq k+2$. We can now choose a vertex $u \in N(v) - pn(v, S)$ and let $S'' = (S - \{v\}) \cup \{u\}$. Then, v and its neighbours except for possibly k are all monitored by S''. These vertices, if they exist, are then monitored by v. Thus, S'' is a $\gamma_{p,k}(G)$-set of G with $\omega(G[S'']) < \omega(G[S])$, which contradicts the choice of S. Therefore, our assumption is wrong, hence each vertex v in S has at least $k+1$ S-pn's.

Therefore, $|N[S]| \geq (k+2)|S|$, so G has at least $(k+2)|S|$ vertices which implies that $\gamma_{p,k}(G) \leq \frac{n}{k+2}$.

By Lemma 4.3.2 we know all graphs G in $\mathcal{T}_k \cup \{K_{k+2,k+2}\}$ with order n satisfy $\gamma_{p,k}(G) = \frac{n}{k+2}$. Conversely, let G be a connected graph with order n and $\gamma_{p,k}(G) = \frac{n}{k+2}$. Then, $n \geq k+2$ and so by Lemma 3.3.6, G contains a $\gamma_{p,k}(G)$-set in which every vertex has degree at least $k+2$. Let S be a $\gamma_{p,k}(G)$-set such that all vertices in S have degree at least $k+2$ and $\omega(G[S])$ is minimised. By Claim 1, $|pn(v, S) - S| \geq k+1$ for all $v \in S$ and therefore $|N[S]| \geq (k+2)|S| = n$. Therefore $N[S] = V(G)$. Since $|S| = \frac{n}{k+2}$ we must have that $|N[S]| \leq n$, so $|pn(v, S) - S| = k+1$ for all $v \in S$. This implies that $G[S]$ has no isolated vertices, as any vertex in S has degree at least $k+2$. Let $S = \{v_1, v_2, ..., v_{\frac{n}{k+1}}\}$, and let $N(v_i) = \{u_{i1}, u_{i2}, ...u_{i\frac{n}{k+1}}\}$ for $i \in \{1, 2, ..., \frac{n}{k+1}\}$. Since $\gamma_{p,k}(G) = \frac{n}{k+2}$, if no vertex in $N(v_i) - S$ is adjacent to a vertex in $N(v_j) - S$ for all $i \neq j$, then $G \in \mathcal{T}_k$ and we are done. We now assume that $|S| > 1$ and that $u_{i1}u_{j1} \in E$ for some $i \neq j$.

Claim (2)**.** *The graph $G[\{v_i, v_j\}]$ is a component in $G[S]$.*

Proof of claim. Assume to the contrary that $G[\{v_i, v_j\}]$ is not a component in $G[S]$. Then, without loss of generality, we can let v_i be adjacent to some $v_r \in S - \{v_j\}$. This is shown in Figure 4.4. Let $S' = (S - \{v_i, v_j\}) \cup \{u_{j1}\}$. Then, $P_k^0(S')$ monitors all vertices except possibly some vertices in $N(\{v_i, v_j\})$. We note that S', monitors at least u_{i1}, u_{j1}, v_i, v_j. Then, both v_i and v_j have at most k unmonitored neighbours and so propagates to all the remaining vertices. Consequently S' is a $\gamma_{p,k}(G)$-set, but $|S'| < |S|$, which contradicts our choice of S. Thus, our assumption is wrong and $G[\{v_i, v_j\}]$ is a component in $G[S]$.

Claim (3)**.** *The cardinality of S is at most 2.*

Proof of claim. Assume to the contrary that $|S| > 2$. Since G is connected, without loss of generality, we can state that there exists a $v_r \in S - \{v_i, v_j\}$ such

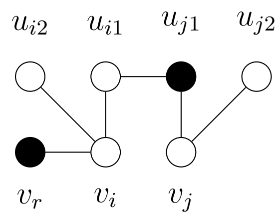

Figure 4.4: Diagram showing that if $G[v_i, v_j]$ is not a component in $G[S]$ then, $S' = (S - \{v_i, v_j\}) \cup \{u_{j1}\}$ is a k-PDS of G. The vertices in S' are shown in black.

that $u_{il}u_{r1} \in E(G)$ for some $u_{il} \in pn(v_i, S)$. Now let $S' = (S - \{v_i, v_r\}) \cup \{u_{r1}\}$, then $N[S']$ monitors S and all vertices in $N[S] - (pn(v_i, S) \cup pn(v_r, S))$. Since v_i and v_r have at most k unmonitored neighbours they will both propagate. This means that S' is a k-PDS of size smaller than S, which contradicts our choice of S. Thus, $|S| \leq 2$.

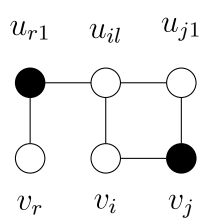

Figure 4.5: Diagram showing that if $|S| > 2$ then $S' = (S - \{v_i, v_r\}) \cup \{u_{r1}\}$ is a k-PDS of G. The vertices in S' are shown in black.

Since $|S| > 1$, we have that $S = \{v_i, v_j\}$, and so $V(G) = N[v_i] \cup N[v_j]$. Since $G[S]$ has no isolated vertices, G must at least have the edges shown in Figure 4.6. Now v_i and v_j must have degree $k+2$, since each has $k+1$ S-pn's. For the rest of the proof let $r = k+1$, $V_1 = \{v_i, u_{j1}, u_{j2}, ..., u_{jr}\}$ and $V_2 = \{v_j, u_{i1}, u_{i2}, ..., u_{ir}\}$.

Assume that $u_{i1}u_{ix} \in E$ for some $u_{ix} \in V_2$ and let $S' = \{u_{i1}\}$. Then, $P^0(S') \supseteq \{v_i, u_{j1}, u_{i1}, u_{ix}\}$. Since u_{i1} and u_{i1} are monitored v_i has at most k unmonitored neighbours and so v_i propagates. Then, $P^1(S') \supseteq V_2 \cup \{v_i, v_{j1}\}$, therefore since v_j has at most k unmonitored neighbours it will propagate and so $P^2(S') \supseteq V_1 \cup V_2 = V$. Thus, S' is a k-PDS that is smaller than S, which contradicts our choice of S. Therefore, our assumption is wrong and u_{i1} is isolated in $G[V_2]$. By symmetry we also have that u_{j1} is isolated in $G[V_1]$.

Now assume that $N[u_{j1}] < k+2$ and let $S' = \{v_j\}$. Then, $P^0(S') = V_1 \cup \{v_j\}$ and since v_j is monitored u_{j1} has at most k unmonitored neighbours, so u_{j1} will propagate. Consequently, $P^1(S') \supseteq V_1 \cup \{v_j, v_{i1}\}$. Since v_i now has at most k unmonitored neighbours it will also propagate and $P^2(S') \supseteq V_1 \cup V_2 = V$. Thus, S' is a k-PDS that is smaller than S which contradicts our choice of S. Therefore, our assumption is wrong and $N(u_{j1}) = V_2$, by symmetry we also have that $N(u_{i1}) = V_1$.

Assume now, that $xy \in E$ for some $x, y \in V_1$ and let $S' = \{x\}$. Then, $P^0(S') \supseteq \{x, y, v_j, v_{i1}\}$ and since $|N[v_j] - P^0(S')| \leq k + 2 - 2 = k$, v_j will propagate. Thus, $|N[v_i] - P^1(S')| \leq k + 2 - 2 = k$, so v_i will propagate. Hence, S' is a k-PDS that is smaller than S which contradicts our choice of S. Therefore, our assumption is wrong and V_1 is an independent set. By symmetry we also have that V_2 is an independent set. Therefore, G is a bipartite graph with $\{V_1, V_2\}$ a bipartition of G.

Assume that G is not a complete bipartite graph. Then, without loss of generality we can let $x \in V_1$ have degree less than $k+2$. Let $S' = \{v_j\}$, then $P^0(S') = V_1 \cup \{v_j\}$ and since v_j is monitored x has at most $k + 1 - 1 = k$ unmonitored neighbours, so x will propagate. Hence $P^1(S') \supseteq V_1 \cup \{v_j, v_{i1}\}$. Since v_i now has at most $k+2-2 = k$ unmonitored neighbours it will also propagate and $P^2(S') \supseteq V_1 \cup V_2 = V$. Thus, S' is a k-PDS that is smaller than S which contradicts our choice of S. Therefore, our assumption is wrong and G is a complete bipartite graph. Since $|V_1| = |V_2| = k + 2$, we have that $G \cong K_{k+2,k+2}$.

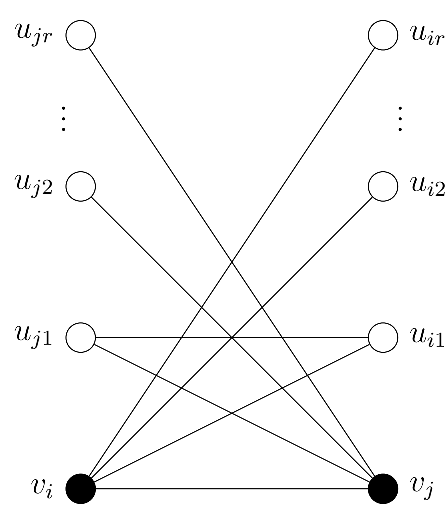

Figure 4.6: Diagram showing a spanning subgraph of G. The vertices of S are shown in black.

□

4.4 Bounds in claw-free cubic graphs

Cubic graphs have been studied extensively in relation to most aspects of graph theory. They are important as they often seem to be the simplest class of graphs for which a problem remains as difficult to solve as for a general graph [12].

In this section, we present an upper bound on $\gamma_p(G)$ for connected claw-free cubic graphs and characterise the graphs where the bound is sharp as proved in [5]. A *claw* is the complete bipartite graph $K_{1,3}$ and a *claw-free graph* is any graph G, such that there is no induced subgraph of G that is isomorphic to a claw.

In order to characterise the graphs that attain this bound, we define the following family of graphs. Note that the *diamond graph* is the graph obtained by removing one edge from K_4.

Definition 4.4.1 (D_k)**.** Let D be a diamond graph and let k be a positive integer. We denote by D_k the connected claw-free cubic graph formed from k disjoint copies of D by joining pairwise $2k$ vertices of degree two. That is,

$$V(D_k) = \{a_i, b_i, c_i, d_i | i \in \{1, ..., k\}\}$$
$$E(D_k) = \{a_ib_i, a_ic_i, b_ic_i, b_id_i, c_id_i, d_ia_{a+1} | i \in \{1, ..., k-1\}\} \cup \{d_ka_1\}$$

We note that D_1 is K_4. We denote by $\mathcal{D}$ the family of graphs D_k for all positive integers k.

The following theorem was first proved in [5].

Theorem 4.4.2 ([5])**.** *Let G be a connected claw-free cubic graph of order n. Then, $\gamma_p(G) \leq \frac{n}{4}$ with equality if and only if $G \in \mathcal{D}$.*

Proof. Since G is 3-regular it must have order $n \geq 4$. For the case where $n = 4$, we must have $G = K_4$ and so the theorem holds. We may now assume that $n \geq 5$.

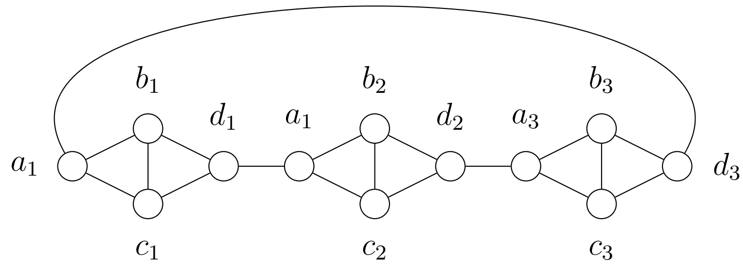

Figure 4.7: The graph $D_3 \in \mathcal{D}$.

Let S be a $\gamma_p(G)$-set of G such that the size of $G[S]$ is minimised. Amongst all such $\gamma_p(G)$-sets, let S also be such that $|N[S]|$ is maximised.

Claim (1)**.** *The set S is an independent set in G.*

Proof of claim. Suppose to the contrary that there exists a $uv \in E$ for vertices $u, v \in S$. If $|pn(v, S)| = 0$ then $N[S - \{v\}] = N[S]$ since $uv \in E$. Therefore $S - \{v\}$ is a PDS. If $pn(v, S) = \{z\}$ then $N[S - \{v\}] = N[S] - \{z\}$. Then v has one unmonitored neighbour and so v propagates to z. Then $S - \{v\}$ is a PDS, a contradiction. Therefore, $|pn(v, S)| \geq 2$. In fact we may let $pn(v, S) - S = \{x, y\}$, where $N(v) = \{u, x, y\}$ since $u \in S$. Since u is not adjacent to x or y, we must have $xy \in E$, otherwise $G[N[v]]$ is a claw. Now let $S' = (S - \{v\}) \cup \{x\}$. Then, x dominates y and u dominates v, therefore S' is a $\gamma_p(G)$-set. Since x is a external S-pn of v, x is isolated in S'. Therefore, $G[S']$ has less edges than $G[S]$. But this contradicts our choice of S, hence our assumption is false and S is an independent set in G.

We now categorise the vertices of G into two types. For any vertex $v \in S$, we label vertices in $N(v)$ as $N(v) = \{v_1, v_2, v_3\}$. By the claw-freeness, $G[N(v)]$ has at least one edge. Also it cannot have three edges, otherwise $G[N[v]]$ is a component of 4 vertices in G which contradicts the connectedness of G. A vertex v is called type 1 if $G[N(v)]$ has exactly one edge and it is called type 2 if $G[N(v)]$ has two edges.

Claim (2)**.** *Let $u, v \in S$. Then $|N[u] \cap N[v]| \leq 1$.*

Proof of claim. Let $u, v \in S$. By Claim 1, S is an independent set and $uv \notin E$. Now assume to the contrary that $|N[u] \cap N[v]| \geq 2$. Let $u_1 = v_1$ and $u_2 = v_2$. By the claw-freeness, $N(v)$ is not an independent set. Without loss of generality we can let v_1 be adjacent to at least v_2 or v_3. Then, either way $N(v_1) \subseteq N[S]$, since v_1 is adjacent to u and v. The two possibilities are shown in Figure 4.8. Let $S' = S - \{u\}$. Then, v dominates u_2 and u_1, and u_1 will propagate to u. Then, u will propagate to u_3. Therefore, S' is a PDS of G, which contradicts our choice S. Consequently, our assumption is wrong and $|N[u] \cap N[v]| \leq 1$ for any two distinct vertices $u, v \in S$.

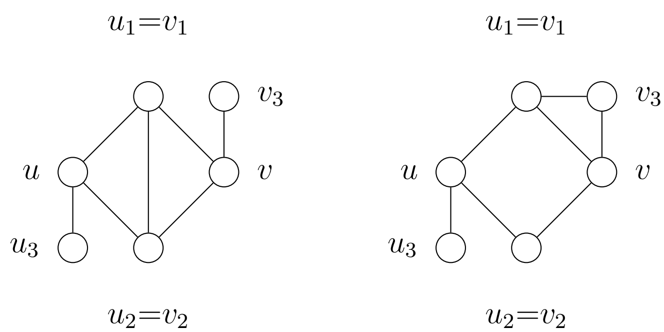

Figure 4.8: The graph on the left shows the case where $v_1v_2 \in E$ and the graph on the right shows the case where $v_1v_3 \in E$ from the proof of Claim 2.

Claim (3)**.** *Let u and v be two distinct vertices in S such that $u_1 = v_3$ and $u_1u_2 \in E$. Then, v is a type 1 vertex such that $v_1v_2 \in E$, $\{v_1, v_2\} \subseteq pn(v, S)$ and $N(v_1) \subseteq N[S]$.*

Proof of claim. We first prove that v is a type 1 vertex, $v_1v_2 \in E$, and $\{v_1, v_2\} \subseteq pn(v, S)$. Since $v_3 = u_1$ is adjacent to u, u_2 and v already, it is not adjacent to v_1 or v_2. But since $G[N[v]]$ is not a claw, we have $v_1v_2 \in E$. Then v is a type 1 vertex. See Figure 4.9. Suppose v_1 is not a S-pn of v and let $S' = S - \{v\}$. Then, u dominates v_3, which propagates to v. Now v can propagate to v_2 since v_1 is already monitored. Thus, S' is a PDS of G that is smaller than S, which contradicts our choice of S. By symmetry, we must also have that v_2 is a S-pn of v. So, $\{v_1, v_2\} \subseteq pn(v, S)$. We now prove that $N(v_1) \subseteq N[S]$. Let $S' = (S - \{v\}) \cup \{v_1\}$. Since u dominates v_3 and v_1 dominates $\{v, v_1, v_2\}$, we have that S' is a $\gamma_p(G)$-set of G. Since v_1 is an S-pn of v, the number of edges of $G[S']$ is no more than that of $G[S]$. Also, $N[S] \subseteq N[S']$. By the maximality condition on S, we must have $N[S] = N[S']$. This means that $N(v_1) \subseteq N[S]$. We note that the third neighbour of v_1, other than v and v_2, is forced to be in $N(S)$ since $N(v_1) \subseteq N[S]$ and v_1 is a S-pn of v.

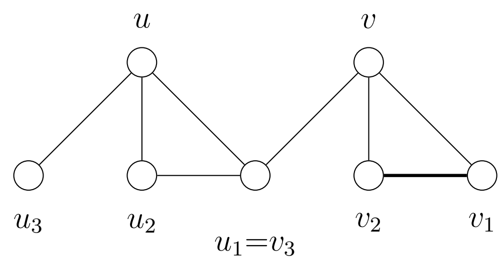

Figure 4.9: Diagram showing that the edge v_1v_2 must necessarily exist from the proof of Claim 3.

Claim (4)**.** *Let u and v be two distinct vertices in S such that $u_1v_3 \in E$ and $u_1u_2 \in E$. Furthermore let u_3 be adjacent to two vertices x and y not in $N[u]$ and let $N[x] \subseteq N[S] - \{u, u_2\}$. Then, $u_2v_3 \in E$ and v is a type 1 vertex such that $v_1v_2 \in E$, $\{v_1, v_2\} \subseteq pn(v, S)$ and $N(v_1) \subseteq N[S]$.*

Proof of claim. We first show that $u_2v_3 \in E$. Assume to the contrary that $u_2v_3 \notin E$. Note that v and u can not be adjacent as they both already have 3 neighbours. since $G[N(v_3)]$ is not a claw and $v, u \in N(v_3)$ are not adjacent, the third vertex in $N(v_3)$ must be adjacent to u or v. By our assumption u_2 is not adjacent to v_3, and by our setup u_1 is not adjacent to v_3. Therefore v_3 is adjacent to either v_2 or v_1. By the claw-freeness of $G[N[u_3]]$ we must have $xy \in E$, since neither x nor y are in $N[u]$. Then, $S' = S - \{u\}$ is a PDS as u_1 can be monitored by v_3 and u_3 by x. Then, u is monitored by u_3 and u_2 by u. This contradicts our choice of S, so $u_2v_3 \in E$. Now by the claw-freeness of $G[N[v]]$ and v_3 having degree three, we have $v_1v_2 \in E$ and v is a type 1 vertex. See Figure 4.10. The proof of the rest of the claim is the same as that for Claim 3, except that u dominates u_2 which then propagates to v_3, instead of u dominating v_3 directly as in Claim 3.

Claim (5)**.** *Let $x, y \in S$. Then $N[x] \cap N[y] = \emptyset$.*

Proof of claim. Assume to the contrary that there exists distinct vertices $u, v \in S$ such that $N[u] \cap N[v] \neq \emptyset$. By Claim 2, $|N[u] \cap N[v]| \leq 1$, and therefore $|N[u] \cap N[v]| = 1$. Let $u_1 = v_3$ then, by the claw-freeness we know that $N(u_1)$ is not independent, and by Claim 1 we know that $u, v \in N(u_1)$ are not adjacent. This implies that u_1 must have a neighbour in $N(u) \cup N(v)$, which without loss of generality we can say is u_2. Then, by Claim 3, we may assume that u and v are configured as in Figure 4.9. That is, $u_1 = v_3$, $u_1u_2 \in E$, $v_1v_2 \in E$ and $N(v_1) \subseteq N[S]$. We

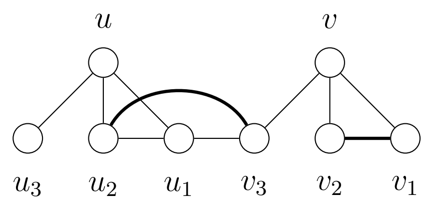

Figure 4.10: Diagram showing that the edges u_2v_3 and v_1v_2 must necessarily exist, and the third neighbour of v_1 must be in $N(S)$ from the proof of Claim 4.

now let $w^0 = u$ and $w^1 = v$. Choose a maximum r such that there are distinct vertices $w^0, w^1, ..., w^r \in S$ with the property that $w_1^i w_2^i \in E$ for $0 \leq i \leq r$ and $\{w_1^i w_3^{i+1}, w_2^i w_3^{i+1}\} \subseteq E$ for $1 \leq i \leq r-1$ and $N(w_1^r) \subseteq N[S]$. We note that such r exists as we can at least choose $r = 1$. By Claim 4, $\{w_1^r, w_2^r\} \subseteq pn(w^r, S)$, so we can let the third neighbour of w_1^r, other than w^r and w_2^r, be w_3^{r+1} for some $w^{r+1} \in S$. By Claim 3 (when $r = 1$) or Claim 4 (when $r \geq 2$), using w^{r-1} and w^r as our distinct vertices in S, we have a longer sequence $w^0, w^1, ..., w^r, w^{r+1}$. Then, the maximality of r implies that $w^{r+1} = w^i$ for some $0 \leq i \leq r-1$. By the construction of $w^0, w^1, ..., w^r$, w_3^0 is the only vertex in $N(\{w^0, ..., w^r\})$ where all of its neighbours has not already been determined, thus $w^{r+1} = w^0$. We can now apply Claim 4 again by using w^r and $w^{r+1} = w^0 = u$, which gives that $u_1, u_2 \subseteq pn(u, S)$ and $N(v_1) \subseteq N[S]$. Consequently the third neighbour of u_1 besides u and u_2 is in $N(S)$. Then $y \in N(S)$, which contradicts that S is an independent set. Therefore, our assumption is wrong and $N[x] \cap N[y] = \emptyset$.

We can now complete the proof of the theorem. By Claim 5 we can say that $|N[S]| = 4|S|$ since each $N[v]$ has 4 vertices and $N[u] \cap N[v] = \emptyset$ for all $u, v \in S$. Therefore, G has at least $4|S|$ vertices and so $\gamma_p(G) \leq \frac{n}{4}$. On the other hand, $\gamma_p(D_k) \leq k = \frac{n}{4}$ as we can choose a vertex in each diamond of D_k to power dominate D_k. Assume S is a PDS of D_k with $|S| < k$. Then by the pigeonhole principle, there exists an i such that $a_i, b_i, c_i, d_i \notin S$. Without loss of generality assume a_i monitors c_i. Then d_i must have monitored b_i in a previous step. But then d_i was adjacent to two unmonitored vertices when it propagated to b_i, contradicting the rules of propagation. Therefore $\gamma_p(D_k) = k = \frac{n}{4}$.

Conversely, let $\gamma_p(G) = \frac{n}{4} = k$. Again, let S be a $\gamma_p(G)$-set of G such that the size of $G[S]$ is minimised. amongst all such S, let S also be chosen such that $|N[S]|$ is maximised. By Claim 5, $|N[S]| = 4|S| = |V|$. Therefore $\{N[v] : v \in S\}$ is a partition of V. For a type 2 vertex $v \in S$, its closed neighbourhood $N[v]$ induces a diamond. So, if all vertices in S are of type 2, then each diamond must be connected to each other as in D_k since G is regular. Thus, because $|V| = 4|S|$ we have that $G \cong D_k$.

Now assume that S has at least one type 1 vertex. Choose a maximum r such that there are distinct type 1 vertices $w^1, w^2, ..., w^r$ with $w_1^i w_2^i \in E$ for $1 \leq i \leq r$ and $\{w_1^i w_3^{i+1}, w_2^i w_3^{i+1}\} \subseteq E$ for $1 \leq i \leq r-1$. Now w_1^r is adjacent to w^r and w_2^r. Additionally there is some vertex $w^{r+1} \in S - \{w^r\}$ such that W_1^r is adjacent to w_3^{r+1}. By Claim 4, w^{r+1} is a type 1 vertex in S with $w_1^{r+1} w_2^{r+1} \in E$ and $w_1^r w_3^{r+1} \in E$. This is shown in Figure 4.11. By the maximality of r, the vertex w^{r+1} is equal to some w^i. Since w_3^0 is the only vertex in $N(\{w^0, ..., w^r\})$ that does not already have its neighbour's determined by the construction, we must have that $w^{r+1} = w^0$. This means that $G[N[\{w^1, w^2, ..., w^r\}]]$ induces a D_r. Since G is connected and $G[N[\{w^1, w^2, ..., w^r\}]]$ is a component, we have $G \cong D_r$.

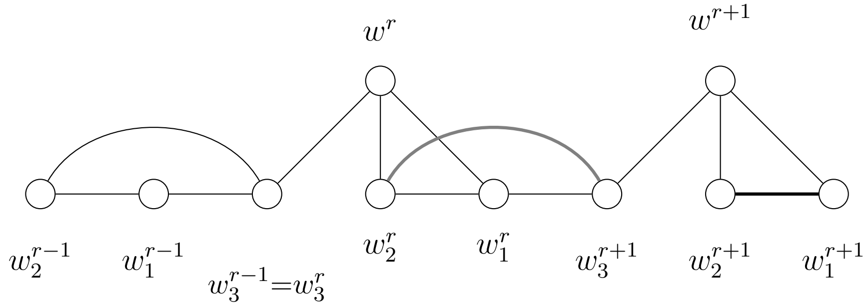

Figure 4.11: Diagram showing that the edges $w_2^r w_3^{r+1}$ and $w_1^{r+1} w_2^{r+1}$ must necessarily exist, and the third neighbour of w_1^{r+1} must be in $N(S)$.

□

Chapter 5

Effect of small graph changes

5.1 Introduction

We now investigate how small changes in a graph can affect the power domination number of the resulting graph. We will look at edge removal, vertex removal and edge contraction. We will also present some example graphs that show how the domination number and power domination number can change under these small changes. The examples will prove that all the bounds in this chapter are sharp. They will also show that the ratio of the change in power domination number to change in domination number can be arbitrarily large or small.

In order to efficiently deal with edge removal, vertex removal and edge contraction, we first prove a technical lemma that will be very useful. The lemma essentially states that for two graphs which differ only on parts that are already monitored, the propagation in the unmonitored parts behaves the same. For a graph G and two subsets X and Y of $V(G)$, we denote by $E_G(X, Y)$ the set of edges $uv \in E(G)$ such that $u \in X$ and $v \in Y$. We note that if $X \subseteq Y$, $E_G(X, Y)$ contains in particular all edges of the induced subgraph $G[X]$.

5.2 Useful lemma

Definition 5.2.1. Let G and H be graphs such that $V(H) \subseteq V(G)$. Then, the *natural mapping* $f : V(H) \to V(G)$ is defined by $f(u) = u$ for all $u \in V(H)$.

Lemma 5.2.2 ([10])**.** *Let G and H be graphs, $S \subseteq V(G)$ and let i be a non-negative integer. Let $X = V(G) - P_G^i(S)$ and G' be the subgraph with vertex set $N_G[X]$ and edge set $E_G(X, N_G[X])$. Let $Y \subseteq V(H)$ such that $H' = (N_H[Y], E_H(Y, N_H[Y]))$ is isomorphic to G'. Let $\psi : N_G[X] \to N_H[Y]$ be an isomorphism such that $Y = \psi(X)$. Let $T \subseteq V(H)$ and j be a non-negative integer such that $Y \subseteq V(H) - P_H^j(T)$. If T is a PDS of H then S is a PDS of G.*

Proof. For $l \geq 0$, let $X^l = X \cap P_G^{l+i}(S)$ and $Y^l = Y \cap P_H^{l+j}(T)$. We prove by induction that for all l, $Y^l \subseteq \psi(X^l)$.

For $l = 0$, $X^0 = X \cap P_G^i(S) = (V(G) - P_G^i(S)) \cap P_G^i(S) = \emptyset$. Similarly $Y^0 = Y \cap P_H^j(T) = (V(H) - P_H^j(T)) \cap P_H^j(T) = \emptyset$. Since $\emptyset \subseteq \emptyset$, the property is true for $l = 0$.

We now assume that $Y^l \subseteq \psi(X^l)$ for some $l \geq 0$. Assume that some vertex $v \in N_H[Y]$ satisfies the conditions for propagation in H at step $j + l$, i.e. $v \in P_H^{j+l}(T)$ and $|N_H[v] - P_H^{j+l}(T)| \leq 1$. Then, v has at most one neighbour in $Y - Y^l$. Let $v = \psi(u)$. We show that u also satisfies the conditions for propagation in G.

First, we show that u is monitored at step $l + i$. If $u \notin X$ then, by definition of X, $u \in P_G^i(S) \subseteq P_G^{l+i}(S)$. If $u \in X$ then, $v \in Y$ since $\psi(X) = Y$. Therefore, $v \in Y \cap P_H^{l+j}(T) = Y^l$. By the inductive hypothesis $\psi(u) = v \in Y^l \subseteq \psi(X^l)$. Therefore, $u \in X^l \subseteq P_G^{l+i}(S)$. That is, u is unmonitored at step $l + i$.

Assume now that u has a neighbour, u', that is not monitored at step $P_G^{l+i}(S)$. Then, $u' \notin X^l$ and by the inductive hypothesis $\psi(u') \notin Y^l$ since $Y^l \subseteq \psi(X^l)$. Therefore, $u' \in X - X^l$ and $\psi(u') \in Y - Y^l$. Since G' and H' are isomorphic, $\psi(u')$ is adjacent to $v = \psi(u)$. But v has only one neighbour in $Y - Y^l$, therefore u has at most one unmonitored neighbour. Then, u propagates in G. Then, $Y^{l+1} \subseteq \psi(X^{l+1})$. By induction $Y^l \subseteq \psi(X^l)$ for all $l \geq 0$.

Since T is a PDS of H, $P_H^\infty(T) = V(H)$. Therefore, $Y^\infty = Y \cap P_H^\infty(T) = Y = \psi(X)$. Then, by induction $Y^\infty = \psi(X) \subseteq \psi(X^\infty)$. But by definition $X^l \subseteq X$, so $X^\infty = X$.

We want to show that $P_G^{\infty}(S) = V(G)$. To do this we manipulate $P_G^{i+l}(S)$.

$$\begin{aligned}
P_G^{i+l}(S) &= P^i \cup (P^{i+l} - P^i) \\
&= P^i \cup (P^{i+l} \cap (V(G) - P^i)) \\
&= P^i \cup (P^{i+l} \cap X) \\
&= P^i \cup X^l \\
&= (V(G) - X) \cup X^l
\end{aligned}$$

If we let $l = \infty$ then, we have that

$$\begin{aligned}
P_G^{\infty}(S) &= (V(G) - X) \cup X^{\infty} \\
&= (V(G) - X) \cup X \\
&= V(G)
\end{aligned}$$

Therefore, S is a PDS of G. $\square$

5.3 Edge removal

In this section we present an upper and lower bound for the power domination number of a graph when a single edge is removed. These bounds will be in terms of the power domination number of the original graph. We will show examples where these bounds are sharp. Let G be a graph and uv an edge of G. To get a lower bound on $\gamma_p(G - uv)$ we take a $\gamma_p(G)$-set and by carefully choosing to add u or v, we get a PDS of $G - uv$.

To get an upper bound on $\gamma_p(G - uv)$ we follow a similiar procedure. We take a $\gamma_p(G - uv)$-set and by carefully choosing to add u or v, we obtain a PDS of G.

We now prove an upper and lower bound for $\gamma_p(G-uv)$. This proof was first shown in [10].

Theorem 5.3.1 ([10])**.** *Let G be a graph and $e \in E(G)$. Let $H = G - e$. Then, $\gamma_p(G) - 1 \leq \gamma_p(H) \leq \gamma_p(G) + 1$.*

Proof. First we prove that $\gamma_p(H) \leq \gamma_p(G) + 1$. Let G be a graph, $e \in E(G)$ and $H = G - e$. Let T be a $\gamma_p(G)$-set. If T is a PDS of H then, $\gamma_p(H) \leq |T| = \gamma_p(G)$. We now assume that T is not a PDS of H.

Let j be the smallest integer such that $P_H^j(T) \subset P_G^j(T)$. We can be sure that j exists since $P_G^\infty(T) = V(G)$ and $P_H^\infty(T) \subset V(G)$. Let $v \in P_G^j(T) - P_H^j(T)$, such that $v \notin P_G^{j-1}(T)$. Then, v has some neighbour $u \in P_G^{j-1}$ such that $|N_G[u] - P_G^{j-1}(T)| \leq 1$. By our choice of j we must have that $P_G^{j-1}(T) = P_H^{j-1}(T)$ and so $|N_H[u] - P_H^{j-1}(T)| \leq 1$. Since $v \in P_G^j(T) - P_H^j(T)$ we have that $v \notin P_H^j(T)$. Then, u does not propagate to v in $P_H^j(T)$. Thus, $uv \notin E(H)$ and $uv \in E(G)$, so $e = uv$. Since $v \notin P_H^j(T)$, we must have that $v \notin T$.

Let $S = T \cup \{v\}$. Then, $P_H^j(T) \subseteq P_H^j(S)$. Therefore, $u \in P_H^j(S)$. Let $X = V(H) - P_H^j(S) = Y$. Then, since $u, v \notin X$, $H[X] \cong G[Y]$. We also have that $Y = V(H) - P_H^j(S) = V(G) - P_G^j(S) \subseteq V(G) - P_G^j(T)$. Then, by Lemma 5.2.2 S is a PDS of H. Then, $\gamma_p(H) \leq |S| = |T| + 1 = \gamma_p(G) + 1$.

We now prove that $\gamma_p(G) - 1 \leq \gamma_p(H)$. Let T be a $\gamma_p(H)$-set and $e = uv$. Let $S = T \cup \{v\}$. Then, $u, v \in P_G^0(S)$ since, $v \in S$ and $u \in N_G(v)$. Let $X = V(G) - P_G^0(S) = Y$. Since $u, v \notin X$, $G[X] \cong H[Y]$. We also have that $Y = V(G) - P_G^0(S) \subseteq V(G) - P_G^0(T)$. Then, by Lemma 5.2.2 S is a PDS of G. Then, $\gamma_p(G) \leq |S| \leq |T| + 1 = \gamma_p(H) + 1$. If we rearrange this we get $\gamma_p(G) - 1 \leq \gamma_p(H)$. $\square$

We now give some of our own examples to illustrate how edge removal can change the domination and power domination numbers of specific graphs.

Example 5.3.2. In this example we show that removing an edge can leave the power domination number unchanged. Let $G = C_n$. Then, $\gamma_p(G) = 1$. If we remove any edge $e \in E(C_n)$ we get that $G - e = P_n$ and $\gamma_p(G - e) = 1 = \gamma_p(G)$. This is shown in Figure 5.1 which depicts C_6 and $C_6 - e$.

Figure 5.1: The graph C_6 is shown on the left and $C_6 - xy$ is shown on the right. We can see that $\gamma_p(C_6 - xy) = 1 = \gamma_p(C_6)$ and $\gamma(C_6) = 2 = \gamma_p(C_6 - xy)$. The black vertices show a minimum PDS while the union of black and grey vertices show a minimum dominating set.

Example 5.3.3. In this example we show that by removing an edge the power domination number can increase by one while the domination number remains the same. This example also shows that the upper bound in Theorem 5.3.1 is sharp. Let G be the graph shown on the left in Figure 5.2. Let $(G - xy)$ be the graph on the right in Figure 5.2. A minimum PDS is highlighted in black for both graphs. It is clear that $\gamma_p(G - xy) = \gamma_p(G) + 1$. However, we see that the domination number is unchanged as shown by the minimum dominating sets which are made of the union of the black and gray vertices.

Figure 5.2: The graph G is shown on the left and $G - xy$ is shown on the right. We can see that $\gamma_p(G - xy) = 1 = \gamma_p(G) + 1$. However $\gamma(G) = 2 = \gamma_p(G - xy)$. The black vertices show a minimum PDS while the union of black and grey vertices show a minimum dominating set.

Example 5.3.4. In this example we show that removing an edge can decrease the power domination number by one. This also shows that the upper bound in Theorem 5.3.1 is sharp. Let G be $K_{2,2}$ and $G' = G - e$ see Figure 5.3. A minimum PDS is highlighted in black for both graphs. A minimum dominating set is shown as the union of black and gray vertices for both graphs. By inspection we see that $\gamma_p(G) = 2$, $\gamma(G) = 2$, $\gamma_p(G') = 2$ and $\gamma(G') = 1$.

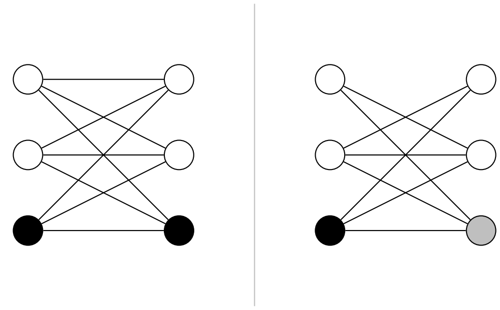

Figure 5.3: The graph G is shown on the left and G' is shown on the right. We can see that $\gamma_p(G') = 1 = \gamma_p(G) + 1$. However, $\gamma(G) = 2 = \gamma_p(G - xy)$. The black vertices show a minimum PDS while the union of black and grey vertices show a minimum dominating set.

5.4 Vertex removal

In this section we present a lower bound for the power domination number of a graph when a single vertex is removed. This bound will be in terms of the power domination number of the original graph. We will also show that there is no upper bound in terms of the power domination number of the original graph. We will show examples where the lower bound is sharp and why it has no upper bound. Let G be a graph and v a vertex of G. To get a lower bound on $\gamma_p(G-v)$ we take a $\gamma_p(G)$-set and by adding v, we get a PDS of $G - uv$. To show there is no upper bound we show an example where the power domination number becomes the order of the new graph.

We will then show an original proof for an upper bound in terms of the neighbourhood of the removed vertex. To get this upper bound on $\gamma_p(G-v)$, we iteratively remove edges incident to v from G.

We begin by showing a proof of a lower bound. This proof was first shown in [10].

Theorem 5.4.1 ([10])**.** *Let G be a graph and let $v \in V(G)$. Then, $\gamma_p(G) - 1 \leq \gamma_p(G - v)$. Moreover there is no function f such that $\gamma_p(G - v) \leq f(\gamma_p(G))$.*

Proof. We first prove the lower bound. Let G be a graph and $v \in V(G)$. Let $H = G - v$ and let T be a $\gamma_p(H)$-set. Let $S = T \cup \{v\}$. We note that $v \notin T$ since $v \notin V(H)$. Let $X = V(G) - P^0_G(S) = Y$. Therefore, since $v \in P^0_G(S)$,

$v \notin X$ and hence $G[X] \cong H[Y]$. We also notice that T is a $\gamma_p(H)$-set and $Y = V(G) - P_G^0(S) \subseteq V(H) - P_H^0(T)$. Thus, by Lemma 5.2.2 S is a PDS of G. Then, $\gamma_p(G) \leq |S| \leq |T| + 1 = \gamma_p(H) + 1$. This implies that $\gamma_p(G) - 1 \leq \gamma_p(G - v)$.

We now prove that there is no upper bound to $\gamma_p(G-v)$ in terms of $\gamma_p(G)$. Consider the star $G = K_{1,n}$. Since the central vertex is adjacent to every other vertex, $\gamma_p(G) = 1$. Let c be the central vertex. Then, since every edge is incident to c, $G - c \cong \overline{K_n}$. Therefore, $\gamma_p(G - c) = n$.

Assume that there is some function f such that $\gamma_p(G-v) \leq f(\gamma_p(G))$. Since $\gamma_p(K_{1,n})$ is always one, we have that $f(\gamma_p(K_{1,n})) = f(1)$. Now let $n = f(1) + 1$ and let c be the centre of $K_{1,n}$. Then $\gamma_p(K_{1,n} - c) = n = f(1) + 1 > f(1) = f(\gamma_p(K_{1,n}))$. This contradicts the definition of f and therefore f does not exist. $\square$

We now present an original upper bound for the power domination number of the graph $G - v$ in terms of the neighbourhood of v in G.

Corollary 5.4.2. *Let G be a graph and let $v \in V(G)$. Then, $\gamma_p(G - v) \leq \gamma_p(G) + |N_G(v)| - 1$.*

Proof. Let G be a graph and let $v \in V(G)$. Then, you can construct $G-v$ by deleting every edge incident to v from G and then removing v. We know from Theorem 5.3.1 that removing one edge from a graph can increase the power domination number by at most one. Let G' be the graph obtained by repeatedly removing edges incident to v from G until v is an isolated vertex. Then, $\gamma_p(G') \leq \gamma_p(G) + |N(v)|$. Since v is an isolated vertex of G', we know that v must be an element of any PDS of G'. Therefore, $\gamma_p(G' - v) = \gamma_p(G') - 1$. Thus, $\gamma_p(G - v) \leq \gamma_p(G) + |N(v)| - 1$. $\square$

We now give some examples of how the power domination number and the domination number can change when a single vertex is removed from a graph.

Example 5.4.3. In this example we show that the upper bound in Corollary 5.4.2 is sharp. We see that it is possible for γ_p to increase up to the upper bound, while γ decreases by one. Let G_n be the graph obtained by adding two vertices v' and v'' for each $v \in C_n$ and then adding a vertex w and all edges of the form wv' where $v \in C_n$. Then the set $\{c_1, c_2, ..c_n\} \cup \{w\}$ is a γ-set of G_n. The singleton $\{w\}$ is a γ_p set of G_n. We also see that $\{c_1, c_2, ..c_n\}$ is a γ-set and a γ_p-set of $G_n - w$. This is shown in Figure 5.4.

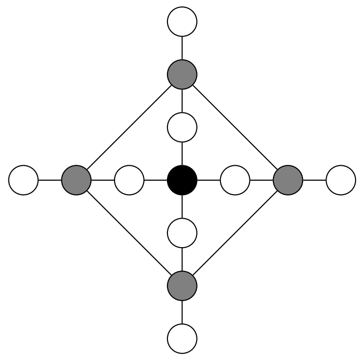

Figure 5.4: The graph on the left is G_4 with $\gamma_p(G_4) = 1$, $\gamma(G_4) = 5$. The graph on the right is $G_4 - v$ with $\gamma_p(G_4 - v) = \gamma_p(G_4) + |N(v)| - 1 = 4$, $\gamma(G_4 - v) = 4$. We note that $\gamma_p(G_4 - v)$ also reaches the bound in 4.2.3. The black vertices show a minimum PDS while the union of black and grey vertices show a minimum dominating set.

Example 5.4.4. In this example we show that by removing a vertex the power domination number can decrease by one while the domination number increases arbitrarily high. This example also shows that the lower bound in Theorem 5.4.1 is sharp. Let W_{3k} be the wheel graph with central vertex c and cycle $c_1, c_2, ..., c_{3k}$. Let G_{3k} be the graph where

$$V(G_{3k}) = V(W_{3k}) \cup \{u, v, w\}, \text{and}$$

$$E(G_{3k}) = E(W_{3k}) \cup \{uv, uw, uc_1, uc_2\}$$

Then, for any k, $\gamma_p(G_{3k}) = \gamma(G_{3k}) = 2$. For $G_{3k} - c$, we have that $\gamma_p(G_{3k} - c) = 1$ and $\gamma(G_{3k} - c) = k + 1$. An example is shown in Figure 5.5.

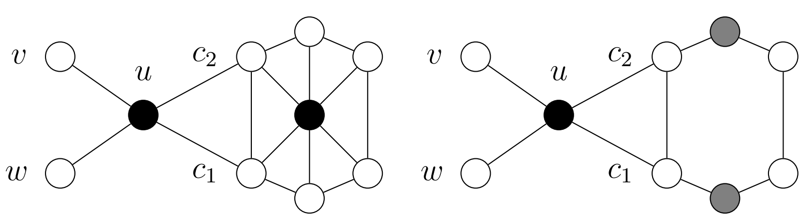

Figure 5.5: The graph G_6 is shown on the left and $G_6 - c$ is shown on the right. We note that $k = 2$. We can see that $\gamma_p(G_6) = 2 = \gamma_p(G_6 - c) + 1$. However $\gamma(G_6) = 2$ and $\gamma(G_6 - c) = 1 + k = 3$. The black vertices show a minimum PDS while the union of black and grey vertices show a minimum dominating set.

Example 5.4.5. In this example we show that by removing a vertex the power domination number can remain the same while the domination number decreases by one. Let $G = P_{3k+1}$ where $k \geq 1$. Then, $\gamma(G) = \lceil \frac{3k+1}{3} \rceil = k + 1$ and $\gamma_p(G) = 1$. Let v be an end vertex of G. Let $H = G - v = P_{3k}$. Then, $\gamma(H) = \lceil \frac{3k}{3} \rceil = k$ and $\gamma_p(H) = 1$. Therefore, the domination number decreased but the power domination number was unchanged. An example is shown in Figure 5.6.

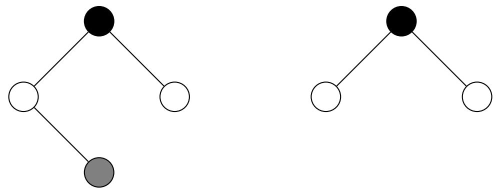

Figure 5.6: The graph $G = P_4$ is shown on the left and $G - v = P_3$ is shown on the right. The black vertices show a minimum PDS while the union of black and grey vertices show a minimum dominating set.

5.5 Edge contraction

In this section we present upper and lower bounds for the power domination number of a graph when a single edge is contracted. These bounds will be in terms of the power domination number of the original graph. Let w be the vertex created by contracting $uv \in E(G)$. To get a lower bound on $\gamma_p(G \sim uv)$ we take a $\gamma_p(G)$-set and by adding w, we get a PDS of $G \sim uv$.

To get an upper bound on $\gamma_p(G \sim uv)$ we follow a similiar procedure. We take a $\gamma_p(G \sim uv)$-set and by carefully choosing to add u or v, we get a PDS of G.

We now show a proof of an upper and lower bound for $\gamma_p(G \sim uv)$. This proof was first shown in [10].

Theorem 5.5.1 ([10])**.** *Let G be a graph and $e \in E(G)$. Let $H = G \sim e$. Then, $\gamma_p(G) - 1 \leq \gamma_p(H) \leq \gamma_p(G) + 1$.*

Proof. We will first prove that $\gamma_p(G) - 1 \leq \gamma_p(H)$. Let $e = uv$ be an arbitrary edge in G and let $H = G \sim e$. Let w_{uv} be the vertex obtained by contracting e. Let T be a minimum PDS of H. We consider 2 cases:

1. *Let* $w_{uv} \in T$. Let $S = (T - \{w_{uv}\}) \cup \{u, v\}$. Then, $N_G(S) - S = N_H(T) - T$. Let $X = V(G) - P_G^0(S)$ and let f be the natural mapping from $V(G) - \{u, v\}$ to $V(H) - \{w_{uv}\}$. Let $Y = f(X)$. We know that $S \cap N_G(X) = \emptyset$ and $T \cap N_H(Y) = \emptyset$ by the definition of X and Y. Then, $u, v \notin N_G(X)$ and $w_{uv} \notin N_H(Y)$, since $u, v \in S$ and $w_{uv} \in T$. Let G' be the graph $G' = (N_G[X], E_G(X, N_G[X])$ and H' be the graph $H' = (N_H[Y], E_H(Y, N_H[Y])$. Then $G' \cong H'$ via f. Then, by Lemma 5.2.2, S is a PDS of G. Therefore, $\gamma_p(H) = |T| = |S| - 1 \geq \gamma_p(G) - 1$. If we rearrange this we get $\gamma_p(G) - 1 \leq \gamma_p(H)$.

2. *Let* $w_{uv} \notin T$. Let j be the smallest integer such that $w_{uv} \in P_H^j(T)$. Let q be a neighbour of w_{uv} that propagates to w_{uv} in $P_H^j(T)$. that is, $q \in P_H^{j-1}(T)$ and $|N_H[q] - P_H^{j-1}(T)| = 1$. By the definition of edge contraction, at least one of qu, qv must exist in $E(G)$. We can assume without loss of generality that $qv \in E(G)$. We now let $S = T \cup \{u\} \subseteq V(G)$. Let $X = V(G) - P_G^j(S)$. We now further split this into two sub cases.

   (a) *Let* $v \notin N_G(X)$. Let f be the natural mapping from $V(G) - \{u, v\}$ to $V(H) - \{w_{uv}\}$. Let $Y = f(X)$. Then, by the definition of S and Y, $Y \subseteq V(H) - P_H^j(T)$. We know that $S \cap N_G[X] = \emptyset$ by the definition of X. Then, $u \notin N_G[X]$ since $u \in S$. Since Y is the image of X and $u, v \notin N_G[X]$, we must have that $w_{uv} \notin N_H[Y]$. Let G' be the graph $G' = (N_G[X], E_G(X, N_G[X])$ and H' be the graph $H' = (N_H[Y], E_H(Y, N_H[Y])$. Then $G' \cong H'$ via f. Then, by Lemma 5.2.2, S is a PDS of G. Therefore, $\gamma_p(H) = |T| = |S| - 1 \geq \gamma_p(G) - 1$. If we rearrange this we get $\gamma_p(G) - 1 \leq \gamma_p(H)$.

   (b) *Let* $v \in N_G(X)$[1]. Let f be the map from $V(G) - \{v\}$ to $V(H)$ such that $f(z) = z$ for all $z \neq u$ and $f(u) = w_{uv}$. Let $Y = f(X)$. Then, by the definition of S and Y, $Y \subseteq V(H) - P_H^j(T)$. We know that $S \cap N_G[X] = \emptyset$ by the definition of X. Then, $u \notin N_G[X]$ since $u \in S$. Since $v \in S$, $N_G(v) \subseteq P_G^0(S)$. Then, $N_G(v) \cap X = \emptyset$. Let G' be the graph $G' = (N_G[x], E_G(X, N_G[X])$ and H' be the graph $H' = (N_H[X], E_H(Y, N_H[Y])$. Then $G' \cong H'$ via f. Then, by Lemma 5.2.2, S is a PDS of G. Therefore, $\gamma_p(H) = |T| = |S| - 1 \geq \gamma_p(G) - 1$. If we rearrange this we get $\gamma_p(G) - 1 \leq \gamma_p(H)$.

We now prove that $\gamma_p(H) \leq \gamma_p(G) + 1$. Let T be a minimum PDS of G and let $S = (T - \{u, v\}) \cup \{w_{uv}\}$. Let j be the smallest integer such that $(N_G[u] \cup N_G[v]) \subseteq$

[1] In the original paper [10], this case is not mentioned.

$P_G^j(T)$. Then, we can use Lemma 5.2.2 with the natural mapping from $H - w_{uv}$ to $G - \{u, v\}$ and $i = j$, and get that S is PDS of H. Therefore, $\gamma_p(H) \leq |S| = |T| + 1 = \gamma_p(G) + 1$. □

We now give some of our own examples to illustrate how edge contraction can change the domination and power domination numbers of specific graphs.

Example 5.5.2. For the path graph P_n, contracting any edge $e \in E(P_n)$ results in $P_n \sim e = P_{n-1}$. Therefore, contracting any edge does not change the power domination number. Similarly for $n \geq 4$ $C_n \sim e = C_{n-1}$ and for $n \geq 2$, $K_n \sim e = K_{n-1}$ so the power domination number of all cycles and complete graphs is unchanged by edge contraction. Therefore, by induction the power domination number of any path, cycle or complete graph is unchanged by contracting an arbitrary number of edges.

Example 5.5.3. In this example we show that by contracting an edge the power domination number can decrease by one. This also shows that the lower bound in Theorem 5.5.1 is sharp. Let $\mathcal{T}$ be the family of graphs as described in Definition 4.2.1. Let $G \in \mathcal{T}$. Let G' be a graph which is obtainable by contracting an edge of G. Then $\gamma_p(G') = \gamma_p(G) - 1$. An example is shown in Figure 5.7.

Example 5.5.4. In this example we show that by contracting an edge the power domination number can increase by one while the domination number decreases by one. This also shows that the upper bound in Theorem 5.8 is sharp. Let G_n be the graph obtained by taking the wheel graph W_n and appending adding a vertex v' and edge vv' for each vertex $v \in V(W_n)$ that is not the central vertex of W_n. Let G' be the graph shown obtained by contracting a cycle edge from G_n. Then, we can see that $\gamma_p(G') = \gamma_p(G) + 1$ and $\gamma(G') = \gamma(G) - 1$. An example is shown in Figure 5.8.

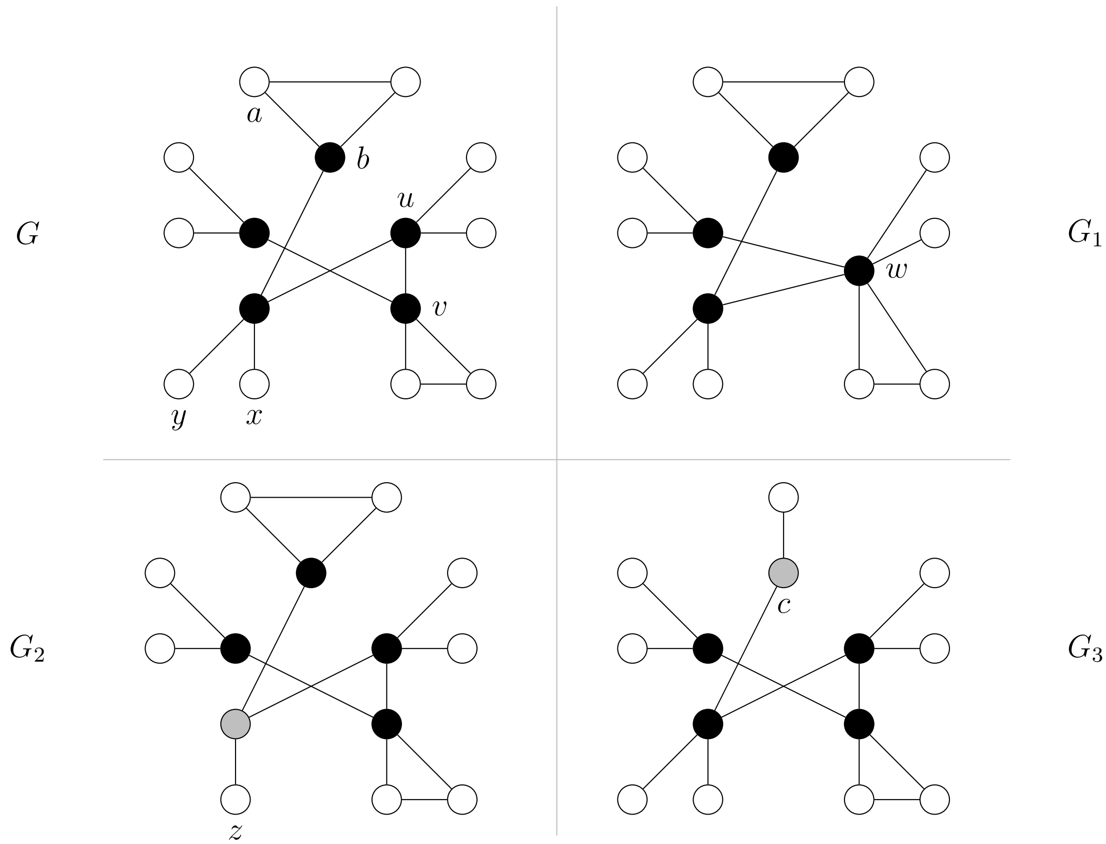

Figure 5.7: The graph G is an element of $\mathcal{T}$. It has order 15 and by Lemma 4.2.2, $\gamma_p(G) = 5$. The other three graphs shown are $G_1 = G \sim uv$, $G_2 = G \sim xy$, and $G_3 = G \sim ab$. The black vertices show a minimum PDS while the union of black and grey vertices show a minimum dominating set.

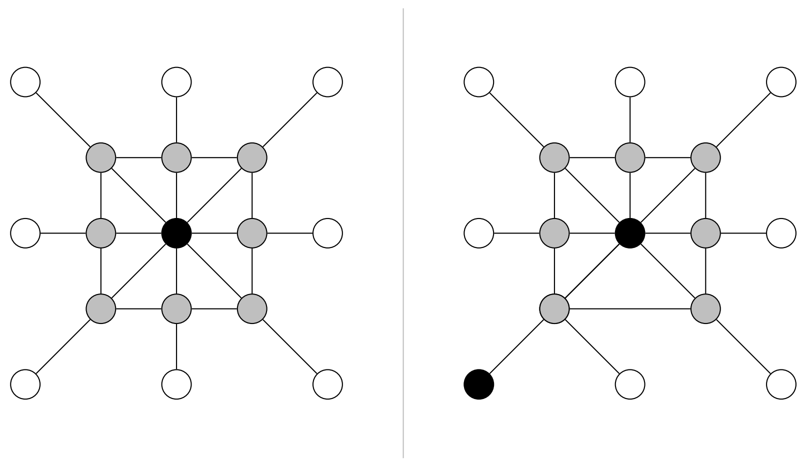

Figure 5.8: Diagram showing an example of the family described in Example 5.5.4. The graph G_9 is shown on the left and the graph G'_9 is shown on the right. The black vertices form a γ_p-set and the grey vertices show a γ-set. Note that unlike the other examples the γ-set is formed by only the grey vertices.

Chapter 6

Trees and unicyclic graphs

6.1 Introduction

In this chapter we will focus on trees and connected unicyclic graphs. Trees are acyclic connected graphs. Many graph algorithms that are NP-hard on general graphs are much easier on trees, often having linear time algorithms. In particular it is NP-hard to find the power domination number of a graph, but the problem becomes linear when restricted to finding the power domination number of a tree [15]. Connected unicylic graphs are a class of graphs that are very similiar to trees. This is because for every connected unicyclic graph G, removing any edge from the cycle in G, results in a tree.

6.2 Trees

Trees have a lot of properties that general graphs do not. One important property of trees is that there is only one path between any two vertices in a tree. This allows us to talk about the unique path between any two vertices. This leads us into our next definition.

Definition 6.2.1 (root)**.** Let T be a tree. The *root* of T is a single chosen vertex, denoted by r. Let $u, v \in T$ and let T have root r. If u is on the $r - v$ path in T, then u is an *ancestor* of v and v is a *descendant* of u. If u is the closest ancestor to v on the $r-v$ path, Then u is the youngest ancestor of v. If u is on the $r-v$ path in T and $uv \in E(T)$, then u is the *parent* of v and v is a *child* of u. Since there is only one path between any two vertices in T, every vertex except r has exactly one parent.

Definition 6.2.2 (Branching vertex)**.** Let G be a graph and $v \in V(G)$. If $deg(v) \geq 3$, then v is a *branching vertex* of G.

Definition 6.2.3 (Segment)**.** Let G be a graph. A segment is an induced subgraph H of G such that H is a path, the endpoints of H do not have degree 2 in G and all other vertices in $V(H)$ have degree 2 in G. A *branching segment of* G is a segment that contains two branching vertices of G. A *leaf segment of* G is a segment that contains at least one leaf vertex of G.

We now define the spider number of a graph. Recall that a spider graph is a tree with at most one vertex of degree greater than 2.

Definition 6.2.4 (spider partition)**.** Let G be a graph and let $\mathcal{X} = \{X_1, ..., X_n\}$ be a partition of $V(G)$. Then $\mathcal{X}$ is a *spider partition* of $V(G)$ if $G[X_i]$ is a spider for all $X_i \in \mathcal{X}$. The *spider number* of G, denoted as $sp(G)$, is the minimum cardinality of a spider partition of $V(G)$.

Remark 6.2.5. Let G be a graph with order n. Since the trivial graph K_1 is itself a spider, it follows immediately that any graph G admits a spider partition and $sp(G) \leq n$.

It turns out that the problem of determining the power domination number of a tree can be translated to determining the spider partition number of a tree. This is true, because $\gamma_p(T) = sp(T)$, as proved in [15]. For the remainder of this section, we present our own proof of this result. The first step is proving the following Lemma.

Lemma 6.2.6 ([15])**.** *Let T be a tree. Then T is a spider if and only if $\gamma_p(T) = 1$.*

Proof. Let T be a spider. If $\Delta(T) < 3$ then by Theorem 3.3.1, $\gamma_p(T) = 1$. We may then assume that $\Delta(T) \geq 3$. Then, by Theorem 3.3.6 there is a $\gamma_p(T)$-set, S, such that every vertex in S has degree at least three. Since there is only one vertex in T with degree at least three, we have that $|S| = 1 = \gamma_p(T)$.

Let T be a tree with $\gamma_p(T) = 1$. Assume to the contrary that T is not a spider. Then, T has at least two branching vertices u and v. Then, by Theorem 3.3.6 there is a $\gamma_p(T)$-set, S, such that every vertex in S has degree at least three. Without loss of generality let $S = \{u\}$ be a $\gamma_p(T)$ set and let u be the root of T. Let v be monitored at step i. Then since v has degree greater than three, v has at least two children, x and y, who are not monitored. Then v can never propagate because it has two unmonitored neighbours and x, y can only be monitored by v since it is on the unique path from u to x and from u to y. This contradicts that S is a PDS. Therefore our assumption is wrong and T is a spider. $\square$

In order to show how spider partitions are related to power dominating sets in trees, we will show how a PDS can induce a spider partition of equal cardinality and vice versa. In order to show the first part, we now introduce an original algorithm that generates a spider partition from a PDS of a tree. The algorithm requires a tree T and S a PDS of T. The algorithm then produces a spider partition $\mathcal{X}$ of T such that $|\mathcal{X}| = |S|$.

Algorithm 6.2.7.
Input: G, a graph; $S = \{v_1, v_2, ..., v_k\}$, a PDS of T.

Output: $\mathcal{X}$, a partition of G with $|\mathcal{X}| = |S|$.

1. Let $\mathcal{X} = \{X_1, X_2, ..., X_k\}$, where $X_i = \{v_i\}$ for $i \in \{1, 2, ..., k\}$. Let $X = \bigcup_{X_i \in X} X_i$. For each X_i label its elements as $X_i = \{v_i = v_i^1, v_i^2, ...\}$.
2. Do the following for each j, starting at $j = 0$ and proceeding sequentially.
   (a) If there exists no $u \in P^j(S) - X$, end
   (b) For each $u \in P^j(S) - X$, do the following
   - If v_i^k is the only vertex in X which propagates to u, then add u to X_i.
   - If there is more than one vertex in X that propagates to u, then add v to X_i where i is the largest subscript among all vertices in X which propagate to u.

We now prove that our Algorithm 6.2.7 produces a spider partition $\mathcal{X}$ such that $|\mathcal{X}| = |S|$.

Theorem 6.2.8 ([15])**.** *Let T be a tree and S a PDS of T. Then Algorithm 6.2.7 with inputs T and S outputs a spider partition of T, $\mathcal{X}$, such that $|\mathcal{X}| = |S|$.*

Proof. Let T be a tree and $S = \{v_1, \ldots, v_k\}$ be a PDS of T. We will use $\mathcal{X}^j$ to denote the set $\mathcal{X}$ after $j+1$ iterations of step 2 in Algorithm 6.2.7. I.e $\mathcal{X}^0$ refer to the set $\mathcal{X}$ after step 2 has been done once and $\mathcal{X}^1$ refer to the value of $\mathcal{X}$ after step 2 has been done twice. We use X_i^j to denote the set X_i after $j+1$ iterations of step 2 in Algorithm 6.2.7.

We show via induction that $\mathcal{X}^j$ is a spider partition of $T[P^j(S)]$. We note that by the construction of Algorithm 6.2.7, $\mathcal{X}^j$ is a partition of $P^j(S)$ for all j. For the base case consider $\mathcal{X}^0$ and X_i^0. Every vertex in X_i^0 that is not v_i is adjacent to v_i. Then, since T is acyclic, $T[X_i^0]$ is acyclic and so $T[X_i^0]$ is a star. Thus, every set in $\mathcal{X}^0$ induces a spider. We also note that every vertex in X_i^0 that does not propagate in $P^0(S)$ has degree one in $T[X_i^0]$ for $i \in \{1, \ldots, k\}$. Thus, $\mathcal{X}^0$ is a spider partition of $P^0(S)$.

Now assume that $\mathcal{X}^j$ is a spider partition of $P^j(S)$ and that every vertex $v \in X_i^j$ that has not propagated in $P^j(S)$ has degree one in $T[X_i^j]$, for $i \in \{1, \ldots, k\}$. We want to show that every set in $\mathcal{X}^{j+1}$ is a spider partition of $P^{j+1}(S)$ and that every vertex $v \in X_i^{j+1}$ that does not propagate in $P^{j1}(S)$ has degree one in $T[X_i^{j+1}]$, for $i \in \{1, \ldots, k\}$. To do this, consider X_i^{j+1} and X_i^j. Let $u \in X_i^{j+1} - X_i^j$. Then, by the construction of Algorithm 6.2.7, u must be adjacent to some w such that $w \in X_i^j$. Therefore, u has degree at least one in $T[X_i^{j+1}]$. Then, every vertex in $T[X_i^{j+1}]$ is adjacent to at least one vertex in $T[X_i^j]$.

Assume that there is a $u \in X_i^{j+1} - X_i^j$ that has degree at least two in $T[X_i^{j+1}]$. Let u_1 and u_2 be adjacent to u in $T[X_i^{j+1}]$. Since both u_1 and u_2 are adjacent to a vertex in X_i^j and $T[X_i^j]$ is connected, there is a $u_1 - u_2$ path in $T[X_i^{j+1}]$ that does not contain u. Let this path be $u_1 = p_1, p_2, \ldots, p_t = u_2$. But then $u_1, p_2, \ldots, p_t, u, u_1$ is a cycle. This contradicts that T is a tree. Thus, every $u \in X_i^{j+1} - X_i^j$ has degree one in $T[X_i^{j+1}]$.

Assume there is a $u \in X_i^{j+1}$ that has degree at least 2 in $T[X_i^{j+1}]$ and does not propagate in $P^{j+1}(S)$. Since every $v \in X_i^{j+1} - X_i^j$ has degree one in $T[X_i^{j+1}]$, and u has degree greater than one, we have that $u \in X_i^{j+1} \cap X_i^j$. Since u does not propagate in X_i^{j+1}, it also does not propagate in X_i^{j+1}. Then by the inductive hypothesis, u has degree one in X_i^j. Since u has degree at least 2 in $T[X_i^{j+1}]$, u must have propagated. This contradicts our choice of u. Therefore, every $u \in X_i^{j+1}$ that has not propagated in $P^{j+1}(S)$ has degree one.

Thus, by the Principle of Mathematical Induction for every non-negative integer j, X^j induces a spider partition of $P^j(S)$ and every vertex $v \in X_i^j$ that has not propagated in $P^j(S)$ has degree one in $T[X_i^j]$, for $i \in \{1, \ldots, k\}$. Therefore, X is a spider partition of $P^\infty(S) = V(T)$. Thus, every PDS S of T, induces a spider partition $\mathcal{X}$ of T such that $|S| = |\mathcal{X}|$ by means of Algorithm 6.2.7. □

An important thing to note with Algorithm 6.2.7 is that the PDS that it generates is entirely dependent on how we choose to label the vertices in our PDS. This is shown in Figure 6.1 and Figure 6.2, where we have the same PDS but use a different labelling of the vertices to produce different partitions.

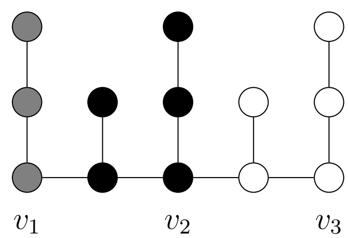

Figure 6.1: The graph showing the spider partition induced by the PDS $S = \{v_1, v_2, v_3\}$. Here, each colour represents an element of the spider partition.

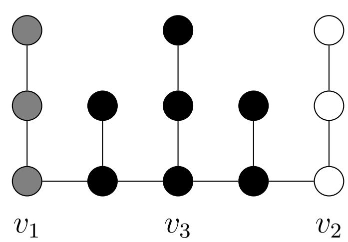

Figure 6.2: The graph showing the spider partition induced by the PDS $S = \{v_1, v_2, v_3\}$. Here, each colour represents an element of the spider partition.

Corollary 6.2.9. *Let T be a tree. Then, $\gamma_p(T) \geq sp(T)$.*

Proof. Let T be a tree and S be a γ_p-set of T. Then, by Theorem 6.2.8, Algorithm 6.2.7 produces a spider partition of T, $\mathcal{X}$ such that $|\mathcal{X}| = |S|$. Therefore, $\gamma_p(T) = |S| = |\mathcal{X}| \geq sp(T)$. □

We have now shown how an arbitrary PDS can induce a spider partition of equal cardinality. We next show how an arbitrary spider partition can induce a PDS of equal cardinality. To do this, we first define a graph that shows how the spider partitions of a tree are connected.

Definition 6.2.10 (Spider partition graph)**.** Let T be a tree and $\mathcal{X} = \{X_1, ..., X_k\}$ a spider partition of T. Let G be a graph with $V(G) = \mathcal{X}$. Let two vertices X_i, X_j be adjacent in G if $T[X_i \cup X_j]$ is connected. We call G the *spider partition graph of T induced by $\mathcal{X}$*.

If T is a tree and G is a spider partition graph of T then it is easy to see that since T is a tree, G must also be a tree. Let $X_iX_j \in E(G)$. Then there must be unique $u \in X_i$, $v \in X_j$ such that $uv \in T$. We say that the edge $uv \in E(T)$ is *paired* with the edge $X_iX_j \in E(G)$.

Theorem 6.2.11. *Let T be a tree and $\mathcal{X} = \{X_1, ..., X_k\}$ a spider partition of T. Let each $T[X_i]$ have a root r_i such that r_i has maximum degree in $T[X_i]$. Let $S = \{r_1, r_2, ..., r_k\}$. Then S is a PDS of T and $|\mathcal{X}| = |S|$.*

Proof. Let G be the spider partition graph of T induced by $\mathcal{X}$. Since S has one vertex in each X_i, S monitors at least one vertex in each partite set. Assume to the contrary that S is not a PDS of T. Then without loss of generality there exists a $u_1 \in X_1$ that is not monitored by S. Then, u_1 must have at least one ancestor in X_1 that is monitored. Let v_1 be the youngest ancestor of u_1 in $T[X_1]$ that is monitored. Now v_1 cannot propagate, otherwise v_1's child would be the youngest ancestor of u that is monitored. Since v_1 does not propagate it must have degree at least three in T. Since v_1 has an ancestor and a child in $T[X_1]$, it must have degree two in $T[X_1]$. Therefore, v_1 must be adjacent to an unmonitored vertex not in X_1. Let $P = u_1, v_1, u_2, v_2, ..., u_t, v_t$ be the longest sequence such that: $u_i, v_i \in X_i$ for all i, $u_i \neq v_i$ for all i, $v_iu_{i+1} \in E(T)$ for all $i \in \{1, ..., t-1\}$, u_i is unmonitored and v_i is the youngest ancestor of u_i that is monitored. Now v_t cannot propagate, otherwise v_t's child would be the youngest ancestor of u that is monitored. Since v_t does not propagate it must have degree at least 3 in T. Since v_t has an ancestor and a child in $T[X_t]$, it must have degree 2 in $T[X_t]$. Therefore, v_t must be adjacent to an unmonitored vertex not in X_t. By the construction of our sequence we have that $X_1, X_2, ..., X_t$ is a path in G. Therefore, v_t cannot be adjacent to a vertex in any X_i, $i \in \{1, .., t\}$, otherwise $X_i, ..., X_t, X_i$ would be a cycle in G. Then v_t is adjacent to some unmonitored vertex $u_{t+1} \in X_{t+1}$. Let v_{t+1} be the youngest ancestor of u_{t+1} in

$T[X_{t+1}]$ that is monitored. Then $P' = u_1, v_1, u_2, v_2, ..., u_{t+1}, v_{t+1}$ is a longer sequence than P which contradicts our choice of P. Therefore our assumption is wrong and S is a PDS of T. □

Corollary 6.2.12. *Let T be a tree. Then, $sp(T) \geq \gamma_p(T)$*

Proof. Let T be a tree and S be a minimum spider partition of T. Then, by Theorem 6.2.8, there is a PDS of T, S such that $|\mathcal{X}| = |S|$. Therefore, $sp(T) = |\mathcal{X}| = |S| \geq \gamma_p(T)$. Thus, $sp(T) \geq \gamma_p(T)$. □

Theorem 6.2.13 ([15])**.** *Let T be a tree. Then, $sp(T) = \gamma_p(T)$.*

Proof. It follows from Corollary 6.2.12 and Corollary 6.2.9 that $sp(T) = \gamma_p(T)$. □

We now use Theorem 6.2.13 to present a proof of the lower bound for γ_p in terms of the number of branching vertices. This proof was first shown in [15].

Theorem 6.2.14 ([15])**.** *Let T be a tree with b branching vertices, then*

$$\frac{b+2}{3} \leq \gamma_p(T)$$

and this bound is sharp.

Proof. Let $\gamma_p(T) = m$. Then, by Theorem 6.2.13, $sp(T) = m$. Let $\mathcal{X} = \{X_1, X_2, ..., X_m\}$ be a minimum spider partition of T. Since each $T[X_i]$ is a spider, it contains at most one branching vertex v, such that $N_T[v] \subseteq X_i$, which we choose to be the root of X_i.

Let G be the spider partition graph of T induced by $\mathcal{X}$. Thus, we can say that every branching vertex in T is either the root of some X_i or is paired with some edge in G. Thus, T contains at most $sp(T) + 2|E(G)| = m + 2(m-1) = 3m - 2$ branching vertices. Thus, $b \leq 3m - 2$ or equivalently, $\gamma_p(T) = m \geq \frac{b+2}{3}$.

We now prove that this bound is sharp. Let T be the corona of a path on $3n$ vertices. Let the path be denoted by $v_1, v_2, ..., v_{3n}$. Then, $\{v_2, ..., v_{3n-1}\}$ will be the only branching vertices of T. Thus, $b = 3n - 2$ or equivalently $n = \frac{b+2}{3}$. Now let $S = \{v_{3i-1} : 1 \leq i \leq n\}$. Then, S is a PDS of T of cardinality n, and so $\gamma_p(T) \leq n$. But T has b branching vertices so $\gamma_p(T) \geq \frac{b+2}{3} = n$. Consequently, $\gamma_p(T) = n = \frac{b+2}{3}$. □

6.3 Unicyclic graphs

We now extend our findings on power domination in trees to power domination in unicylic graphs. Recall that a unicylic graph is a graph with exactly one cycle.

First, we prove an original theorem that characterises all unicyclic graphs that have a power domination number of one.

Theorem 6.3.1. *Let U be a unicylic graph. Then, $\gamma_p(U) = 1$ if and only if U satisfies all of the following criteria:*

i) The graph U is connected.

ii) Every branching vertex of U is in the cycle.

iii) There are at most 3 branching vertices in U.

iv) There is one branching vertex in U that is adjacent to 2 or more leaf segments.

v) If U has 3 branching vertices then, 2 of the branching vertices with degree three must be adjacent.

Proof. Let U be a unicyclic graph with $\gamma_p(U) = 1$.

i) Then U can only have one component and so U is connected.

ii) Assume to the contrary that U has a branching vertex u not in the cycle. Let $v \in V(U)$ be a PDS of U and let v be in the cycle. Assume that u is first monitored at step i. Then, at least two of u's neighbours are not monitored at step i since the unique path from v to the neighbours contain u. Thus, u can not propagate as it has two unmonitored neighbours. But then, u's neighbours will not be monitored which contradicts our choice of v. Therefore, v must be a vertex not on the cycle. Let w be the first vertex on the cycle that v monitors. Then, w has at least two unmonitored neighbours since every path from v to w's unmonitored neighbours contain w. But then, w's neighbours will not be monitored which contradicts our choice of v. Therefore, U can not have a branching vertex that is not on the cycle.

iii) Assume to the contrary that U has at least four branching vertices. Let U $\{c_1, ..., c_n\}$ be the cycle of U. Let c_u, c_v, c_w, c_x be branching vertices of U with $u < v < w < x$. We know from Theorem 3.3.6 that there is a branching vertex that is a PDS of U. Without loss of generality let c_v be a PDS of U. Let c_x be first monitored at step k. Then, since every $c_v - c_x$ path contains either c_u or c_w, at least one of c_u, c_w propagated at step $j \leq k$. Without loss of generality let c_w propagate at step j. Then, at least two of c_w's neighbours were monitored at step $i < j$. Then, there must be two paths of monitored vertices between c_v and c_w. But since there are only two $c_v - c_w$ paths they must both be monitored at step i. This means that the entire cycle is monitored at step i which contradicts that c_x is first monitored at step $k > i$. Therefore, our assumption is false and U has at most 3 branching vertices.

iv) Assume to the contrary that U has at least two branching vertices that are adjacent to two or more leaf segments. Let these branching vertices be u and v. We know from Theorem 3.3.6 that there exists a $\gamma_p(U)$-set S, such that every vertex in S has degree at least three. Assume that $v \notin$ and let S first monitor v at step i. Then, no vertex in a leaf segment adjacent to v can be monitored at step i except for v. This is because every path from a vertex of degree three and a vertex in one of these leaf segments contain v. Then, at step i, v has at least two unmonitored neighbours, and v is in every path from S to these vertices. Therefore, v must propagate before these vertices are unmonitored. So v propagates when at least two of its vertices are unmonitored which is a contradiction. Therefore, $v \in S$. We can then, use a similiar argument to find that $u \in S$. Then, $u, v \in S$ which contradicts that S is a $\gamma_p(U)$-set, thus U can have at most one branching vertex that is adjacent to 2 or more leaf segments.

v) Let U have 3 branching vertices u, v, w. Assume to the contrary that U has no vertices of degree three that are adjacent to each other. We know from property (ii) that u, v, w all lie on the cycle of U. We also know from Theorem 3.3.6 that there exists a $\gamma_p(U)$-set S, such that every vertex in S has degree at least three. Therefore, at least one of u, v or w forms a $\gamma_p(U)$-set. Without loss of generality let $S = \{v\}$ be a $\gamma_p(U)$-set. Then, u must propagate before the leaf segments adjacent to u are monitored. Therefore, u is adjacent to one leaf segment and similarly w is adjacent to one leaf segment. Now in order for u to propagate all but one of its neighbours must be monitored. since the neighbour of u that is in a leaf segment can not be monitored until u propagates, we must have that u's two neighbours that are in the cycle are monitored before u propagates. Therefore, the whole cycle must be monitored before u can propagate. Now let x be a vertex in the $u-w$ path the does not contain v. Then, x is a vertex in the cycle. Let $u \neq x \neq w$. Then, every $v-x$ path contains either u or w, so at least one of u or w must propagate so that x can be monitored. But this contradicts that the whole cycle must be monitored before u and w can propagate. Therefore, our assumption is false and so if U has three branching vertices then, two of the branching vertices with degree three must be adjacent.

Now let U be a unicyclic graph and let U satisfy $i)$ to $v)$. We separate the proof that $\gamma_p(U) = 1$ into 4 cases:

1. *U has 0 branching vertices.* Then U is a cycle and by Theorem 3.3.4, $\gamma_p(U) = 1$.

2. *U has 1 branching vertex v.* Let $S = \{v\}$. Then, S monitors all the leaf segments adjacent to v and S monitors all the vertices in the cycle. Therefore, $\gamma_p(U) = 1$.

3. *U has two branching vertices v and u.* Let $deg(v) \geq deg(u)$ and let $S = \{v\}$. Then, S monitors all the leaf segments adjacent to v. Let P_1 and P_2 be the two unique paths from u to v. Then, since every internal vertex in P_1 has degree two, they all propagate. By a similiar argument the same is true for P_2. Therefore, every vertex in the cycle is monitored. Since U has at most one branching vertex of degree > 3 and $deg(v) \geq deg(u)$, u must have degree three. Then, u and all but one of its neighbours is monitored. Therefore, u propagates and all leaf vertices adjacent to u are monitored. Thus, $\gamma_p(U) = 1$.

4. *U has three branching vertices v, u, w.* Then, at least two of the branching vertices must have degree three and be adjacent to each other. Let u, w be these vertices with degree three that are adjacent to each other and let $S = \{v\}$. Then, S monitors all the leaf segments adjacent to v. Let P_{vu} be the $v-u$ path that does not contain w. Then, since every internal vertex in P_{vu} has degree two, S monitors P_{vu}. similarly S monitors the $v-w$ path that does not contain u. Then, u and all but one of its neighbours is monitored. Therefore, u propagates and all leaf vertices adjacent to u are monitored. Similarly w propagates and all leaf vertices adjacent to w are monitored. Thus, $\gamma_p(U) = 1$.

□

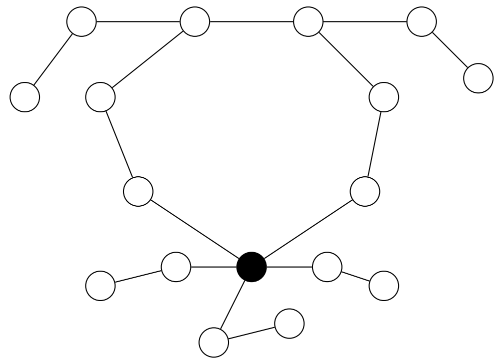

Figure 6.3: A unicyclic graph G with $\gamma_p(G) = 1$. A $\gamma_p(G)$-set is shown in black.

An example of a unicyclic graph with power domination number one is shown in Figure 6.3.

We now extend our results on the relationship between the spider number and the power domination number in trees to connected unicyclic graphs. In particular we will show that while the spider number and power domination number are not equal in general for connected unicyclic graphs, their difference is bounded and this bound is tight.

Theorem 6.3.2. *Let U be a connected unicyclic graph and let $S = \{v_1, ..., v_k\}$ be a PDS of U. Then there exists a spider partition, $\mathcal{X}$, of U such that $|\mathcal{X}| \leq |S| + 1$.*

Proof. To prove this result we first prove a claim about Algorithm 6.2.7.

Claim. *Let $S = \{v_1, ..., v_k\}$ be a PDS of U and let $\mathcal{X} = \{X_1, ..., X_k\}$ be the output of Algorithm 6.2.7 on S and U. Furthermore let X_i be a tree for all $i \in \{1, .., k\}$. Then $\mathcal{X}$ is a spider partition of U.*

Proof of claim. Assume to the contrary that $\mathcal{X}$ is not a spider partition of U. Then there exists a X_i such that $U[X_i]$ is not a spider. Let $u \in X_i$ be a branching vertex in $U[X_i]$ and let v_i be the root of $U[X_i]$. Then v must have at least two children x, y in $U[X_i]$. Then v must propagate to both of them which contradicts the rules of propagation. Therefore our assumption is wrong and $\mathcal{X}$ is a spider partition of U.

Now let $\mathcal{X} = \{X_1, ..., X_k\}$ be the output of Algorithm 6.2.7. Now, by the definition of Algorithm 6.2.7, $|\mathcal{X}| = |S|$. If $\mathcal{X}$ is a spider partition of U then the result follows.

Assume instead that $\mathcal{X}$ is not a spider partition of U. By the definition of Algorithm 6.2.7 we know that $U[X_i]$ is connected for all i. Then by claim 1, there must be a X_i such that $U[X_i]$ is not a tree. Then, since $U[X_i]$ is connected, it must be a connected unicyclic graph. Then there must exist a $v \in X_i$ such that v is on the cycle and $v \notin S$. Let $S' = S \cup \{v\}$ and let $\mathcal{X}'$ be the output of Algorithm 6.2.7 on U and S'. Then every partite set in $\mathcal{X}$ must induce a tree since no partite set has all the cycle vertices. Then, by Claim 1, $\mathcal{X}'$ is a spider partition of U. Therefore $\mathcal{X}'$ is a spider partition of U such that $|\mathcal{X}'| \leq |S| + 1$. $\square$

Corollary 6.3.3. *Let U be a connected unicyclic graph. Then $sp(U) - \gamma_p(U) \leq 1$*

Proof. Let U be a connected unicyclic graph and let S be a $\gamma_p(U)$-set. Then by Theorem 6.3.2, there exist a spider partition, $\mathcal{X}$, of U such that $|\mathcal{X}| \leq |S|+1$. Then, $sp(U) \leq |\mathcal{X}| \leq |S| + 1 = \gamma_p(U) + 1$. Therefore $sp(U) - \gamma_p(U) \leq 1$. $\square$

Theorem 6.3.4. *Let U be a connected unicyclic graph. Then $\gamma_p(U) - sp(U) \leq 1$.*

Proof. Let C be the unique cycle in U. Let $\mathcal{X} = \{X_1, \ldots, X_k\}$ be a spider partition of U such that $|\mathcal{X}| = sp(U)$. Assume that each $e \in E(C)$ is contained in some $U[X_i]$ (though not necessarily the same X_i), then there exists no two vertices in $V(C)$ which are contained in two different partite sets of $\mathcal{X}$. This implies that C is contained in some $U[X_i]$, implying that $U[X_i]$ contains a cycle. This contradicts that $\mathcal{X}$ is a spider partition of U. Therefore, there exists some $e \in E(C)$ such that $e \notin E(U[X_i])$ for any i. This implies that C is not contained in X, so that $\mathcal{X}$ is also a spider partition of the tree $U - e$. Therefore, $sp(U - e) \leq sp(U)$. Then, by Theorem 5.3.1 and Theorem 6.2.13 we have that,

$$\gamma_p(U) - 1 \leq \gamma_p(U - e) = sp(U - e) \leq sp(U)$$

which implies that

$$\gamma_p(U) - sp(U) \leq 1$$

. $\square$

Theorem 6.3.5. *Let U be a connected unicyclic graph. Then $|sp(U) - \gamma_p(U)| \leq 1$.*

Proof. Let U be a connected unicyclic graph. Then it follows from Corollary 6.3.3 and Theorem 6.3.4 that $-1 \leq sp(U) - \gamma_p(U) \leq 1$. $\square$

In Figure 6.5 and Figure 6.4 we show that the bound in Theorem 6.3.5 is sharp from above and below.

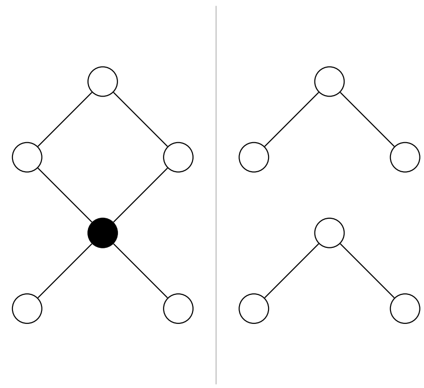

Figure 6.4: A connected unicyclic graph U such that $sp(U) - \gamma_p(U) = -1$. A $\gamma_p(U)$-set is shown in black on the left. A minimum spider partition is shown on the right.

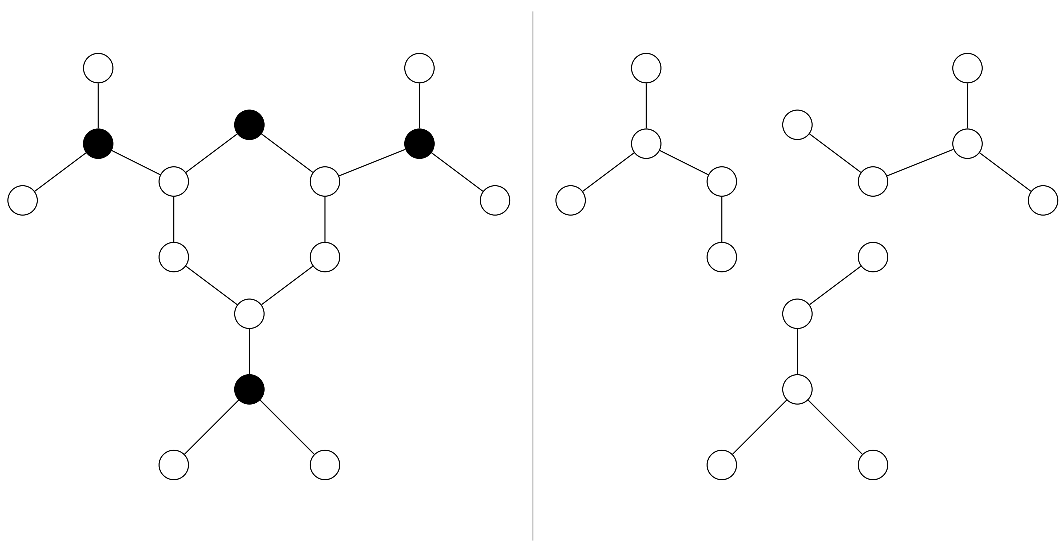

Figure 6.5: A connected unicyclic graph U such that $sp(U) - \gamma_p(U) = 1$. A $\gamma_p(U)$-set is shown in black on the left. A minimum spider partition is shown on the right.

Chapter 7

Propagation radius

7.1 Introduction

In this chapter will present some results on the propagation radius. Specifically we show an upper bound on the propagation radius in terms of δ and γ_p. We also show that the bound is sharp. In order to prove the results on the propagation radius we need a result from zero forcing. Zero forcing is essentially power domination without the domination step. Therefore, zero forcing is very similiar to power domination and there are many results between the two that are analogous. First, we define the propagation radius.

Definition 7.1.1 (Propagation radius)**.** Let G be a graph and S be a PDS of G. Then, the *propagation radius of S with respect to G*, denoted $rad_p(G, S)$, is the number of steps it takes for S to monitor G. That is,

$$rad_p(G, S) = 1 + \min\{i : P^i(S) = V(G)\}.$$

The *propagation radius of* G, denoted as $rad_p(G)$, is the minimum propagation radius of any $\gamma_p(G)$-set. That is,

$$rad_p(G) = \min\{rad_p(G, S) : S \text{ a } \gamma_p(G)\text{-set of } G\}.$$

Definition 7.1.2 (Live set)**.** Let G be a graph and let S be a PDS of G. Then the *live set of $P^i(S)$*, denoted $L^i(S)$, is the set of vertices in $P^i(S)$ that have at least one unmonitored neighbour. These are all the vertices that could possibly propagate in the next step.

Remark 7.1.3. Let G be a graph and let S be a PDS of G. Since a vertex can propagate to at most one vertex, we have that the sequence $|L^0(S)|, |L^1(S)|, \dots$ is non-increasing.

7.2 Zero forcing

We now divert from power domination briefly, to explain the basics of zero forcing. Zero forcing was first defined in [1] to bound the minimum rank of various families of graphs. The process of zero forcing is almost identical to power domination and therefore the results of zero forcing can be useful in a power domination context.

Definition 7.2.1 (Forcing rule)**.** Let G be a graph where every vertex is either monitored or unmonitored. The *forcing rule* is as follows: If $v \in V(G)$ is monitored and has exactly one unmonitored neighbour u, then v *forces* u and u is forced by v, denoted by $v \to u$, and u becomes monitored.

Definition 7.2.2 (Zero forcing set)**.** Let G be a graph and $S \subseteq V$ be a set of monitored vertices. If repeatedly applying the forcing rule results in all vertices of G being monitored, then S is a *zero forcing set (ZFS)* of G.

From the definitions it is clear that zero forcing is equivalent to power domination without the initial domination step. The rules of propagation after the first step are identical to the forcing rule. This mean that if S is a PDS then $N[S]$ is a ZFS.

Definition 7.2.3 (Forcing sequence)**.** Let G be a graph and Z be a ZFS of G. Apply the forcing rule one vertex at a time until the whole graph is monitored, recording the forces as you do so. This list of forces is called a *forcing sequence.*

Definition 7.2.4 (Forcing chain)**.** Let G be a graph, Z a ZFS of G and T a forcing sequence of Z. A *forcing chain with respect to* T is a sequence of vertices $v_1, v_2, ..., v_k$ such that for $i = 1, ..., k-1$, v_i forces v_{i+1}. Given a forcing sequence T, a *maximal forcing chain* is a forcing chain with respect to T which is not a proper subsequence of some forcing chain with respect to T. When there is no confusion we will say forcing chain instead of forcing chain with respect to T.

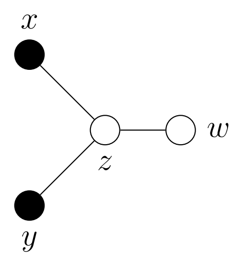

Figure 7.1: There are two possible forcing sequences of Z. One is $y \to z \to w$. Another is $x \to z \to w$. In the former, the set of maximal forcing chains is $x, y \to z \to w$ and the reversal with respect to $x, y \to z \to w$ is $\{x, w\}$. In the latter, the set of maximal forcing chains is $y, x \to z \to w$ and the reversal with respect to $y, x \to z \to w$ is $\{y, w\}$. By Theorem 7.2.7, both reversals are ZFSs.

Remark 7.2.5. If there exists two distinct vertices u and v, both of which force a third vertex w by the forcing rule, then we consider only one of u and v to force w. Thus, if u forces w, then v becomes a terminal vertex in some forcing chain. Since either of u or v could have forced w, we note that a ZFS may have multiple forcing sequences and therefore multiple possible sets of forcing chains.

Definition 7.2.6 (Reversal)**.** Let G be a graph and Z a ZFS of G. A *reversal* of Z is the set of terminal vertices of all the maximal zero forcing chains with respect to some forcing sequence of Z.

In Figure 7.1 we show an example of a ZFS along with its possible forcing sequences, its possible maximal forcing chains, and its possible reversals.

The following theorem is useful for our purposes. We present it without proof. The proof can be found in [11].

Theorem 7.2.7. *[11] Let G be a graph and let Z be a zero forcing set of G. Then any reversal of Z is a zero forcing set of G.*

7.3 Upper bounds for rad_p

Let G be a graph and let S be a $\gamma_p(G)$-set. Consider the propagation sequence in G that begins with S. In the first step, S will propagate to its neighbourhood $N(S)$.

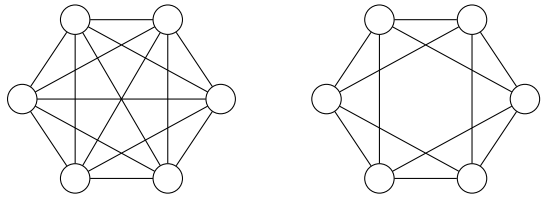

Figure 7.2: The two graphs of order 6 such that $rad_p(G) = n - \delta$.

Each propagation step after that monitors at least one new vertex. Thus, we make the following observation.

Observation 7.3.1. Suppose G is a graph with order n, and S is a $\gamma_p(G)$-set that monitors G in p steps. Then

$$n \geq |N[S]| + (p - 1)$$

Lemma 7.3.2. *Let G be a graph with order n and minimum degree δ. Then $rad_p(G) \leq n - \delta$ and this bound is attained by K_n and $K_n - M$ where M is a perfect matching.*

Proof. From Observation 7.3.1 it is clear that $rad_p(G) \leq n-|N[S]|-1$. By our choice of G, S must have at least one vertex with degree at least δ. Therefore $|N[S]| \geq \delta+1$. Then $rad_p(G) \leq n - (\delta + 1) + 1 = n - \delta$.

Let $G = K_n$. Then it is clear from inspection that $\gamma_p(G) = 1$ and $rad_p(G) = 1$. Then it is clear that $n - \delta(G) = n - n - 1 = 1 = rad_p(G)$.

Let $G = K_n - M$ where M is a perfect matching. In this case n must be even. Let $v \in V(G)$ and let $S = \{v\}$. Then $N[v] = V(G)-u$ where $uv \in M$. Since there is only one vertex that is unmonitored, it is monitored in the next step. Then $rad_p(G) = 2$. Therefore $n - \delta = n - (n - 2) = 2 = rad_p(G)$. □

Theorem 7.3.3. *Let G be a graph with order n that is neither K_n nor $K_n - M$ where M is a perfect matching. Then $rad_p(G) \leq n - \delta(G) - 1$.*

Proof. We note, in general, that $n \geq \delta + 1$. We have three cases to consider:

1. *Let $n = \delta(G) + 1$.* Then G is a complete graph and so it is not considered.

2. *Let $n = \delta(G) + 2$.* If $\Delta(G) = \delta(G)$ then G is $K_n - M$ where M is a perfect matching and so it is not considered. If $\Delta(G) \neq \delta(G)$, then $\Delta(G) = n - 1$ and $rad_p(G) = 1$. Then $rad_p(G) = 1 = n - \delta(G) - 1$.

3. *Let $n \geq \delta(G) + 3$.* Then G cannot be K_n or $K_n - M$, where M is a perfect matching. We know from Lemma 7.3.2 that $rad_p(G) \leq n - \delta$. Assume to the contrary that $rad_p(G) = n - \delta$. By Observation 7.3.1 we have that $rad_p(G) \leq n - |N[S]| - 1$. Therefore $|N[S]| \leq \delta + 1$. For any $v \in V$ we have $|N[v]| \geq \delta + 1$. Then there exists a power dominating set $S = \{v\}$ such that $rad_p(G, S) = n - \delta$ and $deg(v) = \delta$. Then, S has propagation sequence $P^0(S), P^1(S), ..., P^k(S)$ where $k = n - \delta + 1$. Recall as well that $N[S]$ is a ZFS. Let $T = v, p_1, p_2, \ldots, p_k$ be such that p_i forces x_i in $P^i(S)$ for $i \in \{1, ..., k\}$. Then T is a forcing sequence of $N[S]$. Since $n \geq \delta(G) + 3$ and $rad_p(G) = n - \delta$, we must have that $rad_p = n - \delta \geq \delta(G) + 3 - \delta(G) = 3$.

   Claim (1)**.** *For all $i \in \{0, 1, ..., k - 1\}$, $|L^i(S)| = \delta$.*

   Proof of claim. Since $|P^0(S)| = \delta + 1$ and $|S| = 1$, we must have $|L^0(S)| = \delta$. Since x_k is the last vertex to be monitored we must have that all of its neighbours propagate to it. Then since it has at least δ neighbours and all of them are in $L^{k-1}(S)$ we have $|L^{k-1}(S)| \geq \delta$. Since $L^i(S)$ is non-increasing we have that $|L^i(S)| = \delta$ for all $i \in \{0, ..., k - 1\}$.

   Claim (2)**.** *For any forcing sequence of $N[S]$, The terminal vertex of every maximal forcing chain is in $N[x_k] \cup \{v\}$.*

   Proof of claim. Assume to the contrary that $u \notin N[x_k] \cup \{v\}$ is the terminal vertex of a maximal forcing chain. Then u cannot be in $L^{k-1}(S)$ since the only vertices in $L^{k-1}(S)$ are those which have not yet propagated and would therefore propagate to x_k, implying that they are in $N[x_k]$. Let i be the greatest integer such that $u \in L^i(S)$. There must exist some $w \in L^i(S)$ that propagates to some vertex in $P^{i+1}(S)$ such that $w \neq u$. Then $L^{i+1}(S) \subseteq (L^i(S) - \{u, w\}) \cup \{x_{i+1}\}$. But then $|L^{i+1}(S)| < |L^i(S)|$ which contradicts Claim 1.

   Claim (3)**.** *The set $\{x_k\}$ is a power dominating set of G.*

   Proof of claim. Since the maximal forcing chain containing v is just v itself we can say that $N_G(v)$ is a zero forcing set of $G - v$ and that by Claim 2 the terminal vertex of every maximal forcing chain is in $N[x_k]$. By Theorem

7.2.7 any reversal of $N_G(v)$ in $G - v$ is a zero forcing set of $G - v$. Let Z' a reversal of $N_G(v)$ in $G - v$. Then $N_G(v)$ is a reversal of Z'. Thus by definition of the reversal, $N_G(v)$ is a set of the terminal vertices of the maximal forcing chains of some forcing sequence T' of Z' in $G - v$. Therefore Z' monitors at least $V(G) - v$ in G. Then, since only one vertex is unmonitored in G, it is monitored at the next step. Therefore $N[x_k]$ is a zero forcing set of G, which implies that $\{x_k\}$ is a PDS of G.

Claim (4)**.** *For all* $i \in \{0, ..., k\}$*,* $|P^i(S) - \{v, p_1, ..., p_i\}| = \delta$*.*

Proof of claim. Since $rad_p(G) = n - \delta$ we must have that $|P^0(S)| = \delta + 1$ and $|P^i(S) - P^{i-1}(S)| = 1$ for all $i \in \{1, ..., k\}$. Therefore $|P^i(S)| = |P^0(S)| + |P^1(S) - P^0(S)| + ... + |P^i(S) - P^{i-1}(S)| = \delta + 1 + i$. Thus $|P^i(S) - \{v, p_1, ..., p_i\}| = \delta + 1 + i - (1 + i) = \delta$.

Claim (5)**.** *The vertex* x_{k-1} *is adjacent to* x_k*.*

Proof of claim. Assume to the contrary that x_{k-1} is not adjacent to x_k. Then $x_{k-1} \notin \{v, p_1, ..., p_k\}$. Since every neighbour of x_k propagates to x_k we have that $|N(x_k) \cap \{v, p_1, ..., p_k\}| = 1$. Then $|P^k(S) - \{v, p_1, ..., p_k\}| \geq |N(x_k) - 1| + |\{x_{k-1}, x_k\}| = \delta - 1 + 2 = \delta + 1$. But this contradicts Claim 4. Therefore our assumption is wrong and x_{k-1} is adjacent to x_k.

Since x_{k-1} propagates to x_k, we can choose p_k such that $p_k = x_{k-1}$. If any vertex in $N(x_k) - \{x_{k-1}\}$ is not adjacent to x_{k-1}, it will propagate to x_k before the last step, contradicting the propagation radius of G. Therefore $N[x_k] \subseteq N[x_{k-1}]$ and by Claim 3 it follows that $\{x_{k-1}\}$ is a power dominating set of G. Recall that x_k is monitored after x_{k-1}. Then p_{k-1} cannot be adjacent to x_k otherwise it would be adjacent to two unmonitored vertices, x_k and x_{k-1}, when it propagates to x_{k-1}. Therefore $|N[x_{k-1}]| \geq |N[x_k]| + |\{p_{k-1}\}| = (\delta + 1) + 1$. Then $rad_p(G, \{x_{k-1}\}) \leq n - |N[x_{k-1}]| + 1 \leq n - \delta - 2 + 1 = n - \delta - 1$. But this contradicts the propagation radius of G. Therefore our assumption is wrong and $rad_p(G) \leq n - \delta - 1$.

□

The upper bound in Theorem 7.3.3 is sharp. We give some infinite families of graphs to demonstrate this.

To show these infinite families we will first show a small but useful lemma.

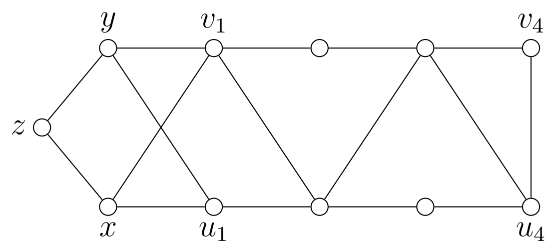

Figure 7.3: The graph $G(2,4)$ as per the proof of Theorem 7.3.5.

Lemma 7.3.4. *Let G be a graph, u and v be vertices in G such that $N(u) - \{v\} = N(v) - \{u\}$, and let S be any PDS of G. Then, $N[S] \cap \{u, v\} \neq \emptyset$.*

Proof. Assume to the contrary that there exists a PDS S such that $N[S] \cap \{u, v\} \neq \emptyset$. Then $V(G) \cap \{u, v\}$ is a zero forcing set of G. Without loss of generality let u be monitored at the last step. Let w propagate to v. Then $w \in N(v) - \{u\}$ since u is monitored at the last step. Then $w \in N(u) - \{v\}$. But then w is adjacent to u and v which are both unmonitored when w propagates to v. This contradicts the rules of propagation and so our assumption is false. Thus $N[S] \cap \{u, v\} \neq \emptyset$ for any PDS S of G. □

Theorem 7.3.5. *For each positive integer $\delta \geq 2$, there exists infinitely many graphs with order n, minimum degree δ, and propagation radius $rad_p(G) = n - \delta - 1$.*

Proof. We divide the proof into cases:

1. $\delta = 2$. For $k \geq 3$, we construct a graph $G(2,k)$. Let $G(2,k)$ have vertices u_i and v_i for each $1 \leq i \leq k$, and three extra vertices x, y and z. Let $G(2,k)$ have edges: xu_1, xv_1, xz incident to x; edges yu_1, yv_1, yz incident to y; all possible edges of the form u_iu_{i+1} and v_iv_{i+1}; for i an even integer, all possible edges of the form u_iv_{i+1} and u_iv_{i-1}; and u_kv_k (see Figure 7.3). It's clear that $G(2,k)$ has minimum degree two and order $n = 2k + 3$. By inspection we see that the singleton sets $\{x\}, \{y\}$ are both PDSs with propagation radius $2k = n - \delta - 1$. To show that $rad_p(G(2,k)) = 2k$, it suffices to show that no other singleton set is a PDS. Note $N(x) = N(y)$. Thus by Lemma 7.3.4, the only other singletons to consider are $\{z\}, \{u_1\}$ and $\{v_1\}$, all of which, by inspection, we see are not PDSs.

2. $\delta = 3$. For $k \geq 3$, we construct a graph $G(3, k)$. Let the graph have vertices x and y, as well as vertices for all ordered pairs (i, j), where $i \in [1, k]$ and $j \in [1, 3]$. Let x and y be adjacent to every vertex of the form $(1, j)$. For $i \geq 2$, let the vertices $(i, 1)$, $(i, 2)$ and $(i, 3)$ all be adjacent. Further, include all possible edges of the form $(i, j)(i + 1, j)$ and $(i, j)(i + 1, j - 1)$ (see Figure 7.4). The resulting graph $G(3, k)$ has order $n = 3k + 2$. The vertices $(1, 1)$ and $(k, 3)$ have degree three, every other vertex has higher degree. It's easy to check that the singletons $\{x\}$ and $\{y\}$ are PDSs with propagation radius $3(k-1)+1 = n-\delta-1$. Per Lemma 7.3.4, it suffices to show that no neighbour of x or y is a PDS. By inspection we see that for all $j \in \{1, 2, 3\}$, the singleton $\{(1, j)\}$ is not a PDS.

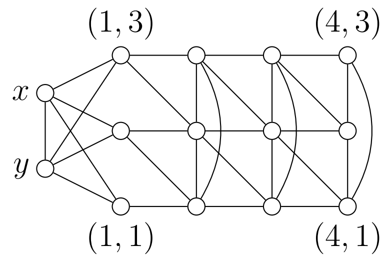

Figure 7.4: The graph $G(3, 4)$ as per the proof of Theorem 7.3.5.

3. $\delta \geq 4$. Let $k \geq 3$, and construct $G(\delta, k)$ as follows. Let $G(\delta, k)$ have vertices u, v, x and y, as well as a vertex for each ordered pair (i, j), where $i \in [1, k]$ and $j \in [1, \delta - 2]$. Let all four vertices u, v, x and y be adjacent; have u, v adjacent to all vertices $(1, j)$; have x, y adjacent to all vertices (k, j); let all vertices $(i, 1), (i, 2), \ldots, (i, \delta - 2)$ be adjacent to each other; and include all possible edges of the form $(i, j)(i + 1, j)$ and $(i, j)(i + 1, j - 1)$ (see Figure 7.5). The graph $G(\delta, k)$ has $n = (\delta - 2)k + 4$ vertices, and minimum degree δ (attained by the vertex $(1, 1)$). Each of the singletons $\{u\}, \{v\}, \{x\}$ and $\{y\}$ is a PDS with propagation radius $(k - 1)(\delta - 2) + 1 = n - \delta - 1$. Note that $N(u) - \{v\} = N(v) - \{u\}$, and that $N(x) - \{y\} = N(y) - \{x\}$, so by Lemma 7.3.4, no other singleton is a PDS.

□

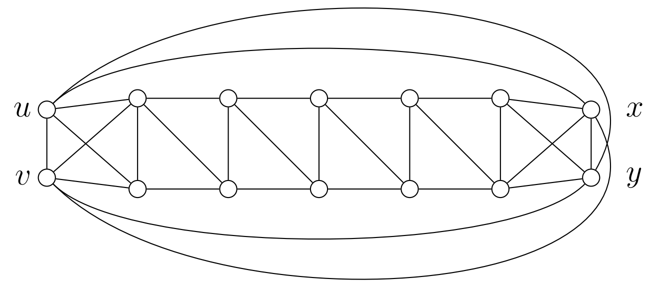

Figure 7.5: The graph $G(4, 5)$ as per the proof of Theorem 7.3.5.

7.4 The sharp lower bound

In [17], Liao determines a lower bound on the power domination number in terms of the propagation radius:

Theorem 7.4.1. [17] *Let G be a connected graph with order n, maximum degree Δ and propagation radius rad_p. Then,*

$$\gamma_p(G) \geq \frac{n}{rad_p \cdot \Delta + 1}$$

This can be easily re-arranged to give a lower bound for the propagation radius:

Corollary 7.4.2. *Let G be a connected graph of order n, maximum degree Δ and power domination number γ_p. Then,*

$$rad_p(G) \geq \frac{n - \gamma_p}{\gamma_p \cdot \Delta}$$

In [17], Theorem 7.4.1 and Corollary 7.4.2 are shown to be sharp for all possible values of rad_p and γ_p respectively. However, $\Delta = 4$ in all of the extremal examples. We show that for all $\Delta \geq 3$, $rad_p \geq 1$ and $\gamma_p \geq 1$, there is a graph $H(\Delta, \gamma_p, rad_p)$

with maximum degree Δ, power domination number γ_p, propagation radius rad_p and order $n = \gamma_p \cdot (rad_p \cdot \Delta + 1)$.

Theorem 7.4.3. *Let $\Delta \geq 3$, $rad_p \geq 1$, and $\gamma_p \geq 1$. Then, there exists a graph $H(\Delta, \gamma_p, rad_p)$ with maximum degree Δ, power domination number γ_p, and propagation radius rad_p.*

Proof. To construct $H(\Delta, \gamma_p, rad_p)$, begin with γ_p disjoint copies of the star $K_{1,\Delta}$. Replace every edge with a path of length rad_p. For $i \in [1, \gamma_p]$, let u_i denote the vertex of degree Δ in the i^{th} subdivided star, and let v_i denote some leaf of the i^{th} subdivided star. Add all possible edges of the form $v_i v_{i+1}$ to complete the construction (see Figure 7.6).

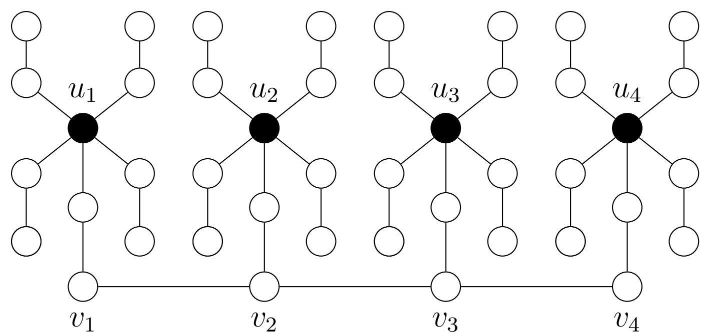

Figure 7.6: The graph $H(5, 4, 2)$ as per the proof of Theorem 7.4.3.

It's clear from inspection that the set $\{u_1, u_2, \ldots, u_{\gamma_p}\}$ is a minimum PDS with γ_p vertices and propagation radius rad_p. □

7.5 A δ regular graph that achieves $rad_p = n - \delta - 1$

Theorem 7.5.1. *For each integer $\delta \geq 2$ there exists a δ-regular graph, G such that $rad_p(G) = n - \delta - 1$.*

Proof. Let $n = \delta + 3$. Let K_n and C_n have vertex set $V = \{v_1, ..., v_n\}$. Let C_n have edge set $E(C_n) = \{v_1v_2, v_2v_3, ..., v_nv_1\}$. We define G as follows: $V(G) = V(K_n)$ and $E(G) = E(K_n) - E(C_n)$. Since $\delta \geq 2$, we must have $n \geq 5$. Let $S = \{v_3\}$.

Then $N[S] = V - \{v_2, v_4\}$. Since $v_1 \in N[S]$ is adjacent to v_4 and $v_1v_2 \notin E(G)$, v_1 propagates to v_4 in step 2. By symmetry we also have that v_5 propagates to v_2 in step 2. Thus, S is a $\gamma_p(G)$-set with propagation radius 2. By the symmetry of G, every singleton is a PDS with propagation radius 2. Therefore $rad_p(G) = 2 = n-\delta-1$. $\square$

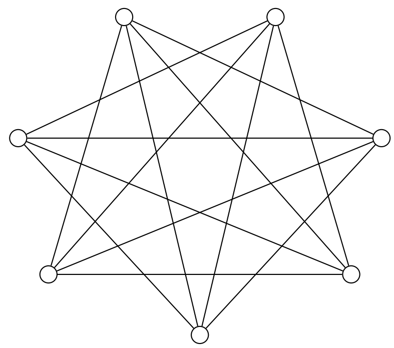

Figure 7.7: A regular graphs attaining the $n - \delta - 1$ bound where δ is even. Here the graph has $\delta = 4$ and $n = 7$.

Chapter 8

Concluding remarks

8.1 Conjectures and further research

During the course of this thesis research, a number of interesting conjectures and ideas for further research were identified. We give a list of ideas for potential further research along with a brief description of why it might be interesting.

Conjecture 8.1.1. Let k be positive integer and let $\mathcal{G}_k$ be set of all connected graphs G such that $\gamma_{p,k}(G \sim uv) = \gamma_{p,k}(G) - 1$ for every $uv \in E(G)$. Then $\mathcal{G}_k = T_k \cup \{K_{k+2,k+2}\}$.

We have shown in Example 5.5.3 that $\mathcal{T} \subseteq \mathcal{G}$. It is easy to verify that $K_{k+2,k+2} \in \mathcal{G}$. Using a similiar argument it could be possible to show that $T_k \cup \{K_{k+2,k+2}\} \subseteq \mathcal{G}_k$.

Conjecture 8.1.2. Let G be a graph with t cycles. Then $-t \leq sp(G) - \gamma_p(G) \leq t$ and these bounds are sharp.

This bound was proved for $t \in \{0, 1\}$ in Chapter 6. This would be a natural extension of those results which could be proven by repeatedly applying the arguments in Theorem 6.3.2 and Theorem 6.3.4.

8.2 Final remarks

In this thesis we have shown multiple bounds on the power domination number and k-power domination number of graphs. We have presented original proofs for some of the results in [15] on trees, specifically that $sp(T) = \gamma_p(T)$ for any tree T. We presented original research on unicyclic graphs and proved that $|sp(U) - \gamma_p(U)| \leq 1$ for any unicyclic graph U. We have also done original research into the propagation radius of graphs in Chapter 7.

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