Optimal Liquidation Strategies

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Abstract

Liquidation strategies consider the problem of minimising transaction costs occurring in a portfolio liquidation. Transaction costs are the difference between current market value and the realised value after the liquidation. A strategy to follow to perform a liquidation is especially important to institutional investors due the large size of their trades. Large trades can have a significant effect on the price of a security which can impact the realised returns of the liquidation. These models solve for trading trajectories that maximise this. The models investigated do this in a mean-variance framework where the expected return of the strategy is constrained by its variance and the investors risk preference. Parameters used in liquidity functions are estimated for securities on the South African JSE Securities Exchange. The effects of security liquidity, volatility, stock correlation and length of liquidation horizon on the optimal strategy are investigated. There is little or no existing literature that attempts to model these functions in the South African market. Due to the smaller size of the South African market as well as the number of thinly traded shares compared to most markets studied in the literature, many securities are highly illiquid. We investigate relationships between firm size and daily traded value and these liquidity parameters. General rules are presented to help traders improve a liquidation strategy without the need to estimate all parameters needed to calculate an optimal strategy using one of these models.

Afrikaans Abstract

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Mathematical Symbols

n_k	number of shares to trade at time k
x_k	number of shares held at time k
S_k	share price at time k
X	initial number of shares in the portfolio
ν_k	rate of trading at time k
σ	volatility of the share returns
μ	mean of the share returns
τ	equal time between trades in the Almgren and Chriss model
τ_i	unequal time between trades in the Mönch model
t_k	time of the k^{th} trade
ξ	standard normal random variable
$\tilde{S}_k$	average realised share price between times $k-1$ and k
T	total trading time
T^*	maximum trading time

Stock Ticker Data

Ticker	**Full Name**
AGL	Anglo American Plc
SLM	Sanlam Limited
MPC	Mr Price Group Limited
SYC	Sycom Property Fund Managers Limited

Chapter 1

Introduction

In 1987 Perold wrote the first paper to distinguish between the paper value of a portfolio when liquidating and the realised value. He called this gap the implementation shortfall. This was essentially the start of investigations into the loses occurred by not being able to trade any amount of shares at any one time at the current market value. He divides the cost into two components, execution cost and opportunity cost. Execution costs are fixed costs together with the price impact, and opportunity costs are related to the transactions an investor fails to execute. In this thesis we investigate strategies to liquidate a portfolio of shares that will minimise the execution costs.

An investor is often faced with liquidating a large position in a security, or basket of securities, and one of the toughest decision he faces is how to structure the sale thereof. This is not a straightforward task since markets are not perfectly liquid. Many institutional investors hold significantly large positions in a number of securities and liquidating such a position can cause significant costs which directly influence the return on the investment (Mönch, 2004).

A market is liquid when traders are able to quickly buy and sell a significant volume of shares without causing significant movements in the share price. In a liquid market a trader is free to trade a large number of shares at once without the problem of selling some of these shares for a lower price than the current market value that he receives for the first shares sold. If a share has low liquidity a transaction of a large number of these shares shifts the share price significantly. Liquidity is effected by the number of traders and the number of shares in the market. A liquid share has many traders and many shares available for trade. If there are few people willing to trade, the depth of the order book will not extend very far and it will only be possible to trade a small number of shares at a time while the order book recovers.

An investor wishing to minimise the total risk of a sale could sell the entire position at once. This eliminates the market risk as the return from the sale is known with certainty. This strategy would also minimise any fixed trading costs. The liquidity discounts resulting from such a strategy would presumably be very large, and this liquidity discount increases with the decrease in liquidity of the security. As found by Chan & Lakonishok (1993, 1995), investors are more inclined to break the trade up into smaller trades, or 'packages', and found over 55% of large trades are spread over four days or more, and only 20% are completed within a single day. As the trading time and number of trades increase so does exposure to market risk and fixed trading costs, but the costs of liquidity discounts are reduced due to the smaller trade sizes.

To optimize a liquidation is to find a strategy that results in the maximum expected return from the liquidation of a position in one or more shares. An optimisation strategy might also solve for the minimum expected transaction or execution cost of the strategy. Some models incorporate the standard deviation and an investors risk preferences into this optimisation.

The aim of the thesis is to compare static liquidation models and to parameterise data from the South African JSE Stock Exchange across various market caps to determine if the form of these functions fit that found in other cases in the literature, and to see if there is any link between the market capitalisation and shares liquidity. We also attempt to derive general trading rules for large blocks of shares, so that traders without knowledge of specific liquidity parameters can still maximise the return of the transaction, while keeping the volatility as low as possible.

Two sets of data are used in this thesis. The main set is a series of snapshots of the order book at 10 minute intervals over a 3 day period for a sample of shares with various market capitalisations. We also make use of 5 second bid-ask spread data for a 2 week period. The order book data was collected using a AutoIT script written to capture I-Net financial data program and collect it throughout the day in Microsoft Excel. VBA programs were used to clean, process and interpolate the raw captured data into a usable form. The bid-ask data was collected by directly interacting with I-Net and storing the bid and ask values in a text file every 5 seconds.

We show that the form of impact functions for South African shares are consistent with the findings in the literature, and that there is a correlation between a shares liquidity and its market capitalisation. We also show that the simpler Almgren & Chriss (2000) model is sufficient to handle single stock and multi-security liquidations without any restrictions or introducing an intraday impact function, as well as introducing simple portfolio liquidation rules that can be applied without detailed liquidity analysis. General rules are also found.

This thesis is organised as followed. Chapter 2 discusses dynamic models, and gives a detailed presentation of the two static models used throughout this thesis. Chapter 3 investigates the optimal trading horizon. These models are programmed in MatLab and the optimisation carried out using the built-in FMINCON optimisation function.

Chapter 4 presents the findings in the literature about the functional forms of the liquidity functions. Chapter 5 adjusts the 2 main liquidation model used to handle comparable inputs and return similar outputs. Chapter 6 estimates liquidity parameters from our data for use in these models. The parameters are estimated from the data using EViews and Microsoft Excel.

Chapter 7 compares the 2 models for a liquidation over a single day, as well as investigating the effect of varying other parameters on the optimal strategies found by finding these models. Chapter 8 applies a multi-security model to investigate portfolio liquidation. Finally, in Chapter 9 we look for relationships and restrictions that might be found or usefully applied when solving for an optimal liquidation strategy.

Since one of the main aims of this thesis is to form general trading rules for traders with an understanding of a stocks liquidity, but might not have access to exact parameter estimates, many of the results are presented graphically. This is also helpful when comparing strategies found by different models.

Chapter 2

Optimal Liquidation Models

We first give a detailed description and derivation of the two static models used in the calculations in this thesis, those being the Almgren & Chriss (2000) model, and the Mönch (2004) model, followed by a summary of some of the dynamic models found in the literature.

The common problem modelled in both static and dynamic optimal liquidation models can be described as one where a trader holding a block of X shares wants to completely liquidate the position by time T^*. T^* is divided into N intervals, not necessarily equal, of length τ_i. The solution to this problem solves for the either the next trade to execute in a dynamic model, or the entire trade list $n_1, ..., n_N$ in a static model where n_k is the number of shares sold between times t_{k-1} and t_k.

2.1 Almgren and Chriss Model

The Almgren & Chriss (2000) model incorporates many of the important components in optimal liquidation. It includes both temporary and permanent impact functions, as well as the variance of the implementation shortfall. It solves for the optimal trajectory given the problem of a trader holding a block of X shares who wants to completely liquidate the position by a time T^* which is exogenously determined. T is divided into N equal intervals of length τ, with discrete trading times $t_k = k\tau$ for $k = 0, ..., N$. A trading strategy is the list $x_0, ..., x_N$ where x_k is the number of shares held at time t_k, $x_0 = X$ is the initial holding and liquidation requires $x_N = 0$ as the final holding.

The trade list $n_1, ..., n_N$, where $n_k = x_{k-1} - x_k$, is the number of shares sold between times t_{k-1} and t_k. The relationship between x_k and n_k is given by

$$x_k = X - \sum_{j=1}^{k} n_j = \sum_{j=k+1}^{N} x_j \qquad k = 1, 2, ..., N \tag{2.1}$$

It is however more convenient to work with the instantaneous *rate* of trading in shares per unit time. During the time interval t_{k-1} to t_k this rate is

$$\nu_k = \frac{n_k}{\tau} = \frac{1}{\tau}(x_{k-1} - x_k) \qquad k = 1, 2, ..., N \tag{2.2}$$

The model assumes stock prices evolve according to the discrete arithmetic Brownian random walk

$$\begin{aligned} S_k &= S_{k-1} + \sigma\tau^{\frac{1}{2}}\xi_k + \mu\tau \\ &= S_0 + \sigma\sum_{j=1}^{k}\tau^{\frac{1}{2}}\xi_j + \mu t_k \qquad k = 1, 2, ..., N \end{aligned} \tag{2.3}$$

The ξ_k are independent random draws from the standard normal distribution. Arithmetic Brownian motion was chosen instead of geometric since total changes in the share price would be small over the short-term. The assumption that μ and σ are constant is a reasonable one since the difference between arithmetic and geometric Brownian motions is negligible over short time horizons. This assumption also makes the model analytically simpler.

In this model it is important to note that the μ and σ used in calculations are divided by the reference price S_0 in order to give rates of return in the standard sense. To find the appropriate parameters for use in the model, we start with the standard volatility and drift as parameterised in an ordinary Brownian motion and multiply by the reference price. It is then divided by the reference parameter in the model resulting in the correct μ and σ being used, and the total cost and volatility equations returning a comparable monetary value. This is explained further later.

An additional factor used to model stock price movements is market impact. The stock price movements can then be seen to evolve according to the following dynamics

$$\begin{aligned} S_k &= S_{k-1} + \sigma\tau^{\frac{1}{2}}\xi_k + \mu\tau - \tau g(\nu_k) \\ &= S_0 + \sigma\sum_{j=1}^{k}\tau^{\frac{1}{2}}\xi_j + \mu t_k - \sum_{j=1}^{k}\tau g(\nu_j) \qquad k = 1, 2, ..., N \end{aligned} \tag{2.4}$$

Where $g(\nu)$ is a function of the rate of trading ν. The last term is the result of trading at rate ν_j in each time j up to time k. The function g of the rate ν_j multiplied by the interval τ gives the effective share price impact for each time period, and these are then summed to give an overall price reduction up to k.

As found by Holthausen *et al.* (1987) the market impact can be divided into two parts - a permanent impact and a temporary impact. Permanent in this sense refers to the life of the liquidation, and temporary refers to the period until the next trade in the liquidation. Both these functions are assumed linear in their initial model, although Almgren (2001) later investigates non-linear impact functions. We assume a general form initially, then show the linear case and later modify the model to handle the form of the impact function found to best fit our data.

Letting the permanent impact function be $g(\nu_k)$, the temporary impact function be $h(\nu_k)$, and $\tilde{S}_k$ the effective price per share received on the sale, we get an expression for the total capture of the strategy

$$\sum_{k=1}^{N} n_k \tilde{S}_k = XS_0 + \sigma\tau^{\frac{1}{2}} \sum_{k=1}^{N} x_k \xi_k + \mu \sum_{k=1}^{N} \tau x_k - \tau \sum_{k=1}^{N} x_k g(\nu_k) - \tau \sum_{k=1}^{N} \nu_k h(\nu_k) \quad (2.5)$$

By defining the total cost of trading as the difference $XS_0 - X\tilde{S}$ between the initial book or market value and the capture we get expressions for the total capture and the variance of the strategy

$$E(x) = -\mu \sum_{k=1}^{N} \tau x_k - \tau \sum_{k=1}^{N} x_k g(\nu_k) - \tau \sum_{k=1}^{N} \nu_k h(\nu_k) \quad (2.6)$$

$$V(x) = \sigma^2 \sum_{k=1}^{N} \tau x_k^2 \quad (2.7)$$

The optimal strategy is then found by solving for the minimal trading costs subject to the following constraints. The solution is one that satisfies:

$$\min_x \left(E[x] + \lambda V[x]\right) \quad (2.8)$$

subject to

$$\begin{aligned} n_{t_i} &> 0 \\ \sum_{i=1}^{n} n_{t_i} &= X \end{aligned}$$

Special Case - Linear Impact Functions

A special case is where both the temporary and permanent impact functions are linear. The permanent impact function is then

$$g(\nu) = \gamma\nu \tag{2.9}$$

where $g(\nu)$ is the drop in price per share per unit time. The constant γ has units of (\$ /share)/share. Each n shares sold reduces the share price by γn, regardless of the time taken to sell the n shares.

The temporary impact function $h(\nu)$ only impacts the next sale, so the actual price received on sale k is

$$\tilde{S}_k = S_{k-1} - h(\nu_k) \tag{2.10}$$

The linear model used is

$$h(\nu) = \alpha + \eta\nu \tag{2.11}$$

In the linear case the constant α can be treated as the fixed costs of selling, such as half the bid ask spread plus fees. This model for transaction costs is often called a *quadratic* cost model because the total cost incurred by trading n shares in a single unit of time is

$$nh\left(\frac{n}{\tau}\right) = \alpha n + \frac{\eta}{\tau}n^2 \tag{2.12}$$

Putting these impact functions together with the model of stock price movements we get a model for the price received on the sale between t_{k-1} and t_k

$$\tilde{S}_k = S_0 + \sigma\sum_{j=1}^{k-1}\tau^{\frac{1}{2}}\xi_j + \mu t_{k-1} - \gamma(X - x_{k-1}) - \alpha - \eta\nu_k \tag{2.13}$$

and the capture and variance of the strategy can be written as

$$E(x) = -\mu\sum_{k=1}^{N}\tau x_k + \frac{1}{2}\gamma X^2 + \alpha X + \left(\eta - \frac{1}{2}\gamma\tau\right)\sum_{k=1}^{N}\tau\nu_k^2 \tag{2.14}$$

$$V(x) = \sigma^2\sum_{k=1}^{N}\tau x_k^2 \tag{2.15}$$

2.2 Mönch Model

The model derived by Mönch (2004) may be the most detailed of the static optimal liquidation models. It includes a temporary impact function as well a factor to model the liquidity throughout the day. This however limits the model to solving for a execution list for a single day. In such a short time periods it is convenient to omit a permanent impact factor. The model also includes the variance of the strategy, as well as methods to calculate optimal trading times and the optimal number of trades to execute in the day.

A trader holding a block of X shares wants to completely liquidate the position by time T^*. Unlike the Almgren & Chriss model, Mönch also solves for trading times t_i, where $T \leq T^*$ and

$$T = t_N = \sum_{i=0}^{N-1} \tau_i \tag{2.16}$$

The trading times τ_i do not have be equal and are part of the optimisation solution based on a market recovery factor. Similar to the Almgren & Chriss model, the trading strategy can be defined as x_k and the corresponding trade list as n_k.

The price dynamics are assumed to follow the stochastic differential equation

$$dS_t = \sigma S_{t-} dW_t - S_{t-} d\Psi_t^+ \tag{2.17}$$

where

$$S_{t-} = \lim_{s \nearrow t} S_t$$

and $d\Psi_t^+$ is the price impact of the trading strategy. The price impact is composed of a temporary price impact function $\rho(n_t)$ and function for intraday stock market liquidity $\vartheta(\tilde{t})$. This intraday liquidity factor is unique to this liquidation model, and allows for a much more complex strategy, however if available data is limited this factor becomes difficult to accurately parameterise, especially in stocks with sporadic and low liquidity levels. Extending the model to handle more than a single day would also require significant changes to the model.

We then have a model of the stock price before and after the sale

$$S_{t_i} = [1 - \rho(n_{t_i})\vartheta(t_i)]S_{t_{i-1}} = \gamma_i S_{t_{i-1}} \tag{2.18}$$

Where $\gamma_i = 1 - \rho(n_{t_i})\vartheta(t_i)$. In this model the price impact is a relative change, a fractional decrease in share price. In the Almgren & Chriss model, the impact is an absolute effect. Care must be taken when estimating parameters for use in both models as adjustments are necessary. This is dealt with later.

We want to find the average amount received on all shares sold in the liquidation period. The average price realised by an investor liquidating n_{t_i} stocks at time t_i is bounded by

$$S_{t_{i-}} \geq \bar{S}_{t_i} \geq S_{t_i}$$

In the Almgren & Chriss model this was done by summing the product of the time interval τ and the price impact of trading at rate ν_j in time j. Here, $\bar{S}_{t_i}$ is the average price, and is found by dividing the total amount received, calculated as an integral, by the number of shares sold in that period. We can simplify this calculation as follows

$$\begin{aligned}\bar{S}_{t_i} &= \frac{1}{n_{t_i}} \int_0^{n_{t_i}} (1 - \rho(u)\vartheta(t_i))\, S_{t_i-} du \\ &= \left[1 - \frac{\vartheta(t_i)}{n_{t_i}} \int_0^{n_{t_i}} \rho(u) du\right] S_{t_i-} \\ &= \frac{1 - \frac{\vartheta(t_i)}{n_{t_i}} \int_0^{n_{t_i}} \rho(u) du}{1 - \rho(n_{t_i})\vartheta(t_i)} S_{t_i} \\ &= \delta_i S_{t_i} \end{aligned} \tag{2.19}$$

where

$$\delta_i = \frac{1 - \frac{\vartheta(t_i)}{n_{t_i}} \int_0^{n_{t_i}} \rho(u) du}{1 - \rho(n_{t_i})\vartheta(t_i)} \tag{2.20}$$

Adding in fixed transaction costs per trade we get an expected net liquidation value, $E(x)$ of

$$E(x) = -Nk + \sum_{i=1}^{N} n_{t_i} \delta_i E(S_{t_i}) \tag{2.21}$$

Using the price dynamics above we can derive the following expressions for the expected value of the liquidation

$$E(x) = -Nk + S_{t_0} \sum_{i=1}^{n} n_{t_i} \delta_i \prod_{j=1}^{i} \gamma_j \tag{2.22}$$

$$V(x) = \sum_{i=1}^{N} (n_{t_i}\delta_i)^2 var(S_{t_i}) + 2 \sum_{i<j}^{N} n_{t_i}\delta_i n_{t_j}\delta_j cov(S_{t_i}, S_{t_j}) \tag{2.23}$$

where

$$cov(S_{t_i}, S_{t_j}) = S_{t_0}^2 \left(exp[\sigma^2(t_i - t_0)] - 1\right) \prod_{m=1}^{i} \gamma_m^2 \prod_{l=i+1}^{j} \gamma_l \qquad (2.24)$$

This is the total received from the sale of the shares, as opposed to the Almgren & Chriss model which minimises the expected loss of the liquidation. We must therefore maximise this expected value while constraining the volatility. The equation to maximise is

$$\max_{\tau_i, n_{t_i}, N} [E(x) - \lambda V(x)]$$

The solution is threefold. It solves for the optimal strategy that returns the highest expected value for the given level of risk, the times of the day to execute those trades, and the optimal number of trades to execute in the day that returns the highest expected value out of all optimal strategies for a given number of trades.

The added solutions require additional constraints in the optimisation. The solution is one that satisfies :

$$\max_{\tau_i, n_{t_i}} [E(x) - \lambda V(x)] \qquad (2.25)$$

subject to

$$\begin{aligned} n_{t_i} &> 0 \\ \sum_{i=1}^{N} n_{t_i} &= X \\ \tau_i &\geq \beta n_{t_i} \\ \tau_0 &> 0 \\ \sum_{i=0}^{N-1} \tau_i &\leq T^* \end{aligned}$$

The optimisation is repeated for $N = 1, ...N^*$. Where N is the number of trades. By repeating the optimisation for each N we can calculate the optimal number of trades to execute by observing the solution with the highest expected return. The solution is still a static one, but the calculation time is greatly increased when solving for optimal N as well as the optimal trade size and trade times. The differences have no effect on the trading pattern, and for ease of comparison this additional calculation has been omitted and an equal number of trades has been used. Figure 2.1 shows this. The difference in the expected return are minimal after 20 trades, which is executing a trade every 24 minutes. In a market with

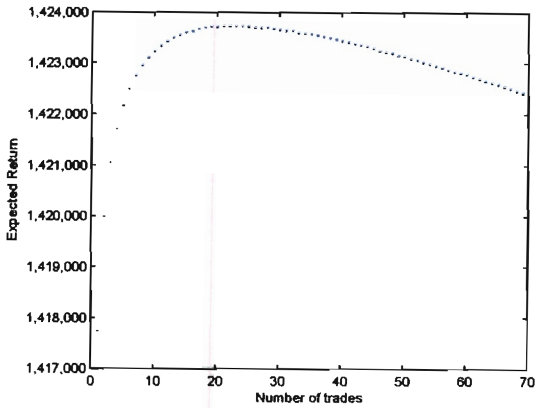

Figure 2.1: Mönch example - Optimal number of trades. This shows optimal expected return for $N = 1, ..., N^*$. In this example, 23 is the optimal number of trades to execute.

low liquidity such as South Africa's, trading more frequently than that will not allow enough time for the order book to recover. We therefore choose not to use more than 20 trades in a day in our calculations.

2.3 Other Optimal Liquidation Models

A commonly referenced dynamic model is that of Bertsimas & Lo (1998), who investigate optimal execution as a dynamic programming approach that minimise the implementation shortfall. The optimal trajectory is found recursively through the use of the Bellman equation. They conclude that the naïve strategy of equal packages over the liquidation period is only optimal in a special case. To better understand how market information might influence the optimal strategy, they use a serially correlated information variable in the price process and find the optimal strategy is then a linear combination of the naïve strategy and the change in the information variable. This model is extended in Bertsimas *et al.* (1999) to handle multiple assets.

Subramanian & Jarrow (2001) use both an execution lag and liquidity discount in their liquidation model. The lags are imposed by setting a rule that the trader may not place any additional sell orders before the previous order has been completed. This waiting time, as in Mönch is predetermined and deterministic. As in Almgren & Chriss a sale generates a permanent price drop determined by the size of the sale. Lau & Kwok (2006) derive numerical algorithms based on this model to calculate the probability that the level at cash at the end of the liquidation is above a target level.

Butenko *et al.* (2005) applies a stochastic programming technique to solve the optimal liquidation problem and consider the case for a risk neutral and risk averse trader using decision trees. For a solution without risk constraints they are able to reduce the problem to a linear programing one. Their algorithms provide path dependent strategies which determine the size of the sell order depending upon the sample-path of the price of the security up to that time.

Krokhmal & Uryasev (2006) also apply a sample-path approach to derive a truly dynamic model allowing for a response at each to changing market conditions. They include a temporary and permanent market impact, as well as a lag in the permanent market impact that prevents small trades from causing permanent price changes. The lag also allows a trader to increase the size of the initial trades.

Chapter 3

Optimal Trading Horizon

In many of the models discussed in this thesis the trading horizon is not determined within the model, but rather an exogenous factor. This may lead to a situation where the strategy may be optimal for that trading horizon, but there may be strategies with higher expected returns available should the trading horizon be changed. Here we consider a model to estimate the optimal trading time, T^*.

3.1 Dubil

A paper by Dubil (2002) on optimal liquidation, adapts the Almgren and Chriss model to continuous time and solves for optimal trading times for various forms of the impact functions. He also works in terms of dollars rather than volume.

Letting W_t be the time t dollar amount of the investment. The initial exposure is W_0 and the final exposure $W_T = 0$. By reducing the dollar holdings to a ratio relative to the initial exposure $x_t = \frac{W_t}{W_0}$, we have $x_0 = X = 1$ and $x_T = 0$. In each interval $(t - dt, t)$, $n_t = x_{t-dt} - x_t = -dx_t$ of his holdings are sold. The speed of trading is then $\nu_t = \frac{n_t}{dt} = -\frac{dx_t}{dt}$. R_t is the cumulative equilibrium return on the asset and follows an arithmetic Brownian motion process with the only drift due to the accumulated permanent impact on the price. Defining the permanent impact as before as $g(\nu_t)$, we can then write R_t as

$$R_t = \sigma\xi_t - \int_0^t g(\nu_s)ds \tag{3.1}$$

The cumulative trading return $\tilde{R}_t$ between $(t - dt, t)$ is also subject to the temporary impact $h(\nu_t)$ and can be defined as

$$\tilde{R}_t = R_t - h(\nu_t) \tag{3.2}$$

Using the assumption of constant speed $\nu_t \equiv \nu$, so that

$$n_t = -dx_t = \nu dt \tag{3.3}$$

The excess profit or capture of the strategy, which is most likely negative, as a fraction of the original wealth is

$$\Pi = \int_0^T \tilde{R}_t(-dx_t) = \int_0^T \tilde{R}_t \nu dt \tag{3.4}$$

Starting with the general case of a power function market impact, we define

$$g(\nu_t) = \gamma \nu_t^G \tag{3.5}$$

$$h(\nu_t) = \varepsilon \nu_t + \eta \nu_t^H \tag{3.6}$$

The trading return in the interval $(t - dt, t)$ is

$$\tilde{R}_t = \sigma \xi_t - \varepsilon - \eta_n u_t^H - \gamma \int_0^t \nu_s^G ds \tag{3.7}$$

We can then extend this to the profit function for a constant trading strategy and write it as

$$\Pi = \nu\sigma \int_0^T \xi_t dt - \varepsilon\nu \int_0^T dt - \eta\nu^{H+1} \int_0^T dt - \gamma\nu^{G+1} \int_0^T \left(\int_0^t ds \right) dt \tag{3.8}$$

which reduces to

$$\Pi = \nu\sigma \int_0^T \xi dt - \varepsilon\nu T - \eta\nu^{H+1} T - \frac{1}{2}\gamma\nu^{G+1}T^2 \tag{3.9}$$

The expected value of the profit is then equal to

$$E[\Pi] = -\varepsilon\nu T - \eta\nu^{H+1}T - \frac{1}{2}\gamma\nu^{G+1}T^2 \tag{3.10}$$

and the variance equal to

$$V[\Pi] = \frac{1}{3}T^3\sigma^2\nu^2 \tag{3.11}$$

Using $X = \nu T$ the mean and variance can be written as

$$E[\Pi] = -\varepsilon X - \eta X^{H+1}T^{-H} - \frac{1}{2}\gamma X^{G+1}T^{-G+1} \tag{3.12}$$

$$V[\Pi] = \frac{1}{3}TX^2\sigma^2 \tag{3.13}$$

Bringing in the traders risk preference factor λ to setup the optimisation problem in terms of T gives

$$\max_T \left(-\varepsilon X - \eta X^{H+1} T^{-H} - \frac{1}{2}\gamma X^{G+1} T^{-G+1} - \lambda \sqrt{\frac{1}{3} T X^2 \sigma^2} \right) \tag{3.14}$$

The first order condition for this optimisation is

$$-H\eta X^{H+1} T^{-H-1} - \frac{1}{2}(G-1)\gamma X^{G+1} T^{-G} - \frac{1}{2\sqrt{3}} \lambda \sigma X T^{-\frac{1}{2}} = 0 \tag{3.15}$$

In general numerical methods are necessary to solve this equation, but closed-form solutions are obtainable to some special cases.

If we assume the permanent impact is linear so that $G = 1$ and the temporary impact is defined as above then we can rewrite the first order condition as

$$-H\eta X^{H+1} T^{-H-1} - \frac{1}{2\sqrt{3}} \lambda \sigma X T^{-\frac{1}{2}} = 0 \tag{3.16}$$

Solving for T:

$$T^* = \left(\frac{2\sqrt{3} H \eta X^H}{\lambda \sigma} \right)^{\frac{1}{H+\frac{1}{2}}} \tag{3.17}$$

Adding the assumption that the temporary impact is also linear so that $G = H = 1$ gives an optimal time T of

$$T^* = \left(\frac{2\sqrt{3} \eta X}{\lambda \sigma} \right)^{\frac{2}{3}} \tag{3.18}$$

3.2 Almgren

Almgren, in a 2001 paper extending his work with Chriss, investigates non-linear impact functions and optimal trading times. He also works in continuous time, but his method to determine optimal trading times differ and allows for closed form solutions for liquidations with impacts following a power function.

Starting with continuous forms of the capture and volatility of the strategy, and with no drift or constant in the temporary impact function, they get

$$E(x) = \int_0^T \Big(x(t)g(\nu_t) + \nu_t h(\nu_t)\Big)dt \tag{3.19}$$

$$V(x) = \int_0^T \sigma^2 x(t)^2 dt \tag{3.20}$$

Introducing as before the risk preference parameter λ the optimisation equation is

$$U(x) = E(x) + \lambda V(x)$$

Minimizing $U(x)$ is a problem relating to the calculus of variations:

$$\min_x(t) \int_0^T F\Big(x(t), -\dot{x}(t)\Big)dt \tag{3.21}$$

with

$$F(x,\nu) = xg(\nu) + \nu h(\nu) + \lambda\sigma^2 x^2 \tag{3.22}$$

This is solved using the Euler-Lagrange equation.
Applying a temporary impact function of the form

$$h(\nu) = \eta\nu^k \tag{3.23}$$

results in the quadrature problem where v_0 is the constant of integration

$$\int_{x(t)}^X \Big(\frac{\lambda\sigma^2}{k\eta} + v_0^{k+1}\Big)^{-\frac{1}{k+1}} dx = t \tag{3.24}$$

Taking $v_0 = 0$ allows us to solve explicitly for the longest optimal trajectories:

$$\frac{x(t)}{X} = \begin{cases} \left(1 - \frac{1-k}{1+k}\frac{t}{T^*}\right)^{-(1+k)/(1-k)} & if 0 < k < 1 \\ \exp\left(-\frac{t}{T^*}\right) & k = 1 \\ \left(1 - \frac{k-1}{k+1}\frac{t}{T^*}\right)^{(k+1)/(k-1)} & k > 1 \end{cases}$$

in which case the "characteristic time" is

$$T^* = \left(\frac{k\eta X^{k-1}}{\lambda\sigma^2}\right)^{1/(k+1)} \tag{3.25}$$

Here in the linear case where $k = 1$ the optimal time T^* is independent of the portfolio size X.

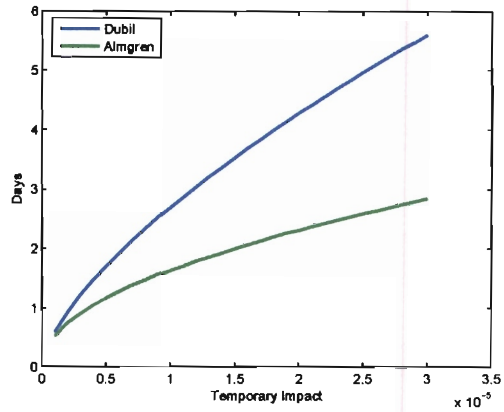

Figure 3.1: Change in optimal time as the impact parameter increases

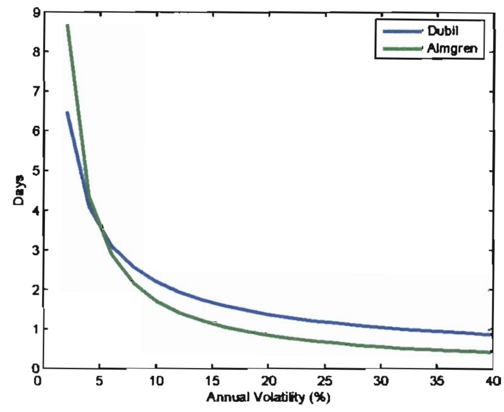

Figure 3.2: Change in optimal time as the annual volatility increases

3.3 Numerical Comparison

There are two important factors we wish to investigate when comparing optimal trading horizon, the effect of the shares liquidity and the volatility of the share. We would hope both models correspond in their solutions since their derivations are based on the same source, although the path taken is different.

We see in Figure 3.1 where a portfolio of 100,000 shares are to be liquidated that the Dubil model suggests a longer trading time, although they both increase as the liquidity decreases. This gap is probably due to the large trade size, since the Almgren solution with linear parameters is independent of the portfolio size X. Figure 3.2 shows the optimal liquidation as the shares annual volatility increases. As would be expected, for a risk averse trader the optimal trading time decreases as volatility increases.

3.4 Naïve Strategy vs Optimal Strategy

Before analysing the optimal strategy as found by the models described in the previous chapter, it is important to determine their advantages over a naïve strategy, the naïve strategy being trading equal amounts at equally spaced intervals. In Figure 3.3 the expected loss of the naïve strategy is compared over time to the optimal strategy, as found with the linear Almgren and Chriss model for a risk averse trader. Figure 3.4 shows the difference in the standard deviation of the expected loss over time between the 2 strategies.

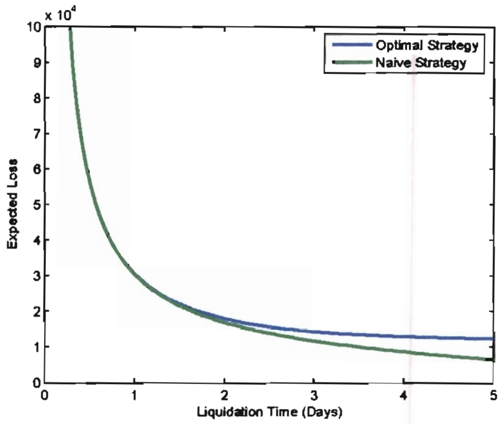

Figure 3.3: Naïve strategy vs Optimal strategy - expected loss

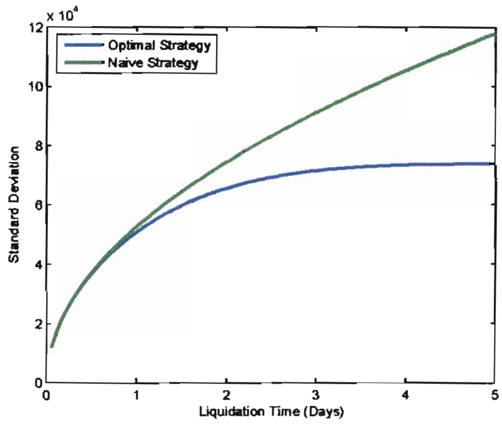

Figure 3.4: Naïve strategy vs Optimal strategy - standard deviation

In a single day liquidation, both the naïve and optimal strategies have the same expected loss and standard deviation. The expected loss decreases faster and the standard deviation increases slower using an optimised strategy. This would indicate that should a trader have a time horizon of greater than one day, an optimal strategy can greatly reduce the volatility of the strategy, as well as reducing the execution costs.

Chapter 4

Parameter Estimation

4.1 Impact Functions

4.1.1 Theoretical Basis for Choice of Functions

The price effects of trades, particularly large block and institutional trades, has been well documented in the literature. Kraus & Stoll (1972) investigated the price effects of block trades on the New York Stock Exchange and found evidence of a temporary price impact. Holthausen *et al.* (1987, 1990) divide the effect onto a temporary and permanent impact, finding strong support for a temporary impact as well as for a permanent impact increasing with block size.

Lillo *et al.* (2003) analyse the form of the impact function and found that the impact is concave, more specifically a power function. They also collapse the curves from stocks of varying liquidity by scaling according to their market cap to derive a master curve. The usefulness of the collapse is that it shows how an easily observable factor such as market cap can explain liquidity, which is difficult to observe. This is useful when all stocks have at least some base level of liquidity, but in the South African market where many stocks have little to no liquidity from either the bid or the ask side on any given day, it would be difficult to extend their master curve theory based purely on market cap and adding other factors such as daily traded volume might provide a better answer.

Almgren & Chriss (2000) paper they assume linear temporary and permanent impact functions. Mönch (2004) shows through empirical data that a power function is a much better fit to the temporary impact function. With little to no empirical studies into short term permanent market impact functions their use of a linear model is as good as any other guess, and allows for easy of estimation and use in calculations. They also provide a suggestion as to how to estimate this parameter from readily available market data. The choice of permanent impact function is backed up by the findings of Huberman & Stanzl (2004), who find that the model is only arbitrage free with linear permanent impact functions.

Almgren *et al.* (2005) use proprietary Citigroup data giving them the number of executions used to complete a transaction, as well as if the sale was buyer or seller initiated. We do not have access to this level of data, but their application of the power law to the impact functions is typical in the literature.

4.1.2 Parameter Specification

We collected snapshots of the bid book at 10 minute intervals throughout the day for 3 consecutive days, 31 May, 1 and 2 June 2006. This gave us a list of all bids and offers' prices and volumes in the market at each 10 minute point. Working with the bid side of the data, we transformed it into a list of percentage decreases from the midpoint of the best bid and offer for different volumes of transactions. Doing this for each time period left irregular data, and to be able to aggregate the 10 minute snapshots the impact had to be found for regular volumes of trades. Since orders are placed in the order book for unequal numbers of shares and we want to average the price impact across times and days we need to have impacts at regularly spaced volume intervals. To do this the price discount or impact was found using linear interpolation. Once we had the data in this form it was possible to carry out further analysis.

Temporary Impact Function

Due to the small size of our sample it was necessary to average over all points to parameterise the impact functions. Following the literature, convex curves were fitted to each stock. Although the data looked likely to fit a power function, the lack of depth in the tail distorted the least squares calculations and accurate fits could not be made, as shown in Figure 4.1. The lack of depth is further illustrated by Figure 4.4 where incomplete data in the average of higher volumes causes jumps. The impact functions were most accurately fitted to the portion of the curve up to the first jump. Attempting to fit a quadratic function were more successful, and all stocks were fitted to this function with adjusted R-squared values over 0.97.

We therefore fitted

$$\rho(n) = an^2 + bn + c \tag{4.1}$$

This curve is fitted to percentage price changes, as is used in Mönch's model, where he defines the price after a trade as

$$\tilde{S}_{t_i} = [1 - \rho(n_{t_i})\vartheta(t_i)]S_{t_{i-1}} \tag{4.2}$$

where ϑ is the factor adjusting the impact for the time of day. To use this estimation in the Almgren and Chriss model the impact must be adjusted to an absolute one to conform to their definition of temporary impact which is

$$\tilde{S}_{t_i} = S_{t_{t-1}} - h(\nu_{t_i}) \tag{4.3}$$

We therefore adjust the effect by multiplying by the stock price S_{t_0} at the beginning of the liquidation. We must also note that in the Almgren & Chriss model the impact is a function of the rate of trade ν, while in Mönch it is a function of the trade size n. We have $n = \nu\tau$. We then get an impact function in absolute terms

$$h(\nu) = S_{t_0}a(\nu\tau)^2 + S_{t_0}b(\nu\tau) + S_{t_0}c \tag{4.4}$$

where $S_{t_0}c$ has the same interpretation as ϵ in Almgren & Chriss's model. Due to the small sample size it is possible to replace $S_{t_0}c$ with half the bid-ask spread as more accurately estimated from a larger sample of market data, while keeping the other coefficients as previously found using least squares as this will not be change the convex shape of the impact function.

Note: The quadratic function fitted often gave a turning point to the right of the y-axis, resulting in a negative impact for the first portion of the curve. This was fixed by setting $b = 0$, forcing the turning point of the quadratic to be on the y-axis. The effect of this adjustment on the adjusted R-squared value was minimal - approximately 0.01. Figure 4.2 shows the decrease on the left of the y-axis before following the impact curve upward.

Permanent Impact Function

With little to no research conducted on permanent impact functions in the very short term as it applies in these models, we can only base our estimates on the same rule of thumb adopted by Almgren & Chriss, which is that price effects become significant when 10% of the daily volume is sold. They suppose that 'significant' might mean a decrease in price of one bid-ask spread. They also assume this effect is linear in trade size.

To carry out this estimation requires the shares' daily volume traded and its bid-ask spread, both of which are readily available. Since we are dealing with short-term transactions in these models, these figures were taken as the average over the last 6 months. The bid-ask spread at the close of each trading day in the period was used.

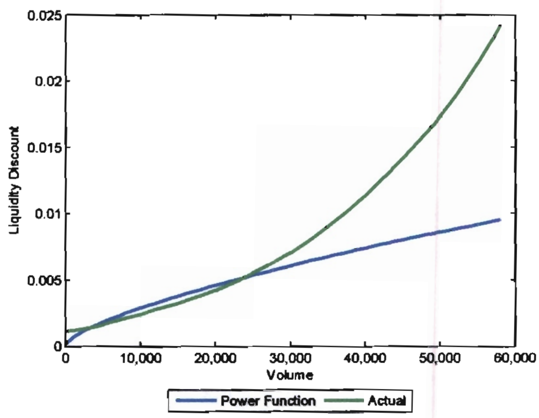

Figure 4.1: Power function approximation of the form $ax^b + c$

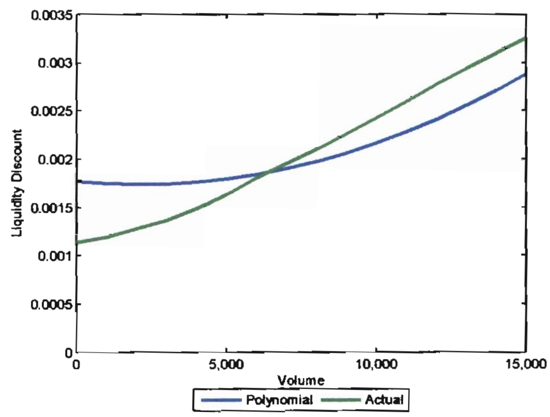

Figure 4.2: Quadratic approximation of the form $ax^2 + bx + c$

We therefore easily fit the linear permanent impact function

$$g(\nu) = \gamma\nu \tag{4.5}$$

where γ is calculated as described above.

4.2 Intraday Trading

4.2.1 Theoretical Basis for Choice of Functions

The Mönch (2004) model used later requires a factor to model liquidity throughout the day. Harris (1986) first found evidence of higher returns at the beginning and end of the day, indicated by a larger bid-ask spread.

McInish & Wood (1992) analyse intraday patterns in the bid/ask spread in NYSE stocks. They find a U-shaped or J-shaped pattern evident in the mean bid-ask spreads for each minute of the trading day over a 6 month period, where spreads are higher at the beginning and end of the day.

Mönch (2004) study of a 62 day period concurs with these findings, showing a evidence and parameterisations of a U-shaped liquidity discount for volumes up to 30 000 shares, compared to the just the bid-ask spread analysis of McInish and Wood.

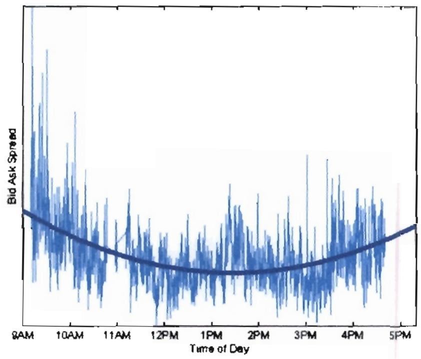

Figure 4.3: 5 second Bid-Ask spread 2 week average, with a fitted quadratic

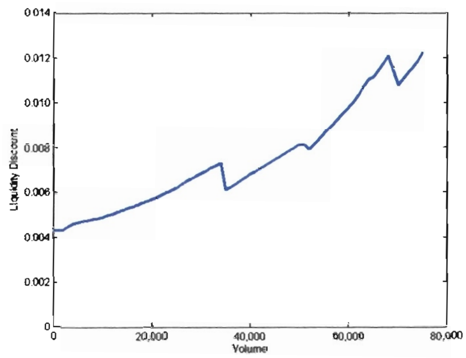

Figure 4.4: Average temporary impact over all volumes with jumps

4.2.2 Parameter Specification

Plotting 5 second bid-ask spreads over a 2 week period in a similar method to McInish and Wood provided justification to pursue a J or U shaped intraday liquidity model. This was done by plotting $(ask - bid)/(\frac{1}{2}(ask + bid))$. It would not be accurate enough to fit a curve to the bid-ask spread data since the spread did not contain enough data to reflect the actual drop off in liquidity for a larger number of shares. It only gives some indication as to the pattern we can expect in liquidity discounts for higher volumes. The 5 second bid-ask spread for AGL is shown in Figure 4.3.

Using the same data as that in the temporary impact function estimation, we can also find a function for impact against time for a given volume, instead of impact against volume averaged over all time periods. Due to the small sample size, it was necessary to average the data in hourly groups. This meant 18 discount values for a given volume at each hour of the day, using all 3 days in the sample. Due to the low levels of liquidity in the South African market there were not always 18 points available to average, resulting in jumps in the curves. To avoid these jumps distorting the data an impact was only calculated when all points existed. This left shortened impact curves at certain time periods.

To accurately fit a curve for impact against time we needed an average over each hour. The lowest level of liquidity for which every point existed gave us a reference for that stock as to what maximum volume readily available to trade at any time in the sample period. We then took the impacts at half this volume and fitting them to the number of minutes elapsed since the market opened. This can be seen in Figure 4.4 where only the data up to the first jump is useful. We take the impacts across time at the volume of half this point.

Mönch fitted a U-shaped function of the form

$$\vartheta(\tilde{t}) = \frac{d}{\tilde{t} + e} + \frac{f}{g - \tilde{t}} \tag{4.6}$$

was not as successful as we would have like due to the small sample with 4 parameters to estimate. Reducing the number of parameters to estimate while still keeping the shape was attempted with a quadratic function. This halved the number of free parameters, while including a constant term. The results were much more favourable here and for most of the stocks The adjusted R-squared was above 0.65. Although not ideal, the evidence indicating a J or U-shaped intraday pattern was justified and the small sample size cannot be expected to give a perfect fit, but rather provide justification for the choice of function.

We therefore prefer

$$\vartheta(\tilde{t}) = d\tilde{t}^2 + e\tilde{t} + f \tag{4.7}$$

where $\tilde{t}$ is the number of minutes since the market opened that day.

Chapter 5

Model Adjustments

In this chapter we consolidate the two main models examined in this thesis and modify the formulas to accept the same inputs and perform the optimisation so that the returned results are directly comparable. They are already essentially the same, but these adjustments are necessary to carry out meaningful comparison between their core differences. The differences we are able to consolidate include the lack of a permanent impact function in the Mönch Model, different forms of the fixed costs and impact functions, and the form of the solution returned by the Almgren and Chriss model minimises the loss of the strategy while that of Mönch maximises the total return of the strategy. The differences are summarised in Table 5.1.

5.1 Almgren and Chriss adjustments

Temporary Impact Function

As explained previously, this model uses an absolute change in the impact functions. Our data, however, is parameterised to percentage changes in the price. We must therefore define the temporary impact function h in terms of the current share price S_{t_0} and the estimated quadratic parameters a, b and c

$$h(\nu) = S_{t_0}a(\nu\tau)^2 + S_{t_0}b(\nu\tau) + S_{t_0}c \tag{5.1}$$

The cost of trading n shares in a time period is then

$$nh(\frac{n}{\tau}) = S_{t_0}an^3 + S_{t_0}bn^2 + S_{t_0}cn \tag{5.2}$$

Fixed Costs

Factor	Almgren & Chriss	Mönch
Absolute Temporary Impact	Yes	No
Relative Temporary Impact	No	Yes
Permanent Impact	Yes	No
Intraday Impact	No	Yes
Fixed costs per share	Yes	No
Fixed costs per trade	No	Yes
Calculates expected loss	Yes	No
Calculates total return	No	Yes

Table 5.1: Model Differences

We wish to assign a fixed cost to each trade and to each share. Since these are independent of the strategy we can add them to the expenses occurred without any problems. We let f be the fixed cost per trade, and g be the fixed cost per share. The terms to add to the cost will then be

$$fN + gX \tag{5.3}$$

Final Solution

Keeping the forms of the other parameters the same we get a new equation for the capture of the strategy.

$$\begin{aligned} X\bar{S} = \sum_{k=1}^{N} n_k \tilde{S}_k &= XS_0 + \sigma\tau^{\frac{1}{2}} \sum_{k=1}^{N} x_k \xi_k + \mu \sum_{k=1}^{N} \tau x_k - \gamma\tau \sum_{k=1}^{N} x_k \nu_k \\ &- S_0 a\tau \sum_{k=1}^{N} n_k^3 - S_0 b\tau \sum_{k=1}^{N} n_k^2 - S_0 cX - fN - gX \end{aligned} \tag{5.4}$$

Using summation by parts on the summation for the permanent impact we can simplify

$$\begin{aligned} \sum_{k=1}^{N} \tau x_k \nu_k = \sum_{k=1}^{N} x_k (x_{k-1} - x_k) &= \frac{1}{2} \sum_{k=1}^{N} \left(x_{k-1}^2 - x_k^2 - (x_k - x_{k-1})^2 \right) \\ &= \frac{1}{2} X^2 - \frac{1}{2} \sum_{k=1}^{N} n^2 \end{aligned} \tag{5.5}$$

We write the capture as

$$X\bar{S} = XS_0 + \sigma\tau^{\frac{1}{2}}\sum_{k=1}^{N} x_k\xi_k + \mu\sum_{k=1}^{N}\tau x_k - \frac{1}{2}\gamma X^2 - fN$$
$$- (S_0c + g)X - \left(S_0an + S_0b - \frac{1}{2}\gamma\right)\sum_{k=1}^{N} n^2 \tag{5.6}$$

Using the total cost of trading as $XS_0 - X\bar{S}$, the expected value and the variance of the strategy are then

$$E(x) = -\mu\sum_{k=1}^{N}\tau x_k + \frac{1}{2}\gamma X^2 + fN + (S_0c + g)X + \left(S_0an + S_0b - \frac{1}{2}\gamma\right)\sum_{k=1}^{N} n^2 \tag{5.7}$$

$$V(x) = \sigma^2\sum_{k=1}^{N}\tau x_k^2 \tag{5.8}$$

5.2 Mönch Model

Impact Functions

The temporary impact function in already in the correct form to use the estimated data properly, but there is no permanent impact function in use in this model. Our assumption is that the permanent impact is linear, and so we can easily add this to the model using the same notation as Almgren and Chriss. The term to add is

$$g(\nu) = \gamma\tau\sum_{k=1}^{N} x_k\nu_k = \gamma\frac{1}{2}X^2 - \gamma\frac{1}{2}\sum_{k=1}^{N} n^2 \tag{5.9}$$

Fixed Costs

The model includes fixed costs per trade, so we only need to add a term for fixed costs per share sold, which is gX.

Model Output

As stated previously, the model differs from Almgren and Chriss in the form of the output of the strategy. With the changes made above the total return from the strategy is

$$E(x) = -fN - gX - \gamma\frac{1}{2}X^2 + \gamma\frac{1}{2}\sum_{k=1}^{N} n^2 + S_{t_0}\sum_{i=1}^{n} n_{t_i}\delta_i\prod_{j=1}^{i}\gamma_j \tag{5.10}$$

Deducting the initial value of the portfolio, XS_{t_0} we get a equation for the capture of the strategy

$$E(x) = -XS_{t_0} - fN - gX + \gamma\frac{1}{2}X^2 - \gamma\frac{1}{2}\sum_{k=1}^{N} n^2 + S_{t_0}\sum_{i=1}^{n} n_{t_i}\delta_i\prod_{j=1}^{i}\gamma_j \tag{5.11}$$

We want to minimise this, therefore the final expected return and variance of the strategy become

$$E(x) = S_{t_0}\left(\sum_{i=1}^{n} n_{t_i}\delta_i\prod_{j=1}^{i}\gamma_j - X\right) - fN - gX - \gamma\frac{1}{2}X^2 + \gamma\frac{1}{2}\sum_{k=1}^{N} n^2 \tag{5.12}$$

$$V(x) = \sum_{i=1}^{N}(n_{t_i}\delta_i)^2 var(S_{t_i}) + 2\sum_{i<j}^{N} n_{t_i}\delta_i n_{t_j}\delta_j cov(S_{t_i}, S_{t_j}) \tag{5.13}$$

Where now we wish to minimise this to return results of the same form as Almgren and Chriss. Including the investor risk preference parameter λ we must solve, for each $n = 1, ..., N$:

$$\min_{\tau_i, n_{t_i}}[E(x) + \lambda V(x)] \tag{5.14}$$

$$\begin{aligned} st. \quad n_{t_i} &> 0 \\ \sum_{i=1}^{n} n_{t_i} &= X \\ \tau_i &\geq \beta n_{t_i} \\ \tau_0 &> 0 \\ \sum_{i=0}^{n-1} \tau_i &\leq T^* \end{aligned}$$

Chapter 6

Liquidity-Specific Parameterisation On Four SA Stocks

6.1 Data for Four South Africa Stocks

Four stocks were chosen, 2 in the top 40 and 2 mid cap stocks. The stocks general data is shown in Table 6.1. The values for the bid, ask and price are at the market close on 2 June 2006. The average spread is a 3 day average from 5 second tick data, and the annual growth, annual volatility and daily volume traded are estimated from the 6 months up to 2 June 2006. Using this data together with the estimated liquidity specific parameters we are able to calculate optimal liquidation trading strategies.

Name	**AGL**	**SLM**	**MPC**	**SYC**
Type	Top 40	Top 40	Mid Cap	Mid Cap
Free Float Market Cap	R 393,323 M	R 38,080 M	R 4,979 M	R 3,502 M
Bid	R 262.05	R 15.81	R 23.10	R 19.10
Ask	R 262.50	R 15.91	R 23.15	R 19.38
Price	R 262.28	R 15.86	R 23.13	R 19.24
Ave Spread	R 0.53	R 0.05	R 0.15	R 0.23
Annual Growth	0.2151	0.0398	0.1390	0.1102
Volatility	0.1540	0.1485	0.1062	0.0639
Daily Volume Traded	3,500,000	6,700,000	700,000	450,000

Table 6.1: General Stock Data

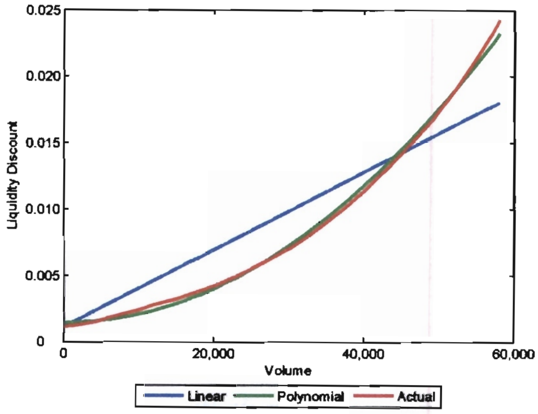

Figure 6.1: AGL fitted temporary impact

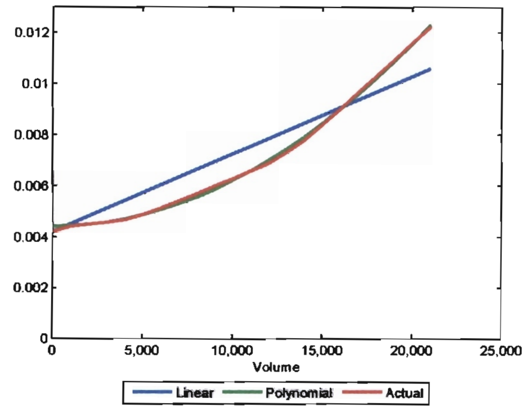

Figure 6.3: MPC fitted temporary impact

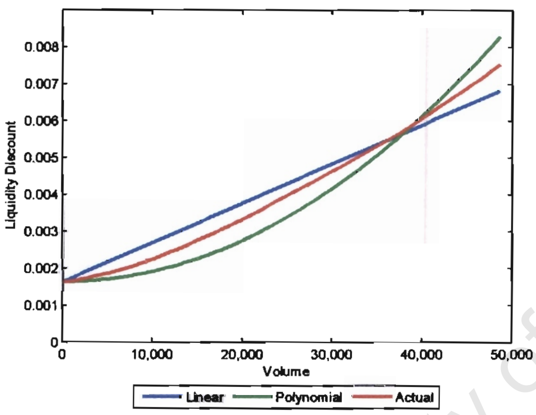

Figure 6.2: SLM fitted temporary impact

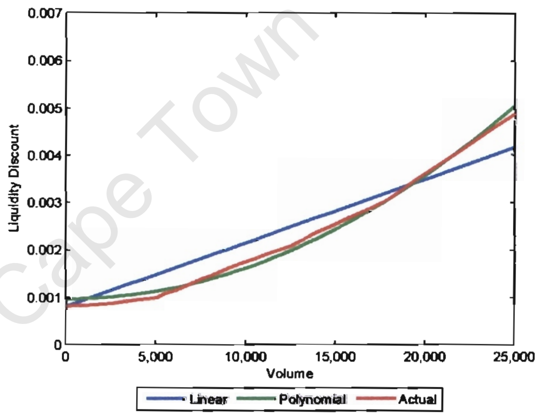

Figure 6.4: SYC fitted temporary impact

6.2 Temporary Impact

Figures 6.1 through 6.4 show linear, quadratic and power functions fitted to the order book data of the 4 shares. The adjusted quadratic with the constant for the linear term removed is also shown. It is clear a quadratic is the best fit to the available data. For many of the stocks the linear fit was best when the y-axis intercept was negative. This will obviously lead to problems, as would any y-axis intercept less than the bid ask spread. The slopes of the linear functions plotted are those where the y-axis intercept has been forced to be the y-axis intercept of the original data. This provides better results than shifting the optimal gradient found without any constraints on the estimation.

Tables 6.2 to 6.4 show the results of the regression to determine the parameters. We can see in Table 6.2 a high R-squared for all the stocks using a linear model.

Name	AGL	SLM	MPC	SYC
a	2.91E-07	5.34E-08	3.02E-07	6.70E-08
Std. Error	9.54E-09	6.45E-10	1.48E-08	1.78E-09
t-Statistic	30.55037	82.83787	20.36773	37.58225
Prob.	0.000000	0.000000	0.000000	0.000000
R-squared	0.863679	0.959604	0.88614	0.914698
Adjusted R-squared	0.863679	0.959604	0.88614	0.914698
S.E. of regression	0.002464	0.000358	0.000853	0.000369
b fixed	0.001142	0.001638	0.004208	0.003808

Table 6.2: Linear regression estimates, constant term fixed

The numbers however do not tell the full story, as they are only estimated up to a cutoff point, as was shown in Figure 4.4. We see a good linear fit up to this point, but with incomplete data beyond this point a linear function may underestimate the impact of a larger transaction.

Tables 6.3 shows the results of the quadratic least squares analysis. A quadratic function does a better job in modelling the actual shape of the impact function beyond the available data, than that done using a linear function, especially for a large trade. Since we are interested in breaking down and limiting the size of the trades, this is important. The results are shown where there is no linear term in the equation, the R-squared for the unrestricted model are also shown for comparison. The importance of the restriction was previously shown in Figure 4.2. We can see by comparing the R-squared values that there is no significant change when using the restricted model, and it allows us to maintain the integrity of the solution since there are no turnings points to the right of the y-axis. The turning point for a quadratic is $\frac{-b}{2a}$, so we would not want a model where b is negative.

Table 6.4 is included to show that even though a power function as suggested in the literature returns a good fit statistically, Figure 4.1 shows that these functions can have the wrong shape when the order book does not have enough depth from which to estimate accurate parameters.

6.3 Intraday Liquidity

We again used a quadratic function to model the U-shape of the intraday liquidity discount, and the results are shown in Figures 6.5 to 6.8, and the regression results in Table 6.5. Even though we have greatly smoothed the results by averaging over each hour of the trading day, we still see a useful pattern emerging in each share.

AGL				
	Coefficient	Std. Error	t-Statistic	Prob.
a	6.47E-12	4.35E-14	148.7469	0.0000
c	0.001454	6.63E-05	21.93196	0.0000
R-squared	0.997430			
Adjusted R-squared	0.997385			
S.E. of regression	0.000341			
Adj R-squared without restrictions	0.997733			
SLM				
	Coefficient	Std. Error	t-Statistic	Prob.
a	6.19E-13	9.01E-15	68.78923	0.0000
c	0.00213	3.82E-05	55.76981	0.0000
R-squared	0.980116			
Adjusted R-squared	0.979909			
S.E. of regression	0.000253			
Adj R-squared without restrictions	0.999590			
MPC				
	Coefficient	Std. Error	t-Statistic	Prob.
a	1.79E-11	1.40E-13	128.0036	0.0000
c	0.004399	2.85E-05	154.1035	0.0000
R-squared	0.998781			
Adjusted R-squared	0.99872			
S.E. of regression	9.05E-05			
Adj R-squared without restrictions	0.998777			
SYC				
	Coefficient	Std. Error	t-Statistic	Prob.
a	1.64E-12	1.91E-14	85.91972	0.0000
c	0.003959	2.17E-05	182.8491	0.0000
R-squared	0.993406			
Adjusted R-squared	0.993272			
S.E. of regression	0.000104			
Adj R-squared without restrictions	0.998938			

Table 6.3: Quadratic regression results, $ax^2 + c$

AGL				
	Coefficient	Std. Error	t-Statistic	Prob.
a	1.28E-05	1.70E-05	0.752981	0.4546
b	0.681762	0.116241	5.865075	0.0000
c	-0.004635	0.001747	-2.652341	0.0104
R-squared	0.852422			
Adjusted R-squared	0.847151			
S.E. of regression	0.002609			
SLM				
	Coefficient	Std. Error	t-Statistic	Prob.
a	2.40E-06	1.56E-06	1.542538	0.1263
b	0.687279	0.054393	12.63547	0.0000
c	0.000277	0.000233	1.189805	0.2371
R-squared	0.945054			
Adjusted R-squared	0.943897			
S.E. of regression	0.000422			
MPC				
	Coefficient	Std. Error	t-Statistic	Prob.
a	5.90E-06	1.02E-05	0.580788	0.5682
b	0.725815	0.168586	4.305301	0.0004
c	0.002437	0.000802	3.038853	0.0068
R-squared	0.874193			
Adjusted R-squared	0.86095			
S.E. of regression	0.000943			
SYC				
	Coefficient	Std. Error	t-Statistic	Prob.
a	1.69E-06	1.77E-06	0.952552	0.3456
b	0.726018	0.093704	7.747973	0.0000
c	0.002824	0.000247	11.41624	0.0000
R-squared	0.918446			
Adjusted R-squared	0.915048			
S.E. of regression	0.000368			

Table 6.4: Power function regression results, $ax^b + c$

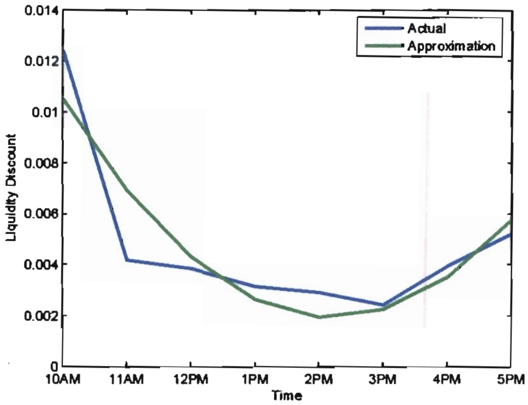

Figure 6.5: AGL intraday liquidity, 20,000 Shares

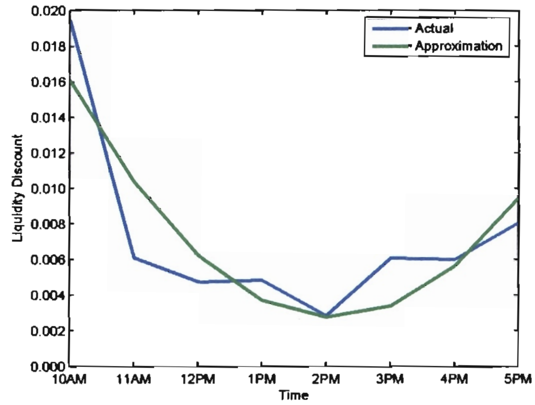

Figure 6.7: MPC intraday liquidity, 10,000 Shares

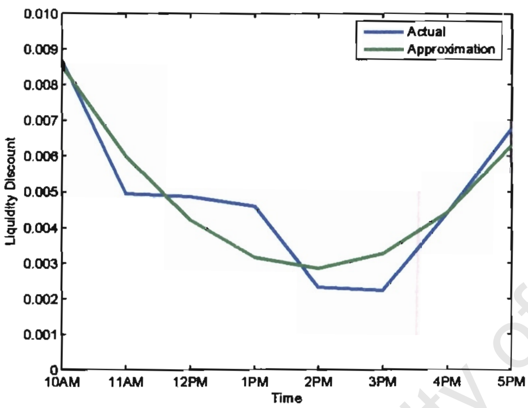

Figure 6.6: SLM intraday liquidity, 50,000 Shares

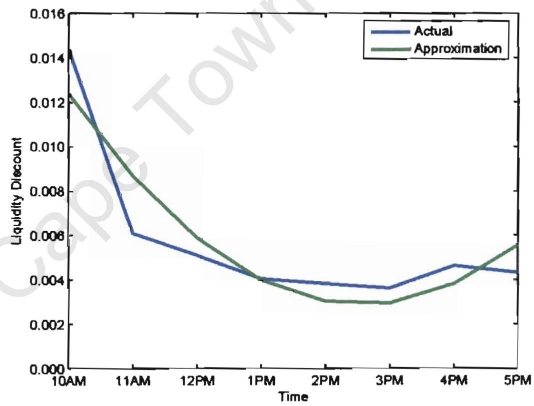

Figure 6.8: SYC intraday liquidity, 17,000 Shares

As found throughout the literature, a J- or U-shaped liquidity discount can be seen. A quadratic was again the best fit and although the R-squared values not as high, it gives us a good indication of the shape of the intraday impact, and supporting the results found in other markets. An interesting observation is that the minimum point on all these models is between 2:00PM and 2:30PM. This would suggest that the most liquid time of the day to trade on the South African JSE Securities Exchange is when the US market opens.

AGL				
	Coefficient	Std. Error	t-Statistic	Prob.
a	1.36E-07	3.49E-08	3.910383	0.0113
b	-8.47E-05	1.93E-05	-4.391119	0.0071
c	0.015111	0.00227	6.657196	0.0012
R-squared	0.816775			
Adjusted R-squared	0.743485			
S.E. of regression	0.001627			
SLM				
	Coefficient	Std. Error	t-Statistic	Prob.
a	1.02E-07	2.18E-08	4.677314	0.0054
b	-6.03E-05	1.21E-05	-4.994178	0.0041
c	0.011762	0.001421	8.278204	0.0004
R-squared	0.837527			
Adjusted R-squared	0.772538			
S.E. of regression	0.001018			
MPC				
	Coefficient	Std. Error	t-Statistic	Prob.
a	2.21E-07	6.24E-08	3.541837	0.0165
b	-0.000135	3.45E-05	-3.905242	0.0113
c	0.02341	0.004061	5.76429	0.0022
R-squared	0.770692			
Adjusted R-squared	0.678969			
S.E. of regression	0.002911			
SYC				
	Coefficient	Std. Error	t-Statistic	Prob.
a	1.26E-07	3.65E-08	3.446235	0.0183
b	-8.42E-05	2.02E-05	-4.171606	0.0087
c	0.016988	0.002377	7.148118	0.0008
R-squared	0.837295			
Adjusted R-squared	0.772214			
S.E. of regression	0.001704			

Table 6.5: Intraday regression results, $ax^2 + bx + c$

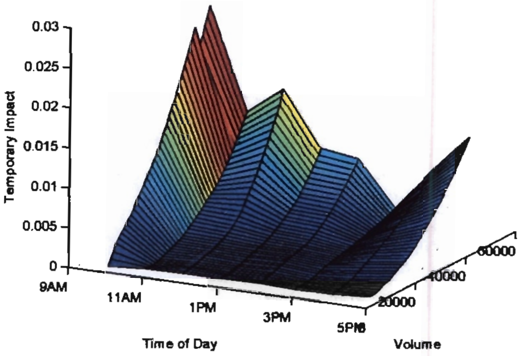

Figure 6.9: AGL actual combined impacts

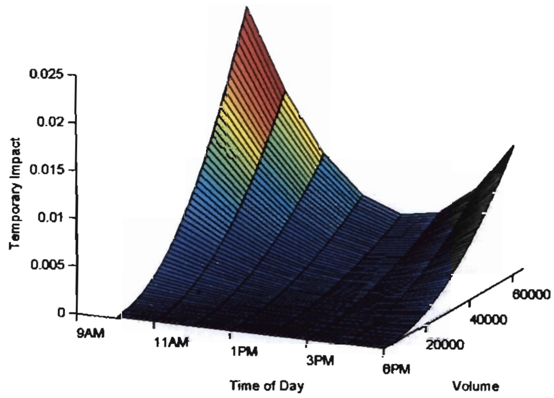

Figure 6.10: AGL estimated combined impacts

6.4 Combined Estimations

As can be seen for AGL, Figure 6.9 shows the average impacts across time and different volumes, and Figure 6.10 shows the combined estimates from the two quadratic functions estimated from the data.

Chapter 7

Daily Liquidation Comparison

7.1 Comparison Between Models

In this section we attempt to compare the optimal strategies found using the consolidated Almgren & Chriss and Mönch models. By utilizing the same temporary impact function we can then distinguish between the models based on their unique characteristics.

The only liquidity specific parameter not yet estimated is time to recover. This is a very difficult characteristic of a security to define, let alone find. The time to recover is used as a constraint in the Mönch model when solving for the times between trades τ_i which, unlike in Almgren & Chriss, are not necessarily equal. Since we use the constraint $\tau_i \geq \beta n_{t_i}$ in the optimisation we also have

$$\sum_{i=1}^{N} \tau_i \geq \beta X$$

Where β is the time to recover and X is the number shares. The initial time to trade, τ_1 is the time from the start of the day to wait before beginning the sequence of trades and is derived from the intraday liquidity function and βX, the total length of the trading strategy.

Given this, it can be seen that βX is equivalent to the liquidation time in Almgren & Chriss, if this liquidation time is less than 1 since the Mönch model solves for a single days strategy. So, for example, if we want to liquidate a position of $100,000$ shares over a period no longer than half the day, or 240 minutes, we would set β to 0.0024 in the Mönch model and set the liquidation time to 0.5 in the Almgren & Chriss model. The only difference would be that the solution found by Mönch could begin at any time of the day, and the Almgren & Chriss models starting point would not be defined by the solution.

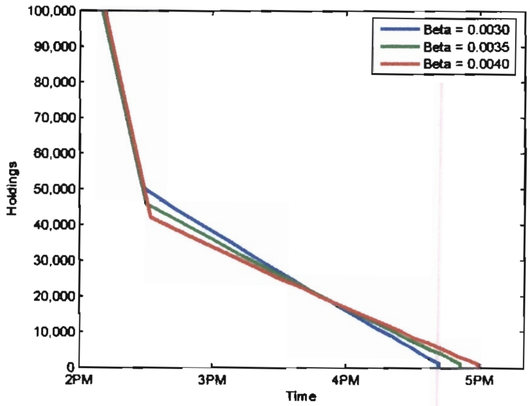

Figure 7.1: AGL holdings, various market recovery parameters, $\lambda = 1.645$

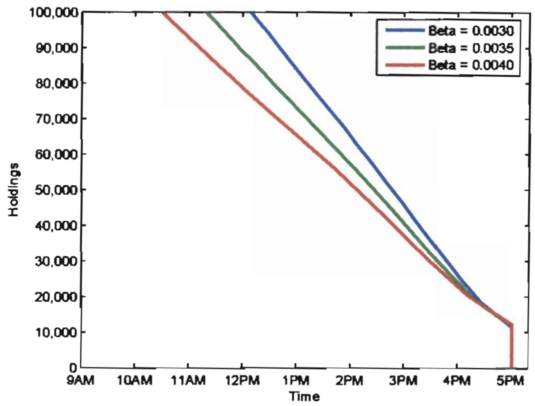

Figure 7.2: AGL holdings with varied market recovery parameters and $\lambda = 0$

As can be seen in Figure 7.1, by adjusting the market recovery parameter we change the length of time between the first and last trade since β is the slope of the holdings curve and there is a set end time. For the risk averse trader, a large initial trade is followed by a series of smaller trades. The trades are not equal in size or time between trades, but each trade is related to the time by the equation $\tau_i \geq \beta n_{t_i}$, excluding the initial trade. When dropping the risk tolerance constraint in Figure 7.2 we see a series of increasing trades followed a a final large trade, the opposite of that used by the risk averse investor. This is because the extra risk of holding more shares for longer is not included in the optimisation constraints and only the expected loss due to liquidation is minimised. These patterns are also observed in the Almgren & Chriss model as can be seen in Figure 7.3.

Figure 7.4 shows the portion of the intraday impact curve used by the optimal liquidation strategy. This shows the reliance of optimal trading time on this function. By increasing trading time the earlier the strategy must start to finish by the end of the day. This graph shows a situation where the recovery parameter is large. For small values, Figure 7.5 shows the optimisation chooses the lowest portion of the intraday liquidity function with the length related to the market recovery parameter as explained above. As can be seen in Figures 7.6 and 7.7, when a trader is more risk averse, the length of time of the strategy decreases to reduce the risk of the strategy. The optimal strategy is then to begin trading at the most liquid time of the time for a shorter time. Looking at this with the trading strategy, we see that the first transaction is also large, as this minimises the risk associated with the returns from the future transactions.

The expected return of the strategy relies heavily on this recovery parameter.

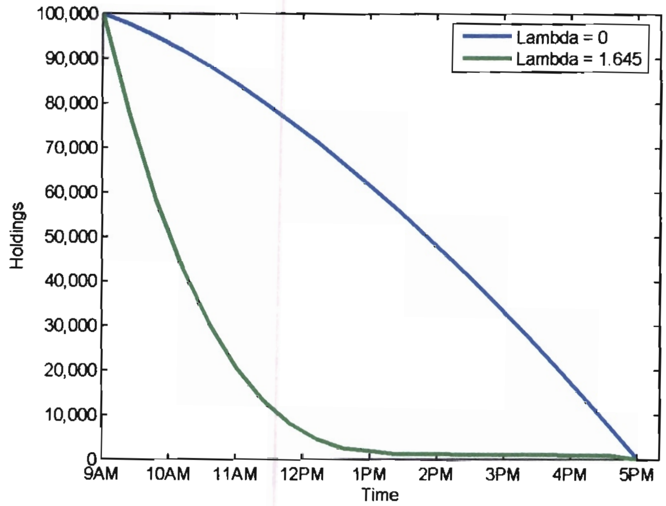

Figure 7.3: AGL holdings, with and without a risk constraint using the Almgren & Chriss model

In practice, a trader would not need to estimate this function to determine an optimal strategy if he can rather apply fixed time constraints and a general understanding of the shape of the intraday impact function. As seen in the estimates of the intraday liquidity function, the minimum for all 4 functions is between 2:00PM and 2:30PM. This can be of great help to a trader. If the model is rather used to determine theoretical values for a liquidity adjusted Value-at-Risk calculation, then this function is more important since the expected return generated by the model is more important than the use of the trading strategy that is most likely to generate the highest return.

Given that the market recovery parameter β is difficult to estimate with the available market data, and is closely linked to the length of time to execute the series of trades, it might be better to create an alternative use for β when comparing these two models in this situation. As seen in the intraday trading function, there is a period of lower liquidity in the middle of the day, and the first and last hour of the day are subject to low liquidity levels and irregular trading due to the closing auction and the time needed to create a liquid order book when the market opens. We therefore choose to solve for optimal strategies with the Mönch model to complete in 480 minutes, but with a trading time of no more than 240 minutes after the first trade, implying a β of 0.0024 for a sale of $100,000$ shares. We also set the total trade length to not exceed 420 minutes for the Mönch model to exclude any trades between 4pm and 5pm. For the Almgren & Chriss model we set $T = 0.5$, giving 240 minutes of equally spaced trades. Since these trades can start at anytime of the day, it is possible to line them up with the start of

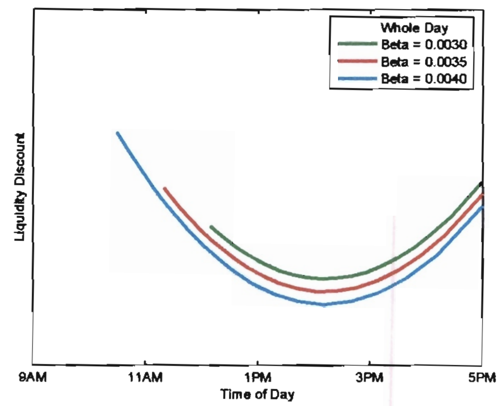

Figure 7.4: Time of day of optimal liquidation, $\lambda = 0$

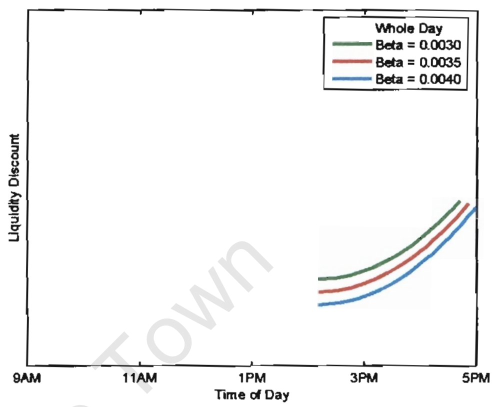

Figure 7.6: Time of day of optimal liquidation, $\lambda = 1.645$

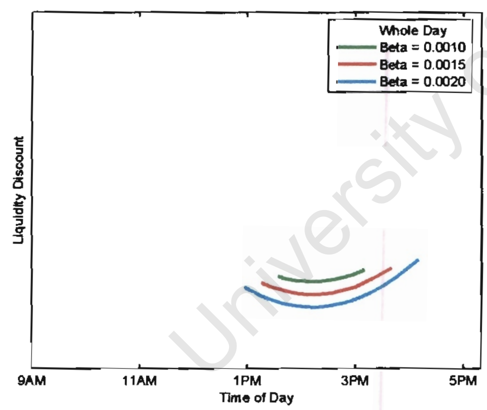

Figure 7.5: Time of day of optimal liquidation, $\lambda = 0$

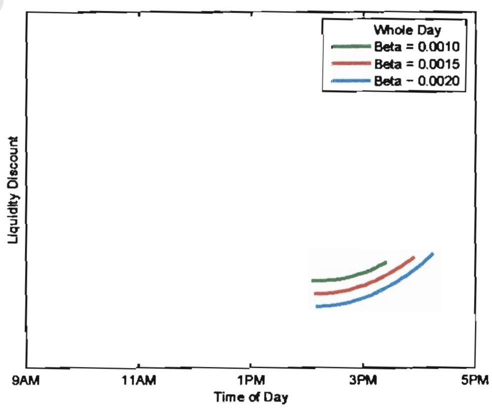

Figure 7.7: Time of day of optimal liquidation, $\lambda = 1.645$

this strategy with that of the strategy found using the Mönch model.

Now that we have equivalent parameters for both models, we can compare their core differences in terms of the strategies expected return, volatility and trading pattern. The strategies found are also compared across the range of market caps available. The majority of the calculations are done using AGL, the largest and second most liquid share in our sample as determined by our parameters. The strategies for a risk averse investor with $\lambda = 1.645$ is compared to a risk neutral investor, which corresponds to the lower 5% quantile of the liquidation value of the expected returns are normally distributed. The use of λ as a constraint in the optimisation function in the two models can be seen in equations 2.8 and 2.25. Using $\lambda = 1.645$ gives us an expected return that this strategy would yield 95% of the time. Both solutions solve for a strategy to liquidate 100,000 AGL shares with a price of R262.28 within a single day over 20 trades.

These models are being compared without any fixed costs, since they would be the same for both models. The Mönch model also contains an optimisation for the exact number of trades in a given time period. This has been omitted and the same number of trades has been used in both models. This does not have a large effect on the shape of the trading pattern, but is rather a refinement to further maximise the expected return of the strategy. To find the correct volatility for the Almgren & Chriss model, it needs to be adjusted. A share price of 20 and a volatility of 20% means the σ used in the model is $\sigma = \frac{20\times 20\%}{\sqrt{250}}$. Doing the same with the mean where the annual drift is 10 gives $\mu = \frac{20\times 10\%}{250}$

Figure 7.8 compares optimal strategies with no risk aversion constraint using both models. They are almost identical, and the expected values are also very close, with the Mönch model giving R26,219,000 while Almgren & Chriss give R26,183,754. In Figure 7.9 the strategies for a risk averse trader can be seen. These also follow a similar pattern of selling faster and selling more early on to reduce the overall risk of the strategy. This is done through large initial trades, slowing down over the liquidation period. The Almgren & Chriss strategy has no set starting point given in the solution, and for purposes of comparisons has been shifted to coincide with the starting point given by the Mönch model's solution. Here it does extend past the close of business, but that is less that 5% of the total holdings. The expected returns are similarly close, with the Mönch strategy giving R26,181,000 and Algrem and Chriss giving R26,156,454.

Comparing these findings to those in the literature, we see some correlations. Dufour and Engle (2000) find a large trade is generally followed by traders revising their beliefs upward. This causes activity in the market that makes them less liquid and trades will now have a greater impact on the price. For a risk neutral

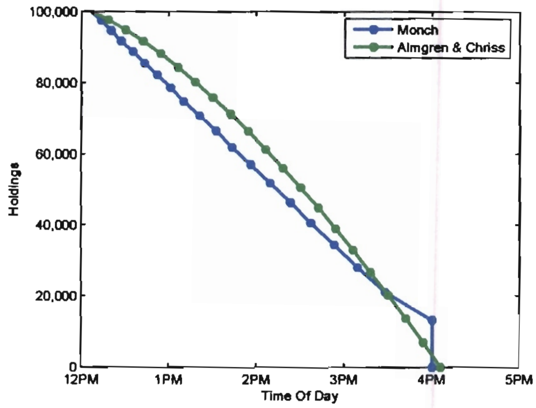

Figure 7.8: AGL comparison between the Mönch model and the Chriss Almgren model with $\lambda = 0$

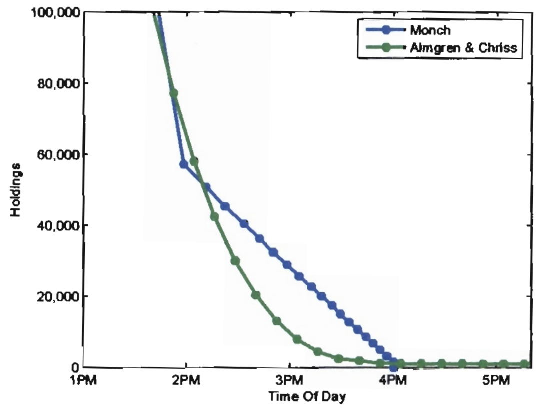

Figure 7.9: AGL comparison between the Mönch model and the Chriss Almgren model with $\lambda = 1.645$

trader maximizing his return, this is done by only entering into a large trade at the end of the liquidation period, all prior trades are small and have little effect on the price. The extended holding time of a large portion of his portfolio does increase the volatility of the price he sells them for, but being risk neutral this volatility is not part of the optimisation. For risk averse investors the volatility of his returns becomes important and a large initial trade may cause significantly lower prices to be realised for all later sales, but this is outweighed by the reduction in volatility.

7.2 Comparison Across Market Capitalisations

In this section compare how trading strategies found by the 2 models differ across different sizes of market capitalisations. The 4 share are those used in Chapter 6, and the models are the same as used in the previous section. We know that for a given liquidity level the models produce similar liquidation strategies, but we now wish to determine how the strategies for more or less liquid shares.

7.2.1 Almgren and Chriss Model

For risk neutral traders, the 2 stocks with the lowest temporary impact function trade slightly slower, and the 2 with the higher impact functions trade at close to a constant rate. In general, a risk neutral trader can obtain a close to optimal liquidation using a constant trading strategy without any knowledge of the liquidity characteristics of the shares order book if the time horizon is less than a day.

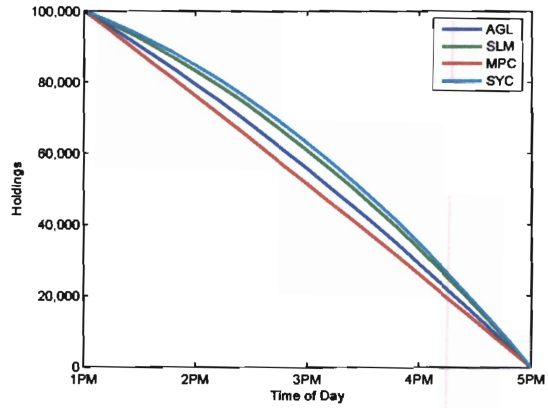

Figure 7.10: Comparison of 4 Stocks, $\lambda = 0$

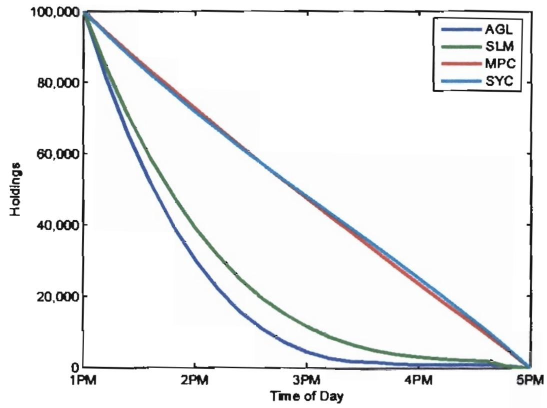

Figure 7.11: Comparison of 4 Stocks, $\lambda = 1.645$

For a risk averse trader we see 2 different groupings. The only change is the addition of the volatility of the share price, and the model uses this to minimise the net cost of the liquidation. The 2 shares that now trade the fastest are those with the highest volatility in their daily returns. The volatility is however not a function of the order book, but it is a common parameter in share price analysis. The shares have different impact functions, and despite this we are still able to observe a general trading rule using only the volatility of the share. This makes optimal liquidation a much easier task for the risk averse trader. Shares with higher volatilities should be traded through a series of trades decreasing in size, starting with a relatively large trade, something between 20% and 30% of the initial portfolio.

If we instead compare percentage holdings with all stocks' initial portfolio consisting of the same value of each share, we see the importance of the permanent impact function since this is determined from the average number of shares traded daily. These comparisons are done for a risk averse trader, a risk neutral trader would have similar constant strategies as shown in Figure 7.10. For a portfolio value of over R10 million, shown in Figure 7.12, the number of shares in the SYC portfolio is more than the average daily traded number of shares, resulting in a constant strategy, and similarly for MPC where the number of shares is over 50% of the average (the SYC strategy is also constant and identical to the MPC strategy so cannot be seen in the figure). For SLM, it is close to 10% of the daily traded value and the optimal strategy is now slower than for a portfolio of a smaller number of shares due to the increase in the effect of the permanent impact. For AGL on the other hand which has a price of R262.28 per share resulting in a comparatively low number of shares to liquidate, the optimal strategy

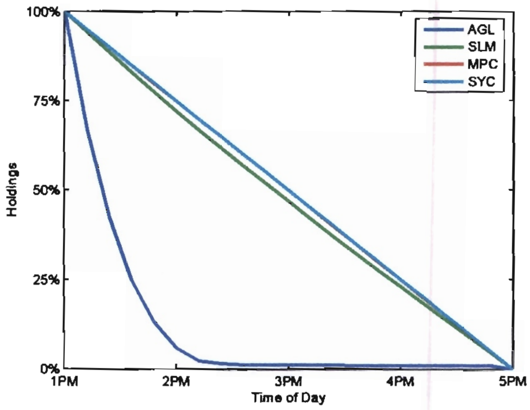

Figure 7.12: Comparison of 4 Stocks, R10 million of shares

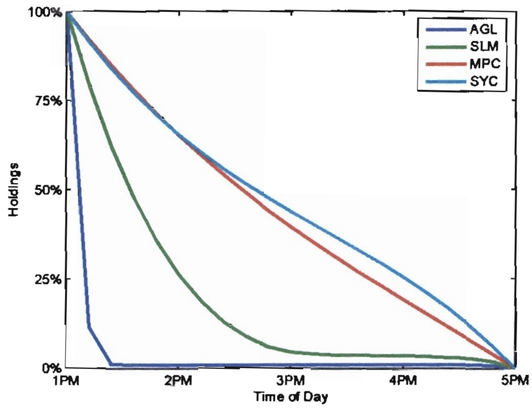

Figure 7.13: Comparison of 4 Stocks, R1 million of shares

Share	Volatility	Average Daily Volume Traded	R1 Million Ex. Loss	Std. Dev.	R10 Million Ex. Loss	Std. Dev.
AGL	15.40%	3,500,000	R1,519	R192	R20,703	R13,008
SLM	14.85%	6,700,000	R2,314	R1,829	R42,022	R35,498
MPC	10.62%	700,000	R6,384	R2,326	R326,070	R26,357
SYC	6.39%	450,000	R10,770	R1,443	R738,970	R15,872

Table 7.1: Equivalent portfolio values expected loss and volatility results

approaches that of immediate liquidation.

Figure 7.13 shows optimal strategies for a portfolio of only R1 million for each stock. Here no stock has more than 12% of their average daily traded shares to liquidate. The strategies are similar to those with the same initial number of shares rather than the equivalent value of the holdings, but shows evidence of the effects described for a R10 million portfolio. From these figures we see that the average daily traded volume or value of a share is important to the optimal strategy.

Table 7.1 shows the expected loss and volatility from the optimal strategies. The closer the total portfolio to liquidate is to the average total traded in a day, the closer the optimal strategy is to a constant one. The expected loss is then very high due to the depth of the order book needed to complete all the transactions. Conversely the smaller the portfolio relative to the average number of shares traded in a day the closer the optimal strategy is to a single trade of the full portfolio. The expected loss is then relatively low. The volatility of all strategies is closely linked to the volatility of the share price movements, especially when a strategy is close to constant.

7.2.2 Mönch Model

We now calculate the optimal strategies using Mönch's model. The only difference being the introduction of the intraday impact function. For a better comparison we must also look at the size of each trade. Given a trade size, the recovery parameter tells the model how long to wait before executing the next trade. The relationship is exact and trading rates may appear linear when plotting time against number of shares still held, but the time between trades varies, unlike the Almgren & Chriss model.

Figure 7.14 shows the holdings over the liquidation period for a risk neutral trader. The strategy appears linear, but looking at the holdings in Table 7.2 we see a series of increases trades followed by a large final trade. The most liquid shares start off with the smallest initial trade, while the least liquid share has an almost constant trade size and time between trades throughout the liquidation. The size of the final trade reduces as the liquidity of the stock decreases.

Figure 7.15 shows the holdings over the liquidation period for a risk averse trader, and Table 7.3 shows the times and size of the trades to execute. The strategy again appears linear, but we can see a large initial trade rather than a large final trade as for the risk neutral trader. The large initial trade reduces the risk of the remainder of the liquidation, therefore reducing the risk of the entire liquidation. We observed in the Almgren & Chriss model that for a risk averse trader the inclusion of the shares volatility the shares with the highest volatility should be traded the fastest, with large initial trades. The same can be seen here, and the shares liquidity is not as important in the optimal trading strategy as the volatility. The more volatile the shares daily returns the more a trader should sell in his first trade, with the amount and time between trades both decreasing over the liquidation period.

For a risk neutral and risk averse trader the inclusion of the shares intraday impact function has little effect without an estimate for the market recovery parameter. We have opted to derive this parameter from the total time over which we would like to execute the liquidation and use the same one for all shares, meaning the timing of the trades is linked to the intraday impact function and the choice of maximum trading time, in this case half the time the market is open. We know how to vary the size and time between trades based an a shares liquidity, we must now decided how to determine when to start trading. For a risk neutral trader, the higher the liquidity the larger the size of the final trade. This allows the trader to start trading later. If the share has low liquidity, the final trade is small and the trader should start trading earlier. For a risk averse trader, the higher the shares volatility the larger the size of the initial trade. This also allows the trader to begin trading later since there are fewer remaining

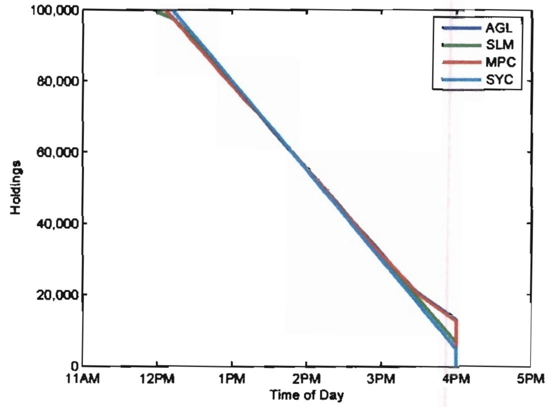

Figure 7.14: Comparison of 4 Stocks, $\lambda = 0$

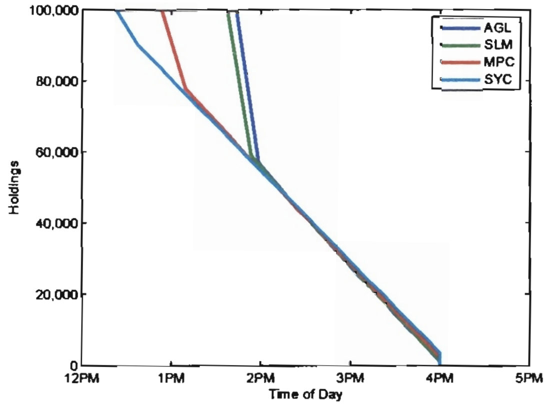

Figure 7.15: Comparison of 4 Stocks, $\lambda = 1.645$

shares to sell. As the shares volatility decreases, the smaller the traders initial trade and the earlier he must begin the liquidation.

7.3 Effect of Volatility on the Optimal Strategy

We have observed in actual stock data that a risk averse trader should trade the more volatile shares quicker. Volatility has no effect on the optimal strategy of risk neutral trader since it is not included in the optimisation. To formalise this result we now simulate 4 optimal solutions for a stock with different volatilities for both the Almgren & Chriss and Mönch models.

We see that in both Figure 7.16 and Figure 7.17 that an increase in volatility increases the size of the initial trade as we expected from the actual data. In the Mönch solution, the volatility has no effect on the timing of the trading, but the higher initial trade means there are fewer shares remaining in the portfolio so the strategy can complete the liquidation sooner.

7.4 Effect of a Shares Temporary Impact on the Optimal Strategy

Even though a trader may not be able to calculate a share temporary impact function, he may know if it is more or liquid than another he might know. This will help in deciding how to alter his trading strategy for the share with the un-

AGL t_i	AGL n_i	SLM t_i	SLM n_i	MPC t_i	MPC n_i	SYC t_i	SYC n_i
12:06:23	2653	11:55:42	2358	12:06:24	2662	12:11:47	4963
12:13:01	2769	12:12:26	2821	12:13:04	2781	12:23:51	4984
12:19:58	2890	12:20:18	3273	12:20:06	2932	12:35:52	4996
12:27:16	3047	12:29:04	3657	12:27:26	3051	12:47:52	4997
12:34:55	3184	12:38:41	4009	12:35:12	3240	12:59:52	5000
12:42:59	3362	12:48:56	4271	12:43:28	3441	01:11:52	5000
12:51:33	3571	12:59:52	4555	12:52:10	3629	01:23:52	5000
01:00:39	3787	01:11:16	4747	01:01:24	3841	01:35:52	5001
01:10:16	4012	01:23:10	4959	01:11:15	4104	01:47:52	5003
01:20:31	4270	01:35:29	5130	01:21:49	4403	01:59:53	5005
01:31:29	4564	01:48:11	5292	01:33:06	4707	02:11:54	5005
01:43:10	4875	02:01:12	5428	01:45:06	4997	02:23:54	5003
01:55:37	5183	02:14:36	5583	01:57:45	5270	02:35:55	5003
02:08:46	5481	02:28:17	5701	02:11:01	5528	02:47:55	5006
02:22:37	5772	02:42:24	5879	02:24:52	5771	02:59:56	5003
02:37:11	6070	02:56:54	6044	02:39:18	6016	03:11:56	5005
02:52:35	6412	03:11:53	6240	02:54:26	6303	03:23:57	5007
03:09:08	6900	03:27:24	6465	03:10:37	6745	03:35:58	5006
03:28:12	7939	03:43:27	6691	03:29:15	7765	03:47:59	5004
04:00:01	13259	04:00:01	6899	04:00:00	12814	04:00:00	5010

Table 7.2: Trading sizes with $\lambda = 0$

AGL t_i	AGL n_i	SLM t_i	SLM n_i	MPC t_i	MPC n_i	SYC t_i	SYC n_i
13:42:31	42714	13:37:21	40560	12:52:53	22032	12:23:08	9635
13:57:44	6338	13:52:47	6431	13:08:56	6683	12:37:13	5868
14:10:53	5480	14:06:13	5595	13:23:45	6180	12:50:55	5708
14:22:27	4817	14:18:05	4945	13:37:32	5743	13:04:15	5557
14:32:45	4292	14:28:42	4424	13:50:24	5360	13:17:14	5413
14:42:01	3866	14:38:18	3999	14:02:28	5022	13:29:54	5276
14:50:27	3514	14:47:03	3647	14:13:48	4723	13:42:15	5146
14:58:11	3218	14:55:05	3349	14:24:29	4455	13:54:18	5022
15:05:18	2968	15:02:31	3095	14:34:36	4216	14:06:04	4904
15:11:54	2752	15:09:25	2875	14:44:12	4001	14:17:34	4792
15:18:03	2565	15:15:51	2684	14:53:20	3805	14:28:49	4684
15:23:49	2400	15:21:54	2516	15:02:03	3628	14:39:49	4581
15:29:14	2256	15:27:34	2368	15:10:22	3466	14:50:34	4483
15:34:20	2127	15:32:56	2235	15:18:20	3317	15:01:06	4388
15:39:10	2012	15:38:01	2117	15:25:58	3181	15:11:25	4297
15:43:45	1908	15:42:50	2010	15:33:17	3055	15:21:31	4210
15:48:06	1814	15:47:26	1913	15:40:20	2938	15:31:25	4127
15:52:15	1729	15:51:49	1824	15:47:08	2830	15:41:08	4046
15:56:13	1652	15:56:00	1744	15:53:41	2730	15:50:39	3969
16:00:00	1581	16:00:00	1670	16:00:01	2636	16:00:00	3895

Table 7.3: Trading sizes with $\lambda = 1.645$

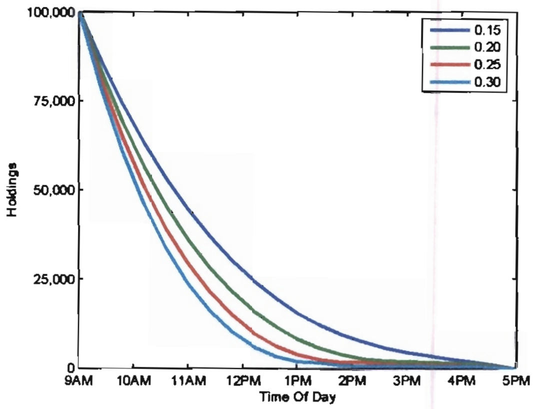

Figure 7.16: Almgren & Chriss - volatility comparison

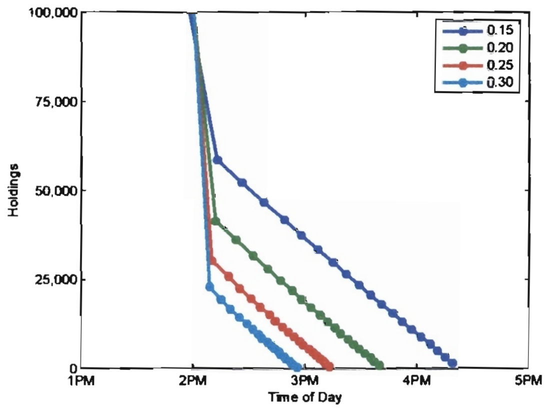

Figure 7.17: Mönch - volatility comparison

known impact parameter. To do this we again simulate 4 identical shares, this time with different temporary impact parameters. This is only properly comparable using the Almgren & Chriss model, since the temporary impact function is only part of the market impact function, the intraday impact function being the other. In the Mönch model, when the size of the first trade changes, as we would expect with a change in volatility, the timing of this trade also changes. This timing is based on the intraday impact function, and all other trades are based on the first trades size and timing. Since we have no substitute for this factor in the Almgren & Chriss model, the Mönch solutions are of no use here.

We see that in Figure 7.18 that as the temporary impact parameter increases, the share becomes less liquid, the more smaller the initial trade and the slower the trader reduces the size of each subsequent trade. The higher the impact parameter, the less liquid the market for the share, so to reduce the volatility of the liquidation more shares must be sold early to reduce the risk of low returns of later sales.

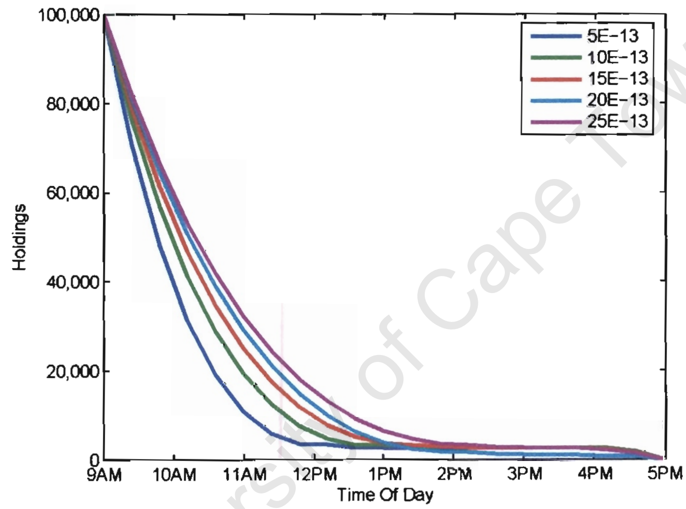

Figure 7.18: Almgren & Chriss - difference between impact parameters

Chapter 8

Multiple-Security Portfolios

In this chapter we use the extended Almgren & Chriss (2000) model for portfolios of multiple security to determine characteristics of portfolio liquidation where assets are correlated and their trading times coincide. This approaches a real world portfolio transition situation. Since these transitions are large, a single days trading will rarely be enough to liquidate the entire position. The Almgren & Chriss model is better suited than the Mönch model for this purpose.

8.1 Multiple-Security Model

Extending the single security model to a solve for an optimal trading strategy with m shares with linear temporary impact function $h(\nu)$ and permanent impact function $g(\nu)$ we have

$$g(\nu) = \Gamma\nu, \qquad h(\nu) = \varepsilon + HV \tag{8.1}$$

where Γ and H are $m \times m$ matrices, and ε is a $m \times 1$ column vector. A requirement to obtain an appropriate solution is that H is positive-definite. If there were a nonzero ν with $\nu^T H \nu \leq 0$ then by selling at rate ν we would obtain a net benefit, or at least lose nothing, from instantaneous market impact. We do not assume H and Γ are symmetric. We have a market value of our initial portfolio of $X^T S_0$. The loss in value from our strategy $x_1, ..., x_N$ is

$$E(x) = \varepsilon|X| + \sum_{k=1}^{N} \tau x_k^T \Gamma \nu_k + \sum_{k+1}^{N} \tau \nu_k^T H \nu_k \tag{8.2}$$

$$= \varepsilon|X| + \frac{1}{2} X^T \Gamma^S X + \sum_{k=1}^{N} \tau \nu_k^T \tilde{H} \nu_k + \sum_{k+1}^{N} \tau x_k^T \Gamma * A \nu_k \tag{8.3}$$

$$V(x) = \sum_{k=1}^{N} \tau x_k^T C x_k \tag{8.4}$$

with $\tilde{H} = H^S - \frac{1}{2}\tau\Gamma^S$. The superscripts S and A are used to denote the symmetric and antisymmetric parts, respectively, so $H = H^S + H^A$ and $\Gamma = \Gamma^S + \Gamma^A$,with

$$H^S = \frac{1}{2}\left(H + H^T\right), \qquad \Gamma^S = \frac{1}{2}\left(\Gamma + \Gamma^T\right), \qquad \Gamma^A = \frac{1}{2}\left(\Gamma - \Gamma^T\right) \tag{8.5}$$

Note that H^S is positive-definite as well as symmetric. We assume τ is sufficiently small enough so that $\tilde{H}$ is positive-definite and hence invertible. We also assume each component of ν has a consistent sign throughout the liquidation. The set of all outcomes can then be completely described by the two scalar functions $E(x)$ and $V(x)$.

8.2 Numerical Computation and Analysis

We begin by creating 3 shares with different levels of liquidity. To avoid any other differences effecting the optimisation, their initial price, mean and volatility are the same. We then give the shares estimated parameters for temporary impact, bid-ask spread and daily volume traded. The bid-ask spread and daily volume traded allow for a permanent market impact as used in the Almgren & Chriss (2000) numerical example. This assumes that sales of 10% of the shares daily volume decreases the price by one bid-ask spread for the duration of the trading strategy. We can then conduct simulations to compare the effects of different levels of correlation between shares.

The estimates used can be seen in Table 8.1. Figure 8.1 illustrates the different temporary impact functions. These values are actual Rand values, not as a percentage, since the Almgren & Chriss models use actual not relative impacts in their model as discussed earlier.

Now comparing the stocks with high and low liquidity, we set the correlation to -0.5 and $+0.5$, again with equal volumes to liquidate but for a risk averse investor. The results are in Figure 8.3 and Figure 8.4. Figure 8.2 has no correlation. These figures show very little movement in the stock with low liquidity,

Liquidity	High	Small
Price	50	50
Annual Growth	10%	10%
Volatility	20%	20%
Daily Volume Traded	2.5 M	0.75 M
Ave Spread	0.25	0.75
Temporary Impact	1×10^{-6}	1×10^{-5}
Permanent Impact	1×10^{-6}	1×10^{-5}

Table 8.1: Multiple Security Data

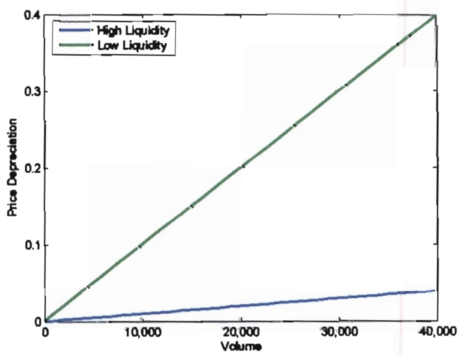

Figure 8.1: Temporary impact functions for various liquidity levels

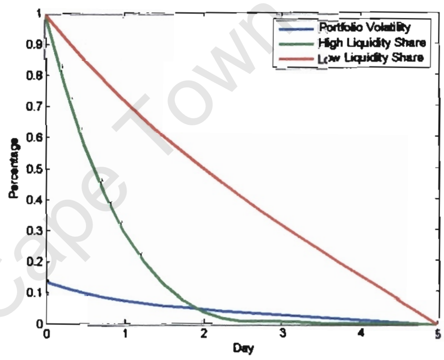

Figure 8.2: High and low liquidity shares with $\rho = 0$

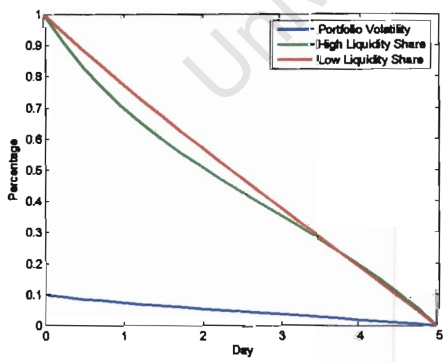

Figure 8.3: High and low liquidity shares with $\rho = -0.5$

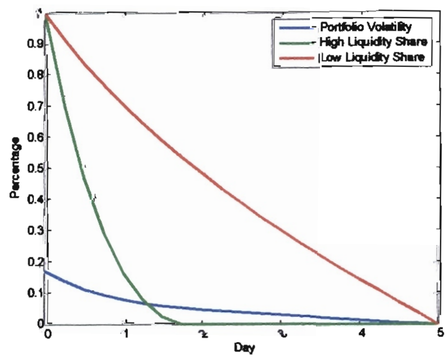

Figure 8.4: High and low liquidity shares with $\rho = 0.5$

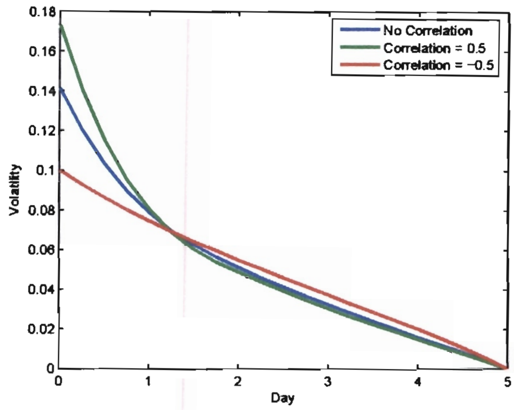

Figure 8.5: Comparison of portfolio volatilities over the duration of the liquidation

but the rate of trading used to liquidate the highly liquid stock increases as the correlation increases. This is because the expected return from liquidating stocks with low liquidity decreases considerably if the rate of trading increases, and the expected return from highly liquid stocks decreases less as the rate of trading increases. To counter the inflexibility of the low liquidity stock and the high correlation, the high liquidity stock is sold off as soon as possible to minimise the effect on the sale of the less liquid stock. If the stocks have negative correlation a higher return for the portfolio liquidation can be gained by extending the time over which the highly liquid stock is liquidated. This is done by reducing the rate at which those shares are sold.

8.3 Portfolio Volatility

The volatility of a portfolio of assets is very important to a trader, and he would like to execute trades while keeping this as low as possible. We investigate how a the volatility of a portfolio consisting of a high liquidity share, a low liquidity share and cash changes during a liquidation. Figures 8.2 to 8.4 also show the volatility of the portfolio, assuming the proceeds from the sale of the shares are invested in a risk free asset. These portfolio volatility curves are shown again on the same axis in Figure 8.5.

We see that the strategy to liquidate the portfolio with the highest volatility enters into the most aggressive early trading to reduce the portfolio volatility. This is to be expected for a risk averse trader. After the first day of the 5 day liquidation period the volatility of the portfolios with no correlation or a positive correlation drops to that of the negatively correlated portfolio for the remainder of the liquidation. The provides further justification for the risk averse trader to sell the high liquidity shares early on, as it reduces the volatility of the return of the liquidation and that of the portfolio during the trade.

Chapter 9

Liquidity and Capitalisation: Relation and Restrictions

Due to the difficulties in estimating the liquidity parameters, we would like to determine if a relationship between the liquidity and a more observable parameter such as market capitalisation exists. Any relationship of this kind would also be very useful in better understanding the nature of market liquidity.

9.1 Relation between Liquidity and Capitalisation

We take the 9 shares for which we have estimated linear and quadratic market liquidity parameters. These parameters are based on the percentage change in share price from the midpoint, but investigating a relationship between a Rand value of the market capitalisation might be better determined if the impact is also a Rand value, and has been therefore multiplied by the share price. We would not expect much of a relationship between the price depreciation and the market cap for low volumes, since a sale of a small number of shares is not a reflection of the stocks liquidity but rather the bid-ask spread. To investigate market liquidity, it is more appropriate to investigate the behaviour of the price impact on large transactions where significant depreciations beyond the bid-ask spread are observable.

Figure 9.1 shows the relationship between the price depreciation on a sale of shares valued at R100,000 based on the quadratic impact functions. AGL has also been left out of this plot, since it has a share price and market capitalisation 10 times the nearest share in our sample and due to its overseas listing is subject to different liquidity pressures in relation to the rest of the sample. The plot of the 8 remaining shares show a good fit using a log function, with an R-squared of

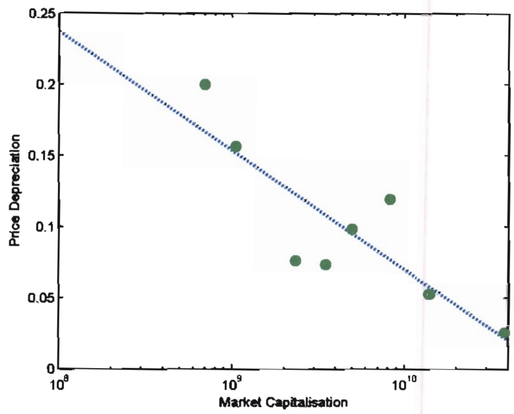

Figure 9.1: Relationship between market capitalisation and price depreciation for R100,000 worth of shares based on a quadratic price impact function.

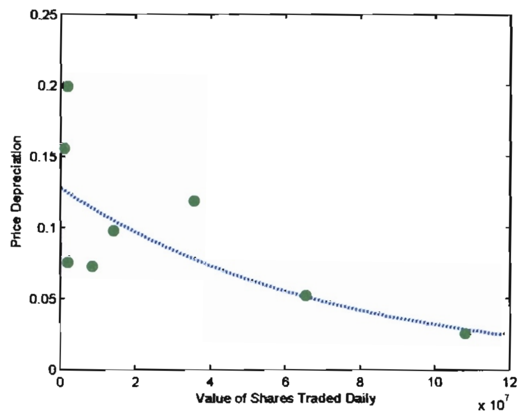

Figure 9.2: Relationship between average daily traded value and price depreciation for R100,000 worth of shares based on a quadratic price impact function.

0.71. The market capitalisation on the x-axis is plotted on log scale. Even though this is a very small sample, this R-squared value is still large enough to determine that there is some relationship between the 2 factors. Breen *et al.* (2000) also find that price impact, defined in terms of number of shares, is negatively related to the size of the market capitalisation.

In the South African market there may also be shares with large market capitalisations that do not trade often and have low liquidity. Another, possibly better, indicator of liquidity might be the average value of shares traded daily. This indicates the size of the market for a particular show, the implication being the larger the market the more buyers and sellers there are and the higher the liquidity. We again plot the price depreciation for shares valued at R100,000, but this time against the average valued traded daily. This is shown in Figure 9.2. Here, an exponential function is the best fit with an R-squared of 0.69. This again indicates a relationship between a freely available share attribute and the more difficult to estimate impact parameter. We should now be able to obtain good estimates of the liquidity parameters of other shares based on their market capitalisation and the average value of shares traded daily.

9.2 Liquidity Restrictions

The calculations done so far all assume any number of shares can be traded at a given time or in a given day, regardless of liquidity. The functions may indicate a large discount for a large transaction, but it is still possible and sometimes

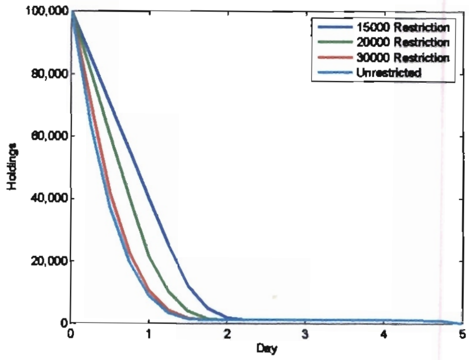

Figure 9.3: AGL with restrictions, minimum liquidity is 58,000

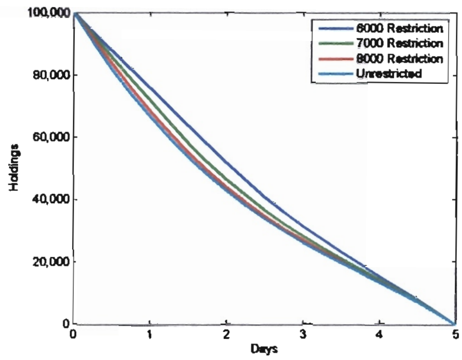

Figure 9.4: MPC with restrictions, minimum liquidity is 21,000

part of the solution if the trader is very risk averse. In a market such as South Africa's, there are many shares with very low liquidity, resulting in not enough bids available to fulfill a large order at one time. This makes these unrestricted solutions unpractical. By adding a restriction onto the number of shares traded at any one time, we are able to improve the results of the model when applied to shares with low liquidity.

We apply a restriction to the maximum number of shares to trade in any single transaction. The time between trades should also be appropriately large to allow for market recovery. To allow for comparability we continue with the parameters used above and liquidate 100,000 shares over 5 days with 20 equally spaced transactions using the Almgren & Chriss model. The calculations were performed with a risk aversion value of 1.645. A risk neutral trader has a strategy much closer to the naïve one and therefore has little chance of going above the restriction.

As shown previously in Figure 4.4, there is clear cutoff giving the minimum depth of the bid book observed at all 10 minutes interval in the sample period. For one liquid and one less liquid share the restricted strategy set the maximum number of shares allowed in a single transaction to various lower share volumes, but higher than a restriction that would permit no more than a naïve strategy. Figures 9.3 and 9.4 show the restricted and unrestricted strategy. They show that there is very little difference between the strategies.

As shown earlier, a risk averse investor will liquidate a large portion of the portfolio early on to minimise the risk from future sales since the future holdings are smaller. Since the strategies take the temporary impact into account and this is linked to the liquidity of the share, the size of this initial trade is linked to the liquidity. The more liquid shares have high minimum bid book depths as well as more liquid impact functions. This results in larger initial trades for the more liquid shares. The conclusion can be drawn that the restrictions are unnecessary and that the impact functions do a good job of restricting the speed of liquidation for the less liquid shares without the need to estimate additional market depth characteristics.

Chapter 10

Conclusion

High frequency order book data is analysed in the liquidation framework to estimate the impact parameters used optimal liquidation models. We find evidence to support the findings in the literature with regard to the functional form of the intraday and temporary impact functions, and parameterise these functions for shares on the JSE Securities Exchange.

An optimal trading strategy was found to increase in effectiveness compared to a constant or naïve strategy as the total liquidation time increases since an optimal strategy can greatly reduce the volatility of the overall return of the liquidation. Over a single day there is no significant difference in expected loss or volatility between an optimal or a naïve strategy. Optimal trading time increases with a decrease in a shares liquidity, and with an increase in the shares volatility.

We also compared optimal strategies found using two models over a single day and find that the introduction of the intraday impact function is not critical to the solution found in the model. The intraday impact showed that a risk averse trader will start with a large trade a the most liquid time of the day and decrease the size of the remaining trades over the rest of the day. The most liquid time of the market was found to be the same for all shares - at 2PM, which coincides with the start of trading on the NYSE. Without the intraday impact the strategy remains the same. The size of the initial trade is determined by the volatility of the shares price movements and temporary impact function. We find that higher the volatility the larger the initial trade, and the more liquid the share the larger the initial trade. The optimal strategy of a risk neutral trader is close to the naïve strategy regardless of changes in volatility or in the impact function.

For portfolio liquidation an optimal trading strategy can significantly reduce the portfolio volatility. Positively correlated shares result in the more liquid shares trading faster to reduce the volatility of returns realised from later sales on the less liquid share. For a negative correlation the more liquid share trades slowly to increase the returns on the later sales of the less liquid share. For correlated shares the portfolio volatility will initially be high, but an optimal strategy quickly reduces this to the same level as that of a portfolio of negatively correlated shares.

A relationship is found to exist between the value of a quadratic impact of a large sale of shares and the shares market capitalisation, as well as the value of shares traded each day. Restrictions on maximum trade size are found to be unnecessary as the temporary impact function limits the trade size to increase the expected return of the strategy.

Appendix A

Captured Data

14:00	Bid			Offer		
	t_i	Size	Bid	Offer	Size	t_i
Total		98969	25172	26997	44292	
1st	13:59:54	1500	26051	26100	1500	13:58:47
2nd	13:58:55	5000	26050	26100	6655	13:59:12
3rd	13:59:45	500	26045	26145	1000	13:58:41
4th	13:34:20	5000	25975	26150	1500	13:48:23
5th	13:33:51	5000	25951	26150	500	13:49:08
6th	13:48:49	2500	25875	26150	1000	13:54:33
7th	13:13:20	10000	25850	26197	500	12:01:08
8th	13:17:35	10000	25830	26200	500	12:50:44
9th	10:19:42	2000	25801	26245	2500	12:12:49
10th	10:18:45	10000	25800	26250	2000	13:53:32
11th	13:00:28	500	25750	26275	500	13:59:53
12th	10:10:53	2500	25740	26299	2000	13:26:26
13th	10:18:58	2000	25700	26300	500	11:48:46
14th	09:12:47	2500	25625	26330	500	11:48:59
15th	06:00:18	190	25500	26340	3000	11:47:16
16th	08:51:49	200	25500	26415	2500	10:59:03
17th	13:48:26	5000	25500	26445	2000	08:57:12
18th	12:13:08	4000	25400	26700	300	10:17:28
19th	06:00:18	200	25250	26950	510	06:00:22
20th	06:00:18	1500	25200	26950	500	08:36:23
21st	08:46:59	2200	25200	27000	100	06:00:22
22nd	09:24:48	100	25200	27000	400	06:00:22
23rd	12:42:49	1877	25200	27000	300	06:00:22
24th	06:00:18	200	25111	27000	500	06:00:23
25th	06:00:18	1000	25000	27000	500	06:00:23
26th	06:00:18	250	25000	27000	50	06:00:23
27th	06:00:18	120	25000	27480	400	06:00:23
28th	08:42:52	240	25000	27480	200	06:00:23
29th	10:37:30	826	25000	27500	85	06:00:23
30th	12:16:42	1327	25000	27500	500	06:00:23
31st	12:26:43	85	25000	27500	568	06:00:23
32nd	06:00:18	210	24800	27500	500	10:18:12
33rd	06:00:18	1000	24750	28000	256	06:00:23
34th	06:00:18	50	24500	28025	100	06:00:23
35th	06:00:18	200	24500	28250	3000	06:00:23

Table A.1: First 35 lines of order book at 14:00 as captured data from I-Net using AutoIT script

Bid		Offer		Cumulative	BP Diff	Cumulative	BP Diff
Size	Price	Size	Price	Bid Value	From Mid	Offer Value	From Mid
0	26051	0	26100	0	-9.40	0	9.40
1500	26051	1500	26100	391133	-9.40	391133	9.40
6500	26050	8155	26100	1694908	-9.69	2126457	9.40
7000	26050	9155	26105	1825285	-9.83	2387212	11.28
12000	26019	10655	26111	3129060	-21.80	2778345	13.71
17000	25999	11155	26113	4432835	-29.43	2908722	14.38
19500	25983	12155	26116	5084723	-35.51	3169477	15.55
29500	25938	12655	26119	7692273	-52.79	3299855	16.77
39500	25911	13155	26122	10299823	-63.26	3430232	17.95
41500	25905	15655	26142	10821333	-65.29	4082120	25.47
51500	25885	17655	26154	13428883	-73.12	4603630	30.16
52000	25884	18155	26157	13559260	-73.62	4734007	31.44
54500	25877	20155	26172	14211148	-76.15	5255517	36.82
56500	25871	20655	26175	14732658	-78.55	5385895	38.02
59000	25860	21155	26178	15384545	-82.54	5516272	39.43
59190	25859	24155	26198	15434088	-82.98	6298537	47.13
59390	25858	26655	26219	15486239	-83.45	6950425	54.92
64390	25830	28655	26234	16790014	-94.11	7471935	60.98
68390	25805	28955	26239	17833034	-103.75	7550161	62.83
68590	25803	29465	26252	17885185	-104.37	7683146	67.54
70090	25790	29965	26263	18276318	-109.33	7813524	72.01
72290	25772	30065	26266	18849979	-116.22	7839599	72.95
72390	25772	30465	26275	18876054	-116.52	7943901	76.65
74267	25757	30765	26282	19365492	-122.06	8022128	79.36
74467	25755	31265	26294	19417643	-122.73	8152505	83.76
75467	25745	31765	26305	19678398	-126.57	8282883	88.02
75717	25743	31815	26306	19743586	-127.51	8295920	88.44
75837	25742	32215	26321	19774877	-127.96	8400222	94.03
76077	25739	32415	26328	19837458	-128.86	8452373	96.77
76903	25732	32500	26331	20052842	-131.90	8474538	97.95
78230	25719	33000	26349	20398864	-136.66	8604915	104.74
78315	25718	33568	26368	20421028	-136.96	8753024	112.21
78525	25716	34068	26385	20475786	-137.90	8883401	118.58
79525	25704	34324	26397	20736541	-142.56	8950155	123.21
79575	25703	34424	26401	20749579	-142.85	8976230	125.02
79775	25700	37424	26550	20801730	-144.01	9758495	181.85
84775	25617	37924	26572	22105505	-175.68	9888873	190.44
85275	25609	39324	26651	22235883	-178.86	10253930	220.86
85471	25606	39692	26673	22286991	-180.19	10349887	229.21
85624	25603	39992	26691	22326886	-181.29	10428114	236.19
85724	25601	40254	26710	22352962	-182.01	10496432	243.20
85924	25597	40354	26717	22405113	-183.44	10522507	245.86
86074	25594	40554	26730	22444226	-184.50	10574658	251.12
86893	25579	40583	26732	22657784	-190.27	10582220	251.88
87013	25577	42583	26865	22689075	-191.37	11103730	302.63
87173	25573	42868	26885	22730796	-192.83	11178045	310.63
89673	25515	43272	26915	23382683	-214.99	11283390	321.78
89773	25513	43292	26916	23408759	-215.85	11288605	322.32
89873	25510	44292	26997	23434834	-216.71	11549360	353.36

Table A.2: First step of processing of captured data - first cumulative trades and weighted average prices, then cumulative value with basis point change from the midpoint

	10:00	11:00	12:00	13:00	14:00	15:00	16:00	17:00
0	-6.723	-9.334	-9.894	-10.070	-10.316	-9.112	-9.719	-12.252
1000	-13.874	-9.334	-10.864	-10.167	-10.538	-9.245	-9.991	-12.398
2000	-21.488	-9.452	-11.625	-10.523	-10.685	-9.462	-10.402	-12.633
3000	-29.775	-9.806	-12.233	-10.949	-11.226	-9.738	-11.147	-13.398
4000	-35.176	-10.232	-12.896	-11.358	-11.879	-10.113	-11.970	-15.101
5000	-39.637	-10.935	-13.560	-12.057	-12.776	-10.647	-12.730	-16.948
6000	-44.043	-12.249	-14.645	-13.037	-14.057	-11.412	-13.774	-18.726
7000	-48.087	-13.809	-15.898	-14.223	-15.357	-12.219	-14.761	-19.863
8000	-52.387	-15.474	-16.990	-15.275	-16.676	-13.194	-15.888	-21.192
9000	-56.274	-17.079	-18.057	-16.433	-17.804	-14.318	-17.282	-23.192
10000	-61.698	-18.663	-19.083	-17.675	-18.845	-15.421	-18.893	-25.446
11000	-68.292	-20.272	-20.086	-18.942	-19.982	-16.482	-20.867	-27.731
12000	-74.840	-21.879	-21.398	-20.183	-20.967	-17.512	-23.110	-29.987
13000	-80.946	-23.485	-22.649	-21.464	-21.797	-18.401	-25.405	-32.200
14000	-86.395	-25.115	-23.902	-22.619	-22.653	-19.332	-27.611	-34.338
15000	-92.876	-26.782	-25.574	-23.796	-23.529	-20.240	-29.695	-36.589
16000	-99.259	-29.117	-27.422	-25.054	-24.490	-21.073	-31.692	-39.450
17000	-105.396	-32.113	-29.722	-26.266	-25.473	-21.871	-33.664	-42.361
18000	-111.444	-35.155	-32.270	-27.785	-26.619	-22.709	-35.663	-45.602
19000	-117.560	-38.156	-35.135	-29.485	-27.790	-23.541	-37.602	-48.823
20000	-124.260	-41.476	-38.195	-31.481	-29.236	-24.389	-39.592	-52.152
21000	-131.844	-44.893	-41.937	-33.475	-30.835	-25.290	-41.645	-55.362
22000	-139.565	-48.111	-46.383	-35.596	-32.826	-26.210	-43.980	-58.719
23000	-147.474	-51.275	-51.315	-38.012	-35.193	-27.115	-46.307	-62.129
24000	-155.833	-54.389	-56.809	-40.532	-37.806	-28.056	-48.865	-66.351
25000	-164.120	-57.458	-62.430	-43.353	-40.709	-29.040	-51.429	-70.637
26000	-172.424	-60.787	-68.246	-46.123	-43.576	-30.033	-54.094	-75.131
27000	-181.159	-64.409	-74.221	-49.027	-46.549	-31.029	-56.764	-79.674
28000	-190.337	-68.099	-80.515	-52.028	-49.442	-32.045	-59.490	-84.309
29000	-199.743	-72.665	-87.082	-55.156	-52.506	-33.098	-62.185	-88.927
30000	-209.556	-77.524	-94.202	-58.494	-55.784	-34.171	-65.100	-93.629
31000	-219.400	-82.750	-101.667	-61.995	-59.435	-35.232	-68.010	-98.454
32000	-229.459	-88.142	-109.646	-65.905	-63.483	-36.273	-70.981	-103.256
33000	-239.704	-94.165	-117.644	-70.361	-67.853	-37.369	-74.260	-108.261
34000	-250.039	-100.419	-125.763	-74.991	-72.380	-38.470	-77.665	-113.651
35000	-215.884	-106.988	-134.200	-80.078	-76.974	-39.589	-81.337	-119.196
36000	-225.524	-113.568	-142.691	-85.447	-81.554	-40.787	-84.948	-124.838
37000	-235.745	-120.095	-151.324	-91.027	-86.262	-42.064	-88.467	-130.573
38000	-246.215	-126.775	-160.032	-96.690	-90.980	-43.452	-91.923	-136.497
39000	-257.134	-133.809	-168.505	-102.455	-96.198	-44.992	-95.874	-142.569
40000	-268.717	-141.208	-177.085	-108.238	-101.731	-46.576	-99.877	-148.836

Table A.3: Intraday impact - average of impacts in each hourly interval

Volume Bid	Interpolated BP Difference from Mid	Value Bid	Interpolated BP Difference from Mid
0	-9.40	0.00	-9.40
1000	-9.40	500000	-9.42
2000	-9.43	1000000	-9.53
3000	-9.48	1500000	-9.65
4000	-9.54	2000000	-11.44
5000	-9.60	2500000	-16.02
6000	-9.66	3000000	-20.61
7000	-9.83	3500000	-23.97
8000	-12.23	4000000	-26.89
9000	-14.62	4500000	-30.06
10000	-17.01	5000000	-34.72
11000	-19.40	5500000	-38.26
12000	-21.80	6000000	-41.58
13000	-23.32	6500000	-44.89
14000	-24.85	7000000	-48.20
15000	-26.38	7500000	-51.52
16000	-27.90	8000000	-54.03
17000	-29.43	8500000	-56.03
18000	-31.86	9000000	-58.04
19000	-34.30	9500000	-60.05
20000	-36.38	10000000	-62.06
21000	-38.10	10500000	-64.04
22000	-39.83	11000000	-65.82
23000	-41.56	11500000	-67.33
24000	-43.29	12000000	-68.83
25000	-45.02	12500000	-70.33
26000	-46.74	13000000	-71.83
27000	-48.47	13500000	-73.40
28000	-50.20	14000000	-75.33
29000	-51.93	14500000	-77.48
30000	-53.31	15000000	-80.19
31000	-54.36	15500000	-83.56
32000	-55.41	16000000	-87.65
33000	-56.45	16500000	-91.74
34000	-57.50	17000000	-96.05
35000	-58.55	17500000	-100.67
36000	-59.60	18000000	-105.83
37000	-60.64	18500000	-112.01
38000	-61.69	19000000	-117.92
39000	-62.74	19500000	-123.94
40000	-63.77	20000000	-131.16
41000	-64.78	20500000	-138.34
42000	-65.68	21000000	-148.82
43000	-66.46	21500000	-160.97
44000	-67.25	22000000	-173.11
45000	-68.03	22500000	-186.01
46000	-68.81	23000000	-201.98
47000	-69.60	23500000	-218.84
48000	-70.38	24000000	-234.86
49000	-71.16	24500000	-264.13
50000	-71.95	25000000	-292.64

Table A.4: Bid size and bid volume interpolated to give regularly spaced basis point impacts across all time periods and days

Appendix B

Optimal Time Results

η	**Dubil**	**Almgren**
0.000001	0.58	0.52
0.000002	0.92	0.74
0.000003	1.21	0.90
0.000004	1.47	1.04
0.000005	1.70	1.16
0.000006	1.92	1.27
0.000007	2.13	1.38
0.000008	2.33	1.47
0.000009	2.52	1.56
0.000010	2.70	1.64
0.000011	2.88	1.72
0.000012	3.05	1.80
0.000013	3.22	1.87
0.000014	3.38	1.94
0.000015	3.54	2.01
0.000016	3.70	2.08
0.000017	3.85	2.14
0.000018	4.00	2.21
0.000019	4.14	2.27
0.000020	4.29	2.32
0.000021	4.43	2.38
0.000022	4.57	2.44
0.000023	4.71	2.49
0.000024	4.84	2.55
0.000025	4.98	2.60
0.000026	5.11	2.65
0.000027	5.24	2.70
0.000028	5.37	2.75
0.000029	5.49	2.80
0.000030	5.62	2.85

Table B.1: Effect of change in temporary impact parameter on optimal trading t_i

Annual σ	**Effective σ**	**Dubil**	**Almgren**
0.02	0.0632	6.52	8.72
0.04	0.1265	4.11	4.36
0.06	0.1897	3.13	2.91
0.08	0.2530	2.59	2.18
0.10	0.3162	2.23	1.74
0.12	0.3795	1.97	1.45
0.14	0.4427	1.78	1.25
0.16	0.5060	1.63	1.09
0.18	0.5692	1.51	0.97
0.20	0.6325	1.40	0.87
0.22	0.6957	1.32	0.79
0.24	0.7589	1.24	0.73
0.26	0.8222	1.18	0.67
0.28	0.8854	1.12	0.62
0.30	0.9487	1.07	0.58
0.32	1.0119	1.03	0.54
0.34	1.0752	0.99	0.51
0.36	1.1384	0.95	0.48
0.38	1.2017	0.92	0.46
0.40	1.2649	0.88	0.44

Table B.2: Effect of change in temporary impact parameter on optimal trading t_i

t_i	**Optimal**	**Naïve**
0.10	256160	256160
0.20	131060	131060
0.30	89303	89298
0.40	68382	68370
0.50	55798	55775
0.60	47387	47347
0.70	41362	41299
0.80	36832	36740
0.90	33302	33173
1.00	30474	30300
1.10	28160	27932
1.20	26233	25943
1.30	24605	24246
1.40	23215	22777
1.50	22017	21492
1.60	20976	20355
1.70	20066	19341
1.80	19265	18429
1.90	18557	17603
2.00	17930	16850
2.10	17371	16160
2.20	16872	15524
2.30	16425	14935
2.40	16024	14387
2.50	15664	13875
2.60	15339	13395
2.70	15046	12944
2.80	14781	12519
2.90	14541	12116
3.00	14324	11733
3.10	14126	11370
3.20	13946	11023
3.30	13783	10691
3.40	13633	10373
3.50	13497	10068
3.60	13372	9774
3.70	13258	9492
3.80	13153	9219
3.90	13056	8955
4.00	12967	8700
4.10	12885	8453
4.20	12809	8212
4.30	12739	7979
4.40	12674	7752
4.50	12613	7531
4.60	12557	7315
4.70	12504	7104
4.80	12455	6898
4.90	12409	6697
5.00	12366	6500

t_i	**Optimal**	**Naïve**
0.10	16687	16693
0.20	23572	23607
0.30	28816	28913
0.40	33189	33386
0.50	36984	37326
0.60	40352	40889
0.70	43383	44164
0.80	46132	47214
0.90	48641	50078
1.00	50937	52787
1.10	53044	55363
1.20	54980	57826
1.30	56759	60186
1.40	58395	62459
1.50	59899	64651
1.60	61279	66771
1.70	62546	68826
1.80	63706	70821
1.90	64770	72762
2.00	65741	74652
2.10	66627	76496
2.20	67436	78296
2.30	68171	80056
2.40	68839	81777
2.50	69445	83464
2.60	69993	85116
2.70	70488	86738
2.80	70934	88329
2.90	71333	89893
3.00	71692	91430
3.10	72013	92941
3.20	72299	94428
3.30	72553	95892
3.40	72778	97334
3.50	72975	98755
3.60	73148	100155
3.70	73299	101538
3.80	73429	102903
3.90	73540	104245
4.00	73634	105575
4.10	73712	106888
4.20	73776	108180
4.30	73827	109462
4.40	73865	110725
4.50	73893	111978
4.60	73911	113217
4.70	73920	114438
4.80	73921	115650
4.90	73913	116850
5.00	73900	118034

Table B.3: Differences in expected return (left), and standard deviation (right)

Appendix C

Parameter Estimation Results

Dependent Variable: AGL
Included observations: 59
AGL = C(1)*VOL + 0.0011420180

	Coefficient		Std. Error	t-Statistic	Prob.
C(1)	2.91E-07		9.54E-09	30.55037	0.0000
R-squared	0.863679	Mean dependent var			0.008771
Adjusted R-squared	0.863679	S.D. dependent var			0.006672
S.E. of regression	0.002464	Akaike info criterion			-9.157621
Sum squared resid	0.000352	Schwarz criterion			-9.122408
Log likelihood	271.1498	Durbin-Watson stat			0.012035

Dependent Variable: SLM
Included observations: 98
SLM = C(1)*VOL + 0.0016377180

	Coefficient		Std. Error	t-Statistic	Prob.
C(1)	5.34E-08		6.45E-10	82.83787	0.0000
R-squared	0.959604	Mean dependent var			0.004082
Adjusted R-squared	0.959604	S.D. dependent var			0.001783
S.E. of regression	0.000358	Akaike info criterion			-13.01936
Sum squared resid	1.25E-05	Schwarz criterion			-12.99298
Log likelihood	638.9487	Durbin-Watson stat			0.003225

Dependent Variable: MPC
Included observations: 22
MPC = C(1)*VOL + 0.0042079140

	Coefficient		Std. Error	t-Statistic	Prob.
C(1)	3.02E-07		1.48E-08	20.36773	0.0000
R-squared	0.88614	Mean dependent var			0.007093
Adjusted R-squared	0.88614	S.D. dependent var			0.002529
S.E. of regression	0.000853	Akaike info criterion			-11.25031
Sum squared resid	1.53E-05	Schwarz criterion			-11.20072
Log likelihood	124.7535	Durbin-Watson stat			0.064097

Dependent Variable: SYC
Included observations: 51
SYC = C(1)*VOL + 0.0038081040

	Coefficient		Std. Error	t-Statistic	Prob.
C(1)	6.70E-08		1.78E-09	37.58225	0.0000
R-squared	0.914698	Mean dependent var			0.005339
Adjusted R-squared	0.914698	S.D. dependent var			0.001264
S.E. of regression	0.000369	Akaike info criterion			-12.95089
Sum squared resid	6.82E-06	Schwarz criterion			-12.91301
Log likelihood	331.2476	Durbin-Watson stat			0.012601

Table C.1: Linear least squares analysis with fixed constants. Without fixed constants least squares could find solution with negative constant is best fit. but for use here that is undesirable

Dependent Variable: AGL				
Method: Least Squares				
Sample(adjusted): 1 59				
Included observations: 59				
AGL = C(1)*VOL^2 + C(2)				
	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	6.47E-12	4.35E-14	148.7469	0.0000
C(2)	0.001454	6.63E-05	21.93196	0.0000
R-squared	0.99743	Mean dependent var		0.008771
Adjusted R-squared	0.997385	S.D. dependent var		0.006672
S.E. of regression	0.000341	Akaike info criterion		-13.09499
Sum squared resid	6.64E-06	Schwarz criterion		-13.02457
Log likelihood	388.3023	Durbin-Watson stat		0.045556

Dependent Variable: SLM				
Method: Least Squares				
Sample(adjusted): 1 98				
Included observations: 98				
SLM = C(1)*VOL^2 + C(2)				
	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	6.19E-13	9.01E-15	68.78923	0.0000
C(2)	0.00213	3.82E-05	55.76981	0.0000
R-squared	0.980116	Mean dependent var		0.004082
Adjusted R-squared	0.979909	S.D. dependent var		0.001783
S.E. of regression	0.000253	Akaike info criterion		-13.70777
Sum squared resid	6.14E-06	Schwarz criterion		-13.65501
Log likelihood	673.6806	Durbin-Watson stat		0.004878

Dependent Variable: MPC				
Method: Least Squares				
Sample(adjusted): 1 22				
Included observations: 22				
MPC = C(1)*VOL^2 + C(2)				
	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	1.79E-11	1.40E-13	128.0036	0.0000
C(2)	0.004399	2.85E-05	154.1035	0.0000
R-squared	0.998781	Mean dependent var		0.007093
Adjusted R-squared	0.99872	S.D. dependent var		0.002529
S.E. of regression	9.05E-05	Akaike info criterion		-15.69623
Sum squared resid	1.64E-07	Schwarz criterion		-15.59704
Log likelihood	174.6585	Durbin-Watson stat		0.500281

Dependent Variable: SYC				
Method: Least Squares				
Sample(adjusted): 1 51				
Included observations: 51				
SYC = C(1)*VOL^2 + C(2)				
	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	1.64E-12	1.91E-14	85.91972	0.0000
C(2)	0.003959	2.17E-05	182.8491	0.0000
R-squared	0.993406	Mean dependent var		0.005339
Adjusted R-squared	0.993272	S.D. dependent var		0.001264
S.E. of regression	0.000104	Akaike info criterion		-15.47173
Sum squared resid	5.27E-07	Schwarz criterion		-15.39597
Log likelihood	396.5292	Durbin-Watson stat		0.036781

Table C.2: Quadratic least squares results with $b = 0$ fixed. If $b \leq 0$ then turning point is on left of y-axis and produces inconsistent results in the optimisation programs

Dependent Variable: AGL
Method: Least Squares
Sample: 1 8
Included observations: 8
AGL = C(1) * T^2 + C(2) * T + C(3)

	Coefficient		Std. Error	t-Statistic	Prob.
C(1)	1.36E-07		3.49E-08	3.910383	0.0113
C(2)	-8.47E-05		1.93E-05	-4.391119	0.0071
C(3)	0.015111		0.00227	6.657196	0.0012
R-squared	0.816775	Mean dependent var			0.00476
Adjusted R-squared	0.743485	S.D. dependent var			0.003212
S.E. of regression	0.001627	Akaike info criterion			-9.724205
Sum squared resid	1.32E-05	Schwarz criterion			-9.694414
Log likelihood	41.89682	Durbin-Watson stat			2.264753

Dependent Variable: SLM
Method: Least Squares
Sample: 1 8
Included observations: 8
SLM = C(1) * T^2 + C(2) * T + C(3)

	Coefficient		Std. Error	t-Statistic	Prob.
C(1)	1.02E-07		2.18E-08	4.677314	0.0054
C(2)	-6.03E-05		1.21E-05	-4.994178	0.0041
C(3)	0.011762		0.001421	8.278204	0.0004
R-squared	0.837527	Mean dependent var			0.004853
Adjusted R-squared	0.772538	S.D. dependent var			0.002135
S.E. of regression	0.001018	Akaike info criterion			-10.6611
Sum squared resid	5.19E-06	Schwarz criterion			-10.63131
Log likelihood	45.64442	Durbin-Watson stat			1.997911

Dependent Variable: MPC
Method: Least Squares
Sample: 1 8
Included observations: 8
MPC = C(1) * T^2 + C(2) * T + C(3)

	Coefficient		Std. Error	t-Statistic	Prob.
C(1)	2.21E-07		6.24E-08	3.541837	0.0165
C(2)	-0.000135		3.45E-05	-3.905242	0.0113
C(3)	0.02341		0.004061	5.76429	0.0022
R-squared	0.770692	Mean dependent var			0.007307
Adjusted R-squared	0.678969	S.D. dependent var			0.005138
S.E. of regression	0.002911	Akaike info criterion			-8.560637
Sum squared resid	4.24E-05	Schwarz criterion			-8.530846
Log likelihood	37.24255	Durbin-Watson stat			2.107613

Dependent Variable: SYC
Method: Least Squares
Sample: 1 8
Included observations: 8
SYC = C(1) * T^2 + C(2) * T + C(3)

	Coefficient		Std. Error	t-Statistic	Prob.
C(1)	1.26E-07		3.65E-08	3.446235	0.0183
C(2)	-8.42E-05		2.02E-05	-4.171606	0.0087
C(3)	0.016988		0.002377	7.148118	0.0008
R-squared	0.837295	Mean dependent var			0.005792
Adjusted R-squared	0.772214	S.D. dependent var			0.003569
S.E. of regression	0.001704	Akaike info criterion			-9.632266
Sum squared resid	1.45E-05	Schwarz criterion			-9.602475
Log likelihood	41.52906	Durbin-Watson stat			2.048043

Table C.3: Quadratic least squares results to estimate intraday impact function

Appendix D

Single Day Results

Model Data

		AGL	SLM	MPC	SYC
Stock Information	Type	**AGL**	**SLM**	**MPC**	**SYC**
	Free-Float Market Cap	R 393,323m	R 38,080m	R 4,979m	R 3,502m
	Bid	R 262.05	R 15.81	R 23.10	R 19.10
	Ask	R 262.50	R 15.91	R 23.15	R 19.38
	Price	R 262.28	R 15.86	R 23.13	R 19.24
	Ave Spread	R 0.53	R 0.05	R 0.15	R 0.23
	Annual Growth	0.2151	0.0398	0.1390	0.1102
	Volatility	0.1540	0.1485	0.1062	0.0639
	Daily Volume x_id	3,500,000	6,700,000	700,000	450,000
Almgren & Chriss Temporary volume impact	a	6.47E-12	6.19E-13	1.79E-11	1.64E-12
	b	0.00	0.00	0.00	0.00
	c	0.00145	0.00213	0.004399	0.003959
Permanent Impact	Gamma	0.0000015143	0.0000000746	0.0000021429	0.0000051111
Mönch Temp Impact	a	6.470	6.190	1.790	1.64
Time Impact	d	1.36E-17	1.02E-18	2.21E-16	1.26E-19
	e	-8.47E-15	-6.03E-16	-1.35E-13	-8.42E-17
	f	1.5111E-12	1.1762E-13	2.341E-11	1.6988E-14

Table D.1: Summary of market and impact characteristics of the stocks used for the single day comparison.

For the Monch model, the impact is a product of the t_i impact and the temporary impact. To get the correct number of decimal places after the multiplication, the t_i impact parameters and temporary impact have been modified from the seperately estimated parameters. The 100 is to account for the double counting of the percentage impact. The constant is not required in the Monch model, the constant is part of the estimation in the t_i impact function.

Looking at AGL for example:

		Original	**Modifcation**	**Modified value**
Temporary	a	6.47E-12	x E12	6.470
t_i	d	1.36E-07	x E-12 x 100	1.36E-17
	e	-8.47E-05	x E-12 x 100	-8.47E-15
	f	0.015111	x E-12 x 100	1.5111E-12

β	0.0030		β	0.0035		β	0.0040	
x_i	τ_i	t_i	x_i	τ_i	t_i	x_i	τ_i	t_i
49559	310	310	54147	310	310	57959	311	311
5970	18	328	5669	20	330	5387	22	332
5055	15	343	4731	17	347	4441	18	350
4371	13	356	4047	14	361	3763	15	365
3844	12	368	3527	12	373	3257	13	378
3425	10	378	3121	11	384	2866	11	389
3085	9	387	2796	10	394	2555	10	400
2805	8	396	2530	9	403	2303	9	409
2570	8	403	2308	8	411	2094	8	417
2370	7	411	2121	7	418	1919	8	425
2198	7	417	1961	7	425	1770	7	432
2048	6	423	1823	6	432	1642	7	439
1917	6	429	1703	6	438	1531	6	445
1802	5	434	1597	6	443	1433	6	450
1699	5	440	1503	5	448	1347	5	456
1607	5	444	1419	5	453	1271	5	461
1525	5	449	1344	5	458	1202	5	466
1450	4	453	1277	4	463	1140	5	470
1382	4	457	1215	4	467	1085	4	475
1320	4	461	1160	4	471	1034	4	479
0	0	461	0	0	471	0	0	479

Table D.2: Mönch optimal strategies for various β's with $\lambda = 1.645$

β	0.0030		β	0.0035		β	0.0040	
x_i	τ_i	t_i	x_i	τ_i	t_i	x_i	τ_i	t_i
100000	188	188	100000	138	138	100000	89	89
97213	9	197	97578	9	147	97806	9	98
94278	9	207	95030	9	157	95506	10	108
91162	10	216	92341	10	167	93101	10	118
87853	11	227	89484	11	177	90538	11	129
84324	11	238	86447	11	189	87867	12	140
80556	12	250	83188	12	201	84988	12	152
76523	13	263	79680	13	214	81888	13	166
72202	14	277	75884	15	229	78589	14	180
67597	15	292	71728	16	245	74982	16	196
62722	15	307	67159	18	262	70941	18	214
57617	16	323	62156	19	281	66451	20	234
52336	16	339	56724	20	302	61470	23	257
46934	16	356	50910	21	323	55801	26	283
41453	17	372	44829	22	345	49370	28	310
35908	17	389	38598	22	367	42405	29	339
30276	17	407	32289	22	389	35210	29	368
24489	18	425	25879	23	413	27934	30	398
18384	21	446	19219	26	439	20484	33	431
11468	34	480	11826	41	480	12309	49	480
0	0	480	0	0	480	0	0	480

Table D.3: Mönch optimal strategies for various β's with $\lambda = 0$

All	Day	β	0.001	β	0.0015	β	0.002
t_i	ϑ_i	t_i	ϑ_i	t_i	ϑ_i	t_i	ϑ_i
24	0.1315	273	0.0212	254	0.0236	235	0.0270
48	0.1135	276	0.0208	259	0.0229	242	0.0258
72	0.0971	279	0.0205	264	0.0222	248	0.0245
96	0.0822	283	0.0202	269	0.0216	255	0.0234
120	0.0689	286	0.0200	274	0.0210	263	0.0224
144	0.0572	290	0.0197	280	0.0205	270	0.0214
168	0.0471	294	0.0195	286	0.0200	278	0.0206
192	0.0385	298	0.0194	292	0.0197	286	0.0200
216	0.0315	302	0.0193	298	0.0194	295	0.0195
240	0.0261	306	0.0192	304	0.0192	304	0.0192
264	0.0222	310	0.0191	311	0.0191	313	0.0191
288	0.0199	315	0.0191	318	0.0192	323	0.0193
312	0.0191	319	0.0192	325	0.0194	333	0.0198
336	0.0199	324	0.0193	333	0.0197	343	0.0205
360	0.0223	329	0.0196	341	0.0203	354	0.0216
384	0.0263	335	0.0199	349	0.0210	365	0.0230
408	0.0318	341	0.0203	358	0.0220	377	0.0249
432	0.0389	347	0.0209	367	0.0234	389	0.0273
456	0.0476	355	0.0217	379	0.0253	404	0.0307
480	0.0578	369	0.0237	399	0.0296	429	0.0379

Table D.4: Mönch optimal strategies times and impacts for various β's with $\lambda = 0$

All	Day	β	0.001	β	0.0015	β	0.002
t_i	ϑ_i	t_i	ϑ_i	t_i	ϑ_i	t_i	ϑ_i
24	0.1315	303	0.0192	307	0.0192	309	0.0191
48	0.1135	310	0.0191	317	0.0192	322	0.0193
72	0.0971	316	0.0192	326	0.0194	333	0.0198
96	0.0822	322	0.0193	334	0.0198	344	0.0205
120	0.0689	327	0.0195	342	0.0204	353	0.0215
144	0.0572	333	0.0197	349	0.0210	361	0.0225
168	0.0471	337	0.0200	355	0.0218	369	0.0236
192	0.0385	342	0.0204	361	0.0225	376	0.0248
216	0.0315	346	0.0208	367	0.0234	383	0.0260
240	0.0261	351	0.0212	373	0.0242	389	0.0273
264	0.0222	354	0.0216	378	0.0251	395	0.0285
288	0.0199	358	0.0221	382	0.0260	400	0.0298
312	0.0191	362	0.0226	387	0.0269	405	0.0310
336	0.0199	365	0.0231	391	0.0278	410	0.0323
360	0.0223	369	0.0236	395	0.0287	414	0.0336
384	0.0263	372	0.0241	399	0.0296	419	0.0348
408	0.0318	375	0.0247	403	0.0305	423	0.0360
432	0.0389	378	0.0252	407	0.0315	427	0.0373
456	0.0476	381	0.0257	410	0.0324	431	0.0385
480	0.0578	384	0.0263	413	0.0333	434	0.0397

Table D.5: Mönch optimal strategies times and impacts for various β's with $\lambda = 1.645$

All	Day	β	0.003	β	0.0035	β	0.004
t_i	ϑ_i	t_i	ϑ_i	t_i	ϑ_i	t_i	ϑ_i
24	0.1315	188	0.0397	139	0.0598	89	0.0865
48	0.1135	197	0.0369	147	0.0557	98	0.0811
72	0.0971	207	0.0341	157	0.0516	108	0.0756
96	0.0822	216	0.0314	167	0.0475	118	0.0701
120	0.0689	227	0.0288	177	0.0435	129	0.0646
144	0.0572	238	0.0264	189	0.0395	140	0.0590
168	0.0471	250	0.0242	201	0.0357	152	0.0535
192	0.0385	263	0.0223	214	0.0319	166	0.0480
216	0.0315	277	0.0207	229	0.0284	180	0.0425
240	0.0261	292	0.0196	245	0.0251	196	0.0372
264	0.0222	307	0.0191	263	0.0224	214	0.0320
288	0.0199	323	0.0193	282	0.0203	234	0.0272
312	0.0191	339	0.0202	302	0.0192	257	0.0231
336	0.0199	356	0.0218	323	0.0193	283	0.0202
360	0.0223	372	0.0242	345	0.0207	311	0.0191
384	0.0263	389	0.0273	367	0.0233	339	0.0202
408	0.0318	407	0.0314	390	0.0274	368	0.0236
432	0.0389	425	0.0366	413	0.0331	398	0.0294
456	0.0476	446	0.0436	439	0.0412	430	0.0384
480	0.0578	480	0.0578	480	0.0578	480	0.0578

Table D.6: Mönch optimal strategies times and impacts for various β's with $\lambda = 0$

All	Day	β	0.003	β	0.0035	β	0.004
t_i	ϑ_i	t_i	ϑ_i	t_i	ϑ_i	t_i	ϑ_i
24	0.1315	310	0.0191	310	0.0191	311	0.0191
48	0.1135	328	0.0195	330	0.0196	332	0.0197
72	0.0971	343	0.0205	347	0.0208	350	0.0211
96	0.0822	356	0.0219	361	0.0225	365	0.0230
120	0.0689	368	0.0234	373	0.0243	378	0.0252
144	0.0572	378	0.0252	384	0.0263	389	0.0274
168	0.0471	387	0.0270	394	0.0284	400	0.0297
192	0.0385	396	0.0288	403	0.0305	409	0.0320
216	0.0315	403	0.0306	411	0.0326	417	0.0344
240	0.0261	411	0.0325	418	0.0347	425	0.0367
264	0.0222	417	0.0343	425	0.0367	432	0.0389
288	0.0199	423	0.0361	432	0.0388	439	0.0411
312	0.0191	429	0.0379	438	0.0408	445	0.0433
336	0.0199	434	0.0397	443	0.0427	450	0.0454
360	0.0223	440	0.0415	448	0.0447	456	0.0475
384	0.0263	444	0.0432	453	0.0465	461	0.0495
408	0.0318	449	0.0448	458	0.0484	466	0.0515
432	0.0389	453	0.0465	463	0.0502	470	0.0535
456	0.0476	457	0.0481	467	0.0520	475	0.0554
480	0.0578	461	0.0497	471	0.0537	479	0.0572

Table D.7: Mönch optimal strategies times and impacts for various β's with $\lambda = 1.645$

Mönch			**Almgren &**	**Chriss**	
E(x)	26181000		**E(x)**	26156454	
StdDev	2953		**StdDev**	52871	
λ	1.645		λ	1.645	
T^*	420.000		T	0.500	(from 280min)
x_i	τ	t_i	x_i	τ	t_i
100000	283	283	100000	0.00	280
57285	15	298	81560	0.05	292
50947	13	311	65518	0.10	304
45468	12	322	51753	0.15	316
40650	10	333	40115	0.20	328
36359	9	342	30425	0.25	340
32493	8	350	22493	0.30	352
28980	8	358	16127	0.35	364
25761	7	365	11137	0.40	376
22794	7	372	7353	0.45	388
20042	6	378	4624	0.50	400
17477	6	384	2825	0.55	412
15077	5	389	1863	0.60	424
12821	5	394	1678	0.65	436
10695	5	399	1158	0.70	448
8683	5	404	1088	0.75	460
6775	4	408	1078	0.80	472
4961	4	412	1049	0.85	484
3232	4	416	1021	0.90	496
1580	4	420	605	0.95	508
0	0	420	0	1.00	520

Table D.8: Optimal solution comparison between the two models, $\lambda = 1.645$

Mönch			**Almgren &**	**Chriss**	
E(x)	26219000		**E(x)**	26183754	
StdDev	6283		**StdDev**	106541	
λ	0.000		λ	0.000	
T^*	420.000		T	0.500	(from 186min)
x_i	τ	t_i	x_i	τ	t_i
100000	186	186	100000	0.00	186
97349	7	193	96132	0.05	198
94583	7	200	92129	0.10	210
91693	7	207	87995	0.15	222
88644	8	215	83732	0.20	234
85466	8	223	79342	0.25	246
82097	9	232	74828	0.30	258
78508	9	241	70192	0.35	270
74738	10	250	65438	0.40	282
70727	10	261	60566	0.45	294
66452	11	272	55581	0.50	306
61871	12	283	50484	0.55	318
56994	12	296	45278	0.60	330
51807	13	309	39965	0.65	342
46327	14	323	34548	0.70	354
40557	15	337	29030	0.75	366
34486	15	353	23412	0.80	378
28080	17	369	17698	0.85	390
21176	19	388	11889	0.90	402
13243	32	420	5989	0.95	414
0	0	420	0	1.00	426

Table D.9: Optimal solution comparison between the two models, $\lambda = 0$

Time	AGL	SLM	MPC	SYC
240	100000	100000	100000	100000
252	96132	97051	95247	97455
264	92129	93886	90468	94652
276	87995	90506	85663	91590
288	83732	86909	80832	88271
300	79342	83096	75975	84693
312	74828	79068	71092	80857
324	70192	74823	66183	76763
336	65438	70363	61248	72411
348	60566	65690	56287	67801
360	55581	60799	51300	62933
372	50484	55691	46287	57805
384	45278	50367	41248	52420
396	39965	44826	36183	46777
408	34548	39070	31092	40868
420	29030	33099	25975	34708
432	23412	26911	20832	28291
444	17698	20507	15663	21599
456	11889	13887	10468	14666
468	5989	7052	5247	7474
480	0	0	0	0

Table D.10: Almgren & Chriss comparison of the 4 stocks, $T = \frac{1}{2}$ and $\lambda = 0$

Time	AGL	SLM	MPC	SYC
240	100000	100000	100000	100000
252	81560	84463	94313	93879
264	65518	70771	88743	88034
276	51753	58781	83282	82438
288	40115	48356	77923	77065
300	30425	39365	72659	71890
312	22493	31683	67482	66885
324	16127	25189	62384	62024
336	11137	19769	57359	57281
348	7353	15312	52398	52630
360	4624	11712	47495	48045
372	2825	8867	42643	43498
384	1863	6678	37833	38964
396	1678	5048	33058	34417
408	1158	3886	28311	29830
420	1088	3101	23584	25177
432	1078	2602	18871	20432
444	1049	2304	14163	15568
456	1021	2123	9454	10558
468	605	501	4735	5378
480	0	0	0	0

Table D.11: Almgren & Chriss comparison of the 4 stocks, $T = \frac{1}{2}$ and $\lambda = 1.645$

	AGL		**SLM**		**MPC**		**SYC**	
i	x_i	% Holdings	x_i	% Holdings	x_i	% Holdings	x_i	% Holdings
0	3813	1.0000	63052	1.0000	43234	1.0000	51975	1.0000
1	444	0.1165	50087	0.7944	39881	0.9224	47594	0.9157
2	46	0.0120	39148	0.6209	36694	0.8487	43653	0.8399
3	40	0.0106	30034	0.4763	33705	0.7796	40108	0.7717
4	40	0.0106	22557	0.3578	30899	0.7147	36920	0.7103
5	40	0.0106	16537	0.2623	28263	0.6537	34044	0.6550
6	40	0.0106	11807	0.1873	25783	0.5964	31440	0.6049
7	40	0.0106	8212	0.1302	23447	0.5423	29064	0.5592
8	40	0.0106	5612	0.0890	21241	0.4913	26873	0.5170
9	40	0.0106	3877	0.0615	19151	0.4430	24828	0.4777
10	40	0.0106	2892	0.0459	17164	0.3970	22885	0.4403
11	40	0.0106	2557	0.0406	15267	0.3531	21004	0.4041
12	40	0.0106	2319	0.0368	13446	0.3110	19143	0.3683
13	40	0.0106	2319	0.0368	11687	0.2703	17261	0.3321
14	40	0.0106	2268	0.0360	9978	0.2308	15316	0.2947
15	40	0.0106	2268	0.0360	8304	0.1921	13267	0.2553
16	40	0.0106	2119	0.0336	6652	0.1539	11073	0.2130
17	40	0.0106	1906	0.0302	5010	0.1159	8691	0.1672
18	40	0.0106	1830	0.0290	3362	0.0778	6081	0.1170
19	34	0.0088	1255	0.0199	1697	0.0392	3198	0.0615
20	0	0.0000	0	0.0000	0	0.0000	0	0.0000

Table D.12: Almgren & Chriss comparison for R1 million portfolio

	AGL		**SLM**		**MPC**		**SYC**	
i	x_i	% Holdings	x_i	% Holdings	x_i	% Holdings	x_i	% Holdings
0	38127	1.0000	630520	1.0000	432340	1.0000	519750	1.0000
1	25579	0.6709	593690	0.9416	410550	0.9496	493540	0.9496
2	16149	0.4236	557700	0.8845	388790	0.8993	467390	0.8993
3	9458	0.2481	522490	0.8287	367060	0.8490	441280	0.8490
4	5000	0.1311	488020	0.7740	345350	0.7988	415220	0.7989
5	2273	0.0596	454240	0.7204	323670	0.7486	389200	0.7488
6	884	0.0232	421110	0.6679	302010	0.6985	363210	0.6988
7	618	0.0162	388590	0.6163	280370	0.6485	337250	0.6489
8	457	0.0120	356620	0.5656	258750	0.5985	311320	0.5990
9	452	0.0119	325160	0.5157	237140	0.5485	285400	0.5491
10	428	0.0112	294170	0.4666	215550	0.4986	259490	0.4993
11	428	0.0112	263600	0.4181	193970	0.4487	233590	0.4494
12	428	0.0112	233410	0.3702	172400	0.3988	207700	0.3996
13	415	0.0109	203550	0.3228	150840	0.3489	181800	0.3498
14	368	0.0096	173980	0.2759	129280	0.2990	155890	0.2999
15	368	0.0096	144650	0.2294	107730	0.2492	129970	0.2501
16	368	0.0096	115510	0.1832	86187	0.1994	104040	0.2002
17	364	0.0096	86519	0.1372	64643	0.1495	78074	0.1502
18	359	0.0094	57635	0.0914	43098	0.0997	52085	0.1002
19	359	0.0094	28811	0.0457	21551	0.0498	26062	0.0501
20	0	0.0000	0	0.0000	0	0.0000	0	0.0000

Table D.13: Almgren & Chriss comparison for R10 million portfolio

AGL t_i	AGL x_i	SLM t_i	SLM x_i	MPC t_i	MPC x_i	SYC t_i	SYC x_i
283	100000	277	100000	233	100000	203	100000
298	57286	293	59440	249	77968	217	90365
311	50948	306	53009	264	71286	231	84498
322	45468	318	47414	278	65105	244	78790
333	40651	329	42469	290	59363	257	73233
342	36359	338	38045	302	54002	270	67820
350	32494	347	34045	314	48980	282	62544
358	28980	355	30399	324	44257	294	57398
365	25762	363	27050	335	39802	306	52375
372	22794	369	23956	344	35586	318	47471
378	20043	376	21080	353	31585	329	42679
384	17478	382	18396	362	27780	340	37996
389	15078	388	15880	370	24152	351	33415
394	12822	393	13512	378	20687	361	28932
399	10695	398	11277	386	17369	371	24544
404	8684	403	9161	393	14189	382	20247
408	6776	407	7151	400	11134	391	16037
412	4961	412	5238	407	8196	401	11910
416	3232	416	3414	414	5366	411	7864
420	1581	420	1670	420	2636	420	3894
420	0	420	0	420	0	420	0

Table D.14: Mönch comparison of the 4 stocks, $T^* = 420$ and $\lambda = 1.645$

AGL t_i	AGL x_i	SLM t_i	SLM x_i	MPC t_i	MPC x_i	SYC t_i	SYC x_i
186	100000	176	100000	186	100000	192	100000
193	97347	192	97642	193	97338	204	95037
200	94578	200	94822	200	94557	216	90053
207	91688	209	91549	207	91625	228	85057
215	88640	219	87892	215	88575	240	80060
223	85456	229	83884	223	85335	252	75060
232	82094	240	79612	232	81894	264	70060
241	78523	251	75057	241	78265	276	65060
250	74736	263	70310	251	74423	288	60058
261	70724	275	65351	262	70319	300	55056
271	66454	288	60222	273	65916	312	50051
283	61890	301	54930	285	61210	324	45045
296	57016	315	49502	298	56213	336	40042
309	51833	328	43919	311	50942	348	35039
323	46352	342	38218	325	45415	360	30034
337	40579	357	32339	339	39643	372	25031
353	34510	372	26296	354	33627	384	20026
369	28098	387	20055	371	27324	396	15019
388	21198	403	13590	389	20579	408	10014
420	13259	420	6899	420	12814	420	5009
420	0	420	0	420	0	420	0

Table D.15: Mönch comparison of the 4 stocks, $T^* = 420$ and $\lambda = 0$

Trade no.	15% x_i	20% x_i	25% x_i	30% x_i
0	100000	100000	100000	100000
1	86401	83597	81045	78736
2	74149	69089	64708	60799
3	63152	56390	50768	45861
4	53333	45397	39013	33611
5	44623	35989	29242	23760
6	36961	28040	21263	16040
7	30288	21419	14896	10205
8	24546	15995	9975	6032
9	19673	11643	6342	3321
10	15602	8243	3855	1894
11	12261	5683	2379	1595
12	9573	3863	1794	883
13	7450	2694	1714	680
14	5801	2095	1232	680
15	4528	1997	1054	676
16	3525	1629	1032	669
17	2684	1474	935	668
18	1892	1283	752	613
19	1036	836	703	507
20	0	0	0	0

Table D.16: Almgren & Chriss volatility comparison

15%		20%		25%		30%	
t_i	x_i	t_i	x_i	t_i	x_i	t_i	x_i
298	100000	300	100000	300	100000	300	100000
313	58886	312	41683	310	30481	309	23173
326	52477	323	36325	319	26117	315	19588
338	46909	332	31913	325	22664	320	16841
349	41995	340	28177	331	19825	325	14630
358	37604	347	24946	336	17423	329	12790
367	33638	353	22104	341	15347	332	11219
375	30026	358	19572	344	13522	335	9851
382	26711	363	17290	348	11896	337	8643
389	23649	368	15216	351	10431	340	7561
395	20806	372	13315	354	9100	342	6582
401	18153	376	11563	357	7881	344	5690
407	15667	379	9937	359	6757	346	4871
412	13329	383	8422	362	5714	347	4114
417	11122	386	7003	364	4742	349	3410
422	9033	389	5670	366	3833	350	2753
427	7051	392	4413	368	2978	352	2137
431	5164	395	3224	370	2172	353	1557
435	3365	397	2095	371	1410	354	1010
439	1646	400	1023	373	687	356	492

Table D.17: Mönch volatility comparison

Trade no.	5.00E-13	1.00E-12	1.50E-12	2.00E-12	2.50E-12
0	100000	100000	100000	100000	100000
1	70803	76287	79003	80790	82065
2	48242	56858	61341	64345	66527
3	31314	41206	46674	50416	53190
4	19122	28864	34680	38766	41865
5	10875	19408	25061	29167	32369
6	5889	12452	17541	21407	24528
7	3586	7655	11867	15287	18178
8	3490	4710	7808	10622	13162
9	2784	3353	5156	7241	9333
10	2724	3351	3726	4987	6555
11	2635	3173	3351	3716	4700
12	2635	2795	3053	3295	3648
13	2619	2795	2684	2953	3287
14	2613	2795	2656	2896	2823
15	2611	2571	2656	2650	2737
16	2590	2571	2654	2649	2657
17	2586	2571	2613	2647	2578
18	2570	2571	2359	2315	2197
19	1891	1844	1583	1545	1394
20	0	0	0	0	0

Table D.18: Almgren & Chriss temporary impact comparison

Appendix E

Multi-security Results

	Stock 1	Stock 2
Price	50	50
Volatility	20%	20%

The value of each stocks holdings were calculated as the t_0 price times number of shares held at that time and the cash value as the t_0 price's times the number of shares sold up to that point for each share. The portfolio holdings are calculated as the sum of these, which is constant since no price changs were used. This would only slightly effect the volatility.

i	Stock 1 x_i	Stock 2 x_i	Stock 1 Value	Stock 2 Value	Cash Value	Portfolio Value	Portfolio σ	% x_i Stock 1	Stock 2
0	100000	100000	5000000	5000000	0	10000000	14.14%	100.00%	100.00%
1	76848	92400	3842400	4620000	1537600	10000000	12.02%	76.85%	92.40%
2	57827	85319	2891350	4265950	2842700	10000000	10.31%	57.83%	85.32%
3	42403	78689	2120150	3934450	3945400	10000000	8.94%	42.40%	78.69%
4	30101	72451	1505050	3622550	4872400	10000000	7.85%	30.10%	72.45%
5	20496	66557	1024800	3327850	5647350	10000000	6.96%	20.50%	66.56%
6	13212	60967	660600	3048350	6291050	10000000	6.24%	13.21%	60.97%
7	7915	55645	395750	2782250	6822000	10000000	5.62%	7.92%	55.65%
8	4310	50560	215500	2528000	7256500	10000000	5.07%	4.31%	50.56%
9	2133	45687	106650	2284350	7609000	10000000	4.57%	2.13%	45.69%
10	1151	41002	57550	2050100	7892350	10000000	4.10%	1.15%	41.00%
11	1151	36484	57550	1824200	8118250	10000000	3.65%	1.15%	36.48%
12	1078	32113	53900	1605650	8340450	10000000	3.21%	1.08%	32.11%
13	918	27869	45900	1393450	8560650	10000000	2.79%	0.92%	27.87%
14	622	23734	31100	1186700	8782200	10000000	2.37%	0.62%	23.73%
15	366	19689	18300	984450	8997250	10000000	1.97%	0.37%	19.69%
16	366	15711	18300	785550	9196150	10000000	1.57%	0.37%	15.71%
17	366	11778	18300	588900	9392800	10000000	1.18%	0.37%	11.78%
18	366	7866	18300	393300	9588400	10000000	0.79%	0.37%	7.87%
19	225	3949	11250	197450	9791300	10000000	0.40%	0.23%	3.95%
20	0	0	0	0	10000000	10000000	0.00%	0.00%	0.00%

Table E.1: Portfolio holdings and volatility with $\rho = 0$

i	Stock 1 x_i	Stock 2 x_i	Stock 1 Value	Stock 2 Value	Cash Value	Portfolio Value	Portfolio σ	% x_i Stock 1	Stock 2
0	100000	100000	5000000	5000000	0	10000000	17.32%	100.00%	100.00%
1	70242	91525	3512100	4576250	1911650	10000000	14.05%	70.24%	91.53%
2	47040	83893	2352000	4194650	3453350	10000000	11.49%	47.04%	83.89%
3	29436	76933	1471800	3846650	4681550	10000000	9.51%	29.44%	76.93%
4	16598	70509	829900	3525450	5644650	10000000	8.01%	16.60%	70.51%
5	7812	64518	390600	3225900	6383500	10000000	6.88%	7.81%	64.52%
6	2462	58880	123100	2944000	6932900	10000000	6.01%	2.46%	58.88%
7	23	53538	1150	2676900	7321950	10000000	5.35%	0.02%	53.54%
8	23	48450	1150	2422500	7576350	10000000	4.85%	0.02%	48.45%
9	23	43589	1150	2179450	7819400	10000000	4.36%	0.02%	43.59%
10	0	38939	0	1946950	8053050	10000000	3.89%	0.00%	38.94%
11	0	34488	0	1724400	8275600	10000000	3.45%	0.00%	34.49%
12	0	30220	0	1511000	8489000	10000000	3.02%	0.00%	30.22%
13	0	26119	0	1305950	8694050	10000000	2.61%	0.00%	26.12%
14	0	22165	0	1108250	8891750	10000000	2.22%	0.00%	22.17%
15	0	18332	0	916600	9083400	10000000	1.83%	0.00%	18.33%
16	0	14596	0	729800	9270200	10000000	1.46%	0.00%	14.60%
17	0	10926	0	546300	9453700	10000000	1.09%	0.00%	10.93%
18	0	7292	0	364600	9635400	10000000	0.73%	0.00%	7.29%
19	0	3661	0	183050	9816950	10000000	0.37%	0.00%	3.66%
20	0	0	0	0	10000000	10000000	0.00%	0.00%	0.00%

Table E.2: Portfolio holdings and volatility with $\rho = 0.5$

i	Stock 1 x_i	Stock 2 x_i	Stock 1 Value	Stock 2 Value	Cash Value	Portfolio Value	Portfolio σ	% x_i Stock 1	Stock 2
0	100000	100000	5000000	5000000	0	10000000	10.00%	100.00%	100.00%
1	91045	94225	4552250	4711250	736500	10000000	9.27%	91.05%	94.23%
2	83269	88596	4163450	4429800	1406750	10000000	8.61%	83.27%	88.60%
3	76448	83105	3822400	4155250	2022350	10000000	8.00%	76.45%	83.11%
4	70395	77744	3519750	3887200	2593050	10000000	7.43%	70.40%	77.74%
5	64956	72504	3247800	3625200	3127000	10000000	6.90%	64.96%	72.50%
6	60005	67374	3000250	3368700	3631050	10000000	6.40%	60.01%	67.37%
7	55436	62342	2771800	3117100	4111100	10000000	5.92%	55.44%	62.34%
8	51162	57396	2558100	2869800	4572100	10000000	5.45%	51.16%	57.40%
9	47108	52523	2355400	2626150	5018450	10000000	5.00%	47.11%	52.52%
10	43211	47710	2160550	2385500	5453950	10000000	4.56%	43.21%	47.71%
11	39412	42944	1970600	2147200	5882200	10000000	4.13%	39.41%	42.94%
12	35657	38211	1782850	1910550	6306600	10000000	3.70%	35.66%	38.21%
13	31894	33498	1594700	1674900	6730400	10000000	3.27%	31.89%	33.50%
14	28070	28791	1403500	1439550	7156950	10000000	2.84%	28.07%	28.79%
15	24127	24078	1206350	1203900	7589750	10000000	2.41%	24.13%	24.08%
16	20000	19347	1000000	967350	8032650	10000000	1.97%	20.00%	19.35%
17	15617	14585	780850	729250	8489900	10000000	1.51%	15.62%	14.59%
18	10893	9780	544650	489000	8966350	10000000	1.04%	10.89%	9.78%
19	5727	4922	286350	246100	9467550	10000000	0.54%	5.73%	4.92%
20	0	0	0	0	10000000	10000000	0.00%	0.00%	0.00%

Table E.3: Portfolio holdings and volatility with $\rho = -0.5$

Appendix F

Restriction Calculations

A					x	b	Ax
-1	0	0	0	0	0	0	0
0	-1	0	0	0	25	0	-25
0	0	-1	0	0	25	0	-25
0	0	0	-1	0	25	0	-25
0	0	0	0	-1	25	0	-25

A					x	b	Ax
1	0	0	0	0	0	MAX	0
0	1	0	0	0	25	MAX	25
0	0	1	0	0	25	MAX	25
0	0	0	1	0	25	MAX	25
0	0	0	0	1	25	MAX	25

Table F.1: Restriction calculations

This is a modification of the constraint that ensures no short sales. This is not a complete solution for restricted sales. For some shares short sales no occur in the solution. Our findings show no need for restrictions so a complete restriction does not appear necessary.

The **FMINCON** function in MatLab uses the constraint $Ax \leq b$. The first equation of tables shows the unrestricted constraint, and the second shows the changes to A and b when the restriction is MAX. Each member of x is the number of shares to trade at that point.

Time	Restriction			Unrestricted
	15000	20000	30000	
0	100000	100000	100000	100000
1	85000	80000	70000	63322
2	70000	60000	41743	37090
3	55000	40000	22533	19529
4	40000	21429	10614	8890
5	25000	9950	4220	3438
6	12085	3923	1668	1437
7	4916	1569	1078	1046
8	1876	1026	1067	1066
9	1078	1045	1082	1081
10	1038	1096	1059	1060
11	1088	1087	1043	1049
12	1086	1051	1047	1055
13	1058	1023	1058	1057
14	1039	1015	1059	1058
15	1030	1026	1057	1039
16	1038	1057	1053	1030
17	1068	1100	1074	1051
18	1086	1101	1093	1069
19	900	879	920	900
20	0	0	0	0

Table F.2: Comparison of strategy for AGL with restrictions

Time	Restriction			Unrestricted
	6000	7000	8000	
0	100000	100000	100000	100000
1	94000	93000	92000	90692
2	88000	86000	84000	82079
3	82000	79000	76000	74118
4	76000	72000	68502	66777
5	70000	65000	61588	60025
6	64000	58343	55239	53835
7	58000	52222	49431	48177
8	52000	46614	44132	43018
9	46248	41486	39303	38319
10	40971	36795	34898	34037
11	36108	32492	30864	30119
12	31596	28519	27140	26507
13	27370	24813	23663	23135
14	23363	21302	20362	19935
15	19506	17911	17163	16829
16	15728	14563	13990	13741
17	11960	11177	10765	10591
18	8129	7673	7409	7301
19	4166	3972	3846	3794
20	0	0	0	0

Table F.3: Comparison of strategy for MPC with restrictions

Appendix G

AutoIT Script

```
; variables
dim $winpos
dim $timeperiods
dim $i           ; counter for number time periods
dim $j           ; counter for number of shares
dim $k           ; counter for number of shares
dim $numshares
dim $sharenames[11]
dim $timegap
dim $file
dim $dir
dim $columns
; before running:
    ; (1) open market depth in INet and open all shares required
    ; (2) create blank Excel files in a directory for each share
    ; (3) fill in data below

; ******************* DATA TO FILL IN ********************
; this data is needed before you start the program
; Excel files in the $dir variable must already be created eg.  "c:\MarketDepth\.xls"
$dir ="c:\MarketDepth\"
;names to use for the stocks - same as filenames
    $sharenames[1] = "AGL"
    $sharenames[2] = "BIL"
    $sharenames[3] = "SOL"
    $sharenames[4] = "RCH"
    $sharenames[5] = "SAB"
    $sharenames[6] = "SBK"
    $sharenames[7] = "OML"
    $sharenames[8] = "MTN"
    $sharenames[9] = "FSR"
    $sharenames[10] = "GFI"
$numshares = 10  ;number of shares in the list
$timeperiods = 48;number of periods to capture
$timegap = 600000;time gap in milliseconds -> 1000 = 1 second, 1800000 = 30minutes
$columns = 4     ;number of columns to move across
```

```
;loop for each time period
For $i = 1 to $timeperiods
    ;loop for each share
    For $j =1 to $numshares
        ;activates the market depth window and gets its position
        WinActivate("Market Depth")
        $winpos = WinGetPos("Market Depth")

        ;selects first tab
        MouseClick("left", $winpos[0] + 20 , $winpos[1] + 50, 1,10)

        ;presses right until at correct share
        If $j> 1 Then
             For $k = 1 to ($j-1)
                 send("{RIGHT}")
             Next
        EndIf

        ;right clicks on the selects the table
        MouseClick("right", $winpos[0] + 20 , $winpos[1] + 100, 1 , 10)

        ;sends copy command -> Tools, copy to clipboard
        send("T")
        send("C")
        send("{ENTER}")

        ;minimizes the market depth windows after getting data
        WinSetState ("Market Depth", "", @SW_MINIMIZE)

        ;runs Excel
        Run ("C:\Program Files\Microsoft Office\OFFICE10\Excel.exe", $dir , @SW_SHOWMAXIMIZED)
        winwait("Microsoft Excel - Book1")
        WinActivate("Microsoft Excel - Book1")

        ;opens file using alt+f then o, waits for "Open" window, opens file name
        send("!f")
        send("o")
        WinWait("Open")
        $file = $dir & $sharenames[$j] & ".xls"
        send($file)
        send("{ENTER}")

        ;waits for Excel to open file, then activates window
        WinWait("Microsoft Excel - " & $sharenames[$j])
        WinActivate("Microsoft Excel - " & $sharenames[$j])

        ;clicks somewhere on the excel sheet
        MouseClick("left", 200 , 200, 1, 1)
```

```
        ;goes to cell A1 by pressing ctrl+home
        send("^HOME")

        presses right until past previous entry
        If $i > 1 Then
             For $k = 1 to ($i-1) * $columns
                 send("{PGDN}")
             Next
        EndIf

        ;enters time of entry in first row
        send(@hour)
        send(":")
        send(@MIN)
        send("{ENTER}")

        ;pastes the data from the clipboard using ctrl+v
        send("^v")

        ;saves the file using ctrl+s
        $file = "Microsoft Excel - " & $sharenames[$j]
        WinActivate($file)
        send("^s")

        ;closes Excel
        $file = "Microsoft Excel - " & $sharenames[$j]
        WinClose($file)

    Next
    ;closes book 1 (excel opens by default sometimes, might be able to omit)
    $file = "Microsoft Excel - Book1"
    WinClose($file)

    ;waits until next capture
    Sleep($timegap)
Next
```

Appendix H

Almgren and Chriss Modified MatLab Code

External Functions

RunOpti

```
function [StratEx StratVol Strat] = RunOpti(Model, Parameters);
```

```
%  Almgren and Chriss optimal liquidation model
%  with quadratic temporary impact functions
%  Inputs
%  -------------------------------------------------------------
%  | Model(1)         T               Trading period (days)     |
%  | Model(2)         N               Number of trades          |
%  | Model(3)         lambda          Risk aversion             |
%  | Model(4)         FixedPerTrade   Fixed costs per trade     |
%  | Model(5)         FixedPerShare   Fixed costs per share     |
%  | Model(6)         X               Number of share           |
%  |                                                            |
%  | Parameters(1)    So      Initial share price               |
%  | Parameters(2)    mu      Drift                             |
%  | Parameters(3)    sigma   Volatility                        |
%  | Parameters(4)    a       Temporary impact parameter 1      |
%  | Parameters(5)    b       Temporary impact parameter 2      |
%  | Parameters(6)    c       Temporary impact parameter 3      |
%  | Parameters(7)    gamma   Permanent impact                  |
%  -------------------------------------------------------------
%
%  Outputs
%  -------------------------------------------------------------
%  |    StratEx     Expected return of strategy                 |
%  |    StratVol    Volatility of strategy                      |
%  |    Strat       Optimal trading strategies                  |
%  -------------------------------------------------------------
```

```
X = Model(6);
lambda = Model(3);
StratX = zeros(Model(2)+1,1);
% runs a single optimization for a given lambda
n = length(StratX);
% initiates blank strategy vector
StratN = XtoN(StratX);
% Expected return, volatility and utility function in terms of the strategy
ExVal = @(StratN) Ex(NtoX(StratN, X),Model, Parameters);
VolVal = @(StratN) Vol(NtoX(StratN, X), Model, Parameters);
Opti = @(StratN) ExVal(StratN) + lambda * VolVal(StratN);
% optimization constraints
A = diag(zeros(1,n)-1);
b = zeros(n,1);
Aeq = zeros(2,n);
Aeq(1,1) = 1;
for i = 1:n
        Aeq(2,i) = 1;
end;
beq = zeros(2,1);
beq(2,1) = X;
% runs the optimization
options = optimset('Display','off', 'LargeScale', 'off');
OptiStrat = fmincon(Opti,StratN,A,b,Aeq,beq,[],[],[],options);
% calculates expected value
StratEx = ExVal(OptiStrat);
% calculates volatility
StratVol = VolVal(OptiStrat);
% returns optimal strategy
Strat = NtoX(OptiStrat,X);
```

Ex

```
function y = Ex(StratX,Model, Parameters);
% Calculates the expected return of a strategy
Tau = Model(1)/Model(2);
N = Model(2);
X = Model(6);
So = Parameters(1);
gamma = Parameters(7);
mu = So * Parameters(3) / 250;
a = So * Parameters(4);
b = So * Parameters(5);
c = So * Parameters(6);
FixedShares = Model(4);
FixedTrades = Model(5);
sumcubed=0;
sumsqr=0;
for i = 2:length(StratX)
        n = (StratX(i-1,1) - StratX(i,1));
        sumcubed = sumcubed + n^3;
```

```
        sumsqr = sumsqr + n^2;
end;
y = -mu*Tau*sum(StratX(2:length(StratX),1)) + FixedTrades*N + 0.5*gamma*X^2
    + (c+FixedShares)*X + a*sumcubed + (b-0.5*gamma*Tau)* sumsqr;
```

Vol

```
function y = Vol(StratX, Model, Parameters)
% calculates the volatility of a strategy
Tau = Model(1)/Model(2);
sigma = Parameters(1) * Parameters(3) / sqrt(250);
y = sigma^2 * Tau * sum(StratX(2:length(StratX),1).^2);
```

Internal Functions

NtoX

```
function y = NtoX(StratN, X);
% internal function that converts strategy in terms of individual
% trades to a strategy in terms of holdings
n = length(StratN);
StratX = zeros(n,1);
sum = 0;
for i = 1:n
        sum = sum + StratN(i,1);
        StratX(i,1) = X - sum;
end;
y = StratX;
```

XtoN

```
function y = XtoN(StratX);
% internal function that converts strategy in terms of
% holdings to a strategy in terms of individual trades
n = length(StratX);
StratN = zeros(n,1);
for i = 2:n;
        StratN(i,1) = StratX(i-1,1) - StratX(i,1);
end;
y = StratN;
```

Appendix I

Almgren and Chriss Multisecurity Code

External Functions

RunOpti

```
function [StratEx StratVol Strat] = RunOpti(T,M,N,X,gamma,mu,H,eps,CovMat,lambda);
```

```
%  Almgren and Chriss multiple security optimal liquidation model
%  with linear impact functions
%  Inputs
%  -----------------------------------------------------------------------------
%  |    T           Trading period (days)                       1 x 1     |
%  |    M           Number of stocks                            1 x 1     |
%  |    N           Number of trades                            1 x 1     |
%  |    X           Initial number of shares for each stock     M x 1     |
%  |    Gamma       Permanent impact function                   M x M     |
%  |    mu          Stock drift                                 M x 1     |
%  |    H           Temporary impact function coefficient       M x M     |
%  |    eps         Temporary impact function constant          M x 1     |
%  |    CovMat      Covariance matrix                           M x M     |
%  |    lambda      Risk aversion                               1 x 1     |
%  -----------------------------------------------------------------------------
%
%  Outputs\\
%  -----------------------------------------------------------------------------
%  |    StratEx     Expected return of strategy                 1 x 1     |
%  |    StratVol    Volatility of strategy                      1 x 1     |
%  |    Strat       Optimal trading strategies                  M x N + 1 |
%  -----------------------------------------------------------------------------
```

```
% initiates blank strategy matrix
StratX = zeros(M,N+1);
% converts strategy matrix to a vector
StratVec = MatToVec(XtoN(StratX, M, N), M, N);
% Expected return, volatility and utility function in terms of the strategy
ExVal = @(StratVec) Ex(NtoX(VecToMat(StratVec,M,N)',X,M,N),T/N,X,gamma,mu,H,eps,M,N);
VolVal = @(StratVec) Vol(CovMat,T/N, NtoX(VecToMat(StratVec,M,N)', X, M, N));
Opti = @(StratVec) ExVal(StratVec) + lambda * VolVal(StratVec);
% optimization constraints
Aeq = AeqMat(M,N);
beq = beqMat(M,N,X);
A = diag(zeros(1,(N+1)*M)-1);
b = zeros((N+1)*M,1);
% optimization settings
options = optimset('Display','off', 'LargeScale', 'off');
% runs optimization
OptiStrat = fmincon(Opti,StratVec,A,b,Aeq,beq,[],[],[],options);
% Expected return of optimal strategy
StratEx = ExVal(OptiStrat);
%volatility of optimal strategy
StratVol = VolVal(OptiStrat);
%optimal strategy
Strat = round(NtoX(VecToMat(OptiStrat,M,N)', X, M, N))';
```

Ex

```
function y = Ex(StratX,Tau,X,gamma,mu,H,eps, M, N);
% Calculates the expected return of a multiple security strategy
gammaS = 0.5 * (gamma + gamma');
gammaA = 0.5 * (gamma - gamma');
HS = 0.5 * (H + H');
term1 = 0;
for i = 2:N+1;
        term1 = term1 + StratX(:,i);
end;
term1 = -Tau * mu' * term1;
term2 = eps' * X;
term3 = 0.5 * X' * gammaS * X;
nu = zeros(M, N);
term4 = 0;
for j = 2:N+1;
        for i = 1:M;
                nu(i,j) = (StratX(i,j-1) - StratX(i,j))/Tau;
        end;
        term4 = term4 + Tau * nu(:,j)' * (HS - 0.5 * Tau * gammaS) * nu(:,j);
end;
term5 = 0;
for i = 2:N+1;
        term5 = term5 + Tau * StratX(:,i)' * gammaA * nu(:,i);
end;
y = term1 + term2 + term3 + term4 + term5;
```

Vol

```
function y = Vol(sigma,Tau,StratX)
% calculates the volatility of a strategy
C = sigma*sigma';
VX = 0;
for i = 2:length(StratX);
      VX = VX + Tau * StratX(:,i)' * C * StratX(:,i);
end;
y = VX;
```

Internal Functions

Aeq

```
function y = AeqMat(M, N);
% internal function to construct Aeq constraint matrix
Aeq = zeros(2*M,M*(N+1));
for i = 1:M
      Aeq((2*i)-1,(i-1)*(N+1)+1) = 1;
      Aeq(2*i,(i-1)*(N+1)+1:i*(N+1)) = 1;
end;
y = Aeq;
```

beqMat

```
function y = beqMat(M, N, X);
% internal function to construct beq constraint matrix
beq = zeros(2*M,1);
for i = 1:M
      beq(2*i,1) = X(i,1);
end;
y = beq;
```

MatToVec

```
function y = MatToVec(X, M, N);
% Converts (N+1)xM trade matrix to a (N+1)*Mx1
% solution vector to use in the optimization
X = X';
Mat = zeros((N+1)*M,1);
for i = 1:M*(N+1)
      Mat(i) = X(i);
end;
y = Mat;
```

VecToMat

```
function y = VecToMat(X,M,N);
% internal function to convert single (N+1)*Mx1 solution vector
% from optimization to (N+1)xM solution matrix
Vec = zeros(N+1,M);
for i = 1:M*(N+1)
      Vec(i) = X(i);
end;
y = Vec;
```

NtoX

```
function y = NtoX(StratN, X, M, N);
% internal function that converts strategy in terms of individual
% trades to a strategy in terms of holdings
StratX = zeros(M,N+1);
for j = 1:M
      sum = 0;
      for i = 1:N+1
           sum = sum + StratN(j,i);
           StratX(j,i) = X(j, 1) - sum;
      end;
end;
y = StratX;
```

XtoN

```
function y = XtoN(StratX, M, N);
% internal function that converts strategy in terms of
% holdings to a strategy in terms of individual trades
StratN = zeros(M,N+1);
for j = 1:M;
      for i = 2:N+1;
           StratN(j,i) = StratX(j,i-1) - StratX(j,i);
      end;
end;
y = StratN;
```

Appendix J

Mönch MatLab Code

External Functions

Max

```
function [ExVal, tau, trades] = Max(Parameters)

% Monch optimal liquidation model
%  Inputs
%  ----------------------------------------------------------------
%  | Parameters(1)   a      Price impact parameter 1               |
%  | Parameters(2)   b      Price impact parameter 2               |
%  | Parameters(3)   c      Price impact parameter 3               |
%  | Parameters(4)   d      Intraday impact parameter 1            |
%  | Parameters(5)   e      Intraday impact parameter 2            |
%  | Parameters(6)   f      Intraday impact parameter 3            |
%  | Parameters(7)   g      Intraday impact parameter 4            |
%  | Parameters(8)   Beta   Market recovery parameter              |
%  | Parameters(9)   n      Number of trades                       |
%  | Parameters(10)  X      Initial number of shares               |
%  | Parameters(11)  So     Initial share price                    |
%  | Parameters(12)  Fixed  Fixed costs per trade                  |
%  | Parameters(13)  N      Maximum number of trades to calculate  |
%  | Parameters(14)  sigma  Stock volatility                       |
%  | Parameters(15)  alpha  Risk aversion                          |
%  ----------------------------------------------------------------
%
%  Outputs
%  ----------------------------------------------------------------
%  |     StratEx     Expected return of strategy                   |
%  |     StratVol    Volatility of strategy                        |
%  |     Strat       Optimal trading strategies                    |
%  ----------------------------------------------------------------
```

```
X = Parameters(10);
n = Parameters(9);
alpha = Parameters(15);
% initiates blank strategy
tradelist = zeros(2*n,1)+0.000001;
% optimization constraints
A = MakeA(n, Parameters(8));
b = zeros(n+2,1);
b(n+2,1) = Parameters(13);
Aeq = MakeAeq(n);
beq = X;
% function to optimize
Opti = @(tradelist) (alpha*sqrt(Var(tradelist,Parameters))-Ex(tradelist,Parameters));
%runs optimization
options = optimset('Display','off', 'LargeScale', 'off', 'Diagnostics', 'off');
OptiStrat = fmincon(Opti,tradelist,A,b,Aeq,beq,0,[],[], options);
% returns trade times
tau = OptiStrat(1:n,1);
% returns trade sizes
trades = OptiStrat(n+1:2*n,1);
% returns expected value of strategy
ExVal = -Opti(OptiStrat);
```

Ex

```
function y = Ex(tradelist, Parameters)
% calculates the expected return given a tradelist (trades and times)
total = 0;
n = Parameters(9);
So = Parameters(11);
FixedCost = Parameters(12);
t(1,1) = tradelist(1,1);
trades(1,1) = tradelist(1+n,1);
for i = 2:n
      t(i,1) = t(i-1,1) + tradelist(i,1);
      trades(i,1) = tradelist(n+i,1);
end;
for i = 1:n
      sum = 0;
      product = 1;
      sum = trades(i) * delta(trades(i),t(i),Parameters);
      for j = 1:i
           temp = gamma(trades(j), t(j), Parameters);
           product = product * temp;
      end;
      total = total + sum * product;
end;
y = - n * FixedCost + So * total;
```

Vol

```
function y = Vol(tradelist, Parameters)
% calculates the volatility given a tradelist (trades and times)
sigma = Parameters(14);
n = Parameters(9);
So = Parameters(11);
sum1 = 0;
sum2 = 0;
t(1,1) = tradelist(1,1);
trades(1,1) = tradelist(1+n,1);
for i = 2:n
      t(i,1) = t(i-1,1) + tradelist(i,1);
      trades(i,1) = tradelist(n+i,1);
end;
Cov = CovMatrix(tradelist,Parameters);
for i = 1:n
      g(i) = gamma(trades(i,1), t(i), Parameters);
      d(i) = delta(trades(i,1), t(i), Parameters);
end;
for i = 1:n
      sum1 = sum1 + (trades(i) * d(i))^2 * Cov(i,i);
      for j = i+1:n
            sum2 = sum2 + Cov(i,j);
      end;
end;
y = sum1 + 2*sum2;
```

Internal Functions

CovMatrix

```
function y = CovMatrix(tradelist, Parameters)
% creates the covariance matrix for use in the volatility calculation
n = Parameters(9);
So = Parameters(11);
sigma = Parameters(14);
t(1,1) = tradelist(1,1);
trades(1,1) = tradelist(1+n,1);
for i = 2:n
      t(i,1) = t(i-1,1) + tradelist(i,1);
      trades(i,1) = tradelist(n+i,1);
end;
cov = zeros(n,n);
for i = 1:n
      g(i) = gamma(trades(i,1), t(i), Parameters);
end;
for i = 1:n
      for j = i:n
            product1 = 1;
```

```
            product2 = 1;
            for m = 1:i
                product1 = product1 * g(m)^2;
            end;
            for l = i+1:j
                product2 = product2 * g(l);
            end;
            cov(i,j) = So^2 * exp(sigma^2*(t(i) - t(1))-1) * product1 * product2;
        end;
end;
y = cov;
```

delta

```
function y = delta(tradesize, t, Parameters)
% calculates the delta term
a = Parameters(1);
b = Parameters(2);
t1 = Intraday(t, Parameters)/(tradesize);
t2 = (a/(b+1)) * tradesize^(b+1);
t3 = gamma(tradesize, t, Parameters);
y = (1- t1 * t2) / t3;
```

gamma

```
function y = gamma(tradesize, t, Parameters)
% calculates the combined impact function
y = 1 - PriceImpact(tradesize, Parameters) * Intraday(t, Parameters);
```

Intraday

```
function y = Intraday(t, Parameters);
% intraday impact function
d = Parameters(4);
e = Parameters(5);
f = Parameters(6);
y = d * t^2 + e * t + f;
```

PriceImpact

```
function y = PriceImpact(tradesize, Parameters)
% calculates impact function for a given volume
a = Parameters(1);
b = Parameters(2);
c = Parameters(3);
y = a * vol^2 + b * vol + c;
```

MakeA

```
function y = MakeA(n, Beta)
% constructs constraint matrix A
```

```
A = zeros(n + 1, 2 * n);
for i = 1:n
        A(i,i) = -1;
        A(i, i + n) = Beta;
        A(n+1,i+n) = -1;
        A(n+2,i) = 1;
end;
y = A;
```

MakeAeq

```
function y = MakeAeq(n);
% constructs constraint matrix Aeq
Aeq = zeros(1,2*n);
for i = n+1:2*n
        Aeq(1,i) = 1;
end;
y = Aeq;
```

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