ACCELERATION WAVES IN CONSTRAINED THERMOELASTIC MATERIALS

by

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A thesis submitted in fulfillment of the requirements for the degree Doctor of Philosophy

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August 1989

ACKNOWLEDGEMENTS

I wish to express my deep gratitude to Professor Daya Reddy for supervising this work. I am also extremely grateful to Shirley Breed for her invaluable assistance in typing the manuscript under the very disruptive conditions under which this thesis was completed, and to Mike Jaklitsch for his assistance with the printing and binding.

It is also a pleasure to acknowledge the support provided by my colleagues in both the Department of Applied Mathematics and the Applied Mechanics Research Unit.

Finally, I wish to thank the Atomic Energy Corporation for their financial assistance.

ABSTRACT

We study the propagation and growth of acceleration waves in isotropic thermoelastic media subject to a broad class of thermomechanical constraints. The work is based on an existing thermodynamic theory of constrained thermoelastic materials presented by Reddy (1984) for both definite and non-conductors, but we differ by adopting a new definition of a constrained non-conductor and by investigating the consequences of isotropy. The set of constraints considered is not arbitrary but is large enough to include most constraints commonly found in practice. We also extend Reddy's (1984) work by including consideration of sets of constraints for which a set of vectors associated with the constraints is linearly dependent. These vectors play a significant role in the propagation conditions and in the growth equations described below.

Propagation conditions (of Fresnel-Hadamard type) are derived for both homothermal and homentropic waves, and solutions for longitudinal and transverse principal waves are discussed. The derivations involve the determination of jumps in the time derivative of constraint multipliers which are required in the solution of the corresponding growth equations, and it is found that these multipliers cannot be separately determined if the set of constraint vectors mentioned above is linearly dependent. This difficulty forces us to restrict the constraint set for which the growth equations for homothermal and homentropic waves can be derived. The growth of plane, cylindrical and spherical waves is considered and solutions are discussed, concentrating on the influence of the constraints on the results.

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CHAPTER 1

INTRODUCTION

One approach to the investigation of finite amplitude waves in nonlinear materials is to study the behaviour of propagating singular surfaces, in which the wave is a disturbance of arbitrary magnitude localized on a surface which propagates through the material (see review in Truesdell and Toupin (1960), Chapter C). This approach, developed mainly by Christoffel (1877), Hugoniot (1885) and Hadamard (1903), has the advantage that apart from assumptions about conditions ahead of the singular surface, only conditions across the surface itself need be considered. One implication of this is that it is not necessary to solve initial-boundary value problems involving the equations of motion. The power and elegance of the method is referred to in the review article of waves in solids by Chen (1973), who notes for instance that the method is able to illuminate common features of the behaviour of waves in various types of materials without having to appeal to any explicit representation for the constitutive relations of the materials.

We focus attention on acceleration waves, and their behaviour in thermoelastic media. An acceleration wave is an example of a singular surface across which the motion, velocity and deformation gradient are continuous, and for which the acceleration and second and third derivatives of the motion have finite discontinuities across the singular surface, but are continuous everywhere else (Chen (1973); see also Truesdell and Noll (1965), Section 71)).

Truesdell (1961) discussed the purely <u>mechanical</u> theory of acceleration waves in elastic media and obtained the Fresnel-Hadamard theorem, which requires the acceleration amplitude of a wave to be a proper vector of a second order tensor called the acoustic tensor. The speed of propagation of the wave is then the square root of the corresponding proper number, which must be real and positive for the wave to exist. Although the Fresnel-Hadamard theorem determines the direction of the acceleration amplitude(s) corresponding to a given direction of propagation, it is unable to predict the magnitude of the acceleration amplitude. W.A. Green (1964,5) however, obtained a differential equation for the magnitude of the acceleration amplitude in the case of plane acceleration waves in homogeneously deformed isotropic materials; the magnitude was shown either to grow to infinity within a finite time t, decay to zero as $t \to \infty$, or to remain constant, depending on conditions ahead of the wave. Extensions of this result have been made by (amongst others) Chen (1968a,b), who considered acceleration waves of arbitrary form, Chadwick and Ogden (1971a), who removed the restriction of isotropy, and by Bowen and Wang (1970), who discussed acceleration waves of arbitrary form in inhomogeneous isotropic bodies. Eringen and Suhubi (1975) reviewed acceleration wave propagation and growth (including Suhubi's results (1970) for hyperelastic materials) in purely mechanical elastic media.

We now consider the behaviour of acceleration waves in thermoelastic media. Truesdell (1961) (after Duhem (1903,6)) showed that the Fresnel-Hadamard theorem applies (but with different acoustic tensors) both to materials that conduct heat according to Fourier's law with a positive definite thermal conductivity tensor, and to non-conductors of heat. In the former case such waves are <u>homothermal</u>; that is, the first spatial and temporal derivatives of the temperature across the wavefront are continuous. In the latter, acceleration waves are <u>homentropic</u>; that is, the first spatial and temporal derivatives of the entropy across the wavefront are continuous. Chen (1968c) used the concept of a definite conductor (rather than Fourier's heat conduction law) from the thermodynamical theory of Coleman and Gurtin (1965) (see also Chapter D III of Truesdell and Noll, (1965)) to derive results for propagation and growth of acceleration waves in both definite and non-conductors of heat; in each case, the material was assumed to be isotropic and homogeneously deformed. Bowen and Wang (1971) extended their earlier results for inhomogeneous isotropic elastic materials by considering thermodynamic influences and including internal state variables. Chadwick and Currie (1972) removed the restriction of isotropy; furthermore, they showed that heat-conductors are more appropriately classified by the heat flux vector than by the thermal conductivity tensor. They defined normal and anomalous conductors by considering the conditions under which the dissipation inequality is satisfied as an equality, and derived results for the propagation and growth of acceleration waves for normal, anomalous and non-conductors.

The question of wave propagation and growth in elastic materials subject to one or more <u>internal constraints</u> has also been considered by various authors. Truesdell and Noll (1965) discussed the propagation of waves in incompressible materials, and referred to the work of Ericksen (1953), who restricted attention to isotropic hyperelastic materials. Ogden (1974) considered both the propagation <u>and</u> growth equations for waves in incompressible media, and presented a detailed investigation of the case of plane waves in homogeneously deformed materials. Scott (1975) developed a theory of arbitrarily constrained elastic materials, adopting the ray-theory approach. The same author applied this theory (Scott (1976)) to the propagation and growth of waves in incompressible elastic solids for plane waves and certain cylindrical waves. Chen and Gurtin (1974) considered wave propagation in inextensible elastic bodies; these results were extended by Chen and Nunziato (1975) to include the constraint of perfect conductivity in the fibre directions, but in neither case was wave growth treated. Borejko and Chadwick (1980) investigated energy relations for arbitrarily constrained elastic materials, and Whitworth (1982) considered the related problem of the behaviour of simple waves in such materials. Whitworth's contribution, and that of Whitworth and Chadwick (1984) on surface waves, are unusual in that they include discussion of the situation in which a set of vectors associated with the constraints is linearly dependent. This situation is usually ignored but is easily encountered in practice, especially in the case of material isotropy.

We turn finally to the behaviour of acceleration waves in thermoelastic materials which are subject to arbitrary thermomechanical constraints. Reddy (1984) (hereafter referred to as (I)) has developed a theory of constrained elastic materials and derived propagation and growth equations for acceleration waves in such materials. The thermodynamic theory presented in (I) incorporates features of the earlier theories developed by Green, Naghdi and Trapp (1970) and Gurtin and Podio-Guidugli (1973). These theories are distinguished by the fact that the constraints make no contribution to the production of entropy, and that they reduce for purely mechanical constraints to the theory of Noll (see Truesdell and Noll, (1965)); they are therefore to be preferred to the more restrictive theory of Andreussi and Podio-

Guidugli (1973) which does not reduce to Noll's formulation (see review in (I)). Reddy revised the general formulation of the constraint equation presented by Trapp (1971) by restating it in the two alternative forms in which it is generally found in practice. Constraints that can be represented as scalar-valued functions of the deformation gradient and temperature are called <u>type I constraints</u>, and those that can be represented as vector-valued functions of the deformation gradient and temperature are called <u>type II constraints</u>. (Examples of type I constraints are temperature-dependent compressibility and temperature-dependent extensibility in a given direction, and an example of a type II constraint is temperature-dependent conductivity in a given direction). Reddy defines an augmented free energy function which incorporates the contributions of the type I constraints, and uses the augmented free energy function to rephrase the constitutive equations in a particularly concise form. Advantages of this approach are apparent both in the discussion of waves in definite conductors, where temperature is used as an independent variable, and of waves in non-conductors, where entropy is used in place of temperature as an independent variable. It is shown that for constrained materials, every acceleration wave in a definite conductor is homothermal and every acceleration wave in a non-conductor is homentropic, generalizing an earlier result of Coleman and Gurtin (1965) for unconstrained materials. Reddy derives necessary conditions to be satisfied by the constraints when either homothermal or non-homothermal waves are present and then shows that the propagation conditions for homothermal and homentropic waves are both of Fresnel-Hadamard type. Finally, attention is restricted to plane waves propagating into static homogeneously deformed regions and the growth equations for homothermal and homentropic waves are derived; then results show similarities with the corresponding unconstrained results derived by Chadwick and Currie (1972). In a subsequent paper (hereafter referred to as (II)), Reddy (1985) has made a preliminary investigation of the consequences of material isotropy for the theory developed in (I). Constraints are classified as isotropic if the scalar constraint equation is a function of temperature and of the principal stretches only, or as directional if the constraint equation is a function of temperature, the principal stretches, and of scalar invariants of vectors which endow the constraints with preferred directions. The propagation conditions for homothermal and homentropic principal waves are derived and, unlike (I), linear dependence of a set of vectors associated with the constraints is considered. The growth equation however, is not treated. A final noteworthy feature of (I) and (II) is that all variables are treated as functions referred to the reference configuration, which is taken to be the undeformed configuration. Such an approach, based on material coordinates, is generally physically meaningless, but an exception is the case when the region ahead of the wave is at rest (as assumed in (I) and (II)), and the approach then makes for a less cluttered analysis.

Aim of the thesis

The aim of this contribution is to extend the analysis presented in (I) and (II); in a sense we re-examine and extend the analysis in (I) but with the restriction of isotropy imposed as in the preliminary study made in (II). We impose a further restriction by considering only a subset of the type I (scalar) constraints investigated in (I); this restriction is however broad enough to encompass most constraints likely to be encountered in practice. We focus attention on each of the following topics in turn:

(i) the theory of thermodynamically constrained materials developed in (I), and its immediate consequences for the nature of acceleration waves in both definite and non-conductors;

(ii) the propagation conditions for homothermal and homentropic acceleration waves in definite conductors and non-conductors respectively;

(iii) the growth equations and solutions for homothermal and homentropic waves.

We now comment on the way in which the topics (i) - (iii) are approached in the thesis and what is achieved. A detailed description of the contents of Chapters 2-9 is not presented here (since that would essentially involve a repetition of the introduction given for each chapter), but rather a brief appraisal is given of what the investigation has yielded by way of new results, insight into the topic, and the delineation of areas worthy of further study.

(i) In Chapter 2 the theory of thermodynamically constrained materials is presented in a form that obeys the principle of material frame-indifference and the restriction of isotropy. The type I (scalar) constraints of (I) are presented with their directionality made explicit as in (II); this directionality characterizes any anisotropy due to the constraints. Constraints for which such directionality is absent (resp. present) are termed <u>isotropic</u> (resp. <u>directional</u>), and particular examples of these are introduced, namely temperature-dependent compressibility (isotropic), temperature-dependent extensibility in each of two orthogonal directions, and temperature-dependent shearing (all directions).

A new definition for constrained materials of a non-conductor of heat is adopted here in preference to that given in (I). There, a material is regarded as a non-conductor if the heat flux vector is identically zero, and we note that the heat flux vector for constrained materials contains contributions from the type II constraints. In the revised definition of a non-conductor adopted here, these contributions from the type II constraints are unrestricted. So for instance, a material that has zero heat flux everywhere except for non-zero heat flow through perfectly-conducting fibres imbedded in it would be regarded as a (constrained) non-conductor in the new definition, irrespective of the heat flow through the fibres. In consequence, acceleration waves in non-conductors are no longer homentropic in general, but are so if type II constraints are absent, or if all type II constraints present are orthogonal to the wave normal (see Chapter 3).

A central issue raised in the discussion in Chapters 3, 5 and 8 of homentropic waves is the question of whether to work in the <u>thermal formulation</u> of the constitutive equations, employing temperature as an independent variable (as is done for homothermal waves), or whether to adopt the <u>entropic formulation</u>, with entropy as an independent variable as is usually done in work on homentropic waves in unconstrained non-conductors (and also utilized by Reddy in (I), (II)). The entropic formulation has the advantage of conciseness for homentropic waves. However, the definition of type I and II constraints employs temperature and not entropy as an independent variable (since this is the way in which constraint properties are usually determined experimentally) and as a result the influence of the constraints cannot be as easily separated out in the entropic formulation as in the thermal formulation, so that valuable simplifications can be obscured. These problems would fall away if constraints were specified in terms of entropy, in which case the homentropic treatment would be a close parallel of the thermal one. We choose to employ the entropic formulation when brevity is an asset (as in the derivations of the propagation conditions and the growth equations), and turn to the thermal formulation in order to facilitate comparison with homothermal results. In situations where waves are both homothermal and homentropic (i.e. are generalized transverse waves, as defined by Chadwick and Currie (1972,4)), we choose the thermal formulation.

We extend in Chapter 2 the analyses of (I) and (II) by considering a set of N type I constraints for which the associated set of constraint vectors is not necessarily linearly independent. A linearly independent subset is isolated and labelled $1, \ldots, M$, so that $M \leq N$. We then have constraints $M+1, \ldots, P$ whose constraint vectors are non-zero and expressible as linear combinations of the vectors making up the linearly independent subset, and constraints P+1, ..., N whose constraint vectors are all zero.

The corresponding entropic formulation of the type I constraint vectors is also presented in Chapter 2, but it is noted that the constraint vectors corresponding to the constraints 1, ..., M are <u>not</u> necessarily linearly independent, unlike the situation in the thermal formulation. An alternative set of constraint vectors is accordingly introduced in Chapter 5 that overcomes this difficulty in the entropic formulation.

(ii) The propagation conditions for homothermal and homentropic principal waves that are longitudinal or transverse are derived in Chapter 4 and 5 respectively. In both cases, we are able to extend the treatment in (I) and (II) to include the possibility of type I constraints with linearly dependent constraint vectors, for which case $M < N$. (Note that constraints whose constraint vectors are <u>collinear</u> are also treated in (II)). For both homothermal and homentropic waves, we obtain a propagation condition of Fresnel-Hadamard type, with a symmetric acoustic tensor in each case.

A second propagation condition is obtained in the two situations by evaluating the jump of the time derivative of the type I constraint definition across the wavefront; unlike the first propagation condition, there is no equivalent of this condition for unconstrained waves.

For homothermal waves (for which the second propagation condition yields the restriction $M \leq 2$), we discuss longitudinal and transverse principal wave solutions for the three possibilities $M = 0,1,2$. In particular we discuss the influence of directional and isotropic constraints on both the strong ellipticity condition for the acoustic tensor and on the speed of propagation. It is found that directional constraints which allow longitudinal waves (these are not possible if isotropic constraints are present) have <u>no effect</u> on the wave speed.

We then consider plane, cylindrically symmetric and spherically symmetric waves in materials subject to irrotational plane, cylindrical and spherically symmetric deformations respectively. We then use both the definitions of the constraints as well as the second propagation condition in a detailed study of the restrictions (if any) placed on the deformations by the constraints, acting singly or in combination. It is found that the deformation is often restricted to either homogeneous deformation or uniform dilatation. These homothermal results are illustrated using the four constraint examples mentioned in (i) acting singly or in combinations of two, three or four; it is seen that linear dependence of the constraints is a common occurrence.

We now turn to a central problem of this investigation, which is revealed in the derivation of the first propagation condition, although it does not cause a difficulty there and is accordingly only discussed in Chapter 6. Each constraint has an associated scalar multiplier (such as the arbitrary hydrostatic pressure in the constraint of incompressibility), and the jump of the time derivative of these multipliers for the type I constraints 1, ..., P appears in the derivation. Although an expression is obtained for the sum of these jumps, expressions for the individual jumps are obtainable in the event that only the constraints 1, ..., M whose constraint vectors are linearly independent are present. No information is obtainable for the jumps corresponding to the type I constraints P+1, ..., N . This has ramifications for the derivation of the growth equation in Chapter 6.

The derivation of the first propagation condition for homentropic waves is given in Chapter 5 in the entropic formulation and, as in the homothermal case, expressions for the jumps in the time derivatives of the scalar multipliers are individually obtainable if only the constraints 1, ..., M are present, and obtainable as a sum if constraints 1,...,P are present. (The details of the derivation of these results differs, however, from the homothermal case). Nevertheless we are able to obtain the first propagation condition and discuss results for longitudinal and transverse principal waves for M = 0,1,2,3. It is noted that constraints are often required to be mechanical, and in some cases, this leads to the result that transverse waves are necessarily both homothermal and homentropic. They are therefore examples of generalized transverse waves and are most conveniently treated using a thermal formulation. Lastly, in a discussion of the influence of isotropic and directional constraints on the propagation conditions, we demonstrate the use of the thermal formulation of the propagation conditions for homentropic waves; this facilitates comparison (for $M \leq 2$) with the homothermal results and displays more easily the influence of the constraints on the solution.

(iii) The derivation of the growth equation for homothermal longitudinal and transverse waves is given in Chapter 6, assuming the material ahead of the wave to be at rest and at constant temperature. We remove the restriction of homogeneous deformation adopted in (I) (except where this is required by the presence of a particular constraint) and present results for plane, cylindrical and spherical waves in materials subject to the plane, cylindrical and spherically symmetric deformations specified at the end of Chapter 4. (Only plane waves are treated in (I)). We are able to obtain the growth equation for <u>longitudinal</u> principal waves in the presence of the thermomechanical constraints 1,...,N if type II constraints are present and at least one type II constraint is <u>not</u> orthogonal to the wave normal. If type II constraints are <u>absent</u>, then the difficulty of evaluating the jumps discussed in (ii) forces us to restrict attention to the following situations:

if the constraints M+1, ..., P are present, then <u>all</u> constraints present must be mechanical;

if the constraints P+1, ..., N are present, these particular constraints must always be mechanical;

if the constraints 1, ..., M are present, these may be thermomechanical (as long as constraints M+1, ..., P are absent).

For transverse waves, evaluation of these particular jumps is not required and consequently only a relatively minor restriction is placed on the thermomechanical behaviour of the constraints M+1, ..., P . In all cases the analysis represents an improvement on (I), where only constraints for which the associated vectors are linearly independent are considered.

Solutions to the growth equations for longitudinal and transverse principal waves are shown in Chapter 7 to be a Bernoulli and linear first order equation respectively, in close analogy with corresponding results in (I) and in earlier investigations for materials subject to mechanical constraints or unconstrained. The analysis of the Bernoulli equation in the context of acceleration waves (see review by Chen (1973)) is appropriate for longitudinal waves and is briefly presented. We are particularly concerned in ascertaining the general nature of the constraint influence on both the longitudinal wave and transverse wave solutions; for longitudinal waves, type I constraints (which must be directional) affect only the curvature terms for cylindrical and spherical waves. For transverse waves, isotropic constraints are permitted, and directional constraints influence both the wave speed and the curvature terms. For homogeneous deformation, plane waves have constant amplitude and are <u>independent</u> of the constraints. In certain circumstances the growth of cylindrical waves in homogeneously deformed media and spherical waves in a situation of uniform dilatation is also independent of constraint influence, and of material properties.

We present in Chapter 8 a derivation of the <u>homentropic</u> growth equation and its solution for longitudinal and transverse principal waves travelling in media that are at rest and in a state of constant entropy ahead of the wave. No restriction to homogeneous deformation or to plane waves is made, as in (I), and we are able to find an alternative to the method employed in (I) to remove terms involving a particular higher-order jump. As usual, we do not initially restrict attention to the type I constraints 1, ..., M as in (I), but we find that the growth equation involves many terms containing jumps in the time derivative of the type I constraint multipliers. A detailed discussion of the conditions under which these jumps can be evaluated is made for M = 0,1,2,3 using results from Chapter 5; we find that although no solutions are obtainable if the constraints M+1, ..., P occur, solutions are always obtainable if the linearly independent constraints 1,...,M are present and, in certain cases, if the set P+1, ..., N is present (as in for instance, the case M = 0, when these are the only constraints present). In some situations it is found that homentropic waves are necessarily homothermal as well, and are therefore generalized transverse waves. They are most easily treated by employing the thermal formulation of Chapters 6 and 7 instead.

A similar difficulty is found regarding the type II constraint multipliers, which (unlike the homothermal case) occur in the homentropic growth equation for curved wavefronts. By restricting the set of type II constraints under consideration, we are able to remove the difficulty; this restriction is not so severe as to exclude the type II constraint examples of either perfect conductivity or perfect conductivity in a particular direction, as considered by Gurtin and Podio-Guidugli (1973). It should be noted however, that this problem with the type II constraints does not arise in (I), due to the different definition of a non-conductor adopted there.

Use of the above results leads to the growth equations for longitudinal and transverse waves, both of which are seen to be Bernoulli equations. The analysis given in Chapter 7 of the solution applies and further discussion is restricted to the nature of the influences of the constraints on the solution. The equations are more cumbersome than their homothermal counterparts, partly because the jumps in the constraint parameter derivatives are now non-zero in general, and also because the type I constraints are present in many terms. Because of this, we do not present a discussion of solutions for particular wavefronts or constraints (although it is clearly possible to obtain such solutions) for the case $M \geq 1$; we conclude the chapter though by showing that for $M = 0$, the longitudinal growth equation is considerably simplified (transverse waves are both homentropic and homothermal and are therefore not treated here), and we obtain the result that the growth of plane waves and spherical waves is <u>not</u> influenced by the constraints. This result also holds for cylindrical waves if the (directional) constraints present obey a particular criterion.

We note finally that the subject of Appendix A is the derivation of the fourth- and sixth-order tensors of elastic moduli when isotropic and directional constraints are present. We employ a method due to Durban (1978) for the differentiation of tensor functions: this method is considerably simpler than those of Chadwick and Ogden (1971a,b) and Bowen and Wang (1970,2) (as used in (II) for the fourth-order tensor case) when directional constraints are present. Appendix B is devoted to a derivation of the physical components of the gradient of the deformation tensor.

In this discussion of the thesis, for the sake of brevity few references have been given to earlier work other than that of (I); many results obtained by the authors mentioned in the review given earlier are retrieved as special cases of the present work.

A slightly earlier version of the work on homothermal waves in definite conductors presented in Chapters 2-4, 6-7 of this thesis has been published (Bleach and Reddy (1987)).

<u>Notation</u>

The space of tensors of rank m is denoted by $\mathcal{T}$: the special cases of scalars (m=0) and vectors (m=1) are denoted by $\mathbb{R}$ and $\mathbb{V}$, respectively. We seldom need to be specific about the smoothness properties of <u>fields</u> of tensors defined on the unbounded domain Ω and the time interval $[0,T]$, so by a corruption of notation we employ the symbol $\mathcal{T}^m$ to denote also the space of tensors of rank m , whose components are functions defined on $\Omega \times [0,T]$.

A tensor $\mathbf{T}$ of rank m is an m-linear function on $V^m \equiv V\times \ldots \times V$ (m times). For any $\mathbf{T} = \mathbf{u} \otimes \mathbf{v} \otimes \ldots$ (m times) $\in \mathcal{T}^m$ we define the m-linear map by

$$T : V^m \to R \quad , \quad \mathbf{u} \otimes \mathbf{v} \otimes \ldots\ldots\ldots(\mathbf{a},\mathbf{b},\ldots) = (\mathbf{u}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{b})\ldots$$

Relative to an arbitrary basis $\mathbf{g}_i$ we than have $\mathbf{T} = T^{ij\cdots}\ \mathbf{g}_i \otimes \mathbf{g}_j \otimes \ldots$ where the contravariant components $T^{ij\cdots}$ of $\mathbf{T}$ are found from

$$T^{ij\cdots} = \mathbf{T}(\mathbf{g}^i,\mathbf{g}^j,\ldots) \text{ (m indices)}$$

$\mathbf{g}^i$ being the reciprocal basis of $\mathbf{g}_i$. We also employ covariant components $T_{ij\ldots}$ relative to the basis $\mathbf{g}_i$, and mixed components $T^{ij\ldots}_{k\ell\ldots}$ relative to combinations of $\mathbf{g}^i$, $\mathbf{g}_k$. Tensors are also conveniently regarded as linear vector- or tensor-valued maps. For example, the second-rank tensor $\mathbf{u} \otimes \mathbf{v}$ and the fourth-rank tensor $\mathbf{S} = \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \otimes \mathbf{z}$ may be defined by

$$T : V \to V \quad , \quad (\mathbf{u} \otimes \mathbf{v})(\mathbf{a}) = \mathbf{u}(\mathbf{v}\cdot\mathbf{a}) \quad ,$$

$$\mathbf{S} : \mathcal{T}^2 \to \mathcal{T}^2 \quad , \quad \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \otimes \mathbf{z}(\mathbf{a} \otimes \mathbf{b}) = (\mathbf{w}\cdot\mathbf{a})(\mathbf{z}\cdot\mathbf{b})\mathbf{u} \otimes \mathbf{v} \quad .$$

We employ both definitions of tensors, making clear if necessary which representation is being used.

Summation convention

We employ the following form of the summation convention for vector and tensor indices (Roman miniscules are summed over 1,2,3; Greek majuscules over 1,2):

(i) if an index appears on both sides of an equation it is not summed;

(ii) indices that occur twice in an expression on only one side of an equation are summed;

(iii) indices occurring more than twice in an expression have the summation explicitly shown, where such summation is appropriate.

So, for example, in equation (2.3) no summation is implied over i or j , but in (2.4), summation is implied over both i and j :

$$x^i = \bar{x}^i(X^j, t) \quad , \qquad (2.13\ \text{bis})$$

$$\mathbf{F} = \text{Grad}\ \bar{x} = \frac{\partial \bar{x}^i}{\partial X^j}\, g_i \otimes G^j \quad . \qquad (2.46\ \text{bis})$$

In (2.7) however, the summation is explicitly shown:

$$\mathbf{F} = \sum_i a_i\, \mathbf{q}_i \otimes \mathbf{p}_i \quad . \qquad (2.7\ \text{bis})$$

The summation convention presented above is also used for the indices (Greek miniscules) labelling the type I and type II constraints, so that in (2.45) and $(2.46)_4$ for example, we sum over $\alpha = 1, \ldots, N$ and $\beta = 1, \ldots, L$ respectively:

$$\psi = \psi^0(\mathbf{F}, \theta) + \lambda_\alpha \phi^\alpha(\mathbf{F}, \theta, \mathbf{e}_A) \quad , \qquad (2.45 \text{ bis})$$

$$\mathbf{q} = \mathbf{q}^0(\mathbf{F}, \theta, \text{Grad } \theta) + \gamma_\beta \mathbf{z}^\beta(\mathbf{F}, \theta, \mathbf{e}_A) \quad . \qquad ((2.46)_4 \text{ bis})$$

Finally, we also use the above convention for the indices (Roman majuscules) labelling the vectors $\mathbf{e}_A$ that characterize the directionality of the constraints, so that, for example, we sum over A, B in

$$c^\alpha = \beta^\alpha_{AB} \sum_i a_i(\mathbf{p}_i \cdot \mathbf{e}_A)(\mathbf{n} \cdot \mathbf{e}_B)q_i \quad , \qquad (2.89 \text{ bis})$$

where clearly we also sum over i but not over α in terms of the summation convention. We note finally that case (iii) above only occurs in what follows for the vector and tensor indices i,j, ..., and not the constraint indices α,β, .. or A,B, ...

CHAPTER 2

CONSTITUTIVE EQUATIONS FOR CONSTRAINED ISOTROPIC MATERIALS

We review the description of a body and its motion, and describe the theory of unconstrained thermoelastic materials in isotropic media, following the approach of Coleman and Noll (1963), Chadwick and Seet (1971) and Gurtin (1974). Attention is then focussed on constitutive theories for elastic materials subject to internal thermoelastic constraints. The theories of Green, Naghdi and Trapp (1970) and of Gurtin and Podio-Guidugli (1973) are considered and then the alternative theory proposed by Reddy in (I) is dealt with in detail. It is presented here in the form appropriate to isotropic materials, taking into account the work of Reddy in (II) and with further modifications. As in (I), we classify the constraints as being of type I or type II according to whether they are defined by a scalar- or vector-valued function respectively. Attention is restricted here to a particular subset of type I constraints which is broad enough to cover those constraints commonly found in practice.

We follow (I) and consider constrained materials that are either <u>definite</u> or <u>non-conductors</u>, but adopt a different definition of a non-conductor for constrained materials to that given by Reddy in (I). It will often (though not invariably) prove convenient when considering wave behaviour in non-conductors to employ entropy rather than temperature as an independent variable in the constitutive equations and this form of the constitutive equations (the entropic formulation) is accordingly presented here.

The chapter concludes with the definition of a constraint vector associated with each of the type I (scalar) constraints. We extend the treatments in (I,II) by allowing the set of constraint vectors to be either fully active (if the set of constraint vectors is linearly independent) or partially active (the constraint vectors form a linearly dependent set) or inactive (each constraint vector has the value zero). The corresponding set of constraint vectors in the entropic formulation is also introduced, but it is noted that a further modification of this set (presented in Chapter 5) proves to be more convenient when the entropic formulation is to be employed.

Description of the motion

We consider a body B and identify the position of each particle of B by its position vector $\mathbf{X}$ in a fixed and undeformed reference configuration at time t=0 . The subsequent position at time t of a particle initially located at $\mathbf{X}$ is found from the motion $\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X},t)$, where $\mathbf{x}(\mathbf{X},0) = \mathbf{X}$. For each value of t , the function $\bar{\mathbf{x}}(\mathbf{X},t)$ is invertible, so that $\mathbf{X} = \bar{\mathbf{x}}^{-1}(\mathbf{x},t)$ where $\bar{\mathbf{x}}^{-1}$ is the inverse of $\bar{\mathbf{x}}$.

An arbitrary set of orthogonal curvilinear coordinates X^i (i = 1,2,3) is chosen and the position vector $\mathbf{X}$ of particles in the reference configuration is a function of these coordinates, that is,

$\mathbf{X} = \bar{\mathbf{X}}(X^i)$. The tangent basis vectors $\mathbf{G}_i$ are thus given by

$$\mathbf{G}_i = \frac{\partial \bar{\mathbf{X}}}{\partial X^i} \quad . \tag{2.1}$$

The dual basis vectors $\mathbf{G}^i$ are uniquely defined by

$$\mathbf{G}_i \cdot \mathbf{G}^j = \delta_i^j \quad ,$$

δ_i^j being the Kronecker delta,

and components of the metric tensor G_{ij} and its inverse G^{ij} relative to $\{\mathbf{G}_i\}$ and $\{\mathbf{G}^i\}$ respectively are

$$G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j \quad , \quad G^{ij} = \mathbf{G}^i \cdot \mathbf{G}^j \quad .$$

A co-ordinate system $\{x^i\}$, generally distinct from $\{X^i\}$, is used to locate particles in the <u>current configuration</u>, so that

$$\mathbf{x} = \bar{\mathbf{x}}(x^i) \quad .$$

The corresponding tangent basis vectors $\mathbf{g}_i$, dual basis vectors $\mathbf{g}^i$, metric tensor components g_{ij} and its inverse g^{ij} are defined respectively by

$$\mathbf{g}_i = \frac{\partial \bar{\mathbf{x}}}{\partial x^i} \quad ,$$

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j \quad ,$$

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \quad , \quad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j \quad . \tag{2.2}$$

The motion of the body can be described alternatively by the set of functions

$$x^i = \bar{x}^i(X^j, t) \quad . \tag{2.3}$$

Any field variable χ , say, associated with the body may be represented either in the <u>spatial form</u>

$$\chi = \bar{\chi}(\mathbf{x}, t) \quad \text{or} \quad \hat{\chi}(x^i, t) \quad ,$$

or using (2.3), it may be represented in the <u>material form</u> (Truesdell and Noll, (1965), Section 66)

$$\chi = \bar{\bar{\chi}}(\mathbf{X}, t) \quad \text{or} \quad \hat{\hat{\chi}}(X^i, t) \quad .$$

The deformation gradient $\mathbf{F}$ is defined by

$$\mathbf{F} = \text{Grad}\ \bar{\mathbf{x}} = \frac{\partial \bar{x}^i}{\partial X^j}\, \mathbf{g}_i \otimes \mathbf{G}^j \quad . \tag{2.4}$$

According to the polar decomposition theorem, $\mathbf{F}$ can be written as

$$\mathbf{F} = \mathbf{R}\,\mathbf{U} = \mathbf{V}\,\mathbf{R} \quad , \tag{2.5}$$

where $\mathbf{R}$ is a proper orthogonal tensor known as the rotation tensor and $\mathbf{U}$ and $\mathbf{V}$ are the (positive definite) symmetric right and left stretch tensors respectively. The stretch tensors have the spectral representations (see for instance Chadwick (1976)):

$$\mathbf{U} = \sum_i a_i \, \mathbf{p}_i \otimes \mathbf{p}_i \quad , \quad \mathbf{V} = \sum_i a_i \, \mathbf{q}_i \otimes \mathbf{q}_i \quad ,$$

where a_i are proper numbers of $\mathbf{U}$ and $\mathbf{V}$ and are known as the principal stretches; the triad of proper vectors $\{\mathbf{p}_i\}$ defines locally a set of principal axes in the <u>reference</u> configuration and the corresponding triad $\{\mathbf{q}_i\}$ defines locally a set of principal axes in the <u>current</u> configuration. These two sets of principal axes will be used extensively later; they are related by

$$\mathbf{q}_i = \mathbf{R} \, \mathbf{p}_i \quad . \tag{2.6}$$

The rotation tensor $\mathbf{R}$ is expressible as

$$\mathbf{R} = \mathbf{q}_i \otimes \mathbf{p}_i$$

and so we can write the deformation gradient $\mathbf{F}$ as

$$\mathbf{F} = \sum_i a_i \, \mathbf{q}_i \otimes \mathbf{p}_i \quad . \tag{2.7}$$

The (symmetric) right and left Cauchy-Green strain tensors are defined respectively by

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \quad , \quad \mathbf{B} = \mathbf{F}\,\mathbf{F}^T \quad ; \tag{2.8}$$

here and henceforth a superscript T denotes the transpose. Using (2.7) we find that $\mathbf{C}$ and $\mathbf{B}$ have the spectral representations

$$\mathbf{C} = \mathbf{U}^2 = \sum_i a_i^2\, \mathbf{p}_i \otimes \mathbf{p}_i \quad , \quad \mathbf{B} = \mathbf{V}^2 = \sum_i a_i^2\, \mathbf{q}_i \otimes \mathbf{q}_i \quad .$$

Thermodynamics of unconstrained elastic materials

We characterize a thermodynamic process (see Chadwick and Seet (1971), Gurtin (1974)) by the motion $\mathbf{x}$, the temperature θ, the free energy ψ, the first Piola-Kirchhoff stress tensor $\mathbf{S}$, the specific entropy η, the referential heat flux vector $\mathbf{q}$, the external body force $\mathbf{b}$ and the rate of heat supply r. The local form of the laws of balance of linear and angular momentum and of energy are respectively

$$\mathrm{Div}\, \mathbf{S} + \rho\, \mathbf{b} = \rho\, \ddot{\mathbf{x}} \quad , \tag{2.9}$$

$$\mathbf{S}\, \mathbf{F}^T = \mathbf{F}\, \mathbf{S}^T \quad , \tag{2.10}$$

$$-\, \rho(\dot{\psi} + \eta\, \dot{\theta} + \dot{\eta}\, \theta) + \mathbf{S}\cdot\dot{\mathbf{F}} - \mathrm{Div}\, \mathbf{q} + \rho\, r = 0 \quad , \tag{2.11}$$

where ρ is the density in the reference configuration, Div is the divergence operator relative to $\mathbf{X}$ and a superimposed dot denotes differentiation with respect to time holding $\mathbf{X}$ constant. The corresponding local form of the entropy production inequality is

$$-\, \rho(\dot{\psi} + \eta\, \dot{\theta}) + \mathbf{S}\cdot\dot{\mathbf{F}} - \theta^{-1}\, \mathbf{q}\cdot\mathrm{Grad}\, \theta \geq 0 \quad . \tag{2.12}$$

The thermodynamic processes that the material can undergo are restricted by the set of constitutive equations for that material. For the case of an <u>unconstrained</u> thermoelastic material, the equations are assumed to have the form

$$\psi = \psi^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) \quad , \quad \mathbf{S} = \mathbf{S}^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) \ ,$$

$$\eta = \eta^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) \quad , \quad \mathbf{q} = \mathbf{q}^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) \ . \qquad (2.13)$$

However, every admissible thermodynamic process must satisfy the reduced dissipation inequality (2.12). Necessary and sufficient conditions that this is so for an unconstrained thermoelastic material are firstly that ψ^0, $\mathbf{S}^0$ and η^0 be independent of Grad θ , i.e.

$$\psi = \psi^0(\mathbf{F},\ \theta) \ , \ \mathbf{S} = \mathbf{S}^0(\mathbf{F},\ \theta) \ , \ \eta = \eta^0(\mathbf{F},\ \theta) \quad ; \qquad (2.14)$$

secondly that ψ^0 determines $\mathbf{S}$ and η by

$$\mathbf{S} = \rho\,\frac{\partial\psi^0}{\partial\mathbf{F}} \quad , \qquad (2.15)$$

$$\eta = -\,\frac{\partial\psi^0}{\partial\theta} \quad ; \qquad (2.16)$$

and finally that $\mathbf{q}^0$ obeys the heat conduction inequality

$$\mathbf{q}^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta)\cdot\mathrm{Grad}\ \theta \le 0 \quad . \qquad (2.17)$$

These results were obtained for the general case with mutual body forces present by Gurtin and Williams (1971); see also the review by

Gurtin (1974). We consider throughout only the influence of external body forces, however.

Some consequences of (2.14) - (2.17) are as follows. The energy equation (2.11) takes the form

$$\rho\ \theta\ \dot{\eta} = -\ \mathrm{Div}\ \mathbf{q} + \rho\ r \tag{2.18}$$

and the stress $\mathbf{S}$ and entropy η obey the Maxwell relation

$$\frac{\partial \mathbf{S}^0}{\partial \theta} = -\ \rho\ \frac{\partial \eta^0}{\partial \mathbf{F}} \quad . \tag{2.19}$$

The absence of a piezo-caloric effect is made manifest through the fact that $\mathbf{q}$ must obey

$$\mathbf{q}^0(\mathbf{F},\ \theta,\ 0) = 0 \tag{2.20}$$

and finally the thermal conductivity tensor $\mathbf{K}$, defined by

$$\mathbf{K} = \left.\frac{-\ \partial \mathbf{q}^0}{\partial(\mathrm{Grad}\ \theta)}\right|_{\mathrm{Grad}\ \theta\ =\ 0} \quad , \tag{2.21}$$

is positive semi-definite, i.e. $\mathbf{v}\cdot\mathbf{K}\ \mathbf{v} \geq 0$ for all vectors $\mathbf{v}\neq 0$.

Further restrictions on the constitutive equations arise as a result of the principle of <u>material frame-indifference</u> of the observer, that is, invariance under every transformation of the form

$$\mathbf{x} \rightarrow \mathbf{Q}\ \mathbf{x} + \mathbf{c} \quad , \tag{2.22}$$

where the orthogonal tensor $\mathbf{Q}$ and vector $\mathbf{c}$ are in general time-dependent. Under a change of observer as expressed by (2.22), the variables characterizing the process transform as (see Gurtin (1974)) :

$$\psi \rightarrow \psi \quad , \quad \theta \rightarrow \theta \quad , \quad \eta \rightarrow \eta \quad ,$$

$$\mathbf{q} \rightarrow \mathbf{q} \quad , \quad \mathbf{b} \rightarrow \mathbf{Q}\,\mathbf{b} \quad ,$$

$$\mathbf{S} \rightarrow \mathbf{Q}\,\mathbf{S} \quad . \tag{2.23}$$

(Note that the heat flux vector is invariant since it is measured per unit area in the reference configuration). In addition, (2.22) and (2.23) imply that $\mathbf{F}$ and Grad θ transform according to

$$\mathbf{F} \rightarrow \mathbf{Q}\,\mathbf{F} \quad , \quad \mathrm{Grad}\ \theta \rightarrow \mathrm{Grad}\ \theta \quad , \tag{2.24}$$

the latter being invariant since it is the gradient relative to the reference configuration.

Material frame-indifference dictates that the functions appearing in (2.14), (2.17) satisfy

$$\psi^{0}(\mathbf{F},\ \theta) = \psi^{0}(\mathbf{Q}\,\mathbf{F},\ \theta) \quad ,$$

$$\mathbf{S}^{0}(\mathbf{F},\ \theta) = \mathbf{Q}^{T}\mathbf{S}^{0}(\mathbf{Q}\,\mathbf{F},\ \theta) \quad ,$$

$$\eta^{0}(\mathbf{F},\ \theta) = \eta^{0}(\mathbf{Q}\,\mathbf{F},\ \theta) \quad ,$$

$$\mathbf{q}^{0}(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) = \mathbf{q}^{0}(\mathbf{Q}\,\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) \quad . \tag{2.25}$$

We make use of the polar decomposition theorem and follow a standard procedure (Carlson (1972)) to obtain the reduced form of the constitutive equations. These are:

$$\psi = \bar{\psi}^0(\mathbf{C}, \theta) \quad ,$$

$$\mathbf{S} = \bar{\mathbf{S}}^0(\mathbf{C}, \theta) \quad ,$$

$$\eta = \bar{\eta}^0(\mathbf{C}, \theta) \quad ,$$

$$\mathbf{q} = \bar{\mathbf{q}}^0(\mathbf{C}, \theta, \mathrm{Grad}\ \theta) \quad , \tag{2.26}$$

where $\mathbf{C}$ is the right Cauchy-Green strain tensor defined in (2.8).

We conclude the discussion of unconstrained thermoelastic materials by imposing the condition of isotropy and finding the appropriate form of the constitutive equations. An unconstrained material is <u>isotropic</u> if it possesses a reference configuration (called an undistorted state) for which the isotropy group of the material contains the full orthogonal group (see for example Chadwick (1976)). We require therefore that the response to a deformation $\mathbf{F}$ from an undistorted state be indistinguishable from the response to a deformation $\mathbf{F}\ \mathbf{Q}$ from that state. The condition of isotropy when applied to $\bar{\psi}^0$ and $\bar{\mathbf{q}}^0$ (see $(2.26)_{1,4}$), yields the following relations

$$\bar{\psi}^0(\mathbf{C}, \theta) = \bar{\psi}^0(\mathbf{Q}\ \mathbf{C}\ \mathbf{Q}^T, \theta)$$

and

$$\mathbf{Q}\ \bar{\mathbf{q}}^0(\mathbf{C}, \theta, \mathrm{Grad}\ \theta) = \bar{\mathbf{q}}^0(\mathbf{Q}\ \mathbf{C}\ \mathbf{Q}^T, \theta, \mathbf{Q}\ \mathrm{Grad}\ \theta) \quad . \tag{2.27}$$

Hence the scalar $\bar{\psi}^0$ and vector $\bar{q}^0$ are <u>isotropic functions</u> (Truesdell and Noll (1965), Sections 8, 47; Chadwick and Seet (1971)). Note that only ψ and $\mathbf{q}$ are considered here, since $\mathbf{S}$ and η are obtainable from ψ by (2.15) and (2.16) respectively.

The representation theorems for isotropic functions given by Truesdell and Noll (1965, Sections 10-13) and by Wang (1969,70) (see also Smith (1970)) can then be used to write $\bar{\psi}^0$ as

$$\psi = \tilde{\psi}^0(\iota_{\mathbf{C}}, \theta) \quad , \tag{2.28}$$

where $\iota_{\mathbf{C}} = (I_1, I_2, I_3)$ is the set of scalar invariants of $\mathbf{C}$: $I_1 = \mathrm{tr}\,\mathbf{C}$, $I_2 = \frac{1}{2}\left\{(\mathrm{tr}\,\mathbf{C})^2 - \mathrm{tr}\,\mathbf{C}^2\right\}$, and $I_3 = \det \mathbf{C}$.

The corresponding representation for $\mathbf{q}$ is (see also Chadwick and Seet (1971))

$$\mathbf{q} = \tilde{\mathbf{q}}^0(\iota_{\mathbf{C}}, \theta, \kappa_1, \kappa_2, \kappa_3) \quad , \tag{2.29}$$

where

$$\kappa_1 = |\mathrm{Grad}\,\theta|, \ \kappa_2 = |\mathbf{F}\,\mathrm{Grad}\,\theta|, \ \kappa_3 = |\mathbf{C}\,\mathrm{Grad}\,\theta| \quad .$$

<u>Thermomechanically constrained elastic materials</u>

The original theory of Noll (Truesdell and Noll (1965), Section 30) for mechanically constrained elastic materials expressed the total stress in a constrained material as the sum of an undetermined reaction plus a determinate stress, and this general approach underlies all the later theories described below. Cohen and Wang (1987) claim that such theories are not entirely satisfactory since the determinate stress is not unique and so should not be used to formulate intrinsic properties or conditions on the material model such as material frame-indifference and material symmetry. They develop a theory for mechanically constrained elastic materials that is independent of the concept of determinate stress, and note that it is theoretically possible that for some constrained materials, a determinate response function that specifies both the conditions of material frame-indifference and material symmetry cannot be found. They are, however, unable to furnish an example of a constraint having this property; indeed, their analysis of commonly used constraints, acting singly and in combination, shows that these constraints do not cause such problems, and accordingly we see no need to abandon the approaches described below.

We develop a constitutive theory for elastic materials subject to a set of internal thermomechanical constraints. The first such theory was that of Green, Naghdi and Trapp (1970), with further contributions by Trapp (1971). They assume the constraint equations to be of the form (here presented in our notation, relative to the reference configuration and incorporating the dependence on $\dot{\theta}$ used by Trapp (1971)):

$$\mathbf{A}\cdot\dot{\mathbf{F}} + \beta\,\dot{\theta} + \mathbf{c}\cdot\mathrm{Grad}\,\theta = 0 \quad . \tag{2.30}$$

Here $\mathbf{A} = \bar{\mathbf{A}}(\mathbf{F},\ \theta)$ is a second order tensor, $\mathbf{c} = \bar{\mathbf{c}}(\mathbf{F},\ \theta)$ is a vector and $\beta = \bar{\beta}(\mathbf{F},\ \theta)$ is a scalar.

They further assume that constraint contributions $\mathbf{S}^c$, η^c, $\mathbf{q}^c$ to the stress, entropy and heat flux respectively are such that there is no entropy production due to the constraints, that is,

$$\mathbf{S}^c \cdot \dot{\mathbf{F}} - \rho\, \eta^c\, \dot{\theta} - \theta^{-1}\, \mathbf{q}^c \cdot \mathrm{Grad}\, \theta = 0 \quad , \tag{2.31}$$

where here and henceforth a superscript c denotes the contribution from the constraints. Both of these assumptions are the appropriate generalizations of the general <u>mechanical</u> theory of internal constraints developed by Noll.

Gurtin and Podio-Guidugli (1973) develop a general thermodynamic theory of constrained materials which is essentially based on constitutive equations of the form

$$\psi = \psi^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) \quad ,$$

$$\mathbf{S} = \mathbf{S}^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) + \mathbf{S}^c \quad ,$$

$$\eta = \eta^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) + \eta^c \quad ,$$

$$\mathbf{q} = \mathbf{q}^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) + \mathbf{q}^c \quad , \tag{2.32}$$

where here and henceforth a superscript zero denotes a function with no explicit constraint dependence, such dependence being contained in the functions with superscript c . (We note, however, that functions such as ψ^0, $\mathbf{S}^0$, η^0, $\mathbf{q}^0$ <u>are</u> influenced by the constraints through the effect of the constraints on $\mathbf{F}$, θ, Grad θ). Gurtin and Podio-Guidugli assume the existence of a reaction set $(\mathbf{S}^c,\ -\ \eta^c,\ -\ \theta^{-1}\ \mathbf{q}^c)$ and use the entropy production inequality (2.12) to obtain as a consequence of their theory the assumption (2.31) of Green, Naghdi and Trapp. Furthermore, Gurtin and Podio-Guidugli are able to strengthen this result to

$$\mathbf{S}^c \cdot \dot{\mathbf{F}} - \rho\, \eta^c\, \dot{\theta} = 0 \quad ,$$

$$\mathbf{q}^c \cdot \mathrm{Grad}\ \theta = 0 \quad . \tag{2.33}$$

Since (2.31) holds, the entropy production inequality (2.12) is independent of the constraints and it reduces to

$$- \rho(\dot{\psi}^o + \eta^o\, \dot{\theta}) + \mathbf{S}^o \cdot \dot{\mathbf{F}} - \theta^{-1}\, \mathbf{q}^o \cdot \mathrm{Grad}\ \theta \geq 0 \quad . \tag{2.34}$$

A third theory, that of Andreussi and Podio-Guidugli (1973), imposes the additional restriction that the constraints make zero contribution to the energy equation. A disadvantage of this restriction is that the resulting theory does not reduce to Noll's formulation for purely mechanical constraints. These authors also add a constraint term to the free energy function ψ , so that

$$\psi = \psi^o(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) + \psi^c \quad . \tag{2.35}$$

Note however that Gurtin and Podio-Guidugli omit the term ψ^c, following Green, Naghdi and Trapp, on the grounds that ψ^c , if included, is eventually found to be constant in every process.

We now develop the constitutive theory for constrained thermoelastic materials proposed by Reddy (I,II). This theory is a development of that of Green, Naghdi and Trapp in the way the constraint equations are defined and exploits the results (2.33) of Gurtin and Podio-Guidugli. Instead of assuming the constraint equations to be of the form (2.30) as proposed by Green, Naghdi and Trapp, Reddy makes the following subdivision. The material is assumed to be subject to N internal constraints of the form

$$\phi^{\alpha}(\mathbf{F},\ \theta,\ \mathbf{e}_A) = 0 \quad , \quad \alpha = 1,2,\ \ldots,\ N \tag{2.36}$$

and L internal constraints of the form

$$\mathbf{z}^{\beta}(\mathbf{F},\ \theta,\ \mathbf{e}_A)\cdot \mathrm{Grad}\ \theta = 0 \quad , \quad \beta = 1,2,\ \ldots,\ L \tag{2.37}$$

where ϕ^{α} and $\mathbf{z}^{\beta}$ are respectively scalar- and vector-valued functions. The vector fields $\mathbf{e}_A$ $(A = 1,2,\ldots)$ are assumed to be time independent and of unit length, so

$$\mathbf{e}_A = \mathbf{e}_A(\mathbf{X}) \quad , \quad |\mathbf{e}_A| = 1 \quad . \tag{2.38}$$

These vectors characterize the <u>directionality</u> of the constraints, as for example in the constraints of temperature-dependent extensibility (which obeys (2.36)) and of perfect conductivity in some direction $\mathbf{e}$ (which obeys (2.37)). Of course, not all the $\mathbf{e}_A$ need appear in each constraint. We will return to the case of particular constraints later, but here it suffices to note that an advantage of the above subdivision is that constraints commonly fall into either of the categories (2.36) or (2.37), rather than appear in the more general form (2.30). Following Reddy, we call constraints expressible in the form (2.36) and (2.37) <u>type I and type II constraints</u> respectively.

The explicit inclusion of possible constraint directionality through the $\mathbf{e}_A$ in the definitions (2.36,7) differs from the original development in (I) - (2.36) though, is used in its present form in (II). The inclusion of such vectors is important here because we shall be concerned with isotropic materials and the vectors $\mathbf{e}_A$ characterize any anisotropy due to the constraints. As we will see, the presence of the $\mathbf{e}_A$ in (2.36,7) does not require substantial modification of the constitutive theory presented in (I).

The set of constitutive equations obeyed by a material subject to type I and type II constraints is given below. This is a slight modification of the set (2.32) proposed by Gurtin and Podio-Guidugli (1973) and is

$$\psi = \psi^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) + \lambda_\alpha\ \bar{\phi}^\alpha(\mathbf{F},\ \theta,\ \mathbf{e}_A) \quad ,$$

$$\mathbf{S} = \mathbf{S}^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) + \lambda_\alpha\ \mathbf{S}^\alpha(\mathbf{F},\ \theta,\ \mathbf{e}_A) \quad ,$$

$$\eta = \eta^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) + \lambda_\alpha\ \eta^\alpha(\mathbf{F},\ \theta,\ \mathbf{e}_A) \quad ,$$

$$\mathbf{q} = \mathbf{q}^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) + \gamma_\beta\ \mathbf{q}^\beta(\mathbf{F},\ \theta,\ \mathbf{e}_A) \quad , \qquad (2.39)$$

where $\lambda_\alpha(\mathbf{X},t)$ and $\gamma_\beta(\mathbf{X},t)$ are arbitrary scalar fields; here we assume summation over α and β (for N type I and L type II constraints respectively) from the outset. Substitution of $(2.39)_{1-4}$ and the type II constraint equation (2.37) in the entropy production inequality (2.12) and use of the fact that λ_α and γ_β are arbitrary yields

$$\phi^\alpha = 0 \quad , \qquad (2.40)$$

and

$$- \rho \, \eta^{\alpha} \, \dot{\theta} + \mathbf{S}^{\alpha} \cdot \dot{\mathbf{F}} = 0 \quad , \tag{2.41}$$

corresponding to Gurtin and Podio-Guidugli's assumption $(2.32)_1$ and their result $(2.33)_1$ respectively. Equation (2.40) is also consistent with the definition (2.36) of type I constraints. Now the rate form of (2.36) is

$$\frac{\partial \phi^{\alpha}}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}} + \frac{\partial \phi^{\alpha}}{\partial \theta} \, \dot{\theta} = 0 \quad , \tag{2.42}$$

so for consistency with (2.39) and (2.41) we must have

$$\mathbf{S}^{\alpha} = \rho \, \frac{\partial \phi^{\alpha}}{\partial \mathbf{F}} \quad ,$$

$$\eta^{\alpha} = - \frac{\partial \phi^{\alpha}}{\partial \theta} \quad ,$$

and

$$\mathbf{q}^{\beta} = \mathbf{z}^{\beta} \quad . \tag{2.43}$$

Since the entropy production inequality (2.34) is not affected by the constraints, it can be used as described in the unconstrained case (see (2.13) - (2.18)) to obtain restricted forms of ψ^{0}, $\mathbf{S}^{0}$, η^{0} and $\mathbf{q}^{0}$.

The constitutive equations (2.39) then reduce to

$$\psi = \psi^{0}(\mathbf{F}, \, \theta) \quad ,$$

$$\mathbf{S} = \rho \left[\frac{\partial \psi^{0}}{\partial \mathbf{F}} + \lambda_{\alpha} \, \frac{\partial \phi^{\alpha}}{\partial \mathbf{F}} \right] \quad ,$$

$$\eta = - \left[\frac{\partial \psi^{0}}{\partial \theta} + \lambda_{\alpha} \frac{\partial \phi^{\alpha}}{\partial \theta} \right] \quad ,$$

$$\mathbf{q} = \mathbf{q}^{0}(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) + \gamma_{\beta}\ \mathbf{z}^{\beta} \quad . \tag{2.44}$$

A more convenient representation of (2.44) is obtained by defining an augmented free energy function ψ by

$$\psi = \psi^{0}(\mathbf{F},\ \theta) + \lambda_{\alpha}\ \phi^{\alpha}(\mathbf{F},\ \theta,\ \mathbf{e}_{A}) \quad . \tag{2.45}$$

The set of constitutive equations (2.44) can then be written more concisely as

$$\psi = \psi'(\mathbf{F},\ \theta,\ \lambda_{\alpha},\ \mathbf{e}_{A}) = \psi^{0}(\mathbf{F},\ \theta) + \lambda_{\alpha}\ \phi^{\alpha}(\mathbf{F},\ \theta,\ \mathbf{e}_{A}) \quad ,$$

$$\mathbf{S} = \mathbf{S}'(\mathbf{F},\ \theta,\ \lambda_{\alpha},\ \mathbf{e}_{A}) = \rho\ \frac{\partial \psi'}{\partial \mathbf{F}} \quad ,$$

$$\eta = \eta'(\mathbf{F},\ \theta,\ \lambda_{\alpha},\ \mathbf{e}_{A}) = - \frac{\partial \psi'}{\partial \theta} \quad ,$$

$$\mathbf{q} = \mathbf{q}'(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta,\ \gamma_{\beta},\ \mathbf{e}_{A})$$

$$= \mathbf{q}^{0}(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) + \gamma_{\beta}\ \mathbf{z}^{\beta}(\mathbf{F},\ \theta,\ \mathbf{e}_{A}) \quad , \tag{2.46}$$

and we also have the relation

$$\phi^{\alpha} = \partial \psi' / \partial \lambda_{\alpha} \quad . \tag{2.47}$$

The set of constitutive equations in this form will be particularly convenient later when we construct a conjugate set with η replacing θ as an independent variable.

The fact that the entropy production inequality (2.34) has no contribution from the constraints has the following further consequences, which parallel those for unconstrained materials quoted in (2.18) - (2.21). Specifically, the energy equation and entropy production inequality take the respective forms

$$\rho\ \theta\ \dot{\eta} = -\ \mathrm{Div}\ \mathbf{q} + \rho\ r \tag{2.48}$$

and

$$\mathbf{q}^{0}\cdot \mathrm{Grad}\ \theta \le 0 \quad , \tag{2.49}$$

while the Maxwell relation can be expressed in terms of $(2.46)_{2,3}$ as

$$\frac{\partial \mathbf{S}'}{\partial \theta} = -\ \rho\ \frac{\partial \eta'}{\partial \mathbf{F}} \quad . \tag{2.50}$$

The results (2.20) and (2.21) that express respectively the absence of a piezo-caloric effect and the positive semi-definiteness of the thermal conductivity tensor carry over unchanged.

As in the unconstrained case, the set of constitutive equations (2.46) must obey the principle of material frame-indifference. We note that under transformations of the form (2.22), $\mathbf{e}_A$ and $\mathbf{q}$ are invariant as they are defined relative to the reference configuration; this, together with the use of standard procedures for invoking frame-indifference (recall (2.22) - (2.26), see also Carlson (1972)) leads to the following reduced forms of the constitutive equations:

$$\psi'(\mathbf{F},\ \theta,\ \lambda_\alpha,\ \mathbf{e}_A) = \bar{\psi}(\mathbf{C},\ \theta,\ \lambda_\alpha,\ \mathbf{e}_A) = \bar{\psi}^0(\mathbf{C},\ \theta) + \lambda_\alpha\ \bar{\phi}^\alpha(\mathbf{C},\ \theta,\ \mathbf{e}_A)\ ,$$

$$\mathbf{S}'(\mathbf{F},\ \theta,\ \lambda_\alpha,\ \mathbf{e}_A) = \bar{\mathbf{S}}(\mathbf{C},\ \theta,\ \lambda_\alpha,\ \mathbf{e}_A) = \bar{\mathbf{S}}^0(\mathbf{C},\ \theta) + \lambda_\alpha\ \bar{\mathbf{S}}^\alpha(\mathbf{C},\ \theta,\ \mathbf{e}_A)$$

$$\eta'(\mathbf{F},\ \theta,\ \lambda_\alpha,\ \mathbf{e}_A) = \bar{\eta}(\mathbf{C},\ \theta,\ \lambda_\alpha,\ \mathbf{e}_A) = \bar{\eta}^0(\mathbf{C},\ \theta) + \lambda_\alpha\ \bar{\eta}^\alpha(\mathbf{C},\ \theta,\ \mathbf{e}_A),$$

$$\mathbf{q}'(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta,\ \gamma_\beta,\ \mathbf{e}_A) = \bar{\mathbf{q}}(\mathbf{C},\ \theta,\ \lambda_\alpha,\ \mathbf{e}_A)$$

$$= \bar{\mathbf{q}}^0(\mathbf{C},\ \theta,\ \mathrm{Grad}\ \theta) + \gamma_\beta\ \bar{\mathbf{z}}^\beta(\mathbf{C},\ \theta,\ \mathrm{Grad}\ \theta,\ \mathbf{e}_A)\ . \tag{2.51}$$

We now define the two classes of constrained heat-conducting materials with which we will be concerned for the remainder of this thesis.

A <u>definite conductor</u> of heat is a material for which

$$\mathbf{v}\cdot(\mathrm{sym}\ \mathbf{K})\mathbf{v} > 0 \tag{2.52}$$

for all non-zero vectors $\mathbf{v}$ and where sym $\mathbf{K}$ denotes the symmetric part of the thermal conductivity tensor (2.21) (see also the discussion following (2.50)).

<u>A non-conductor</u> of heat is defined for constrained materials to be one for which

$$\mathbf{q}^0(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta) = 0\ , \tag{2.53}$$

i.e.

$$\mathbf{q} = \gamma_\beta\ \mathbf{z}^\beta(\mathbf{F},\ \theta,\ \mathbf{e}_A)\ . \tag{2.54}$$

This definition is in contrast with that used in (I), where a non-conductor was defined to be a constrained material for which

$$\mathbf{q} = \mathbf{q}^0 + \gamma_\beta \mathbf{z}^{\beta'} = 0 \quad .$$

The present definition is to be preferred as it implies that the material is a non-conductor in all directions except $\mathbf{z}^\beta$. For example, heat-conducting fibres passing through an otherwise non-conducting material in some direction $\mathbf{e}$ give the constraint of perfect conductivity in the direction $\mathbf{e}$.

We conclude this discussion of the constitutive equations (2.39) proposed by Reddy in (I) for constrained thermoelastic materials by imposing (as in (II)) the condition of isotropy. As in the unconstrained case it is only necessary to obtain the appropriate form of ψ' and $\mathbf{q}'$, since $\mathbf{S}'$ and η' can be found from ψ' through $(2.46)_2$ and $(2.46)_3$ respectively.

The constraint vectors $\mathbf{e}_A$ characterize the directionality of the constraints and this directionality must be accommodated within our definition of an isotropic material. If we refer back to the unconstrained case, an analogous situation exists in the case of the temperature gradient Grad θ , which also defines a field of directions within the body. In that case, the material is regarded as <u>isotropic</u> if after an arbitrary rotation $\mathbf{Q}$ of the reference configuration, the material response is unchanged provided that the vector Grad θ has also been rotated through $\mathbf{Q}$,

$$\text{i.e.} \quad \text{Grad}\,\theta \rightarrow \mathbf{Q}\,\text{Grad}\,\theta \quad .$$

The constraint vectors e_A define particular (fixed) directions in the body which is otherwise isotropic. We require therefore that the material response be unaffected by an arbitrary rotation $\mathbf{Q}$ provided that the vectors e_A are also rotated through $\mathbf{Q}$, that is,

$$e_A \rightarrow \mathbf{Q}\, e_A \quad .$$

This approach is also adopted by Spencer (1972) in a discussion of the purely mechanical case of fibre-reinforced materials. We note that Gurtin and Podio-Guidugli (1973), however, exclude the directional vectors e_A completely from their discussion of isotropy.

We require for an isotropic constrained thermoelastic material that

$$\psi'(\mathbf{F},\ \theta,\ \lambda_\alpha,\ e_A) = \psi'(\mathbf{F}\,\mathbf{Q},\ \theta,\ \lambda_\alpha,\ \mathbf{Q}\, e_A)$$

and

$$q'(\mathbf{F},\ \theta,\ \mathrm{Grad}\ \theta,\ \gamma_\beta,\ e_A) = q'(\mathbf{F}\,\mathbf{Q},\ \mathbf{Q}\ \mathrm{Grad}\ \theta\ ,\ \gamma_\beta,\ \mathbf{Q}\, e_A) \quad .$$

(We restrict atention to ψ', q', since S' and η' are obtainable from ψ' by $(2.46)_{2,3}$). Consequently, the functions $\bar{\psi}$ and $\bar{q}$ defined in $(2.51)_1$ and $(2.51)_4$ respectively satisfy the following relations:

$$\bar{\psi}(C,\ \theta,\ \lambda_\alpha,\ e_A) = \bar{\psi}(\mathbf{Q}\ C\ \mathbf{Q}^T,\ \theta,\ \lambda_\alpha,\ \mathbf{Q}\, e_A) \tag{2.55}$$

and

$$\mathbf{Q}\ \bar{q}(C,\ \theta,\ \mathrm{Grad}\ \theta\ ,\ \gamma_\beta,\ e_A) = \bar{q}(\mathbf{Q}\ C\ \mathbf{Q}^T,\ \theta,\ \mathbf{Q}\ \mathrm{Grad}\ \theta\ ,\ \gamma_\beta,\ \mathbf{Q}\, e_A)\ ; \tag{2.56}$$

that is, $\bar{\psi}$ and $\bar{q}$ are isotropic scalar- and vector-valued functions, respectively (Truesdell and Noll (1965), Section 8). Results on the representation of isotropic functions ((II), Truesdell and Noll (1965), Sections 10-13, Wang (1969,70), Smith (1970)) can be invoked and we find

$$\bar{\psi}(\mathbf{C},\ \theta,\ \lambda_\alpha,\ \mathbf{e}_A) = \bar{\bar{\psi}}^0(\iota_{\mathbf{C}},\ \theta) + \lambda_\alpha\ \bar{\bar{\phi}}^\alpha(\iota_{\mathbf{C}},\ \theta,\ e_{AB},\ f_{AB},\ k_{AB}), \tag{2.57}$$

where

$$e_{AB} = \mathbf{e}_A\cdot\mathbf{e}_B\ ,\ \mathbf{f}_A = \mathbf{F}\ \mathbf{e}_A\ ,\ f_{AB} = \mathbf{F}\ \mathbf{e}_A\cdot\ \mathbf{F}\ \mathbf{e}_B\ ,$$

$$\mathbf{k}_A = \mathbf{C}\ \mathbf{e}_A\ ,\ k_{AB} = \mathbf{k}_A\cdot\mathbf{k}_B\ ; \tag{2.58}$$

$$\bar{q}(\mathbf{C},\ \theta,\ \mathrm{Grad}\ \theta\ ;\ \gamma_\beta,\ \mathbf{e}_A) = \bar{\bar{q}}^0(\iota_{\mathbf{C}},\ \theta,\ \kappa_1,\ \kappa_2,\ \kappa_3)$$

$$+\ \gamma_\beta\ \bar{\bar{\mathbf{z}}}^\beta(\iota_{\mathbf{C}},\ \theta,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ e_{AB},\ f_{AB},\ k_{AB})\ , \tag{2.59}$$

where κ_1, κ_2, κ_3 are as defined following (2.29).

Use of $(2.46)_{2,3}$, (2.57) and (2.59) together with the definitions of $\mathbf{S}'$ and η' from $(2.51)_{2,3}$ enables us to write the constitutive equations for isotropic constrained thermoelastic materials in their final form as follows:

$$\psi = \psi(a_i,\ \theta,\ e_{AB},\ f_{AB},\ k_{AB})$$

$$= \psi^0(a_i,\ \theta) + \lambda_\alpha\ \phi^\alpha(a_i,\ \theta,\ e_{AB},\ f_{AB},\ k_{AB})\ , \tag{2.60}$$

$$\rho^{-1}\,\mathbf{S} = \sum_i \frac{\partial\psi}{\partial a_i}\,\mathbf{q}_i \otimes \mathbf{p}_i + \lambda_\alpha \frac{\partial\phi^\alpha}{\partial f_{AB}}\,(\mathbf{f}_A \otimes \mathbf{e}_B + \mathbf{f}_B \otimes \mathbf{e}_A)$$

$$+ \lambda_\alpha \frac{\partial\phi^\alpha}{\partial k_{AB}}\,(\mathbf{F}\,\mathbf{k}_A \otimes \mathbf{e}_B + \mathbf{F}\,\mathbf{k}_B \otimes \mathbf{e}_A + \mathbf{f}_A \otimes \mathbf{k}_B + \mathbf{f}_B \otimes \mathbf{k}_A) \quad , \tag{2.61}$$

$$\eta = -\frac{\partial\psi}{\partial\theta} \quad , \tag{2.62}$$

$$\mathbf{q} = \mathbf{q}^0(a_i,\ \theta,\ \kappa_1,\ \kappa_2,\ \kappa_3)$$
$$+ \gamma_\beta\,\mathbf{z}^\beta(a_i,\ \theta,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ e_{AB},\ f_{AB},\ k_{AB}) \quad . \tag{2.63}$$

Equations (2.60-63) are presented in the notation to be used in the remainder of the thesis, and for convenience the same symbol is used to denote the function ψ and its value. The derivation of the final form of (2.61) is given in (II) (see (2.22) there), and summation is assumed here and henceforth over repeated indices A, B. In (2.60-63) the constraints satisfy

$$\frac{\partial\psi}{\partial\lambda_\alpha} = \phi^\alpha(a_i,\ \theta,\ e_{AB},\ f_{AB},\ k_{AB}) = 0 \tag{2.64}$$

or

$$\mathbf{z}^\beta(a_i,\ \theta,\ e_{AB},\ f_{AB},\ k_{AB})\cdot\mathrm{Grad}\ \theta = 0 \tag{2.65}$$

in the case of type I and II constraints respectively, with the dependence on a_i in (2.60) - (2.65) being <u>symmetric</u>.

Isotropic and directional type I constraints

We now turn our attention to the general representation (2.36) of type I constraints and define restricted subsets of this set of constraints that are still sufficiently general to accommodate a wide variety of constraints found in practice. In particular, we restrict attention to the following two classes:

(i) Isotropic constraints, which are those expressible in the form

$$\phi^{a}(a_i,\ \theta) = \phi_1^{a}(a_i) + \phi_2^{a}(\theta) = 0 \quad . \tag{2.66}$$

An example of such a constraint is temperature-dependent compressibility (Trapp (1971), Gurtin and Podio-Guidugli (1973)):

$$a_1\ a_2\ a_3 - \xi_1(\theta) = 0 \quad , \quad \xi_1(\theta) = 0 \quad , \tag{2.67}$$

where $\xi_1(\theta)$ is a scalar-valued function of temperature.

(ii) Directional constraints, which are of the form

$$\phi^{a}(e_{AB},\ f_{AB},\ \theta) = \phi_1^{a}(f_{AB}) + \phi_2^{a}(e_{AB},\ \theta) \quad , \tag{2.68}$$

and which are further restricted by the requirement that dependence on f_{AB} is to be linear, so that

$$\phi_1^{a}(f_{AB}) = \frac{1}{2}\ \beta_{AB}^{a}\ f_{AB} \quad , \tag{2.69}$$

where for each a , β_{AB}^{a} is a symmetric matrix of constants. An example of a directional constraint is temperature-dependent extensibility in a direction e_1 (Trapp (1971)) defined by the condition

$$f_{11} - \xi_2(\theta) = 0 \quad , \quad \xi_2(\theta) > 0 \quad , \tag{2.70}$$

where $\xi_2(\theta)$ is a scalar-valued function of temperature.

The constraint of temperature-dependent shearing is now considered in the context of the above definition of directional constraints. This constraint is a generalization of the orthogonality-preserving (mechanical) constraint (Gurtin and Podio-Guidugli (1973)), and constrains the angle β between two directions $\mathbf{F}\, e_1$ and $\mathbf{F}\, e_2$ in the current configuration to be a function of both the angle α between e_1 and e_2 in the reference configuration and the temperature. Since

$$\cos \alpha = e_1 \cdot e_2 = e_{12}$$

and

$$\cos \beta = (\mathbf{F}\, e_1 \cdot \mathbf{F}\, e_2)/|\mathbf{F}\, e_1||\mathbf{F}\, e_2| = f_{12}/\sqrt{f_{11}\, f_{22}} \quad ,$$

the constraint can be written as

$$f_{12} - \bar{\xi}_3(\theta,\, e_{12})\sqrt{f_{11}\, f_{22}} = 0 \quad , \tag{2.71}$$

where $\bar{\xi}_3$ is a scalar-valued function of temperature and the angle α .

This clearly does not obey (2.68) or (2.69), but there are two special cases of (2.71) which do. Firstly, if the temperature-dependent extensibility constraint (2.70) holds in the two directions e_1 and e_2

then (2.71) reduces to

$$f_{12} - \xi_3(\theta, e_{12}) = 0 \quad ; \tag{2.72}$$

where ξ_3 is a scalar-valued function of temperature and the angle α .

Secondly, the mechanical orthogonality-preserving constraint discussed by Gurtin and Podio-Guidugli (1973) (for which e_{12} and f_{12} are zero) can be written simply as

$$f_{12} = 0 \quad . \tag{2.73}$$

Entropic formulation of the constitutive equations

For the analysis of wave propagation and growth in isotropic definite conductors, the set of constitutive equations is appropriately taken to be (2.60-3) with the constraints obeying (2.64) or (2.65). In the case of non-conductors though, it proves more convenient to construct a set with the specific internal energy ϵ and η replacing ψ and θ as the independent variables. This results for the unconstrained case in constitutive equations of the form

$$\begin{aligned}
\epsilon &= \bar{\epsilon}(\mathbf{F}, \eta) \\
\mathbf{S} &= \rho \frac{\partial \bar{\epsilon}}{\partial \mathbf{F}} \\
\theta &= \frac{\partial \bar{\epsilon}}{\partial \eta} \\
\mathbf{q} &= \bar{\mathbf{q}}(\mathbf{F}, \eta, \text{Grad } \theta) \quad ,
\end{aligned} \tag{2.74}$$

where

$$\epsilon = \psi^0 + \eta\ \theta \tag{2.75}$$

is the specific internal energy (see Chadwick and Currie (1972), (I), (II)).

A corresponding set for <u>constrained</u> elastic materials was derived in (I). An augmented internal energy function $\hat{\epsilon}$ is defined by

$$\epsilon = \hat{\epsilon}(\mathbf{F},\ \eta,\ \lambda_a,\ \mathbf{e}_A) = \psi + \eta\ \theta \quad . \tag{2.76}$$

Differentiation of $\hat{\epsilon}$ with respect to $\mathbf{F}$, η, λ_a then yields

$$\mathbf{S} = \rho\,\frac{\partial\hat{\epsilon}}{\partial\mathbf{F}} \quad , \tag{2.77}$$

$$\theta = \frac{\partial\hat{\epsilon}}{\partial\eta} \quad , \tag{2.78}$$

$$\phi^a = \frac{\partial\hat{\epsilon}}{\partial\lambda_a} \quad , \tag{2.79}$$

and we complete the set by writing the heat flux $\mathbf{q}$ in the form

$$\mathbf{q} = \hat{\mathbf{q}}(\mathbf{F},\ \eta,\ \lambda_a,\ \gamma_\beta,\ \mathbf{e}_A) \quad . \tag{2.80}$$

A superimposed caret will be used throughout to denote the use of η rather than θ as one of the independent variables.

We choose the above approach for incorporating the constraints rather than for example expressing constraints in the form

$$\phi^{\alpha}(\mathbf{F},\ \eta,\ \mathbf{e}_A) = 0$$

because constraints do not usually appear in the above form, but rather as in the definitions (2.36,7) of type I and II constraints respectively. We now comment on a significant difference between the two sets of constitutive equations which have θ and η respectively as variables. In (2.46), for instance, the constrained part of ψ is <u>explicitly</u> separated from the unconstrained term, that is,

$$\psi(\mathbf{F},\ \theta,\ \lambda_\alpha,\ \mathbf{e}_A) = \psi^0(\mathbf{F},\ \theta) + \lambda_\alpha\ \phi^\alpha(\mathbf{F},\ \theta,\ \mathbf{e}_A) \qquad ((2.46)_1 \text{ bis})$$

but in the corresponding equation for ϵ , we have

$$\hat{\epsilon}(\mathbf{F},\ \eta,\ \lambda_\alpha,\ \mathbf{e}_A) = \psi^0(\mathbf{F},\ \theta) + \lambda_\alpha\ \phi^\alpha(\mathbf{F},\ \theta,\ \mathbf{e}_A) + \eta\ \theta \quad , \qquad (2.76 \text{ bis})$$

where throughout we must insert $\theta = \bar{\theta}(\mathbf{F},\ \eta,\ \lambda_\alpha,\ \mathbf{e}_A)$. Consequently it is not generally possible to disengage the constraint contributions in the same manner as in $(2.46)_1$.

The restrictions on the set (2.76) - (2.80) due to material frame-indifference can be obtained by arguments similar to those used for the thermal formulation; details are therefore omitted and the results are found to be (see also (II))

$$\epsilon = \hat{\epsilon}(a_i,\ \eta,\ \lambda_\alpha,\ e_{AB},\ f_{AB},\ k_{AB}) \qquad (2.81)$$

$$\rho^{-1}\,\mathbf{S} = \sum_{i} \left[\frac{\partial\hat{\epsilon}}{\partial a_i}\right] \mathbf{q}_i \otimes \mathbf{p}_i + \left[\frac{\partial\hat{\epsilon}}{\partial f_{AB}}\right](\mathbf{f}_A \otimes \mathbf{e}_B + \mathbf{f}_B \otimes \mathbf{e}_A)$$

$$+ \left[\frac{\partial\hat{\epsilon}}{\partial k_{AB}}\right](\mathbf{F}\,\mathbf{k}_A \otimes \mathbf{e}_B + \mathbf{F}\,\mathbf{k}_B \otimes \mathbf{e}_A + \mathbf{f}_A \otimes \mathbf{k}_B + \mathbf{f}_B \otimes \mathbf{k}_A) \quad , \tag{2.82}$$

$$\theta = \frac{\partial\hat{\epsilon}}{\partial\eta} \quad , \tag{2.83}$$

$$\mathbf{q} = \hat{\mathbf{q}}(a_i,\ \eta,\ \kappa_1,\ \kappa_2,\ \kappa_3,\ e_{AB},\ f_{AB},\ k_{AB}) \quad . \tag{2.84}$$

Constraint vectors for type I constraints

In later chapters we will make extensive use of sets of vectors associated with the constraints. For convenience we describe these vectors and their properties here.

Given a set of type I constraints

$$\phi^{a}(\mathbf{F},\ \theta,\ \mathbf{e}_A) = 0 \quad , \quad a = 1,\ \ldots,\ N \tag{2.36 bis}$$

and a unit vector $\mathbf{n}$, we define the set $\left\{\mathbf{c}^{a}(\mathbf{n})\right\}_{a=1}^{N}$ of vectors by

$$\mathbf{c}^{a}(\mathbf{n}) = \frac{\partial\phi^{a}}{\partial\mathbf{F}}\,\mathbf{n} \quad , \quad a = 1,\ \ldots,\ N \quad . \tag{2.85}$$

In subsequent chapters the vector $\mathbf{n}$ appearing in the definition of $\mathbf{c}^{a}(\mathbf{n})$ will be the normal to a singular surface or wavefront.

It was assumed in (I) that the set $\{\mathbf{c}^{\alpha}\}$ was linearly independent. For numerous combinations of the constraint examples mentioned earlier (recall (2.67,70,72,73)), however, the corresponding vectors $\mathbf{c}^{\alpha}$ form a linearly dependent set, some often having the value zero. We accordingly allow for this possibility in general (see also Whitworth (1982), Whitworth and Chadwick (1984) and Chadwick, Whitworth and Borejko (1985) for similar discussions relating to simple waves, surface waves and small-amplitude waves respectively).

We assume that

$$\dim \operatorname{span} \{\mathbf{c}^{\alpha}\} = M \le N \quad ,$$

and order the constraints in the following way:

(i) the subset $\left\{\mathbf{c}^{\sigma}(\mathbf{n})\right\}_{\sigma=1}^{M}$ is linearly independent.

(ii) the subset $\left\{\mathbf{c}^{M+\mu}(\mathbf{n})\right\}_{\mu=1}^{P-M}$ $(P \le N)$ consists of vectors which are non-zero, and which are linear combinations of the subset (i), so that

$$\bar{\mathbf{c}}^{\mu}(\mathbf{n}) \equiv \mathbf{c}^{M+\mu}(\mathbf{n}) = D^{\mu}_{\sigma}\, \mathbf{c}^{\sigma}(\mathbf{n}) \quad , \quad \sigma = 1 \; , \; \ldots , \; M \qquad (\mu = 1, \; \ldots, \; P-M) \quad , \tag{2.86}$$

where D^{μ}_{σ} is a matrix of rank M and summation on σ is implied.

(iii) the subset $\left\{c^{P+\eta}(\mathbf{n})\right\}_{\eta=1}^{N-P}$ consists of the remaining (N-P) constraints, each satisfying

$$c^{P+\eta}(\mathbf{n}) \equiv 0 \quad , \quad \eta = 1, \ldots, N-P \quad . \tag{2.87}$$

We adopt the following terminology, introduced by Chadwick, Whitworth and Borejko (1985) : the type I constraints ϕ^a, $a = 1, \ldots, N$ defined by (2.36) are said to be <u>fully active</u> in the direction $\mathbf{n}$ if the vectors $\mathbf{c}^a(\mathbf{n})$ are linearly independent (so that dim span $\{\mathbf{c}^a(\mathbf{n})\} = M = N$). The constraints are <u>partially active</u> if dim span $\{\mathbf{c}^a(\mathbf{n})\} = M < N$, and are <u>inactive</u> if $\mathbf{c}^a(\mathbf{n}) \equiv 0$, $a = 1, \ldots, N$ (so that P=0 in (2.87)).

The vectors $\mathbf{c}^a(\mathbf{n})$ take particular forms for <u>isotropic</u> materials subject to the restricted classes of type I constraints defined by (2.66,8). For <u>isotropic constraints</u> (we henceforth drop the explicit indication of dependence of $\mathbf{c}^a$ on $\mathbf{n}$ except where this would cause confusion)

$$\mathbf{c}^a = \sum_i \frac{\partial\phi^a}{\partial a_i} (\mathbf{p}_i \cdot \mathbf{n})\, \mathbf{q}_i \tag{2.88}$$

and for <u>directional constraints</u>

$$\mathbf{c}^a = \beta^a_{AB} \sum_i a_i\, (\mathbf{p}_i \cdot \mathbf{e}_A)(\mathbf{n} \cdot \mathbf{e}_B)\, \mathbf{q}_i \quad . \tag{2.89}$$

We will require later an alternative to the set $\{\mathbf{c}^a\}$ that arises from the entropic formulation (2.81-4), with the type I constraints satisfying (2.36) in the form

$$\hat{\phi}^a(\mathbf{F},\ \theta(\mathbf{F},\ \eta,\ \lambda_\gamma,\ \mathbf{e}_{AB})) = 0 \quad , \quad a = 1, \ldots, N \quad . \tag{2.90}$$

A set of entropic type I constraint vectors $\left\{\hat{\mathbf{c}}^{a}(\mathbf{n})\right\}_{a=1}^{N}$ is defined in terms of (2.90) by (see Reddy (I)):

$$\hat{\mathbf{c}}^{a}(\mathbf{n}) \equiv \frac{\partial\hat{\phi}^{a}}{\partial\mathbf{F}}\,\mathbf{n} = \mathbf{c}^{a} - \mu^{-1}\,\omega^{a}\,\mathbf{M}\,\mathbf{n} \quad , \quad a = 1, \ldots, N \tag{2.91}$$

where $\mathbf{M}$ is the second-order tensor

$$\mathbf{M} = \rho\,\frac{\partial^{2}\psi}{\partial\mathbf{F}\partial\theta} \quad , \tag{2.92}$$

and the scalars μ and ω^{a} are defined by

$$\mu = \rho\,\frac{\partial^{2}\psi}{\partial\theta^{2}} \tag{2.93}$$

and

$$\omega^{a} = \frac{\partial\phi^{a}}{\partial\theta} \quad , \quad a = 1, \ldots, N \tag{2.94}$$

respectively. As before, in definition (2.85) of the set $\{\mathbf{c}^{a}(\mathbf{n})\}$, the unit vector $\mathbf{n}$ appearing in (2.91) will be the normal to a singular surface or wavefront.

For the restricted subsets of type I constraints defined by (2.66) and (2.68,9) respectively, $\mathbf{M}$ takes the form

$$\mathbf{M} = \mathbf{M}^{0}(a_{i},\ \theta) = \sum_{i} \rho\,\frac{\partial^{2}\psi^{0}}{\partial a_{i}\partial\theta}\,\mathbf{q}_{i} \otimes \mathbf{p}_{i} \tag{2.95}$$

where $\psi^{0}(a_{i},\ \theta)$ is as defined in (2.60).

We can write the set $\{\hat{c}^{\alpha}\}$ (we suppress as earlier the explicit dependence of $\hat{c}^{\alpha}$ on $\mathbf{n}$ from now on) as:

$$\hat{c}^{\sigma} = c^{\sigma} - \rho\mu^{-1}\omega^{\sigma} \sum_i \frac{\partial^2\psi^0}{\partial a_i \partial\theta} (\mathbf{n}.\mathbf{p}_i)\ \mathbf{q}_i \ , \ \sigma = 1, \ldots, M \ ,$$

$$\hat{c}^{M+\mu} = c^{M+\mu} - \rho\mu^{-1}\omega^{M+\mu} \sum_i \frac{\partial^2\psi^0}{\partial a_i \partial\theta} (\mathbf{n}.\mathbf{p}_i)\ \mathbf{q}_i \ , \ \mu = 1, \ldots, P-M \ ,$$

$$\hat{c}^{P+\eta} = c^{P+\eta} - \rho\mu^{-1}\omega^{P+\eta} \sum_i \frac{\partial^2\psi^0}{\partial a_i \partial\theta} (\mathbf{n}.\mathbf{p}_i)\ \mathbf{q}_i \ , \ \eta = 1, \ldots, N-P, \tag{2.96}$$

where the subsets $\{c^{\sigma}\}$, $\{c^{M+\mu}\}$ and $\{c^{P+\eta}\}$ are as defined following (2.85). Although there are some similarities between the entropic constraint vectors $\{\hat{c}^{\sigma}\}$, $\{\hat{c}^{M+\mu}\}$ and $\{\hat{c}^{P+\eta}\}$ and their thermal counterparts, the subset $\{\hat{c}^{\sigma}\}$ is <u>not</u> necessarily linearly independent. It is therefore convenient to introduce another set of entropic constraint vectors which may be conveniently partitioned as was $\{c^{\alpha}\}$, and this point is taken up in Chapter 5.

We conclude this discussion of the constrained vectors associated with each of the type I constraints with a remark pertaining to the corresponding situation for type II constraints. Here the vectors $\left\{\mathbf{z}^{\beta}\right\}_{\beta=1}^{L}$ will play a role equivalent to the $\left\{c^{\alpha}\right\}_{\alpha=1}^{N}$. Since the vectors $\mathbf{z}^{\beta}$ appear far less frequently than the c^{α} in both the propagation conditions and the growth equation for acceleration waves, we neither restrict the set $\{\mathbf{z}^{\beta}\}$ nor subdivide it into linear independent-dependent subsets (as was done with $\{\phi^{\alpha}\}$ and $\{c^{\alpha}\}$) at this stage. A relatively minor restriction on $\{\mathbf{z}^{\beta}\}$ is imposed in Chapter 8 in the derivation of the growth equation for homentropic waves.

CHAPTER 3

SINGULAR SURFACES IN CONSTRAINED THERMOELASTIC MATERIALS

In this chapter we review aspects of the theory of singular surfaces relevant to our work. Most of the results in this chapter are not new but are in some cases clarified by their presentation here in coordinate-free form.

We begin by defining a singular surface and discuss the geometrical and kinematic compatibility conditions to be satisfied across the surface (see Truesdell and Toupin, (1960), Sections 173-181). Acceleration waves are then defined following Chen (1973) and the compatibility conditions are exploited to find expressions for the required non-zero jumps of second and higher order derivatives of the motion $\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X},t)$. We give the definitions of principal waves and of longitudinal and transverse waves; we will often be considering principal waves that are either longitudinal or transverse later in the thesis. We then deal with the behaviour at the wavefront of the remaining variables characterizing the thermodynamic process and in particular define homothermal and homentropic waves.

The chapter concludes with an investigation of acceleration waves in definite and non-conductors. We discuss Reddy's result in (I) that all acceleration waves in constrained definite conductors are <u>homothermal</u> and use our revised definition (2.53,4) of a constrained non-conductor to show that waves in constrained non-conductors are <u>homothermal</u> if the material is subject to at least one type II constraint for which $\mathbf{z}^{\beta}\cdot\mathbf{n} \neq 0$. If there are no type II constraints, or if all type II constraints present satisfy $\mathbf{z}^{\beta}\cdot\mathbf{n} = 0$, then all waves in the constrained non-conductor are shown to be homentropic.

Singular surfaces

We consider a one-parameter family of surfaces S_t (a moving surface) in the reference configuration

$$\Sigma(\mathbf{X},t) = 0 \tag{3.1}$$

parametrised by time, and continuously differentiable but of arbitrary curvature. The surface has an alternative representation in terms of a pair of surface coordinates Y^{Γ} ($\Gamma = 1,2$), so that the position vector of a point on the surface is

$$\mathbf{X} = \Xi(Y^{\Gamma},t) \quad , \tag{3.2}$$

with the corresponding tangent basis at $\mathbf{X}$ given by

$$\mathbf{H}_{\Gamma} = \frac{\partial\Xi}{\partial Y^{\Gamma}} \quad , \tag{3.3}$$

and the unit normal to the surface $\mathbf{n}$ given by

$$\mathbf{n} = \frac{\text{Grad }\Sigma}{|\text{Grad }\Sigma|} \quad . \tag{3.4}$$

The components $H_{\Gamma\Delta}$ and $\Omega_{\Gamma\Delta}$ relative to $\{\mathbf{H}_\Gamma\}$ of the surface tensor and curvature tensor are then

$$H_{\Gamma\Delta} = \mathbf{H}_\Gamma \cdot \mathbf{H}_\Delta \tag{3.5}$$

and

$$\Omega_{\Gamma\Delta} = \mathbf{n} \cdot \frac{\partial \mathbf{H}_\Gamma}{\partial Y^\Delta} \quad . \tag{3.6}$$

The velocity $\mathbf{u}$ of a point on the surface is

$$\mathbf{u} = \frac{\partial}{\partial t} (\Xi(Y^\Gamma, t)) \tag{3.7}$$

and the <u>speed of propagation</u> of the surface ν (Truesdell and Toupin (1960), Section 183) is defined by

$$\nu = \mathbf{u} \cdot \mathbf{n} \quad ; \tag{3.8}$$

ν (unlike $\mathbf{u}$) is independent of the choice of coordinates Y^Γ and is therefore an intrinsic property of the surface. It is a measure of the speed with which the surface S_t traverses the material.

We now discuss the representation of the surface in the current configuration. The image of the surface S_t in the current configuration s_t is defined by

$$\sigma(\mathbf{x}, t) \equiv \Sigma(\bar{\mathbf{X}}(\mathbf{x}, t), t) = 0 \quad , \tag{3.9}$$

using the representation for the motion introduced in Chapter 2.

Alternatively, from (3.2) we may characterize s_t by

$$\mathbf{x} = \bar{\mathbf{x}}(\Xi(Y^\Gamma, t), t) \equiv \bar{\bar{\mathbf{x}}}(Y^\Gamma, t) \quad . \tag{3.10}$$

The tangent basis vectors $\mathbf{h}_\Gamma$ on s_t are given by

$$\mathbf{h}_\Gamma = \frac{\partial \bar{\bar{\mathbf{x}}}}{\partial Y^\Gamma} \tag{3.11}$$

and the unit normal to s_t is

$$\mathbf{m} = \frac{\text{grad } \sigma}{|\text{grad } \sigma|} \quad ,$$

where grad is the gradient operator in the current configuration.

The components $h_{\Gamma\Delta}$ and $\omega_{\Gamma\Delta}$ relative to $\{\mathbf{h}_\Gamma\}$ of the surface metric tensor and curvature tensor are

$$h_{\Gamma\Delta} = \mathbf{h}_\Gamma \cdot \mathbf{h}_\Delta$$

and

$$\omega_{\Gamma\Delta} = \mathbf{n} \cdot \frac{\partial \mathbf{H}_\Gamma}{\partial Y^\Delta} \quad .$$

We will not make use of the speeds corresponding to (3.7,8) for the surface in the current configuration and accordingly omit their details, which are given, for example, in Chen ((1973), Section 4).

Although the two descriptions (3.1) and (3.9) describe the same surface, they are fundamentally different in that the spatial description (3.9) gives the geometry of the surface at time t whereas

(3.1) is the locus of <u>initial positions</u> of the particles that are on the surface Σ at time t (Truesdell and Toupin, (1960), Section 182). This means that in general the properties of the surface S_t (such as the curvature or the wave normal) may be very different to those viewed by an observer of the actual surface s_t , and in cases where this is so, the material description of the surface loses much of its appeal. We shall be concerned however, with deformations of the material that are static ahead of the wave and for which material wavefronts that are plane (resp. cylindrical, spherical) correspond to plane (resp. cylindrical, spherical) wavefronts in the spatial description. With these assumptions, the wave normals $\mathbf{n}$ and $\mathbf{m}$ in the material and spatial descriptions respectively coincide. In such cases, the material description of the wave surface <u>is</u> easily interpreted; we accordingly henceforth adopt the material description and thereby take advantage of the less cluttered analysis which it allows for. (The relative simplicity of the material description in the context of acceleration waves is also apparent in Eringen and Suhubi (1975), where the spatial description is only introduced at a penultimate stage of the analysis of wave growth).

The concept of a <u>singular surface</u> is now introduced; an extensive treatment of this topic is in Truesdell and Toupin ((1960), Chapter C), and we give merely a summary. A smooth surface S_t defined by (3.1) divides the body B into two regions B^+ and B^- , the surface forming the common boundary between B^+ and B^- . The unit normal $\mathbf{n}$ to S_t is directed towards B^+. We consider a function f(X,t) (which may be scalar-, vector- or tensor-valued), continuous within B^+ and B^- and with definite limits f^+ and f^- as S_t is approached from within B^+ and

B^- respectively. The jump of f at $X \in S_t$ is denoted by

$$[f(X)] \equiv f^+(X) - f^-(X)$$

and the surface S_t is singular with respect to f at time t if $[f] \neq 0$. (We subsequently drop the explicit dependence of jump quantities on their position on the surface, as this is unlikely to cause confusion). A surface S_t that is singular with respect to some quantity and has non-zero normal velocity, that is,

$$\nu(X,t) \neq 0 \quad ,$$

is said to be a wave (Truesdell and Toupin (1960), Section 183).

There are a number of conditions to be satisfied across S_t . These follow from applications of Hadamard's lemma (Truesdell and Toupin (1960), Sections 174,5), according to which a scalar-valued function χ and vector- or tensor-valued function V obey

$$\frac{d}{ds}[\chi] = [\text{Grad } \chi]\cdot\frac{d\ell}{ds}$$

and

$$\frac{d}{ds}[V] = [\text{Grad } V]\cdot\frac{d\ell}{ds} \tag{3.12}$$

respectively, where $\ell(s)$ is a curve on S_t .

If we choose for $\ell(s)$ the coordinate curves Y^{Γ} = constant, then the equations (3.12) take the form

$$\frac{\partial}{\partial Y^{\Gamma}}[\chi] = [\mathrm{Grad}\ \chi]\cdot \mathbf{H}_{\Gamma} \quad ,$$

$$\frac{\partial}{\partial Y^{\Gamma}}[\mathbf{V}] = [\mathrm{Grad}\ \mathbf{V}]\mathbf{H}_{\Gamma} \quad . \tag{3.13}$$

These results can in turn be expressed (Truesdell and Toupin (1960), Section 175) in the form:

$$[\mathrm{Grad}\ \chi] = [\mathrm{Grad}\ \chi\cdot\mathbf{n}]\mathbf{n} + \frac{\partial}{\partial Y^{\Gamma}}[\chi]\mathbf{H}^{\Gamma} \quad ,$$

$$[\mathrm{Grad}\ \mathbf{V}] = [(\mathrm{Grad}\ \mathbf{V})\mathbf{n}] \otimes \mathbf{n} + \frac{\partial}{\partial Y^{\Gamma}}[\mathbf{V}] \otimes \mathbf{H}^{\Gamma} \quad . \tag{3.14}$$

Summation over Γ (Γ = 1,2) is implied from (3.14) onwards and also implied for Δ = 1,2 from (3.15) onwards.

The results (3.14) are known as the <u>geometrical conditions of compatibility</u>, and the reduced forms that (3.14) take when $[\chi] = 0$ or when $[\mathbf{V}] = \mathbf{0}$ are known as Maxwell's theorem.

Since the conditions (3.14) are merely identities expressing the jump of a derivative in terms of the jump of the normal derivative and the tangential derivatives of the jump of the function, they can be iterated and expressions for jumps in second derivatives obtained. The conditions obtained in this way are known as <u>iterated geometrical</u>

conditions of compatibility, and are given by

$$[\mathrm{Grad}(\mathrm{Grad}\ \chi)] = [\mathbf{n}\cdot\mathrm{Grad}(\mathrm{Grad}\ \chi)\mathbf{n}]\ \mathbf{n}\otimes\mathbf{n}$$

$$+ (\mathbf{n}\otimes\mathbf{H}^{\Gamma} + \mathbf{H}^{\Gamma}\otimes\mathbf{n})[\mathrm{Grad}\ \chi\cdot\mathbf{n}]_{,\Gamma} - [\mathrm{Grad}\ \chi\cdot\mathbf{n}]\Omega^{\Gamma\Delta}\ \mathbf{H}_{\Gamma}\otimes\mathbf{H}_{\Delta}\ ,$$

$$[\mathrm{Grad}(\mathrm{Grad}\ \mathbf{V})] = [\mathrm{Grad}(\mathrm{Grad}\ \mathbf{V})(\mathbf{n},\mathbf{n})]\otimes\mathbf{n}\otimes\mathbf{n}$$

$$+ [(\mathrm{Grad}\ \mathbf{V})\mathbf{n}]_{,\Gamma}\otimes(\mathbf{n}\otimes\mathbf{H}^{\Gamma} + \mathbf{H}^{\Gamma}\otimes\mathbf{n})$$

$$- \Omega^{\Gamma\Delta}[\mathrm{Grad}(\mathrm{Grad}\ \mathbf{V})(\mathbf{n},\mathbf{n})]\otimes\mathbf{H}_{\Gamma}\otimes\mathbf{H}_{\Delta} \tag{3.15}$$

for the case in which $[\chi] = 0$ and $[\mathbf{V}] = \mathbf{0}$.

We require that the moving singular surface persist in time, that is, that discontinuities do not appear or disappear. This requirement is expressed in the kinematical condition of compatibility, which we now discuss. The rate of change of functions $\chi(\mathbf{X},t)$ and $\mathbf{V}(\mathbf{X},t)$ seen by an observer moving with the normal velocity $\nu\ \mathbf{n}$ (see (3.7,8)) is given by the displacement derivative $\delta/\delta t$ defined by

$$\frac{\delta\chi}{\delta t} = \dot{\chi} + \nu(\mathrm{Grad}\ \chi)\cdot\mathbf{n}$$

for scalars, and

$$\frac{\delta\mathbf{V}}{\delta t} = \dot{\mathbf{V}} + \nu(\mathrm{Grad}\ \mathbf{V})\mathbf{n}$$

for vectors and tensors. (3.16)

By an application of Hadamard's lemma along the path tangent to $\mathbf{n}$ the kinematical conditions of compatibility

$$\frac{\delta}{\delta t}[\chi] = [\dot{\chi}] + \nu[\mathrm{Grad}\ \chi\cdot\mathbf{n}]$$

and

$$\frac{\delta}{\delta t}[\mathbf{V}] = [\dot{\mathbf{V}}] + \nu[(\mathrm{Grad}\ \mathbf{V})\mathbf{n}] \qquad (3.17)$$

are obtained.

As with the geometrical compatibility conditions, the results (3.17) can be iterated, yielding the <u>iterated kinematical conditions of compatibility</u>

$$[\mathrm{Grad}\ \dot{\chi}] = [(\mathrm{Grad}\ \dot{\chi})\cdot\mathbf{n}]\mathbf{n} + [\dot{\chi}],_{\Gamma}\ \mathbf{H}^{\Gamma} \quad ,$$

$$[\ddot{\chi}] = -\ \nu[(\mathrm{Grad}\ \dot{\chi})\mathbf{n}] + \frac{\delta}{\delta t}[\dot{\chi}] \quad ,$$

$$[\mathrm{Grad}\ \dot{\mathbf{V}}] = [(\mathrm{Grad}\ \dot{\mathbf{V}})\mathbf{n}] \otimes \mathbf{n} + [\dot{\mathbf{V}}],_{\Gamma} \otimes \mathbf{H}^{\Gamma} \quad .$$

and

$$[\ddot{\mathbf{V}}] = -\ \nu[(\mathrm{Grad}\ \dot{\mathbf{V}})\mathbf{n}] + \frac{\delta}{\delta t}[\dot{\mathbf{V}}] \quad . \qquad (3.18)$$

It is possible to find expressions for $[(\mathrm{Grad}\ \dot{\chi})\cdot\mathbf{n}]$ and $[(\mathrm{Grad}\ \dot{\mathbf{V}})\mathbf{n}]$ (using for example $(3.17)_1$ with $\dot{\chi}$ replacing χ) to obtain the following alternatives to (3.18). These alternatives are known as Thomas's iterated kinematical conditions of compatibility. $[\chi]$ is taken as zero in $(3.19)_{1,2}$ and $[\mathbf{V}]$ is zero in $(3.19)_{3,4}$.

$$[\mathrm{Grad}\ \dot{\chi}] = -\ \nu[\mathbf{n}\cdot\mathrm{Grad}(\mathrm{Grad}\ \chi)\mathbf{n}]\mathbf{n} + \frac{\delta}{\delta t}[\mathrm{Grad}\ \chi\cdot\mathbf{n}]\mathbf{n}$$
$$-\ (\nu[\mathrm{Grad}\ \chi\cdot\mathbf{n}]),_{\Gamma}\ \mathbf{H}^{\Gamma}$$

and

$$[\ddot{\chi}] = \nu^2[\mathbf{n}\cdot\text{Grad}(\text{Grad }\chi)\mathbf{n}] - 2\nu \frac{\delta}{\delta t} [\text{Grad }\chi\cdot\mathbf{n}] - [\text{Grad }\chi\cdot\mathbf{n}] \frac{\delta\nu}{\delta t} ;$$

$$[\text{Grad }\dot{\mathbf{V}}] = -\nu[\text{Grad}(\text{Grad }\mathbf{V})(\mathbf{n},\mathbf{n})] \otimes \mathbf{n} + \frac{\delta}{\delta t} [(\text{Grad }\mathbf{V})\mathbf{n}] \otimes \mathbf{n}$$
$$- (\nu[(\text{Grad }\mathbf{V})\mathbf{n}])_{,\Gamma} \otimes \mathbf{H}^{\Gamma} ,$$

and

$$[\ddot{\mathbf{V}}] = \nu^2[\text{Grad}(\text{Grad }\mathbf{V})(\mathbf{n},\mathbf{n})] - 2\nu \frac{\delta}{\delta t} [(\text{Grad }\mathbf{V})\mathbf{n}] - [(\text{Grad }\mathbf{V})\mathbf{n}] \frac{\delta\nu}{\delta t}. \tag{3.19}$$

We conclude this discussion of the geometric and kinematic conditions of compatibility by giving special cases that will be useful later. When χ and $\mathbf{V}$ are continuous equations (3.14) take the forms

$$[\text{Grad }\chi] = [\text{Grad }\chi\cdot\mathbf{n}]\mathbf{n}$$

and

$$[\text{Grad }\mathbf{V}] = [(\text{Grad }\mathbf{V})\mathbf{n}] \otimes \mathbf{n} . \tag{3.20}$$

Furthermore, equations (3.17) with $[\chi] = 0$ and $[\mathbf{V}] = 0$ yield

$$[\text{Grad }\chi\cdot\mathbf{n}] = -\nu^{-1}[\dot{\chi}]$$

and

$$[(\text{Grad }\mathbf{V})\mathbf{n}] = -\nu^{-1}[\dot{\mathbf{V}}] . \tag{3.21}$$

Substitution for $[\text{Grad }\chi\cdot\mathbf{n}]$ and $[(\text{Grad }\mathbf{V})\mathbf{n}]$ from $(3.21)_{1,2}$ in $(3.20)_{1,2}$ respectively provide the required identities

$$[\text{Grad }\chi] = -\nu^{-1}[\dot{\chi}]\mathbf{n}$$

and

$$[\text{Grad }\mathbf{V}] = -\nu^{-1}[\dot{\mathbf{V}}] \otimes \mathbf{n} . \tag{3.22}$$

Acceleration waves

We follow Chen (1973) and define an acceleration wave to be a propagating singular surface for which

> $\mathbf{x}(\mathbf{X},t)$, $\dot{\mathbf{x}}(\mathbf{X},t)$, $\mathbf{F}(\mathbf{X},t)$ are continuous functions everywhere;
> $\ddot{\mathbf{x}}$, $\dot{\mathbf{F}}$, Grad $\ddot{\mathbf{F}}$, ${}^{\cdot}\dot{\mathbf{x}}^{\cdot}$, $\ddot{\mathbf{F}}$, Grad(Grad $\mathbf{F}$) have non-zero jumps across the singular surface but are continuous everywhere else.

The jump $[\ddot{\mathbf{x}}]$ in acceleration is known as the amplitude and we write

$$[\ddot{\mathbf{x}}] = \mathbf{s} \quad . \tag{3.23}$$

Now by making use of the compatibility conditions $(3.18)_3$ and $(3.18)_4$ with $\mathbf{V} = \ddot{\mathbf{x}}(\mathbf{X},t)$,

$$[\dot{\mathbf{F}}] = -\,\nu^{-1}\,\mathbf{s} \otimes \mathbf{n} \quad , \tag{3.24}$$

and by (3.15) and $(3.19)_3$ with $\mathbf{V} = \bar{\mathbf{x}}(\mathbf{X},t)$,

$$[\mathrm{Grad}\ \mathbf{F}] = -\,\nu^{-1}[\dot{\mathbf{F}}] \otimes \mathbf{n} = \nu^{-2}\,\mathbf{s} \otimes \mathbf{n} \otimes \mathbf{n} \quad . \tag{3.25}$$

The jumps in the third-order derivations of $\mathbf{x}$ that will be required later are found similarly and are:

$$[{}^{\cdot}\dot{\mathbf{x}}^{\cdot}] = 2\,\frac{\delta \mathbf{s}}{\delta t} - \nu^{-1}\,\frac{\delta \nu}{\delta t}\,\mathbf{s} + \nu^{2}\,\mathbf{w} \quad , \tag{3.26}$$

$$\begin{aligned}[\mathrm{Grad}\ \dot{\mathbf{F}}] = {} & \mathbf{w} \otimes \mathbf{n} \otimes \mathbf{n} - (\nu^{-1}\,\mathbf{s})_{,\Gamma} \otimes (\mathbf{n} \otimes \mathbf{H}^{\Gamma} + \mathbf{H}^{\Gamma} \otimes \mathbf{n}) \\ & + \nu^{-1}\Omega^{\Gamma\Delta}\,\mathbf{s} \otimes \mathbf{H}_{\Gamma} \otimes \mathbf{H}_{\Delta} \quad , \end{aligned} \tag{3.27}$$

where the vector $\mathbf{w}$ satisfies

$$\mathbf{w}\cdot\mathbf{a} = [\mathrm{Grad}\,\dot{\mathbf{F}}](\mathbf{a},\ \mathbf{n},\ \mathbf{n}) \qquad (3.28)$$

for an arbitrary vector $\mathbf{a}$.

We will often in this thesis be concerned with <u>principal waves</u>, which are waves travelling in the direction of one of the proper vectors $\mathbf{p}_i$ (defined following (2.5)). The principal directions are numbered in such a way that

$$\mathbf{n} = \mathbf{p}_3 \quad . \qquad (3.29)$$

We will also be considering <u>longitudinal waves</u>, whose defining property is that

$$\mathbf{s} \wedge \mathbf{m} = 0 \quad , \qquad (3.30)$$

and <u>transverse waves</u>, defined by

$$\mathbf{s}\cdot\mathbf{m} = 0 \quad . \qquad (3.31)$$

It is convenient to choose the material coordinates X^i in such a way that at time t the wavefront is on the surface X^3 = constant = C , say, and $X^\Gamma = Y^\Gamma$ so that

$$\mathbf{H}_\Gamma(X^\Delta) = \mathbf{G}_\Gamma(X^\Delta,\ C) \quad . \qquad (3.32)$$

Later we will focus attention on a class of irrotational deformations for which the principal directions $\{\mathbf{p}_i\}$ are tangent to the coordinate curves and we can write

$$\mathbf{p}_i = \frac{\mathbf{G}_i}{|\mathbf{G}_i|} \quad ;$$

since the motion is irrotational, $\mathbf{R} = \mathbf{I}$ and so

$$\mathbf{q}_i = \mathbf{p}_i \quad . \tag{3.33}$$

We also denote the magnitude of the amplitude $\mathbf{s}$ by σ :

$$\sigma = \sqrt{\mathbf{s}\cdot\mathbf{s}} \quad . \tag{3.34}$$

(The symbol σ was used previously in (3.9) to denote the surface s_t . No confusion is likely to arise from this in future).

In the event that a principal wave is a longitudinal wave, its amplitude will satisfy

$$\mathbf{s} = \sigma\,\mathbf{m} = \sigma\,\mathbf{q}_3 \quad . \tag{3.35}$$

Similarly, if a principal wave is a transverse wave its amplitude will satisfy

$$\mathbf{s} = \sigma\,\mathbf{q}_\Delta \quad , \quad \Delta = 1 \text{ or } 2 \quad . \tag{3.36}$$

We note, however, that in general it is possible to have longitudinal and transverse waves which are not principal waves.

Homothermal and homentropic waves in definite and non-conductors

We now consider the behaviour at the singular surface of the remaining variables characterizing the thermodynamic process. Attention is restricted to processes for which the body force **b** , the heat supply r and the density ρ are continuous, as are their first temporal and spatial derivatives. The constraint multipliers λ_a and γ_β in (2.39) are assumed continuous, but may have discontinuous derivatives at the wavefront.

The temperature θ and entropy η are assumed to be continuous at the wavefront, but their derivatives are not necessarily so. Since $[\theta] = 0$, the identities $(3.20)_1$ and $(3.21)_1$ with θ replacing χ apply and we obtain

$$[\mathrm{Grad}\ \theta] = T\mathbf{n} \quad , \quad [\dot{\theta}] = -\nu T \quad , \tag{3.37}$$

where

$$T = [\mathrm{Grad}\ \theta\cdot\mathbf{n}] \quad . \tag{3.38}$$

An acceleration wave is called homothermal if

$$T \equiv 0 \quad , \tag{3.39}$$

in which case the first derivatives of θ are continuous by (3.37).

Since η is continuous, $[\eta] = 0$ and use of $(3.20)_1$ and $(3.21)_1$ leads to the following results analogous to (3.37-9):

$$[\mathrm{Grad}\ \eta] = \mathrm{H}\mathbf{n} \quad , \quad [\dot{\eta}] = -\nu \mathrm{H} \quad , \tag{3.40}$$

where

$$\mathrm{H} = [\mathrm{Grad}\ \eta\cdot\mathbf{n}] \quad . \tag{3.41}$$

An acceleration wave is called homentropic if

$$\mathrm{H} \equiv 0 \quad . \tag{3.42}$$

Waves that are both homothermal and homentropic ($\mathrm{H} = \mathrm{T} = 0$) are called generalized transverse waves (see Chadwick and Currie (1974)).

We now investigate the conditions under which acceleration waves in definite and non-conductors are homothermal or homentropic.

Coleman and Gurtin (1965) used the entropy production inequality to establish that all acceleration waves in unconstrained definite conductors are homothermal. Reddy showed in (I) that this result also holds for constrained definite conductors, noting that the entropy production inequality (2.49) for constrained materials has no contributions from the constraints.

We now turn to the case of acceleration waves in constrained non-conductors. Reddy showed in (I) that the Coleman-Gurtin result (1965) for unconstrained materials that all acceleration waves in non-conductors are homentropic is also valid for materials subject to type I and II constraints. Reddy however, defined a non-conductor for constrained materials to be a material for which $\mathbf{q} = 0$ and we accordingly reconsider his analysis with the definition of a non-conductor as a material for which $q^0 = 0$ and $\mathbf{q} = \gamma_\beta \mathbf{z}^\beta$ by (2.53,4). We assume throughout that <u>entropy is constant ahead of the wave</u> for non-conductors, and use the energy equation (2.48) to determine the entropy jump H across the singular surface. We have at the wavefront

$$\rho \theta [\dot{\eta}] = - [\mathrm{Div}\ \mathbf{q}] \quad , \tag{3.43}$$

assuming the body force r to be continuous. Use of $(3.40)_2$ and (2.54) in (3.43) yields

$$- \rho \theta \nu H = - [\mathrm{Div}\ (\gamma_\beta \hat{\mathbf{z}}^\beta)] \quad , \tag{3.44}$$

where the superposed caret denotes the use of η as an independent variable, so $\mathbf{z}^\beta = \hat{\mathbf{z}}^\beta(\mathbf{F}, \theta(\mathbf{F}, \eta, \lambda_\alpha, \mathbf{e}_A), \mathbf{e}_A)$ (see (2.80) and following remark).

Clearly, if type II constraints are <u>absent</u>, then $H = 0$ by (3.44).

We now consider the situation in which at least one type II constraint is present, and evaluate the jump of (2.37) across the singular surface. After employing the identity $(3.37)_1$, we obtain the <u>type II constraint equation</u>

$$T\ \hat{\mathbf{z}}^\beta \cdot \mathbf{n} = 0 \quad , \quad \beta = 1, \ldots, L \quad . \tag{3.45}$$

The condition (3.45) suggests that we consider the following two cases separately in order to evaluate H in (3.44):

(i) $\hat{\mathbf{z}}^{\beta}\cdot\mathbf{n} \neq 0$

By (3.45), this situation is only possible for <u>homothermal</u> waves (for which $T \equiv 0$ by (3.39)). An expression for H is obtained by evaluating (3.44) in the thermal formulation (using θ and not η as an independent variable). We find that

$$- \rho\, \theta\, \nu\, H = - \,[(\gamma_\beta\, z^{\beta i})_{;i}] = - \,[\gamma_{\beta,i}]\, z^{\beta i} - \gamma_\beta\, [z^{\beta i}{}_{;i}]$$

$$= - \,[\gamma_{\beta,i}]\, z^{\beta i} - \gamma_\beta \left[\frac{\partial z^{\beta i}}{\partial F^k{}_\ell}\, [F^k{}_{\ell;i}] + \frac{\partial z^{\beta i}}{\partial \theta}\, [\theta_{,i}]\right] \quad , \tag{3.46}$$

where the vector and tensor components are relative to the basis vectors defined in (2.1,2), and the semi-colon denotes the covariant derivative.

Now $[\theta_{,i}] = 0$ for homothermal waves by $(3.37)_1$, (3.39), and expressions for $[\gamma_{\beta,i}]$ and $[F^k{}_{\ell;i}]$ are obtained from $(3.22)_1$ and (3.25) respectively. With these results, (3.46) yields

$$H = \rho^{-1}\theta^{-1}\nu^{-2}\left[[\dot{\gamma}_\beta]\, z^{\beta i} n_i - \nu^{-1}\, \gamma_\beta\, \frac{\partial z^{\beta i}}{\partial F^k{}_\ell}\, s^k n_\ell n_i\right] \quad , \tag{3.47}$$

and such waves are clearly not in general homentropic.

(ii) $\hat{\mathbf{z}}^{\beta}\cdot\mathbf{n} = 0$

Here, unlike case (i), no restriction is placed on T by (3.41). In evaluating (3.44), it is convenient to choose orthogonal curvilinear coordinates X^i with corresponding basis vectors $\mathbf{G}_i$ defined by (2.1) such that $\mathbf{n} = \mathbf{G}_3/|\mathbf{G}_3|$. That is, the wavefront coincides with the surface X^3 = constant. Then the condition $\hat{\mathbf{z}}^{\beta}\cdot\mathbf{n} = 0$ implies that

$$\hat{\mathbf{z}}^{\beta} = \hat{z}^{\beta\Gamma}\,\mathbf{G}_{\Gamma} \quad , \quad \Gamma = 1,2 \quad . \tag{3.48}$$

Equation (3.44) is now evaluated in the entropic formulation and we obtain

$$\begin{aligned}
-\rho\,\theta\,\nu\,H = &- [\gamma_{\beta,\Gamma}]\hat{z}^{\beta\Gamma} \\
&- \gamma_{\beta}\left[\frac{\partial\hat{z}^{\beta\Gamma}}{\partial F^{k}{}_{\ell}}\,[F^{k}{}_{\ell;\Gamma}] + \frac{\partial\hat{z}^{\beta\Gamma}}{\partial\eta}\,[\eta_{,\Gamma}] + \frac{\partial\hat{z}^{\beta\Gamma}}{\partial\lambda_{\alpha}}\,[\lambda_{\alpha,\Gamma}]\right] \\
= &\left[\nu^{-1}\,[\dot{\gamma}_{\beta}]\hat{z}^{\beta\Gamma}\right. \\
&- \gamma_{\beta}\left\{\nu^{-2}\,\frac{\partial\hat{z}^{\beta\Gamma}}{\partial F^{k}{}_{\ell}}\,s^{k}n_{\ell} + \frac{\partial\hat{z}^{\beta\Gamma}}{\partial\eta}\,H\right. \\
&\left.\left. - \nu^{-1}\,\frac{\partial\hat{z}^{\beta\Gamma}}{\partial\lambda_{\alpha}}\,[\dot{\lambda}_{\alpha}]\right\}\right]n_{\Gamma} \quad ,
\end{aligned} \tag{3.49}$$

where in evaluating the jumps $[\gamma_{\beta,\Gamma}]$, $[\lambda_{\alpha,\Gamma}]$, $[F^{k}{}_{\ell;\Gamma}]$ and $[\eta_{,\Gamma}]$, the identities $(3.22)_1$ (twice), (3.25) and $(3.40)_1$ respectively have been used.

Now $\mathbf{n} = \mathbf{G}_3/|\mathbf{G}_3|$ implies that $n_\Gamma = 0$, $\Gamma = 1,2$ in (3.49) and consequently we have the result that

$$H = 0 \quad , \tag{3.50}$$

so that the waves are homentropic by (3.42).

The above results for acceleration waves in constrained definite and non-conductors can be summarized as follows:

(i) All waves in definite conductors are <u>homothermal</u>.

(ii) Waves in constrained non-conductors (as defined by (2.53,4)) are <u>homothermal</u> if the material is subject to at least one type II constraint for which $\mathbf{z}^\beta \cdot \mathbf{n} \neq 0$, and such waves are not in general homentropic.

(iii) Waves in constrained non-conductors are <u>homentropic</u> if all type II constraints present satisfy $\mathbf{z}^\beta \cdot \mathbf{n} = 0$, or if type II constraints are absent.

CHAPTER 4

PROPAGATION CONDITIONS FOR HOMOTHERMAL WAVES

Introduction

We begin by deriving the first and second propagation conditions for homothermal waves. These conditions are obtained from an evaluation across the wavefront of the equation of motion and the time derivative of the type I constraint equation (2.36) respectively. The derivations given here extend the treatment in (I) to include type I constraints whose corresponding constraint vectors $\mathbf{c}^{a}$ are linearly dependent. The first propagation condition is found to be of Fresnel-Hadamard type (Truesdell and Noll (1965, Section 71)), and is a modification of the corresponding propagation condition for unconstrained materials. The second propagation condition, however, has no parallel in the case of unconstrained materials. Both the first and second propagation conditions involve only the type I constraints; the type II constraints yield instead the condition (3.45) discussed previously in the context of homothermal and homentropic wave propagation in non-conductors.

After derivation of the propagation conditions, the material is taken to be isotropic and we investigate both longitudinal and transverse principal wave solutions of the propagation conditions. The speed and amplitude of the waves are obtained from the proper numbers and proper vectors respectively of the acoustic tensor that appears in the first propagation condition, and the second propagation condition restricts the amplitude to the subspace orthogonal to $\mathbf{c}^{a}$. The solutions for the all-embracing cases $M = 0,1,2$ are discussed in turn; for $M = 0$ it is shown that the constraints have <u>no effect</u> on the wave speed. We then discuss the influence of isotropic and directional constraints on the strong ellipticity condition; this condition when applied to the acoustic tensor (Scott (1975), Ogden (1984)) ensures that the proper numbers of the acoustic tensor (and hence the squares of the corresponding wave speeds) are positive. This is followed by a further analysis of the propagation conditions that focuses on the influence of isotropic and directional constraints on longitudinal and transverse principal waves.

Firstly, we consider homothermal wave propagation in definite conductors that are subject to a particular class of irrotational deformations and are assumed to be at rest and at constant temperature ahead of the wave. (Later, in Chapters 6 and 7, we will treat wave growth in materials subject to the above conditions). We treat three distinct situations: plane waves (resp. cylindrical, spherical) propagating in materials subject to plane (resp. cylindrical, spherical) deformations where the directional constraints present are so configured that they make a constant angle with the plane (resp. cylindrical, spherical) coordinate curves. (Note that there is an entirely different constraint configuration in general when different waveforms are considered; i.e. we are not making a general comparison of the behaviour of plane, cylindrical and spherical waves in a given material situation). The analysis yields the range of deformations, compatible with the constraints, corresponding to which the propagation of waves is possible. These results are illustrated by the analysis of longitudinal and transverse wave propagation in materials subject to the four constraint examples introduced in Chapter 2 (see (2.67,70, 72,73)). The constraints are treated singly and in combinations of two, three or four; it is to be noted that in a number of cases the constraint vectors c^a take the value zero, or are found to be linearly dependent when the constraints act in combination. In each case, the restrictions (if any) on the deformation are presented.

We differ in this work from most authors by restricting attention to principal waves, which at times clarifies the details of the constraint contributions. Apart from this difference, there are close analogies with the results of Ogden (1974) for acceleration waves in incompressible materials and those of Scott (1975,76,85) for acceleration waves in materials subject to linearly independent mechanical constraints. Whitworth (1982) and Chadwick, Whitworth and Borejko (1985) obtain corresponding solutions for simple waves and small-amplitude waves respectively in materials subject to arbitrary mechanical constraints. Chen and Nunziato (1975) treat acceleration waves in inextensible elastic bodies which are also subject to the thermomechanical constraint of perfect heat conduction in the direction of inextensibility.

Derivation of the propagation conditions for homothermal waves

We now derive the propagation conditions for acceleration waves for a material subject to type I constraints (and possibly also type II constraints, although these do not appear in the propagation conditions). We begin with the definition (2.36) of type I constraints and consider the jump in $\dot{\phi}^a$ at the wavefront: from (2.36),

$$0 = [\dot{\phi}^a] = \frac{\partial \phi^a}{\partial \mathbf{F}} \cdot [\dot{\mathbf{F}}] + \frac{\partial \phi^a}{\partial \theta} [\dot{\theta}]$$

$$= -\nu^{-1} \frac{\partial \phi^a}{\partial \mathbf{F}} \cdot (\mathbf{s} \otimes \mathbf{n}) - \nu \frac{\partial \phi^a}{\partial \theta} \mathrm{T}$$

using (3.24) and $(3.37)_2$.

With the aid of (2.85) and (2.94) we obtain

$$\mathbf{c}^a \cdot \mathbf{s} = - \nu^2 \omega^a \mathrm{T} \quad , \quad a=1,\ldots,N \quad . \tag{4.1}$$

Since we restrict attention in this chapter to homothermal waves, $\mathrm{T} \equiv 0$ by (3.39). Thus (4.1) becomes

$$\mathbf{c}^a \cdot \mathbf{s} = 0 \quad , \tag{4.2}$$

so that a necessary condition for the existence of homothermal waves is that

$$\dim \operatorname{span} \{\mathbf{c}^a\} = M \leq 2 \quad . \tag{4.3}$$

We recall from (2.88) that for isotropic constraints the constraint vectors $\mathbf{c}^a$ take the form

$$\mathbf{c}^a = \sum_i \frac{\partial \phi^a}{\partial a_i} (\mathbf{p}_i \cdot \mathbf{n}) \, \mathbf{q}_i \tag{2.88 bis}$$

so that for these constraints (4.2) becomes

$$\sum_i \frac{\partial \phi^a}{\partial a_i} (\mathbf{p}_i \cdot \mathbf{n})(\mathbf{q}_i \cdot \mathbf{s}) = 0 \quad . \tag{4.4}$$

For directional constraints, the vectors $\mathbf{c}^a$ take the form

$$\mathbf{c}^a = \beta^a_{AB} \sum_i a_i (\mathbf{p}_i \cdot \mathbf{e}_A)(\mathbf{n} \cdot \mathbf{e}_B)\ \mathbf{q}_i \tag{2.89 bis}$$

(recall that we sum on repeated indices A,B) and in this case (4.2) becomes

$$\beta^a_{AB} \sum_i a_i (\mathbf{p}_i \cdot \mathbf{e}_A)(\mathbf{n} \cdot \mathbf{e}_B)(\mathbf{q}_i \cdot \mathbf{s}) = 0 \quad . \tag{4.5}$$

Clearly equations (4.4,5) establish necessary conditions to be satisfied by the wave amplitude $\mathbf{s}$. We will explore further aspects of (4.4,5) later in this chapter.

We now turn our attention to the equation of motion. The local form of the equation for the balance of linear momentum is

$$\text{Div}\ \mathbf{S} + \rho \mathbf{b} = \rho \ddot{\mathbf{x}} \quad . \tag{2.9 bis}$$

At the singular surface S_t , (2.9) together with the assumption that the body force vector $\mathbf{b}$ is continuous imply that

$$[\dot{\mathbf{S}}]\mathbf{n} = -\ \rho\ \nu\ \mathbf{s} \quad , \tag{4.6}$$

where we have made use of the identity $[\mathrm{Div}\ \mathbf{S}] = -\ \nu^{-1}[\dot{\mathbf{S}}]\mathbf{n}$ which follows from $(3.22)_2$.

Now the use of $(2.46)_2$ for $\mathbf{S}$ together with the definitions (2.85) for $\mathbf{c}^a$ and (2.92) for $\mathbf{M}$ yield the expression

$$[\dot{\mathbf{S}}]\mathbf{n} = (\mathbf{A}[\dot{\mathbf{F}}])\mathbf{n} + \mathbf{M}\ \mathbf{n}[\dot{\theta}] + \rho[\dot{\lambda}_a]\ \mathbf{c}^a \quad , \tag{4.7}$$

where $\mathbf{A}$ is the fourth-order tensor of elastic moduli defined by

$$\mathbf{A} = \rho\ \frac{\partial^2\psi}{\partial\mathbf{F}\partial\mathbf{F}} \quad . \tag{4.8}$$

Now (3.24) provides an expression for $[\dot{\mathbf{F}}]$, and $[\dot{\theta}] = 0$ for homothermal waves by $(3.37)_2$, (3.39). With the aid of these results, the substitution of (4.7) in (4.6) yields

$$(\rho\nu^2\ \mathbf{I} - \mathbf{Q})\mathbf{s} = -\ \rho\nu\ [\dot{\lambda}_a]\ \mathbf{c}^a \quad , \tag{4.9}$$

where the second-order <u>acoustic tensor</u> $\mathbf{Q}$ is defined by

$$\mathbf{Q}(\mathbf{u},\mathbf{v}) = \mathbf{A}(\mathbf{u},\mathbf{n},\mathbf{v},\mathbf{n}) \qquad \text{for all } \mathbf{u},\mathbf{v} \in V \quad , \tag{4.10}$$

with components $\quad Q_{ik} = A_i{}^j{}_k{}^\ell n_j n_\ell \qquad (4.11)$

relative to the tangent bases $\mathbf{G}^j$ and $\mathbf{g}_i$, and $\mathbf{I}$ is the identity tensor. In view of the definition (4.8) of $\mathbf{A}$ we have $A_i{}^j{}_k{}^\ell = A_k{}^\ell{}_i{}^j$ so that $\mathbf{Q}$ is <u>symmetric</u>.

Equation (4.9) was examined in (I) for linearly independent $\mathbf{c}^a$ and in (II) for collinear $\mathbf{c}^a$. Here these treatments are extended to allow for the linearly dependent set of constraint vectors introduced in (2.86,7).

We first eliminate the jumps $[\dot{\lambda}_a]$ in (4.9). The right-hand side of (4.9) may be written as

$$- \rho \, \nu[\dot{\lambda}_a]\mathbf{c}^a = - \rho \, \nu([\dot{\lambda}_\sigma] + D^\mu_\sigma \, [\dot{\lambda}_{M+\mu}]) \, \mathbf{c}^\sigma \tag{4.12}$$

where, as in (2.86), $\sigma = 1, \ldots, M$; $\mu = 1, \ldots, P\text{-}M$, and we note that the set $\left\{[\dot{\lambda}_{P+\eta}]\right\}_{\eta=1}^{N-P}$ corresponding to the vectors $\mathbf{c}^{P+\eta}$ in (2.87) which are identically zero does not appear.

A set $\left\{\mathbf{d}_\tau\right\}_{\tau=1}^{M}$ of vectors is defined that is reciprocal to $\mathbf{c}^\sigma$ in the sense that

$$\mathbf{c}^\sigma \cdot \mathbf{d}_\tau = \delta^\sigma_\tau \quad , \quad \mathbf{d}_\tau \in \text{span} \, \{\mathbf{c}^\sigma\} \quad . \tag{4.13}$$

For homothermal waves, $\mathbf{c}^\sigma \cdot \mathbf{s} = 0$ by (4.2) and hence $\mathbf{d}_\tau \cdot \mathbf{s} = 0$; the scalar product of (4.9) with $\mathbf{d}_\tau$ together with the use of (4.12) yields the expression

$$\mathbf{d}_\tau \cdot \mathbf{Q} \, \mathbf{s} = \rho \, \nu([\dot{\lambda}_\tau] + D^\mu_\tau \, [\dot{\lambda}_{M+\mu}]) \quad , \tag{4.14}$$

so that (4.9) can be written as

$$(\rho\nu^2 \, \mathbf{I} - \mathbf{Q})\mathbf{s} = - \, (\mathbf{d}_\sigma \cdot \mathbf{Q} \, \mathbf{s}) \, \mathbf{c}^\sigma \quad . \tag{4.15}$$

Rearrangement of (4.15) yields the equation

$$(\rho v^2 \mathbf{I} - \mathbf{P}\,\mathbf{Q})\mathbf{s} = 0 \quad , \tag{4.16}$$

where the projection tensor $\mathbf{P}$ is defined by

$$\mathbf{P} = \mathbf{I} - \mathbf{c}^{\sigma} \otimes \mathbf{d}_{\sigma} \quad . \tag{4.17}$$

Clearly $\mathbf{P}$ leaves invariant vectors $\mathbf{s}$ which are orthogonal to $\mathbf{c}^{a}$ (and hence $\mathbf{d}_{a}$) since

$$\mathbf{P}\,\mathbf{s} = \mathbf{s} - (\mathbf{d}_{\sigma}\cdot\mathbf{s})\,\mathbf{c}^{\sigma} = \mathbf{s} \quad . \tag{4.18}$$

We adopt the terminology of Chadwick, Whitworth and Borejko (1985) introduced in their study of small-amplitude waves in a constrained elastic body, and refer to the equations (4.16) and (4.2) as the <u>propagation conditions</u> of the wave. For ease of reference, these equations are collected here and renumbered:

$$(\rho v^2 \mathbf{I} - \mathbf{P}\,\mathbf{Q})\mathbf{s} = 0 \quad , \tag{4.19}$$

$$\mathbf{c}^{a}\cdot\mathbf{s} = 0 \quad , \qquad a=1,\dots,N \quad . \tag{4.20}$$

We will refer to (4.19) and (4.20) as the <u>first</u> and <u>second</u> propagation conditions respectively, and note that in the corresponding unconstrained situation, (4.20) does not occur, so that (4.19) with $\mathbf{P} = \mathbf{I}$ is the sole propagation condition for the wave. We also note that, in view of (4.18), the first propagation condition (4.19) implies the condition (4.20), but that the converse is not true.

We now proceed with the investigation of the propagation conditions by imposing the condition of isotropy, and deriving the form of $\mathbf{A}$ (and thereafter of $\mathbf{Q}$) for isotropic materials subject to type I constraints. (Type II constraints, if present, do not enter into the expression for $\mathbf{A}$). We restrict attention to isotropic or directional type I constraints, for which $\phi^a = \phi^a(a_i, \theta)$ by (2.66) and $\phi^a = \phi^a(e_{AB}, f_{AB}, \theta)$ by (2.68,9) respectively. It is therefore sufficient in deriving the required expression for $\mathbf{A}$ to assume that $\phi^a = \phi^a(a_i, \theta, e_{AB}, f_{AB})$ and to ignore the dependence which ϕ^a may have on k_{AB} (see (2.60-3)). (The expression for $\mathbf{A}$ that results if such dependence on k_{AB} is included is given in (II), and is found to be considerably more cumbersome than that given below). The analysis of Appendix A yields the required expression for $\mathbf{A}$, which is found to be

$$\rho^{-1}\mathbf{A} = \sum_{i,j}\Bigg[q_i \otimes p_j \otimes \Bigg\{\frac{a_i\partial\psi/\partial a_i - a_j\partial\psi/\partial a_j}{a_i^2 - a_j^2}(1-\delta_{ij})q_i \otimes p_j + \frac{a_j\partial\psi/\partial a_i - a_i\partial\psi/\partial a_j}{a_i^2 - a_j^2}(1-\delta_{ij})q_j \otimes p_i\Bigg\} + \frac{\partial^2\psi}{\partial a_i\partial a_j} q_i \otimes p_i \otimes q_j \otimes p_j + 2\lambda_a \frac{\partial\phi^a}{\partial f_{AB}} q_i \otimes e_A \otimes q_i \otimes e_B\Bigg] . \tag{4.21}$$

Here it is assumed that $a_i \neq a_j$; the slight modifications for $a_i = a_j$ follow those given by Chadwick and Ogden (1971b).

Since we will be primarily concerned with the propagation of <u>principal waves</u> we specialize here for convenience and derive the acoustic tensor for the case in which the elastic moduli are given by (4.21), for principal waves propagating in the direction $\mathbf{n} = \mathbf{p}_3$ (recall 3.29)). Substitution of the expression (4.21) for $\mathbf{A}$ and (3.29) in the definition (4.10) for $\mathbf{Q}$ yields

$$\rho^{-1}\,\mathbf{Q} = \sum_i \Bigg[(1 - \delta_{i3}) \left\{ \frac{a_i \partial\psi/\partial a_i - a_3 \partial\psi/\partial a_3}{a_i^2 - a_3^2} \right\} \mathbf{q}_i \otimes \mathbf{q}_i + \frac{\partial^2 \psi}{\partial a_3^2}\, \mathbf{q}_3 \otimes \mathbf{q}_3$$

$$+ 2\,\lambda_\alpha \frac{\partial \phi^\alpha}{\partial f_{AB}} (\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B)\, \mathbf{q}_i \otimes \mathbf{q}_i \Bigg] \qquad (4.22)$$

when $a_\Gamma \neq a_3$, $\Gamma = 1$ or 2;

$$\rho^{-1}\,\mathbf{Q} = \sum_i \Bigg[(1 - \delta_{i3}) \left\{ \frac{1}{a}\frac{\partial\psi}{\partial a_i} + \frac{\partial^2\psi}{\partial a_i^2} - \frac{\partial^2\psi}{\partial a_i \partial a_3} \right\} \mathbf{q}_i \otimes \mathbf{q}_i + \frac{\partial^2 \psi}{\partial a_3^2}\, \mathbf{q}_3 \otimes \mathbf{q}_3$$

$$+ 2\,\lambda_\alpha \frac{\partial \phi^\alpha}{\partial f_{AB}} (\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B)\, \mathbf{q}_i \otimes \mathbf{q}_i \Bigg] \qquad (4.23)$$

when $a_\Gamma = a_3$, $\Gamma = 1$ or 2 , or when $a_1 = a_2 = a_3 = a$.

Clearly $\mathbf{Q}$ is in <u>spectral form</u> in both (4.22) and (4.23), with proper vectors $\mathbf{q}_i$ and corresponding proper numbers given by

$$Q_i = \mathbf{Q}(\mathbf{q}_i\ ,\ \mathbf{q}_i) \qquad \text{(no sum on i)} \quad , \qquad (4.24)$$

and we also introduce the notation

$$Q_i = Q_i^0 + \lambda_\alpha\, Q_i^\alpha \quad , \qquad (4.25)$$

where $\quad \mathbf{Q}_i^0 = \mathbf{Q}^0(\mathbf{q}_i\ ,\ \mathbf{q}_i) = \rho\ \dfrac{\partial^2\psi^0}{\partial\mathbf{F}\partial\mathbf{F}}\ (\mathbf{q}_i,\mathbf{n},\mathbf{q}_i,\mathbf{n})$

and $\quad \mathbf{Q}_i^\alpha = \mathbf{Q}^\alpha(\mathbf{q}_i\ ,\ \mathbf{q}_i) = \rho\ \dfrac{\partial^2\phi^\alpha}{\partial\mathbf{F}\partial\mathbf{F}}\ (\mathbf{q}_i,\mathbf{n},\mathbf{q}_i,\mathbf{n})$.

It will be found that for waves to propagate with non-zero speed ν in the direction $\mathbf{q}_i$, the corresponding proper number $\mathbf{Q}_i$ must necessarily be positive. A sufficient condition for the proper numbers $\mathbf{Q}_i$ to be positive is that $\mathbf{A}$ satisfies the strong ellipticity condition

$$\mathbf{A}(\mathbf{v},\ \mathbf{u},\ \mathbf{v},\ \mathbf{u}) > 0 \qquad \text{for all } \mathbf{u},\ \mathbf{v} \in V$$

in which case

$$\mathbf{Q}(\mathbf{v}\ ,\ \mathbf{v}) > 0 \quad , \quad \text{for all } \mathbf{v} \in V \tag{4.26}$$

as obtained by Truesdell and Toupin, (1965, equation (71.15)) for unconstrained materials.

The influence of the constraints on (4.26) will be discussed later, but here it is simply assumed that conditions are such that (4.26) holds and consequently that the $\mathbf{Q}_i$ are positive.

We examine now whether longitudinal and transverse waves are possible. In these cases $\mathbf{s} = \sigma\ \mathbf{q}_i$, i = 1,2, or 3. By (4.22,3) then, $\mathbf{Q}\ \mathbf{s}$ is parallel to $\mathbf{s}$ since from (4.22)

$$\rho^{-1}\ \mathbf{Q}\ \mathbf{q}_i = \left[(1 - \delta_{i3})\ \frac{a_i\partial\psi/\partial a_i - a_3\partial\psi/\partial a_3}{a_i^2 - a_3^2}\ (\mathbf{q}_i\cdot\mathbf{q}_i) + \frac{\partial^2\psi}{\partial a_3^2}\ (\mathbf{q}_3\cdot\mathbf{q}_i)\right.$$
$$\left. + 2\ \lambda_\alpha\ \frac{\partial\phi^\alpha}{\partial f_{AB}}\ (\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B)(\mathbf{q}_i\cdot\mathbf{q}_i)\right]\mathbf{q}_i \quad ; \tag{4.27}$$

and a similar result follows from (4.23). The effect of $\mathbf{P}$ on $\mathbf{Q}$ s is that

$$\mathbf{P}\,\mathbf{Q}\,\mathbf{s} = \mathbf{Q}\,\mathbf{s} \text{ for } \mathbf{s} \text{ such that } \mathbf{c}^{a}\cdot\mathbf{s} = 0 \quad ,$$

and the first propagation condition (4.19) is therefore equivalent to the condition

$$(\rho\nu^2\,\mathbf{I} - \mathbf{Q})\mathbf{s} = 0 \quad , \tag{4.28}$$

for all $\mathbf{s}$ that satisfy the second propagation condition (4.20), namely that $\mathbf{c}^{a}\cdot\mathbf{s} = 0$.

Longitudinal and transverse principal wave solutions of the propagation conditions

We now obtain longitudinal and transverse principal wave solutions to the propagation conditions (4.19,20). For homothermal waves, dim span $\{\mathbf{c}^{a}\} = M \leq 2$ by (4.3) and we consider the three cases $M = 0$, 1,2 in turn. Wave speeds ν_1, ν_2, ν_3 corresponding to the wave amplitudes $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ respectively are obtained where these are not precluded by the constraints.

(i) $M = 0$

The constraint set $\{\mathbf{c}^{a}\}$ is inactive when $M = 0$ by the definition following (2.87). Clearly $\mathbf{P} = \mathbf{I}$ by (4.17) and the vectors $\mathbf{c}^{a}$ have no effect on the propagation conditions (4.19,20). The discussion of Truesdell and Noll (1965, Section 71) applies; transverse waves with amplitudes $\mathbf{q}_1$, $\mathbf{q}_2$ and longitudinal waves with amplitude $\mathbf{q}_3$ are possible with corresponding wave speeds ν_i given by

$$\nu_i^2 = \rho^{-1} Q_i \quad , \qquad i = 1,2,3 \quad , \tag{4.29}$$

where Q_i are as defined in (4.24). Clearly, the Q_i must be positive for non-zero propagation speeds ν_i to exist, and the strong ellipticity condition (4.26) ensures that this is so. Although the form of (4.19) (with $\mathbf{P} = \mathbf{I}$) and (4.29) is identical to the unconstrained case discussed by Truesdell and Noll (1965), the values of Q_i (and hence ν_i) are in general influenced by the type I constraints and we return to this point later.

(ii) <u>$M = 1$</u>

Equation (2.86) with $M = 1$ implies that the set $\{c^a\}$ is fully active if $N = 1$ and partially active if $N > 1$. In the latter situation the constraints c^1, ..., c^N are collinear since $M = 1$ and the second propagation condition (4.20) takes the form

$$c^1 \cdot s = 0 \quad . \tag{4.30}$$

The projection tensor $\mathbf{P}$ defined by (4.17) now takes the form

$$\mathbf{P} = \mathbf{I} - \mathbf{c} \otimes \mathbf{c} \qquad \text{where } \mathbf{c} \text{ is the unit vector } \mathbf{c}^1/|\mathbf{c}^1| \quad . \tag{4.31}$$

We now distinguish two further cases:

(ii_a) $\quad$ $\mathbf{c}$ is parallel to one of the proper vectors $\mathbf{q}_i$;

(ii_b) $\quad$ $\mathbf{c}$ lies in the plane of $\mathbf{q}_i$ and $\mathbf{q}_j$. (Since we are dealing with principal waves that are either longitudinal or transverse, the case of $\mathbf{c}$ not orthogonal to any of the $\mathbf{q}_i$ does not arise.)

For case (ii_a) we take $\mathbf{c} = \mathbf{q}_1$ for definiteness, so $\mathbf{P}$ takes the form

$$\mathbf{P} = \mathbf{q}_2 \otimes \mathbf{q}_2 + \mathbf{q}_3 \otimes \mathbf{q}_3$$

and

$$\mathbf{P}\,\mathbf{Q} = Q_2\, \mathbf{q}_2 \otimes \mathbf{q}_2 + Q_3\, \mathbf{q}_3 \otimes \mathbf{q}_3 \quad . \tag{4.32}$$

Hence for $\mathbf{c}$ parallel to $\mathbf{q}_1$ (resp. $\mathbf{q}_2$, $\mathbf{q}_3$) , there are two proper values of $\mathbf{P}\,\mathbf{Q}$, namely Q_2 , Q_3 (resp. Q_3 , Q_1 ; Q_1 , Q_2) where the Q_i are as defined in (4.24), with corresponding proper vectors $\mathbf{q}_2$, $\mathbf{q}_3$ (resp. $\mathbf{q}_3$, $\mathbf{q}_1$; $\mathbf{q}_1$, $\mathbf{q}_2$) .

For case (ii_b) we consider for definiteness $\mathbf{c} = \alpha\, \mathbf{q}_1 + \beta\, \mathbf{q}_2$ where α and β are scalars such that $\alpha^2 + \beta^2 = 1$.

By (4.31),

$$\mathbf{P} = \mathbf{I} - \mathbf{c} \otimes \mathbf{c}$$

$$= (1-\alpha^2)\mathbf{q}_1 \otimes \mathbf{q}_1 + (1-\beta^2)\mathbf{q}_2 \otimes \mathbf{q}_2 + \mathbf{q}_3 \otimes \mathbf{q}_3 - \alpha\,\beta(\mathbf{q}_1 \otimes \mathbf{q}_2 + \mathbf{q}_2 \otimes \mathbf{q}_1),$$

so

$$\mathbf{P}\,\mathbf{Q} = (1-\alpha^2)Q_1\, \mathbf{q}_1 \otimes \mathbf{q}_1 + (1-\beta^2)Q_2\, \mathbf{q}_2 \otimes \mathbf{q}_2 + Q_3\, \mathbf{q}_3 \otimes \mathbf{q}_3$$
$$- \alpha\,\beta(Q_2\, \mathbf{q}_1 \otimes \mathbf{q}_2 + Q_1\, \mathbf{q}_2 \otimes \mathbf{q}_1) \quad . \qquad (4.33)$$

Since we consider only longitudinal or transverse principal waves, we are interested in proper vectors of $\mathbf{P}\,\mathbf{Q}$ that are parallel to one of the principal axes $\mathbf{q}_i$. Now $\mathbf{c}\cdot\mathbf{s} = 0$ by (4.30) and we are considering a constraint vector $\mathbf{c} = \alpha\, \mathbf{q}_1 + \beta\, \mathbf{q}_2$. Hence $\mathbf{s} = \sigma\, \mathbf{q}_3$ is the only possibility for a proper vector of $\mathbf{P}\,\mathbf{Q}$. In general then, for $\mathbf{c} \in$ span $\{\mathbf{q}_1 , \mathbf{q}_2\}$ (resp. $\{\mathbf{q}_2 , \mathbf{q}_3\}$, $\{\mathbf{q}_3 , \mathbf{q}_1\}$) there is one proper value Q_3 (resp. $\{Q_1 , Q_2\}$ with corresponding proper vector $\mathbf{q}_3$ (resp. $\mathbf{q}_1$, $\mathbf{q}_2$) .

The possibility of a <u>zero</u> proper number for $\mathbf{P}\,\mathbf{Q}$ is spurious, for then we would have

$$\mathbf{P}\,\mathbf{Q}\,\mathbf{s} = 0$$

and so by the discussion following (4.27)

$$\mathbf{Q}\,\mathbf{s} = 0 \quad . \qquad (4.34)$$

Since $\mathbf{Q}$ is positive definite by (4.25), it is non-singular and so (4.34) implies

$$\mathbf{s} = 0 \quad . \qquad (4.35)$$

(iii) $\underline{M = 2}$

Since $M = 2$, we see from (2.86) that the set $\{c^a\}$ is fully active if $N = 2$ and partially active if $N > 2$. We employ $\{c^1, c^2, \mathbf{a}\}$ as a basis, where $\mathbf{a}$ is a unit vector satisfying $c^a \cdot \mathbf{a} = 0$ for all a . Since only principal waves that are longitudinal or transverse are considered, $\mathbf{a}$ parallel to a principal axis q_i is the only possibility if the second propagation condition (4.20) is to hold. We take $\mathbf{a} = q_k$ for k = 1, 2 or 3 in which case

$$\mathbf{I} = c^\gamma \otimes d_\gamma + q_k \otimes q_k \quad , \quad \gamma = 1,2, \tag{4.36}$$

where $c^\sigma \cdot d_\tau = \delta^\sigma_\tau$ as in (4.13).

Now $\mathbf{P} = q_k \otimes q_k$ by (4.17) and (4.36) and

$$\mathbf{Q} = \bar{Q}_\gamma{}^\delta \, c^\gamma \otimes d_\delta + \bar{Q}_\gamma{}^k \, c^\gamma \otimes q_k + \bar{Q}_k{}^\gamma \, q_k \otimes d_\gamma + Q_k \, q_k \otimes q_k \quad .$$

Hence

$$\mathbf{P}\,\mathbf{Q} = \bar{Q}_k{}^\gamma \, q_k \otimes d_\gamma + Q_k \, q_k \otimes q_k \quad , \quad \gamma, \delta = 1,2 \quad , \tag{4.37}$$

where the $\bar{Q}_i{}^j$ are components of $\mathbf{Q}$ relative to the basis vectors c^γ, d^γ, q_k . We therefore have the same result as for case (ii$_b$) and the discussion there applies; once again, the possibility $\mathbf{P}\,\mathbf{Q}\,\mathbf{s} = \mathbf{0}$ is <u>spurious</u>.

The results of case (i), (ii) and (iii) have close parallels to those given by previous authors. Ogden (1974) considers acceleration waves in incompressible elastic materials and Scott (1975,6) treats acceleration waves in elastic materials subject to mechanical constraints for which the vectors $\mathbf{c}^{\alpha}$ are linearly independent (see also Scott and Hayes (1976) for small amplitude sinusoidal waves). Whitworth (1982) considers the related case of mechanically constrained simple waves and includes in his analysis the situation where the constraint vectors are linearly dependent. As far as thermomechanical constraints are concerned, Chen and Nunziato (1975) treat acceleration waves in perfectly heat conducting inextensible elastic bodies, thereby extending the earlier results of Chen and Gurtin (1974) for the single mechanical constraint of inextensibility. We have considered the propagation conditions only for principal waves that are longitudinal or transverse (a restriction not generally imposed by the above authors) but there is no intrinsic difficulty in extending the present analysis to cater for non-principal waves (see (I) for the case of thermomechanical constraints with linearly independent vectors $\mathbf{c}^{\alpha}$), and details are omitted.

<u>Influence of constraints on the strong ellipticity condition for $\mathbf{Q}$</u>

We have assumed so far that the strong ellipticity condition (4.26) holds. We now discuss the influence of the constraints on this condition. It has been shown that for principal waves, the proper values of $\mathbf{Q}$ corresponding to proper vectors $\mathbf{s}$ of $\mathbf{Q}$ which satisfy $\mathbf{c}^{\alpha}\cdot\mathbf{s} = 0$ are also the required proper values of $\mathbf{P}\,\mathbf{Q}$ in the first propagation condition (4.19). Hence in considering the strong ellipticity condition (4.26) we can restrict attention to vectors $\mathbf{v}$ such that $\mathbf{c}^{\alpha}\cdot\mathbf{v} = 0$, so that (4.26) may be replaced by the weaker condition

$$\mathbf{Q}(\mathbf{v}\ ,\ \mathbf{v}) > 0 \qquad \text{for all } \mathbf{v} \text{ such that } \mathbf{c}^{a}\cdot\mathbf{v} = 0 \quad . \tag{4.38}$$

Scott (1975) considered the direct equivalent of (4.38) in his corresponding analysis of mechanical constraints with linearly independent vectors $\mathbf{c}^{a}$.

Now (4.38) can be written as:

$$\mathbf{Q}(\mathbf{v}\ ,\ \mathbf{v}) = \mathbf{Q}^{0}(\mathbf{v}\ ,\ \mathbf{v}) + \lambda_{a}\ \mathbf{Q}^{a}(\mathbf{v}\ ,\ \mathbf{v}) > 0 \tag{4.39}$$

with $\mathbf{Q}^{0}$ and $\mathbf{Q}^{a}$ as defined following (4.25).

Clearly the requirement that $\mathbf{Q}(\mathbf{v}\ ,\ \mathbf{v})$ be positive is influenced by the values of the constraint parameters λ_{a} , but we concentrate particularly on investigating the conditions under which $\mathbf{Q}^{a}(\mathbf{v},\mathbf{v}) = 0$, in which case the a^{th} type I constraint has <u>no effect</u> on the positive definiteness of the proper numbers of $\mathbf{Q}$.

Equation (4.22) provides the required expression for $\mathbf{Q}^{a}$ in the situation when $a_{\Gamma} \neq a_{3}$, $\Gamma = 1$ or 2 , and so

$$\mathbf{Q}^{a}(\mathbf{v}\ ,\ \mathbf{v}) = \sum_{i} \left\{ (1-\delta_{i3}) \left[\frac{a_i \partial\phi^{a}/\partial a_i - a_3 \partial\phi^{a}/\partial a_3}{a_i^2 - a_3^2} \right] (\mathbf{v}\cdot\mathbf{q}_i)^2 + \frac{\partial^2\phi^{a}}{\partial a_3^2}\ (\mathbf{v}\cdot\mathbf{q}_3)^2 \right.$$

$$\left. + \beta^{a}_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B)(\mathbf{v}\cdot\mathbf{q}_i)^2 \right\} \tag{4.40}$$

where $\mathbf{v}\cdot\mathbf{c}^{\alpha} = 0$; the expression corresponding to (4.40) which is obtained by using the expression (4.23) for $\mathbf{Q}^{\alpha}$ (when $a_{\Gamma} = a_3$, $\Gamma = 1,2,$ or $a_1 = a_2 = a_3$) is omitted.

The constraint contributions $\mathbf{Q}^{\alpha}(\mathbf{v} , \mathbf{v})$ to the strong ellipticity condition (4.38) can now be studied by using (4.40). Both longitudinal and transverse waves are treated, and in each case the contributions of isotropic and directional constraints are considered separately. We also make use of the second propagation condition (4.20) which for isotropic constraints takes the form

$$\frac{\partial\phi^{\alpha}}{\partial a_3}(\mathbf{q}_3\cdot\mathbf{s}) = 0 \quad , \quad \text{by (4.4) with } \mathbf{n} = \mathbf{p}_3 \quad , \tag{4.41}$$

where the constraint vector $\mathbf{c}^{\alpha}$ is given by

$$\mathbf{c}^{\alpha} = \frac{\partial\phi^{\alpha}}{\partial a_3}\mathbf{q}_3 \qquad \text{from (2.88) with } \mathbf{n} = \mathbf{p}_3 \quad . \tag{4.42}$$

For directional constraints we recall that (4.20) takes the form

$$\beta^{\alpha}_{AB}\sum_i a_i(\mathbf{p}_i\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B)(\mathbf{q}_i\cdot\mathbf{s}) = 0 \qquad \text{(4.5 bis)}.$$

Longitudinal waves

Longitudinal waves have $\mathbf{s} = \sigma\,\mathbf{q}_3$ by (3.35) and we turn our attention first to isotropic constraints. Now $\dfrac{\partial\phi^{\alpha}}{\partial a_3} \neq 0$ in (4.41,2) since ϕ^{α} depends symmetrically on a_1, a_2, a_3 for isotropic constraints, and so we have the result from the propagation condition (4.41) and

(3.35) that <u>isotropic constraints preclude the existence of homothermal longitudinal waves</u>. Consequently, the question of the contribution $\mathbf{Q}^{\alpha}(\mathbf{v}, \mathbf{v})$ due to the α^{th} isotropic constraint does not arise.

For <u>directional</u> constraints (4.5) with $\mathbf{s} = \sigma\, \mathbf{q}_3$ implies that the constraint vectors present satisfy

$$\beta^{\alpha}_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B) = 0 \quad . \tag{4.43}$$

Now the contribution of directional constraints (for which $\phi^{\alpha} = \phi^{\alpha}(e_{AB}, f_{AB}, \theta)$ by (2.68)) to the strong ellipticity condition (4.38) is found from (4.40) to be

$$\mathbf{Q}^{\alpha}(\mathbf{v}\ ,\ \mathbf{v}) = \beta^{\alpha}_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B)(\mathbf{v}\cdot\mathbf{q}_i)^2\ , \qquad \mathbf{v}\cdot\mathbf{c}^{\alpha} = 0 \quad ,$$

and by (4.43)

$$\mathbf{Q}^{\alpha}(\mathbf{v}\ ,\ \mathbf{v}) = 0 \quad . \tag{4.44}$$

Hence <u>directional constraints have no effect</u> on the positive definiteness of $\mathbf{Q}$ for longitudinal waves.

Turning now to <u>transverse</u> waves, $\mathbf{s} = \sigma\, \mathbf{q}_\Delta$ for $\Delta = 1$ or 2 by (3.36).

For <u>isotropic</u> constraints, $\mathbf{c}^{\alpha} \cdot \mathbf{q}_3 = 0$ by (4.42) and so $\mathbf{v}\cdot\mathbf{q}_3 = 0$.

Hence (4.40) reduces to

$$\mathbf{Q}^{\alpha}(\mathbf{v}\ ,\ \mathbf{v}) = \left[\frac{a_{\Gamma}\partial\phi^{\alpha}/\partial a_{\Gamma} - a_3\partial\phi^{\alpha}/\partial a_3}{a_{\Gamma}^2 - a_3^2}\right](\mathbf{v}\cdot\mathbf{q}_{\Gamma})^2 \quad , \quad \Gamma = 1,2 \quad , \tag{4.45}$$

and there is nothing further to be gained from this without considering particular examples of isotropic constraints.

In the case of <u>directional</u> constraints, (4.40) reduces to

$$\mathbf{Q}^{\alpha}(\mathbf{v}\ ,\ \mathbf{v}) = \beta^{\alpha}_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B)(\mathbf{v}\cdot\mathbf{q}_{\Delta})^2 \quad , \tag{4.46}$$

where $\mathbf{v}\cdot\mathbf{q}_{\Delta} \neq 0$. A necessary and sufficient condition for $\mathbf{Q}^{\alpha}(\mathbf{v}\ ,\ \mathbf{v})$ to be zero for arbitrary $\mathbf{v}$ is

$$\beta^{\alpha}_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B) = 0 \quad , \tag{4.47}$$

so we have the following two possibilities:

(a) If β^{α}_{AB} is <u>positive- or negative-definite</u>, then (4.47) holds if and only if $(\mathbf{n}\cdot\mathbf{e}_A) = 0$ for all vectors $\mathbf{e}_A$. In this situation $\mathbf{c}^{\alpha} = 0$ by (2.89) and so only the case (i) considered earlier is possible. Such constraints therefore have no effect on either the strong ellipticity of $\mathbf{Q}$ or on the propagation condition.

(b) If β^{α}_{AB} is neither positive- nor negative-definite, then (4.47) can be satisfied for constraints for which $(\mathbf{n}\cdot\mathbf{e}_A) \neq 0$ for at least one $\mathbf{e}_A$. The form of the corresponding vectors $\mathbf{c}^{\alpha}$ is given by

$$\mathbf{c}^{\alpha} = \beta^{\alpha}_{AB}\sum_i a_i(\mathbf{p}_i\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B)\ \mathbf{q}_i \quad . \tag{2.89 bis}$$

Now (4.47) is also the criterion for $c^{a}\cdot q_3 = 0$ by (4.43) and we are considering transverse homothermal waves with $s = \sigma\, q_{\Delta}$, so that the second propagation condition (4.20) takes the form

$$c^{a}\cdot q_{\Delta} = 0 \quad . \tag{4.48}$$

Hence any non-zero c^{a} must be parallel to $q_{\underline{\Delta}}$, where $\underline{\Delta} = 1$ or 2 according as $\Delta = 2$ or 1, so that from (2.89)

$$c^{a} = \beta^{a}_{AB}\, a_{\underline{\Delta}}(p_{\underline{\Delta}}\cdot e_A)(n\cdot e_B)\; q_{\underline{\Delta}} \quad , \tag{4.49}$$

and all such c^{a} are therefore examples of case (ii_a) considered earlier.

Influence of isotropic and directional constraints on the propagation conditions

Solutions of the propagation conditions (4.19,20) have been derived earlier in this chapter for homothermal principal waves for the three cases (i), (ii), (iii) for which dim span $\{c^{a}\} = M = 0,1,2$ respectively. The influence of the constraints on the wave speeds was not discussed in detail, but it was noted that for case (i) the constraints have no effect on wave propagation. We now reconsider the propagation conditions and investigate separately the influence of isotropic or directional constraints on the solutions; the results have close parallels with those for the positive definiteness of $\mathbf{Q}(v\ , v)$. We present these results under the headings of longitudinal and transverse waves rather than isotropic and directional constraints since this serves to emphasize the significant result that for these constraints, the speed of propagation of longitudinal homothermal waves in unaffected by the presence of constraints. (Of course, in many cases the constraints do not allow the possibility of longitudinal waves.)

Longitudinal waves

We consider solutions $\mathbf{s} = \sigma\, \mathbf{q}_3$ as in (3.35) to the propagation conditions (4.19,20). The first propagation condition now takes the form

$$(\rho\nu_3^2\, \mathbf{I} - \mathbf{P}\,\mathbf{Q})\, \mathbf{q}_3 = 0 \tag{4.50}$$

where ν_3 is defined in (4.29), and the second propagation condition is

$$\mathbf{c}^{\alpha}\cdot\mathbf{q}_3 = 0 \quad . \tag{4.51}$$

Isotropic constraints are incompatible with (4.51), as remarked in the discussion of longitudinal waves following (4.42).

Directional constraints must satisfy (4.51) and consequently

$$\beta^{\alpha}_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B) = 0 \qquad \text{(4.43 bis)},$$

so that from (2.89),

$$\mathbf{c}^{\alpha} = \beta^{\alpha}_{AB} \sum_{\Gamma} a_{\Gamma}(\mathbf{p}_{\Gamma}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B)\, \mathbf{q}_{\Gamma} \quad , \quad \Gamma = 1,2 \quad . \tag{4.52}$$

It is clear from (4.52) for directional constraints that the three situations dim span $\{\mathbf{c}^a\} = M = 0$, 1, and 2 are all possible. As a result the discussion of solutions to (4.50) for the cases (i) - (iii) given earlier is applicable and the wave speed ν_3 for longitudinal waves is obtained from

$$\nu_3^2 = \rho^{-1} \mathrm{Q}_3 \qquad ((4.29)_3 \text{ bis})$$

where

$$\mathrm{Q}_3 = \mathbf{Q}(\mathbf{q}_3 , \mathbf{q}_3)$$

by (4.24), and $\mathbf{Q}$ is found from (4.22) or (4.23) as appropriate.

The contribution from the directional constraints to Q_3 is $\check{\lambda}_a \mathrm{Q}_3^a$ by (4.25), and with the aid of (4.22-4) and (4.43) we find that

$$\rho^{-1} \mathrm{Q}_3^a = \rho^{-1} \mathrm{Q}^a(\mathbf{q}_3 , \mathbf{q}_3) = \beta_{AB}^a(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B) = 0 \quad . \qquad (4.53)$$

Consequently, the speed of propagation of longitudinal waves is unaffected by the directional constraints.

Transverse waves

We consider solutions $\mathbf{s} = \sigma\, \mathbf{q}_\Delta$ for $\Delta = 1$ or 2 as in (3.36) to the propagation conditions (4.19,20) which take the forms

$$(\rho\nu_\Delta^2\, \mathbf{I} - \mathbf{P}\, \mathbf{Q})\, \mathbf{q}_\Delta = \mathbf{0} \qquad (4.54)$$

and

$$\mathbf{c}^a\cdot\mathbf{q}_\Delta = 0 \quad . \qquad (4.48 \text{ bis})$$

<u>Isotropic</u> constraints have constraint vectors of the form

$$c^\alpha = \frac{\partial \phi^\alpha}{\partial a_3} q_3 \quad . \tag{4.42 bis}$$

Clearly, if only isotropic constraints are present, dim span $\{c^\alpha\}$ = $M = 1$ and the constraint vectors c^α are collinear. Since $c^\alpha \cdot q_3 = 0$ for isotropic constraints by (4.42), we see that for transverse waves, the second propagation condition (4.48) places no restriction on the isotropic constraints. Turning now to the first propagation condition (4.54) we see that the earlier discussion of case (ii_a) is therefore appropriate and the comments following (4.32) apply : waves with amplitudes q_1 and q_2 are possible with corresponding speeds ν_1 and ν_2 defined by $(4.29)_{1,2}$. The constraint contributions to ρ^{-1} Q_1 and ρ^{-1} Q_2 (and hence ν_1 , ν_2) are given by

$$\rho^{-1}\, Q^\alpha_\Delta = \rho^{-1}\, Q^\alpha(q_\Delta \ , \ q_\Delta) = \frac{a_\Delta \partial\phi^\alpha/\partial a_\Delta - a_3 \partial\phi^\alpha/\partial a_3}{a^2_\Delta - a^2_3} \quad ,$$

or

$$\rho^{-1}\, Q^\alpha_\Delta = \left[\frac{1}{a}\frac{\partial\phi^\alpha}{\partial a_\Delta} + \frac{\partial^2\phi^\alpha}{\partial a^2_\Delta} - \frac{\partial^2\phi^\alpha}{\partial a_\Delta \partial a_3}\right] \quad , \quad \Delta = 1 \text{ or } 2 \quad , \tag{4.55}$$

according as (4.22) or (4.23) is appropriate. In general then, isotropic constraints <u>do</u> influence the speed of transverse waves. The constraint of temperature-dependent compressibility (recall (2.67)), however, is a notable exception in this respect, as for this constraint $\rho^{-1}\, Q^\alpha_\Delta$ is easily seen to be zero. If transverse waves exist, then from

(4.48) the <u>directional</u> constraints must obey

$$\beta^{a}_{AB}(\mathbf{p}_{\Delta}\cdot\mathbf{e}_{A})(\mathbf{n}\cdot\mathbf{e}_{B}) = 0 \quad , \tag{4.56}$$

where (4.5) with $\mathbf{n} = \mathbf{p}_3$ has been used.

Note the contribution $\mathbf{Q}^{a}_{\Delta}$ of the a^{th} directional constraint to $\mathbf{Q}_{\Delta}$ in (4.25) is found from (4.22) or (4.23) to be

$$\rho^{-1}\,\mathbf{Q}^{a}_{\Delta} = \beta^{a}_{AB}(\mathbf{n}\cdot\mathbf{e}_{A})(\mathbf{n}\cdot\mathbf{e}_{B}) \quad . \tag{4.57}$$

Constraints for which $\mathbf{Q}^{a}_{\Delta} = 0$ in (4.57), so that

$$\beta^{a}_{AB}(\mathbf{n}\cdot\mathbf{e}_{A})(\mathbf{n}\cdot\mathbf{e}_{B}) = 0 \quad , \tag{4.58}$$

have no effect on the wave speed ν_{Δ} . In such cases, (4.56) and (4.58) hold and the discussion following (4.47) (for $\mathbf{Q}^{a}(\mathbf{v}\ ,\ \mathbf{v}) = 0$) applies:

(a) If β^{a}_{AB} is positive- or negative-definite, then $\mathbf{c}^{a} = \mathbf{0}$. If all constraints present satisfy this condition, then $\mathbf{M} = 0$ and case (i) applies.

(b) If β^{a}_{AB} is neither positive- nor negative-definite, then non-zero vectors $\mathbf{c}^{a}$ for which (4.56,8) hold must be of the form

$$\mathbf{c}^{a} = \beta^{a}_{AB}\, a_{\underline{\Delta}}(\mathbf{p}_{\underline{\Delta}}\cdot\mathbf{e}_{A})(\mathbf{n}\cdot\mathbf{e}_{B})\mathbf{q}_{\underline{\Delta}} \quad , \tag{4.49 bis}$$

and consequently the discussion of case (ii_a) applies.

Propagation of plane, cylindrical and spherical waves in irrotationally deformed materials

We now consider some particular examples of wave propagation, namely the propagation of plane, cylindrical and spherical waves in irrotationally deformed materials that are subject to certain configurations of the directional constraints. The analysis serves to delineate the range of deformations, compatible with the constraints, corresponding to which the propagation of waves is possible. The results obtained here are used later in the analysis of the growth of plane, cylindrical and spherical waves in constrained materials.

Attention is restricted to definite conductors which are taken to be at rest and at constant temperature ahead of the wave, and which are subject to the class of irrotational deformations specified below. In each case the orthogonal curvilinear coordinates X^i are chosen so that their tangent basis vectors are aligned with the principal directions; that is

$$\mathbf{p}_i = \mathbf{G}_i/|\mathbf{G}_i|$$

(recall the discussion preceding (3.33)).

Furthermore, since we consider irrotational deformations, $\mathbf{R} = \mathbf{I}$ and so

$$\mathbf{q}_i = \mathbf{p}_i \quad . \qquad (3.33\ \text{bis}).$$

The directional constraint vectors $\mathbf{e}_A$ are assumed to make a constant angle with the coordinate curves so that

$$\mathbf{e}_A \cdot \hat{\mathbf{G}}_i = \text{constant} \quad , \tag{4.59}$$

where

$$\hat{\mathbf{G}}_i \equiv \mathbf{G}_i / |\mathbf{G}_i| = \mathbf{p}_i \quad .$$

We will consider three classes of problems: the propagation of plane waves in material which is subject to a plane deformation, cylindrical waves in material subject to a cylindrical deformation, and spherical waves in material subject to a spherical deformation. This approach is similar in spirit to that of Bowen and Wang (1970) (see also Wang and Truesdell (1973)), in their analysis of acceleration waves in laminated bodies. Since (4.59) applies in all the three cases given above, it is clear that we are considering materials that are differently configured in the three cases, since the directional vectors $\mathbf{e}_A$ (if present) make constant angles with the plane, cylindrical and spherical coordinate curves in each of the three cases. The use of $\hat{\mathbf{G}}_i$ rather than $\mathbf{p}_i$ in stating (4.59) is simply to emphasize that we are in fact considering three distinct situations corresponding to distinct constraint configurations.

The situations that we will be considering are:

(A) <u>Plane deformations</u>

We adopt a fixed rectangular cartesian coordinate system with the coordinates of a particle in the reference configuration denoted by $X^i = (Y, Z, X)$. The corresponding orthonormal basis is denoted by

$\mathbf{E}_i$, so that in the reference configuration,

$$\mathbf{X} = X^i \mathbf{E}_i \quad . \tag{4.60}$$

We also have that $\mathbf{G}_i = \mathbf{E}_i$, where the tangent basis vectors $\mathbf{G}_i$ are as defined by (2.1). In the current configuration, the coordinates are given by $x^i = (y, z, x)$ and

$$\mathbf{x} = x^i \mathbf{E}_i \quad ,$$

with the tangent basis vectors $\mathbf{g}_i$ obeying $\mathbf{g}_i = \mathbf{E}_i$.

We will be considering irrotational plane deformations specified by

$$\frac{y}{Y} = \text{constant}, \quad \frac{z}{Z} = \text{constant}, \quad x = x(X) \quad , \tag{4.61}$$

for which the principal stretches are given by

$$a_1, \; a_2 = \text{constant}, \; a_3 = \frac{dx}{dX} \quad . \tag{4.62}$$

Directional vectors (if present) obey (4.59), as discussed above.

(B) Cylindrical deformations

Here, a fixed cylindrical coordinate system is adopted, with the coordinates in the reference configuration denoted by $X^i = (\Theta, Z, R)$, so that

$$\mathbf{X} = R(\sin\Theta \; \mathbf{E}_1 + \cos\Theta \; \mathbf{E}_3) + Z \; \mathbf{E}_2 \quad , \tag{4.63}$$

where $\mathbf{E}_i$ are as defined preceding (4.60); the corresponding cylindrical tangent basis vectors $\mathbf{G}_i$ are given by

$$\mathbf{G}_1 \equiv \mathbf{G}_\Theta = R(\cos\Theta\ \mathbf{E}_2 + \sin\Theta\ \mathbf{E}_3) \quad ,$$

$$\mathbf{G}_2 \equiv \mathbf{G}_Z = \mathbf{E}_2 \quad ,$$

$$\mathbf{G}_3 \equiv \mathbf{G}_R = \sin\Theta\ \mathbf{E}_1 + \cos\Theta\ \mathbf{E}_3 \quad . \tag{4.64}$$

The cylindrical coordinates in the current configuration are $x^i = (\theta, z, r)$ and

$$\mathbf{x} = r(\sin\theta\ \mathbf{E}_1 + \cos\theta\ \mathbf{E}_3) + z\ \mathbf{E}_2 \quad ;$$

the tangent basis vectors $\mathbf{g}_\theta$, $\mathbf{g}_z$, $\mathbf{g}_r$ coinciding with $\mathbf{G}_\Theta$, $\mathbf{G}_Z$, $\mathbf{G}_R$ respectively.

The cylindrically symmetric deformations to be considered are specified by

$$\theta = \Theta \quad , \quad \frac{z}{Z} = \text{constant}, \quad r = r(R) \tag{4.65}$$

and the principal stretches are given by

$$a_1 = \frac{r}{R}\ , \ a_2 = \text{constant}, \ a_3 = \frac{dr}{dR} \quad . \tag{4.66}$$

Directional constraints (if present) satisfy (4.59), where now the $\hat{\mathbf{G}}_i$ are the cylindrical unit basis vectors obtained from $\mathbf{G}_\Theta$, $\mathbf{G}_Z$, $\mathbf{G}_R$.

(C) Spherical deformations

A fixed spherical coordinate system is employed with spherical coordinates in the reference configuration $X^i = (\Theta\ ,\ \Phi,\ R)$, so that

$$\mathbf{X} = R(\cos\Phi(\sin\Theta\ \mathbf{E}_1 + \cos\Theta\ \mathbf{E}_3) + \sin\Phi\ \mathbf{E}_2) \quad , \qquad (4.67)$$

and the spherical tangent basis vectors are given by

$$\mathbf{G}_1 \equiv \mathbf{G}_\Theta = R\cos\Phi(\cos\Theta\ \mathbf{E}_1 - \sin\Theta\ \mathbf{E}_3) \quad ,$$

$$\mathbf{G}_2 \equiv \mathbf{G}_\Phi = R(-\sin\Phi(\sin\Theta\ \mathbf{E}_1 + \cos\Theta\ \mathbf{E}_3) + \cos\Phi\ \mathbf{E}_2) \quad ,$$

$$\mathbf{G}_3 \equiv \mathbf{G}_R = \cos\Phi(\sin\Theta\ \mathbf{E}_1 + \cos\Theta\ \mathbf{E}_3) + \sin\Phi\ \mathbf{E}_2 \quad . \qquad (4.68)$$

Once again $\mathbf{E}_i$ in (4.67,8) are the orthonormal basis vectors defined preceding (4.60), and the angles Θ , Φ are as defined in Flugge (1972). In the current configuration, the spherical coordinates are $x^i = (\theta,\ \phi,\ r)$ and the tangent basis vectors $\mathbf{g}_\theta$, $\mathbf{g}_\phi$, $\mathbf{g}_r$ coincide with $\mathbf{G}_\Theta$, $\mathbf{G}_\Phi$ and $\mathbf{G}_R$ respectively.

The spherical symmetric deformations considered here are specified by

$$\theta = \Theta \quad , \quad \phi = \Phi \quad , \quad r = r(R) \quad , \qquad (4.69)$$

and the principal stretches are

$$a_1 = a_2 = \frac{r}{R} \quad ; \quad a_3 = \frac{dr}{dR} \quad . \qquad (4.70)$$

Any directional constraints present satisfy (4.59) with $\hat{G}_i$ being the spherical unit basis vectors obtained from G_θ , G_Φ, G_R .

(The symbols θ, ϕ and r appearing in (4.65) and (4.69) above have been used previously to denote a type I constraint, the temperature, and the rate of heat supply respectively. No confusion is likely to result from these ambiguities, however).

We will require later the components $\Omega_\Gamma{}^\Delta$ of the surface curvature tensor defined by (3.6) in the reference configuration for the three cases presented above, and the results are accordingly presented here:

$$\Omega_\Gamma{}^\Delta = - R_\Gamma^{-1}\delta_\Gamma{}^\Delta \quad , \quad \Gamma, \Delta, = 1,2 \quad ,$$

where

$$R_1 = R_2 = \infty \quad \text{for plane waves} \quad , \tag{4.71}$$

$$R_1 = R \ , R_2 = \infty \ \text{for cylindrical waves} \quad , \tag{4.72}$$

$$R_1 = R_2 = R \quad \text{for spherical waves} \quad . \tag{4.73}$$

<u>Longitudinal wave propagation for the cases A - C</u>

We assume the existence of a homothermal principal wave with $n = p_3$ that is longitudinal, so that $s = \sigma\, q_3$. The previous discussion of (4.50-53) of the propagation conditions is applicable; isotropic constraints are incompatible with longitudinal waves, and any directional constraints present must obey

$$\beta^a_{AB}(n\cdot e_A)(n\cdot e_B) = 0 \quad . \tag{4.43 bis}$$

The definition (2.68,9) of directional type I constraints together with (4.43) yields

$$\phi^{\alpha} = \frac{1}{2}\,\beta^{\alpha}_{AB} \sum_{\Gamma} a^2_{\Gamma}(p_{\Gamma}\cdot e_A)(p_{\Gamma}\cdot e_B) + \phi^{\alpha}_2(e_{AB},\, \theta) = 0 \quad . \tag{4.74}$$

Now the temperature θ is constant ahead of the wave by assumption, the components $(\hat{G}_i \cdot e_A) = (p_i \cdot e_A)$ of e_A do not vary with position and the β^{α}_{AB} are also constant, so (4.74) furnishes necessary conditions on the deformation through the principal stretches a_{Γ} .

For plane waves with the situation as described in (A), no restriction is imposed on $(4.61)_3$ by $(4.62)_{1,2}$ and (4.74).

For cylindrical waves propagating in the situation described in (B), there are two possibilities:

firstly, if

$$\beta^{\alpha}_{AB}(p_1\cdot e_A)(p_1\cdot e_B) = 0 \quad , \tag{4.75}$$

then no restriction is placed on $(4.65)_3$ by $(4.66)_2$ and (4.74).

Secondly, if

$$\beta^{\alpha}_{AB}(p_1\cdot e_A)(p_1\cdot e_B) \neq 0 \quad , \tag{4.76}$$

then (4.74) implies

$$a_1 = \frac{r}{R} = \text{constant} \quad , \tag{4.77}$$

so longitudinal waves require that the material be in a state of <u>homogeneous deformation</u>.

For <u>spherical waves</u> in materials as specified in (C), (4.74) takes the form

$$\zeta\left[\frac{r}{R}\right]^2 - \phi_2^a(e_{AB}, \theta) = 0 \qquad (4.78)$$

where

$$\zeta = \frac{1}{2}\beta_{AB}^a \sum_\Gamma \left[\frac{r}{R}\right]^2 (p_\Gamma \cdot e_A)(p_\Gamma \cdot e_B) \quad ,$$

and $(4.70)_1$ has been used.

If $\zeta = 0$, the deformation is <u>unrestricted</u>. If however, both ζ and ϕ_2^a are <u>non-zero</u> then r/R = constant and the state of the material is necessarily one of <u>uniform dilatation</u> since $a_3 = dr/dR = r/R$. We note that the situation of $\zeta \neq 0$ and $\phi_2^a = 0$ is not compatible with longitudinal wave propagation.

<u>Transverse wave propagation for the cases A - C</u>

We now assume a transverse principal wave exists with $\mathbf{s} = \sigma\, \mathbf{q}_\Delta$ so that the previous discussion of (4.54 - 58) is applicable. Isotropic constraints satisfy the second propagation condition (4.48) identically and directional constraints obey (4.48) if

$$\beta_{AB}^a(p_\Delta \cdot e_A)(\mathbf{n} \cdot e_B) = 0 \quad . \qquad (4.56\text{ bis})$$

Substitution of (4.56) into the definition (2.68,9) of directional constraints yields

$$\phi^{a} = \frac{1}{2}\,\beta^{a}_{AB}\left\{a^{2}_{\underline{\Delta}}(p_{\underline{\Delta}}\cdot e_{A})(p_{\underline{\Delta}}\cdot e_{B}) + a^{2}_{3}(n\cdot e_{A})(n\cdot e_{B})\right\} - \phi^{a}_{2}(e_{AB},\,\theta) = 0 \quad , \tag{4.79}$$

where $\underline{\Delta} = 1$ or 2 according as $\Delta = 2$ or 1 .

We proceed in a similar manner to that for longitudinal waves, and find that the following two situations arise (the quantities ζ_1 and ζ_2 are constants):

(i) $n\cdot e_A = 0$ for <u>all</u> vectors e_A :

Case (A)	plane waves	: motion unrestricted
Case (B)	cylindrical waves	: $\Delta = 1$ motion unrestricted
		: $\Delta = 2$ homogeneous deformation (motion unrestricted if $\phi^{a}_{2} = 0$)
Case (C)	spherical waves	: homogeneous deformation (motion unrestricted if $\phi^{a}_{2} = 0$) .

(4.80)

(ii) $n\cdot e_A \neq 0$ for <u>at least one vector</u> e_A :

Case (A)	plane waves	: homogeneous deformation
Case (B)	cylindrical waves	: $\Delta = 1$ homogeneous deformation
		: $\Delta = 2$ $dr/dR = \left\{\zeta_1\left[\frac{r}{R}\right]^2 + \zeta_2\right\}^{1/2}$
Case (C)	spherical waves	: $dr/dR = \left\{\zeta_1\left[\frac{r}{R}\right]^2 + \zeta_2\right\}^{1/2}$.

(4.81)

Plane, cylindrical and spherical wave propagation for combinations of four constraints

We illustrate the above results by considering the propagation of plane, cylindrical and spherical waves in materials subject to the deformations of cases (A) - (C) , but restrict attention to the four constraints specified in Chapter 2 (see (2.67,70,72,73), but here we change the numbering of the functions $\xi(\theta)$ for convenience). The constraints are considered both singly and in combinations of two, three or four; permissible deformations are calculated and the cases for which constraint combinations are linearly dependent are noted. The four constraints and their associated vectors $\mathbf{c}^{\alpha}$ are as follows, and the results appear in Table 4.1. It should be emphasized that the results presented in Table 4.1 under the headings of plane, cylindrical and spherical deformations refer in general to materials with entirely different configurations of directional constraints (recall (4.59) and following discussion).

(1) temperature-dependent extensibility in direction $\mathbf{e}_1$:

$$\phi^1 = \frac{1}{2} f_{11} - \xi_1(\theta) = \frac{1}{2} a_i^2(\mathbf{e}_1 \cdot \mathbf{p}_i)^2 - \xi_1(\theta) \quad , \qquad (4.82)$$

$$\mathbf{c}^1 = \sum_i a_i(\mathbf{e}_1 \cdot \mathbf{n})(\mathbf{e}_1 \cdot \mathbf{p}_i)\, \mathbf{q}_i \quad ; \qquad (4.83)$$

(2) temperature-dependent extensibility in direction $\mathbf{e}_2$ with $\mathbf{e}_1 \perp \mathbf{e}_2$ (i.e. $e_{12} = 0$) :

$$\phi^2 = \frac{1}{2} f_{22} - \xi_2(\theta) = \frac{1}{2} a_i^2(\mathbf{e}_2 \cdot \mathbf{p}_i)^2 - \xi_2(\theta) \quad , \qquad (4.84)$$

$$\mathbf{c}^2 = \sum_i a_i(\mathbf{e}_2 \cdot \mathbf{n})(\mathbf{e}_2 \cdot \mathbf{p}_i)\, \mathbf{q}_i \quad ; \qquad (4.85)$$

(3) temperature-dependent shearing with respect to e_1 and e_2 ($e_{12} = 0$):

$$\phi^3 = f_{12} - \xi_3(\theta) = \sum_i a_i^2(e_1 \cdot p_i)(e_2 \cdot p_i) - \xi_3(\theta) \quad , \tag{4.86}$$

where <u>$\xi_3(\theta) = 0$ if ϕ^1 and ϕ^2 are not both present as well</u>; recall (2.72,73)).

$$c^3 = \sum_i a_i \Big\{ (e_2 \cdot n)(e_1 \cdot p_i) + (e_1 \cdot n)(e_2 \cdot p_i) \Big\} \, q_i \quad ; \tag{4.87}$$

(4) temperature-dependent compressibility:

$$\phi^4 = a_1 \, a_2 \, a_3 - \xi_4(\theta) \quad , \tag{4.88}$$

$$c^4 = a_1 \, a_2 \, q_3 \quad . \tag{4.89}$$

Table 4.1 : Permissible deformations of the form (4.61,5,9) for various combinations of the constraints (4.82,4,6,8).

	CONSTRAINT	Long (L) Trans (T) Both (B)	dim $\{\mathbf{c}^\alpha\}$	dir^n of, or plane of, $\{\mathbf{c}^\alpha\}$	lin dep of $\{\mathbf{c}^\alpha\}$	$\mathbf{c}^\alpha\cdot\mathbf{q}_3 = 0$ (L) OR $\mathbf{c}^\alpha\cdot\mathbf{q}_\Delta = 0$ (T)	PERMISSIBLE DEFORMATION FOR MATERIALS SPECIFIED IN CASE A,B,C — PLANE $x(X)$	CYLINDRICAL $r(R)$	SPHERICAL $r(R)$
(i)	1	B	0	$\mathbf{c}^\alpha = \mathbf{0}$		$\mathbf{e}_1\cdot\mathbf{n} = 0$	$x(X)$	$r(R)$ $(\Delta = 1)$ hom $(\Delta = 2)$, (L)	dilatation
(ii)		T	1	$\mathbf{q}_\Delta, \mathbf{q}_3$		$\mathbf{e}_1\cdot\mathbf{p}_\Delta = 0$	hom	hom $(\Delta = 1)$ $r' = F(r/R)$ $(\Delta = 2)$	$r' = F(r/R)$
(iii)	3	L	1	$\mathbf{q}_1, \mathbf{q}_2$		$(\mathbf{e}_1\cdot\mathbf{n})(\mathbf{e}_2\cdot\mathbf{n}) = 0$	$x(X)$	hom	dilatation
(iv)		T	0	$\mathbf{c}^\alpha = \mathbf{0}$		$\mathbf{e}_1\cdot\mathbf{n} = \mathbf{e}_2\cdot\mathbf{n} = 0$	$x(X)$	$r(R)$	$r(R)$
(v)		T	1	$\mathbf{q}_\Delta, \mathbf{q}_3$		$(\mathbf{e}_2\cdot\mathbf{n})(\mathbf{e}_1\cdot\mathbf{p}_\Delta) + (\mathbf{e}_1\cdot\mathbf{n})(\mathbf{e}_2\cdot\mathbf{p}_\Delta) = 0$	hom	$r' = F(r/R)$	$r' = F(r/R)$
(vi)	1+2	B	0	$\mathbf{c}^\alpha = \mathbf{0}$	*	$\mathbf{e}_1\cdot\mathbf{n} = \mathbf{e}_2\cdot\mathbf{n} = 0$		as (i)	
(vii)		T	2	$\mathbf{q}_\Delta, \mathbf{q}_3$		$\mathbf{e}_1\cdot\mathbf{p}_\Delta = \mathbf{e}_2\cdot\mathbf{p}_\Delta = 0$	hom	hom $(\Delta = 1)$ $r' = F(r/R)$ $(\Delta = 2)$	$r' = F(r/R)$
(viii)	1+3	L	1	$\mathbf{q}_1, \mathbf{q}_2$	*	$\mathbf{e}_1\cdot\mathbf{n} = 0$		as (iii)	
(ix)		T	0	$\mathbf{c}^\alpha = \mathbf{0}$	*	$\mathbf{e}_1\cdot\mathbf{n} = \mathbf{e}_2\cdot\mathbf{n} = 0$ OR		as (i)	
(x)		T	2	$\mathbf{q}_\Delta, \mathbf{q}_3$		$\mathbf{e}_1\cdot\mathbf{p}_\Delta = \mathbf{e}_2\cdot\mathbf{p}_\Delta = 0$ OR		as (vii)	
(xi)		T	1	$\mathbf{c}^\alpha \parallel \mathbf{q}_\Delta$	*	$\mathbf{e}_1\cdot\mathbf{n} = \mathbf{e}_2\cdot\mathbf{p}_\Delta = 0$	hom	$r(R)^\dagger$ $(\Delta = 1)$ hom† $(\Delta = 2)$	dilatation†
(xii)	1+2+3	B	0	$\mathbf{c}^\alpha = \mathbf{0}$	*	$\mathbf{e}_1\cdot\mathbf{n} = \mathbf{e}_2\cdot\mathbf{n} = 0$		as (i)	
(xiii)		T	2	$\mathbf{q}_\Delta, \mathbf{q}_3$	*	$\mathbf{e}_1\cdot\mathbf{p}_\Delta = \mathbf{e}_2\cdot\mathbf{p}_\Delta = 0$		as (vii)	
(xiv)	4	T	1	$\mathbf{c}^\alpha \parallel \mathbf{q}_3$		-	hom	$r^2 - r_o^2 = f(\theta)(R^2 - R_o^2)$	$r^3 - r_o^3 = f(\theta)(R^3 - R_o^3)$
(xv)	4+1	T	1	$\mathbf{c}^\alpha \parallel \mathbf{q}_3$	*	$\mathbf{e}_1\cdot\mathbf{n} = 0$	hom	not possible	not possible
(xvi)		T	1	$\mathbf{c}^\alpha \parallel \mathbf{q}_3$	**	$\mathbf{e}_1 = \mathbf{n}$			
(xvii)	4+3	T	1	$\mathbf{c}^\alpha \parallel \mathbf{q}_3$	*	$\mathbf{e}_1\cdot\mathbf{n} = \mathbf{e}_2\cdot\mathbf{n} = 0$		as (xv)	
(xviii)		T	1	$\mathbf{c}^\alpha \parallel \mathbf{q}_3$	**	$(\mathbf{e}_2\cdot\mathbf{n})(\mathbf{e}_1\cdot\mathbf{p}_\Delta) + (\mathbf{e}_1\cdot\mathbf{n})(\mathbf{e}_2\cdot\mathbf{p}_\Delta) = 0$ in addition to (v)		as (xv)	
(xix)	4+1+2, 4+1+3, 4+1+2+3	T	2 OR 1	$\mathbf{q}_\Delta, \mathbf{q}_3$ OR $\mathbf{c}^\alpha \parallel \mathbf{q}_3$	*, ** OR *, **	see (vi), (vii), (ix) - (xiii)		as (xv)	

KEY :

*	Linear dependent, since at least one $\mathbf{c}^\alpha$ is always zero.
**	Linear dependence possible, all vectors $\mathbf{c}^\alpha$ involved are non-zero.
hom	Homogeneous deformation.
$F(r/R)$	Function of the form $F(Z) = \sqrt{aZ^2 + b}$, a and b are constants such that the constraint equation $\phi = 0$ holds.
†	Only permissible if $\mathbf{e}_1\cdot\mathbf{p}_\Delta = 0$ as well.

CHAPTER 5

PROPAGATION CONDITIONS FOR HOMENTROPIC WAVES

Introduction

We investigate only non-homothermal waves in this chapter, so that $T \equiv [\mathrm{Grad}\,\theta\cdot\mathbf{n}] \neq 0$ by (3.38,9). The necessary condition (4.1) for the existence of non-homothermal waves is employed to derive results for the thermal properties of the type I constraints, and we find that the constraints $\phi^{P+\eta}$, $\eta = 1, \ldots, N-P$ (whose constraint vectors satisfy $\mathbf{c}^{P+\eta} \equiv 0$ by (2.87)) are all mechanical, so that $\omega^{P+\eta} = 0$. An alternative form of (4.1) is then presented that replaces the set of constraint vectors, $\left\{\mathbf{c}^{\alpha}\right\}_{\alpha=1}^{N}$ in (4.1) by the set $\left\{\mathbf{c}^{M}, \mathbf{m}^{\kappa}\right\}$ where $\kappa = 1, \ldots, M-1, M+1, \ldots, N$, and where $\mathbf{m}^{\kappa} \equiv \omega^{M}\mathbf{c}^{\kappa} - \omega^{\kappa}\mathbf{c}^{M}$. Equation (4.1) now yields the necessary condition that $\dim\operatorname{span}\{\mathbf{m}^{\kappa}\} \leq 2$ for non-homothermal waves to exist. We also present (4.1) in the entropic formulation (with η replacing θ as an independent variable) and note that it is advantageous to work with the vectors $\hat{\mathbf{m}}^{\kappa}$ rather than with the vectors $\hat{\mathbf{c}}^{\alpha}$ in that the subset $\left\{\hat{\mathbf{m}}^{\zeta}\right\}_{\zeta=1}^{M-1}$ is linearly independent, whereas we recall from the discussion following (2.96) that this is not necessarily so for the subset $\left\{\hat{\mathbf{c}}^{\sigma}\right\}_{\sigma=1}^{M}$.

Attention is then restricted to non-homothermal waves which are homentropic, so that $H \equiv [\mathrm{Grad}\,\eta\cdot\mathbf{n}] = 0$ by (3.41,2). We recall (see summary following (3.50)) that waves for which $T \neq 0$ and $H = 0$ occur in non-conductors when the type II constraints present satisfy $\mathbf{z}^{\beta}\cdot\mathbf{n} = 0$, $\beta, \ldots, L$, or when type II constraints are absent. The first propagation condition is derived in the entropic formulation, and the set $\left\{\hat{c}^M , \hat{m}^\kappa\right\}$ is used to remove the jumps $[\dot{\lambda}_a]$. Although (unlike (I)) we permit linear dependence of the vectors $\hat{m}^\kappa$, the final result involves only the linearly independent subset $\{\hat{m}^\zeta\}$, a situation reminiscent of the corresponding homothermal result in which only the linearly independent subset $\left\{c^\sigma\right\}_{\sigma=1}^M$ appeared. The special form of the first propagation condition when all type I constraints are mechanical ($\hat{\omega}^a = \omega^a = 0$, $a = 1, \ldots, N$) is also discussed, since $\hat{\omega}^a = 0$ is found to be a necessary condition for the existence of non-homothermal waves in a number of cases.

Solutions of the propagation conditions for longitudinal and transverse principal waves are discussed for the all-embracing cases $M = 0,1,2,3$, and in each case the wave speeds are found from the proper values of the appropriate form of the acoustic tensor. When $M = 0$ (all type I constraints are inactive), the constraints are necessarily <u>mechanical</u> ($\hat{\omega}^a = 0$, $a = 1, \ldots, N$). Longitudinal waves with $H = 0$, $T \neq 0$ are compatible with these constraints, but transverse waves have $H = T = 0$ and are therefore generalized transverse waves. For $M \geq 1$, acceleration waves are compatible with the situation when thermo-mechanical constraints are present, and for $M = 1$ or 2, longitudinal waves are also compatible with the situation when all constraints are mechanical. Generalized transverse waves are not possible when $M = 3$ because $M \leq 2$ is a necessary condition for homothermal waves by (4.3).

We then reconsider the propagation conditions when only isotropic constraints are present ($M = 1$), or when directional constraints are present ($M = 0,1,2,3$). The thermal formulation is employed here since

$$c^{P+\eta} \equiv 0 \Rightarrow$$

$$c^{P+\eta} \cdot s = - \nu^2 \omega^{P+\eta} T = 0 \quad , \qquad \eta = 1, \ldots, N-P \quad . \tag{5.1}$$

Substitution for $c^{\sigma} \cdot s$ from $(5.1)_1$ in $(5.1)_2$ yields the following necessary condition for the existence of waves when $T \neq 0$:

$$\omega^{M+\mu} = D^{\mu}_{\sigma} \omega^{\sigma} \quad , \quad \sigma = 1, \ldots, M \quad ,$$
$$\mu = 1, \ldots, P-M \quad ,$$

and we also have

$$\omega^{P+\eta} = 0 \quad , \qquad \eta = 1, \ldots, N-P \quad , \tag{5.2}$$

where this last result is obtained from $(5.1)_3$ if we recall from (2.87) that $c^{P+1}, \ldots, c^{N}$ are all zero and $T \neq 0$. It is therefore a necessary condition for the existence of acceleration waves that the constraints $\phi^{P+\eta}$ be <u>mechanical</u>.

It will prove useful later to employ an alternative set to (5.1) that involves a set of N-1 vectors m^{κ} defined by

$$m^{\kappa} = \omega^{M} c^{\kappa} - \omega^{\kappa} c^{M} \quad , \quad \kappa = 1, \ldots, M-1, M+1, \ldots, N \quad , \tag{5.3}$$

where c^{M} is the M^{th} and last member of the subset $\{c^{\sigma}\}$ of linearly independent constraint vectors in $(5.1)_1$. Before deriving the alternative to (5.1), we investigate the properties of the m^{κ} a little further. The set $\{\mathbf{m}^\kappa\}$ is first subdivided as follows:

$$\mathbf{m}^\zeta = \omega^M \mathbf{c}^\zeta - \omega^\zeta \mathbf{c}^M \quad , \qquad \zeta = 1, \ldots, M-1 \quad ,$$

$$\mathbf{m}^{M+\mu} = \omega^M \mathbf{c}^{M+\mu} - \omega^{M+\mu} \mathbf{c}^M \quad , \quad \mu = 1, \ldots, P-M \quad ,$$

$$\mathbf{m}^{P+\eta} = \omega^M \mathbf{c}^{P+\eta} - \omega^{P+\eta} \mathbf{c}^M \quad , \quad \eta = 1, \ldots, N-P \quad , \qquad (5.4)$$

where M and P have the same meanings as in (2.86,7).

In $(5.4)_1$ the vectors $\mathbf{m}^\zeta$ are only defined for $M \geq 2$. The vectors $\mathbf{m}^{M+\mu}$ are defined for $M \geq 1$ however, but we have from $(5.4)_2$ that for $M = 1$,

$$\begin{aligned} \mathbf{m}^{1+\mu} &= \omega^1 \mathbf{c}^{1+\mu} - \omega^{1+\mu} \mathbf{c}^1 \\ &= \omega^1(D_1^\mu \mathbf{c}^1) - (D_1^\mu \omega^1) \mathbf{c}^1 \text{ by (2.86) and (5.2) when } T \neq 0 \quad , \end{aligned}$$

and so for $T \neq 0$,

$$\mathbf{m}^{1+\mu} = 0 \quad , \quad \mu = 1, \ldots, P-1 \quad . \qquad (5.5)$$

For $M \geq 2$, $(5.4)_2$ and $(5.2)_1$ yield the relation ($T \neq 0$)

$$\begin{aligned} \mathbf{m}^{M+\mu} &= \omega^M \mathbf{c}^{M+\mu} - \omega^{M+\mu} \mathbf{c}^M \\ &= \omega^M D_\zeta^\mu \mathbf{c}^\zeta - D_\zeta^\mu \omega^\zeta \mathbf{c}^M \\ &= D_\zeta^\mu \mathbf{m}^\zeta \quad . \end{aligned}$$

Turning now to $(5.4)_3$, we recall that $\mathbf{c}^{P+\eta}$ and $\omega^{P+\eta}$ are all zero by (2.87) and by $(5.2)_2$ respectively, so $(5.4)_3$ becomes

$$\mathbf{m}^{P+\eta} = 0 \quad , \quad \eta = 1, \ldots, N-P \quad .$$

The set $\{\mathbf{m}^{\kappa}\}$ appearing in (5.4) can be written for $T \neq 0$ with the aid of the above results as

$$\mathbf{m}^{\zeta} = \omega^{M}\,\mathbf{c}^{\zeta} - \omega^{\zeta}\,\mathbf{c}^{M} \quad , \qquad \zeta = 1, \ldots, M-1 \ , \ (M \geq 2)$$

$$\begin{aligned} \mathbf{m}^{M+\mu} &= \omega^{M}\,\mathbf{c}^{M+\mu} - \omega^{M+\mu}\,\mathbf{c}^{M} \\ &= \begin{cases} 0 & (M = 1) \\ D^{\mu}_{\zeta}\,\mathbf{m}^{\zeta} & (M \geq 2) \end{cases} \\ &\qquad\qquad \mu = 1, \ldots, P-M \quad , \end{aligned}$$

$$\mathbf{m}^{P+\eta} = 0 \quad , \qquad \eta = 1, \ldots, N-P \quad . \tag{5.6}$$

It is easily shown that since the set $\{\mathbf{c}^{\sigma}\}$ is linearly independent (recall (2.86)) so is $\left\{\mathbf{m}^{\zeta}\right\}^{M-1}_{\zeta}$.

We now derive the alternative set to (5.1), and follow Reddy (I) by writing (4.1) in the following form

$$\mathbf{c}^{M}\cdot\mathbf{s} = -\,\nu^{2}\,\omega^{M}\,T \quad , \tag{5.7}$$

and

$$\mathbf{m}^{\kappa}\cdot\mathbf{s} = 0 \quad , \quad \kappa = 1, \ldots, M-1, M+1, \ldots, N \quad , \tag{5.8}$$

where, as before, c^M is the M^{th} and last member of the set $\{c^\sigma\}$ of linearly independent constraint vectors in $(5.1)_1$. Equation (5.8) immediately provides a necessary condition for the existence of non-homothermal waves (corresponding to (4.3) for homothermal waves) and requires that

$$\dim \operatorname{span} \{m^\kappa\} \leq 2 \quad . \tag{5.9}$$

Entropic formulation of type I constraint vectors

It will prove convenient in the derivation and analysis of the propagation conditions for homentropic waves to employ the entropic formulation. (The thermal formulation introduced above will also be useful, however, especially in facilitating comparisons with the corresponding homothermal results). The type I constraint vectors in the entropic formulation, $\{\hat{c}^\alpha\}$, together with the subsets $\{\hat{c}^\sigma\}$, $\{\hat{c}^{M+\mu}\}$ and $\{\hat{c}^{P+\eta}\}$ were introduced in (2.91) and $(2.96)_{1-3}$ respectively. We also recall that the set $\{\hat{c}^\sigma\}$ is not necessarily linearly independent, and this prompts us to introduce the set of vectors $\{\hat{m}^\kappa\}$ (analogous to the set $\{m^\kappa\}$ in the thermal formulation) which will be found to have the desired properties; namely that the subset $\left\{\hat{m}^\zeta\right\}_{\zeta=1}^{M-1}$ is linearly independent.

Returning now to the subsets of $\{\hat{c}^\alpha\}$ defined in (2.96), we find with the aid of the definitions (2.86) for $\{c^{M+\mu}\}$ and the condition

(5.2) that for <u>non-homothermal</u> waves,

$$\hat{c}^{M+\mu} = D^{\mu}_{\sigma}\left[c^{\sigma} - \rho\ \mu^{-1}\ \omega^{\sigma} \sum_i \frac{\partial^2\psi^0}{\partial a_i \partial\theta}\ (n.p_i)\ q_i\right] \quad , \quad \sigma = 1, \ldots, M \quad ,$$
$$\mu = 1, \ldots, P-M \quad ,$$

$$= D^{\mu}_{\sigma}\ \hat{c}^{\sigma} \qquad \text{for } M \geq 1 \quad . \tag{5.10}$$

Also, (2.93) yields

$$\hat{c}^{P+\eta} = -\ \rho\ \mu^{-1}\ \omega^{P+\eta} \sum_i \frac{\partial^2\psi^0}{\partial a_i \partial\theta}\ (n.p_i)\ q_i \quad , \qquad \eta = 1, \ldots, N-P \quad ,$$

where we have made use of the fact that $c^{P+\eta} \equiv 0$ by (2.87).

Furthermore, $\omega^{P+\eta} = 0$ for non-homothermal waves from $(5.2)_2$, so finally

$$\hat{c}^{P+\eta} = c^{P+\eta} \equiv 0 \quad , \quad \eta = 1, \ldots, N-P \quad . \tag{5.11}$$

The entropic quantities $\hat{\mu}$ and $\hat{\omega}$ are defined by

$$\hat{\mu} = \rho\ \frac{\partial^2\epsilon}{\partial\eta^2}$$

and

$$\hat{\omega}^a = \frac{\partial^2\epsilon}{\partial\eta\partial\lambda_a} = \frac{\partial\hat{\phi}^a}{\partial\eta} \quad ; \tag{5.12}$$

they are related to their thermal counterparts μ and ω^a (defined by (2.93) and (2.94) respectively) by

$$\hat{\mu} = -\ \rho^2\ \mu^{-1} \quad , \quad \hat{\omega}^a = -\ \rho\ \mu^{-1}\ \omega^a \quad . \tag{5.13}$$

Now $\hat{\omega}^{M+\mu} = - \rho \mu^{-1} \omega^{M+\mu} = - \rho \mu^{-1} D^{\mu}_{\sigma} \omega^{\sigma}$

$$= D^{\mu}_{\sigma} \hat{\omega}^{\sigma} \tag{5.14}$$

where $(5.2)_1$ has been used;

furthermore, $\hat{\omega}^{P+\eta} = - \rho \mu^{-1} \omega^{P+\eta}$

$$= 0 \tag{5.15}$$

from $(5.2)_2$ for $T \neq 0$.

We now use the entropic formulation and obtain the equivalent of the condition

$$\mathbf{c}^{a} \cdot \mathbf{s} = - \nu^2 \omega^{a} T \quad . \tag{4.1 bis}$$

The jump of the time derivative of the type I constraint definition (2.36) yields

$$0 = [\dot{\phi}^{a}] = \frac{\partial \hat{\phi}^{a}}{\partial \mathbf{F}} [\dot{\mathbf{F}}] + \frac{\partial \hat{\phi}^{a}}{\partial \eta} [\dot{\eta}] + \frac{\partial \hat{\phi}^{a}}{\partial \lambda_{\gamma}} [\dot{\lambda}_{\gamma}] \quad , \quad \gamma = 1, \ldots, N \quad . \tag{5.16}$$

Use of the expressions (3.24) for $[\dot{\mathbf{F}}]$ and $(3.40)_2$ for $[\dot{\eta}]$ plus the definitions (2.91) for $\hat{\mathbf{c}}^{a}$ and $(5.12)_2$ for $\hat{\omega}^{a}$ in (5.16) gives

$$\hat{\mathbf{c}}^{a} \cdot \mathbf{s} = \nu^2 \hat{\omega}^{a} H + \nu \frac{\partial \hat{\phi}^{a}}{\partial \lambda_{\gamma}} [\dot{\lambda}_{\gamma}] \quad . \tag{5.17}$$

We will be restricting attention to homentropic waves from now on, in which case $H = 0$ by (3.42). We also make use of the identity

$$\frac{\partial \hat{\phi}^{a}}{\partial \lambda_{\gamma}} = \rho\ \hat{\mu}^{-1}\ \hat{\omega}^{a}\ \hat{\omega}^{\gamma} \tag{5.18}$$

given in (I), and consequently (5.17) can be written for homentropic waves in the form

$$\hat{\mathbf{c}}^{a}\cdot\mathbf{s} = \rho\ \nu\ \hat{\mu}^{-1}\ \hat{\omega}^{a}\ \hat{\omega}^{\gamma}\ [\dot{\lambda}_{\gamma}]\quad ,\quad \gamma = 1,\ \ldots,\ N\quad . \tag{5.19}$$

Use of the results (5.10,11,14,15) enables us to write (5.19) when $T \neq 0$ as

$$\begin{aligned} \hat{\mathbf{c}}^{\sigma}\cdot\mathbf{s} &= \rho\ \nu\ \hat{\mu}^{-1}\ \hat{\omega}^{\sigma}\ \hat{\omega}^{\tau}\left[[\dot{\lambda}_{\tau}] + D^{\mu}_{\tau}\ [\dot{\lambda}_{M+\mu}]\right] \qquad \sigma,\tau = 1,\ \ldots,\ M\quad ,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mu = 1,\ \ldots,\ P-M\quad ,\\ \hat{\mathbf{c}}^{M+\mu}\cdot\mathbf{s} &= D^{\mu}_{\sigma}\ \hat{\mathbf{c}}^{\sigma}\cdot\mathbf{s}\quad ,\\ \hat{\mathbf{c}}^{P+\eta}\cdot\mathbf{s} &= 0\quad . \end{aligned} \tag{5.20}$$

Elimination of the jumps $[\dot{\lambda}_{a}]$ appearing in (5.19,20) will be dealt with later in the derivation of the first propagation condition for homentropic waves.

The set $\{\hat{\mathbf{c}}^{a}\}$ suffers from the disadvantage that the subset $\left\{\hat{\mathbf{c}}^{\sigma}\right\}_{\sigma=1}^{M}$ is not necessarily linearly independent, as has been remarked previously (discussion following (2.96)). We therefore construct an alternative set to (5.20) that is analogous to the set (5.7,8) by writing

$$\hat{\mathbf{c}}^{M} \cdot \mathbf{s} = \rho \, \nu \, \hat{\mu}^{-1} \, \hat{\omega}^{M} \, \hat{\omega}^{\gamma} \Big[[\dot{\lambda}_{\gamma}] + D^{\mu}_{\tau} \, [\dot{\lambda}_{M+\mu}] \Big] \qquad \tau = 1, \ldots, M \quad , \quad \mu = 1, \ldots, P-M \quad , \tag{5.21}$$

$$\hat{\mathbf{m}}^{\kappa} \cdot \mathbf{s} = 0 \quad ; \tag{5.22}$$

where (5.14,15) have been used in (5.21) and where the vectors $\hat{\mathbf{m}}^{\kappa}$ are defined by

$$\hat{\mathbf{m}}^{\kappa} = \hat{\omega}^{M} \, \hat{\mathbf{c}}^{\kappa} - \hat{\omega}^{\kappa} \, \hat{\mathbf{c}}^{M} \quad , \quad \kappa = 1, \ldots, M-1, M+1, \ldots, N \quad .$$

We also note that the $\hat{\mathbf{m}}^{\kappa}$ are related to their thermal counterparts $\mathbf{m}^{\kappa}$ by

$$\hat{\mathbf{m}}^{\kappa} = - \rho \, \mu^{-1} \, \mathbf{m}^{\kappa} \quad . \tag{5.23}$$

The set $\{\hat{\mathbf{m}}^{\kappa}\}$ can be rewritten as follows after the use of the expressions (5.10,11) for $\hat{\mathbf{c}}^{a}$, (5.14,15) for $\hat{\omega}^{a}$ and (5.23) for $\hat{\mathbf{m}}^{\kappa}$ in terms of $\mathbf{m}^{\kappa}$: we have

$$\begin{aligned} \hat{\mathbf{m}}^{\zeta} &= \hat{\omega}^{M} \, \hat{\mathbf{c}}^{\zeta} - \hat{\omega}^{\zeta} \, \hat{\mathbf{c}}^{M} \\ &= - \rho \, \mu^{-1} \, \mathbf{m}^{\zeta} \quad , \qquad \zeta = 1, \ldots, M-1 \quad , \quad (M \geq 2) \quad ; \end{aligned}$$

$$\begin{aligned} \hat{\mathbf{m}}^{M+\mu} &= D^{\mu}_{\zeta} \, \hat{\mathbf{m}}^{\zeta} \quad , \\ &= - \rho \, \mu^{-1} \, D^{\mu}_{\zeta} \, \mathbf{m}^{\zeta} \quad , \qquad \mu = 1, \ldots, P-M \quad , \quad (M \geq 2) \quad ; \end{aligned}$$

$$\hat{\mathbf{m}}^{P+\eta} = - \rho \, \mu^{-1} \, \mathbf{m}^{P+\eta} = 0 \quad , \quad \eta = 1, \ldots, N-P \quad . \tag{5.24}$$

Because of the relations $(5.13)_2$ for $\hat{\omega}^\alpha$ and (5.23) for $\hat{\mathbf{m}}^\kappa$ in terms of the corresponding thermal variables, the subset $\{\hat{\mathbf{m}}^\zeta\}$ is linearly independent.

Propagation conditions for homentropic waves

We begin the derivation of the first propagation condition for homentropic waves with the local form of the balance of linear momentum equation (2.9) as was done in Chapter 4 for homothermal waves. At the singular surface, (2.9) takes the form

$$[\dot{\mathbf{S}}]\mathbf{n} = -\rho\,\nu\,\mathbf{s} \quad . \tag{4.6 bis}$$

The entropic formulation is employed here and by (2.76,7) we have

$$[\dot{\hat{\mathbf{S}}}]\mathbf{n} = \hat{\mathbf{A}}[\dot{\mathbf{F}}]\mathbf{n} + \hat{\mathbf{M}}\mathbf{n}[\dot{\eta}] + \rho[\dot{\lambda}_\alpha]\hat{\mathbf{c}}^\alpha \quad , \tag{5.25}$$

where the definition (2.91) for $\hat{\mathbf{c}}^\alpha$ has been used and where the second order tensor $\hat{\mathbf{M}}$ and fourth order tensor $\hat{\mathbf{A}}$ are defined respectively by

$$\hat{\mathbf{M}} = \rho\,\frac{\partial^2\epsilon}{\partial\mathbf{F}\partial\eta} \quad , \tag{5.26}$$

and

$$\hat{\mathbf{A}} = \rho\,\frac{\partial^2\epsilon}{\partial\mathbf{F}\partial\mathbf{F}} \quad ; \tag{5.27}$$

these quantities being related to their thermal counterparts $\mathbf{M}$ (see (2.92,5) and $\mathbf{A}$ (see (4.8)) by

$$\hat{\mathbf{M}} = -\rho\,\mu^{-1}\,\mathbf{M}^0 \tag{5.28}$$

and

$$\hat{\mathbf{A}} = \mathbf{A} - \mu^{-1}\, \mathbf{M}^0 \otimes \mathbf{M}^0 \quad . \tag{5.29}$$

For homentropic waves $[\dot{\eta}] = 0$ in (5.25) by (3.40,2). We use (3.24) to substitute for $[\dot{\mathbf{F}}]$ in (5.25) and use (5.25) in turn to substitute for $[\dot{\mathbf{S}}]\mathbf{n}$ in (4.6). This yields

$$(\rho\, \nu^2\, \mathbf{I} - \hat{\mathbf{Q}})\ \mathbf{s} = -\ \rho\ \nu[\dot{\lambda}_\alpha]\hat{\mathbf{c}}^\alpha \quad , \tag{5.30}$$

where the homentropic acoustic tensor $\hat{\mathbf{Q}}$ is defined by

$$\hat{\mathbf{Q}}(\mathbf{u},\ \mathbf{v}) = \hat{\mathbf{A}}(\mathbf{u},\ \mathbf{n},\ \mathbf{v},\ \mathbf{n}) \qquad \text{for all } \mathbf{u},\ \mathbf{v},\ \in \mathrm{V} \quad , \tag{5.31}$$

and is related to the acoustic tensor $\mathbf{Q}$ (recall (4.10,11)) in the thermal formulation by

$$\hat{\mathbf{Q}} = \mathbf{Q} - \mu^{-1}\, \mathbf{M}^0\mathbf{n} \otimes \mathbf{M}^0\mathbf{n} \quad . \tag{5.32}$$

The set of equations (5.21,2) is now employed in (5.30), following the general approach given in (I). There, however, only the type I constraint subset $\left\{\phi^\sigma\right\}_{\sigma=1}^{\mathrm{M}}$ with the corresponding linearly independent set of vectors $\left\{\hat{\mathbf{m}}^\kappa\right\}_{\zeta=1}^{\mathrm{M}-1}$ was treated; here we allow also the subsets $\left\{\phi^{\mathrm{M}+\mu}\right\}_{\mu=1}^{\mathrm{P}-\mathrm{M}}$ and $\left\{\phi^{\mathrm{P}+\eta}\right\}_{\eta=1}^{\mathrm{N}-\mathrm{P}}$ with the corresponding vectors $\hat{\mathbf{m}}^{\mathrm{M}+\mu}$ and $\hat{\mathbf{m}}^{\mathrm{P}+\eta}$ given by $(5.24)_{2,3}$ respectively.

First we assume $\hat{\omega}^{\mathrm{M}} \neq 0$ (the case $\hat{\omega}^{\mathrm{M}} = 0$ will be discussed later) in (5.21) to obtain

$$[\dot{\lambda}_{\mathrm{M}}] = \rho^{-1}\ \nu^{-1}\ \hat{\mu}(\hat{\omega}^{\mathrm{M}})^{-2}\ \hat{\mathbf{c}}^{\mathrm{M}}\cdot\mathbf{s} - \hat{\omega}^\kappa(\hat{\omega}^{\mathrm{M}})^{-1}[\dot{\lambda}_\kappa] \quad ,$$
$$\kappa = 1,\ \ldots,\ \mathrm{M}-1,\ \mathrm{M}+1,\ \ldots,\ \mathrm{N} \quad . \tag{5.33}$$

Equation (5.30) can then be written

$$(\rho\, \nu^2\, \mathbf{I} - \hat{\mathbf{Q}})\ \mathbf{s} = -\ \mu(\hat{\omega}^{\mathrm{M}})^{-2}(\hat{\mathbf{c}}^{\mathrm{M}}\cdot\mathbf{s})\hat{\mathbf{c}}^{\mathrm{M}} + \rho\ \nu\ \hat{\omega}^{\kappa}(\hat{\omega}^{\mathrm{M}})^{-1}[\dot{\lambda}_{\kappa}]\hat{\mathbf{c}}^{\mathrm{M}} - \rho\ \nu[\dot{\lambda}_{\kappa}]\hat{\mathbf{c}}^{\kappa} \quad ,$$

and with the aid of the definition for $\hat{\mathbf{m}}^{\kappa}$ following (5.22) we find that

$$\begin{aligned}(\rho\, \nu^2\, \mathbf{I} - \hat{\mathbf{Q}}^*)\ \mathbf{s} &= -\ \rho\ \nu(\hat{\omega}^{\mathrm{M}})^{-1}[\dot{\lambda}_{\kappa}]\ (\hat{\omega}^{\mathrm{M}}\ \hat{\mathbf{c}}^{\kappa} - \hat{\omega}^{\kappa}\ \hat{\mathbf{c}}^{\mathrm{M}}) \\ &= -\ \rho\ \nu(\hat{\omega}^{\mathrm{M}})^{-1}[\dot{\lambda}_{\kappa}]\hat{\mathbf{m}}^{\kappa} \quad ,\end{aligned} \tag{5.34}$$

where the modified acoustic tensor $\hat{\mathbf{Q}}^*$ is defined by

$$\hat{\mathbf{Q}}^* = \hat{\mathbf{Q}} - \hat{\mu}(\hat{\omega}^{\mathrm{M}})^{-2}\ \hat{\mathbf{c}}^{\mathrm{M}} \otimes \hat{\mathbf{c}}^{\mathrm{M}} \quad . \tag{5.35}$$

Since $\hat{\mathbf{Q}}$ is symmetric by (5.32), $\hat{\mathbf{Q}}^*$ is clearly also symmetric.

Now from (5.24) we have, when $\mathrm{T} \neq 0$,

$$\begin{aligned} &\hat{\mathbf{m}}^{\mathrm{M}+\mu} = \mathrm{D}^{\mu}_{\zeta}\ \hat{\mathbf{m}}^{\zeta} \quad , && \zeta = 1, \ldots, \mathrm{M}-1 \quad , \\ & && \mu = 1, \ldots, \mathrm{P}-\mathrm{M} \quad , \\ &\hat{\mathbf{m}}^{\mathrm{P}+\eta} = 0 \quad , && \eta = 1, \ldots, \mathrm{N}-\mathrm{P} \quad , \end{aligned}$$

so (5.34) can be written as

$$(\rho\, \nu^2\, \mathbf{I} - \hat{\mathbf{Q}}^*)\ \mathbf{s} = -\ \rho\ \nu(\hat{\omega}^{\mathrm{M}})^{-1}\ \hat{\mathbf{m}}^{\zeta}\Big[[\dot{\lambda}_{\zeta}] + \mathrm{D}^{\mu}_{\zeta}\ [\dot{\lambda}_{\mathrm{M}+\mu}]\Big] \quad . \tag{5.36}$$

We recall from (4.15) that in the derivation of the first homothermal propagation condition, only the linearly independent set of constraint vectors $\left\{\mathbf{c}^{\sigma}\right\}_{\sigma=1}^{\mathrm{M}}$ appeared. Their role is played in the homentropic case by the set $\left\{\hat{\mathbf{m}}^{\zeta}\right\}_{\zeta=1}^{\mathrm{M}-1}$; in both cases the jumps $[\dot{\lambda}_{\mathrm{P}+\eta}]$ corresponding to the constraint vectors $\left\{\mathbf{c}^{\mathrm{P}+\eta}\right\}_{\eta=1}^{\mathrm{N}-\mathrm{P}}$ (or equivalently, $\left\{\hat{\mathbf{c}}^{\mathrm{P}+\eta}\right\}_{\eta=1}^{\mathrm{N}-\mathrm{P}}$) , are absent from the first propagation condition. We recall from the discussion preceding (4.29) that the subset $\left\{\mathbf{c}^{\sigma}\right\}_{\sigma=1}^{\mathrm{M}}$ has no effect on homothermal propagation if $\mathrm{M} = 0$; correspondingly, the subset $\left\{\hat{\mathbf{m}}^{\zeta}\right\}_{\zeta=1}^{\mathrm{M}-1}$ has no effect on homentropic propagation if $\mathrm{M} = 0$ or 1 by (5.24).

For $\mathrm{M} \geq 2$, the remaining jumps in $[\dot{\lambda}_{\alpha}]$ on the right-hand side of (5.36) are eliminated by defining a set $\left\{\hat{\boldsymbol{\ell}}_{\zeta}\right\}_{\zeta=1}^{\mathrm{M}-1}$ of vectors reciprocal to $\hat{\mathbf{m}}^{\zeta}$, in the sense that

$$\hat{\mathbf{m}}^{\zeta}\cdot\hat{\boldsymbol{\ell}}_{\epsilon} = \delta^{\zeta}_{\epsilon} \quad , \quad \hat{\boldsymbol{\ell}}_{\zeta} \in \text{ span } \{\hat{\mathbf{m}}^{\zeta}\} \quad . \tag{5.37}$$

The scalar product of (5.35) with $\hat{\boldsymbol{\ell}}_{\epsilon}$ yields

$$\hat{\boldsymbol{\ell}}_{\zeta}\cdot\hat{\mathbf{Q}}^{*}\ \mathbf{s} = -\ \rho\ \nu(\hat{\omega}^{\mathrm{M}})^{-1}\Big[[\dot{\lambda}_{\zeta}] + \mathrm{D}^{\mu}_{\zeta}\ [\dot{\lambda}_{\mathrm{M}+\mu}]\Big] \quad , \quad \zeta = 1, \ldots, \mathrm{M}-1 \quad , \quad \mu = 1, \ldots, \mathrm{P}-\mathrm{M} \quad . \tag{5.38}$$

The <u>first propagation condition for homentropic waves</u> when $\mathrm{T} \neq 0$ is obtained from (5.36) with the aid of (5.38) and is found to be

$$(\rho\ \nu^{2}\ \mathbf{I} - \hat{\mathbf{P}}\ \hat{\mathbf{Q}}^{*})\ \mathbf{s} = 0 \quad , \tag{5.39}$$

where the projection operator $\hat{\mathbf{P}}$ is defined by

$$\hat{\mathbf{P}} = \mathbf{I} - \hat{\mathbf{m}}^{\zeta} \otimes \boldsymbol{\ell}_{\zeta} \quad , \quad \zeta = 1, \ldots, \mathbf{M}\text{-}1 \quad , \tag{5.40}$$

and we recall that the $\{\hat{\mathbf{m}}^{\zeta}\}$ are only defined for $\mathbf{M} \geq 2$ by (5.24), so for $\mathbf{M} < 2$, $\mathbf{P} = \mathbf{I}$.

The derivation given above for the first propagation condition (5.39) requires $\hat{\omega}^{\mathbf{M}} \neq 0$ in the expression (5.33) for $[\dot{\lambda}_{\mathbf{M}}]$. We will find that in some cases, it is a necessary condition for the existence of non-homothermal waves that all the type I constraints be mechanical, so that $\hat{\omega}^{a} \equiv 0$, $a = 1, \ldots, \mathrm{N}$. We therefore rederive the first propagation condition under these circumstances.

We begin by noting that $\hat{\omega}^{a} = 0 \Rightarrow \omega^{a} = 0$ by the transformation $(5.13)_2$.

Now

$$\hat{\mathbf{c}}^{a} = \mathbf{c}^{a} - \mu^{-1}\, \hat{\omega}^{a}\, \mathbf{M}\, \mathbf{n} \quad , \tag{2.91 bis}$$

and so $\omega^{a} = 0$ implies that

$$\hat{\mathbf{c}}^{a} = \mathbf{c}^{a} \tag{5.41}$$

and in consequence the subset $\{\hat{\mathbf{c}}^{\sigma}\}$ is linearly independent. This is not normally the case for $\{\hat{\mathbf{c}}^{\sigma}\}$, (although it is for $\{\mathbf{c}^{\sigma}\}$ by the definition (2.86)), as will be recalled from the discussion following (2.96). Furthermore, we have from (5.1) and (5.4) that when $\omega^{a} = 0$,

($T \neq 0$ still holds)

$$\hat{c}^{\sigma} \cdot s = c^{\sigma} \cdot s = 0 \quad , \qquad \sigma = 1, \ldots, M \quad ,$$
$$\hat{c}^{M+\mu} \cdot s = c^{M+\mu} \cdot s = D^{\mu}_{\sigma} \, c^{\sigma} \cdot s = 0 \quad , \qquad \mu = 1, \ldots, P-M \quad ,$$

and we also recall the result (5.11) that

$$\hat{c}^{P+\eta} = c^{P+\eta} \equiv 0 \quad , \qquad \eta = 1, \ldots, N-P \quad . \tag{5.42}$$

The above results (5.41,2) mean that for $\omega^{a} = 0$ and $T \neq 0$, the analogy with the derivation of the homothermal propagation condition (equations (4.9-19)) is immediate, and we follow the procedure given there. We begin with (5.30) (the homentropic equivalent of (4.9)) which is

$$(\rho \, \nu^2 \, I - \hat{Q}) \, s = - \rho \, \nu [\dot{\lambda}_a] \hat{c}^{a} \quad , \tag{5.30 bis}$$

and write

$$- \rho \, \nu [\dot{\lambda}_a] \, \hat{c}^{a} = - \rho \, \nu \Big[[\dot{\lambda}_{\sigma}] + D^{\mu}_{\sigma} \, [\dot{\lambda}_{M+\mu}] \Big] \hat{c}^{\sigma} \quad , \tag{5.43}$$

by analogy with (4.12).

A set $\left\{\hat{d}_{\tau}\right\}_{\tau=1}^{M}$ is now defined such that $\hat{c}^{\sigma} \cdot \hat{d}_{\tau} = \delta^{\sigma}_{\tau}$, $\hat{\delta}_{\tau} \in \operatorname{span} \{\hat{c}^{\sigma}\}$ and the scalar product of (5.30) with $\hat{d}_{\tau}$ yields

$$\hat{d}_{\tau} \cdot \hat{Q} \, s = \rho \, \nu \Big[[\dot{\lambda}_{\tau}] + D^{\mu}_{\tau} \, [\dot{\lambda}_{M+\mu}] \Big] \quad . \tag{5.44}$$

The first propagation condition for homentropic waves when $\hat{\omega}^{a} = 0$ and $T \neq 0$ is then found from (5.30) with the aid of (5.43,4) to be

$$(\rho\, \nu^2\, \mathbf{I} - \tilde{\mathbf{P}}\, \hat{\mathbf{Q}})\, \mathbf{s} = 0 \quad , \tag{5.45}$$

where the projection tensor $\tilde{\mathbf{P}}$ is defined by

$$\tilde{\mathbf{P}} = \mathbf{I} - \hat{\mathbf{c}}^{\sigma} \otimes \hat{\mathbf{d}}_{\sigma} \quad . \tag{5.46}$$

<u>Propagation conditions for longitudinal and transverse principal waves in isotropic materials</u>

We continue this investigation of the propagation conditions for homentropic waves by imposing the condition of <u>isotropy</u> and assume that $\phi^{a} = \phi^{a}(a_i,\ \theta,\ e_{AB},\ f_{AB})$ for type I constraints as in the homothermal case (discussion following (4.20)). Attention is also restricted to <u>principal waves</u> that are longitudinal ($\mathbf{s} = \sigma\, \mathbf{q}_3$) or transverse ($\mathbf{s} = \sigma\, \mathbf{q}_{\Delta}$, Δ = 1 or 2) and the waves are assumed to propagate in the direction $\mathbf{n} = \mathbf{p}_3$.

Since the analysis of homentropic wave propagation is somewhat more cumbersome than it is for homothermal waves, it is advantageous to identify situations in which waves are found to be both homentropic and homothermal ($H = T = 0$) and so are generalized transverse waves as defined following (3.42). In such cases the analysis of homothermal wave propagation presented in Chapter 4 can be adopted in prefence to that given here. An expression that will enable the value of T to be determined is now derived.

We begin by evaluating the jump $[\dot{\eta}]$ using the expression (2.62) for η and find that

$$[\dot{\eta}] = - \left[\left[\frac{\partial\psi}{\partial\theta}\right]^{\cdot}\right]$$

$$= - \frac{\partial^2\psi}{\partial\mathbf{F}\partial\theta} [\dot{\mathbf{F}}] - \frac{\partial^2\psi}{\partial\theta^2} [\dot{\theta}] - \frac{\partial\phi^a}{\partial\theta} [\dot{\lambda}_a] \quad , \tag{5.47}$$

where the result $\frac{\partial\psi}{\partial\lambda_a} = \phi^a$ has been used in the last term on the right-hand side of (5.47). With the aid of the definitions (2.93-5) and the identities (3.24), (3.37), (3.40), equation (5.47) yields, after some rearrangement,

$$T = - \rho\, \mu^{-1}\, H - \mu^{-1}\, \nu^{-2}\, \mathbf{s}\cdot\mathbf{M}^0\mathbf{n} + \rho\, \mu^{-1}\, \nu^{-1}\, \omega^a\, [\dot{\lambda}_a] \quad . \tag{5.48}$$

Now $H \equiv 0$ for homentropic waves by (3.42) and $\mathbf{s}\cdot\mathbf{M}^0\mathbf{n}$ can be simplified with the aid of (2.95), since $\mathbf{n} = \mathbf{p}_3$ and we are considering isotropic media. Hence (5.48) reduces to

$$T = \rho\, \mu^{-1}\, \nu^{-1} \left[- \nu^{-1}\, \frac{\partial^2\psi^0}{\partial a_3 \partial\theta} (\mathbf{s}\cdot\mathbf{q}_3) + \omega^a\, [\dot{\lambda}_a] \right] \quad . \tag{5.49}$$

When $T \neq 0$, we can use the results for $\omega^{M+\mu}$, $\omega^{P+\eta}$ given in (5.2) and write (5.49) as

$$T = \rho\, \mu^{-1}\, \nu^{-1} \left[\nu^{-1}\, \frac{\partial^2\psi^0}{\partial a_3 \partial\theta} (\mathbf{s}\cdot\mathbf{q}_3) + \omega^\sigma \left[[\dot{\lambda}_\sigma] + D^\mu_\sigma\, [\dot{\lambda}_{M+\mu}] \right] \right] \quad . \tag{5.50}$$

The equivalents of (5.49,50) in the entropic formulation are easily obtained from (5.47) with the aid of the transformations (5.13) and (5.28) and are

$$T = \nu^{-1}\left[\rho^{-1}\, \Psi(\mathbf{s}\cdot\mathbf{q}_3) - \hat{\omega}^{a}[\dot{\lambda}_a]\right] \tag{5.51}$$

and, when $T \neq 0$,

$$T = \nu^{-1}\left[\rho^{-1}\, \Psi(\mathbf{s}\cdot\mathbf{q}_3) - \hat{\omega}^{\sigma}([\dot{\lambda}_\sigma] + D^{\mu}_{\sigma}\, [\dot{\lambda}_{M+\mu}])\right] \quad , \tag{5.52}$$

where $\Psi(\mathbf{s}\cdot\mathbf{q}_3) = \mathbf{s}\cdot\hat{\mathbf{M}}\mathbf{n}$ and Ψ contains contributions from the constraints, unlike $\mathbf{s}\cdot\mathbf{M}^0\mathbf{n}$.

Two results are immediate from (5.49) or (5.51): when $\hat{\omega}^a = \omega^a = 0$ and all type I constraints are <u>mechanical</u>, then

$T \neq 0$ for longitudinal waves;

$T = 0$ for transverse waves, so that these waves are generalized transverse waves ($H = T = 0$). (5.53)

We follow our previous approach in Chapter 4 and collect together the <u>propagation conditions</u> that are satisfied across the wavefront when homentropic waves exist. Both the thermal and entropic formulations of the conditions are given where appropriate.

The first propagation condition is given by (5.39) or (5.45) (we renumber the equations for convenience):

$$(\rho\, \nu^2\, \mathbf{I} - \hat{\mathbf{P}}\, \hat{\mathbf{Q}}^*)\, \mathbf{s} = 0 \quad , \quad (\hat{\omega}^M \neq 0) \tag{5.54}$$

$$(\rho\, \nu^2\, \mathbf{I} - \tilde{\mathbf{P}}\, \hat{\mathbf{Q}})\, \mathbf{s} = 0 \quad , \quad (\hat{\omega}^a = 0) \quad . \tag{5.55}$$

The second propagation condition is given in the entropic formulation by the set (5.21,2):

$$\hat{c}^{M}\cdot s = \rho\ \nu\ \hat{\mu}^{-1}\ \hat{\omega}^{M}\ \hat{\omega}^{\tau}\Big[[\dot{\lambda}_{\tau}] + D^{\mu}_{\tau}\ [\dot{\lambda}_{M+\mu}]\Big] \quad \tau = 1, \ldots, M \quad ,$$
$$\mu = 1, \ldots, P\text{-}M \quad ,$$

$$\hat{m}^{\kappa}\cdot s = 0 \quad , \qquad \kappa = 1, \ldots, M\text{-}1, M{+}1, \ldots, N \quad . \tag{5.56}$$

The thermal formulation of the second propagation condition is given either by (4.1), which is

$$\hat{c}^{a}\cdot s = -\ \nu^{2}\ \omega^{a}\ T \quad , \quad a = 1, \ldots, N \quad ; \tag{5.57}$$

or by (5.7,8):

$$\hat{c}^{M}\cdot s = -\ \nu^{2}\ \omega^{M}\ T \tag{5.58}$$

and

$$m^{\kappa}\cdot s = 0 \quad , \quad \kappa = 1, \ldots, M\text{-}1, M{+}1, \ldots, N \quad . \tag{5.59}$$

It will at times prove useful to express the type I constraint vectors $\hat{c}^{a}$, m^{κ} and the acoustic tensor $\hat{Q}$ in the thermal formulation with the aid of the transformations

$$\hat{c}^{a} = c^{a} - \rho\ \mu^{-1}\ \omega^{a}\ \frac{\partial^{2}\psi^{o}}{\partial a_{3}\partial\theta}\ q_{3} \quad ; \quad a = 1, \ldots, N \quad , \tag{5.60}$$

which follows from (2.92), (2.95) and (3.29), and

$$\hat{\mathbf{m}}^{\kappa} = - \rho \, \mu^{-1} \, \hat{\mathbf{m}}^{\kappa} \quad , \quad \kappa = 1, \ldots, M-1, M+1, \ldots, N \quad , \tag{5.61}$$

as in (5.23).

Finally (5.32) with the aid of (2.95) and (3.29) yields

$$\hat{\mathbf{Q}} = \mathbf{Q} - \rho^2 \, \mu^{-1} \left[\frac{\partial^2 \psi^0}{\partial a_3 \partial \theta}\right]^2 \mathbf{q}_3 \otimes \mathbf{q}_3 \tag{5.62}$$

so that $\hat{\mathbf{Q}}$, like $\mathbf{Q}$, is in spectral form (recall 4.22-5) with proper vectors $\mathbf{q}_i$ and corresponding proper numbers

$$\begin{aligned} \hat{Q}_{\Delta} &= \hat{\mathbf{Q}}(\mathbf{q}_{\Delta} \, , \, \mathbf{q}_{\Delta}) = Q_{\Delta} \quad , \qquad \Delta = 1 \text{ or } 2 \quad , \\ \hat{Q}_3 &= \hat{\mathbf{Q}}(\mathbf{q}_3 \, , \, \mathbf{q}_3) = Q_3 - \rho^2 \, \mu^{-1} \left[\frac{\partial^2 \psi^0}{\partial a_3 \partial \theta}\right]^2 \quad . \end{aligned} \tag{5.63}$$

<u>Longitudinal and transverse principal wave solutions of the propagation conditions</u>

We investigate solutions to the propagation conditions for homentropic waves for the wave speeds ν_1, ν_2, ν_3 corresponding to the wave amplitudes $\sigma \, \mathbf{q}_1$, $\sigma \, \mathbf{q}_2$, $\sigma \, \mathbf{q}_3$ respectively, where such solutions are not precluded by the constraints. The propagation conditions are employed in the entropic formulation, but it will prove convenient later to employ the thermal formulation when investigating the influence of the constraints on the solutions for ν_i and to facilitate comparison with the corresponding homothermal results.

We have the result from (5.9) that for homentropic waves, dim span $\{\mathbf{m}^{\kappa}\} \leq 2$. We treat the four all-embracing cases dim span $\{\mathbf{c}^{a}\} = M = 0,1,2,3$ in turn. It will be recalled that $M \leq 2$ for homothermal waves by (4.3), and where appropriate we relate the homentropic results obtained here to the homothermal results obtained previously. It is assumed throughout that conditions are such that

$$\hat{\mathbf{Q}}^{*}(\mathbf{v}, \mathbf{v}) > 0$$

for all non-zero $\mathbf{v}$ for which $\hat{\mathbf{m}}^{\kappa}\cdot\mathbf{v} = 0$, so that the proper numbers of $\hat{\mathbf{Q}}^{*}$ are real and positive. Since we restrict attention to principal waves that are longitudinal or transverse, we will only be concerned with situations in which there exist one or more proper vectors of $\hat{\mathbf{Q}}^{*}$ that are parallel to $\mathbf{q}_i$, $i = 1,2$, or 3, with corresponding proper numbers

$$\hat{Q}_i^{*} = \hat{\mathbf{Q}}^{*}(\mathbf{q}_i \, , \, \mathbf{q}_i) \quad , \quad i = 1,2, \text{ or } 3 \quad .$$

In each of the following cases, $M = 0,1,2,3$, we begin by assuming the existence of a homentropic wave that is non-homothermal, so that the previous expression (5.52) for T is applicable.

<u>$M = 0$</u>

All type I constraints are inactive and $\mathbf{c}^{\eta} \equiv 0$, $\eta = 1, \ldots, N$ from (2.87) with $P = 0$. Since $T \neq 0$ we have, from (5.2) and (5.15), $\hat{\omega}^{\eta} = \omega^{\eta} = 0$, so that all type I constraints are <u>mechanical</u>, and $\hat{\mathbf{c}}^{\eta} = \mathbf{c}^{\eta} \equiv 0$ from (5.11). The results (5.53) for mechanical constraints are recalled; only <u>longitudinal</u> waves with $T \neq 0$ are compatible with mechanical constraints, and transverse waves (for which $\mathbf{H} = \mathrm{T} = 0$) are best treated as in Chapter 4. For waves with mechanical constraints, the form of the first propagation condition given in (5.55) is appropriate:

$$(\rho\, \nu^2\, \mathbf{I} - \tilde{\mathbf{P}}\, \hat{\mathbf{Q}})\, \mathrm{s} = 0 \quad . \tag{5.55 bis}$$

For $\mathbf{M} = 0$, $\tilde{\mathbf{P}} = \mathbf{I}$ from (5.46), and (5.55) can be rewritten as

$$\hat{\mathbf{Q}}\, \mathrm{s} = \rho\, \nu^2\, \mathrm{s} \quad , \tag{5.64}$$

and from (5.62,3) we see that longitudinal waves $\mathrm{s} = \sigma\, \mathrm{q}_3$ satisfy (5.64) with wavespeed ν_3 given by

$$\begin{aligned} \nu_3^2 &= \rho^{-1}\, \hat{\mathbf{Q}}_3 \\ &= \rho^{-1}\left[\mathbf{Q}_3 - \rho^2\, \mu^{-1}\left[\frac{\partial^2 \psi^{\mathrm{o}}}{\partial \mathrm{a}_3 \partial \theta}\right]^2\right] \quad . \end{aligned} \tag{5.65}$$

<u>$\mathbf{M} = 1$</u>

We begin by considering the situation in which $\hat{\omega}^1 \neq 0$, and since $\mathrm{T} \neq 0$ is assumed, we have from (5.52) that

$$\mathrm{T} = \nu^{-1}\left[\rho^{-1}\, \mathbf{\Psi}(\mathrm{s}\cdot\mathrm{q}_3) - \hat{\omega}^1([\dot{\lambda}_1] + \mathrm{D}_1^{\mu}\, [\dot{\lambda}_{1+\mu}])\right] \quad . \tag{5.66}$$

The first propagation condition (5.54) is appropriate when $\hat{\omega}^1 \neq 0$ and can be rewritten in the form

$$\hat{\mathbf{Q}}\, \mathrm{s} = \hat{\mu}(\hat{\omega}^1)^{-2}\, (\hat{\mathrm{c}}^1\cdot\mathrm{s})\, \hat{\mathrm{c}}^1 = \rho\, \nu^2\, \mathrm{s} \quad , \tag{5.67}$$

where we have made use of the definition (5.35) of $\hat{\mathbf{Q}}^{*}$, so that for $\mathbf{M} = 1$,

$$\hat{\mathbf{Q}}^{*} = \hat{\mathbf{Q}} - \hat{\mu}(\hat{\omega}^{1})^{-2}\, \hat{\mathbf{c}}^{1} \otimes \hat{\mathbf{c}}^{1} \quad . \tag{5.68}$$

The second propagation condition is found from (5.56) to be

$$\hat{\mathbf{c}}^{1}\cdot\mathbf{s} = \rho\, \nu\, \hat{\mu}^{-1}(\hat{\omega}^{1})^{2}\Big[[\dot{\lambda}_{1}] + D_{1}^{\mu}\, [\dot{\lambda}_{1+\mu}]\Big] \quad . \tag{5.69}$$

The first propagation condition (5.67) has a proper vector in the direction of $\mathbf{q}_i$, i = 1,2, or 3 if either

$$\hat{\mathbf{c}}^{1}\cdot\mathbf{s} = 0 \tag{5.70}$$

or

$$\hat{\mathbf{c}}^{1} \wedge \mathbf{s} = 0 \quad . \tag{5.71}$$

These situations are considered in turn for longitudinal and transverse waves.

For <u>longitudinal</u> waves $\mathbf{s} = \sigma\, \mathbf{q}_3$ and if (5.70) holds, the first propagation condition reduces to

$$\hat{\mathbf{Q}}\, \mathbf{q}_3 = \rho\, \nu^{2}\, \mathbf{q}_3 \quad , \tag{5.72}$$

with the wavespeed ν_3 given by

$$\nu_3^{2} = \rho^{-1}\, \hat{Q}_3 \tag{5.73}$$

with $\hat{Q}_3$ as in (5.63).

The second propagation condition (5.56) reduces to

$$([\dot{\lambda}_1] + D_1^{\mu}\,[\dot{\lambda}_{1+\mu}]) = 0 \quad , \tag{5.74}$$

and we note that substitution of (5.74) in (5.66) yields that $T \neq 0$, consistent with our assumption there. We will in Chapter 8 be making use of the results such as (5.74) for $M = 0,1,2,3$ in order to evaluate terms involving $[\dot{\lambda}_a]$.

When (5.71) holds, the first propagation condition (5.67) reveals that the wavespeed ν_3 is given by

$$\nu_3^2 = \rho^{-1}(\hat{Q}_3 - \hat{\mu}(\hat{\omega}^1)^{-2}(\hat{c}^1\cdot q_3)) \quad . \tag{5.75}$$

The second propagation condition (5.69) now yields

$$\sigma\ \hat{c}^1\cdot q_3 = \rho\ \nu\ \hat{\mu}^{-1}(\hat{\omega}^1)^2\Big[[\dot{\lambda}_1] + D_1^{\mu}\,[\dot{\lambda}_{1+\mu}]\Big] \quad , \tag{5.76}$$

and so

$$[\dot{\lambda}_1] + D_1^{\mu}\,[\dot{\lambda}_{1+\mu}] = \rho^{-1}\ \nu^{-1}\ \hat{\mu}(\hat{\omega}^1)^{-2}\ \sigma(\hat{c}^1\cdot q_3) \neq 0 \quad . \tag{5.77}$$

We see from (5.66) that (5.77) is consistent with the assumption that $T \neq 0$, as long as

$$\Psi - \nu^{-1}\ \hat{\mu}(\hat{\omega}^1)^{-1}(\hat{c}^1\cdot q_3) \neq 0 \quad .$$

For <u>transverse</u> waves $s = \sigma\, q_\Delta$, $\Delta = 1$ or 2 , and when (5.70) holds, the second propagation condition takes the form

$$0 = \hat{c}^1 \cdot q_\Delta = \rho\, \nu\, \hat{\mu}^{-1} (\hat{\omega}^1)^2 \Big[[\dot{\lambda}_1] + D_1^\mu\, [\dot{\lambda}_{1+\mu}] \Big] \quad . \qquad (5.78)$$

Since $\hat{\omega}^1 \neq 0$ is assumed, we must have $[\dot{\lambda}_1] + D_1^\mu\, [\dot{\lambda}_{1+\mu}] = 0$ from (5.78), but then $T = 0$ in (5.66), contradicting our assumption there. Consequently, this constraint configuration is <u>not</u> compatible with transverse waves for which $T \neq 0$.

Conversely, when (5.71) holds, we find from (5.67) that transverse waves propagate with wavespeed

$$\nu_\Delta^2 = \rho^{-1} (\hat{Q}_\Delta - \hat{\mu} (\hat{\omega}^1)^{-2} (\hat{c}^1 \cdot q_\Delta)^2) \qquad \Delta = 1 \text{ or } 2 \quad , \qquad (5.79)$$

and the second propagation condition (5.59) yields

$$[\dot{\lambda}_1] + D_1^\mu\, [\dot{\lambda}_{1+\mu}] = \rho^{-1}\, \nu^{-1}\, \hat{\mu} (\hat{\omega}^1)^{-2}\, \sigma (\hat{c}^1 \cdot q_\Delta) \neq 0 \quad ; \qquad (5.80)$$

use of (5.80) in (5.66) yields a non-zero value for T as required.

The situation of $\hat{\omega}^1 = 0$ is now considered; since $\hat{\omega}^{1+\mu} = D_1^\mu\, \hat{\omega}^1$ and $\hat{\omega}^{P+\eta} = 0$ for $T \neq 0$ by (5.2), all type I constraints are mechanical. The results (5.53) apply; <u>longitudinal</u> waves with $T \neq 0$ are compatible with mechanical constraints, but transverse waves are not.

For <u>longitudinal</u> waves, $s = \sigma\, q_3$ and the first propagation condition for $\hat{\omega}^a = 0$ takes the form

$$(\rho\, \nu^2\, I - \tilde{P}\, \hat{Q}) s = 0 \quad , \qquad (5.55 \text{ bis})$$

where $\tilde{\mathbf{P}} = \mathbf{I} - \hat{c} \otimes \hat{c}$ from (5.46), and $\hat{c} = \hat{c}^1/|\hat{c}^1|$. Hence (5.55) can be written in the form

$$\hat{\mathbf{Q}}\, s - (\hat{c}\cdot\hat{\mathbf{Q}}\, s)\hat{c} = \rho\, \nu^2\, s \quad , \tag{5.81}$$

and since $\hat{\omega}^1 = 0$, the second propagation condition (5.58) yields

$$\hat{c}^1\cdot s = 0 \quad . \tag{5.82}$$

Now $\hat{\mathbf{Q}}$ s is parallel to s from (5.62,3) for $s = \sigma\, \mathbf{q}_i$, and so we have, with the aid of (5.82),

$$\hat{c}\cdot \hat{\mathbf{Q}}\, s = 0 \quad . \tag{5.83}$$

Now finally (5.81) with (5.83) yields the result that for $s = \sigma\, \mathbf{q}_3$, waves propagate with speed

$$\nu_3^2 = \rho^{-1}\, \hat{\mathbf{Q}}_3 \quad . \tag{5.84}$$

<u>M = 2</u>

We begin by assuming $\hat{\omega}^2 \neq 0$ and (5.52) takes the form

$$T = \nu^{-1}\Big[\rho^{-1}\, \mathbf{\Psi}(s\cdot\mathbf{q}_3) - \hat{\omega}^{\sigma}([\dot{\lambda}_{\sigma}] + D^{\mu}_{\sigma}\, [\dot{\lambda}_{2+\mu}])\Big] \neq 0 \quad . \tag{5.85}$$

Since $\hat{\omega}^2 \neq 0$ the first propagation condition (5.54) is appropriate and with the aid of (5.35) we find

$$(\rho\, \nu^2\, \mathbf{I} - \hat{\mathbf{P}}(\hat{\mathbf{Q}} - \hat{\mu}(\hat{\omega}^2)^{-2}\, \hat{c}^2 \otimes \hat{c}^2))s = 0 \quad . \tag{5.86}$$

The second propagation condition is found from (5.56), and with the aid of the results (5.24) for $\{\hat{\mathbf{m}}^{\kappa}\}$ it takes the form

$$\hat{\mathbf{c}}^2 \cdot \mathbf{s} = \rho\ \nu\ \hat{\mu}^{-1}\ \hat{\omega}^2\ \hat{\omega}^{\tau}\Big[[\dot{\lambda}_{\tau}] + D^{\mu}_{\tau}\ [\dot{\lambda}_{2+\mu}]\Big] \qquad \tau = 1,2 \quad ,$$

$$\hat{\mathbf{m}}^1 \cdot \mathbf{s} = 0 \quad ,$$

$$\hat{\mathbf{m}}^{2+\mu} \cdot \mathbf{s} = D^{\mu}_1\ \hat{\mathbf{m}}^1 \cdot \mathbf{s} = 0 \quad , \qquad \mu = 1, \ldots, P-2 \quad ,$$

$$\hat{\mathbf{m}}^{P+\eta} = 0 \quad , \qquad \eta = 1, \ldots, N-P \quad . \tag{5.87}$$

Since $\hat{\mathbf{m}}^1 \cdot \mathbf{s} = 0$ by (5.87), we have $\hat{\mathbf{m}}^1 \cdot \hat{\mathbf{Q}}\ \mathbf{s} = 0$, as $\hat{\mathbf{Q}}\ \mathbf{s}$ is parallel to $\mathbf{s}$ (recall the discussion preceding (5.83)). For $M = 2$, we have $\hat{\mathbf{P}} = \mathbf{I} - \hat{\mathbf{m}} \otimes \hat{\mathbf{m}}$ from (5.40) with $\hat{\mathbf{m}} = \hat{\mathbf{m}}^1/|\hat{\mathbf{m}}^1|$, and with the aid of these results the first propagation condition can be written as

$$(\hat{\mathbf{c}}^2 \cdot \mathbf{s})(\hat{\mathbf{c}}^2 - (\hat{\mathbf{m}} \cdot \hat{\mathbf{c}}^2)\hat{\mathbf{m}}) = \hat{\mu}^{-1}(\hat{\omega}^2)^2(\hat{\mathbf{Q}} - \rho\ \nu^2\ \mathbf{I})\mathbf{s} \quad . \tag{5.88}$$

We are considering only longitudinal and transverse principal waves, so that $\mathbf{s} = \sigma\ \mathbf{q}_i$, $i = 1,2$ or 3. We recall from (5.62,3) that $\hat{\mathbf{Q}}$ is in spectral form, and so (5.88) yields the result that either

$$\hat{\mathbf{c}}^2 \cdot \mathbf{s} = 0 \tag{5.89}$$

or

$$(\hat{\mathbf{c}}^2 - (\hat{\mathbf{m}} \cdot \hat{\mathbf{c}}^2)\hat{\mathbf{m}}) \wedge \mathbf{s} = 0 \quad ,$$

and, since $\hat{\mathbf{m}} \cdot \mathbf{s} = 0$ by (5.87), this latter condition is equivalent to

$$\hat{\mathbf{c}}^2 \wedge \mathbf{s} = 0 \quad . \tag{5.90}$$

Longitudinal and transverse wave solutions are now investigated when (5.89,90) hold, following the previous approach for $M = 1$.

For longitudinal waves $\mathbf{s} = \sigma\, \mathbf{q}_3$ and if (5.89) holds, then from (5.88) we see that longitudinal waves propagate with speed

$$\nu_3^2 = \rho^{-1}\, \hat{\mathbf{Q}}_3 \quad . \tag{5.73 bis}$$

The second propagation condition (5.87) yields firstly that

$$\hat{\omega}^{\tau}([\dot{\lambda}_{\tau}] + D_{\tau}^{\mu}\, [\dot{\lambda}_{2+\mu}]) = 0 \quad . \tag{5.91}$$

Secondly,
$$\hat{\mathbf{m}}^1 \cdot \mathbf{q}_3 = 0$$
$$\Rightarrow \hat{\omega}^2\, \hat{\mathbf{c}}^1 \cdot \mathbf{q}_3 - \hat{\omega}^1 \hat{\mathbf{c}}^2 \cdot \mathbf{q}_3 = 0$$

by the definition of $\hat{\mathbf{m}}^{\kappa}$ following (5.22), and so

$$\hat{\mathbf{c}}^1 \cdot \mathbf{q}_3 = 0 \tag{5.92}$$

since (5.89) holds.

No further information is gained from the expressions involving $\hat{\mathbf{m}}^{1+\mu}$, $\hat{\mathbf{m}}^{P+\eta}$ in (5.87). If instead (5.90) holds, then (5.88) yields

$$\nu_3^2 = \rho^{-1}(\hat{\mathbf{Q}}_3 - \hat{\mu}(\hat{\omega}^2)^{-2}(\hat{\mathbf{c}}^2 \cdot \mathbf{q}_3)^2) \quad , \tag{5.93}$$

and from (5.87),

$$\hat{\omega}^{\tau}([\dot{\lambda}_{\tau}] + D^{\mu}_{\tau}\ [\dot{\lambda}_{2+\mu}]) = \sigma(\hat{c}^2\cdot q_3)\ \rho^{-1}\ \nu^{-1}\ \hat{\mu}(\hat{\omega}^2)^{-2} \quad . \tag{5.94}$$

Furthermore,

$$\hat{m}^1\cdot q_3 = 0$$
$$\Rightarrow \hat{\omega}^2\ \hat{c}^1\cdot q_3 - \hat{\omega}^1\ \hat{c}^2\cdot q_3 = 0$$

by the definition of $\hat{m}^{\kappa}$ following (5.22), and so

$$\hat{c}^1\cdot q_3 = \hat{\omega}^1(\hat{\omega}^2)^{-1}\ \hat{c}^2\cdot q_3 \quad . \tag{5.95}$$

For <u>transverse</u> waves $s = \sigma\ q_{\Delta}$, $\Delta = 1$ or 2 , and we find by an analysis similar to that used for $M = 1$ (see discussion following (5.78)) that a constraint obeying (5.89) is <u>not</u> compatible with transverse waves for which $T \neq 0$.

If (5.90) holds then (5.88) and (5.87) are easily shown to yield the results

$$\nu_{\Delta}^2 = \rho^{-1}(\hat{Q}_{\Delta} - \hat{\mu}(\hat{\omega}^2)^{-2}(\hat{c}^2\cdot q_{\Delta})^2) \quad , \tag{5.96}$$

$$\hat{\omega}^{\tau}([\dot{\lambda}_{\tau}] + D^{\mu}_{\tau}\ [\dot{\lambda}_{2+\mu}]) = \sigma(\hat{c}^2\cdot q_{\Delta})\ \rho^{-1}\ \nu^{-1}\ \hat{\mu}(\hat{\omega}^2)^{-2} \quad , \tag{5.97}$$

$$\hat{c}^1\cdot q_{\Delta} = \hat{\omega}^1(\hat{\omega}^2)^{-1}\ \hat{c}^2\cdot q_{\Delta} \quad . \tag{5.98}$$

The situation $\hat{\omega}^1 = \hat{\omega}^2 = 0$ is now treated. (The case $\hat{\omega}^1 \neq 0$, $\hat{\omega}^2 = 0$ can be treated by the above procedure by relabelling the constraints). All constraints are mechanical since $\hat{\omega}^{2+\mu} = D^{\mu}_{\sigma}\ \hat{\omega}^{\sigma}$ and $\hat{\omega}^{P+\eta} \equiv 0$ when $T \neq 0$ by (5.14,15). As in the situation when $M = 1$, the results (5.53) apply and only longitudinal waves are compatible with mechanical constraints when $T \neq 0$.

We proceed as before for $M = 1$; the first propagation condition (5.55) applies, but now

$$\tilde{\mathbf{P}} = \mathbf{I} - \hat{\mathbf{c}}^{\sigma} \otimes \hat{\mathbf{d}}_{\sigma} \quad , \qquad \sigma = 1,2 \quad , \tag{5.99}$$

and so (5.55) can be written as

$$\hat{\mathbf{Q}}\, \mathbf{s} - (\hat{\mathbf{d}}_{\sigma} \cdot \hat{\mathbf{Q}}\, \mathbf{s}) = \rho\, \nu^2\, \mathbf{s} \quad . \tag{5.100}$$

The second propagation condition is best used in the form (5.19), since $\hat{\mathbf{m}}^1 \cdot \mathbf{s} = \hat{\omega}^2\, \hat{\mathbf{c}}^1 \cdot \mathbf{s} - \hat{\omega}^1\, \hat{\mathbf{c}}^2 \cdot \mathbf{s} = 0$ is trivially satisfied when $\hat{\omega}^1 = \hat{\omega}^2 = 0$, and we find from (5.19) that

$$\begin{aligned} \hat{\mathbf{c}}^2 \cdot \mathbf{s} &= 0 \quad , \\ \hat{\mathbf{c}}^1 \cdot \mathbf{s} &= 0 \quad . \end{aligned} \tag{5.101}$$

Consequently $\hat{\mathbf{d}}_{\sigma} \cdot \hat{\mathbf{Q}}\, \mathbf{s} = 0$, $\sigma = 1,2$ since $\hat{\mathbf{d}}_{\sigma} \cdot \mathbf{s} = 0$ by (5.101) and $\hat{\mathbf{Q}}\mathbf{s}$ is parallel to $\mathbf{s}$ (recall discussion preceding (5.83)). Longitudinal waves therefore propagate with speed ν_3 where

$$\nu_3^2 = \rho^{-1}\, \hat{Q}_3 \tag{5.103}$$

from (5.100).

$\underline{M = 3}$

The possibility of homothermal waves does not arise here since $M \leq 2$ for homothermal waves from (4.3). Furthermore, when $M = 3$ the case $\hat{\omega}^a = 0$ (all constraints mechanical) is <u>not</u> compatible with homentropic wave propagation since then

$$\hat{c}^\sigma \cdot s = 0 \ , \qquad \sigma = 1,2,3 \ , \qquad \text{by (5.42)},$$

which is clearly impossible for non-zero s as the set $\{\hat{c}^\sigma\}$ is linearly independent by the discussion following (5.41). Consequently we assume $\hat{\omega}^3 \neq 0$ and proceed as before for $M = 1,2$; we find from (5.54) and (5.35) that the first propagation condition can be written as

$$(\rho \ \nu^2 - I - \hat{P}(\hat{Q} - \hat{\mu}(\hat{\omega}^3)^{-2} \ \hat{c}^3 \otimes \hat{c}^3))s = 0 \tag{5.103}$$

and the second propagation condition is found from (5.56) and (5.24) to be

$$\begin{aligned}
&\hat{c}^3 \cdot s = \rho \ \nu \ \hat{\mu}^{-1} \ \hat{\omega}^3 \ \hat{\omega}^\tau([\dot{\lambda}_\tau] + D^\mu_\tau \ [\dot{\lambda}_{3+\mu}]) \ , && \tau = 1,2,3 \ , \\
&\hat{m}^\zeta \cdot s = 0 \ , && \zeta = 1,2 \ , \\
&\hat{m}^{3+\mu} \cdot s = D^\mu_\zeta \ \hat{m}^\zeta \cdot s = 0 \ , && \mu = 1, \ldots, P-3 \ , \\
&\hat{m}^{P+\eta} = 0 \ , && \eta = 1, \ldots, N-P \ .
\end{aligned} \tag{5.104}$$

The projection $\hat{\mathbf{P}}$ in (5.103) is given by

$$\hat{\mathbf{P}} = \mathbf{I} - \hat{\mathbf{m}}^{\zeta} \otimes \hat{\boldsymbol{\ell}}_{\zeta} \quad , \quad \zeta = 1,2 \quad , \tag{5.105}$$

from (5.40) and we also note that

$$\hat{\boldsymbol{\ell}}_{\zeta} \cdot \hat{\mathbf{Q}}\ \mathbf{s} = 0 \tag{5.106}$$

since $\hat{\mathbf{Q}}\ \mathbf{s}$ is parallel to $\mathbf{s}$ by the discussion preceding (5.83) and $\hat{\boldsymbol{\ell}}_{\zeta} \cdot \mathbf{s} = 0$ by (5.37) and (5.104). With the aid of (5.105,6) the first propagation condition can be rewritten as

$$(\hat{\mathbf{c}}^3 \cdot \mathbf{s})(\hat{\mathbf{c}}^3 - (\hat{\boldsymbol{\ell}}_{\zeta} \cdot \hat{\mathbf{c}}^3)\hat{\mathbf{m}}^{\zeta}) = \hat{\mu}^{-1}(\hat{\omega}^3)^2(\hat{\mathbf{Q}} - \rho\ \nu^2\ \mathbf{I})\mathbf{s} \quad , \quad \zeta = 1,2 \quad . \tag{5.107}$$

The argument used earlier for $M = 2$ in analyzing (5.88) is again applicable: we are considering solutions with $\mathbf{s} = \sigma\ \mathbf{q}_i$, $i = 1,2$ or 3 and $\hat{\mathbf{Q}}$ is in spectral form by (5.62,3), so (5.107) yields the result that either

$$\hat{\mathbf{c}}^3 \cdot \mathbf{s} = 0 \tag{5.108}$$

or

$$(\hat{\mathbf{c}}^3 - (\hat{\boldsymbol{\ell}}_{\zeta} \cdot \hat{\mathbf{c}}^3)\hat{\mathbf{m}}^{\zeta}) \wedge \mathbf{s} = 0 \quad ,$$

and since $\hat{\mathbf{m}}^{\zeta} \cdot \mathbf{s} = 0$ by (5.104), this latter condition is equivalent to

$$\hat{\mathbf{c}}^3 \wedge \mathbf{s} = 0 \quad . \tag{5.109}$$

The two possibilities (5.108,9) are considered for longitudinal and transverse waves; the analysis is closely related to that given for $M = 2$ of the conditions (5.89,90) and only a summary is given.

For <u>longitudinal</u> waves $\mathbf{s} = \sigma\,\mathbf{q}_3$, and if $\hat{\mathbf{c}}^3\cdot\mathbf{s} = 0$ then it is found from (5.103) that

$$\nu_3^2 = \rho^{-1}\,\hat{Q}_3 \quad . \tag{5.110}$$

If $\hat{\mathbf{c}}^3 \wedge \mathbf{s} = 0$ then

$$\nu_3^2 = \rho^{-1}(\hat{Q}_3 - \hat{\mu}(\hat{\omega}^3)^{-2}(\hat{\mathbf{c}}^3\cdot\mathbf{q}_3)^2) \quad . \tag{5.111}$$

For <u>transverse</u> waves, $\mathbf{s} = \sigma\,\mathbf{q}_\Delta$, $\Delta = 1$ or 2 and if $\hat{\mathbf{c}}^3\cdot\mathbf{s} = 0$ then

$$\nu_\Delta^2 = \rho^{-1}\,\hat{Q}_\Delta \quad ; \tag{5.112}$$

if on the other hand $\hat{\mathbf{c}}^3 \wedge \mathbf{s} = 0$, then (5.103) yields

$$\nu_\Delta^2 = \rho^{-1}(\hat{Q}_\Delta - \hat{\mu}(\hat{\omega}^3)^{-2}(\hat{\mathbf{c}}^3\cdot\mathbf{q}_\Delta)^2) \quad . \tag{5.113}$$

<u>Influence of isotropic and directional constraints on the homentropic propagation conditions</u>

We now consider the propagation conditions (5.54-6) when either isotropic constraints or directional constraints are present. The results (5.64-113) are employed where appropriate, but are presented here in the thermal formulation. This facilitates comparison with the corresponding homothermal results and also enables the influence or otherwise of the constraints on the solution to be more easily seen.

Longitudinal waves

In the case of isotropic constraints only, $M = 1$ and all constraints are collinear:

$$\mathbf{c}^1 = \frac{\partial \phi^1}{\partial a_3} \mathbf{q}_3$$

from (4.42) , and

$$\mathbf{c}^{1+\mu} = D_1^\mu \, \mathbf{c}^1 \quad , \quad \mu = 1, \ldots, N \quad , \text{ from (2.86) with } P=N \; .$$

The constraint vector $\hat{\mathbf{c}}^1$ in the entropic formulation is found from the transformation (5.60) with the aid of (4.42) to be

$$\hat{\mathbf{c}}^1 = \frac{\partial \phi^1}{\partial a_3} \mathbf{q}_3 - \rho \, \mu^{-1} \, \omega^1 \, \frac{\partial^2 \psi^o}{\partial a_3 \partial \theta} \mathbf{q}_3 \quad . \tag{5.114}$$

Clearly $\hat{\mathbf{c}}^1$ is parallel to $\mathbf{q}_3$ and so $\hat{\mathbf{c}}^1$ is parallel to $\mathbf{s}$ since $\mathbf{s} = \sigma \, \mathbf{q}_3$; consequently the condition (5.71) is satisfied and the longitudinal wave speed is given by (5.75), which takes the form (thermal formulation):

$$\nu_3^2 = \rho^{-1} \left[Q_3 - \rho^2 \, \mu^{-1} \left[\frac{\partial^2 \psi^o}{\partial a_3 \partial \theta} \right]^2 + \mu (\omega^1)^{-2} \left[\frac{\partial \phi^1}{\partial a_3} - \rho \, \mu^{-1} \, \omega^1 \, \frac{\partial^2 \psi^o}{\partial a_3 \partial \theta} \right]^2 \right] \quad . \tag{5.115}$$

The transformations $(5.13)_2$, (5.60) and (5.62) have been used in obtaining (5.115) and we note that ω^1 (and $\hat{\omega}^1$) is non-zero (constraints for which $\hat{\omega}^1 = \omega^1 = 0$ when $M = 1$ must obey $\hat{\mathbf{c}}^1 \cdot \mathbf{s} = 0$ by (5.82) to be compatible with non-homothermal waves, and this is not possible here by (5.114)).

The wave speed ν_3 is clearly influenced by the constraint ϕ^1 through the terms $\frac{\partial\phi^1}{\partial a_3}$ and ω^1 ; furthermore, all isotropic constraints contribute non-zero terms $\lambda_a Q_3^a$ to Q_3 . (This is easily seen by substituting the results (4.42) and (2.86) quoted above into (4.22,3) and evaluating (4.24) with the aid of (4.25)). Finally, we note that

$$\mu = \rho \frac{\partial^2\psi^0}{\partial\theta^2} + \rho \lambda_a \frac{\partial\omega^a}{\partial\theta} \qquad (5.116)$$

by (2.60), (2.93,4). We will not be evaluating $\frac{\partial\omega^a}{\partial\theta}$, but we note for future reference that constraints for which $\omega^a = 0$ do not contribute to μ .

When only <u>directional</u> constraints are present, the situations dim span $\{c^a\} = M = 0,1,2,3$ are all possible and are now dealt with in turn. We recall from the discussion following (4.53) that the corresponding <u>homothermal</u> wave speeds ν_3 ($M \leq 2$) are <u>unaffected</u> by the presence of directional constraints and are given by

$$\nu_3^2 = \rho^{-1} Q_3 \quad , \qquad (4.29 \text{ bis})$$

where Q_3 is obtained from (4.22-5) and $Q_3^a = 0$ by (4.53). In the <u>homentropic</u> situation, it will be found that the constraint influence on the longitudinal wave speed is non-zero in general for $M \geq 1$. Detailed results are given for the cases $M = 0,1$; results for $M = 2,3$ have similarities to those for $M = 1$ and are accordingly dealt with in brief.

M = 0

Now M = P = 0 and we have, by the definition (2.89) of c^a ,

$$c^\eta = 0 \Rightarrow \beta^\eta_{AB} \sum_i a_i(p_i \cdot e_A)(n \cdot e_B)q_i = 0 \quad ; \qquad (5.117)$$

these constraints are mechanical by the previous discussion in this chapter of the case M = 0.

The wave speed is given by

$$\nu_3^2 = \rho^{-1}\left[Q_3 - \rho^2\ \mu^{-1}\left[\frac{\partial^2\psi^0}{\partial a_3 \partial\theta}\right]^2\right] \qquad (5.65\ \text{bis})$$

and the constraint contributions to Q_3 are found from (4.25), (2.89) and (5.117) to be

$$\lambda_\eta\ Q_3^\eta = \lambda_\eta \beta^\eta_{AB}(n \cdot e_A)(n \cdot e_B)$$

$$= 0 \quad . \qquad (5.118)$$

Since all the constraints are mechanical, μ in (5.65) is unaffected by the constraints (recall (5.116)). Consequently the directional constraints have no effect on the wave speed. We recall from the discussion following (4.53) that directional constraints have no effect on the corresponding result for homothermal waves, M = 0,1, or 2, which is

$$\nu_3^2 = \rho^{-1}\ Q_3 \quad . \qquad (4.29\ \text{bis})$$

<u>M = 1</u>

The constraint vector $\hat{c}^1$ is required to satisfy either (5.70) or (5.71) when $\hat{\omega}^1$ (and ω^1) is non-zero and we consider these in turn:

When (5.70) holds, we have

$$\hat{c}^1 \cdot q_3 = 0 \quad ,$$

so that (5.60) yields

$$c^1 \cdot q_3 = \rho \; \mu^{-1} \; \omega^1 \; \frac{\partial^2 \psi^o}{\partial a_3 \partial \theta} \neq 0 \quad ,$$

and hence from (2.89),

$$\beta^1_{AB} \; a_3 (n \cdot e_A)(n \cdot e_B) = \rho \; \mu^{-1} \; \omega^1 \; \frac{\partial^2 \psi^o}{\partial a_3 \partial \theta} \neq 0 \quad . \tag{5.119}$$

The wave speed is given by

$$\nu_3^2 = \rho^{-1} \left[Q_3 - \rho^2 \; \mu^{-1} \left[\frac{\partial^2 \psi^o}{\partial a_3 \partial \theta} \right]^2 \right] \quad , \tag{5.65 bis}$$

as it was for $M = 0$, but now the constraint contribution from ϕ^1 to $\rho^{-1} Q_3$ is found with the aid of (2.69), (4.22-5) to be

$$\rho^{-1} \; \lambda_1 \; Q_3^1 = \lambda_1 \; \beta^1_{AB} (n \cdot e_A)(n \cdot e_B) \quad , \tag{5.120}$$

and this is non-zero by (5.119). Similar contributions to Q_3 arise from the constraints $\phi^{1+\mu}$, $\mu = 1, \ldots, P-1$, and details are omitted.

Alternatively, if (5.71) holds, $\hat{\mathbf{c}}^1$ is parallel to $\mathbf{q}_3$ and so therefore is $\mathbf{c}^1$; consequently we have, from (5.60), that

$$\beta^1_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B) \neq 0 \tag{5.121}$$

(its value, however, differs from that in (5.119)).

The wave speed ν_3 is given by (5.75), which when transformed (as in (5.115) for isotropic constraints) to the thermal formulation takes the form

$$\nu_3^2 = \rho^{-1}\left\{Q_3 - \rho^2\,\mu^{-1}\left[\frac{\partial^2\psi^0}{\partial a_3\partial\theta}\right]^2 + \mu(\omega^1)^{-2}\left[\beta^1_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B)\right.\right.$$
$$\left.\left. - \rho\,\mu^{-1}\,\omega^1\,\frac{\partial^2\psi^0}{\partial a_3\partial\theta}\right]^2\right\} \tag{5.122}$$

with the aid of (2.89). Once again the contribution $\rho^{-1}\,\lambda_1\,Q^1_3$ is non-zero from (5.120,1), as is the contribution to μ .

If we consider the case when all constraints are mechanical, then (5.82) holds and this together with the transformation (5.60) and (2.89) yields

$$\beta^1_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B) = 0 \quad . \tag{5.123}$$

The wave speed is given by (5.84), which transforms as

$$\nu_3^2 = \rho^{-1}\left[Q_3 - \rho^2\ \mu^{-1}\left[\frac{\partial^2\psi^0}{\partial a_3 \partial\theta}\right]^2\right] \quad , \tag{5.124}$$

and from (5.120) and (5.123) we see that the constraint contribution Q_3^1 to Q_3 is <u>zero</u>, as is that from the constraints $\phi^{1+\mu}$, $\mu = 1, \ldots, P-1$, since $c^{1+\mu} = D_1^\mu\ c^1$, and so from (5.123)

$$\beta_{AB}^{1+\mu}(n\cdot e_A)(n\cdot e_B) = 0 \quad . \tag{5.125}$$

Furthermore, μ in (5.124) is unconstrained since $\omega^a = 0$ in (5.116), and consequently the directional constraints have <u>no effect</u> on the wave speed ν_3 in (5.124) when all constraints are <u>mechanical</u>.

<u>M = 2</u>

The constraint vector $\hat{c}^2$ is required when $\hat{\omega}^2 \neq 0$ to satisfy either (5.89) or (5.90) and wave speeds are given by (5.73) and (5.93) respectively. The analysis in each situation is very similar to that given above for M = 1; in essence we proceed as before and obtain equivalents of (5.116-119) and (5.65) but with ϕ^2 (resp. c^2, ω^2, Q_3^2) replacing ϕ^1 (resp. c^1, ω^1, Q_3^1) throughout. Consequently, details are omitted. Contributions to Q_3 from the constraint labelled ϕ^1 when M = 2 are similar to those from ϕ^2 , and contributions from the constraints $\phi^{2+\mu}$, $\mu = 1, \ldots, P-2$ can then be deduced with the aid of the result $(5.1)_2$ that $c^{2+\mu}\cdot s = D_\sigma^\mu\ c^\sigma\cdot s$, $\sigma = 1,2$.

$\underline{M = 3}$

The previous analysis of this situation noted the similarity to the case when $M = 2$; the constraint vector $\hat{c}^3$ (when $\hat{\omega}^3 \neq 0$) must satisfy (5.108) or (5.109) and the corresponding wave speeds are given by (5.110) and (5.111) respectively; the analysis with the results for $M = 1,2$ are immediate.

One significant difference between the situation when $M = 3$ and those when $M = 0,1,2$ is that comparisons with homothermal waves are only possible for $M \leq 2$, since $M \leq 2$ is a necessary condition for the existence of homothermal waves by (4.3). We recall that situations when $M = 3$ and only mechanical constraints are present are not compatible with waves for which $T \neq 0$.

Transverse waves

Since the analysis is similar to that presented above for longitudinal waves, it is given in succinct form. Once again reference is made to the earlier analysis (5.64-110) of solutions to the propagation conditions for principal waves that are homentropic and non-homothermal.

When only isotropic constraints are present, $M = 1$ and we have from (4.42) and (2.86) with $P = N$ the results

$$c^1 = \frac{\partial \phi^1}{\partial a_3} q_3 \quad ,$$

$$c^{1+\mu} = D_1^\mu c^1 \quad , \quad \mu = 1, \ldots, N \quad .$$

Now $c^1 \cdot s = 0$ for $s = \sigma\, q_\Delta$, $\Delta = 1$ or 2 from (4.42) and with the aid of the transformation (5.60) we find that $\hat{c}^1 \cdot s = 0$. We recall the previous analysis of this situation: transverse waves with $T \neq 0$ are <u>not</u> compatible with a constraint for which $\hat{c}^1 \cdot s = 0$ (<u>irrespective</u> of the value of $\hat{\omega}^1$) from the discussion following (5.78) and the discussion for $\hat{\omega}^1 = 0$ following (5.80).

Turning now to the situation in which only <u>directional</u> constraints are present, we recall from (4.56-8) that in the corresponding homothermal situation for transverse wave propagation, directional constraints do in general influence the wave speed, unlike the situation for homothermal longitudinal waves.

<u>$M = 0$</u>

The previous discussion of principal wave solutions to the propagation conditions when $M = 0$ is immediately applicable: transverse homentropic waves with $T \neq 0$ are not compatible with this constraint configuration.

<u>$M = 1$</u>

We recall that when $\hat{\omega}^1 \neq 0$, transverse waves with $T \neq 0$ are compatible with directional constraints only when (5.71) holds, so that $\hat{c}^1$ is parallel to q_Δ . The situation $\hat{\omega}^a = 0$ is not allowed by (5.53) and the discussion following (5.80). The wave speed ν_Δ is found from (5.79), and in thermal variables takes the form

$$\nu_\Delta^2 = \rho^{-1}(Q_\Delta + \mu(\omega^1)^{-2}(c^1 \cdot q_\Delta)^2) \quad . \tag{5.126}$$

The influence of the constraint ϕ^1 on ν_Δ is manifest through $\mathbf{c}^1 \cdot \mathbf{q}_\Delta$ ($\neq 0$) and ω^1, but is also present in Q_Δ (since $\mathbf{c}^1 \cdot \mathbf{q}_3 \neq 0$ by (5.60)) and μ. The constraints $\phi^{1+\mu}$, $\mu = 1, \ldots, P-1$ (for which $\mathbf{c}^{1+\mu}$ are collinear with $\mathbf{c}^1$) influence Q_Δ ; μ in a similar fashion to ϕ^1. Furthermore, the directional constraints $\phi^{P+\eta}$, $\eta = 1, \ldots, N-P$ (for which $\mathbf{c}^{P+\eta} \equiv 0$) <u>are</u> compatible with non-homothermal transverse waves when $M = 1$, unlike the previous case when $M = 0$, since ϕ^1 unlike $\phi^{P+\eta}$ is able to provide the necessary non-zero contributions to T by (5.80). The constraints $\phi^{P+\eta}$ do not influence ν_Δ, however.

<u>$M = 2,3$</u>

The results for $M = 2,3$ have close parallels with those for $M = 1$ and are merely summarized here. In each case, $\hat{\omega}^M \neq 0$ and $\hat{\mathbf{c}}^M$ is parallel to $\mathbf{q}_\Delta$; the wave speed ν_Δ in the thermal formulation is given by

$$\nu_\Delta^2 = \rho^{-1}(Q_\Delta + \mu(\omega^M)^{-2}(\mathbf{c}^M \cdot \mathbf{q}_\Delta)^2) \tag{5.127}$$

from (5.96) and (5.113) according as $M = 2$ or 3 respectively. The contribution of the M^{th} constraint to ν_Δ parallels that of ϕ^1 when $M = 1$. The contributions of ϕ^ζ, $\zeta = 1, \ldots, M-1$ to Q_Δ are found from an analysis of (5.98) in the case when $M = 2$, and similar results are obtainable from $\hat{\mathbf{m}}^\zeta \cdot \mathbf{s} = 0$ in (5.104) when $M = 3$. Contributions from the remaining N-M constraints are similar to those discussed above in the equivalent situation when $M = 1$.

Temperature gradient ahead of homentropic waves in non-conductors

It has been noted that when $H = T = 0$, the analysis of homothermal wave propagation presented in Chapter 4 can be used in prefererence to that for homentropic waves developed in this chapter. A similar situation is found to occur in the investigation of wave growth; we treat the growth of homentropic waves in non-conductors in Chapter 8, but when $H = T = 0$, the corresponding analysis of homothermal wave growth in Chapters 6 and 7 is to be preferred. In Chapter 4 (section following (4.58)) and Chapter 6, however, it is assumed that the waves propagate in definite conductors and that the temperature is constant ahead of the wave. By contrast, the homentropic waves treated in this chapter and in Chapter 8 are propagating in non-conductors for which any type II constraints present satisfy $\mathbf{z}^{\beta}\cdot\mathbf{n} = 0$ (recall discussion following (3.50)), and in Chapter 8 it is assumed that the entropy is constant ahead of the wave. We therefore investigate the temperature gradient ahead of a generalized transverse wave ($H = T = 0$) in a non-conductor for which entropy is constant ahead of the wave.

We recall that $\eta = -\frac{\partial\psi}{\partial\theta}$ as in (2.62), and so since the entropy is assumed to be constant ahead of the wave,

$$0 = \text{Grad}\ \eta^{+} = -\ \text{Grad}\left[\frac{\partial\psi}{\partial\theta}\right]^{+} \quad , \qquad (5.128)$$

where a superscript plus sign denotes evaluation just ahead of the wave.

Equation (5.128) is evaluated in the thermal formulation and we obtain

$$\sum_j \mathbf{M}^0_{<jj>} F^+_{<jj;i>} + \mu \, \theta^+_{,<i>} + \rho \, \omega^a \, \lambda^+_{a,<i>} = 0 \tag{5.129}$$

with the aid of (2.92-5) for $\mathbf{M}^0$, μ and ω^a . The angle brackets in (5.129) denote "physical components" with respect to an orthonormal basis of vectors $\mathbf{p}_i$ or $\mathbf{q}_i$ or combinations of these. (We will make extensive use of such components in Chapters 6-8).

We now assume that the temperature is constant ahead of the wave, and so (5.129) reduces to

$$\sum_j \mathbf{M}^0_{<jj>} F^+_{<jj;i>} + \rho \, \omega^a \, \lambda^+_{a,<i>} = 0 \quad . \tag{5.130}$$

The condition (5.130) must be satisfied whenever the analysis of Chapters 4, 6, 7 is adopted for waves satisfying H = T = 0 with Grad θ^+ = Grad $\eta^+ = 0$.

Two special cases of (5.130) are worth noting. Firstly, when all type I constraints are mechanical, $\omega^a = 0$ for $a = 1, \ldots, N$ and (5.130) reduces to

$$\sum_j \mathbf{M}^0_{<jj>} F^+_{<jj;i>} = 0 \quad . \tag{5.131}$$

Secondly, if we assume that plane, cylindrical or spherical waves are propagating in irrotationally deformed materials as specified by

(4.60-70), (this assumption is also made in Chapters 6-8), then the results of Appendix B for Grad $\mathbf{F}$ yield that the only non-zero components $F^{+}_{\langle jj;i\rangle}$ are

$$F^{+}_{\langle jj;3\rangle} = a_{j,3} \quad , \tag{5.132}$$

(see also (6.34)), and so (5.130) takes the form

$$\sum_{i} \mathbf{M}^{0}_{\langle ii\rangle}\, a_{i,3} + \rho\, \omega^{\alpha}\, \lambda^{+}_{\alpha,\langle 3\rangle} = 0 \quad ,$$

$$\rho\, \omega^{\alpha}\, \lambda^{+}_{\alpha,\langle\Gamma\rangle} = 0 \quad ; \quad \Gamma = 1,\, 2 \quad . \tag{5.133}$$

CHAPTER 6

GROWTH EQUATIONS FOR HOMOTHERMAL WAVES

The propagation conditions (4.19,20) are unable to predict the change in magnitude σ of a homothermal acceleration wave as it passes through the material. In this chapter we derive the growth equation for homothermal waves in isotropic media subject to both type I constraints satisfying (2.66) or (2.68,9), and to type II constraints; this is a differential equation whose solution provides information about the behaviour of σ with time.

Reddy in (I) derived the growth equation for plane waves in thermoelastic media subject to type I and type II constraints, and assumed homogeneous deformation. We extend these results by removing the restriction of homogeneous deformation and also by considering cylindrical and spherical waves as well as plane waves; these are taken to be propagating in definite conductors subject to the deformations specified in cases (A) - (C) at the end of Chapter 4.

We further extend the results of (I) by treating the type I constraint subsets $\left\{\phi^{M+\mu}\right\}_{\mu=1}^{P-M}$ and $\left\{\phi^{P+\eta}\right\}_{\eta=1}^{N-P}$ (see (2.86,7)) as well as the linearly independent subset $\left\{\phi^{\sigma}\right\}_{\sigma=1}^{M}$ dealt with in (I). It is found that when we consider longitudinal waves and when type II constraints are absent, the type I constraints must satisfy certain restrictions as detailed in (6.29) if either or both of the subsets $\left\{\phi^{M+\mu}\right\}$, $\left\{\phi^{P+\eta}\right\}$ are present. For transverse waves a minor restriction is imposed on the subset $\left\{\phi^{M+\mu}\right\}$ in the discussion following (6.32). The restrictions imposed above are due to the fact that solutions for the jumps $[\dot{\lambda}_a]$ are not separately available when linearly dependent constraints are present; this difficulty manifests itself again in the investigation of homentropic wave growth in Chapter 8.

The growth equation is derived for both longitudinal and transverse principal waves and is shown to be a Bernoulli equation and a linear first-order equation respectively. Solutions to these equations are dealt with in Chapter 7.

Our equations have close analogues in the literature on unconstrained materials; Chen (1968c), Bowen and Wang (1971), and Chadwick and Currie (1972) consider thermodynamic influences on such materials; see also the review by Chen (1973). For constrained elastic materials, Ogden (1974) treats the growth of plane acceleration waves in incompressible media, Scott (1975) discusses propagation and growth of acceleration waves in elastic materials subject to arbitrary mechanical constraints, and provides an application (Scott (1976)) to incompressible materials. Reddy (I) however, appears to be the first to provide a general theory for the propagation and growth of acceleration waves subject to arbitrary <u>thermomechanical</u> constraints.

The derivation of the growth equation begins with the calculation of the jump in the time derivative of the equation of motion (2.9) across the wavefront. The time derivative of the body force **b** is assumed to be continuous, and we find that

$$[\mathrm{Div}\ \dot{\mathrm{S}}] = \rho[\dddot{\mathbf{x}}] \quad . \tag{6.1}$$

Now (3.26) provides an identity for $[\dot{\mathbf{x}}]$, and this introduces the displacement derivative of $\mathbf{s}$. We find after substitution of (3.26) in (6.1) and some rearrangement that

$$2\ \rho\ \frac{\delta \mathbf{s}}{\delta t} = \rho\ \nu^{-1}\ \frac{\delta \nu}{\delta t}\ \mathbf{s} - \rho\ \nu^{2}\ \mathbf{w} + [\mathrm{Div}\ \dot{\mathbf{S}}] \quad , \qquad (6.2)$$

where the vector $\mathbf{w}$ satisfies $\mathbf{w}\cdot\mathbf{a} = [\mathrm{Grad}\ \dot{\mathbf{F}}](\mathbf{a},\ \mathbf{n},\ \mathbf{n})$ as in (3.28). After taking the dot product of (6.2) with $\mathbf{s}$ and using the definition (3.34) of the magnitude σ of the wave amplitude, we find that

$$2\ \rho\ \frac{\delta \sigma}{\delta t} = \rho\ \nu^{-1}\ \sigma\ \frac{\delta \nu}{\delta t} - \rho\ \nu^{2}\ \sigma^{-1}\ \mathbf{s}\cdot\mathbf{w} + \sigma^{-1}\mathbf{s}\cdot[\mathrm{Div}\ \dot{\mathbf{S}}] \quad . \qquad (6.3)$$

The expression $\mathbf{s}\cdot[\mathrm{Div}\ \dot{\mathbf{S}}]$ is now evaluated. We have from $(2.46)_{1,2}$ and (2.47) that

$$\dot{S}_{i}{}^{j} = A_{i}{}^{j}{}_{k}{}^{\ell}\ \dot{F}^{k}{}_{\ell} + M^{0}_{i}{}^{j}\ \dot{\theta} + \dot{S}^{\alpha}_{i}{}^{j}\ \lambda_{\alpha} \quad , \qquad (6.4)$$

where the vector and tensor components in (6.4) are relative to the basis vectors defined by (2.1,2), and $\mathbf{A}$, $\mathbf{M}^{0}$ and $\mathbf{S}^{\alpha}$ are defined by (4.8), (2.92,5) and $(2.43)_1$ respectively.

The divergence of $\dot{\mathbf{S}}$ is found from (6.4) to be

$$\dot{S}_{i}{}^{r}{}_{;r} = \left[\mathcal{A}_{i}{}^{j}{}_{k}{}^{\ell}{}_{m}{}^{n}\ F^{m}{}_{n;j} + \frac{\partial}{\partial \theta}\left[A_{i}{}^{j}{}_{k}{}^{\ell}\right]\theta_{,j} + A^{\alpha}_{i}{}^{j}{}_{k}{}^{\ell}\ \lambda_{\alpha,j}\right]\dot{F}^{k}{}_{\ell}$$

$$+ A_{i}{}^{j}{}_{k}{}^{\ell}\ \dot{F}^{k}{}_{\ell;j}$$

$$+ \left[\frac{\partial}{\partial F^k{}_\ell}\left[M_i^{o\ j}\right] F^k{}_{\ell;j} + \frac{\partial}{\partial\theta}\left[M_i^{o\ j}\right]\theta_{,j}\right]\theta + M_i^{o\ j}\ \dot{\theta}_{,j}$$

$$+ A_i^{\alpha\ j}{}_k{}^\ell\ F^k{}_{\ell;j}\ \dot{\lambda}_\alpha + S_i^{\alpha\ j}\ \dot{\lambda}_{\alpha,j} \quad . \tag{6.5}$$

In (6.5), the sixth-order tensor of elastic moduli $\boldsymbol{\mathcal{A}}$ is defined by

$$\boldsymbol{\mathcal{A}} = \rho\ \frac{\partial^3\psi}{\partial\mathbf{F}\partial\mathbf{F}\partial\mathbf{F}} \quad , \tag{6.6}$$

and the constraint term $\mathbf{A}^\alpha$ is given by

$$\mathbf{A}^\alpha = \rho\ \frac{\partial^3\psi}{\partial\mathbf{F}\partial\mathbf{F}\partial\lambda_\alpha} = \rho\ \frac{\partial^2\phi^\alpha}{\partial\mathbf{F}\partial\mathbf{F}}$$

from $(2.46)_1$ and (4.8). We have also used the result that for type I constraints considered here,

$$\mathbf{M}^\alpha = \rho\ \frac{\partial^2\phi^\alpha}{\partial\mathbf{F}\partial\theta} = 0 \qquad\qquad \text{by } (2.92,5).$$

The jump $[\dot{S}_i{}^r{}_{;r}]$ is now evaluated using (6.5) in conjunction with the identity (Wang and Truesdell (1973), p.456)

$$[ab] = -\ [a][b] + a^+[b] + [a]b^+ \quad , \tag{6.7}$$

where a,b are scalar-, vector-, or tensor-valued quantities and the superscript plus sign denotes evaluation just ahead of the wave. We assume the material ahead of the wave to be <u>at rest</u> and at <u>constant temperature</u>, so that $\dot{\mathbf{F}}^+$, Grad $\dot{\mathbf{F}}^+$, $\dot{\lambda}_\alpha^+$, Grad $\dot{\lambda}_\alpha^+$, $\dot{\theta}^+$ and Grad θ^+ are all zero, and we find that

$$[\dot{S}_{i}{}^{r}{}_{;r}] = \Big[- \mathcal{A}_{i}{}^{j}{}_{k}{}^{\ell}{}_{m}{}^{n}[F^{m}{}_{n;j}] + E^{+}_{ik}{}^{\ell} - \frac{\partial}{\partial\theta} A_{i}{}^{j}{}_{k}{}^{\ell}[\theta_{,j}]$$

$$- A^{\alpha}_{i}{}^{j}{}_{k}{}^{\ell}[\lambda_{\alpha,j}] + A^{\alpha}_{i}{}^{j}{}_{k}{}^{\ell}\lambda^{+}_{\alpha,j}\Big][\dot{F}^{k}{}_{\ell}] + A_{i}{}^{j}{}_{k}{}^{\ell}[\dot{F}^{k}{}_{\ell;j}]$$

$$+ \left[- \frac{\partial}{\partial F^{k}{}_{\ell}}\left[M^{0j}_{i}\right][F^{k}{}_{\ell;j}] + \frac{\partial}{\partial F^{k}{}_{\ell}}\left[M^{0}_{i}{}^{j}\right]F^{+}{}^{k}{}_{\ell;j} - \frac{\partial}{\partial\theta}\left[M^{0}_{i}{}^{j}\right][\theta_{,j}]\right][\dot{\theta}]$$

$$+ M^{0}_{i}{}^{j}[\dot{\theta}_{,j}] + \left[- A^{\alpha}_{i}{}^{j}{}_{k}{}^{\ell}[F^{k}{}_{\ell;j}] + A^{\alpha}_{i}{}^{j}{}_{k}{}^{\ell}F^{+}{}^{k}{}_{\ell;j}\right][\dot{\lambda}_{\alpha}]$$

$$+ S^{\alpha}_{i}{}^{j}[\dot{\lambda}_{\alpha,j}] \quad . \tag{6.8}$$

In (6.8), the components of the third-order tensor $\mathbf{E}$ are defined by

$$E_{ik}{}^{\ell} = \mathcal{A}_{i}{}^{j}{}_{k}{}^{\ell}{}_{m}{}^{n}\, F^{m}{}_{n;j} \quad .$$

The jumps involving derivatives of $\mathbf{F}$ in (6.8) are evaluated using (3.24,5,7), and we also employ the result for <u>homothermal</u> waves that $[\mathrm{Grad}\ \theta] = [\dot{\theta}] = 0$ from (3.37,9). The identity $(3.22)_1$ is then used to express $[\mathrm{Grad}\ \lambda_{\alpha}]$ in terms of $[\dot{\lambda}_{\alpha}]$.

An expression for $\mathbf{s}\cdot[\mathrm{Div}\ \dot{\mathbf{S}}]$ can now be obtained from (6.8) with the aid of the above results from Chapter 3, and we find that

$$\mathbf{s}\cdot[\mathrm{Div}\ \dot{\mathbf{S}}] = \nu^{-3}\, \mathcal{A}(\mathbf{s},\mathbf{n},\mathbf{s},\mathbf{n},\mathbf{s},\mathbf{n}) - \nu^{-1}\, \mathbf{E}^{+}(\mathbf{s},\mathbf{s},\mathbf{n})$$

$$- \nu^{-1}\, \mathbf{A}^{\alpha}(\mathbf{s},\ \mathrm{Grad}\ \lambda^{+}_{\alpha},\ \mathbf{s},\mathbf{n})$$

$$+ \mathbf{Q}(\mathbf{s},\mathbf{w}) - \mathbf{A}(\mathbf{s},\mathbf{n}, (\nu^{-1}\mathbf{s})_{,\Gamma}, \mathbf{H}^{\Gamma}) - \mathbf{A}(\mathbf{s},\mathbf{H}^{\Gamma},(\nu^{-1}\mathbf{s})_{,\Gamma},\mathbf{n})$$

$$+ \nu^{-1} \Omega^{\Gamma\Delta} \mathbf{A}(\mathbf{s},\mathbf{H}_{\Gamma},\mathbf{s},\mathbf{H}_{\Delta}) + \mathbf{M}^{0}(\mathbf{s}, [\mathrm{Grad}\ \dot{\theta}])$$

$$- 2\nu^{-2} \mathbf{Q}^{\alpha}(\mathbf{s},\mathbf{s})[\dot{\lambda}_{\alpha}]$$

$$+ \mathbf{k}^{\alpha+}\cdot\mathbf{s}[\dot{\lambda}_{\alpha}] + \mathbf{S}^{\alpha}(\mathbf{s}, [\mathrm{Grad}\ \dot{\lambda}_{\alpha}]) \quad . \tag{6.9}$$

Here $\mathbf{k}^{\alpha}$ is defined to be the vector with components

$$k^{\alpha}_{i} = A^{\alpha}_{i}{}^{j}{}_{k}{}^{\ell}\, F^{k}{}_{\ell;j} \quad , \tag{6.10}$$

$\mathbf{Q}$ is as defined in (4.10,11), and $\mathbf{Q}^{\alpha}(\mathbf{s},\mathbf{s}) = \mathbf{A}^{\alpha}(\mathbf{s},\mathbf{n},\mathbf{s},\mathbf{n})$ (see (4.10,11,25)).

The expression (6.9) for $\mathbf{s}\cdot[\mathrm{Div}\ \dot{\mathbf{S}}]$ is used in (6.3), and the remaining jumps in (6.3) are evaluated in turn. A term $\sigma^{-1}(\mathbf{Q}(\mathbf{s},\mathbf{w}) - \rho\, \nu^{2}\, \mathbf{s}\cdot\mathbf{w})$ is now present on the right-hand side of (6.3), and with the aid of (4.15) we find that

$$\mathbf{Q}(\mathbf{s},\mathbf{w}) - \rho\, \nu^{2}\, \mathbf{s}\cdot\mathbf{w} = - (\mathbf{d}_{\sigma}\cdot\mathbf{Q}\ \mathbf{s})\mathbf{c}^{\sigma}\cdot\mathbf{w} \quad , \qquad \sigma = 1, \ldots, M \quad . \tag{6.11}$$

We are concerned with homothermal principal waves for which $\mathbf{s} = \sigma\, \mathbf{q}_{i}$, $i = 1,2$ or 3, in which case $\mathbf{Q}\mathbf{s}$ is parallel to $\mathbf{s}$ by (4.27). Now $\mathbf{d}_{\sigma}\cdot\mathbf{s} = 0$ by the discussion following (4.13), and consequently

$$\mathbf{d}_{\sigma}\cdot\mathbf{Q}\mathbf{s} = 0 \quad , \tag{6.12}$$

so that the expression on the left-hand side of (6.11) vanishes. (The result (6.12) will again be useful later on in the evaluation of the jumps $[\dot{\lambda}_a]$) .

The term $\mathbf{M}^0(\mathbf{s}, [\mathrm{Grad}\ \dot{\theta}])$ in (6.9) is now evaluated. Thomas's iterated kinematical condition of compatibility $(3.19)_1$ yields for homothermal waves the identity

$$[\mathrm{Grad}\ \dot{\theta}] = -\ \nu\ \Xi\ \mathbf{n} \tag{6.13}$$

where $\Xi = [\mathbf{n}\cdot(\mathrm{Grad}\ (\mathrm{Grad}\ \theta)\mathbf{n}]$. (The scalar Ξ is not related to the vector Ξ introduced in (3.2), and no confusion is likely to arise between the two).

From (6.13) then,

$$\mathbf{M}^0(\mathbf{s}, [\mathrm{Grad}\ \dot{\theta}]) = -\ \nu\ \Xi\ \mathbf{M}^0(\mathbf{s},\mathbf{n}) \quad ,$$

and since we assume the waves to be propagating with $\mathbf{n}\cdot\mathbf{p}_3$ in isotropic media, we have from (2.95) that

$$\mathbf{M}^0(\mathbf{s},\mathbf{n}) = \rho\ \frac{\partial^2\psi^0}{\partial a_3 \partial\theta}\ (\mathbf{s}\cdot\mathbf{q}_3) \quad .$$

Consequently,

$$\mathbf{M}^0(\mathbf{s}, [\mathrm{Grad}\ \dot{\theta}]) = -\ \rho\ \nu\ \Xi\ \frac{\partial^2\psi^0}{\partial a_3 \partial\theta}\ (\mathbf{s}\cdot\mathbf{q}_3) \quad . \tag{6.14}$$

For <u>transverse</u> waves $\mathbf{s} = \mathbf{q}_\Delta$, $\Delta = 1$ or 2, and so

$\mathbf{M}^0(\mathbf{s}, [\mathrm{Grad}\ \dot{\theta}]) = 0$ from (6.14). (6.15)

For longitudinal waves, we follow (I) and find an expression for Ξ and thence $[\text{Grad}\ \dot{\theta}]$, but are able to show that the restriction imposed there of homogeneous deformation is unnecessary.

We first assume that type II constraints defined by (2.37) are present. The time derivative of (2.37) is

$$\dot{\mathbf{z}}^{\beta}\cdot\text{Grad}\ \theta + \mathbf{z}^{\beta}\cdot\text{Grad}\ \dot{\theta} = 0 \tag{6.16}$$

and the jump of (6.16) is found with the aid of the identity (6.7) to be

$$-\ [\dot{\mathbf{z}}^{\beta}]\cdot[\text{Grad}\ \theta] + \dot{\mathbf{z}}^{\beta+}\cdot[\text{Grad}\ \theta] + [\text{Grad}\ \dot{\mathbf{z}}^{\beta}]\cdot\text{Grad}\ \theta^{+} + \mathbf{z}^{\beta}\cdot[\text{Grad}\ \dot{\theta}] = 0 \quad . \tag{6.17}$$

Since the material ahead of the wave is assumed to be at constant temperature, $\text{Grad}\ \theta^{+} = 0$, and we also have $[\text{Grad}\ \theta] = 0$ for homothermal waves by (3.37,9). Hence (6.17) reduces to

$$\mathbf{z}^{\beta}\cdot[\text{Grad}\ \dot{\theta}] = 0 \quad . \tag{6.18}$$

Now (6.18) can be written with the aid of (6.13) as

$$-\ \nu\ \Xi\ \mathbf{z}^{\beta}\cdot\mathbf{n} = 0 \quad . \tag{6.19}$$

If $\mathbf{z}^{\beta}\cdot\mathbf{n} = 0$ for all type II constraints present, we are unable to determine Ξ from (6.19). We therefore assume (as in (I)) that $\mathbf{z}^{\beta}\cdot\mathbf{n} \neq 0$

for at least one type II constraint, in which case (6.19) yields

$$\Xi = 0 \quad . \tag{6.20}$$

Consequently $[\mathrm{Grad}\ \dot{\theta}] = 0$ from (6.13), so that

$$\mathbf{\Gamma}^0(s,\ [\mathrm{Grad}\ \dot{\theta}]) = 0 \quad . \tag{6.21}$$

When type II constraints are absent, we consider the jump in the heat flux $\mathbf{q}$, where $\mathbf{q}$ is defined by $(2.46)_4$:

$$[\mathrm{Grad}\ \mathbf{q}] = \frac{\partial \mathbf{q}}{\partial \mathbf{F}}\,[\mathrm{Grad}\ \mathbf{F}] + \frac{\partial \mathbf{q}}{\partial(\mathrm{Grad}\ \theta)}\,[\mathrm{Grad}\ (\mathrm{Grad}\ \theta)] + \frac{\partial \mathbf{q}}{\partial \theta} \otimes [\mathrm{Grad}\ \theta] \quad . \tag{6.22}$$

For homothermal waves Grad θ is continuous at the wavefront by (3.37,9). Since Grad $\theta^+ = 0$ by the assumption of constant temperature ahead of the wave we have Grad $\theta = 0$ at the wavefront. Equation (6.22) now reduces to

$$[\mathrm{Grad}\ \mathbf{q}] = -\ \Xi\ (\mathbf{K}\mathbf{n})\Big|_{\mathrm{Grad}\ \theta\ =\ 0} \quad , \tag{6.23}$$

where the thermal conductivity tensor $\mathbf{K}$ for constrained materials is defined as in (2.21) to be

$$\mathbf{K} = \frac{-\ \partial \mathbf{q}^0}{\partial(\mathrm{Grad}\ \theta)}\Bigg|_{\mathrm{Grad}\ \theta\ =\ 0} \qquad (2.21\ \mathrm{bis}),$$

and where we have used the result (Chadwick and Currie (1972), equation (3.1)) that

$$\left.\frac{\partial \mathbf{q}}{\partial \mathbf{F}}\right|_{\text{Grad } \theta = 0} = 0 \quad .$$

From (6.23), the identity Div $\mathbf{q}$ = tr (Grad $\mathbf{q}$) implies

$$[\text{Div } \mathbf{q}] = - \Xi \kappa \quad , \tag{6.24}$$

where $\kappa = \left.\mathbf{K}(\mathbf{n},\mathbf{n})\right|_{\text{Grad } \theta = 0}$,

and so the jump of the energy equation (2.48) can be written as

$$\rho \ \theta \ \nu \ \mathrm{H} = - \Xi \kappa \quad , \tag{6.25}$$

where the entropy jump H is defined by

$$\mathrm{H} = [\text{Grad } \eta \cdot \mathbf{n}] \quad . \tag{3.42 bis}$$

An expression for H is obtained from (5.48), and for homothermal waves we have

$$\rho \ \nu \ \mathrm{H} = - \rho \ \nu^{-1} \frac{\partial^2 \psi^o}{\partial a_3 \partial \theta} (\mathbf{s} \cdot \mathbf{q}_3) + \rho \ \omega^a [\dot{\lambda}_a] \quad , \tag{6.26}$$

where we have used the fact that $\mathbf{s} \cdot \mathbf{M}^o \mathbf{n} = \rho \dfrac{\partial^2 \psi^o}{\partial a_3 \partial \theta} (\mathbf{s} \cdot \mathbf{q}_3)$ as in the derivation of (6.14).

We now evaluate $\rho\omega^{\alpha}[\dot{\lambda}_{\alpha}]$ in (6.26). The only solutions available for $[\dot{\lambda}_{\alpha}]$ are those obtained from (4.14) with the aid of (6.12); we recall that

$$\mathbf{d}_{\sigma}\cdot\mathbf{Q}\ \mathbf{s} = \rho\ \nu\left[[\dot{\lambda}_{\sigma}] + D^{\mu}_{\sigma}\ [\dot{\lambda}_{M+\mu}]\right] \quad , \quad \begin{array}{l}\sigma = 1, \ldots, M \quad . \\ \mu = 1, \ldots, P\text{-}M \quad ,\end{array} \tag{4.14 bis}$$

and for homothermal principal waves with $\mathbf{s} = \sigma\ \mathbf{q}_i$, i = 1,2 or 3,

$$\mathbf{d}_{\sigma}\cdot\mathbf{Q}\mathbf{s} = 0 \quad , \tag{6.12 bis}$$

so we have

$$[\dot{\lambda}_{\sigma}] + D^{\mu}_{\sigma}\ [\dot{\lambda}_{M+\mu}] = 0 \quad . \tag{6.27}$$

Clearly, when only the subset $\{\phi^{\sigma}\}$ is present (we recall from (2.86) that these constraints have constraint vectors $\mathbf{c}^{\sigma}$ which form a linearly independent set), then

$$[\dot{\lambda}_{\sigma}] = 0 \quad , \qquad \sigma = 1, \ldots, M \quad . \tag{6.28}$$

When the subset $\left\{\sigma^{M+\mu}\right\}$ is present, (6.27) is unable to provide solutions for the individual jumps $[\dot{\lambda}_{\sigma}]$, $[\dot{\lambda}_{M+\mu}]$. Furthermore, there is no information regarding the jumps $[\dot{\lambda}_{P+\eta}]$, η = 1, ..., N-P , corresponding to the constraint subset $\left\{\phi^{P+\eta}\right\}$ corresponding to which $\mathbf{c}^{P+\eta} \equiv \mathbf{0}$.

Consequently the term $\rho\omega^{\alpha}[\dot{\lambda}_{\alpha}]$ can only be evaluated in the following situations:

(i) when the subset $\left\{\phi^{P+\eta}\right\}$ is present, if these constraints are mechanical, so that $\omega^{P+\eta} \equiv 0$;

(ii) when the subset $\left\{\phi^{M+\mu}\right\}$ is present, if all constraints are mechanical, so that $\omega^{\alpha} \equiv 0$, $\alpha = 1, \ldots, N$.

(6.29)

When (6.29) holds, $\rho\omega^{\alpha}[\dot{\lambda}_{\alpha}] = 0$ and from (6.25) with the aid of (6.26) we have for $\mathbf{s} = \sigma\, \mathbf{q}_3$ that

$$\Xi = \rho\, \nu^{-1}\, \kappa^{-1}\, \theta\, \sigma \frac{\partial^2 \psi^0}{\partial a_3 \partial \theta} \quad . \tag{6.30}$$

Finally we obtain from (6.14), for the case in which type II constraints are absent,

$$\mathbf{I}^0(\mathbf{s},\ [\mathrm{Grad}\ \dot{\theta}]) = -\ \kappa^{-1}\ \theta\ \sigma^2 \left[\rho\ \frac{\partial^2 \psi^0}{\partial a_3 \partial \theta}\right]^2 \quad . \tag{6.31}$$

We collect together the results (6.15), (6.21) and (6.31) and find that for longitudinal waves ($\mathbf{s} = \sigma\, \mathbf{q}_3$) ,

$\mathbf{I}^0(\mathbf{s},\ [\mathrm{Grad}\ \dot{\theta}]) = 0$ when type II constraints are present (recall that we assume that at least one type II constraint obeys $\mathbf{z}^{\beta}\cdot\mathbf{n} \neq 0$) ,

$= \kappa^{-1}\ \theta\ \sigma^2 \left[\rho\ \frac{\partial^2 \psi^0}{\partial a_3 \partial \theta}\right]^2$ when type II constraints are absent;

$$c^{M+\mu} = \beta^{a}_{AB}\, a_{\underline{\Delta}}(p_{\underline{\Delta}}\cdot e_A)(n\cdot e_B)q_{\underline{\Delta}} \qquad (4.49\text{ bis})$$

where $\underline{\Delta} = 1$ or 2 according as $\Delta = 2$ or 1,

and $\beta^{M+\mu}_{AB}$ is required to be neither positive- nor negative-definite.

(It is noteworthy that the term $Q^{M+\mu}(s,s)$ in (4.55) is not necessarily non-zero for isotropic constraints; we recall from the discussion following (4.55) that it is zero for temperature-dependent compressibility, for instance).

We therefore assume that for transverse waves, constraints for which $Q^{M+\mu}(s,s) \neq 0$ are <u>absent</u>.

Turning now to the term $k^{a+}\cdot s$, we note that for waves with $s = q_i$, $i = 1,2$ or 3, the definition (6.10) for k^{a} leads to the following expression for $k^{a+}\cdot s$:

$$k^{a+}\cdot s = \sigma\, A^{a}_{<ij\ k\ell>}\, F^{+}_{<k\ell;j>} \qquad (6.33)$$

(We recall from (5.129) that the subscripts enclosed in angle brackets denote "physical components" relative to an orthonormal basis of proper vectors p_i or q_i , or combinations of these. We will make extensive use of this component representation in future).

We now investigate (6.33) for isotropic materials subject to the class of <u>irrotational deformations</u> introduced in Chapter 4, and show that for longitudinal and transverse principal waves propagating under these conditions, the term $k^{a+}\cdot s$ is always zero. It will be recalled that we considered plane waves (resp. cylindrical, spherical) in materials subject to a plane deformation (resp. cylindrically symmetric, spherically symmetric), with the deformation and principal stretches specified for plane deformations by (4.61,2) (resp. (4.65,6) for cylindrical deformations, (4.69,70) for spherical deformations).

Under these situations, the only non-zero components $F^{+}_{<k\ell;j>}$ are found with the aid of the results in Appendix B to be

$$\begin{aligned}
&\text{plane deformation} &&: F^{+}_{<33;3>} \quad ; \\
&\text{cylindrical deformation} &&: F^{+}_{<33;3>} \,,\; F^{+}_{<11;3>} \,,\; F^{+}_{<31;1>} \,,\; F^{+}_{<13;1>} \;; \\
&\text{spherical deformation} &&: F^{+}_{<33;3>} \,,\; F^{+}_{<\Gamma\Gamma;3>} \,,\; F^{+}_{<3\Gamma;\Gamma>} \,,\; F^{+}_{<\Gamma 3;\Gamma>} \,, \\
& && \qquad\qquad \Gamma = 1,2 \quad .
\end{aligned} \tag{6.34}$$

Now for <u>longitudinal</u> waves we may have only directional constraints (recall discussion preceding (4.43)) and, with $s = q_3$, we have from (4.21) for A , (2.68,9) for c^{a} and (6.33,4)

$$k^{a+}\cdot q_3 = \beta^{a}_{AB}(p_j\cdot e_A)(p_\ell\cdot e_B)F^{+}_{<3\ell;j>}$$

$$= \begin{cases} \beta^{a}_{AB}(n\cdot e_A)(n\cdot e_B)F^{+}_{<33;3>} \\ \beta^{a}_{AB}\left[(n\cdot e_A)(n\cdot e_B)F^{+}_{<33;3>} + (p_1\cdot e_A)(p_1\cdot e_B)F^{+}_{<31;1>}\right] \\ \beta^{a}_{AB}\left[(n\cdot e_A)(n\cdot e_B)F^{+}_{<33;3>} + \sum\limits_{\Gamma}(p_\Gamma\cdot e_A)(p_\Gamma\cdot e_B)F^{+}_{<3\Gamma;\Gamma>}\right] \end{cases} . \tag{6.35}$$

for plane, cylindrical and spherical waves, respectively. But we recall from (4.43) that a necessary condition for the existence of longitudinal waves is that

$$\beta^{\alpha}_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B) = 0 \quad , \tag{4.43 bis}$$

and so the terms in (6.35) involving $F^{+}_{<33;3>}$ are zero. For cylindrical waves we see from (4.75-77) that either $\beta^{\alpha}_{AB}(\mathbf{p}_1\cdot\mathbf{e}_A)(\mathbf{p}_1\cdot\mathbf{e}_B) = 0$ or, if it is non-zero then the deformation must necessarily be homogeneous. For this latter case $F^{+}_{<31;1>} = R^{-1}(a_3 - a_1) = 0$, so that either way $(6.35)_2$ is zero. In the same way we can establish that $(6.35)_3$ is zero.

For transverse waves we have, for $\mathbf{s} = \sigma\,\mathbf{q}_\Delta$, $\Delta = 1$ or 2 , that

$$\begin{aligned}\mathbf{k}^{\alpha+}\cdot\mathbf{q}_\Delta &= A^{\alpha}_{<\Delta j\ k\ell>}\ F^{+}_{<k\ell;j>} \\ &= A^{\alpha}_{<\Delta 3\ 33>}\ F^{+}_{<33;3>} + \sum_{\Gamma}\Big[A^{\alpha}_{<\Delta 3\ \Gamma\Gamma>}\ F^{+}_{<\Gamma\Gamma;3>} \\ &\qquad + A^{\alpha}_{<\Delta\Gamma\ 3\Gamma>}\ F^{+}_{<3\Gamma;\Gamma>} + A^{\alpha}_{<\Delta\Gamma\ \Gamma 3>}\ F^{+}_{<\Gamma 3;\Gamma>}\Big]\end{aligned} \tag{6.36}$$

in which only the first term is non-zero for plane waves, while for cylindrical waves the remaining terms are zero for $\Gamma = 2$. By inspection the first term is zero for an isotropic constraint since the only non-zero components of $\mathbf{A}^{\alpha}$ are then $A^{\alpha}_{<ii\ jj>}$, $A^{\alpha}_{<ij\ ij>}$ and $A^{\alpha}_{<ij\ ji>}$ from (4.21). Hence we now look at directional constraints, and we recall from (4.56) that

$$\beta^{\alpha}_{AB}(\mathbf{p}_\Delta\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B) = 0 \tag{4.56 bis}$$

must be satisfied. Also, from (4.21) we have

$$A^{a}_{\langle\Delta j\ k\ell\rangle} = \rho\ \beta^{a}_{AB}(\mathbf{p}_j\cdot\mathbf{e}_A)(\mathbf{p}_\ell\cdot\mathbf{e}_B)\delta_{\Delta k} \quad ,$$

so that $A^{a}_{\langle\Delta 3\ 33\rangle} = A^{a}_{\langle\Delta\Gamma\ 3\Gamma\rangle} = 0$. For cylindrical waves we are left with the terms

$$A^{a}_{\langle\Delta 3\ 11\rangle}\ F^{+}_{\langle 11;3\rangle} + A^{a}_{\langle\Delta 1\ 13\rangle}\ F^{+}_{\langle 13;1\rangle}$$

$$= \beta^{a}_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{p}_1\cdot\mathbf{e}_B)\left[\frac{da_1}{dR} + \frac{a_3 - a_1}{R}\right]\delta_{\Delta 1}$$

which is zero from (4.56), so that $\mathbf{k}^{a+}\cdot\mathbf{s} = 0$. In the same way we can argue that $\mathbf{k}^{a+}\cdot\mathbf{s} = 0$ for spherical waves. In all circumstances then, we have the result

$$\mathbf{k}^{a+}\cdot\mathbf{s} = 0 \quad . \tag{6.37}$$

We turn finally to the term $\mathbf{S}^{a}(\mathbf{s}, [\mathrm{Grad}\ \dot{\lambda}_a])$ that occurs on the right-hand side of (6.3) from (6.9). The compatibility condition $(3.18)_1$ enables us to write

$$[\mathrm{Grad}\ \dot{\lambda}_a] = \Lambda_a\ \mathbf{n} + [\dot{\lambda}_a]_{,\Gamma}\ \mathbf{H}^{\Gamma} \quad , \tag{6.38}$$

where $\Lambda_a = [(\mathrm{Grad}\ \dot{\lambda}_a)\cdot\mathbf{n}]$.

Since $S^{\alpha} = \rho \frac{\partial\phi^{\alpha}}{\partial \mathbf{F}}$ from $(2.43)_1$, we find with the aid of (6.38) that

$$S^{\alpha}(\mathbf{s},\ [\mathrm{Grad}\ \dot{\lambda}_{\alpha}]) = \rho \frac{\partial\phi^{\alpha}}{\partial \mathbf{F}} \left[\mathbf{s},\ \Lambda_{\alpha}\ \mathbf{n} + [\dot{\lambda}_{\alpha}],_{\Gamma}\ \mathbf{H}^{\Gamma}\right]$$

$$= \Lambda_{\alpha}\ \mathbf{c}^{\alpha}\cdot\mathbf{s} + S^{\alpha}(\mathbf{s},\mathbf{H}^{\Gamma})[\dot{\lambda}_{\alpha}],_{\Gamma} \quad , \tag{6.39}$$

where we have used the definition (2.85) for $\mathbf{c}^{\alpha}$.

Now for homothermal waves we recall from (4.20) that the second propagation condition takes the form $\mathbf{c}^{\alpha}\cdot\mathbf{s} = 0$, and so we obtain from (6.39)

$$S^{\alpha}(\mathbf{s},\ [\mathrm{Grad}\ \dot{\lambda}_{\alpha}]) = S^{\alpha}(\mathbf{s},\mathbf{H}^{\Gamma})[\dot{\lambda}_{\alpha}],_{\Gamma} \quad . \tag{6.40}$$

The term $S^{\alpha}[\mathbf{s},\mathbf{H}^{\Gamma}]$ can be shown to vanish in certain circumstances, but is in general non-zero, and so the terms $[\dot{\lambda}_{\alpha}],_{\Gamma}$ need to be evaluated. The jumps $[\dot{\lambda}_{\alpha}]$ in (6.40) can be determined in the situation when only the subset $\{\phi^{\sigma}\}$ is present, and then $[\dot{\lambda}_{\sigma}] = 0$, $\sigma = 1, \ldots, M$ by (6.28). If either or both of the subsets $\left\{\phi^{M+\mu}\right\}$ and $\left\{\phi^{P+\eta}\right\}$ are also present, the jumps $[\dot{\lambda}_{\alpha}]$ can no longer be determined, as will be recalled from the discussion following (6.28).

We accordingly adopt another approach and recall that the material is subject to the class of irrotational deformations described in Chapter 4, and that for these deformations there is dependence on the X^3 coordinate only, so that $x = x(X)$ (resp. $r = r(R)$) for plane (resp. cylindrically and spherically symmetric) deformations by (4.61) (resp. (4.65,9). We also recall that $\mathbf{e}_A\cdot\mathbf{p}_i$ is independent of position by

(4.59), and that we have earlier assumed the temperature ahead of the wave to be constant. We further assume that the body force **b** is a function of X^3 only, and that in the unbounded medium all other data (such as prescribed surface tractions on boundary surfaces, if such surfaces exist) depend at most on X^3 . It is not difficult to show that the equations of equilibrium under these circumstances admit solutions λ_α which are functions of X^3 only. Rather than furnish full details, for an example of the outcome of such a calculation we refer the reader to work by Beskos (1973) on universal solutions for fibre-reinforced incompressible isotropic elastic materials. Beskos obtained solutions for stresses and constraint multipliers λ_α as functions of X^3 for a variety of situations which include the class of deformations considered here. With the above assumption for λ_α we find that $[\dot{\lambda}_\alpha]$ is independent of X^Γ and consequently (6.40) reduces to

$$\mathbf{S}^\alpha(\mathbf{s},\ [\mathrm{Grad}\ \dot{\lambda}_\alpha]) = 0 \quad . \tag{6.41}$$

This completes the evaluation of the remaining terms involving jumps in (6.9). We have finally from (6.3) and (6.9) together with the results (6.11,12), (6.32), (6.37) and (6.41) the equation for the amplitude:

$$\begin{aligned} 2\rho\ \mathbf{s}\cdot\frac{\delta\mathbf{s}}{\delta t} = {} & \rho\ \nu^{-1}\ \frac{\delta\nu}{\delta t}\ \mathbf{s}\cdot\mathbf{s} + \nu^{-3}\ \mathcal{A}(\mathbf{s},\mathbf{n},\mathbf{s},\mathbf{n},\mathbf{s},\mathbf{n}) - \nu^{-1}\ \mathbf{E}^{+}(\mathbf{s},\mathbf{s},\mathbf{n}) \\ & - \mathbf{A}(\mathbf{s},\mathbf{n},\ (\nu^{-1}\mathbf{s})_{,\Gamma},\mathbf{H}^\Gamma) - \mathbf{A}(\mathbf{s},\mathbf{H}^\Gamma,\ (\nu^{-1}\mathbf{s})_{,\Gamma},\mathbf{n}) \\ & + \nu^{-1}\ \mathbf{A}(\mathbf{s},\mathbf{H}_\Gamma,\mathbf{s},\mathbf{H}_\Delta)\Omega^{\Gamma\Delta} - \nu\ \lambda^{+}_{\sigma,3}\ \mathbf{Q}^\sigma(\mathbf{s},\mathbf{s}) \\ & - \underline{\nu\ \Xi\ \mathbf{M}^0(\mathbf{s},\mathbf{n})} \quad , \qquad \sigma = 1,\ \ldots,\ M \quad . \end{aligned} \tag{6.42}$$

The underlined term in (6.42) is zero unless $s = \sigma\, q_3$ and type II constraints are absent by (6.32). We have simplified the term $- \nu^{-1}$ $A^{a}(s, \text{Grad}\ \lambda_{a}^{+}, s, n)$ in (6.9) and (6.3) by using the assumption that $\lambda_{a}^{+} = \lambda_{a}^{+}(X^3)$ and the result $A^{M+\xi}(s,n,s,n) = Q^{M+\xi}(s,s) = 0$, $\xi = 1, \ldots,$ $N\text{-}M$, as in the discussion of $Q^{a}(s,s)$ following (6.32).

<u>Growth equation for longitudinal waves</u>

Our aim now is to investigate further equation (6.42) for longitudinal waves. Since longitudinal waves are incompatible with isotropic constraints by the discussion preceding (4.43), it is only necessary to consider <u>directional</u> constraints. The growth equation becomes, with $s = \sigma\, q_3$ and $n = p_3$,

$$
\begin{aligned}
2\,\rho\,\sigma \frac{\delta\sigma}{\delta t} &= \rho\,\nu^{-1}\,\sigma^2 \frac{\delta\nu}{\delta t} + \nu^{-3}\,\sigma^3\,\mathcal{A}^0(q_3,p_3,q_3,p_3,q_3,p_3) \\
&\quad - \nu^{-1}\,\sigma^2\,E^{+}(q_3,q_3,p_3) \\
&\quad - \sigma\,A(q_3,p_3,(\nu^{-1}\sigma\, q_3)_{,\Gamma}, H^{\Gamma}) - \sigma\,A(q_3, H^{\Gamma},(\nu^{-1}\sigma\, q_3)_{,\Gamma}, p_3) \\
&\quad + \nu^{-1}\,\sigma^2\,\Omega^{\Gamma\Delta}\,A(q_3, H_{\Delta}, q_3, H_{\Gamma}) - \underline{\kappa^{-1}\,\theta\,\sigma^2\,M^{0}(q_3,p_3)}
\end{aligned}
\qquad (6.43)
$$

and we note that the term involving $Q^{\sigma}(s,s)$ in (6.42) is zero for longitudinal waves by (4.53), and that the superscript zero on $\mathcal{A}$ indicates that the term contains no quantities associated with constraints (see comment at the end of Appendix A).

We now specialize (6.43) for the set of irrotational deformations described by (4.61,5,9). Making use of the results in Appendices A and

B, the term $E^+(q_3,q_3,p_3)$ is

$$E^+(q_3,q_3,p_3) = \mathcal{A}^0_{\langle 3j\ 33\ mn\rangle} F^+_{\langle mn;j\rangle}$$

$$= \sum_m \mathcal{A}^0_{\langle 33\ 33\ mm\rangle} a_{m,3} + \sum_\Gamma (\mathcal{A}^0_{\langle 3\Gamma\ 33\ 3\Gamma\rangle} + \mathcal{A}^0_{\langle 3\Gamma\ 33\ \Gamma 3\rangle} R_\Gamma^{-1} (a_3 - a_\Gamma), \tag{6.44}$$

the remaining terms being zero.

We consider next the terms involving the fourth-order tensor $\mathbf{A}$ in (6.43). For the class of deformations (4.61,5,9) and corresponding wavefronts, ν and σ depend only on X^3 and so do not vary along the wavefront. Hence

$$(\nu^{-1}\ \sigma\ q_3)_{,\Gamma} = \nu^{-1}\sigma \frac{\partial q_3}{\partial x^i}\frac{\partial x^i}{\partial X^\Gamma} = \nu^{-1}\sigma\ \gamma^\ell_{3\Gamma}(1 - \delta_{\ell 3})\ a_\Gamma |g_\Gamma|^{-1}|G_\Gamma| g_\ell$$

$$= -\ \nu^{-1}\sigma\ r_\Gamma^{-1}\ a_\Gamma |G_\Gamma| q_\Gamma = -\ \nu^{-1}\sigma\ R_\Gamma^{-1}\ |G_\Gamma| q_\Gamma \quad , \tag{6.45}$$

where we have used expressions for $\dfrac{\partial q_3}{\partial x^i}\dfrac{\partial x^i}{\partial X^\Gamma}$ from Appendix B, and the fact that $a_\Gamma = r_\Gamma R_\Gamma^{-1}$ whenever $R_\Gamma^{-1} \neq 0$.

The terms on the right-hand side of (6.43) involving $\mathbf{A}$ now reduce to

$$-\ \nu^{-1}\ \sigma^2 \sum_\Gamma R_\Gamma^{-1} \left[A^0_{\langle 33\ \Gamma\Gamma\rangle} + A^0_{\langle 3\Gamma\ \Gamma 3\rangle} - A^0_{\langle 3\Gamma\ 3\Gamma\rangle} - \lambda_a\ A^a_{\langle 3\Gamma\ 3\Gamma\rangle}\right] \quad .$$

Finally, the growth equation for longitudinal waves is found from (6.43) with the aid of the above results to be

$$2\rho\,\frac{\delta\sigma}{\delta t} = \nu^{-3}\,\sigma^2\,\mathcal{A}^0_{<33\ 33\ 33>} + \nu^{-1}\sigma\left\{\rho\,\frac{\delta\nu}{\delta t} - \sum_m \mathcal{A}^0_{<33\ 33\ mm>}\,a_{m,3}\right.$$

$$- \sum_\Gamma R_\Gamma^{-1}\left\{\left[\mathcal{A}^0_{<3\Gamma\ 33\ 3\Gamma>} + \mathcal{A}^0_{<3\Gamma\ 33\ \Gamma 3>}\right](a_3 - a_\Gamma)\right.$$

$$\left. + A^0_{<33\ \Gamma\Gamma>} + A^0_{<3\Gamma\ \Gamma 3>} - A^0_{<3\Gamma\ 3\Gamma>} - \lambda_a\,A^a_{<3\Gamma\ 3\Gamma>}\right\}$$

$$\left. + \underline{\kappa^{-1}\,\theta\left[\rho\,\frac{\partial^2\psi^0}{\partial a_3\partial\theta}\right]^2}\right\} \quad , \tag{6.46}$$

where the underlined term is zero unless type II constraints are absent (recall (6.32)). Note that the influence of type I constraints is manifested solely in the term involving λ_a ; we recall that the wave speed ν is unaffected by the constraints from (4.53).

Growth equation for transverse waves

We begin again with (6.42) and set $s = \sigma\, q_\Delta$ with $\Delta = 1$ or 2. The growth equation is then

$$2\rho\,\frac{\delta\sigma}{\delta t} = \rho\,\nu^{-1}\,\sigma\,\frac{\delta\nu}{\delta t} + \nu^{-3}\,\sigma^2\,\mathcal{A}(q_\Delta,n,q_\Delta,n,q_\Delta,n)$$

$$+ \nu^{-1}\Big\{-\sigma\,E^+(q_\Delta,q_\Delta,n) - A(q_\Delta,n,s_{,\Gamma},H^\Gamma) - A(q_\Delta,H^\Gamma,s_{,\Gamma},n)$$

$$+ \sigma\sum_\Gamma A(q_\Delta,H_\Gamma,q_\Delta,H_\Gamma)\Omega^{\Gamma\Gamma}\Big\} \quad . \tag{6.47}$$

We now evaluate the terms in (6.45). First,

$$\nu^{-3}\,\sigma^2\,\mathcal{A}(\mathbf{q}_\Delta,\mathbf{n},\mathbf{q}_\Delta,\mathbf{n},\mathbf{q}_\Delta,\mathbf{n}) = \nu^{-3}\,\sigma^2\,\mathcal{A}_{<\Delta 3\ \Delta 3\ \Delta 3>} = 0 \quad , \tag{6.48}$$

using the results of Appendix A. Next,

$$\begin{aligned} \mathbf{E}^{+}(\mathbf{q}_\Delta,\mathbf{q}_\Delta,\mathbf{n}) &= \mathcal{A}_{<\Delta j\ \Delta 3\ mn>}\,F^{+}_{<mn;j>} \\ &= \sum_i \mathcal{A}_{<\Delta 3\ \Delta 3\ ii>}\,a_{i,3} + \sum_\Gamma \left[\mathcal{A}_{<\Delta\Gamma\ \Delta 3\ \Gamma 3>} + \mathcal{A}_{<\Delta\Gamma\ \Delta 3\ 3\Gamma>}\right] R_\Gamma^{-1}\,(a_3 - a_\Gamma) \quad ; \end{aligned} \tag{6.49}$$

the remaining terms are found from Appendices A and B to be zero for the irrotational deformations described by (4.61,5,9).

We turn now to the terms involving $\mathbf{A}$. The expression $s_{,\Gamma} = \sigma\,q_{\Delta,\Gamma}$ is simplified using

$$\mathbf{q}_\Delta = \mathbf{p}_\Delta = \mathbf{H}_\Delta/(\mathbf{H}_\Delta\cdot\mathbf{H}_\Delta)^{1/2} \quad \text{and} \quad \frac{\partial\mathbf{H}_\Delta}{\partial Y^\Gamma} = \Gamma^\Lambda_{\Delta\Gamma}\,\mathbf{H}_\Lambda + \Omega_{\Gamma\Delta}\mathbf{n}$$

where $\Gamma^\Lambda_{\Delta\Gamma}$ is the Christoffel symbol, and we get, after some manipulation,

$$\begin{aligned} &-\nu^{-1}\,\sigma\left\{\mathbf{A}(\mathbf{q}_\Delta,\mathbf{n},\mathbf{q}_{\Delta,\Gamma},\mathbf{H}^\Gamma) + \mathbf{A}(\mathbf{q}_\Delta,\mathbf{H}^\Gamma,\mathbf{q}_{\Delta,\Gamma},\mathbf{n}) - \sum_\Gamma \mathbf{A}(\mathbf{q}_\Delta,\mathbf{H}_\Gamma,\mathbf{q}_\Delta,\mathbf{H}_\Gamma)\Omega^{\Gamma\Gamma}\right\} \\ &\quad = -\,\nu^{-1}\,\sigma\sum_\Gamma\Big\{2(-\,H_{\Gamma\Gamma}^{-1/2}\,\Gamma^\Delta_{\Delta\Gamma} + H_{\Delta\Delta}^{-1}\,\Gamma_{\Delta\Gamma\Delta})\mathbf{A}_{<\Delta 3\ \Delta\Gamma>} \end{aligned}$$

$$- R_\Delta^{-1}(A_{<\Delta 3\ 3\Delta>} + A_{<\Delta\Delta\ 33>}) + R_\Gamma^{-1} A_{<\Delta\Gamma\ \Delta\Gamma>}\Big\}$$

$$= - \nu^{-1} \sigma\Big\{- R_\Delta^{-1}(A_{<\Delta 3\ 3\Delta>} + A_{<\Delta\Delta\ 33>}) + \sum_\Gamma R_\Gamma^{-1} A_{<\Delta\Gamma\ \Delta\Gamma>}\Big\} \quad , \tag{6.50}$$

the terms involving the Christoffel symbols being zero for the types of wavefronts considered here.

With the aid of (6.49,50) in (6.47) the growth equation for transverse waves reduces to

$$2\rho \frac{\delta\sigma}{\delta t} = \nu^{-1} \sigma\Big\{\rho \frac{\delta\nu}{\delta t} - \sum_i \mathcal{A}(_{\Delta 3\ \Delta 3\ ii>} a_{i,3} - \lambda^+_{\tau},_{<3>} Q^\tau_{<\Delta\Delta>}$$

$$- \sum_\Gamma R_\Gamma^{-1}\Big[(\mathcal{A}_{<\Delta\Gamma\ \Delta 3\ \Gamma 3>} + \mathcal{A}_{<\Delta\Gamma\ \Delta 3\ 3\Gamma>} (a_3 - a_\Gamma) + A_{<\Delta\Gamma\ \Delta\Gamma>}\Big]$$

$$+ R_\Delta^{-1}(A_{<\Delta 3\ 3\Delta>} + A_{<\Delta\Delta\ 33>})\Big\} \quad , \qquad \tau = 1, \ldots, M \ . \tag{6.51}$$

CHAPTER 7

SOLUTIONS OF THE GROWTH EQUATIONS FOR HOMOTHERMAL WAVES

We now present solutions to the growth equations (6.46) and (6.51), which are shown to be a Bernoulli equation and a linear first-order equation respectively. We discuss the general behaviour of the solutions, following Bailey and Chen (1971) and the review by Chen (1973), then present results for plane, cylindrical and spherical waves. Throughout the chapter we recover as special cases the well-known results for unconstrained materials, in particular those given by Chen (1968 a,b,c), Chadwick and Currie (1972) and by Bowen and Wang (1970,1971). (We differ slightly from Bowen and Wang (1972) however, in that our expression for transverse waves in non-homogeneously deformed media involves the wavefront curvature components $\Omega_{\langle\Gamma\Delta\rangle}$ rather than the mean curvature $\frac{1}{2}\,\Omega^{\Gamma}_{\Gamma})$.

In the case of constrained materials our general result for transverse waves includes the result for plane waves in an incompressible isotropic solid given by Ogden (1974) and confirmed by Scott (1976), who also discusses the cylindrical case. Reddy in (I) provides results for plane waves in homogeneously deformed thermoelastic media subject to type I and type II constraints, but does not treat the case in which the constraint vectors $\mathbf{c}^{a}$ are linearly dependent. Our results for plane waves are special cases of his in the sense that we assume material isotropy and the restrictions (2.66,8,9) on type I constraints, but are also slightly more general in that we allow in the case of longitudinal (resp. transverse) waves, the presence of mechanical (resp. thermomechanical) constraints with linearly dependent vectors c^{α} , and do not assume homogeneous deformation a priori. (Of course, the results of Chapter 4 indicate that in certain circumstances the constraints do not admit non-homogeneous deformations.) Our approach in this chapter is not to concentrate on results for specific constraints (although this can easily be done using, for example, the constraints described in Chapter 4), but rather to show the general influence (or otherwise) of the constraints on the solution.

We begin by recording the identity

$$\mathcal{A}_{\langle i3\ i3\ kk\rangle} = \frac{\partial}{\partial a_k} (\rho \nu_i^2) \tag{7.1}$$

which follows from (4.29), (4.21-4), and the results of Appendix A. Since the material is assumed to be <u>at rest ahead of the wave</u>, the displacement derivative defined by (3.16) takes the respective forms

$$\frac{\delta \chi}{\delta t} = \nu_i \frac{d\chi}{dn}$$

and

$$\frac{\delta \mathbf{V}}{\delta t} = \nu_i \frac{d\mathbf{V}}{dn} \tag{7.2}$$

for arbitrary scalar χ and vector or tensor $\mathbf{V}$, where n measures displacement in the normal direction.

Longitudinal waves

With the aid of (7.1) and (7.2) the growth equation (6.46) reduces to

$$\frac{d\sigma}{dn} = a\sigma^2 - (\beta + \gamma_\Gamma R_\Gamma^{-1})\sigma \tag{7.3}$$

where

$$a(n) = \nu_3^{-3} \frac{\partial \nu_3}{\partial a_3} , \tag{7.4}$$

$$\beta(n) = \frac{1}{2} \nu_3^{-1}\left[\frac{\partial \nu_3}{\partial n} + \underline{\kappa^{-1} \theta\left[\rho \frac{\partial^2 \psi^0}{\partial a_3 \partial \theta}\right]^2}\right] \tag{7.5}$$

$$\gamma_\Gamma(n) = \begin{cases} 0 \qquad \text{for plane waves,} \\ \frac{1}{2} \rho^{-1} \nu_3^{-2}\Big\{(A^0_{<3\Gamma\ 33\ 3\Gamma>} + A^0_{<3\Gamma\ 33\ \Gamma 3>})(a_3 - a_\Gamma) \\ - A^0_{<33\ \Gamma\Gamma>} - A^0_{<3\Gamma\ \Gamma 3>} + A^0_{<3\Gamma\ 3\Gamma>} + \lambda_a \beta^a_{AB}(p_\Gamma \cdot e_A)(p_\Gamma \cdot e_B)\Big\} \\ \text{for cylindrical or spherical waves.} \end{cases} \tag{7.6}$$

The underlined term in (7.5) is zero unless type II constraints are absent. We note immediately that $a(n)$ in (7.4) has <u>no contribution from the constraints</u>. This is because ν_3 is constraint-independent by (4.53), and so is $\partial\nu_3/\partial a_3$, since for longitudinal waves isotropic constraints are absent and A in (7.1) has no contributions from the directional constraints (see conclusions to Appendix A). Furthermore, $a(n) > 0$ (resp. < 0) implies that $\partial\nu_3/\partial a_3 > 0$ (resp. < 0) since we have assumed that $\nu_3 > 0$. We note that for homogeneous deformation, $\beta(n)$ in (7.5) is independent of the constraints, α, β, γ_Γ are all constant, and $\beta > 0$.

The growth equation (7.3) is a Bernoulli equation and its solutions in the context of acceleration waves were discussed by Bailey and Chen (1971), Bowen and Chen (1972) and reviewed by Chen (1973). Since we treat plane, cylindrical and spherical waves propagating in regions subject to plane, cylindrically symmetric and spherically symmetric deformations respectively, the waves propagate as families of <u>parallel surfaces</u>. Furthermore the principal curvatures for these waves are non-positive by (4.71-3), so that the waves are <u>diverging waves</u> in the sense defined by Bowen and Chen (op.cit.). The following analysis (due to Bailey and Chen (op.cit.), but see also Chen's review (Chen (1973), Section 13) therefore applies: we assume that

(i) α, β, γ_Γ are defined and integrable on every finite subinterval of $[0,\infty)$;

(ii) $\alpha(n)$ is of fixed sign on $[0,\infty)$;

(iii) $\sigma_0 > 0$, where $\sigma_0 = \sigma|_{t=0}$, (Bailey and Chen treat also $\sigma_0 < 0$) ; we note also from (7.3) that if $\sigma_0(n_1) = 0$ then $\sigma_0(n) = 0$ for all $n > n_1$.

(iv) $\liminf\limits_{n\to\infty} |\alpha(n)| \neq 0$ (in one case, (see Chen (1973), page 340) the weaker condition $\int_{n_0}^{\infty} |\alpha(n)|\, dn = \infty$, where $n_0 = n|_{t=0}$, is sufficient).

(7.7)

The local, global and asymptotic behaviour of the amplitude was discussed by Bailey and Chen (op. cit.) and the results are conveniently presented by Chen ((1973), Theorems 13.1-13.5). We outline here the essential features of these theorems, then comment in more detail on the influence of the constraints on the solutions for the particular waveforms specified.

(a) $\underline{\alpha > 0}$

Local behaviour of $\sigma(n)$:

$$\text{If } \sigma \text{ is } \left\{\begin{array}{l}\text{greater than}\\ \text{equal to}\\ \text{less then}\end{array}\right\} \frac{\beta + \gamma_\Gamma R_\Gamma^{-1}}{\alpha} \text{ then } \frac{d\sigma}{dn} \text{ is } \left\{\begin{array}{l}\text{greater than}\\ \text{equal to}\\ \text{less then}\end{array}\right\} 0 . \tag{7.8}$$

Global behaviour of $\sigma(n)$:

$$\text{If } \sigma_0 > \sigma_{cr} \text{ then } \lim_{n \to n_\infty} \sigma(n) = \infty \quad , \tag{7.9}$$

$$\text{and if } \sigma_0 < \sigma_{cr} \text{ then } \liminf_{n \to \infty} \sigma(n) = 0 \quad , \tag{7.10}$$

where the critical amplitude σ_{cr} and the finite distance n_∞ are defined by

$$\sigma_{cr}^{-1} = \int_{n_0}^{\infty} \alpha(n) \exp\left[- \int_{n_0}^{n} (\beta(\zeta) + \gamma(\zeta)\zeta^{-1}) \, d\zeta\right] dn \tag{7.11}$$

and

$$\int_{n_0}^{n_\infty} \alpha(n) \exp\left[- \int_{n_0}^{n} (\beta(\zeta) + \gamma(\zeta)\zeta^{-1}) \, d\zeta\right] dn = \sigma_0^{-1} \quad , \tag{7.12}$$

respectively, and where $\gamma = 0$ for plane waves, $\gamma = \gamma_1$ for cylindrical waves and $\gamma = \gamma_1 + \gamma_2$ for spherical waves (see (7.6)).

Asymptotic behaviour of $\sigma(n)$:

If $\sigma_0 > \sigma_{cr}$ and a is continuous from below at n_∞ , then

$$\sigma(n) = \{a(n_\infty)(n - n_0)\}^{-1}(1 + 0(1)) \text{ as } n \to n_\infty$$

where

$$a(n_\infty) = \lim_{n \to n_\infty} a(n) \quad .$$

If $\sigma_0 < \sigma_{cr}$ then

$$\sigma(n) = \sigma_0 \exp\left[- \int_{n_0}^{n} (\beta(\zeta) + \gamma(\zeta)\zeta^{-1}) \, d\zeta\right](1 - \sigma_0 \, \sigma_{cr}^{-1})^{-1}(1 + 0(1))$$

as $n \to \infty$ or as $\sigma(n) \to 0$.

We note that for $\sigma_0 > \sigma_{cr}$, the asymptotic behaviour of σ is due solely to the unconstrained term $a(n)$.

(b) $\underline{a < 0}$

Local behaviour of $\sigma(n)$:

$$\text{If } \sigma \text{ is} \left\{\begin{array}{l}\text{greater than}\\ \text{equal to}\\ \text{less then}\end{array}\right\} \frac{\beta + \gamma_\Gamma \, R_\Gamma^{-1}}{a} \text{ then } \frac{d\sigma}{dn} \text{ is} \left\{\begin{array}{l}\text{greater than}\\ \text{equal to}\\ \text{less then}\end{array}\right\} 0 \; . \tag{7.13}$$

Global behaviour of $\sigma(n)$:

(i) If $\dfrac{\beta + \gamma_\Gamma R_\Gamma^{-1}}{\alpha}$ is bounded above or tends to a finite or infinite limit $L \geq 0$, the same is true for any solution $\sigma(n)$.

(ii) Let $\sigma_1(n)$, $\sigma_2(n)$ be any two solutions with $\sigma < 0$.

If $\lim_{n \to \infty} \sigma_1(n) = \infty$ then $\lim_{n \to \infty} \sigma_2(n) = \infty$.

If $\sigma_1(n)$ is bounded then $\sigma_2(n)$ is bounded.

If $\lim_{n \to \infty} \sigma_1(n) = 0$, then $\lim_{n \to \infty} \sigma_2(n) = 0$ and $\lim_{n \to \infty} (\sigma_1(n) - \sigma_2(n)) = 0$.

Theorem (i) says that if $\alpha^{-1}(\beta + \gamma_\Gamma R_\Gamma^{-1})$ is well behaved, then the solutions $\sigma(n)$ will eventually behave the same way as $\alpha^{-1}(\beta + \gamma_\Gamma R_\Gamma^{-1})$, and theorem (ii) says that the eventual behaviour of all solutions is the same even if the behaviour of $\alpha^{-1}(\beta + \gamma_\Gamma R_\Gamma^{-1})$ is not known (Chen (1973)).

We now investigate in more detail the solution of (7.3) for particular waveforms.

(i) Plane waves

Equation (7.3) reduces to

$$d\sigma/dX = \alpha(X)\sigma^2 - \beta(X)\sigma \quad , \qquad (7.14)$$

with $\alpha(X)$ and $\beta(X)$ as in (7.4,5) with X replacing n . Since both α and β are unaffected by the constraints, the growth of plane waves is <u>constraint-independent</u>. The solution of (7.14) is

$$\sigma = \begin{cases} \dfrac{\sigma_0 \exp(-\beta X)}{\{1 - (\alpha \sigma_0 \beta^{-1})(1 - \exp(-\beta X))\}} & \text{for } \alpha \neq 0, \beta \neq 0 \quad , \\ \sigma_0 (1 - \alpha \sigma_0 X)^{-1} & \text{for } \alpha \neq 0, \beta = 0 \quad , \\ \sigma_0 \exp(-\beta X) & \text{for } \alpha = 0, \beta \neq 0 \quad . \end{cases} \tag{7.15}$$

For $\alpha > 0$, $\beta \neq 0$, the critical amplitude σ_{cr} in (7.11) takes the form

$$\sigma_{cr}^{-1} = \int_{X_0}^{\infty} \alpha(X) \exp\left[- \int_{X_0}^{X} \beta(\zeta)\, d\zeta\right] dX$$

and the finite distance X_∞ is defined by

$$\int_{X_0}^{X_\infty} \alpha(X) \exp\left[- \int_{X_0}^{X} \beta(\zeta)\, d\zeta\right] dX = \sigma_0^{-1} \quad .$$

In the special case of homogeneous deformation,

$$\sigma_{cr} = \frac{\beta}{\alpha} \quad .$$

There are then three possibilities for $\sigma(X)$:

$\sigma_0 > \sigma_{cr}$ => $\sigma \to \infty$ in a distance X_∞ ,

$\sigma_0 = \sigma_{cr} = \beta/\alpha$ => σ remains constant

$\sigma_0 < \sigma_{cr}$ => σ decreases monotonically and approaches zero exponentially as $X \to \infty$.

The solution (7.15) reduces to that given by Chadwick and Currie (1972) for unconstrained materials subject to homogeneous deformations and also to that given by Bowen and Wang (1971) for inhomogeneous media (note, however, that we are dealing with homogeneous media subject to non-homogeneous deformations).

(ii) Cylindrical waves

For this case equation (7.3) is

$$\frac{d\sigma}{dR} = \alpha(R)\sigma^2 - (\beta(R) + \gamma_1(R)R^{-1})\sigma \quad . \tag{7.16}$$

The constraints influence the result only through the term $\lambda_\alpha \beta^\alpha_{AB}(p_1 \cdot e_A)(p_1 \cdot e_B)$ appearing in γ_1 , and we recall from (4.75-7) that the deformation is unrestricted unless $\lambda_\alpha\beta^\alpha_{AB}(p_1 \cdot e_A)(p_1 \cdot e_B) \neq 0$ for at least one α , for which case we the material must necessarily be in a state of homogeneous deformation when longitudinal waves are present.

For homogeneous deformation, the principal stretches are of course constant and so α, β (>0) and γ are no longer functions of R . The solution in this case is then

$$\sigma(R) = \sigma_0 \exp(-\beta(R - R_0))\{\sigma_0 \sigma^{-1}\}^{\gamma_1}(1 - \alpha \sigma_0 \mu)^{-1} \quad , \tag{7.17}$$

where

$$\mu(R) = \int_{R_0}^{R} \{R_0 \zeta^{-1}\}^{\gamma_1} \exp(-\beta((\zeta - R_0)) \, d\zeta \tag{7.18}$$

and

$$R_0 = R|_{t=0} \quad .$$

The behaviour of the solution follows the general result given earlier with σ_{cr} (>0) and R_∞ being given by (Bowen and Wang (1971))

$$\sigma_{cr}^{-1} = \alpha R_0 \exp(\beta R)(\pi \beta^{-1} R_0^{-1})^{1/2} \operatorname{erfc}(\beta^{1/2} R_0^{1/2}) \tag{7.19}$$

and

$$\operatorname{erfc}(\beta^{1/2} R_\infty^{1/2}) = (1 - \sigma_{cr} \sigma_0^{-1}) \operatorname{erfc}(\beta^{1/2} R_0^{1/2}) \quad , \tag{7.20}$$

where

$$\operatorname{erfc} z = 2 \pi^{-1/2} \int_z^\infty \exp(-t^2) \, dt \quad . \tag{7.21}$$

Finally, we note that in the case $a_1 = a_3$ (which includes as a special case homogeneous deformation) , γ_1 takes the simple form

$$\gamma_1 = \frac{1}{2}\left[1 + \rho^{-1} \nu_3^{-2} \lambda_\alpha \beta^\alpha_{AB}(p_1 \cdot e_A)(p_1 \cdot e_B)\right] \tag{7.22}$$

after using the limiting forms of $A^0_{<33\ 11>}$ and $A^0_{<31\ 13>}$ (see note following (4.21)).

(iii) Spherical waves

The growth equation is identical to that for cylindrical waves with γ_1 in (7.16) replaced by $(\gamma_1 + \gamma_2)$. We recall from the discussion following (4.78) that when spherical waves are present, the material is necessarily in a state of uniform dilatation if both $\beta^{\alpha}_{AB}(p_\Gamma \cdot e_A)(p_\Gamma \cdot e_B)$ and $\phi^{\alpha}_2(e_{AB}, \theta)$ are non-zero. In this case, the expression for $(\gamma_1 + \gamma_2)$ takes the special form

$$\gamma_1 + \gamma_2 = 1 + \frac{1}{2}\rho^{-1}\nu_3^{-2}\lambda_\alpha\beta^{\alpha}_{AB}(p_\Gamma \cdot e_A)(p_\Gamma \cdot e_B) \quad . \tag{7.23}$$

The solution to the growth equation and the corresponding results for σ_{cr} and R_∞ are immediate on replacing γ_1 in (7.17) by the above expression.

Transverse waves

We simplify (6.51) using (7.2) and obtain, for a wave with $s = \sigma\, q_\Delta$, $\Delta = 1$ or 2, the growth equation

$$\frac{d\sigma}{dn} = (\delta + \epsilon_\Gamma R_\Gamma^{-1})\sigma \quad , \tag{7.24}$$

where

$$\delta = -\frac{1}{2}\nu_\Delta^{-1}\left[\frac{d\nu_\Delta}{dn} + Q^{\tau}_{<\Delta\Delta>}\frac{d\lambda^{+}_{\tau}}{dn}\right] \quad , \quad \tau = 1, \ldots, M \quad , \tag{7.25}$$

and

$$\epsilon_\Gamma = \begin{cases} 0 & \text{for plane waves} \\ \frac{1}{2}\rho^{-1}\nu_\Delta^{-2}(\omega_\Gamma - A_{<\Delta\Gamma\,\Delta\Gamma>}) & \text{otherwise,} \end{cases} \tag{7.26}$$

with

$$\omega_\Gamma = (a_\Gamma - a_3)(A_{<\Delta\Gamma\ \Gamma3\ \Gamma3>} + A_{<\Delta\Gamma\ \Gamma3\ 3\Gamma>}) + A_{<\Delta3\ 3\Delta>} + A_{<\Delta\Delta\ 33>} \ . \tag{7.27}$$

Clearly δ vanishes for homogeneous deformation. The influence of directional constraints is through the term $A_{<\Delta\Gamma\ \Delta\Gamma>}$ in ϵ_Γ and through $Q^\tau_{<\Delta\Delta>}$. We note that $Q^\tau_{<\Delta\Delta>}$ can also be written as Q^τ_Δ by (4.24,5), and recall from (4.57) and the discussion following (6.32) that ν^2_Δ has a contribution $\rho^{-1}\ Q^\tau_\Delta$ from the constraint subset $\left\{\phi^\tau\right\}^M_{\tau=1}$. The terms making up ω_Γ are free from directional constraints by (4.21) and the conditions of Appendix A.

The general solution of (7.24) is

$$\sigma = \begin{cases} \left[\left\{\rho^{-1}\ \nu_\Delta^{-1}\ \exp\left[\left(\int_{R_0}^{\infty} 2\epsilon R^{-1}\right)\ dR\right]\right\}\right]^{1/2} & \delta \neq 0,\ \epsilon \neq 0 \\ \sigma_0 (RR_0^{-1})^\epsilon & \delta = 0,\ \epsilon \neq 0 \\ c\ \nu_\Delta^{-1/2}\ , \quad c \text{ an integration constant} & \delta \neq 0,\ \epsilon = 0 \\ \sigma_0 & \delta = 0,\ \epsilon = 0\ , \end{cases} \tag{7.28}$$

where $\epsilon = 0$ for plane waves, $\epsilon = \epsilon_1$ for cylindrical waves and $\epsilon = \epsilon_1 + \epsilon_2$ for spherical waves.

The presence of constraints restricts the permissible deformation in many cases to homogeneous deformation (see (4.80,1) for directional constraints and also Table 4.1 for the specific constraints mentioned in the text). In such cases, the form of the solution with $\delta = 0$ is appropriate. We now comment on the form of the solution for particular wavefronts.

(i) Plane waves

By (7.28) for $\epsilon = 0$, we see that for homogeneous deformation ($\delta = 0$) the wave amplitude is constant and independent of the constraints. This agrees with Ogden (1974), who considered the single constraint of incompressibility. When $\delta \neq 0$ wave growth is influenced by the constraints through the value of ν_Δ .

(ii) Cylindrical waves

We recall from (4.72) that $R_1 = R$, $R_2 = \infty$ for cylindrical waves and consequently $\epsilon_2 R_2^{-1} = 0$ in (7.24). For waves in the circumferential direction ($\Delta = 1$) , we have from (7.26) that ϵ_1 takes the form

$$\epsilon_1 = 1/2\, \rho^{-1}\, \nu_1^{-2}(\omega_1 - A_{<11\ 11>}) \quad . \tag{7.29}$$

In the special case where $a_1 = a_3$,

$$\omega_1 - A_{<11\ 11>} = -\rho\left\{(\nu_1^I)^2 + \frac{1}{2}\lambda_\alpha \beta^\alpha_{AB}(p_1 \cdot e_A)(p_1 \cdot e_B)\right\} \tag{7.30}$$

where ν_1^I is the wave speed in the absence of directional constraints. Hence if $\mathbf{e}_A \perp \mathbf{p}_1$ (or if directional constraints are absent) then the solution is

$$\sigma = c(\nu_1^I)^{-1/2}\, R^{-1/2} \quad . \tag{7.31}$$

If in addition the deformation is homogeneous, then $\delta = 0$, $\epsilon = -1/2$ and we have

$$\sigma = \sigma_0 R_0^{-3/2} R^{-1/2} \tag{7.32}$$

so that the amplitude is <u>independent of the constraints and of the material properties.</u>

For waves in the axial direction ($\Delta = 2$) we have from (7.26) that

$$\epsilon_1 = 1/2 \rho^{-1} \nu_2^{-2} (\omega_1 - A_{\langle 21\ 21 \rangle}) \quad . \tag{7.33}$$

(iii) <u>Spherical waves</u>

We choose $\Delta = 1$ (for $\Delta = 2$, the behaviour is identical) and the solution is given by (7.28) with $\epsilon = \epsilon_1 + \epsilon_2$.

The appropriate restrictions on the deformation due to the constraints are given by (4.80,1), and we note that for homogeneous deformation, the appropriate form of (7.28) is

$$\sigma = \sigma_0 (R R_0^{-1})^{\epsilon} \quad . \qquad ((7.28)_2 \text{ bis}).$$

CHAPTER 8

DERIVATION AND SOLUTION OF THE GROWTH EQUATIONS FOR HOMENTROPIC WAVES

We investigate the growth of homentropic acceleration waves in non-conducting isotropic media subject to isotropic and directional type I constraints and to type II constraints. Reddy in (I) derived the growth equation for constrained homentropic waves in non-conductors; there however, attention was restricted to plane waves in elastic media subject to homogeneous deformation, and a non-conductor was defined to be a material for which $\mathbf{q} \equiv \mathbf{q}^0 + \gamma_\beta\, \mathbf{z}^\beta = 0$, rather than the definition (2.53) adopted here that $\mathbf{q}^0 = 0$. Since (2.53) applies, we recall from (3.50) that all type II constraints must satisfy $\mathbf{z}^\beta \cdot \mathbf{n} = 0$ when homentropic waves are present. The analysis of (I) is extended here by treating the subsets $\left\{\phi^{M+\mu}\right\}_{\mu=1}^{P-M}$ and $\left\{\phi^{P+\eta}\right\}_{\eta=1}^{N-P}$ of type I constraints (see (2.86,7)); we find however that the required solutions for terms involving $[\dot{\lambda}_a]$ are only possible if the subset $\left\{\phi^{M+\mu}\right\}_{\mu=1}^{P-M}$ is <u>absent</u>. In order to evaluate a term involving $[\mathrm{Grad}\ \dot{\eta}]$, we restrict attention to the subset of type II constraints for which $\left[\mathrm{Grad}(\mathrm{Div}(\gamma_\beta\ \hat{\mathbf{z}}^\beta))\right] = 0$; this subset nevertheless contains the constraints of perfect conductivity in all directions and of perfect conductivity in a particular direction $\mathbf{e}$ as investigated by Gurtin and Podio-Guidugli (1973). The treatment of (I) is further extended by considering cylindrical and spherical wavefronts in addition to plane waves, and the final form of the growth equation is somewhat simplified compared to the corresponding result in (I) by the use of an alternative technique in dealing with terms involving $[\mathbf{n} \cdot \mathrm{Grad}\ \dot{\lambda}_a]$.

We find that both the longitudinal and transverse growth equations are of Bernoulli type, as for the case of longitudinal homothermal waves (see (7.3)). Both homentropic growth equations are more complicated than their homothermal counterparts; this is due to the presence of constraint terms throughout the equations and because the expressions for the $[\dot{\lambda}_a]$ - where these are obtainable at all - are usually non-zero, in contrast to the homothermal result. As a consequence, for $M \geq 1$ we do not investigate particular results for different waveforms, etc., as was done in Chapter 7; the procedure involves a considerable amount of tedious substitution and the results are not especially illuminating as constraint influences are generally non-zero and so remain in the equations but simply take different forms. We rely rather on the general analysis of the Bernoulli equation given in Chapter 7 to give the nature of the solution; particular results can be obtained from the growth equations using the procedures given here for evaluating $[\dot{\lambda}_a]$ together with the results given earlier for dealing with the particular constraints. For the case $M = 0$, however, numerous simplifications occur and we show that plane waves and spherical waves are then unaffected by the constraints, as are certain cylindrical waves. Since the corresponding homothermal approach is simpler, we indicate throughout situations in which only generalized transverse waves are possible, i.e. the waves are both homothermal and homentropic and are therefore best treated using the thermal formulation of Chapters 6 and 7.

Results in the literature are essentially restricted to the case of unconstrained materials (Chen (1968c), Chadwick and Currie (1972), Bowen and Wang (1971)), apart from the investigation of Reddy (I) as discussed above.

Derivation of the growth equation

We begin as in Chapter 6 with the time derivative of the equation of motion (2.9) and obtain

$$2\,\rho\,\frac{\delta\sigma}{\delta t} = \rho\,\nu^{-1}\,\sigma\,\frac{\delta\nu}{\delta t} - \rho\,\nu^{2}\,\sigma^{-1}\,\mathbf{s}\cdot\mathbf{w} + \sigma^{-1}\,\mathbf{s}\cdot[\mathrm{Div}\,\dot{\mathbf{S}}] \quad , \tag{8.1}$$

as in (6.3), where the vector $\mathbf{w}$ satisfies $\mathbf{w}\cdot\mathbf{a} = [\mathrm{Grad}\,\dot{\mathbf{F}}](\mathbf{a},\mathbf{n},\mathbf{n})$ as in (3.28). The expression $\mathbf{s}\cdot[\mathrm{Div}\,\dot{\mathbf{S}}]$ is now evaluated in the entropic formulation by proceeding similarly to the derivation of (6.9). We have, from (2.76,7,9),

$$\dot{S}_i^{\ j} = \hat{A}_i^{\ j}{}_k^{\ \ell}\,\dot{F}^k{}_\ell + \hat{M}_i^{\ j}\,\dot{\eta} + \hat{S}^{\alpha}{}_i^{\ j}\,\dot{\lambda}_\alpha \quad , \tag{8.2}$$

where $\hat{\mathbf{A}}$ and $\hat{\mathbf{M}}$ are defined by (5.27) and (5.26) respectively, and where

$$\hat{\mathbf{S}}^{\alpha} = \rho\,\frac{\partial^2\hat{\epsilon}}{\partial\mathbf{F}\,\partial\lambda_\alpha} = \rho\,\frac{\partial\hat{\phi}^{\alpha}}{\partial\mathbf{F}} \quad ,$$

since

$$\phi^{\alpha} = \frac{\partial\hat{\epsilon}}{\partial\lambda_\alpha} \quad . \tag{2.79 bis}$$

The divergence of $\mathbf{S}$ is found with the aid of (8.2) to be

$$\dot{S}_i^{\ r}{}_{;r} = \left[\mathcal{A}_i^{\ j}{}_k^{\ \ell}{}_m^{\ n}\,\dot{F}^m{}_{n;j} + \frac{\partial}{\partial\eta}\left[\hat{A}_i^{\ j}{}_k^{\ \ell}\right]\eta_{,j} + \left[\hat{A}^{\alpha}{}_i^{\ j}{}_k^{\ \ell}\,\lambda_{\alpha,j}\right]\dot{F}^k{}_\ell\right.$$

$$+ \hat{A}_i^{\ j}{}_k^{\ \ell}\,\dot{F}^k{}_{\ell;j}$$

$$+ \left[\frac{\partial}{\partial F^k{}_\ell}\left[\hat{M}_i{}^j\right]F^k{}_{\ell;j} + \frac{\partial}{\partial\eta}\left[\hat{M}_i{}^j\right]\eta_{,j} + \hat{M}^a{}_i{}^j\,\dot{\lambda}_{a,j}\right.$$

$$+ \hat{M}_i{}^j\dot{\eta}_{,j}$$

$$+ \left[\hat{A}^a{}_i{}^j{}_k{}^\ell\,F^k{}_{\ell;j} + \hat{M}^a{}_i{}^j\eta_{,j} + \frac{\partial\hat{S}^a{}_i{}^j}{\partial\lambda_\xi}\,\lambda_{\xi,j}\right]\dot{\lambda}_a$$

$$+ \hat{S}^a{}_i{}^j\,\dot{\lambda}_{a,j} \quad . \tag{8.3}$$

In (8.3) we have, by analogy with the definitions of $\mathbf{A}$, $\mathbf{A}^a$ and $\mathbf{M}^a$ following (6.5), that the sixth-order tensor of elastic moduli $\hat{\mathbf{A}}$ is defined by

$$\hat{\mathbf{A}} = \rho\,\frac{\partial^3\epsilon}{\partial\mathbf{F}\partial\mathbf{F}\partial\mathbf{F}} \quad ,$$

and the constraint terms $\hat{\mathbf{A}}^a$ and $\hat{\mathbf{M}}^a$ are given by

$$\hat{\mathbf{A}}^a = \rho\,\frac{\partial^3\epsilon}{\partial\mathbf{F}\partial\mathbf{F}\partial\lambda_a} = \rho\,\frac{\partial^2\hat{\phi}^a}{\partial\mathbf{F}\partial\mathbf{F}}$$

and

$$\hat{\mathbf{M}}^a = \rho\,\frac{\partial^3\epsilon}{\partial\mathbf{F}\partial\eta\partial\lambda_a} = \rho\,\frac{\partial^2\hat{\phi}^a}{\partial\mathbf{F}\partial\eta} \quad . \tag{8.4}$$

We note that $\hat{\mathbf{M}}^a$, unlike $\mathbf{M}^a$, is non-zero (recall the discussion of the entropic formulation following (2.80)).

The material is now taken to be <u>at rest</u> ahead of the wave, as in Chapter 6, but is assumed to be at <u>constant entropy</u> ahead of the wave. With the aid of the identity (6.7) and the above assumptions, the following expression for the jump in the divergence of **S** is obtained from (8.3):

$$\left[\dot{S}_i{}^r{}_{;r}\right] = \left[- \hat{\mathcal{A}}_i{}^j{}_k{}^\ell{}_m{}^n\left[F^m{}_{n;j}\right] + \hat{E}^+_{ik}{}^\ell - \frac{\partial}{\partial\eta}\hat{A}_i{}^j{}_k{}^\ell[\eta_{,j}]\right.$$

$$- \hat{A}^\alpha{}_i{}^j{}_k{}^\ell[\lambda_{\alpha,j}] + \left.\hat{A}^\alpha{}_i{}^j{}_k{}^\ell\,\lambda^+_{\alpha,j}\right]\left[\dot{F}^k{}_\ell\right] + \hat{A}_i{}^j{}_k{}^\ell\left[\dot{F}^k{}_{\ell;j}\right]$$

$$+ \left[- \frac{\partial}{\partial F^k{}_\ell}\left[\hat{M}_i{}^j\right]\left[F^k{}_{\ell;j}\right] + \frac{\partial}{\partial F^k{}_\ell}\left[\hat{M}_i{}^j\right]F^{+k}{}_{\ell;j} - \frac{\partial}{\partial\eta}\hat{M}_i{}^j[\eta_{,j}]\right][\dot{\eta}]$$

$$+ \hat{M}_i{}^j[\eta_{,j}] + \left[- \hat{A}^\alpha{}_i{}^j{}_k{}^\ell\left[\dot{F}^k{}_{\ell;j}\right] + \hat{A}^\alpha{}_i{}^j{}_k{}^\ell\,F^{+k}{}_{\ell;j} + \hat{M}^\alpha{}_i{}^j[\eta_{,j}]\right.$$

$$\left.- \frac{\partial\hat{S}^\alpha{}_i{}^j}{\partial\lambda_\xi}[\lambda_{\xi,j}] + \frac{\partial\hat{S}^\alpha{}_i{}^j}{\partial\lambda_\xi}[\lambda^+_{\xi,j}]\right][\dot{\lambda}_\alpha] + \hat{S}^\alpha{}_i{}^j[\dot{\lambda}_{\alpha,j}] \quad . \tag{8.5}$$

The components of the third-order tensor $\hat{\mathbf{E}}$ appearing in (8.5) are defined by

$$\hat{E}_{ik}{}^\ell = \hat{\mathcal{A}}_i{}^j{}_k{}^\ell{}_m{}^n\,F^m{}_{n;j} \quad .$$

We proceed as before in the evaluation of (6.8), and employ the results (3.24,5,7) to eliminate the jumps involving derivatives of **F** in (8.5). For <u>homentropic waves</u>, [Grad η] = [$\dot{\eta}$] = 0 from (3.40,2) and [Grad λ_α] = $-\ \nu^{-1}[\dot{\lambda}_\alpha]\mathbf{n}$ from $(3.22)_1$. With the aid of these results, (8.5) can be used to obtain the required expression for $\mathbf{s}\cdot[\mathrm{Div}\ \dot{\mathbf{S}}]$ and we find that

$$\mathbf{s}\cdot[\mathrm{Div}\ \dot{\mathbf{S}}] = \nu^{-3}\ \hat{\mathcal{A}}(\mathbf{s},\mathbf{n},\mathbf{s},\mathbf{n},\mathbf{s},\mathbf{n}) - \nu^{-1}\ \hat{\mathbf{E}}^{+}(\mathbf{s},\mathbf{s},\mathbf{n}) - \nu^{-1}\ \hat{\mathbf{A}}^{a}(\mathbf{s},\mathrm{Grad}\ \lambda^{+}_{a},\mathbf{s},\mathbf{n})$$

$$+ \hat{\mathbf{Q}}(\mathbf{s},\mathbf{w}) - \hat{\mathbf{A}}(\mathbf{s},\mathbf{n},(\nu^{-1}\mathbf{s})_{,\Gamma},\mathbf{H}^{\Gamma}) - \hat{\mathbf{A}}(\mathbf{s},\mathbf{H}^{\Gamma},(\nu^{-1}\mathbf{s})_{,\Gamma},\mathbf{n})$$

$$+ \nu^{-1}\Omega^{\Gamma\Delta}\ \hat{\mathbf{A}}(\mathbf{s},\mathbf{H}_{\Gamma},\mathbf{s},\mathbf{H}_{\Delta}) + \hat{\mathbf{M}}(\mathbf{s},[\mathrm{Grad}\ \dot{\eta}]) + [\dot{\lambda}_{a}]\Big[- 2\nu^{-2}\ \hat{\mathbf{Q}}^{a}(\mathbf{s},\mathbf{s})$$

$$+ \hat{\mathbf{k}}^{a+}\cdot\mathbf{s} + \rho\nu^{-1}[\dot{\lambda}_{\gamma}]\ \frac{\partial\hat{\mathbf{c}}^{\gamma}}{\partial\lambda_{a}}\cdot\mathbf{s} + \frac{\partial\hat{\mathbf{S}}^{a}}{\partial\lambda_{\xi}}\left[\mathbf{s},\mathrm{Grad}\ \lambda^{+}_{\xi}\right]\Big]$$

$$+ \rho\Lambda_{\gamma}\ \hat{\mathbf{c}}^{\gamma}\cdot\mathbf{s} + [\dot{\lambda}_{a}]_{,\Gamma}\ \hat{\mathbf{S}}^{a}(\mathbf{s},\mathbf{H}^{\Gamma}) \quad . \tag{8.6}$$

Here a, $\xi = 1, \ldots, N$ and $\gamma = 1, \ldots, P$, and we have used the result (5.11) that when $T \neq 0$, $\hat{c}^{P+\eta} = 0$, $\eta = 1, \ldots, N-P$.

In (8.6), $\hat{\mathbf{k}}^{a}$ is defined to be the vector with components

$$\hat{k}^{a+}_{i} = \hat{A}^{a}{}_{i}{}^{j}{}_{k}{}^{\ell}\ F^{+k}{}_{\ell;j} \quad ,$$

and

$$\hat{\mathbf{Q}}^{a}(\mathbf{s},\mathbf{s}) = \hat{\mathbf{A}}^{a}(\mathbf{s},\mathbf{n},\mathbf{s},\mathbf{n}) \quad .$$

We recall that $\hat{\mathbf{Q}}$, $\hat{\mathbf{M}}$, $\hat{\mathbf{c}}^{a}$ and $\hat{\mathbf{w}}$ in (8.6) are defined by (5.31), (5.26), (2.91) and (5.12) respectively. The last two terms in (8.6) are obtained from the last term in (8.3) with the aid of the compatibility condition (6.38); $\Lambda_{a} = [(\mathrm{Grad}\ \dot{\lambda}_{a})\cdot\mathbf{n}]$ as in the definition following (6.38).

We use (8.6) to substitute for $\mathbf{s}\cdot[\mathrm{Div}\,\dot{\hat{\mathbf{S}}}]$ in (8.1), and investigate the remaining jumps in (8.1). A term $\sigma^{-1}(\hat{\mathbf{Q}}(\mathbf{s},\mathbf{w}) - \rho\,\nu^2\,\mathbf{s}\cdot\mathbf{w})$ occurs on the right-hand side of (8.1), and by (5.30) together with (5.11) we have

$$\hat{\mathbf{Q}}(\mathbf{s},\mathbf{w}) - \rho\nu^2\mathbf{s}\cdot\mathbf{w} = \rho\nu[\dot{\lambda}_\gamma]\hat{\mathbf{c}}^\gamma\cdot\mathbf{w} \quad , \quad \gamma = 1, \ldots, P \quad . \tag{8.7}$$

An expression for $\rho\,\hat{\mathbf{c}}^\gamma\cdot\mathbf{w}$ can be found from the condition $[\mathbf{n}\cdot\mathrm{Grad}\,(\dot{\phi}^\gamma)] = 0$. We begin with the definition (2.36) of type I constraints in the entropic formulation (recall (2.76,9)) and find that

$$\begin{aligned}\rho\,\dot{\phi}^\gamma{}_{,m} = {} & \left[\hat{A}^{\gamma}{}_{i}{}^{j}{}_{k}{}^{\ell}\,F^k{}_{\ell;m} + \hat{M}^\gamma{}_i{}^j\eta_{,m} + \frac{\partial\hat{S}^\gamma{}_i{}^j}{\partial\lambda_a}\,\lambda_{a,m}\right]\dot{F}^i{}_j \\ & + \hat{S}^\gamma{}_i{}^j\,\dot{F}^i{}_{j;m} + \left[\hat{M}^\gamma{}_i{}^j\dot{F}^i{}_j + \frac{\partial}{\partial\eta}\,\hat{\omega}^\gamma\eta_{,m} + \rho\,\frac{\partial}{\partial\lambda_a}\,\hat{\omega}^\gamma\lambda_{a,m}\right]\dot{\eta} + \rho\,\hat{\omega}^\gamma\eta_{,m} \\ & + \left[\frac{\partial\hat{S}^\gamma{}_i{}^j}{\partial\lambda_a}\,\dot{F}^i{}_{j;m} + \rho\,\frac{\partial\hat{\omega}^\gamma\eta_{,m}}{\partial\lambda_a}\left[\rho\,\hat{\mu}^{-1}\hat{\omega}^\gamma\hat{\omega}^\xi\right]\lambda_{\xi,m}\right] + \rho\,\hat{\mu}^{-1}\,\hat{\omega}^\gamma\hat{\omega}^a\lambda_{a,m} \quad ,\end{aligned} \tag{8.8}$$

where $\gamma = 1, \ldots, N\text{-}P$ and $a,\ \xi = 1, \ldots, N$, and we have made use of the identity (5.18) to substitute for terms of the form $\partial\phi^\gamma/\partial\lambda_\xi$. We now use (8.8) to evaluate $[\mathbf{n}\cdot\mathrm{Grad}\,\dot{\phi}^\gamma] = 0$ with the aid of the jump conditions (3.22,4,5,7) and (6.7,38); the approach is similar to that used in deriving (8.6) from (8.5) and details are omitted. The assumptions given following (8.3) and (8.5) are invoked and we also employ the result (5.18). After some minor rearrangement we obtain

$$\rho\hat{\mathbf{c}}^{\gamma}\cdot\mathbf{w} = -\ \nu^{-3}\hat{\mathbf{Q}}(\mathbf{s},\mathbf{s}) + \nu^{-1}\hat{\mathbf{A}}^{\gamma}(\mathbf{s},\mathbf{n},\ \mathrm{Grad}\ \mathbf{F}^{+}\cdot\mathbf{n}) + \rho\nu^{-1}\left[\frac{\partial\hat{\mathbf{c}}^{\gamma}}{\partial\lambda_{a}}\cdot\mathbf{s}\right](\mathrm{Grad}\ \lambda_{a}^{+}\cdot\mathbf{n})$$

$$+\ \hat{\mathbf{S}}^{\gamma}((\nu^{-1}\mathbf{s})_{,\Gamma},\mathbf{H}^{\Gamma}) + [\dot{\lambda}_{a}]\Bigg[2\rho\nu^{-2}[\dot{\lambda}_{a}]\ \frac{\partial\hat{\mathbf{c}}^{\gamma}}{\partial\lambda_{a}}\cdot\mathbf{s} - \frac{\partial\hat{\mathbf{S}}^{\gamma}}{\partial\lambda_{a}}\ (\mathrm{Grad}\ \mathbf{F}^{+}\cdot\mathbf{n})$$

$$-\ \rho\ \frac{\partial}{\partial\lambda_{a}}\ (\rho\hat{\mu}^{-1}\hat{\omega}^{\gamma}\hat{\omega}^{\delta})(\mathrm{Grad}\ \lambda_{a}^{+}\cdot\mathbf{n})\{\nu^{-1}[\dot{\lambda}_{\delta}] + (\mathrm{Grad}\ \lambda_{a}^{+}\cdot\mathbf{n})\}\Bigg]$$

$$-\ \rho\hat{\omega}^{\gamma}[\mathrm{Grad}\ \dot{\eta}] - \rho^{2}\hat{\mu}^{-1}\hat{\omega}^{\gamma}\hat{\omega}^{\delta}\Lambda_{\delta}\quad . \tag{8.9}$$

In (8.9) $a = 1, \ldots, N$; $\gamma, \delta = 1, \ldots, P$ and we have also used the result (5.15) that for $T \neq 0$, $\hat{\omega}^{P+\eta} = 0$, $\eta = 1, \ldots, N\text{-}P$.

If we substitute for $\rho\hat{\mathbf{c}}^{\gamma}\cdot\mathbf{w}$ from (8.9) in (8.7), we find that (8.1) has on the right-hand side the expressions

$$\rho\Lambda_{\gamma}\hat{\mathbf{c}}^{\gamma}\cdot\mathbf{s} - \rho\nu[\dot{\lambda}_{\delta}]\rho\hat{\mu}^{-1}\hat{\omega}^{\delta}\hat{\omega}^{\gamma}\Lambda_{\gamma}\qquad ,\qquad \gamma,\ \delta = 1,\ \ldots,\ P\quad ,$$

involving Λ_{γ} , and since

$$\hat{\mathbf{c}}^{\gamma}\cdot\mathbf{s} = \rho\nu\hat{\mu}^{-1}\hat{\omega}^{\gamma}\hat{\omega}^{\delta}\Lambda_{\gamma}[\dot{\lambda}_{\delta}]\qquad\qquad \text{by (5.19)},$$

the terms involving Λ_{γ} <u>vanish</u> in (8.1). This treatment of the Λ_{γ} represents an improvement on the procedure adopted in (I).

With the elimination of Λ_{γ} , the only remaining jumps in (8.1) are [Grad $\dot{\eta}$] and $[\dot{\lambda}_{a}]$, and we now investigate the former. The gradient of the energy equation (2.48) yields

$$\dot{\eta}\ \mathrm{Grad}(\rho\theta) + \rho\theta\mathrm{Grad}\ \dot{\eta} = -\ \mathrm{Grad}(\mathrm{Div}\ \mathbf{q}) + \mathrm{Grad}(\rho r) \quad , \qquad (8.10)$$

so for non-conductors (for which $\mathbf{q} = \gamma_\beta\ \hat{\mathbf{z}}^\beta$, $\beta = 1, \ldots, L$ by (2.54)), we have at the wavefront that

$$\dot{\eta}^+[\mathrm{Grad}\ (\rho\theta)] + [\dot{\eta}]\mathrm{Grad}\ (\rho\theta)^+ - [\dot{\eta}][\mathrm{Grad}\ (\rho\theta)] + (\rho\theta)^+[\mathrm{Grad}\ \dot{\eta}]$$

$$= -\ [\mathrm{Grad}(\mathrm{Div}(\gamma_\beta\ \hat{\mathbf{z}}^\beta))] \quad , \qquad (8.11)$$

after using the identity (6.7) and the assumption that r is continuous. Since we have assumed the entropy to be constant ahead of the wave, we have from (8.11) for homentropic waves that

$$(\rho\theta)^+[\mathrm{Grad}\ \dot{\eta}] = -\ [\mathrm{Grad}(\mathrm{Div}(\gamma_\beta\ \hat{\mathbf{z}}^\beta))] \quad . \qquad (8.12)$$

Clearly, <u>in the absence of type II constraints</u>, $[\mathrm{Grad}\ \dot{\eta}] = 0$.

When type II constraints are present, the expression $[\mathrm{Grad}(\mathrm{Div}(\gamma_\beta\ \hat{\mathbf{z}}^\beta))]$ must be evaluated. Use of the definition (2.37) of type II constraints and the entropic formulation (with $\hat{\mathbf{z}}^\beta = \hat{\mathbf{z}}^\beta(\mathbf{F},\theta(\mathbf{F},\eta,\lambda_\alpha,\mathbf{e}_A),\mathbf{e}_A)$ plus the methods of Appendix A for derivatives of $\hat{\mathbf{z}}^\beta$ yields the required expression. The result is found to be somewhat cumbersome and, more seriously, is found to involve $[\dot{\gamma}_\beta]$ for curved wavefronts if no restrictions are imposed on either the deformation or the form of $\hat{\mathbf{z}}^\beta$.

It will be recalled from Chapter 5 that for type I constraints, the corresponding solutions for $[\dot{\lambda}_a]$ were obtained during the derivation of the propagation condition, that is, from an analysis of the equation of motion (2.9). The type II constraints do not appear in the equation of motion and consequently the analysis of Chapter 5 yields no information about $[\dot{\gamma}_\beta]$. In fact, it is only the energy equation (2.48) that involves the type II constraints through the term Div $\mathbf{q}$; it will be recalled that the entropy production inequality (2.49) involves only $\mathbf{q}^0$. The energy equation was examined in Chapter 3 and we recall from (3.49) that the sole term involving $[\dot{\gamma}_\beta]$ is $\nu^{-1}[\dot{\gamma}_\beta]\ \hat{\mathbf{z}}^\beta\cdot\mathbf{n}$. Since $\hat{\mathbf{z}}^\beta\cdot\mathbf{n} = 0$ for non-homothermal waves by (3.45), the energy equation also yields no information about $[\dot{\gamma}_\beta]$.

We proceed further despite the indeterminacy of $[\dot{\gamma}_\beta]$ by restricting attention to the subset of type II constraints for which

$$[\mathrm{Grad}(\mathrm{Div}(\gamma_\beta\ \hat{\mathbf{z}}^\beta))] = 0 \quad . \tag{8.13}$$

This set includes the following important type II constraints investigated by Gurtin and Podio-Guidugli (1973):

(a) Perfect conductivity in all directions;

(b) Perfect conductivity in a particular direction $\mathbf{e}$.

Now if (a) holds, then Grad $\theta = 0$ in every direction. Hence at the wavefront,

$$[\mathrm{Grad}\ \theta] = 0$$

and so by (3.37,9), such waves are necessarily homothermal and are best treated by the analysis given in Chapters 6 and 7.

In the situation (b), the constraint of perfect conductivity in a direction $\mathbf{e}_1$ is expressed by

$$\hat{\mathbf{z}}^1 = \mathbf{e}_1 \quad , \tag{8.14}$$

where by (3.45) we must have $\mathbf{e}_1 \in \{\mathbf{p}_1, \mathbf{p}_2\}$ when non-homothermal waves with $\mathbf{n} = \mathbf{p}_3$ are present. For perfect conductivity in a second direction in the $(\mathbf{p}_1, \mathbf{p}_2)$ plane,

$$\hat{\mathbf{z}}^2 = \mathbf{e}_2 \quad , \tag{8.15}$$

and any further constraints expressing perfect conductivity in some direction in the $(\mathbf{p}_1, \mathbf{p}_2)$ plane can be described in terms of the linearly independent subset $\{\hat{\mathbf{z}}^1, \hat{\mathbf{z}}^2\}$, by

$$\hat{\mathbf{z}}^{2+\nu} = Z^{\nu}_{\chi} \, \hat{\mathbf{z}}^{\chi} \qquad \chi = 1,2 \quad , \qquad \nu = 3, \ldots, L-2 \quad , \tag{8.16}$$

where Z^{ν}_{χ} is a matrix of constants.

For type II constraints such as (8.14-16) that obey (8.13), terms involving [Grad $\dot{\eta}$] vanish in (8.6,9). These type II constraints therefore have no effect on the growth of homentropic waves. We recall from the discussion following (2.54) that Reddy in (I) obtained this result for type II constraints in general, but did so by employing a different definition of a non-conductor.

After making use of the results just described, the growth equation is found from (8.1) to be

$$
\begin{aligned}
2\rho\sigma \frac{\delta\sigma}{\delta t} &= \rho\nu^{-1}\sigma^2 \frac{\delta\nu}{\delta t} + \nu^{-3}\hat{\mathbf{A}}(\mathbf{s},\mathbf{n},\mathbf{s},\mathbf{n},\mathbf{s},\mathbf{n}) - \nu^{-1}\hat{\mathbf{E}}^{+}(\mathbf{s},\mathbf{s},\mathbf{n}) - \nu^{-1}\hat{\mathbf{A}}^{a}(\mathbf{s}, \text{Grad } \lambda_a^{+},\mathbf{s},\mathbf{n}) \\
&- \hat{\mathbf{A}}(\mathbf{s},\mathbf{n},(\nu^{-1}\mathbf{s})_{,\Gamma} ,\mathbf{H}^{\Gamma}) - \hat{\mathbf{A}}(\mathbf{s},\mathbf{H}^{\Gamma},(\nu^{-1}\mathbf{s})_{,\Gamma} ,\mathbf{n}) + \bar{\nu}^{1}\Omega^{\Gamma\Delta}\, \hat{\mathbf{A}}(\mathbf{s},\mathbf{H}_{\Gamma},\mathbf{s},\mathbf{H}_{\Delta}) \\
&+ [\dot{\lambda}_a]\Bigg[- 2\nu^{-2}\hat{\mathbf{Q}}^{a}(\mathbf{s},\mathbf{s}) + \hat{\mathbf{k}}^{a+}\cdot\mathbf{s} + \frac{\partial\hat{\mathbf{S}}^{a}}{\partial\lambda_\xi}(\mathbf{s}, \text{Grad } \lambda_\xi^{+}) + [\dot{\lambda}_\gamma]\Bigg\{3\rho\nu^{-1} \frac{\partial\hat{\mathbf{c}}^{\gamma}}{\partial\lambda_a}\cdot\mathbf{s} \\
&- \nu \frac{\partial\hat{\mathbf{S}}^{\gamma}}{\partial\lambda_a}(\mathbf{s}, \text{Grad } \mathbf{F}^{+}\cdot\mathbf{n}) - \rho \frac{\partial}{\partial\lambda_a}(\rho\hat{\mu}^{-1}\hat{\omega}^{\gamma}\hat{\omega}^{\delta})\Big[[\dot{\lambda}_\delta] + \nu(\text{Grad } \lambda_\delta^{+}\cdot\mathbf{n})\Big]\Bigg\}\Bigg] \\
&+ [\dot{\lambda}_\gamma]\Big[- \nu^{-2}\hat{\mathbf{Q}}^{\gamma}(\mathbf{s},\mathbf{s}) + \hat{\mathbf{A}}^{\gamma}(\mathbf{s},\mathbf{n}, \text{Grad } \mathbf{F}^{+}\cdot\mathbf{n}) + \nu\, \hat{\mathbf{S}}^{\gamma}((\nu^{-1}\mathbf{s})_{,\Gamma} ,\mathbf{H}^{\Gamma})\Big] \\
&+ [\dot{\lambda}_\gamma]_{,\Gamma}\, \hat{\mathbf{S}}^{a}(\mathbf{s},\mathbf{H}^{\Gamma}) \quad , \qquad (8.17)
\end{aligned}
$$

where as usual, $a,\ \xi = 1\ ,\ \ldots,\ N\ ;\ \gamma,\ \delta = 1\ ,\ \ldots,\ P$, and $\Gamma = 1,2$.

We turn now to the evaluation of the remaining jumps $[\dot{\lambda}_a]$ in (8.17) and begin by briefly recalling the corresponding situation for homothermal waves. In the discussion of the homothermal growth equation (6.3), we require solutions for the jumps in order to evaluate

$\rho\omega^a\ [\dot{\lambda}_a]$ for longitudinal waves (see (6.26)). We know only that the $[\dot{\lambda}_a]$ obey

$$[\dot{\lambda}_\sigma] + D^\mu_\sigma\ [\dot{\lambda}_{M+\mu}] = 0 \quad , \quad \sigma = 1, \ldots, M \quad (M \geq 1) \quad ,$$
$$\mu = 1, \ldots, P\text{-}M \quad , \qquad (6.27\ \text{bis})$$

and so we can determine $\rho\omega^a[\dot{\lambda}_a]$ in the following situations (recall (6.29)):

<u>$M = 0$</u>

if all constraints present are mechanical (recall that $P = 0$ and only the subset $\left\{\phi^\eta\right\}_{\eta=1}^N$ is present);

<u>$M = 1,2$</u>

(i) if only the linearly independent subset $\left\{\phi^\sigma\right\}_{\sigma=1}^M$ is present;

(ii) if the subsets $\{\phi^\sigma\}$, $\left\{\phi^{M+\mu}\right\}_{\mu=1}^{P-M}$ and $\left\{\phi^{P+\eta}\right\}_{\eta=1}^{N-P}$ are all present and <u>all</u> constraints are mechanical;

(iii) if the subset $\{\phi^{M+\mu}\}$ is absent and constraints $\phi^{P+\eta}$ are mechanical.

We now return to the homentropic growth equation (8.17) and note that $[\dot{\lambda}_a]$, $a = 1, \ldots, N$ and $[\dot{\lambda}_\gamma]$, $\gamma = 1, \ldots, P$ occur in many terms. Most of the expressions multiplying these jumps do not vanish for longitudinal or transverse principal waves and so the results for the $[\dot{\lambda}_a]$ from Chapter 5 must be used. As in the homothermal case, solutions are not available for all of the $[\dot{\lambda}_a]$ separately; we only know from (5.21) and (5.38) respectively that when $\hat{\omega}^{M} \neq 0$,

$$[\dot{\lambda}_M] = \rho^{-1}\nu^{-1}\hat{\mu}(\hat{\omega}^M)^{-2}\ \hat{\mathbf{c}}^M\cdot\mathbf{s}$$

$$-\ (\hat{\omega}^M)^{-1}\left\{\hat{\omega}^\zeta[\dot{\lambda}_\zeta] + \hat{\omega}^\sigma\ D^\mu_\sigma\ [\dot{\lambda}_{M+\mu}]\right\}\ , \quad (M \geq 1) \qquad (8.18)$$

and

$$[\dot{\lambda}_\zeta] = -\ \rho^{-1}\nu^{-1}\hat{\omega}^M\ \hat{\boldsymbol{\ell}}_\zeta\cdot\hat{\mathbf{Q}}^*\mathbf{s} - D^\mu_\zeta\ [\dot{\lambda}_{M+\mu}]\ , \quad (M \geq 2) \qquad (8.19)$$

where $\quad \zeta = 1, \ldots, M-1\ ;\ \sigma = 1, \ldots, M\ ;\ \mu = 1, \ldots, P-M$.

Alternatively, when $\hat{\omega}^a = 0$ and all constraints are <u>mechanical</u>, we obtain from (5.44) by using (5.62) and proceeding as in the derivation of (6.27) the result

$$[\dot{\lambda}_\sigma] + D^\mu_{\cdot\sigma}\ [\dot{\lambda}_{M+\mu}] = 0\ , \quad (M \geq 1) \qquad (8.20)$$

where again $\sigma = 1, \ldots, M\ ;\ \mu = 1, \ldots, P-M$.

For $M \geq 1$, (8.18-20) yield expressions for the $[\dot{\lambda}_\sigma]$ if the subset $\{\phi^{M+\mu}\}$ is <u>absent</u>. We are unable to obtain expressions for the $[\dot{\lambda}_{M+\mu}]$ separately and have no information about the $[\dot{\lambda}_{P+\eta}]$. This of course limits the circumstances under which we can solve the growth equation (8.17), but we will see that by making use of some of the special cases discussed in Chapters 5 (in which, for instance, waves are both homentropic and homothermal) we can obtain solutions in a wider variety of situations than initially expected. It again proves convenient to consider the situations $M = 0,1,2,3$ separately.

Growth equation for longitudinal waves

We begin the derivation of the longitudinal wave growth equation by discussing the form of the constraints ϕ^{α} and the solutions for $[\dot{\lambda}_{\alpha}]$ that arise from the results of Chapter 5 and the use of (8.18-20).

M = 0

Now P = 0 and only the subset $\left\{\phi^{\eta}\right\}_{\eta=1}^{N}$ is present; we recall from (5.11) and (5.15) respectively that $\hat{c}^{\eta} = c^{\eta} \equiv 0$ and $\hat{\omega}^{\eta} = \omega^{\eta} = 0$, so all type I constraints present are mechanical. Only directional constraints are possible when M = 0 and we have from (5.117) that they obey

$$\beta^{\eta}_{AB}(\mathbf{p}_i \cdot \mathbf{e}_A)(\mathbf{n} \cdot \mathbf{e}_B) = 0 \quad , \quad i = 1,2,3 \quad . \tag{8.21}$$

Since (8.18-20) yield no information about $[\dot{\lambda}_{\eta}]$, terms involving these jumps can only be eliminated in (8.17) if the quantities

$$\hat{\mathbf{Q}}^{\eta}(\mathbf{q}_3,\mathbf{q}_3) \ , \ \frac{\partial \hat{S}^{\eta}}{\partial \lambda_{\xi}}(\mathbf{q}_3, \ \mathrm{Grad}\ \lambda^{+}_{\xi}) \ , \ \hat{\mathbf{k}}^{\eta+} \cdot \mathbf{q}_3 \ , \ [\dot{\lambda}_{\eta}]_{,\Gamma}\ \hat{S}^{\eta}(s, \ \mathbf{H}^{\Gamma}) \quad ,$$

are all zero, where in (8.17) we have used the results $\hat{\omega}^{\eta} = 0$, $\hat{c}^{\eta} = 0$ and noted that the terms involving summations over γ, δ disappear since M = P = 0 . Firstly, we have from (5.63), (5.118), and the fact that μ

is unconstrained for mechanical constraints by (5.116) the result

$$\hat{Q}^{\eta}(q_3,q_3) = \frac{\partial}{\partial\lambda_{\eta}}\left\{Q_3 - \rho^2\mu^{-1}\left[\frac{\partial^2\psi^0}{\partial a_3\partial\theta}\right]^2\right\}$$

$$= Q_3^{\eta} = \rho\beta^{\eta}_{AB}(n\cdot e_A)(n\cdot e_B)$$

$$= 0 \quad . \tag{8.22}$$

Next, we note the transformation (see also equation (6.21) of (I))

$$\hat{S}^{a} = S^{a} - \rho\mu^{-1}\omega^{a}\,\mathbf{M}^0 \quad , \quad a = 1, \ldots, N \tag{8.23}$$

and since $\omega^{\eta} = 0$,

$$\hat{S}^{\eta} = S^{\eta} = \rho\beta^{\eta}_{AB}\sum_{i} a_i(p_i\cdot e_A)(q_i \otimes e_B) \qquad \text{by (2.61,9),}$$

and in (2.61) we have ignored dependence of ϕ^{a} on k_{AB}, as in Chapter 4. Now Grad λ^{+}_{ξ} is parallel to q_3 if we impose the assumption that $\lambda^{+}_{\xi} = \lambda^{+}_{\xi}(X^3)$, as in Chapter 6, and so

$$\hat{S}^{\eta}(q_3,\ \text{Grad}\ \lambda^{+}_{\xi}) = \rho\beta^{\eta}_{AB}a_3(n\cdot e_A)(n\cdot e_B)\,|\text{Grad}\ \hat{\lambda}^{+}_{\xi}|$$

$$= 0 \qquad \text{by (8.21);}$$

hence

$$\frac{\partial\hat{S}^{\eta}}{\partial\lambda\xi}\,(q_3,\ \text{Grad}\ \lambda^{+}_{\xi}) = 0 \quad . \tag{8.24}$$

We now recall the definition of $\hat{\mathbf{k}}^{\alpha}$ following (8.6) and evaluate $\hat{\mathbf{k}}^{\eta+}\cdot\mathbf{q}_3$. We have

$$\hat{k}^{\eta+}_{<3>} = \hat{A}^{\eta+}_{<3j\ k\ell>} F^{+}_{<k\ell;j>}$$

$$= \frac{\partial}{\partial\lambda_\eta}\left\{A_{<3j\ k\ell>} - \mu^{-1}\delta_{3j}\,\delta_{k\ell}\,M^{0}_{<3j>}\,M^{0}_{<k\ell>}\right\} F^{+}_{<k\ell;j>} = k^{\eta+}_{<3>} \quad ,$$

where we have used the transformation (5.29) for $\hat{\mathbf{A}}$ to convert to the thermal formulation, then used the definition (6.10) for $\mathbf{k}^{\alpha}$ plus the fact that μ is unconstrained for mechanical constraints by (5.116). As in (6.35), we investigate $k^{\eta+}_{<3>}$ for plane, cylindrical and spherical waves. Equation (6.35) was previously analysed with the aid of the propagation condition $\mathbf{c}^{\alpha}\cdot\mathbf{s} = 0$. For homentropic waves with $\mathbf{c}^{\eta} = \mathbf{0}$ and $T \neq 0$, we again have $\mathbf{c}^{\eta}\cdot\mathbf{s} = 0$ (recall (5.1) with $P = 0$) and so the previous analysis applies, except as noted below. Since (8.21) applies, the terms involving $F^{+}_{<33;3>}$ in (6.35) vanish and we have the result

$$\hat{\mathbf{k}}^{\eta+}\cdot\mathbf{q}_3 = 0 \qquad \text{for plane waves.} \tag{8.25}$$

For cylindrical and spherical waves we have respectively

$$k^{\eta+}_{<3>} = \begin{cases} \beta^{\eta}_{AB}(\mathbf{p}_1\cdot\mathbf{e}_A)(\mathbf{p}_1\cdot\mathbf{e}_B)F^{+}_{<31;1>} \\ \\ \sum\limits_{\Gamma} \beta^{\eta}_{AB}(\mathbf{p}_\Gamma\cdot\mathbf{e}_A)(\mathbf{p}_\Gamma\cdot\mathbf{e}_B)F^{+}_{<3\Gamma;\Gamma>} \end{cases} \tag{8.26}$$

where $F^{+}_{<3\Gamma;\Gamma>} = R^{-1}(a_3 - a_\Gamma)$ from Appendix B.

In (6.35), the term $(8.26)_1$ was shown to be zero since either $\beta^{\eta}_{AB}(\mathbf{p}_1\cdot\mathbf{e}_A)(\mathbf{p}_1\cdot\mathbf{e}_B)$ was zero or, if $\beta^{\eta}_{AB}(\mathbf{p}_1\cdot\mathbf{e}_A)(\mathbf{p}_1\cdot\mathbf{e}_B)$ was non-zero, then the deformation was homogeneous by (4.77) so $F^{+}_{<31;1>}$ was zero. The analysis of (6.35) also applies here, since (4.77) still holds for mechanical constraints. For homentropic cylindrical waves we therefore obtain the required result that $\hat{\mathbf{k}}^{\eta+}\cdot\mathbf{q}_3 = 0$.

For spherical waves, we have from (4.78) and the discussion following it that for mechanical constraints we must have

$$\beta^{\eta}_{AB}(\mathbf{p}_\Gamma\cdot\mathbf{e}_A)(\mathbf{p}_\Gamma\cdot\mathbf{e}_B) = 0 \quad ,$$

so that $(8.26)_2$ is zero, and so $\hat{\mathbf{k}}^{\eta+}\cdot\mathbf{q}_3 = 0$.

Finally, we recall from the analysis of (6.40) that when $\lambda^{+}_{a} = \lambda_{a}(X^3)$, we have that $[\dot{\lambda}_{a}]_{,\Gamma} = 0$ and so $[\dot{\lambda}_{a}]_{,\Gamma}\ \hat{S}^{a}(s,\ \mathbf{H}^{\Gamma}) = 0$ $(M = 0,1,2,3)$.

<u>$M = 1$</u>

We have from (8.18) with $s = \sigma\ \mathbf{q}_3$ and $\hat{\omega}^1 \neq 0$ that

$$[\dot{\lambda}_1] = \rho^{-1}\nu^{-1}\ \hat{\mu}(\hat{\omega}^1)^{-2}\ \sigma\ \hat{\mathbf{c}}^1\cdot\mathbf{q}_3 - D^{\mu}_{1}[\dot{\lambda}_{1+\mu}] \quad . \tag{8.27}$$

Since we have no further information about the $[\dot{\lambda}_{1+\mu}]$, (8.27) only provides a solution for $[\dot{\lambda}_1]$ when the subset $\{\phi^{1+\mu}\}$ is absent, in which case

$$[\dot{\lambda}_1] = \rho^{-1}\nu^{-1}\,\mu(\hat{\omega}^1)^{-2}\,\sigma\,\hat{c}^1\cdot q_3 \quad . \tag{8.28}$$

We recall from (5.70,1) that the presence of longitudinal waves requires that the constraint vector $\hat{c}^1$ in (8.28) satisfies either $\hat{c}^1\cdot q_3 = 0$ or $\hat{c}^1 \wedge q_3 = 0$.

When $\hat{\omega}^a = 0$ and all type I constraints are mechanical, we have from (8.20) with $s = \sigma\, q_3$ that when the subset $\{\phi^{1+\mu}\}$ is absent,

$$[\dot{\lambda}_1] = 0 \quad . \tag{8.29}$$

<u>M = 2</u>

We can only solve for $[\dot{\lambda}_\sigma]$, $\sigma = 1,2$, when the subset $\{\phi^{2+\mu}\}$ is absent, in which case we have from (8.18,19) when $s = \sigma\, q_3$ and $\hat{\omega}^2 \neq 0$ the results

$$[\dot{\lambda}_2] = \rho^{-1}\nu^{-1}\,\hat{\mu}(\hat{\omega}^2)^{-2}\,\sigma\,\hat{c}^2\cdot q_3 - (\hat{\omega}^2)^{-1}\,\hat{\omega}^1[\dot{\lambda}_1] \tag{8.30}$$

and

$$[\dot{\lambda}_1] = -\,\rho^{-1}\nu^{-1}\,\hat{\omega}^2\,\sigma\,\hat{\boldsymbol{\ell}}_1\cdot\hat{\mathbf{Q}}^*\,q_3 \quad ; \tag{8.31}$$

$\hat{c}^2$ in (8.30) obeys either $\hat{c}^2\cdot q_3 = 0$ or $\hat{c}^2 \wedge q_3 = 0$ by (5.89,90).

In the situation when $\hat{\omega}^a \neq 0$ we find from (8.20) with $s = \sigma\, q_3$ when $\{\phi^{2+\mu}\}$ is absent that

$$[\dot{\lambda}_\sigma] = 0 \quad , \quad \sigma = 1,2 \quad . \tag{8.32}$$

<u>$M = 3$</u>

We can solve (8.18,19) for $[\dot{\lambda}_\sigma]$, $\sigma = 1,2,3$, if the subset $\{\phi^{3+\mu}\}$ is absent and obtain for $\mathbf{s} = \sigma\, \mathbf{q}_3$ and $\hat{\omega}^3 \neq 0$ that

$$[\dot{\lambda}_3] = \rho^{-1}\nu^{-1}\, \mu(\hat{\omega}^3)^{-2}\, \sigma\, \hat{\mathbf{c}}^3\cdot\mathbf{q}_3 - (\hat{\omega}^3)^{-1}\, \hat{\omega}^\zeta[\dot{\lambda}_\zeta] \quad , \tag{8.33}$$

where $[\dot{\lambda}_\zeta] = -\, \rho^{-1}\nu^{-1}\, \hat{\omega}^3\, \sigma\, \boldsymbol{\ell}_\zeta\cdot\hat{\mathbf{Q}}^*\, \mathbf{q}_3 \quad , \quad \zeta = 1,2 \quad ;$ (8.34)

$\hat{\mathbf{c}}^3$ in (8.33) obeys either $\hat{\mathbf{c}}^3\cdot\mathbf{q}_3 = 0$ or $\hat{\mathbf{c}}^3 \wedge \mathbf{q}_3 = 0$ by (5.108,9).

The situation $\hat{\omega}^\alpha = 0$ is <u>not</u> compatible with homentropic waves when $M = 3$ from the discussion preceding (5.103).

Before leaving this evaluation of the jumps $[\dot{\lambda}_\alpha]$ we note that for $M \geq 1$, the jumps $[\dot{\lambda}_{P+\eta}]$ appear in the following terms as part of the summations over $\alpha = 1, \ldots, N$ in (8.17):

$$[\dot{\lambda}_{P+\eta}]\Big[3\rho\nu^{-1}[\dot{\lambda}_\gamma]\, \frac{\partial\hat{\mathbf{c}}^\gamma}{\partial\lambda_{P+\eta}}\cdot\mathbf{s} - \nu\, \frac{\partial\hat{\mathbf{S}}^\gamma}{\partial\lambda_{P+\eta}}\, (\mathbf{s},\ \mathrm{Grad}\ \mathbf{F}^+\cdot\mathbf{n})$$

$$-\, \rho\, \frac{\partial}{\partial\lambda_{P+\eta}}\, (\rho\hat{\mu}^{-1}\hat{\omega}^\gamma\hat{\omega}^\delta)([\dot{\lambda}_\delta] + \nu(\mathrm{Grad}\ \dot{\lambda}^+_\delta\cdot\mathbf{n}))\Big] \quad .$$

We recall from the discussions following (8.17) and (8.21) that we have no information regarding $[\dot{\lambda}_{P+\eta}]$. We can, however, proceed as in (8.22-4) for $M = 0$: we transform $\hat{\mu}$, $\hat{\omega}^\gamma$, $\hat{\mathbf{c}}^\gamma$ and $\hat{\mathbf{S}}^\gamma$ to the thermal formulation using $(5.13)_{1,2}$, (5.60) and (8.23), then evaluate the derivatives $\partial/\partial\lambda_{P+\eta}$ required above. With the aid of the fact that μ has no contributions from the constraints $\phi^{P+\eta}$ since $\omega^{P+\eta} = 0$, (recall

(5.116)), we find that the aforementioned derivatives vanish, and consequently the terms listed above make no contribution in (8.17).

This completes the analysis of the jumps appearing in (8.17). The remaining terms can be treated by the methods used in deriving the homothermal growth equation (7.3) for longitudinal waves from (6.43,6), and we therefore give merely an outline of the corresponding analysis required here.

We take $s = \sigma\, q_3$ in (8.17) and denote the longitudinal wave speed by ν_3. The displacement derivatives $\delta\sigma/\delta t$ and $\delta\nu_3/\delta t$ are evaluated with the aid of (7.2), $\hat{\mathbf{E}}^+(\mathbf{s},\mathbf{n},\mathbf{n})$ is treated as was $\mathbf{E}^+(\mathbf{s},\mathbf{n},\mathbf{n})$ in (6.44), and the terms involving $(\nu^{-1}s)_{,\Gamma}$ are evaluated with the aid of the result (6.45) for $(\nu^{-3}\sigma\, q_3)_{,\Gamma}$. We simplify terms involving Grad λ_a^+ by using the assumption that $\lambda_a^+ = \lambda_a^+(X^3)$, and employ the results given in (6.34) and in Appendix B to evaluate Grad $\mathbf{F}^+$ for the plane, cylindrical and spherically symmetric deformations described in Chapter 4.

With the aid of these results, plus those for $[\dot{\lambda}_a]$ given in (8.18-34) when the type I constraint subset $\{\phi^{M+\mu}\}$ is <u>absent</u>, the homentropic growth equation for longitudinal waves is found from (8.17) to be

$$\frac{d\sigma}{dn} = a\sigma^2 - (\beta + \gamma_\Gamma\, R_\Gamma^{-1})\sigma \tag{8.35}$$

where

$$\alpha(n) = \frac{1}{2}\rho^{-1}\nu_3^{-1}\left[\nu_3^{-3}\hat{\mathcal{A}}_{<33\ 33\ 33>} - \chi_a\left\{3\nu_3^{-2}\chi_a\hat{Q}^a_{<33>}\right.\right.$$

$$\left.\left. + \chi_\gamma\left[3\rho\nu_3^{-1}\frac{\partial\hat{c}^\gamma_{<3>}}{\partial\lambda_a} - \chi_\delta\frac{\partial}{\partial\lambda_a}(\rho\,\hat{\mu}^{-1}\hat{\omega}^\gamma\hat{\omega}^\delta)\right]\right\}\right] \quad ,$$

(8.36)

$$\beta(n) = \frac{1}{2}\rho^{-1}\nu_3^{-1}\left[-\rho\frac{d\nu_3}{dn} + \nu_3^{-1}\sum_i\hat{\mathcal{A}}_{<33\ 33\ ii>}a_{i,3}\right.$$

$$+ \nu_3^{-1}\hat{A}^a_{<33\ 33>}\lambda^+_{a,<3>}$$

$$- \chi_a\left\{2\sum_i\hat{A}^a_{<33\ ii>}a_{i,3} + \frac{\partial\hat{S}^a_{<33>}}{\partial\lambda_\gamma}\lambda^+_{\gamma,<3>}\right.$$

$$\left.\left. - \chi_\gamma\nu_3\left\{\sum_i\frac{\partial\hat{S}^\gamma_{<ii>}}{\partial\lambda_a}a_{i,3} + \rho\frac{\partial}{\partial\lambda_a}\left[\rho\,\hat{\mu}^{-1}\hat{\omega}^\gamma\hat{\omega}^\delta\right]\lambda^+_{\delta,<3>}\right\}\right\}\right] \quad ,$$

(8.37)

$$\gamma_\Gamma(n) = \begin{cases} 0 \quad \text{for plane waves,} \\ \frac{1}{2}\rho^{-1}\nu_3^{-1}\Big[-\nu_3^{-1}\Big\{\hat{A}_{<33\ \Gamma\Gamma>} + \hat{A}_{<3\Gamma\ \Gamma 3>} - \hat{A}_{<3\Gamma\ 3\Gamma>} \\ \qquad - (\hat{\mathcal{A}}_{<3\Gamma\ 33\ 3\Gamma>} + \hat{\mathcal{A}}_{<3\Gamma\ 33\ \Gamma 3>})(a_3 - a_\Gamma)\Big\} \\ \quad - \chi_a(\hat{A}^a_{<3\Gamma\ \Gamma 3>} + \hat{A}^a_{<3\Gamma\ 3\Gamma>})(a_3 - a_\Gamma) + \chi_a\,\hat{S}^a_{<\Gamma\Gamma>}\Big] \\ \text{for cylindrical or spherical waves.} \end{cases}$$

(8.38)

In (8.35-37), the notation

$$\chi_a = \sigma^{-1}[\dot{\lambda}_a] \tag{8.39}$$

has been used and we note that a, γ, δ, are now summed over 1, ..., M ($M \geq 1$) only; the appropriate results for $[\dot{\lambda}^a]$ when M = 0,1,2,3 are inserted from (8.21) -(8.34). The solution of (8.35) is discussed later in the chapter.

Growth equation for transverse waves

We return to (8.17) and begin by discussing solutions for $[\dot{\lambda}_a]$, making use of the results of Chapter 5 together with (8.18-20) for waves with $s = \sigma\, q_\Delta$ where Δ = 1 or 2. A significant feature of the transverse wave situation is that there are numerous instances in which waves are both homentropic and homothermal; in such cases the thermal formulation of transverse wave growth given in Chapters 6 and 7 is to be preferred (recall also the discussion of (5.130-3)).

M = 0

Now P = 0 and only the constraint subset $\left\{\phi^\eta\right\}_{\eta=1}^{N}$ is present, with $\hat{c}^\eta \equiv 0$.

By the discussion of this case preceding (5.64), all transverse homentropic waves are also homothermal and so need not be considered here.

$\underline{M = 1}$

When the subset $\{\phi^{1+\mu}\}$ is absent, (8.18) with $s = \sigma\, q_\Delta$ where $\Delta = 1$ or 2 gives the following solution for $[\dot{\lambda}_1]$ when $\hat{\omega}^1 \neq 0$:

$$[\dot{\lambda}_1] = \rho^{-1}\nu^{-1}\,\hat{\mu}(\hat{\omega}^1)^{-2}\,\sigma\,\hat{c}^1\cdot q_\Delta \quad , \tag{8.40}$$

and the vector $\hat{c}^1$ satisfies $\hat{c}^1\cdot q_\Delta = 0$ or $\hat{c}^1 \wedge q_\Delta = 0$ by (5.70,1). We recall from (5.78-80) however, that only $\hat{c}^1 \wedge q_\Delta = 0$ is compatible with transverse waves that are non-homothermal. Furthermore, $\hat{\omega}^1 = 0$ is not compatible with non-homothermal transverse waves by the discussion following (5.80).

$\underline{M = 2,3}$

Equations (8.18,19) with $s = \sigma\, q_\Delta$, $\Delta = 1$ or 2, yield the following solutions for $[\dot{\lambda}_\sigma]$ when the subset $\{\phi^{M+\mu}\}$ is absent and $\hat{\omega}^M \neq 0$:

$$[\dot{\lambda}_M] = \rho^{-1}\nu^{-1}\,\hat{\mu}(\hat{\omega}^M)^{-2}\,\sigma\,\hat{c}^M\cdot q_\Delta - (\hat{\omega}^M)^{-1}\,\hat{\omega}^\zeta[\dot{\lambda}_\zeta] \quad , \tag{8.41}$$

$$[\dot{\lambda}_\zeta] = -\,\rho^{-1}\nu^{-1}\hat{\omega}^M\,\sigma\,\hat{\boldsymbol{\ell}}_\zeta\cdot\hat{\mathbf{Q}}^* q_\Delta \quad , \qquad \zeta = 1, \ldots, M-1 \ . \tag{8.42}$$

For $\underline{M = 2}$, non-homothermal transverse waves require that the vector c^2 satisfies $c^2 \wedge q_\Delta = 0$ by the discussion following (5.95). The case $\hat{\omega}^1 = \hat{\omega}^2 = 0$ is not allowed by the discussion following (5.98).

For $\underline{M = 3}$, both the situations $\hat{c}^3 \cdot q_\Delta = 0$ and $\hat{c}^3 \cdot q_\Delta = 0$ are compatible with non-homothermal transverse waves by (5.112,3) but the case $\hat{\omega}^a = 0$ is not allowed by the discussion following (5.102).

This completes the discussion of the jumps $[\dot{\lambda}_\sigma]$; $\sigma = 1, \ldots, M$ appearing in the growth equation for transverse principal waves. Since the subset $\left\{\phi^{M+\mu}\right\}_{\mu=1}^{P-M}$ is assumed absent, the jumps $[\dot{\lambda}_{M+\mu}]$ do not appear. We noted above that the constraint subset $\left\{\phi^{P+\eta}\right\}_{\eta=1}^{N-P}$ is not compatible with non-homothermal transverse waves when $M = P = 0$, but the same constraints are allowed when $M \geq 1$, provided that the subset $\left\{\phi^\sigma\right\}_{\sigma=1}^{M}$ is such that $\hat{\omega}^\sigma[\dot{\lambda}_\sigma] \neq 0$ so that $T \neq 0$ in (5.52). In such circumstances, the jumps $[\dot{\lambda}_{P+\eta}]$ occur in the following terms in (8.17) ($s = \sigma\, q_\Delta$, $\Delta = 1$ or 2) :

$$[\dot{\lambda}_{P+\eta}]\left[-\,2\nu_\Delta^{-2}\,\hat{Q}^{P+\eta}_{<\Delta\Delta>} + \frac{\partial S^{P+\eta}_{<\Delta 3>}}{\partial \lambda_\xi}\left(\lambda^{+}_{\xi,<3>}\right) + \hat{k}^{+}_{<\Delta>} + [\dot{\lambda}_\gamma]\left\{3\rho\nu_\Delta^{-1}\,\frac{\partial \hat{c}^{\gamma}_{<\Delta>}}{\partial \lambda_{P+\eta}} - \rho\,\frac{\partial}{\partial \lambda_{P+\eta}}\left(\rho\hat{\mu}^{-1}\hat{\omega}^\gamma\hat{\omega}^\delta\right)\left[[\dot{\lambda}_\delta] + \nu_\Delta\left(\dot{\lambda}^{+}_{\delta,<3>}\right)\right]\right\}\right] \quad , \qquad (8.43)$$

where $a,\xi = 1, \ldots, N$ and $\gamma,\delta = 1, \ldots, M$, $\eta = 1, \ldots, N-P$, and the subset $\left\{\phi^{M+\mu}\right\}_{\mu=1}^{P-M}$ is absent so we take $P = M$.

It is a straightforward procedure to show that the terms in (8.43) are zero, as was the case for longitudinal waves (recall the discussion of $[\dot{\lambda}_{P+\eta}]$ following (8.34)). The (directional) constraints $\phi^{P+\eta}$ satisfy (8.21), $\omega^{P+\eta} = 0$ by (5.2) since $T \neq 0$, and μ therefore contains no contributions from these constraints by (5.116).

Firstly, with the aid of (5.63), (4.24), (4.22) and (8.21) in turn,

$$\hat{Q}^{P+\eta}_{<\Delta\Delta>} = \frac{\partial Q_\Delta}{\partial \lambda_{P+\eta}}$$

$$= \rho\beta^{\eta}_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B)$$

$$= 0 \quad . \tag{8.44}$$

We now proceed as in (8.23,4) and have

$$\hat{S}^{P+\eta}_{<\Delta 3>} = \rho\beta^{P+\eta}a_\Delta(\mathbf{p}_\Delta\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B)$$

$$= 0 \quad \text{by (8.21)} \quad . \tag{8.45}$$

Furthermore, by proceeding as before (following (8.24)) for $\hat{k}^{\eta+}_{<3>}$, we find

$$\hat{k}^{P+\eta^+}_{<\Delta>} = \hat{A}^{P+\eta}_{<\Delta j\ k\ell>} F^+_{<k\ell;j>}$$

$$= A^{P+\eta}_{<\Delta j\ k\ell>} F^+_{<k\ell;j>} = k^{P+\eta^+}_{<\Delta>} \quad ,$$

with the aid of the transformation (5.29) for $\hat{\mathbf{A}}$ and the fact that μ is independent of $\phi^{P+\eta}$ since $\omega^{P+\eta} = 0$. Now from the expression (4.21) for $\mathbf{A}$ and the results of Appendix B, we find that the plane, cylindrical and spherical deformations considered here,

$$k^{P+\eta^+}_{<\Delta>} = \overset{P+\eta}{A}_{<\Delta 3\ \Delta\Delta>}\, a_{\Delta,3}$$

$$= \beta^{P+\eta}_{AB}(p_\Delta \cdot e_A)(n \cdot e_B)\ a_{\Delta,3}$$

$$= 0 \quad \text{by (8.21)} \quad . \tag{8.46}$$

The remaining terms in (8.43) are found to vanish by proceeding similarly; the transformations (5.60) for $\hat{c}^\alpha$ and $(5.13)_{1,2}$ for $\hat{\mu}$, $\hat{\omega}^\alpha$ are used and the derivatives $\partial/\partial\lambda_{P+\eta}$ are found to vanish since all terms (including μ) are independent of $\lambda_{P+\eta}$. (Recall the similar discussion following (8.34) for longitudinal waves).

The remaining terms in (8.17) can be evaluated for $s = \sigma\, q_\Delta$, Δ = 1 or 2, by proceeding similarly to the evaluation of corresponding terms in the derivation of the growth equation (8.35) for homentropic longitudinal waves, and in the derivation of the growth equation (7.24) for homothermal transverse waves from the preliminary forms (6.47,51). Consequently, only a summary is given here.

The displacement derivatives $\delta/\delta t$ are evaluated using (7.2), and we then make use of the transformation

$$\hat{\mathcal{A}} = \mathcal{A} - \mu^{-1}\frac{\partial \mathbf{A}}{\partial \theta} \otimes \mathbf{M}^0 - \mu^{-1}\frac{\partial}{\partial \mathbf{F}}(\mathbf{M}^0 \otimes \mathbf{M}^0) + \mu^{-2}\frac{\partial}{\partial \theta}(\mathbf{M}^0 \otimes \mathbf{M}^0 \otimes \mathbf{M}^0)$$

$$- \mu^{-3}\frac{\partial \mu}{\partial \theta}(\mathbf{M}^0 \otimes \mathbf{M}^0 \otimes \mathbf{M}^0) \tag{8.47}$$

to show that

$$\hat{A}_{<\Delta 3\ \Delta 3\ \Delta 3>} = 0 \tag{8.48}$$

(recall $A_{<\Delta 3\ \Delta 3\ \Delta 3>} = 0$ in (6.48), and also that $M^0_{<\Delta 3>} = 0$ by (2.95)).

The transformation (8.47) and results of Appendix A are again used to evaluate $\hat{E}^+_{<\Delta\Delta 3>}$, proceeding by analogy with the treatment of $E^+_{<\Delta\Delta 3>}$ in (6.49), and the terms involving $(\nu^{-1}s)_{,\Gamma}$ are evaluated using (6.50) as well as the transformation (5.29) for $\hat{A}$. The assumption $\lambda^+_a = \lambda^+_a(X^3)$ is used to simplify Grad λ^+_a, and the results (6.34) plus those of Appendix B are used to evaluate Grad F^+.

The growth equation for transverse homentropic waves ($s = \sigma\, q_\Delta$, $\Delta = 1$ or 2) is found from (8.17) with the aid of the above procedures, and, with $\left\{\phi^{M+\mu}\right\}_{\mu=1}^{P-M}$ absent, takes the form

$$\frac{d\sigma}{dn} = a\sigma^2 - (\delta + \epsilon_\Gamma\, R_\Gamma^{-1})\sigma \tag{8.49}$$

where

$$a(n) = \frac{1}{2}\,\rho^{-1}\nu_\Delta^{-1}\left[-\ \chi_a\left\{3\nu_\Delta^{-2}\ \hat{Q}^a_{<\Delta\Delta>}\right.\right.$$

$$\left.\left. +\ \chi_\gamma\left[3\rho\nu_\Delta^{-1}\ \frac{\partial\hat{c}^\gamma_{<\Delta>}}{\partial\lambda_a} - \rho\chi_\delta\,\frac{\partial}{\partial\lambda_a}\,(\rho\hat{\mu}^{-1}\hat{\omega}^\gamma\hat{\omega}^\delta)\right]\right\}\right] \quad , \tag{8.50}$$

$$\delta(n) = \frac{1}{2}\rho^{-1}\nu_\Delta^{-1}\Bigg[-\rho\frac{d\nu_\Delta}{dn} + \nu_\Delta^{-1}\sum_i \hat{\mathcal{A}}_{<\Delta 3\ \Delta 3\ ii>} a_{i,3}$$

$$+ \nu_\Delta^{-1}\hat{A}^{\alpha}_{<\Delta 3\ \Delta 3>}\lambda^{+}_{\alpha,<3>}$$

$$- \chi_\alpha\Bigg\{2\hat{A}^{\alpha}_{<\Delta 3\ \Delta\Delta>} a_{\Delta,3} + \frac{\partial\hat{S}^{\alpha}_{<\Delta 3>}}{\partial\lambda_\gamma}\lambda^{+}_{\gamma,<3>}$$

$$- \chi_\gamma\nu_\Delta\Bigg\{\sum_i \frac{\partial\hat{S}^{\gamma}_{<ii>}}{\partial\lambda_\alpha} a_{i,3} + \rho\frac{\partial}{\partial\lambda_\alpha}(\rho\hat{\mu}^{-1}\hat{\omega}^{\gamma}\hat{\omega}^{\delta})\lambda^{+}_{\delta,<3>}\Bigg\}\Bigg\}\Bigg] \quad , \tag{8.51}$$

$$\epsilon_\Gamma(n) = \begin{cases} 0 \quad \text{for plane waves,} \\ \frac{1}{2}\rho^{-1}\nu_\Delta^{-1}\Bigg[-\nu_\Delta^{-1}\Big\{\hat{A}_{<\Delta\Delta\ 33>} + \hat{A}_{<\Delta 3\ 3\Delta>})\delta_{\Delta\Gamma} + \hat{A}_{<\Delta\Gamma\ \Delta\Gamma>} \\ \qquad - (\hat{\mathcal{A}}_{<\Delta\Gamma\ \Delta 3\ \Gamma 3>} + \hat{\mathcal{A}}_{<\Delta\Gamma\ \Delta 3\ 3\Gamma>})(a_3 - a_\Gamma)\Big\} \\ \qquad - \chi_\alpha\Big\{(\hat{A}^{\alpha}_{<\Delta\Delta\ \Delta 3>} + \hat{Q}^{\alpha}_{<\Delta\Delta>})(a_3 - a_\Delta)\delta_{\Delta\Gamma} + \hat{S}^{\alpha}_{<\Gamma\Gamma>}\Big\}\Bigg] \\ \quad \text{for cylindrical or spherical waves.} \end{cases} \tag{8.52}$$

We have employed the definition (8.39) of χ_α in (8.50-52) with α, γ, $\delta = 1, \ldots, M$, and the results (8.40-42) are inserted as appropriate.

<u>Solutions to the growth equation for homentropic waves</u>

Both the equations (8.35,49) for the growth of longitudinal and transverse principal waves are Bernoulli equations, in contrast to the homothermal situation where only the longitudinal equation (7.3) was of Bernoulli form; the transverse equation (7.24) being a linear first order equation. The analysis of the Bernoulli equation in Chapter 7 following (7.6) is therefore applicable here and details are omitted except to quote the general form of the solution, which is

$$\sigma(n) = \sigma_0 \exp\left[-\int_{n_0}^{n} (\beta + \gamma_\Gamma \zeta_\Gamma^{-1}) d\zeta\right] \left\{1 - \sigma_0 \int_{n_0}^{n} \alpha \exp\left[-\int_{N_0}^{N} (\beta + \gamma_\Gamma \zeta_\Gamma^{-1}) d\zeta\right] dN\right\}^{-1} \tag{8.53}$$

where as before, n measures distance in the normal direction and $\sigma_0 = \sigma(n_0)$.

In Chapter 7 attention was focussed on the nature of the constraint influence on the solution; in certain cases the constraints had no effect or influenced only one part of the equation. This is generally not so for homentropic waves when $M \geq 1$; both (8.35) and (8.49) are then considerably more cumbersome than their thermal counterparts and the constraints influence α, β , γ_Γ , δ and ϵ_Γ . In addition the solutions for $[\dot{\lambda}_\alpha]$, where they are obtainable at all, are often non-zero unlike the homothermal case. In view of this, there seems little to be gained from discussing particular solutions when $M \geq 1$ for the various wavefronts as was done in Chapter 7; the general features of the solution are clear from the previous analysis of the Bernoulli equation and particular solutions are obtainable with the results of Chapter 5 plus the expressions for $[\dot{\lambda}_\alpha]$ determined earlier in this chapter.

In the case when $\mathbf{M} = 0$, and only the type I constraints ϕ^η for which $\hat{c}^\eta = c^\eta = 0$, $\eta = 1, \ldots, N$ are present, the influence of the constraints is considerably less complicated, and is now described. We consider only longitudinal waves, since for $\mathbf{M} = 0$ transverse waves that are non-homothermal are not compatible with these constraints, as noted previously. With the results given earlier during the discussion of the longitudinal growth equation for $\mathbf{M} = 0$ (see (8.21-6)), the expressions (8.36-8) for α, β, γ_Γ in the growth equation (8.35) reduce respectively to:

$$\alpha(n) = \frac{1}{2}\rho^{-1}\nu_3^{-4}\,\hat{\mathcal{A}}_{\langle 33\ 33\ 33\rangle} \tag{8.54}$$

$$\beta(n) = \frac{1}{2}\rho^{-1}\nu_3^{-1}\left[-\rho\frac{d\nu_3}{dn} + \nu_3^{-1}\sum_i \hat{\mathcal{A}}_{\langle 33\ 33\ ii\rangle}\, a_{i,3}\right] \tag{8.55}$$

$$\gamma_\Gamma(n) = \begin{cases} 0 \quad \text{for plane waves,} \\ -\frac{1}{2}\rho^{-1}\nu_3^{-1}\Big\{\hat{A}_{\langle 33\ \Gamma\Gamma\rangle} + \hat{A}_{\langle 3\Gamma\ \Gamma 3\rangle} - \hat{A}_{\langle 3\Gamma\ 3\Gamma\rangle} \\ \qquad - (\hat{\mathcal{A}}_{\langle 3\Gamma\ 33\ 3\Gamma\rangle} + \hat{\mathcal{A}}_{\langle 3\Gamma\ 33\ \Gamma 3\rangle})(a_3 - a_\Gamma)\Big\} \\ \text{for cylindrical or spherical waves.} \end{cases} \tag{8.56}$$

We now establish whether or not the terms on the right-hand sides of (8.54-6) are dependent on the constraints ϕ^η. Firstly,

$$\nu_3^2 = \rho^{-1}\hat{Q}_3 = \rho^{-1}\left[Q_3 - \rho^2\mu^{-1}\left[\frac{\partial^2\psi^o}{\partial a_3\partial\theta}\right]^2\right], \tag{5.65 bis}$$

and the contribution of the η^{th} constraint to Q_3 is

$$Q_3^\eta = \rho\, \beta^\eta_{AB}(\mathbf{n}\cdot\mathbf{e}_A)(\mathbf{n}\cdot\mathbf{e}_B) = 0$$

by (5.118) or (8.21), so ν_3 is independent of these constraints.

We now investigate $\hat{\mathcal{A}}$ with the aid of the transformation (8.47) to thermal variables. $\mathcal{A}$ is unconstrained, by the conclusions to Appendix A, as is $\mathbf{\Gamma}^0$ by (2.95), and so is μ by (5.116), since $\omega^\eta = 0$. Finally, we require an expression for $\frac{\partial}{\partial\theta}\,(A^\eta)$; this is also zero by (4.21) and (2.68). Hence $\hat{\mathcal{A}}$ is unconstrained and so $\alpha(n)$ is also unconstrained. Clearly $\beta(n)$ is also unconstrained, since ν_3 and $\hat{\mathbf{A}}$ are unconstrained.

Finally, we investigate $\gamma_\Gamma(n)$ and find with the aid of the transformation (5.29) for $\hat{\mathbf{A}}$ and the expression (4.21) for $\mathbf{A}$ that the only constraint terms in the expression (8.56) for $\gamma_\Gamma(n)$ are

$$\beta^\eta_{AB}(\mathbf{p}_\Gamma\cdot\mathbf{e}_A)(\mathbf{p}_\Gamma\cdot\mathbf{e}_B) \quad , \tag{8.57}$$

where $\Gamma = 1$ (resp. 1,2) for cylindrical (resp. spherical) waves. The expression in (8.57) is not necessarily zero (see the discussion of (8.26)), and so we have the result for homentropic longitudinal waves propagating in media subject to constraints ϕ^η only that:

> plane waves and spherical waves are unconstrained, and cylindrical waves are unconstrained if all constraints ϕ^η obey
>
> $$\beta^\eta_{AB}(\mathbf{p}_1\cdot\mathbf{e}_A)(\mathbf{p}_1\cdot\mathbf{e}_B) = 0 \quad .$$

CHAPTER 9

CONCLUSIONS

This work is an extension of the analysis of plane waves in homogeneously deformed thermoelastic media subject to linearly independent type I and type II constraints presented in (I), but adopts the restriction of isotropy imposed in the preliminary investigation made in (II). The extensions made have been: the introduction of curved wavefronts and the removal of the general restriction to homogeneous deformation in the discussion of the growth equation; an extensive treatment of type I constraints for which the vectors $\mathbf{c}^a$ are linearly dependent; a new definition of a constrained non-conductor, and the recognition that for thermodynamic constraints specified with temperature as an independent variable, the thermal formulation is often appropriate to investigations of homentropic waves in constrained non-conductors.

These extensions have been achieved at the cost of a restriction imposed throughout on the set of arbitrary type I constraints from (I) namely that we consider only isotropic and directional constraints as defined by (2.66) and (2.68,9) respectively. A restriction is also imposed on the type II constraints set in the derivation of the homentropic growth equation, where we restrict attention to type II constraints for which $[\text{Grad }(\text{Div }(\gamma_\beta\ \hat{\mathbf{z}}^\beta))] = \mathbf{0}$. These restrictions are, however, not severe in the sense that most constraints commonly encountered in practice can be accommodated within the restricted sets; for instance we mention the constraint examples of Gurtin and Podio-

Guidugli (1973). We give four examples of type I constraints (similar to those of Gurtin and Podio-Guidugli referred to above), and present results for homothermal wave propagation in media subject to these constraints acting singly or in combination. Apart from the value of the results obtained as they stand, these constraints illustrate the fact that linear dependence of such constraints cannot be ignored, and in more general terms, illustrate the ease with which combinations of constraints can be studied within the general theory presented both here and in (I).

We note that underlying the above results is an extensive treatment for isotropic materials of the thermodynamic theory introduced in (I); we also incorporate the revised definition of a constrained non-conductor mentioned above. The use of an approach due to Durban (1978) for the derivation of the fourth- and sixth-order moduli of elasticity for constrained materials is also to be noted.

The work just described has highlighted the following areas as being worthy of further investigation. The definition of the constraints in (2.36,7) with temperature as an independent variable is ideally suited to the use of the thermal formulation, which is the natural choice for homothermal waves. For homentropic waves though, the entropic formulation is to be preferred for unconstrained materials. The possibility of treating homentropic waves in materials subject to type I and II constraints in either formulation therefore arises; the thermal formulation yields a more cumbersome treatment but has the advantage that constraint contributions are explicit, whereas the more concise entropic formulation tends to obscure constraint details that may very well be significant. Such difficulties are resolved if the constraints are presented with entropy as an independent variable, but it is debatable whether this is compatible with experimental techniques for the range of constraints considered here. It seems then, that the hybrid approach employed here is perhaps the most reasonable (though not ideal) approach at present.

We now turn to the questions raised by the presence of type I constraints for which the corresponding constraint vectors $\mathbf{c}^{\alpha}$ are dependent. The investigation of the propagation conditions proceeds relatively unimpeded for both homothermal and homentropic waves, but for the growth equations, the indeterminacy of the jumps $[\dot{\lambda}_{\alpha}]$ is a major difficulty. These problems are less severe for homothermal waves, but for homentropic waves these terms proliferate in both the longitudinal and transverse growth equations. A similar difficulty arises in the case of type II constraints, but is relatively easily removed by considering only the subset of the type II constraints mentioned above, with little loss of generality. It is difficult to see where one could obtain further information regarding $[\dot{\lambda}_{\alpha}]$ and $[\dot{\gamma}_{\beta}]$, since we have already made use of the constraint equations, the equations of motion and the energy equation, as well as first derivatives of these.

Apart from the above topics, at least two further areas of investigation would be of interest. It would be valuable to have a restricted version of the theory that assumed constraints to be <u>mechanical</u> from the outset, among the many simplifications which would follow from this is the consequence that all non-homothermal waves would also obey the type I propagation condition in the form $\mathbf{c}^{\alpha}\cdot\mathbf{s} = 0$. In this thesis of course, we have assumed constraints to be thermomechanical unless forced to do otherwise. Finally, it would be valuable to extend the investigation of the growth equation to allow wavefronts of arbitrary curvature.

It seems then that the original theory developed by Reddy (I) has provided a substantial and adaptable base on which to build the work of this investigation, and that there is considerable potential for further development.

APPENDIX A

Derivations of the components of the tensors $\mathbf{A}$ and $\mathcal{A}$ relative to the basis of proper vectors have been given by Chadwick and Ogden (1971a,b), (see also Ogden (1984)), and by Bowen and Wang (1970,1972) for unconstrained materials. Durban (1978) has given an alternative treatment, as an application of his results pertaining to the differentiation of tensor functions. While the methods of Chadwick and Ogden and of Bowen and Wang may be trivially extended to cover <u>isotropically</u> constrained materials, for directional constraints the procedure becomes extremely cumbersome. Durban's method, on the other hand, appears more suitable, and we use his formalism as a basis for deriving representations for tensors $\mathbf{A}$ and $\mathcal{A}$ associated with arbitrarily constrained materials. The procedure will be to derive the components of the tensors $\mathbf{L}$ and $\mathcal{L}$, defined by

$$\mathbf{L} = \partial \mathrm{T}/\partial \mathrm{C} \quad , \quad \mathcal{L} = \partial^2 \mathrm{T}/\partial \mathrm{C} \partial \mathrm{C} \quad , \tag{A.1}$$

and then to obtain the components of $\mathbf{A}$ and $\mathcal{A}$ from the identities (Chadwick and Ogden (1971a,b), Marsden and Hughes (1983))

$$A_{<ijkl>} = T_{<jl>} \, \delta_{<ik>} + 2a_i a_k \, L_{<ijkl>} \tag{A.2}$$

and

$$\mathcal{A}_{<ijklmn>} = 4 \, a_i a_k a_m \, \mathcal{L}_{<ijklmn>} + 2 \, a_i L_{<ijnl} \, \delta_{km}$$

$$+ 2 \, a_k L_{<njkl>} \, \delta_{im} + 2 \, a_m L_{<ljmn>} \, \delta_{ik} \tag{A.3}$$

where <...> denotes components relative to the principal basis. Here $\mathbf{T}$ and $\mathbf{C}$ are, respectively, the second Piola-Kirchhoff stress and right Cauchy-Green tensors, defined by

$$\mathbf{T} = \mathbf{S}^T\mathbf{F}^{-T} \quad , \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} \quad ; \tag{A.4}$$

relative to the principal basis $\mathbf{p}_i$ these tensors are given by

$$\frac{1}{2}\rho^{-1}\mathbf{T} = \sum_i \frac{\partial\psi}{\partial c_i}\, \mathbf{p}_i \otimes \mathbf{p}_i + \frac{\partial\psi}{\partial f_{AB}}\, \mathbf{e}_A \otimes \mathbf{e}_B \quad , \quad \mathbf{C} = \sum_i c_i\, \mathbf{p}_i \otimes \mathbf{p}_i \quad , \tag{A.5}$$

where $c_i = a_i^2$ and the strain energy function ψ is assumed <u>independent</u> of k_{AB} .

We start by setting

$$\mathbf{N}_i = \mathbf{p}_i \otimes \mathbf{p}_i \quad , \quad \mathbf{S}_i = \mathbf{p}_j \otimes \mathbf{p}_k + \mathbf{p}_k \otimes \mathbf{p}_j \quad ,$$

$$\mathbf{E}_{AB} = \mathbf{e}_A \otimes \mathbf{e}_B \quad , \tag{A.6}$$

where in the second equation i,j,k form a cyclic permutation. Then with these tensors available we define the identity tensor $\mathbf{I} \in \mathcal{T}^4$ by

$$\mathbf{I} : \mathcal{T}^2 \to \mathcal{T}^2 \quad , \quad \mathbf{IT} \text{ for all } \mathbf{T} \in \mathcal{T}^2 \quad ,$$
$$\mathbf{I} = \mathbf{N}_i \otimes \mathbf{N}_i + \frac{1}{2}\mathbf{S}_i \otimes \mathbf{S}_i \quad . \tag{A.7}$$

From $(A.5)_1$ we have

$$\frac{1}{2}\,\rho^{-1}\dot{\mathbf{T}} = \frac{\partial^2\psi}{\partial c_i\partial c_j}\,\dot{c}_j\mathbf{N}_i + \frac{\partial\psi}{\partial c_i}\,\dot{\mathbf{N}}_i + \frac{\partial^2\psi}{\partial c_i\partial f_{AB}}\,\dot{f}_{AB}\dot{\mathbf{N}}_i$$

$$+ \frac{\partial^2\psi}{\partial c_i\partial f_{AB}}\,\dot{c}_i\mathbf{E}_{AB} + \frac{\partial^2\psi}{\partial f_{AB}\partial f_{CD}}\,\dot{f}_{CD}\mathbf{E}_{AB} \quad . \tag{A.8}$$

where we have used the fact that $\dot{\mathbf{e}}_A = 0$, so $\dot{\mathbf{E}}_{AB} = 0$.

Now

$$\dot{\mathbf{C}} = \dot{c}_j\mathbf{N}_i + c_j\dot{\mathbf{N}}_i \tag{A.9}$$

and it can be shown (Durban (1978)) that

$$\dot{c}_i = \mathbf{N}_i\cdot\dot{\mathbf{C}} \quad , \quad \dot{f}_{AB} = \mathbf{E}_{AB}\cdot\dot{\mathbf{C}} \quad , \tag{A.10}$$

$$\dot{\mathbf{N}}_i = 1/2\left[\frac{\mathbf{S}_j\otimes\mathbf{S}_j}{c_i - c_k} + \frac{\mathbf{S}_k\otimes\mathbf{S}_k}{c_i - c_j}\right]\dot{\mathbf{C}} \quad , \tag{A.11}$$

$$\dot{\mathbf{S}}_i = \left\{-\left[\frac{\mathbf{N}_j\otimes\mathbf{N}_k}{c_j - c_k}\right]\otimes\mathbf{S}_i - 1/2\left[\frac{\mathbf{S}_j\otimes\mathbf{S}_k}{c_i - c_j} + \frac{\mathbf{S}_k\otimes\mathbf{S}_j}{c_i - c_k}\right]\right\}\dot{\mathbf{C}} \tag{A.12}$$

where i,j,k are cyclic and there is no summation on repeated indices. Hence (A.8) is easily rewritten as

$$\dot{\mathbf{T}} = \mathbf{L}(\dot{\mathbf{C}}) \tag{A.13}$$

where

$$\rho^{-1}\mathbf{L} \equiv \rho^{-1}\frac{\partial \mathbf{T}}{\partial \mathbf{C}} = 2\frac{\partial^2\psi}{\partial c_i \partial c_j}\mathbf{N}_i \otimes \mathbf{N}_j + 1/2\sum_i \sigma_i \mathbf{S}_i \otimes \mathbf{S}_i$$

$$+ 2\frac{\partial^2\psi}{\partial c_i \partial f_{AB}}(\mathbf{N}_i \otimes \mathbf{E}_{AB} + \mathbf{E}_{AB} \otimes \mathbf{N}_i)$$

$$+ \frac{\partial^2\psi}{\partial f_{AB}\partial f_{CD}}\mathbf{E}_{AB} \otimes \mathbf{E}_{CD} \quad ,$$

and where

$$\sigma_i = \frac{T_{\langle jj\rangle} - T_{\langle kk\rangle}}{c_j - c_k} \quad . \tag{A.14}$$

Here, as before, i,j,k form a cyclic permutation. Using (A.2) we recover the expression (4.21) for $\mathbf{A}$. For the restricted class of type I constraints defined in (2.68,9), none of the directional constraint terms in (A.14) survive. The directional constraint terms appearing in the expression (4.21) for $\mathbf{A}$ arise from the term $T_{\langle j\ell\rangle}\delta_{ik}$ in the transformation (A.2) .

To obtain the tensor of sixth order moduli, we use the identity

$$\dot{\mathbf{L}} = \frac{\partial \mathbf{L}}{\partial \mathbf{C}}\dot{\mathbf{C}} = \mathcal{L}(\dot{\mathbf{C}}) \quad ;$$

$\dot{\mathbf{L}}$ is found by differentiating (A.14) and is

$$\rho^{-1}L \equiv 2(\partial^3\psi/\partial c_i\partial c_j\partial c_k)\dot{c}_k\, N_i \otimes N_j + 2(\partial^3\psi/\partial c_i\partial c_j\partial f_{AB})\dot{f}_{AB}\, N_i \otimes N_j$$

$$+ 2\ \partial^2\psi/\partial c_i\partial c_j\ (N_i \otimes N_j)^{\cdot} + 1/2\sum_i \dot{\sigma}_i\, S_i \otimes S_i + \sum_i \sigma_i\, (S_i \otimes S_i)^{\cdot}$$

$$+ 2(\partial^3\psi/\partial c_i\partial c_j\partial f_{AB})\dot{c}_j(N_i \otimes E_{AB} + E_{AB} \otimes N_i)$$

$$+ 2(\partial^3\psi/\partial c_i\partial f_{AB}\partial f_{CD})\dot{f}_{CD}(N_i \otimes E_{AB} + E_{AB} \otimes N_i)$$

$$+ 2\ \partial^2\psi/\partial c_i\partial f_{AB})(\dot{N}_i \otimes E_{AB} + E_{AB} \otimes \dot{N}_i)$$

$$+ 2(\partial^3\psi/\partial c_i\partial f_{AB}\partial f_{CD})\dot{c}_i(E_{AB} \otimes E_{CD})$$

$$+ 2(\partial^3\psi/\partial f_{AB}\partial f_{CD}\partial f_{EF})\dot{f}_{EF}(E_{AB} \otimes E_{CD}) \quad . \qquad (A.15)$$

We now evaluate the time derivatives using (A.10) - (A.12) and the definition of σ_i to obtain the following expression for $\mathcal{L}$:

$$\rho^{-1}\mathcal{L} = 2\,\frac{\partial^3\psi}{\partial c_i\partial c_j c_k}\, N_i \otimes N_j \otimes N_k$$

$$+ \frac{1}{2}\sum_i \frac{\partial\sigma_i}{\partial c_m}\,(N_m \otimes S_i \otimes S_i + S_i \otimes N_m \otimes S_i + S_i \otimes S_i \otimes N_m)$$

$$+ \frac{1}{2}\sum_i a_{im}\, N_m \otimes S_i \otimes S_i + \frac{1}{4}\,\beta \sum_{i,j,k\neq} S_i \otimes S_j \otimes S_k$$

$$+ 2\,\frac{\partial^3\psi}{\partial c_i\partial c_j\partial f_{AB}}\,(N_i \otimes N_j \otimes E_{AB} + N_i \otimes E_{AB} \otimes N_j + E_{AB} \otimes N_i \otimes N_j)$$

$$+ 2\,\frac{\partial^3\psi}{\partial c_i\partial f_{AB}\partial f_{CD}}\,(N_i \otimes E_{AB} \otimes E_{CD} + E_{CD} \otimes N_i \otimes E_{AB} + E_{AB} \otimes E_{CD} \otimes N_i)$$

$$+ 2 \frac{\partial^3 \psi}{\partial f_{AB} \partial f_{CD} \partial f_{EF}} E_{AB} \otimes E_{CD} \otimes E_{EF}$$

$$+ 2 \frac{\partial^2 \psi}{\partial c_i \partial f_{AB}} \left\{ \left[\frac{S_j \otimes S_j}{c_i - c_k} + \frac{S_k \otimes S_k}{c_i - c_j} \right] \otimes E_{AB} \right.$$

$$+ \frac{S_j \otimes E_{AB} \otimes S_j}{c_i - c_k} + \frac{S_k \otimes E_{AB} \otimes S_k}{c_i - c_j}$$

$$\left. + E_{AB} \otimes \left[\frac{S_j \otimes S_j}{c_i - c_k} + \frac{S_k \otimes S_k}{c_i - c_j} \right] \right\}$$

in which

$$\alpha_{im} = \frac{\frac{\partial T_{<mm>}}{\partial c_j} - \frac{\partial T_{<jj>}}{\partial c_m} - \frac{\partial T_{<mm>}}{\partial c_k} - \frac{\partial T_{<kk>}}{\partial c_m}}{c_j - c_k}$$

and

$$\beta = \sum_i \frac{T_{<ii>}}{(c_i - c_j)(c_i - c_k)} \quad ,$$

$$i,j,k \text{ cyclic.} \tag{A.16}$$

As in the case of the expression for **L** , there are limiting values for those terms involving $(c_i - c_j)^{-1}$ when $c_i = c_j$; such modifications follow those given by Chadwick and Ogden (1971b) and by Bowen and Wang (1970). The components of $\mathcal{A}$ relative to the principal basis are obtained using the transformation (A.3) plus the appropriate terms from (A.14), but for the sake of brevity we do not quote the expression for $\mathcal{A}_{<ijklmn>}$ here.

Finally, we observe that for the restricted subset of type I directional constraints defined by (2.68,9), $\mathcal{L}$ and $\mathcal{A}$ contain no terms due to directional constraints.

APPENDIX B

In this Appendix we present details of the components of the tensor Grad $\mathbf{F}$, where $\mathbf{F}$ is the deformation gradient tensor.

By definition

$$\text{Grad } \mathbf{F} = \frac{\partial \mathbf{F}}{\partial X^j} \otimes \mathbf{G}^j$$

and since $\mathbf{F} = \sum_i a_i \, \mathbf{q}_i \otimes \mathbf{p}_i$,

$$\frac{\partial \mathbf{F}}{\partial X^j} = \sum_i \left\{ \frac{\partial a_i}{\partial X^j} \mathbf{q}_i \otimes \mathbf{p}_i + a_i \left[\frac{\partial \mathbf{q}_i}{\partial x^k} \frac{\partial x^k}{\partial X^j} \otimes \mathbf{p}_i + \mathbf{q}_i \otimes \frac{\partial \mathbf{p}_i}{\partial X^j} \right] \right\} .$$

Assuming the coordinates X^i to be principal coordinates and the deformation to be irrotational (these assumptions are made in the main part of the thesis) we have

$$\mathbf{q}_i = \frac{\mathbf{g}_i}{|\mathbf{g}_i|} \Rightarrow \frac{\partial \mathbf{q}_i}{\partial x^k} = |\mathbf{q}_i|^{-1} \, \gamma_{ij}^{\ell} \mathbf{g}_{\ell} (1 - \delta_{i\ell})$$

using the fact that

$$\partial \mathbf{g}_i / \partial x^j = \gamma_{ijk} \mathbf{g}^k = \gamma_{ij}^k \mathbf{g}_k \quad .$$

γ_{ijk} and γ_{ij}^k being Christoffel symbols relative to the coordinates x^i .

Similarly,

$$\frac{\partial \mathbf{p}_i}{\partial X^j} = |\mathbf{G}_i|^{-1}\Gamma^{\ell}_{ij}\mathbf{G}_{\ell}(1-\delta_{i\ell})$$

where Γ^k_{ij} and Γ_{ijk} are Christoffel symbols relative to the coordinates X^i . It is not difficult to work out that

$$\frac{\partial x^k}{\partial X^k} = a_k|\mathbf{g}_k|^{-1}|\mathbf{G}_k| \quad , \quad \frac{\partial x^k}{\partial X^{\ell}} = 0 \quad \text{for } k \neq \ell \quad ,$$

so that

$$\text{Grad } \mathbf{F} = \sum_{i,j} \left\{ \frac{\partial a_i}{\partial X^j} \mathbf{q}_i \otimes \mathbf{p}_i + a_i a_j |\mathbf{g}_j|^{-1}|\mathbf{G}_j|(|\mathbf{g}_i|^{-1}|\mathbf{g}_{\ell}|\gamma^{\ell}_{ij}\mathbf{q}_{\ell} \right.$$

$$\left. - \gamma^i_{ij}\mathbf{q}_i) \otimes \mathbf{p}_i + a_i\mathbf{q}_i \otimes (|\mathbf{G}_i|^{-1}|\mathbf{G}_{\ell}|\Gamma^{\ell}_{ij}\mathbf{p}_{\ell} - \Gamma^i_{ij}\mathbf{p}_i)\right\} \otimes \mathbf{p}_j|\mathbf{G}_j|^{-1}$$

and

$$F_{\langle mn;j\rangle} = \text{Grad } \mathbf{F}(\mathbf{q}_m,\mathbf{p}_n,\mathbf{p}_j)$$

$$= \frac{\partial a_m}{\partial X^j}\delta_{mn}|\mathbf{G}_i|^{-1} + a_n a_j|\mathbf{g}_j|^{-1}(|\mathbf{g}_n|^{-1}|\mathbf{g}_m|\gamma^m_{nj}$$

$$- \gamma^n_{nj}\delta_{mn}) + a_m|\mathbf{G}_j|^{-1}(|\mathbf{G}_m|^{-1}|\mathbf{G}_n|\Gamma^n_{mj} - \Gamma^m_{mj}\delta_{mn}) \quad .$$

The components of $F^{+}_{<mn;j>}$ required in (6.34) are now easily obtained: for example, for spherically symmetric deformation we have (see (4.69,70))

$$F_{<13;1>} = a_3 a_1 |g_3|^{-1} \gamma^1_{31} + a_1 |G_1^{-2}| |G_3| \Gamma^3_{11}$$

$$= a_3 a_1 r^{-1} + a_1 (R\cos\Phi)^{-2} (-R\cos^2\Phi)$$

$$= (a_3 - a_1)/R \quad .$$

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