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UNIVERSITY OF CAPE TOWN

Department of Mathematics
and Applied Mathematics

Function Spaces and a Problem of Banach

by

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To my Parents

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Declaration

The work done in this thesis is of original work by the author and concepts, ideas and results of others are duly expressed.

Abstract

Function spaces have been a useful tool in probing the convergence of sequences of functions. The theory seems to have been triggered off by the works of Ascoli [36], Arzelà [37] and Hadamard [38]. In this thesis, we consider the space of continuous functions from a topological space X into the reals $\mathbb{R}$, which we denote $C(X)$. Natural topologies such as the compact–open and the topology of pointwise convergence, i.e. $C_k(X)$ and $C_p(X)$ respectively, have been extensively studied. Fox [39] introduced the compact–open topology yet it was actually first studied by Arens [40] who called it the k–topology.

An interesting component of Function Space Theory is the interaction of the properties of the base space X with those of $C(X)$. More specifically, the characterization of the properties of $C(X)$ in terms of the topological properties of X. A study of these interactions in a general setting can be found in [23]. In this thesis, we study three function space topologies on $C(X)$: The topology of pointwise convergence on a dense set D in X, the lower semifinite and the weak compact–open topology denoted as $C_D(X)$, $C_l(X)$, $C_\omega(X)$ respectively.

The motivation for the investigation of these weak function space topologies was to consider a problem of Banach on a function space level. For a metric space X it can be stated as follows (see [17], [41], [42]): When does a metric space have a coarser compact metrizable topology? Function spaces are in general non–compact. Consequently, we would need to weaken these topologies to study the problem. Trivially, the indiscrete topology is a partial solution yet clearly an unsatisfactory one. Interestingly, most function spaces carrying completeness properties reduce to $\mathbb{R}^\omega$ which has a compact metrizable contraction.

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We begin in Chapter 1 by looking at initial topologies and their connection to complete regularity. As Banach's problem is one of our main concerns, which entails the search for a coarser compact metrizable topology from a given topology, it makes sense to look into this separation property. We also touch on compactness. Compactness is used as a tool to define the compact–open topology on $C(X)$ as our topologies are weaker than the compact–open topology. This chapter serves as an overture to $C(X)$ and centres around some classical results.

Chapter 2 is an investigation into the topology of pointwise convergence determined on a set D in X i.e. $C_D(X)$. Among various other topological properties, we look at separability, countability and networks to mention a few.

Chapter 3 is also investigative on the lower semifinite topology on $C(X)$ i.e. $C_l(X)$. Topological properties are studied.

Chapter 4 is along the same line as the previous two chapters but on the weak topology of $C_k(X)$.

Chapter 5 revolves on Banach's problem. We explore systematically a few results pertaining to it on topological spaces. Kulpa showed that every homogenous Polish space has a compact metrizable contraction. Armed with this and also with various other results, we move to function spaces to see if it is feasible to obtain spaces that have compact metrizable contractions.

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CHAPTER 1

Initial Topologies, Complete Regularity and Compactness

1.1 Initial Topologies generated by a family of maps

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As an introductory chapter, we lay out classical results which show how the separation property of complete regularity is inextricably connected to the spaces of continuous functions. Since one of the main themes of this thesis is to investigate Banach's problem which involves the discovery of a coarser compact metrizable topology from a given metrizable topology, we take the opportunity to discuss compactness. We begin by looking at initial topologies.

Let X be a non-empty set, $\{X_\alpha \mid \alpha \in \Gamma\}$ a family of topological spaces and $\{f_\alpha : X \to X_\alpha \mid \alpha \in \Gamma\}$ a family of maps from X to X_α where Γ is an index set. Define a topology τ_ω on X by taking as a subbase for the open sets the collection $\mathcal{S} = \{f_\alpha^{-1}(V_\alpha) \mid \alpha \in \Gamma\}$, where V_α is open in X_α for every $\alpha \in \Gamma$. It has the following useful property.

1.1.1 Proposition. τ_ω is the coarsest topology on X that makes each f_α continuous.

Keeping the above proposition in mind, the following definition makes sense:

1.1.2 Definition. The topology τ_ω generated by the maps $\{f_\alpha : X \longrightarrow X_\alpha\}$ is called the initial topology induced on X by the maps $\{f_\alpha : X \longrightarrow X_\alpha\}$.

The topology τ_ω is often referred to as the weak topology generated by the family of maps $\{f_\alpha \mid \alpha \in \Gamma\}$. This conflicts with the functional analysts notion of "weak topology"and hence we will therefore use the term "initial topology" instead. Another way of looking at τ_ω is that it is the intersection of all topologies on X with respect to which all the f_α's are continuous mappings.

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A useful result:

1.1.3 Proposition. Let X have the initial topology τ_ω induced by the family of maps $\{f_\alpha : X \longrightarrow X_\alpha \mid \alpha \in \Gamma\}$ and let Y be a topological space. Then a function $g: Y \longrightarrow X$ is continuous iff $f_\alpha \circ g$ is continuous for every $\alpha \in \Gamma$.

Let $\{X_\alpha \mid \alpha \in \Gamma\}$ be a collection of non-empty sets where Γ is an index set. The cartesian product of the X_α's denoted by $\prod_{\alpha\in\Gamma} X_\alpha$ or $\prod\{X_\alpha \mid \alpha \in \Gamma\}$ is defined by $\prod_{\alpha\in\Gamma} X_\alpha = \{x : \Gamma \longrightarrow \bigcup_{\alpha\in\Gamma} X_\alpha \mid x(\alpha) \in X_\alpha \text{ for every } \alpha \in \Gamma\}$. We will denote the functional value $x(\alpha)$ by x_α and call it the α^{th} coordinate of x. It is known that the statement "$\prod_{\alpha\in\Gamma} X_\alpha \neq \emptyset$" is equivalent to the axiom of choice. Thus we assume that the products are non–empty. The map $\pi_\beta : \prod_{\alpha\in\Gamma} X_\alpha \longrightarrow X_\beta$ defined by $\pi_\beta(\langle x_\alpha \rangle_\alpha) = x_\beta$ is called the β^{th}–projection. Note that each projection is onto.

We now consider the product topology which is a special kind of topology generated by maps:

1.1.4 Definition. Let $\prod_{\alpha\in\Gamma} X_\alpha$ denote the cartesian product of the family of sets $\{X_\alpha : \alpha \in \Gamma\}$,where Γ is an index set and $X_\alpha \neq \emptyset$ for each $\alpha \in \Gamma$. For each $\beta \in \Gamma$ denote the β-th projection by π_β. If $\{X_\alpha : \alpha \in \Gamma\}$ is a family of topological spaces, the (cartesian) produact topology on $\prod X_\alpha$ is the initial topology induced by the projection maps π_β.

Note that if X is a non-empty set and Y is a topological space , then the set Y^X of all functions $f : X \longrightarrow Y$ is simply the product topology $\prod_{x\in X} Y_x$, where $Y_x = Y$ for every $x \in X$.

Convergence in Y^X is simply pointwise convergence as the next stated proposition conveys.

1.1.5 Proposition. A net $(f_\alpha)_\alpha$ in Y^X converges to some $f \in Y^X$ in the product topology iff $(f_\alpha(x))_\alpha$ converges pointwise to $f(x)$ in Y for every $x \in X$.

It is no surprise that from proposition 1.1.5, the product topology on Y^X is called the topology of pointwise convergence.

Clearly $C(X)$ is a subspace of $\mathbb{R}^{\mathbb{X}}$. It would be fitting to introduce a set–open

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topology here before we proceed. The point–open topology on $C(X)$, denoted $C_p(X)$, is given by subbasic open sets of the form $[A, V] = \{f \in C(X) : f(A) \subset V\}$ where A is finite in X and V open in $\mathbb{R}$. It can be shown easily that $C_p(X)$ is actually the product topology that $C(X)$ inherits from $\mathbb{R}^X$. In fact we know more as the following theorem states.

1.1.6 Definition. A subspace A of X is said to be C–embedded (C^*–embedded) in X iff every continuous function $f : A \longrightarrow \mathbb{R}$ ($f : A \longrightarrow I$) can be extended to a continuous function $F : X \longrightarrow \mathbb{R}$ ($F : X \longrightarrow I$) where $I = [0, 1]$. We say that F is an extension of f.

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1.1.7 Theorem. Let X be a space in which every finite subset is C–embedded. Then $C_p(X)$ is a dense subspace of $\mathbb{R}^X$.

It can be proved that in any completely regular space (i.e a T_1-space such that if for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exits a continous function $f : X \longrightarrow [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in F$), the compact sets are C–embedded. Using this we have the following result:

1.1.8 Corollary. If X is completely regular then $C_p(X)$ is dense in $\mathbb{R}^X$. □

Let $\{f_\alpha \mid \alpha \in \Gamma\}$ be a family of maps such that $f_\alpha : X \longrightarrow X_\alpha$ with X_α a topological space. Form the product space $\prod X_\alpha$. It would be interesting to know the conditions that should prevail on X to embed it in $\prod X_\alpha$. Firstly we need two definitions:

1.1.9 Definition. The evaluation function generated by the f_α's is defined as $\epsilon \colon X \longrightarrow \prod_{\alpha \in \Gamma} X_\alpha$ operating as $\epsilon(x) = \langle f_\alpha(x) \rangle_\alpha$.

Observe that $f_\alpha = \pi_\alpha \circ \epsilon$ for every $\alpha \in \Gamma$.

1.1.10 Definition. A collection M of maps defined on a set X is said to **separate points** of X iff whenever $x \neq y$ in X, there exist $f \in$ M such that $f(x) \neq f(y)$.

We are now in a position to answer the question posed above.

1.1.11 Theorem. $\epsilon : X \longrightarrow \prod_{\alpha \in \Gamma} X_\alpha$ is an embedding iff X has the initial topology induced by the f_α's and the family $\{f_\alpha \mid \alpha \in \Gamma\}$ separates points of X.

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1.2 Completely Regular Spaces

In this section, we show that a completely regular topology is an initial topology. Recall that a space X is said to be completely regular (or Tychonoff) if for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in F$. Note that this can be generalised to a function $f : X \to [a, b]$, $a, b \in \mathbb{R}$. A Tychonoff space is one that is T_1 and completely regular. We start by giving a result that shows the relationship between $C(X)$ and $C^*(X)$ (i.e. the space of continuous and bounded functions on X) via the initial topologies they generate on the base space X.

1.2.1 Lemma. Let X be a topological space. The families of maps $C(X)$ and $C^*(X)$ generate the same initial topology on X.

Since the bounded open intervals form a base for the natural topology of $\mathbb{R}$, a basic neighbourhood $W(x)$ for $x \in X$ in the initial topologies of X induced by $C(X)$ (or $C^*(X)$) can be represented as follows:

Let $\epsilon > 0$ and $U = \{t \in \mathbb{R} : | f(x) - t | < \epsilon\}$ which is open in $\mathbb{R}$. Then $W(x) = f^{-1}(U) = \{y \in X : | f(x) - f(y) | < \epsilon\} = \{y \in X : f(y) \in (f(x) - \epsilon, f(x) + \epsilon)\}$.

We now turn to complete regularity:

1.2.2 Theorem. Let X be a T_1 topological space with topology τ. Then τ is a Tychonoff topology on X iff τ is the initial topology generated by $C(X)$.

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A useful concept that links the space $C(X)$ and X are zero–sets of elements in $C(X)$ (see [11]) and it turns out to be another powerful tool in describing complete regularity as we will see.

1.2.3 Definition. Let X be a topological space and let $f \in C(X)$. The zero–set of f, denoted by $Z(f)$ is the set $Z(f) = \{x \in X : f(x) = 0\} = f^{-1}(0)$. The complement of $Z(f)$ in X is called the cozero–set of f. A subset A of X is called a zero–set of X (a cozero–set of X) iff $A = Z(f)\,(A = X \backslash Z(f))$ for some $f \in C(X)$.

We give the following notations that will make the forthcoming discussion more fluid.

Notations:

1. If $C \subset C(X)$ then $Z(C) := \{Z(f) : f \in C\}$
2. For any topological space X, $Z(X) := \{Z(f) : f \in C(X)\}$ and $Z^*(X) := \{Z(f) : f \in C^*(X)\}$. Let $f, g \in C(X)$. Then for any $x \in X$, we have the following:
3. $(f \vee g)(x) := \max\{f(x), g(x)\}$ and $(f \wedge g)(x) := \min\{f(x), g(x)\}$
4. $\mid f \mid (x) := \mid f(x) \mid$
5. Let $c \in \mathbb{R}$. Then c_X denotes the constant function that maps X to the real c.

We gather the following indispensable properties on zero–sets:

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1.2.4 Proposition. Let $f, g, c_X \in C(X)$ where X is a topological space and $x \in X$. Then

1. If $f(x) \neq 0$ for each $x \in X$, then $\frac{1}{f} \in C(X)$

2. $\mid f \mid \in C(X)$

3. $(f \vee g) = \frac{1}{2}(f + g + \mid f - g \mid)$ and $(f \wedge g) = \frac{1}{2}(f + g - \mid f - g \mid)$. Hence $(f \vee g), (f \wedge g) \in C(X)$.

4. $Z(f) = Z(\mid f \mid) = Z(f^n)$ for every $n \in \mathbb{N}$

5. $Z(fg) = Z(f) \cup Z(g)$

6. $Z(f^2 + g^2) = Z(\mid f \mid + \mid g \mid) = Z(f) \cap Z(g)$

7. $Z(f) = \bigcap_{n \in \mathbb{N}} \{x \in X : \mid f(x) \mid < \frac{1}{n}\}$. Hence every zero–set is a (closed) G_δ–set in X

8. $Z(X) = Z^*(X)$

9. $Z(f_0) = X$, and $Z(c_X) = \emptyset$ iff $c_X \neq 0$, where f_0 is the zero function.

10. $Z(f \wedge f_0) = Z(f - \mid f \mid) = \{x \in X : f(x) \geq 0\}$

11. $Z(f \vee f_0) = Z(f + \mid f \mid) = \{x \in X : f(x) \leq 0\}$

Zero sets are closed and for that very reason we confront topological spaces in terms of their respective bases formed by closed sets to make the interaction between the two concepts.

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1.2.5 Definition. A collection $\mathcal{J}$ of closed subsets of a topological space X is a base for the closed sets of X if and only if every closed set in X is an intersection of some members of $\mathcal{J}$. So, $\mathcal{J}$ is a base for the closed sets of X iff $\mathcal{J}^C = \{X \backslash F : F \in \mathcal{J}\}$ is a base for the open sets of X.

In fact, $\mathcal{J}$ is a base for the closed sets of X iff for each pair (x, C) where $x \in X$, $C \subset X$ closed and $x \notin C$, there exists a $F \in \mathcal{J}$ such that $C \subset F$ and $x \notin F$.

Back to complete regularity.

1.2.6 Theorem. Let X be a T_1–space. Then X is Tychonoff iff $Z(X)$ is a base for the closed sets of X.

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We now want to look at Tychonoff spaces from a geometric point of view.

1.2.7 Theorem. A topological space X is Tychonoff iff X is homeomorphic to a subspace of some cube (i.e a product of closed bounded intervals of $\mathbb{R}$).

To avoid certain pathologies such as the function space being isomorphic to the space of reals $\mathbb{R}$, we will always assume that the base space is completely regular. This assumption is supported by the following result.

1.2.8 Theorem. [31]. Let X be a topological space. Then there exists a completely regular space Y such that $C(X)$ is isomorphic to $C(Y)$.

Proof: Let X be as in the premise. Define an equivalence relation $\sim$ on X as follows: $x \sim y$ iff $f(x) = f(y)$ for every $f \in C(X)$, $x, y \in X$. Let $\widetilde{X}$ denote the collection of equivalence classes induced by $\sim$ and $\pi : X \longrightarrow \widetilde{X}$ be the canonical projection that takes any element $x \in X$ to its equivalence class $[x]$. Observe that every function $f \in C(X)$ generates a function $\tilde{f} \in C(\widetilde{X})$ such that $\tilde{f}([x]) = f(x)$, $[x] \in \widetilde{X}$. Induce the initial topology on $\widetilde{X}$ generated by the family $\{\tilde{f} : f \in C(X)\}$ and denote this topological space by Y. By theorem 1.2.2 we have Y to be completely regular. Define the function $\pi^* : C(Y) \longrightarrow C(X)$ by $\pi^*(\tilde{g}) = \tilde{g} \circ \pi$. It is easy to see that π^* is an isomorphism and we are finished. □

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1.3 Compactness and Filters

Using complete regularity, we now study compactness. However, we firstly introduce filters as a tool to study convergence.

1.3.1 Definition. Let X be a set and $\mathcal{J} \subset \mathcal{P}(X)$. Then $\mathcal{J}$ is a filter on X iff the following three conditions hold:

1. $\phi \notin \mathcal{J}$
2. $F_1, F_2 \in \mathcal{J} \Rightarrow F_1 \cap F_2 \in \mathcal{J}$
3. $F \in \mathcal{J}$ and $F \subset F' \Rightarrow F' \in \mathcal{J}$

Property (2) maintains that $\mathcal{J}$ is closed under finite intersections and (3) maintains that supersets of members of $\mathcal{J}$ are in $\mathcal{J}$.

1.3.2 Definition. Let X, $\mathcal{J}$ be as above.

$\mathcal{J}$ has the finite (countable) intersection property iff the intersection of a finite (countable) subfamily of $\mathcal{J}$ is non-empty. These properties will be denoted as f. i. p (c. i. p).

We continue to lay the foundation:

1.3.3 Definition. Let $\mathcal{J}$ and $\mathcal{J}'$ be filters on a topological space X and $x \in X$. If $\mathcal{J} \subset \mathcal{J}'$, we say that $\mathcal{J}'$ is finer than $\mathcal{J}$ or that $\mathcal{J}$ is coarser than $\mathcal{J}'$. An ultrafilter $\mathcal{J}$ is a maximal filter i.e. there is no filter strictly finer than $\mathcal{J}$.

$\mathcal{J}$ converges to x (i.e. $\mathcal{J} \longrightarrow x$) iff every neighbourhood of x is a member of $\mathcal{J}$. Example: $\mathcal{J}_x = \{U \subset X : U \text{ is a neighbourhood of } x\}$ is a filter on X converging to x.

Let $\mathcal{J}' \subset \mathcal{J}$. Then $\mathcal{J}'$ is a filterbase for $\mathcal{J}$ iff every element of $\mathcal{J}$ contains some element of $\mathcal{J}'$ i.e. $\mathcal{J} = \{F : F_0 \subset F \text{ for some } F_0 \in \mathcal{J}'\}$. Note that $\mathcal{J}'$ is a filter base for some filter on X iff (1) $\emptyset \notin \mathcal{J}'$ and (2) $F_1, F_2 \in \mathcal{J}' \Rightarrow \exists F_3 \in \mathcal{J}$ such that $F_3 \subset F_1 \cap F_2$. In this case the filter is given by $\mathcal{J} = \{F \subset X : F' \subset F \text{ for some } F' \in \mathcal{J}'\}$. Example: Let $\mathcal{J}$ be a filter on X and $f : X \longrightarrow Y$ a function. Then $f(\mathcal{J}) = \{f(F) \mid F \in \mathcal{J}\}$ is a filter base for some filter on Y.

Inevitably, the properties of filters in any topological space must not be overlooked:

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1. If $A \subset X$ then $x \in \bar{A}$ iff there exists a filter $\mathcal{J}$ on X such that $A \in \mathcal{J}$ and $\mathcal{J} \longrightarrow x$.

2. A filter $\mathcal{J}$ on X is an ultrafilter iff for every $A \subset X$, either $A \in \mathcal{J}$ or $X \backslash A \in \mathcal{J}$.

3. Every filter on X is contained in some ultrafilter on X.

4. Let X and Y be topological spaces and $f : X \longrightarrow Y$ a function, $x_0 \in X$. Then f is continuous at x_0 iff for each filter $\mathcal{J}$ in X converging to x_0 the filter generated by $f(\mathcal{J}) = \{f(F) : F \in \mathcal{J}\}$ converges to $f(x_0)$ in Y.

5. Let $\{X_\alpha : \alpha \in \Gamma\}$ be a family of topological spaces and $\prod_\alpha X_\alpha$ their Tychonoff product. Then a filter $\mathcal{J}$ converges to x in $\prod_\alpha X_\alpha$ iff the filter generated by $\pi_\alpha(\mathcal{J})$ converges to $\pi_\alpha(x) = x_\alpha$ in X_α for every $\alpha \in \Gamma$.

We are in a position to describe compactness using filters:

1.3.4 Theorem. Let X be a Hausdorff space. Then the following are equivalent.

1. X is a compact space

2. Every family of closed sets in X with the f.i.p has non–empty intersection

3. Every ultrafilter in X converges

Another important consequence is the following celebrated result:

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1.3.5 Theorem. (Tychonoff) Let $\{X_\alpha : \alpha \in \Gamma\}$ be a family of topological spaces. Then $\prod_{\alpha\in\Gamma} X_\alpha$ is compact iff X_α is compact for every $\alpha \in \Gamma$.

We now use compact sets to introduce another set–open topology on $C(X)$ called the compact–open topology $C_k(X)$. This is a topology with the subbasic open sets of the form $[A, V] = \{f \in C(X) : f(A) \subset V\}$, where A is compact in X and V open in $\mathbb{R}$. Note that $C_k(X)$ is also called the topology of uniform convergence on compact sets. We now look at compact sets in Tychonoff spaces. Firstly, the notion of complete separation.

1.3.6 Definition. Two sets A and B are completely separated in X iff there is $f\colon X \longrightarrow [0, 1]$ continuous such that $f(A) = 0$ and $f(B) = 1$.

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Take note that any two sets are completely separated in X iff their closures are completely separated too. Interestingly enough we have the following connection between zero–sets and complete separation:

1.3.7 Theorem. Two sets are completely separated iff they are contained in two disjoint zero–sets.

1.3.8 Corollary. Let X be Tychonoff. Then any two disjoint compact subspaces of X are completely separated.

To finish off we state the following two important results.

1.3.9 Theorem. (Urysohn's Extension Theorem) A subspace A of a space X is C^*–embedded in X iff every two sets completely separated in A are completely separated in X.

1.3.10 Theorem. Let X be Tychonoff. Then any compact subspace is C–embedded (C^*–embedded) in X.

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CHAPTER 2

A Function Space With The Pointwise Topology Determined by A Set

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The pointwise topology on $C(X)$ has been studied extensively and a very good overview can be found in [23]. As a generalisation of this topology, it would be interesting to see how the function space properties behave in relation to the base space X, when the function space topology is determined by a subset of X. A lot of results have been obtained in [6], where the determining set in X is a countable dense set, which mostly initiated an investigation on the Borel complexity and topological classification of such spaces. The work done in this chapter proceeds with the investigation of the topological properties on this function space. Let D be any subset of X where we take X to be Tychonoff. We define a topology on $C(X)$ by describing the basic open sets as sets of the form $\langle f, A, \epsilon\rangle := \{g \in C(X) : | f(x) - g(x) | < \epsilon \ \forall x \in A\}$, where A is a finite set in D, $f \in C(X)$ and $\epsilon > 0$; or by using subbasic open sets of the form

$[A, V] = \{g \in C(X) : g(x) \in V \; \forall x \in A\} = \bigcap_{a \in A} [\{a\}, V]$ where A is a finite subset in D and V open in $\mathbb{R}$. We will have $C_D(X)$ to denote the topological space with the underlying set $C(X)$ with the topology just described. In fact, $C_D(X)$ can be generated by semi–norms $P_A : C(X) \to \mathbb{R}$ defined by $P_A(f) := \sup\{| f(x) | : x \in A\}$ and so $C_D(X)$ is a locally convex topological vector space. Observe that $C_X(X) = C_p(X)$. We can view $C_D(X)$ as the topology of pointwise convergence on D. It is clear that for any $D \subset X$, we have that $C_D(X)$ is coarser than $C_p(X)$ and $C_p(X)$ is coarser than $C_k(X)$. It can be shown that $C_D(X) = C_p(X)$ if and only if $D = X$ and $C_D(X) = C_k(X)$ if and only if each compact set in X is finite and $X = D$.

As a generalisation of the topology described in the previous paragraph we have the following: If X and Y are topological spaces, $D \subset X$ and $F \subset Y^X$, define a topology on F by taking as a subbase all sets of the form $[A, U]$ where $A \subset D$ is finite and U is open. We denote this topological space by $F_D(X, Y)$.

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2.1 Separation Property

The compact–open and point–open topologies on $C(X)$, i. e. $C_k(X)$ and $C_p(X)$ respectively, are Tychonoff spaces irrespective of the base space X. As $C_D(X)$ is coarser than $C_p(X)$, it could well be expected that the separation properties of $C_D(X)$ could well begin to part from those of $C_p(X)$ as we see in the following proposition.

2.1.1 Proposition. The space $C_D(X)$ is a Hausdorff space iff D is dense in X.

Proof: Assume D is dense in X. Let $f, g \in C(X)$ such that $f \neq g$. So the set $U := \{x \in X : f(x) \neq g(x)\}$ is a nonempty open set. Hence there is $x_0 \in U \cap D$. Let V and W be disjoint open subsets of $\mathbb{R}$ such that $f(x_0) \in V$ and $g(x_0) \in W$. Then $f \in [\{x_0\}, V]$, $g \in [\{x_0\}, W]$ and $[\{x_0\}, V] \cap [\{x_0\}, W] = \emptyset$.

Conversely, suppose that $C_D(X)$ is Hausdorff and let U be an open subset of X containing some point x. Let $f \in C(X)$ such that $f(x) = 1$ and $f(X \setminus U) = 0$. Since f is not the zero function f_0, there is a finite set A in D and $\epsilon > 0$ such that $f \notin \langle f_0, A, \epsilon \rangle$. Since $f|_A \neq 0$, it means $U \cap A \neq \emptyset$ and so D is dense in X. □

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Henceforth, due to Proposition 2.1.1, we take D to be dense in X in order to obtain reasonable separation properties on $C(X)$.

2.2 Separability

We now determine a necessary and sufficient condition for $C_D(X)$ to be a separable space. It is known that the topology of pointwise convergence, i.e. $C_p(X)$, is separable iff there exists a one–to–one continuous function from X into a separable metric space [23, 35]. For our situation, we need a property that is weaker than one–to–one and so to this end we define the following notion.

2.2.1 Definition. Let X and Y be topological spaces and D be dense in X. We say a function $\Phi\colon X \to Y$ is D–injective if Φ restricted to D is injective.

For each continuous function $\Phi\colon X \to Y$, the induced map $\Phi^*\colon C(Y) \to C(X)$ is defined by $\Phi^*(f) = f \circ \Phi$ for each $f \in C(Y)$. This map serves as a tool in studying separability. Its relationship with the mother map $\Phi\colon X \to Y$ is as follows:

2.2.2 Proposition. A continuous function $\Phi : X \to Y$ is D–injective iff $\Phi^*(C(Y))$ is dense in $C_D(X)$.

Proof: Suppose Φ is D–injective. Let $W = \langle h, A, \epsilon\rangle$ be a subbasic open set in $C_D(X)$ where $h \in C(X)$, A is a finite set in $D \subset X$ and $\epsilon > 0$. Let $\Phi(A) := B$. Then $\Phi\,|_A\colon A \to B$ is bijective and since A is finite, $(h\,|_A) \circ (\Phi\,|_A)^{-1}$ has an extension $g \in C(Y)$ (Theorem 1.3.10). Now $\Phi^*(g) \in W$ since for each $a \in A$ we have $\Phi^*(g)(a) = g(\Phi(a)) = h(a)$. Hence $W \cap \Phi^*(C(Y)) \neq \emptyset$.
Suppose $\Phi^*(C(Y))$ is dense in $C_D(X)$ and let $x, x_0 \in D \subset X$ such that $x \neq x_0$. Choose $f \in C(X)$ such that $f(x) \neq f(x_0)$ and let V, V_0 be disjoint open subsets of $\mathbb{R}$ that contain $f(x), f(x_0)$ respectively. Since $f \in [\{x\}, V] \cap [\{x_0\}, V_0]$ in $C_D(X)$ there is a $g \in C(Y)$ such that $\Phi^*(g)(x) \in V$ and $\Phi^*(g)(x_0) \in V_0$ implying $\Phi(x) \neq \Phi(x_0)$. So Φ is D–injective. □

The following theorem characterizes the separability of $C_D(X)$ in a way very similar to that of $C_p(X)$ but now using the notion of D–injectivity of a map $\Phi\colon X \to Y$ where D is dense in X.

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2.2.3 Theorem. The space $C_D(X)$ is separable iff there exists a D–injective continuous function from X into a separable metric space.

Proof: Let $\Phi: X \to Y$ be a D–injective continuous function from X into a separable metric space Y. Since $\Phi^*: C_p(Y) \to C_p(X)$ is continuous, $\Phi^*(C(Y))$ is a separable subspace of $C_p(X)$. Since $C_D(X)$ is coarser than $C_p(X)$ we have $C_D(X)$ is separable.

Conversely, let F be a countable dense subset of $C_D(X)$. Define $\Phi: X \to \mathbb{R}^{\mathbb{F}}$ by $\Phi(x)(f) = f(x)$ for $x \in X, f \in F$. Note that Φ is a continuous function into the separable metrizable space $\mathbb{R}^{\mathbb{F}}$ since each f is continuous. We finish if we show that Φ is D–injective. Let x, x_0 be distinct points in D. Let $g \in C(X)$ such that $g(x) \neq g(x_0)$ and V, V_0 be disjoint open sets of $\mathbb{R}$ containing $g(x)$, $g(x_0)$ respectively. So there is a $f \in F \cap [\{x\}, V] \cap [\{x_0\}, V_0]$. Then $f(x) \neq f(x_0)$ and so $\Phi(x) \neq \Phi(x_0)$. □

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2.3 Embedding Theorems

The following results involving $\Phi : X \to Y$ and its induced map with several other conditions serve to be very useful in studying embeddings between function spaces with topologies determined by dense sets.

2.3.1 Proposition. Let D and D' be dense in X and Y respectively and let $\Phi: X \to Y$ be continuous. Then $\Phi^*: C_{D'}(Y) \to C_D(X)$ is an embedding iff $\Phi(D) = D'$.

Proof: Suppose $\Phi(D) = D'$. We show that $\Phi^* : C_{D'}(Y) \to \mathcal{R}$ is a homeomorphism where $\mathcal{R} = \Phi^*(C_{D'}(Y))$. Let $[A, V]$ be a non–empty subbasic open set in $\mathcal{R}$ where A is a finite set in D and V is open in $\mathbb{R}$. Then $(\Phi^*)^{-1}([A, V]) = [\Phi(A), V]$ which is open in $C_{D'}(V)$ since $\Phi(A) \subset D'$. We show injectivity. Suppose $f, g \in C(Y)$ are such that $\Phi^*(f) = \Phi^*(g)$. Then $f(\Phi^*(x)) = g(\Phi^*(x))$ for all $x \in X$, and so f and g agree on the dense subset D' of Y. Consequently $f = g$, and thus Φ^* is injective. Hence Φ^* and Φ^{*-1} are continuous. Consequently, Φ^* is an embedding.

Conversely, suppose that $\Phi^*: C_{D'}(Y) \to \mathcal{R}$, where $\mathcal{R} = \Phi^*(C_{D'}(Y))$, is a homeomorphism. We first show that $\Phi(D) \subset D'$. If $x \in D$, then $[\{x\}, (-1, 1)]$ is a neighbourhood of the zero function f_0 in $C_D(X)$. As Φ^* is continuous, there is a finite $B \subset D'$ and $\epsilon > 0$ such that $\Phi^*(\langle 0_Y, B, \epsilon\rangle) \subset [\{x\}, (-1, 1)]$ where 0_Y is the zero function in $C_{D'}(Y)$. We finish if we show that $\Phi(x) \in B$. Assume the contrary. Then there is a $g \in C(Y)$ such that $g(\Phi(x)) = 1$ and $g(B) = 0$. This means that $g \in \langle 0_Y, B, \epsilon\rangle$ and $\Phi^*(g) \not\in [\{x\}, (-1, 1)]$ which is a contra-

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diction to $g(\Phi(x)) = 1$. Hence $\Phi(D) \subset D'$. Now we show that $D' \subset \Phi(D)$. Well, let $y \in D'$. The open set $[\{y\}, (-1, 1)]$ is a neighbourhood of the the 0_Y in $C_{D'}(Y)$. Let $W := \langle f_0, A, \epsilon \rangle$, A a finite set in D, be a basic neighbourhood of the zero function f_0 in $C_D(X)$ such that $W \cap \Phi^*(C_{D'}(Y)) \subset \Phi^*([\{y\}, (-1, 1)])$. We finish if we show that $y \in \Phi(A)$. Assume the contrary and let $q \in C_D(Y)$ such that $q(y) = 1$ and $q(\Phi(A)) = 0$. Since $\Phi^*(q) \in \Phi^*([\{y\}, (-1, 1)]$, we have that there is a $h \in [\{y\}, (-1, 1)]$ such that $\Phi^*(h) = \Phi^*(q)$ and because Φ^* is injective we have $h = q$. Then $q \in [\{y\}, (-1, 1)]$ which is a contradiction. Hence $D' \subset \Phi(D)$. □

2.3.2 Corollary. Let D be dense in X and let $\Phi\colon X \to Y$ be a continuous dense map. Then $\Phi^*\colon C_{\Phi(D)}(Y) \to C_D(X)$ is an embedding.

For $Y \subset X$, we define $C(X \mid Y) := \{f \in C(Y) : f$ has a continuous extension to $X\}$.

2.3.3 Corollary. The space $C_D(X)$ is linearly homeomorphic to the dense subspace $C(X \mid D)$ of $C_p(D)$.

Proof: Let D be dense in X and $i : D \to X$ be the inclusion map which is a dense map. Then $i^*\colon C_D(X) = C_{i(D)}(X) \to C_D(D) = C_p(D)$ is a dense embedding of $C_D(X)$ into $C_p(D)$ (Proposition 2.2.2 and Corollary 2.3.2). Since $i^*(C(X)) = C(X \mid D)$ we have our result. □

On a similar note, we may extend Corollary 2.3.3 to the Hewitt realcompactification νX of a space X. Consider the inclusion map $j\colon X \to \nu X$. Then

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$j^*: C_X(\nu X) \to C_p(X)$ is an embedding. In fact, j^* is a homeomorphism since $j^*(C(\nu X)) = C(\nu X \mid X) = C(X)$. □

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2.4 Countability and Metrizability

Looking at the construction of the topology of $C_D(X)$, it is clear that $C_D(X)$ is second countable if D is countable. The converse is true as we will see later but first let us investigate a few countability properties.

2.4.1 Definition. A space X is said to have **countable tightness** if for each $x \in X$ and $A \subset X$ with $x \in \overline{A}$, there is a countable $B \subset A$ such that $x \in \overline{B}$.

2.4.2 Definition. A space X is said to be a **q–space** if each point of X has a sequence $(U_n)_{n \in \mathbb{N}}$ of neighbourhoods such that if $x_n \in U_n$ for each $n \in \mathbb{N}$, then the sequence $(x_n)_{n \in \mathbb{N}}$ has a cluster point in X.

2.4.3 Definition. Let D be a subset of X. An **ω_D–cover** of a space X is a family of subsets of X such that every finite subset of D is contained in some member of this family.

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We now proceed to investigate countable tightness.

2.4.4 Theorem. Let D be a subset of X. $C_D(X)$ has countable tightness iff every open ω_D–cover of X has a countable ω_D–subcover.

Proof: Assume $C_D(X)$ has countable tightness and let φ be an open ω_D–cover of X. Then for each finite $A \subset D$ there is a $U_A \in \varphi$ such that $A \subset U_A$. Choose a continuous function $f_A \in C(X)$ such that $f_A(A) = 0$ and $f_A(X \backslash U_A) = 1$. Let $F = \{f_A : A \text{ finite in } D\}$. Then the zero–function f_0 is contained in the closure of F in $C_D(X)$. Since $C_D(X)$ has countable tightness, there is a countable subset $F^* \subset F$ such that $f_0 \in \overline{F^*}$. Now let $S := \{U_A : f_A \in F^*\}$. We show that S is an ω_D–subcover of φ. Well, let A_1 be any finite set in D and $W := [A_1, (-1, 1)]$ which is a neighbourhood of f_0. Then for some B finite in D, there is an $f_B \in F^* \cap W$. It suffices to show that $A_1 \subset U_B$. For any $x \in A_1$, $| f_B(x) | < 1$ and for $x \in X \backslash U_B$, $f_B(x) = 1$ and so by the construction of f_B, we have $A_1 \subset U_B$. Hence S is a countable ω_D–subcover of φ.

Conversely, assume every open ω_D–cover has a countable ω_D–subcover and let G be a subset of $C_D(X)$ containing the zero–function f_0 in its closure. For each $n \in \mathbb{N}$ and finite $A \subset D$, choose $g_{n,A} \in G \cap [A, (-\frac{1}{n}, \frac{1}{n})]$ and set $W(n, A) = \{x \in X : | g_{n,A}(x) | < \frac{1}{n}\}$. So $W_n := \{W(n, A) : A \text{ finite}\}$ is an open ω_D–cover in X. Consequently, W_n has a countable ω_D–subcover V_n. Define $G^* = \{g_{n,A} : n \in \mathbb{N} \text{ and } W(n, A) \in V_n\}$. Clearly $G^* \subset G$ and $f_0 \in \overline{G^*}$. Hence $C_D(X)$ has countable tightness. □

As for the q–space property, we have the following lemma.

2.4.5 Lemma. If $C_D(X)$ is a q–space, then D is countable.

Proof: Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of neighbourhoods of the zero function f_0 in $C_D(X)$ that satisfies the q–space property at f_0. For each $n \in \mathbb{N}$ there exists a finite $A_n \subset D$ and $\epsilon_n > 0$ such that $\langle f_0, A_n, \epsilon_n \rangle \subset W_n$. Define $A := \bigcup_{n\in\mathbb{N}} A_n$ and assume there is an $x \in D \backslash A$. Then $x \notin A_n$ for each $n \in \mathbb{N}$ and there is an $f_n \in C(X)$ with $f_n(x) = n$ and $f_n(A_n) = 0$. So each $f_n \in W_n$ and hence the sequence $(f_n)_{n\in\mathbb{N}}$ must cluster in $C_D(X)$. This is a contradiction since

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$f_n(x) = n$ for each $n \in \mathbb{N}$. Hence $D = A$ and hence is countable. $\square$

2.4.6 Theorem. For any space X, $C_D(X)$ has G_δ–points iff D is separable.

Proof: Assume D is separable and let $A = \{x_1, ..., x_n, ...\}$ be a countable dense subset of D. Let $W_n := \langle f_0, \{x_n\}, \frac{1}{n}\rangle$ where f_0 is the zero function. Clearly W_n is a neighbourhood of f_0 in $C_D(X)$ and so we finish if we show $\{f_0\} = \bigcap_{n\in\mathbb{N}} W_n$. Assume there exists $f \in \bigcap_{n\in\mathbb{N}} W_n$ such that $f \neq f_0$. Then the set $U = \{x \in X \mid f(x) \neq 0\}$ is open in X and so there exists $x_m \in A \cap U$. Without loss of generality assume $f(x_m) > \frac{1}{n}$. Then $f \notin W_n$. So $f \notin \bigcap_{n\in\mathbb{N}} W_n$ and hence f can only be the zero function.

Conversely, assume $C_D(X)$ has G_δ–points. Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of $C_D(X)$ such that $\{f_0\} = \bigcap_{n\in\mathbb{N}} W_n$. For each $n \in \mathbb{N}$, there is a finite set $A_n \subset D$ and $\epsilon_n > 0$ such that $\langle f_0, A_n, \epsilon_n\rangle \subset W_n$. Define $A := \bigcup_{n\in\mathbb{N}} A_n$. Suppose there is a $x \in D\backslash\overline{A}$. Then choose a function $f \in C(X)$ such that $f(x) = 1$ and $f(\overline{A}) = 0$. Then $f \in \langle f_0, A_n, \epsilon_n\rangle$ for each $n \in \mathbb{N}$ and so $f \in \bigcap_{n\in\mathbb{N}} W_n = \{f_0\}$ which is a contradiction to $f(x) = 1$. So A is a countable dense set in D. $\square$

The following is an equivalence theorem analoguous to a theorem for pointwise convergence in [23].

2.4.7 Corollary. For any space X and any dense set D in X, the following are equivalent.

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(a) $C_D(X)$ is second countable

(b) $C_D(X)$ is separable metrizable

(c) $C_D(X)$ is a q–space

(d) D is countable.

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2.5 Completeness Properties

Since the topology of $C_D(X)$ is determined by the uniformity of pointwise convergence on a dense set D in X, we would like to find what property D must have to have the function space uniformly complete.

2.5.1 Theorem. For any Tychonoff space X and any dense D in X, the following are equivalent:

(a) $C_D(X)$ is uniformly complete

(b) $C_D(X)$ is uniformly homeomorphic to a power of $\mathbb{R}$

(c) $C(X \mid D)$ is homeomorphic to $\mathbb{R}^D$

(d) D is C–embedded and discrete in X

Proof: $(d) \Rightarrow (c)$: $C(X \mid D) = C(D) = \mathbb{R}^D$.

$(c) \Rightarrow (b)$: Follows from Corollary 2.3.3.

$(b) \Rightarrow (a)$: Clear.

$(a) \Rightarrow (d)$: Assume $C_D(X)$ is uniformly complete and let $f \in C(D)$. For each finite set A in D, the restriction of f to A has an extension $f_A \in C_D(X)$. So the net $(f_A)_A$ is Cauchy and by uniform completeness, there exists a limit $g \in C_D(X)$ which is the required extension of f to X. Hence D is C–embedded. Now $C_D(X)$ is linearly homeomorphic to the dense subspace $C(X \mid D)$ of $C_p(D) \subset \mathbb{R}^D$ (Corollary 2.3.3) and since $C_D(X)$ is complete we have that $C_p(D) = \mathbb{R}^D$. But this is only possible when D is discrete. □

Moving to complete metrizability, we state a few definitions.

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2.5.2 Definition . ([10],[23]). A **Čech–complete** space is a Tychonoff space that is a G_δ–subset of any Hausdorff space in which it is densely embedded.

2.5.3 Definition. ([23]). An **almost Čech–complete** space is a Tychonoff space having a dense Čech–complete subspace.

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2.5.4 Theorem. For any space X and any dense set D in X, the following are equivalent.

(a) $C_D(X)$ is almost Čech–complete

(b) $C_D(X)$ is Čech–complete

(c) $C_D(X)$ is separable and completely metrizable

(d) X is countable and discrete.

Proof: (d) $\Rightarrow$ (c): Clearly if X is countable and discrete then so is D and so $C_D(X) = C_p(D) = \mathbb{R}^D$.

(c)$\Rightarrow$ (b) $\Rightarrow$(a): (see [10]).

(a) $\Rightarrow$ (d): Assume $C_D(X)$ is almost Čech–complete. As $C(X \mid D)$ is densely embedded in $C_p(D)$ as a G_δ–subset ([10]), we have $C_p(D)$ almost Čech–complete. Consequently, D is countable and discrete [23]. Moreover since $C_D(X)$ is uniformly complete, we have that D is C–embedded in X (Theorem 2.5.1). From [11], a C–embedded countable set is closed and so $D = X$. □

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2.6 Networks on $C_D(X)$

We can get to understand the structure of $C_D(X)$ by studying networks on it. The following two definitions can be found in [23].

2.6.1 Definition. A ***k*–network** for a topological space X is any family τ of subsets in X such that if A is any compact set in X and U open in X such that $A \subset U$, then there exists an $N \in \tau$ with $A \subset N \subset U$. If 'compact' is replaced by 'singleton' we obtain the definition of **network**.

2.6.2 Definition. A regular space with a countable k–network (network) is called an **$\mathcal{X}_0$–space** (**cosmic space**).

It is interesting to note that cosmic spaces are precisely the continuous images of separable metrizable spaces and $\mathcal{X}_0$–spaces are compact–covering images of separable metrizable spaces (see [24]). (A map $f \in C(X, Y)$ is said to be compact– covering if each compact subset of Y is equal to the image of some compact subset of X). Note that subspaces of cosmic spaces are cosmic. Also if a topological space X is coarser than a cosmic space Y, then X is cosmic. From ([23],[24]), $C_p(X)$ is cosmic iff X is cosmic. For $C_D(X)$, we have the following analoguous result.

2.6.3 Theorem. For any Tychonoff space X and any dense set D in X, $C_D(X)$ is cosmic iff D is cosmic.

Proof: If D is cosmic, then $C_p(D)$ is cosmic. Since $C_D(X)$ is homeomorphic to $C(X \mid D) \subset C_p(D)$, we have that $C_D(X)$ is cosmic.

Conversely, assume $C_D(X)$ is cosmic and let $\mathcal{B}$ be a countable network in $C_D(X)$. For each $B \in \mathcal{B}$, define

$$B^* := \{x \in X : f(x) > 0 \text{ for all } f \in B\}$$

$$\text{and} \quad \mathcal{B}^* := \{B^* : B \in \mathcal{B}\}$$

Now for any $x \in D$ and U open in X with $x \in U$, we show there is a $B^* \in \mathcal{B}^*$ such that $x \in B^* \subset U$. Well, let f be a continuous function such that $f(x) = 1$ and $f(X \backslash U) = 0$. Then there is a $B \in \mathcal{B}$ with $f \in B \subset [\{x\}, (0, \infty)]$. We show $x \in B^* \subset U$: Clearly $x \in B^*$ since every function in B is positive on x.

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Now assume there is $x_0 \in B^* \backslash U$. Then $f(x_0) = 0$ since $x_0 \in X \backslash U$. However, $f(x_0) > 0$ since $x_0 \in B^*$ which is a contradiction. So $B^* \subset U$. Since D is a subspace of X the result follows. $\square$

To see when $C_D(X)$ is an $\mathcal{X}_0$–space, we need the following lemmas.

2.6.4 Lemma. For any topological space X and D dense in X, we let $\pi: \mathbb{R}^{X \times X} \longrightarrow (\mathbb{R}^X)^X$ be defined as follows: If $f \in \mathbb{R}^{X \times X}$ then $\pi(f) : X \longrightarrow \mathbb{R}^X$ is given by $\pi(f)(x) = f(x, \cdot)$ for each $x \in X$. Let $S = \pi^{-1}(C_D(X, C_D(X))$. Then $\psi : S_{D \times D}(X \times X, \mathbb{R}) \longrightarrow C_D(X, C_D(X))$ is a homeomorphism where ψ is the restriction of π to S.

Proof: Clearly ψ is bijective. We show that ψ is continuous. Well, let $[\{a\}, [\{a^*\}, V^*]]$ be a subbasic open set in $C_D(X, C_D(X))$ where a, a^* are points in D and V^* is an open interval in $\mathbb{R}$. Then

$$\begin{aligned} \psi^{-1}([\{a\}, [\{a^*\}, V^*]]) &= \{g \in S_{D \times D}(X \times X, \mathbb{R}) : \psi(g) \in [\{a\}, [\{a^*\}, V^*]]\} \\ &= \{g \in S_{D \times D}(X \times X, \mathbb{R}) : g(a, \cdot) \in [\{a^*\}, V^*]\} \\ &= \{g \in S_{D \times D}(X \times X, \mathbb{R}) : g(a, a^*) \in V^*\}. \\ &= [\{(a, a^*)\}, V^*] \cap S_{D \times D}(X \times X, \mathbb{R}) \text{ which is open.} \end{aligned}$$

Hence ψ and ψ^{-1} are continuous and we have our result. $\square$

In [24] it is given that $C_p(X)$ is an $\mathcal{X}_0$–space iff X is countable. As for $C_D(X)$, the situation is similar.

2.6.5 Lemma. If X is a cosmic space, then $C_D(X, C_D(X))$ is cosmic iff D is countable.

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Proof: If D is countable, $C_D(X)$ is separable metrizable (Theorem 2.4.7) and so a $\mathcal{X}_0$–space. So $C_p(X, C_D(X))$ is cosmic (Prop. 10.4 [24]) and so $C_D(X, C_D(X))$ is cosmic since $C_D(X, C_D(X))$ is coarser than $C_p(X, C_D(X))$. For the converse assume D is uncountable and we proceed to show that $C_D(X, C_D(X))$ is not hereditarily separable (i.e not every subspace is necessarily separable). As $C_D(X, C_D(X))$ is homeomorphic to $S_{D\times D}(X \times X, \mathbb{R})$ (Theorem 2.6.4), it suffices to show that for each $z \in D$, there is an $f_z \in S$ such that $f_z(z, z) = 1$ and $f_z(x, x) = 0$ for $x \neq z$ since then the collection $\mathcal{P} = \{f_z : z \in D\}$ is uncountable and a discrete subset of $S_{D\times D}(X\times X, \mathbb{R})$. Now as X is cosmic, we have that $X \times X$ is normal and the diagonal $\triangle$ is a closed G_δ–subset of $X \times X$ and so there is a continuous function $u : X \times X \longrightarrow [0, 1]$ such that $u^{-1}(0) = \Delta$ (see [10]). Now for each $z \in D$, define

$$f_z(x, y) = 2u(x, y)\backslash\{u(x, y) + u(x, z) + u(z, y)\}$$

$$\text{if } (x, y) \neq (z, z).$$

$$f_z(z, z) = 1.$$

We show that $f_z \in S$, i.e $\psi(f_z) \in C(X, C_D(X))$. Let $U := [\{x\}, V]$ where $x \in D$ and V is open in $\mathbb{R}$. Then $f_z^{-1}(U) = \{y \in X : f_z(y, x) \in V\} = (f_z(\cdot, x))^{-1}(V)$, which is open in X provided $f_z(\cdot, x) : X \longrightarrow \mathbb{R}$ is continuous. This follows, since by construction, $f_z(\cdot, x)$ is continuous everywhere if $x \neq z$, while $f_z(\cdot, z)$ is the constant function with value 1.

□

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2.6.6 Theorem. $C_D(X)$ is an $\aleph_0$–space iff D is countable.

Proof: $C_D(X)$ is homeomorphic to a subspace of $C_p(D)$ which is an $\aleph_0$–space if D is countable ([23]). Hence $C_D(X)$ is $\aleph_0$–space.

For the converse, if $C_D(X)$ is an $\aleph_0$–space, then $C_D(X)$ is cosmic. So D is cosmic (Theorem 2.6.3). By [24], $C_p(D, C_D(X))$ is cosmic and so $C_D(X, C_D(X))$ is cosmic since it is coarser than $C_p(D, C_D(X))$. By Lemma 2.6.5, D is countable. □

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2.7 Equicontinuity

Let D be a dense subset of X. We now investigate compact sets in $C_D(X)$ by establishing an Ascoli type of theorem. To this end, we give a definition.

2.7.1 Definition. A subset $\mathcal{F}$ of $C(X)$ is **equicontinuous** if and only if for each $x \in X$ and $\epsilon > 0$, there is a neighbourhood U of x such that $| f(y) - f(x) | < \epsilon$ for all $f \in \mathcal{F}$ and $y \in U$.

Given $\mathcal{F} \subset C(X)$, let $\mathcal{F}_p$ and $\mathcal{F}_D$ represent $\mathcal{F}$ together with the point–open topology and point–open topology induced by a dense set D in X respectively on it. Then we have the following result.

2.7.2 Theorem. Let $\mathcal{F}$ be an equicontinuous subset of $C(X)$. Then $\overline{\mathcal{F}_D} = \overline{\mathcal{F}_p}$. In addition if $\mathcal{F}$ is pointwise bounded, then $\overline{\mathcal{F}_D}$ is compact.

Proof: It is already known that $\overline{\mathcal{F}}_D \leq \overline{\mathcal{F}}_p$. To show the reverse inequality, we show that for any net $(f_\alpha)_\alpha \subset \mathcal{F}$ such that $(f_\alpha)_\alpha \to f$ in $\mathcal{F}_D$, then the net $(f_\alpha)_\alpha$ converges to f in $\mathcal{F}_p$. So let us take a net $(f_\alpha)_\alpha$ converging to f in $\mathcal{F}_D$ and let $W_p = \langle f, A, \epsilon \rangle \cap \mathcal{F}$ be a basic neighbourhood of f in $\mathcal{F}_p$, where A is finite in X and $\epsilon > 0$. For each $x_i \in A$, there is a neighbourhood U_i of x_i such that $| g(x_i) - g(y) | < \frac{\epsilon}{3}$ for all $g \in \mathcal{F}$ and $y \in U_i$. For each $i \in \{1, ..., | A |\}$, choose $y_i \in U_i \cap D$. Then $W_D := \langle f, A^*, \frac{\epsilon}{3} \rangle \cap \mathcal{F}$ is a neighbourhood of f in $\mathcal{F}_D$ where $A^* := \bigcup_{i=1}^{|A|} \{y_i\}$ and so there is a α_0 such that for $\alpha \geq \alpha_0$, we have $f_\alpha \in W_D$. Now for each $i \in \{1, ..., | A |\}$ and $\alpha \geq \alpha_0$, we have

$$\begin{aligned} |\, f(x_i) - f_\alpha(x_i) \,| &\leq |\, f_\alpha(x_i) - f_\alpha(y_i) \,| + |\, f_\alpha(y_i) - f(y_i) \,| \\ &\qquad + |\, f(y_i) - f(x_i) \,| \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Hence each $f_\alpha \in W_p$ for $\alpha \geq \alpha_0$ and (f_α) converges to f in $\mathcal{F}_p$. So $\overline{\mathcal{F}_D} = \overline{\mathcal{F}_p}$.

The rest follows from Ascoli's theorem. □

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2.8 Spread

We introduce a structural notion (see [1]).

2.8.1 Definition. For any topological space X, the **spread** denoted by $s(X)$, is given by $s(X) := \sup\{| A |: A \text{ is a discrete subspace of } X\}$.

It is known from [1] that $\sup\{s(X^n) : n \in \mathbb{N}\} = \sup\{s((C_p(X))^n) : n \in \mathbb{N}\}$. We give an analoguous result for $C_D(X)$.

2.8.2 Theorem. For any space X and any dense $D \subset X$,

$$\sup\{s(D^n) : n \in \mathbb{N}\} = \sup\{s((C_D(X))^n) : n \in \mathbb{N}\}.$$

Proof: Since $C_D(X)$ is a subspace of $C_p(D)$ we have that $s(C_D(X)) \leq s(C_p(D))$. So $s((C_D(X))^n) \leq s((C_p(D))^n)$ for $n \in \mathbb{N}$. By the quoted result above (from [1]),

$$\begin{aligned}\sup\{s((C_D(X))^n) : n \in \mathbb{N}\} &\leq \sup\{s((C_p(D))^n) : n \in \mathbb{N}\} \\ &= \sup\{s(D^n) : n \in \mathbb{N}\}.\end{aligned}$$

For the converse inequality, D is embedded in $C_p(C_D(X))$ and so $s(D) \leq s(C_p(C_D(X)))$which implies $s(D^n) \leq s((C_p(C_D(X)))^n)$.

Consequently, using the quoted theorem above it follows that

$$\begin{aligned}\sup\{s(D^n) : n \in \mathbb{N}\} &\leq \sup\{s((C_p(C_D(X)))^n) : n \in \mathbb{N}\} \\ &= \sup\{s((C_D(X))^n) : n \in \mathbb{N}\}.\end{aligned}$$

Hence, we have our result. □

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2.9 Hereditary Density and Hereditary Lindelöf Number of $C_D(X)$

2.9.1 Definition. The **hereditary density** of a topological space X is defined by $hd(X) = \sup\{d(A) : A \subset X\}$ where $d(A) = \omega + \min\{|B| : B$ is dense in $A\}$.

The **hereditary Lindelöf number** of X is $hl(X) = \sup\{l(A) : A \subset X\}$ where $l(A) = \omega + \min\{\tau :$ every open cover of A has a subcover of cardinality $\leq \tau\}$.

For any space X, it is known that

$\sup\{hd(X^n) : n \in \mathbb{N}\} = \sup\{hl((C_p(X))^n) : n \in \mathbb{N}\}$ and $\sup\{hl(X^n) : n \in \mathbb{N}\} = \sup\{hd((C_p(X))^n) : n \in \mathbb{N}\}$ which illustrates the duality of heredity density and hereditary Lindelöfness in conjunction with powers of X and $C_p(X)$ ([23]).

We investigate this duality in $C_D(X)$:

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2.9.2 Theorem.

(a) $\sup\{hd(D^n) : n \in \mathbb{N}\} = \sup\{hl((C_D(X))^n) : n \in \mathbb{N}\}$.

(b) $\sup\{hl(D^n) : n \in \mathbb{N}\} = \sup\{hd((C_D(X))^n) : n \in \mathbb{N}\}$.

Proof: Let $\tau_1 := \sup\{hd(D^n) : n \in \mathbb{N}\}$, $\lambda_1 := \sup\{hl((C_D(X))^n) : n \in \mathbb{N}\}$, $\tau_2 := \sup\{hl(D^n) : n \in \mathbb{N}\}$ and $\lambda_2 := \sup\{hd((C_D(X))^n) : n \in \mathbb{N}\}$.

First we show the inequalities $\lambda_1 \leq \tau_1$ and $\lambda_2 \leq \tau_2$.

For a fixed $n \in \mathbb{N}$,

$$\begin{aligned} hl((C_D(X))^n) &= \sup\{l(Y)\colon Y \subset (C_D(X))^n\} \\ &\leq \sup\{l(Y)\colon Y \subset (C_p(D))^n\} \text{ (since } C_D(X) \subset C_p(D)) \\ &= hl((C_p(D))^n). \end{aligned}$$

Therefore

$$\begin{aligned} \lambda_1\colon &= \sup\{hl((C_D(X))^n)\colon n \in \mathbb{N}\} \\ &\leq \sup\{hl((C_p(D))^n)\colon n \in \mathbb{N}\} \\ &= \sup\{hd(D^n)\colon n \in \mathbb{N}\} = \tau_1. \end{aligned}$$

Similarly, for a fixed $n \in \mathbb{N}$,

$$\begin{aligned} hd((C_D(X))^n) &= \sup\{d(Y)\colon Y \subset (C_D(X))^n\} \\ &\leq \sup\{d(Y)\colon Y \subset (C_p(D))^n\} \text{ (since } C_D(X) \subset C_p(D)) \\ &= hd((C_p(D))^n). \end{aligned}$$

Therefore

$$\begin{aligned} \lambda_2\colon &= \sup\{hd((C_D(X))^n)\colon n \in \mathbb{N}\} \\ &\leq \sup\{hd((C_p(D))^n)\colon n \in \mathbb{N}\} \\ &= \sup\{hl(D^n)\colon n \in \mathbb{N}\} = \tau_2. \end{aligned}$$

For the reverse inequalities, since $C_D(X) \subset C_p(D)$ we have that $D \subset C_p(C_D(X))$ and so $D^n \subset (C_p(C_D(X)))^n$ for $n \in \mathbb{N}$. Then

$$\begin{aligned} \tau_2\colon &= \sup\{hl(D^n)\colon n \in \mathbb{N}\} \\ &\leq \sup\{hl((C_p(C_D(X)))^n)\colon n \in \mathbb{N}\} \\ &= \sup\{hd((C_D(X))^n)\colon n \in \mathbb{N}\} = \lambda_2. \end{aligned}$$

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Similarly,

$$\begin{aligned}\tau_1 : \; &= \sup\{hd(D^n) \colon n \in \mathbb{N}\} \\ &\leq \sup\{hd((C_p(C_D(X)))^n) \colon n \in \mathbb{N}\} \\ &= \sup\{hl((C_D(X))^n) \colon n \in \mathbb{N}\} \\ &= \lambda_1.\end{aligned}$$

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2.10 Analyticity

The results in this section can be found in [6] and [27]. We give them here for completeness.

2.10.1 Definition. A Tychonoff space X is **analytic** if it is a continuous image of a Polish space, where a Polish space is a completely metrizable separable space.

2.10.2 Definition. A $\boldsymbol{k_{\sigma\delta}}$**–space** is a space representable as the intersection of a sequence of σ–compact subspaces of some embracing space. A $\boldsymbol{k}$**–analytic** space is a continuous image of a $k_{\sigma\delta}$– space.

A k–analytic space with a countable network is an analytic space, and every analytic space is k–analytic.

2.10.3 Lemma. For a separable metrizable space X which is σ–compact, we have that $C_D(X, I)$ is a $k_{\sigma\delta}$–space where D is a countable dense set in X and I is the closed unit interval.

Proof: Let $X = \bigcup\{A_l \colon l \in \mathbb{N}\}$ where each A_l is compact. Let p be a compatible metric on X, and then set $M_{kln} := \{f \in I^D \colon | f(x) - f(y) | \leq \frac{1}{k}$ whenever $p(x, z) < \frac{1}{n}$ and $p(y, z) < \frac{1}{n}$ for some $z \in A_l\}$, where $k, l, n \in \mathbb{N}$.

Firstly, we show that $C(X \mid D, I) = \bigcap_{k\in\mathbb{N}} \bigcap_{l\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} M_{kln}$:

Let $f \in \bigcap_{k\in\mathbb{N}} \bigcap_{l\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} M_{kln}$. We show that f has a continuous extension over X.

Well, choose $z \in X$ and find $l \in \mathbb{N}$ such that $z \in A_l$. Since $f \in \bigcap_{k\in\mathbb{N}} \bigcap_{l\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} M_{kln}$,

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we have that f has an arbitrarily small oscillation near z (i.e For every $\epsilon > 0$ there is a neighbourhood U of z such that Diameter$(f(D \cap U)) < \epsilon$) and so f has a continuous extension over X (see [10]). Thus $\bigcap_{k \in \mathbb{N}} \bigcap_{l \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} M_{kln} \subset C(X \mid D, I)$.

For the reverse inclusion, let $f \in C(X \mid D, I)$ and f^* be the continuous extension of f over X. Fix k and l in $\mathbb{N}$. Since A_l is compact, f^* is uniformly continuous on A_l, i. e. there is a $n \in \mathbb{N}$ such that $\mid f^*(x) - f^*(y) \mid \leq \frac{1}{k}$ when $p(x, z) < \frac{1}{n}$ and $p(y, z) < \frac{1}{n}$ for some $x \in A_l$. So $f \in M_{kln}$.

Secondly, we show that each M_{kln} is compact. Since I^D is compact, it suffices to show that each M_{kln} is closed in I^D. Since the evaluation functional e_x is continuous whenever $x \in D$, we have that $I^D \setminus M_{kln} = \{ f \in I^D : \mid f(x) - f(y) \mid = \mid e_x(f) - e_y(f) \mid > \frac{1}{k}$ for some $x, y \in D$ such that $p(x, z) < \frac{1}{n}$ and $p(y, z) < \frac{1}{n}$ for some $z \in A_l \}$ is open in I^D. Thus M_{kln} is compact in I^D. So $C_D(X, I) = \bigcap_{k \in \mathbb{N}} \bigcap_{l \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} M_{kln}$ where each M_{kln} is compact. Hence $C_D(X, I)$ is a $k_{\sigma\delta}$–space. □

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A family of Baire sets in a space X is defined as the minimal family of sets that contains all zero-sets of continuous real functions and is closed with respect to countable unions and complements. A mapping $f : X \longrightarrow Y$ is called measurable if the preimage of any Baire set in Y is a Baire set in X.

Before we proceed to obtain a more general result we state the following result found in [27]:

2.10.4 Lemma. Let $f : X \to Y$ be a measurable mapping of a k–analytic space X onto a cosmic space Y. Then Y is analytic.

The following result can be found in [27] and [6]:

2.10.5 Theorem. Let X be a separable metrizable space. Then the following are equivalent.

(i) X is σ–compact

(ii) $C_D(X)$ is a $k_{\sigma\delta}$–space

(iii) $C_D(X)$ is analytic.

Proof: (i) $\Rightarrow$ (ii): From Lemma 2.10.3 we have $C_D(X, I)$ is a $k_{\sigma\delta}$–space. If $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is the two point compactification of $\mathbb{R}$, then $C_D(X, \overline{\mathbb{R}})$ is homeomorphic to $C_D(X, I)$ and so $C_D(X, \overline{\mathbb{R}})$ is a $k_{\sigma\delta}$–space. Now put $S = \bigcap_{l \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \{f \in \overline{\mathbb{R}}^D : \mid f(x) \mid \leq n, \forall x \in D \cap A_l\}$ where A_l is as in the proof of Lemma 2.10.3 . Clearly, S is an $F_{\sigma\delta}$ set in $\overline{\mathbb{R}}^D$ and so $C(X \mid D) = S \cap C_D(X, \overline{\mathbb{R}})$. Since $C_D(X)$ is an intersection of two $k_{\sigma\delta}$ spaces, we have $C_D(X)$ to be a $k_{\sigma\delta}$ space.

(ii) $\Rightarrow$ (iii): Since D is cosmic, we have $C_D(X)$ to be cosmic (Theorem 2.6.3). Since every k–analytic cosmic space is analytic ([30]), we have $C_D(X)$ to be analytic.

(iii) $\Rightarrow$ (i): Define $\varphi: C(X \mid D) \to C_p(X)$ to be the extension map inverse to the restriction map $r_D : C_p(X) \to C(X \mid D)$. As X is cosmic, $C_p(X)$ is cosmic (see [23]). We show that $C_p(X)$ is analytic using Lemma 2.10.4. So it suffices to show that φ is measurable: Let $V = [\{x\}, U]$ be open in $C_p(X)$, $x \in X$, U open in $\mathbb{R}$. We show that $\varphi^{-1}(V)$ is a Baire set in $C_D(X)$. Choose a sequence $\{x_n : n \in \mathbb{N}\} \subset D$ such that (x_n) converges to x. Then we have

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$$V = \bigcup_{n\in\mathbb{N}} \bigcap_{k>n} [\{x_k\}, U]$$

and so

$$\varphi^{-1}(V) = \bigcup_{n\in\mathbb{N}} \bigcap_{k>n} [\{x_k\}, U]$$

is a $G_{\delta\sigma}$ set in $C_D(X)$ with $[\{x_k\}, U]$ open in $C_D(X)$. Therefore, φ is measurable and $C_p(X)$ is analytic. By a theorem in [23, 5], X is σ–compact.

□

2.11 Z_σ–property of $C_D(X)$

We start off with a few definitions (see [34]).

2.11.1 Definition.

Let X be a separable metrizable space. Let $Q = I^\omega$ represent the Hilbert cube, where $I = [-1, 1]$. A closed subset $A \subset X$ is called a **Z – set** in X provided that for every open cover $\mathcal{U}$ of X and every function $f \in C(Q, X)$, there is a function $g \in C(Q, X)$ such that (1) f and g are U-close (i.e for any $x \in X$, there is a $U \in \mathcal{U}$ such that $f(x), g(x) \in U$) , and (2) $g(Q) \cap A = \emptyset$. A space which is a countable union of its own Z–sets is called a $\mathbf{Z_\sigma}$ **– space.**

We state a Lemma to put the above concept into " metric " perspective. To this end if ρ is a metric on Y then define the metric d by $d(f, g) = \sup\{\rho(f(x), g(x)) : x \in X\}$. If X is compact, it can be shown that for all admissible topologies on Y, the topologies on $C(X, Y)$ coincide (see [34]).

2.11.2 Lemma. Let X be a separable metrizable space and $A \subset X$ be closed. Let d be a metric on $C(Q, X)$. Then A is a Z–set if and only if for every $\epsilon > 0$ and for every $f \in C(Q, X)$ there exists a $g \in C(Q, X)$ such that $d(f, g) < \epsilon$ and $g(Q) \cap A = \emptyset$.

Before we proceed, note that $C_D(X)$ is separable metrizable if and only if D is a dense countable set in X (Corollary 2.4.7).

2.11.3 Lemma ([9]). Let X and Y be dense linear subspaces of $\mathbb{R}^\omega$ such that $X \subset Y$ and Y is a Z_σ–space. Then X is a Z_σ–space.

Also formulated in [9] is the following result which links $C_p(X)$ and the Z_σ property.

2.11.4 Lemma. For every Tychonoff infinite countable space X the function space $C_p^*(X)$ is a Z_σ–space.

Also from [9]:

2.11.5 Proposition. For each infinite countable and dense set D in X, $C_D^*(X)$ is a Z_σ–space.

Then we have the following.

2.11.6 Corollary. If X is compact and D infinite countable, then $C_D(X)$ is a Z_σ–space.

Proof: This follows since $C_D(X) = C_D^*(X)$. □

Proposition 2.11.5 and Corollary 2.11.6 are mentioned in [6].

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There is a link between analyticity and the Z_σ–property.

2.11.7 Proposition. Let D be a countable nondiscrete dense set in X such that $C_p(D)$ is analytic. Then $C_D(X)$ is a Z_σ–space.

Proof: If D is a countable nondiscrete space and $C_p(D)$ is analytic, then $C_p(D)$ is a Z_σ–space (Corollary 3.6 [9]). Since $C_D(X)$ is a dense linear subspace of $C_p(D)$ (Corollary 2.3.3), it follows from Lemma 2.11.3 that $C_D(X)$ is also a Z_σ–space. □

2.11.8 Theorem. Let X be a countable nondiscrete space and D dense in X. If $C_D(X)$ is analytic, then

(a) $C_D(X)$ and $C_p(X)$ are Z_σ–spaces

(b) $C_p(D)$ is analytic.

Proof:

(a) This is Proposition 4.2 [6]

(b) $C_D(X)$ analytic $\Longrightarrow C_p(D)$ is analytic (Remark 2.10 [6]) □

2.12 Dual of $C_D(X)$

For a dense subset D of X, $C_D(X)$ is a locally convex topological vector space. In this section, we identify the dual $M_D(X)$ of $C_D(X)$. The dual of $C_p(X)$, denoted $M_p(X)$, is the space $M_p(X) = \{\lambda_1 e_{x_1} + \ldots + \lambda_n e_{x_n} : x_1, \ldots, x_n \in X, \lambda_1, \ldots, \lambda_n \in \mathbb{R}, n \in \mathbb{N}\}$ where each $e_{x_i}(f) = f(x_i)$ for any $f \in C_D(X)$. The dual is the smallest linear subspace of the linear topological space $C_p(C_p(X))$

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containing X (see [1]). We would expect a similar space for the dual of $C_D(X)$ and to see this we need the following lemma which we state([12]).

2.12.1 Lemma. A linear functional λ on $C(X)$ is a linear combination of a finite set $\lambda_{1,\dots}\lambda_n$ of linear functionals on $C(X)$ if and only if

$$\bigcap_{i=1}^{n} \mathrm{Ker}\lambda_i \subset \mathrm{Ker}\lambda.$$

We now describe $M_D(X)$:

2.12.2 Theorem. $M_D(X)$ is generated by the basis $\mathcal{B} := \{e_x : x \in D\}$.

Proof: Since $M_p(X)$ is generated by the basis $\{e_x : x \in X\}$, it follows that $\mathcal{B}$ is a linearly independent set. It remains to show that $\mathcal{B}$ spans $M_D(X)$: Well, let λ be a non–zero linear functional in $M_D(X)$. Since λ is continuous at the zero function f_0, there is a finite set $F \subset D$ and $\delta > 0$ such that

$$\lambda(\langle f_0,\, F,\, \delta\rangle) \subset (-1,\, 1).$$

We show that λ is a linear combination of $\{e_x : x \in F\}$. Now, let $f \in \bigcap_{x\in F} \mathrm{Ker}\, e_x$ and $\epsilon > 0$. Then since $f(x) = 0$ for $x \in F$, we have $\frac{1}{\epsilon} f(F) = 0$. Hence $\frac{1}{\epsilon} f \in \langle f_0,\, F,\, \delta\rangle$ and so $\mid \lambda(\frac{1}{\epsilon} f) \mid < 1$. Hence $\mid \lambda(f) \mid < \epsilon$. Since ϵ is arbitrary, $f \in \mathrm{Ker}\lambda$. By Lemma 2.12.1 there are scalars $\alpha_x \in \mathbb{R}$ such that $\lambda = \sum_{x\in F} \alpha_x e_x$.

□

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CHAPTER 3

Lower Semifinite Topology

We can obtain a function space topology on $C(X)$ with the subbasic open sets of the form

$$[A] = \{f \in C(X): \Gamma_f \cap A \neq \emptyset\}$$

where A is an open set in $X \times \mathbb{R}$ and Γ_f represents the graph of f (see [29]). We denote the space $C(X)$ with this topology by $C_l(X)$. An alternative notational approach to a subbasic open set of $C_l(X)$ is $[U \times (a, b)] = \{f \in C(X): f(x) \in (a, b)$ for some $x \in U\}$ where U is open in X and $a < b$ for some $a, b \in \mathbb{R}$. Clearly $[U \times (a, b)] = \bigcup_{x \in U} [x, (a, b)]$ where $[x, (a, b)] = \{f \in C(X): f(x) \in (a, b)\}$. From this we see that $C_l(X)$ is coarser than $C_p(X)$ for any space X. We investigate the topological properties of $C_l(X)$.

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3.1 Basic Properties

3.1.1 Definition. A **π–base** for X is a family $\mathcal{P}$ of nonempty open subsets of X such that every nonempty open subset of X contains some member of $\mathcal{P}$.

With this we have the following lemma.

3.1.2 Lemma. If $\mathcal{P}$ is a π–base for X and D is a dense subset of $\mathbb{R}$, then the family

$$\{[P \times (p,\, q)] \colon P \in \mathcal{P} \text{ and } p,\, q \in D\}$$

is a subbase for $C_l(X)$.

Proof: Let $[U \times (a,\, b)]$ be a subbasic open set in $C_l(X)$ where U is open in X and $a,\, b \in \mathbb{R}$ with $a < b$. Let $f \in [U \times (a,\, b)]$. We need to show that there is a $P \in \mathcal{P}$ and $s,\, t \in D$ such that $f \in [P \times (s,\, t)] \subset [U \times (a,\, b)]$. To this end, since $f \in [U \times (a,\, b)] := \{g \in C(X) : g(x) \in (a,\, b) \text{ for some } x \in U\}$, there is an $x \in U$ such that $(x,\, f(x)) \in \Gamma_f \cap (U \times (a,\, b))$. Since $a < f(x) < b$, let $s, t \in D$ such that $a < s < f(x) < t < b$. We have f continuous and so there is an open set W in X such that $x \in W$ and $s < f(y) < t$ for all $y \in W$. Let $P \in \mathcal{P}$ such that $P \subset W \cap U$. Then we clearly have $f \in [P \times (s,\, t)] \subset [U \times (a,\, b)]$. □

It also turns out that $C_l(X)$ is a homogeneous space i.e we can seek out the local properties of $C_l(X)$ by working with the zero function f_0 in $C_l(X)$. We use the shift map to show homogeneity.

3.1.3 Theorem. For $f \in C(X)$, the shift map $S_f \colon C_l(X) \to C_l(X)$ defined

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by $S_f(g) = f + g$ is a homeomorphism.

Proof: Let $g \in C_l(X)$ and $S := [U \times (a,\ b)]$ be a subbasic open set of $C_l(X)$ such that $f + g \in S$ where U is an open set in X and $a < b$, where $a,\ b \in \mathbb{R}$. Then there is an $x \in U$ such that $f(x) + g(x) \in (a,\ b)$. Set $\epsilon_b = b - f(x) - g(x)$ and $\epsilon_a = f(x) + g(x) - a$. By the continuity of f, we can find a neighbourhood V of x such that $f(x) - \frac{\epsilon_a}{2} < f(y) < f(x) + \frac{\epsilon_b}{2}$ for each $y \in V$. Define $q := g(x) + \frac{\epsilon_b}{2}$ and $p := g(x) - \frac{\epsilon_a}{2}$. Then $g \in [V \times (p,\ q)]$. We finish if we show that $S_f([V \times (p,\ q)]) \subset [U \times (a,\ b)]$. Now let $h \in [V \times (p,\ q)]$. Then there is a $t \in V$ such that $h(t) \in (p,\ q)$. So $f(t) + h(t) < f(t) + q = f(t) + g(x) + \frac{\epsilon_b}{2} < f(x) + g(x) + \epsilon_b = b$. Also $f(t) + h(t) > f(t) + p = f(t) + g(x) - \frac{\epsilon_a}{2} > f(x) + g(x) - \epsilon_a = a$. Hence $S_f(h) \in (a,\ b)$. □

By the above result $C_l(X)$ is homogeneous. We end this section by observing how the evaluation functionals e_x, $x \in X$, act on $C_l(X)$.

3.1.4 Lemma. $[\{x\} \times (a,\ b)]$ is open in $C_l(X)$ iff x is isolated in X.

Proof: If x is isolated in X, then the result is clear.

For the converse, assume $[\{x\},\ (a,\ b)]$ is open and let $f \in [\{x\} \times (a,\ b)]$. Then there exist open sets U_i and reals a_i, b_i in X with $a_i < b_i$ such that

$$f \in [U_1 \times (a_1,\ b_1)] \cap ... \cap [U_n \times (a_n,\ b_n)] \subset [\{x\} \times (a,\ b)]$$

where $i \in I := \{1, ..., n\}$. Let $U^i := U_i \cap f^{-1}(a_i, b_i)$. If x is not isolated, there is a $x_i \in U^i \backslash \{x\}$ for each $i \in I$. Choose $g : X \to \mathbb{R}$ continuous such that $g(x_i) = f(x_i)$ and $g(X \backslash U^i) \subset \mathbb{R} \backslash (a,\ b)$. Then $g \in \bigcap_{i \in I} [U_i \times (a_i,\ b_i)]$ yet $g \notin [\{x\} \times (a,\ b)]$ which yields a contradiction. Thus x must be isolated. □

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3.1.5 Theorem. For each $x \in X$, the evaluation functional $e_x \colon C_l(X) \to \mathbb{R}$ defined by $e_x(f) = f(x)$ is continuous iff $\{x\}$ is open in X.

Proof: Let x be isolated in X and $(a, b) \subset \mathbb{R}$. Then $e_x^{-1}(a, b) = \{f \in C(X) \colon f(x) \in (a, b)\} = [\{x\} \times (a, b)]$, which is open in $C_l(X)$. So $e_x \colon C_l(X) \to \mathbb{R}$ is continuous.

For the converse assume $e_x : C_l(X) \to \mathbb{R}$ is continuous. Then $e_x^{-1}(a, b) = [\{x\} \times (a, b)]$ is open in $C_l(X)$ and so x is isolated in X by Lemma 3.1.4. □

It turns out that $C_l(X)$ will be a topological semi–group only under strong conditions on X.

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3.1.6 Theorem. The following are equivalent:

(a) $C_l(X)$ is a topological semigroup with respect to addition.

(b) Addition is continuous at some point of $C_l(X) \times C_l(X)$.

(c) X has a dense set of isolated points.

Proof: (a) $\Longrightarrow$ (b) : Immediate.

(b) $\Longrightarrow$ (c): Assume that X does not have a dense set of isolated points. Let U be a nonempty open set in X with no isolated points. Let $x_0 \in U$ and by complete regularity choose $f_1 : X \longrightarrow [0, 1]$ continuous such that $f_1(x_0) = 1$ and $f_1(X \backslash U) = 0$. Then $f_1 + f_0 \in [U \times (-2, 2)]$ where f_0 is the zero function. Let

$$B_k := [U_1^k \times (a_1^k, b_1^k)] \cap \ldots \cap [U_{n_k}^k \times (a_{n_k}^k, b_{n_k}^k)]$$

be open neighbourhoods of f_0, f_1 for $k = 0, 1$ where U_i^k is open in X and

$a_i^k < b_i^k$ for $i \in I_k := \{1, ..., n_k\}, n_k \in \mathbb{N}$. Let us consider the first case when $U_i^0 \cap U \neq \emptyset$ for some $i \in I_0$ and $U_j^1 \cap U \cap f_1^{-1}(a_j^1, b_j^1) \neq \emptyset$ for some $j \in I_1$. Let $J_0 = \{i \in I_0 : U_i^0 \cap U \neq \emptyset\}$ and $J_1 = \{j \in I_1 : U_j^1 \cap U \cap f_1^{-1}(a_j^1, b_j^1) \neq \emptyset\}$ and let the cardinalities of I_k be m_k for $k = 0, 1$. Choose $x_i^0 \in U_i^0 \cap U$ for each $i \in J_0$, and $x_j^1 \in U_j^1 \cap U \cap f_1^{-1}(a_j^1, b_j^1)$ for each $j \in J_1$. Let V be an open set in X such that $\{x_i^0 : i \in J_0\} \cup \{x_j^1 : j \in J_1\} \subset V \subset \overline{V} \subset U$. Let V_i^k be open sets such that $x_i^k \in V_i^k \subset V$ and $V_i^k \cap V_j^{k'} = \emptyset$ for $i \neq j$ or $k \neq k'$.

For $k = 0, 1$, let $\alpha_k : X \longrightarrow [0, 2]$ be continuous such that $\alpha_k(x_i^k) = 0$ and $\alpha_k(x) = 2$ for $x \in \overline{V} \backslash (V_1^k \cup ... \cup V_{m_k}^k)$. Define $g_0 = \alpha_0$ and $g_1 = f_1 + \alpha_1$. Clearly $g_0 \in B_0$ and since $g_1(x_j^1) = f_1(x_j^1) + \alpha_1(x_j^1) = f_1(x_j^1) \in (a_j^1,\, b_j^1)$ for each $j \in J_1$ we have $g_1 \in B_1$. We show that $g_0 + g_1 \notin [U \times (-2,\, 2)]$. Well, let $x \in U$. If $x \in V_j^1$ for some $i \in J_1$, then $x \notin V_i^0$ for any $i \in I_0$. Then by construction $g_0(x) = 2$ and $g_1(x) \geq 0$. So $g_0(x) + g_1(x) \geq 2$. Now we proceed to the case when $U_i^0 \cap U = \emptyset$ and $U_i^1 \cap U \cap f_1^{-1}(a_i^1,\, b_i^1) = \emptyset$ for all $i \in I_0 \cup I_1$. Choose $x_i^k \in U_i^k$ for all $i \in I_k, k = 0, 1$. Let $\beta : X \longrightarrow [0, 2]$ be continuous such that $\beta(x_i^k) = 0$ and $\beta(X \backslash W) = 2$ where W is the union of U_i^0 and U_j^1 for all the $i, j \in I_0, I_1$ respectively. Define $g_0 = \beta$ and $g_1 = f_1 + \beta$. Clearly $g_0 \in B_0$ and since $g_1(x_i^1) = f_1(x_i^1) + \beta(x_i^1) = f_1(x_i^1) = 0$ for each $i \in I_1$ we have that $g_1 \in B_1$. Now let $x \in U$. Then $g_0(x) + g_1(x) = \beta(x) + g_1(x) = 2 + f_1(x) + \beta(x) > 2$. Hence $g_0 + g_1 \notin [U \times (-2,\, 2)]$. The other cases follow similarly.

(c) $\Longrightarrow$ (a): Suppose that X has a dense set of isolated points and $f_1 + f_2 \in [U \times (a,\, b)]$ for some open set U in X and $a, b \in \mathbb{R}$ with $a < b$. Then $(f_1 + f_2)^{-1}(a,\, b) \cap U \neq \emptyset$ and so there is an isolated point x_0 in U such that $(f_1 + f_2)(x_0) \in (a,\, b)$. Define $\epsilon_a = f_1(x_0) + f_2(x_0) - a$ and $\epsilon_b = b - f_1(x_0) -$

$f_2(x_0)$. Also define $t_k^a = f_k(x_0) - \frac{\epsilon_a}{2}$, $t_k^b = f_k(x_0) + \frac{\epsilon_b}{2}$ for $k = 1, 2$. Then $f_k \in [\{x_0\} \times (t_k^a, t_k^b)]$ which is open in $C_l(X)$. Now for $k = 1, 2$ let $g_k \in [\{x_0\} \times (t_k^a, t_k^b)]$. Then $g_1(x_0) + g_2(x_0) > t_1^a + t_2^a = f_1(x_0) + f_2(x_0) - \epsilon_a = a$ and $g_1(x_0) + g_2(x_0) < t_1^b + t_2^b = f_1(x_0) + f_2(x_0) + \epsilon_b = b$. Hence $g_1 + g_2 \in [U \times (a, b)]$.

□

3.2 Separation Properties

We have that $C_l(X)$ is not necessarily Hausdorff for an arbitrary space X (See Theorem 3.2.2.) However, we have the following theorem.

3.2.1 Theorem. For all X, $C_l(X)$ is a T_1-space.

Proof: Let $f \in C(X)$. We show that $\{f\}$ is closed in $C_l(X)$. Let $g \in C(X) \backslash \{f\}$. Then there exists an $x_0 \in X$ such that $g(x_0)$ and $f(x_0)$ are distinct. Now without loss of generality let $\alpha \in \mathbb{R}$ such that $g(x_0) < \alpha < f(x_0)$. By continuity, let U be an open set such that $\alpha < f(x)$ for all $x \in U$. Then $g \in [U \times (-\infty, \alpha)] \subset C(X) \backslash \{f\}$. Hence $\{f\}$ is closed. □

We proceed to investigate further the other separation properties.

We have the following theorem which was contributed by an anonymous external examiner.

3.2.2 Theorem. The following are equivalent:

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(a) $C_l(X)$ is Hausdorff

(b) $C_l(X)$ is completely regular

(c) $C_l(X) = C_D(X)$ for some dense set D of isolated points in X.

(d) $C_l(X)$ is linearly homeomorphic to a dense subspace of a power of $\mathbb{R}$

(e) X has a dense set of isolated points.

Proof: (d)⇒(b)⇒(a): Clear.

(c)⇒(d): Corollary 2.3.3.

(a)⇒(e): Assume X does not have a dense set of isolated points. So there exists a nonempty open set W in X which has no isolated points. Let $x \in W$ and f_0 represent the zero function. Let $g \in C(X)$ be such that $g(x) = 1$ and $g(X \backslash W)$=0. Also let $B_{f_0} := [U_1 \times (a_1, b_1)] \cap \ldots \cap [U_n \times (a_n, b_n)]$ and B_g:=$[V_1 \times (c_1, d_1)] \cap \ldots \cap [V_m \times (c_m, d_m)]$ where U_i, V_j are open sets in X, Y respectively with i, j representing elements from their respective index sets and $(a_i, b_i), (c_j, d_j) \subset \mathbb{R}$, be basic open neighbourhoods of f_0 and g respectively. Let $I = \{i \in \{1, \ldots, n\} : U_i \cap W \neq \emptyset\}$ and $J = \{j \in \{1, \ldots, m\} : V_j \cap W \neq \emptyset\}$. If I or J is empty then $C_l(X)$ is not Hausdorff. For instance if $I = \emptyset$ then $g \in B_{f_0} \cap B_g$ and we would be finished. Hence assume $I \neq \emptyset$ and $J \neq \emptyset$. Choose $x_i \in U_i \cap W$ for each $i \in I$ and $y_j \in V_j \cap W$ for each $j \in J$ such that $x_i \neq y_j$ and this can be done since W has no isolated point. By the Tychonoff property there exists an $h \in C_l(X)$ such that $h(x_i) \in (a_i, b_i)$ for each $i \in I$ and $h(y_j) \in (c_j, d_j)$ for each $j \in J$ and $h(X \backslash W) = 0$. Then $h \in B_{f_0} \cap B_g$ and so $C_l(X)$ is not Hausdorff.

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(e)⇒(c): Let D be a dense set of isolated points in X. Let $[U \times (a,b)]$ be an open set in $C_l(X)$ where U is an open set in X and $a, b \in \mathbb{R}$ such that $a < b$. For each $f \in [U \times (a,b)]$ there exists an $x \in U$ such that $f(x) \in (a,b)$. Since f is continuous, the nonempty set $U \cap f^{-1}(a,b)$ is open in X and so there exists $x_f \in U \cap D$ such that $f(x_f) \in (a,b)$. Now $[U \times (a,b)] = \bigcup\{[\{x_f\},(a,b)] : f \in [U \times (a,b)]\}$ and so $[U \times (a,b)]$ is open in $C_D(X)$.

Conversely consider $[\{x\}, V]$ where $x \in D$ and V is open in $\mathbb{R}$. Now if $f \in [\{x\}, V]$, then there are $a_f, b_f \in \mathbb{R}$ with $a_f < b_f$ and $f(x) \in (a_f, b_f) \subset V$. Then $[\{x\}, V] = \bigcup\{[\{x\} \times (a_f, b_f) : f \in [\{x\}, V]\}$ which shows that $[\{x\}, V]$ is open in $C_l(X)$ since $\{x\}$ is open in X. □

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3.3 Induced Functions

Recall that if $\phi \colon X \to Y$ is a continuous function, then the induced function $\phi^* : C(Y) \to C(X)$ is defined by $\phi^*(g) = g \circ \phi$ for each $g \in C(Y)$. We have already seen that the induced map serves as a tool to understand function spaces. Now ϕ^* is continuous as a map from $C_k(Y), C_p(Y)$ and $C_D(Y)$ to $C(X)$ with their respective topologies (see [23]). For $C_l(X)$ the situation looks different. To see this we need the following definition.

3.3.1 Definition. A function $\phi \colon X \to Y$ is **weakly open** provided that for every nonempty open set U in X, there is a nonempty open set V in Y such that $V \subset \overline{\phi(U)}$.

3.3.2 Theorem. Let $\phi\colon X \to Y$ be a continuous function.

Then $\phi^*\colon C_l(Y) \to C_l(X)$ is continuous iff ϕ is a weakly open map.

Proof: Assume ϕ is a weakly open map. Suppose $g \in C(Y)$ such that $\phi^*(g) \in [U \times (a, b)]$ where U is open in X and $a, b \in \mathbb{R}$ such that $a < b$. Then there is some $x_0 \in U$ such that $g \circ \phi(x_0) \in (a, b)$. By the continuity of $g \circ \phi$, there is an open neighbourhood W of x_0 contained in U such that $g \circ \phi(x) \in (a, b)$ for all $x \in W$. By the premise there is a nonempty open set V in Y such that $V \subset \overline{\phi(W)}$. As $V \cap \phi(W) \neq \emptyset$ there is an $x_1 \in W$ such that $\phi(x_1) \in V$. Since $g \circ \phi(x_1) \in (a, b)$ we have $g \in [V \times (a, b)]$. We claim that $\phi^*([V \times (a, b)]) \subset [U \times (a, b)]$. Well, let $h \in [V \times (a, b)]$. Then there is a $y \in V$ such that $h(y) \in (a, b)$. Now $V \cap h^{-1}(a,b) \neq \emptyset$ and since $V \cap h^{-1}(a,b) \subset \overline{\phi(W)}$ there exists an $x \in W$ such that $h(\phi(x)) \in (a, b)$ i. e. $(h \circ \phi)(x) \in (a, b)$ implying $\phi^*(h) \in (a, b)$. Hence ϕ^* is continuous.

Conversely, assume that ϕ is not a weakly open map. Then there is a nonempty open set U in X such that $\overline{\phi(U)}$ contains no interior in Y. Note that the zero function g_0 in $C(Y)$ gets mapped to the zero function f_0 in $C(X)$ under ϕ^* . Consider the neighbourhood $[U \times (-1, 1)]$ of f_0 in $C_l(X)$.

Let

$$B := [U_1 \times (a_1, b_1)] \cap ... \cap [U_n \times (a_n, b_n)]$$

be an open neighbourhood of g_0 where U_i $(i \in \{1, ..., n\})$ is open in Y and $a_i, b_i \in \mathbb{R}$ such that $a_i < 0$, $b_i > 0$. Now for each i choose a point $y_i \in U_i \backslash \overline{\phi(U)}$ and let $g_i : Y \to [0, 1]$ be continuous such that $g_i(y_i) = 0$ and $g_i(\overline{\phi(U)}) = 1$. Let $g(y) := \min\{g_1(y), ..., g_n(y)\}$, $y \in Y$. Then $g \in C(Y)$ and $g \in B$ but

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$\phi^*(g) \notin [U \times (-1, 1)]$. So ϕ^* is not continuous since B is arbitrary. □

3.4 Separability

As mentioned in the previous chapter, $C_p(X)$ is separable if and only if there exists a continuous one–to–one map from X onto a separable metric space (see [23],[35]). This is also true for $C_k(X)$ ([23]). For much weaker topologies on $C(X)$, we need a stronger property to replace injectivity and we give one in the following definition as advised by an anonymous examiner.

3.4.1 Definition. A map $\phi\colon X \to Y$ is said to be **weakly injective** if for every pairwise disjoint family $\{U_1, ..., U_n\}$ of nonempty open subsets of X, there exists $x_i \in U_i$, for $i \in \{1, ..., n\}$ such that $\phi(x_i) \neq \phi(x_j)$ for $i \neq j$.

Analogous to Proposition 2.3.2 we have the following proposition.

3.4.2 Proposition. Let $\phi\colon X \to Y$ be a continuous map. Then $\phi^*(C(Y))$ is dense in $C_l(X)$ if and only if ϕ is weakly injective.

Proof: $\Rightarrow$: Suppose that ϕ is not a weakly injective map. Then there exists two nonempty open disjoint sets U and V such that $\phi(x) = \phi(y)$ for all x, y in U, V respectively. Let $x_0 \in U$ and $f : X \longrightarrow \mathbb{R}$ be continuous such that $f(x_0) = 0$ and $f(X \backslash U) > 1$. Let $W = [U \times (-\infty, 1)] \cap [V \times (1, \infty)]$, which is open in $C_l(X)$. Since $f \in W$ we have that W is nonempty. Now suppose that there is a $g \in C(Y)$ such that $\phi^*(g) \in W$. Then there is a $x_1 \in U$ such that $g \circ \phi(x_1) \in (-\infty, 1)$. Also there is a $x_2 \in V$ such that $g \circ \phi(x_2) \in (1, \infty)$. But

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this contradicts the fact that $\phi(x_1) = \phi(x_2)$ and we are finished.

$\Leftarrow$: Assume ϕ is a weakly injective map and let $B := [U_1 \times (a_1, b_1)] \cap ... \cap [U_n \times (a_n, b_n)]$ be a nonempty open set in $C_l(X)$ where the U_i are open sets X and $a_i, b_i \in \mathbb{R}$ such that $a_i < b_i$. We must find a $g \in C(Y)$ such that $\phi^*(g) \in B$. Since $B \neq \emptyset$, let $f \in B$ and without loss of generality we assume $U_i \cap U_j = \emptyset$ for $i \neq j$. Now by the premise there exists $x_i \in U_i$ for $i \in I := \{1, ..., n\}$ such that $\phi(x_i) \neq \phi(x_j)$ for $i \neq j$. Now for each $i \in I$, let U_i^* be open sets of $\phi(x_i)$ in Y such that $U_i^* \cap U_j^* = \emptyset$ for $i \neq j$. Then by continuity there exist open sets W_i^* in X such that $x_i \in W_i^*$ and $\phi(x) \in U_i^*$ for all $x \in W_i^*$. Then let $g_i : Y \longrightarrow \mathbb{R}$ be a continuous map such that $g_i(\phi(x_i)) = \alpha_i$ for some $\alpha_i \in (a_i, b_i)$ and $g_i(Y \backslash U_i^*) = 0$. Define $\Theta = g_1 + g_2 + ... + g_n$. We finish if we show that $\phi^*(\Theta) \in B$. Well for each $x_i \in U_i$ we have that $\phi^*(\Theta(x_i)) = \Theta(\phi(x_i)) = g_i(\phi(x_i)) = \alpha_i \in (a_i, b_i)$ and we are finished. □

The following result was advised by the same anonymous external examiner.

3.4.3 Theorem. $C_l(X)$ is separable iff X admits a weakly injective continuous map onto a separable metrizable space.

Proof: Suppose $\phi: X \to Y$ is a weakly injective continuous map where Y is a separable metrizable space. Then $C_k(Y)$ is separable and $\phi^*: C_k(Y) \to C_k(X)$ is continuous (from [23]). Hence $\phi^*: C_k(Y) \to C_l(X)$ is continuous and $C_l(X)$ is separable.

Conversely, suppose $C_l(X)$ is separable and let D be a countable dense set in $C_l(X)$. Define $\phi : X \to \mathbb{R}^D$ by $\pi_f \circ \phi(x) = f(x)$ where $\pi_f : \mathbb{R}^D \to \mathbb{R}$

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is the f^{th}–projection. Observe that $\mathbb{R}^D$ is separable metrizable since D is countable and ϕ is continuous since each $f \in D$ is continuous. To show that ϕ is weakly injective, it suffices to show that $\phi^* : C(\mathbb{R}^D) \to C_l(X)$ is a dense map (Proposition 3.4.2). Well, let W be a nonempty open set in $C_l(X)$. Since D is dense in $C_l(X)$, let $g \in D \cap W$. So $\pi_g \in C(\mathbb{R}^D)$ and we have that $\phi^*(\pi_g) = \pi_g \circ \phi = g$. Hence $\phi^*(C(\mathbb{R}^D))$ is dense in $C_l(X)$ and we have our result. □

3.5 Networks in $C_l(X)$

Let us now determine when $C_l(X)$ has a countable network.

3.5.1 Theorem. If X is separable, then $C_l(X)$ has a countable network.

Proof: Let D be a countable dense set in X. Then $C_p(D)$ has a countable network ([23]). Hence $C_D(X)$ has a countable network by Corollary 2.3.3. Since $C_l(X)$ is coarser than $C_D(X)$ (See the first part of proof of (e)⇒(c) in Theorem 3.2.2), it has a countable network. □

We then have the following for a space X which is not necessarily regular.

3.5.2. Corollary. If X has a countable k–network, then $C_l(X)$ has a countable network.

Proof: If X has a countable k–network, then $C_p(X)$ has a countable network and so $C_l(X)$ has a countable network. □

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3.6 Countability

3.6.1 Theorem. The following are equivalent:

(a) $C(X)$ has a second countable topology which is coarser $C_l(X)$.

(b) $C(X)$ has a first countable topology which is coarser than $C_l(X)$.

(c) The points of $C_l(X)$ are G_δ-points.

(d) $C_l(X)$ is second countable.

(e) $C_l(X)$ is first countable.

(f) X has a countable $\pi-$base.

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Proof: (f) $\Rightarrow$(d): If X has a countable $\pi-$base $\mathcal{P}$ then $\{[P \times (a, b)] : P \in \mathcal{P}, a, b \in \mathbb{Q}\}$ is a countable subbase for $C_l(X)$ by Lemma 3.1.2.

$(c) \Rightarrow (f)$: Let us assume the zero-function f_0 is a G_δ-point i.e. $\{f_0\} = \bigcap\{B_n : n \in \mathbb{N}\}$ where

$B_n := [U_{n_1} \times (a_{n_1}, b_{n_1})] \cap ... \cap [U_{n_i} \times (a_{n_i}, b_{n_i})] \cap ... \cap [U_{n_{m_n}} \times (a_{n_{m_n}}, b_{n_{m_n}})]$, $a_{n_i} < 0$, $b_{n_i} > 0$ for $1 \leq i \leq m_n$, where U_{n_i} are open sets in X, $m_n \in \mathbb{N}$. We show that $\{U_{n_i} : n \in \mathbb{N}, 1 \leq i \leq m_n\}$ is a $\pi-$base for X. Let U be any open set in X. Assume each $U_{n_i} \backslash U \neq \emptyset$. Choose $x \in U$ and a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(X \backslash U) = 0$. Then $f \in \bigcap\{B_n : n \in \mathbb{N}\} = \{f_0\}$ which is a contradiction

(d)$\Rightarrow$ (e)$\Rightarrow$(b): This is clear.

(a)$\Rightarrow$ (b)$\Rightarrow$(c):Clear.

(d)$\Rightarrow$ (a): Clear. □

3.7 Baire Properties

3.7.1 Definition. A **Baire** space is one in which every sequence of open dense subsets has a dense intersection. If we require only a nonempty intersection, then the space is said to be of **second category in itself**.

Firstly a lemma before we proceed.

3.7.2 Lemma. Let U be an infinite open subset of X and $t \in \mathbb{R}$. Then the set $W = [U \times (t, t + \alpha_0)]$,where $\alpha_0 \in \mathbb{R}$ is fixed and $\alpha_0 > 0$, is dense and open in $C_l(X)$.

Proof: Clearly W is open in $C_l(X)$. We show that W is dense in $C_l(X)$. Let $W' = [U_1 \times (a_1, b_1)] \cap ... \cap [U_n \times (a_n, b_n)]$ be a nonempty basic open set in $C_l(X)$ where U_i is open in X and $a_i, b_i \in \mathbb{R}$ such that $a_i < b_i$ for each $i \in I := \{1, ..., n\}$. We show that $W \cap W' \neq \emptyset$. Since $W' \neq \emptyset$, let $f \in W'$. Then $f \in [U_i \times (a_i, b_i)]$ and so there is a $x_i \in U_i$ such that $f(x_i) \in (a_i, b_i)$, for each $i \in I$. Since U is infinite choose $x_0 \in U \backslash \{x_1, ..., x_n\}$. Choose open sets $V_0, V_i \subset X$ such that $x_0 \in V_0$, $x_i \in V_i$ and $V_i \cap V_j = \emptyset$ when $i \neq j$, and furthermore $V_0 \cap V_i = \emptyset$ for each $i \in I$. Choose $n_0 \in (t, t + \alpha_0)$ and let $m = \min\{n_0, f(x_1), f(x_2), ..., f(x_n)\}$. Now choose $g_i : X \longrightarrow [m, f(x_i)]$ continuous such that $g_i(x_0) = n_0$, $g_i(x_i) = f(x_i)$ and $g_i(X \backslash \{V_0 \cup V_i\}) = m$, for each $i \in I$. Define $\kappa = \max\{g_i : i \in I\}$, which is in $C(X)$. We finish if we show that $\kappa \in W \cap W'$. Clearly, by construction of κ we have that $\kappa \in W'$. Also $\kappa(x_0) = \max\{g_i(x_0) : i \in I\} = n_0$. Hence $\kappa \in W \cap W'$. □

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3.7.3 Corollary. If $C_l(X)$ is a Baire space then every compact set in X has a finite interior.

Proof: Let K be a compact set in X with infinite interior. For each $n \in \mathbb{N}$ let $W_n := \{f \in C(X) : f(x) > n \text{ for some } x \in \text{int}K\}$, which is open and dense in $C_l(X)$ by Lemma 3.7.2. Since $C_l(X)$ is a Baire space we have that $\bigcap W_n \neq \emptyset$. Let $h \in \bigcap W_n$. Then there is a sequence $(z_n)_n \subset K$ such that $h(z_n) > n$ for each $n \in \mathbb{N}$. Since the sequence $(z_n)_n$ clusters in the compact set K, the sequence $(h(z_n))_n$ should cluster since h is continuous. This is impossible since $h(z_n) > n$ for each $n \in \mathbb{N}$. Consequently K has a finite interior. □

3.7.4 Corollary.(a) If X is locally compact, then $C_l(X)$ is Baire iff X is discrete.

(b) If X is compact, then $C_l(X)$ is Baire iff X is finite.

Proof: (a): If X is discrete then $\mathbb{R}^X$ is Baire. Assume that $C_l(X)$ is Baire. Let $x \in X$. Then since X is locally compact there is an open set U such that $x \in U$ and $\overline{U}$ is compact. Well, since $\overline{U}$ has a finite interior we have $\{x\}$ to be open and so X is discrete.

(b): Assume $C_l(X)$ is Baire. Since X is locally compact we have X to be discrete from (a). But X is compact and this is only possible if X is finite. The converse follows from Corollary 3.7.3. □

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CHAPTER 4

The Weak Topology of $C_k(X)$

In this chapter we give an outline of results on the weak topology of $C_k(X)$, denoted by $C_\omega(X)$. The results are mainly from [26, 18]. Let $M(X)$ denote the dual of $C_k(X)$ (i.e continuous linear functionals on $C_k(X)$). Then a subbasic open set in $C_\omega(X)$ is of the type

$$\lambda^{-1}(V) = \{f \in C(X) : \lambda(f) \in V\}$$

where $\lambda \in M(X)$ and V is open in $\mathbb{R}$.

Equivalently, $C_\omega(X)$ can be generated by a collection of seminorms: Define a seminorm as $P_F(f) = \max\{| \lambda(f) | : \lambda \in F\}$ for each $f \in C(X)$, F a finite subset of $M(X)$. Then the collection of seminorms $\Gamma = \{P_F : F$ is a finite subset of $M(X)\}$ generates a locally convex topology on $C_k(X)$ i.e. the weak topology denoted as $C_\omega(X)$. Hence this topology is determined by neighbourhoods at the zero function f_0.

Let $\lambda \in M(X)$ and $A \subset X$. Then λ is said to be supported on A if, whenever $f \in C(X)$ with $f|_A = 0$, then $\lambda(f) = 0$. Then the support of λ, i.e. $\operatorname{supp} \lambda$,

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is the smallest compact subset of X on which λ is supported. Note that $\lambda \in M_p(X)$ iff $\operatorname{supp} \lambda$ is a finite set, where $M_p(X)$ denotes the dual of the point–open topology $C_p(X)$. In comparison to other topologies of $C(X)$, we have $C_p(X) \leq C_\omega(X) \leq C_k(X)$.

4.1 Comparison of Topologies

Clearly if X is discrete we would have that $C_p(X) = C_\omega(X) = C_k(X) = \mathbb{R}^X$. A pseudofinite space is one where every compact set is finite. From [18] we have the following result:

4.1.1 Theorem. For any space X, $C_p(X) = C_\omega(X)$ if and only if X is pseudofinite.

Proof: If X is pseudofinite, then the result follows since $C_p(X) = C_k(X)$. For the converse, see [18]. □

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4.2 Metrizability

A subset H of a topological vector space X is said to be **total** if for each $\lambda \neq \lambda_0$, where λ_0 is the zero functional of the dual of X, there is a $f \in H$ such that $\lambda(f) \neq 0$. This is equivalent to the vector subspace span H generated by H being dense in X. It follows that since $C_\omega(X)$ is generated by collections of continuous linear functionals on $C_k(X)$ that are total, then $C_\omega(X)$ is metrizable if and only if $M(X)$ has a countable Hamel basis (see p. 157 in [33]).

In fact the metrizability of $C_\omega(X)$ is similar to $C_p(X)$. We know from [23] that $C_p(X)$ is metrizable if and only if X is countable.

4.2.1 Theorem. The following are equivalent:

(a) $C_\omega(X)$ is a q–space

(b) $C_\omega(X)$ is first countable

(c) $C_\omega(X)$ is (separable) metrizable

(d) $M(X)$ has a countable Hamel basis

(e) X is a countable pseudofinite space.

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Proof: For brevity we give references to portions of the proof.

(b) $\iff$ (c) $\iff$ (d) $\iff$ (e): See [18] and pg. 159 in [33].

(b)$\Rightarrow$(a): Clear.

(a)$\Rightarrow$(b): We give the proof from [26]. Assume that $C_\omega(X)$ is a q–space and let $(U_n)_n$ be a sequence of basic neighbourhoods of f_0, the zero function, in $C_\omega(X)$ that satisfy the q–space property at f_0. For each n, let $\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{k_n n}$ be in $M(X)$ and $\epsilon_n > 0$ such that $U_n = \bigcap\{\lambda_{in}^{-1}(-\epsilon_n, \epsilon_n) : 1 \leq i \leq k_n\}$. Let $H = \{\lambda_{in} : n \in \mathbb{N},\ k_n \in \mathbb{N} \text{ and } 1 \leq i \leq k_n\}$. We show that H is a total subset of $M(X)$. Well, assume the contrary and let $f \neq f_0$ such that $\lambda(f) = 0$ for all $\lambda \in H$. For $n \in \mathbb{N}$, let $f_n = nf$. Since $f_n \in C_\omega(X)$ and $\lambda_{in}(f_n) = 0$ for each

$1 \leq i \leq k_n$ it follows that $f_n \in U_n$. As this holds for each n, the sequence (f_n) must cluster in $C_\omega(X)$. Now let $\mu \in M(X)$ such that $\mu(f) = 1$. Now $\mu((f_n)_n)$ clusters in $\mathbb{R}$. But $\mu(f_n) = \mu(nf) = n\mu(f) = n$ implying $\mu((f_n)_n)$ does not cluster in $\mathbb{R}$. Contradiction and so $\lambda(f) \neq 0$ for some $\lambda \in H$. Hence H is total. From theorem 2.1 [26], $C_\omega(X)$ is then a subcosmic space (i.e $C_\omega(X)$ has a coarser cosmic topology). We then have our result since a subcosmic q–space is first countable (see [32]). □

Let us investigate when $C_\omega(X)$ is complete. Again, $C_\omega(X)$ behaves similar to $C_p(X)$ with respect to completeness. In fact $C_p(X)$ is complete if and only if X is a discrete space. Moreover $C_p(X)$ is completely metrizable if and only if X is a countable discrete space.

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The following theorem is from [18]:

4.2.2 Theorem. For any space X, the following are equivalent:

(a) $C_\omega(X)$ is completely metrizable

(b) $C_\omega(X)$ is metrizable and $C_p(X)$ is complete

(c) $C_\omega(X)$ is complete and $C_p(X)$ is metrizable

(d) $C_\omega(X) = C_p(X) = \mathbb{R}^X$ and X is countable

(e) X is a countable discrete space.

Proof: (e) $\Longrightarrow$ (a): Follows from $C_\omega(X) = \mathbb{R}^X$.

(e) $\Longleftrightarrow$(d) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (b): Clear.

(a) $\Rightarrow$ (e): Well, if $C_\omega(X)$ is complete then $C_\omega(X)$ is a product of lines (see pg.

248 in [16]). So $C_\omega(X)$ is barrelled (i.e $C_\omega(X)$ has a base of neighbourhoods which are closed, absolutely convex and absorbing) and hence it is a Mackey topology (i.e the finest locally convex topology on $C(X)$ such that $M(X)$ is the dual of $C(X)$). Then $C_\omega(X) = C_k(X) = C_p(X)$ by Theorem 4.1.1, so we have that X is countable and discrete. □

We have the following result on when $C_\omega(X)$ is Baire.

4.2.3. Theorem. $C_\omega(X)$ is a Baire space if and only if $C_p(X)$ is Baire and X is pseudofinite.

Proof: Let X be a pseudofinite space such that $C_p(X)$ is Baire. Now $C_\omega(X) = C_p(X)$ by theorem 4.1.1 and so $C_\omega(X)$ is Baire.

For the converse, if $C_\omega(X)$ is Baire then it is barrelled and has a Mackey topology. So $C_\omega(X) = C_p(X)$ and we have $C_p(X)$ to be Baire with X pseudofinite (Theorem 4.1.1). □.

4.3 Netweight

There is an interesting relationship between the netweights of the weak topology and weak* topology on the dual of a topological vector space E (denoted E_ω and E^* respectively). The netweight of a space X is defined by $n\omega(X) := \omega +$ min$\{| \mathcal{B} |: \mathcal{B}$ is a network for $X\}$. A k-netweight is defined by $kn\omega$:=ω+min$\{| \mathcal{B} |: \mathcal{B}$ is a k-network of $X\}$. A subbasic open set in the weak* topology E^* has the form

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$e_x^{-1}(V) = \{f \in E^* \mid f(x) \in V\}$ where $x \in E$, $V \subset \mathbb{R}$ and e_x is an evaluation functional such that $e_x(f) = f(x)$. Similarly, a subbasic open set in E_ω is of the type $f^{-1}(V) = \{x \in E \mid f(x) \in V\}$ where $f \in E^*$ and V open in $\mathbb{R}$. The following result is from [26].

4.3.1. Theorem. $n\omega(E_\omega) = n\omega(E^*)$

Proof: We begin by showing that $n\omega(E_\omega) \leq n\omega(E^*)$.

Let $\mathcal{B}$ be a network for E^* such that $\mid \mathcal{B} \mid = n\omega(E^*)$ and let $\mathcal{V}$ be a countable base for $\mathbb{R}$. For each $B \in \mathcal{B}$ and $V \in \mathcal{V}$, set $[B, V] = \{x \in E : f(x) \in V$ for all $f \in B\}$ and let $\mathcal{N}$ denote the collection of all finite intersections of elements of the form $[B, V]$. Clearly $\mid \mathcal{N} \mid \leq \mid \mathcal{B} \mid$. We show that $\mathcal{N}$ is a network for E_ω: Let $x \in E_\omega$ and let U be a basic neighbourhood of x in E_ω. There is a finite subset F of E^* and $\epsilon > 0$ such that $U = \{y \in E : \mid f(y - x) \mid < \epsilon$ for all $f \in F\}$. We must find $N \in \mathcal{N}$ such that $x \in N \subset U$. Fix f in F and consider $f(x) \in (f(x) - \epsilon, f(x) + \epsilon)$. Choose $V_f \in \mathcal{V}$ such that $f(x) \in V_f \subset (f(x) - \epsilon, f(x) + \epsilon)$. Since $f \in e_x^{-1}(V_f)$ and $e_x^{-1}(V_f)$ open in E^*, there is a $B_f \in \mathcal{B}$ such that $f \in B_f \subset e_x^{-1}(V_f)$. Clearly, $x \in [B_f, V_f]$. Since this holds for each $f \in F$, let $N = \bigcap\{[B_f, V_f] : f \in F\}$. Then $x \in N \subset U$ and so $\mathcal{N}$ is a network for E_ω. Therefore $n\omega(E_\omega) \leq \mid \mathcal{N} \mid \leq n\omega(E^*)$.

To show $n\omega(E^*) \leq n\omega(E_\omega)$, let $\mathcal{F}$ be a network for E_ω such that $\mid \mathcal{F} \mid = n\omega(E_\omega)$. Let $\mathcal{U}$ be a countable base for $\mathbb{R}$. For each $F \in \mathcal{F}$ and $U \in \mathcal{U}$, set $[F, U] = \{f \in E^* : f(x) \in U$ for all $x \in F\}$ and let $E^*_{\mathcal{F}}$ denote E^* with the topology having subbasic open sets of the form $[F, U]$ where $F \in \mathcal{F}$ and $U \in \mathcal{U}$. It is obvious that $\omega(E^*_{\mathcal{F}}) \leq \mid \mathcal{F} \mid \leq n\omega(E_\omega)$. We finish if we show that

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the topology of $E^*_{\mathcal{F}}$ is finer than the weak* topology on E^*. To show this, let W be a non–empty subbasic open set in E^* with the weak*–topology. Let $x \in E$ and $U \in \mathcal{U}$ such that $W = e_x^{-1}(U)$. Let $f \in W$. Since $f^{-1}(U)$ is open in E_ω and since $x \in f^{-1}(U)$, there is $F \in \mathcal{F}$ such that $x \in F \subset f^{-1}(U)$. One sees that $f \in [F, U] \subset W$. Clearly then $f \in W$ in $E^*_{\mathcal{F}}$. This shows that E^* is coarser than $E^*_{\mathcal{F}}$ and $n\omega(E^*) \leq n\omega(E^*_{\mathcal{F}}) \leq \omega(E^*_{\mathcal{F}}) \leq |\, \mathcal{F} \,| \leq n\omega(E_\omega)$. □

4.3.2. Corollary. $n\omega(C_\omega(X)) = n\omega(M(X))$.

4.3.3. Problem. Is $n\omega C_\omega(X) = kn\omega X$?

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CHAPTER 5

A Function Space Approach To A Problem Of Banach

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This is a problem posed by Banach on July 17, 1935 and which can be found in the 'Scottish Book' (edited by R. Daniel Mauldin in 1981). In essence, the problem can be stated as follows: 'When can a metric space have a continuous one–to–one map onto a compact metric space?' We say that a metric space X has a contraction to a compact metric space Y (or has a compact metrizable contraction to Y) iff there exists a continuous bijection from X onto Y. In connection with this problem we begin by looking first at topological spaces having compact metric contractions before studying function spaces having the same property. We will in particular look at the c_0 of sequences of $\mathbb{R}^\omega$ that converge to zero.

5.1 Theorem ([28]). Every locally compact space X has a contraction to a compact space.

As a corollary one has the following result.

5.2 Corollary ([17]). Let X be a locally compact separable metric space. Then X has a contraction to a compact metric space Y.

Proof: Since X is locally compact, by Theorem 5.1 let $f : X \longrightarrow Y$ be a contraction map where Y is compact. Since X is separable metric, it is cosmic. Hence Y is cosmic since it is a continuous image of a cosmic space. The result then follows from the fact that a compact cosmic space is metrizable. □

5.3 Corollary. $\mathbb{R}$ has a compact metrizable contraction.

The next result is an extension of Corollary 5.2.

5.4 Lemma ([17]). Let $X = \prod_{i=0}^{\infty} X_i$ where X_i is locally compact separable metrizable. Then X has a compact metrizable contraction.

Proof: Let $f_i \colon X_i \longrightarrow Y_i$ be a contraction map from X_i to a compact metric space Y_i (Corollary 5.2). Define $\varphi : \prod_{i=1}^{\infty} X_i \longrightarrow \prod_{i=1}^{\infty} Y_i$ by $\varphi(x_0, x_1, ..., x_i) = (f_0(x_0), f_1(x_1), \ldots)$. Let $x, y \in X$ such that $x \neq y$. Then there is a j$^{\text{th}}$ term such that $x_j \neq y_j$. So $f_j(x_j) \neq f_j(y_j)$ since f_j is injective. It follows that $\varphi(x) \neq \varphi(y)$ and φ is injective. Since each f_i is continuous, φ is continuous. Clearly φ is onto and $\prod_{i=1}^{\infty} Y_i$ is compact.

5.5 Corollary. $\mathbb{R}^{\omega}$ has a compact metrizable contraction.

Our next goal is to show that every homogeneous Polish space has a compact metrizable contraction. Firstly, it can be shown that every homogeneous Polish space is either locally compact or non σ–compact (see [17]). We move to show that each non σ–compact Polish space has a continuous one–to–one map onto $\mathbb{R}^{\omega}$. We set out the notation:

Let $\mathbb{R}^{\omega}(m) := \{(x_0, x_1, x_2, \ldots) \in \mathbb{R}^{\omega} \colon x_n \neq m$ for finitely many $n \in \omega\}$, $C_m := \mathbb{R}^{\omega} \times \mathbb{R}^{\omega}(m), \mathbb{R}^{\omega} \times \mathbb{R}^{\omega} = \prod_{i=0}^{\infty} \mathbb{R}_i^1 \times \prod_{i=0}^{\infty} \mathbb{R}_i^2$, where $\mathbb{R}_i^1 = \mathbb{R}_i^2 = \mathbb{R} = \mathbb{R}_i$ and let $\pi_i : \mathbb{R}^{\omega} \longrightarrow \mathbb{R}_i$, $\pi_i^{\alpha} : \mathbb{R}^{\omega} \times \mathbb{R}^{\omega} \longrightarrow \mathbb{R}_i^{\alpha} (\alpha = 1, 2)$ be projections.

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5.6 Lemma ([17]). Let $m \in \omega$ and $f: M \longrightarrow \mathbb{R}^\omega \times \mathbb{R}^\omega \backslash C_m$ be a continuous one–to–one map from a closed subspace M of a separable metric space X. Then there exists a continuous one–to–one extension $f^*: X \longrightarrow \mathbb{R}^\omega \times \mathbb{R}^\omega$ of f such that $f^*(X\backslash M) \subset C_m$.

Proof: Since M is a G_δ–subset, let $X\backslash M = \bigcup_{i=0}^{\infty} M_j$ where M_j is a closed subset of X and $M_j \subset M_{j+1}$, $j \in \omega$. Let $h: X \longrightarrow \mathbb{R}^\omega$ be an embedding such that for each $n \in \omega$, the set $\{i \in w: \pi_n h = \pi_i h\}$ is infinite. Define continuous maps $f_j^\alpha: M \cup M_j \longrightarrow \mathbb{R}_j^\alpha$ $(\alpha = 1, 2)$ by

$$f_j^\alpha(x) = \begin{cases} \pi_j^\alpha f(x), & x \in M, \quad \alpha = 1, 2 \\ \pi_j h(x), & x \in M_j, \quad \alpha = 1 \\ m, & x \in M_j, \quad \alpha = 2 \end{cases}$$

Let $(f_j^\alpha)^*: X \longrightarrow \mathbb{R}_j^\alpha$ be a continuous extension of f_j^α. The map $f^*: X \longrightarrow \mathbb{R}^\omega \times \mathbb{R}^\omega$, defined by $\pi_j^\alpha f^* = (f_j^\alpha)^*$, is an extension of the map f. We show $f^*(X\backslash M) \subset C_m$: Observe that if $x \in X\backslash M$, then there is an $n \in \omega$ such that $x \in M_i$ for $i \geq n$. So by construction of f_j^α, $\pi_i^2 f^*(x) = m$ for each $i \geq n$ and so $f^*(x) \in C_m$. Now, our premise states that $f^* |_M = f$ is one–to–one. We show that $f^* |_{X\backslash M}$ is one–to–one. Let $x, y \in X\backslash M$ and $x \neq y$. Then there is an $i \in \omega$ such that $x, y \in M_i$ and $\pi_i h(x) \neq \pi_i h(y)$ since h is an embedding. Then $\pi_i^1 f^*(x) = \pi_i h(x) \neq \pi_i h(y) = \pi_i^1 f^*(y)$. Hence f^* is one–to–one. □

For the main theorem we need the following: A metric space is a Borel space iff it is homeomorphic to a Borel set of a Polish space. It is known that if X is a Borel space and is non σ–compact, then there is a closed subset $P \subset X$ homeomorphic to the space of irrational numbers [19]. A continuous one–to–one image of a Borel space is a Borel space [21]. Each Borel space is a continuous one–to–one image of a closed subspace of the space of irrational numbers (see [21]).

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5.7 Theorem. ([17]). Let X be a non σ-compact Borel space, $P \subset X$ be a closed copy of the space of irrational numbers, and let $h\colon X \longrightarrow \mathbb{R}^\omega \times \mathbb{R}^\omega(0)$ be an embedding. Then for each open set $U \subset X$ such that $P \subset U$ there exists a continuous bijective map $f\colon X \longrightarrow \mathbb{R}^\omega \times \mathbb{R}^\omega$ such that

$$f\mid_{(X\backslash U)} = h\mid_{(X\backslash U)}$$

Proof: Assume $P = \bigcup_{i=1}^{\infty} P_i$, where $\{P_i\}$ is a family of pairwise disjoint closed subsets of P homeomorphic to P. Set $P_0 = X\backslash U$. Let $\{F_i,\ i \in \omega\}$ be a locally finite closed covering of X such that $P_i \subset F_i \backslash \bigcup_{j\neq i} F_j$ for each $i \in \omega$. Then we set

$$B_n := \mathbb{R}^\omega \times \mathbb{R}^\omega \backslash \bigcup_{j>n} C_j,\ A_n := \bigcup_{i=0}^{n} F_i$$

Then we have $B_n \subset B_{n+1}$, $B_{n+1} = B_n \cup C_{n+1}$, $\bigcup_{n=0}^{\infty} B_n = \mathbb{R}^\omega \times \mathbb{R}^\omega$, $A_n \subset A_{n+1}$, $\bigcup_{n=0}^{\infty} A_n = X$. We define inductively injective continuous maps $f_n : A_n \longrightarrow \mathbb{R}^\omega \times \mathbb{R}^\omega$ such that $f_{n+1}\mid_{A_n} = f_n$ and $B_n \subset f_{n+1}(A_{n+1}) \subset B_{n+1}$. Let $f_0 := h\mid_{A_0} = h\mid_{F_0}$, $X\backslash U \subset F_0$. Assume the map f_n is defined. Let $f_{n+1}\colon A_{n+1} \longrightarrow B_{n+1}$ be a continuous injective extension of a map $f^*_{n+1}\colon A_n \cup D_{n+1} \longrightarrow B_n$ where D_{n+1} is a closed subset of P_{n+1} and the map $f^*_{n+1}\mid_{D_{n+1}} \longrightarrow B_n \backslash f_n(A_n)$ is bijective (Corollary 5.7). Now, define $f : X \longrightarrow \mathbb{R}^\omega \times \mathbb{R}^\omega$ by $f(x) := f_n(x)$ iff $x \in A_n$. We see that f is continuous since $f\mid_{F_n} = f_n\mid_{F_n}$ is continuous and $\{F_n : n \in w\}$ is locally finite. As each f_n is injective, f is injective since $A_n \subset A_{n+1}$ and $X = \bigcup_{n=0}^{\infty} A_n$. It also follows that f is surjective because $\bigcup_{n=0}^{\infty} B_n = \mathbb{R}^\omega \times \mathbb{R}^\omega$ and $B_n \subset f_{n+1}(A_{n+1})$. Observe $f\mid_{(X\backslash U)} = f_0\mid_{(X\backslash U)} = h\mid_{(X\backslash U)}$ by construction. □

As a consequence, we have our important result from [17]:

5.8 Theorem. Every homogeneous Polish space has a compact metrizable contraction.

Proof: If X is a homogeneous Polish space, then X is locally compact or X is non σ-compact [17]. For the first case the result follows from Corollary 5.2. In the latter case, from Theorem 5.7 there is a continuous bijective map from

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X to $\mathbb{R}^\omega$. □

Let us proceed to investigate Banach's problem as applied to function spaces. It was pointed out by an anonymous external examiner that all Borel metrizable function spaces $C_p(X)$ admit a contraction if X is not discrete: for such spaces contain a closed copy of the irrationals and are not σ-compact. The desired consequences follow from Theorem 5.7 and Corollary 5.5. We assume that each of these function spaces is metrizable and we want to find weaker compact metric topologies $C_\alpha(X)$. Now, $C_k(X)$ is metrizable $\iff$ X is hemicompact[1]([23]), $C_p(X)$ is metrizable $\iff$ X is countable ([23]), $C_D(X)$ is metrizable $\iff$ D is countable (Corollary 2.4.7) and $C_\omega(X)$ is metrizable $\iff$ X is countable and pseudofinite(Theorem 4.2.1).

Each of the spaces $C_k(X), C_\omega(X), C_p(X), C_D(X)$ contains closed copies of $\mathbb{R}$. In fact $C_\ell(X)$ too contains a closed copy of $\mathbb{R}$. We show the proof : Consider the map $i : \mathbb{R} \to C_\ell(X)$ defined by $i(c) = c_X$ where c_X represents the constant function with the value c over X. Firstly we show continuity. Let $[U \times (a,b)]$ be an open set in $C_\ell(X)$ where U is open in X and $a, b \in \mathbb{R}$. Then $i^{-1}([U \times (a,b)]) = (a,b)$ which is open in $\mathbb{R}$. We now show that $i(\mathbb{R})$ is closed in $C_\ell(X)$. Well, let $f \in C_\ell(X)\backslash i(\mathbb{R})$. Choose $x, y \in X$ such that $f(x) < s < f(y)$, where $s \in \mathbb{R}$. Let U_x, U_y be open sets in X such that $U_x \cap U_y = \emptyset$ and $x, y \in U_x, U_y$ respectively. Then $f \in [U_x \times (-\infty, s)] \cap [U_y \times (s, \infty)]$ which is open in $C_\ell(X)$ and is contained in $C_\ell(X)\backslash i(\mathbb{R})$.

Clearly, having copies of the reals in these function spaces makes the investigation of Banach's problem on function spaces interesting since we know that the reals have compact metrizable contractions (Corollary 5.3).

The next step would be to weaken the function space topologies to generate compact topologies. We could weaken them to the indiscrete topology but this is clearly not interesting. The ideal approach to Banach's problem is to find a

[1] A space X is hemicompact if there is a countable family of compact sets $\{A_i\}$satisfying the condition that for any compact set K there is a A_j in the family such that $K \subset A_j$.

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property $\mathcal{P}$ on X such that the function space $C_k(X), C_p(X), C_\omega(X)$ or $C_D(X)$ admit a coarser compact metrizable topology. We do have for instance if X is a hemicompact $\aleph_0$ k–space[2], then $C_k(X)$ is Polish (see [23]) and consequently has a compact metrizable contraction (Theorem 5.8). In [23], it was also shown that $C_p(X)$ is Polish iff X is countable and discrete. The same situation holds for $C_D(X)$ (Theorem 2.5.4). As for $C_\omega(X)$, $C_\omega(X)$ is homeomorphic to $\mathbb{R}^X$ iff X is countable and discrete (Theorem 4.2.2). Hence, by Corollary 5.5 and Theorem 5.8, if X is countable and discrete then $C_p(X)$, $C_D(X)$, and $C_\omega(X)$ have compact metrizable contractions.

We proceed to study various specialized cases for certain function spaces that have compact metrizable contractions. To this end we denote $c_0 = \{(x_i) \in \mathbb{R}^\omega : \lim_{i\to\infty} \mid x_i \mid = 0\}$ with the subspace topology and $c = \{(x_i) \in \mathbb{R}^\omega : (x_i)$ converges$\}$ also with the subspace topology. We let l^p denote all p^{th}power summable sequences for $1 \leq p < \infty$. Let σ^ω be the countable product of the pre–Hilbert space $l^2_f := \{(x_i) \in l^2 : x_i = 0 \text{ a.e}\}$.

Let us consider the Banach space $(c_0, \|\|)$ under the sup norm. Klee [13] showed that $(c_0, \|\|)$ possesses a compact metrizable contraction. We give Klee's proof. Note that this result can be obtained using Theorem 5.8 since c_0 is a separable Banach space.

5.9 Theorem. The Banach space $(c_0, \|\|)$ has a compact metrizable contraction.

Proof: Kadec showed that c_0 is homeomorphic to l^1 (see [13]) and by a theorem of Mazur [22], l^1 is homeomorphic to the Hilbert space l^2 which in turn is homeomorphic to the unit ball $S_n = \{x : \|x\| \leq 1\}$ under the norm topology (see [14]). Now the unit ball under the weak topology S_ω is coarser than S_n and admits a homeomorphism to the Hilbert Cube ([15]). Hence c_0 has a compact metrizable contraction. □

[2] A space X is a k–space provided that if A is a subset of X such that the intersection of A with each compact subset of X is closed, then A must be closed.

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From now on we consider c_0 with the product topology and view c_0 as a subspace of $\mathbb{R}^\omega$. We know the following from [8].

5.10 Theorem. c_0 and c are linearly homeomorphic.

Proof: Define $h\colon c \longrightarrow c_0$ by $h(x) = (f_0(x), f_1(x), ..., f_i(x), ...)$ where $f_0(x) = \lim x$ and $f_i(x) = \lim x - x_i$, $i > 0$, where $x \in c$. We show that h is injective. Let $x, y \in c$ and $x \neq y$. So there is an i$^{\text{th}}$ term such that $x_i \neq y_i$. Consequently, $h(x) \neq h(y)$ since $f_i(x) \neq f_i(y)$. Clearly h is surjective since $c_0 \subset c$. Also for $x, y \in c$, $h(x) + h(y) = (f_0(x), f_1(x), ..., f_i(x)) + (f_0(y), ..., f_i(y), ...) = (f_0(x+y), ..., f_i(x+y), ...) = h(x+y)$. For $\alpha \in \mathbb{R}$, $h(\alpha x) = \alpha h(x)$. So h is linear. We are finished if we show that h is an open map: Let W be a subbasic open set in c. Then for some V_i open in $\mathbb{R}$, $W = \pi_i^{-1}(V_i) = h^{-1} \circ \pi_i^{-1}|_{c_0}(V_i)$ since h is injective and where $\pi_i : \mathbb{R}^\omega \longrightarrow \mathbb{R}^i$ is the natural i$^{\text{th}}$ projection. Then $h(W) = \pi_i^{-1}|_{c_0}(V_i) = c_0 \cap \pi_i^{-1}(V_i)$ which is open in c_0. $\square$

5.11 Remark. It has also been shown that c_0 is homeomorphic to σ^ω ([3, 8]).

Let us now consider the Hilbert space l^2 in c_0. The next theorem shows that l^2 is homeomorphic to the countable infinite product of lines. But before we proceed we state the following lemma [34].

5.12 Lemma. Let A be a σ–compact subset of $\mathbb{R}^\omega$. Then $\mathbb{R}^\omega \backslash A$ is homeomorphic to $\mathbb{R}^\omega$.

The proof of the next theorem is from [34]. We give the main ingredients of the proof for brevity.

5.13 Theorem (Anderson). l^2 is homeomorphic to $\mathbb{R}^\omega$.

Proof: Let K be a compactification of l^2 such that K is homeomorphic to the Hilbert cube I^ω where $I = [0, 1]$ (see [34]). The subspace $E := \{x \in K : \sum\limits_{i=1}^{\infty} x_i^2 = 1\}$ of K is homeomorphic to $\mathbb{R}^\omega$ (Corollary 6.6.10 [34]). But $E \simeq S := \{x \in l^2 : \|x\| = 1\}$ (see [34]). Consequently $S \simeq \mathbb{R}^\omega$. Let

$v = (-1, 0, 0, \ldots) \in S$. We first show that $l^2 \simeq S\backslash\{v\}$. To this end, let $l_0^2 := \{x \in l^2 : x_1 = 0\}$ and define $g : l^2 \longrightarrow l_0^2$ by $g(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ which is an isometry. So $l^2 \simeq l_0^2$. We now show that $S\backslash\{v\} \simeq l_0^2$. Define $h : S\backslash\{v\} \longrightarrow l_0^2$ by

$$h(x) = \begin{cases} (0, x_2, x_3, \ldots)] & (x_1 \geq 0) \\ \frac{(0, x_2, x_3, \ldots)}{||0, x_2, x_3, \ldots||} & (x_1 \leq 0) \end{cases}$$

which is a homeomorphism. Hence $l^2 \simeq S\backslash\{v\}$. Since $\{v\}$ is σ–compact, $l^2 \simeq S\backslash\{v\} \simeq \mathbb{R}^\omega\backslash\{v\} \simeq \mathbb{R}^\omega$ (Lemma 5.12). Consequently $l^2 \simeq \mathbb{R}^\omega$. □

A generalization of Theorem 5.13 is the celebrated Anderson–Kadec Theorem: Every infinite dimensional separable Fréchet space (i.e. a locally convex completely metrizable and separable space) is homeomorphic to $\mathbb{R}^\omega$.

5.14 Corollary. The Hilbert space l^2 has a compact metrizable contraction.

The pre–Hilbert space l_f^2 is a subspace of l^2 and consequently a subspace of c_0. In fact we show the following.

5.15 Theorem. l_f^2 is dense in c_0 under the product topology.

Proof: Let $W := c_0 \cap \langle f, A, \epsilon \rangle$ be a basic open set in c_0 where A finite in $\mathbb{N}$. We show that $W \cap l_f^2 \neq \emptyset$. To this end, let $g \in W$ and let us say that $A = \{a_1, a_2, ..., a_N\}$. Let $h(1) = g(a_1), h(2) = g(a_2), ..., h(N) = g(N), h(N+1) = 0, h(N+2) = 0, ...$ Then $h(n) \in l_f^2 \cap c_0$ and $h \in W$. Hence l_f^2 is dense in c_0. □

We now pose the following question.

5.16 Problem. Does c_0 under the product topology have a compact metrizable contraction?

If the answer to the Problem 5.16 is in the affirmative, then we are in a position to discover function spaces with compact metrizable contractions. This is

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because there are a number of function spaces that are homeomorphic to c_0 that have appeared in the literature. For completeness, we give some of them here.

5.17 Definition. A space X is an **absolute** $F_{\sigma\delta}$**–set** if whenever X is a subset of a metric space Y, then $X = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} Y_{nk}$, where Y_{nk} is a closed subset of Y for n, $k = 1, 2, \ldots$

In [9], it was shown that if X is a countable non-discrete completely regular space such that $C_p(X)$ is an absolute $F_{\sigma\delta}$–set, then $C_p(X)$ is homeomorphic to σ^ω. However in [3], it was shown that if X is a countable nondiscrete metric space, then $C_p(X)$ and $C_p(X)^*$ are both homeomorphic to σ^ω. We proceed to elaborate on the latter point.

5.18 Lemma. ([3]). Let $C_0 = \{0\} \cup \{n^{-1} : n \geq 1\}$. Then $C_p(C_0)$ is homeomorphic to c_0.

Proof: For a continuous function f on C_0 we have the sequence $\{f(\frac{1}{n})\}_n$ convergent to $f(0)$. Then the map $\varphi : C_p(C_0) \longrightarrow c_0$ defined by $\varphi(f) = (f(0), f(1) - f(0), \ldots, f(\frac{1}{n}) - f(0), \ldots)$ is a homeomorphism. □

Note that C_0 is a countable and first countable compact space of which every nondiscrete space contains a copy. As a consequence, we have the following result from [3].

5.19 Theorem. Let X be a countable nondiscrete metric space. Then $C_p(X)$ and $C_p(X)^*$ are homeomorphic to σ^ω.

Proof: Assume X is nondiscrete. We have that X contains a closed C homeomorphic to C_0. Since X is countable, it is of dimension zero which ensures the existence of a retraction r from X to C (see [20] pp. 265–268). Let A be a subspace of $C_p(X)$ formed by functions which are zero on C_0. Then $C_p(X)$ is homeomorphic to $C_p(C) \times A$ by the homeomorphism
$\psi(f) = (f|_C, f - (f|_C) \circ r)$. Then we have that $C_p(X) \simeq C_p(C) \times A \simeq$

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$C_p(C_0) \times A \simeq c_0 \times A \simeq \sigma^\omega$. Hence, we have our result. □

5.20 Corollary. $c_0 \simeq C_p(\mathbb{Q}) \simeq \sigma^\omega$.

Proof: From Theorem 5.19 and Remark 5.11. □

We take a brief moment to comment on the search for an answer to Problem 5.16. One might hope to show that c_0 is homeomorphic to $\mathbb{R}^\omega$. Unfortunately this is not true as a consequence of the following theorem due to Chigogidze ([4]).

5.21 Theorem. The following conditions are equivalent for each locally convex linear topological space E.

(1) E is homeomorphic to the countable (finite or infinite) power of the real line.

(2) E is a Polish space.

From Theorem 5.21 we see that if $c_0 \simeq \mathbb{R}^\omega$ then $C_p(\mathbb{Q})$ is Polish. But this is only possible if $\mathbb{Q}$ is discrete (see [23]) yielding a contradiction. Hence c_0 is not homeomorphic to $\mathbb{R}^\omega$.

We now proceed with our investigation into spaces homeomorphic to c_0. Before we move to $C_D(X)$ we state without proof the following imperative result from [9].

5.22 Lemma. Let X be a countable nondiscrete completely regular space. Then one of the following conditions holds:

(i) There exists a clopen subset Y of X with exactly one accumulation point.

(ii) There exists a decomposition $X = \bigcup_{n\in\omega} X_n$ where $\{X_n\}_{n\in\omega}$ is a pairwise disjoint sequence of nondiscrete clopen sets.

We then have the following for $C_D(X)$ from [6].

5.23 Theorem. Let X be a countable nondiscrete space and D a dense subset of X such that $C_D(X)$ is a $F_{\sigma\delta}$–set. Then $C_D(X)$ is homeomorphic to σ^ω.

Proof: From Lemma 5.22, we must consider the two conditions that could prevail: Condition (i): $C_D(X)$ is homeomorphic to $C_{D\cap Y}(Y) \times C_{D\setminus Y}(X\setminus Y)$ where Y is a clopen set in X with exactly one nonisolated point. Now Y can be identified with the space $N_F = \mathbb{N} \cup \{\infty\}$ topologized by isolating points of $\mathbb{N}$ and $\{A \cup \{\infty\} \mid A \in F\}$ as a neighbourhood base at ∞, where F is a filter on $\mathbb{N}$. However, we know from the consequence of Theorem 1.1 in [9], that $C_{D\cap Y}(Y)$ is homeomorphic to c_0 and hence to σ^ω. By Corollary 5.4. [2], $C_D(X)$ is homeomorphic to σ^ω. Condition (ii): $C_D(X)$ can be identified with $\prod_{n\in\omega} C_{D\cap X_n}(X_n)$. Since X_n is nondiscrete and countable, $C_{D\cap X_n}(X)$ is analytic and so a Z_σ–space (Proposition 4.2 [6]). From Corollary 2.7 [7], $\prod_{n\in\omega} C_{D\cap X_n}(X_n)$ is homeomorphic to σ^ω. □

We proceed to discuss a few examples of linear subspaces of F^ω, where F is a separable Banach space, which are homeomorphic to c_0.

A generalization of the convex structure of linear spaces is the notion of an equiconnecting map.

5.24 Definition. Let E be a separable complete metric space. A map $\lambda\colon E\times E\times[0,\,1] \longrightarrow E$ is called an **equiconnecting map** on E if $\lambda(x,\,y,\,0) = x$, $\lambda(x,\,y,\,1) = y$ and $\lambda(x,\,x,\,t) = x$ for every $x,\,y \in E$ and $t \in [0,\,1]$. A subset X of E is **λ-convex** if $\lambda(X \times X \times [0,\,1]) \subseteq X$ where λ is an equiconnecting map.

5.25 Definition. Let E be a separable complete metric space with λ as an equiconnecting map. Then a λ–convex subset X of E is called a **λ–convex $F_{\sigma\delta}$–nest** in E if $X = \bigcap_{i=1}^{\infty} E_i$, where E_i are F_σ subsets of E such that there exist a sequence $\{C_i\}$ of closed λ–convex subsets of E and a homeomorphism Φ of X into X^ω satisfying for $i = 1,\,2, \ldots$

(1) X is an absolute retract dense in E and $C_i \cap X$ is an absolute retract dense

in C_i

(2) for each compact set $K \subseteq X$ there is a $c_i \in C_i \backslash E_{i+1}$ such that $\lambda(K \times \{c_i\} \times (0, 1])$, the open λ–cone, is contained in $E \backslash E_{i+1}$,

(3) $\{(x_i) \subset X^\omega : x_i = 0 \text{ a.e}\} \cup \prod_{i=1}^{\infty} (X \cap C_i) \subseteq \Phi(X)$.

In [7], it is shown that if E is a separable complete metric space with an equiconnecting map, then every λ–convex $F_{\sigma\delta}$–nest X in E is homeomorphic to σ^ω. Hence these spaces are homeomorphic to c_0 (Remark 5.11).

A special λ–convex $F_{\sigma\delta}$–nest is given in [7] for linear spaces:

5.26 Definition. Let E be a separable complete metric linear space. A linear subspace X of E is a **linear $F_{\sigma\delta}$–nest** in E if $X = \bigcap_{i=1}^{\infty} E_i$, where $\{E_i\}$ is a sequence of F_σ–linear subspaces of E such that there exist a sequence $\{C_i\}$ of closed convex subsets in E and a homeomorphism Φ of X into X^ω satisfying:

(1^*) : X is an absolute retract dense in E and $C_i \cap X$ is an absolute retract dense in C_i

(2^*) : $C_i \backslash E_{i+1} \neq \emptyset$ for $i = 1, 2, ...$

(3^*) : $\{(x_i) \in X^\omega : x_i = 0 \text{ a.e}\} \cup \prod_{i=1}^{\infty} (X \cap C_i) \subseteq \Phi(X)$.

Again in [7], Proposition 3.3 states that every linear $F_{\sigma\delta}$–nest is homeomorphic to σ^ω.

We will use these notions for our forthcoming discussion. For a separable Banach space $(F, \| \|)$, for $0 < q < \infty$ and $0 \leq p < \infty$, set

$$l^q(F) = \{(x_i) \in F^\omega : \sum_{i=1}^{\infty} \|x_i\|^q < \infty\},$$

$$\hat{l}^p(F) = \bigcap_{q>p} l^p(F).$$

For sequences $(x_i) \in F^\omega$, set $\|(x_i)\|_q = (\sum_{i=1}^{\infty} \|x_i\|^q)^{\min(1, 1/q)}$.

The following result has its proof from [7]:

5.27 Theorem. $\hat{l}^p(F)$ is homeomorphic to σ^ω for $0 \leq p < \infty$.

Proof: We show that $X = \hat{l}^p(F)$ is a linear $F_{\sigma\delta}$–nest in $E = F^\omega$. Now $\hat{l}^p(F)$

is an absolute retract since it is a dense linear subspace of F^ω which is a locally convex metric linear space. Choose a strictly decreasing sequence $\{q_n\}$ converging to p and let $E_i = l^{q_i}(F) = \{(x_i) \in F^\omega \colon \sum_{i=1}^{\infty} \|x_i\|^{q_i} < \infty\}$.

Then $E_i = \bigcup_{k=1}^{\infty} B_i(k)$ where $B_i(r) = \{x \in E_i : \|x\|_{q_i} \leq r\}$ and observe that $B_i(r) = \bigcap_{k=1}^{\infty} \{(x_n) \in F^\omega \colon \sum_{n=1}^{\infty} \|x_n\| \leq r^{\max(1,\, q_i)}\}$

Therefore $B_i(r)$ is closed and E_i is a F_σ–subset in F^ω.

Set $x_i = (x_n^i) \in B_i(2^{-i}) \backslash l^{q_i+1}(F)$ and set $C_i = \{(y_n) \in F^\omega : \|y_n\| \leq \|x_n^i\|\}$. It follows that $C_i \cap X$ is dense in C_i. Now consider the linear isomorphism $\phi \colon F^\omega \to (F^\omega)^\omega$ defined by $\phi((x_n)) = ((s_i))_i$ where $s_i = (x_n)_{n \in \mathbb{N}}$, where $\{\mathbb{N}_i\}$ is a family of infinite disjoint sequence of subsets of $\mathbb{N}$. Now

$$\phi(X) = \phi(\hat{l}^p(F)) = \{(x_i) \in (\hat{l}^p(F))^\omega \colon \forall\, q > p,\ \sum_{i=1}^{\infty} \|x_i\|_q^{\max(1,\, q)} < \infty\}.$$

So we have that

$$\{(x_i) \in (\hat{l}^p(F))^\omega \colon x_i = 0 \text{ a.e}\} \subset \phi(\hat{l}^p(F)).$$

By construction of C_i, we have $C_i \cap X \subset B_i(2^{-i})$. Choose $q > p$ and pick i_0 such that $q_i \leq q$ for $i \geq i_0$ as $\{q_i\}$ is a decreasing sequence. Let $x_i \in C_i \cap \hat{l}^p(F)$. Therefore $\|x_i\|_{q_i} \leq 2^{-i}$ and so for $i \geq i_0$ we have $\|x_i\|_q \leq \|x_i\|_{q_i} \leq 2^{-i}$. So $\sum_{i=1}^{\infty} \|x_i\|_q^{\max(1,\, q)} = \sum_{i=1}^{i_0-1} \|x_i\|_q^{\max(1,\, q)} + \sum_{i=i_0}^{\infty} \|x_i\|_{q_i}^{\max(1,\, q)} < \infty$ and so $\phi\{(x_i) \in X : (x_i) = 0 \text{ a.e}\} \subset \phi(X)$. Hence $\hat{l}^p(F)$ is a linear F_σ–nest. Thus $\hat{l}^p(F)$ is homeomorphic to σ^ω. □

We discuss one more example. Let $L^0([a,\, b],\, F)$ denote the space of equivalence classes of Lebesque measurable functions $x : [a,\, b] \longrightarrow F$ endowed with the topology of convergence on measure. For $q > 0$, let

$L^q([a,\, b],\, F) = \{x \in L^0([a,\, b],\, F) \colon \int_a^b \|x(t)\|^q \, dt < \infty\} \subseteq L^0([a,\, b],\, F)$. Endow $L^q([a,\, b],\, F)$ with the subspace topology. Set $\hat{L}^p([a,\, b],\, F) = \bigcap_{q<p} L^p([a,\, b],\, F)$ for $0 < p \leq \infty$.

If G is a subset of F, then $L^q([a,\, b],\, G) \subseteq L^q([a,\, b],\, F)$. Lemma 5.1 ([7]) states that if G is an unbounded set of F and $p > q$, then $L^p([0,\, 1],\, G) \neq L^q([0,\, 1],\, G)$. For $x \in L^q([a,\, b],\, F)$, let $\|x\|_q = (\int_a^b \|x(t)\|^q \, dt)^{\min(1,\, 1/q)}$.

Let $L^q(F) := L^q([0,\, 1],\, F)$ and clearly $L^q(F)$ has a natural linear isomorphism to $L^q([a,\, b],\, F)$. Nhu [25] proved that every λ–convex subset of $L^0(F)$ is an

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absolute retract. The following Theorem is from [7].

5.28 Theorem. If G is a closed unbounded subset of F and $0 < p \leq \infty$, then $\hat{L}^p(G)$ is homeomorphic to σ^ω.

Proof: We show that $\hat{L}^p(G)$ is a λ–convex $F_{\sigma\delta}$–nest in $L^0(G)$. Choose a strictly increasing sequence (q_i) converging to p. Then
$\hat{L}^p(G) = \bigcap_{q_i<p} L^{q_i}(G) = \bigcap_{q_i<p} \{x \in L^0(G) : \int_0^1 \|x(t)\|\, dt < \infty\}$. Now
$L^{q_i}(G) = \bigcup_{k=1}^{\infty} B_{q_i}(k) := \bigcup_{k=1}^{\infty} \{x \in L^0(G) : \int_0^1 \|x(t)\|^{q_i}\, dt \leq k\}$ and so the $L^{q_i}(G)$ are F_σ–subsets of $L^0(G)$. Let $x_i \in B_{q_i}(2^{-i}) \cap L^0(G)$ be such that $\chi_{[0,\,t]} \cdot x_i \notin L^{q_{i+1}}(G)$ and $\chi_{[t,\,1]} \cdot x_i$ is a step function for all $t > 0$. Set $C_i = \{x \in L^0(G) : \|x(t)\| \leq \|x_i(t)\| \text{ a.e}\}$. Now for $x,\, y \in C_i$, $\|(1-t)\,x + ty\| \leq \|x_i(t)\|$ and so C_i is a λ–convex, closed subset of $L^0(G)$. Also $\hat{L}^p(G)$ contains all step functions and is dense in $L^0(G)$. So $\hat{L}^p(G)$ is an absolute retract since $L^0(G)$ is. By choice of x_i, the step functions of C_i are dense in C_i and so property (1) is satisfied. Since $\chi_{[t,\,1]} \cdot x + \chi_{[0,\,t]} \cdot x_i \in C_i \backslash L^{q_{i+1}}(G)$ for all $x \in \hat{L}^p(G)$ and $t > 0$, (2) is satisfied since $(2^*) \Rightarrow (2)$ by using Nhu's result. Consider the linear isomorphism $\Psi : L^0(G) \longrightarrow (L^0(G))^\omega$ such that $\Psi(\hat{L}^p(G)) = \{(x_n) \in \prod_{n=1}^{\infty} L^p([2^{-n},\, 2^{-n+1}],\, G) : \forall\ q < p,\ \sum_{n=1}^{\infty} \|x_n\|_q^{\max(1,\,q)} < \infty\}$. The rest of the argument for property (3) follows from the fact that $C_i \cap \hat{L}^p(G) \subset C_i$ like Theorem 5.27. So $\hat{L}^p(G)$ is a λ–convex $F_{\sigma\delta}$–nest homeomorphic to σ^ω and we have our result. □

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