Pricing Equity Options on Multiple Underlyings in the South African Context

Bradley Preston

supervised by

Petrus Bosman, Cadiz Financial Services Group

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Declaration

I know the meaning of Plagarism and declare that all of the work in the document, save for that which is properly acknowledged, is my own.

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Chapter 1

Introduction

It is well documented that financial asset prices returns are not normally distributed. Historical return distributions exhibit fatter tails and positive skewness that is not explained by a normal distribution. Moreover the standard Black-Scholes option pricing framework that assumes that asset prices follow geometric Brownian Motion does not explain option prices observed in the market. In particular much work has been done trying to explain the volatility skew. Most of this work focuses on a single asset. When considering more than one asset not only does the assumption of normally distributed returns imply that each individual asset's returns are normally distributed it also implies that the dependence structure between the asset returns is Gaussian. Copulas allow us to separate a multivariate distribution into the marginals that describe each variables behavior independent of the other variables and a dependence structure which describes how the variables depend on each other. The copula describes the dependence structure and assumption of a normal joint distribution assumes normal marginals as well as a normal copula. In the case of a normal or Gaussian copula, the copula is uniquely determined by a correlation matrix, but this is not always the case. When the dependence structure between a number of variables is not given by a Gaussian copula knowledge of the marginals and the correlation matrix may not be sufficient to determine the entire joint distribution.

The goal of this thesis is to investigate various methods for pricing derivatives on multiple underlyings that take into account the volatility skew of the underlying securities as well as a more flexible dependence structure between the individual asset returns.

The structure of this thesis is as follows. We first define copulas and present a number of important results. We introduce some copula families and discuss the estimation of copulas from data as well as methods of simulating drawings from a copula.

Secondly we consider two different methods of modeling the implied volatility skew for a single security. Finally we consider various approaches to modeling derivatives on multiple assets that allow us to take into account the volatility skew and our choice of copula for the dependence structure of the asset returns.

Chapter 2

Copulas

2.1 Copula Basics

A copula is a function that joins univariate marginal distributions together to form a multivariate distribution. Given a multivariate distribution the copula determines the dependence structure while the behavior of each variable in isolation is governed by the corresponding marginal distribution. Copulas allow us to completely separate these two aspects of any joint distribution, so we can create a multivariate distribution by specifying the marginals and the marginals and then combining these to obtain the joint distribution. In this section we define a Copula and state a number of important results that we will use. We present a very brief introduction to Copulas and omit most of the proofs of these results. For most of this section we follow Cherubini et al. [2004] quite closely and refer readers to that text for omitted details and proofs. We also recommend Embrechts et al. [2003], Embrechts et al. [2002] as summaries of the use of copulas in financial modeling and risk management applications.

Definition 1. Given non-empty $A_1, A_2 \in \mathbb{R} \cup \{-\infty, \infty\}$ and a function $G : A_1 \times A_2 \rightarrow \mathbb{R}$. Then G is called **grounded** if $\mathcal{C}(v, 0) = \mathcal{C}(0, z) = 0$ for all $(v, z) \in A_1 \times A_2$.

Definition 2. Given non-empty $A_1, A_2 \in \mathbb{R} \cup \{-\infty, \infty\}$ and a function $G : A_1 \times A_2 \rightarrow \mathbb{R}$. Then G is called **2-increasing** if for every $v_1, v_2 \in A_1, z_1, z_2 \in A_2$ such that $v_1 \leq v_2$ and $z_1 \leq z_2$ we have

$$G(v_2, z_2) - G(v_2, z_1) - G(v_1, z_2) + G(v_1, z_1) \geq 0$$

By setting $v_1 = a_1$ and $z_1 = a_2$ it is easy to see that if a function $G(v, z)$ is both 2-increasing and grounded then it is increasing in both v and z.

Definition 3. A 2-dimensional **subcopula** $\mathcal{C}$ is a function $\mathcal{C} : A_1 \times A_2 \to \mathbb{R}$, where $A, B \in [0, 1]$ are nonempty and both contain $\{0, 1\}$. satisfying

1. $\mathcal{C}$ is grounded.
2. $\mathcal{C}(v, 1) = v, \mathcal{C}(1, z) = z$ for all $(v, z) \in A \times B$.
3. $\mathcal{C}$ is 2-increasing.

Definition 4. A 2-dimensional **copula** is a 2-dimensional subcopula with $A = B = [0, 1]$.

Next we present the result that allows us to use Copulas so effectively. This result says that we can consider any joint distribution as a set of marginal distributions and a copula. Any joint distribution can be split into a copula and a set of marginal univariate distributions. Also given any set of marginal distributions and any copula these can be combined to form a joint distribution.

Theorem 1 (Sklar's Theorem). Let $F_1(x), F_2(x), \dots, F_n(x)$ be univariate distribution functions, with range $A_1, A_2, \dots, A_n$ respectively. If $\mathcal{C}$ is a copula with $A_1, A_2, \dots, A_n \subseteq \text{Dom}\mathcal{C}$ then

$$\mathcal{C}(F_1(x), F_2(x), \dots, F_n(x))$$

is a joint distribution function with margins $F_1(x), F_2(x), \dots, F_n(x)$.

The converse also holds. If F is a joint distribution function with marginals $F_1(x), F_2(x), \dots, F_n(x)$ there exists a unique subcopula $\mathcal{C}$ with domain $\text{Ran}F_1 \times \text{Ran}F_2 \times \dots \times \text{Ran}F_n$ such that

$$F(x) = \mathcal{C}(F_1(x), F_2(x), \dots, F_n(x))$$

This copula is given by

$$\mathcal{C}(u_1, u_2, \dots, u_n) = F(F_1^{-1}(x), F_2^{-1}(x), \dots, F_n^{-1}(x))$$

If the $F_1(x), F_2(x), \dots, F_n(x)$ are all continuous then $\mathcal{C}$ is a copula, otherwise $\mathcal{C}$ can be extended (non-uniquely) to a copula.

Finally, the following theorem will be useful in a number of places. It says that copulas are invariant under monotone transformations.

Theorem 2. Let α_i $i = 1, 2, \ldots, n$ be n increasing functions from the reals to the reals. Let $X_1, X_2, \ldots, X_n$ be random variables with marginal distribution functions $F_1, F_2, \ldots, F_n$ and copula $\mathcal{C}$. Then the transformed variables

$$H_i = F_i\left(\alpha_i^{-1}\right) \quad i = 1, 2, \ldots, n$$

also have the copula $\mathcal{C}$.

Proof. Let $\hat{\mathcal{C}}$ be the copula of $H_1, H_2, \ldots, H_n$. Now for any $v \in \mathbb{R}^n$ let $x = (F_1^{-1}(v_1), F_2^{-1}(v_2), \ldots, F_n^{-1}(v_n))$, then

$$\begin{aligned}
\mathcal{C}(v) &= \mathcal{C}(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)) \\
&= F(x_1, x_2, \ldots, x_n) \\
&= \mathbb{P}[X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n] \\
&= \mathbb{P}[\alpha_1(X_1) \leq \alpha_1(x_1), \alpha_2(X_2) \leq \alpha_2(x_2), \ldots, \alpha_n(X_n) \leq \alpha_n(x_n)] \\
&= H(\alpha_1(x_1), \alpha_2(x_2), \ldots, \alpha_n(x_n)] \\
&= \hat{\mathcal{C}}(H_1(\alpha_1(x_1)), H_2(\alpha_2(x_2)), \ldots, H_n(\alpha_n(x_n))] \\
&= \hat{\mathcal{C}}(\mathbb{P}[\alpha_1(X_1) \leq \alpha_1(x_1))], \mathbb{P}[\alpha_2(X_2) \leq \alpha_2(x_2))], \ldots, \mathbb{P}[\alpha_n(X_n) \leq \alpha_n(x_n))]] \\
&= \hat{\mathcal{C}}(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)] \\
&= \hat{\mathcal{C}}(v)
\end{aligned}$$

2.2 Measures of association

Next we present a number of measures of the association between two or more random variables that are not independent. Two random variables are said to be **associated** if they are not independent. The following measures of association give some information about the copula of the random variables, but don't necessarily define the copula entirely. The most well known measure of association is linear correlation. Unfortunately due to the common English use of the word, correlation is often misunderstood or misused and very often used erroneously to mean association. In particular it is very easy to fall into the trap of thinking that a correlation matrix completely defines the dependence structure between two variables, this is the case for example in the case of a Gaussian copula, but is not the case when considering the Student-T copula where two different copulas can share the same correlation matrix, but different degrees of freedom. We present these measures in the case of 2 variables, some of them may be extended to the general case.

2.2.1 Tail dependence

Given a copula C, let

$$\lambda_L = \lim_{v\to 0^+} \frac{C(v,v)}{v} \tag{2.1}$$

C is said to have lower tail dependence if and only if $\lambda_L \in (0,1]$ and no lower tail dependence if and only if $\lambda_L = 0$. Upper tail dependence is defined in a similar way.

Now

$$\begin{aligned}\lambda_L = \lim_{v\to 0^+} \frac{C(v,v)}{v} &= \lim_{v\to 0^+} \frac{\mathbb{P}(U_1 < v, U_2 < v)}{\mathbb{P}(U_2 < v)} \\ &= \lim_{v\to 0^+} \mathbb{P}(U_1 < v | U_2 < v)\end{aligned}$$

Upper tail dependence is defined similarly. Thus the existence of tail dependence implies a positive probability that extreme values will occur simultaneously. This is an important consideration when modeling returns on financial securities. It is well documented that extreme moves in security prices often occur simultaneously, especially during market crashes. When pricing derivatives on a basket of securities, or dealing with risk management it becomes important to be able to capture this simulate this behavior and so it may be desirable to use a copula that exhibits tail dependence when modeling codependency of security returns.

2.2.2 Kendall's tau

Given two random variables X and Y with Copula C, Kendall's tau is

$$\tau = 4 \int\int_{I^2} C(v,z) dC(v,z) - 1$$

An unbaised estimator of the Kendalls coefficient is Kendall's sample τ:

$$\hat{\tau} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j>i} \operatorname{sgn}(x_i - x_j)(y_i - y_j)$$

Spearman's rho

Given two random variables X and Y with Copula C, Spearman's rho is

$$\rho = 12 \int\int_{I^2} C(v,z)dvdz - 3 = 12 \int\int_{I^2} C(v,z)dvdz - 3$$

An unbaised estimator is Spearman's sample ρ

$$\frac{\sum_{i=1}^{n}(R_i - \bar{R})(S_i - \bar{S})}{\sqrt{\sum_{j=1}^{n}(R_j - \bar{R})^2 \sum_{k=1}^{n}(S_k - \bar{S})}}$$

Where R_i denotes the rank of x_i and S_i denotes the rank of y_i

2.2.3 Linear Correlation

For two random variables X and Y, the linear correlation coefficient is given by

$$\rho_{XY} = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}$$

Now

$$\begin{aligned} \text{cov}(X,Y) &= \int\int_D (F(x,y) - F_1(x)F_2(y))dxdy \\ &= \int\int_D (C(F_1(x), F_2(y)) - F_1(x)F_2(y))dxdy \end{aligned}$$

and

$$\text{var}(X) = \int_{D_1} (F_1(x))^2 dx$$

It is also useful to recall that $\rho_{XY} = 0$ does not imply independence of X and Y unless X and Y have a normal copula.

2.3 Copula families

We now introduce a number of copula families and discuss some of their properties. We focus most of our attention on the Gaussian and Student T copulas as these will be the most commonly used in our applications. We will consider two broad classes of copulas, namely the Elliptical copulas and Archimedian copulas. The elliptical copulas that we will consider have larger number of parameters than the Archimedian copulas, which are typically parameterised by one factor. This makes the copula family more flexible but makes estimation more difficult especially in higher dimensions.

2.3.1 Elliptical copulas

Elliptical copulas are those copulas that exhibit rotational symmetry, i.e. the copulas that satisfy the property $C(x_1, x_2, \ldots, x_n) = C(-x_1, -x_2, \ldots, -x_n)$. Empirical data for financial timeseries suggest that elliptical copulas may be useful for modeling financial return series, but the assumption of strict rotational symmetry may be too restrictive. For example, the case of widespread contagion during a market crash or correction that is not seen as often witnessed during market rallies suggests that returns may not be entirely rotationally symmetrical.

Gaussian copula

The Gaussian copula is the copula associated with the multivariate Gaussian distribution. We obtain the copula using Sklar's theorem. If ϕ_R is the standard multivariate normal distribution function with correlation matrix R,

$$\phi_R(\mathbf{x}) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} \frac{1}{(2\pi)^{\frac{n}{2}} |R|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{u}^T R^{-1} \mathbf{u}\right) \mathrm{d}u_1 \mathrm{d}u_2 \ldots \mathrm{d}u_n$$

and ϕ^{-1} is the inverse of the standard univariate normal distribution function ϕ. Then

$$C_R^{Ga}(\mathbf{u}) = \phi_R\left(\phi^{-1}(x_1), \phi^{-1}(x_2), \ldots, \phi^{-1}(x_n)\right)$$

In the bivariate case we can show that C^{Ga} with a correlation coefficient of ρ does not exhibit tail dependence unless $\rho = 1$. Using the following representation of C^{Ga} due to Roncalli

$$C^{Ga}(v, z) = \int_0^v \Phi\left(\frac{\Phi^{-1}(z) - \rho\Phi^{-1}(t)}{\sqrt{1-\rho^2}}\right) dt \qquad (2.2)$$

and applying l'Hôpital's rule we obtain the coefficient of tail dependence for the Gaussian copula. First assume $\rho < 1$

$$\begin{aligned}
\lambda_L &= \lim_{v\to 0^+} \frac{C^{Ga}(v,v)}{v} \\
&= \lim_{v\to 0^+} \frac{\int_0^v \Phi\left(\frac{\Phi^{-1}(v)-\rho\Phi^{-1}(t)}{\sqrt{1-\rho^2}}\right) dt}{v} \\
&= \lim_{v\to 0^+} \frac{\Phi\left(\frac{\Phi^{-1}(v)-\rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right)}{1} \\
&= \Phi\left(\lim_{v\to 0^+} \frac{(1-\rho)\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right) \\
&= 0
\end{aligned}$$

And for $\rho = 1$, $\mathbb{P}(U_1 < v|u_2 < v) = 1$ for all v. So,

$$\lambda_L = \begin{cases} 1 & \text{if } \rho = 1 \\ 0 & \text{if } \rho \neq 1 \end{cases}$$

Throughout this thesis we will use the terms Gaussian copula and normal copula interchangeably. It is important to note that the Gaussian Copula does not exhibit tail dependence and is rotationally symmetric. This limits the ability of this copula to model the occurrence of extreme events coinciding.

Finally, using the definition of Kendall's τ and Spearman's ρ, Lindskog et al. Embrechts et al. [2002] prove the following relationship between these two measures and the linear correlation coefficient ρ, for all elliptical copulas.

$$\begin{aligned}
\rho &= \sin\left(\frac{\tau\pi}{2}\right) \\
\rho &= 2\sin\left(\frac{\rho_S\pi}{6}\right)
\end{aligned}$$

Student T copula

We obtain the Student T copula in a similar way to the Gaussian. If $t_{R,\nu}$ is the standard multivariate student's t distribution function with correlation

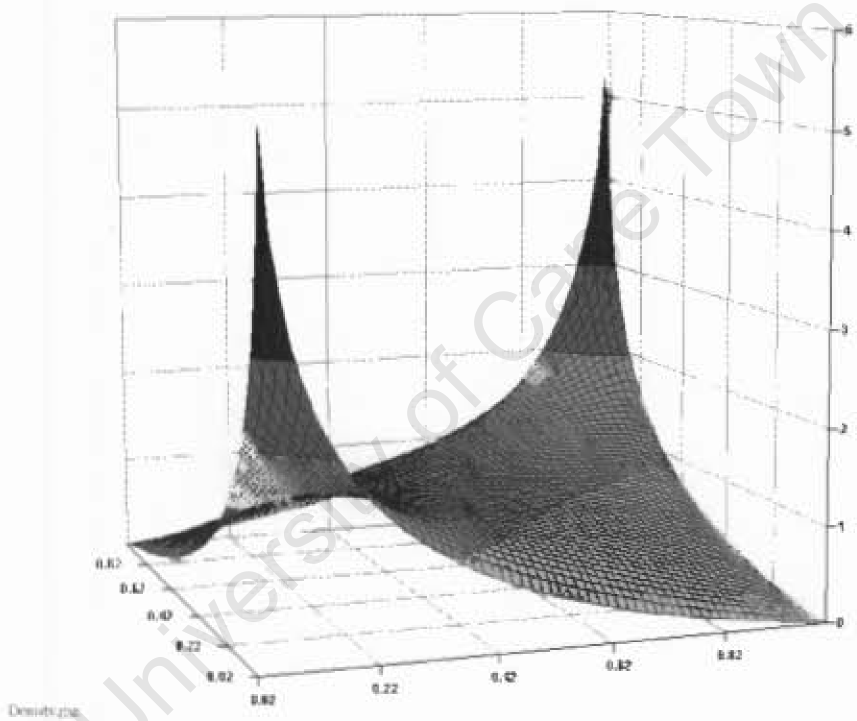

Figure 2.1: Density of a Gaussian copula $\rho = 0.5$

matrix R, and ν degrees of freedom

$$t_{R,\nu}(\mathbf{x}) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_n} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{(\nu\pi)^{\frac{n}{2}}\Gamma\left(\frac{\nu}{2}\right)|R|^{\frac{1}{2}}}\left(1+\frac{1}{\nu}\mathbf{u}^T R^{-1}\mathbf{u}\right)^{-\frac{\nu+n}{2}} \mathrm{d}u_1\mathrm{d}u_2\ldots\mathrm{d}u_n$$

and t_ν^{-1} is the inverse of the standard univariate student's t distribution function

$$t_\nu(x) = \int_{-\infty}^{x} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{u^2}{\nu}\right)^{-\frac{\nu+1}{2}} \mathrm{d}u$$

Then

$$\mathcal{C}^T_{R,\nu}(\mathbf{u}) = t_{\nu,R}\left(t_\nu^{-1}(x_1), t_\nu^{-1}(x_2), \ldots, t_\nu^{-1}(x_n)\right)$$

The density of the Student T copula is

$$c^T_{R,\nu}(x) = |R|^{-\frac{1}{2}}\frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\right)^n \frac{\left(1+\frac{1}{\nu}\xi^T R^{-1}\xi\right)^{-\frac{\nu+n}{2}}}{\prod_{j=1}^n\left(1+\frac{\xi_j^2}{\nu}\right)^{-\frac{\nu+1}{2}}}$$

When ν grows large, the Student T copula converges to the Gaussian copula.

Again we use an expression due to Roncalli for $C^T_{\rho,\nu}(v,z)$ to investigate tail dependence in the bivariate case

$$T_{\rho,\nu}(v,z) = \int_0^v t_{\nu+1}\left(\sqrt{\frac{\nu+1}{\nu+t_\nu^{-1}(s)^2}}\frac{t_\nu^{-1}(z)-\rho t_\nu^{-1}(s)}{\sqrt{1-\rho^2}}\right) ds \qquad (2.3)$$

again applying l'Hôpital's rule we obtain the coefficient of tail dependence.

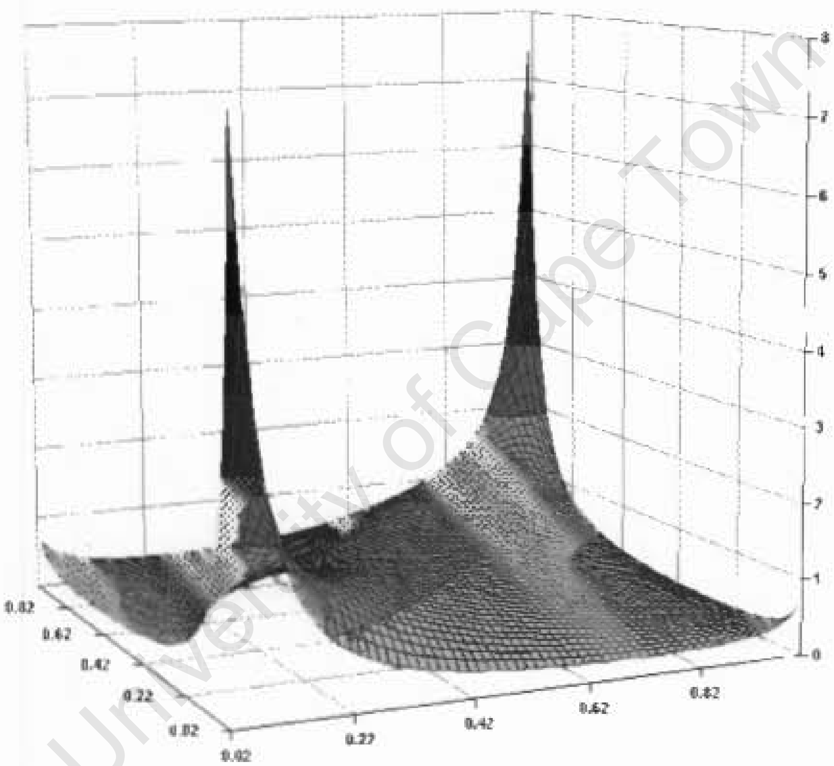

Figure 2.2: Density of a Student T copula $\nu = 3, \rho = 0.5$

Assume $\rho > -1$

$$
\begin{aligned}
\lambda_L &= \lim_{v \to 0^+} \frac{T_{\rho,\nu}(v,v)}{v} \\
&= \lim_{v \to 0^+} \frac{\int_0^v t_{\nu+1}\left(\sqrt{\frac{\nu+1}{\nu+t_\nu^{-1}(s)^2}} \frac{t_\nu^{-1}(v)-\rho t_\nu^{-1}(s)}{\sqrt{1-\rho^2}}\right) ds}{v} \\
&= \lim_{v \to 0^+} \frac{t_{\nu+1}\left(\sqrt{\frac{\nu+1}{\nu+t_\nu^{-1}(v)^2}} \frac{t_\nu^{-1}(v)-\rho t_\nu^{-1}(v)}{\sqrt{1-\rho^2}}\right)}{1} \\
&= t_{\nu+1}\left(\sqrt{\frac{\nu+1}{1-\rho^2}}(1-\rho) \lim_{v \to 0^-} \frac{t_\nu^{-1}(v)}{\sqrt{\nu + t_\nu^{-1}(v)^2}}\right) \\
&= t_{\nu+1}\left(\sqrt{\frac{\nu+1}{1-\rho^2}}(1-\rho) \lim_{v \to 0^-} \frac{1}{\sqrt{\frac{\nu}{t_\nu^{-1}(v)^2}+1}}\right) \\
&= 2t_{\nu+1}\left(\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}(1-\rho)\right) \\
&> 0
\end{aligned}
$$

Thus we have

$$
\lambda_L = \begin{cases} 0 & \text{if } \rho = -1 \\ > 0 & \text{if } \rho > -1 \end{cases}
$$

Further more λ_L is decreasing in ν and increasing in ρ.

The same relationship between the linear correlation coefficient and Kendall's τ and Spearman's ρ holds as in the Gaussian case.

$$
\begin{aligned}
\rho &= \sin\left(\frac{\tau\pi}{2}\right) \\
\rho &= 2\sin\left(\frac{\rho_S\pi}{6}\right)
\end{aligned}
$$

At this stage it is important to note that the main difference that we have found so far between the Gaussian and Student t copulas is the occurrence of tail dependence. In almost all cases the T copula exhibits tail dependence while the Gaussian does not. When modeling financial asset returns we are often interested in accurate modeling of the occurrence of extreme events. Tail dependence is directly related to simultaneous occurrence of extreme events and thus is of practical importance. The fact that a T copula is able to model tail dependence while a Gaussian copula is unable is an important consideration when selecting a copula in the modeling of financial assets.

Again using we can determine Kendall's tau for the Clayton copula.

$$\begin{aligned}
\phi(t) &= \frac{t^{-\alpha} - 1}{\alpha} \\
\therefore \quad \phi'(t) &= -t^{-\alpha-1} \\
\therefore \quad \tau &= 1 + 4\int_0^1 \frac{t^{\alpha+1} - t}{\alpha} dt \\
\therefore \quad \tau &= \frac{\alpha}{\alpha + 2}
\end{aligned}$$

Definition 7. Frank copula

If

$$\phi(u) = \ln\left(\frac{e^{-\alpha u} - 1}{e^{-\alpha} - 1}\right)$$

and

$$\phi^{-1}(t) = -\frac{1}{\alpha}\ln(1 + e^{t}(e^{-\alpha} - 1)$$

then the Frank copula is

$$\mathcal{C}^{Frank}(u_1, u_2, \ldots, u_n) = -\frac{1}{\alpha}\left(1 + \frac{\prod_{i=1}^{n}(e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)^{n-1}}\right)$$

1. Let $\hat{R}_1$ be the estimator of R for a Gaussian copula

2.
$$\hat{R}_{m+1} = \frac{1}{T}\left(\frac{\nu+n}{\nu}\right)\sum_{t=1}^{T}\frac{\xi_t^T\xi_t}{1+\frac{1}{\nu}\xi\hat{R}_m^{-1}\xi_t^T}$$

3. The previous step is repeated until convergence is achieved.

Mashal and Zeevi, Mashal and Zeevi [2002], suggest the following method to find ν and R:

1. Transform the variates to uniform variates using the empirical marginal transformation.

2. Estimate $\hat{R}$ using Kendall's τ,
$$\hat{R}_{ij} = \sin\left(\frac{\pi\hat{\tau}_{ij}}{2}\right)$$

3. Estimate ν by numerically maximizing the log-likelihood with respect to ν.

Unfortunately both of these methods have disadvantages. The first requires an initial estimate of ν. The second method estimates R without taking into account the effect of ν on R. Mashal and Zeevi argue that the difference between the results of the two approaches is small and that their approach is numerically more stable. If speed of implementation is not a requirement it is possible to use both approaches, either to check the stability of the result or using the Mashal-Zeevi approach as the seed for the Bouye approach.

Stock 1	Stock 2	Normal	Student T		Clayton
		ρ	ν	ρ	α
AGL	BIL	0.772	5.510	0.794	2.802
AGL	SOL	0.455	18.001	0.475	0.920
IMP	AMS	0.601	3.114	0.626	1.515
FSR	MTN	0.433	6.741	0.451	0.849
ANG	HAR	0.587	5.231	0.613	1.447
FSR	SBK	0.697	3.816	0.719	2.092
SAB	RCH	0.380	8.204	0.404	0.720
REM	RCH	0.262	52.210	0.274	0.429
ANG	AGL	0.403	6.206	0.424	0.772
BIL	SOL	0.428	4.059	0.453	0.854
GFI	HAR	0.616	22.425	0.633	1.548

Table 3.1: Results of Copula estimation. 355 data points. 30 July 2006 to 31 December 2007

Basket	ν	R
AGL, BIL, SOL	6.02	$\begin{pmatrix} 1.00 & 0.79 & 0.48 \\ 0.79 & 1.00 & 0.45 \\ 0.48 & 0.45 & 1.00 \end{pmatrix}$
FSR, SBK, ASA	4.80	$\begin{pmatrix} 1.00 & 0.72 & 0.65 \\ 0.72 & 1.00 & 0.69 \\ 0.65 & 0.69 & 1.00 \end{pmatrix}$
IMP, AMS, REM	5.76	$\begin{pmatrix} 1.00 & 0.63 & 0.29 \\ 0.63 & 1.00 & 0.33 \\ 0.29 & 0.33 & 1.00 \end{pmatrix}$
ANG, HAR, GFI	13.09	$\begin{pmatrix} 1.00 & 0.61 & 0.71 \\ 0.61 & 1.00 & 0.63 \\ 0.71 & 0.63 & 1.00 \end{pmatrix}$

Table 3.2: Results of Copula estimation. 355 data points, 30 July 2006 to 31 December 2007

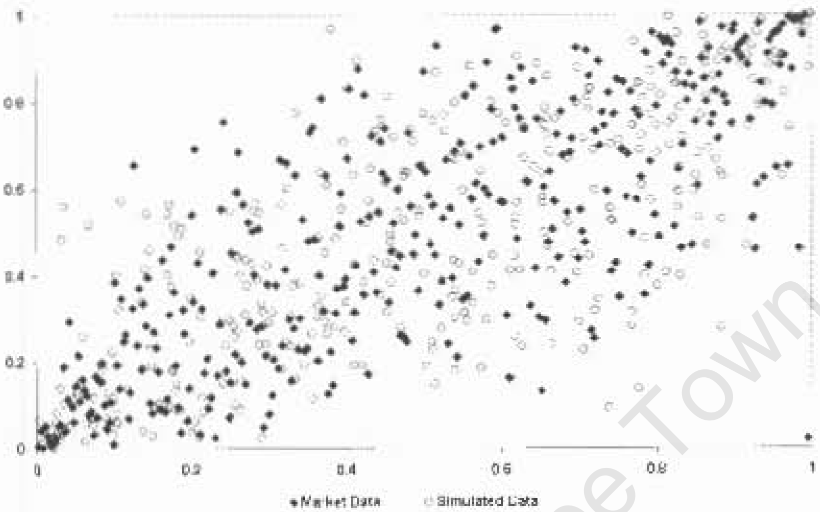

Figure 3.1: Copula of returns AGL vs. BIL compared to Gaussian Copula

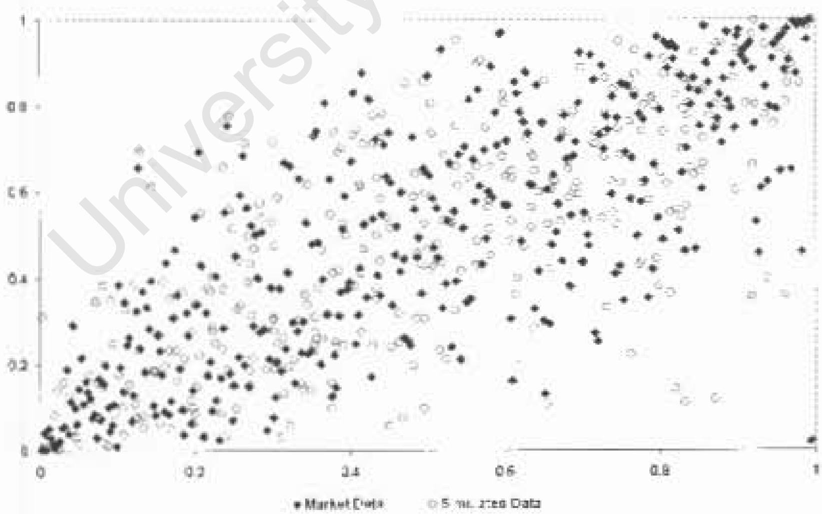

Figure 3.2: Copula of returns AGL vs. BIL compared to Student Copula

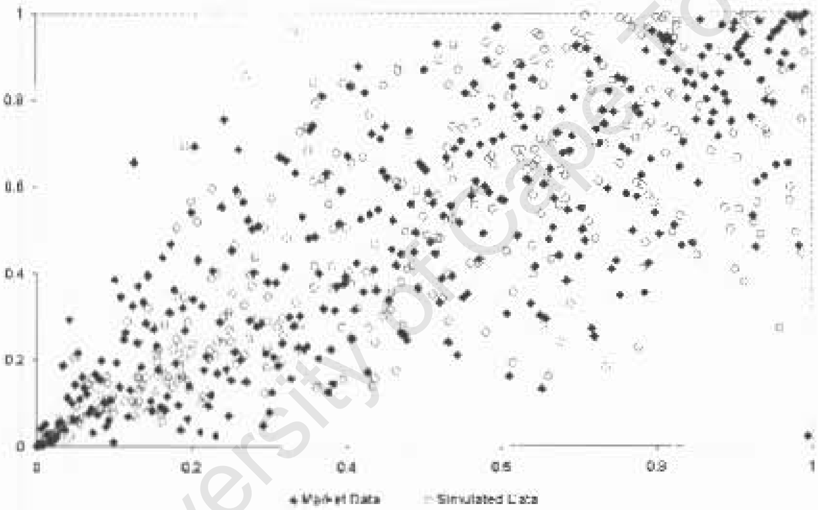

Figure 3.3: Copula of returns AGL vs. BIL compared to Clayton Copula

3.2 Generating drawings from Copulas

Once we have estimated the multivariate risk neutral density for the asset returns we need to be able to draw random variates from this distribution. In general this can be achieved by first drawing an n-dimensional vector from the copula given by the distribution.

$$F(x_1, x_2, \ldots, x_n) = \mathcal{C}(x_1, x_2, \ldots, x_n)$$

These variates would have a uniform marginals but would have the required dependence structure. Then we convert each variate to a drawing from the correct marginal by applying the inverse distribution function of the required marginal to that variate.

3.2.1 Drawing from a Gaussian copula

The most common approach to generating variates with a required correlation structure is to simulate a vector of independent normally distributed random variables and multiply it by the Cholesky decomposition of the correlation matrix. This is equivalent to simulating n variates from an n dimensional multivariate normal distribution. Care must be taken with this approach as most implementations of Cholesky decomposition assume a positive definite matrix while a correlation matrix is in general only positive semi-definite. See Rebonato and Jackel [1999] and Higham [2002] for methods for finding a positive definite matrix from a general correlation matrix.

1. Find the Cholesky decomposition A of R
2. Simulate n independent random variates $\mathbf{z} = (z_1, \ldots, z_n)^T$ from N(0,1)
3. Set $\mathbf{x} = A\mathbf{z}$
4. Set $u_i = \Phi(x_i)$ for $i = 1, 2, \ldots, n$

The $\mathbf{u}$ generated by this algorithm will have a joint distribution with uniform marginals and a Gaussian copula with correlation matrix R.

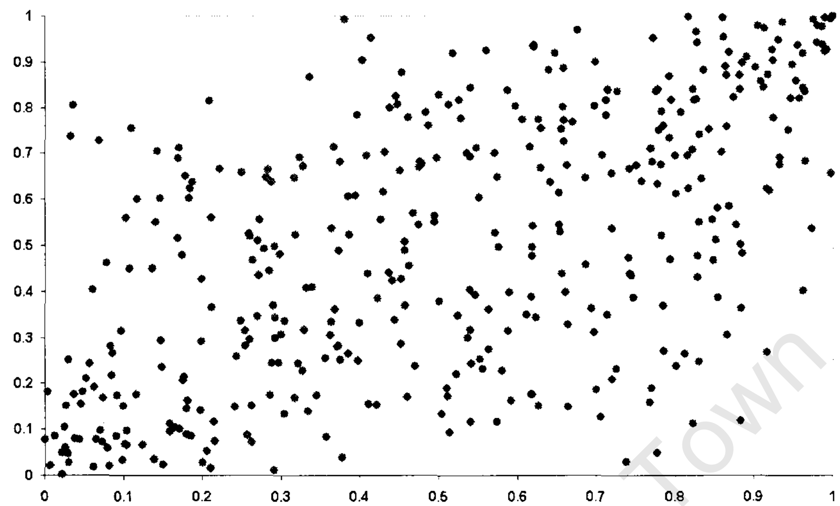

Figure 3.4: Random samples from a Normal Copula $\rho = 0.6$

3.2.2 Drawing from a Student T copula

1. Find the Cholesky decomposition A of R
2. Simulate n independent random variates $\mathbf{z} = (z_1, \ldots, z_n)^T$ from N(0,1)
3. Simulate a random variate s from χ^2_ν independent of $\mathbf{z}$
4. Set $\mathbf{y} = A\mathbf{z}$
5. Set $\mathbf{x} = \sqrt{\frac{\nu}{s}}\mathbf{y}$
6. Set $u_i = T_\nu(x_i)$ for $i = 1, 2, \ldots, n$

The **u** generated by this algorithm will have a joint distribution with uniform marginals and a Student t copula with parameters ν and R.

3.2.3 Drawing from an Archimedean copula

In Cherubini et al. [2004] the authors propose two methods for simulating from a Clayton copula. The first is a conditional sampling approach. Conditional sampling is a simple approach that is useful when simulating from Archimedian copulas. The general algorithm is as follows

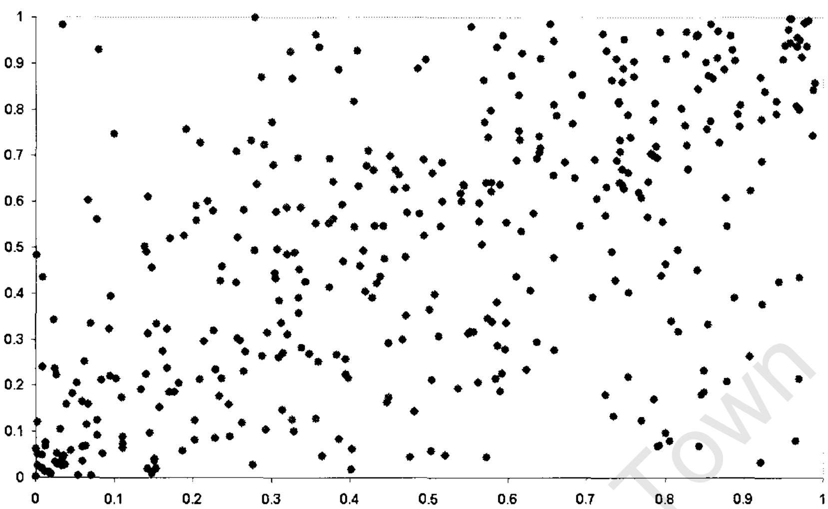

Figure 3.5: Random samples from a Student Copula $\nu = 4, \rho = 0.6$

- Simulate a random variate u_1 from $U(0, 1)$
- Simulate a random variate u_1 from $C_2(\cdot|u_1)$
- ...
- Simulate a random variate u_n from $C_n(\cdot|u_1, \ldots, u_{n-1})$

To simulate u_k from $C_k(\cdot|u_1, \ldots, u_{k-1})$ we need to solve $u_k = C_k^{-1}(v|u_1, \ldots, u_{k-1})$ where $v \sim U(0, 1)$.

In Cherubini et al. [2004] the authors state and prove the following theorem which, in some cases, helps in solving this equation.

Theorem 4. Let $C(u_1, \ldots, u_n) = \phi^{-1}(\phi(u_1) + \ldots + \phi(u_n))$ be an Archimedean copula with generator ϕ, then for $k = 2, \ldots, n$

$$C_k(u_k|u_1, \ldots, u_{k-1}) = \frac{\phi^{-1(k-1)}(\phi(u_1) + \ldots + \phi(u_k))}{\phi^{-1(k-1)}(\phi(u_1) + \ldots + \phi(u_{k-1}))} \tag{3.1}$$

Where $\phi^{-1(k-1)}$ denotes the $k-1$th derivative of ϕ^{-1}.

Now recall that for the Clayton copula $\phi(u) = u^{-\alpha} - 1$, hence $\phi^{-1}(t) = (t+1)^{-\frac{1}{\alpha}}$ and the copula is

$$\mathcal{C}(u_1, u_2, \ldots, u_n) = \left(\sum_{i=1}^{n} u_i^{-\alpha} - n + 1 \right)^{-\frac{1}{\alpha}}$$

For $k = 1, \ldots, n$ the derivatives of ϕ^{-1} are

$$\begin{aligned}
\phi^{-1(1)}(t) &= -\frac{1}{\alpha}(t+1)^{-\frac{1}{\alpha}-1} \\
\phi^{-1(2)}(t) &= \frac{1}{\alpha}\frac{\alpha+1}{\alpha}(t+1)^{-\frac{1}{\alpha}-2} \\
&\vdots \\
\phi^{-1(k)}(t) &= (-1)^k \frac{(\alpha+1)\ldots(\alpha+k-1)}{\alpha^k}(t+1)^{-\frac{1}{\alpha}-k}
\end{aligned}$$

Thus the procedure to generate a drawing from an n-dimensional Clayton copula is the following:

- Simulate n independent random variables $(v_1, v_2, \ldots, v_n)$from $U(0,1)$
- Then $u_1 = v_1$
- $$\begin{aligned}
v_2 &= C_2(u_2|v_1) \\
&= \frac{\phi^{-1(1)}(\phi(u_1)+\phi(u_2))}{\phi^{-1(1)}(\varphi(u_1))} \\
&= \frac{(u_1^{-\alpha}+u_2^{-\alpha}-1)^{-\frac{1}{\alpha}-1}}{(u_1^{-\alpha})^{-\frac{1}{\alpha}-1}}
\end{aligned}$$

  Therefore
  $$u_2 = \left(u_1^{-\alpha}\left(v_2^{-\frac{\alpha}{\alpha+1}}-1\right)+1\right)^{-\frac{1}{\alpha}} \tag{3.2}$$
- ...
- $$\begin{aligned}
v_n &= \left(\frac{u_1^{-\alpha}+u_2^{-\alpha}+\cdots+u_n^{-\alpha}-n+1}{u_1^{-\alpha}+u_2^{-\alpha}+\cdots+u_{n-1}^{-\alpha}-n+2}\right)^{-\frac{1}{\alpha}-n+1} \\
u_n &= \left[(u_1^{-\alpha}+u_2^{-\alpha}+\cdots+u_{n-1}^{-\alpha}-n+2)\left(v_n^{\frac{\alpha}{\alpha(1-n)-1}}-1\right)+1\right)^{-\frac{1}{\alpha}}
\end{aligned}$$

Despite the complexity of the expressions above, this method is relatively simple to implement numerically as the expressions $(u_1^{-\alpha}+u_2^{-\alpha}+\cdots+u_{n-1}^{-\alpha}-n+2)$ can be calculated iteratively.

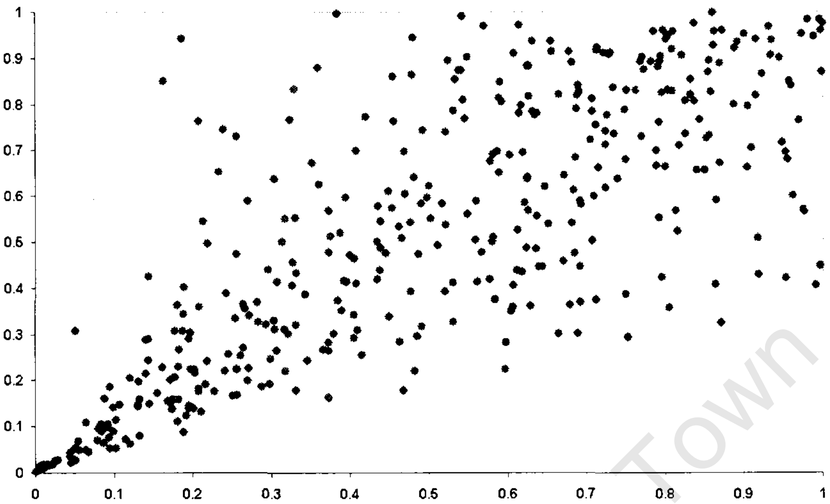

Figure 3.6: Random samples from a Clayton Copula $\alpha = 3$

3.3 Implementation

We have implemented the methods of this chapter with in an object oriented c# library. In most cases where multi-dimensional optimisation is required we use the Nelder-Mead or downhill simplex method as described in W. Press and Flannery [2000] and Nelder and Mead [1965]. The simulation of drawings from any copula requires random drawings from $U(0, 1)$ at some stage. The quality and speed of the random number generator is an important consideration when performing simulations. We have implemented the Mersenne twister of Matsumoto and Nishimura [1998] for this purpose. The algorithms above also require the inverse of the cumulative normal distribution function. We have used the method of Acklam [2004] which uses rational approximations to obtain an approximation and then uses Halley's method to obtain machine precision. See also Moro [1995] as well as West [2005b]. The method we use to generate drawings from a Student T copula, requires the simulation of a variable with a χ^2 distribution. We use the method of Marsaglia and Tsang [2000] for simulating from a Gamma distribution of which the χ^2 distribution is a special cases. Finally, calculating the density function of the Student T copula requires the calculation of the inverse of the cumulative distribution function of the Student's t-distribution. We have not been able to find much written on this topic in the literature and thus we used a simple numerical approach. The steps we take are as follows. First, we determine, for a number of values of ν the value of $sigma(\nu)$ that minimizes the sum of squares distance between the Student's T distribution function with ν degrees of freedom and the normal distribution function with standard deviation $sigma(\nu)$. These values are calculated once and stored, or hard coded into the algorithm. We only store values for $nu < 30$ as $sigma(\nu)$ approaches 1 as ν increases. To calculate the inverse of the Student's T distribution we use the rational approximation of Acklam [2004] to invert a normal distribution with standard deviation $sigma(\nu)$ to obtain an approximation to the inverse for the Student's T distribution and then use Halley's method to numerically obtain a more accurate answer.

Chapter 4

Modeling single stock options

We now leave copulas behind for a while and turn our attention to pricing and modeling options on a single underlying. This problem that has been the focus of large amount of work over the past few decades that started with the famous work of Black and Scholes in the 1970s. The majority of the research into pricing and modeling single stock options has been focused on modeling the volatility skew effect. Two classes of models have been developed to deal with this effect. The first is local volatility models which assume that the volatility of the underlying is a deterministic function of time and the price of the underlying. Local volatility models have received much criticism and many have been shown to be inconsistent with observed market behaviour.

The second class of models is stochastic volatility models, which assume that the volatility of the underlying security price is driven by an additional source of randomness different to that driving the price process. We consider two approaches to modeling the single stock option volatility skew. First we consider the log-normal mixture model approach of Brigo and Mercurio [2002] as well as the popular SABR model of Hagan et al. [2002]. In addition the popular Heston model could also be considered, see Heston [1993] and Mikhailov and Nogel [2003].

4.1 Log-Normal Mixture Local Volatility Model

We consider the approach of Brigo and Mercurio Brigo and Mercurio [2002] and their Log-normal Mixture Local Volatility (LMLV) model. They model the terminal risk-neutral distribution of S_t conditional on S_0 as a mixture of log-normal distributions and solve for the local volatility model that produces this distribution. The authors develop the model in a number of stages of increasing flexibility. They start by modeling the risk neutral distribution as a sum of lognormals each with zero mean, they then allow the individual distributions to be shifted by a common mean and finally they generalize their approach to allow each individual distribution to have a different non-zero mean. We will follow the general approach of a mixture of log normals with different means, as it allows us to fit the model to a wider range of volatility skews. The advantages of this model are that it produces closed form solutions for option prices and greeks and it explicitly gives the dynamics of the underlying process. It must be noted that this is a local volatility model and thus it is open to the criticisms of local volatility models in general. In further work Brigo et al. [2004a] the authors have generalized this mixture model even further in developing a stochastic version of it.

The choice of this model is driven by a number of factors. Firstly, its simplicity is appealing, the fact that the model gives a closed form solution for option price allows us to calibrate the model to market data relatively easily: secondly, this model provides dynamics for the underlying price series; finally, the model can be generalised to the multi-asset case.

4.1.1 Model derivation

We follow Brigo and Mercurio [2002] and refer readers to Brigo and Mercurio [2002] and Brigo et al. [2003] for more details and omitted proofs. As usual we work for time t to T. We assume that the T-forward risk-adjusted measure Q^T exists. Let the dynamics of S under Q^T be

$$dS_t = (r - q)S_t dt + \sigma(t, S_t)S_t dW_t \tag{4.1}$$

Where W is a standard Brownian motion under Q^T. Now we assume that the Q^T density f can be expressed as a weighted sum of N lognormal densities for each time t.

$$f_t(x) = \sum_{i=1}^{N} \omega_i f_t^i(x) \tag{4.2}$$

where

$$\sum_{i=1}^{N} \omega_i = 1$$

and f^i are the density functions of the solutions to

$$dS_t^i = \mu(t)S_t^i dt + \sigma(t)S_t^i dW_t \qquad S_0^i = S_0 \tag{4.3}$$

where σ and μ are deterministic functions of t on $[0, T]$. Then

$$f_t^i(x) = \frac{1}{xV_i(t)\sqrt{2\pi}} \exp\left[-\frac{1}{2V_i^2(t)}\left(\ln\left(\frac{x}{S_0}\right) - M_i(t) + \frac{1}{2}V_i^2(t)\right)^2\right]$$

Where

$$M^i(t) = \int_0^t \mu_i(u)du \qquad V_i(t) = \sqrt{\int_0^t \sigma_i^2(u)du}$$

We also need

$$\sum_{i=1}^N \omega_i e^{M_i(t)} = e^{(r-q)t}$$

to ensure risk-neutrality.

We now use the Fokker-Planck equations on 4.1 and 4.3 to solve for $\sigma(t, S_t)$.

$$\frac{\partial}{\partial t}f_t(y) = -\frac{\partial}{\partial y}\left[\mu y f_t(y)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\sigma(t, y)y^2 f_t(y)\right] \tag{4.4}$$

(4.5)

and

$$\frac{\partial}{\partial t}f_t^i(y) = -\frac{\partial}{\partial y}\left[\mu_i(t)y f_t^i(y)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\sigma(t, y)y^2 f_t^i(y)\right] \tag{4.6}$$

(4.7)

Now substituting these two equations into 4.2 we get

$$\begin{aligned} & \frac{\partial}{\partial y}\left[\mu y f_t(y)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\sigma(t, y)y^2 f_t(y)\right] \\ = \; & \sum_{i=1}^N \omega_i \frac{\partial}{\partial y}\left[\mu_i(t) y f_t^i(y)\right] \\ - \; & \sum_{i=1}^N \omega_i \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\sigma(t, y)y^2 f_t^i(y)(\mu - \sigma_i^N \omega_i \mu_i(t))\right] \frac{\partial}{\partial y}\left[\mu y \sum_{i=1}^N \omega_i f_t^i(y)\right] \\ + \; & \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\sum_{i=1}^N \omega_i \sigma(t, y) y^2 f_t^i(y)\right] \end{aligned}$$

Brigo and Mercurio now solve for $\sigma(t, S_t)$ so that 4.3 has a solution with density f_t. They obtain

$$
\begin{aligned}
\sigma(t,y)^2 &= \frac{\sum_{i=1}^{N} \omega_i \sigma_i(t)^2 f_t^i(y)}{\sum_{i=1}^{N} \omega_i f_t^i(y)} \\
&+ \frac{2S_0 \sum_{i=1}^{N} \omega_i(\mu_i(t) - (r-q)) e^{M_i(t)} \Phi\left(\frac{\ln\left(\frac{S_0}{y}\right) - M_i(t) - \frac{1}{2}V_i^2(t)}{V_i(t)}\right)}{y^2 \sum_{i=1}^{N} \omega_i f_t^i(y)}
\end{aligned}
$$

which is strictly positive on $(0, T]$ given that the following conditions hold: there exists $n \in \{1, 2, 3, \ldots, N\}$ such that,

1. for each $t \in [0, T]$, $\mu_i(t) \geq (r-q)$ for each $i = 1, 2, 3, \ldots, N$, $i \neq n$, and $\mu_n(t) \leq (r-q)$,
2. $$\frac{V_i^2(t)}{2} - \frac{2V_i^2(t)}{\sigma_i^2(t)}(\mu_i(t) - (r-q)) > \frac{V_n^2(t)}{2} - \frac{2V_n^2(t)}{\sigma_n^2(t)}(\mu_n(t) - (r-q))$$
   for all $t \in (0, T]$, and $i \neq n$,
3. each σ_i is continuous and bounded from below by a positive constant,
4. each μ_i is continuous,
5. there exists a $\epsilon > 0$ such that $\sigma_i(t) = \sigma_0 > 0$ and $\mu_i(t) = r - q$ for each $t \in [0, \epsilon]$ and $i = 1, \ldots, N$

Pricing options under these dynamics is quite simple with the option price a convex combination of Black-Scholes prices.

$$
\begin{aligned}
C(K,T) &= P(0,T) \sum_{i=1}^{N} \omega_i \left[S_0 e^{M_i(T)} N(d_i^1) - K N(d_i^2) \right] \\
d_i^1 &= \frac{\ln \frac{S_0}{K} + M_i(T) + \frac{1}{2}\left(\frac{V_i(t)}{\sqrt{T}}\right)^2 T}{\left(\frac{V_i(t)}{\sqrt{T}}\right)\sqrt{T}} \\
d_i^2 &= d_i^1 - \left(\frac{V_i(t)}{\sqrt{T}}\right)\sqrt{T}
\end{aligned}
$$

Similarly the greeks are also given by a combination of Black-Scholes greeks.

4.1.2 Parameterisation and Simulation

One advantage of this model is that it gives an explicit formula for the daily dynamics of the stock price process. To simulate sample paths of this process all that we still need is to determine σ_i and μ_i. We assume $e^M(t)$ is linear in t. Given that

$$M^i(t) = \int_0^t \mu_i(u)du$$

and

$$\mu_i(0) = (r - q)$$

we get

$$\lim_{t \to 0^+} e^{M(t)} = e^{r-q}$$

It is easy to see that

$$e^{M(t)} = e^{(r-q)t} + \frac{e^{M(T)} - e^{(r-q)t}}{T} t \tag{4.8}$$

satisfies the condition $\sum_{i=1}^N \omega_i e^{M_i(t)} = e^{(r-q)t}$ required to ensure risk-neutrality. Now differentiating gives

$$\nu(t) = \frac{d}{dt}\left[\ln\left[e^{(r-q)t}\left((1 - \frac{t}{T}) + e^{M(t)-(r-q)t}\frac{t}{T}\right)\right]\right] \tag{4.9}$$

$$= r - q + \frac{e^{M(t)-(r-q)t} - 1}{T - t + te^{M(t)-(r-q)t}} \tag{4.10}$$

$$\tag{4.11}$$

Next we follow Brigo and Mercurio [2002] and use a parameterization suggested by Nelson and Siegel [1987] to model σ_i as follows

$$\frac{V_i(T)}{\sqrt{T}} = \sqrt{\frac{\int_0^T \sigma_i^2(t)dt}{T}} = \eta_i(T; a_i, b_i, c_i, \tau_i)$$

where

$$\eta_i(T; a_i, b_i, c_i, \tau_i) = a_i + b_i\left[1 - \exp\left(-\frac{T}{\tau_i}\right)\right]\frac{\tau_i}{T} + c_i \exp\left(-\frac{T}{\tau_i}\right)$$

Then rearranging and differentiating by T gives us $\sigma_i^2(t)$

$$\begin{aligned}
\int_0^T \sigma_i^2(t)dt &= T\eta_i(T;a_i,b_i,c_i,\tau_i)^2 \\
\therefore \sigma_i^2(t) &= \frac{d}{dt}\left[t\eta_i(t;a_i,b_i,c_i,\tau_i)^2\right] \\
&= t\frac{d}{dt}\left[t\eta_i(t;a_i,b_i,c_i,\tau_i)\right] + \eta_i(t;a_i,b_i,c_i,\tau_i)^2 \\
&= \eta_i(t;a_i,b_i,c_i,\tau_i)\left[2t\frac{d}{dt}\eta_i(t;a_i,b_i,c_i,\tau_i) + \eta_i(t;a_i,b_i,c_i,\tau_i)\right]
\end{aligned}$$

Now

$$\frac{d}{dt}\eta_i(t;a_i,b_i,c_i,\tau_i) = -b_i\left[1-\exp\left(-\frac{t}{\tau_i}\right)\right]\frac{\tau_i}{t^2} + b_i\left[\exp\left(-\frac{t}{\tau_i}\right)\right]\frac{1}{t} - c_i\frac{1}{\tau_i}\exp\left(-\frac{t}{\tau_i}\right) \tag{4.12}$$

Finally simplifying gives:

$$\sigma_i^2(t) = \eta_i(T;a_i,b_i,c_i,\tau_i)\left[a_i - b_i\frac{\tau_i}{t} + \exp\left(-\frac{t}{\tau_i}\right)\left(2b_i + b_i\frac{\tau_i}{t} - 2c_i\frac{t}{\tau_i} + c_i\right)\right] \tag{4.13}$$

Now

$$\lim_{t\to 0}\left[1-\exp\left(-\frac{t}{\tau_i}\right)\right]\frac{\tau_i}{t} = \lim_{t\to 0}\frac{\frac{1}{\tau_i}\exp\left(-\frac{t}{\tau_i}\right)}{\frac{1}{\tau_i}} = 1 \tag{4.14}$$

Which gives

$$\lim_{t\to 0}\eta_i(t;a_i,b_i,c_i,\tau_i) = a_i + b_i + c_i \tag{4.15}$$

and substituting gives

$$\lim_{t\to 0}\sigma_i^2(t) = (a_i + b_i + c_i)^2 \tag{4.16}$$

To ensure a valid parameterisation we need to ensure that $\sigma_i^2(t) > 0$ for all t. Unfortunately this isn't always easy to achieve and unless one is very careful about the choice of σ_0 and bounds on a, b and c is is possible to end up with a calibration of the model which fits market data well but results in non-real $\sigma_i(t)$ for some t.

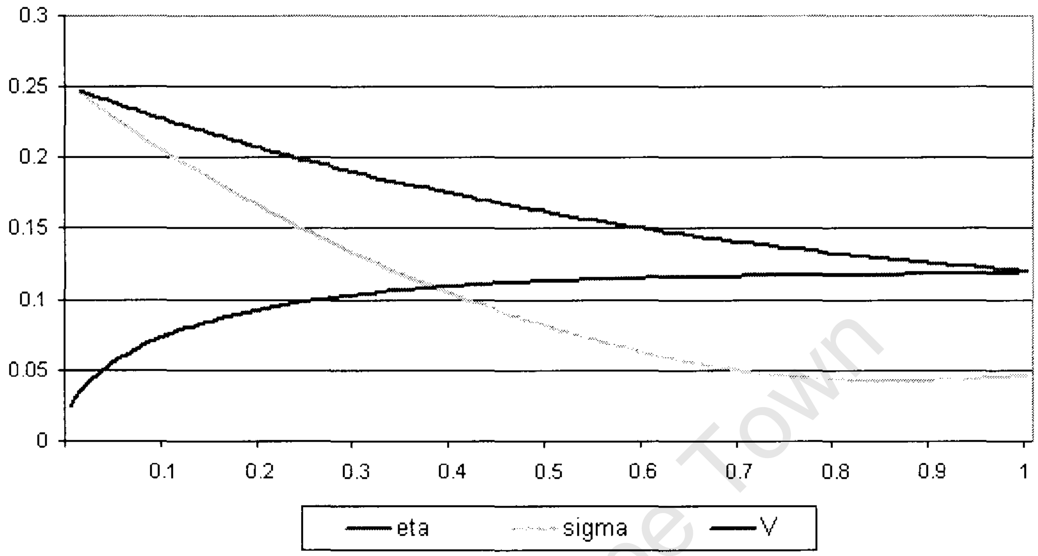

Figure 4.1: $V(t)$, $\sigma(t)$ and $\eta(t)$ for $a = 0.25, b = -0.49, c = 0.49$ and $\tau = 1$

As a result of these difficulties we also consider a simpler parameterisation that enables us to easily ensure that $\sigma_i^2(t) > 0$.

Set

$$\sigma^2(t) = \sigma(0)^2 + \left(\sigma(T)^2 - \sigma(0)^2\right)\left(\frac{t}{T}\right)^2 \tag{4.17}$$

$$V^2(T) = \int_0^T \sigma^2(t)dt = \sigma(0)^2 T + \left(\sigma(T)^2 - \sigma(0)^2\right)\frac{T}{3} \tag{4.18}$$

(4.19)

Which gives

$$\left(\sigma(T)^2 - \sigma(0)^2\right) = 3\left(\frac{V(T)^2}{T} - \sigma(0)^2\right) \tag{4.20}$$

Now $\sigma(t)^2 > 0$ as long as

$$\frac{3V(T)^2}{2T} > \sigma(0)^2 \tag{4.21}$$

Moments of the marginal distribution

We calculate the moments of the marginal distribution of the log process

$$\hat{S}_t = \frac{\ln(S_t)}{\ln(S_0)}$$

Now we have from above that the density of $\hat{S}_t$ is given by

$$\hat{f}_t(x) = \sum_{i=1}^{N} w_i \frac{1}{xV_i(t)\sqrt{2\pi}} \exp\left[-\frac{1}{2V_i^2(t)}\left(x - M_i(t) + \frac{1}{2}V_i^2(t)\right)^2\right]$$

Now given that the central moments of a linear combination of densities is given by the linear combination of the moments of each density we have

$$\begin{aligned}
\mathbb{E}^T\left[\hat{S}_t\right] &= \mathcal{M}(t) - \frac{1}{2}\mathcal{V}_2(t) \\
\mathbb{E}^T\left[\hat{S}_t^2\right] &= \mathcal{M}_2(t) + \mathcal{V}_2(t) - \mathcal{M}\mathcal{V}_2(t) + \frac{1}{4}\mathcal{V}_4(t) \\
\mathbb{E}^T\left[\hat{S}_t^3\right] &= \mathcal{M}_3(t) - \frac{3}{2}\mathcal{M}_2\mathcal{V}_2(t) + \frac{3}{4}\mathcal{M}\mathcal{V}_4(t) - \frac{1}{8}\mathcal{V}_6(t) + 3\mathcal{M}\mathcal{V}_2(t) - \frac{3}{2}\mathcal{V}_4(t) \\
\mathbb{E}^T\left[\hat{S}_t^4\right] &= \mathcal{M}_4(t) - 2\mathcal{M}_3\mathcal{V}_2(t) + \frac{3}{2}\mathcal{M}_2\mathcal{V}_4(t) - \frac{1}{2}\mathcal{M}\mathcal{V}_6(t) + \frac{1}{16}\mathcal{V}_8(t) + 6\mathcal{M}_2\mathcal{V}_2(t) \\
&- 6\mathcal{M}\mathcal{V}_4(t) + \frac{3}{2}\mathcal{V}_6(t) + 3\mathcal{V}_4(t)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{V}_k(t) &= \sum_{i=1}^{N} \lambda_i V_i^k(t) \\
\mathcal{M}_k(t) &= \sum_{i=1}^{N} \lambda_i M_i^k(t) \\
\mathcal{M}_k\mathcal{V}_j(t) &= \sum_{i=1}^{N} \lambda_i M_i^k(t) V_i^j(t)
\end{aligned}$$

While it is possible to obtain expressions for the variance, skewness and kurtosis of $\hat{S}_t$ it is simpler to calculate these numerically using the following simple properties of raw, μ'_k, and centered, μ_k moments.

$$\begin{aligned}\mu_3 &= \mu_3' - 3\mu_2'\mu_1 + 2(\mu_1)^3 \\ \mu_4 &= \mu_4' - 4\mu_3'\mu_1 + 6\mu_2'(\mu_1)^2 - 3(\mu_1)^4\end{aligned}$$

4.1.3 Log-Normal Mixture Uncertain Volatility Model

Due to the difficulties encountered parameterising and simulating the LMLV model we consider the Log-Normal Mixture Uncertain Volatility (LMUV) Model introduced in Brigo et al. [2004a].

$$dS(t) = \begin{cases} S(t)[r(t)dt + \sigma_0 dW(t)] & t \in [0, \epsilon] \\ S(t)[q(t)dt + \xi(t)dW(t)] & t > \epsilon \end{cases} \tag{4.22}$$

where (q, ξ) is a random pair that is drawn at time ϵ independently of W and takes values in a set of N deterministic functions:

$$(t \mapsto (q(t), \xi(t)) = \begin{cases} (t \mapsto (r_1(t), \sigma_1(t)) & \text{with probability } w_1 \\ (t \mapsto (r_2(t), \sigma_2(t)) & \text{with probability } w_2 \\ \vdots & \vdots \\ (t \mapsto (r_N(t), \sigma_N(t)) & \text{with probability } w_N \end{cases} \tag{4.23}$$

This model produces the same option prices as the LMLV model for vanilla options and thus is calibrated in exactly the same manner. This model lends itself to an easy Monte Carlo implementation as it manages to side step some of the issues that arose with the parametrisation of the LMLV model and ensuring that the local volatility remains positive.

4.1.4 Calibration and implementation

The above model gives us the dynamics of the stock price process at each time t given that we have the prices of options that expire at t. We can solve for $M_i(t)$ and $V_i(t)$ where t corresponds to the maturity date of quoted options by fitting the prices given by 4.8 to the quoted prices. This fitting needs to be done numerically. There are two approaches that can be taken. The model can be fitted to data for a single maturity by numerically solving directly for $w_i(t)$, $M_i(t)$ and $V_i(t)$. Alternatively the model can be fitted to the whole volatility surface, using prices of options with various terms to maturity. This approach requires a parameterisation of $\sigma_i(t)$. We then solve for these parameters instead of solving for $V_i(t)$ directly. We also have that $\mu_i(0) = r - q$ and $\sigma_i(0) = \sigma_0$ for some $\sigma_0 > 0$.

We follow the approach used in West [2005a] to calibrate the SABR model in the South African market. Equity derivatives in South Africa are traded on the South African Futures Exchange (SAFEX). Options are traded on indices and single stocks. The contract on the ALSI index is the most liquid contract while the single stock contracts are much less liquid. SAFEX publishes data of trades but there are a few issues that need to be kept in mind when using this data. Up until recently market participants have not been required to enter the spot price of the underlying that the option was traded at or the volatility that the trade was completed at, furthermore trades are often booked as structures and so the trade price of various legs of the structures may not be accurate as long as the price of the structure as a whole is correct. At the time that is analysis was conducted the exchange had started to take steps to overcome these issues in an effort to improve liquidity in the market and the quality of data available, by implementing a trading front end that requires market participants to enter the spot price and volatility that the trade was completed at. Unfortunately these issues still apply to the historic trade data that is published by the exchange. For the purposes of the analysis in this thesis we have collected actual trade data from various market participants as well as historic bid offer quotes from various market makers.

We consider the two approaches detailed in West [2005a] to calibrate the SABR and the LMUV model to market data. The first is to minimise the sum of squares difference between the model volatilities and the quoted implied volatilities at various strikes. This method does not enable us to overcome the issues mentioned above as it requires us to have accurate volatility data for each option traded and can not be used to calibrate to data where we only have accurate price data for a structure of derivatives.

The second method is to minimise the sum of squares difference between the model prices and the traded prices at various strikes. This method can be used to calibrate a model to prices of structures and not just single options and is also more efficient in the LMUV case as the previous method requires the model implied vol to be obtained numerically for strike at each step of the optimisation.

Again we follow West [2005a] and use the Nelder-Mead optimization method to solve this optimization numerically.

While the Nelder-Mead algorithm is quite robust we are using it with higher dimension than for the SABR model where the optimization is typically done in two dimensions. One must also be aware that we are using a local optimisation method and that we result we find may only be a local minimum. For these reasons and also to speed up the calibration process, the starting point that we give the algorithm is important. Ideally we want to start the search algorithm at a point that corresponds to a typically shaped volatility skew with ATM volatility that is close to the ATM vol that is present in the market.

4.2 SABR model

We briefly introduce the Stochastic $\alpha\beta\rho$ model of Hagan et al. [2002]. This model has been used by practitioners internationally and we implement it mainly to provide a comparison and as an alternative for the LMLV model.

We provide a brief introduction to the model but refer the reader to West [2005a],Hagan et al. [2002] and Majmin [2005] for further detail.

The SABR is a stochastic volatility model given by:

$$\begin{aligned} dF &= \alpha F^{\beta} dW_1 \\ d\alpha &= \nu \alpha dW_2 \\ dW_1 dW_2 &= \rho dt \end{aligned}$$

where $\beta \in [0, 1]$,$\rho \in [-1, 1]$ and $\alpha > 0$. The model is driven by two correlated Brownian motions which introduces correlation between the futures price and the implied volatility. In this model α is a stochastic variable which is directly related to the ATM volatility. The constants β, ρ and ν effect the shape of the volatility skew. In Hagan et al. [2002] the authors refer to the path that the ATM volatility traverses as F changes as the *backbone* while the volatility skew or smile refers to the volatility as a function of strike for a fixed F. β determines the shape of the backbone, with $\beta = 1$ producing a flat backbone and a downward sloping backbone for $\beta < 1$. A negatively sloping backbone means that the ATM implied volatility will decrease as the spot price increases, a behavior that is consistent with observed market prices. For the same reason ρ is generally expected to be negative for equity markets as it gives the correlation between the spot price and the volatility of the underlying process. A larger negative value for ρ causes a steeper volatility skew. ν can be thought of as the volatility of volatility and determines the curvature of the vol skew, as ν increases the volatility skew takes on a more curved shape and starts to approach a smile like shape.

Now an option price is given by the Black-Scholes price with the following volatility.

$$
\begin{aligned}
\sigma(X, F) &= \frac{\alpha\left(1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(FX)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(FX)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right)\tau\right)}{(FX)^{(1-\beta)/2}\left[1+\frac{(1-\beta)^2}{24}\ln^2\frac{F}{X}+\frac{(1-\beta)^4}{1920}\ln^4\frac{F}{X}\right]}\frac{z}{\Xi(z)} \\
z &= \frac{\nu}{\alpha}(FX)^{(1-\beta)/2}\ln\frac{F}{X} \\
\Xi(z) &= \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)
\end{aligned}
$$

In Hagan et al. [2002] it is suggested that its more convenient to fit the model to $\sigma_{ATM}, \beta, \rho, \nu$ where σ_{ATM} is the at-the-money volatility. It is possible to determine α from σ_{ATM} by inverting the following formula, which is a cubic in α. This can be done easily using Tartaglia's method (also sometimes referred to as Cardano's method) see West [2005a].

$$
\sigma_{ATM} = \sigma(F, F) = \frac{\alpha}{F^{(1-\beta)}}\left[1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{F^{2-2\beta}}+\frac{1}{4}\frac{\rho\beta\alpha\nu}{F^{1-\beta}}+\frac{2-3\rho^2}{24}\nu^2\right]t_{ex}\right] \tag{4.24}
$$

4.2.1 Monte Carlo Simulation

It is shown in Majmin [2005] that the SDE of S under the SABR model is given by

$$
\begin{aligned}
d\hat{S} &= (r-q)\hat{S}dt + \hat{\alpha}e^{(r-q)(\beta-1)(T-t)}\hat{S}^{\beta}dW_1 \\
d\hat{\alpha} &= \nu\hat{\alpha}dW_2 \\
dW_1 dW_2 &= \rho dt
\end{aligned}
$$

The discretized version of the SDE is then

$$
\begin{aligned}
\hat{S}_j &= \hat{S}_{j-1} + (r-q)\hat{S}_{j-1}\Delta t + \hat{\alpha}_{j-1}e^{(r-q)(\beta-1)(t_{ex}-j\Delta t)}\hat{S}_{j-1}^{\beta}u_j\sqrt{\Delta t} \\
\hat{\alpha}_j &= \hat{\alpha}_{j-1} + \nu\hat{\alpha}_{j-1}\left(\rho u_j + \sqrt{1-\rho^2}v_j\right)\sqrt{\Delta t}
\end{aligned}
$$

4.2.2 Calibration

As mentioned above we follow West [2005a] regarding the calibration of the model. We estimate β using a regression of $\ln\sigma(F, F)$ on $\ln F$. This is possible as Hagan et al. [2002] shows that $\ln\sigma(F,F) = \ln\alpha - (1-\beta)\ln F + \ldots$. It is mentioned in West [2005a] that some empirical evidence suggests that β varies with time to maturity but that maintaining a constant β over the life of the option does not seem to impact the values of other parameters materially. The approach we take is to estimate β using the regression method mentioned above using historical ATM volatility data. Once we have β we then use the Nelder-Mead algorithm to find ν and ρ as described above in 4.1.4.

4.3 Results

We fit both the SABR and the LMUV models to the eleven sets of data for SAFEX listed ALSI contracts. The data sets have been obtained from various sources and are either actual interbank trades, indicative skews published by various banks equity derivate trading desks or the indicative skew published by SAFEX. The trade sets are contained in the period from February 2007 to February 2008. The results of the SABR calibration are presented in 3.1.2 and the results of the LMUV calibration in 4.3. Selected graphs of the fitted skews are also presented.

Both models fit certain datasets very closely, these are generally datasets with a wide range of strikes provided by a single source. Data sets that contain a number of data points clustered around a narrow range of strikes and which contain a number of trades from different sources provide a challenge to both the models but particularly the LMUV model. Due to the higher number of parameters the LMUV model is more flexible and thus fits the data sets closer with an average sum of squares error of 0.00015, while the SABR had an average error of 0.00031, but this is probably more due to over fitting. The SABR model was faster on average taking just under 7.5 seconds to fit all eleven data sets while the LMUV took just over 40. The main difference between the models is the stability of the SABR model, particularly in the case of difficult datasets. The final data set demonstrates how the LMUV can produce a concave skew shape for certain difficult to fit data sets.

Trade Date	Maturity	ATM Vol	β	ν	ρ	# Points	Time (ms)	Score
2007-07-12	2007-12-20	21.00%	0.53	0.81	-0.75	9	125	0.000012
2007-07-12	2008-03-20	22.00%	0.53	0.64	-0.78	9	31	0.000002
2007-07-12	2008-06-19	22.00%	0.53	0.55	-0.75	9	31	0.000008
2007-02-12	2007-05-12	20.10%	0.86	1.63	-0.65	6	7062	0.000450
2007-02-12	2007-08-12	21.38%	0.86	1.09	-0.70	5	7031	0.000001
2007-02-12	2007-11-12	21.52%	0.86	0.90	-0.74	5	7156	0.000001
2008-02-07	2008-06-18	32.00%	0.68	1.028	-0.53	9	656	0.000017
2008-02-07	2008-09-18	28.50%	0.68	0.78	-0.62	9	343	0.000007
2008-02-07	2008-12-18	27.50%	0.68	0.73	-0.63	9	343	0.000008
2007-03-07	2007-06-21	22.15%	0.73	1.69	-0.42	17	3078	0.002582
2007-03-07	2007-09-20	21.95%	0.73	0.55	-0.99	5	3093	0.000253

Table 4.1: Results of SABR calibration to SAFEX ALSI Index Skew

Trade Date	Maturity	W1	W2	W3	M1	M2	M3	V1	V2	V3	# Points	Time (ms)	Score
2007-07-12	2007-12-20	0.233	0.624	0.142	0.167	0.047	-0.206	0.052	0.094	0.207	9	515	4.767E-05
2007-07-12	2008-03-20	0.427	0.426	0.146	0.191	0.044	-0.267	0.087	0.129	0.255	9	2765	8.366E-07
2007-07-12	2008-06-19	0.210	0.655	0.135	0.231	0.131	-0.379	0.079	0.164	0.207	9	1937	1.808E-06
2007-02-12	2007-05-12	0.530	0.264	0.206	0.091	-0.013	-0.123	0.051	0.022	0.173	6	7562	5.893E-05
2007-02-12	2007-08-12	0.238	0.573	0.189	0.146	0.068	-0.172	0.054	0.105	0.254	5	8531	4.228E-12
2007-02-12	2007-11-12	0.268	0.601	0.130	0.186	0.096	-0.375	0.069	0.138	0.205	5	7375	4.792E-08
2008-02-07	2008-06-18	0.349	0.420	0.231	0.115	-0.010	-0.182	0.100	0.137	0.316	9	1015	6.646E-06
2008-02-07	2008-09-18	0.247	0.533	0.220	0.140	0.029	-0.274	0.103	0.162	0.337	9	531	3.424E-06
2008-02-07	2008-12-18	0.370	0.396	0.233	0.150	0.016	-0.337	0.131	0.173	0.378	9	1031	1.373E-06
2007-03-07	2007-06-21	0.329	0.424	0.247	0.045	0.068	-0.074	0.039	0.116	0.217	17	6140	1.397E-03
2007-03-07	2007-09-20	0.180	0.416	0.404	0.193	0.125	-0.108	0.038	0.068	0.135	5	3203	1.288E-04

Table 4.2: Results of LMUV calibration to SAFEX ALSI Index Skew

			Implied			Historic		
Underlying	**Date**	**Maturity**	**Volatility**	**Skewness**	**Kurtosis**	**Volatility**	**Skewness**	**Kurtosis**
TOPI	2007-07-12	2007-12-20	24.26%	-1.012	1.487	17.07%	-0.043	0.002
TOPI	2007-07-12	2008-03-20	25.87%	-1.250	2.124	17.07%	-0.043	0.002
TOPI	2007-07-12	2008-06-19	25.19%	-1.156	1.194	17.07%	-0.043	0.002
TOPI	2007-02-12	2007-05-12	25.30%	-0.786	0.908	23.58%	-0.027	0.014
TOPI	2007-02-12	2007-08-12	25.87%	-1.158	1.783	23.58%	-0.027	0.014
TOPI	2007-02-12	2007-11-12	25.68%	-1.264	1.717	23.58%	-0.027	0.014
TOPI	2008-02-07	2008-06-18	37.24%	-0.745	0.924	23.86%	-0.002	0.007
TOPI	2008-02-07	2008-09-18	33.38%	-1.001	1.316	23.86%	-0.002	0.007
TOPI	2008-02-07	2008-12-18	33.30%	-1.261	1.965	23.86%	-0.002	0.007
TOPI	2007-03-07	2007-06-21	27.16%	-0.647	0.818	22.94%	-0.031	0.015
TOPI	2007-03-07	2007-09-20	22.00%	-0.559	-0.163	22.94%	-0.031	0.015

Table 4.3: Results of LMUV calibration to SAFEX ALSI Index Skew

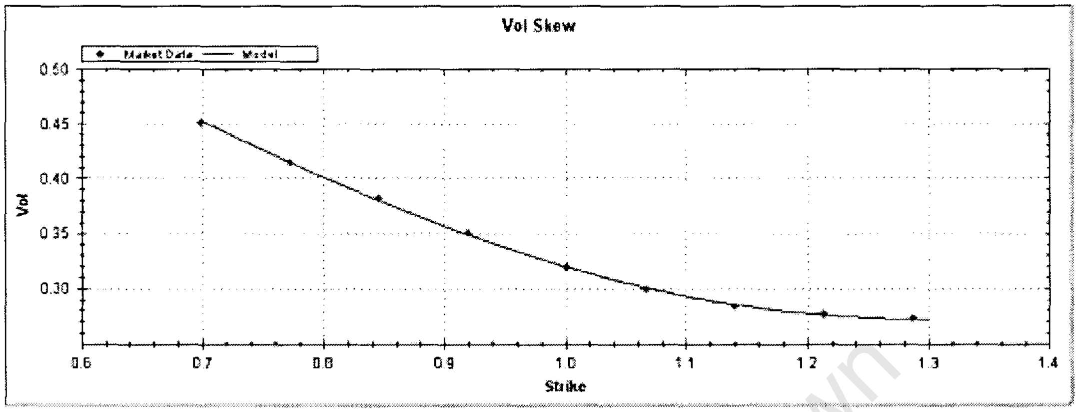

Figure 4.2: SABR model fitted to the SAFEX ALSI Index Skew 7 Feb 2008, expiry 18 June 2008

4.3.1 Fitting single stock volatility skew

We now compare three different methods of calibrating a volatility skew for a single stock option using the ATM quoted volatility for the single stock and the index volatility skew. We assume that we have an ATM volatility quote for the single stock and that we have enough volatility data across a number of strikes to fit a volatility skew model to the benchmark index that contains the stock. In the South African market it may not be the case that we always have traded ATM option prices for all single stock options but a number of market makers publish indicative bid offer spreads on ATM vol for the top 40 largest cap stocks in the market on a weekly or fortnightly basis.

Fitting the SABR model

We fit the SABR model to single stocks by fitting the model to the index skew for the corresponding maturity. We then use the ρ and ν parameters from the index model to calibrate the single stock model. Quoted ATM implied volatility for the single stock is used to determine β and α. As before we estimate β using a regression of $\ln \sigma(F, F)$ on $\ln F$ using historic ATM volatility data for the stock and use the current ATM volatility to obtain α. This is consistent with the advice of the authors in Hagan et al. [2002] where they suggest that ρ and ν are relatively stable and can be calibrated less frequently while the ATM volatility parameter is need to update the model

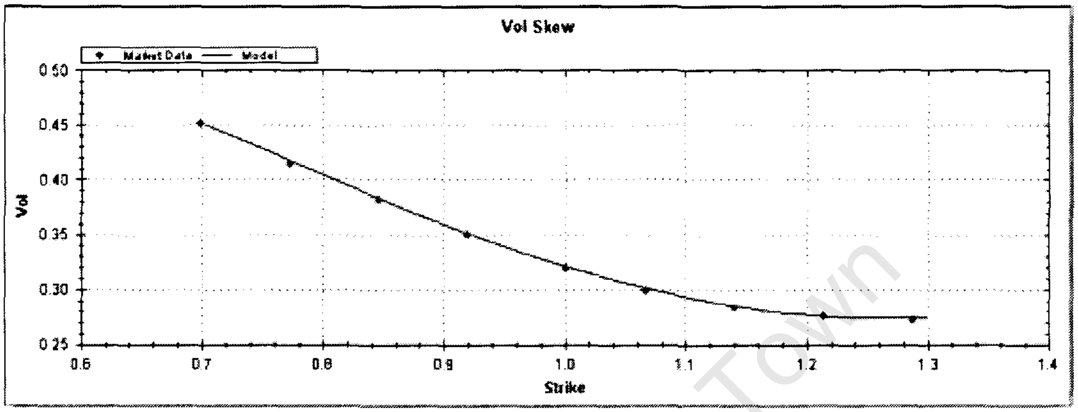

Figure 4.3: LMUV model fitted to the SAFEX ALSI Index Skew 7 Feb 2008, expiry 18 June 2008

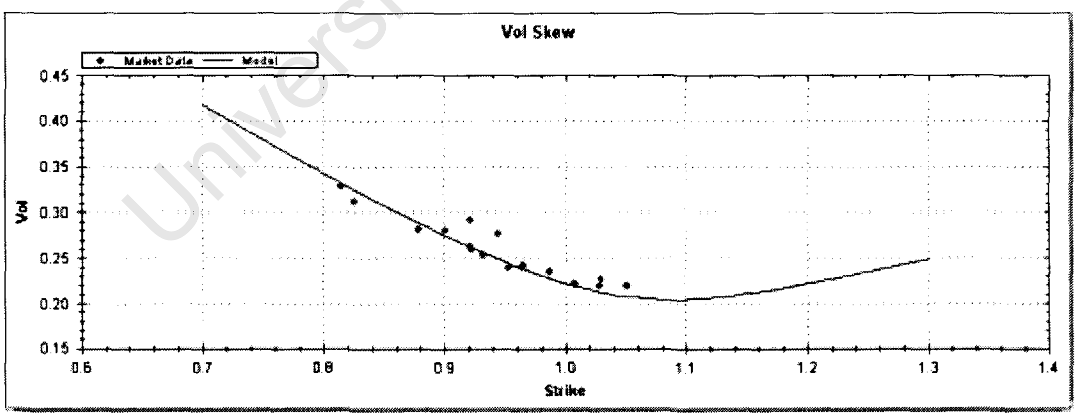

Figure 4.4: SABR model fitted to the SAFEX ALSI Index Skew 7 March 2007, expiry 21 June 2007

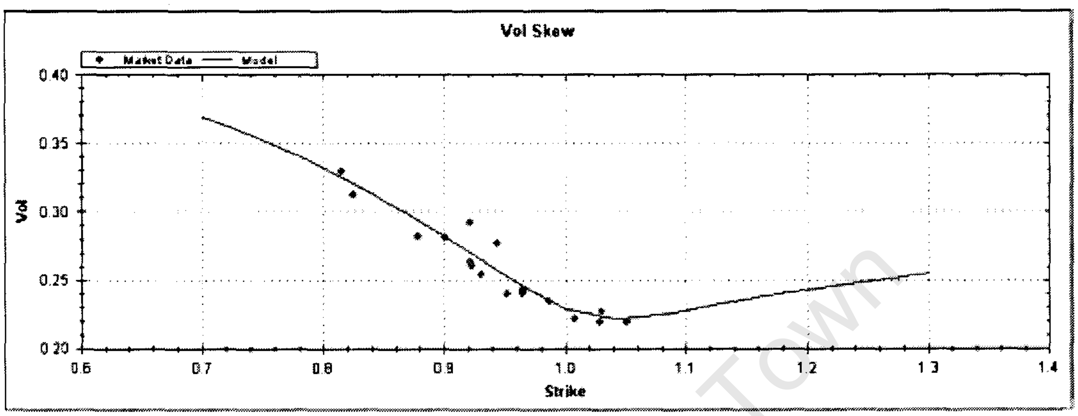

Figure 4.5: LMUV model fitted to the SAFEX ALSI Index Skew 7 March 2007, expiry 21 June 2007

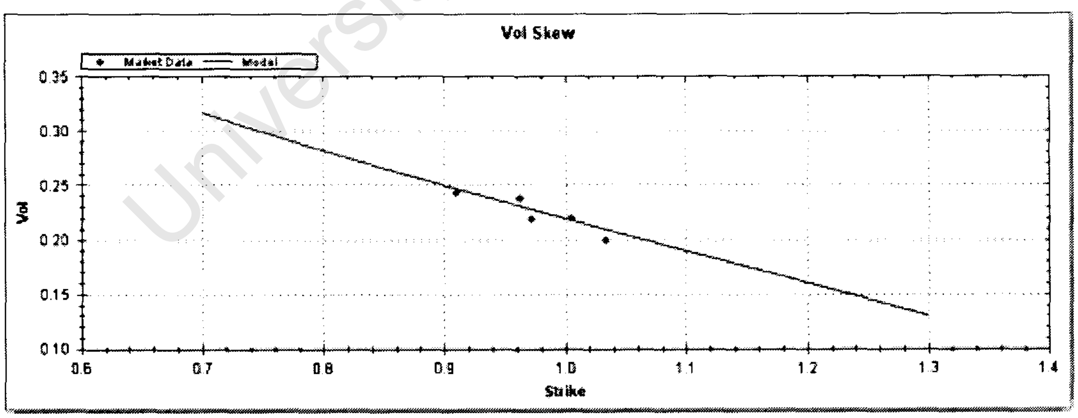

Figure 4.6: SABR model fitted to the SAFEX ALSI Index Skew 7 March 2007, expiry 20 Sep 2007

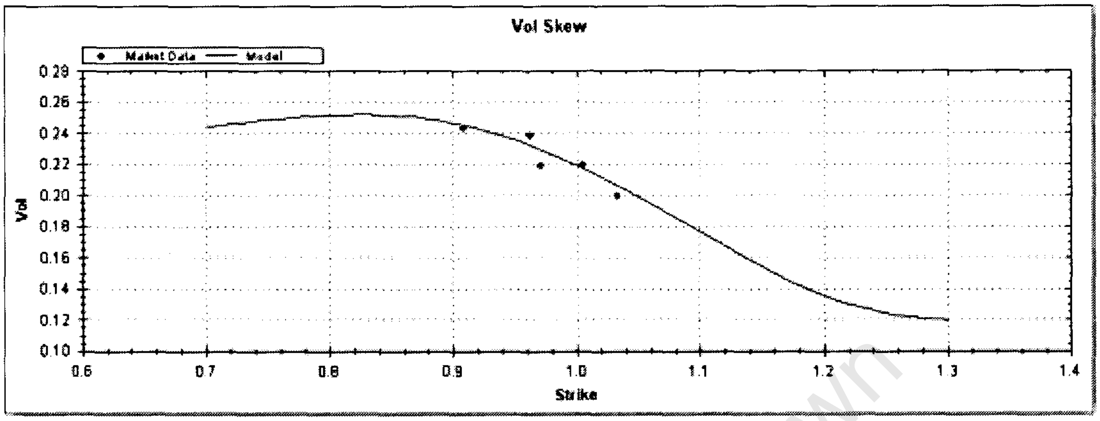

Figure 4.7: LMUV model fitted to the SAFEX ALSI Index Skew 7 March 2007, expiry 20 Sep 2007

Underlying	Trade	Expiry	ATM Vol	β	ν	ρ
AGL	2008-02-07	2008-06-18	44.80%	0.977	1.096	-0.582
GFI	2008-02-07	2008-06-18	49.30%	0.874	1.096	-0.582
REM	2008-02-07	2008-06-18	31.20%	0.719	1.096	-0.582
SBK	2008-02-07	2008-06-18	40.30%	0.626	1.096	-0.582

Table 4.4: SABR model used to create a single stock volatility skew using index skew data

as this parameter changes. As can be seen for the results this method results in a single stock skew that is very similar in shape to the index skew but is shifted vertically to adjust for the different ATM volatility of the underlying. This method is very simple to achieve and produces intuitively reasonable results.

Fitting the LMUV factor model

The second method we consider is similar to that used in Carr and Madan [2000]. The approach is to model the return on the stock with a CAPM model using on a liquid financial index as the market portfolio. This approach was developed to provide quotes for options on single stock names in the Tokyo and Hong Kong markets in the early 2000s. In these markets as in South Africa single stock options were thinly traded, while index options were liquid and a implied volatility skew was available. In Carr and Madan

[2000] the authors apply this methodology to the CEV skew model and allow the residual volatility of the single stock to be non-normally distributed. We follow a slightly simpler approach applying the idea to the LMUV model but assuming that the skew is driven completely by the index and that the residual volatility of the single stock is normally distributed.

We assume that we have an index I that has the risk neutral dynamics

$$\frac{dI}{I} = rdt + \sigma_1 dW_1 \tag{4.25}$$

now given a stock S we model S using a CAPM approach.

$$\begin{aligned} \frac{dS}{S} &= \alpha dt + \beta\frac{dI}{I} + \hat{\sigma} dW_2 && (4.26)\\ &= (\alpha + \beta r)dt + \sqrt{\beta^2\sigma_1^2 + \sigma_2^2}dW && (4.27)\\ &= rdt + \sqrt{\beta^2\sigma_1^2 + \sigma_2^2}dW && (4.28) \end{aligned}$$

Here W_1 and W_2 are independent and α is determined by the risk neutral requirement. We determine σ_2 by using the implied ATM volatility for the stock σ_{ATM} and the index σ^I_{ATM},

$$\sigma_2 = \sqrt{\max(0, \sigma^2_{ATM} - \beta^2 * (\sigma^I_{ATM})^2)}$$

Thus if we have fitted a LMUV model to the index skew with parameters $V_i^I[T]$ and $M_i^I[T]$ we are able to obtain V_i and M_i for S as follows

$$\begin{aligned} V_i[T] &= \sqrt{\beta^2(V_i^I[T])^2 + \sigma_2{}^2T} \\ M_i[T] &= \beta M_i^I[T] + \ln\left(\frac{\sum_i w_i e^{M_i[t]}}{\sum_i w_i e^{\beta M_i[t]}}\right) \end{aligned}$$

			Implied			Historic		
Underlying	**Date**	**Maturity**	**Volatility**	**Skewness**	**Kurtosis**	**Volatility**	**Skewness**	**Kurtosis**
AGL	2008-02-07	2008-06-18	53.78%	-0.771	0.946	0.401	0.025	0.007
GFI	2008-02-07	2008-06-18	50.81%	-0.087	0.055	0.359	-0.015	0.008
REM	2008-02-07	2008-06-18	33.03%	-0.222	0.193	0.245	-0.009	0.004
SBK	2008-02-07	2008-06-18	44.01%	-0.368	0.368	0.326	0.002	0.002

Table 4.5: LMUV factor model used to create volatility skews for single stock options.

				Implied Moments		
Underlying	**Trade**	**Expiry**	**ATM Vol**	**Vol**	**Skewness**	**Kurtosis**
AGL	2008-02-07	2008-06-18	44.80%	55.00%	-0.74	0.92
GFI	2008-02-07	2008-06-18	49.30%	61.50%	-0.74	0.92
REM	2008-02-07	2008-06-18	31.20%	36.00%	-0.74	0.92
SBK	2008-02-07	2008-06-18	40.30%	48.00%	-0.74	0.92

Table 4.6: Moment matching approach used to create volatility skew for single stock options.

Moment matching

Our final approach is to fit the skew by matching the implied moments of the risk neutral distribution to some target moments. We use the implied skewness and kurtosis of the index model to obtain targets for the skewness and kurtosis. We then use the ATM vol for the stock to obtain a target volatility. If we naively set the implied volatility of the distribution to the ATM implied volatility the model skew does not necessarily fit through the ATM vol from the market. Instead we need to ensure that the fit the model by ensuring that the implied skewness and kurtosis match those of the index model and the ATM volatility of the model fits the quoted ATM volatility of the stock. This results in the implied volatility of the distribution being higher than the ATM Vol as can be seen in 4.3.1.

Comparison of results

The results of the three methods of calibration for four stocks are presented in tables 4.3.1, 4.3.1, 4.3.1 and in figures 4.8. 4.9. 4.10. The SABR and moment matching methods produce quite similar skews while the LMUV factor model approach can produce skews that are quite different from the other two methods. The LMUV factor model method generally seems to produces skews that are flatter than the other models.

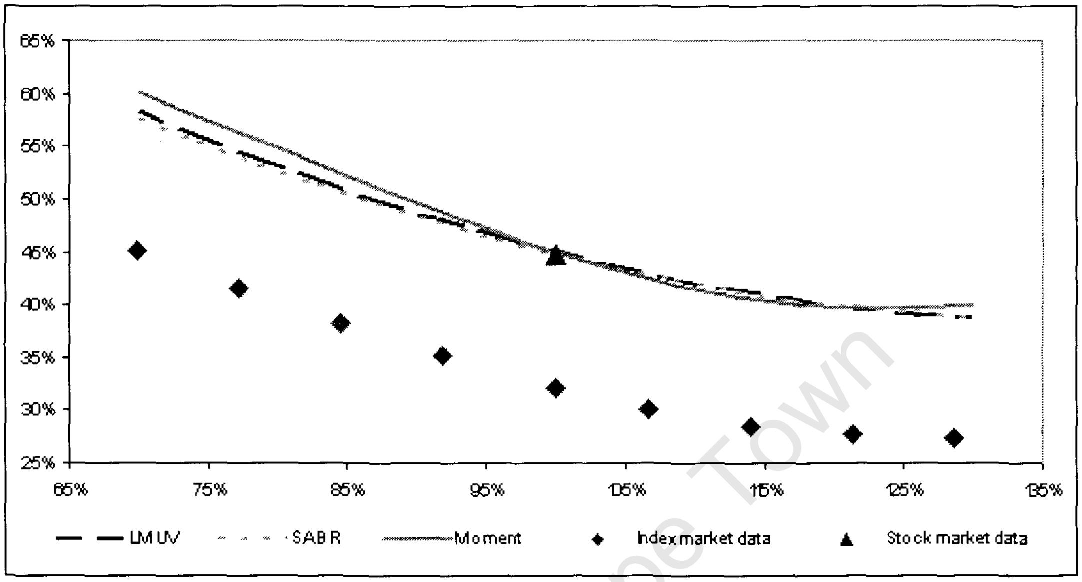

Figure 4.8: Results of the three methods for calibrating a single stock volatility skew for AGL

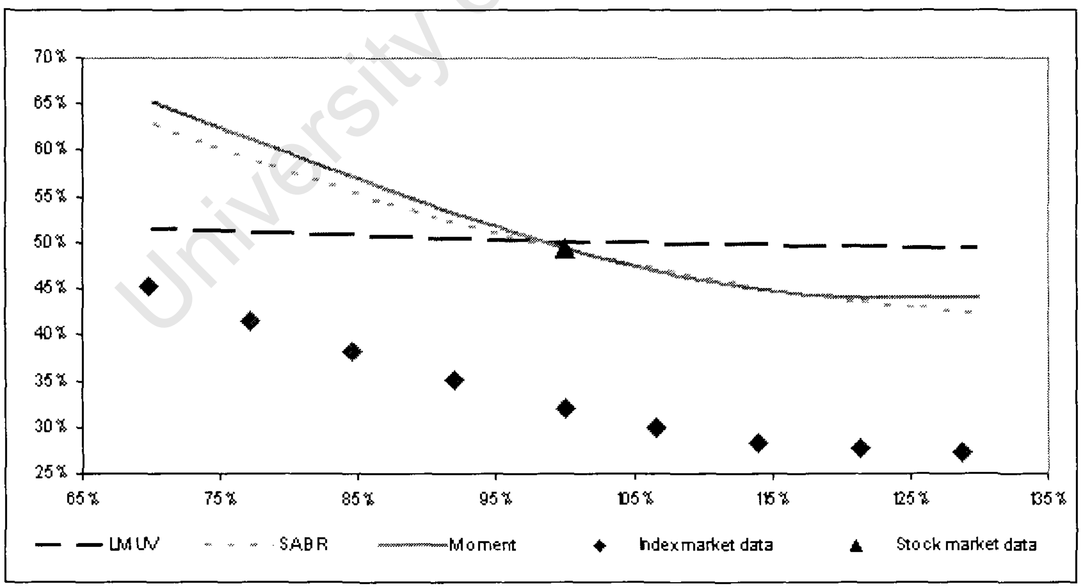

Figure 4.9: Results of the three methods for calibrating a single stock volatility skew for GFI

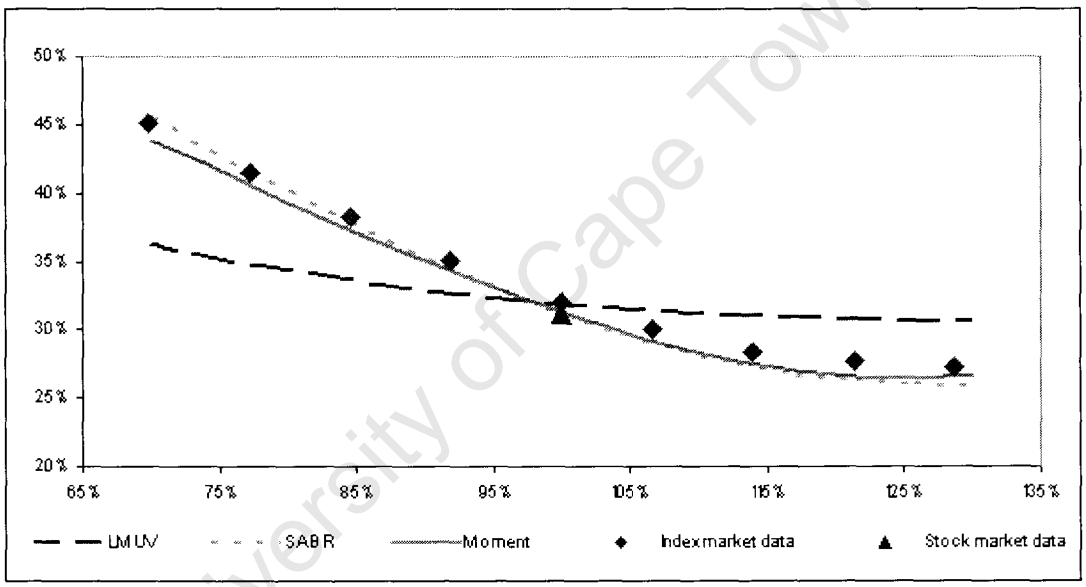

Figure 4.10: Results of the three methods for calibrating a single stock volatility skew for REM

Chapter 5

Pricing multi-asset options

Now we are ready to bring the previous work together and tackle the problem of pricing basket options. The first challenge that arises when pricing basket options is that the basket price does not have lognormal dynamics. This makes it difficult to arrive at a closed form solution to the option pricing problem using standard theory that has been developed in the univariate case. The problem lends itself to Monte Carlo simulation, but this can become computationally intensive, especially in high dimensions. Practitioners are often required to use closed form approximations that are computationally efficient especially when positions are valued many times over during risk calculations. An example of this approach that uses the machinery of the LMLV model mentioned previously is examined in Brigo et al. [2004c] . The authors approximate the terminal density of the basket price using a mixture of lognormal densities. The parameters of the mixture model are found by matching the moments of the model with the moments of the actual basket price process. This model provides a closed form approximation for the basket option price without having to assume that the basket price process follows lognormal dynamics. Unfortunately this model assumes that the single stocks all follow a standard Black Scholes model and thus individual volatility skews can not be taken into account.

We consider two approaches and compare these to the standard multivariate Black Scholes approach which does not take into account the volatility skew of the single stocks and assumes a Gaussian copula. First we consider a multivariate extension of the LMLV. This method models the volatility skews of the individual stocks consistently with the univariate LMLV model. Second we attempt to bring together the univariate LMLV or SABR models by using copulas to model the dependence structure of the stocks in the basket.

5.1 The standard Black-Scholes Approach

The standard Black-Scholes approach assumes that the securities are driven by a normal multivariate Brownian motion

$$d\bar{S}_t = \bar{\mu} dt + \Sigma d\bar{W}_t \tag{5.1}$$

where $\bar{\mu}$ is the risk free rate, Σ is the covariance matrix of returns of the securities and $\bar{W}$ is an n-dimensional Brownian motion. Any derivative on $\bar{S}$ can be valued by Monte-Carlo simulation, but this can become computationally expensive, especially as n grows large. Much work has been done on developing approximate closed form pricing formulas for multivariate options in this framework, for example Brigo et al. [2004b]. We implement a Monte-Carlo approach to pricing options in this framework as a benchmark to consider other methods against. We have shown that the assumption of joint normally distributed returns fails to account for a number of features of observed security returns and we are interested in seeing how much effect taking these features into account has on the value of option prices.

5.2 The multivariate Log-Normal Mixture model

In Brigo et al. [2004c] the authors propose a multivariate extension of the LMLV (Multivariate LM) model . This model extends the LMLV to the multi-asset case while maintaining consistency with the single asset LMLV model for each individual asset. This model requires that a LMLV model be calibrated to each of the underlying stocks and then mixes in all possible ways the component densities of the individual stocks while applying an instantaneous correlation matrix to these densities.

5.2.1 Model Derivation

Consider an n-dimensional stochastic process $\mathbf{x}(t)$ with each component following the SDE

$$\frac{dx_i(t)}{x_i(t)} = \mu_i(t)dt + \mathbf{C}_i(x,t) \cdot d\mathbf{W}_t \tag{5.2}$$

Here $\mathbf{W}_t$ is a d-dimensional Brownian motion with $d \leq n$.

Now we proceed as we did in the univariate case. We assume that the density $p_t(\mathbf{x})$ of $\mathbf{x}$ is equal to the weighted sum of densities $p_t^{(k)}$

$$p_t(\mathbf{x}) = \sum_{i=1}^{N} \lambda_k p_t^{(k)}(\mathbf{x}) \tag{5.3}$$

where $p_t^{(k)}$ corresponds to the dynamics

$$\frac{dx_i(t)}{x_i(t)} = \mu_i(t)dt + \mathbf{C}_i(x,t) \cdot d\mathbf{W}_t \tag{5.4}$$

The multivariate forward Kolmogorov equation gives

$$\frac{\partial p_t}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\mu_i(t)x_i p_t] - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}[\mathbf{C}_i \cdot \mathbf{C}_j x_i x_j p_t] = 0 \tag{5.5}$$

and for each k

$$\frac{\partial p_t^{(k)}}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\mu_i(t)x_i p_t^{(k)}] - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}[\sigma_{ij}^{(k)}(\mathbf{x},t) x_i x_j p_t^{(k)}] = 0 \tag{5.6}$$

This gives

$$\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[\left(C_{ij}(x,t)p_t - \sum_{k=1}^{N} \lambda_k \sigma_{ij}^{(k)}(x,t)p_t^{(k)}\right) x_i x_j\right] = 0 \tag{5.7}$$

This PDE has solution

$$\frac{\sum_{k=1}^{N} \lambda_k \sigma_{ij}^{(k)}(x,t)p_t^{(k)}}{\sum_{k=1}^{N} \lambda_k p_t^{(k)}} \tag{5.8}$$

Now we show that under certain assumptions the multivariate dynamics are consistent with the dynamics of the univariate log normal mixture model.

First we assume that for all k, $\sigma_{ij}^{(k)}$ is a function of t, independent of x and of the form

$$\sigma_{ij}^{(k)} = \sigma_i^{(k)}(t) \cdot \sigma_j^{(k)}(t) \tag{5.9}$$

Now we make the further assumption that $\sigma_i^{(k)}(t) \cdot \sigma_j^{(k)}(t) = \sigma_i^{(k)}(t)\sigma_j^{(k)}(t)\rho_{ij}$

Under this assumption

$$p_t^{(k)}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \Xi^{(k)}(t)}\prod_{i=1}^n x_i} \exp\left[-\frac{\tilde{\mathbf{x}}(\Xi^{(k)}(t))^{-1}\tilde{\mathbf{x}}}{2}\right] \tag{5.10}$$

where $\Xi^{(k)}(t)$ is the integrated covariance matrix of returns for the components of $\mathbf{X}$:

$$\Xi_{ij}^{(k)}(t) = \int_0^t \sigma_i^{(k)}(t) \cdot \sigma_j^{(k)}(t)ds \tag{5.11}$$

and

$$\tilde{x}_i = \ln x_i - \ln x_i(0) - \int_0^t \left(\mu_s^{(i)} - \frac{{\sigma_i^{(k)}}^2(s)}{2}\right) ds \tag{5.12}$$

Now lets assume that we have a calibrated a univariate mixture model for each stock. Let $\pi_t^{(i)}$ be the density of S_i. Then

$$\pi_t^{(i)} = \sum_{k=1}^{\nu_i} \lambda_{ik}\pi_t^{(ik)}(x) \tag{5.13}$$

Where each $\pi_t^{(ik)}(x)$ is a shifted log normal density and $\sum_{k=1}^{\nu_i} \lambda_{ik} = 1$

Now we make the following choice for the base densities $p_t(\mathbf{x})$

$$p_t(\mathbf{x}) = \sum_{i_1,i_2,\ldots,i_n=1}^{\nu} \lambda_{1,i_1} \cdots \lambda_{1,i_n} p_t^{(i_1\cdots i_n)}(\mathbf{x}) \tag{5.14}$$

where

$$p_t^{(i_1\cdots i_n)}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \Xi^{(i_1\cdots i_n)}(t)}\prod_{i=1}^n x_i} \exp\left[-\frac{\tilde{\mathbf{x}}^{(i_1\cdots i_n)}(\Xi^{(i_1\cdots i_n)}(t))^{-1}\tilde{\mathbf{x}}^{(i_1\cdots i_n)}}{2}\right] \tag{5.15}$$

where $\Xi^{(i_1\cdots i_n)}(t)$ is the integrated covariance matrix of returns with (l, m) element

$$\Xi_{lm}^{(i_1\cdots i_n)}(t) = \int_0^t \sigma_{l.i_l}(t)\sigma_{m.i_m}(t)\rho_{lm}ds \tag{5.16}$$

and

$$\tilde{x}_l^{(i_1\cdots i_n)} = \ln x_l - \ln x_l(0) - \int_0^t \left(\mu_s^{(l.i_l)} - \frac{{\sigma_s^{(l.i_l)}}^2}{2}\right) ds \tag{5.17}$$

It can be shown that this model is consistent with the individual LMLV models for each underlying in that all moments of the individual asset price densities as well as the unconditional dynamics of all the individual assets are reproduced.

It is important to note that the correlation parameter in the model ρ is not the instantaneous correlation that is actually exhibited by the process, but the average correlation felt by the process can be determined analytically Brigo et al. [2004c].

Finally, it is instructive to consider an example to clarify the notation. Let $n = 2$ and $\nu = 2$.

Then we have two univariate mixture models

$$\begin{aligned}\pi_t^{(1)} &= \lambda_{1,1}\pi_t^{(1,1)}(x) + \lambda_{1,2}\pi_t^{(1,2)}(x)\\ \pi_t^{(2)} &= \lambda_{2,1}\pi_t^{(2,1)}(x) + \lambda_{2,2}\pi_t^{(2,2)}(x)\end{aligned}$$

and $\sigma_{1,1}$ is the volatility coefficient of $\pi_t^{(1,1)}(x)$, $\sigma_{1,2}$ the volatility coefficient of $\pi_t^{(1,1)}(x)$, etc.

$$\begin{aligned}p_t(\mathbf{x}) &= \lambda_{1,1}\lambda_{2,1}p_t^{(1,1)}(\mathbf{x})\\ &+ \lambda_{1,1}\lambda_{2,2}p_t^{(1,2)}(\mathbf{x})\\ &+ \lambda_{1,2}\lambda_{2,1}p_t^{(2,1)}(\mathbf{x})\\ &+ \lambda_{1,2}\lambda_{2,2}p_t^{(2,2)}(\mathbf{x})\end{aligned}$$

and

$$\begin{aligned}p_t^{(1,1)}(\mathbf{x}) &= \frac{1}{(2\pi)^{\frac{2}{2}}\sqrt{\det \Xi^{(1,1)}(t)}x_1x_2}\exp\left[-\frac{\tilde{\mathbf{x}}^{(1,1)}(\Xi^{(1,1)}(t))^{-1}\tilde{\mathbf{x}}^{(1,1)}}{2}\right]\\ p_t^{(1,2)}(\mathbf{x}) &= \frac{1}{(2\pi)^{\frac{2}{2}}\sqrt{\det \Xi^{(1,2)}(t)}x_1x_2}\exp\left[-\frac{\tilde{\mathbf{x}}^{(1,2)}(\Xi^{(1,2)}(t))^{-1}\tilde{\mathbf{x}}^{(1,2)}}{2}\right]\\ p_t^{(2,1)}(\mathbf{x}) &= \frac{1}{(2\pi)^{\frac{2}{2}}\sqrt{\det \Xi^{(2,1)}(t)}x_1x_21}\exp\left[-\frac{\tilde{\mathbf{x}}^{(2,1)}(\Xi^{(2,1)}(t))^{-1}\tilde{\mathbf{x}}^{(2,1)}}{2}\right]\\ p_t^{(2,2)}(\mathbf{x}) &= \frac{1}{(2\pi)^{\frac{2}{2}}\sqrt{\det \Xi^{(2,2)}(t)}x_1x_2}\exp\left[-\frac{\tilde{\mathbf{x}}^{(2,2)}(\Xi^{(2,2)}(t))^{-1}\tilde{\mathbf{x}}^{(2,2)}}{2}\right]\end{aligned}$$

$$
\begin{aligned}
\Xi^{(1,1)} &= \begin{pmatrix} \int_0^t \sigma_{1,1}^2 \rho_{1,1} ds & \int_0^t \sigma_{1,1}\sigma_{2,1}\rho_{1,2} ds \\ \int_0^t \sigma_{2,1}\sigma_{1,1}\rho_{2,1} ds & \int_0^t \sigma_{2,1}^2 \rho_{2,2} ds \end{pmatrix} \\
\Xi^{(2,1)} &= \begin{pmatrix} \int_0^t \sigma_{1,2}^2 \rho_{1,1} ds & \int_0^t \sigma_{1,2}\sigma_{2,1}\rho_{1,2} ds \\ \int_0^t \sigma_{2,1}\sigma_{1,2}\rho_{2,1} ds & \int_0^t \sigma_{2,1}^2 \rho_{2,2} ds \end{pmatrix} \\
\Xi^{(1,2)} &= \begin{pmatrix} \int_0^t \sigma_{1,1}^2 \rho_{1,1} ds & \int_0^t \sigma_{1,1}\sigma_{2,2}\rho_{1,2} ds \\ \int_0^t \sigma_{2,2}\sigma_{1,1}\rho_{2,1} ds & \int_0^t \sigma_{2,2}^2 \rho_{2,2} ds \end{pmatrix} \\
\Xi^{(2,2)} &= \begin{pmatrix} \int_0^t \sigma_{1,2}^2 \rho_{1,1} ds & \int_0^t \sigma_{1,2}\sigma_{2,2}\rho_{1,2} ds \\ \int_0^t \sigma_{2,2}\sigma_{1,2}\rho_{2,1} ds & \int_0^t \sigma_{2,2}^2 \rho_{2,2} ds \end{pmatrix}
\end{aligned}
$$

here we haven't shown the dependence on t for notational simplicity. Finally

$$
\begin{aligned}
\tilde{x}_1^{(1,1)} &= \tilde{x}_1^{(1,2)} = \ln x_1 - \ln x_1(0) - \int_0^t \left(\mu_{(1,1)} - \frac{{\sigma_{(1,1)}}^2}{2} \right) ds \\
\tilde{x}_1^{(2,1)} &= \tilde{x}_1^{(2,2)} = \ln x_1 - \ln x_1(0) - \int_0^t \left(\mu_{(1,2)} - \frac{{\sigma_{(1,2)}}^2}{2} \right) ds \\
\tilde{x}_2^{(1,1)} &= \tilde{x}_2^{(2,1)} = \ln x_2 - \ln x_2(0) - \int_0^t \left(\mu_{(2,1)} - \frac{{\sigma_{(2,1)}}^2}{2} \right) ds \\
\tilde{x}_2^{(1,2)} &= \tilde{x}_2^{(2,2)} = \ln x_2 - \ln x_2(0) - \int_0^t \left(\mu_{(2,2)} - \frac{{\sigma_{(2,2)}}^2}{2} \right) ds
\end{aligned}
$$

5.2.2 Option Pricing

Now that we have an expression for $p_t(\mathbf{x})$ we are able to price derivatives with payouts based on $\mathbf{x}$

$$
\begin{aligned}
\mathbb{E}[f(\mathbf{x})] &= \int f(\mathbf{x}) p_t(\mathbf{x}) d\mathbf{x} \\
&= \sum_{i_1, i_2, \ldots, i_n = 1}^{\nu} \lambda_{1,i_1} \cdots \lambda_{1,i_n} \int f(\mathbf{x}) p_t^{(i_1 \cdots i_n)}(\mathbf{x}) d\mathbf{x}
\end{aligned}
$$

So we can compute the option price by $\sum_{i=1}^{n} \nu_i$ single step Monte Carlo integrations, one for each combination $(i_1, i_2, \ldots, i_n)$. This provides a computational benefit as we don't have to simulate the discretized process for each asset price path and only need one single step simulation for each asset price path.

5.3 Pricing using copulas

We now attempt to combine the volatility skew models that we discussed earlier with copulas to create an option pricing framework that captures both the volatility skew effect in the individual securities as well as the non-Gaussian dependence structure between the individual security returns. There are a number of considerations that need to be taken into account that can complicate this approach. Firstly we need to determine the frequency at which we will estimate and model the dynamics of the securities. This problem is encountered because there is no simple connection between, for example, the copula that governs daily returns and the copula that governs monthly returns. If we assume a Gaussian copula we have the convenient fact that independent normal variates are additive and so we are able to estimate the correlation matrix using daily or weekly data and the use the same copula to simulate returns over the life of the option using a single step Monte Carlo integration. Unfortunately if we don't choose a Gaussian copula we can't be sure of this, which leads us to make a choice between two approaches. If we attempt to model the distribution of asset price returns over the life of the contract then we need to estimate the copula of those returns over the same time period, which requires long histories of data to achieve. Alternatively if we use higher frequency data to estimate the copula of asset price returns we need to model the dynamics of these returns with a similar frequency. This requires us to chose a model with explicit dynamics for the underlyings but also increases the computational intensity of any numerical pricing approach.

Secondly, if we wish to model the volatility skew of the individual assets then it seems that we have to allow the instantaneous volatility of the individual prices processes to be variable over time. In fact to be able to use a stochastic volatility model we need to allow this volatility to be stochastic. We have spent a lot of energy on modeling the dependence structure of the underlying security returns but haven't considered the dependence structure of the individual securities volatility processes. It is clear that the volatilities of individual equities are not independent, volatilities of stocks in the same market very often move higher or lower at the same time due to a global risk factor, similarly stocks in a similar sector or exposed to a similar risk factor may have dependent volatilities. We have not explicitly taken this into account in our approach. This effect is indirectly taken into account when using the SABR model for the individual securities as the spot level and the stock volatility are negatively correlated by construction so if S_1 and S_2 are both driven by a SABR process with $\rho_1, \rho_2 < 0$ and S_1 and S_2 have some dependence structure between them then σ_1 and σ_2 have some dependence structure.

Finally, movements in the volatility may inadvertently influence the co-dependence of the individual security returns. This has already been seen in the case of the multivariate LMLV model.

The next two approaches we consider are to simulate the sample paths of each underlying security using either an SABR model or a LMUV model but to apply a dependence structure to the Brownian motions that are driving these process.

5.3.1 The SABR and copulas approach

Given N stocks $S_1, S_2, \ldots, S_N$ we assume that we have calibrated an SABR model for each S_i and have the corresponding parameters $(\alpha_i, \rho_i, \beta_i, \nu_i)$ for each S_i. Then we have the dynamics of each stock.

$$\begin{aligned} d\hat{S}_i &= (r-q_i)\hat{S}_i dt + \hat{\alpha}_i e^{(r-q_i)(\beta_i-1)(T-t)}\hat{S}_i^{\beta} dW_{i,1} \\ d\hat{\alpha}_i &= \nu_i \hat{\alpha}_i dW_{i,2} \\ dW_{i,1} dW_{i,2} &= \rho_i dt \end{aligned}$$

and $dW_{1,1}, \ldots, dW_{n,1}$ have a copula $\mathcal{C}$. This allows us to price derivatives on $S_1, S_2, \ldots, S_N$ by Monte Carlo simulation, by simulating each S_i individually but by drawing $dW_{1,1}, \ldots, dW_{n,1}$ from a multivariate distribution with normal marginals and copula $\mathcal{C}$. It is important to note that we don't explicitly model the dependence between the $\hat{\alpha}_i$ which gives the dependence between the volatilities of the individual stocks. There is an implicit relationship due to the fact that the brownian motions $W_{i,1}$ and $W_{i,2}$ have correlation ρ_i and $W_{j,1}$ and $W_{j,2}$ have correlation ρ_j while the relationship between $W_{i,1}$ and $W_{j,1}$ is specified by the copula $\mathcal{C}$. While we accept this weakness for the sake of simplicity, there is no guarantee that this implicit relationship will be strong enough to model the behavior observed in markets.

5.3.2 The LMUV and copulas approach

Given N stocks $S_1, S_2, \ldots, S_N$ we assume that we have calibrated an LMUV model for each S_i

$$dS_i(t) = \begin{cases} S_i(t)[r_i(t)dt + \sigma_0 dW_i(t)] & t \in [0, \epsilon] \\ S_i(t)[q_i(t)dt + \xi_i(t)dW_i(t)] & t > \epsilon \end{cases} \tag{5.18}$$

where (q_i, ξ_i) is a random pair that is drawn at time ϵ independently of W_i and (q_j, ξ_j) where $i \neq j$ and takes values in a set of N_i deterministic functions;

$$(t \mapsto (q(t), \xi(t)) = \begin{cases} (t \mapsto (r_{i,1}(t), \sigma_{i,1}(t)) & \text{with probability } w_{i,1} \\ (t \mapsto (r_{i,2}(t), \sigma_{i,2}(t)) & \text{with probability } w_{i,2} \\ \vdots & \vdots \\ (t \mapsto (r_{i,N_i}(t), \sigma_{i,N_i}(t)) & \text{with probability } w_{i,N_i} \end{cases} \tag{5.19}$$

and $dW_{1,1}, \ldots, dW_{n,1}$ have a copula $\mathcal{C}$. Again we are able to use Monte Carlo simulation to price derivatives under this model in a similar manner to the previous approach. Also the same criticism holds for this model as we are not explicitly modeling any dependence between the volatilities of the individual stocks in this model.

5.4 Results

5.4.1 Instruments

We price three multi asset derivatives using the methods presented above. Given a set of n securities $S_1, \ldots, S_n$ with weights $w_1, \ldots, w_n$.

Basket options

A basket option pays the return on a weighted basket of securities minus the strike of the option.

$$\text{Call Payoff} = \max\left[0, \frac{\sum_i^n w_i S_i(T)}{\sum_i^n w_i S_i(0)} - K\right]$$

Best-of options

A best-of option pays the return on the best performing asset in a basket of assets over the life of the option minus the strike of the option.

$$\text{Call Payoff} = \max\left[0, \frac{S_1(T)}{S_1(0)} - K, \ldots, \frac{S_n(T)}{S_n(0)} - K\right]$$

Dispersion Trades

A dispersion trade pays out the average of the absolute difference between the performance of each individual asset and the performance of the basket as a whole.

$$\text{Payoff} = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{S_i(T)}{S_i(0)} - \frac{A(T)}{A(0)}\right|$$

$$A(T) = \sum_{i=1}^{n} w_i \frac{S_i(T)}{S_i(0)}$$

5.4.2 Three underlyings

We price all three different instruments on a basket of three stocks listed on the JSE. The basket of stocks consists of the three stocks ASA,BIL and MTN with weights 30%, 30% 40%. The derivatives were priced on trade date 19 September 2007 with maturity date 20 March 2008. In the case of the basket option and best-of option all results refer to the price of a put option. We have ignored any dividends in this analysis for the sake of simplicity. Both the SABR Copula and LMUV Copula methods were used assuming a Normal copula and a Student T copula.

These methods required us to estimate the dependence structure for the basket in three different ways. Estimating the correlation matrix for a joint normal distribution gave the following correlation matrix.

$$\begin{pmatrix} 1.000 & 0.381 & 0.483 \\ 0.381 & 1.000 & 0.361 \\ 0.483 & 0.361 & 1.000 \end{pmatrix}$$

Estimating the normal copula gave the following matrix.

$$\begin{pmatrix} 1.000 & 0.286 & 0.448 \\ 0.286 & 1.000 & 0.331 \\ 0.448 & 0.331 & 1.000 \end{pmatrix}$$

This is different to the matrix for the joint normal distribution as in the first case we assumed a a normal distribution for the marginal distributions while in this case we don't make any assumption about the marginals but rather apply the probability integral transform of the empirical distribution to each marginal using the transform $x \mapsto rank(x)$ and then estimate the copula from the transformed data. The first case is equivalent to the Exact Maximum Likelihood method or the Inference For the Margins (IFM) method under the assumption of normal marginals and a normal copula. The second estimation method is the Canonical Maximum Likelihood method, see chapter 5 of Cherubini et al. [2004] for more details and formal definitions of the different inference methods.

Estimating the Student T copula gave a copula with degrees of freedom = 9.4237 and the following correlation matrix.

$$\begin{pmatrix} 1.000 & 0.302 & 0.471 \\ 0.302 & 1.000 & 0.351 \\ 0.471 & 0.351 & 1.000 \end{pmatrix}$$

The results of the pricing for the basket option are presented in 5.1. The Brigo-Mercurio multivariate approach was the fasted method to run. The speed advantage of the Brigo-Mercurio multivariate method over the Black Scholes Monte Carlo was less than expected. The LMUV Copula and SABR Copula methods were considerably slower than the other two methods. While we have presented results of the time taken to run the various pricing methods this is not necessarily an indication of the speed of an optimal implementation of these methods. The code that was developed to implement these models was designed with flexibility and speed of development as the first priority rather than speed of execution. Thus it is the relative performance of the methods that is important as the same code library was used for all the models.

The same result set is present in a more compact format in 5.2. The main conclusion that can be drawn from these results is that it is the choice of marginal distributions and not the copula that results in different prices from these different approaches. Changing the copula from a Normal to a Student T copula in the case of the SABR Copula and LMUV Copula approaches does not result in a significant difference in price. In general the SABR Copula model produces prices that are the most different from a standard Black-Scholes approach, while the LMUV Copula and Brigo-Mercurio multivariate methods seem to give results closer to that of the Black-Scholes model. In particular these models give similar prices to the Black-Scholes prices for deep

					Time taken (ms)	
Method	**Copula**	**Strike**	**Value**	**Std Error**	**Calibrate**	**Run**
Black Scholes	Normal	120%	17.11%	0.07%	953	24,578
Black Scholes	Normal	110%	10.44%	0.06%	937	24,843
Black Scholes	Normal	100%	5.48%	0.04%	875	24,062
Black Scholes	Normal	90%	2.22%	0.02%	921	24,328
Black Scholes	Normal	80%	0.65%	0.01%	921	23,687
Multivariate LM	Normal	120%	18.91%	0.06%	2,828	18,625
Multivariate LM	Normal	110%	11.10%	0.05%	3,203	18,093
Multivariate LM	Normal	100%	5.04%	0.03%	3,515	20,421
Multivariate LM	Normal	90%	1.74%	0.02%	2,281	17,859
Multivariate LM	Normal	80%	0.40%	0.01%	3,046	17,937
LMUV	Normal	120%	17.30%	0.06%	2,312	79,296
LMUV	Normal	110%	10.28%	0.05%	2,359	62,765
LMUV	Normal	100%	5.06%	0.04%	2,515	70,796
LMUV	Normal	90%	1.73%	0.02%	1,921	71,750
LMUV	Normal	80%	0.52%	0.01%	3,015	72,468
SABR	Normal	120%	16.23%	0.08%	984	74,546
SABR	Normal	110%	9.91%	0.06%	968	74,031
SABR	Normal	100%	5.41%	0.05%	890	73,609
SABR	Normal	90%	2.78%	0.03%	921	72,937
SABR	Normal	80%	1.25%	0.02%	953	75,781
LMUV	Student T	120%	17.30%	0.06%	3,203	82,531
LMUV	Student T	110%	10.31%	0.05%	2,750	72,062
LMUV	Student T	100%	5.00%	0.04%	2,234	84,531
LMUV	Student T	90%	1.80%	0.02%	1,765	86,250
LMUV	Student T	80%	0.51%	0.01%	2,796	84,828
SABR	Student T	120%	16.28%	0.08%	1,078	100,421
SABR	Student T	110%	10.04%	0.06%	1,062	100,343
SABR	Student T	100%	5.45%	0.05%	1,046	100,546
SABR	Student T	90%	2.81%	0.04%	953	97,734
SABR	Student T	80%	1.33%	0.02%	953	95,109

Table 5.1: Results of calibration and pricing of basket option on ASA,BIL,MTN basket

		Strike				
Method	**Copula**	**80%**	**90%**	**100%**	**110%**	**120%**
Black Scholes	Normal	0.65%	2.22%	5.48%	10.44%	17.11%
Multivariate LM	Normal	0.40%	1.74%	5.04%	11.10%	18.91%
LMUV	Normal	0.52%	1.73%	5.06%	10.28%	17.30%
SABR	Normal	1.25%	2.78%	5.41%	9.91%	16.23%
LMUV	Student T	0.51%	1.80%	5.00%	10.31%	17.30%
SABR	Student T	1.33%	2.81%	5.45%	10.04%	16.28%

Table 5.2: Prices of basket option on ASA,BIL,MTN basket

Method	**Copula**	**Strike**				
		80%	**90%**	**100%**	**110%**	**120%**
Black Scholes	Normal	3.52%	7.89%	14.19%	21.74%	30.45%
Multivariate LM	Normal	4.72%	9.49%	16.62%	25.54%	34.61%
LMUV	Normal	3.55%	8.83%	15.54%	22.84%	32.54%
SABR	Normal	6.99%	11.24%	16.70%	23.84%	32.28%
LMUV	Student T	3.73%	9.38%	16.31%	24.75%	33.37%
SABR	Student T	7.12%	11.17%	16.67%	23.66%	32.16%

Table 5.3: Prices of best-of-option on ASA,BIL,MTN basket

out the money put options and deviate more for in the money puts, while the SABR Copula prices out of the money puts quite significantly higher than the Black Scholes prices, for example in the case of the three stock basket the SABR Copula approach gives a price double that of the Black Scholes price for a 80% strike put and about 25% higher for a 90% strike put. For this reason the SABR Copula seems to produce the most intuitively correct results. We would expect that due to the fact that the Black-Scholes method ignores the single stock volatility skew that it should under price out of the money puts. It is clear to see from these results that modeling the volatility skew of the individual stocks has a larger effect on the pricing results than changing the copula. For the 80% strike puts the change of copula does seem to have an influence but this difference is much less than the difference between the SABR and LMUV models. In the case of the best-of option the LMUV approach using a Student T copula differs from the LMUV approach using a Gaussian copula as can be seen in 5.4.

Method	Copula	Value	Std Error
Black Scholes	Normal	4.34%	0.01%
Multivariate LM	Normal	4.45%	0.01%
LMUV	Normal	4.81%	0.01%
SABR	Normal	4.65%	0.01%
LMUV	Student T	4.70%	0.01%
SABR	Student T	4.56%	0.01%

Table 5.4: Prices of dispersion trade on ASA,BIL,MTN basket

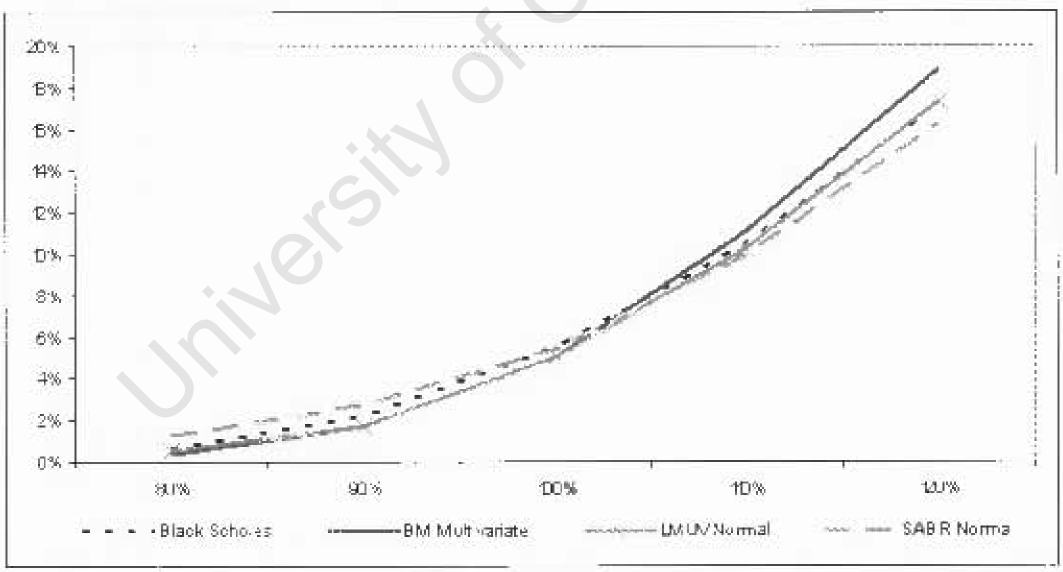

Figure 5.1: Results of pricing of basket option on ASA,BIL,MTN basket assuming a normal copula

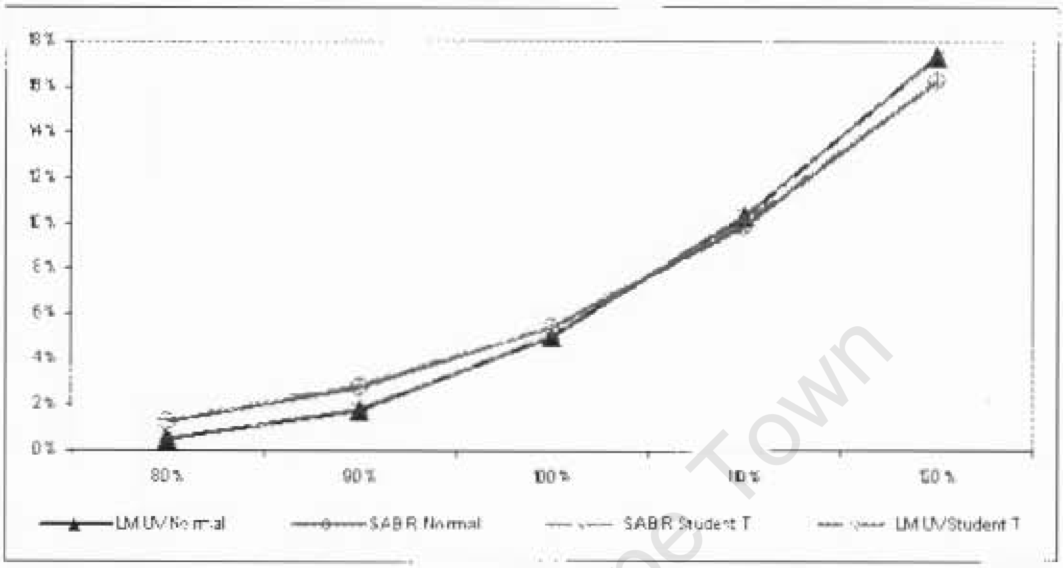

Figure 5.2: Pricing a basket option on ASA,BIL,MTN basket using two different copulas

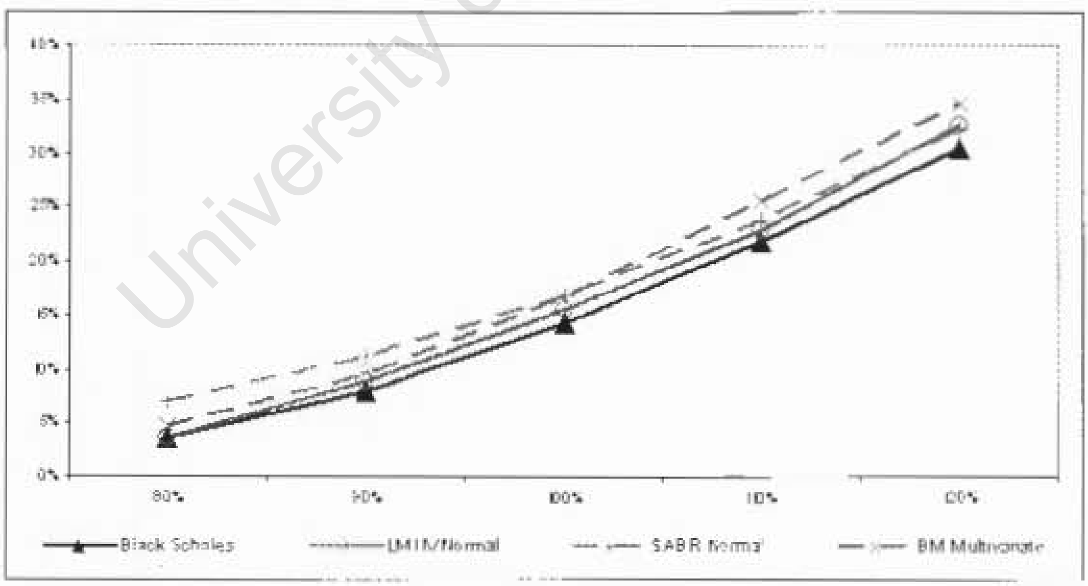

Figure 5.3: Results of pricing of best-of option on ASA,BIL,MTN basket assuming a normal copula

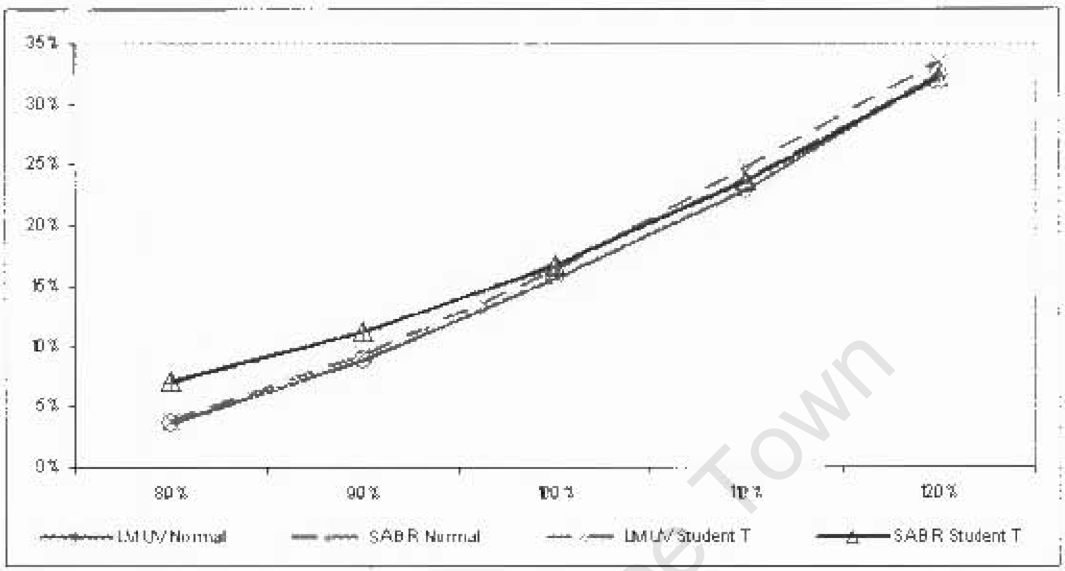

Figure 5.4: Pricing a best-of option on ASA,BIL,MTN basket using two different copulas

5.4.3 Two underlyings

We also show the results of using the various methods to price instruments on an basket of two equally weighted stocks, IMP and AMS. This allowed us to test the use of a Clayton copula to model the dependence structure of the two stocks. While the Clayton copula is not necessarily a good model of the dependence of the returns two equities we include this copula to try and see if using a different dependence structure, specifically one that is rotationally asymmetric, may change the pricing more than the change from a Gaussian copula to a Student T copula.

Estimating the correlation matrix for a joint normal distribution gave the following correlation matrix.

$$\begin{pmatrix} 1.000 & 0.624 \\ 0.624 & 1.000 \end{pmatrix}$$

Estimating the normal copula resulted in the following correlation matrix.

$$\begin{pmatrix} 1.000 & 0.605 \\ 0.605 & 1.000 \end{pmatrix}$$

		Strike				
Method	**Copula**	**80%**	**90%**	**100%**	**110%**	**120%**
Black Scholes	Normal	1.80%	4.17%	7.91%	13.21%	19.55%
Multivariate LM	Normal	1.50%	3.33%	7.63%	13.26%	20.53%
LMUV	Normal	1.59%	4.04%	7.70%	13.56%	20.03%
SABR	Normal	3.08%	5.09%	8.02%	12.36%	18.22%
LMUV	Student T	1.69%	4.18%	7.85%	13.45%	20.16%
SABR	Student T	3.26%	5.08%	8.12%	12.40%	18.24%
LMUV	Clayton	1.87%	4.11%	8.25%	13.57%	20.10%
SABR	Clayton	3.30%	5.12%	8.03%	12.45%	18.32%

Table 5.5: Prices of basket option on IMP.AMS basket

		Strike				
Method	**Copula**	**80%**	**90%**	**100%**	**110%**	**120%**
Black Scholes	Normal	3.26%	6.87%	12.09%	18.49%	26.20%
Multivariate LM	Normal	3.93%	7.66%	13.95%	21.42%	30.18%
LMUV	Normal	3.93%	7.63%	13.29%	20.28%	28.16%
SABR	Normal	6.39%	9.27%	13.78%	19.45%	26.64%
LMUV	Student T	3.73%	7.65%	13.19%	20.53%	28.94%
SABR	Student T	6.37%	9.40%	13.48%	19.17%	26.42%
LMUV	Clayton	3.55%	7.73%	13.05%	20.45%	28.53%
SABR	Clayton	6.21%	9.20%	13.41%	19.31%	26.35%

Table 5.6: Prices of best-of-option on IMP.AMS basket

The estimated Student T copula had degrees of freedom = 5.2608 and the following correlation matrix.

$$\begin{pmatrix} 1.000 & 0.631 \\ 0.631 & 1.000 \end{pmatrix}$$

Finally we also estimated the Clayton copula and obtained $\alpha = 1.5399$.

Method	Copula	Value	Std Error
Black Scholes	Normal	4.85%	0.02%
Multivariate LM	Normal	5.80%	0.02%
LMUV	Normal	5.66%	0.02%
LMUV	Student T	5.51%	0.02%
LUMV	Clayton	5.64%	0.02%
SABR	Normal	5.49%	0.02%
SABR	Student T	5.37%	0.02%
SABR	Clayton	5.49%	0.02%

Table 5.7: Prices of dispersion trade on IMP,AMS basket

The results of this set of simulations is very similar to the previous set. Again the volatility skew seems to account for most of the difference in prices generated by the different approaches. In the case of the basket option the Student T and Clayton copula models both generated higher prices on average than the corresponding Gaussian copula model. This is consistent with our expectations as these models allow for lower tail dependence that is not taken into account by the Gaussian copula approach and so they should give a higher price for put options. It is important to note that the choice of the Clayton copula, arguably not a good model for the data that we are trying to model, does not cause the result to deviate more than a change in the volatility skew model for the individual stocks. This suggests that it is more important to model the dynamics of the underlying stocks correctly before worrying about the correct dependence structure for the basket. It is also interesting to note the different results produced by the LMUV approach and the SABR approach. While both of these models have been shown to fit the volatility skew well and to produce very similar fits in the presence of sufficient data the difference in the dynamics under the different models can result in quite different option prices when pricing using simulation. It is also notable that these differences occur for derivatives that are not path dependent.

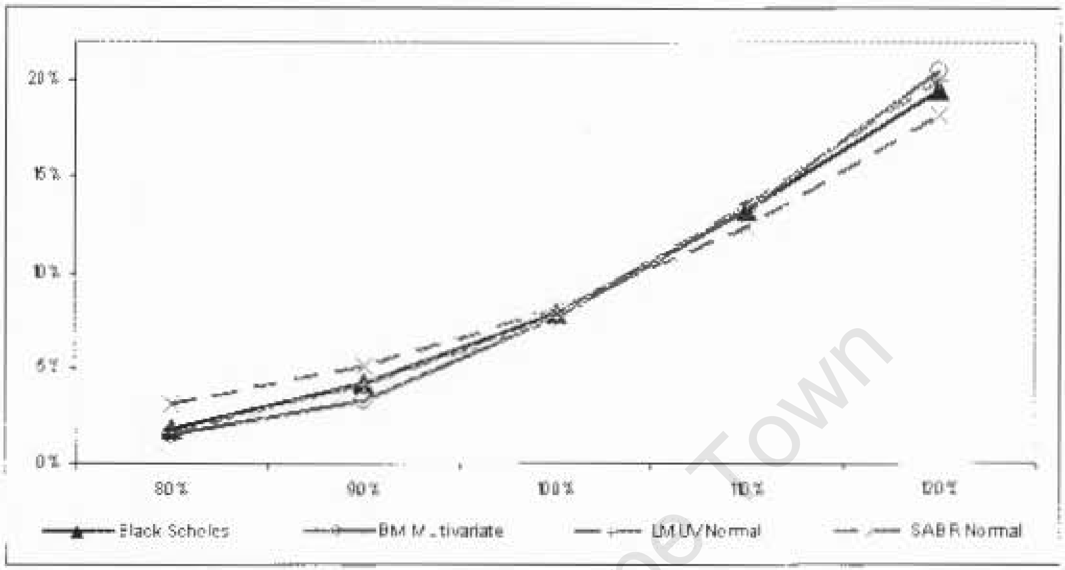

Figure 5.5: Results of pricing basket option on IMP.AMS basket assuming a normal copula

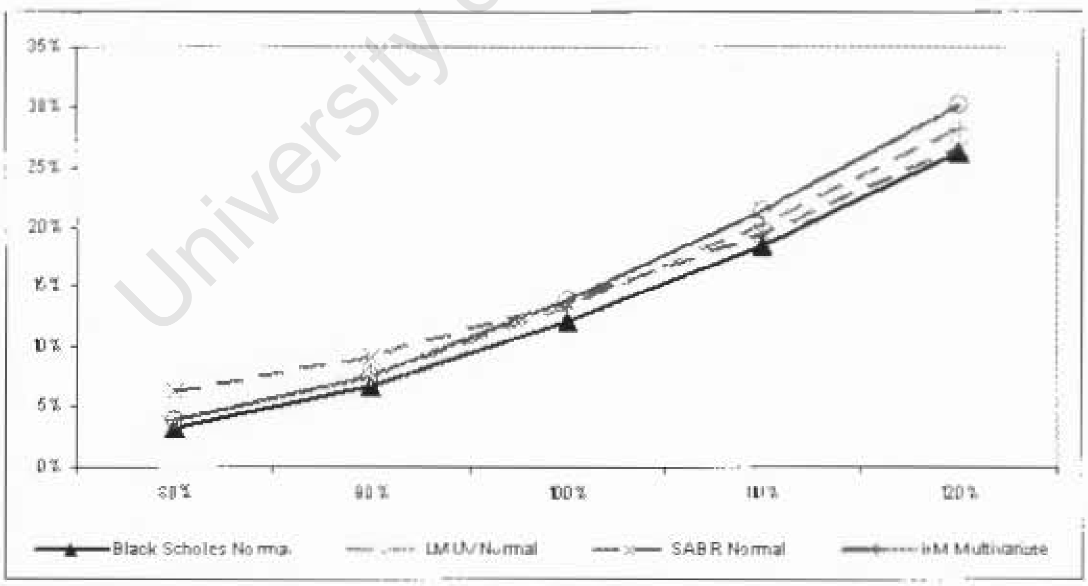

Figure 5.6: Results of pricing best-of option on IMP.AMS basket assuming a normal copula

Chapter 6

Conclusion

In this work we have considered two approaches to modeling the volatility skew for a single asset equity option. We found that in most cases the two approaches produced similar results. We also considered three possible approaches to generating a single stock volatility skew given an index skew and an ATM volatility quote for the single stock option. Finally we considered various approaches to pricing equity options on multiple underlyings. We introduced two methods that attempt to take into account the implied volatility skew of the underlying assets that constitute the basket as well as a general dependence structure modeled by a copula. We compared these two methods with the standard Black-Scholes approach as well as the multivariate log-normal mixture model of Brigo et al. [2004c]. Our results show that these methods do produce significantly different prices from the Black-Scholes approach. In the case of basket options this difference can be as big as 90% for out-the-money options and more than 5% in some cases for in-the-money options. For dispersion trades the difference can also be as large as 10% in some cases. The main finding of this work has been the importance of modeling the volatility skew of the underlying options compared to modeling a more general copula. While changing the copula does influence the pricing this influence is far less than that of a change in the single asset dynamics. The influence of a change in copula also dependents on the type of derivative being priced and would be more important in the case where the occurrence of extreme events coinciding is very important. In our results deep out-of-the money basket options seemed to be influenced the most by a change in copula. Given this finding the multivariate log-normal mixture model proves to be an appealing model given the fact that it takes the volatility skew into account in a way consistent with the LMUV model and it is numerically more efficient than the other approaches. The SABR approach also showed promise as it produced results that were consistent with intuition and also proves to be a very easy and stable model to calibrate. As we have pointed out our implementation of this approach had a few weakness that would need to be addressed, namely the dependence structure of the underlying stocks volatilities as well as the effect that the stochastic volatility of the underlying dynamics has on the dependence structure actually felt by the process.

In conclusion we have highlighted a number of important issues regarding the pricing of multi asset derivative contracts and introduced some possible approaches to solving this problem that improve on a standard Black-Scholes approach.

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