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# Normal Bases and Compactifications of Frames

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A dissertation prepared under the supervision of Dr C.R.A. Gilmour  
in fulfilment of the requirements of the degree of Master of Science  
in Mathematics.

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## Introduction

A topic of interest in the realm of topological spaces is the possibility of densely embedding a given space inside a compact Hausdorff extension space. Such a compact Hausdorff space is called a *compactification* of the given space.

The first person to consider extending a topological space was Carathéodory [13] in 1913, who did so in the context of subsets of the plane. The first to formally study compactifications of more general topological spaces was Tietze [26] in 1923. In 1924, Alexandroff [1] established the existence of the minimal compactification of a locally compact, non-compact Hausdorff space. In the paper [27] of 1930 Tychonoff showed that the topological spaces having Hausdorff compactifications are precisely those which are now known as Tychonoff spaces. In 1937, Čech [14], and later M.H. Stone [25], constructed the largest compactification of a Tychonoff space. The existence of the Stone-Čech compactification is also a corollary to later work by Lubben [22] in 1941, in which he introduced the familiar order on the family of all compactifications of a topological space and showed that every non-empty subfamily of this family has a least upper bound.

It is also possible to construct non-Hausdorff compactifications of topological spaces. In 1938, Wallman [28] described how to construct a compactification of a  $T_1$  topological space. He showed that this compactification is Hausdorff if and only if the original space is normal. Further work on Wallman-type extensions has been carried out by Fan and Gottesman [16], Banaschewski [5] and [6], Frink [18], Njastad [24] and others.

Frames are lattice-theoretic generalisations of topological spaces. It is possible to define the notion of a compactification of a frame in such a way that it mimics that of a topological space. This dissertation is concerned with the frame-theoretic analogues of certain classical compactifications of topological spaces.

Compactifications of frames were first considered by Banaschewski, see [7] and [9]. In [12], Banaschewski and Mulvey obtained the compact (completely) regular coreflection

of a frame and hence established the existence of a frame-theoretic analogue of the Stone-Čech compactification of a topological space. Johnstone [21] constructed a frame counterpart to the Wallman compactification by means of a nucleus defined on the frame of ideals of a distributive lattice. In [11], Banaschewski and Harting generalised Johnstone's construction by means of a nucleus defined on an arbitrary compact frame. In [2], Baboolal defined the Fan Gottesman compactification of a frame. He also showed that the minimal (Alexandroff) compactification of a regular continuous frame can be obtained by means of a Fan Gottesman normal base.

The initial aim of this dissertation was to provide a frame-theoretic analogue of Banaschewski's *normal systems of sets* in [5] as well as a frame counterpart to their associated compactifications. Having completed this part of the task, it seemed natural to investigate the relationship between this compactification and those mentioned above. Hence the first five chapters of the dissertation are devoted to the study of the frame counterparts to six well-known compactifications in the category of topological spaces. For each compactification studied, we give some motivation as to why it should be regarded as a frame-theoretic analogue of its classical counterpart. The sixth chapter is concerned with the relationships between the compactifications: in particular we are interested in conditions under which the different constructions give rise to the same compactification.

We now give a detailed outline of the dissertation:

## CHAPTER 0

Chapter Zero contains those definitions and results with which the reader should be familiar.

## CHAPTER 1

In this chapter, we introduce the concept of a *normal base* for a regular frame  $L$ . This definition is motivated by Banaschewski's *normal systems of sets* in [5]. A normal base for  $L$  is a non-empty subset of  $L$ , which join-generates  $L$ , and satisfies certain normality and regularity properties. If  $L$  is a regular frame with normal base  $N$ , then it is possible to construct a compactification of  $L$ , by considering the frame  $\mathfrak{R}$  of all *regular filters* on  $N$ , together with a map  $h: \mathfrak{R} \rightarrow L$ . We are able to show that our method of compactification of a frame by means of a normal base includes Banaschewski's compactification of a topological space by means of a normal system.

The results of Chapter One are to the best of my knowledge my own.

## CHAPTER 2

In Chapter Two we first present Baboolal's construction [2] of the *Fan Gottesman* compactification of a regular frame. The construction makes use of a *Fan Gottesman normal base* which is the frame-theoretic analogue of the classical normal base used by Fan and Gottesman in [16]. As for spaces (see [5]), we are able to relate the Fan Gottesman normal bases to the normal bases of Chapter One, and we show that every Fan Gottesman compactification can be obtained by means of a compactification relative to a naturally associated normal base. The final result of this Chapter is, as far as I am aware, original: Baboolal's Fan Gottesman compactification is indeed a generalisation of its classical counterpart in the category of topological spaces.

## CHAPTERS 3 – 5

Chapters Three to Five are entirely expository and we do not claim any originality. The material is a synthesis from various sources.

In Chapter Three, we give the frame counterpart to the *Wallman* compactification. The presentation is due to Banaschewski, Harting and Johnstone ([11] and [21]). The compactification is obtained via a nucleus defined on the frame of ideals of a subfit frame  $L$ . We give some motivation as to why the compactification should be called the Wallman compactification.

Chapter Four concerns the *zero-dimensional* compactifications of spaces and frames, due to Banaschewski ([4] and [8]).

In Chapter Five we present frame-theoretic analogues of the minimal (*Alexandroff*) and maximal (*Stone-Čech*) compactifications. Following Baboolal [2], it is shown that given a regular continuous frame, it is possible to define a smallest strong inclusion, which corresponds to the smallest compactification of that frame, thus providing a frame counterpart to the Alexandroff compactification. The Stone-Čech compactification is obtained as the compact (completely) regular coreflection of a completely regular frame; this result is due to Banaschewski and Mulvey [12].

## CHAPTER 6

In the final chapter of the dissertation, we examine some relationships which exist between the compactifications which we have studied. We are interested in conditions imposed on the frame under which the different constructions give rise to the same compactification. As far as I know, the results relating our normal base compactification to the other compactifications (the appropriate equivalences contained in Propositions 6.1, 6.2 and 6.4) are original.

## Chapter 0. Preliminaries

In this chapter, we briefly recall some of the basic concepts of frame theory. We also introduce some terminology and several elementary results which will be used throughout the thesis. For a general background of frame theory, see [19].

### THE CATEGORIES **Frm** AND **Top**

A *frame* is a complete lattice  $L$  in which

$$x \wedge \bigvee_I x_i = \bigvee_I x \wedge x_i$$

for any  $x \in L$  and family  $(x_i)_I$  contained in  $L$ . A *frame homomorphism* is a map  $h: M \rightarrow L$ , between frames  $M$  and  $L$ , preserving finite meets (including the top  $e$ ) and arbitrary joins (including the bottom  $0$ ). We denote the category of frames and frame homomorphisms by **Frm**.

The open set lattice  $\mathcal{O}X$  of a topological space  $X$  is a typical example of a frame. In fact, we have a functor  $\mathcal{O}: \mathbf{Top} \rightarrow \mathbf{Frm}$ , called the *open set functor*, taking a topological space  $X$  to the frame  $\mathcal{O}X$ , and a continuous map  $f: X \rightarrow Y$  to the frame homomorphism  $\mathcal{O}f = f^{-1}: \mathcal{O}Y \rightarrow \mathcal{O}X$ .

On the other hand, we have the *spectrum functor*  $\Sigma: \mathbf{Frm} \rightarrow \mathbf{Top}$ , which assigns to every frame  $L$  the topological space  $\Sigma L$ , called the *spectrum* of  $L$ . There are several ways of describing the points of the space  $\Sigma L$ ; for our purposes, it is most convenient to think of these points as the prime elements of  $L$ . (An element  $p \in L$  is called *prime* if  $p = a \wedge b$  implies  $p = a$  or  $p = b$ .) The basic open sets of  $\Sigma L$  are of the form  $\Sigma_a = \{x \in \Sigma L \mid x \not\leq a\}$ , where  $a \in L$ .

Given a frame homomorphism  $h: L \rightarrow M$ , we obtain a continuous map  $\Sigma h: \Sigma M \rightarrow \Sigma L$  by defining  $\Sigma h(p) = \{x \in L \mid h(x) \leq p\}$  for each element  $p \in \Sigma M$ .

The functors  $\mathcal{D}$  and  $\Sigma$  are adjoint on the right with unit

$$\begin{aligned} \eta_L: L &\rightarrow \mathcal{D}\Sigma L \\ a &\mapsto \Sigma_a \end{aligned}$$

and co-unit

$$\begin{aligned} \varepsilon_X: X &\rightarrow \Sigma\mathcal{D}X \\ x &\mapsto \bigcup\{U \in \mathcal{D}X \mid x \notin U\}. \end{aligned}$$

The frame  $L$  is said to be *spatial* if  $\eta_L$  is an isomorphism and the space  $X$  is said to be *sober* if  $\varepsilon_X$  is a homeomorphism. The spatial frames are those which are isomorphic to the open set lattice of some topological space, while the sober spaces are those which are isomorphic to the spectrum of some frame.

## THE CATEGORY **RegFRM**

For elements  $a$  and  $b$  in a bounded distributive lattice  $A$ , we say that  $a$  is *rather below*  $b$ , written  $a \prec b$ , if there exists a *separating* element  $s \in A$  such that  $a \wedge s = \mathbf{0}$  and  $s \vee b = \mathbf{e}$ . Note that if  $L$  is a frame, we have  $a \prec b$  if and only if  $a^* \vee b = \mathbf{e}$ , where  $a^*$  is the *pseudocomplement* of  $a$  in  $L$ , given by

$$a^* = \bigvee\{x \in L \mid x \wedge a = \mathbf{0}\}.$$

A frame is said to be *regular* if  $x = \bigvee\{y \in L \mid y \prec x\}$  for each  $x \in L$ . The category of regular frames and frame homomorphisms is denoted by **RegFRM**. An element  $a$  of a frame is said to be *regular* if it is equal to its double pseudocomplement  $a^{**}$ . It can be seen that the regular open sets of a topological space  $X$  are the regular elements of the frame  $\mathcal{D}X$ .

We shall often make use of the following result:

**Lemma 0.1.** *In a regular frame  $L$ , the prime elements are precisely the maximal elements.*

*Proof.* That a maximal element is prime, is true in any frame. For the converse, we assume that  $L$  is regular and that  $a \in L$  is prime. To show that  $a$  is maximal, we suppose that there is an element  $m \in L$  with  $a \prec m$  and deduce that  $m = \mathbf{e}$ :

$$\begin{aligned} a < m &\implies \exists x \in L \text{ with } x \prec m \text{ and } x \not\leq a \text{ since } L \text{ is regular} \\ &\implies \exists s \in L \text{ with } x \wedge s = \mathbf{0}, s \vee m = \mathbf{e} \text{ and } a < a \vee x \end{aligned}$$

Hence  $a = a \vee \mathbf{0} = a \vee (x \wedge s) = (a \vee x) \wedge (a \vee s)$ . Since  $a$  is prime, we have that  $a = a \vee s$  or  $a = a \vee x$ . The latter is false since  $a < a \vee x$ , so we must have that  $a = a \vee s$  or  $s \leq a$ .

Hence  $m = m \vee a \geq m \vee s = \mathbf{e}$ , as required.  $\square$

A frame homomorphism  $h: M \rightarrow L$  is said to be

- (i) *dense* if  $h(x) = \mathbf{0}_L \implies x = \mathbf{0}_M$  for  $x \in M$ ,
- (ii) *codense* if  $h(x) = \mathbf{e}_L \implies x = \mathbf{e}_M$  for  $x \in M$ .

We shall also need the following result:

**Lemma 0.2.** *In the category  $\mathbf{RegFRM}$ , a dense homomorphism  $h: M \rightarrow L$  is a monomorphism.*

*Proof.* Suppose that for homomorphisms  $g, k: N \rightarrow M$  in  $\mathbf{RegFRM}$  we have

$$h(g(x)) = h(k(x)) \quad \text{for each } x \in N.$$

For each  $a \in N$ , note the following:

$$\begin{aligned} g(a) &= g\left(\bigvee\{x \in N \mid x \prec a\}\right) \quad \text{since } N \text{ is regular} \\ &= \bigvee\{g(x) \in M \mid x \in N \text{ and } x \prec a\}. \end{aligned}$$

Let  $x \in N$  and suppose that  $x \prec a$  with separating element  $s$ . Then  $x \wedge s = \mathbf{0}$  and  $s \vee a = \mathbf{e}$ . Now

$$\begin{aligned} x \wedge s = \mathbf{0} &\implies g(x) \wedge g(s) = \mathbf{0} \\ &\implies h(g(x)) \wedge h(g(s)) = \mathbf{0} \\ &\implies h(g(x)) \wedge h(k(s)) = \mathbf{0} \quad \text{since } hg = hk \\ &\implies h(g(x) \wedge k(s)) = \mathbf{0} \\ &\implies g(x) \wedge k(s) = \mathbf{0} \quad \text{since } h \text{ is dense,} \end{aligned}$$

and

$$s \vee a = \mathbf{e} \implies k(s) \vee k(a) = \mathbf{e}.$$

This shows that  $g(x) \prec k(a)$ , and hence in particular  $g(x) \leq k(a)$ .

So  $g(a) = \bigvee\{g(x) \mid x \in N \text{ and } x \prec a\} \leq k(a)$ . Since the argument is symmetrical, the result follows.  $\square$

## NORMAL LATTICES

A bounded distributive lattice  $A$  is said to be *normal* if for elements  $a, b \in A$ ,

$$a \vee b = \mathbf{e} \implies \exists c, d \in A \text{ such that } c \wedge d = \mathbf{0} \text{ and } a \vee c = d \vee b = \mathbf{e}.$$

We include the following useful Lemma:

**Lemma 0.3.** *If  $L$  is a normal frame, then we have the following result:*

$$a \prec b \vee c \implies \exists x \prec b \text{ such that } a \prec x \vee c.$$

*Proof.* Suppose that  $a \prec b \vee c$ . Then there exists a separating element  $s$  such that  $a \wedge s = \mathbf{0}$  and  $s \vee b \vee c = \mathbf{e}$ . Since  $L$  is normal, there exist elements  $x$  and  $y$  such that  $x \wedge y = \mathbf{0}$  and  $x \vee s \vee c = y \vee b = \mathbf{e}$ . So  $x \prec b$  and  $a \prec x \vee c$ .  $\square$

## COMPACTIFICATIONS

An element  $a$  in a frame  $L$  is said to be *compact* if whenever  $a = \bigvee S$  for an arbitrary subset  $S \subseteq L$ , then  $a = \bigvee T$  for a finite subset  $T \subseteq S$ . A frame  $L$  is said to be *compact* if its top element  $\mathbf{e}$  is compact. It is easily seen that a subframe of a compact frame is compact.

A *compactification* of a frame  $L$  is a dense onto homomorphism  $h: M \rightarrow L$ , where  $M$  is a compact regular frame. If  $k: X \rightarrow Y$  is a compactification in **Top** (i.e.,  $X$  is densely embedded in the compact Hausdorff space  $Y$ ), then the frame homomorphism  $\mathcal{D}k: \mathcal{D}Y \rightarrow \mathcal{D}X$  is a compactification in **Frm**. Conversely, if  $X$  is a  $T_0$  topological space and  $h: K \rightarrow \mathcal{D}X$  is a compactification in **Frm**, then the composition  $\Sigma h \circ \varepsilon_X: X \rightarrow \Sigma K$  is a compactification of  $X$  in **Top**. The latter result uses the fact that every compact regular frame is spatial, which can be shown to be equivalent to the Boolean Ultrafilter Theorem.

THE CATEGORY **CRegFRM**

In any frame  $L$ , we say that  $a$  is *completely below*  $\overset{b}{B}$ , written  $a \prec\prec b$ , if there exists a family  $(c_{i,k})$ , where  $i = 0, 1, 2, \dots$  and  $k = 0, 1, \dots, 2^i$ , such that for all  $i$  and  $k$ ,

$$c_{i,0} = a, \quad c_{i,2^i} = b, \quad c_{i,k} = c_{i+1,2k} \quad \text{and} \quad c_{i,k} \prec c_{i,k+1}.$$

The frame  $L$  is said to be *completely regular* if  $a = \bigvee \{x \in L \mid x \prec\prec a\}$  for each  $a \in L$ . The category of completely regular frames and frame homomorphisms is denoted **CRegFRM**. In the presence of the Axiom of Countably Dependent Choice, every normal regular (and hence every compact regular) frame is completely regular.

## CONTINUOUS LATTICES

For elements  $x$  and  $y$  in a complete lattice  $L$ , we say that  $x$  is *way below*  $y$ , written  $x \ll y$ , if whenever  $y \leq \bigvee S$  for some  $S \subseteq L$ , there exists a finite subset  $F$  of  $S$  such that  $x \leq \bigvee F$ . A frame  $L$  is said to be *continuous* if  $a = \bigvee \{x \in L \mid x \ll a\}$  for each  $a \in L$ .

## FRAMES OF FILTERS AND IDEALS

Let  $A$  be a bounded distributive lattice.

An *ideal* on  $A$  is a non-empty subset  $\mathcal{J} \subseteq A$  which satisfies the following properties:

- $0 \in \mathcal{J}$ .
- $a, b \in \mathcal{J} \implies a \vee b \in \mathcal{J}$ .
- $a \in \mathcal{J} \ \& \ a \geq b \in A \implies b \in \mathcal{J}$ .

An ideal  $\mathcal{J}$  on a frame  $L$  is said to be *regular* (*completely regular*) if for each  $x \in \mathcal{J}$  there exists a  $y \in \mathcal{J}$  such that  $x \prec y$  ( $x \prec\prec y$ ). The *principal* ideals of a frame  $L$  are the ideals of the form  $\downarrow a = \{x \in L \mid x \leq a\}$  for  $a \in L$ .

An *filter* on  $A$  is a non-empty subset  $\mathcal{F} \subseteq A$  which satisfies properties dual to those above:

- $e \in \mathcal{F}$ .
- $a, b \in \mathcal{F} \implies a \wedge b \in \mathcal{F}$ .
- $a \in \mathcal{F} \ \& \ a \leq b \in A \implies b \in \mathcal{F}$ .

A filter  $\mathcal{F}$  on a frame  $L$  is said to be *regular* (*completely regular*) if for each  $x \in \mathcal{F}$  there exists a  $y \in \mathcal{F}$  such that  $y \prec x$  ( $y \prec\prec x$ ).

The ideal  $\mathcal{J}$  (filter  $\mathcal{F}$ ) is said to be *proper* if  $e \notin \mathcal{J}$  ( $0 \notin \mathcal{F}$ ).

The collection of all ideals (filters) on  $A$  forms a frame denoted  $\mathcal{J}(A)$  ( $\mathfrak{F}(A)$ ). Note that for ideals  $\mathcal{J}, \mathcal{J}' \in \mathcal{J}(A)$ ,

$$\mathcal{J} \wedge \mathcal{J}' = \mathcal{J} \cap \mathcal{J}' \text{ and } \mathcal{J} \vee \mathcal{J}' = \{i \vee j \mid i \in \mathcal{J} \text{ and } j \in \mathcal{J}'\},$$

while for filters  $\mathcal{F}, \mathcal{G} \in \mathfrak{F}(A)$ ,

$$\mathcal{F} \wedge \mathcal{G} = \{a \vee b \mid a \in \mathcal{F} \text{ and } b \in \mathcal{G}\} \text{ and } \mathcal{F} \vee \mathcal{G} = \{a \wedge b \mid a \in \mathcal{F} \text{ and } b \in \mathcal{G}\}.$$

The arbitrary join of a collection  $(\mathcal{J}_i)_I$  of ideals is given by

$$\bigvee_I \mathcal{J}_i = \{x_{i_1} \vee x_{i_2} \vee \dots \vee x_{i_n} \mid x_{i_k} \in \mathcal{J}_{i_k}\},$$

whilst the arbitrary join of a collection  $(\mathcal{F}_i)_I$  of filters is given by

$$\bigvee_I \mathcal{F}_i = \{x_{i_1} \wedge x_{i_2} \wedge \dots \wedge x_{i_n} \mid x_{i_k} \in \mathcal{F}_{i_k}\}.$$

It is easily seen that the frames  $\mathcal{J}(A)$  and  $\mathfrak{F}(A)$  are compact.

## STRONG INCLUSIONS

In [9], Banaschewski introduces the notion of a strong inclusion on a frame  $L$ , and shows that there is an intimate relationship between the set  $K(L)$  of all compactifications on  $L$ , and the set  $S(L)$  of all strong inclusions on  $L$ . The following summary is extracted from [9]:

A *strong inclusion* on a frame  $L$  is a binary relation  $\triangleleft$  on  $L$  satisfying the following properties:

- (S1)  $x \leq a \triangleleft b \leq y \implies x \triangleleft y$ .
- (S2)  $\triangleleft$  is a sublattice of  $L \times L$  (i.e.,  $\mathbf{0} \triangleleft \mathbf{0}$ ,  $e \triangleleft e$ ,  $a, b \triangleleft x \implies a \vee b \triangleleft x$  and  $a \triangleleft x, y \implies a \triangleleft x \wedge y$ ).
- (S3)  $a \triangleleft b \implies a \prec b$ .
- (S4)  $\triangleleft$  interpolates (i.e.,  $a \triangleleft b \implies \exists c \in L$  such that  $a \triangleleft c \triangleleft b$ ).
- (S5)  $a \triangleleft b \implies b^* \triangleleft a^*$ .
- (S6) For each  $a \in L$ ,  $a = \bigvee \{x \in L \mid x \triangleleft a\}$ .

Given a strong inclusion  $\triangleleft$  on  $L$ , we obtain a compactification on  $L$  as follows: Let  $\mathfrak{S}$  denote the collection of all strongly regular ideals on  $L$ . (An ideal  $\mathcal{J}$  on  $L$  is *strongly regular* if  $x \in \mathcal{J} \implies \exists y \in \mathcal{J}$  such that  $x \triangleleft y$ .) Then  $\mathfrak{S}$  is a compact regular frame, and the join map  $\bigvee: \mathfrak{S} \rightarrow L$ , taking an ideal of  $\mathfrak{S}$  to its join in  $L$ , is a compactification of  $L$ .

Conversely, given a compactification  $h: M \rightarrow L$ , the associated strong inclusion is given by

$$a \triangleleft_h b \iff h_*(a) \prec h_*(b),$$

where  $h_*: L \rightarrow M$  is the right adjoint of  $h$ , satisfying  $h(a) \leq b \iff a \leq h_*(b)$ .

Banaschewski shows in [9] that the above correspondence between the compactifications and strong inclusions on  $L$  is an isomorphism. In particular, this means that if we start off with a strong inclusion  $\triangleleft$  on  $L$ , and we form the corresponding compactification  $\bigvee: \mathfrak{S} \rightarrow L$  and then determine the associated strong inclusion  $\triangleleft_{\bigvee}$ , we will find that  $\triangleleft = \triangleleft_{\bigvee}$ . Similarly, if we start off with a compactification  $h: M \rightarrow L$ , and then determine the associated strong inclusion  $\triangleleft_h$ , we shall find that there exists an isomorphism  $f: M \rightarrow \mathfrak{S}$  such that  $\bigvee \circ f = h$ .

It is worthwhile to note that if  $h: L \rightarrow M$  is onto, then  $h_*(a) = \bigvee \{x \in M \mid h(x) = a\}$  and  $h \circ h_* = \text{id}_M$ . We shall make use of the following Lemma in one of our later results:

**Lemma 0.4.** *If  $h: L \rightarrow M$  is a dense onto homomorphism, then  $h_*(a^*) = (h_*(a))^*$ .*

*Proof.* We have that

$$\begin{aligned} h(h_*(a^*) \wedge h_*(a)) &= h(h_*(a^*)) \wedge h(h_*(a)) \\ &= a^* \wedge a \quad (h \circ h_* = \text{id}_M \text{ since } h \text{ is onto}) \\ &= \mathbf{0}. \end{aligned}$$

Since  $h$  is dense, it follows that  $h_*(a^*) \wedge h_*(a) = \mathbf{0}$ . So we have  $h_*(a^*) \leq h_*(a)^*$ .

For the reverse inequality we argue as follows:

$$\begin{aligned} h_*(a) \wedge h_*(a)^* = \mathbf{0} &\implies h(h_*(a)) \wedge h(h_*(a)^*) = \mathbf{0} \\ &\implies a \wedge h(h_*(a)^*) = \mathbf{0} \quad (h \circ h_* = \text{id}_M \text{ since } h \text{ is onto}) \\ &\implies h(h_*(a)^*) \leq a^* \\ &\implies h_*(a^*) \leq h_*(a)^*. \quad \square \end{aligned}$$

## NUCLEI

We define a *nucleus* on a frame  $L$  to be a map  $n: L \rightarrow L$  which satisfies the following properties:

- $n(a \wedge b) = n(a) \wedge n(b)$ .
- $a \leq n(a)$ .
- $n(n(a)) = n(a)$ .

It can be shown that given a nucleus  $n$  on a frame  $L$ , the set

$$L_n := \{a \in L \mid n(a) = a\},$$

of elements which are fixed under  $n$  is a frame, and the map  $n: L \rightarrow L_n$  is a frame homomorphism. The meet of two elements in  $L_n$  is the same as their meet in  $L$ , whilst the join of an arbitrary collection  $(a_i)_I$  of elements of  $L_n$  is given by  $n(\bigvee_L a_i)$ .

## Chapter 1. Normal Bases

In [5], Banaschewski introduces the concept of a *normal system* on a set, with the purpose of generalising the normal bases introduced by Fan and Gottesman in [16]. Banaschewski shows that every topological space having a normal system amongst its bases can be densely embedded in a compact Hausdorff space. We briefly outline the procedure below, occasionally modifying the notation so that it is consistent with ours. For a more detailed account, see [5].

A *normal system* on a set  $E$  is a non-empty collection  $\mathcal{N}$  of subsets of  $E$  satisfying the following properties:

- (NS1)  $\mathcal{N}$  is closed under finite intersections.
- (NS2) For each  $X \in \mathcal{N}$ ,  $X^* = \bigcup\{Y \in \mathcal{N} \mid X \cap Y = \emptyset\} \in \mathcal{N}$  and  $X^{**} = X$ .
- (NS3) If  $X \cup Y = E$  for  $X, Y \in \mathcal{N}$ , then there exists a  $Z \in \mathcal{N}$  with  $X \cup Z = Z^* \cup Y = E$ .
- (NS4) For each  $X \in \mathcal{N}$ ,  $X = \bigcup\{Y \in \mathcal{N} \mid Y^* \cup X = E\}$ .

Banaschewski then defines the concept of a regular filter (he uses the term ideal) on a normal system. A *regular filter* on a normal system  $\mathcal{N}$  is a subset  $\mathcal{F}$  of  $\mathcal{N}$  satisfying the usual filter properties as well as the following regularity condition:

$$X \in \mathcal{F} \implies \exists Y \in \mathcal{F} \text{ such that } Y^* \cup X = E.$$

Let  $\mathfrak{M}$  denote the set of all maximal regular filters on  $\mathcal{N}$ . A topology  $\mathcal{D}_{\mathfrak{M}}$  is defined on  $\mathfrak{M}$  to consist of all sets of the form  $\mathfrak{M}_F := \{\mathcal{F} \in \mathfrak{M} \mid F \in \mathcal{F}\}$ , where  $F \in \mathcal{N}$ . The space  $(\mathfrak{M}, \mathcal{D}_{\mathfrak{M}})$  is seen to be compact Hausdorff.

By virtue of (NS1), every normal system  $\mathcal{N}$  on a set  $X$  forms a basis for a topology  $\mathcal{D}X$  on  $X$ , and the map  $\varphi: (X, \mathcal{D}X) \longrightarrow (\mathfrak{M}, \mathcal{D}_{\mathfrak{M}})$  that takes an element  $x \in X$  to the maximal regular filter  $M_x := \{X \in \mathcal{N} \mid x \in X\}$  is a homeomorphic dense embedding.

We now give a frame-theoretic analogue of the above. In what follows,  $L$  will be a regular frame.

**Definition 1.0.** We define a *normal base* for  $L$  to be a non-empty subset  $N$  that join-generates  $L$  and satisfies the following conditions :

- (N1)  $N$  is closed under finite meets.
- (N2) For each  $x \in N$ ,  $x^* \in N$  and  $x^{**} = x$ .
- (N3) If  $a, b \in N$  and  $a \prec b$ , then there exists a  $c \in N$  with  $a \prec c \prec b$ .

*Remark.* Note that the fact that  $N$  generates the regular frame  $L$  implies that for each  $n \in N$ ,

$$n = \bigvee \{x \in N \mid x \prec n\}.$$

**Example 1.1.** Every normal system  $\mathcal{N}$  on a set  $E$  is a normal base for the frame of open sets which it generates.

We shall adopt the convention that  $N$  denotes a normal base and  $\mathcal{N}$  denotes a normal system.

**Definition 1.2.** A non-empty subset  $\mathcal{F} \subseteq N$  is called a *regular filter* on  $N$  if it satisfies the following conditions :

- (F1)  $\mathcal{F}$  is closed under finite meets.
- (F2)  $a \in \mathcal{F}$  &  $a \leq b \in N \implies b \in \mathcal{F}$ .
- (F3)  $a \in \mathcal{F} \implies \exists b \in \mathcal{F}$  such that  $b \prec a$ .

Let  $\mathfrak{K}$  denote the collection of all regular filters on  $N$ .

**Lemma 1.3.**  $\mathfrak{K}$  is a compact regular subframe of the frame  $\mathfrak{F}(N)$  of all filters on  $N$ .

*Proof.* That  $\mathfrak{K}$  is a subframe of  $\mathfrak{F}(N)$  follows easily from the properties of the rather below relation and the fact that updirected join in  $\mathfrak{F}(N)$  is given by union. Since  $\mathfrak{K}$  is a subframe of the compact frame  $\mathfrak{F}(N)$ , we have that  $\mathfrak{K}$  is compact so it remains to show that  $\mathfrak{K}$  is regular. For this purpose we introduce the map

$$\begin{aligned} s: N &\longrightarrow \mathfrak{K} \\ a &\longmapsto \{x \in N \mid a \prec x\} \end{aligned}$$

Note the following:

- For each  $a \in N$ ,  $s(a)$  is indeed a regular filter:

That  $s(a)$  is a filter follows easily from the properties of the rather below relation. For regularity, the argument is as follows:

$$\begin{aligned} x \in s(a) &\implies a \prec x \ \& \ x \in N \\ &\implies \exists b \in N \text{ such that } a \prec b \prec x \quad \text{by (N3)} \\ &\implies \exists b \in s(a) \text{ such that } b \prec x. \end{aligned}$$

- $\mathcal{F} \in \mathfrak{K} \implies \mathcal{F} = \bigcup_{a \in \mathcal{F}} s(a)$ :

$$\begin{aligned} \subseteq : \quad & x \in \mathcal{F} \implies \exists y \in \mathcal{F} \text{ such that } y \prec x \quad \text{by (F3)} \\ & \implies x \in s(y) \text{ for } y \in \mathcal{F}. \\ \supseteq : \quad & x \in s(y) \text{ for some } y \in \mathcal{F} \implies y \prec x \text{ for some } y \in \mathcal{F} \\ & \implies x \in \mathcal{F} \quad \text{by (F2)}. \end{aligned}$$

- If  $a \prec b$  then  $s(b) \prec s(a)$ :

$$\begin{aligned} a \prec b &\implies \exists c, d \in N \text{ such that } a \prec c \prec d \prec b \quad \text{applying (N3) twice} \\ & \text{and hence } b^* \prec d^* \prec c^* \prec a^* \\ & \implies s(b) \wedge s(d^*) = \{\mathbf{e}\} \text{ and } s(d^*) \vee s(a) = N \\ & \text{since } b \vee d^* = \mathbf{e} \text{ and } c^* \wedge c = \mathbf{0} \text{ respectively} \\ & \implies s(b) \prec s(a) \end{aligned}$$

- For each  $a \in N$ ,  $s(a) = \bigcup \{s(x) \mid a \prec x\}$ :

From the previous observation it follows that  $s(a)$  contains all the filters  $s(x)$  with  $a \prec x$ . On the other hand,

$$\begin{aligned} b \in s(a) &\implies a \prec b \\ &\implies \exists c \in N \text{ such that } a \prec c \prec b \\ &\implies b \in s(c) \text{ with } a \prec c \\ &\implies b \in \bigcup \{s(x) \mid a \prec x\}. \end{aligned}$$

Finally, to show that  $\mathfrak{K}$  is regular, we take  $\mathcal{F} \in \mathfrak{K}$ . Then

$$\mathcal{F} = \bigcup_{a \in \mathcal{F}} s(a) = \bigcup_{a \in \mathcal{F}} \bigcup_{a \prec x} s(x) \subseteq \bigcup_{s(x) \prec \mathcal{F}} s(x)$$

which is the non-trivial inclusion.  $\square$

We shall use the following result in Lemma 1.9:

**Lemma 1.4.** *Let  $\mathcal{G}$  be a maximal regular filter on a normal base  $N$ . If  $x, y \in N$  satisfy*

$$x \vee y = \mathbf{e} \ \& \ x \notin \mathcal{G},$$

*then  $y \in \mathcal{G}$ .*

*Proof.* Since  $x \notin \mathcal{G}$ , it follows that  $\mathcal{G} \vee s(x^*)$  is a proper regular filter:

If  $\mathbf{0} \in \mathcal{G} \vee s(x^*)$ , then  $\mathbf{0} = p \wedge q$  where  $p \in \mathcal{G}$  and  $q \in s(x^*)$ . Since  $x^* \prec q$  and  $p \wedge q = \mathbf{0}$ , we have  $p \prec x$ , which implies that  $x \in \mathcal{G}$ , a contradiction.

Now  $\mathcal{G} \subseteq \mathcal{G} \vee s(x^*)$ , so by the maximality of  $\mathcal{G}$  we have  $\mathcal{G} = \mathcal{G} \vee s(x^*)$ . Since  $y \vee x = \mathbf{e}$ , it follows that  $x^* \prec y$  or  $y \in s(x^*) \subseteq \mathcal{G}$ .  $\square$

We now introduce the map

$$\begin{aligned} h: \mathfrak{K} &\longrightarrow L \\ \mathcal{F} &\longmapsto \bigvee \{x \in N \mid x^* \in \mathcal{F}\}, \end{aligned}$$

which provides us with a compactification of  $L$ .

**Lemma 1.5.** *The map  $h$  is a frame homomorphism.*

*Proof.* That  $h$  preserves the top,  $N$ , and the bottom,  $\{\mathbf{e}\}$ , of  $\mathfrak{K}$  is clear. It is also easily seen that  $h$  preserves order.

- $h(\mathcal{F} \wedge \mathcal{G}) = h(\mathcal{F}) \wedge h(\mathcal{G})$  for any two filters  $\mathcal{F}$  and  $\mathcal{G}$  of  $\mathfrak{K}$ :

$\leq$ :  $h$  preserves order.

$\geq$ : Suppose that  $z \geq h(\mathcal{F} \wedge \mathcal{G})$  — i.e.,  $z \geq x$  whenever  $x^* \in \mathcal{F} \wedge \mathcal{G}$ . Then  $z \geq a \wedge b$  for any  $a, b \in N$  with  $a^* \in \mathcal{F}$  and  $b^* \in \mathcal{G}$ , since  $(a \wedge b)^* \geq a^* \vee b^* \in \mathcal{F} \wedge \mathcal{G}$ . That is,  $z \geq h(\mathcal{F}) \wedge h(\mathcal{G})$ .

- $h(\mathcal{F} \vee \mathcal{G}) = h(\mathcal{F}) \vee h(\mathcal{G})$  for any two filters  $\mathcal{F}$  and  $\mathcal{G}$  of  $\mathfrak{K}$ :

$\leq$ : Take  $x \in N$  with  $x^* \in \mathcal{F} \vee \mathcal{G}$  so that  $x^*$  is of the form  $x_1 \wedge x_2$  with  $x_1 \in \mathcal{F}$  and  $x_2 \in \mathcal{G}$ . By regularity of  $\mathcal{F}$  and  $\mathcal{G}$  respectively there exists  $y_1 \in \mathcal{F}$  and  $y_2 \in \mathcal{G}$  such that  $y_1 \prec x_1$  and  $y_2 \prec x_2$ . We show that  $x^* \vee (h(\mathcal{F}) \vee h(\mathcal{G})) = \mathbf{e}$  which shows that  $x \leq (h(\mathcal{F}) \vee h(\mathcal{G}))$ : Since  $y_1 = y_1^{**} \in \mathcal{F}$ , we have  $y_1^* \leq h(\mathcal{F})$  and similarly  $y_2^* \leq h(\mathcal{G})$ . Hence  $y_1^* \vee y_2^* \leq h(\mathcal{F}) \vee h(\mathcal{G})$ . So

$$\begin{aligned} x^* \vee h(\mathcal{F}) \vee h(\mathcal{G}) &\geq x^* \vee y_1^* \vee y_2^* \\ &= (x_1 \wedge x_2) \vee y_1^* \vee y_2^* \\ &= (x_1 \vee y_1^* \vee y_2^*) \wedge (x_2 \vee y_1^* \vee y_2^*) \\ &= \mathbf{e} \wedge \mathbf{e} \quad \text{since } y_i \prec x_i \text{ for } i = 1, 2 \\ &= \mathbf{e} \end{aligned}$$

$\geq$ :  $h$  preserves order.

- $h\left(\bigvee_{i \in I} \mathcal{F}_i\right) = \bigvee_{i \in I} h(\mathcal{F}_i)$  for any updirected family  $I$  of regular filters on  $N$ :

Since the join of an updirected family of filters is given by union, we have

$$\begin{aligned}
h\left(\bigvee_{i \in I} \mathcal{F}_i\right) &= h\left(\bigcup_{i \in I} \mathcal{F}_i\right) \\
&= \bigvee \{x \in L \mid x^* \in \bigcup_{i \in I} \mathcal{F}_i\} \\
&= \bigvee \bigcup_{i \in I} \{x \in L \mid x^* \in \mathcal{F}_i\} \\
&= \bigvee_{i \in I} \bigvee \{x \in L \mid x^* \in \mathcal{F}_i\} \\
&= \bigvee_{i \in I} h(\mathcal{F}_i). \quad \square
\end{aligned}$$

**Proposition 1.6.**  *$h$  is a compactification of  $L$ .*

*Proof.* We have seen already that  $\mathfrak{K}$  is a compact regular frame and that  $h$  is a frame homomorphism, so it remains to show that  $h$  is dense and onto:

- $h$  is dense:

Suppose that  $h(\mathcal{F}) = \mathbf{0}$  for some regular filter  $\mathcal{F} \in \mathfrak{K}$

$$\text{i.e., } \bigvee \{x \in N \mid x^* \in \mathcal{F}\} = \mathbf{0}$$

That means that if  $x^*$  belongs to  $\mathcal{F}$ , then  $x = \mathbf{0}$ , so it follows that  $\mathcal{F} = \{\mathbf{e}\}$ .

That  $h$  is onto follows from the fact that  $N$  generates  $L$  and the following useful observation:

- $h(s(a^*)) = a$  for any  $a \in N$ :

$$\begin{aligned}
h(s(a^*)) &= \bigvee \{x \in N \mid x^* \in s(a^*)\} \\
&= \bigvee \{x \in N \mid a^* \prec x^*\} \\
&= \bigvee \{x \in N \mid x \prec a\} \\
&= a \quad \text{by (N4)}. \quad \square
\end{aligned}$$

For the sake of our comparisons in Chapter 6, it is useful to determine the strong inclusion which gives rise to the compactification  $h: \mathfrak{K} \rightarrow L$ .

**Lemma 1.7.** *Let  $N$  be a normal base for  $L$ . For elements  $a, b \in L$ , the relation*

$$a \triangleleft b \iff \exists c, d \in N \text{ such that } a \leq c \prec d \leq b$$

*determines a strong inclusion on  $L$ .*

*Proof.* We have to verify (S1) — (S6) in the definition of a strong inclusion.

(S1)  $x \leq a \triangleleft b \leq y \implies x \triangleleft y$  is clear.

(S2)  $\triangleleft$  is a sublattice of  $L \times L$ :

That  $0 \triangleleft 0, e \triangleleft e$  and  $x \triangleleft a, b \implies x \triangleleft a \wedge b$  all follow easily from the properties of the rather below relation, so it remains to show that  $a, b \triangleleft x \implies a \vee b \triangleleft x$ .

$$\begin{aligned}
a, b \triangleleft x &\implies \exists c_1, d_1, c_2, d_2 \in N \text{ with } a \leq c_1 \prec d_1 \leq x \text{ and } b \leq c_2 \prec d_2 \leq x. \\
&\implies \exists s_1, s_2 \in N \text{ with } a \leq c_1 \prec s_1 \prec d_1 \leq x \text{ and } b \leq c_2 \prec s_2 \prec d_2 \leq x \\
&\quad \text{by (N3)} \\
&\implies a \vee b \leq c_1 \vee c_2 \prec s_1 \vee s_2 \prec d_1 \vee d_2 \leq x \\
&\implies a \vee b \leq c_1 \vee c_2 \leq (c_1 \vee c_2)^{**} \prec s_1 \vee s_2 \leq (s_1 \vee s_2)^{**} \prec d_1 \vee d_2 \leq x \\
&\implies a \vee b \triangleleft x \\
&\quad \text{since } (c_1 \vee c_2)^{**} = (c_1^* \wedge c_2^*)^* \text{ and } (s_1 \vee s_2)^{**} = (s_1^* \wedge s_2^*)^* \text{ both belong to } N.
\end{aligned}$$

(S3)  $a \triangleleft b \implies a \prec b$  is clear.

(S4)  $\triangleleft$  interpolates:

$$\begin{aligned}
a \triangleleft b &\implies \exists c, d \in N \text{ such that } a \leq c \prec d \leq b \\
&\implies \exists s \in N \text{ such that } a \leq c \prec s \prec d \leq b \quad \text{by (N3).} \\
&\implies a \triangleleft s \triangleleft b
\end{aligned}$$

(S5)  $a \triangleleft b \implies b^* \triangleleft a^*$ :

$$\begin{aligned}
a \triangleleft b &\implies \exists c, d \in N \text{ such that } a \leq c \prec d \leq b \\
&\implies b^* \leq d^* \prec c^* \leq a^* \text{ and } c^*, d^* \in N \quad \text{by (N2).} \\
&\implies b^* \triangleleft a^*
\end{aligned}$$

(S6) For each  $a \in L$ ,  $a = \bigvee \{x \in L \mid x \triangleleft a\}$ :

$$\begin{aligned}
a &= \bigvee \{n \in N \mid n \leq a\} \quad \text{since } N \text{ generates } L \\
&= \bigvee_{n \leq a} \bigvee \{m \in N \mid m \prec n\} \quad \text{since } L \text{ is regular} \\
&\leq \bigvee \{m \in N \mid m \triangleleft a\} \quad \text{since } m \in N \ \& \ m \prec n \leq a \implies m \triangleleft a \\
&\leq \bigvee \{x \in L \mid x \triangleleft a\} \\
&\leq a \quad \square
\end{aligned}$$

We now show that the strong inclusion  $\triangleleft_h$  on  $L$  arising from  $h$  is the strong inclusion  $\triangleleft$  of the above Lemma. In [9], Banaschewski shows that

$$a \triangleleft_h b \iff h_*(a) \prec h_*(b) \quad \text{for all } a, b \in L,$$

where  $h_*$  denotes the right adjoint of  $h$ .

**Lemma 1.8.** *For  $a, b \in L$ ,  $a \triangleleft_h b$  if and only if  $a \triangleleft b$ . In other words,*

$$a \triangleleft_h b \iff \exists c, d \in N \text{ such that } a \leq c \prec d \leq b.$$

*Proof.*

$\implies$ : Suppose that  $a \triangleleft_h b$  — i.e.,  $h_*(a)^* \vee h_*(b) = N$ . By Lemma 0.4 we have that  $h_*(a)^* = h_*(a^*)$  so that

$$h_*(a^*) \vee h_*(b) = N$$

This means that there exists  $x \in h_*(a^*) = \bigvee \{\mathcal{F} \in \mathfrak{K} \mid h(\mathcal{F}) = a^*\}$  and there exists  $y \in h_*(b) = \bigvee \{\mathcal{G} \in \mathfrak{K} \mid h(\mathcal{G}) = b\}$  such that  $x \wedge y = \mathbf{0}$ .

Now  $x \in h_*(a^*)$  implies that  $x$  is of the form  $x = x_1 \wedge x_2 \wedge \dots \wedge x_n$  where  $x_i \in \mathcal{F}_i$  with  $h(\mathcal{F}_i) = a^*$ . Further,  $x_i \in \mathcal{F}_i$  implies that there exists  $z_i \in \mathcal{F}_i$  such that  $z_i \prec x_i$  since  $\mathcal{F}_i$  is a regular filter. Since  $h(\mathcal{F}_i) = \bigvee \{z \in N \mid z^* \in \mathcal{F}_i\}$  and  $z_i = z_i^{**} \in \mathcal{F}_i$ , we have  $z_i^* \leq a^*$  for each  $i$ . So  $a \leq a^{**} \leq z_i$  for each  $i$  and hence  $a \leq \bigwedge z_i = z$ , say, with  $z \prec x$ .

Similarly,  $y \in h_*(b)$  implies that  $y$  is of the form  $y = y_1 \wedge y_2 \wedge \dots \wedge y_m$  where  $y_i \in \mathcal{G}_i$  with  $h(\mathcal{G}_i) = b$ . As above, there exists  $w_i \in \mathcal{G}_i$  with  $w_i \prec y_i$  and  $w_i^* \leq b$  for each  $i$ .

Now,

$$\begin{aligned} y \vee b &= (y_1 \wedge y_2 \wedge \dots \wedge y_m) \vee b \\ &= (y_1 \vee b) \wedge (y_2 \vee b) \wedge \dots \wedge (y_m \vee b) \\ &\geq (y_1 \vee w_1^*) \wedge (y_2 \vee w_2^*) \wedge \dots \wedge (y_m \vee w_m^*) \\ &= \mathbf{e} \wedge \mathbf{e} \wedge \dots \wedge \mathbf{e} \\ &= \mathbf{e} \end{aligned}$$

We have shown that  $a \leq z \prec x \leq y^* \leq b$  for  $z$  and  $x$  belonging to  $N$ . Thus  $a \triangleleft b$ .

$\impliedby$ : Suppose that  $a \triangleleft b$  — i.e.,

$$\exists c, d \in N \text{ such that } a \leq c \prec d \leq b.$$

Now  $c \prec d \implies d^* \prec c^* \implies s(c^*) \prec s(d^*)$ , (see proof of Lemma 1.3), so there exists a regular filter  $\mathcal{F} \in \mathfrak{K}$  such that  $s(c^*) \cap \mathcal{F} = \{\mathbf{e}\}$  and  $\mathcal{F} \vee s(d^*) = N$ .

$$\begin{aligned} h(s(d^*)) = d &\implies s(d^*) \leq h_*(d) \quad (\text{property of the adjoint}) \\ &\implies s(c^*) \prec h_*(d) \end{aligned}$$

Since  $h$  is dense and

$$\begin{aligned} h(h_*(c) \cap \mathcal{F}) &= h(h_*(c)) \wedge h(\mathcal{F}) \\ &= c \wedge h(\mathcal{F}) \\ &= h(s(c^*)) \wedge h(\mathcal{F}) \\ &= h(s(c^*) \cap \mathcal{F}) \\ &= h(\{\mathbf{e}\}) \\ &= \mathbf{0}, \end{aligned}$$

we have  $h_*(c) \cap \mathcal{F} = \{e\}$ . This together with  $\mathcal{F} \vee s(d^*) = N$  yields  $h_*(c) \prec h_*(d)$ . So  $c \triangleleft_h d$  and hence  $a \triangleleft_h b$ , as required.  $\square$

We now show that the compactification which we have constructed in the category of frames captures the original compactification in the category of topological spaces presented by Banaschewski in [5].

That is, given a topological space  $X$ , where  $\mathcal{D}X$  is generated by a normal system  $\mathcal{N}$  on  $X$ , we have the compactification of the frame  $\mathcal{D}X$  constructed above

$$\begin{aligned} h: \mathfrak{K} &\longrightarrow \mathcal{D}X \\ \mathcal{F} &\longmapsto \bigcup \{X \in N \mid X^* \in \mathcal{F}\}, \end{aligned}$$

where  $\mathfrak{K}$  is the collection of all regular filters of basic open sets belonging to  $\mathcal{N}$ .

We apply the spectrum functor,  $\Sigma: \mathbf{Frm} \longrightarrow \mathbf{Top}$  to  $h$  to obtain

$$\Sigma h: \Sigma \mathcal{D}X \longrightarrow \Sigma \mathfrak{K}.$$

Our aim is to show that  $\Sigma \mathfrak{K}$  is the compactification  $(\mathfrak{M}, \mathcal{D}_{\mathfrak{M}})$  of  $X$  obtained by Banaschewski in [5].

Recall that  $\mathfrak{M}$  is the collection of all maximal regular filters on the normal system  $\mathcal{N}$  and  $\mathcal{D}_{\mathfrak{M}}$  is generated by sets of the form

$$\mathfrak{M}_F := \{\mathcal{F} \in \mathfrak{M} \mid F \in \mathcal{F}\}, \quad \text{where } F \in \mathcal{N}.$$

In Lemma 0.1, we proved that in a regular frame, the prime elements are precisely the maximal elements. Hence in general, given any normal base  $N$  for a frame  $L$ , the spectrum  $\Sigma \mathfrak{K}$  of  $\mathfrak{K}$  consists of all maximal regular filters on  $N$  with basic open sets of the form

$$\Sigma_{\mathcal{F}} := \{\mathcal{G} \in \Sigma \mathfrak{K} \mid \mathcal{F} \not\subseteq \mathcal{G}\}, \quad \text{where } \mathcal{F} \in \mathfrak{K}.$$

In particular, this description of  $\Sigma \mathfrak{K}$  holds in the context of a normal system  $\mathcal{N}$  on  $X$  that generates the frame  $\mathcal{D}X$ .

**Lemma 1.9.** *For each  $F \in \mathcal{N}$ ,  $\mathfrak{M}_F$  is open in the topology of  $\Sigma \mathfrak{K}$ .*

*Proof.* We show that  $\{\mathcal{F} \in \mathfrak{M} \mid F \in \mathcal{F}\} = \{\mathcal{G} \in \Sigma \mathfrak{K} \mid s(F^*) \not\subseteq \mathcal{G}\}$ :

- $\subseteq$ : If  $\mathcal{F}$  is a maximal regular filter containing  $F \in \mathcal{N}$ , then  $\exists G \in \mathcal{F}$  such that  $G \prec F$  by (F3). Therefore  $F^* \prec G^*$  or  $G^* \in s(F^*)$ . This shows that  $s(F^*) \not\subseteq \mathcal{F}$ , for if  $G^* \in \mathcal{F}$ , then  $G \cap G^* = \emptyset \in \mathcal{F}$  which is not allowed.
- $\supseteq$ : Suppose that  $\mathcal{G}$  is a maximal regular filter such that  $s(F^*) \not\subseteq \mathcal{G}$ . Then there exists an element  $G \in \mathcal{N}$  such that  $F^* \prec G$ , but  $G \notin \mathcal{G}$ . Now  $F \cup G = X$  and  $G \notin \mathcal{G}$  so by Lemma 1.4 it follows that  $F \in \mathcal{G}$ .  $\square$

**Lemma 1.10.**  $\mathcal{D}_{\mathfrak{M}}$  is a basis for the topology of  $\Sigma\mathfrak{K}$ .

*Proof.* Let  $\Sigma_{\mathcal{F}}$  be an open set in  $\Sigma\mathfrak{K}$ . We show that  $\Sigma_{\mathcal{F}} = \bigcup_{S^* \in \mathcal{F}} \mathfrak{M}_S$ .

$$\begin{aligned}
\subseteq: \quad \mathcal{G} \in \Sigma_{\mathcal{F}} &\implies \mathcal{F} \not\subseteq \mathcal{G} \\
&\implies \exists A \in \mathcal{F} \text{ such that } A \notin \mathcal{G} \\
&\implies \exists B \in \mathcal{F} \text{ with } B \prec A \text{ and } A \notin \mathcal{G} \\
&\implies \exists B \in \mathcal{F} \text{ with } B^* \cup A = X \text{ and } A \notin \mathcal{G} \\
&\implies \exists B \in \mathcal{F} \text{ with } B^* \in \mathcal{G} \quad \text{by Lemma 1.4} \\
&\implies \mathcal{G} \in \mathfrak{M}_{B^*} \text{ and } B \in \mathcal{F} \\
&\implies \mathcal{G} \in \bigcup_{S^* \in \mathcal{F}} \mathfrak{M}_S \\
\supseteq: \mathcal{G} \in \bigcup_{S^* \in \mathcal{F}} \mathfrak{M}_S &\implies S \in \mathcal{G} \text{ for some } S^* \in \mathcal{F} \\
&\implies \mathcal{F} \not\subseteq \mathcal{G} \\
&\implies \mathcal{G} \in \Sigma_{\mathcal{F}} \quad \square
\end{aligned}$$

Thus far we have established that the spaces  $\Sigma\mathfrak{K}$  and  $(\mathfrak{M}, \mathcal{D}_{\mathfrak{M}})$  coincide. To show that the compactifications are equivalent, it remains to show that the space  $X$  is embedded into these two spaces in the same way — i.e., that the following diagram commutes:

$$\begin{array}{ccc}
(X, \mathcal{D}) & \xrightarrow{\varphi} & (\mathfrak{M}, \mathcal{D}_{\mathfrak{M}}) \\
\epsilon_X \downarrow & & \parallel \\
\Sigma\mathcal{D}X & \xrightarrow{\Sigma h} & \Sigma\mathfrak{K}
\end{array}$$

$$\begin{aligned}
\Sigma h(\epsilon_X(x)) &= \bigvee \{ \mathcal{F} \in \mathfrak{K} \mid \epsilon_X(x) \circ h(\mathcal{F}) = \mathbf{0} \} \\
&= \bigvee \{ \mathcal{F} \in \mathfrak{K} \mid x \notin h(\mathcal{F}) \} \\
&= M_x :
\end{aligned}$$

- $M_x$  is an upper bound for  $\{ \mathcal{F} \in \mathfrak{K} \mid x \notin h(\mathcal{F}) \}$ .

Suppose that  $\mathcal{F} \in \mathfrak{K}$  and  $x \notin h(\mathcal{F}) = \bigcup \{ Z \in \mathcal{N} \mid Z^* \in \mathcal{F} \}$  — i.e., if  $Z \in \mathcal{N}$  and  $Z^* \in \mathcal{F}$ , then  $x \notin Z$ . Let  $A \in \mathcal{F}$ . Since  $\mathcal{F}$  is a regular filter, there exists  $B \in \mathcal{F}$  such that  $B^* \cup A = X$ . Now  $B = B^{**} \in \mathcal{F}$  and so by our assumption above,  $x \notin B^*$  which yields  $x \in A$  or  $A \in M_x$ , as required.

- $M_x$  is the least upper bound.

Suppose that the regular filter  $\mathcal{G}$  is also an upper bound for  $\{\mathcal{F} \in \mathfrak{K} \mid x \notin h(\mathcal{F})\}$  — i.e., if  $x \notin h(\mathcal{F})$ , then  $\mathcal{F} \subseteq \mathcal{G}$ . To see that  $M_x \subseteq \mathcal{G}$ , it suffices to show that  $x \notin h(M_x)$ . This is clear since

$$h(M_x) = \bigcup \{Z \in \mathcal{N} \mid Z^* \in M_x\} = \bigcup \{Z \in \mathcal{N} \mid x \in Z^*\}$$

and  $Z \cap Z^* = \emptyset$ .

Thus we have that the two compactifications

$$\varphi : (X, \mathcal{D}) \longmapsto (\mathfrak{M}, \mathcal{D}_{\mathfrak{M}}) \text{ and } \Sigma h \circ \varepsilon_X : (X, \mathcal{D}) \longmapsto \Sigma \mathfrak{K}$$

are equivalent and our compactification in the frame setting captures the compactification in the topological setting.

## Chapter 2. Fan–Gottesman Normal Bases

In [17] Freudenthal provides a method of constructing a Hausdorff compactification of any rim compact space. Fan and Gottesman generalise Freudenthal's construction in [16] to include all regular spaces with a *normal basis* — hereafter Fan–Gottesman normal bases to distinguish them from the normal bases of Chapter 1. We briefly summarise the method below. The details can be found in [16].

Let  $R$  be a regular topological space. A *Fan–Gottesman normal basis* for  $R$  is a basis  $\mathcal{B}$  for the open sets of  $R$  satisfying the following conditions:

- (B1)  $\mathcal{B}$  is closed under finite intersections.
- (B2) For each  $A \in \mathcal{B}$ ,  $\mathbb{C} \text{Cl} A \in \mathcal{B}$ .
- (B3) If  $A \in \mathcal{B}$  and  $U \in \mathfrak{D}R$  satisfy  $\text{Cl} A \subseteq U$  then there exists  $B \in \mathcal{B}$  such that  $\text{Cl} A \subseteq B \subseteq \text{Cl} B \subseteq U$ .

A binding family on  $\mathcal{B}$  is a non-empty family  $\mathcal{A}$  of sets belonging to  $\mathcal{B}$  such that for any finite subcollection  $A_1, A_2, \dots, A_n \in \mathcal{A}$ ,

$$\text{Cl} A_1 \cap \text{Cl} A_2 \cap \dots \cap \text{Cl} A_n \neq \emptyset.$$

The following Lemma is proved in [16]:

**Lemma 2.1.** *If an open set  $U$  of  $R$  and a finite number of sets  $A_i \in \mathcal{B}$  ( $1 \leq i \leq n$ ) satisfy*

$$\text{Cl} A_1 \cap \text{Cl} A_2 \cap \dots \cap \text{Cl} A_n \subseteq U,$$

*then there exists  $B \in \mathcal{B}$  such that*

$$\text{Cl} A_1 \cap \text{Cl} A_2 \cap \dots \cap \text{Cl} A_n \subseteq B \subseteq \text{Cl} B \subseteq U.$$

By Zorn's Lemma, every binding family is contained in a maximal binding family. Denote by  $\mathfrak{R}^*$  the set of all such maximal binding families. For each  $A \in \mathcal{B}$  we define

$$\psi(A) = \{\mathcal{A} \in \mathfrak{R}^* \mid \exists X \in \mathcal{A} \text{ with } \text{Cl} X \subseteq A\}.$$

The collection  $\{\psi(A) \mid A \in \mathcal{B}\}$  forms a basis for a compact Hausdorff topology on  $\mathfrak{R}^*$ . The map  $\varphi: R \rightarrow \mathfrak{R}^*$  sending an element  $x \in R$  to the maximal binding family  $\{B \in \mathcal{B} \mid x \in \text{Cl} B\}$  is a dense embedding of  $R$  into  $\mathfrak{R}^*$  and is known as the Fan-Gottesman compactification of  $R$ .

In [2], Baboolal presents the notion of a Fan-Gottesman normal base for a regular frame  $L$ . This definition is entirely analogous to the classical definition given by Fan and Gottesman in [16] and leads to a compactification which Baboolal calls the Fan-Gottesman compactification.

The results up to and including Proposition 2.4 are those of Baboolal. The rest of the chapter is original and is devoted to showing that Baboolal's construction, when applied to the frame  $\mathcal{O}R$  of open sets of a regular space  $R$ , yields the classical Fan-Gottesman compactification.

In what follows,  $L$  will be a regular frame.

**Definition 2.2.** We define a *Fan-Gottesman normal base* for  $L$  to be a non-empty subset  $B$  that join generates  $L$  and satisfies the following conditions :

- (FG1)  $B$  is closed under finite meets.
- (FG2) For each  $b \in B$ ,  $b^* \in B$ .
- (FG3) If  $a \in B$  &  $c \in L$  satisfy  $a \prec c$ , then there exists  $b \in B$  such that  $a \prec b \prec c$ .

Hereafter  $B$  will be a Fan-Gottesman normal base for the regular frame  $L$ .

**Proposition 2.3.** For  $x, y \in L$ , the relation

$$x \triangleleft y \iff \exists b \in B \text{ such that } x \prec b \prec y$$

defines a strong inclusion on  $L$ .

*Proof.* We have to verify (S1) – (S6) in the definition of a strong inclusion.

- (S1)  $a \leq x \triangleleft y \leq b \implies a \triangleleft b$  is clear.

(S2)  $\triangleleft$  is a sublattice of  $L \times L$ :

That  $\mathbf{0} \triangleleft \mathbf{0}$ ,  $\mathbf{e} \triangleleft \mathbf{e}$  and  $x \triangleleft a, b \implies x \triangleleft a \wedge b$  all follow easily from the properties of the rather below relation, so it remains to show that

$$a, b \triangleleft x \implies a \vee b \triangleleft x$$

$$\begin{aligned} a, b \triangleleft x &\implies \exists u, v \in B \text{ with } a \prec u \prec x \text{ and } b \prec v \prec x. \\ &\implies a \vee b \prec u \vee v \prec x \\ &\implies a \vee b \prec (u \vee v)^{**} \prec x \\ &\implies a \vee b \triangleleft x \end{aligned}$$

since  $(u \vee v)^{**} = (u^* \wedge v^*)^* \in B$  by (FG1) and (FG2).

(S3)  $a \triangleleft b \implies a \prec b$  is clear.

(S4)  $\triangleleft$  interpolates:

$$\begin{aligned} a \triangleleft b &\implies a \prec u \prec b \text{ for some } u \in B \\ &\implies a \prec u \prec v \prec w \prec b \text{ for some } v, w \in B \quad (\text{applying (FG3) twice}). \\ &\implies a \triangleleft v \triangleleft b \end{aligned}$$

(S5)  $a \triangleleft b \implies b^* \triangleleft a^*$ :

$$\begin{aligned} a \triangleleft b &\implies a \prec u \prec b \text{ for some } u \in B \\ &\implies b^* \prec u^* \prec a^* \text{ with } u^* \in B \quad \text{by (FG2)} \\ &\implies b^* \triangleleft a^* \end{aligned}$$

(S6) For each  $a \in L$ ,  $a = \bigvee \{x \in L \mid x \triangleleft a\}$ :

Since  $L$  is regular and  $B$  generates  $L$ , we have

$$a = \bigvee \{z \in B \mid z \prec a\}.$$

Now if  $z \in B$  and  $z \prec a$ , then there exists  $x \in B$  such that  $z \prec x \prec a$  by (FG3). So  $z \triangleleft a$  and the result follows.  $\square$

Let  $\gamma_B L$  denote the compactification of  $L$  which arises from the strong inclusion. Baboolal calls  $\gamma_B L$  the *Fan-Gottesman compactification* of  $L$  and any compactification of  $L$  isomorphic to  $\gamma_B L$  is said to be of *Fan-Gottesman type*.

**Proposition 2.4.** *Let  $B$  be a Fan-Gottesman normal base for  $L$  and let  $N_B$  be the collection of regular elements of  $B$ . That is,  $N_B = \{b \in B \mid b = b^{**}\}$ . Then  $N_B$  is a Fan-Gottesman normal base for  $L$  and moreover  $\gamma_B L \cong \gamma_{N_B} L$ .*

*Proof.* We first show that  $N_B$  is a Fan-Gottesman normal base:

$N_B$  generates  $L$ : Take  $a \in L$ . Since  $B$  generates  $L$ , we have that  $a = \bigvee \{b \in B \mid b \prec a\}$ .

Now  $b \prec a$  implies  $b \leq b^{**} \prec a$  and since  $b^{**} \in N_B$ , we have that

$$a = \bigvee \{r \in N_B \mid r \prec a\}.$$

(FG1) If  $a, b \in N_B$ , then  $(a \wedge b)^{**} = a^{**} \wedge b^{**} = a \wedge b$ , so  $a \wedge b \in N_B$ .

(FG2)  $a \in N_B \implies a^* \in N_B$  since  $(a^*)^{**} = (a^{**})^* = a^*$ .

(FG3) Suppose that  $a \in N_B$  &  $c \in L$  satisfy  $a \prec c$ . By (FG3) there exists  $b \in B$  such that  $a \prec b \prec c$  and hence  $a \prec b^{**} \prec c$  with  $b^{**} \in N_B$ .

To see that  $\gamma_B L \cong \gamma_{N_B} L$ , we show that the following strong inclusions coincide:

$$(1) \ x \triangleleft_{N_B} y \implies x \triangleleft_B y \text{ since } N_B \subseteq B$$

$$(2) \ x \triangleleft_B y \implies x \triangleleft_{N_B} y:$$

$$\begin{aligned} x \triangleleft_B y &\implies \exists b \in B \text{ with } x \prec b \prec y \\ &\implies x \prec b^{**} \prec y \\ &\implies x \triangleleft_{N_B} y \quad \text{since } b^{**} \in N_B. \quad \square \end{aligned}$$

*Remark.* Proposition 2.4 essentially says that there is no harm in restricting our Fan-Gottesman normal basis  $B$  to the subcollection  $N_B$  of its regular elements, for their respective compactifications are the same. It is clear that the subcollection  $N_B$  is a normal basis in the sense of Chapter 1, so it would of interest to investigate the relationship between the Fan-Gottesman compactification,  $\gamma_{N_B} L$  and the normal base compactification constructed in Chapter one. It turns out that the two compactifications coincide. In order to show this, we prove in the following Lemma that the strong inclusions determined by the compactifications are equivalent.

**Lemma 2.5.** *For  $a, b \in L$  and  $B$  and  $N_B$  as above, the following are equivalent:*

- (i)  $\exists s \in B$  such that  $a \prec s \prec b$ .
- (ii)  $\exists c, d \in N_B$  such that  $a \leq c \prec d \leq b$ .

*Proof.*

- (i)  $\implies$  (ii) : Suppose that (i) holds. Then by (FG3) there exists an element  $t \in B$  such that  $a \prec s \prec t \prec b$ . Hence  $a \leq s^{**} \prec t^{**} \leq b$  where  $s^{**}$  and  $t^{**} \in N_B$ .
- (ii)  $\implies$  (i) : Suppose that (ii) holds. Again using (FG3), since  $N_B \subseteq B$ , there exists an element  $s \in B$  such that  $a \leq c \prec s \prec d \leq b$  which yields  $a \prec s \prec b$  for an element  $s \in B$ .  $\square$

In view of the previous two results, we can follow the route taken in Chapter 1 to arrive at the Fan–Gottesman compactification. That is, we can consider the collection  $\mathfrak{K}$  of all regular filters on  $N_B$  together with the map

$$\begin{aligned} h: \mathfrak{K} &\longrightarrow L \\ \mathcal{F} &\longmapsto \bigcup \{x \in N_B \mid x^* \in \mathcal{F}\}. \end{aligned}$$

We now set out to show that Baboolal’s Fan–Gottesman compactification of frames captures the classical Fan–Gottesman compactification of spaces. For this purpose, let  $R$  be a regular topological space with Fan–Gottesman normal basis  $B$  and associated normal basis  $N_B$ . We form the compactification

$$\begin{aligned} h: \mathfrak{K} &\longrightarrow \mathcal{D}R \\ \mathcal{F} &\longmapsto \bigcup \{X \in N_B \mid X^* \in \mathcal{F}\}, \end{aligned}$$

where  $\mathfrak{K}$  is the collection of all regular filters of basic open sets belonging to  $N_B$ . We apply the spectrum functor,  $\Sigma: \mathbf{Frm} \longrightarrow \mathbf{Top}$  to  $h$  to obtain

$$\Sigma h: \Sigma \mathcal{D}R \longrightarrow \Sigma \mathfrak{K},$$

and our aim is to show that  $\Sigma \mathfrak{K} \cong R^*$ , the classical Fan–Gottesman compactification of the topological space  $R$ .

For this purpose we introduce the map

$$\begin{aligned} \rho: R^* &\longrightarrow \Sigma \mathfrak{K} \\ \mathcal{A} &\longmapsto \{C \in N_B \mid \exists A_1, A_2, \dots, A_n \in \mathcal{A} \text{ with } \text{Cl} A_1 \cap \text{Cl} A_2 \cap \dots \cap \text{Cl} A_n \subseteq C\} \end{aligned}$$

**Proposition 2.6.**  *$\rho$  is an isomorphism between  $R^*$  and  $\Sigma \mathfrak{K}$ .*

*Proof.*

- $\rho$  is well defined.

It is easily seen that  $\rho(\mathcal{A})$  is a regular filter.

For maximality we suppose that  $\rho(\mathcal{A})$  is properly contained in a regular filter  $\mathcal{G}$ . This means that there exists an open set  $Z \in \mathcal{G}$  such that  $Z \notin \rho(\mathcal{A})$ . Since  $\mathcal{G}$  is regular, there exist open sets  $X, Y \in \mathcal{G}$  such that  $\text{Cl} X \subseteq Y \subseteq \text{Cl} Y \subseteq Z$ .

We show that  $\mathcal{A} \cup \{\text{C Cl} Y\}$  is binding so that by maximality of  $\mathcal{A}$  it follows that  $\text{C Cl} Y \in \mathcal{A}$ .

Suppose not. Then there exists  $A_1, A_2, \dots, A_n \in \mathcal{A}$  such that

$$\text{Cl } A_1 \cap \text{Cl } A_2 \cap \dots \cap \text{Cl } A_n \cap \text{Cl } C \text{Cl } Y = \emptyset.$$

This means that

$$\text{Cl } A_1 \cap \text{Cl } A_2 \cap \dots \cap \text{Cl } A_n \subseteq C \text{Cl } C \text{Cl } Y \subseteq \text{Cl } Y \subseteq Z,$$

contradicting the fact that  $Z \notin \rho(\mathcal{A})$ .

We now show that  $\mathcal{G}$  is not proper:

$$\begin{aligned} \text{Cl } X \subseteq Y \subseteq \text{Cl } Y \subseteq Z &\implies C Z \subseteq C \text{Cl } Y \subseteq C Y \subseteq C \text{Cl } X \\ &\implies C Z \subseteq C \text{Cl } C \text{Cl } Y \subseteq C Y \subseteq C \text{Cl } X \\ &\quad \text{taking closures} \\ &\implies C \text{Cl } X \in \rho(\mathcal{A}) \quad \text{since } C \text{Cl } Y \in \mathcal{A} \\ &\implies C \text{Cl } X \in \mathcal{G} \\ &\implies C \text{Cl } X \cap X = \emptyset \in \mathcal{G} \end{aligned}$$

So  $\mathcal{G}$  is not proper and this completes the proof that  $\rho$  is well defined.

- $\rho$  is one-to-one.

Suppose that  $\rho(\mathcal{A}_1) = \rho(\mathcal{A}_2)$ . By symmetry we need only show that  $\mathcal{A}_1 \subseteq \mathcal{A}_2$ . Let  $X \in \mathcal{A}_1$ . To show that  $X \in \mathcal{A}_2$ , it suffices to show that  $\mathcal{A}_2 \cup \{X\}$  is binding:

Suppose not. Then there exists  $A_1, A_2, \dots, A_n \in \mathcal{A}_2$  such that

$$\text{Cl } A_1 \cap \text{Cl } A_2 \cap \dots \cap \text{Cl } A_n \cap \text{Cl } X = \emptyset,$$

and hence

$$\text{Cl } A_1 \cap \text{Cl } A_2 \cap \dots \cap \text{Cl } A_n \subseteq C \text{Cl } X.$$

So  $C \text{Cl } X \in \rho(\mathcal{A}_1) = \rho(\mathcal{A}_2)$ . Hence there exist elements  $B_1, B_2, \dots, B_m$  of  $\mathcal{A}_1$  such that

$$\begin{aligned} \text{Cl } B_1 \cap \text{Cl } B_2 \cap \dots \cap \text{Cl } B_m &\subseteq C \text{Cl } X \\ \text{i.e., } \text{Cl } B_1 \cap \text{Cl } B_2 \cap \dots \cap \text{Cl } B_m \cap \text{Cl } X &= \emptyset \end{aligned}$$

which contradicts the fact that  $X \in \mathcal{A}_1$ .

- $\rho$  is onto.

Let  $\mathcal{F}$  be a maximal regular filter belonging to  $\Sigma\mathcal{R}$ . Then  $\mathcal{F}$  is certainly a binding family and by Zorn's Lemma,  $\mathcal{F}$  is contained in a maximal binding family  $\mathcal{G}$ , say. We show that  $\rho(\mathcal{G}) = \mathcal{F}$ :

By maximality of the regular filter  $\mathcal{F}$ , it suffices to show that  $\mathcal{F} \subseteq \rho(\mathcal{G})$ . Now

$$\begin{aligned} X \in \mathcal{F} &\implies \exists Y \in \mathcal{F} \text{ with } \text{Cl } Y \subseteq X \quad \text{since } \mathcal{F} \text{ is regular} \\ &\implies X \in \rho(\mathcal{G}) \quad \text{since } \mathcal{F} \subseteq \mathcal{G} \text{ implies } Y \in \mathcal{G}. \end{aligned}$$

- $\rho$  is continuous.

Recall that the basis for the topology of  $R^*$  is given by  $\{\psi(A) \mid A \in N_B\}$  where

$$\psi(A) = \{\mathcal{A} \in R^* \mid \exists C \in \mathcal{A} \text{ with } \text{Cl } C \subseteq A\},$$

while by Lemma 1.9, the basic open sets of  $\Sigma\mathfrak{K}$  are of the form

$$M_F := \{\mathcal{F} \in \Sigma\mathfrak{K} \mid F \in \mathcal{F}\} \quad \text{for } F \in N_B.$$

Let  $M_F$  be an open set of  $\Sigma\mathfrak{K}$ . We show that  $\rho^{-1}(M_F) = \psi(F)$ :

$$\begin{aligned} \mathcal{A} \in \rho^{-1}(M_F) &\iff \rho(\mathcal{A}) \in M_F \\ &\iff F \in \rho(\mathcal{A}) \\ &\iff \exists C_1, C_2, \dots, C_n \in \mathcal{A} \text{ with } \text{Cl } C_1 \cap \text{Cl } C_2 \cap \dots \cap \text{Cl } C_n \subseteq F \\ &\iff \exists S \in \mathcal{A} \text{ with } \text{Cl } S \subseteq F \\ &\quad \text{using Lemma 2.1 and maximality of } \mathcal{A} \\ &\iff \mathcal{A} \in \psi(F) \quad \square \end{aligned}$$

Thus far, we have established that the spaces  $\mathfrak{K}^*$  and  $\Sigma\mathfrak{K}$  are isomorphic, so it remains to show that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{K} & \xrightarrow{\varphi} & \mathfrak{K}^* \\ \varepsilon_R \downarrow & & \parallel \\ \Sigma\mathcal{D}\mathfrak{K} & \xrightarrow{\Sigma h} & \Sigma\mathfrak{K} \end{array}$$

Recall from Chapter one that  $\Sigma h(\varepsilon_R(x)) = \bigvee \{\mathcal{F} \in \mathfrak{K} \mid x \notin h(\mathcal{F})\}$ . We have to show that

$$\bigvee \{\mathcal{F} \in \mathfrak{K} \mid x \notin h(\mathcal{F})\} = \rho(\varphi(x)) :$$

- $\rho(\varphi(x))$  is an upper bound for  $\{\mathcal{F} \in \mathfrak{K} \mid x \notin h(\mathcal{F})\}$ :  
Suppose that  $\mathcal{F} \in \mathfrak{K}$  and  $x \notin h(\mathcal{F})$ . We observe that this means that if  $Z \in N_B$  and  $Z^* \in \mathcal{F}$  then  $x \notin Z$ . We show that  $\mathcal{F} \subseteq \rho(\varphi(x))$ :

$$\begin{aligned} A \in \mathcal{F} &\implies \exists B \in \mathcal{F} \text{ with } B \prec A && \text{since } \mathcal{F} \in \mathfrak{K} \\ &\implies x \notin B^* && \text{since } B = B^{**} \text{ and } x \notin h(\mathcal{F}) \\ &\implies x \in \text{Cl } B && \text{since } B^* = \mathbb{C} \text{Cl } B \\ &\implies x \in \varphi(B) \\ &\implies A \in \rho(\varphi(x)) && \text{since } \text{Cl } B \subseteq A \end{aligned}$$

- $\rho(\varphi(x))$  is the least upper bound:

Suppose that the regular filter  $\mathcal{G}$  is also an upper bound for  $\{\mathcal{F} \in \mathfrak{R} \mid x \notin h(\mathcal{F})\}$ . This means that if  $x \notin h(\mathcal{F})$  then  $\mathcal{F} \subseteq \mathcal{G}$ . To show that  $\rho(\varphi(x)) \subseteq \mathcal{G}$ , it suffices to show that  $x \notin h(\rho(\varphi(x)))$ . However, this is clear since

$$\begin{aligned} h(\rho(\varphi(x))) &= \bigcup \{Z \in N_B \mid Z^* \in \rho(\varphi(x))\} \\ &= \bigcup \{Z \in N_B \mid \exists A_1, A_2, \dots, A_n \in \varphi(x) \\ &\quad \text{with } \text{Cl } A_1 \cap \text{Cl } A_2 \cap \dots \cap \text{Cl } A_n \subseteq Z^*\}, \end{aligned}$$

and  $Z$  and its pseudocomplement  $Z^*$  are disjoint.

This concludes the proof that Baboolal's Fan-Gottesman compactification of frames captures the classical Fan-Gottesman compactification of spaces.  $\square$

## Chapter 3. The Wallman Compactification

In [28], Wallman uses the set of all ultrafilters on the family of all closed sets of a  $T_1$  topological space to construct a compact  $T_1$  extension of that space. He shows that the compactification is Hausdorff if and only if the original space is normal. Several authors have refined this procedure, by considering the set of all ultrafilters on a certain subcollection of the family of closed sets. In particular, normal (Wallman) bases were introduced as a generalization of Wallman's original construction by Frink in [18] to construct Hausdorff compactifications of Tychonoff spaces.

In the setting of frames, it is more natural to consider the dual construction, working rather with maximal ideals of open sets. The following description of Wallman's method of compactification, with some notational changes, is taken from [19].

Let  $X$  be a topological space. A sublattice  $\mathcal{B}$  of  $\mathcal{O}X$  which is a base for  $\mathcal{O}X$  and satisfies the property

$$U \in \mathcal{B} \text{ and } x \in U \implies \exists V \in \mathcal{B} \text{ with } x \notin V \text{ and } U \cup V = X,$$

is called a *Wallman base* for  $X$ .

Let  $\text{Max}\mathcal{J}(\mathcal{B})$  denote the space of all maximal ideals on  $\mathcal{B}$  with open sets

$$\Sigma_{\mathcal{J}} := \{\mathcal{M} \in \text{Max}\mathcal{J}(\mathcal{B}) \mid \mathcal{J} \not\subseteq \mathcal{M}\} \quad \text{for } \mathcal{J} \in \mathcal{J}(\mathcal{B}).$$

$\text{Max}\mathcal{J}(\mathcal{B})$  is a compact  $T_1$  topological space and the map

$$\begin{aligned} \Gamma_{\mathcal{B}}: X &\longrightarrow \text{Max}\mathcal{J}(\mathcal{B}) \\ x &\longmapsto M_x := \{U \in \mathcal{B} \mid x \notin U\} \end{aligned}$$

is a dense embedding.  $\text{Max}\mathcal{J}(\mathcal{B})$  is called the Wallman compactification relative to  $\mathcal{B}$ , denoted  $\omega_{\mathcal{B}}X$ . This compactification is shown to be Hausdorff if  $\mathcal{B}$  is normal.

In this section we present some of the material covered by Bernhard Banaschewski during a series of seminars at the University of Cape Town during 1993. The material is based on a section of the work of Banaschewski and Harting in [11].

In what follows,  $L$  denotes an arbitrary frame.

**Definition 3.0.**

- (i) For  $x, a \in L$ , we say that  $x$  is  $a$ -small if for all  $y \in L$ ,

$$x \vee y = \mathbf{e} \implies a \vee y = \mathbf{e}$$

- (ii) For  $a \in L$ ,  $S_a := \{x \in L \mid x \text{ is } a\text{-small}\}$ .  
 (iii) For  $a \in L$ ,  $s(a) := \bigvee S_a$  is called the *saturation* of  $a$ , and  $a$  is said to be *saturated* if  $s(a) = a$ .

*Remark.*

- (i) For each  $a \in L$ ,  $S_a$  is an ideal of  $L$ .  
 (ii) If we consider the frame  $\mathfrak{O}X$  of open sets of a topological space  $X$ , then for elements  $U, V \in \mathfrak{O}X$ :

$$U \text{ is } V\text{-small} \iff \text{every closed set contained in } U \text{ is contained in } V.$$

**Lemma 3.1.** *If  $L$  is a compact frame, then the following are true:*

- (i)  $s(a) \in S_a$ .  
 (ii)  $s : L \longrightarrow L$  is a codense nucleus on  $L$ .

*Proof.*

- (i) To show that  $s(a)$  is  $a$ -small, we take an element  $y \in L$  such that  $s(a) \vee y = \mathbf{e}$  and show that  $a \vee y = \mathbf{e}$ :

$$\begin{aligned} \text{Now } s(a) \vee y = \mathbf{e} &\implies \bigvee \{x \vee y \mid x \in S_a\} = \mathbf{e} \\ &\implies x \vee y = \mathbf{e} \text{ for some } x \in S_a \text{ since } L \text{ is compact} \\ &\implies a \vee y = \mathbf{e} \text{ since } x \text{ is } a\text{-small.} \end{aligned}$$

- (ii) We first show that  $s$  is a nucleus:

- $a \leq s(a)$  since  $a \in S_a$ .

- $s(s(a)) = s(a)$ :
  - $\leq$ : From (i)  $S_{s(a)} \subseteq S_a$  and hence  $s(s(a)) \leq s(a)$ .
  - $\geq$ : This follows from the first condition.
- $a \leq b \implies S_a \subseteq S_b \implies s(a) \leq s(b)$ .
- $s(a \wedge b) = s(a) \wedge s(b)$ : We show that  $s(a) \wedge s(b)$  is  $(a \wedge b)$ -small and hence  $s(a) \wedge s(b) \leq s(a \wedge b)$  which is the non-trivial inequality.
  - Suppose that  $(s(a) \wedge s(b)) \vee y = \mathbf{e}$  for some  $y \in L$ . Then we have that  $s(a) \vee y = \mathbf{e}$  and  $s(b) \vee y = \mathbf{e}$  and hence  $a \vee y = \mathbf{e} = b \vee y$  using Lemma 3.1 (i). Since  $(a \wedge b) \vee y = (a \vee y) \wedge (b \vee y)$ , we have  $(a \wedge b) \vee y = \mathbf{e}$  as required.

It remains to show that  $s$  is codense. Suppose that for some  $a \in L$ ,  $s(a) = \mathbf{e}$ . That is,  $\bigvee S_a = \mathbf{e}$ . Since  $L$  is compact, this yields  $\mathbf{e} \in S_a$ . Therefore in particular, we have  $\mathbf{e} \vee \mathbf{0} = \mathbf{e} \implies a \vee \mathbf{0} = \mathbf{e}$ , which shows that  $a = \mathbf{e}$ .  $\square$

Denote by  $L_s$  the frame of saturated elements of  $L$ . That is,

$$L_s = \{a \in L \mid s(a) = a\}.$$

Then as we observed in Chapter 0,  $L_s$  is a frame and

$$\begin{aligned} s : L &\longrightarrow L_s \\ a &\longmapsto s(a) \end{aligned}$$

is a frame homomorphism. The meet of two elements in  $L_s$  is the same as that in  $L$  and the arbitrary join of a family of elements is given by  $s(\bigvee_L)$ . We will sometimes distinguish this join by using the notation  $\sqcup$  and  $\bigsqcup$ .

**Lemma 3.2.** *If  $L$  is compact, then  $L_s$  is compact.*

*Proof.* Let  $\mathcal{J}$  be an ideal of  $L_s$  whose join in  $L_s$  is  $\mathbf{e}$ . That is,  $s(\bigvee \mathcal{J}) = \mathbf{e}$ . Since  $s$  is codense it follows that  $\bigvee \mathcal{J} = \mathbf{e}$ , where the join is taken in  $L$ . Since  $L$  is compact and  $\mathcal{J}$  is updirected, we see that  $\mathbf{e} \in \mathcal{J}$ . Hence  $L_s$  is compact.  $\square$

We now aim to give a characterisation, of the frame  $L_s$  for a compact frame  $L$ . First, we need the following Lemma:

**Lemma 3.3.** *Let  $L$  be a compact frame, and let  $\Sigma_C L$  denote the subspace of  $\Sigma L$  consisting of all closed points. Then the following are equivalent:*

- (i)  $p \in \Sigma_C L$
- (ii)  $p$  is maximal in  $\Sigma L$ .
- (iii)  $p$  is maximal in  $L$ .

*Proof.* The equivalence of the first two statements follows from the following observation:

$$\begin{aligned}
q \in \text{Cl}\{p\} &\iff \text{every open set containing } q \text{ contains } p \\
&\iff q \in \Sigma_a \implies p \in \Sigma_a \text{ for all } a \in L \\
&\iff q \not\geq a \implies p \not\geq a \text{ for all } a \in L \\
&\iff p \geq a \implies q \geq a \\
&\iff p \leq q
\end{aligned}$$

This shows that  $p$  is closed in  $\Sigma L$  if and only if  $p$  is maximal in  $\Sigma L$ .

(ii)  $\implies$  (iii): trivial

(iii)  $\implies$  (ii): Suppose that  $p$  is maximal in  $\Sigma L$ . This means that if  $q \in \Sigma L$  and  $q \geq p$ , then  $q = p$ . To show that  $p$  is maximal in  $L$ , we take  $s \in L$  with  $s \geq p$  and show that  $s = p$ . By Zorn's Lemma and compactness of  $L$ , there exists a maximal, and hence prime element  $m \in L$  such that  $m \geq s$ . Hence  $m \geq p$ , but by maximality of  $p$  in  $\Sigma L$ , we have that  $m = p$  and hence  $s = p$ .  $\square$

**Proposition 3.4.** *Assuming the Axiom of Choice,  $s(a)$  is the meet of all maximal elements greater than or equal to  $a$ .*

*Proof.* We have to show that

$$s(a) = \bigwedge \{p \in L \mid p \text{ is maximal and } p \geq a\}.$$

It is clear that  $s(a)$  is a lower bound, for if  $p$  is maximal and  $p \geq a$ , then  $s(a) \leq s(p) = p$ . To see that  $s(a)$  is the greatest lower bound, we consider another lower bound  $b$ , say. If  $s(a) < b$ , then we have  $S_{s(a)} \subsetneq S_b$  and hence there exists an element  $x$  such that  $x \in S_b$ , but  $x \notin S_{s(a)}$ . This means that for some element  $y \in L$ ,

$$x \vee y = e \implies b \vee y = e \text{ \& } s(a) \vee y < e.$$

Using Zorn's Lemma and compactness of  $L$  there exists a maximal element  $p$  such that  $p \geq s(a) \vee y$ . So  $p \geq s(a) \geq a$ , but  $p \not\geq b$  — (if  $b \leq p$ , then  $e = b \vee y \leq p$ ). This contradicts the fact that  $b$  is a lower bound for all maximal elements of  $L$  greater than or equal to  $a$ . So it follows that  $s(a)$  is indeed the greatest lower bound.  $\square$

*Remark.* In view of Proposition 3.4 we are able to give the following characterisation of  $L_s$  :

$$a \in L_s \iff a \text{ is a meet of maximal elements of } L$$

**Corollary 3.5.**  $L_s \cong \mathcal{D}(\Sigma_C L)$ , the topology of the maximal spectrum of  $L$ .

*Proof.* The isomorphism is given by

$$\begin{aligned}\omega: L_s &\longrightarrow \mathcal{D}(\Sigma_C L) \\ a &\longmapsto \Sigma_a \cap \Sigma_C L\end{aligned}$$

$\omega$  is clearly an onto and order preserving homomorphism. The fact that  $\omega$  reverses order follows from the remark above.  $\square$

**Definition 3.6.** A frame  $L$  is said to be *subfit* if for any pair of elements  $a, b \in L$  the following is true:

$$a < b \implies \exists c \in L \text{ such that } a \vee c < e \text{ \& } b \vee c = e.$$

A frame which is both compact and subfit is called a *Wallman frame*.

**Lemma 3.7.** *The following are equivalent:*

- (a)  $L$  is subfit.
- (b)  $L = L_s$ .
- (c) For  $a, b \in L$ ,  $a \not\leq b \implies \exists c \in L$  such that  $a \vee c = e$  and  $b \vee c < e$ .
- (d) For  $a, b \in L$ ,  $a < b \implies \exists c \in L$  such that  $c \vee b = e$  and  $a \leq c < e$ .
- (e) For  $a, b, y \in L$ , if  $a \vee y = e \implies b \vee y = e$  then  $a \leq b$ .

*Proof.* The equivalence of (b), (c) and (d) is shown in [11].

- (a)  $\implies$  (b): Suppose that  $L$  is subfit. We have to show the non-trivial inclusion  $L \subseteq L_s$ . Let  $a \in L$ . To show that  $a \in L_s$  we must show that  $s(a) < a$ . In other words  $b \in S_a \implies b \leq a$ .  
If  $b \not\leq a$  then  $a < a \vee b$  and since  $L$  is subfit there exists a  $c \in L$  such that  $a \vee c < e$  and  $a \vee b \vee c = e$ . Now the latter together with the fact that  $b \in S_a$  implies that  $a \vee c = e$  — a contradiction. Hence  $b \leq a$  as required.
- (d)  $\implies$  (e): similar
- (e)  $\implies$  (a): Suppose that (e) holds and that  $a < b$ . Then  $b \not\leq a$  Using the contrapositive of (e), there exists an element  $y \in L$  such that  $b \vee y = e$ , but  $a \vee y < e$ .  $\square$

*Remark.* The following are easily verified:

- (i) If  $X$  is a  $T_1$  topological space, then  $\mathcal{D}X$  is a subfit frame.
- (ii) Every regular frame is subfit.

**Proposition 3.8.**  *$L$  is subfit if and only if every principal ideal of  $L$  is a saturated element of  $\mathfrak{J}(L)$ .*

*Proof.*

$\implies$  : It is our aim to show that for each  $a \in L$ ,  $s(\downarrow a) \subseteq \downarrow a$  which is the non-trivial inclusion. For this purpose we introduce the following notation which is used in [21].

$$j(\downarrow a) := \{x \in L \mid \forall b \in L \quad x \vee b = e \implies c \vee b = e \text{ for some } c \in \downarrow a\},$$

and we show that

$$s(\downarrow a) = j(\downarrow a)$$

i.e.,  $\bigvee \{\mathfrak{J} \in \mathfrak{J}(L) \mid \mathfrak{J} \text{ is } \downarrow a\text{-small}\} = \{x \in L \mid \forall b \in L \quad x \vee b = e \implies c \vee b = e \text{ for some } c \in \downarrow a\}$ :

To show that  $j(\downarrow a)$  is an upper bound, we take an ideal  $\mathfrak{J}$  of  $L$  which is  $\downarrow a$ -small and show that  $\mathfrak{J} \subseteq j(\downarrow a)$ . To this end, let  $x \in \mathfrak{J}$  and suppose that  $x \vee b = e$  for some element  $b \in L$ . Then  $\downarrow x \vee \downarrow b = \downarrow e$  which implies that  $\mathfrak{J} \vee \downarrow b = \downarrow e$ . Since  $\mathfrak{J}$  is  $\downarrow a$ -small, we have that  $\downarrow a \vee \downarrow b = \downarrow e$ . This means that there exists an element  $c \in \downarrow a$  such that  $c \vee b = e$ . That is,  $x \in j(\downarrow a)$ .

To see that  $j(\downarrow a)$  is the least upper bound, we consider another upper bound  $\mathfrak{K}$ , and show that  $j(\downarrow a) \subseteq \mathfrak{K}$ . Since  $\mathfrak{K}$  is an upper bound for all the  $\downarrow a$ -small ideals, it suffices to show that  $j(\downarrow a)$  is  $\downarrow a$ -small. So suppose that for some ideal  $\mathfrak{J}$  of  $L$  we have that  $j(\downarrow a) \vee \mathfrak{J} = \downarrow e$ . This means that there exist elements  $x \in j(\downarrow a)$  and  $j \in \mathfrak{J}$  such that  $x \vee j = e$ . Since  $x \in j(\downarrow a)$ , it follows that there exists an element  $c$  in the ideal  $\downarrow a$  such that  $c \vee j = e$  which shows that  $\downarrow a \vee \mathfrak{J} = \downarrow e$ . This concludes the proof that  $j(\downarrow a)$  is  $\downarrow a$ -small.

It is now easily seen that  $s(\downarrow a) \subseteq \downarrow a$ . For if  $x \in s(\downarrow a) = j(\downarrow a)$  then  $x$  is  $c$ -small for some element  $c \in \downarrow a$ . Since  $L$  is subfit, we have that  $x \leq c$  by Lemma 3.7 (e). So  $x \leq c \leq a$  and  $x$  belongs to  $\downarrow a$  as required.

$\impliedby$  : Conversely, suppose that every principal ideal of  $L$  is saturated. In order to see that  $L$  is subfit we suppose that  $x$  is  $y$ -small for elements  $x, y \in L$  and show that  $x \leq y$ . Now if  $x$  is  $y$ -small, then  $x \in j(\downarrow y)$ , but by our assumption,  $j(\downarrow y) = \downarrow y$ , so  $x \leq y$  and  $L$  is subfit.  $\square$

**Lemma 3.9.** *A compact frame  $L$  with basis  $B$  is normal if and only if  $B$  is normal.*

*Proof.*

$\implies$  : Suppose that  $a, b \in B$  satisfy  $a \vee b = e$ . Since  $L$  is normal, there exist elements  $x = \bigvee x_i$  and  $y = \bigvee y_j$  in  $L$  with  $x_i, y_j \in B$  such that

$$x \wedge y = \mathbf{0} \text{ and } a \vee x = b \vee y = e.$$

Since  $L$  is compact there exist elements  $u, v \in B$  such that

$$u \wedge v = \mathbf{0} \text{ and } a \vee u = b \vee v = \mathbf{e}.$$

So  $B$  is normal.

$\Leftarrow$  : Suppose that  $x, y \in L$  satisfy  $x \vee y = \mathbf{e}$ . Since  $B$  generates  $L$ , we may rewrite this as  $\bigvee x_i \vee \bigvee y_j = \mathbf{e}$  where  $x_i, y_j \in B$ . Since  $L$  is compact, we can find elements  $a, b \in B$  such that  $a \vee b = \mathbf{e}$  with  $a \leq x$  and  $b \leq y$ . Since  $B$  is normal, there exist elements  $u, v \in B$  satisfying

$$u \wedge v = \mathbf{0} \text{ and } a \vee u = b \vee v = \mathbf{e}.$$

Hence  $x \vee u = y \vee v = \mathbf{e}$  and  $L$  is normal.  $\square$

**Lemma 3.10.** *If  $L$  is a compact frame then  $L_s$  is subfit.*

*Proof.* Suppose that  $a < b$  in  $L_s$ . Then  $S_a \subsetneq S_b$ , so there exists  $x \in L$  which is  $b$ -small, but not  $a$ -small. That is, there exists an element  $z \in L$  such that  $x \vee z = \mathbf{e}$ ,  $b \vee z = \mathbf{e}$  and  $a \vee z < \mathbf{e}$ . So

$$s(b \vee s(z)) = \mathbf{e} > s(a \vee s(z)).$$

Putting  $c = s(z)$ , we have an element  $c \in L_s$  with  $a \sqcup c < \mathbf{e} = b \sqcup c$  as required.  $\square$

**Lemma 3.11.** *A normal subfit frame is regular.*

*Proof.* Let  $x \in L$ . We have to show that  $x = \bigvee \{y \in L \mid y \prec x\}$ . It is clear that  $x$  is an upper bound, so suppose that  $a$  is also an upper bound. i.e., if  $y \prec x$ , then  $y \leq a$ . We have to show that  $x \leq a$ :

Using the fact that  $L$  is subfit, it suffices to show that  $x$  is  $a$ -small. So let  $x \vee z = \mathbf{e}$  for some element  $z \in L$ . By normality of  $L$ , there exist elements  $u, v \in L$  such that

$$u \wedge v = \mathbf{0} \text{ and } x \vee u = z \vee v = \mathbf{e}.$$

This shows that  $v \prec x$  with separating element  $u$ . By our assumption above, this implies that  $v \leq a$  which yields  $z \vee a = \mathbf{e}$ .  $\square$

**Proposition 3.12.** *If  $L$  is a normal subfit frame, then the join map*

$$\bigvee : (\mathcal{J}(L))_s \longrightarrow L$$

*is a subfit compactification of  $L$ , called the Wallman compactification.*

*Proof.* Since the frame  $\mathcal{J}(L)$  of all ideals on  $L$  is always compact, it follows from Lemma 3.2 that  $(\mathcal{J}(L))_s$  is compact. To show that  $(\mathcal{J}(L))_s$  is regular, we use Lemma 3.11:

Since  $L$  is normal, it follows that  $\{\downarrow a \mid a \in L\}$  is normal, and since the latter generates the compact frame  $(\mathcal{J}(L))_s$ , we have by Lemma 3.9 that  $(\mathcal{J}(L))_s$  is normal.

That  $(\mathcal{J}(L))_s$  is subfit follows from Lemma 3.10.

The join map is clearly dense, and that it is onto follows from the fact that  $L$  is subfit and hence every principal ideal of  $L$  is saturated. (Proposition 3.8.)  $\square$

One obtains a relativised version of the Wallman compactification as follows:

**Definition 3.13.** Let  $A$  be a normal sublattice generating  $L$ .  $L$  is called  $A$ -subfit if for any pair of elements  $x, y \in L$ , the following holds:

$$(x \vee a = e \implies y \vee a = e \quad \forall a \in A) \implies x \leq y.$$

$A$  is called a *subfitting basis* if  $L$  is  $A$ -subfit.

**Proposition 3.14.** For any normal subfitting basis  $A$  of a frame  $L$ , the join map

$$\bigvee: (\mathcal{J}(A))_s \longrightarrow L$$

is an  $A$ -subfit compactification of  $L$  called the *Wallman compactification relative to  $A$* .

*Proof.* The proof is essentially the relativised analogue of the proof of Proposition 3.12 and is therefore omitted.

We now explore the connection between the Wallman compactification of a  $T_1$  topological space  $X$ , as outlined at the beginning of this chapter, and that of the associated frame  $\mathfrak{D}X$ . First we need the following Lemma.

**Lemma 3.15.** Let  $X$  be a  $T_1$  topological space. Then

- (i)  $\mathfrak{D}X$  is a Wallman basis for the space  $X$ .
- (ii)  $\mathfrak{D}X$  is a subfitting basis for the frame  $\mathfrak{D}X$ .

*Proof.*

- (i) That  $\mathfrak{D}X$  is a basis for the space  $X$  is trivial, so suppose that  $x \in A$  for some open set  $A$ . Since  $X$  is  $T_1$ , we have that  $\mathbb{C}\{x\}$  is an open set. Moreover  $x \notin \mathbb{C}\{x\}$  and  $\mathbb{C}\{x\} \cup A = X$ , so  $\mathfrak{D}X$  is a Wallman basis for  $X$ .
- (ii) Again it is trivial that  $\mathfrak{D}X$  generates  $\mathfrak{D}X$ , so suppose that there are open sets  $U$  and  $V$  such that  $U \subsetneq V$ . Then there exists an element  $v \in V$  such that  $v \notin U$  and since  $X$  is  $T_1$ , we have  $\mathbb{C}\{v\} \in \mathfrak{D}X$  satisfying  $U \cup \mathbb{C}\{v\} \subsetneq X$  and  $V \cup \mathbb{C}\{v\} = X$ .  $\square$

In view of Lemma 3.15 (ii), we have the compactification

$$\cup: (\mathcal{J}(\mathfrak{D}X))_s \longrightarrow \mathfrak{D}X \quad \text{in Frm.}$$

By Lemma 3.15 (i) and Lemma 3.3, we have the compactification

$$\Gamma_{\mathfrak{D}X}: X \longrightarrow \text{Max}\mathcal{J}(\mathfrak{D}X) = \Sigma_C\mathcal{J}(\mathfrak{D}X) \quad \text{in Top.}$$

Applying the open set functor to the latter, we obtain the compactification

$$\mathfrak{D}\Gamma_{\mathfrak{D}X}: \mathfrak{D}\Sigma_C\mathcal{J}(\mathfrak{D}X) \longrightarrow \mathfrak{D}X \quad \text{in Frm.}$$

Now by Corollary 3.5, we have that  $\mathfrak{D}\Sigma_C\mathcal{J}(\mathfrak{D}X) \cong (\mathcal{J}(\mathfrak{D}X))_s$ , so to show that the compactifications are equivalent, we must show that the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{J}(\mathfrak{D}X))_s & \xrightarrow{\cup} & \mathfrak{D}X \\ \downarrow \omega & \nearrow \mathfrak{D}\Gamma_{\mathfrak{D}X} & \\ \mathfrak{D}\Sigma_C\mathcal{J}(\mathfrak{D}X) & & \end{array}$$

Let  $\mathcal{J} \in (\mathcal{J}(\mathfrak{D}X))_s$ . Then we have

$$\begin{aligned} (\mathfrak{D}\Gamma_{\mathfrak{D}X} \circ \omega)(\mathcal{J}) &= \Gamma_{\mathfrak{D}X}^{-1}(\omega(\mathcal{J})) \\ &= \{x \in X \mid \Gamma_{\mathfrak{D}X}(x) \in \omega(\mathcal{J})\} \\ &= \{x \in X \mid \Gamma_{\mathfrak{D}X}(x) \in \Sigma_{\mathcal{J}} \cap \Sigma_C\mathcal{J}(\mathfrak{D}X)\} \\ &= \{x \in X \mid \Gamma_{\mathfrak{D}X}(x) \supseteq \mathcal{J}\} \\ &= \{x \in X \mid \exists A \in \mathcal{J} \text{ with } x \in A\} \\ &= \bigcup \mathcal{J}. \end{aligned}$$

Hence the Wallman compactification of the frame  $\mathfrak{D}X$  is a generalisation of that of the topological space  $X$ .

## Chapter 4. The Zero-Dimensional Compactification

Banaschewski has given a construction of a universal zero-dimensional Hausdorff compactification of a  $T_0$  zero-dimensional topological space (see [4] or [6]). We follow [6] in giving a brief outline of the construction.

Let  $X$  be a zero-dimensional  $T_0$  topological space with basis  $\mathcal{Z}$  of clopen sets. Let  $\Omega$  denote the collection of all ultrafilters on  $\mathcal{Z}$ , and for each  $A \in \mathcal{Z}$ , let

$$\Omega_A = \{\mathcal{U} \in \Omega \mid A \in \mathcal{U}\}.$$

It is easily seen that  $\{\Omega_A \mid A \in \mathcal{Z}\}$  can be taken as a basis for the closed sets of a topology on  $\Omega$ . The set  $\Omega$  equipped with this topology is a compact Hausdorff, zero-dimensional space which contains  $X$  as a dense subspace. Moreover the associated embedding

$$\begin{aligned} \Phi: X &\longrightarrow \Omega \\ x &\longmapsto \{A \in \mathcal{Z} \mid x \in A\} \end{aligned}$$

is the compact Hausdorff, zero-dimensional reflection.

Following [8], we give a frame-theoretic analogue of the above construction, i.e., we construct the zero-dimensional coreflection of a zero-dimensional frame. Given a frame  $L$ , we denote by  $\mathcal{C}L$  the sublattice of complemented elements of  $L$ ,

$$\text{i.e., } \mathcal{C}L = \{x \in L \mid x \vee x^* = \mathbf{e}\}.$$

**Definition 4.0.** A frame  $L$  is said to be *zero-dimensional* if it is generated by  $\mathcal{C}L$ . A zero-dimensional, compact frame is called a *Stone frame*.

**Lemma 4.1.** *For any frame  $L$ ,  $\mathfrak{J}(\mathcal{C}L)$  is a Stone frame.*

*Proof.* That  $\mathfrak{J}(\mathcal{C}L)$  is compact follows from the fact that it is a subframe of the compact frame  $\mathfrak{J}(L)$ . To show that  $\mathfrak{J}(\mathcal{C}L)$  is zero-dimensional, we use the fact that the principal ideals  $\downarrow x$  for  $x \in \mathcal{C}L$  are complemented in  $\mathfrak{J}(\mathcal{C}L)$  and every ideal  $\mathfrak{J} \in \mathfrak{J}(\mathcal{C}L)$  can be written in the form:

$$\mathfrak{J} = \bigvee \{\downarrow x \mid x \in \mathfrak{J}\}. \quad \square$$

**Corollary 4.2.** *For any frame  $L$ ,  $\mathfrak{J}(\mathcal{C}L)$  is a regular frame.*

*Proof.* An element is complemented if and only if it is rather below itself. Hence every zero-dimensional frame is regular. Since we proved above that  $\mathfrak{J}(\mathcal{C}L)$  is zero-dimensional, the result follows.  $\square$

**Lemma 4.3.** *The join map  $\bigvee: \mathfrak{J}(\mathcal{C}L) \longrightarrow L$  is an isomorphism if and only if  $L$  is a Stone frame.*

*Proof.*

$\implies$  : This direction is clear since  $\mathfrak{J}(\mathcal{C}L)$  is a Stone frame.

$\impliedby$  : In the category of regular frames, every codense map is a monomorphism. Therefore to establish that  $\bigvee: \mathfrak{J}(\mathcal{C}L) \longrightarrow L$  is one-to-one, it suffices to show that it is codense. Suppose that  $\bigvee \mathfrak{J} = \mathbf{e}$  for some ideal  $\mathfrak{J}$  of  $\mathfrak{J}(\mathcal{C}L)$ . Since  $\mathfrak{J}(\mathcal{C}L)$  is compact, it follows that  $\mathbf{e} \in \mathfrak{J}$  or  $\mathfrak{J} = \mathcal{C}L$ .

To show that  $\bigvee: \mathfrak{J}(\mathcal{C}L) \longrightarrow L$  is onto, we take an element  $x \in L$ , and show that there is an ideal in  $\mathfrak{J}(\mathcal{C}L)$  whose join is  $x$ . Since  $L$  is zero-dimensional, we have that  $x = \bigvee x_i$  with  $x_i \in \mathcal{C}L$ . Now  $\downarrow x \cap \mathcal{C}L \in \mathfrak{J}(\mathcal{C}L)$  and  $\bigvee (\downarrow x \cap \mathcal{C}L) = \bigvee x_i = x$ .  $\square$

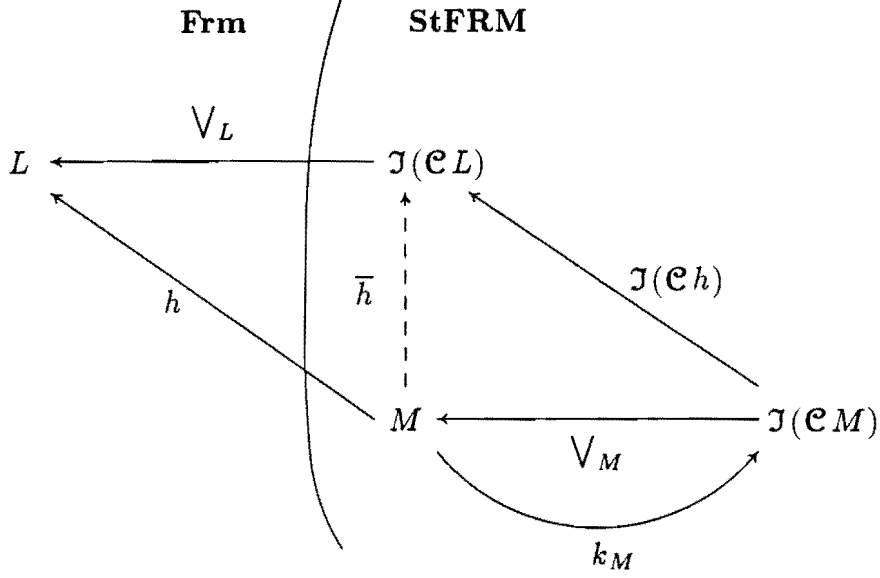
**Proposition 4.4.** *The category  $\mathbf{StFRM}$  of Stone frames and frame homomorphisms is coreflective in  $\mathbf{FRM}$ , with coreflection map given by join.*

*Proof.* Let  $L$  be a frame. In Corollary 4.2, we proved that  $\mathfrak{J}(\mathcal{C}L) \in \mathbf{StFRM}$ , so consider a morphism  $h: M \longrightarrow L$  with  $M \in \mathbf{StFRM}$ . We have to show that  $h$  factors through  $\mathfrak{J}(\mathcal{C}L)$ . Note that we have the outer commuting square in the diagram below, where  $\mathfrak{J}(\mathcal{C}h)$  takes an ideal  $\mathfrak{J}$  of complemented elements of  $M$  to the ideal of complemented elements of  $L$  generated by the image  $h(\mathfrak{J})$ . From Lemma 4.3, since  $M$

is a Stone frame, we have that the join map  $\bigvee_M: \mathcal{J}(\mathcal{C}M) \longrightarrow M$  is an isomorphism with inverse  $k_M$ , say, where

$$\begin{aligned} k_M: M &\longrightarrow \mathcal{J}(\mathcal{C}M) \\ a &\longmapsto \downarrow a \cap \mathcal{C}M. \end{aligned}$$

We define  $\bar{h}: M \longrightarrow \mathcal{J}(\mathcal{C}L)$  by  $\bar{h} = \mathcal{J}(\mathcal{C}h) \circ k_M$ . Then  $\bigvee_L \circ \bar{h} = \bigvee_L \circ \mathcal{J}(\mathcal{C}h) \circ k_M = h \circ \bigvee_M \circ k_M = h \circ \text{id}_M = h$ . So the left-hand triangle in the following diagram commutes:



All that remains to be seen is that  $\bar{h}$  is unique. To this end, suppose that there is a morphism  $g: M \longrightarrow \mathcal{J}(\mathcal{C}L)$  such that  $h = \bigvee_L \circ g$ . We show that  $\bar{h} = g$ :

Let  $x \in \mathcal{C}M$ . Then  $g(x)$  is a complemented element of  $\mathcal{J}(\mathcal{C}L)$ . It is easily proved that in a Stone frame, the complemented elements are precisely the compact elements. Hence  $g(x)$  is compact. Since every compact ideal is principal, we may write  $g(x)$  in the form  $g(x) = \downarrow b \cap \mathcal{C}L$  for some element  $b \in \mathcal{C}L$ .

Now

$$h(x) = \bigvee_L (g(x)) = \bigvee_L (\downarrow b \cap \mathcal{C}L) = b, \text{ so } g(x) = \downarrow h(x) \cap \mathcal{C}L.$$

However,

$$\begin{aligned} \bar{h}(x) &= \mathcal{J}(\mathcal{C}h) \circ k_M \\ &= \bigcup \{ \downarrow h(y) \cap \mathcal{C}L \mid y \in \downarrow x \cap \mathcal{C}M \} \\ &= \downarrow h(x) \cap \mathcal{C}L \end{aligned}$$

because  $x$  occurs as one of the  $y$ 's, and  $h$  preserves order. Since  $\bar{h}$  and  $g$  agree on the generators of  $M$ , it follows that  $\bar{h} = g$ .  $\square$

**Corollary 4.5.** *For any zero-dimensional frame  $L$ ,  $\bigvee_L: \mathcal{J}(\mathcal{C}L) \longrightarrow L$  is the universal zero-dimensional compactification of  $L$ .*

*Proof.* The result follows from Proposition 4.4 and the proof of Lemma 4.3. (The latter shows that the join map is onto.)

We now give a description of the strong inclusion which gives rise to the compactification described in Corollary 4.5. The right adjoint to the join map  $\bigvee: \mathcal{J}(\mathcal{C}L) \longrightarrow L$  is the downset map,  $\downarrow(-) \cap \mathcal{C}L: L \longrightarrow \mathcal{J}(\mathcal{C}L)$ . Thus using the result of Banaschewski in [9] we know that the strong inclusion is

$$a \triangleleft b \text{ in } L \iff \downarrow a \cap \mathcal{C}L \prec \downarrow b \cap \mathcal{C}L \text{ in } \mathcal{J}(\mathcal{C}L).$$

This description of the strong inclusion will be used in Chapter 6.

## Chapter 5. Minimal and Maximal Compactifications

For any topological space  $X$ , let  $\mathfrak{C}X$  denote the family of ~~all~~ all ordered pairs of the form  $(K, h)$ , where  $K$  is a compact Hausdorff space, and  $h: X \rightarrow K$  is a homeomorphic dense embedding. We can define an order, due to Lubben [22], on  $\mathfrak{C}X$  as follows:  $(K_1, h_1) \leq (K_2, h_2)$  if and only if there exists a continuous map  $f: K_2 \rightarrow K_1$  such that  $f \circ h_2 = h_1$ .

Lubben [22] shows that every non-empty subfamily of  $\mathfrak{C}X$  has a least upper bound with respect to the order  $\leq$ . As a corollary to this result, we have that for every Tychonoff space  $X$ , the *Stone-Ćech* or *maximal* compactification of  $X$ , denoted  $\beta X$ , is the largest element of  $\mathfrak{C}X$ . The compactification  $\beta X$  is the reflection of the category of topological spaces in the category of compact Hausdorff spaces.

A natural question then arises as to which spaces have minimal compactifications. This question was answered by Alexandroff in [1] with the construction of the *one point compactification* of a locally compact, non-compact Hausdorff space  $X$ . The compactification is obtained by adjoining to  $X$  a “point at infinity” as outlined below.

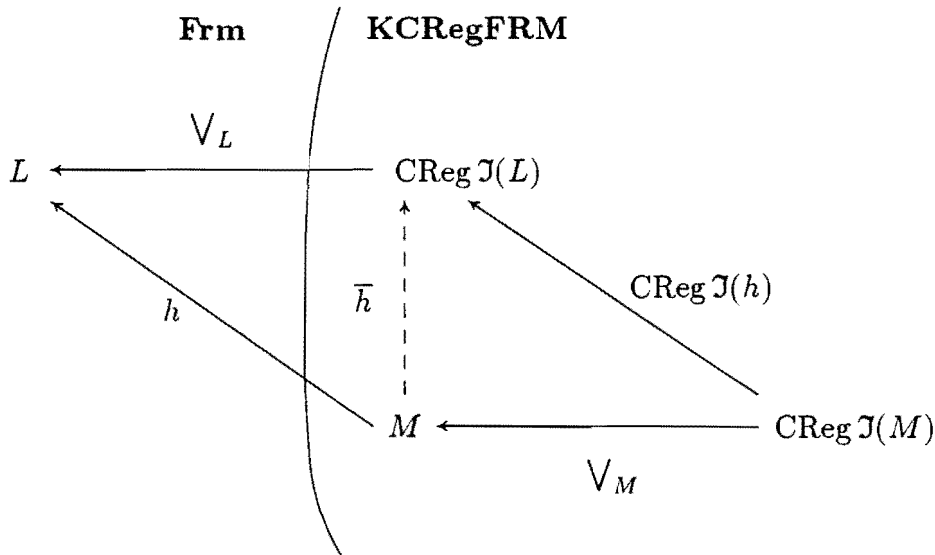
Let  $X$  be a locally compact, non-compact Hausdorff space and let  $\infty$  be a point which is not an element of  $X$ . Denote by  $\alpha X$  the set  $X \cup \{\infty\}$ . The open sets of  $\alpha X$  are the open sets of  $X$ , together with the sets of the form  $\{\infty\} \cup (X \setminus F)$  where  $F$  is a compact subset of  $X$ . It can be shown that  $\alpha X$  is a compact Hausdorff space and the mapping  $\alpha: X \rightarrow \alpha X$ , given by inclusion, is a homeomorphic dense embedding. The compactification is called the *Alexandroff, one point* or *minimal* compactification of  $X$ , and is the smallest element in  $\mathfrak{C}X$  with respect to the order  $\leq$ . The details of this construction can be found in [15].

We now investigate the frame counterparts of the maximal and minimal compactifications.

## THE STONE-ČECH COMPACTIFICATION OF A FRAME

In [12], Banaschewski and Mulvey show that the categories **RegFRM**, of regular frames, and **KCRegFRM**, of compact completely regular frames, are coreflective subcategories of **Frm**.

The coreflector for the compact completely regular coreflection is  $\text{CReg } \mathcal{J}(L)$ , which associates with each frame  $L$ , the frame of completely regular ideals on  $L$ , and with each map  $h: M \rightarrow L$ , the map  $\text{CReg } \mathcal{J}(h): \text{CReg } \mathcal{J}(M) \rightarrow \text{CReg } \mathcal{J}(L)$  which takes an ideal  $\mathcal{J} \in \text{CReg } M$  to the ideal generated by  $h(\mathcal{J})$  in  $L$ . The coreflection arrow is given by the join map  $\bigvee: \text{CReg } \mathcal{J}(L) \rightarrow L$  which is onto if and only if  $L$  is completely regular (see [3]). Thus, for a completely regular frame  $L$ , the compact completely regular coreflection provides us with a frame-theoretic analogue of the Stone-Čech compactification of spaces. This coreflection is illustrated below:



The coreflector for the compact regular coreflection is  $\text{Reg } \mathcal{J}$  which assigns to every frame  $L$ , the frame  $\text{Reg } \mathcal{J}(L)$ , of regular ideals on  $L$ , and to any map  $h: M \rightarrow L$ , the map  $\text{Reg } \mathcal{J}(h): \text{Reg } \mathcal{J}(M) \rightarrow \text{Reg } \mathcal{J}(L)$  which takes a regular ideal  $\mathcal{J}$  on  $M$  to the ideal generated by  $h(\mathcal{J})$  in  $L$ .

It is well known that in the presence of the axiom of countably dependent choice, a normal regular frame is completely regular, and in this case, the compact regular and compact completely regular coreflections coincide. In [3], Baboolal and Banaschewski give the following characterisation of the compact regular coreflection of a normal regular frame:

**Proposition 5.1.** *If  $h: M \longrightarrow L$  is a compactification such that its right adjoint  $q: L \longrightarrow M$  is a lattice homomorphism, then  $L$  is normal regular and  $h: M \longrightarrow L$  is isomorphic to the compact regular coreflection and conversely.*

*Proof.* If  $a \vee b = \mathbf{e}$  in  $L$  then  $q(a) \vee q(b) = q(a \vee b) = \mathbf{e}$  in  $M$ . By the normality of compact regular frames, there exist elements  $s$  and  $t$  in  $M$  for which  $q(a) \vee t = \mathbf{e} = s \vee q(b)$  and  $s \wedge t = \mathbf{0}$ . Then, since  $h \circ q = \text{id}_L$ ,  $u = h(s)$  and  $v = h(t)$  satisfy the conditions  $a \vee v = u \vee b = \mathbf{e}$  and  $u \wedge v = \mathbf{0}$ , showing that  $L$  is normal.  $L$  is regular, being the image of a regular frame.

Since  $L$  is normal and regular, it follows that  $\bigvee: \text{Reg } \mathcal{J}(L) \longrightarrow L$  is the universal compactification of  $L$  and hence there exists a homomorphism  $\bar{h}: M \longrightarrow \text{Reg } \mathcal{J}(L)$  such that the following diagram commutes:

$$\begin{array}{ccc} \text{Reg } \mathcal{J}(L) & \xrightarrow{\bigvee} & L \\ & \nwarrow \bar{h} & \nearrow h \\ & M & \end{array}$$

We show that  $\bar{h}$  is an isomorphism: Since  $h$  is dense, it follows that  $\bar{h}$  is dense, and thus  $\bar{h}$  is one-to-one by Lemma 0.2. It remains to show that  $\bar{h}$  is onto.

For each  $a \in L$ , let  $r(a) = \{x \in L \mid x \prec a\}$ . Since the collection  $\{r(a) \mid a \in L\}$  generates the frame  $\text{Reg } \mathcal{J}(L)$ , it suffices to show that for each  $a \in L$ ,

$$\bar{h}(q(a)) = r(a).$$

- $\subseteq$  Suppose that  $x$  belongs to  $\bar{h}(q(a))$ . Then there exists an element  $y \in \bar{h}(q(a))$  such that  $x \prec y$ . Since  $y \leq \bigvee \bar{h}(q(a)) = a$ , we have  $x \prec a$ . So  $x \in r(a)$ .
- $\supseteq$  Conversely, suppose that  $x \prec a$  with separating element  $y$ . Then  $x \wedge y = \mathbf{0}$  and  $y \vee a = \mathbf{e}$ . Since  $q$ , and hence  $\bar{h} \circ q$ , is a lattice homomorphism, we have that  $\bar{h}(q(x)) \cap \bar{h}(q(y)) = \{\mathbf{0}\}$  and  $\bar{h}(q(y)) \vee \bar{h}(q(a)) = L$ . The latter implies that there exist elements  $s \in \bar{h}(q(a))$  and  $t \in \bar{h}(q(y))$  such that  $s \vee t = \mathbf{e}$ . Now  $t \leq \bigvee \bar{h}(q(y)) = h(q(y)) = y$ , so  $t \wedge x = \mathbf{0}$ . Hence

$$x = x \wedge \mathbf{e} = x \wedge (s \vee t) = (x \wedge s) \vee (x \wedge t) = x \wedge s \leq s,$$

which implies that  $x \in \bar{h}(q(a))$ .

For the converse, we assume that  $L$  is a normal, regular frame. The right adjoint to the compact regular reflection is given by

$$\begin{aligned} r: L &\longrightarrow \text{Reg } \mathfrak{J}(L) \\ a &\longmapsto \{x \in L \mid x \prec a\}. \end{aligned}$$

That  $r$  preserves finite meets and the top and bottom elements of  $L$  is clear. To see that  $r$  preserves joins, we argue as follows, using Lemma 0.3:

$$\begin{aligned} x \in r(a \vee b) &\implies x \prec a \vee b \\ &\implies x \prec c \vee d \text{ where } c \in r(a) \text{ and } d \in r(b) \\ &\implies x \in r(a) \vee r(b) \quad \square \end{aligned}$$

### THE MINIMAL COMPACTIFICATION OF A FRAME

It is shown in [2] that a regular continuous frame  $L$  has a smallest strong inclusion given by

$$x \triangleleft y \iff x \prec y \text{ and either } \uparrow(x^*) \text{ or } \uparrow y \text{ is a compact frame.}$$

This smallest strong inclusion will give rise to the smallest compactification and hence a frame counterpart to the Alexandroff compactification.

**Lemma 5.2.** *For a regular continuous frame,*

$$x \ll y \iff x \prec y \text{ and } \uparrow(x^*) \text{ is a compact frame.}$$

*Proof.*

$\implies$  Suppose  $x \ll y$ . Since  $L$  is regular, we may express  $y$  as  $y = \bigvee \{z \in L \mid z \prec y\}$ . The set  $\{z \in L \mid z \prec y\}$  is updirected, and since  $x \ll y$  we have that  $x \leq z$  for some  $z \in L$  satisfying  $z \prec y$ . Hence  $x \leq z \prec y$  which yields  $x \prec y$ . To show that  $\uparrow(x^*)$  is compact, we take a subset  $A \subseteq \uparrow(x^*)$  with  $\bigvee A = e$ . Since  $L$  is continuous, the way below relation interpolates, and there exists an element  $s \in L$  such that  $x \ll s \ll y$ . Now  $y \leq \bigvee A \implies s \leq a_1 \vee a_2 \vee \dots \vee a_n$  where  $a_i \in A$ . Further, as we have seen above,  $x \ll s \implies x \prec s$  so that  $x^* \vee s = e$  and hence  $a_1 \vee a_2 \vee \dots \vee a_n \geq x^* \vee s = e$ . So  $\uparrow(x^*)$  is compact.

$\Leftarrow$  Suppose that  $x \prec y$  and  $\uparrow(x^*)$  is compact. Let  $S$  be a subset of  $L$  such that  $y \leq \bigvee S$ . Now

$$\begin{aligned}
x \prec y &\implies x^* \vee y = \mathbf{e} \\
&\implies x^* \vee \bigvee S = \mathbf{e} \\
&\implies \bigvee \{x^* \vee s \mid s \in S\} = \mathbf{e} \\
&\implies s_1 \vee x^* \vee s_2 \vee x^* \vee \dots \vee s_n \vee x^* = \mathbf{e} \quad \text{since } \uparrow(x^*) \text{ is compact} \\
&\implies (s_1 \vee s_2 \vee \dots \vee s_n) \vee x^* = \mathbf{e} \\
&\implies x \leq x^{**} \leq s_1 \vee s_2 \vee \dots \vee s_n \\
&\implies x \ll y \quad \square
\end{aligned}$$

**Lemma 5.3.** *For a regular continuous frame  $L$ , the relation*

$$x \triangleleft y \iff x \prec y \text{ and either } \uparrow(x^*) \text{ or } \uparrow y \text{ is a compact frame}$$

*defines a strong inclusion on  $L$ . Furthermore, this strong inclusion is contained in every other strong inclusion on  $L$ .*

*Proof.*

(S1)  $x \leq a \triangleleft b \leq y \implies x \triangleleft y$ :

Suppose that  $x \leq a \triangleleft b \leq y$ . It is clear that  $x \prec y$ . If  $\uparrow(a^*)$  is compact, then  $\uparrow(x^*)$  is compact, since  $\uparrow(x^*) \subseteq \uparrow(a^*)$ . Similarly, if  $\uparrow b$  is compact, then so is  $\uparrow y$  since  $\uparrow y \subseteq \uparrow b$ . Hence  $x \triangleleft y$ .

(S2)  $\triangleleft$  is a sublattice of  $L \times L$ :

Since  $\uparrow(\mathbf{0}^*) = \uparrow \mathbf{e} = \{\mathbf{e}\}$  is compact, and both  $\mathbf{e} \prec \mathbf{e}$  and  $\mathbf{0} \prec \mathbf{0}$  hold, we have  $\mathbf{e} \triangleleft \mathbf{e}$  and  $\mathbf{0} \triangleleft \mathbf{0}$ . Suppose that  $x \triangleleft a, b$ . Then  $x \prec a \wedge b$ . If  $\uparrow(x^*)$  is compact then  $x \triangleleft a \wedge b$ , otherwise both  $\uparrow a$  and  $\uparrow b$  are compact, which yields that  $\uparrow(a \wedge b)$  is compact. So  $x \triangleleft a \wedge b$ . Finally, suppose that  $x, y \triangleleft a$ . Then  $x \vee y \prec a$ . If  $\uparrow a$  is compact, then  $x \vee y \triangleleft a$ . Otherwise both  $\uparrow(x^*)$  and  $\uparrow(y^*)$  are compact, which yields that  $\uparrow(x^* \wedge y^*) = \uparrow(x \vee y)^*$  is compact and  $x \vee y \triangleleft a$ .

(S3)  $a \triangleleft b \implies a \prec b$  is implicit in the definition of the strong inclusion.

(S4)  $\triangleleft$  interpolates:

Suppose that  $a \triangleleft b$ . If  $\uparrow(a^*)$  is compact, then by the previous Lemma, we have that  $a \ll b$ . Using the fact that the way below relation interpolates for a continuous frame, we obtain an element  $c \in L$  such that  $a \ll c \ll b$ . Hence  $a \prec c$ , which yields  $a \triangleleft c$ . Further, the fact that  $c \prec b$  and  $\uparrow(c^*)$  is compact, implies that  $c \triangleleft b$ . On the other hand, if  $\uparrow b$  is compact, we can find a separating element  $s \in L$  satisfying  $a \wedge s = \mathbf{0}$  and  $s \vee b = \mathbf{e}$ . Since  $L$  is continuous, we may express  $s$  as  $s = \bigvee \{x \in L \mid x \ll s\}$  and hence by compactness of  $\uparrow b$  there exists an element  $z \in L$  with  $z \ll s$  and  $z \vee b = \mathbf{e}$ . So  $z^* \prec b$  and hence  $z^* \triangleleft b$ . Now  $z \ll s$  implies that  $z \prec s$  and  $\uparrow(z^*)$  is compact. Thus we have  $a \wedge s = \mathbf{0}$  and  $z^* \vee s = \mathbf{e}$  which yields  $a \prec z^*$ , and hence  $a \triangleleft z^*$ .

(S5)  $a \triangleleft b \implies b^* \triangleleft a^*$ :

If  $a \triangleleft b$ , then  $a \prec b$ , which implies  $b^* \prec a^*$ . If  $\uparrow(a^*)$  is compact, then  $b^* \triangleleft a^*$ .  
 Otherwise  $\uparrow b$  is compact, which implies that  $\uparrow(b^{**})$  is compact, since  $b \leq b^{**}$ , and  $b^* \triangleleft a^*$ .

(S6) For each  $a \in L$ ,  $a = \bigvee\{x \in L \mid x \triangleleft a\}$ : This follows from the fact that  $L$  is continuous and the observation that  $x \ll a$  implies  $x \triangleleft a$ .

Finally we show that  $\triangleleft$  is the smallest strong inclusion on  $L$ . Let  $\blacktriangleleft$  be another strong inclusion on  $L$  and suppose that  $a \triangleleft b$ . Then  $a \prec b$  and either  $\uparrow(a^*)$  or  $\uparrow b$  is compact. If it is the case that  $\uparrow(a^*)$  is compact, then  $a \ll b$  and we use the property (S6) of  $\blacktriangleleft$  to find an element  $z \in L$  with  $z \blacktriangleleft b$  and  $a \leq z \blacktriangleleft b$ . Hence  $a \blacktriangleleft b$ . If on the other hand, we have that  $\uparrow b$  is compact, then since  $a \prec b$ , there exists an element  $s \in L$  satisfying  $a \wedge s = \mathbf{0}$  and  $s \vee b = \mathbf{e}$ . Again using the property (S6) of  $\blacktriangleleft$  and compactness of  $\uparrow b$ , we find an element  $z \in L$  with  $z \blacktriangleleft s$  and  $z \vee b = \mathbf{e}$ . Hence  $a \leq s^* \blacktriangleleft z^* \leq b$  which establishes the required result.  $\square$

## Chapter 6. A Comparison of the Compactifications

In this chapter, we investigate some of the relationships which exist between the compactifications we have seen. We are interested in conditions imposed on the frame, and (or) conditions imposed on its basis, under which different means of compactification yield the same result.

It was shown in Chapter 2 that every compactification by means of a Fan-Gottesman normal basis  $B$ , can be obtained by means of the naturally associated normal basis  $N_B$ . The converse however is false — see Banaschewski [5] page 55.

In [23], Marcus shows that for a completely regular frame  $L$ , the collection  $\text{Coz } L$ , of cozero elements of  $L$ , (i.e., those elements  $a \in L$  which can be expressed as  $a = \bigvee a_n$  for some sequence  $(a_n)$  in  $L$  with  $a_i \prec\prec a_{i+1}$  for each  $i \in \mathbb{N}$ ), is a normal subfitting base for  $L$  whose associated relativised Wallman compactification is the Stone-Čech compactification.

**Proposition 6.1.** *Let  $L$  be a normal regular frame. Then the following compactifications coincide with the universal compactification:*

- (i) *The compactification  $h: \mathfrak{K} \longrightarrow L$  relative to the normal basis  $N_L = \{a \in L \mid a = a^{**}\}$ .*
- (ii) *The Fan-Gottesman compactification  $\gamma_L L$ .*
- (iii) *The Wallman compactification  $\bigvee: (\mathfrak{J}(L))_s \longrightarrow L$ .*

*Proof.* That (ii) and (iii) coincide with the Stone-Čech compactification is proved in [2]. To prove that (i) is the Stone-Čech compactification, we note that  $N_L$  is a normal base, and we appeal to Proposition 5.1, which states that  $h: \mathfrak{K} \longrightarrow L$  is the Stone-Čech compactification if the right adjoint  $h_*$  preserves finite joins.

It will be useful at this stage to make the following observation: For each  $a \in L$ ,

$$h_*(a) = \bigcup_{\substack{p \in N_L \\ p \prec a}} s(p^*):$$

We know that  $h_*(a) = \bigvee \{\mathcal{F} \in \mathfrak{R} \mid h(\mathcal{F}) = a\}$ . Now since

$$h\left(\bigcup_{\substack{p \in N_L \\ p \prec a}} s(p^*)\right) = \bigvee_{\substack{p \in N_L \\ p \prec a}} h(s(p^*)) = \bigvee_{\substack{p \in N_L \\ p \prec a}} p = a,$$

it follows that  $h_*(a) \supseteq \bigcup_{\substack{p \in N_L \\ p \prec a}} s(p^*)$ .

For the reverse inclusion, we take a regular filter  $\mathcal{F} \in \mathfrak{R}$  with  $h(\mathcal{F}) = a$  and recall from the proof of Lemma 1.3 that  $\mathcal{F}$  may be written as  $\mathcal{F} = \bigcup_{x \in \mathcal{F}} \bigcup_{x \prec y} s(y)$ . Let  $z \in \mathcal{F}$ . Then there exist elements  $x \in \mathcal{F}$  and  $y \in N_L$  such that  $x \prec y \prec z$ . Now  $x \in \mathcal{F}$  implies that  $x^* \leq a$  since  $h(\mathcal{F}) = a$ . Hence  $z \in s(y) = s(y^{**})$  where  $y^* \prec x^* \leq a$  and  $y^* \in N_L$ ,

$$\text{i.e., } z \in \bigcup_{\substack{p \in N_L \\ p \prec a}} s(p^*).$$

Having made this observation, we have to show the non-trivial inclusion:

$$\begin{aligned} h_*(a \vee b) &\subseteq h_*(a) \vee h_*(b) \\ \text{i.e., } \bigvee \{\mathcal{F} \in \mathfrak{R} \mid h(\mathcal{F}) = a \vee b\} &\subseteq \bigcup_{\substack{p \in N_L \\ p \prec a}} s(p^*) \vee \bigcup_{\substack{q \in N_L \\ q \prec b}} s(q^*) \end{aligned}$$

Take  $\mathcal{F} \in \mathfrak{R}$  with  $h(\mathcal{F}) = a \vee b$ . This means that if  $z \in \mathcal{F}$ , then  $z^* \leq a \vee b$ .

$$\begin{aligned} \text{Now } x \in \mathcal{F} &\implies \exists y \in \mathcal{F} \text{ with } y \prec x \quad \text{since } \mathcal{F} \in \mathfrak{R} \\ &\implies x^* \prec y^* \leq a \vee b \\ &\implies x^* \prec a \vee b \\ &\implies x^* \prec s \vee t \text{ where } s \prec c \prec a \text{ and } t \prec d \prec b \\ &\quad \text{by repeated application of Lemma 0.3} \\ &\implies (s \vee t)^* = s^* \wedge t^* \prec x \\ &\quad \text{where } s^{**} \prec c^{**} \prec a \text{ and } t^{**} \prec d^{**} \prec b. \end{aligned}$$

Hence  $s^* \wedge t^* \in \bigcup_{\substack{p \in N_L \\ p \prec a}} s(p^*) \vee \bigcup_{\substack{q \in N_L \\ q \prec b}} s(q^*)$  and since  $x \geq s^* \wedge t^*$  we have that

$$x \in \bigcup_{\substack{p \in N_L \\ p \prec a}} s(p^*) \vee \bigcup_{\substack{q \in N_L \\ q \prec b}} s(q^*) \quad \text{as required.} \quad \square$$

**Proposition 6.2.** *Let  $L$  be a zero-dimensional frame. Then the following compactifications coincide with the zero-dimensional compactification:*

- (i) *The compactification  $h: \mathfrak{K} \rightarrow L$  relative to the normal basis  $N = \mathcal{C}L$ .*
- (ii) *The Fan-Gottesman compactification  $\gamma_{\mathcal{C}L}$ .*
- (iii) *The relativised Wallman compactification  $\downarrow: (\mathcal{J}(\mathcal{C}L))_s \rightarrow L$ .*

*Proof.*

- (i) It is easily seen that  $\mathcal{C}L$  is a normal basis for  $L$ , so to show that the compactification  $h: \mathfrak{K} \rightarrow L$  is the zero-dimensional compactification, we show that the following associated strong inclusions coincide.

- $a \triangleleft_h b \iff \exists c, d \in \mathcal{C}L$  such that  $a \leq c \prec d \leq b$ .
- $a \triangleleft_{\downarrow} b \iff \downarrow a \cap \mathcal{C}L \prec \downarrow b \cap \mathcal{C}L$  in  $\mathcal{J}(\mathcal{C}L)$ .

Suppose that  $a \triangleleft_h b$ . Then  $a \wedge c^c = \mathbf{0}$  and  $c^c \vee d = \mathbf{e}$ . Now  $\downarrow c^c \cap \mathcal{C}L \in \mathcal{J}(\mathcal{C}L)$  and it is clear that  $(\downarrow a \cap \mathcal{C}L) \cap (\downarrow c^c \cap \mathcal{C}L) = \{\mathbf{0}\}$ , and  $(\downarrow c^c \cap \mathcal{C}L) \vee (\downarrow b \cap \mathcal{C}L) = \mathcal{C}L$ . So  $a \triangleleft_{\downarrow} b$ .

Conversely, suppose that  $a \triangleleft_{\downarrow} b$ . Then there exists an ideal  $\mathcal{J} \in \mathcal{J}(\mathcal{C}L)$  such that

$$(\downarrow a \cap \mathcal{C}L) \cap \mathcal{J} = \{\mathbf{0}\} \text{ and } \mathcal{J} \vee (\downarrow b \cap \mathcal{C}L) = \mathcal{C}L.$$

The latter implies that there exist elements  $s \in \mathcal{J}$  and  $t \in (\downarrow b \cap \mathcal{C}L)$  such that  $t \vee s = \mathbf{e}$ . Further,

$$\begin{aligned} s \wedge a &= s \wedge \bigvee \{x \in \mathcal{C}L \mid x \leq a\} && \text{since } L \text{ is generated by } \mathcal{C}L \\ &= \bigvee \{x \wedge s \mid x \in \mathcal{C}L \text{ and } x \leq a\} \\ &= \bigvee \{\mathbf{0}\} && \text{since } (\downarrow a \cap \mathcal{C}L) \cap \mathcal{J} = \{\mathbf{0}\} \\ &= \mathbf{0}. \end{aligned}$$

So  $a \leq s^* \prec t \leq b$  with  $s^*, t \in \mathcal{C}L$  and hence  $a \triangleleft_h b$ .

(ii)  $\mathcal{C}L$  is clearly a Fan–Gottesman normal base for  $L$ , and moreover each element of  $\mathcal{C}L$  is regular, which means that  $\mathcal{C}L$  is a normal basis. So by the remark following Proposition 2.4, it follows that the Fan–Gottesman compactification of  $L$  is the normal base compactification. Hence the result follows from (i).

(iii) We first have to show that  $\mathcal{C}L$  is a normal subfitting basis for  $L$ . The only fact which is not immediately obvious is that  $L$  is  $\mathcal{C}L$ -subfit. So suppose that for all  $x \in \mathcal{C}L$ ,  $a \vee x = e \implies b \vee x = e$ . Then for  $s \in \mathcal{C}L$ ,

$$s \leq a \implies a \vee s^c = e \implies b \vee s^c = e \implies s \leq b.$$

Hence  $a = \bigvee \{s \in \mathcal{C}L \mid s \leq a\} \leq \bigvee \{t \in \mathcal{C}L \mid t \leq b\} = b$  and  $L$  is  $\mathcal{C}L$ -subfit.

To prove that the compactifications coincide, it suffices to show that

$$(\mathcal{J}(\mathcal{C}L))_s = \mathcal{J}(\mathcal{C}L).$$

Let  $\mathcal{J} \in \mathcal{J}(\mathcal{C}L)$ . We have to show that  $s(\mathcal{J}) \subseteq \mathcal{J}$ ,

$$\text{i.e., } \bigvee \{\mathcal{J}_i \in \mathcal{J}(\mathcal{C}L) \mid \mathcal{J}_i \text{ is } \mathcal{J}\text{-small}\} \subseteq \mathcal{J}.$$

Now  $x \in s(\mathcal{J}) \implies x = x_1 \vee x_2 \vee \dots \vee x_n$  where  $x_i \in \mathcal{J}_i$  and  $\mathcal{J}_i$  is  $\mathcal{J}$ -small. So

$$\begin{aligned} x_i \in \mathcal{J}_i &\implies (\downarrow x_i^c \cap \mathcal{C}L) \vee \mathcal{J}_i = \mathcal{C}L \\ &\implies (\downarrow x_i^c \cap \mathcal{C}L) \vee \mathcal{J} = \mathcal{C}L \quad \text{since } \mathcal{J}_i \text{ is } \mathcal{J}\text{-small} \\ &\implies x_i \in \mathcal{J} \end{aligned}$$

It thus follows that  $x = x_1 \vee x_2 \vee \dots \vee x_n \in \mathcal{J}$  and  $s(\mathcal{J}) \subseteq \mathcal{J}$  which is the non-trivial inclusion.  $\square$

That the latter two coincide was suggested to me by Professor Bernard Banaschewski. The equivalence of these two with the remaining compactifications is original.

The following result concerning the relationship between the Fan–Gottesman and minimal (Alexandroff) compactifications is due to Baboolal [2].

**Proposition 6.3.** *Let  $L$  be a regular continuous frame and let*

$$B = \{a \in L \mid \text{either } \uparrow a \text{ or } \uparrow(a^*) \text{ is compact}\}.$$

*Then  $B$  is a Fan–Gottesman normal base for  $L$  and the Fan–Gottesman compactification relative to  $B$  is the minimal compactification of  $L$ .*

*Proof.* We first establish that  $B$  is a Fan–Gottesman normal base.

(FG1) Suppose that  $a, b \in B$ . There are two possible cases:

- If both  $\uparrow a$  and  $\uparrow b$  are compact, then  $\uparrow(a \wedge b)$  is compact and  $a \wedge b \in B$ .
- If  $\uparrow(a^*)$  or  $\uparrow(b^*)$  is compact, then  $\uparrow((a \wedge b)^*)$  is compact and  $a \wedge b \in B$ .

(FG2) Suppose that  $a \in B$ . If  $\uparrow a$  is compact, then  $\uparrow(a^{**})$  is compact, and hence  $a^* \in B$ . If on the other hand  $\uparrow(a^*)$  is compact, then immediately  $a^* \in B$ .

(FG3) Suppose that  $a \prec c$  with  $a \in B$  and  $c \in L$ . We consider two cases:

- If  $\uparrow a$  is compact, then  $\uparrow c$  is compact. Now  $a \prec c$  implies that  $a^* \vee c = \mathbf{e}$  or using the continuity of  $L$ , we have  $\bigvee\{x \vee c \mid x \ll a^*\} = \mathbf{e}$ . Using the compactness of  $\uparrow c$ , we have that there exists an element  $x \in L$  satisfying  $x \ll a^*$  and  $x \vee c = \mathbf{e}$ . Further,

$$\begin{aligned} x \ll a^* &\implies x \prec a^* \text{ and } \uparrow(x^*) \text{ is compact} \\ &\implies a \leq a^{**} \prec x^* \prec c \quad \text{with } x^* \in B. \end{aligned}$$

- Suppose that  $\uparrow(a^*)$  is compact. Then  $a \ll c$  and since  $L$  is continuous, the way below relation interpolates, so there exists an element  $b \in L$  such that  $a \ll b \ll c$ . Now  $b \ll c$  implies that  $b \prec c$  and  $\uparrow(b^*)$  is compact, the latter implying that  $b \in B$ . Hence  $a \prec b \prec c$  as required.

That  $B$  generates  $L$  follows from the fact that  $L$  is continuous and the observation that for any  $a, x \in L$ ,  $x \ll a \implies x \in B$ .

So  $B$  is a Fan–Gottesman normal base for  $L$ . To show that the Fan–Gottesman compactification is the minimal compactification, we prove that the associated strong inclusions  $\triangleleft_B$  and  $\triangleleft$  are equivalent. That is,

$$\exists b \in B \text{ such that } a \prec b \prec c \iff a \prec c \text{ and either } \uparrow(a^*) \text{ or } \uparrow c \text{ is compact.}$$

$\implies$  Suppose that  $a \triangleleft_B c$  so that there exists an element  $b \in B$  such that  $a \prec b \prec c$ . Then we have that  $a \prec c$ . Suppose that  $\uparrow b$  is compact. Then  $\uparrow c$  is compact, since  $b \leq c$ . On the other hand, if  $\uparrow(b^*)$  is compact, then  $\uparrow(a^*)$  is compact since  $b^* \leq a^*$ .

$\impliedby$  Suppose that  $a \triangleleft c$ . Using the fact that the strong inclusion interpolates, there exist elements  $x, y \in L$  such that  $a \triangleleft x \triangleleft y \triangleleft b$ . If  $\uparrow(x^*)$  is compact, then  $x \in B$  and we have  $a \prec x \prec b$ . If on the other hand we have that  $\uparrow y$  is compact, then  $y \in B$  and we have  $a \prec y \prec b$   $\square$

We now examine the relationship between the relativised Wallman compactification and that by means of a normal basis. A normal basis  $N$  for  $L$  need not in general be a normal subfitting basis for  $L$ , since we do not require that  $N$  be closed under finite joins. Similarly, a normal subfitting basis  $A$  for  $L$  need not be a normal basis, since it need not be closed under pseudocomplements, and we do not require that its elements be regular. If, however, we enforce these additional constraints, we find that the respective compactifications coincide:

**Proposition 6.4.**

- (i) Let  $N$  be a normal basis for  $L$ , closed under finite joins. Then  $N$  is a normal subfitting basis for  $L$  whose Wallman compactification coincides with that by means of the normal basis.
- (ii) Let  $A$  be a normal subfitting basis for  $L$  closed under pseudocomplements, and let  $N_A = \{a \in A \mid a = a^{**}\}$ . Then  $N_A$  is a normal basis whose compactification coincides with the Wallman compactification relative to  $A$ .

*Proof.*

- (i) That  $N$  is normal, follows from (N3). To see that  $N$  is subfit, we suppose that for all  $n \in N$ ,

$$a \vee n = e \implies b \vee n = e,$$

and show that  $a \leq b$ . To this end, we take  $x \in N$  with  $x \prec a$ . Then  $x^* \in N$  and  $x^* \vee a = e$  which implies  $x^* \vee b = e$  by our assumption above. Hence we have, as required,

$$a = \bigvee \{x \in N \mid x \prec a\} \leq \bigvee \{y \in N \mid y \prec b\} = b.$$

To show that the compactifications coincide, we show that the following associated strong inclusions coincide.

- $a \triangleleft_h b \iff \exists c, d \in N$  such that  $a \leq c \prec d \leq b$ .
- $a \triangleleft_\vee b \iff \downarrow a \cap N \prec \downarrow b \cap N$  in  $(\mathcal{J}N)_s$ .

Suppose that  $a \triangleleft_h b$ . Then  $(\downarrow a \cap N) \wedge (\downarrow c^* \cap N) = \{\mathbf{0}\}$  and  $(\downarrow c^* \cap N) \vee (\downarrow b \cap N) = N$  since  $a \wedge c^* = \mathbf{0}$  and  $c^* \vee d = e$  respectively. Recall that  $L$  is  $A$ -subfit if and only if every principal ideal of  $A$  is saturated. Hence  $\downarrow c^* \cap N \in (\mathcal{J}(N))_s$ , and we have  $\downarrow a \cap N \prec \downarrow b \cap N$  in  $(\mathcal{J}(N))_s$ , as required.

Suppose that  $a \triangleleft_\vee b$ . Then there exists an ideal  $\mathcal{J} \in (\mathcal{J}(N))_s$  such that

$$(\downarrow a \cap N) \wedge \mathcal{J} = \{\mathbf{0}\} \text{ and } \mathcal{J} \vee (\downarrow b \cap N) = N.$$

The latter implies that there exist elements  $x \in \mathcal{J}$  and  $n \in N$  with  $n \leq b$  such that  $x \vee n = e$ . Further,

$$\begin{aligned} x \wedge a &= x \wedge \bigvee \{y \in N \mid y \prec a\} && \text{since } N \text{ is a normal base for } L \\ &= \bigvee \{x \wedge y \mid y \in N \text{ and } y \prec a\} \\ &= \bigvee \{\mathbf{0}\} && \text{since } (\downarrow a \cap N) \wedge \mathcal{J} = \{\mathbf{0}\} \text{ and } x \in \mathcal{J} \\ &= \mathbf{0} \end{aligned}$$

Hence  $a \leq x^* \prec n \leq b$  with  $x^*, n \in N$  which proves that  $a \triangleleft_h b$ .

(ii) It is clear that  $N_A$  is a normal base for  $L$ . In order to establish that the compactifications are the same, we show once again that the following associated strong inclusions are equivalent.

- $a \triangleleft_h b \iff \exists c, d \in N_A$  such that  $a \leq c \prec d \leq b$ .
- $a \triangleleft_V b \iff \downarrow a \cap N \prec \downarrow b \cap N$  in  $(\mathfrak{J}(N))_s$ .

The proof that  $a \triangleleft_h b$  implies  $a \triangleleft_V b$  is essentially the same as that above.

The proof of the converse, although similar to that above, requires a slight modification at the end, since the elements of  $A$  are not necessarily regular. Suppose that  $a \triangleleft_V b$ . Then there exists an ideal  $\mathfrak{J} \in (\mathfrak{J}(A))_s$  such that

$$(\downarrow a \cap A) \wedge \mathfrak{J} = \{\mathbf{0}\} \text{ and } \mathfrak{J} \vee (\downarrow b \cap A) = A.$$

So there exist elements  $x \in \mathfrak{J}$  and  $s \in A$  with  $s \leq b$  such that  $s \vee x = e$ . Since  $A$  is normal, there exist elements  $p, q \in A$  such that  $p \wedge q = \mathbf{0}$  and  $p \vee s = q \vee x = e$ . So  $x^* \prec q \prec s$  and hence  $x^* \prec q^{**} \prec s$ . Now

$$\begin{aligned} x \wedge a &= x \wedge \bigvee \{y \in A \mid y \leq a\} && \text{since } A \text{ is a base for } L \\ &= \bigvee \{x \wedge y \mid y \in A \text{ and } y \leq a\} \\ &= \bigvee \{\mathbf{0}\} && \text{since } (\downarrow a \cap A) \wedge \mathfrak{J} = \{\mathbf{0}\} \\ &= \mathbf{0} \end{aligned}$$

Hence  $a \leq x^* \prec q^{**} \prec s \leq b$  with  $x^*, q^{**} \in N_A$ .  $\square$

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