Equilibration of a Scalar Field Theory
A Comparison of Methods

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Abstract

This thesis investigates tools to study the extreme and very dynamical environment created after a heavy ion collision. This situation is well described by the Colour Glass Condensate framework. In particular, this thesis will investigate two different methods that are able to work within such an environment, the Classical-Statistical method and the 2 Particle Irreducible Method, and aims to compare and contrast these two methods analytically. A scalar ϕ^4 theory is used to implement the discussion in order to concentrate on core issues without the added complications contained in a gauge theory such as QCD.

Declaration

I know the meaning of plagiarism and declare that all of the work in this thesis, save for that which is properly acknowledged, is my own.

Jason Myers

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Contents

Introduction

The Large Hadron Collider is a huge particle collider that, among other things, collides heavy ions together at massive energies. In such heavy ion collisions a new state of matter called the Quark Gluon Plasma (QGP) is formed where, in this state, quarks are no longer confined to hadrons but are free to move about. Studying this state is expected to lead to a far greater understanding of QCD as this is the only known state where one can find free quarks and gluons, the degrees of freedom of QCD. The QGP is known to be a very hot, very dense medium that is rapidly expanding, so one needs techniques that allow for the study of a Quantum Field Theory in an extreme and very dynamical environment. [1].

One such technique capable of describing a hot, densely populated system is hydrodynamics. The behaviour of the QGP in heavy ion collisions can be well described by hydrodynamical models [1]. This is very interesting as hydrodynamics can only work in specific conditions, particularly one needs to be describing a locally thermalised system [1]. This suggests that the QGP must be in local thermal equilibrium [1, 2]. A very curious fact is that hydrodynamics is successful from very early times ($\sim 1fm/c$) after a heavy ion collision, this implies that local thermalization is achieved in a remarkably short time [1].

The initial state of the collision, before the QGP is formed, is determined by the wave functions of the incoming nuclei which are dominated by small x effects and can be discussed in the Colour Glass Condensate (CGC) framework [3, 4]. A crucial gap in our understanding lies in precisely how the system can evolve from this unthermalized state towards the thermalized state described well by hydrodynamics in such a minuscule time frame. The state in between the CGC and the QGP is known as the Glasma [3, 4].

The CGC framework describes a situation where there are strong physical sources that generate field excitations [4, 5]. This situation leads to complicated behaviour within the system that is vastly different to a system in which there are no sources.

This thesis investigates two different approaches to describing this thermalization process in the presence of physical sources. The first method, based on the work of [2, 5] is known as the Classical-Statistical Method. The second method is known as the 2 Particle Irreducible Method. This method was first developed as an extension to the 1 Particle Irreducible Effective Action [6] and then linked to a finite temperature frame work in [7].

The purpose of this thesis is to ascertain exactly how each of the above methods deals with the exceptionally complicated system found after a heavy ion collision, in particular it will be shown how each of the methods incorporates evolutionary dynamics within the background of physical sources. Furthermore, the difference between how these two methods achieve the required dynamics is investigated.

This thesis will not aim to calculate numerical solutions but rather study the analytical properties of both methods. For the purposes of simplicity and clarity, QCD itself is not used but rather the scalar ϕ^4 theory.

To investigate this scalar field far from equilibrium, this thesis utilises the Schwinger-Keldysh formalism. This formalism may be viewed as a generalization of the real time construction of a finite temperature field theory that easily takes into account both equilibrium and non-equilibrium situations [8].

This thesis will start from an understanding of field theory in the language Bra-Ket notation and work forward. There are 5 chapters where the idea is to move from textbook, zero temperature content through to the two methods under investigation. Throughout this work, diagrammatic notation is regularly used as it is often easier to gain an understanding through this representation.

In chapter 1 the ϕ^4 theory used as a stand-in for QCD (as motivated above) will be defined along with the conventions used throughout the thesis. The path-integral formulation of QFT is derived followed by a discussion on what correlations look like in this formulation. The generating functional for all n-point functions is presented together with an interpretation of Feynmann diagrams. Following this, a derivation of the Schwinger-Dyson equations for the connected one, two and three point functions will be given in the presence of a strong source. Finally there is a discussion on loop expansions and classical vs. quantum objects followed by an argument that shows that classical objects are parametrically dominant within the context of strong physical sources.

Chapter 2 is about effective actions. First the need for an effective action is discussed, then both the 1 Particle Irreducible and 2 Particle Irreducible effective actions are derived in the presence of a strong physical source. Both derivations are followed by discussions on some important features of the respective effective actions.

Chapter 3 investigates a system experiencing collective effects using the Schwinger-Keldysh time contour. First the Schwinger-Keldysh contour is motivated, followed up by a derivation of a Schwinger-Keldysh generating functional. This generating functional will allow for the discussion of an out of equilibrium theory in the presence of physical sources. From here, this chapter presents an alternative choice of fields that are linear combinations of the usual Schwinger-Keldysh fields. These new fields will be shown to have a natural interpretation in terms of classical fields and quantum fluctuations in the presence of strong sources. Finally this chapter will derive a particular relationship that must exist for a system that is in thermal equilibrium.

Chapter 4 discusses the Classical-Statistical method. It is shown that this method is uniquely suited to a non-equlibrium theory with physical sources. What this method does is to identify and track the dominating equilibrating fluctuations in the presence of strong sources. These fluctuations are then resummed and in doing so equilibration is achieved. However, the limiting distribution in equilibration is only a semi-classical approximation to the true quantum distribution, which implies that certain quantum effects are not being taken into account by the resummation.

Chapter 5 discusses the 2 Particle Irreducible method which builds from the work done in chapter 2. By linking the idea of the 2 Particle Irreducible action to the Schwinger-Keldysh formulation, one can find equations of motion that apply in a non-equilibrium situation. It is shown that this method is capable of reproducing the full equilibrium condition. However, this method's predictions of the equilibration process are subject to large uncertainties.

Chapter 1

Introductory Concepts

1.1 Definitions and Conventions

For this thesis a ϕ^4 theory will be used as the model to gain insight into a system that is undergoing rapid thermalization within the presence of strong physical sources. This theory is defined by the following Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi(x)\right)^2 - \frac{1}{2} m^2 \phi^2(x) - \frac{g^2}{4!}\phi^4(x) \; . \tag{1.1}$$

From this the action is given by

$$S[\phi] = \int dx^4 \left\{ \left(\frac{1}{2}\partial_\mu \phi(x)\right)^2 - \frac{1}{2} m^2 \phi^2(x) - \frac{g^2}{4!}\phi^4(x) \right\} \; . \tag{1.2}$$

Using integration by parts, this can also be written as

$$S[\phi] = \int dx^4 \left\{ -\frac{1}{2}\phi(x)\left(\Box_x + m^2\right)\phi(x) - \frac{g^2}{4!}\phi^4(x) \right\} \; . \tag{1.3}$$

The term $-\frac{g^2}{4!}\phi^4(x)$ represents the interactions in this theory. Since this thesis will be exploring techniques that can be applied to the perturbative regime of QCD, g^2 is assumed to be a small parameter such that one can use perturbation theory. This is in accordance with the CGC setting of QCD as discussed in the introduction.

The term $-\frac{1}{2}\phi(x)\left(\Box_x + m^2\right)\phi(x)$ represents the kinetic term in the theory.

In a free field theory there are no interactions and only the kinetic term would be left. In other words a free field theory is the same as above except with $g^2 = 0$.

1.1.1 List of Notations and Comments

- The free propagator is defined as G_0 where,
  - ⋆ $(\Box_x + m^2)G_0(x, y) = i\delta(x - y)$
  - ⋆ $-i(\Box_x + m^2) = G_0^{-1}$
- Space-time points will be introduced in the definition of all objects. Subsequently, for clarity, space-time points may not be presented but will always appear if important to the discussion.
- A functional that depends on a single space-time argument can be thought of a an infinite dimensional vector. A functional that depends on two space-time arguments can be interpreted as an infinite dimensional matrix.
  - ⋆ If A and B are both infinite dimensional matrices then $A \times B = \int d^4z A(x, z)B(z, y)$.
  - ⋆ The the trace of an object $A(x, y)$ is given as $\mathrm{Tr}A(x, y) = \int dx A(x, x)$.

1.2 The Path Integral

The path integral formalism of quantum field theory (QFT) is an alternative to the usual Bra-Ket notation of expressing a transition amplitude. In this new formulation, the transition amplitude becomes an integral over an exponential of the Lagrangian density of the theory. This is not a usual integral as it will turn out to be a functional integral, an integration over functions. For this thesis, it is best to utilize the path integral formulation.

Following [9], the derivation is started for a non-relativistic theory with a time dependent Hamiltonian, the result can then be generalized to a quantum field theory. The proof presented is not mathematically rigorous but the results have been shown to be valid.

The outline of the proof is briefly discussed here. One begins with a NRQM amplitude in the bra-ket notation, then this amplitude is split up into N small amplitudes by considering the time dependent Hamiltonian and using functional $\mathbb{1}$'s (equation (1.10) below). This is followed by simplifying one of the small amplitudes after generalizing the proof to deal with a quantum field theory (equation (1.25) below). The splitting up of the transition amplitude gives the functional integration. Then multiplying the N small amplitudes one finds the path integral representation of a quantum field theory amplitude in equation (1.30) below.

The Proof

Let Q be a generalized co-ordinate and let P be its corresponding conjugate momentum. When dealing with operators for Q and P the obvious notation will be used i.e. the operators are $\hat{Q}$ and $\hat{P}$.

From non- relativistic Quantum Mechanics it is known that

$$|q, t\rangle = e^{i\hat{H}t}|q\rangle_s \tag{1.4}$$

where the s represent the state in the Schrdinger Picture and $\hat{H}$ is the Hamiltonian operator.

Given the above expression, the probability amplitude for a system in state $|q, t\rangle$ (at time t) to go to a state $|q', t'\rangle$ (at time t') is

$$\langle q', t'|q, t\rangle = \langle q'|e^{-i\hat{H}(t'-t)}|q\rangle \tag{1.5}$$

In field theory one deals with the time dependent Hamiltonian. For a time dependent Hamiltonian the above expression becomes

$$\langle q', t'|q, t\rangle = \langle q'|Te^{-i\int\limits_t^{t'} d\tau \hat{H}(\tau)}|q\rangle \tag{1.6}$$

where the T in the above stands for time ordering.

A simplification can be made for the time dependent Hamiltonian such that

$$Te^{-i\int\limits_t^{t'} d\tau \hat{H}(\tau)} \simeq Te^{-i\sum\limits_{i=0}^{N} \Delta t \hat{H}(t_i)} \ . \tag{1.7}$$

Note that $t_0 = t$ and $t_N = t'$ and that$\Delta\ t \equiv \frac{t'-t}{N+1}$. Δt must be very small to make this approximation valid. For the rest of the section let$\Delta\ t \equiv \epsilon$. Since the time slices are equal $\epsilon = t_{i+1} - t_i$.

Now consider that time ordering tells one that objects must be ordered strictly according to their time. One reads time from right to left for an amplitude and t_0 is the earliest time whilst t' is the latest. Thus,

$$Te^{-i\int\limits_t^{t'} d\tau \hat{H}(\tau)} \simeq e^{-i\hat{H}(t')\epsilon}...e^{-i\hat{H}(t_1)\epsilon}e^{-i\hat{H}(t)\epsilon} \tag{1.8}$$

where ϵ goes very small and there are $N + 1$ terms in the product.

The above expression can now be put back into equation (1.6) to get that

$$\langle q', t'|q, t\rangle = \langle q'|e^{-i\hat{H}(t')\epsilon}...e^{-i\hat{H}(t_1)\epsilon}e^{-i\hat{H}(t)\epsilon}|q\rangle \ . \tag{1.9}$$

A complete set of states can be inserted at any position as it acts as a functional $\mathbb{1}$. The complete set of states is given by $\int dq_i|q_i\rangle\langle q_i|$. This can be used to make the above nested convolution of Hamiltonians into a product of transition amplitudes.

Thus

$$\begin{aligned}\langle q', t'|q, t\rangle &= {}_s\langle q'|e^{-i\hat{H}(t')\epsilon}\left(\int dq_{N-1}|q_{N-1}\rangle\langle q_{N-1}|\right)... \\ &\quad \times\left(\int dq_2|q_2\rangle\langle q_2|\right)e^{-i\hat{H}(t_1)\epsilon}\left(\int dq_1|q_1\rangle\langle q_1|\right)e^{-i\hat{H}(t)\epsilon}|q\rangle \\ &= \prod_{i=1}^{N}\int dq_i\langle q'|e^{-i\hat{H}(t')\epsilon}|q_{N-1}\rangle...\langle q_2|e^{-i\hat{H}(t_1)\epsilon}|q_1\rangle\langle q_1|e^{-i\hat{H}(t)\epsilon}|q\rangle \\ &.\end{aligned} \tag{1.10}$$

By looking through the literature, one can show that the above expression for $\langle q', t'|q, t\rangle$ is obtained whether one considers a time dependent Hamiltonian or not [10]. So to derive a path integral expression for equation (1.9) it is sufficient to discuss Hamiltonians of the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \ . \tag{1.11}$$

Although some generality is lost, the Hamiltonians of Quantum Field Theory are all of this general form.

Note that simplifications occur for Hamiltonians where there are no products of $\hat{p}$ with $\hat{q}$. [9]

Now let N go very large, remember that ϵ (the length of the $N+1$ time slices) is very small. One starts by considering only one slice of equation (1.10),

$$\langle q_{j+1}|e^{-i\hat{H}\epsilon}|q_j\rangle \ . \tag{1.12}$$

From here the usual procedure found in most text books such as [9] or [10] is to consider expanding $e^{-i\hat{H}\epsilon}$ around the small ϵ. Then one can act with the second term of the expansion on the position state $|q_j\rangle$. This produces the eigenvalue $-\frac{1}{2m}\frac{\partial^2}{\partial q_j^2} + V(q_j)$ acting on the term $\delta(q_{j+1}-q_j)$. This $\delta(q_{j+1}-q_j)$ is re-written as a momentum integral. Then one can act with $-\frac{1}{2m}\frac{\partial^2}{\partial q_j^2} + V(q_j)$ on the integral and compute the integral.

The method outlined above has a glaring problem. The momentum integral one gets to is given as $-i\int_{-\infty}^{+\infty} dp_j \epsilon \left[\left(\frac{1}{2m}p_j^2 + V(q_j)\right) e^{ip_j(q_{j+1}-q_j)}\right]$. This integral is divergent as the integral over p_j^2 is divergent. The small ϵ isn't enough to control this divergence and thus using the limit of a small ϵ is not sufficient to ensure convergence, despite the popularity of this argument.

A different method will be used here, it is based on [11] but the proof done there is amended slightly to better explain how their solution controls this problem in the momentum integral.

Starting off where the other proofs go awry, a complete set of momentum states is inserted

$$\langle q_{j+1}|e^{-i\hat{H}\epsilon}|q_j\rangle = \int_{-\infty}^{\infty} dp_i \langle q_{j+1}|p_i\rangle\langle p_i|e^{-i\hat{H}\epsilon}|q_j\rangle \ . \tag{1.13}$$

The first term $\langle q_{j+1}|p_i\rangle$ is simply a phase factor well known as $e^{iq_{j+1}p_i}$. Next the term $\langle p_i|e^{-i\hat{H}\epsilon}|q_j\rangle$ is discussed.

It would be great if one could order $e^{-i\hat{H}\epsilon}$ such that only momentum operators acted on the left and position operators on the right. Remember the Hamiltonian under investigation is of the form given in equation (1.11). So what one wants is to write [1]

$$e^{-i\hat{H}\epsilon} = e^{-i\epsilon\frac{\hat{p}^2}{2m}} \left\{1 + A[\hat{p}, V(\hat{q})]\right\} e^{-i\epsilon V(\hat{q})} \ . \tag{1.14}$$

The question is, what is $A[\hat{p}, V(\hat{q})]$. One can find $A[\hat{p}, V(\hat{q})]$ from a power series where

[1] It is interesting to note that the following splitting is reminiscent of the splitting of the Hamiltonian that is used to justify the numerical method known as the leapfrog algorithm. This algorithm is used to numerically solve Hamilton's equations. This same splitting occurs in more sophisticated simplectic solvers too.

$$\sum_{n=0}^{\infty}\frac{(-i\epsilon)^n}{n!}\hat{H}^n=\sum_{k=0}^{\infty}\frac{(-i\epsilon)^n k}{k!}\left(\frac{\hat{p}^2}{2m}\right)^k\left[1+\sum_{l=2}^{\infty}\frac{(-i\epsilon)^l}{l!}\Delta^{(l)}\right]\sum_{m=0}^{\infty}\frac{(-i\epsilon)^m}{m!}\left(V(\hat{q})\right)^n \tag{1.15}$$

One then matches up powers of ϵ and solve for theΔ $^{(l)}$ in this way. This proved to be a bit tricky, it was found that

$$\Delta^{(2)}=-\left[\frac{\hat{p}^2}{2m},V(\hat{q})\right] \tag{1.16}$$

$$\Delta^{(3)}=\left[\frac{\hat{p}^2}{2m},\left[\frac{\hat{p}^2}{2m},V(\hat{q})\right]\right]+\left[\left[\frac{\hat{p}^2}{2m},V(\hat{q})\right],V(\hat{q})\right] \tag{1.17}$$

$$\begin{aligned}\Delta^{(4)}=&\left[\frac{\hat{p}^2}{2m},\left[\frac{\hat{p}^2}{2m},\left[\frac{\hat{p}^2}{2m},V(\hat{q})\right]\right]\right]+\left[\left[\left[\frac{\hat{p}^2}{2m},V(\hat{q})\right],V(\hat{q})\right],V(\hat{q})\right]\\&+\frac{1}{2}\left\{\left[\left[\frac{\hat{p}^2}{2m},\left[\frac{\hat{p}^2}{2m},V(\hat{q})\right]\right],V(\hat{q})\right]+\left[\frac{\hat{p}^2}{2m},\left[\left[\frac{\hat{p}^2}{2m},V(\hat{q})\right],V(\hat{q})\right]\right]\right\}\\&-3\left[\frac{\hat{p}^2}{2m},V(\hat{q})\right]\left[\frac{\hat{p}^2}{2m},V(\hat{q})\right]\end{aligned} \tag{1.18}$$

where in the above $[,]$ represents the commutator.

Although a pattern was not observed forΔ ^{l}, there is enough information here to continue. Importantly $\left[\frac{\hat{p}^2}{2m},V(\hat{q})\right]=-\frac{1}{4m}\left[V''(\hat{q})+2iV'(\hat{q})\hat{p}\right]$ [11].

$V(\hat{q})$ is assumed to be a polynomial as this is almost always the case. So each commutator is simply a polynomial of different degree in $\hat{q}$ as derivatives of a polynomial are also polynomials. There is enough of a pattern to notice that theΔ $^{(l)}$ are just going to be more and more complicated types of commutators. With these terms acting on the states $\langle p|$ and $|q_j\rangle$, one will simply obtain polynomials in p_i and q_j with each order coming with higher powers of $i\epsilon$.

If the polynomials created by the termsΔ $^{(l)}$ are labelled $P^{(l)}(p_i,q_j)$ then

$$\langle p_i|e^{-i\hat{H}\epsilon}|q_j\rangle=e^{-i\epsilon\frac{p_i^2}{2m}}\left(1+\sum_{l=2}^{\infty}\frac{(i\epsilon)^l}{l!}P^{(l)}(p_i,q_j)\right)e^{-i\epsilon V(q_j)}\langle p_i|q_j\rangle\ . \tag{1.19}$$

Putting this back into equation (1.13), and using $\langle p_i|q_j\rangle{=}e^{-ip_iq_j}$

$$\langle q_{j+1}|e^{-i\hat{H}\epsilon}|q_j\rangle=\int_{-\infty}^{\infty}dp_i e^{iq_{j+1}p_i}e^{-ip_iq_j}e^{-i\epsilon\frac{p_i^2}{2m}}\left(1+\sum_{l=2}^{\infty}\frac{(i\epsilon)^l}{l!}P^{(l)}(p_i,q_j)\right)e^{-i\epsilon V(q_j)}\ . \tag{1.20}$$

Now consider that $p_i=i\frac{\partial}{\partial q_j}e^{-ip_iq_j}$. The above expression contains a term $e^{-ip_iq_j}$. Then taking the term $e^{-i\epsilon V(q_j)}$ out of the p integral and allowing the derivative to act only on terms inside the integral gives,

$$
\begin{aligned}
\langle q_{j+1}|e^{-i\hat{H}\epsilon}|q_j\rangle = & \int\limits_{-\infty}^{\infty} dp_i e^{iq_{j+1}p_i} e^{-ip_iq_j} e^{-i\epsilon\frac{p_i^2}{2m}} e^{-i\epsilon V(q_j)} \\
& + \left[\sum_{l=2}^{\infty} \frac{(i\epsilon)^l}{l!} P^{(l)}(i\frac{\partial}{\partial q_j}, q_j) \int\limits_{-\infty}^{\infty} dp_i e^{iq_{j+1}p_i} e^{-ip_iq_j} e^{-i\epsilon\frac{p_i^2}{2m}}\right] e^{-i\epsilon V(q_j)} \; .
\end{aligned}
\tag{1.21}
$$

To make sense of these integrals, one has to take time to have a slightly imaginary part such that these integrals are real. Note that this means ϵ gets a slightly imaginary part. Doing this it is seen that the p_i integral is now well behaved as there are only Gaussian integrals. The divergence seen in the literature has now been contained.

Furthermore, one mustn't forget that equation (1.10) shows that the position space is also integrated over. None of the literature seems to discuss this. With the above structure, the q_j integral is also safe and convergent.

Now let ϵ go very small and discard all terms $O(\epsilon^2)$. It is now valid to discard higher orders of ϵ as the ϵ aren't multiplying divergent terms. Doing this small ϵ approximation gives

$$
\begin{aligned}
\langle q_{j+1}|e^{-i\hat{H}\epsilon}|q_j\rangle &= \int\limits_{-\infty}^{\infty} dp_i e^{iq_{j+1}p_i} e^{-ip_iq_j} e^{-i\epsilon\frac{p_i^2}{2m}} e^{-i\epsilon V(q_j)} \\
&= \int\limits_{-\infty}^{\infty} dp_i e^{ip_i(q_{j+1}-q_j)-i\epsilon H(p_i,q_j)} \\
&= \int\limits_{-\infty}^{\infty} dp_i e^{i\epsilon\left(p_i\frac{(q_{j+1}-q_j)}{\epsilon}\right)-H(p_i,q_j)} \; .
\end{aligned}
\tag{1.22}
$$

Note that the ϵ term is describing the width of the Gaussian integral in p_i.

Equation (1.22) is defined for a quantum theory. What is wanted is the path integral for a field theory. Thus, equation (1.22) is used for a particular choice of quantum theory [10].

For a Quantum Field Theory, the generalized coordinates q_j (remember j labels the time) become field amplitudes $\phi(t_j, \boldsymbol{x}) = \phi_j(\boldsymbol{x})$. The conjugate momentum p_i is labelled $\pi(t_i, \boldsymbol{x}) = \pi_i(\boldsymbol{x})$. Since the states have been upgraded to fields, the multiplication of these objects is given by functional multiplication.

For the ϕ^4 scalar theory that will be used throughout this thesis, the Hamiltonian eigenvalue is given by

$$
H = \int d^3x \left[\frac{1}{2}\pi_i^2(\boldsymbol{x}) + \frac{1}{2}\left(\nabla\phi_i(\boldsymbol{x})\right)^2 + V(\phi_i(\boldsymbol{x}))\right] \tag{1.23}
$$

This choice is the reason this proof started with a time dependent Hamiltonian. This Hamiltonian is time dependent and so the previous result holds. With this choice, the path integral is being derived only for the scalar ϕ^4 theory.

Thus equation (1.22) generalizes to,

$$
\begin{aligned}
\langle \phi_{j+1}(\boldsymbol{x}) | \phi_j(\boldsymbol{x}) \rangle &= \int_{-\infty}^{+\infty} d\pi_i e^{i\epsilon\left(\frac{\pi_i(\phi_{j+1}-\phi_j)}{\epsilon} - H(\pi_i,\phi_j)\right)} \\
&\quad \text{Plugging in the Hamiltonian of equation (1.23)} \\
&= \int_{-\infty}^{+\infty} d\pi_i(\boldsymbol{x}) e^{i\epsilon \int d^3x \left(\frac{\pi_i(\boldsymbol{x})(\phi_{j+1}(\boldsymbol{x})-\phi_j(\boldsymbol{x}))}{\epsilon} - \left[\frac{1}{2}\pi_i^2(\boldsymbol{x}) + \frac{1}{2}(\nabla\phi_j(\boldsymbol{x}))^2 + V(\phi_j(\boldsymbol{x}))\right]\right)} .
\end{aligned}
\tag{1.24}
$$

One more thing to note is that there is a term in the above equation that goes like $\frac{(\phi_{j+1}-\phi_j)}{\epsilon}$. At this point, ϵ has already been taken to be a very small time. The point ϕ_{j+1} is $\phi(\boldsymbol{x})$ at time $t_j + \epsilon$. This is then the definition of a time derivative, thus

$$
\begin{aligned}
\langle \phi_{j+1}(\boldsymbol{x}) | \phi_j(\boldsymbol{x}) \rangle &= \int_{-\infty}^{+\infty} d\pi_i(\boldsymbol{x}) e^{i\epsilon \int d^3x \left(\pi_i(\boldsymbol{x})\dot{\phi}_j(\boldsymbol{x}) - \left[\frac{1}{2}\pi_i^2(\boldsymbol{x}) + \frac{1}{2}(\nabla\phi_j(\boldsymbol{x}))^2 + V(\phi_j(\boldsymbol{x}))\right]\right)} \\
&\quad \text{Completing the square in the } \pi_i \text{ term} \\
&= \int_{-\infty}^{+\infty} d\pi_i(\boldsymbol{x}) e^{\frac{1}{2} i\epsilon \int d^3x \left(-\left(\pi_i(\boldsymbol{x})-\dot{\phi}_j(\boldsymbol{x})\right)^2 + \left(\dot{\phi}_j(\boldsymbol{x})\right)^2 - (\nabla\phi_j(\boldsymbol{x}))^2 - 2V(\phi_j(\boldsymbol{x}))\right)} \\
&\quad \text{Splitting off the terms not dependent on } \pi_i \\
&= e^{i\epsilon \int d^3x \left(\frac{1}{2}\left(\dot{\phi}_j(\boldsymbol{x})\right)^2 - \frac{1}{2}(\nabla\phi_j(\boldsymbol{x}))^2 - V(\phi_j(\boldsymbol{x}))\right)} \int_{-\infty}^{+\infty} d\pi_i(\boldsymbol{x}) e^{\frac{1}{2} i\epsilon \int d^3x \left(-\left(\pi_i(\boldsymbol{x})-\dot{\phi}_j(\boldsymbol{x})\right)^2\right)} \\
&\quad \text{Doing the Gaussian integral} \\
&= \left(\frac{2\pi}{i\epsilon}\right)^{\frac{1}{2}} e^{i\epsilon \int d^3x \left(\frac{1}{2}\left(\dot{\phi}_j(\boldsymbol{x})\right)^2 - \frac{1}{2}(\nabla\phi_j(\boldsymbol{x}))^2 - V(\phi_j(\boldsymbol{x}))\right)} .
\end{aligned}
\tag{1.25}
$$

At this point it is necessary to discuss the $\frac{1}{\epsilon}$ term. This term seems to imply that each little amplitude slice blows up which should result in the product of these terms blowing up as $\epsilon \to \infty$. This is not actually the case, to show this one needs to recall how these terms entered the calculation.

The ϵ term entered through the Gaussian integral in the top line of equation (1.22). Consider if this integral was describing a free theory, this would imply that $V(\hat{q}) = 0$. Then to find the total amplitude would is the matter of multiplying $N+1$ terms that go like $\int_{-\infty}^{\infty} dp_i e^{iq_{j+1}p_i - ip_iq_j - i\epsilon\frac{p^2}{2m}}$ together. Look at the product of two of these terms with the q integrals included, this gives

$$
\begin{aligned}
&\int_{-\infty}^{\infty} dq_j dq_{j-1} dp_i e^{iq_{j+1}p_i - p_iq_j - i\epsilon\frac{p_i^2}{2m}} dp_{i-1} e^{iq_jp_{i-1} - p_{i-1}q_{j-1} - i\epsilon\frac{p_{i-1}^2}{2m}} \\
&= \int_{-\infty}^{\infty} dq_j dq_{j-1} dp_i dp_{i-1} e^{iq_{j+1}p_i} e^{i(p_{i-1}-p_i)q_j} e^{-i\epsilon\frac{p_i^2}{2m}} e^{-p_{i-1}q_{j-1} - i\epsilon\frac{p_{i-1}^2}{2m}} .
\end{aligned}
\tag{1.26}
$$

From the above, one can see that the integral over q_j will give the delta function $\delta(p_{i-1} - p_i)$. Then integrating over one of the momentum integrals removes the delta and sets $p_i = p_{i-1}$. By repeating this procedure for the $N + 1$ integrals one can see that the end result will turn out to be a single p_i integral that looks like

$$\int_{-\infty}^{\infty} dp e^{\epsilon \frac{p^2}{2m} + \epsilon \frac{p^2}{2m} + \epsilon \frac{p^2}{2m} + \ldots} . \tag{1.27}$$

Given the definition of ϵ as small time slices, one would be left with the term $\int_{-\infty}^{\infty} dp e^{(t'-t)\frac{p^2}{2m}}$. Solving this momentum integral results in a finite result where instead of finding an exploding term that goes like $\left(\frac{1}{\epsilon}\right)^{(N+1)/2}$, one is left with a term $\frac{1}{\sqrt{t'-t}}$.

Obviously to solve the interacting theory is a little more difficult but the idea there is the same. Thus, in what follows, instead of referring to a term $\left(\frac{2\pi}{i\epsilon}\right)^{\frac{N+1}{2}}$ this will be called a finite term M as this really isn't an infinite term but an artefact from the splitting up of the transition amplitude. Recall that this proof is not claiming to be mathematically rigorous.

The Lagrangian density of the ϕ^4 theory is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) \\ &= \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi) . \end{aligned} \tag{1.28}$$

To get the path integral for this QFT one must insert equation (1.25) into equation (1.10), where equation (1.10) is put into the QFT notation. Then using equation (1.28) and recalling that the infinite term $\left(\frac{2\pi}{i\epsilon}\right)^{\frac{N+1}{2}}$ should be considered a finite term M one gets,

$$\begin{aligned} \langle \phi(t', \boldsymbol{x}) | \phi(t, \boldsymbol{x}) \rangle &= M \prod_{i=1}^{N} \int d\phi_i(\boldsymbol{x}) e^{i \left(\epsilon \sum_{i=1}^{N+1} \right) \int d^3x \mathcal{L}(\phi_i)} \\ & \text{Taking the limit as } N \text{ gets very large and noting that} \\ & \int dx \approx \sum_{i=1}^{N \to \infty} \Delta_i \\ & \text{(remember the } i \text{ slices are time slices starting at } t \text{ ending at } t') \\ &= M \prod_{i=1}^{N} \int d\phi_i(\boldsymbol{x}) e^{i \int_t^{t'} dt \int d^3x \mathcal{L}(\phi(x))} . \end{aligned} \tag{1.29}$$

Lastly, a new definition needs to be made. The object $M \prod_{i=1}^{N} \int d\phi_i(\boldsymbol{x})$ will be given the label of $\int D[\phi(x)]$. This object represents an integration over all functionals $\phi(x)$.

Thus one finds the interesting result that [10],

$$\langle \phi(t', \boldsymbol{x}) | \phi(t, \boldsymbol{x}) \rangle = \int D[\phi(x)] e^{i \int_t^{t'} dt \int d^3x \mathcal{L}(\phi(x))} . \tag{1.30}$$

This is the path integral formulation of Quantum Field Theory.

1.2.1 Ground State to Ground State

Previously the amplitude considered was $\langle\phi(t',\boldsymbol{x})|\phi(t,\boldsymbol{x})\rangle$. The most useful thing is to find the path integral formulation for the ground state to ground state amplitude i.e. $\langle\Omega|\Omega\rangle$. The reason is that this connects up with the n point Green's functions that contain all the information of the theory.

Start by considering the state $|\phi(t,\boldsymbol{x})\rangle$ as being evolved from some initial time t_0. A very rough justification for considering this is to think about the physical system. One has a populated state at some time t. Presumably in the distant past, before the system had a chance to interact, it was in the ground state. Thus one looks at the time evolution of the occupied state.

The proof presented here follows [10] very closely.

So starting with the time evolution operator on the state, one can insert a complete set of states. Choose the complete set of states to be the energy eigenstates of the full theory such that $\hat{H}|n\rangle = E_n|n\rangle$,

$$\begin{aligned}|\phi(t,\boldsymbol{x})\rangle &= e^{-i\hat{H}(t-t_0)}|\phi(t_0,\boldsymbol{x})\rangle \\ &= \sum_n e^{-iE_n(t-t_0)}|n\rangle\langle n|\phi(t_0,\boldsymbol{x})\rangle\end{aligned} \tag{1.31}$$

where $\hat{H}$ is the full Hamiltonian of the theory.

One must assume that $\langle\Omega|\phi(t_0,\boldsymbol{x})\rangle \neq 0$. This means there is some overlap between the ground state and $|\phi(t_0,\boldsymbol{x})\rangle$. Thus $|\Omega\rangle$ must be contained in the above series. Furthermore one considers the ground state to have energy E_0. This must be the lowest energy from the definition of the ground state.

Thus

$$e^{-i\hat{H}(t-t_0)}|\phi(t_0,\boldsymbol{x})\rangle = e^{-iE_0(t-t_0)}|\Omega\rangle\langle\Omega|\phi(t_0,\boldsymbol{x})\rangle + \sum_{n\neq 0} e^{-iE_n(t-t_0)}|n\rangle\langle n|\phi(t_0,\boldsymbol{x})\rangle \tag{1.32}$$

Energy is always positive, and E_0 is the lowest energy. This means that one can get rid of all the $n \neq 0$ terms by sending t to infinity. Actually, due to the i in the exponential, t is sent to infinity with a small imaginary part, i.e. $t \to +\infty(1-i\epsilon)$. The reason this limit eliminates all $n \neq 0$ terms is all the terms with $n \neq 0$, will decay faster than the term E_0 since the E_n are larger values.

$$\lim_{t\to-\infty(1+i\epsilon)} e^{-i\hat{H}(t-t_0)}|\phi(t_0,\boldsymbol{x})\rangle = \lim_{t\to-\infty(1+i\epsilon)} e^{-iE_0(t-t_0)}\langle\Omega|\phi(t_0,\boldsymbol{x})\rangle|\Omega\rangle \tag{1.33}$$

In the same way one can find that

$$\lim_{t\to+\infty(1+i\epsilon)} \langle\phi(t_0,\boldsymbol{x})|e^{-i\hat{H}(t_0-t)} = \lim_{t\to+\infty(1+i\epsilon)} \langle\Omega|\langle\phi(t_0,\boldsymbol{x})|\Omega\rangle e^{-iE_0(t_0-t)} \tag{1.34}$$

Then putting this all together

$$\begin{aligned}\lim_{\substack{t\to-\infty(1+i\epsilon)\\ t'\to+\infty(1+i\epsilon)}}\langle\phi(t',\boldsymbol{x}')|\phi(t,\boldsymbol{x})\rangle &= \lim_{\substack{t\to-\infty(1+i\epsilon)\\ t'\to+\infty(1+i\epsilon)}}\langle\phi(t_0',\boldsymbol{x})|e^{-i\hat{H}(t_0'-t')-i\hat{H}(t-t_0)}|\phi(t_0,\boldsymbol{x})\rangle\\ &= \lim_{\substack{t\to-\infty(1+i\epsilon)\\ t'\to+\infty(1+i\epsilon)}}\langle\Omega|\langle\phi(t_0',\boldsymbol{x}')|\Omega\rangle e^{-iE_0[(t'+t)-(t_0'+t_0)]}\langle\Omega|\phi(t_0,\boldsymbol{x})\rangle|\Omega\rangle\ .\end{aligned} \tag{1.35}$$

Since $\langle\Omega|\phi(t_0,\boldsymbol{x})\rangle$ is a number this can be considered a normalization issue. Through out this thesis, this extra term is absorbed into a normalization constant i.e.

$$\lim_{\substack{t\to-\infty(1+i\epsilon)\\ t'\to\infty(1+i\epsilon)}}\langle\phi(t_0',\boldsymbol{x}')|\Omega\rangle e^{-iE_0[(t'+t)-(t_0'+t_0)]}\langle\Omega|\phi(t_0,\boldsymbol{x})\rangle \equiv N\ . \tag{1.36}$$

This means that

$$\lim_{\substack{t\to-\infty(1+i\epsilon)\\ t'\to\infty(1+i\epsilon)}}\langle\phi(t',\boldsymbol{x}')|\phi(t,\boldsymbol{x})\rangle = N\langle\Omega|\Omega\rangle\ . \tag{1.37}$$

What this means for the path integral is that the ground state to ground state amplitude is given by

$$\begin{aligned}\langle\Omega|\Omega\rangle &= \lim_{\substack{t\to-\infty(1+i\epsilon)\\ t'\to\infty(1+i\epsilon)}}\frac{1}{N}\langle\phi(t',\boldsymbol{x}')|\phi(t,\boldsymbol{x})\rangle\\ &\quad\text{Using equation (1.30)}\\ &= \lim_{\substack{t\to-\infty(1+i\epsilon)\\ t'\to\infty(1+i\epsilon)}}\frac{1}{N}\int D[\phi(x)]e^{i\int_t^{t'}dt\int d^3x\mathcal{L}(\phi(x))}\\ &= N\int D[\phi(x)]e^{i\int_{-\infty}^{\infty}dt\int d^3x\mathcal{L}(\phi(x))}\ .\end{aligned} \tag{1.38}$$

1.3 n-point Green's Functions and the Path Integral

This section will start with a brief reminder of how a n point function is defined in the bra-ket notation. Following this it will be shown what the n-point function looks like in the path integral representation.

1.3.1 n-Point Functions

The n-point Green's functions are important for any theory as they provide information on the scattering matrix. The n-point Green's function is also known as the n-point correlation function.

Using the field creation operator, ϕ, the 2 point correlator is given by [10]

$$\langle\Omega|T\left\{\phi(x)\phi(y)\right\}|\Omega\rangle \tag{1.39}$$

where Ω represents the ground state of the interacting theory and the field operator $\phi(y)$ 'creates' the field at y. The T denotes the fact that the objects $\phi(x)$ and $\phi(y)$ must be time ordered. The object defined above can be interpreted as the amplitude of propagation of a field excitation (particle) between x and y.

The higher n-point functions have a more difficult interpretation but can be thought of as amplitudes. In general an n-point function is given by

$$\langle\Omega|T\left\{\phi(x_1)...\phi(x_n)\right\}|\Omega\rangle\ . \tag{1.40}$$

The path integral formulation, introduced in the previous section, provides one with an easy way to get the n-point functions.

1.3.2 n-Point Functions from the Path Integral

This section will follow [9] very closely.

It needs to be shown that one can determine the n-point functions directly from the path integral notation. A proof for the two point function will be done explicitly but it will be simple to see how to generalise to get the result that an n-point function is given by $\int D[\phi]\phi(x_1)...\phi(x_n)e^{i\int_{-\infty}^{\infty} d^4x\mathcal{L}(\phi)}$.

The proof utilizes some of the same tricks as were used to derive the path integral representation of an amplitude.

Start with the usual two point function,

$$\langle\Omega|T\left\{\hat{\phi}(x_b)\hat{\phi}(x_a)\right\}|\Omega\rangle\ . \tag{1.41}$$

Consider the case where $x_b^0 > x_a^0$. This discards the need for the time ordering at this point but this issue will be returned to later.

This amplitude can be split up by inserting functional $\mathbb{1}$'s. This gives

$$\begin{aligned}
\langle\Omega|\hat{\phi}(x_b)\hat{\phi}(x_a)|\Omega\rangle &= \int d\phi_1(\boldsymbol{x}_1)...d\phi_n(\boldsymbol{x}_n)\ \langle\Omega|\phi_n(\boldsymbol{x}_n)\rangle...\langle\phi_{b+1}(\boldsymbol{x}_{b+1})|\hat{\phi(x_b)}|\phi_b(\boldsymbol{x}_b)\rangle... \\
&\quad\times\langle\phi_{a+1}(\boldsymbol{x}_{a+1})|\hat{\phi(x_a)}|\phi_a(\boldsymbol{x}_a)\rangle...\langle\phi_2(\boldsymbol{x_2})|\phi_1(\boldsymbol{x}_1)\rangle\langle\phi_1(\boldsymbol{x}_1)|\Omega\rangle \\
&\quad\text{Using } \hat{\phi}(x_a)|\phi_a(\boldsymbol{x}_a)\rangle = \phi(x_a)|\phi_a(\boldsymbol{x}_a)\rangle \\
&= \int d\phi_1(\boldsymbol{x}_1)...d\phi_n(\boldsymbol{x}_n)\ \phi(x_b)\phi(x_a)\langle\Omega|\phi_n(\boldsymbol{x}_n)\rangle...\langle\phi_{b+1}(\boldsymbol{x}_{b+1})|\phi_b(\boldsymbol{x}_b)\rangle... \\
&\quad\langle\phi_{a+1}(\boldsymbol{x}_{a+1})|\phi_a(\boldsymbol{x}_a)\rangle...\langle\phi_2(\boldsymbol{x_2})|\phi_1(\boldsymbol{x}_1)\rangle\langle\phi_1(\boldsymbol{x}_1)|\Omega\rangle\ .
\end{aligned} \tag{1.42}$$

If one considers a time dependent amplitude then each little amplitude above is given as $\langle\phi_{i+1}(\boldsymbol{x}_{i+1})|Te^{-i\int\limits_{t_i}^{t_{i+1}}\hat{H}(t')dt'}|\phi_i(\boldsymbol{x}_i)\rangle$. One can put in $\mathbb{1}$'s such that $t_{i+1}-t_i=\epsilon\to 0$, in this case

$$\langle\phi_{i+1}(\boldsymbol{x}_{i+1})|Te^{-i\int\limits_{t_i}^{t_{i+1}}H(t')dt'}|\phi_i(\boldsymbol{x}_i)\rangle = \lim_{\epsilon\to 0}\langle\phi_{i+1}(\boldsymbol{x}_{i+1})|Te^{-i\hat{H}(t_i)\epsilon}|\phi_i(\boldsymbol{x}_i)\rangle\ . \tag{1.43}$$

Using the above, equation (1.42) is equivalent to equation (1.10) but with two fields out the front. Also notice the order of the fields, they come sequentially in time. Using the fact that this looks like the path integral gives,

$$\langle\Omega|\hat{\phi}(x_b)\hat{\phi}(x_a)|\Omega\rangle = \int D[\phi]\phi(x_b)\phi(x_a)e^{i\int\limits_{-\infty}^{\infty}d^4x\mathcal{L}(x)}\ , \tag{1.44}$$

if $t_b < t_a$ then it would have been found that

$$\langle\Omega|\hat{\phi}(x_a)\hat{\phi}(x_b)|\Omega\rangle = \int D[\phi]\phi(x_a)\phi(x_b)e^{i\int\limits_{-\infty}^{\infty}d^4x\mathcal{L}(x)}\ . \tag{1.45}$$

This means that if the operators were time ordered the correct result is obtained.

This can be generalised by using the same procedure. In general [9]

$$\langle\Omega|T\phi(x_1)...\phi(x_n)|\Omega\rangle = \int D[\phi(x)]\left(\phi(x_1)...\phi(x_n)\right)e^{i\int\limits_{-\infty}^{\infty}d^4x\mathcal{L}(\phi)}\ . \tag{1.46}$$

1.4 The Generating Functional with Vanishing Source

This section will follow [10]. In this section variational derivatives are used. Variational derivatives are discussed in appendix A.1.

The generating functional is an object that can be used to produce the n-point functions of the theory. This is done by exploiting the way n-point functions are represented in the path integral formulation (section 1.3.2). The generating functional is so called as, by using n functional derivatives, one can generate an n-point function in the path integral representation.

It is claimed that the object [9]

$$Z[J] = \int D[\phi]e^{i\int d^4x[\mathcal{L}+J(x)\phi(x)]} \tag{1.47}$$

is the generating functional. Note the relationship to the path integral discussed above.

It can be seen that by varying $Z[J]$ with respect to J n times, one pulls down n fields from the exponential. This is the form of an n point function in the path integral representation up to factors of i and the J in the exponential. To fix the factors of i, one varies with respect to iJ. To fix the left over J in the exponential, the expression is evaluated at $J = 0$.

One can check to see if the one point function is achieved by varying $Z[J]$ with respect to $iJ(y)$ at $J = 0$,

$$\begin{aligned}\frac{\delta Z[J]}{\delta iJ(y)} &= \int D[\phi]\frac{\delta}{\delta iJ(y)}e^{i\int d^4x[\mathcal{L}+J(x)\phi(x)]}\\ &= \int D[\phi]e^{i\int d^4x[\mathcal{L}+J(x)\phi(x)]}i\int d^4x\frac{\delta J(x)}{\delta iJ(y)}\phi(x)\\ &= \int D[\phi]e^{i\int d^4x[\mathcal{L}+J(x)\phi(x)]}i\frac{1}{i}\int d^4x\delta(x-y)\phi(x)\\ &= \int D[\phi]\phi(y)e^{i\int d^4x[\mathcal{L}+J(x)\phi(x)]}\end{aligned} \tag{1.48}$$

now evaluate the above at $J = 0$. Thus

$$\frac{\delta Z[J]}{\delta iJ(y)}\Big|_{J=0} = \int D[\phi]\phi(y)e^{i\int d^4x[\mathcal{L}]} . \tag{1.49}$$

Compare this to equation (1.46). The above is the path integral expression of the one point function.

Each variation with respect to J brings down another factor of the field. As previously stated, the factor of i in the variation cancels with the i in the exponential. Thus one can see that a general n point function is given as [9],

$$\langle\Omega|T\phi(x_1)...\phi(x_n)|\Omega\rangle = \frac{\delta}{\delta iJ(x_n)}...\frac{\delta}{\delta iJ(x_1)}Z[J]\Big|_{J=0} . \tag{1.50}$$

So it has been shown that $Z[J]$ is the generating functional, with it you can generate all n-point functions. Since the n-point functions contain all information on the theory the generating functional does too, thus $Z[J]$ is very useful.

1.5 Finite Source J

This thesis is looking at techniques that can be used to study processes that occur in heavy ion collisions. As stated in the introduction, in such collisions there is a source of particles. The term $J\phi$ in the exponential of equation (1.52), describes some term J coupling to the field where J is a free term. This is interpreted as a source term since the J term 'brings the particles'.

When discussing QFT, one usually sets $J = 0$ after a calculation, as was done in the previous section. However, in the introduction it was stated that this thesis is to study high energy collisions. As such one has to deal with sources that are found in the CGC. Thus in this thesis, J is considered to be a finite, physical source of the fields i.e. $J \neq 0$.

This property leads to interesting outcomes, particularly as the expectation value of the field becomes important even in the case of spontaneous symmetry breaking.

In fact a stronger requirement is made on the source for this thesis. J is assumed to be a large source where large means that

$$J \sim O(\frac{1}{g}) \tag{1.51}$$

where g^2 is the coupling in the ϕ^4 theory. Since g^2 is a small coupling constant, this means $\frac{1}{g}$ is large.

The large source has exceptionally important ramifications and is in fact the main reason for the development of the classical-statistical method, which is discussed in far greater detail later on.

1.6 The Generating Functional and Feynmann Diagrams

There are two ways to use the generating functional to obtain information on the theory being investigated. The first way is to generate the Feynmann diagrams from the generating functional. In the second way a far more abstract but very general set of relationships called the Schwinger-Dyson equations will be introduced. This section will discuss the first way, obtaining Feynmann diagrams.

As a reminder, the generating functional is given by

$$Z[J] = \int D[\phi] e^{i \int d^4x [\mathcal{L} + J(x)\phi(x)]} \tag{1.52}$$

with the Lagrangian of 1.1,

$$\mathcal{L} = -\frac{1}{2} \left(\partial_\mu \phi(x)\right)^2 - m^2 \phi^2(x) - \frac{g^2}{4!} \phi^4(x) \ . \tag{1.53}$$

To get the Feyamnn diagrams from $Z[J]$ is a fairly lengthy process thus an outline of the derivation is given.

Feynmann diagrams can be constructed from $Z[J]$ by considering the interaction terms in the Lagrangian as functional derivatives. These functional derivatives are considered to be acting on a generating functional constructed from only the kinetic term of the theory (see equation (1.56) below). Then one can expand in g^2, the coupling parameter in the interaction term. Since the interaction terms are in powers of functional derivatives, the expansion will become a series of derivatives acting on the newly constructed free generating functional (equation (1.57) below). It turns out that this free generating functional is a solvable integral. The series of derivatives acting on the solved free generating functional will lead to Feynmann diagrams coming from $Z[J]$ in increasing powers of the coupling constant (equation (1.67) below).

Now onto the detailed discussion of the above outline.

Finding the Diagrams

Remember from section 1.1 that $\mathcal{L}$ can be understood as the sum of the kinetic and interacting terms. The generating functional can thus be written as

$$
\begin{aligned}
Z[J] &= \int D[\phi] e^{i\int d^4x[\mathcal{L}+J(x)\phi(x)]} \\
&= \int D[\phi] e^{i\int d^4x[\mathcal{L}_0+J(x)\phi(x)]} e^{i\int d^4x[\mathcal{L}_{\text{int}}]} \\
&\quad \text{expanding the } \mathcal{L}_I \text{ exponential in powers of } g^2 \\
&= \int D[\phi](1+i\int d^4y \frac{g^2}{4!}\phi(y)^4+...)e^{i\int d^4x[\mathcal{L}_0+J(x)\phi(x)]} \ . \qquad (1.54)
\end{aligned}
$$

One can now reverse the logic used in the previous section and see that

$$
i\int d^4y \frac{g^2}{4!}\phi^4(y) e^{i\int d^4x[\mathcal{L}_0+J(x)\phi(x)]} = i\int d^4y \frac{g^2}{4!}\frac{\delta^4}{\delta(iJ(y))^4} e^{i\int d^4x[\mathcal{L}_0+J(x)\phi(x)]} \ . \qquad (1.55)
$$

This observation leads to the conclusion that the exponential of the interaction term can be simply written as the exponential of the J derivatives acting on $e^{i\int d^4x[\mathcal{L}_0+J(x)\phi(x)]}$ [12], [13].

$$
\begin{aligned}
Z[J] &= \int D[\phi] e^{i\int d^4x \mathcal{L}_{\text{int}}[\frac{\delta}{\delta iJ}]} e^{[\mathcal{L}_0+J(x)\phi(x)]} \\
&\quad \text{Pulling out terms that don't depend on } \phi \\
&= e^{i\int d^4x \mathcal{L}_{\text{int}}[\frac{\delta}{\delta iJ}]} \int D[\phi] e^{[\mathcal{L}_0+J(x)\phi(x)]} \\
&= e^{i\int d^4x \mathcal{L}_{\text{int}}[\frac{\delta}{\delta iJ}]} Z_0[J] \qquad (1.56)
\end{aligned}
$$

where in the above $Z_0[J] \equiv \int D[\phi] e^{[\mathcal{L}_0+J(x)\phi(x)]}$.

Equation (1.56) can be used to get a diagrammatic representation of $Z[J]$, in powers of the coupling constant. The next step is to expand in powers of g^2.

The expansion (to first order in g^2) is

$$
\begin{aligned}
Z[J] &= e^{i\int d^4x \mathcal{L}_{\text{int}}[\frac{\delta}{\delta iJ}]} Z_0[J] \\
&= \left\{1+i\int d^4x \mathcal{L}_{\text{int}}[\frac{\delta}{\delta iJ}] + \mathcal{O}(g^4)\right\} Z_0[J] \qquad (1.57)
\end{aligned}
$$

where $\mathcal{L}_{\text{int}}[\frac{\delta}{\delta iJ}] = -\frac{g^2}{4!}\left(\frac{\delta}{\delta iJ}\right)^4$. Thus it is necessary to determine the 4^{th} J derivative of $Z_0[J]$. Note that since this is the interaction term, the derivatives are taken at the same space-time point.

By definition Z_0 is given by:

$$Z_0[J] = \int D[\phi] e^{i \int d^4x [\mathcal{L}_0 + J\phi]}$$

Putting in the form of $\mathcal{L}_0$

$$= \int D[\phi] e^{i\left[-\frac{1}{2}\phi(\Box + m^2)\phi + J\phi\right]} \; . \tag{1.58}$$

The above is actually a solvable integral. It is a Gaussian integral with a linear part. This integral is solved in appendix A.2 where it is shown that,

$$Z_0[J] = N e^{\int d^4y d^4y' \frac{1}{2} J(y) G_0(y-y') J(y')} \; . \tag{1.59}$$

Here, the term G_0 refers to the free propagator. The normalization used was discussed in section 1.1.

In doing this integral, one can recognize that the kinetic term describes the free propagator of the theory. The general result is that the term that connects the two fields in the kinetic term of the action is found to be the inverse free propagator of the theory. This will be used later when defining different generating functionals.

Now to find the derivatives. The first derivative is done explicitly to show how this calculation works, after that the calculation will be completed in appendix A.3. (Note that the i in the variation $\frac{\delta}{\delta i J}$ is left out for now but will be re-introduced.)

The first derivative:

$$\begin{aligned}
\frac{\delta}{\delta J(x)} Z_0[J] &= \frac{\delta}{\delta J(x)} N e^{\int d^4y d^4y' \frac{1}{2} J(y) G_0(y-y') J(y')} \\
&= N e^{\int d^4y d^4y' \frac{1}{2} J(y) G_0(y-y') J(y')} \frac{\delta}{\delta J(x)} \left(\int d^4y d^4y' \frac{1}{2} J(y) G_0(y-y') J(y') \right) \\
&= Z_0[J] \left[\left(\frac{1}{2} \int d^4y d^4y' \delta(x-y) G_0(y-y') J(y') \right) \right. \\
&\quad \left. + \left(\frac{1}{2} \int d^4y d^4y' J(y) G_0(y-y') \delta(x-y') \right) \right] \; .
\end{aligned} \tag{1.60}$$

Then using the integration of the δ followed by a relabelling of integration variables as $y = y' = x'$ gets,

$$\frac{\delta}{\delta J(x)} Z_0[J] = \left[\int d^4x' G_0(x-x') J(x') \right] Z_0[J] \; . \tag{1.61}$$

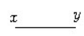
The rest of this calculation is left to appendix A.3, the result is given by equation (A.24). The solution is written in a different notation, where $\int d^4x' G_0(x-x') J(x') \equiv (G_0 \otimes J)_{x'}(x)$. Equation (A.24) is re-written with the i's back in the variations. It turns out that this won't change the result as $\frac{1}{i^4} = 1$.

$$\frac{\delta^4 Z_0[J]}{\delta(iJ(x))^4} = \Big[3(G_0(x-x))^2 + 6G_0(x-x)\,(G_0 \otimes J)_{y'}(x)\,(G_0 \otimes J)_{z'}(x) + (G_0 \otimes J)_{x'}(x)\,(G_0 \otimes J)_{y'}(x)\,(G_0 \otimes J)_{z'}(x)\,(G_0 \otimes J)_{w'}(x)\Big] Z_0[J] \tag{1.62}$$

Now to finally express $Z[J]$ up to 1st order in g^2, insert equation (1.62) into equation (1.57) [12],[13],

$$\begin{aligned} Z[J] \quad = \quad & \Big\{1 + \left(\frac{g^2}{4!}\right) \int d^4x [\\ & +3(G_0(x-x))^2 + 6G_0(x-x)\,(G_0 \otimes J)_{y'}(x)\,(G_0 \otimes J)_{z'}(x) \\ & + (G_0 \otimes J)_{x'}(x)\,(G_0 \otimes J)_{y'}(x)\,(G_0 \otimes J)_{z'}(x)\,(G_0 \otimes J)_{w'}(x)] \\ & + \mathcal{O}(g^4)\Big\}\, Z_0[J] \;. \end{aligned} \tag{1.63}$$

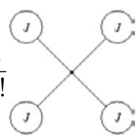
One can now introduce diagrammatic notation for the above. These are the coordinate space Feynmann rules. Remember that $(G_0 \otimes J)_{z'}(x)$ was an integral over the dashed variable, thus this co-ordinate does not actually appear.

$G_0(x-y) \equiv$  (1.64)

$(G_0 \otimes J)_{x'}(x) \equiv$  (1.65)

$$\frac{g^2}{4!} \int d^4x \equiv \bullet \, . \tag{1.66}$$

Equation (1.63) in diagrammatic form reads [12],[13]

$Z[J] = \Big(1 + \frac{3}{4!}$  $+ \frac{6}{4!}$  $+ \frac{1}{4!}$  $+ \mathcal{O}(g^4)\Big) Z_0[J]\;.$ (1.67)

To continue to higher orders in g^2 one would need to do more derivatives. It can be seen that it takes a lot of effort to produce the Feynmann diagrams.

The generating functional turns out to have a very useful by-product, a set of very general relations called the Schwinger-Dyson equations.

1.7 Schwinger-Dyson Equations

This is one of the most important sections and will be referred to often through out this thesis. This is a very powerful consequence of the generating functional notation.

Before the actual Schwinger-Dyson equations are derived, a discussion on the connected diagrams and their generating functional needs to be had.

1.7.1 Connected Diagrams and their Generating Functional

Connected diagrams are diagrams such that if one were to draw the diagram with a pen, the entire diagram could be drawn without ever needing to lift the pen from the page. A disconnected diagram is where one would need to lift the pen. Thus the building blocks of the disconnected diagrams are the connected diagrams. $Z[J]$ produces all diagrams, connected and disconnected. Since all diagrams, connected and disconnected, are built from the connected diagrams, an object that only generates the connected diagrams would be very useful to have.

It will be shown that $W[J] \equiv -i\ln[Z[J]]$ is the sum of all connected diagrams and in fact can be used to generate connected n-point functions.

To show $W[J]$ as defined above is a sum of all connected diagrams, one can use the replica technique where the clear argument of [15] is used. The replica technique is a trick developed in statistical mechanics. $W[J]$ turns out to be the sum of all connected diagrams in the same way that $Z[J]$ was shown to be the sum of all diagrams in equation (1.67).

Then it will be shown that $W[J]$ is capable of generating the connected n-point functions through functional derivatives, as $Z[J]$ produces the full n-point functions.

Defining $W[J]$

One starts by defining a theory governed by the generating functional $Z_N[J]$ [15]. Where

$$\begin{aligned} Z_N[J] &\equiv (Z[J])^N \\ &= \int D[\phi_1]....D[\phi_N]e^{iS[\phi_1]+iJ\phi_1}...e^{iS[\phi_N]+iJ\phi_N} \ . \end{aligned} \tag{1.68}$$

Thus the theory governed by $Z_N[J]$ is made up of N copies of $Z[J]$, where (as can be seen by the interaction terms) the copies can't interact with one another.

Now one can consider how the connected diagrams of $Z[J]$ enter into $Z_N[J]$. In the theory $Z_N[J]$ the connected diagrams of $Z[J]$ become N copies of themselves. Consider a particular connected diagram denoted C_i, in the theory $Z_N[J]$ this diagram will come in as C_i^N.

The trick comes in realising that the disconnected diagrams contained in $Z[J]$ appear in $Z_N[J]$ differently. Consider a disconnected diagram as a multiplication of connected diagrams. Thus a simple disconnected diagram of $Z[J]$ could be defined by two connected diagrams C_i and C_j such that the disconnected diagram is $C_i \times C_j$. A general disconnected diagram of $Z[J]$ is made up of the multiplication of n connected diagrams of $Z[J]$. Each disconnected diagram of $Z[J]$ appears in $Z_N[J]$, N times, thus a general connected diagram coming from disconnected diagrams in $Z[J]$ appears N^n times in $Z_N[J]$.

This gives an order for each type of diagram. The number of connected diagrams (C_i) from $Z[J]$ must be proportional to N in $Z_N[J]$. The number of connected diagrams that make up the disconnected diagrams of $Z[J]$ is proportional to N^n in $Z_N[J]$.

Finally one can expand $(Z[J])^N$ in powers of N,

$$(Z[J])^N = 1 + N \ln[Z[J]] + O(N^2) \; . \tag{1.69}$$

Since $\sum C_i \propto N$, this identifies

$$\sum C_i = \ln[Z[J]] \tag{1.70}$$

or

$$Z[J] = e^{\sum C_i} \; . \tag{1.71}$$

By defining the generating functional of connected diagrams as $iW[J]$, the result is [15]

$$Z[J] = e^{iW[J]} \tag{1.72}$$

Relationship of $W[J]$ to Connected n-point Functions

So it has been shown that $W[J]$ is the sum of connected diagrams in the same sense that $Z[J]$ is the sum of all diagrams. Now it will be shown that $W[J]$ is in fact the generating functional of connected n-point functions just as $Z[J]$ can generate the full n-point functions. The following argument shows this.

It was shown that the full n-point function is given as $\frac{\delta^n Z[J]}{\delta(iJ)^n}$. The full two point function is made of a connected and a disconnected piece where the disconnected piece is made from the one point function squared. This information can be used to find the properties of $W[J]$.

$$\begin{aligned}
\frac{\delta^2 Z[J]}{\delta(iJ)^2} &= \frac{\delta^2}{\delta(iJ)^2} e^{iW[J]} \\
&= \left(\frac{\delta^2 iW[J]}{\delta(iJ)^2} + \frac{\delta iW[J]}{\delta(iJ)} \frac{\delta iW[J]}{\delta(iJ)} \right) e^{iW[J]} \\
&= \left(-i \frac{\delta^2 W[J]}{\delta J^2} + \frac{\delta W[J]}{\delta J} \frac{\delta W[J]}{\delta J} \right) e^{iW[J]} \; .
\end{aligned} \tag{1.73}$$

In the above the disconnected piece of the propagator is given by $\frac{\delta W[J]}{\delta J} \frac{\delta W[J]}{\delta J}$. This then is the one point function squared. The connected one point function is the same thing as the full one point function. This makes sense as there cannot be disconnected n-point functions making up the one point function. The term $-i\frac{\delta^2 W[J]}{\delta J^2}$ must be the connected two point function.

Using this, the connected n-point function is defined as [10]

$$\begin{aligned}\langle\Omega|T\{\phi(x_1)...\phi(x_n)\}|\Omega\rangle_{\text{con}} &= \frac{\delta^n iW[J]}{\delta(iJ)^n} \\ &= (i)^{1-n}\frac{\delta^n W[J]}{\delta J^n} \ . \end{aligned} \tag{1.74}$$

.

Thus the connected propagator, which is the connected two point function, is given by $-i\frac{\delta^2 W[J]}{\delta J^2}$. The connected propagator will be referred to as G in this thesis.

1.7.2 Schwinger-Dyson Equations

The Schwinger-Dyson Equations are equations of motion for the connected n-point functions of the theory. Again, this is a very lengthy derivation so an outline is provided.

To start, one exploits a constant shift in the integration variable of $Z[J]$ and using the full form of the action of the theory an equation of motion for the integration variable of $Z[J]$ is found (equation (1.78) below). Then these fields are considered as functional derivatives acting on $Z[J]$, where $Z[J] = e^{iW[J]}$. Thus the functional derivatives are acting on $e^{iW[J]}$, which allows one to express everything in terms of the connected n point functions (equation (1.79) below). By carrying out this differentiation the equation of motion for the one point function is found , this is known as the Schwinger-Dyson equation of the one point function (equation (1.81) below). To make the result clearer, one multiplies through by the inverse of the differential operator, which is G_0, and everything is put into diagrammatic notation to find equation (1.84) below.

It is found that the S-D equation of the one point function depends on both the connected two and three point functions. To find the diagrammatic Schwinger-Dyson equations of both the connected two and three point functions functional derivatives of both sides are taken. One finds that the S-D equations form an infinite hierarchy.

Onto the detailed derivation of the Schwinger-Dyson equations

The Derivation of the Schwinger-Dyson Equations

The starting point for the Schwinger-Dyson equations is the generating functional where as a reminder,

$$Z[J] = e^{iW[J]} = \int D[\phi]e^{iS[\phi]+iJ\phi} \ . \tag{1.75}$$

The above integral is constant under shifts in the integration variable. Then by considering an infinitesimal shift it is found that ,

$$\int D[\phi] \frac{\delta}{\delta \phi} e^{iS[\phi]+iJ\phi} = 0 \,. \tag{1.76}$$

Thus [13],

$$\int D[\phi] e^{iS[\phi]+iJ\phi} \left(S'[\phi] + J\right) = 0 \,. \tag{1.77}$$

This sets up the identity needed to get the Schwinger-Dyson equations. From here, using the form of the action sets up the Schwinger-Dyson equations for the scalar ϕ^4 theory.

$$0 = \int D[\phi] \left(-\left(\Box + m^2\right)\phi - \tfrac{g^2}{3!}\phi^3 + J \right) e^{iS[\phi]+iJ\phi} \tag{1.78}$$

Using the same trick from before that, $\phi^n e^{iS+iJ\phi} = \frac{\delta^n e^{iS+iJ\phi}}{\delta(iJ)^n}$, equation (1.78) can be re-written. For neatness let $S[\phi, J] = iS + iJ\phi$ then [13],

$$\begin{aligned} 0 &= \int D[\phi] \left(-\left(\Box + m^2\right) \frac{\delta}{\delta(iJ)} - \frac{g^2}{3!} \frac{\delta^3}{\delta(iJ)^3} + J \right) e^{iS[\phi,J]} \\ &\quad \text{The } \frac{\delta^n}{\delta(iJ)^n} \text{ terms don't depend on } \phi \text{ so they come out the integral} \\ &= \left(-\left(\Box + m^2\right) \frac{\delta}{\delta(iJ)} - \frac{g^2}{3!} \frac{\delta^3}{\delta(iJ)^3} + J \right) \int D[\phi] e^{iS[\phi,J]} \\ &\quad \text{Note that } \int D[\phi] e^{iS[\phi,J]} = e^{iW[J]}, \\ &= \left(-\left(\Box + m^2\right) \frac{\delta}{\delta(iJ)} - \frac{g^2}{3!} \frac{\delta^3}{\delta(iJ)^3} + J \right) e^{iW[J]} \,. \end{aligned} \tag{1.79}$$

By carrying out the differentiation, one can obtain the Schwinger-Dyson equation for the 1-point function.

With a bit of work, one can show that

$$\frac{\delta^3 e^{W[J]}}{\delta(iJ)^3} = \left[\frac{\delta^3 iW}{\delta(iJ)^3} + 3 \frac{\delta^2 iW}{\delta(iJ)^2} \frac{\delta iW}{\delta(iJ)} + \left(\frac{\delta iW}{\delta(iJ)} \right)^3 \right] e^{iW[J]} \,. \tag{1.80}$$

Putting this into equation (1.79)

$$\left(-\left(\Box + m^2\right) \frac{\delta iW}{\delta(iJ)} - \frac{g^2}{3!} \frac{\delta^3 iW}{\delta(iJ)^3} - 3 \frac{g^2}{3!} \frac{\delta^2 iW}{\delta(iJ)^2} \frac{\delta iW}{\delta(iJ)} - \frac{g^2}{3!} \left(\frac{\delta iW}{\delta(iJ)} \right)^3 + J \right) e^{iW[J]} = 0 \tag{1.81}$$

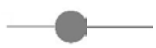
then noting that $G_0\left(\Box + m^2\right) = i\delta$ and multiplying through by G_0 gives

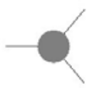
$$\left(\frac{\delta iW}{\delta(iJ)} - \frac{g^2}{3!}iG_0\frac{\delta^3 iW}{\delta(iJ)^3} - 3\frac{g^2}{3!}iG_0\frac{\delta^2 iW}{\delta(iJ)^2}\frac{\delta iW}{\delta(iJ)} - \frac{g^2}{3!}iG_0\left(\frac{\delta iW}{\delta(iJ)}\right)^3 + iG_0 J\right)e^{iW[J]} = 0\ . \tag{1.82}$$

Where, using the definition given in equation (1.74), $\frac{\delta iW}{\delta(iJ)}$ gives the one point function; $\frac{\delta^2 iW}{\delta(iJ)^2}$ gives the connected two point function and $\frac{\delta^3 iW}{\delta(iJ)^3}$ gives the connected three point function. This equation is expressing the one point function in terms of both the two and the three point functions. This is a governing equation of the one point function, in it contains all the information of the one point function. The problem is that the equations of both the connected two and three point functions are necessary to fully describe the equation. To find these equations, its easier to turn to diagrammatic notation.

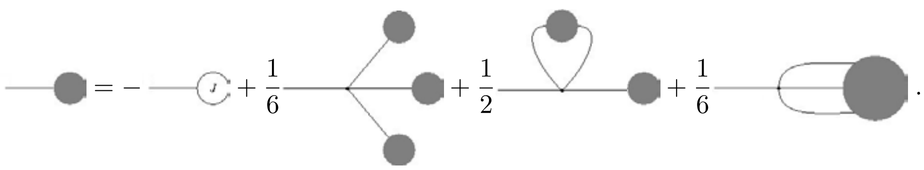
One can assign diagrammatic notation to equation (1.82) where free propagators,the source and vertices are defined as in equations (1.64) - (1.66). The connected n point functions are defined as

$$\frac{\delta iW}{\delta(iJ)} =$$ 

$$\frac{\delta^2 iW}{\delta(iJ)^2} =$$ 

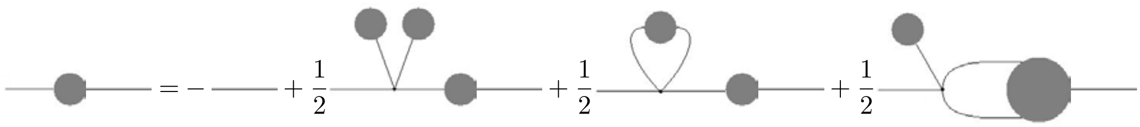
$$\frac{\delta^3 iW}{\delta(iJ)^3} =$$  (1.83)

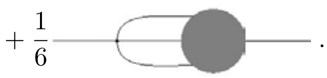
The rest of the connected n-point functions can be defined in the same manner, as n legs from a shaded blob. Thus the Schwinger-Dyson equation of the one point function becomes, [13]:


(1.84)

To get the higher n-point functions' Schwinger-Dyson equations, derivatives of equation (1.82) (or diagrammatically equation (1.84)) with respect to J are taken. Diagrammatically, the J derivative of a shaded in blob amounts to adding another leg to the blob. When varying the term to the left of the equals sign with respect to J, one gets the connected two point function.

Thus varying the equation of the one point function [13],

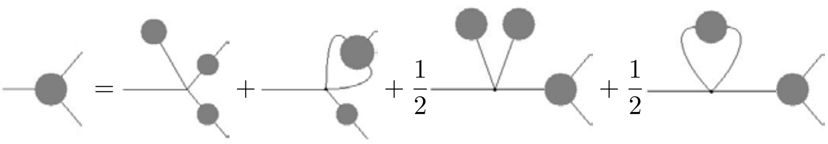


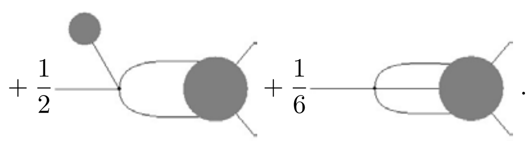


(1.85)

This gives the governing equation of the connected 2 point function (propagator). A problem is now visible, in the process of obtaining the governing equation of the two point function, it is found that this equation depends on both the three and four point functions.

The three point function is given by,





(1.86)

The equation of the three point function in turn depends on the four and five point functions.

Although the equations completely describe the n-point functions, they always depend on the n+2 point function. This has set up an infinite hierarchy of equations, which cannot be solved without an approximation that reduces the system to a finite one.

The Schwinger Dyson equations are useful as they provide one with a sanity check. Whatever is done, the solutions that are found must be contained within the Schwinger Dyson equations.

The One Point Function and Large J

This is the first instance where having a large source J as described by equation (1.51) plays a role.

By looking at the leading term for the Schwinger-Dyson equation of the one point function, one can see that $\frac{\delta W[J]}{\delta J} \sim J$. Since $J \sim O(g^{-1})$ this implies that the one point function is $O(g^{-1})$ i.e.

$$\frac{\delta W[J]}{\delta[J]} \sim O(g^{-1}) \ . \tag{1.87}$$

This implies that fields in the background of a large source are also large. This idea is very important within this thesis.

1.8 A Loop Expansion, Classical Objects and Strong Sources

In this section two different ways of showing that a $\hbar$ expansion is equivalent to a loop expansion will be done.

Then the relationship between a loop expansion and classical objects is discussed.

Before starting with the loop counting, it will turn out to be useful to be able to classify a general vacuum diagram by how many loops, vertices, source inserts and internal and external lines it has. It is useful to have these relationships as the loop counting methods will provide $\hbar$ factors for each of these objects.

First one can determine the relationship between the number of vertices, source inserts and internal and external lines. Following this, the relationship between the number of loops, vertices, and internal lines is determined.

Vertices, Source Inserts and Internal and External Lines

In an arbitrary vacuum diagram in a ϕ^4 theory, one can determine the relationship between vertices, source inserts and internal and external lines by considering the way these objects need to connect. Each vertex has 4 ends, thus each vertex needs four ends to be saturated. Each source term J can saturate one end, each internal propagator line can saturate 2 ends and each external propagator can saturate one end. Thus,

$$4N_v = N_J + 2N_I - N_E \tag{1.88}$$

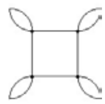
where N_J is the number of J inserts and N_E is the number of external lines, N_I is the number of internal lines and N_v is the number of vertices.

Now onto the general number of loops in a diagram.

Number of Loops in a Diagram

There are two separate arguments that can be made in this light. The first relies on the topology of the diagrams, the second relies on momentum integrals. The common argument is the second one.

The first argument is started by looking at the maximum number of loops a diagram with N_v vertices can have. It turns out the maximum number of loops $N_L^{\max} = N_v + 1$. This can be seen by considering that each vertex can have a maximum number of 1 loop on it, then two free propagator lines that need to connect up. For the diagram to be a connected diagram these extra lines will have link up all the vertices, this means that in joining up the vertices one more loop is created. As an example for 4 vertices,



. (1.89)

By looking at this one can be convinced that even by cutting up and re-joining lines as long as this is the starting diagram, no more loops than 5 can be made from these four vertices.

So the general rule is that $N_L^{\max} = N_v + 1$.

Now if one is to consider a general number of loops, consider what happens when one loop line is removed or 'cut'. This can be replaced by a combination of J and external lines. Specifically it can be seen that $N_L^{\max} = N_L + \frac{N_J}{2} + \frac{N_E}{2}$. This then identifies

$$N_L + \frac{N_J}{2} + \frac{N_E}{2} = N_v + 1\ . \tag{1.90}$$

One can now eliminate N_J and N_E from the above by utilizing equation (1.88), this can then be shown to give [14],

$$N_L = N_I - N_v + 1\ . \tag{1.91}$$

The second argument that can be made is briefly described. It is known that a loop is described as a momentum integral. Now consider that in a diagram, each propagator brings a momentum integral and each vertex brings a delta function that 'kills' an integral and that there is an over all momentum conservation integral for the diagram. So the total number of momentum integrals in a diagram are $N_I - N_v + 1$ where the one is the overall momentum integral. It was stated that the loops are given by momentum integrals. Thus if there are $N_I - N_v + 1$ integrals there must be $N_I - N_v + 1$ loops. This brings one back to the result of equation (1.91) [14].

1.8.1 Intuitive Loop Counting

The argument made in this section is taken from [16].

In the section on path integrals above $\hbar$ was set to 1. Now, the $\hbar$ dependence is re-introduced..

In this case $Z[J] = N \int D[\phi] e^{\frac{i}{\hbar} S[\phi, J]}$.

So in the numerator, for a ϕ^4 theory

$$\frac{i}{\hbar}\left(-\frac{1}{2}\phi\left(\Box + m^2\right)\phi - \frac{g^2}{4!}\phi^4 + J\phi\right) = \phi\frac{G_0^{-1}}{\hbar}\phi - \frac{i}{\hbar}\frac{g^2}{4!}\phi^4 + \frac{iJ}{\hbar}\phi\ . \tag{1.92}$$

Where the fact that the inverse free propagator of a theory is described by the kinetic term of the theory is used.

From this one can read off the effects of the $\hbar$ in the theory. Consider that every factor of G_0^{-1} will come with an $\hbar^{-1}$, this means that each G_0 will be coming with an $\hbar$. Each vertex will bring a factor $\hbar^{-1}$.

Thus in general, using equation (1.91), $\hbar^{N_L} = \hbar^{N_I}\hbar^{-N_v}\hbar\hbar^{N_E}$. For constant external lines, the $\hbar$ power will count the loop power. In the case of the vacuum diagrams (where $N_E = 0$), the power of $\hbar$ will be counting $N_L - 1$ directly.

1.8.2 Alternative Loop Counting Method

One can also use an alternative method to do loop counting, it is less intuitive but will be useful later. In this case one absorbs the $\hbar$ into the integration variables such that only the vertex and J will carry factors of $\hbar$.

Starting with the $\hbar$ exposed $Z[J]$,

$$Z[J] = N \int D[\phi] e^{\frac{i}{\hbar}\left(-\phi G_0^{-1}\phi - \frac{g^2}{4!}\phi^4 + J\phi\right)} . \tag{1.93}$$

Since the integration variable ϕ is integrated over all space, we can get rid of some of the $\hbar$ dependence by re-defining the integration variable [17].

Let $\phi \Rightarrow \sqrt{\hbar}\phi$.

Due to the Jacobian this leads to $D[\phi] \Rightarrow \sqrt{\hbar}D[\phi]$.

Then using this change of variables and absorbing the $\hbar$ of the integration measures into the normalization constant,

$$\begin{aligned} Z[J] &= N \int D[\phi] e^{\frac{i}{\hbar}\left(-(\sqrt{\hbar}\phi)G_0^{-1}(\sqrt{\hbar}\phi) - \frac{g^2}{4!}(\sqrt{\hbar}\phi)^4 + \sqrt{\hbar}J\phi\right)} \\ &= N \int D[\phi] e^{i\left(-\phi G_0^{-1}\phi - \hbar\frac{g^2}{4!}\phi + \frac{1}{\sqrt{\hbar}}J\phi\right)} . \end{aligned} \tag{1.94}$$

Thus it can be seen that each power of the coupling constant brings with it a power of $\hbar$ whilst each power of J will bring down a power of $\hbar^{-1/2}$.

Starting with equation (1.91) one can inert equation (1.88) solved for N_I. Here only vacuum diagrams are considered thus $N_E = 0$. Doing this gives,

$$\begin{aligned} N_L &= (2N_v - \frac{1}{2}N_J) - N_v + 1 \\ &= N_v - \frac{1}{2}N_J + 1 \end{aligned} \tag{1.95}$$

a general diagram in powers of $\hbar$ is given by

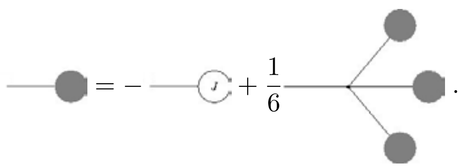
$$\begin{aligned}\hbar^{N_v}\hbar^{-N_J/2} &= \hbar^{N_v - N_J/2} \\ &= \hbar^{N_L - 1} \; . \end{aligned} \tag{1.96}$$

Thus absorption of the $\hbar$'s will lead to the same loop counting.

1.8.3 Classical vs. Quantum Objects Using Loops

Loops are a quantum effect since a loop represents a particle interacting with itself. This sort of behaviour just is not possible on a classical level, classical objects have no way of interacting with themselves. This observation leads to some powerful conclusions, the most important of which is that the leading order term in a loop expansion is a classical term. The rest of the terms provide quantum effects.

Throughout this thesis, classical objects may also be refereed to as tree level objects. The word tree is describing Feynmann diagrams that contain no loops. A good example of the tree behaviour is found by looking at the Schwinger-Dyson equation of the one point function equation (1.84). Since, as was just explained, the loops are quantum effects one should discard these to find the classical solution. So in equation (1.84), dropping all the loop terms leaves one with,



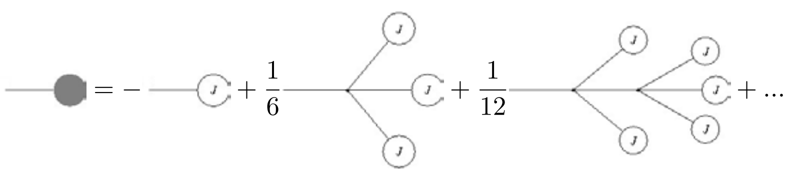
(1.97)

Note that if one is to consider dropping loop terms from the mathematical statement of the S-D equation of the one point function, then equation (1.82) becomes

$$\left(\frac{\delta iW}{\delta(iJ)} - \frac{g^2}{3!}iG_0\left(\frac{\delta iW}{\delta(iJ)}\right)^3 + iG_0 J\right)e^{iW[J]} = 0 \; . \tag{1.98}$$

This is describing the equation of motion for the classical one point function.

Diagrammatically equation (1.97) can be solved recursively to find



(1.99)

where each subsequent term replaces one J blob with three more. Thus the name tree level, the above picture looks like a set of trees with more and more branches. This procedure will not produce any loops.

Finally, classical objects may also be referred to as leading order objects. This must be understood in light of the leading order term of the loop expansion being a classical object.

1.8.4 Loop Expansion and Strong Sources

When dealing with small or no sources, an expansion in g (the coupling constant of the theory) is valid, however, with $J \sim O(g^{-1})$ one must be careful. It will be shown that for the classical field that the number of vertices is independent of the order in g of the diagram [16]. Thus the diagrams don't decrease in size with increasing number of vertices i.e. all the tree diagrams are of the same order.

Look at the expression given in equation (1.99) and consider the order in g of the terms shown. All three terms are $O(g^{-1})$. It will now be shown that every term in this sequence is $O(g^{-1})$ to all orders in the iteration.

One constructs this recursive expression by removing a J bubble, thus removing g^{-1}, then placing a coupling constant which is of order g^2 followed by three more J bubbles thereby multiplying in g^{-3}. The overall effect of removing the g^{-1} is replacing it with $g^2.g^{-3} = g^{-1}$. So each subsequent term in the sequence just removes then replaces the g^{-1}. The overall effect is that each term in the sequence, no matter how many branches it has, is $O(g^{-1})$.

As one adds more branches, more vertices are added. This means that each term in the sequence has more vertices. Since each term in the expansion is $O(g^{-1})$, increasing the number of vertices has no effect on the size of the terms. Thus an expansion in the coupling constant isn't valid as the terms don't decrease in size with the increase in vertices.

This property is true for all tree level objects if $J \sim O(g^{-1})$.

A Loop Expansion with a Strong Source

The fact that an expansion in vertices is not valid leads one to look at the next most natural way to do an expansion, a loop expansion. The idea here is to prove that as one increase loops, the terms become parametrically smaller.

In terms of g a general diagram is given by

$$g^{2N_v} g^{-N_J} = g^{2(N_v - \frac{1}{2}N_J)} \; . \tag{1.100}$$

Refer back to equation (1.88). One would like to consider n-point functions, this means re-introducing the term counting the external lines. This means equation (1.95) becomes

$$N_L = N_v - \frac{1}{2}N_J - \frac{N_E}{2} + 1 \; . \tag{1.101}$$

Putting this result back into equitation (1.100) a general n point function diagram has [16]

$$g^{2N_L + N_E - 2} \; . \tag{1.102}$$

This proves that the size of the terms decrease with increasing loops and thus a loop expansion can work within the background of a large source.

If one considers the one point function, this has $N_E = 1$. Putting this into equation (1.102)gives a result that is consistent with the findings that the tree level one point function is $O(g^{-1})$.

This shows that a loop expansion is a good way of obtaining information on a theory when in the presence of a strong source. It also shows that classical objects will be very important within this thesis as these will make up the leading order terms in any valid expansion, a fact that is heavily relied on within the Classical-Statistical method. [16].

Both the Classical-Statistical method as well as the 2 P.I. method rely on loop expansions and thus both can validly take care of this situation where one has a large source.

As a side note, one can see that the result of equation (1.102) doesn't depend on the number of vertices. This is further proof that a diagram's order in g is unrelated to the number of vertices it contains when one considers $J \sim O(g^{-1})$ [16].

Chapter 2

One and Two Particle Irreducible Action

In classical physics one can vary the action of the theory to get the equation of motion for a degree of freedom. It turns out that the action S of a QFT isn't capable of producing an equation of motion with quantum information. This chapter is about discovering an effective action that, when varied, can provide a fully quantum equation of motion.

In this chapter it will first be proven that S is not capable of producing loops. Then the one Particle Irreducible (1 P.I.) action is introduced, it is shown that this has the properties required of an object that can provide the full quantum equations of motion. Some consequences of the 1 P.I. result are explored, the most important of which is the introduction of the classical propagator. Following this the 2 Particle Irreducible (2 P.I.) action is introduced. This object upgrades the connected propagator to become a degree of freedom in theory. Lastly it is shown that the 2 P.I. action can be used to re-write the Schwinger-Dyson equations of section 1.7.2.

2.1 Equations of Motion due to the Action S

As a reminder, the action under consideration is that of a ϕ^4 theory.

Starting with the ϕ^4 action, the equation of motion for the field ϕ is expected to be given by $\frac{\delta S[\phi]}{\delta \phi}\big|_{\phi=\varphi} = -J$. Where φ is the value of the field that satisfies the equation of motion. Thus the equation of motion according to the action is

$$\left(\Box + m^2\right)\varphi + \frac{g^2}{3!}\varphi^3 = J \; . \tag{2.1}$$

Note that this expression is exactly that of equation (1.98) of section 1.8.3 as long as the above equation is multiplied through by G_0^{-1}. In this section, equation (1.98) was described as the equation of motion for the classical one point function as it contained no loops.

This identifies φ as the classical one point function and shows that varying S produces an equation of motion for the classical one point function only.

From this point the field that satisfies equation (2.1) is called the classical field (φ).

Thus it has been demonstrated that the action S is not enough to describe the full equations of motion of the quantum theory.

2.2 The 1 P.I. Effective Action

2.2.1 Definition of the 1 P.I. effective action

Recall that an object that produces the full equation of motion is desired for the QFT. Thus it would be nice to have an object$\Gamma[\ \phi]$ that behaves as $S[\phi]$ but includes quantum corrections. What is required is an object that gives $\frac{\delta\Gamma}{\delta\phi}|_{\phi=\bar{\phi}}= -J$, where $\bar{\phi}$ is the value of the field where the equation is satisfied.

Exploiting the Legendre transformation, the above requirements can be met.

Recall that $W[J]$ is the generating functional of the connected diagrams. Since the full theory can be described using connected diagrams, $W[J]$ must contain all quantum information of the theory. Thus a good starting point is $W[J]$.

$W[J]$ cannot be used directly as the effective action as it doesn't depend on the field, the physical degree of freedom, in anyway but on the source term J. One can, however, use the Legendre transformation to create an object that depends on the full one point function while eliminating explicit dependence on J.

The following argument is found in [18]:

The Legendre transformation of $W[J]$ is defined as follows;

$$\Gamma[\bar{\phi}] = W[J] - J\frac{\delta W[J]}{\delta J} \tag{2.2}$$

where the following notation is introduced,

$$\bar{\phi}[J] = \frac{\delta W[J]}{\delta J}\ . \tag{2.3}$$

This defines $\bar{\phi}$ as the one point function. Note is has J dependence.

It will now be shown that this Legendre transform has the required property that $\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}} = -J$

$$\begin{aligned}
\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}} &= \frac{\delta}{\delta\bar{\phi}}\left(W[J] - \bar{\phi}[J]J\right) \\
&= \frac{\delta W[J]}{\delta\bar{\phi}} - J - \bar{\phi}[J]\frac{\delta J}{\delta\bar{\phi}} \\
&= \frac{\delta W[J]}{\delta J}\frac{\delta J}{\delta\bar{\phi}} - J - \bar{\phi}[J]\frac{\delta J}{\delta\bar{\phi}} \\
&= \bar{\phi}[J]\frac{\delta J}{\delta\bar{\phi}} - J - \bar{\phi}[J]\frac{\delta J}{\delta\bar{\phi}} \\
&= -J
\end{aligned} \tag{2.4}$$

$\Gamma[\bar{\phi}]$ inherits all quantum effects from $W[J]$ and thus incorporates loop corrections. This is the object that was desired.

In the following section, the explicit form of$\Gamma[\ \bar{\phi}]$ up to 1 loop order will be derived.

2.2.2 Deriving $\Gamma[\bar{\phi}]$ to 1 Loop

$\Gamma[\bar{\phi}]$ is only derived up to 1 loop order as this is all that is necessary for what is required later in this thesis. As was shown in section 1.8.2 an $\hbar$ expansion is the same as doing a loop expansion. Accordingly explicit units of $\hbar$ will be reintroduced as in section 1.8.2.

Once again a very long derivation will be encountered, so what follows is a brief outline of what will be done.

The derivation starts with the $Z[J]$ defined with the $\frac{1}{\hbar}$. Then one shifts the integration variable by $\bar{\phi}$ as this allows one to expand around the large $\bar{\phi}$ (which is $O(g^{-1})$ in the presence of $J \sim O(g^{-1})$ as was discussed in section 1.7.2) (see equation (2.8) below). The end goal is to Legendre transform $W[J]$, thus one can take ln of both sides to get everything in terms of $W[J]$.

Now everything is ready to focus on the J in the exponent of the integrand (equation (2.9) below). This is eliminated by using a very clever trick where $\bar{\phi}$ is split into a classical part φ, and a part that is $O(\hbar)$, this allows one to express the classical equation of motion as the derivative of $S[\bar{\phi}]$ in terms of J and a part that's $O(\hbar)$ (equation (2.11) below). Then the J in the exponential can be cancelled off, while keeping track of the order in $\hbar$ of all terms involved (equation (2.12) below). Keep in mind that one would like to an $\hbar$ expansion so keeping track of the $\hbar$'s is important.

To finally do the loop expansion an $\hbar$ expansion is performed in the same way as section 1.8.2, i.e. by absorbing $\hbar$'s into the integration variables. In the new variables one can split off the leading order term from the $O(\hbar)$ term such that one finds a term $O(\hbar)$ added to a term $O(\hbar^2)$ (equation (2.18) below). This provides a $\hbar$ expansion in up to up to $O(\hbar)$ i.e. one loop. Finally the last J term is eliminated using the Legendre transformation. This then completes the one loop order expression for $\Gamma[\bar{\phi}]$ (equation (2.19) below).

This is the plan for the next few pages. The proof is important to do properly as it is difficult to find a proof done thoroughly in the literature. What follows is the details of the above plan.

The Derivation

Starting with the definition of the generating functional $Z[J]$,

$$Z[J] = e^{\frac{i}{\hbar}W} = N \int D[\phi] e^{\frac{i}{\hbar}(S[\phi]+\int J\phi)} \ . \tag{2.5}$$

The above integral is invariant under constant shifts of the integration variable, thus let $\phi \to \phi + \bar{\phi}$ in the integral in the numerator [17]. Then since $\int D[\phi] = \int D[\phi + \bar{\phi}]$

$$e^{\frac{i}{\hbar}W[J]} = N \int D[\phi] e^{\frac{i}{\hbar}\left(S[\phi+\bar{\phi}]+J\left(\phi+\bar{\phi}\right)\right)} \ . \tag{2.6}$$

The term $S[\phi + \bar{\phi}]$ can be expanded.

$$S[\phi+\bar{\phi}] = S[\bar{\phi}] + \phi S'[\bar{\phi}] + \frac{1}{2}\phi^2 S''[\bar{\phi}] + \underbrace{\frac{1}{3!}\phi^3 S'''[\bar{\phi}] + \frac{1}{4!}\phi^4 S''''[\bar{\phi}]}_{\equiv \mathcal{I}\{\phi,\bar{\phi}\}} \tag{2.7}$$

Then inserting equation (2.7) into equation (2.6) gives

$$e^{\frac{i}{\hbar}W[J]} = N\int D[\phi] e^{\frac{i}{\hbar}\left(S[\bar{\phi}]+\phi S'[\bar{\phi}]+\frac{1}{2}\phi^2 S''[\bar{\phi}]+\mathcal{I}\{\phi,\bar{\phi}\}+J\left(\phi+\bar{\phi}\right)\right)} . \tag{2.8}$$

Putting everything in terms of $W[J]$,

$$\begin{aligned} W[J] &= -i\hbar\ln\left[N\int D[\phi]e^{\frac{i}{\hbar}\left(S[\bar{\phi}]+\phi S'[\bar{\phi}]+\frac{1}{2}\phi^2 S''[\bar{\phi}]+\mathcal{I}\{\phi,\bar{\phi}\}+J\left(\phi+\bar{\phi}\right)\right)}\right] \\ &\quad \text{Pulling out all terms that don't depend on } \phi \text{ from the integral} \\ &= -i\hbar\ln\left[e^{\frac{i}{\hbar}\left(S[\bar{\phi}]+J\bar{\phi}\right)}N\int D[\phi]e^{\frac{i}{\hbar}\left(\phi S'[\bar{\phi}]+\frac{1}{2}\phi^2 S''[\bar{\phi}]+\mathcal{I}\{\phi,\bar{\phi}\}+J\phi\right)}\right] \\ &\quad \text{Using } \ln[ab]=\ln[a]+\ln[b] \\ &= S[\bar{\phi}]+J\bar{\phi}-i\hbar\ln\left[N\int D[\phi]e^{\frac{i}{\hbar}\left(\phi S'[\bar{\phi}]+\frac{1}{2}\phi^2 S''[\bar{\phi}]+\mathcal{I}\{\phi,\bar{\phi}\}+J\phi\right)}\right] \end{aligned} \tag{2.9}$$

Since the goal is to find an object that doesn't depend on J, eliminating J in the exponential is necessary. The $J\bar{\phi}$ term will be dealt with a little later. A clever idea found in [17] is to find the relationship between the tree level approximation to $\bar{\phi}$, φ (see section 2.1) and $\bar{\phi}$. This relationship can be defined as,

$$\varphi = \bar{\phi} + \phi_\hbar \tag{2.10}$$

$\phi_\hbar$ is defined such that it is of order $\hbar$. The justification for equation (2.10) is that the full one point function must contain the classical field (classical 1 point function), as was seen in section 1.8.3. Since an expansion in powers of $\hbar$ gives loop counting (see section 1.8.2), $\bar{\phi}$ should be φ when $\hbar = 0$ as only terms that are classical will survive when $\hbar = 0$.

Remember that φ is defined to be classical if it satisfies the classical equations of motion. Thus $S'[\varphi] = -J$. Plugging in equation (2.10)

$$\begin{aligned} S'[\varphi]+J &= 0 \\ S'[\bar{\phi}+\phi_\hbar]+J &= 0 \\ S'[\bar{\phi}]+\phi_\hbar S''[\bar{\phi}]+\frac{1}{2!}\phi_\hbar^2 S'''[\bar{\phi}]+\frac{1}{3!}\phi_\hbar^3 S''''[\bar{\phi}]+J &= 0 \\ S'[\bar{\phi}]+J &= \mathcal{I}'\{\phi_\hbar,\bar{\phi}\} \end{aligned} \tag{2.11}$$

where $\mathcal{I}'\{\phi_\hbar,\bar{\phi}\} \sim O(\hbar)$, due to $\phi_\hbar \sim O(\hbar)$

Then equation (2.11) can be inserted into equation (2.9).

$$W[J] = S[\bar{\phi}] + J\bar{\phi} - i\hbar \ln \left[N \int D[\phi] e^{\frac{i}{\hbar}\left(\frac{1}{2}\phi^2 S''[\bar{\phi}] + \mathcal{I}\{\phi,\bar{\phi}\} + \phi\mathcal{I}'\{\phi_\hbar,\bar{\phi}\}\right)} \right] . \tag{2.12}$$

The $\hbar$'s

From here the $\hbar$ will be absorbed by the integration variable ϕ in exactly the same way as in section 1.8.2 to allow for a loop expansion by considering powers of $\hbar$.

Thus $\phi \to \sqrt{\hbar}\phi$. Making this change makes the integration measure change, thus $\int D[\phi] \to \sqrt{\hbar}\int D[\phi]$. This factor $\sqrt{\hbar}$ is immediately absorbed into the normalization constant

Thus 2.12 becomes

$$W[J] = S[\bar{\phi}] + J\bar{\phi} - i\hbar \ln \left[N \int D[\phi] e^{\frac{i}{\hbar}\left(\frac{1}{2}\left(\sqrt{\hbar}\phi\right)^2 S''[\bar{\phi}] + \mathcal{I}\{\sqrt{\hbar}\phi,\bar{\phi}\} + \sqrt{\hbar}\phi\mathcal{I}'\{\phi_\hbar,\bar{\phi}\}\right)} \right] . \tag{2.13}$$

Looking back at equation (2.7) it is seen that $\mathcal{I}\{\sqrt{\hbar}\phi, \bar{\phi}\} \sim O(\hbar^{\frac{3}{2}})$. Thus $\frac{1}{\hbar}\mathcal{I}\{\sqrt{\hbar}\phi, \bar{\phi}\} \sim O(\sqrt{\hbar})$.

In (2.11) it was pointed out that $\mathcal{I}'\{\phi_\hbar, \bar{\phi}\} \sim O(\hbar)$. This gives $\frac{1}{\hbar}\sqrt{\hbar}\phi\mathcal{I}'\{\phi_\hbar, \bar{\phi}\} \sim O(\sqrt{\hbar})$

For a loop expansion, an $\hbar$ expansion is desired. This means that these terms can be expanded in powers of $\hbar$, but it is at least $O(\sqrt{\hbar})$, thus these terms are $O(\hbar)$ [17].

For simplicity, the terms $O(\hbar)$ are grouped together.

$$\frac{1}{\hbar}\mathcal{I}\{\sqrt{\hbar}\phi, \bar{\phi}\} + \frac{1}{\hbar}\sqrt{\hbar}\phi\mathcal{I}'\{\phi_\hbar, \bar{\phi}\} = O(\hbar) \tag{2.14}$$

Using this,

$$\begin{aligned} W[J] &= S[\bar{\phi}] + J\bar{\phi} - i\hbar \ln \left[N \int D[\phi] e^{i\left(\frac{1}{2}\phi^2 S''[\bar{\phi}] + O(\hbar)\right)} \right] \\ &= S[\bar{\phi}] + J\bar{\phi} - i\hbar \ln \left[\int D[\phi] e^{i\left(\frac{1}{2}\phi^2 S''[\bar{\phi}] + O(\hbar)\right)} \right] - i\hbar \ln[N] . \end{aligned} \tag{2.15}$$

The plan is now to try isolate a term $O(1)$ from the ln term, this would then become a term $O(\hbar)$. To do this one can multiply and divide by term $O(1)$ in $\hbar$

$$
\begin{aligned}
W[J] &= S[\bar{\phi}] + J\bar{\phi} - i\hbar \ln\left[\frac{\int D[\phi]e^{i\frac{1}{2}\phi^2 S''[\bar{\phi}]}}{\int D[\phi]e^{i\frac{1}{2}\phi^2 S''[\bar{\phi}]}}\int D[\phi]e^{i\left(\frac{1}{2}\phi^2 S''[\bar{\phi}]+O(\hbar)\right)}\right] + \mathcal{N} \\
&= S[\bar{\phi}] + J\bar{\phi} - i\hbar \ln\left[\int D[\phi]e^{i\frac{1}{2}\phi^2 S''[\bar{\phi}]}\right] - i\hbar \ln\left[\frac{\int D[\phi]e^{i\left(\frac{1}{2}\phi^2 S''[\bar{\phi}]+O(\hbar)\right)}}{\int D[\phi]e^{i\frac{1}{2}\phi^2 S''[\bar{\phi}]}}\right] + \mathcal{N}
\end{aligned}
\tag{2.16}
$$

then the Gaussian integral can be solved using appendix A.2 such that [17] ,

$$
W[J] = S[\bar{\phi}] + J\bar{\phi} + i\hbar \ln\left[\left(\det S''[\bar{\phi}]\right)^{\frac{1}{2}}\right] - i\hbar \ln\left[\frac{\int D[\phi]e^{i\left(\frac{1}{2}\phi^2 S''[\bar{\phi}]+O(\hbar)\right)}}{\int D[\phi]e^{i\frac{1}{2}\phi^2 S''[\bar{\phi}]}}\right] + \mathcal{N} \ . \tag{2.17}
$$

Note that some terms have been absorbed into $\mathcal{N}$.

In appendix B.1 it is shown that if $F[\phi,\hbar] = \ln\left[\frac{\int D[\phi]e^{g[\phi]}e^{A(\hbar)h[\phi]}}{\int D[\phi]e^{g[\phi]}}\right]$ then is $F[\phi,\hbar] \sim O(\hbar)$. Thus,

$$
W[J] = S[\bar{\phi}] + J\bar{\phi} + i\hbar \ln\left[\left(\det S''[\bar{\phi}]\right)^{\frac{1}{2}}\right] + O(\hbar^2) + \mathcal{N} \tag{2.18}
$$

All that is left of the explicit J dependence is the term $J\bar{\phi}$. This is where the Legendre transformation comes in. Using equation (2.2), i.e.$\Gamma[\ \bar{\phi}] = W[J] - J\bar{\phi}$ gives [17],

$$
\Gamma[\bar{\phi}] = S[\bar{\phi}] + i\hbar \ln\left[\left(\det S''[\bar{\phi}]\right)^{\frac{1}{2}}\right] + O(\hbar^2) + \mathcal{N} \ . \tag{2.19}
$$

$\Gamma[\bar{\phi}]$ is an object that has no explicit dependence on J but on $\bar{\phi}$ as was required. This is the well known loop expansion of the effective action to first order.

2.3 Self Energy and 1 Particle Irreducible

In this section two concepts are explained, self energy diagrams and 1 Particle Irreducible (1 P.I.) diagrams. It will be shown that these objects can be considered building blocks for the connected propagator.

The importance of this is that (as will be shown in the next section)$\Gamma[\ \bar{\phi}]$ is related to the connected propagator, thus the ideas of 1 P.I. diagrams and self energy are closely related to$\Gamma[\ \bar{\phi}]$.

1 P.I. diagrams are defined to be diagrams that don't split into two disconnected pieces after cutting one internal propagator line.

For example;

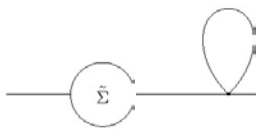
(2.20)

Now it will be shown how the 1 P.I. diagrams can be used to create the connected two point function.

To start consider that the sum of all connected diagrams with 2 external legs will produce the connected 2 point function, i.e. the full connected propagator with all insertions. It clear that this sum of diagrams is infinitely large. This sum of all possible diagrams can be organised in anyway, as long as all possible diagrams are included. A particular organizational choice will now be argued for that involves the 1 P.I. diagrams [11].

The entire sum of possible diagrams that produces the full propagator is required, how can one build up this sum? First start with the obvious, easy choice, the free propagator. One can now add all 1 particle irreducible diagrams to that. Obviously the set of 1 P.I. diagrams is itself, infinite.

Of course one could imagine a diagram that has all 1 P.I. diagrams and attached to that is one loop as follows;



(2.21)

where the $\tilde{\Sigma}$ blob with 2 lines coming from it represents the sum of all 1 P.I. diagrams and, as before, the lines represent free propagators.

Thus simply taking the sum of the free propagator with the 1 P.I. diagrams is not sufficient to include all possible diagrams as the diagram of equation (2.21) needs to be in the sum. Furthermore it is also possible to have a loop on top of loop in equation (2.21). Thus one can add a term to the sum that is 2 sets of 1 P.I. diagrams that are connected by a single free propagator line. Of course the sum is still not complete, the same argument could be made for 3 sets of 1 P.I. diagrams connected by 2 free propagators. One can keep adding terms that are made of multiples of $\tilde{\Sigma}$;

(2.22)

It is easy to see that this sum that has been set up looks like a geometric series. A refinement can be made on the above summation of equation (2.22). An object called the self energy is defined. The self energy is the sum of all 1 P.I. diagrams that have two legs that are amputated. An amputated leg is effectively a delta function that goes from some external space time point to the diagram. The reason this is called the self energy is that if one attached propagators to the amputated parts of the diagram, this diagram will now look like a particle travelling in, interacting through self interactions, as there is nothing else to interact with, then continuing on after the self interactions.

Back to equation (2.22). Since this is a geometric series, there exists a particular solution to this sum. If the free propagator is called G_0, the self energy $\tilde{\Sigma}$ and the connected propagator called G then equation (2.22) can be re-written as the recurring equation [11];

$$G \propto G_0 + G_0 \tilde{\Sigma} G \tag{2.23}$$

This equation has only been derived diagrammatically so, although the structure is correct, the pre-factors may not be. This argument is just to get some idea of the structure of G in terms of this new object called the self energy.

This is an exceptionally important result. It is also consistent with the Schwinger Dyson equation of the two point function.

2.4 $\Gamma[\bar{\phi}]$; the Connected Propagator; Self Energy and the Classical Propagator

In this section$\Gamma[\ \bar{\phi}]$ is shown to be related to the inverse connected propagator G. This can then be used to find the relationship between the self energy and$\Gamma[\ \bar{\phi}]$ by utilizing the expression given in equation (2.23). The eventual use of this discussion finds itself in the introduction of the classical propagator. The classical propagator will play an important role throughout this thesis as classical objects are important in the context of strong sources (see section 1.8.4).

One also finds a relationship between the classical propagator and the connected propagator while introducing the 'strictly' 1 P.I. diagrams. This discussion is vital for the work done in the next section on the 2 P.I. action (see section 2.5.3 below).

2.4.1 $\Gamma[\bar{\phi}]$ and G

The effective action,$\Gamma[\ \bar{\phi}]$ is related to the full propagator by the fact that $\frac{\delta^2\Gamma[\bar{\phi}]}{\delta\phi\delta\phi} = iG^{-1}$. This is now demonstrated.

Starting from equation (2.4) [10]

$$\begin{aligned}
&\frac{\delta^2}{\delta\bar{\phi}}\Gamma[\bar{\phi}] = -J \\
&\qquad \text{Vary both sides with respect to J, and then use chain rule} \\
&\frac{\delta\bar{\phi}}{\delta J}\frac{\delta}{\delta\bar{\phi}\delta\bar{\phi}}\Gamma[\bar{\phi}] = -\delta
\end{aligned} \tag{2.24}$$

Remember (equation (2.3)) $\frac{\delta W[J]}{\delta J} \equiv \bar{\phi}$. Thus

$$\frac{\delta\bar{\phi}}{\delta J} = \frac{\delta^2 W[J]}{\delta J^2} = iG \tag{2.25}$$

.

where, as always, G stands for the full propagator (see section 1.7.2 for justification of i's).

Thus

$$iG\left(\frac{\delta^2\Gamma[\bar{\phi}]}{\delta\bar{\phi}\delta\bar{\phi}}\right) = -\delta\ . \tag{2.26}$$

One can define a G^{-1} such that $GG^{-1} = \delta$. This then identifies [10]

$$\frac{\delta^2\Gamma[\bar{\phi}]}{\delta\bar{\phi}\delta\bar{\phi}} = iG^{-1} \tag{2.27}$$

This is an exceptionally useful result.

2.4.2 The Connection Between the Self Energy and $\Gamma[\bar{\phi}]$

$\Gamma[\bar{\phi}]$ is known as the 1 P.I. action. In this section it is shown that the terms $O(\hbar)$ of$\Gamma[\ \bar{\phi}]$ is the self energy. Since the self energy is constructed from 1 P.I. diagrams one can see how$\Gamma[\ \bar{\phi}]$ is strongly related to the 1 P.I. diagrams. In this section, the idea of the classical propagator is set up, although this is only fully discussed in the next section.

Since this section is interested in the self energy, it makes sense to start with the equation that introduces it, equation (2.23). Then since the relationship between$\Gamma[\ \bar{\phi}]$ and $\tilde{\Sigma}$ is desired, one can use the relationship between$\Gamma[\ \bar{\phi}]$ and G.

So, starting with inverting equation (2.23) to get an expression for G^{-1} in terms of $\tilde{\Sigma}$,

$$G^{-1} \propto G_0^{-1} - \tilde{\Sigma}\ . \tag{2.28}$$

From the previous section $\frac{\delta\Gamma[\bar{\phi}]}{\delta\phi\delta\phi} = iG^{-1}$. Now refer to equation (2.19), and taking the derivative $\frac{\delta}{\delta\phi\delta\phi}$ on both sides of it gives

$$\begin{aligned}
\Gamma[\bar{\phi}] &= S[\bar{\phi}] + i\hbar \ln\left[\left(\det S''[\bar{\phi}]\right)^{\frac{1}{2}}\right] + O(\hbar^2) + \mathcal{N} \\
\frac{\delta^2\Gamma[\bar{\phi}]}{\delta\bar{\phi}^2} &= \frac{\delta^2 S[\bar{\phi}]}{\delta\bar{\phi}^2} + i\frac{\delta^2}{\delta\bar{\phi}^2}\left(\hbar \ln\left[\left(\det S''[\bar{\phi}]\right)^{\frac{1}{2}}\right] + O(\hbar^2)\right) + 0 \\
&\text{Using the fact that this is for a } \phi^4 \text{ theory and that } G_0^{-1} = -i(\Box + m^2) \\
iG^{-1} &= -iG_0^{-1} - \frac{g^2}{2}\bar{\phi}^2 + i\frac{\delta^2}{\delta\bar{\phi}^2}\left(\hbar \ln\left[\left(\det S''[\bar{\phi}]\right)^{\frac{1}{2}}\right] + O(\hbar^2)\right) \\
G^{-1} &= -G_0^{-1} + i\frac{g^2}{2}\bar{\phi}^2 + \frac{\delta^2}{\delta\bar{\phi}^2}\left(\hbar \ln\left[\left(\det S''[\bar{\phi}]\right)^{\frac{1}{2}}\right] + O(\hbar^2)\right) \\
&.
\end{aligned} \tag{2.29}$$

Looking at the above expression and comparing to equation (2.28) gives

$$\tilde{\Sigma} = -i\frac{g^2}{2}\bar{\phi}^2 - i\frac{\delta^2}{\delta\bar{\phi}^2}\left(\hbar \ln\left[\left(\det S''[\bar{\phi}]\right)^{\frac{1}{2}}\right] + O(\hbar^2)\right) \tag{2.30}$$

Thus here lies the connection to of the effective action and it's alternative name as the 1 P.I. effective action. It is this relationship to the self energy, which is made up of 1 P.I. type diagrams. It is useful to use the sign of $\tilde{\Sigma}$ as negative as in equation 2.28 as this leads to certain simplifications along the way.

In the process the pre factors of equitation (2.28) have been fixed.

Thus [1]

$$G^{-1} = -G_0^{-1} - \tilde{\Sigma} \tag{2.31}$$

2.4.3 $\tilde{\Sigma}$ and Classical Propagator

Finally the classical propagator can be found, by exploring the definition of 1 P.I. and carefully examining the terms found in the previous section.

It was claimed that $\tilde{\Sigma}$ is the sum of 1 P.I. diagrams with two amputated legs. The definition of 1 P.I. diagrams are those that don't become disconnected with the cutting of an internal line. The issue is the term $\frac{g^2}{2}\bar{\phi}^2$, seen in equation (2.30), it technically has no internal lines to cut at all. Thus for a true definition of 1 P.I. diagrams, it would be nice to incorporate this term in a different manner, i.e. exclude it from being part of the self energy.

Looking back at equation (2.29),

[1] The sign here may seem unusual, this is purely due to the normalization chosen in section 1.1. If one switches every instance of G_0 to $-G_0$ and every instance of $-G_0^{-1}$ to G_0^{-1} then one gets back to the usual conventions in the literature.

$$\begin{aligned}
G^{-1} &= -G_0^{-1} + i\frac{g^2}{2}\bar{\phi}^2 - \underbrace{\frac{\delta^2}{\delta\bar{\phi}^2}\left(-\hbar\ln\left[\left(\det S''[\bar{\phi}]\right)^{\frac{1}{2}}\right] + O(\hbar^2)\right)}_{\Sigma} \\
&\text{by definition of } G_0^{-1} \\
&= i(\Box + m^2) + i\frac{g^2}{2}\bar{\phi}^2 - \underbrace{\frac{\delta^2}{\delta\bar{\phi}^2}\left(-\hbar\ln\left[\left(\det S''[\bar{\phi}]\right)^{\frac{1}{2}}\right] + O(\hbar^2)\right)}_{\Sigma} \\
&= -i\frac{\delta^2 S}{\delta\bar{\phi}^2} - \underbrace{\frac{\delta^2}{\delta\bar{\phi}^2}\left(-\hbar\ln\left[\left(\det S''[\bar{\phi}]\right)^{\frac{1}{2}}\right] + O(\hbar^2)\right)}_{\Sigma} \ .
\end{aligned} \tag{2.32}$$

So $\frac{\delta^2 S[\bar{\phi}]}{\delta\bar{\phi}^2} = -iG_0^{-1} + \frac{g^2}{2}\bar{\phi}^2$. The Σ term is just a definition, where the Σ term here represents the strictly 1 P.I. diagrams.

One can then re-write

$$G^{-1} = -i\frac{\delta^2 S[\bar{\phi}]}{\delta\bar{\phi}^2} - \Sigma \ . \tag{2.33}$$

In fact it will be shown that $i\frac{\delta^2 S[\bar{\phi}]}{\delta\bar{\phi}^2}$ can be interpreted as the inverse classical propagator such that [2] [8]

$$G^{-1} = -G_t^{-1} - \Sigma \tag{2.34}$$

2.4.4 Classical Propagator

It will now be shown that $i\frac{\delta^2 S[\bar{\phi}]}{\delta\bar{\phi}^2}$ is G_t^{-1}, the inverse classical propagator. The following argument will use diagrammatic notation as things are clearer in this representation.

From the definition of the action it is easy to find that

$$i\frac{\delta^2 S[\bar{\phi}]}{\delta\bar{\phi}^2} = G_0^{-1} - i\frac{g^2}{2}\bar{\phi}^2 \ . \tag{2.35}$$

Now inverting both sides gives,

[2] Again the sign here may seem unusual, this is due to normalizations and if one switches every instance of G_t to $-G_t$ and $-G_t^{-1}$ to G_t^{-1} then one gets back to the usual conventions in the literature.

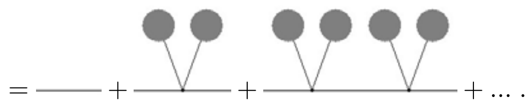
$$
\begin{aligned}
-i\left(\frac{\delta^2 S[\bar{\phi}]}{\delta\bar{\phi}^2}\right)^{-1} =& \frac{1}{G_0^{-1} - i\frac{g^2}{2}\bar{\phi}^2} \\
& \text{Using } G_0^{-1}G_0 = \delta \\
=& \frac{G_0}{\delta - i\frac{g^2}{2}\bar{\phi}^2 G_0} \ .
\end{aligned}
\tag{2.36}
$$

The above can be expanded as a geometric series i.e.

$$
-i\left(\frac{\delta^2 S[\bar{\phi}]}{\delta\bar{\phi}^2}\right)^{-1} = G_0 + G_0\frac{g^2}{2}\bar{\phi}^2 G_0 + ... \tag{2.37}
$$

Remembering that G_0 is represented as a line, that the coupling constant is given by a dot and the field $\bar{\phi}$ is given by a shaded in blob with one leg coming from it diagrammatically this is given by

$$
-i\left(\frac{\delta^2 S[\bar{\phi}]}{\delta\bar{\phi}^2}\right)^{-1} =
$$

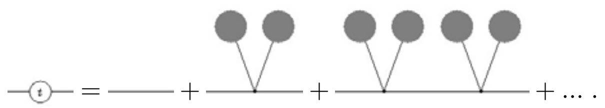


(2.38)

In the above it is easy to see that there are no explicit loops (there are some hidden within the terms $\bar{\phi}$ but the question is more about explicit loops). Thus the term $-i\left(\frac{\delta^2 S[\bar{\phi}]}{\delta\bar{\phi}^2}\right)^{-1}$ is considered to be the classical or tree level propagator. This means that $i\frac{\delta^2 S[\bar{\phi}]}{\delta\bar{\phi}^2}$ is the inverse classical propagator.

The classical propagator is called G_t . Diagrammatically;

$$
G_t =
$$



(2.39)

Classical Propagator and Strong Sources

In section 1.8.4 it was claimed that all the terms that make up a tree level object are the same order in g when there are strong sources. Here this will be confirmed for the tree level propagator.

At the end of section 1.7.2 it was shown that the one point function $\bar{\phi}$ is $O(g^{-1})$. Now look at equation (2.39), each term in the sequence is made from having two $\bar{\phi}$ terms attached to a vertex. The vertex itself brings g^2 whilst each $\bar{\phi}$ brings g^{-1}, so this particular combination of vertex with two $\bar{\phi}$ is $O(1)$. Thus each term in this sequence is of the same order in g. The entire classical propagator is then $O(1)$, this is in agreement with equation (1.102) of section 1.8.4. The strong source plays its role by fixing the size of $\bar{\phi}$.

It is worth emphasising that it is this behaviour which makes tree level objects very important in the presence of a strong source.

2.5 The 2 Particle Irreducible Action

In the 1 P.I. action, there was only one degree of freedom, $\bar{\phi}$. The 2 particle Irreducible (2 P.I.) effective action comes about by introducing a second degree of freedom. Thus the 2 P.I. action is related to that of the 1 P.I. action.

For the 1 P.I. action, the degree of freedom was initially introduced through the source term $J(x)$, then $J(x)$ was eliminated in favour of $\bar{\phi}$. In the same manner a second degree of freedom is introduced by adding a new bi-local source term ($R(x,y)$) into the generating functional such that one has a new generating functional $Z[J,R]$ (equation 2.40 below).

One can then eliminate both J and R in terms of $\bar{\phi}$ and the connected propagator G respectively. This implies that G becomes a degree of freedom. To do this elimination requires a Legendre transformation in two variables. The elimination of J and R in favour of $\bar{\phi}$ and G produces an object$\Gamma[\bar{\phi},G]$ known as the 2 P.I. action.

In this section, after defining and introducing$\Gamma[\bar{\phi},G]$, an expression for$\Gamma[\bar{\phi},G]$ is derived, this is given in equation (2.57) below. Using this result it is found that$\Gamma[\bar{\phi},G]$ is expressed as an infinite sequence of diagrams that are 2 particle irreducible (2 P.I.) i.e. they remain connected diagrams when any two internal lines of a diagram are removed. Finally$\Gamma[\bar{\phi},G]$ is shown to have an impact on the Schwinger-Dyson equations, it turns out that by using$\Gamma[\bar{\phi},G]$, one can derive a set of Schwinger-Dyson equations that are closed. i.e. one doesn't haven't an infinite hierarchy (see section 1.7.2).

2.5.1 Introducing the 2 P.I. Action

In the presence of two sources $J(x)$ and $R(x,y)$, where J is a local source as in the previous section and R is a newly introduced bi-local source, the generating functional is [6]

$$Z[J,R] = N\int D[\phi]e^{i\left[S[\phi]+\int d^4xJ(x)\phi(x)+\frac{1}{2}\int d^4x\int d^4yR(x,y)\phi(x)\phi(y)\right]} \tag{2.40}$$

where, again J is taken to be a physical source of $O(g^{-1})$. The new source term R is taken to be non-physical. This means that R is 0 when working with physical objects.

The 2 P.I. action$\Gamma[\bar{\phi},G]$ is defined as the Legendre transformation of $W[J,R] = -i\ln\left[Z[J,R]\right]$ i.e.

$$\Gamma[\bar{\phi},G] = W[J,R] - \int\frac{\delta W}{\delta J}J - \int\frac{\delta W}{\delta R}R\,. \tag{2.41}$$

The term $\frac{\delta W}{\delta J}$ is the same as before i.e. the one point function $\bar{\phi}$. It is proven bellow that the term $\frac{\delta W}{\delta R}$ is the full two point function with a factor of a half.

$$\begin{aligned}\frac{\delta W}{\delta R(x,y)} &= -i\frac{\delta}{\delta R(x,y)}\ln Z \\ &= N\int D[\phi]\frac{1}{2}\phi(x)\phi(y)e^{i\left[S[\phi]+\int d^4xJ(x)\phi(x)+\frac{1}{2}\int d^4x\int d^4yR(x,y)\phi(x)\phi(y)\right]}\end{aligned} \tag{2.42}$$

where N is a normalization constant. The above is the full two point function multiplied by $\frac{1}{2}$. The full two point function can be split into a connected piece, the connected two point function plus a disconnected piece made of two one point functions. The connected two point function will be labelled G. This G then defines a second degree of freedom in the theory. Thus [6],

$$\frac{\delta W}{\delta R(x,y)} = \frac{1}{2}\left(G(x,y)+\bar{\phi}(x)\bar{\phi}(y)\right) \tag{2.43}$$

So

$$\Gamma[\bar{\phi},G] = W[J,R]-J\bar{\phi}-\frac{1}{2}R\left(G+\bar{\phi}\bar{\phi}\right)\ . \tag{2.44}$$

From this expression we can find the 'stationary conditions' of $\Gamma[\bar{\phi},G]$. The stationary conditions are simply those equations which define the Schwinger-Dyson (S-D) equations of motion for the degrees of freedom in the theory.

$$\frac{\delta\Gamma[\bar{\phi},G]}{\delta\bar{\phi}} = -J-R\bar{\phi} \tag{2.45}$$

and

$$\frac{\delta\Gamma[\bar{\phi},G]}{\delta G} = -\frac{1}{2}R\ . \tag{2.46}$$

2.5.2 Derivation of 2 P.I. Action

The derivation done will closely follow [6] and [8].

The derivation will be done in two parts, the first part will involve removing the J dependence of $W[J,R]$. This is achieved by redefining the action to absorb the new R term. This will allow for the use of the result for the 1 P.I. action from section 2.2.2 as with this redefined action, $Z[J,R]$ will look like $Z[J]$ (equation (2.50) below). The second part will complete the proof by removing the R dependence of $W[J,R]$. This is done by using equation (2.34), but with the redefined action. This results in an expression for R in terms of G, G_t and Σ (equation (2.56) below). Plugging this into the result of removing J gives the required result (equation (2.57) below).

Due to the fact that this proof relies on the result of the 1 P.I. action, this result is presented as a loop expansion until the very end. At the end of the derivation it will turn out to be best to classify all terms $O(\hbar^2)$ as a new term $\Gamma_2[\bar{\phi},G]$. This term is discussed in the following section but it will turn out to be the sum of all 2 P.I. diagrams.

This outlines the proof of the 2 P.I. action. In what follows the details of the above outline are completed.

Removing J Dependence

By making an observation about the exponential of equation (2.40), the proof done for the 1 P.I. action can be very closely followed.

The exponential of equation (2.40) with the ϕ^4 action made explicit looks like

$$-\frac{1}{2}\phi\left(\Box+m^2\right)\phi-\frac{g^2}{4!}\phi^4+J\phi+\frac{1}{2}R\phi^2\ . \tag{2.47}$$

From here one can consider the R term to be some sort of addition to the mass term, where it can be written that $m^R \equiv m^2 - R$ [8]. Thus

$$-\frac{1}{2}\phi\left(\Box+m^2\right)\phi-\frac{g^2}{4!}\phi^4+J\phi+\frac{1}{2}R\phi^2=-\frac{1}{2}\phi\left(\Box+m^R\right)\phi-\frac{g^2}{4!}\phi^4+J\phi. \tag{2.48}$$

Note that eventually the R term in the mass will need to be dealt with as the 2 P.I. action cannot depend on the R term.

To make the connection to the derivation produced for the 1 P.I. action even more explicit, the following identity is made [8];

$$S^R[\phi]=-\frac{1}{2}\phi\left(\Box+m^R\right)\phi-\frac{g^2}{4!}\phi^4\ , \tag{2.49}$$

with this identity

$$W^R[J]\equiv -i\ln\left[\int D\phi e^{i\left[S^R[\phi]+J\phi\right]}\right]\ . \tag{2.50}$$

Finally consider only eliminating the J term for now, this amounts to performing a Legendre transform of $W^R[J]$.

This situation is exactly the situation of the 1 P.I. effective action done in section 2.2.2

Thus one can easily write, directly from equation (2.19) that

$$\begin{aligned}\Gamma[R,\bar{\phi}] &= S^R[\bar{\phi}]+i\hbar\ln\left[\left(\det\left(S^R\right)''[\bar{\phi}]\right)^{\frac{1}{2}}\right]+O(\hbar^2)+\mathcal{N}\\ &= S[\bar{\phi}]+\frac{1}{2}R\bar{\phi}^2+i\hbar\ln\left[\left(\det\left\{S''[\bar{\phi}]+R\right\}\right)^{\frac{1}{2}}\right]+O(\hbar^2)+\mathcal{N}\ .\end{aligned} \tag{2.51}$$

Removing the R dependence

Since $W^R[J] = W[J,R]$ equation (2.41) can be written as

$$\Gamma[\bar{\phi}, G] = \underbrace{W^R[J] - \frac{W^R[J]}{\delta J} J}_{\Gamma[R,\bar{\phi}]} - \frac{W^R[J]}{\delta R} R \tag{2.52}$$

and

$$\frac{W^R[J]}{\delta R} = \frac{W[J,R]}{\delta R} = \frac{1}{2}\left(G + \bar{\phi}\bar{\phi}\right) \; . \tag{2.53}$$

Putting all this together

$$\begin{aligned}\Gamma[\bar{\phi}, G] =& \Gamma[\, R, \bar{\phi}] - \frac{1}{2}\left(G + \bar{\phi}\bar{\phi}\right) R \\ &= S[\bar{\phi}] + \frac{1}{2} R\bar{\phi}^2 + i\hbar \ln\left[\left(\det\left\{S''[\bar{\phi}] + R\right\}\right)^{\frac{1}{2}}\right] + O(\hbar^2) + \mathcal{N} - \frac{1}{2}\left(G + \bar{\phi}\bar{\phi}\right) R \\ &= S[\bar{\phi}] + i\hbar \ln\left[\left(\det\left\{S''[\bar{\phi}] + R\right\}\right)^{\frac{1}{2}}\right] - \frac{1}{2} GR + O(\hbar^2) + \mathcal{N} \; .\end{aligned} \tag{2.54}$$

There is still some dependence on R though

In section 2.4.3 is was shown that $G^{-1} = -i\frac{\delta^2 S[\phi]}{\delta\phi^2} - \Sigma$, this is a defining relationship determined by the theory. In the presence of R the action is changed as seen in equation (2.49). This means that

$$\begin{aligned} G^{-1} &= -i\frac{\delta^2 S^R[\phi]}{\delta\phi^2} - \Sigma^R \\ &= -i\left(-(\Box + m^2) - \frac{g^2}{2}\bar{\phi}^2\right) - iR - \Sigma^R \\ &\text{But } i\frac{\delta^2 S^R[\phi]}{\delta\phi^2} = G_t^{-1} \text{ (section 2.4.4)} \\ &= -G_t^{-1} - iR - \Sigma^R \end{aligned} \tag{2.55}$$

where$\Sigma^{\ R}$ is the self energy diagrams in the presence of the source term R.

Rearranging the above [8]

$$R = iG^{-1} + iG_t^{-1} + i\Sigma^R \; . \tag{2.56}$$

Then plugging in equation (2.56) into (2.54), the explicit R is eliminated in favour of G. The calculation that follows is long but basically just algebra. The only important point to note is that one needs to use the fact that$\Sigma^{\ R}$ is $O(\hbar)$. This allows one to push certain objects into the term classified as $O(\hbar^2)$. The details of the calculation are left to appendix B.2 but note that the identity $\mathrm{Tr}\ln[A] = \ln\det[A]$ is used.

The result from equation (B.3) is that

$$\Gamma[\bar{\phi}, G] = S[\bar{\phi}] + \frac{i}{2}\hbar Tr \ln \left[G^{-1}\right] - \frac{i}{2}Tr\mathbf{1} - \frac{i}{2}Tr\left(G_t^{-1}G\right) + O(\hbar^2) \, .$$

Now group all terms $O(\hbar^2)$ into one unknown term$\Gamma_2[\bar{\phi}, G]$. Thus all two or more loop terms are contained in$\Gamma_2[\bar{\phi}, G]$ [6] .

$$\Gamma[\bar{\phi}, G] = S[\bar{\phi}] + \frac{i}{2}\hbar Tr \ln \left[G^{-1}\right] - \frac{i}{2}Tr\mathbf{1} - \frac{i}{2}Tr\left(G_t^{-1}G\right) + \Gamma_2[\bar{\phi}, G] \tag{2.57}$$

The structure of$\Gamma_2[\bar{\phi}, G]$ is important and is discussed in the next section.

At this point in the discussion, explicit $\hbar$ factors have served their purpose and natural units are re-instated.

2.5.3 $\Gamma_2[\bar{\phi}, G]$ and 2 particle Irreducible

One would like to understand the structure of$\Gamma_2[\bar{\phi}, G]$ of equation (2.57). To do this one can find an expression for$\Gamma_2[\bar{\phi}, G]$ in terms of G from the stationary condition of equation (2.46). This can be compared to the known expression for G^{-1} in terms of Σ given in equation (2.55). It will be in comparing the self energy to the variation of$\Gamma_2[\bar{\phi}, G]$ that will show that$\Gamma_2[\bar{\phi}, G]$ is the sum of all 2 P.I. diagrams. Recall that 2 P.I. diagrams are diagrams that remain connected diagrams after any two internal lines of a diagram are removed

The stationary condition of equation (2.46) shows that $\frac{\delta\Gamma}{\delta G} = -\frac{1}{2}R$.

Referring to equation (2.57),

$$\begin{aligned}\frac{\delta\Gamma[\bar{\phi}, G]}{\delta G} &= \frac{\delta S[\bar{\phi}]}{\delta G} + \frac{i}{2}\frac{\delta Tr \ln \left[G^{-1}\right]}{\delta G} - \frac{i}{2}\frac{\delta Tr\mathbf{1}}{\delta G} - \frac{i}{2}\frac{\delta Tr\left(G_t^{-1}G\right)}{\delta G} + \frac{\delta\Gamma_2[\bar{\phi}, G]}{\delta G} \\ -\frac{1}{2}R &= 0 + \frac{i}{2}\frac{\delta Tr \ln \left[G^{-1}\right]}{\delta G} - 0 - \frac{i}{2}\frac{\delta Tr\left(G_t^{-1}G\right)}{\delta G} + \frac{\delta\Gamma_2[\bar{\phi}, G]}{\delta G} \, .\end{aligned} \tag{2.58}$$

One must take a bit of care in finding $\frac{\delta Tr \ln\left[G^{-1}\right]}{\delta G}$ and $\frac{\delta Tr\left(G_t^{-1}G\right)}{\delta G}$. This will be shown here and not in the appendix as it shows important details on working with functionals.

Starting with the tougher variation: $\frac{\delta Tr \ln\left[G^{-1}\right]}{\delta G}$.

This notation hides information, with space-time coordinates written out fully this reads $\frac{\delta Tr\left\{\ln\left[G^{-1}\right](x',y')\right\}}{\delta G(x,y)}$. In this notation, remember that the Tr term is an integral such that $Tr\left\{\ln\left[G^{-1}\right](x',y')\right\} = \int d^4x' \ln\left[G^{-1}\right](x',x')$.

In full notation what needs to be found is $\frac{\delta \int d^4x' \ln\left[G^{-1}\right](x',x')}{\delta G(x,y)}$. Start by expanding the ln term.

$$
\begin{aligned}
\frac{\delta \int d^4x' \ln\left[G^{-1}\right](x',x')}{\delta G(x,y)} &= -\frac{\delta}{\delta G(x,y)}\left[\int d^4x' \left[\sum_n \frac{(-1)^{n+1}}{n}(G-\delta)^n\right](x',x')\right] \\
&= -\frac{\delta}{\delta G(x,y)}\left[\int d^4x' \left[G-\delta\right](x',x')\right. \\
&-\frac{1}{2}\int d^4x' \int d^4z \left[G-\delta\right](x',z)\left[G-\delta\right](z,x') \\
&\left.+\frac{1}{3}\int d^4x' \int d^4z \int d^4z' \left[G-\delta\right](x',z)\left[G-\delta\right](z,z')\left[G-\delta\right](z',x')\right] + ... \\
&\text{Using } \frac{G(a,b)}{G(x,y)} = \delta(a,x)\delta(b,y) \text{ and product rule} \\
&= -\left[\delta(y,x) - \left[G-\delta\right](x,y) + \int d^4z' \left[G-\delta\right](x,z')\left[G-\delta\right](z',y) + ...\right] \\
&= -\delta(x,y) + \left[G-\delta\right](x,y) - \left[G-\delta\right]^2(x,y) + ... \\
&= -\left[\frac{1}{G}\right](x,y) \\
&= -G^{-1}(x,y)\ . \qquad (2.59)
\end{aligned}
$$

Given the same reasoning as above, it can be shown that

$\frac{\delta Tr\left(\left[G_t^{-1}G\right](x',y')\right)}{\delta G(x,y)} = G_t^{-1}(x,y).$

With this equation (2.58) becomes

$$-\frac{1}{2}R = -\frac{i}{2}G^{-1} - \frac{i}{2}G_t^{-1} + \frac{\delta\Gamma_2[\bar{\phi},G]}{\delta G}\ . \qquad (2.60)$$

It is easy to redefine$\Gamma_2[\bar{\phi},G]$ with a factor of a half. Then rearranging the above gives

$$G^{-1} = -G_t^{-1} - i\frac{\delta\Gamma_2[\bar{\phi},G]}{\delta G} - iR\ . \qquad (2.61)$$

Compare this result with equation (2.55). This identifies [8]

$$i\frac{\delta\Gamma_2[\bar{\phi},G]}{\delta G} = \Sigma^R \qquad (2.62)$$

whereΣ^R is the self energy in the presence of the source term R.

Thus the equation for the full propagator is rediscovered as

$$G^{-1} = -G_t^{-1} - \Sigma^R - iR\ . \qquad (2.63)$$

Remember that the self energy is made up of the sum of 1 P.I. diagrams, even if there is some R dependence. Varying $\Gamma_2[\bar{\phi}, G]$ with respect to G, is like removing a full propagator from $\Gamma_2[\bar{\phi}, G]$. One can think of this removal of a propagator line like snipping one internal line of a diagram in $\Gamma_2[\bar{\phi}, G]$.

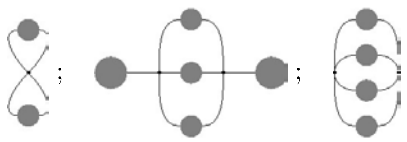
Thus equation (2.62) says that if one cuts a propagator line in $\Gamma_2[\bar{\phi}]$ one is left with a 1 P.I. diagram. If, after snipping one internal line, one is left with a 1 particle irreducible diagram, then there must have originally been a 2 P.I. diagram.

Thus $\Gamma_2[\bar{\phi}, G]$ is actually the sum of all 2 P.I. diagrams. This is where the name comes from, the 2 P.I. effective action is made from a sum of 2 P.I. diagrams.

Furthermore, since $\Gamma_2[\bar{\phi}, G]$ depends on $\bar{\phi}$ and G, the diagrams one draws actually depend on these objects. When seeing diagrams from $\Gamma_2[\bar{\phi}, G]$, the propagators must be full propagators.

2.5.4 Diagrammatic Structure of $\Gamma_2[\bar{\phi}, G]$

Since the structure of $\Gamma_2[\bar{\phi}, G]$ is known, one can construct it diagrammatically. All that is needed is all 2 P.I. diagrams where all propagators are full propagators and all inserts are full one point functions. Up to 3 loops, the diagrams in $\Gamma_2[\bar{\phi}, G]$ are given by [21]

 (2.64)

where it can see that $\Gamma_2[\bar{\phi}, G]$ is made up only from the 2 degrees of freedom, $\bar{\phi}$ and G. In these diagrams the vertices are the free vertices. One can find the pre-factors for these by using the pre factors of the self energy as these come from the understanding of the self energy diagrams.

2.5.5 Consequence for n-Point Functions

In terms of the Schwinger-Dyson equations, what the 2 P.I. formalism does is to trade the infinite hierarchy of Schwinger-Dyson equations (see section 1.7.2) for an infinite equation for each n-point function, i.e. an equation that needs truncating. This will be shown for the easy case of the connected two point function. This will then be extended to the connected three point function by varying the equation for G with respect to J. The pattern that the equations are now closed will become evident at this point.

Each n-point function will now have a Schwinger-Dyson (S-D) equation of motion that can be completely described in terms of $\bar{\phi}$ and G. The S-D equations of motion for $\bar{\phi}$ and G will be discussed in great detail in sections 5.3.1 and 5.3.2 respectively. For now it will suffice to be brief with the details.

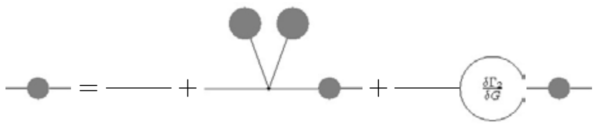
R can be considered to be 0 as these equations of motion are for physical objects, and R is not physical. Starting from equation (2.61), with $R = 0$, multiply both sides by G. Note that $GG^{-1} = \delta$.

$$
\begin{aligned}
\delta &= G_t^{-1}G + \frac{\delta\Gamma_2[\bar{\phi}, G]}{\delta G}G \\
&\text{Multiply through by } G_t\text{, with } G_t G_t^{-1} = \delta \\
G_t\delta &= \delta G + G_t\frac{\delta\Gamma_2[\bar{\phi}, G]}{\delta G}G \\
G &= G_t - G_t\frac{\delta\Gamma_2[\bar{\phi}, G]}{\delta G}G
\end{aligned} \tag{2.65}
$$

G_t refers to the tree level propagator, the above statement is equivalent to

$$
G = G_0 + G_0\frac{g^2}{2!}\bar{\phi}^2 G + G_0\frac{\delta\Gamma_2[\bar{\phi}, G]}{\delta G}G\ . \tag{2.66}
$$

Diagrammatically the above expression is



(2.67)

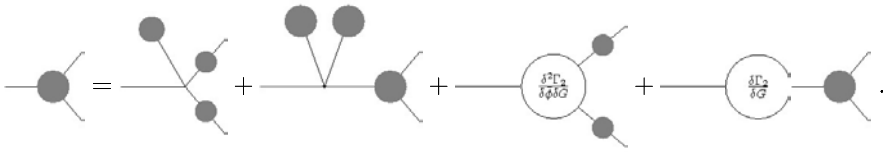
Compare this to equation (1.85), the Schwinger-Dyson equation of the connected two point function. In the above case, since $\frac{\delta\Gamma_2[\bar{\phi},G]}{\delta G}$ only depends on $\bar{\phi}$ and G, this equation doesn't depend on the three or four point functions. Thus the equation of the two point function has lost its dependence on the higher n point functions. The cost is now that one must truncateΓ_2 as this is an infinite sum of diagrams.

Since the two point function has lost its dependence on higher n-point functions, the entire hierarchy is only dependent on a truncation ofΓ_2. This can be seen by considering generating the equation of motion for the three point function from equation (2.67). This amounts to varying equation (2.67) with respect to J. In section 1.7, this variation was explained to be equivalent to adding an extra leg onto the dark blobs. To act on $\frac{\delta\Gamma_2[\bar{\phi},G]}{\delta G}$ is slightly more complicated. One can use the chain rule to find:

$$
\begin{aligned}
\frac{\delta}{\delta J}\frac{\delta\Gamma_2[\bar{\phi}, G]}{\delta G} &= \frac{\delta\bar{\phi}}{\delta J}\frac{\delta}{\delta\bar{\phi}}\frac{\delta\Gamma_2[\bar{\phi}, G]}{\delta G} \\
G\frac{\delta^2\Gamma_2[\bar{\phi}, G]}{\delta\bar{\phi}\delta G}\ .
\end{aligned} \tag{2.68}
$$

The last line used the fact that varying the one point function $\bar{\phi}$ with respect to J gives the two point function.

Thus, diagrammatically, the three point function is given by



(2.69)

Despite the complicated appearance, the above expression is closed. It's a recursive equation for the connected three point function that depends on derivatives of $\Gamma_2[\bar{\phi}, G]$, where $\Gamma_2[\bar{\phi}, G]$ only contains $\bar{\phi}$ and G with no higher n-point functions. This equation means that one can find an expression purely in terms of the one and two point functions for the three point function.

To derive equations for the higher n-point functions, the expressions are varied with respect to J, this process cannot produce an equation for a n-point function that depends on higher order n-point functions. The equations can depend on lower n-point functions, but these can always be expressed in terms of even lower n-point functions, which eventually are expressed in terms of $\bar{\phi}$ and G.

Chapter 3

Collective Effects and Schwinger-Keldysh Formalism

As was mentioned in the introduction, just after a heavy ion collision the system is very densely populated, highly dynamic and very far from equilibrium. To describe a system in this situation requires one to be able to deal collective effects within a field theory. Furthermore, the method used must be able to deal with a system far from equilibrium. This thesis will use a method known as the Schwinger-Keldysh formalism.

A zero temperature field theory is constructed within in the perturbative vacuum which text books often formulate as the in-out formalism for scattering theory. In this in-out formalism one usually takes time to run from $t' \to -\infty$ to $t \to \infty$. The problem with the methodology for the situation of a densely populated medium is that perturbation around the vacuum is extremely cumbersome if not intractable.

The Schwinger-Keldysh formalism allows one to calculate expectation values of operators when one is far from equilibrium by utilizing equal time expectation values. Accordingly, time is taken to run along a closed contour that wraps the real axis, i.e. time is taken to be as shown in figure 3.1 below. The interpretation of this contour will be discussed in the first section.

The equal time expectation value allows for a very general treatment of a system with collective affects. In this formalism, as is usual for a finite temperature field theory (this is a special case of a system expressing collective effects), one finds the expectation values of operators as given by a trace of the operators multiplied by a density matrix ρ. Where the Schwinger-Keldysh formalism differs is that this density matrix ρ can be defined as a distribution that describes a system in equilibrium or far from equilibrium. This is unique to this formalism.

The first half of the chapter will go about building the generating functional for the Schwinger-Keldysh formalism where this generating functional allows one to find expectation values in a field theory that experiences medium effects. From here a discussion will be had on altering the degrees of freedom of the Schwinger-Keldysh formalism. This altered form has the very interesting property that, when considering physical sources, one can find the generating functional (in the altered form) produces a δ function in the classical field. The way this can be made to deal with thermal equilibrium is discussed in the last section of this chapter.

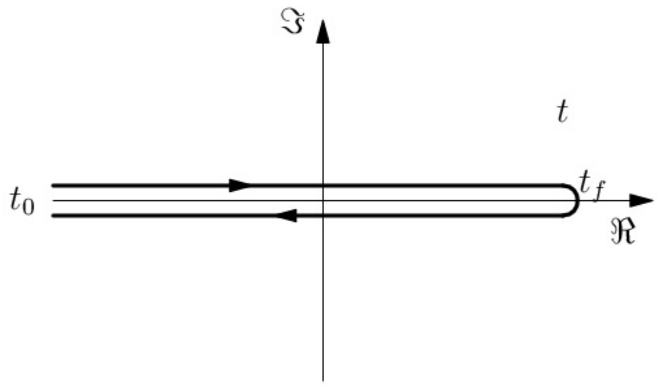


Figure 3.1: Schwinger-Keldysh Contour

3.1 Schwinger-Keldysh Formalism

The main point in the Schwinger-Keldysh formalism is that, contrary to the in-out formalism of text book scattering theory, time is taken to be as shown in figure 3.1. This time contour is briefly described here.

Any times on the lower branch of the contour occur are interpreted to be later than the upper branch. Furthermore, on the bottom branch, a point closer to the left is at a later time than that further from the left. Thus the time ordering along this contour is such that the upper branch is as normal, but the lower branch follows a reverse time order. From now on, any point labelled on the upper branch will come with a '+' and any point living on the bottom will come with a '−', the upper branch will be referred to as the + branch or forward time branch and the lower as the − branch or backward time branch [7].

3.1.1 A Justification of the Schwinger-Keldysh Contour

Objects of interest in a finite temperature theory are expectation values of field inserts, i.e. $\left|\langle out|T\left\{\phi(x_1)...\phi(x_n)\right\}|0_{in}\rangle\right|^2$. Where $T\left\{\phi(x_1)...\phi(x_n)\right\}$ means the time ordered products of the fields. $|0_{in}\rangle$ represents the vacuum of the full theory initially whilst $\langle out|$ represents the final state. To justify the Schwinger-Keldysh contour one needs to investigate these expectation values. In the expectation value it will be found that the complex conjugate fields exist in a way that suggests they are time ordered in the reverse of the usual time ordering. This can provide the justification for the closed time contour.

The expectation value can be expanded out,

$$\left|\langle out|T\left\{\phi(x_1)...\phi(x_n)\right\}|0_{in}\rangle\right|^2 = \langle 0_{in}|\left(T\left\{\phi(y_1)...\phi(y_n)\right\}\right)^\dagger|out\rangle\langle out|T\left\{\phi(x_1)...\phi(x_n)\right\}|0_{in}\rangle \ .$$

The above describes the probability of the system starting at $|0_{in}\rangle$ and ending at a particular state $\langle out|$. This thesis considers finding the expectation value to any out state. This means there is a sum over final states such that,

$$\sum_{out} \left|\langle out|T\left\{\phi(x_1)...\phi(x_n)\right\}|0_{in}\rangle\right|^2 = \sum_{out}\langle 0_{in}|\left(T\left\{\phi(y_1)...\phi(y_n)\right\}\right)^\dagger |out\rangle\langle out|T\left\{\phi(x_1)...\phi(x_n)\right\}|0_{in}\rangle \, . \tag{3.1}$$

This sum creates a sum over states where $\sum\limits_{out} |out\rangle\langle out = \mathbb{1}$, thus

$$\sum_{out} \left|\langle out|T\left\{\phi(x_1)...\phi(x_n)\right\}|0_{in}\rangle\right|^2 = \langle 0_{in}|\left(T\left\{\phi(y_1)...\phi(y_n)\right\}\right)^\dagger T\left\{\phi(x_1)...\phi(x_n)\right\}|0_{in}\rangle \, . \tag{3.2}$$

To carry on with this discussion, the interaction picture will be used.

In the interaction picture $\phi(x) = U^\dagger_{t,t_0}\phi_I(t,\boldsymbol{x})U_{t,t_0}$. Where $U_{t,t_0} = Te^{-i\int\limits_{t_0}^{t} dt' H_I(t')}$. Here H_I represents the interaction part of the Hamiltonian.

Back to equation (3.2), where everything is now time ordered,

$$\begin{aligned}
\sum_{out} \left|\langle 0_{out}|T\left\{\phi(x_1)...\phi(x_n)\right\}|0_{in}\rangle\right|^2 &= \langle 0_{in}|\left(U^\dagger_{t_n,t_0}\phi_I(y_n)U_{t_n,t_0}...U^\dagger_{t_2,t_0}\phi_I(y_2)U_{t_2,t_0}U^\dagger_{t_1,t_0}\phi_I(y_1)U_{t_1,t_0}\right)^\dagger \\
&\quad \times U^\dagger_{t_n,t_0}\phi_I(x_n)U_{t_n,t_0}...U^\dagger_{t_2,t_0}\phi_I(x_2)U_{t_2,t_0}U^\dagger_{t_1,t_0}\phi_I(x_1)U_{t_1,t_0}|0_{in}\rangle \\
&\quad \text{Where it is easily seen that } U_{t_n,t_0}U^\dagger_{t_{n-1},t_0} = U_{t_n,t_{n-1}} \\
&= \langle 0_{in}|\left(U^\dagger_{t_n,t_0}\phi_I(y_n)U_{t_n,t_{n-1}}...U_{t_3,t_2}\phi_I(y_2)U_{t_2,t_1}\phi_I(y_1)U_{t_1,t_0}\right)^\dagger \\
&\quad \times U^\dagger_{t_n,t_0}\phi_I(x_n)U_{t_n,t_{n-1}}...U_{t_3,t_2}\phi_I(x_2)U_{t_2,t_1}\phi_I(x_1)U_{t_1,t_0}|0_{in}\rangle \, .
\end{aligned} \tag{3.3}$$

From the above expression, the term $\left(U^\dagger_{t_n,t_0}\phi_I(y_n)U_{t_n,t_{n-1}}...U_{t_3,t_2}\phi_I(y_2)U_{t_2,t_1}\phi_I(y_1)U_{t_1,t_0}\right)^\dagger$ will be investigated in greater detail.

$$\begin{aligned}
&\left(U^\dagger_{t_n,t_0}\phi_I(y_n)U_{t_n,t_{n-1}}...U_{t_3,t_2}\phi_I(y_2)U_{t_2,t_1}\phi_I(y_1)U_{t_1,t_0}\right)^\dagger = \\
&\qquad U^\dagger_{t_1,t_0}\phi^\dagger_I(y_1)U^\dagger_{t_2,t_1}\phi^\dagger_I(y_2)U^\dagger_{t_3,t_2}...U^\dagger_{t_n,t_{n-1}}\phi^\dagger_I(y_n)U_{t_n,t_0}
\end{aligned} \tag{3.4}$$

to discuss $U^\dagger_{t,t_0}$ one needs the concept of anti-time ordering. Anti-time ordering is just the usual time ordering in reverse. This will be denoted by $\bar{T}$. Using this,

$$\begin{aligned}
U^\dagger_{t,t_0} &= \left(Te^{-i\int\limits_{t_0}^{t} dt' H_I(t')}\right)^\dagger \\
&= \bar{T}e^{i\int\limits_{t_0}^{t} dt' (H_I)^\dagger(t')} \\
&\quad H_I \text{ is Hermetian} \\
&= U_{t_0,t} \, .
\end{aligned} \tag{3.5}$$

Thus, if U_{t,t_0} is interpreted as evolving the free field through interactions from t_0 to t, then $U^\dagger_{t,t_0}$ can have the interpretation of evolving the free field from t to t_0. This is a backward type of time ordering. So,

$$\begin{aligned}\left(U^\dagger_{t_n,t_0}\phi_I(y_n)U_{t_n,t_{n-1}}...U_{t_3,t_2}\phi_I(y_2)U_{t_2,t_1}\phi_I(y_1)U_{t_1,t_0}\right)^\dagger &= \\ U_{t_0,t_1}\phi^\dagger_I(y_1)U_{t_1,t_2}\phi^\dagger_I(y_2)U_{t_2,t_3}...U_{t_{n-1},t_n}\phi^\dagger_I(y_n)U_{t_n,t_0}& \end{aligned} \tag{3.6}$$

Thus, the Schwinger-Keldysh time contour idea has been arrived at. The last equation can be interpreted as fields that are anti-time ordered since reading, as is convention, from right to left there is an evolution from time t_0 to t_n then from time t_n to time t_{n-1} and so forth. This is as if we have the fields $\phi^\dagger(y_i)$ inserted on a time contour that has a reverse time ordering.

Currently the field insertions $\phi^\dagger(y_i)$ depend on the field insertions $\phi(x_i)$. The main departure that the Schwinger-Keldysh formalism makes from what is done above is that it gives freedom to insert fields at will on the path that corresponds to the backward time ordering. Thus a freedom is given to independently insert fields $\phi^+(x_i)$ on the normal, forward time contour and fields $\phi^-(y_i)$ on the backward time contour. The $+$ and $-$ refer to the notation of the Schwinger-Keldysh contour discussed at the start of this section.

3.1.2 Latest Time

This section will briefly discuss the concept of the latest time. This is a very useful and powerful concept. It is shown that if one considers some time t_n to be he latest time a field is inserted, all interactions after that time cancel in the Schwinger-Keldysh formalism [20].

Starting from equation (3.3) and plugging in equation (3.6)

$$\begin{aligned}\left|\langle 0_{out}|T\left\{\phi(x_1)...\phi(x_n)\right\}|0_{in}\rangle\right|^2 &= \langle 0_{in}|U_{t_0,t_1}\phi^\dagger_I(x_1)U_{t_1,t_2}\phi^\dagger_I(x_2)U_{t_2,t_3}...U_{t_{n-1},t_n}\phi^\dagger_I(x_n)\underline{U_{t_n,t_0}} \\ &\quad \underline{\times U^\dagger_{t_n,t_0}}\phi_I(x_n)U_{t_n,t_{n-1}}...U_{t_3,t_2}\phi_I(x_2)U_{t_2,t_1}\phi_I(x_1)U_{t_1,t_0}|0_{in}\rangle \,.\end{aligned} \tag{3.7}$$

If the above now has the interpretation of a forward and backward time contour, after the latest time t_n, all interactions cancel out. Looking at the underlined part of the above equation, the term U_{t_n,t_0} multiplied with $U^\dagger_{t_n,t_0}$ gives a 1. Thus all interactions after the latest time wont contribute to the expectation value.

If the objects inserted on the backward time path aren't related to the objects on the forward time path ,the cancellation mentioned above will still occur. This is easily shown. If the latest time was t_n, and if the term at t_n was a ϕ^+ insertion, then the last evolution operator of the forward branch will have the form $U^\dagger_{t_n,t_0}$. Then let the next insertion, along the contour, be a ϕ^- insertion at a time t_m. Then the first evolution operator will be of the form U_{t_m,t_0} where $U_{t_m,t_0} = U_{t_m,t_n}U_{t_n,t_0}$. In the multiplication the only term that survives is the U_{t_m,t_n}, but this is only looking at backwards evolution (i.e. evolution in the backward time ordered branch) from the final time t_n up to t_m. This means everything after the latest time t_n still cancels out.

3.1.3 Propagators on the Contour

Now that there are two completely different fields, the concept of correlators become different on the Schwinger-Keldysh contour.

In the usual zero temperature situation there is only one type of propagator, the correlator between the only field at two different space time points. In the Schwinger-Keldysh contour, there are two different types of fields, those living on the $+$ contour and those living on the $-$ contour. One can find correlations between fields living on the same contour and one can find correlations between the two different fields. This can be generalized to any correlation function, no matter how many field inserts.

Propagators have obvious importance in a theory so special attention is given to them.

One can start with propagators that live on the above described forward and backward contours. These are simply the usual, time ordered propagator and a similar reverse time ordered propagator [16]:

$$\begin{aligned} G^{++}(x,y) &= \langle T\left\{\phi^{+}(x)\phi^{+}(y)\right\}\rangle \\ &= \Theta\left(x^0-y^0\right)G^{-+}(x,y)+\Theta\left(y^0-x^0\right)G^{+-}(x,y) \end{aligned} \tag{3.8}$$

$$\begin{aligned} G^{--}(x,y) &= \langle \overline{T}\left\{\phi^{-}(x)\phi^{-}(y)\right\}\rangle \\ &= \Theta\left(x^0-y^0\right)G^{+-}(x,y)+\Theta\left(y^0-x^0\right)G^{-+}(x,y) \end{aligned} \tag{3.9}$$

The T is an instruction to do usual, forward time ordering along the $+$ time contour. $\overline{T}$ is an instruction to do backward, anti-time ordering along the $-$ contour. At this point the notation $G^{-+}(x,y)$ and $G^{+-}(x,y)$, while suggestive, should not be taken to have deeper relevance.

To find explicit expressions for G^{+-} and G^{-+} consider G^{++}. The term $G^{++}(x,y)$, is made up of two fields that are time ordered in the usual manner. This propagator is in fact the same object as the usual free propagator in 0 temperature G_0, used previously in this thesis.

This object has a well known structure [10]

$$G_0=\Theta\left(x^0-y^0\right)\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}e^{-ip(x-y)}+\Theta\left(y^0-x^0\right)\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}e^{ip(x-y)}\;. \tag{3.10}$$

Since this object is the same as that found in equation (3.8), one can match up terms to get

$$G_0^{+-}(x,y)=\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}e^{ip(x-y)} \tag{3.11}$$

$$G_0^{-+}(x,y)=\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}e^{-ip(x-y)}\;. \tag{3.12}$$

Now a brief discussion on these two propagators can be had. In the literature, one usually finds that these are introduced and defined without proof as follows,

$$G^{-+}(x,y) = \langle \phi^-(x)\phi^+(y) \rangle \tag{3.13}$$
$$G^{+-}(x,y) = \langle \phi^+(y)\phi^-(x) \rangle \tag{3.14}$$

With this notation, the claim is being made that these propagators connect up the + contour to the − contour. To define G^{-+} and G^{-+} as above one needs to show that the two objects as defined in equations (3.11) and (3.12) are doing the job of connecting up the two different parts of the contour. This is not usually well motivated within the literature. This discussion of this will be provided in section 3.1.6. For now it will be taken as if this is true.

A very useful identity can be found by looking at equations (3.8), (3.9), (3.13) and (3.14).

$$\begin{aligned}
G^{++}(x,y) + G^{--}(x,y) =& \Theta(x^0 - y^0)G^{-+}(x,y) + \Theta(y^0 - x^0)G^{+-}(x,y) \\
& + \Theta(x^0 - y^0)G^{+-}(x,y) + \Theta(y^0 - x^0)G^{-+}(x,y) \\
=& \left(\Theta(x^0 - y^0) + \Theta(y^0 - x^0)\right) G^{-+}(x,y) \\
& + \left(\Theta(y^0 - x^0) + \Theta(x^0 - y^0)\right) G^{+-}(x,y) \\
G^{++}(x,y) + G^{--}(x,y) =& G^{-+}(x,y) + G^{+-}(x,y)
\end{aligned} \tag{3.15}$$

This shows that these objects are not independent.

The Schwinger-Keldysh propagators can be thought of as forming a matrix structure such that

$$\boldsymbol{G}(x,y) = \begin{pmatrix} G^{++}(x,y) & G^{+-}(x,y) \\ G^{-+}(x,y) & G^{--}(x,y) \end{pmatrix} . \tag{3.16}$$

This is an important consequence of the Schwinger-Keldysh formalism, propagators, along with other objects, become matrices. This particular from of the matrix will turn out to be true only in the case of an expectation value of all possible final states. If one were to consider the expectation value to a particular set of out states, this matrix would need to be altered. This discussion is presented within section 3.1.6 under the subsection 'The Sum over final States or, $e^{\mathcal{D}[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}]}$'.

Different Types of Propagators

One can also define a few different types of propagators that will be used extensively,

One can define the F and ρ propagators [8],

$$F(x,y) \equiv \frac{1}{2}\langle \{\phi(x), \phi(y)\} \rangle \tag{3.17}$$
$$\rho(x,y) \equiv i\langle [\phi(x), \phi(y)] \rangle . \tag{3.18}$$

Since this is a real scalar theory, the fields are purely real. Given the above definitions this implies that $F(x,y)$ is purely real while $\rho(x,y)$ is purely imaginary.

These propagators are also the purely symmetric and anti-symmetric parts of the connected propagator G. Their relation to G can be seen as follows [8],

$$G(x,y) = \Theta_{\mathcal{C}}(x^0 - y^0)\langle\phi(x)\phi(y)\rangle + \Theta_{\mathcal{C}}(y^0 - x^0)\langle\phi(y)\phi(x)\rangle \ . \tag{3.19}$$

The$\Theta_{\ c}$ means the theta function works over the whole time contour i.e. over both branches of the Schwinger-Keldysh contour.

Then

$$\begin{aligned} G(x,y) &= \Theta_{\mathcal{C}}(x^0 - y^0)\langle\phi(x)\phi(y)\rangle + \Theta_{\mathcal{C}}(y^0 - x^0)\langle\phi(y)\phi(x)\rangle \\ &= \frac{1}{2}\langle\{\phi(x),\phi(y)\}\rangle \left(\Theta_{\mathcal{C}}(x^0 - y^0) + \Theta_{\mathcal{C}}(y^0 - x^0)\right) \\ &\quad + \frac{1}{2}\langle[\phi(x),\phi(y)]\rangle \left(\Theta_{\mathcal{C}}(x^0 - y^0) - \Theta_{\mathcal{C}}(y^0 - x^0)\right) \ . \end{aligned} \tag{3.20}$$

It is easy to see that$\Theta_{\ \mathcal{C}}(x^0 - y^0) + \Theta_{\mathcal{C}}(y^0 - x^0) = 1$. Looking carefully at$\Theta_{\ \mathcal{C}}(x^0 - y^0) - \Theta_{\mathcal{C}}(y^0 - x^0)$ one can see that this object will always give the sign of $x^0 - y^0$. If $x^0 > y^0$ then it gives $1 - 0 = +1$. If $y^0 > x^0$ then it gives $0 - 1 = -1$. The using $-i^2 = 1$

$$\begin{aligned} G(x,y) &= \frac{1}{2}\langle\{\phi(x),\phi(y)\}\rangle - i\frac{i}{2}\langle[\phi(x),\phi(y)]\rangle \mathrm{sign}_c(x^0 - y^0) \\ &\text{Using equations 3.17 and 3.18} \\ &= F(x,y) - \frac{i}{2}\rho(x,y)\mathrm{sign}_c(x^0 - y^0) \ . \end{aligned} \tag{3.21}$$

Importantly this expression also implies that, since F is real and ρ is imaginary, that F is the real part and ρ is the imaginary part of G. This is will be proven to be very useful in the chapter of the 2 P.I. equations of motion below.

Lastly there are two objects called the advanced propagator (G^A) and retarded propagator (G^R). The retarded propagator(G^R) in particular will be found to have a lot of use. Both these propagators can be defined in terms of the $\rho(x,y)$ propagator [8].

$$\begin{aligned} G^R(x,y) &= \Theta\ (x^0 - y^0)\rho(x^0, y^0) \\ &= i\Theta(x^0 - y^0)\langle[\phi(x),\phi(y)]\rangle \end{aligned} \tag{3.22}$$

$$\begin{aligned} G^A(x,y) &= -\Theta(y^0 - x^0)\rho(x^0, y^0) \\ &= -i\Theta(y^0 - x^0)\langle[\phi(x),\phi(y)]\rangle \end{aligned} \tag{3.23}$$

Using $\Theta(x^0 - y^0) + \Theta\ (y^0 - x^0)$ and the above two definitions, it can be shown that

$$\rho(x,y) = G^R(x,y) - G^A(x,y) \ . \tag{3.24}$$

3.1.4 Generating Functional of + and − Branches

Now that there are two different fields one ϕ^+ and ϕ^- one can think of defining generating functionals for each type of field. One can describe the two separate generating functionals in analogy to $Z[J] = N\int D[\phi]e^{i\left(-\frac{1}{2}\phi(\Box+m^2)\phi-\frac{g^2}{4!}\phi^4+J\phi\right)}$. Since the ϕ^- fields are associated with the complex conjugates of the original fields (see section 3.1.1), this leads to the following definition

$$Z[J_+] = \int D[\phi^+]e^{i\left(-\frac{1}{2}\phi^+(\Box+m^2)\phi^+-\frac{g^2}{4!}(\phi^+)^4+J_+\phi^+\right)} \tag{3.25}$$

$$Z[J_-] = \int D[\phi^-]e^{i\left(\frac{1}{2}\phi^-(\Box+m^2)\phi^-+\frac{g^2}{4!}(\phi^-)^4-J_-\phi^-\right)} \tag{3.26}$$

such that the signs for $Z[J_-]$ are as if it is the complex conjugate of $Z[J_+]$.

There are respective connected generating functionals such that

$$Z[J_+] = e^{iW[J_+]} \tag{3.27}$$

$$Z[J_-] = e^{-iW[J_-]}\ . \tag{3.28}$$

This will be used in the next section.

It is important to note that if one wants the correlations of only ϕ^+ fields, $Z[J_+]$ can be varied with respect to iJ_+. This is as normal as these fields live on the part of the time contour that is the usual time ordering. To get correlations of only ϕ^- fields, one must vary $Z[J_-]$ with respect to $-iJ_-$. This is because of the reverse time ordering of the ϕ^- fields.

3.1.5 Schwinger- Keldysh Generating Functional

In what follows is a fairly long and involved derivation of the Schwinger-Keldysh generating functional. Again, an outline of the derivation is made before actually doing the derivation so as to motivate and clarify the steps that will be taken.

One starts with the equal time expectation value which is given as a trace over the operator of interest (in this case a correlation of + and − fields) and an initial density matrix. From here one interprets the trace in the functional sense i.e. as an integral over fields at some initial time. Then a complete set of states is inserted, also at the initial time, such that the operator is isolated from the initial density matrix $\rho(t_0)$. One of the initial time fields is considered as living on the + contour, the other on the − contour. From this one inserts a complete set of populated states that live out at some final time such that it goes in-between the operators' + and − field inserts (equation (3.32) below).

At this point there will be two sets of correlation functions, one for the + fields and one for the − fields. Both of these can be re-expressed using the LSZ reduction formula in the presence of strong sources to implement a sum over final states. The LSZ reduction allows one to express the two set of correlations as variations of the $e^{iW[J_+]}$ and $e^{-iW[J_-]}$ defined in equations (3.27) and (3.28) with some terms out the front (equations (3.35) and (3.36) below).

Then two things can be noted from the terms that come out the front of the LSZ reduced amplitudes. Firstly some of these terms combine to create G_0^{+-} of equation (3.14). The rest of the terms are grouped together with G_0^{+-} into powers of an operator labelled $\mathcal{D}[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}]$, which accounts for particles in the final state (equation (3.40) below). In a complete sum over all final states $\mathcal{D}[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}]$ exponentiates, which leads to the final result found in equation (3.43) below. The final result is the generating functional of the Schwinger-Keldysh formalism given in equation (3.44).

An important result that comes out of this discussion is that this formalism would need to be modified if the final state is subject to restrictions [25]. This result is discussed thoroughly in the next section.

What follows now is the fully detailed derivation of the Schwinger-Keldysh generating functional.

The Full Derivation

As was explained in the introduction of this chapter, for a system that experiences collective effects the expectation value of an operator $\mathcal{O}$ is given by [8],

$$\langle \mathcal{O} \rangle = Tr\left\{\rho(t_0)\mathcal{O}\right\} \tag{3.29}$$

where $\rho(t_0)$ is an initial time density matrix that encodes the initial medium ensemble as was discussed. If $\rho(t_0)$ is a thermal distribution, i.e. it is in a pure state, then the above average describes a thermal average. If $\rho(t_0)$ is not a thermal distribution, then the average describes an object out of equilibrium [8].

To put this in QFT terms, the trace is given by an integral over initial states, i.e. at states that exist at $t = t_{\text{initial}} = t_0$ [8].

$$\begin{aligned} Tr\left\{\rho(t_0)\mathcal{O}\right\} &= \int D[\phi^{(1)}_{t_0}(\boldsymbol{x})]\langle\phi^{(1)}_{t_0}|\rho(t_0)\mathcal{O}|\phi^{(1)}_{t_0}\rangle \\ &\quad \text{Inserting a complete set of states again at time } t = t_{\text{initial}} = t_0 \\ &= \int D[\phi^{(1)}_{t_0}(\boldsymbol{x})]\int D[\phi^{(2)}_{t_0}(\boldsymbol{x})]\langle\phi^{(1)}_{t_0}|\rho(t_0)|\phi^{(2)}_{t_0}\rangle\langle\phi^{(2)}_{t_0}|\mathcal{O}|\phi^{(1)}_{t_0}\rangle \; . \end{aligned} \tag{3.30}$$

The complete set of states above is inserted such that $\mathcal{O}$ can be isolated from $\rho(t_0)$.

The objects of interest are correlation functions so the only operators $\mathcal{O}$ that will be looked at are those of field insertions. Thus let

$$\begin{aligned} \mathcal{O} =& T_{\mathcal{C}}\left\{\phi^-(y_1)...\phi^-(y_b)\phi^+(x_1)...\phi^+(x_a)\right\} \\ &= \bar{T}\left\{\phi^-(y_1)...\phi^-(y_b)\right\}T\left\{\phi^+(x_1)...\phi^+(x_a)\right\} \\ &= \mathcal{O}[\phi^+]\mathcal{O}[\phi^-] \end{aligned} \tag{3.31}$$

where $T_{\mathcal{C}}$ represents time ordering along the Schwinger-Keldysh time contour, as discussed in the introduction of this section, while T and $\bar{T}$ are the forward and backward time ordering repetitively.

Looking more closely at the object $\langle\phi^{(2)}_{t_0}|\mathcal{O}|\phi^{(1)}_{t_0}\rangle$, it can be seen that this is an expectation value very much related to that discussed in section 3.1.1. The comparison is made clearer if $\phi^{(1)}_{t_0}$ is identified as $\phi^{(+)}_{t_0}$ and $\phi^{(2)}_{t_0}$ is identified as $\phi^{(-)}_{t_0}$. Thus equation (3.30) can be re-written as [16]

$$\begin{aligned}
Tr\left\{\rho(t_0)\mathcal{O}[\phi^+]\mathcal{O}[\phi^-]\right\} &= \int D[\phi^{(+)}_{t_0}(\boldsymbol{x})]\int D[\phi^{(-)}_{t_0}(\boldsymbol{x})]\langle\phi^{(+)}_{t_0}|\rho(t_0)|\phi^{(-)}_{t_0}\rangle \\
&\quad\times\langle\phi^{(-)}_{t_0}|\mathcal{O}[\phi^+]\mathcal{O}[\phi^-]|\phi^{(+)}_{t_0}\rangle \\
&\quad\text{Inserting a complete set of states at a very late time,} \\
&\quad\text{after the latest time of the upper branch} \\
&\quad\text{where } \mathbb{1} = \sum_{n=0}^{\infty}\frac{1}{n!}\int\left[\prod_{i=1}^{n}\frac{d^3\boldsymbol{p}_i}{(2\pi)^3 2E_i}\right]|\boldsymbol{p}_1...\boldsymbol{p}_n\rangle\langle\boldsymbol{p}_1...\boldsymbol{p}_n| \\
&= \int D[\phi^{(+)}_{t_0}(\boldsymbol{x})]\int D[\phi^{(-)}_{t_0}(\boldsymbol{x})]\langle\phi^{(+)}_{t_0}|\rho(t_0)|\phi^{(-)}_{t_0}\rangle\sum_{n=0}^{\infty}\frac{1}{n!}\int\left[\prod_{i=1}^{n}\frac{d^3\boldsymbol{p}_i}{(2\pi)^3 2E_i}\right] \\
&\quad\times\langle\phi^{(-)}_{t_0}|\mathcal{O}[\phi^-]|\boldsymbol{p}_1...\boldsymbol{p}_n\rangle\langle\boldsymbol{p}_1...\boldsymbol{p}_n|\mathcal{O}[\phi^+]|\phi^{(+)}_{t_0}\rangle\ .
\end{aligned} \tag{3.32}$$

The complete set of states inserted in the above is done such that the fields living on the upper part of the contour are isolated from the fields living on the lower part of the contour, i.e. the states are inserted at the final time. This must be stressed, the functional $\mathbb{1}$ is inserted at the final time.

To continue the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula in the presence of a finite source needs to be used. The LSZ reduction formula in the presence of a finite source J is found in [16], it shows that

$$\langle\boldsymbol{p}_1...\boldsymbol{p}_n|0_{in}\rangle[J] = \frac{1}{Z^{n/2}}\int\left[\prod_{i=1}^{n}d^4x_i e^{ip_i\cdot x_i}(\Box_{x_i}+m^2)\frac{\delta}{i\delta J(x_i)}\right]e^{iW[J]}\ . \tag{3.33}$$

Then since $\prod_{i=1}^{n}\frac{\delta}{i\delta J(x_i)}e^{iW[J]} = \langle out|\phi(x_1)...\phi(x_n)|in\rangle[J]$ one can write that,

$$\langle\boldsymbol{p}_1...\boldsymbol{p}_n|0_{in}\rangle[J] = \frac{1}{Z^{n/2}}\int\left[\prod_{i=1}^{n}d^4x_i e^{ip_i\cdot x_i}(\Box_{x_i}+m^2)\right]\langle out|\phi(x_1)...\phi(x_n)|0_{in}\rangle[J]\ . \tag{3.34}$$

Using equation (3.34), but generating the ϕ^+ fields from $e^{iW[J_+]}$ (equation (3.27)) and the ϕ^- fields from $e^{iW[J_-]}$ (equation (3.28)).

$$\begin{aligned}
&\langle\boldsymbol{p}_1...\boldsymbol{p}_n|\mathcal{O}[\phi^+]|\phi^{(+)}_{t_0}\rangle[J] \\
&\quad= \frac{1}{Z^{n/2}}\int\left[\prod_{i=1}^{n}d^4z_i e^{ip_i\cdot z_i}(\Box_{z_i}+m^2)\right]\langle\phi_{out}|\phi^+(z_1)...\phi^+(z_n)\mathcal{O}[\phi^+]|\phi^{(+)}_{t_0}\rangle[J] \\
&\quad= \frac{1}{Z^{n/2}}\int\left[\prod_{i=1}^{n}d^4z_i e^{ip_i\cdot z_i}(\Box_{z_i}+m^2)\frac{\delta}{i\delta J_+(z_i)}\right]\mathcal{O}\left[\frac{\delta}{\delta iJ_+}\right]e^{iW[J^+]} \\
&\quad= \mathcal{O}\left[\frac{\delta}{\delta iJ_+}\right]\frac{1}{Z^{n/2}}\int\left[\prod_{i=1}^{n}d^4z_i e^{ip_i\cdot z_i}(\Box_{z_i}+m^2)\frac{\delta}{i\delta J_+(z_i)}\right]e^{iW[J^+]}
\end{aligned} \tag{3.35}$$

Similarly

$$
\begin{aligned}
&\langle\phi_{t_0}^{(-)}|\mathcal{O}[\phi^-]|\boldsymbol{p}_1...\boldsymbol{p}_n\rangle[J] \\
&= \mathcal{O}\left[\frac{\delta}{\delta(-iJ_-)}\right]\frac{1}{Z^{n/2}}\int\left[\prod_{i=1}^{n} d^4w_i e^{-ip_i\cdot w_i}(\Box_{w_i}+m^2)\frac{\delta}{-i\delta J_-(w_i)}\right]e^{-iW[J^-]}
\end{aligned}
\tag{3.36}
$$

Inserting the above two LSZ reduction objects into 3.32,

$$
\begin{aligned}
Tr\left\{\rho(t_0)\mathcal{O}[\phi^+]\mathcal{O}[\phi^-]\right\} &= \int D[\phi_{t_0}^{(+)}(\boldsymbol{x})]\int D[\phi_{t_0}^{(-)}(\boldsymbol{x})]\langle\phi_{t_0}^{(+)}|\rho(t_0)|\phi_{t_0}^{(-)}\rangle\sum_{n=0}^{\infty}\frac{1}{n!}\int\left[\prod_{i=1}^{n}\frac{d^3\boldsymbol{p}_i}{(2\pi)^3 2E_i}\right]\frac{1}{Z} \\
&\times\mathcal{O}\left[\frac{\delta}{\delta(-iJ_-)}\right]\int\left[\prod_{i=1}^{n} d^4w_i e^{ip_i\cdot w_i}(\Box_{w_i}+m^2)\frac{\delta}{-i\delta J_-(w_i)}\right]e^{-iW[J^-]} \\
&\times\mathcal{O}\left[\frac{\delta}{\delta(iJ_+)}\right]\int\left[\prod_{i=1}^{n} d^4z_i e^{ip_i\cdot z_i}(\Box_{z_i}+m^2)\frac{\delta}{i\delta J_+(z_i)}\right]e^{iW[J^+]}\ .
\end{aligned}
\tag{3.37}
$$

The above expression can be rearranged to clearly show that one has a factor of G_0^{+-} as defined in equation (3.11).

$$
\begin{aligned}
Tr\left\{\rho(t_0)\mathcal{O}[\phi^+]\mathcal{O}[\phi^-]\right\} &= \int D[\phi_{t_0}^{(+)}(\boldsymbol{x})]\int D[\phi_{t_0}^{(-)}(\boldsymbol{x})]\langle\phi_{t_0}^{(+)}|\rho(t_0)|\phi_{t_0}^{(-)}\rangle\mathcal{O}\left[\frac{\delta}{\delta(-iJ_-)}\right]\mathcal{O}\left[\frac{\delta}{\delta(iJ_+)}\right] \\
&\times\sum_{n=0}^{\infty}\frac{1}{n!}\int\left[\prod_{i=1}^{n}\frac{d^3\boldsymbol{p}_i}{(2\pi)^3 2E_i}\right]\int\left[\prod_{i=1}^{n} d^4w_i d^4z_i\right]\frac{1}{Z}e^{ip_i\cdot(z_i-w_i)} \\
&\times(\Box_{w_i}+m^2)(\Box_{z_i}+m^2)\frac{\delta}{-i\delta J_-(w_i)}\frac{\delta}{i\delta J_+(z_i)}e^{-iW[J^-]}e^{iW[J^+]}
\end{aligned}
\tag{3.38}
$$

Looking back to equation (3.14), it is possible to see that the term $\int\left[\prod_{i=1}^{n}\frac{d^3\boldsymbol{p}_i}{(2\pi)^3 2E_i}\right]$ multiplied by the term $e^{ip_i\cdot(z_i-w_i)}$ in the above gives the propagator $G_0^{+-}(z_i,w_i)$ [16].

$$
\begin{aligned}
Tr\left\{\rho(t_0)\mathcal{O}[\phi^+]\mathcal{O}[\phi^-]\right\} &= \int D[\phi_{t_0}^{(+)}(\boldsymbol{x})]\int D[\phi_{t_0}^{(-)}(\boldsymbol{x})]\langle\phi_{t_0}^{(+)}|\rho(t_0)|\phi_{t_0}^{(-)}\rangle\mathcal{O}\left[\frac{\delta}{\delta(-iJ_-)}\right]\mathcal{O}\left[\frac{\delta}{\delta(iJ_+)}\right] \\
&\times\sum_{n=0}^{\infty}\frac{1}{n!}\int\left[\prod_{i=1}^{n} d^4w_i d^4z_i\right]\frac{1}{Z}G_0^{+-}(w_i,z_i) \\
&\times(\Box_{w_i}+m^2)(\Box_{z_i}+m^2)\frac{\delta}{-i\delta J_-(w_i)}\frac{\delta}{i\delta J_+(z_i)}e^{-iW[J^-]}e^{iW[J^+]}
\end{aligned}
\tag{3.39}
$$

[16] then gives a way forward from here, one introduces the following,

$$\mathcal{D}\left[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}\right] \equiv \frac{1}{Z}\int d^4x d^4y G^0_{+-}(x,y)(\Box_x + m^2)(\Box_y + m^2)\frac{\delta}{\delta J_+(x)}\frac{\delta}{\delta J_-(y)} \ . \tag{3.40}$$

Putting this definition into equation (3.39)

$$\begin{aligned} Tr\left\{\rho(t_0)\mathcal{O}[\phi^+]\mathcal{O}[\phi^-]\right\} = &\int D[\phi^{(+)}_{t_0}(\boldsymbol{x})]\int D[\phi^{(-)}_{t_0}(\boldsymbol{x})]\langle\phi^{(+)}_{t_0}|\rho(t_0)|\phi^{(-)}_{t_0}\rangle\mathcal{O}\left[\frac{\delta}{\delta(-iJ_-)}\right]\mathcal{O}\left[\frac{\delta}{\delta(iJ_+)}\right] \\ &\times\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\mathcal{D}\left[\frac{\delta}{\delta J_+(z_i)},\frac{\delta}{\delta J_-(w_i)}\right]e^{-iW[J^-]}e^{iW[J^+]} \end{aligned} \tag{3.41}$$

and observe that

$$\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\mathcal{D}[J_+(z_i),J_-(w_i)] = e^{\mathcal{D}\left[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}\right]} \tag{3.42}$$

such that equation (3.41) becomes,

$$\begin{aligned} Tr\left\{\rho(t_0)\mathcal{O}[\phi^+]\mathcal{O}[\phi^-]\right\} = &\int D[\phi^{(+)}_{t_0}(\boldsymbol{x})]\int D[\phi^{(-)}_{t_0}(\boldsymbol{x})]\langle\phi^{(+)}_{t_0}|\rho(t_0)|\phi^{(-)}_{t_0}\rangle\mathcal{O}\left[\frac{\delta}{\delta(-iJ_-)}\right]\mathcal{O}\left[\frac{\delta}{\delta(iJ_+)}\right] \\ &\times e^{\mathcal{D}^n[J_+,J_-]}e^{-iW[J^-]}e^{iW[J^+]} \ . \end{aligned} \tag{3.43}$$

Thus, one can define the object

$$Z[J_+,J_-,\rho(t_0)] = \int D[\phi^{(+)}_{t_0}(\boldsymbol{x})]D[\phi^{(-)}_{t_0}(\boldsymbol{x})]\langle\phi^{(+)}_{t_0}|\rho(t_0)|\phi^{(-)}_{t_0}\rangle e^{\mathcal{D}\left[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}\right]}e^{iW[J_+]}e^{-iW[J_-]} \tag{3.44}$$

as the Schwinger-Keldysh generating function of all correlation functions for a finite temperature field theory since

$$Tr\left\{\rho(t_0)\mathcal{O}[\phi^+]\mathcal{O}[\phi^-]\right\} = \mathcal{O}\left[\frac{\delta}{\delta(-iJ_-)}\right]\mathcal{O}\left[\frac{\delta}{\delta(iJ_+)}\right]Z[J_+,J_-,\rho(t_0)] \ . \tag{3.45}$$

3.1.6 Discussing the Schwinger-Keldysh Generating Functional

There are three parts to the generating functional $Z[J_+,J_-,\rho(t_0)]$. The first part is the initial density matrix, $\langle\phi^{(+)}_{t_0}|\rho(t_0)|\phi^{(-)}_{t_0}\rangle$ where one sets the initial ensemble of the system. The second part is $e^{iW[J_+]}e^{-iW[J_-]}$ which allows one to generate diagrams on the upper and lower branches. The last part is $e^{\mathcal{D}\left[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}\right]]}$ which creates the connections between the upper and lower branches. Each terms is now discussed in greater detail. There is also the aspect of the sources in this formulation which is also briefly discussed.

Sources

For a physical solution $J_+ = J_- = J$. This is because, the source actually exist at some time. There can't be one source existing at x^0 on the upper branch and a different one existing on the lower branch at x^0 [16]. Although the lower branch is considered to be later in the contour ordering, this is a calculational tool, not a physical reality. The fields can have different values as they are subject to quantum fuzziness, the sources aren't.

One only invokes this fact after the variations have been done., i.e. all variations are with respect to sources are to be understood as

$$\frac{\delta}{\delta J_+}\Big|_{J_+=J} \tag{3.46}$$

$$\frac{\delta}{\delta J_-}\Big|_{J_-=J} \tag{3.47}$$

The Initial Density Matrix $\rho(t_0)$

$\rho(t_0)$ was briefly discussed in the introduction to this chapter but here more detail is given.

The object $\langle\phi^{(+)}_{t_0}|\rho(t_0)|\phi^{(-)}_{t_0}\rangle$ describes the initial ensemble of the theory. For a dense system displaying collective effects, one needs to start with a certain ensemble. This is where it can be chosen whether or not this is describing a theory in thermal equilibrium, or out of equilibrium [8]. To consider a theory that is thermal equilibrium, one must chose a thermal distribution such as $\rho(t_0) \sim e^{\beta H}$ where β is the inverse temperature. The case of equilibrium is discussed in greater detail in the last section of this chapter. The Schwinger-Keldysh formalism is very general in dealing with systems that have collective effects, where the generailty is the strength of the Schwinger-Keldysh formalism.

The Factor $e^{iW[J_+]}e^{-iW[j_-]}$

The term $e^{iW[J_+]}e^{-iW[J_-]}$ will be written out fully as this will be needed in the next section. Using equations (3.27) and (3.28)

$$\begin{aligned} e^{iW[J_+]}e^{-iW[J_-]} &= \left\{\int D[\phi^+]e^{i\left(-\frac{1}{2}\phi^+(\Box+m^2)\phi^+-\frac{g^2}{4!}(\phi^+)^4+J_+\phi^+\right)}\right\} \\ &\quad\times\left\{\int D[\phi^-]e^{i\left(\frac{1}{2}\phi^-(\Box+m^2)\phi^-+\frac{g^2}{4!}(\phi^-)^4-J_-\phi^-\right)}\right\} \\ &= \int D[\phi^+]D[\phi^-]e^{i\left(-\frac{1}{2}\phi^+(\Box+m^2)\phi^+-\frac{g^2}{4!}(\phi^+)^4+J_+\phi^+\right)+i\left(\frac{1}{2}\phi^-(\Box+m^2)\phi^-+\frac{g^2}{4!}(\phi^-)^4-J_-\phi^-\right)} . \end{aligned} \tag{3.48}$$

From this one can then define [7],

$$\begin{aligned} Z_{+-}[J_+,J_-] &= e^{iW[J_+]}e^{-iW[J_-]} \\ &= \int D[\phi^+]D[\phi^-]e^{iS[\phi^+]+iJ_+\phi^+-iS[\phi^-]-iJ_-\phi^-} . \end{aligned} \tag{3.49}$$

From this object, one can vary with respect to J_+ or J_- and pull down ϕ^+ and ϕ^- fields as necessary and create correlations as required. The term $Z_{+-}[J_+, J_-]$ does not, however, contain the free propagators G_0^{+-} and G_0^{-+}.

The way these are seen to be missing is to remember back to section 1.6. Here the Feynmann diagrams of a theory governed by a given generating functional were found. In the process one finds that the object $\phi(\Box + m^2)\phi$, the kinetic term of $\mathcal{L}$, determines the free inverse propagator of the theory. If one were to go through the same process with $Z_{+-}[J_+, J_-]$, it would be found that this object contains only two free propagators. The propagator going from the field ϕ^+ to another field ϕ^+ and one going from the field ϕ^- to another field ϕ^-. This is seen by the terms in the exponentials of equations (3.25) and (3.26). This generating functional is missing the explicit G_0^{+-} and G_0^{-+}.

What this means is that $Z_{+-}[J_+, J_-]$ can be used to generate diagrams on the upper and lower contours but it doesn't produce any terms that can connect the upper branch diagrams to lower branch diagrams. Most of the literature ignores this issue and continues on as if they can create G_0^{+-} and G_0^{-+}.

This is where the unusual term $\mathcal{D}[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}]$ introduced in [16] is exceptionally useful.

The Sum over final States or, $e^{\mathcal{D}[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}]}$

As was written in equation (3.40)

$$\mathcal{D}\left[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}\right] \equiv \frac{1}{Z}\int d^4x d^4y G^0_{+-}(x,y)(\Box_x + m^2)(\Box_y + m^2)\frac{\delta}{\delta J_+(x)}\frac{\delta}{\delta J_-(y)} \ . \tag{3.50}$$

Now reconsider $\mathcal{D}[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}]$. It was stated before that $G_0^{+-}(x,y) = G_0^{-+}(y,x)$. Thus one can start by setting $G_0^{+-}(x,y) = \frac{1}{2}\left(G_0^{+-}(x,y) + G_0^{-+}(y,x)\right)$. Then in $\mathcal{D}$, the arguments of the G_0^{-+} propagator are just integration variables that one can relabel. This leads to

$$\begin{aligned}\mathcal{D}[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}] = \frac{1}{2}\frac{1}{Z}\int d^4x d^4y \Big\{ & G_0^{+-}(x,y)(\Box_x + m^2)(\Box_y + m^2)\frac{\delta}{\delta J_+(x)}\frac{\delta}{\delta J_-(y)} \\ & + G_0^{-+}(x,y)(\Box_x + m^2)(\Box_y + m^2)\frac{\delta}{\delta J_-(x)}\frac{\delta}{\delta J_+(y)}\Big\} \ .\end{aligned} \tag{3.51}$$

Using $\mathcal{D}\left[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}\right]$, this subsection will first show that the integrals of equation (3.11) and (3.12) have the interpretation of objects the literature describes as propagators that can connect vertices on the upper contour to those on the lower contour, particularly this connection happens at latest time. Then it will be shown how $\mathcal{D}\left[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}\right]$ inserts G_0^{-+} and G_0^{+-} into the generating functional. This proves that $\mathcal{D}$ is a sewing operator in the sense that it sews together the upper and lower branches. Finally the importance of $\mathcal{D}$ and the details the literature usually misses will be discussed, i.e. why worry about this term to such an extent.

Firstly, showing that G_0^{+-} and G_0^{-+} connect up the upper and lower contours.

Look at (3.51), reading from right to left one can see what is happening. First diagrams that live on the upper and lower branches have a leg picked out of each by the $\frac{\delta}{\delta J}$ variations. The two inverse propagators then amputate these picked out legs. Finally using $G^0_{+-}(x,y)$ and $G^0_{-+}(x,y)$ one sews these two ends together. This shows that one can interpret the objects defined in equation (3.11) and (3.12) as correlations between the + and − branch. This validates the literature's claims made in equations (3.13) and (3.14) where one interprets the G^{+-} and G^{-+} as a correlation between fields living on the upper and lower contours.

Recall that the momentum integral that allowed the introduction of G_0^{+-} came from inserting a complete set of states at the latest time (see equation (3.32)), thus one can determine that these propagators work by connecting the two branches out at the latest time. This identifies $\mathcal{D}$ as a sewing operator that sews together the upper and lower contours out at the latest time.

Now it will be shown exactly how this operator is managing to insert G_0^{-+} and G_0^{+-} into the generating functional.

To show how $e^{\mathcal{D}[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}]}$ works, it is best to consider it acting on the free version of $Z_{+-}[J_+, J_-]$. The free version means that there is no coupling in the theory. In this case the generating functional becomes $Z^0_{+-}[J_+, J_-]$. Using equation (3.49) this means

$$Z^0_{+-}[J_+, J_-] = \int D[\phi^+]D[\phi^-]e^{-i\frac{1}{2}\phi^+(\Box+m^2)\phi^+ + J_+\phi^+ + i\frac{1}{2}\phi^-(\Box+m^2)\phi^- - J_-\phi^-} \ . \tag{3.52}$$

The above is simply the product of two Gaussian integrals where the solution to the integrals is worked out in appendix A.2.

Thus

$$Z^0_{+-}[J_+, J_-] = Ne^{J_+G_0^{++}J_+\phi^+ + J_-G_0^{--} - J_-\phi^-} \tag{3.53}$$

where N is some normalization constant.

Equivalently one can consider the term in the exponential to have a matrix structure,

$$Z^0_{+-}[J_+, J_-] = Ne^{\frac{1}{2}\int d^4x' \int d^4y' \boldsymbol{J}^T(x')\boldsymbol{G}_0^D(x',y')\boldsymbol{J}(y')} \tag{3.54}$$

where,

$$\boldsymbol{J} = \begin{pmatrix} J_+ \\ J_- \end{pmatrix} \tag{3.55}$$

$$\boldsymbol{G}_0^D = \begin{pmatrix} G_0^{++} & 0 \\ 0 & G_0^{--} \end{pmatrix} \ . \tag{3.56}$$

Note the D in $\boldsymbol{G}_0^D$ stands for diagonal.

Then equation (3.51) can be rewritten using matrix notation such that,

$$\mathcal{D}[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}] = \frac{1}{2}\frac{1}{Z}\int d^4xd^4y \left(\delta[\boldsymbol{J}]\right)^T(x)\boldsymbol{G}^0_{OD}(x,y)\left(\delta[\boldsymbol{J}]\right)(y) \tag{3.57}$$

where

$$(\delta[\boldsymbol{J}]) = \begin{pmatrix} \frac{\delta}{\delta J_+} \\ \frac{\delta}{\delta J_-} \end{pmatrix} \tag{3.58}$$

$$\boldsymbol{G}^0_{OD}(x,y) = \begin{pmatrix} 0 & (\Box_x + m^2)G^0_{+-}(x,y)(\Box_y + m^2) \\ (\Box_x + m^2)G^0_{-+}(x,y)(\Box_y + m^2) & 0 \end{pmatrix} \tag{3.59}$$

where OD stands for off diagonal.

Now the object $e^{\mathcal{D}[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}]} Z^0_{-+}$ can be found.

It is best to look at the above object using equations (3.54) and (3.57) then one can see that

$$\begin{aligned} e^{\mathcal{D}[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}]} Z^0_{-+} &= N e^{\frac{1}{2}\frac{1}{Z}\int d^4x d^4y (\delta[\boldsymbol{J}])^T(x)\boldsymbol{G}^0_{OD}(x,y)(\delta[\boldsymbol{J}])(y)} e^{\frac{1}{2}\int d^4x' \int d^4y' \boldsymbol{J}^T(x')\boldsymbol{G}^D_0(x',y')\boldsymbol{J}(y')} \\ &= \left(1 + \frac{1}{2}\frac{1}{Z}\int d^4x d^4y \,(\delta[\boldsymbol{J}])^T(x)\boldsymbol{G}^0_{OD}(x,y)\,(\delta[\boldsymbol{J}])(y) + ...\right) \\ &\quad \times e^{\frac{1}{2}\int d^4x' \int d^4y' \boldsymbol{J}^T(x')\boldsymbol{G}^D_0(x',y')\boldsymbol{J}(y')} . \end{aligned} \tag{3.60}$$

This differentiation is left for appendix C.2, the result necessary is given in equation (C.16) where

$$\begin{aligned} &\int d^4x d^4y \left[(\delta[\boldsymbol{J}])^a(x)\left(\boldsymbol{G}^0_{OD}\right)^{ab}(x,y)\,(\delta[\boldsymbol{J}])^b(y)\right] e^{\frac{1}{2}\int d^4x' \int d^4y' \boldsymbol{J}^{a'}(x')\left(\boldsymbol{G}^D_0\right)^{a'b'}(x',y')\boldsymbol{J}^{b'}(y')} \\ &= \int d^4x d^4y\,(\boldsymbol{J})^T(x)\begin{pmatrix} 0 & G_0^{+-}(x,y) \\ G_0^{-+}(x,y) & 0 \end{pmatrix}\boldsymbol{J}(y) e^{\frac{1}{2}\int d^4x' \int d^4y' \boldsymbol{J}^{a'}(x')\left(\boldsymbol{G}^D_0\right)^{a'b'}(x',y')\boldsymbol{J}^{b'}(y')} . \end{aligned} \tag{3.61}$$

It will be shown that acting with the differential operator term in the square brackets above on the result of eqaution (3.61) gives 0's unless acting on the exponential term. This will allow an exponentiation of the pre-factors in the result of equation (3.61).

First let

$$\boldsymbol{G}^0_{\pm\mp} = \begin{pmatrix} 0 & G_0^{+-}(x,y) \\ G_0^{-+}(x,y) & 0 \end{pmatrix} \tag{3.62}$$

then consider,

$$\left[\left(\boldsymbol{G}^0_{OD}\right)^{ab}(\delta[\boldsymbol{J}])^b\right]\boldsymbol{J}^{a'}\left(\boldsymbol{G}^0_{\pm\mp}\right)^{a'b'}\boldsymbol{J}^{b'} = \left(\boldsymbol{G}^0_{OD}\right)^{ab}\left(\boldsymbol{G}^0_{\pm\mp}\right)^{bb'}\boldsymbol{J}^{b'} . \tag{3.63}$$

By looking at the definitions of $\boldsymbol{G}^0_{OD}$ and $\boldsymbol{G}^0_{\pm\mp}$ and considering that $\left(\Box + m^2\right) G^{\pm\mp} = 0$ implies that

$$\left[\left(\boldsymbol{G}^0_{OD}\right)^{ab}(\delta[\boldsymbol{J}])^b\right]\boldsymbol{J}^{a'}\left(\boldsymbol{G}^0_{\pm\mp}\right)^{a'b'}\boldsymbol{J}^{b'} = 0 . \tag{3.64}$$

This means this differential operator will move through the pre-exponential terms in equation (3.61) and keep acting on the exponential term. This leads to,

$$
\begin{aligned}
e^{\mathcal{D}\left[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}\right]} Z^0_{-+} = &\Bigg[1 + \frac{1}{2}\frac{1}{Z}\int d^4x d^4y \,(\boldsymbol{J})^T(x)\begin{pmatrix} 0 & G_0^{+-}(x,y) \\ G_0^{-+}(x,y) & 0 \end{pmatrix}\boldsymbol{J}(y) \\
&+\frac{1}{2!}\left(\frac{1}{2}\frac{1}{Z}\int d^4x d^4y \,(\boldsymbol{J})^T(x)\begin{pmatrix} 0 & G_0^{+-}(x,y) \\ G_0^{-+}(x,y) & 0 \end{pmatrix}\boldsymbol{J}(y)\right)^2 + \ldots\Bigg] Z^0_{-+} \\
= &\exp\left\{\frac{1}{2}\frac{1}{Z}\int d^4x d^4y \,(\boldsymbol{J})^T(x)\begin{pmatrix} 0 & G_0^{+-}(x,y) \\ G_0^{-+}(x,y) & 0 \end{pmatrix}\boldsymbol{J}(y)\right\} \\
&\times N e^{\frac{1}{2}\int d^4x' \int d^4y' \boldsymbol{J}^T(x')\boldsymbol{G}_0^D(x',y')\boldsymbol{J}(y')} \\
= &N e^{\frac{1}{2}\int d^4x \int d^4y \boldsymbol{J}^T(x)\boldsymbol{G}_0(x,y)\boldsymbol{J}(y)}
\end{aligned}
\tag{3.65}
$$

where $\boldsymbol{G}_0$ is given in equation (3.16) just with free propagators in all the entries.

This then shows how $\mathcal{D}$ is inserting G_0^{-+} and G_0^{+-} into the generating functional, it is done through $e^{\mathcal{D}\left[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}\right]}$. If one considers switching the interactions back on then only the mathematics gets tougher, the $\mathcal{D}$ term isn't changed with interactions included. Thus it's role remains the same.

The discussion leading up to equation (3.65) sets the baseline for a more detailed discussion of the $\mathcal{D}$ operator.

$\mathcal{D}$ and its interpretation is missing in most of the literature but it is exceptionally important. Understanding this object can lead to being able to find expectation values to particular out states, and not necessarily the sum over all possible out states and over all of phase space [25].

There are three different classes of specific final state expectation values one can look at, one could consider looking at a particular part of phase space i.e. not integrating over all of phase space; one can consider looking at a finite number of particles in the out state i.e. not summing over all possible out states or one could also look at a combination of both situations.

First one can consider not summing over all possible out states, such as to find an expectation value for a finite number of particles in the out state. Recall that only in the summing over all possible out states was the exponential $e^{\mathcal{D}}$ produced (see equation (3.42)). The exponential structure leads directly to the free propagators G_0^{+-} and G_0^{-+} of the Schwinger-Keldysh formalism being inserted into the generating functional (see equation (3.65)). If one were to try find the expectation value for a finite number of particles in the out state then the sum over all out states doesn't occur. This would mean the exponential of $\mathcal{D}$ required wouldn't be there. Thus the existence of the G_0^{+-} and G_0^{-+} required by the literature is only true for the case of the expectation value to all final states, in the case of specific out states there are no off diagonal terms in $\boldsymbol{G}_0$.

For instance the solution obtained in the literature wouldn't be capable of producing a solution for the case covered by the in-out formalism. This is because the in-out formalism assumes one is interested in a single out state. With the previous discussion in mind, it is clear that one can not get G_0^{+-} and G_0^{-+} in this instance. However, the usual treatment would still assume the existence of the propagators G_0^{+-} and G_0^{-+} and thus they wouldn't find the correct solution. The work of [25] shows how one can answer questions on expectation values to particular out states.

The situation where one has a combination of both particular phase space and finite out states will be constrained by the above discussion such that there will be a lack of diagonal elements in the free matrix propagator.

Finally consider the situation of looking at a restricted part of phase space i.e. not integrating over all of phase space. The work done in [25] tackles this exact issue but their work does not investigate the consequences for the free matrix propagator $\boldsymbol{G}_0$. A justification for considering restricted phase space is what experimentalists call a cut, where in the experiment one is only interested in states from a particular part of phase space. The phase space restriction alters the set of states inserted at the final time in equation (3.32) (part of the derivation of the Schwinger-Keldysh generating functional), by modifying the momentum integral by a factor. This would then translate into modifying G_0^{+-} and G_0^{-+} by this factor as these elements are described by the momentum integral inserted with the set of states (see equations (3.11) and (3.12)). Note that G_0^{++} and G_0^{--} remain unchanged in this restricted phase space. The result of this on the free propagator matrix $\boldsymbol{G}_0$ is actually drastic. Since the off diagonal entries G_0^{+-} and G_0^{-+} are inserted through $e^{\mathcal{D}}$ into $\boldsymbol{G}_0$, this phase space constraint means the off diagonal terms of this matrix would be altered. Thus by considering a restricted phase space in the final state results in a change of the off diagonal elements of the free Schwinger-Keldysh propagator matrix i.e. in this case

$$\boldsymbol{G}_0 = \begin{pmatrix} G_0^{++} & \tilde{G}_0^{+-} \\ \tilde{G}_0^{-+} & G_0^{--} \end{pmatrix} \tag{3.66}$$

where the $\tilde{G}_0^{\pm\mp}$ are the modified $G_0^{\pm\mp}$ terms. These $\tilde{G}$ propagators are necessarily different from those used in the literature and specifically different to those defined in the free versions of equations (3.13), (3.14) and (3.16) of section 3.1.3. This is an important result that has not been seen in the literature.

This change of the off-diagonal entries can be considered an instruction on how to set the boundary conditions of the problem. The off diagonal terms set the boundaries since $\mathcal{D}$ inserts objects at the latest time and the off diagonal terms come in through the use of $\mathcal{D}$.

A very simple illustration of the fact that the off diagonal terms of $\boldsymbol{G}_0$ fix boundary values is as follows. The inverse propagator multiplied by the propagator should give a δ or in the case of matrices, a $\mathbb{1}$. Then it is known that $(\Box + m^2)G_0^{\pm\mp} = 0$, thus it can be shown that

$$\begin{pmatrix} \Box + m^2 & 0 \\ 0 & \Box + m^2 \end{pmatrix} \begin{pmatrix} G_0^{++} & G_0^{+-} \\ G_0^{-+} & G_0^{--} \end{pmatrix} = \mathbb{1}\ . \tag{3.67}$$

This is the usual solution used in the literature however, this isn't general. Since $\tilde{G}_0^{\pm\mp}$ is $G_0^{\pm\mp}$ up to some factor, it can be shown that $(\Box + m^2)\tilde{G}_0^{\pm\mp} = 0$. This means that an equally valid solution, of the above is given by

$$\begin{pmatrix} \Box + m^2 & 0 \\ 0 & \Box + m^2 \end{pmatrix} \begin{pmatrix} G_0^{++} & \tilde{G}_0^{+-} \\ \tilde{G}_0^{-+} & G_0^{--} \end{pmatrix} = \mathbb{1}\ . \tag{3.68}$$

The off diagonal terms amount to the freedom in the boundaries of the system. In this case one could even consider the situation where $\tilde{G}_0^{\pm\mp} = 0$. This case amounts to the situation of the in-out formalism discussed above where one is interested in finite particles in the out state.

So the usual treatment of always assuming equation (3.67) is not general and only by understanding $\mathcal{D}$ does one gain insight into how to deal with general expectation values.

A final important note is that this entire discussion was done for the free propagator. The full off diagonal elements may contain slightly different behaviour but, in the case where one doesn't have off diagonal free propagators would imply that there are no off diagonal elements for the full propagator either. No more will be said on this matter as this requires a far more in depth study than is done here.

3.1.7 Closed Time Path and Schwinger-Keldysh Generating Functional

The literature ([7], [8]) makes the claim that the Schwinger-Keldysh generating functional is a normal looking path integral but with the Lagrangian density integrated over a closed time contour. This is not done very well as the terms G_0^{+-} and G_0^{-+} are not dealt with well. One can now make the connection to the literature's presentation by using the $\mathcal{D}[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}]$ operator .

Start by writing,

$$e^{iW[J_+]}e^{-iW[J_-]} = \int D[\phi^+] \int D[\phi^-] e^{i\int_{t_0}^{t_f} dx^0 \int d^3\boldsymbol{x}\left(\mathcal{L}[\phi^+(x)]+J_+\phi^+\right) - i\int_{t_f}^{t_0} dx^0 \int d^3\boldsymbol{x}\left(\mathcal{L}[\phi^-(x)]+J_-\phi^-\right)} \tag{3.69}$$

note the integration of the time x^0 in each case. Instead of splitting up the integral, consider integrating over all time of the Schwinger-Keldysh contour [8].

$$e^{iW[J_+]}e^{-iW[J_-]} = \int D[\phi^+] \int D[\phi^-] e^{i\int_C dx^0 \int d^3\mathbf{x}[\mathcal{L}[\phi(x)]+J(x)\phi(x)]} \tag{3.70}$$

where $\int_C$ is the instruction to integrate over the whole Schwinger-Keldysh time contour.

The problem is the two path's haven't been sewed together, there is no object that links them. Making the claim from the first equation to the second assumes the existence of a way of knitting the + branch and − branch together out at $t = +\infty$.

As was discussed above, the term $e^{\mathcal{D}}$ does this sewing. Thus having the term $e^{\mathcal{D}[\frac{\delta}{\delta J_+}, \frac{\delta}{\delta J_-}]} e^{iW[J_+]}e^{-iW[J_-]}$ means that one can write the full generating functional as an integral over a closed time contour such that

$$\begin{aligned} Z[J_+, J_-, \rho(t_0)] = &\int D[\phi_{t_0}^{(+)}(\boldsymbol{x})] D[\phi_{t_0}^{(-)}(\boldsymbol{x})] \langle \phi_{t_0}^{(+)}|\rho(t_0)|\phi_{t_0}^{(-)}\rangle \\ &\times \int D[\phi^+] \int D[\phi^-] e^{i\int_C dx^0 \int d^3\mathbf{x}[\mathcal{L}[\phi(x)]+J(x)\phi(x)]} \end{aligned} \tag{3.71}$$

where $\int_C$ is the instruction to integrate over the whole Schwinger-Keldysh time contour.

3.2 Schwinger-Keldysh Formlalism in a Transformed Basis

It turns out that to compare the two methods of interest in this thesis, it is a good idea to re-express the Schwinger-Keldysh objects in a different manner, as linear combinations of the original objects.

The proposal is as follows [22], [23];

$$\sigma = \frac{1}{2}(\phi_- + \phi_+) \tag{3.72}$$
$$\eta = \phi_+ - \phi_- \; . \tag{3.73}$$

Then in analogy to equations (3.72) and (3.73);

$$J_\sigma = \frac{1}{2}(J_+ + J_-) \tag{3.74}$$
$$J_\eta = J_+ - J_- \; . \tag{3.75}$$

The repercussions of this change of variables impacts the Lagrangian of the Schwinger-Keldysh formalism. This new Lagrangian will be discussed following section 3.2.1.

3.2.1 Schwinger-Keldysh Propagators after ϕ, η Transformation

In this section the propagators due to the redefined fields are discovered as these propagators are very important. They allow one to compare the two methods under investigation in an easier manner.

To find these rotated propagators the work of [23] is followed. In this source it shows how to consider the field redefinitions of equations (3.72) and (3.73) as a linear map of the vector created from ϕ^+ and ϕ^- by a matrix R (equation (3.77)). To find the propagators in the newly defined frame work, one can map the usual Schwinger-Keldysh propagators $\boldsymbol{G}$ (see equation (3.16)) by the newly found matrix R.

In what follows $\boldsymbol{G}$ is transformed using R. Each entry in the new matrix becomes a linear combination of the usual Schwinger-Keldysh propagators (equation (3.80) below). One can then exploit equations (3.8) and (3.9) to express each entry as combinations of G^{+-} and G^{-+} with Θ functions. It will turn out that these combinations form previously discussed propagators, specifically F, G^R and G^A from section 3.1.3 (equation (3.84) below). Only the proof for the first component is presented here, the rest is left for appendix C.1.

One can then understand each of the usual Schwinger-Keldysh propagators as correlators of the transformed fields σ and η (equations (3.85) - (3.88) below). These are then inserted into the transformed matrix (equation (3.80)), which then expresses the transformed propagator as a matrix of propagators made from σ η correlations (equation (3.91) below).

Finally the two different representations of the matrix are equated. This shows how one should interpret the σ, η correlators in terms of F, G^R and G^A (equation (3.92) below). This will allow one to switch between the two representations which will have great value later on in this thesis.

Defining the Linear Map R

[23] showed that the transformations of the variables ϕ_+ and ϕ_- from equations 3.72 and 3.73 can be given as a linear map of a vector.

$$\begin{pmatrix} \sigma \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\phi_- + \phi_+) \\ \phi_- - \phi_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \tag{3.76}$$

This gives the interpretation of σ and η as obtained by a linear map R of the usual Schwinger-Keldysh basis where [23],

$$R \equiv \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} . \tag{3.77}$$

This linear map can be used to transform other objects in the usual Schwinger-Keldysh basis into a form consistent with the new notation. This is now used to transform $\boldsymbol{G}$.

$R\boldsymbol{G}(x,y)R^T$ in Terms of F, G^R and G^A

As a reminder, section 3.1.3 introduced the idea that the usual Schwinger-Keldysh full propagator is a matrix of the form

$$\boldsymbol{G}(x,y) = \begin{pmatrix} G^{++}(x,y) & G^{+-}(x,y) \\ G^{-+}(x,y) & G^{--}(x,y) \end{pmatrix} . \tag{3.78}$$

(So it is assumed again that one is only interested in expectation values to all final out states, thus this matrix is used).

Using the newly defined linear map R, this propagator can be transformed to get a new form consistent with the σ, η notation.

$$\boldsymbol{G}(x,y) \quad \to \quad R\boldsymbol{G}(x,y)R^T \tag{3.79}$$

where,

$$R\boldsymbol{G}R^T = \begin{pmatrix} \frac{1}{4}(G^{++}+G^{-+}+G^{+-}+G^{--}) & \frac{1}{2}(G^{++}-G^{+-}+G^{-+}-G^{--}) \\ \frac{1}{2}(G^{++}+G^{+-}-G^{-+}-G^{--}) & (G^{++}-G^{+-}-G^{-+}+G^{--}) \end{pmatrix} . \tag{3.80}$$

The first component of the transformed matrix will be discussed in detail here, the rest of the terms are left for appendix C.1 turn.

$\left(R\boldsymbol{G}R^T\right)^{1,1}$: $\frac{1}{4}(G^{++}+G^{-+}+G^{+-}+G^{--})$

Start with expressing G^{++} and G^{--} in terms of G^{+-} and G^{-+} through equations (3.8) and (3.9).

$$
\begin{aligned}
\frac{1}{4}(G^{++}+G^{-+}+G^{+-}+G^{--}) &= \frac{1}{4}\Big(\theta(x^0-y^0)G^{-+}(x,y)+\theta(y^0-x^0)G^{+-}(x,y) \\
&\quad +G^{-+}(x,y)+G^{+-}(x,y)+\theta(x^0-y^0)G^{+-}(x,y) \\
&\quad +\theta(y^0-x^0)G^{-+}(x,y)\Big) \\
&= \frac{1}{4}\Big(\left[\theta(x^0-y^0)+1+\theta(y^0-x^0)\right]G^{-+}(x,y) \\
&\quad +\left[\theta(y^0-x^0)+1+\theta(x^0-y^0)\right]G^{+-}(x,y)\Big) \\
&\quad \text{Using } \theta(x^0-y^0)+\theta(y^0-x^0)=1 \\
&= \frac{1}{4}\left(2(G^{-+}(x,y)+G^{+-}(x,y))\right) \\
&= \frac{1}{2}\left(G^{-+}(x,y)+G^{+-}(x,y)\right)
\end{aligned} \tag{3.81}
$$

Using the definition of $G^{-+}(x,y)$ and $G^{+-}(x,y)$ from 3.13 and 3.14

$$\left(R\boldsymbol{G}R^T\right)^{1,1} = \frac{1}{2}\left(\langle\phi(x)\phi(y)\rangle + \langle\phi(y)\phi(x)\rangle\right) \ . \tag{3.82}$$

The above is the exact definition of the statistical two point function $F(x,y)$ given in equation (3.17) [8]

Thus

$$\left(R\boldsymbol{G}R^T\right)^{1,1} = F(x,y) \ . \tag{3.83}$$

The rest of the terms can be found in a similar way, the details of which are left to appendix C.1. The only point one must be careful of is that one gets the G^R and G^A up to factors of i as compared with equations (3.22) and (3.23). Thus quoting equations (C.4), (C.6) and (C.8), it is found that [23]

$$\left(R\boldsymbol{G}(x,y)R^T\right) = \begin{pmatrix} F(x,y) & -iG^R(x,y) \\ -iG^A(x,y) & 0 \end{pmatrix} \ . \tag{3.84}$$

$R\boldsymbol{G}(x,y)R^T$ in Terms of σ, η Correlators

Lastly, the matrix $R\boldsymbol{G}(x,y)R^T$ needs to be connected to the propagators made from σ and η.

Using equations (3.8) - (3.9) as the starting point for each then using equations (3.98) and (3.99)

$$
\begin{aligned}
G^{++}(x,y) &= \langle\phi^+(x)\phi^+(y)\rangle \\
&= \langle(\sigma(x)+\frac{\eta(x)}{2})(\sigma(y)+\frac{\eta(y)}{2})\rangle \\
&= \langle\sigma(x)\sigma(y)\rangle + \frac{1}{2}\langle\sigma(x)\eta(y)\rangle + \frac{1}{2}\langle\eta(x)\sigma(y)\rangle + \frac{1}{4}\langle\eta(x)\eta(y)\rangle
\end{aligned} \tag{3.85}
$$

similarly

$$G^{+-}(x,y) = \langle\sigma(x)\sigma(y)\rangle - \frac{1}{2}\langle\sigma(x)\eta(y)\rangle + \frac{1}{2}\langle\eta(x)\sigma(y)\rangle - \frac{1}{4}\langle\eta(x)\eta(y)\rangle \tag{3.86}$$

$$G^{-+}(x,y) = \langle\sigma(x)\sigma(y)\rangle + \frac{1}{2}\langle\sigma(x)\eta(y)\rangle - \frac{1}{2}\langle\eta(x)\sigma(y)\rangle - \frac{1}{4}\langle\eta(x)\eta(y)\rangle \tag{3.87}$$

$$G^{--}(x,y) = \langle\sigma(x)\sigma(y)\rangle - \frac{1}{2}\langle\sigma(x)\eta(y)\rangle - \frac{1}{2}\langle\eta(x)\sigma(y)\rangle + \frac{1}{4}\langle\eta(x)\eta(y)\rangle\ . \tag{3.88}$$

Putting all the above into equation (3.80) gives [22],

$$R\boldsymbol{G}(x,y)R^T = \begin{pmatrix} \langle\sigma(x)\sigma(y)\rangle & \langle\sigma(x)\eta(y)\rangle \\ \langle\eta(x)\sigma(y)\rangle & \langle\eta(x)\eta(y)\rangle \end{pmatrix}\ . \tag{3.89}$$

Using the notation

$$\langle\sigma(x)\sigma(y)\rangle = G_{\sigma\sigma} \tag{3.90}$$

and likewise for the other objects in a similar manner gives [22]

$$R\boldsymbol{G}(x,y)R^T = \begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\eta} \\ G_{\eta\sigma} & G_{\eta\eta} \end{pmatrix}\ . \tag{3.91}$$

Equating the Two Representations of $R\boldsymbol{G}(x,y)R^T$

Finally equating (3.84) and equation (3.91) gives

$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\eta} \\ G_{\eta\sigma} & G_{\eta\eta} \end{pmatrix} = \begin{pmatrix} F(x,y) & -iG^R(x,y) \\ -iG^A(x,y) & 0 \end{pmatrix}\ . \tag{3.92}$$

This gives a map between correlations in terms of σ, η and the propagators F, G^R and G^A. This is very useful for simplifying certain arguments made later in the thesis.

3.2.2 Consequence for G^R and G^A

The above relationships also provide a further set of relationships that are useful for later in this thesis. It is useful to express G^R and G^A as linear combinations of the usual Schwinger-Keldysh Propagators. This will allow for a quicker translation between the two representations.

These results are found in [16].

Since $\left(R\boldsymbol{G}R^T\right)^{1,2} = -iG^R(x,y)$

$$
\begin{aligned}
-iG^R &= \frac{1}{2}(G^{++} - G^{+-} + G^{-+} - G^{--}) \\
&\quad \text{Using equation (3.15)} \\
&= G^{++} - G^{+-} \,.
\end{aligned} \tag{3.93}
$$

Similarly since $\left(R\boldsymbol{G}R^T\right)^{2,1} = -iG^A(x,y)$

$$
-iG^A = G^{++} - G^{-+} \,. \tag{3.94}
$$

3.2.3 The Schwinger-Keldysh Generating Functional in Terms of σ and η

In this section it becomes clear why one would like to consider this change in variables. Considering these variables in the generating functional leads to the incredible result that the generating functional gives a δ function in the classical solution for σ [22] . To find this is a long process so an outline of the proof is given first.

One starts by considering the generating functional in terms of the transformed fields σ and η. To do this one transforms each part of equation (3.44) separately. First one considers transforming $e^{iW[J_+]}e^{-iW[J_-]}$. This will turn out to contain the important behaviour that is discussed in a moment. As a side note from this transformation only the $G^{\sigma\eta}$ and $G^{\eta\sigma}$ can be obtained. Next one briefly considers transforming $e^{\mathcal{D}\left[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}\right]}$. The important point here is that this term contains the $G^{\sigma\sigma}$ propagator. Lastly the term $\langle\phi^{(+)}_{t_0}|\rho(t_0)|\phi^{(-)}_{t_0}\rangle$ is mentioned, but it doesn't play a role in what follows.

The important point to pull out of this section is that when looking at the completely transformed generating functional, it is found that one can obtain a constraint integral in the η field (equation (3.114) below). The constraint integral in turn gives a delta function that instructs σ to solve the classical equation of motion, this means that the delta function forces σ to be the classical field φ (equation (3.115) below).

Before starting on the details, a reminder of equation (3.44),

$$
Z[J_+,J_-,\rho(t_0)] = \int D[\phi^{(+)}_{t_0}(\boldsymbol{x})]D[\phi^{(-)}_{t_0}(\boldsymbol{x})]\langle\phi^{(+)}_{t_0}|\rho(t_0)|\phi^{(-)}_{t_0}\rangle e^{\mathcal{D}\left[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}\right]}e^{iW[J_+]}e^{-iW[J_-]} \,. \tag{3.95}
$$

Z_{+-} Part

The following argument follows [22] and [23]

Remember that

$$
Z_{+-}[J_+,J_-] = e^{iW[J_+]}e^{-iW[J_-]} \,. \tag{3.96}
$$

From equation (3.48), the exponential of the above looks like

$$\mathcal{L}[\phi_+,\phi_-] = \left(-\frac{1}{2}\phi^+(\Box+m^2)\phi^+ - \frac{g^2}{4!}(\phi^+)^4 + J_+\phi^+\right) + \left(\frac{1}{2}\phi^-(\Box+m^2)\phi^- + \frac{g^2}{4!}(\phi^-)^4 - J_-\phi^-\right) . \tag{3.97}$$

Equations 3.72 and 3.73 can be rearranged to show that

$$\phi_- = \sigma - \frac{\eta}{2} \tag{3.98}$$

$$\phi_+ = \sigma + \frac{\eta}{2} . \tag{3.99}$$

Putting the above back into equation (3.97) ,

$$\mathcal{L} = -\sigma\left(\Box - m^2\right)\eta - \frac{g^2}{3!}\sigma^3\eta - \frac{g^2}{4!}\sigma\eta^3 \tag{3.100}$$

In the notation of equation (3.100),

$$Z_{\sigma\eta}[J_+,J_-] = \int D[\sigma]D[\eta]e^{i\left[-\sigma\left(\Box-m^2\right)\eta-\frac{g^2}{3!}\sigma^3\eta-\frac{g^2}{4!}\sigma\eta^3+J_+(\phi+\frac{\eta}{2})-J_-(\phi-\frac{\eta}{2})\right]} . \tag{3.101}$$

Note that the Jacobian involved in the change of variables is 1.

The sources still refer to the $+$, $-$ notation. Using equations (3.74) and (3.75) the J_- and J_+ terms can be re-written in terms of variables J_σ and J_η [23].

$$\begin{aligned} Z_{\sigma\eta}[J_\sigma,J_\eta] &= \int D[\sigma]D[\eta]e^{i\left[-\sigma\left(\Box-m^2\right)\eta-\frac{g^2}{3!}\sigma^3\eta-\frac{g^2}{4!}\sigma\eta^3+(J_\sigma+\frac{J_\eta}{2})(\sigma+\frac{\eta}{2})-(J_\sigma-\frac{J_\eta}{2})(\sigma-\frac{\eta}{2})\right]} \\ &\text{Multiplying out the brackets, and simplifying} \\ &= \int D[\sigma]D[\eta]e^{i\left[-\sigma\left(\Box-m^2\right)\eta-\frac{g^2}{3!}\sigma^3\eta-\frac{g^2}{4!}\sigma\eta^3+J_\sigma\eta+J_\eta\sigma\right]} \end{aligned} \tag{3.102}$$

Looking at the above term, one can see that the only free propagators described by this term are the $G_0^{\sigma\eta}$ and $G_0^{\eta\sigma}$. The terms with the g^2 are giving the interaction terms. Thus there are two different vertices, one vertex containing three σ ends and an η end and the other vertex that joins three η ends and a σ end. Finally the two different types of sources, and the fields they couple to are represented.

$\mathcal{D}\left[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}\right]$ Part

This will not be discussed in depth mathematically as the hard work has been done in that respect. One can discuss the term's role by referring to equation (3.65).

Consider the free theory transformed by R defined in equation (3.77). To do this transform the expression given in equation (3.65) using R.

$$e^{\mathcal{D}\left[\frac{\delta}{\delta J_+},\frac{\delta}{\delta J_-}\right]} Z^0_{-+} = N e^{\frac{1}{2}\int d^4x \int d^4y \boldsymbol{J}^T(x) R^{-1} R\boldsymbol{G}_0(x,y) R^T (RT)^{-1} \boldsymbol{J}(y)} \tag{3.103}$$

It is not hard to show that

$$\begin{aligned} \boldsymbol{J}^{Trans} &= (R^T)^{-1}\boldsymbol{J} \\ &= \begin{pmatrix} J_\sigma \\ J_\eta \end{pmatrix} \end{aligned} \tag{3.104}$$

and

$$\begin{aligned} \left(\boldsymbol{J}^T\right)^{Trans} &= \boldsymbol{J}^T(x) R^{-1} \\ &= \begin{pmatrix} J_\sigma \\ J_\eta \end{pmatrix}^T . \end{aligned} \tag{3.105}$$

Finally, $R\boldsymbol{G}_0(x,y)R^T$ is given as in equation (3.91).

So the transformed free theory is given by

$$e^{\mathcal{D}\left[\frac{\delta}{\delta J_\sigma},\frac{\delta}{\delta J_\eta}\right]} Z^0_{\sigma\eta} = N e^{\frac{1}{2}\int d^4x \int d^4y \left(\boldsymbol{J}^T\right)^{Trans}(x) G_0^{Trans}(x,y) J^{Trans}(y)} . \tag{3.106}$$

Since the free part of $Z_{\sigma\eta}$ gives $G_0^{\sigma\eta}$ and $G_0^{\eta\sigma}$, this means that the role played by the term $e^{\mathcal{D}\left[\frac{\delta}{\delta J_\sigma},\frac{\delta}{\delta J_\eta}\right]}$ is to provide $G_0^{\sigma\sigma}$, the $G^{\eta\eta}$ is 0.

The form of $\mathcal{D}\left[\frac{\delta}{\delta J_\sigma},\frac{\delta}{\delta J_\eta}\right]$ is complicated. Despite this complicated form, it has a simple application as shown above. The form can be found by transforming equation (3.57) with R, thus

$$\mathcal{D}\left[\frac{\delta}{\delta J_\sigma},\frac{\delta}{\delta J_\eta}\right] = \frac{1}{2}\frac{1}{Z}\int d^4x d^4y \left(\delta[\boldsymbol{J}]\right)^T(x)(R)^{-1} R\boldsymbol{G}^0_{OD}(x,y) R^T \left(R^T\right)^{-1} \left(\delta[\boldsymbol{J}]\right)(y) \tag{3.107}$$

where using equation (3.55) gives

$$\begin{aligned} R^{T^{-1}}\left(\delta[\boldsymbol{J}]\right)(y) &= \begin{pmatrix} \frac{\delta}{\delta J_+(y)} + \frac{\delta}{\delta J_-(y)} \\ \frac{1}{2}\frac{\delta}{\delta J_+(y)} - \frac{1}{2}\frac{\delta}{\delta J_-(y)} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\delta}{\delta J_\sigma(y)} \\ \frac{\delta}{\delta J_\eta(y)} \end{pmatrix} \end{aligned} \tag{3.108}$$

and using equation (3.59) where for clarity let $\Box_x^{m^2} \equiv \Box_x + m^2$ gives,

$$
\begin{aligned}
R\boldsymbol{G}^0_{OD}(x,y)R^T &= \begin{pmatrix} \frac{1}{4}\overleftarrow{\Box}^{m^2}_x\left(G^{+-}_0 + G^{-+}_0\right)\overrightarrow{\Box}^{m^2}_y & \frac{1}{2}\overleftarrow{\Box}^{m^2}_x\left(-G^{+-}_0 + G^{-+}_0\right)\overrightarrow{\Box}^{m^2}_y \\ \frac{1}{2}\overleftarrow{\Box}^{m^2}_x\left(G^{+-}_0 - G^{-+}_0\right)\overrightarrow{\Box}^{m^2}_y & \overleftarrow{\Box}^{m^2}_x\left(-G^{+-}_0 - G^{-+}_0\right)\overrightarrow{\Box}^{m^2}_y \end{pmatrix} \\
&\quad \text{use equations (3.85) - (3.88)} \\
&= \begin{pmatrix} \frac{1}{2}\overleftarrow{\Box}^{m^2}_x G^{\sigma\sigma}_0 \overrightarrow{\Box}^{m^2}_y & \frac{1}{2}\overleftarrow{\Box}^{m^2}_x\left(G^{\sigma\eta}_0 - G^{\eta\sigma}_0\right)\overrightarrow{\Box}^{m^2}_y \\ \frac{1}{2}\overleftarrow{\Box}^{m^2}_x\left(-G^{\sigma\eta}_0 + G^{\eta\sigma}_0\right)\overrightarrow{\Box}^{m^2}_y & -2\overleftarrow{\Box}^{m^2}_x G^{\sigma\sigma}_0 \overrightarrow{\Box}^{m^2}_y \end{pmatrix} . \qquad (3.109)
\end{aligned}
$$

Again, one need not consider this complicated form as it has been shown the role this operator plays.

The Initial Density Matrix

The integral over the initial density matrix is just a number. It is thus invariant under rotations. Since the particular structure is not important for the following discussion, it will be enough to just refer to the rotated density matrix as

$$
\int D[\sigma_{t_0}(\boldsymbol{x})]D[\eta_{t_0}(\boldsymbol{x})]\langle\sigma_{t_0},\eta_{t_0}|\rho^{Trans}(0)|\sigma_{t_0},\eta_{t_0}\rangle . \qquad (3.110)
$$

Putting It All Together

Thus one has

$$
\begin{aligned}
Z[J_\sigma, J_\eta, \rho^{Trans}(0)] = &\int D[\sigma_{t_0}(\boldsymbol{x})]D[\eta_{t_0}(\boldsymbol{x})]\langle\sigma_{t_0},\eta_{t_0}|\rho^{Trans}(0)|\sigma_{t_0},\eta_{t_0}\rangle e^{\mathcal{D}\left[\frac{\delta}{\delta J_\sigma},\frac{\delta}{\delta J_\eta}\right]]} \\
&\times \int D[\sigma]D[\eta]e^{i\left[-\sigma\left(\Box - m^2\right)\eta - \frac{g^2}{3!}\sigma^3\eta - \frac{g^2}{4!}\sigma\eta^3 + J_\sigma\eta + J_\eta\sigma\right]} . \qquad (3.111)
\end{aligned}
$$

From here one can consider pulling out the interaction term $\sigma\eta^3$ by considering it as variations with respect to J_σ and J_η i.e. this is done to try and isolate a single term that multiplies η. This will then be used to integrate over the η terms to get a delta function in the classical solution.

$$
\begin{aligned}
Z[J_\sigma, J_\eta, \rho^{Trans}(0)] = &\int D[\sigma_{t_0}(\boldsymbol{x})]D[\eta_{t_0}(\boldsymbol{x})]\langle\sigma_{t_0},\eta_{t_0}|\rho^{Trans}(0)|\sigma_{t_0},\eta_{t_0}\rangle e^{\mathcal{D}\left[\frac{\delta}{\delta J_\sigma},\frac{\delta}{\delta J_\eta}\right]} \\
&\times e^{-i\frac{g^2}{4!}\frac{\delta}{\delta J_\sigma}\frac{\delta^3}{\delta (J_\eta)^3}} \int D[\sigma]D[\eta]e^{i\left[-\sigma\left(\Box - m^2\right)\eta - \frac{g^2}{3!}\sigma^3\eta + J_\sigma\eta + J_\eta\sigma\right]} \qquad (3.112)
\end{aligned}
$$

For the remainder of this section, the terms that don't depend on the functional integrals will be defined as $\mathcal{S}\left[\frac{\delta}{\delta J_\sigma},\frac{\delta}{\delta J_\eta}\right]$ i.e.

$$
\mathcal{S}\left[\frac{\delta}{\delta J_\sigma},\frac{\delta}{\delta J_\eta}\right] \equiv \int D[\sigma_{t_0}(\boldsymbol{x})]D[\eta_{t_0}(\boldsymbol{x})]\langle\sigma_{t_0},\eta_{t_0}|\rho^{ROT}(0)|\sigma_{t_0},\eta_{t_0}\rangle e^{\mathcal{D}\left[\frac{\delta}{\delta J_\sigma},\frac{\delta}{\delta J_\eta}\right]]} e^{-i\frac{g^2}{4!}\frac{\delta}{\delta J_\sigma}\frac{\delta^3}{\delta (J_\eta)^3}} . \qquad (3.113)
$$

Thus equation (3.112), becomes

$$Z[J_\sigma, J_\eta, \rho^{Trans}(0)] = \mathcal{S}\left[\frac{\delta}{\delta J_\sigma}, \frac{\delta}{\delta J_\eta}\right] \int D[\sigma] D[\eta] e^{i\left[-\sigma(\Box - m^2) - \frac{g^2}{3!}\sigma^3 + J_\sigma\right]\eta} e^{iJ_\eta \sigma} \tag{3.114}$$

Inspecting the η integration, one encounters a functional δ function $\int D[\eta] e^{i(\dots)\eta} = \delta\,[...]$ so that the following is arrived at [22],

$$Z[J_\sigma, J_\eta, \rho^{Trans}(0)] = \mathcal{S}\left[\frac{\delta}{\delta J_\sigma}, \frac{\delta}{\delta J_\eta}\right] \int D[\sigma] \delta\left[-\sigma\left(\Box - m^2\right) - \frac{g^2}{3!}\sigma^3 + J_\sigma\right] e^{iJ_\eta \sigma} \,. \tag{3.115}$$

The δ function enforces the condition that,

$$-\sigma\left(\Box - m^2\right) - \frac{g^2}{3!}\sigma^3 + J_\sigma = 0 \,. \tag{3.116}$$

This equation is nothing but the classical equation of motion. If any field satisfies the classical equation of motion, then that field is the classical field. Thus it has been shown that the generating functional in the transformed co-ordinates contains a δ function in the classical solution for the field σ. This then identifies σ as φ.

This is an awesome result that also sets up the Classical-Statistical method. This method is discussed in great detail in chapter 4.

3.2.4 Classical-Statistical and Quantum-Statistical Vertices

This discussion requires looking closely at equation (3.102). This object describes the building blocks of the theory in terms of σ and η.

The $\sigma(\Box + m^2)\eta$ term describes the free propagator from σ to η and η to σ. The term $\eta\sigma^3$ is a vertex that has three σ ends and one η end. The term $\eta^3\sigma$ is a vertex in the same way.

The term $\eta\sigma^3$ is named the classical-statistical vertex. The reason for calling it classical is that this is the only vertex left when $O(\eta^2)$ are thrown away. When the $O(\eta^2)$ are thrown away, the solution is a delta function in the classical fields. So the surviving vertex is the only vertex that appears when one gets the classical solutions from $Z[J_\sigma]$. The quantum-statistical vertex is the vertex $\eta^3\sigma$.

The statistical in the names classical-statistical and quantum-statistical vertices is only in reference to the method, Classical-Statistical method. It is not an actual statistical object. The name comes from the fact that the Classical-Statistical method can only produce terms with the classical-statistical vertex.

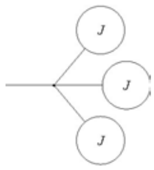
This is all that's necessary to say at this point as these vertices will be discussed in greater detail in the section on the Classical-Statistical method.

3.2.5 The Classical Field in the R Transformed Notation

It is useful to note that the classical field can only end on a σ space on a vertex, i.e. it cannot end on an η space.

This can be shown by using a very clever trick used in [16].

Consider a tree level diagram such as the second term in equation (1.99) which is



. (3.117)

In the Schwinger-Keldysh one sums over the vertices being + or −. Consider the sum for one leg on the 'tree' starting at a + vertex and ending in a blob J of the above. Do this for the situation where there are physical sources, i.e. $J_+ = J_- = J$. Note that the relative minus sign is coming from the fact that in the exponential of the generating functional, the J_+ comes with a + sign and the J^- comes with a − sign. Then,

$$\begin{aligned} G_0^{++}J^+ - G_0^{+-}J^- &= \left(G_0^{++} - G_0^{+-}\right) J \\ &\quad \text{Using equation (3.93)} \\ &= G_0^R J \, . \end{aligned} \tag{3.118}$$

One can continue this logic all the way back to the root of the tree. One ends with a sum of G_0^R propagators. Now looking back to equation (3.92), one can identify, up to factors of i, that $G_0^R = G_0^{\sigma\eta}$. Since this propagator starts at η and ends at σ the classical field is made up of a whole bunch of $G_0^{\sigma\eta}$ propagators such that the last part of the tree ends on a σ. Thus the classical field can only couple to a σ space on the vertex, there is no way it can link to an η space.

3.2.6 Diagrammatic Notation and Translation

It is useful to put some of the above discussion into diagrammatic form.

Let

$G_{\sigma\eta} =$ (3.119)

$G_{\eta\sigma} =$ (3.120)

$G_{\sigma\sigma} =$ (3.121)

Diagrammatically, the classical field insert is given by

$\sigma =$  . (3.122)

It represents the full solution to equation (3.116) and is thus the same as φ.

Note that it ends in a solid line due to the discussion had in the previous section.

Finally, in this notation, one talks about the classical-statistical vertex and the quantum-statistical vertex

classical-s vertex =  (3.123)

quantum-s vertex =  (3.124)

3.3 Tell Tale Signs for Equilibrium in the Schwinger-Keldysh Formalism

This section will show a particular relationship between the F and ρ propagators that occurs when the system is in thermal equilibrium. This relationship can then be used to characterize when a system has reached thermal equilibrium. Both methods under investigation in this thesis attempt to reproduce this relationship.

3.3.1 Kubo-Martin-Schwinger (KMS) Condition

The Kubo-Martin-Schwinger (KMS) condition is a well known condition met when a system is in thermal equilibrium. It allows one to find an expression for the occupation number $f(k)$ in thermal equilibrium. Before the result is presented one must define equilibrium propagators.

A statistical average of an operator is given by the following expression (see section 3.1.5);

$$\langle \hat{A} \rangle = \mathrm{Tr}\left[\rho(t_0)\hat{A}\right] \; . \tag{3.125}$$

In equilibrium objects are different to those out of equilibrium, the difference comes in the use of the initial density matrix. This was discussed in section 3.1.6. Describing a system in equilibrium, amounts to choosing the initial density matrix to be given by a thermal distribution. The thermal distribution that is used is given by $\rho(t_0) = e^{-\beta\hat{H}}$.

Thus a thermal average at some β is given by

$$\langle \hat{A} \rangle_\beta = \mathrm{Tr}\left[e^{-\beta\hat{H}}\hat{A}\right] \; . \tag{3.126}$$

With this in mind one can define thermal propagators. In analogy to the G^{+-} and the G^{-+} defined in section 3.1.3, except with thermal averages one gets

$$G_\beta^{-+}(t,t') = \langle \hat{\phi}(t)\hat{\phi}(t') \rangle_\beta \tag{3.127}$$

$$G_\beta^{+-}(t,t') = \langle \hat{\phi}(t')\hat{\phi}(t) \rangle_\beta = G_\beta^{-+}(t',t) \; . \tag{3.128}$$

With these, the KMS relationship is defined as [27]

$$G_\beta^{+-}(t,t') = G_\beta^{-+}(t - i\beta, t'), \tag{3.129}$$

and equivalently

$$G_\beta^{-+}(t,t') = G_\beta^{+-}(t + i\beta, t'). \tag{3.130}$$

The proof of these expressions is left to appendix C.3.

3.3.2 Using the KMS relationship to Derive Equilibrium Conditions

In this section, one can show that for a momentum k_0, $G_\beta^{-+}(k_0) = i(1 + f(k_0))\rho_\beta(k_0)$ and $G_\beta^{+-}(k_0) = if(k_0)\rho_\beta(k_0)$ where $f(k_0)$ is the Bose-Einstein distribution and ρ_β is the equilibrium ρ propagator.

The proof starts with a Fourier transform of $G_\beta^{-+}(t,t')$ and $G_\beta^{+-}(t,t')$. Then the KMS relationship is used to express $G_\beta^{+-}(k_0)$ in terms of $G_\beta^{-+}(k_0)$, after manipulating the time integral in the Fourier transform (equation (3.134) below).

Then an equilibrium ρ propagator (ρ_β) is defined such that one can rearrange the relationship between $G_\beta^{+-}(k_0)$ and $G_\beta^{-+}(k_0)$ to find expressions for $G_\beta^{+-}(k_0)$ and $G_\beta^{-+}(k_0)$ in terms of $\rho_\beta(k_0)$ and the Bose-Einstein distribution.

Then the equilibrium F propagator is defined in terms of $G_\beta^{+-}(k_0)$ and $G_\beta^{-+}(k_0)$. Using this a relationship between F and ρ in equilibrium can be obtained.

Equilibrium Conditions for $G_\beta^{+-}(k_0)$ and $G_\beta^{-+}(k_0)$

In equilibrium propagators must be invariant under translations in time. This is due to the fact that equilibrium is defined as the situation where over a time span, things look homogeneous. This means that one can think of $G_\beta^{-+}(t,t') = G_\beta^{-+}(t-t') = G_\beta^{-+}(t+t', 2t')$, and the same is true for $G_\beta^{+-}(t,t')$, i.e. the time difference is important, not necessarily the start and end times.

Define the Fourier transformation of $G_\beta^{-+}(t,t')$ and $G_\beta^{+-}(t,t')$

$$G_\beta^{-+}(k_0) = \int_{-\infty}^{\infty} dt e^{ik_0(t-t')} G_\beta^{-+}(t,t') \tag{3.131}$$

$$G_\beta^{+-}(k_0) = \int_{-\infty}^{\infty} dt e^{ik_0(t-t')} G_\beta^{+-}(t,t') \ . \tag{3.132}$$

Due to the KMS relationship of equation (3.129),

$$G_\beta^{+-}(k_0) = \int_{-\infty}^{\infty} dt e^{ik_0(t-t')} G_\beta^{+-}(t,t') = \int_{-\infty}^{\infty} dt e^{ik_0(t-t')} G_\beta^{-+}(t-i\beta,t') \,. \tag{3.133}$$

This equality allows one to make a useful observation, one can shift the integration variable t in the above, to $t + i\beta$. Since this is a shift by a constant amount, and the integral is over the entire area, the integration measure doesn't change [27]

$$\begin{aligned} G_\beta^{+-}(k_0) &= \int_{-\infty}^{\infty} dt e^{ik_0(t-t')} G_\beta^{-+}(t-i\beta,t') \\ &= \int_{-\infty}^{\infty} dt e^{-\beta k_0} e^{ik_0(t-t')} G_\beta^{-+}(t,t') \\ &\text{Using equation (3.131)} \\ &= e^{-\beta k_0} G_\beta^{-+}(k_0) \,. \end{aligned} \tag{3.134}$$

With the same argument the equilibrium condition of G_β^{-+} can be found to be,

$$G_\beta^{-+}(k_0) = e^{\beta k_0} G_\beta^{+-}(k_0) \tag{3.135}$$

Equilibrium Conditions for ρ and F

In this section the equilibrium conditions for ρ and F are found. This is done by first expressing the equilibrium ρ as a combination of $G_\beta^{-+}(k_0)$ and $G_\beta^{+-}(k_0)$. Then from here one can rearrange to find these two propagators in terms of the equilibrium ρ. Finally the equilibrium F is defined in terms of $G_\beta^{-+}(k_0)$ and $G_\beta^{+-}(k_0)$. This can be used to find a relationship in equilibrium for F in terms of ρ.

The equilibrium ρ propagator is defined in a similar fashion to equation (3.24) of section 3.1.3,

$$\rho_\beta(k_0) = -i\left(G_\beta^{-+}(k_0) - G_\beta^{+-}(k_0)\right) \tag{3.136}$$

The plan is to re-express $G_\beta^{-+}(k_0)$ and $G_\beta^{+-}(k_0)$ in terms of the $\rho_\beta(k_0)$.

Starting with $G_\beta^{+-}(k_0)$

$$\begin{aligned} \rho_\beta(k_0) &= -i\left(G_\beta^{-+}(k_0) - G_\beta^{+-}(k_0)\right) \\ &\text{Using equation (3.135)} \\ &= -i\left(e^{\beta k_0} G_\beta^{+-}(k_0) - G_\beta^{+-}(k_0)\right) \\ &= -i\left(e^{\beta k_0} - 1\right) G_\beta^{+-}(k_0) \,. \end{aligned} \tag{3.137}$$

Note that the above represents a condition of ρ in equilibrium. Using this it is found that [27],

$$G_\beta^{+-}(k_0) = \frac{i}{e^{\beta k_0} - 1}\rho_\beta(k_0) \; . \tag{3.138}$$

Then $G_\beta^{-+}(k_0)$ [27],

$$\begin{aligned}
\rho_\beta(k_0) &= -i\left(G_\beta^{-+}(k_0) - G_\beta^{+-}(k_0)\right) \\
&\text{Using equation (3.134)} \\
&= -i\left(G_\beta^{-+}(k_0) - G_\beta^{-+}(k_0)e^{-\beta k_0}\right) \\
&= -i\left(1 - e^{-\beta k_0}\right) G_\beta^{-+}(k_0)
\end{aligned} \tag{3.139}$$

One then reads off that [27],

$$G_\beta^{-+}(k_0) = \frac{i}{1 - e^{-\beta k_0}}\rho_\beta(k_0) \; . \tag{3.140}$$

Consistency is ensured by

$$\frac{1}{1 - e^{-\beta k_0}} = 1 + \frac{1}{e^{\beta k_0} - 1} \; . \tag{3.141}$$

Thus one can summerize the relationships between ρ and G_β^{+-}, G_β^{-+} in equilibrium as

$$G_\beta^{-+}(k_0) = i(1 + f(k_0))\rho_\beta(k_0) \tag{3.142}$$

$$G_\beta^{+-}(k_0) = if(k_0)\rho_\beta(k_0) \tag{3.143}$$

where

$$f(k_0) = \frac{1}{e^{\beta k_0} - 1} \tag{3.144}$$

is the Bose-Einstein distribution.

The only missing ingredient is the relationship between F and ρ in equilibrium.

To this end recall equation (3.17), which states that

$$F(x,y) = \frac{1}{2}\left(G^{-+}(x,y) + G^{+-}(x,y)\right) \; . \tag{3.145}$$

This leads one to use equations (3.142) and (3.143) to define an equilibrium F propagator.

$$\begin{aligned}
F_\beta(k_0) &= \frac{1}{2}\left(G_\beta^{-+}(k_0) + G_\beta^{+-}(k_0)\right) \\
&\qquad \text{Using equations (3.142) and (3.143)} \\
&= i\frac{1}{2}\left[(1+f(k_0))\rho_\beta(k_0) + f(k_0)\rho_\beta(k_0)\right] \\
&= i\left(\frac{1}{2} + f(k_0)\right)\rho_\beta(k_0)\ . \qquad (3.146)
\end{aligned}$$

So [8]

$$F_\beta(k_0) = i\left(\frac{1}{2} + f(k_0)\right)\rho_\beta(k_0)\ . \qquad (3.147)$$

Any method that can produce predictions on the behaviour of F and ρ can look at the late time behaviour of these objects. One can then try and fit the relationship between these two objects. If one gets $\left(\frac{1}{2} + f(k_0)\right)$ where $f(k_0)$ is the Bose Einstein distribution, then the equations of motion in that theory have correctly driven towards thermal equilibrium.

Note that this expression only hold in thermal equilibrium, all of this depends on the fact that the initial density distribution was given as $e^{\beta\hat{H}}$.

Chapter 4

Classical-Statistical Method

The classical-statistical (C-S) method was developed for studying equilibration of a system far from equilibrium when there are large source terms in the set up of the problem.

A large source leads to a large classical field relative to quantum effects as was seen in section 1.8.4. Then in, section 3.2.3, it was shown that by using a linear map R one can find a generating functional that puts the transformed field σ on the classical field through a δ function. The classical-statistical (C-S) method uses the transformed fields, σ and η for the reason that a lot of the discussion works naturally within this frame work.

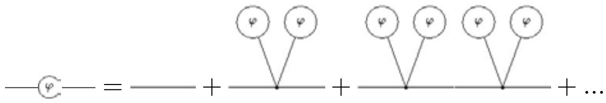
A single classical solution cannot lead to the equilibration of a system, even if the classical solution is the parametrically dominant part. This is because equilibration requires either interactions or it requires fluctuations of a solution. Since this is a quantum system one should consider quantum effects as a mechanism that causes equilibration.

The C-S method begins its approach by considering a small quantum fluctuation ($a_{\pm\boldsymbol{k}}$) off of the classical solution (equation (4.8) below). These small fluctuations are fundamental in constructing a one loop correction operator that can be used to create one loop corrections to both the classical field and the classical propagator (see equations (4.43) and (4.54) below). It will then be shown that the one loop solution diverges in time, as some of the quantum fluctuations $a_{\pm\boldsymbol{k}}$ diverge exponentially with time (equation (4.67) below). These divergent fluctuations are the dominant fluctuations in the system. It is by studying these dominant fluctuations that one can find a solution that leads to equilibration. The solution is a resummation of the dominant fluctuations that will be shown to be equivalent to taking an average of different classical solutions.

It can be shown that this resummation is successful as it shows equilibration behaviour, but it has a particular weaknesses. The resummation only re-sums diagrams that contain the classical-statistical vertices (see section 3.2.4 for a reminder on these vertices). This is assumed to be the reason that the C-S method is only capable of producing the semi-classical approximation to the equilibration condition discussed in section 3.3.2.

4.1 Tree Level Propagators of this Section

In equation (2.39), the classical propagator was defined with full one point function inserts. In this chapter, the tree level propagators will only have classical field inserts. The tree level propagators in this chapter will be labelled as $\mathcal{G}_t$. Diagrammatically,

$\mathcal{G}_t =$  (4.1)

where the φ bubbles refer to classical field, and as always the line are defined as iG_0.

4.2 Defining Objects of Interest

A few objects need to be defined to begin. First a discussion is had on the classical field. Then the discussion will move to the small fluctuation terms described in the introduction to this chapter.

Classical Fields

As was discussed in section 2.1, the classical field obeys the equation governed by $\frac{\delta S[\phi]}{\delta \phi}\Big|_{\phi=\varphi} = -J$. Throughout this chapter, the potential will be kept as general as possible. Hence one uses V to stand for the potential in the theory. The classical equation written out fully is then,

$$\begin{aligned} &\Box\varphi + \frac{\delta V[\varphi]}{\delta\varphi} = J, \\ &\lim_{x^0\to-\infty}\varphi(x^0,x) = 0, \ \lim_{x^0\to-\infty}\dot{\varphi}(x^0,x) = 0\ . \end{aligned} \tag{4.2}$$

The reason that initial conditions are introduced is that this expression is actually an equation of motion. This is a second order differential equation and so needs two initial conditions. The way large sources enter the set up in this chapter is through the initial conditions.

Since the initial conditions are important, one would like to keep this information in the solution of the equation. The way to do this is using the Green's solution. The proof of it is done is appendix D.1. The result is reproduced here.

Given an equation of the form

$$\left(\Box_x + m^2\right) F(x) = J(x) \tag{4.3}$$

one can show the solution is given by

$$F(x) = i\int\limits_{y^0>t_0} d^4y G^0_R(x,y)J(y) + i\int\limits_{y^0=t_0} d^3\boldsymbol{y}\left\{G^R_0(x,y)\left(\overleftarrow{\partial}_{y^0} - \overrightarrow{\partial}_{y^0}\right)F(y)\right\} \tag{4.4}$$

where the arrows on the operators show the direction they act in. Here it is assumed that one is only interested in finding the solution of $F(x)$ after some time t_0. The terms inside $\int\limits_{y^0=t_0} d^3\boldsymbol{y}$ represent the initial conditions of $F(x)$ and $\partial_{x^0}F(x)$ at time t_0. If one were to assume trivial initial conditions, the naive solution to this equation is found.

Thus solving equation (4.2) with the above Green's solution gives [24],

$$\varphi(x) = -i \int\limits_{\Sigma+} d^4y G_0^R(x,y) \frac{\delta V}{\delta\varphi(y)} + i \int\limits_{\Sigma} d^3\boldsymbol{y} \left[\left(\partial_0 G_0^R(x,\boldsymbol{y})\right) \varphi_0(\boldsymbol{y}) + G_0^R(x,\boldsymbol{y}) \left(\partial_0 \varphi_0(\boldsymbol{y})\right) \right] \tag{4.5}$$

where and $\varphi_0(\boldsymbol{y}) \equiv \varphi(y^0 = t_0, \boldsymbol{y})$. Σ represents the $t = 0$ space-time surface andΣ $^+$ represents the space-time volume from Σ to all $t > t_0$.

It is crucial to again point out that in this chapter the source term J is taken to exist before the initial time surface. Thus the effects of the large source are encoded directly into the initial conditions. After the initial time one assumes that the source term switches off. This is a different way to discuss large sources in the set up of the problem.

With the classical field discussed and defined, the next step is to consider small quantum fluctuations on top of the classical field, but before doing so an important discussion is had on initial conditions.

Discussion on Initial Time Surface

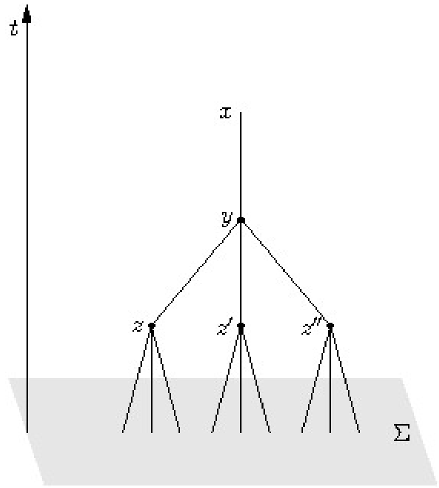
Before continuing with the small fluctuations it is worthwhile to introduce the concept of the initial time surface diagrams. The initial time surface plays an important role within the C-S method as it can be shown that the CS method is built by doing things at this surface alone. The initial time surface also plays an important role with the large sources in the system as the large sources have been removed to the initial time surface. The slightly unusual diagrammatic notation used by the authors of [24] is now introduced.

The equation for the classical field is given by equation (4.5). To be specific, look at the situation where this is a ϕ^4 theory, thus

$$\varphi(x) = \int\limits_{\Sigma+} d^4y G_0^R(x,y) \left(\frac{g^2}{3!} \varphi^3(y) \right) + \int\limits_{\Sigma} d^3\boldsymbol{y} \left[\left(\partial_0 G_0^R(x,\boldsymbol{y})\right) \varphi_0(\boldsymbol{y}) + G_0^R(x,\boldsymbol{y}) \left(\partial_0 \varphi_0(\boldsymbol{y})\right) \right] . \tag{4.6}$$

The retarded propagator has a very specific direction in time due to its definition with the Θ function. If one has $G^R(x,y)$ then information travelled from y in the past, to x. Using this understanding, the non-homogeneous part of $\varphi(x)$ shows that starting from some interaction point at space-time point y, one propagates forward in time, through the retarded propagator $G_0^R(x,y)$, until the space-time point x.

This equation is iterative as each field at point y is itself determined by some previous space-time point. This continues back to the initial time surface denoted as Σ. Thus the classical field can be represented as follows



(4.7)

Here the initial time surface is the shaded in part of the diagram marked Σ. The time axis shows how one should consider time in these diagrams. This is only a particular representation of the classical field, as there are actually an infinite number of lines than will be in contact with the initial time surface due to the iterative nature of the expression.

The G^R in the problem's solutions show that the entire solution rests on the values at the initial time surface.

Small Quantum Fluctuations

This section introduces a small fluctuation in the background of the classical field. Specifically these fluctuations are actually the momentum modes of the tree level propagator. The small fluctuations are labelled $a_{\pm\boldsymbol{k}}$.

The fluctuations follow the equation of motion given by [5],

$$\begin{aligned} \left[\square + m^2 + V''(\varphi)\right] a_{\pm\boldsymbol{k}} &= 0 \\ \lim_{x^0 \to -\infty} a_{\pm\boldsymbol{k}} &= e^{\pm ik\cdot x} \ . \end{aligned} \tag{4.8}$$

The initial conditions are plane waves as these describe small solutions. Recall that these fluctuations are supposed to be parametrically smaller than g^{-1}.

The solution to this equation, using Green's theorem, can be shown to be [16]

$$\begin{aligned} a_{\pm\boldsymbol{k}}(x) &= i \int\limits_{\Sigma} d^3\boldsymbol{y} \left\{ \mathcal{G}_t^R(x,y) \big(\partial_{y^0} a_{\pm\boldsymbol{k}}(y) \big) - \big(\partial_{y^0} \mathcal{G}_t^R(x,y) \big) a_{\pm\boldsymbol{k}}(y) \right\} \\ & \qquad\qquad \text{using the given value of } a_{\pm\boldsymbol{k}} \text{ on the boundary} \\ &= i \int\limits_{\Sigma} d^3\boldsymbol{y} \left\{ \mathcal{G}_t^R(x,y) \big(\partial_{y^0} e^{\pm ik\cdot y} \big) - \big(\partial_{y^0} \mathcal{G}_t^R(x,y) \big) e^{\pm ik\cdot y} \right\} \ . \end{aligned} \tag{4.9}$$

Alternatively equation (4.8) can be re-written in a different form,

$$\begin{aligned}\left[\Box + m^2\right] a_{\pm\boldsymbol{k}} &= -V''(\varphi) a_{\pm\boldsymbol{k}} \\ \lim_{x^0 \to -\infty} a_{\pm\boldsymbol{k}} &= e^{\pm ik\cdot x}\end{aligned} \tag{4.10}$$

This then admits the Green's solution [24],

$$a_{\pm\boldsymbol{k}}(x) = -i\int_{\Sigma+} d^4y G_0^R(x,y)\frac{\delta^2 V}{\delta\varphi(y)^2} a_{\pm\boldsymbol{k}}(y) + i\int_{\Sigma} d^3\boldsymbol{y}\left\{G_0^R(x,y)\left(\partial_{y^0} a_{\pm\boldsymbol{k}}(0,\boldsymbol{y})\right) - \left(\partial_{y^0} G_0^R(x,y)\right) a_{\pm\boldsymbol{k}}(0,\boldsymbol{y})\right\} \ . \tag{4.11}$$

To start building NLO objects from LO objects, the C-S method uses a uniquely defined operator. The operator is defined as, [5]

$$a \cdot \mathbb{T}_{\boldsymbol{u}} \equiv a(0,\boldsymbol{u})\frac{\delta}{\delta\varphi_0(\boldsymbol{u})} + \dot{a}(0,\boldsymbol{u})\frac{\delta}{\delta\partial_0\varphi_0(\boldsymbol{u})} \ . \tag{4.12}$$

This operator can be used to create tree level propagators from the product of two classical fields as will be shown in section 4.3. It does this by inserting the small fluctuation modes at the initial time surface. This is seen by the partial derivatives that act on fields that live at the initial time surface.

This again shows that the initial time surface is crucial in this discussion.

4.3 Creating Tree Level Propagators

The operator defined in equation (4.12) will be used to construct $\mathcal{G}_t^{+-}$ from φ. This is useful later in trying to create the NLO equations.

To show that this is true, two facts are necessary to prove.

1) The small fluctuations $a_{\pm\boldsymbol{k}}(x)$ can create $\mathcal{G}_t^{+-}$. It turns out that $\mathcal{G}_t^{+-}(x,y) = \int \frac{d^4k}{(2\pi)^4} 2\pi\Theta(-k^0)\delta(k^2 - m^2) a_{+\boldsymbol{k}}(x) a_{-\boldsymbol{k}}(y)$.

2) The operator defined in equation (4.12) acting on φ produces $a_{\pm\boldsymbol{k}}$. This is done keeping the initial conditions.

This will then prove that [16],

$$\mathcal{G}_t^{+-}(x,y) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k^0} d^3\boldsymbol{u} d^3\boldsymbol{v}\left(\left[a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}\right]\varphi(x)\right)\left(\left[a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}\right]\varphi(y)\right) \ . \tag{4.13}$$

4.3.1 Generating $\mathcal{G}_t^{+-}$ from the $a_{\pm\boldsymbol{k}}$

It needs to be shown that $\mathcal{G}_t^{+-}(x,y) = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2-m^2) a_{+\boldsymbol{k}}(x) a_{-\boldsymbol{k}}(y)$. The proof of this is again quite involved so a brief outline of what will be done is presented here.

To start one uses an expression derived in appendix D.2 that shows $\mathcal{G}_t^{+-} = \mathcal{G}_t^R \left(G_0^R\right)^{-1} G_0^{+-} \left(G_0^A\right)^{-1} \mathcal{G}_t^A$.

The first step from here is to use the momentum integral expression of $G_0^{+-}(x,y)$. Then the $\left(G_0^R\right)^{-1}$ is a differential operator that one considers to be acting on $\mathcal{G}_t^R$ and the e^{ipx} part of the momentum integral of $G_0^{+-}(x,y)$. Then Stokes theorem can be used to show that the effect of this operator acting on the parts described, produces a term identical to the boundary values of the $a_{+\boldsymbol{k}}$ from equation (4.9) (see equation (4.25) below). The term $\left(G_0^A\right)^{-1} \mathcal{G}_t^A$ is related to $\mathcal{G}_t^R \left(G_0^R\right)^{-1}$ by time ordering. The result is that this term can be considered to give an identical result up to a minus sign. This identifies $\left(G_0^A\right)^{-1}$ acting on $\mathcal{G}_t^A$ and e^{-ipy} of the momentum integral of $G_0^{+-}(x,y)$ as the boundary value of $a_{-\boldsymbol{k}}$. Then since one has the right hand side of equation (4.9), this identifies the two terms due to the operators $\left(G_0^R\right)^{-1}$ and $\left(G_0^A\right)^{-1}$ acting on their respective terms as $a_{+\boldsymbol{k}}$ and $a_{-\boldsymbol{k}}$. This is then proof that $\mathcal{G}_t^{+-}(x,y) = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2-m^2) a_{+\boldsymbol{k}}(x) a_{-\boldsymbol{k}}(y)$.

Now onto the actual calculations

Detailed Proof

Appendix D.2 finds alternative expressions for the Schwinger-Keldysh propagators in terms of F, $\mathcal{G}_t^R$ and $\mathcal{G}_t^A$. equation (D.29) shows that

$$\mathcal{G}_t^{+-} = \mathcal{G}_t^R \left(G_0^R\right)^{-1} G_0^{+-} \left(G_0^A\right)^{-1} \mathcal{G}_t^A \ . \tag{4.14}$$

Starting with the above expression [16],

$$\begin{aligned} \mathcal{G}_t^{+-}(x,y) &= [\mathcal{G}_t^R \left(G_0^R\right)^{-1} G_0^{+-} \left(G_0^A\right)^{-1} \mathcal{G}_t^A](x,y) \\ &= \int d^4u d^4v \left[\mathcal{G}_t^R (G_R^0)^{-1}\right](x,u) G_0^{+-}(u,v) \left[(G_0^A)^{-1} \mathcal{G}_t^A\right](v,y) \ . \end{aligned} \tag{4.15}$$

In section 3.1.3, it was found that $G_0^{+-}(x,y) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 E_{\boldsymbol{p}}} e^{ip\cdot(x-y)}$ (equation (3.11)). This can be re-written to be [16],

$$G_0^{+-}(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} 2\pi\Theta(-p^0)\delta(p^2-m^2) \ . \tag{4.16}$$

Inserting equation (4.16) into equation (4.15)

$$\mathcal{G}_t^{+-}(x,y) = \int \frac{d^4k}{(2\pi)^4} 2\pi\Theta(-k^0)\delta(k^2-m^2) \int d^4u \left[\mathcal{G}_t^R (G_0^R)^{-1}\right](x,u)e^{ik\cdot u}$$
$$\times \int d^4v \left[(G_0^A)^{-1}\mathcal{G}_t^A\right](v,y)e^{-ik\cdot v} \ . \tag{4.17}$$

The term $\int d^4u \left[\mathcal{G}_t^R (G_R^0)^{-1}\right](x,u)e^{ik\cdot u}$ needs to be investigated further.

$$\int d^4u \left[\mathcal{G}_t^R (G_0^R)^{-1}\right](x,u)e^{ik\cdot u} = \int d^4u \left[i(\Box_u + m^2)\mathcal{G}_t^R\right](x,u)e^{ik\cdot u}$$
$$= i\int d^4u \left[(\partial_{u^0}^2 - \nabla_{\boldsymbol{u}}^2 + m^2)\mathcal{G}_t^R(x,u)\right]e^{ik\cdot u} \tag{4.18}$$

<u>The time derivative part:</u>

$$\int d^4u \left(\partial_{u^0}^2 \mathcal{G}_t^R(x,u)\right)e^{ik\cdot u}$$
$$= \int d^4u \left\{\partial_{u^0}\left[\left(\partial_{u^0}\mathcal{G}_t^R(x,u)\right)e^{ik\cdot u}\right] - \left(\partial_{u^0}\mathcal{G}_t^R(x,u)\right)\left(\partial_{u^0}e^{ik\cdot u}\right)\right\}$$
$$= \int d^4u \left\{\partial_{u^0}\left[\left(\partial_{u^0}\mathcal{G}_t^R(x,u)\right)e^{ik\cdot u}\right] - \partial_{u^0}\left[\mathcal{G}_t^R(x,u)\left(\partial_{u^0}e^{ik\cdot u}\right)\right] + \mathcal{G}_t^R(x,u)\left(\partial_{u^0}^2 e^{ik\cdot u}\right)\right\}$$
$$= \int d^4u \left\{\partial_{u^0}\left[\left(\partial_{u^0}\mathcal{G}_t^R(x,u)\right)e^{ik\cdot u}\right] - \partial_{u^0}\left[\mathcal{G}_t^R(x,u)\left(\partial_{u^0}e^{ik\cdot u}\right)\right] + \mathcal{G}_t^R(x,u)\left((ik^0)^2 e^{ik\cdot u}\right)\right\} \tag{4.19}$$

In the above there are two terms that have been expressed as total derivatives. This means that one can use Stoke's theorem where Stoke says that

$$\int\limits_{\Sigma+} d^4x \partial_{x^0}\left(A(x)\right) = \int\limits_{\Sigma} d^3\boldsymbol{x} A(\boldsymbol{x}) \ . \tag{4.20}$$

Using this identity gives,

$$\int d^4u \left(\partial_{u^0}^2 \mathcal{G}_t^R(x,u)\right)e^{ik\cdot u} = \int\limits_{\Sigma} d^3\boldsymbol{u} \left\{\left(\partial_{u^0}\mathcal{G}_t^R(x,u)\right)e^{ik\cdot u} - \mathcal{G}_t^R(x,u)\left(\partial_{u^0}e^{ik\cdot u}\right)\right\}$$
$$+ \int d^4u \mathcal{G}_t^R(x,u)\left((ik^0)^2 e^{ik\cdot u}\right) \ . \tag{4.21}$$

Since $\int d^4u = \int\limits_{-\infty}^{\infty} du^0 \int d^3\boldsymbol{u}$, the boundary of the surface being integrated over is at the time surface where $u^0 \to -\infty$

The Space Derivative part :

This can be done in exact analogy with the above section giving

$$\int d^4u \left(\nabla^2_{\boldsymbol{u}} \mathcal{G}^R_t(x,u)\right) e^{ik\cdot u} = \lim_{\boldsymbol{u}\to-\infty} \int du^0 \left\{ \left(\nabla_{\boldsymbol{u}} \mathcal{G}^R_t(x,u)\right) e^{ik\cdot u} - \mathcal{G}^R_t(x,u) \left(\nabla_{\boldsymbol{u}} e^{ik\cdot u}\right) \right\} + \int d^4u \mathcal{G}^R_t(x,u) \left((-i\boldsymbol{k})^2 e^{ik\cdot u}\right) \ . \tag{4.22}$$

The assumption is that the field and its spacial derivatives vanish fast enough at infinity in the spacial directions [26] thus,

$$\int d^4u \left(\nabla^2_{\boldsymbol{u}} \mathcal{G}^R_t(x,u)\right) e^{ik\cdot u} = 0 + \int d^4u \mathcal{G}^R_t(x,u) \left((-i\boldsymbol{k})^2 e^{ik\cdot u}\right) \ .$$

Putting the results of this time and space part of the differentiation back into equation (4.18) gives

$$\begin{aligned}
\int_\Sigma d^4u \left[\mathcal{G}^R_t (G^R_0)^{-1}\right](x,u) e^{ik\cdot u} &= i \int_\Sigma d^3\boldsymbol{u} \left\{ \left(\partial_{u^0} \mathcal{G}^R_t(x,u)\right) e^{ik\cdot u} - i\mathcal{G}^R_t(x,u) \left(\partial_{u^0} e^{ik\cdot u}\right) \right\} \\
&\quad + i \int d^4u \mathcal{G}^R_t(x,u) \left((ik^0)^2 e^{ik\cdot u}\right) - i \int d^4u \mathcal{G}^R_t(x,u) \left((-i\boldsymbol{k})^2 e^{ik\cdot u}\right) \\
&\quad + i \int d^4u (m^2) \mathcal{G}^R_t(x,u) e^{ik\cdot u} \\
&= i \int_\Sigma d^3\boldsymbol{u} \left\{ \left(\partial_{u^0} \mathcal{G}^R_t(x,u)\right) e^{ik\cdot u} - \mathcal{G}^R_t(x,u) \left(\partial_{u^0} e^{ik\cdot u}\right) \right\} \\
&\quad + i \int d^4u \mathcal{G}^R_t(x,u) \left[-k^2 + m^2\right] e^{ik\cdot u} \ .
\end{aligned} \tag{4.23}$$

From the delta function in equation (4.17), objects are put on mass shell, thus $-k^2 + m^2 = 0$. The means equation (4.23) is,

$$\int_\Sigma d^4u \left[\mathcal{G}^R_t (G^R_0)^{-1}\right](x,u) e^{ik\cdot u} = i \int_\Sigma d^3\boldsymbol{u} \left\{ \left(\partial_{u^0} \mathcal{G}^R_t(x,u)\right) e^{ik\cdot u} - \mathcal{G}^R_t(x,u) \left(\partial_{u^0} e^{ik\cdot u}\right) \right\} \ . \tag{4.24}$$

The term, to the right of the equals sign, is identical to (4.9),the solution of equation (4.8). Thus this object is $a_{+\boldsymbol{k}}$ [16]

$$\int d^4u \left[\mathcal{G}^R_t (G^R_0)^{-1}\right](x,u) e^{ik\cdot u} = a_{+\boldsymbol{k}}(x) \ . \tag{4.25}$$

Due to the time ordering of the objects,

$$\left[\left(G_0^A\right)^{-1}\mathcal{G}_t^A\right](v,x) = \left[\mathcal{G}_t^R(G_0^R)^{-1}\right](x,v) \tag{4.26}$$

Thus the term $\int d^4v\left[\left(G_0^A\right)^{-1}\mathcal{G}_t^A\right](v,y)e^{-ik\cdot v}$ of equation (4.17) can be written in an analogous way to the above results,

$$\int d^4v\left[\left(G_0^A\right)^{-1}\mathcal{G}_t^A\right](v,y)e^{-ik\cdot v} = a_{-\boldsymbol{k}}(y) \tag{4.27}$$

where the fact that this is the $a_{-\boldsymbol{k}}$ obvious due to the term $e^{-ik\cdot v}$.

Putting this information (equation (4.25) and equation (4.27)) back into equation (4.17) gives

$$\mathcal{G}_t^{+-}(x,y) = \int \frac{d^4k}{(2\pi)^4} 2\pi\Theta(-k^0)\delta(k^2-m^2)a_{+\boldsymbol{k}}(x)a_{-\boldsymbol{k}}(y)\ . \tag{4.28}$$

This shows $\mathcal{G}_t^{+-}(x,y)$ is made from the terms $a_{\pm\boldsymbol{k}}$.

4.3.2 $a\cdot\mathbb{T}_{\boldsymbol{u}}$ acting on φ

Here it will be shown that $\int\limits_\Sigma d^3\boldsymbol{u}\,[a\cdot\mathbb{T}_{\boldsymbol{u}}]\,\varphi(x)$ gives an object that satisfies the same equation of motion, with the same boundary conditions, as that in equation (4.8). Doing this identifies $a_{\pm\boldsymbol{k}}(x) = \int\limits_\Sigma d^3\boldsymbol{u}\,[a_{\pm\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{u}}]\,\varphi(x)$ [24].

$$\int\limits_\Sigma d^3\boldsymbol{u}\,[a_{\pm\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{u}}]\,\varphi(x)$$

Using equation (4.5) for $\varphi(x)$

$$\begin{aligned}
&= \int\limits_\Sigma d^3\boldsymbol{u}\,[a_{\pm\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{u}}]\left(-i\int\limits_{\Sigma+} d^4y G_0^R(x,y)\frac{\delta V}{\delta\varphi(y)} + i\int\limits_\Sigma d^3\boldsymbol{y}\left[G_0^R(x,y)\left(\partial_0\varphi_0(\boldsymbol{y})\right) - \left(\partial_0 G_0^R(x,y)\right)\varphi_0(\boldsymbol{y})\right]\right) \\
&= -i\int\limits_\Sigma d^3\boldsymbol{u}\int\limits_{\Sigma+} d^4y G_0^R(x,y)\frac{\delta^2 V}{\delta\varphi(y)^2}\left(a_{\pm\boldsymbol{k}}(0,\boldsymbol{u})\frac{\delta\varphi(y)}{\delta\varphi_0(\boldsymbol{u})} + \dot{a}_{\pm\boldsymbol{k}}(0,\boldsymbol{u})\frac{\delta\varphi(y)}{\delta\partial_0\varphi_0(\boldsymbol{u})}\right) \\
&+ i\int\limits_\Sigma d^3\boldsymbol{u}\int\limits_\Sigma d^3\boldsymbol{y}\left[G_0^R(x,y)\dot{a}_{\pm\boldsymbol{k}}(0,\boldsymbol{u})\left(\frac{\delta\partial_0\varphi_0(\boldsymbol{y})}{\delta\partial_0\varphi_0(\boldsymbol{u})}\right) - \left(\partial_0 G_0^R(x,y)\right)a_{\pm\boldsymbol{k}}(0,\boldsymbol{u})\frac{\delta\varphi_0(\boldsymbol{y})}{\delta\varphi_0(\boldsymbol{u})}\right]
\end{aligned} \tag{4.29}$$

Note that by actually using the differentiation defined in equation (4.12),

$$\int\limits_\Sigma d^3\boldsymbol{u}\,[a_{\pm\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{u}}]\,\varphi(x) = \int\limits_\Sigma d^3\boldsymbol{u}\left(a_{\pm\boldsymbol{k}}(0,\boldsymbol{u})\frac{\delta\varphi(x)}{\delta\varphi_0(\boldsymbol{u})} + \dot{a}_{\pm\boldsymbol{k}}(0,\boldsymbol{u})\frac{\delta\varphi(x)}{\delta\partial_0\varphi_0(\boldsymbol{u})}\right)\ . \tag{4.30}$$

Inserting this into equation (4.29), and carrying out the differentiation in the initial condition terms gives

$$
\begin{aligned}
&\int_{\Sigma} d^3\boldsymbol{u}\,[a_{\pm\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{u}}]\,\varphi(x)\\
&= -i\int_{\Sigma^+} d^4y G_0^R(x,y)\frac{\delta^2 V}{\delta\varphi(y)^2}\int_{\Sigma} d^3\boldsymbol{u}\,[a_{\pm\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{u}}]\,\varphi(y) + i\int_{\Sigma} d^3\boldsymbol{u}\int_{\Sigma} d^3\boldsymbol{y}\left[G_0^R(x,y)\dot{a}_{\pm\boldsymbol{k}}(0,\boldsymbol{u})\left(\delta^{(3)}(\boldsymbol{u}-\boldsymbol{y})\right)\right.\\
&\quad\left. - \left(\partial_0 G_0^R(x,y)\right)a_{\pm\boldsymbol{k}}(0,\boldsymbol{u})\delta^{(3)}(\boldsymbol{u}-\boldsymbol{y})\right]\\
&= -i\int_{\Sigma^+} d^4y G_0^R(x,y)\frac{\delta^2 V}{\delta\varphi(y)^2}\int_{\Sigma} d^3\boldsymbol{u}\,[a_{\pm\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{u}}]\,\varphi(y) + i\int_{\Sigma} d^3\boldsymbol{y}\left[G_0^R(x,y)\dot{a}_{\pm\boldsymbol{k}}(0,\boldsymbol{y})\right.\\
&\quad\left. - \left(\partial_0 G_0^R(x,y)\right)a_{\pm\boldsymbol{k}}(0,\boldsymbol{y})\right]\ .
\end{aligned}
\tag{4.31}
$$

The above equation is identical to equation (4.11). If one identifies,

$$
a_{\pm\boldsymbol{k}}(x) = \int_{\Sigma} d^3\boldsymbol{u}\,[a_{\pm\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{u}}]\,\varphi(x) \tag{4.32}
$$

4.3.3 Putting Together Equations (4.28) and (4.32)

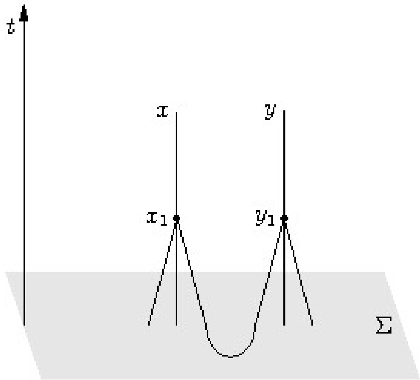
So it has been shown that one can create the tree level propagator using the operator $a_{\pm\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{u}}$ (equation (4.12)) on the classical field φ such that [24].

$$
\mathcal{G}_t^{+-}(x,y) = \int \frac{d^4k}{(2\pi)^4}2\pi\Theta(-k^0)\delta(k^2-m^2)a_{+\boldsymbol{k}}(x)a_{-\boldsymbol{k}}(y) \tag{4.33}
$$

then using equation (4.32), the same object can be re-expressed by varying the initial conditions of the classical fields. This then gives,

$$
\mathcal{G}_t^{+-}(x,y) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k^0}d^3\boldsymbol{u}d^3\boldsymbol{v}\left([a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}]\,\varphi(x)\right)\left([a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}]\,\varphi(y)\right)\ . \tag{4.34}
$$

In terms of the initial time surface what has been done is to link two classical fields at the initial time surface. Each field has had its initial conditions shifted and a momentum mode added in. Then by integrating over this with the $d^3\boldsymbol{k}$ integral, these two modes have been connected up, this has created the propagator. This is best seen diagrammatically,



$$(4.35)$$

where the line connecting up the two parts must be seen as existing below the time surface Σ.

One can see how using the initial time surface has led to a tree level propagator that goes from point y on the right to point x on the left. This particular diagram actually only represents one term in the sum over all possible connections, i.e. one should sum over all ways to connect up the legs at the initial time surface. In this way one can also note that this method can only effect changes bellow the initial time surface, which effectively means that it works by changing initial conditions of these objects.

So the above operator takes two classical fields at x and y, and creates a propagator from x to y. This fact can be exploited to define an operator that acts on classical objects to create one loop corrections to them. The classical objects will have to depend on φ. The loop, in the one loop correction, is made from a tree level propagator.

4.4 Creating a One Loop Correction Using $a_{\pm \boldsymbol{k}} \cdot \mathbb{T}_{\boldsymbol{u}}$

It will be shown that the following object creates one loop corrections to both φ and $\mathcal{G}_t$.

The one loop operator is given by [24],

$$\frac{1}{2} \int d^3\boldsymbol{u} d^3\boldsymbol{v} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k^0} \left[a_{k+} \cdot \mathbb{T}_{\boldsymbol{u}}\right] \left[a_{k-} \cdot \mathbb{T}_{\boldsymbol{v}}\right] \; . \tag{4.36}$$

In this section one loop corrections to both the classical field and the classical propagator are done.

4.4.1 One loop Correction to φ

One now looks at,

$$\left[\frac{1}{2} \int d^3\boldsymbol{u} d^3\boldsymbol{v} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k^0} \left[a_{k+} \cdot \mathbb{T}_{\boldsymbol{u}}\right] \left[a_{k-} \cdot \mathbb{T}_{\boldsymbol{v}}\right]\right] \varphi(x) \; . \tag{4.37}$$

The first part of the proof is very long is mathematically cluttered, but it provides the details of how to work with this operator mathematically. The mathematical result obtained in equation (4.42) is then interpreted diagrammaticality in equation (4.43) as this clearly exposes the result obtained.

Cluttered Mathematical Solution

Start with φ as given in equation (4.5). A simplification is that the boundary terms in this expression only have one power of φ_0. Thus this double differentiation caused by $[a_{k+} \cdot \mathbb{T}_{\boldsymbol{u}}]\,[a_{k-} \cdot \mathbb{T}_{\boldsymbol{v}}]$ make the boundary term 0.

To simplify the calculation let $\frac{1}{2}\int d^3\boldsymbol{u} d^3\boldsymbol{v} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k} = \int\limits_{u,v,k}$.

Thus

$$\left[\frac{1}{2}\int d^3\boldsymbol{u} d^3\boldsymbol{v} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k^0}\,[a_{k+} \cdot \mathbb{T}_{\boldsymbol{u}}]\,[a_{k-} \cdot \mathbb{T}_{\boldsymbol{v}}]\right]\varphi(x)$$

$$= \left[\int\limits_{u,v,k} [a_{k+} \cdot \mathbb{T}_{\boldsymbol{u}}]\,[a_{k-} \cdot \mathbb{T}_{\boldsymbol{v}}]\right] - i\int\limits_{\Sigma+} d^4y G_0^R(x,y)\frac{\delta V}{\delta\varphi(y)}$$

Using the chain rule

$$= -\left[\int\limits_{u,v,k} [a_{k+} \cdot \mathbb{T}_{\boldsymbol{u}}]\right] i\int\limits_{\Sigma+} d^4y G_0^R(x,y)\frac{\delta^2 V}{\delta\varphi(y)^2}\left(a_{k-}(0,\boldsymbol{v})\frac{\delta\varphi(y)}{\delta\varphi_0(\boldsymbol{v})} + \dot{a}_{k-}(0,\boldsymbol{v})\frac{\delta\varphi(y)}{\delta\partial_0\varphi_0(\boldsymbol{v})}\right)$$

And again using the chain rule

$$\begin{aligned}
&= -i\int\limits_{u,v,k}\int\limits_{\Sigma+} d^4y G_0^R(x,y)\Bigg[\frac{\delta^3 V}{\delta\varphi(y)^3}\left(a_{k+}(0,\boldsymbol{u})\frac{\delta\varphi(y)}{\delta\varphi_0(\boldsymbol{u})} + \dot{a}_{k+}(0,\boldsymbol{u})\frac{\delta\varphi(y)}{\delta\partial_0\varphi_0(\boldsymbol{u})}\right)\\
&\times\left(a_{k-}(0,\boldsymbol{v})\frac{\delta\varphi(y)}{\delta\varphi_0(\boldsymbol{v})} + \dot{a}_{k-}(0,\boldsymbol{v})\frac{\delta\varphi(y)}{\delta\partial_0\varphi_0(\boldsymbol{v})}\right)\\
&+ \frac{\delta^2 V}{\delta\varphi(y)^2}\Bigg(a_{k+}(0,\boldsymbol{u})a_{k-}(0,\boldsymbol{v})\frac{\delta^2\varphi(y)}{\delta\varphi_0(\boldsymbol{u})\delta\varphi_0(\boldsymbol{v})} + a_{k+}(0,\boldsymbol{u})\dot{a}_{k-}(0,\boldsymbol{v})\frac{\delta^2\varphi(y)}{\delta\varphi_0(\boldsymbol{u})\delta\partial_0\varphi_0(\boldsymbol{v})}\\
&+\dot{a}_{k+}(0,\boldsymbol{u})a_{k-}(0,\boldsymbol{v})\frac{\delta^2\varphi(y)}{\delta\partial_0\varphi_0(\boldsymbol{v})\delta\varphi_0(\boldsymbol{u})} + \dot{a}_{k+}(0,\boldsymbol{u})\dot{a}_{k-}(0,\boldsymbol{v})\frac{\delta^2\varphi(y)}{\delta\partial_0\varphi_0(\boldsymbol{v})\delta\partial_0\varphi_0(\boldsymbol{u})}\Bigg)\Bigg].
\end{aligned} \tag{4.38}$$

Note that

$$\begin{aligned}
&[a_{k+} \cdot \mathbb{T}_{\boldsymbol{u}}]\,[a_{k-} \cdot \mathbb{T}_{\boldsymbol{v}}]\,\varphi(x)\\
&= \left[a_+(0,\boldsymbol{u})\frac{\delta}{\delta\varphi_0(\boldsymbol{u})} + \dot{a_+}(0,\boldsymbol{u})\frac{\delta}{\delta\partial_0\varphi_0(\boldsymbol{u})}\right]\left[a_-(0,\boldsymbol{v})\frac{\delta}{\delta\varphi_0(\boldsymbol{v})} + \dot{a_-}(0,\boldsymbol{v})\frac{\delta}{\delta\partial_0\varphi_0(\boldsymbol{v})}\right]\varphi(x)\\
&= \Bigg(a_{k+}(0,\boldsymbol{u})a_{k-}(0,\boldsymbol{v})\frac{\delta^2\varphi(y)}{\delta\varphi_0(\boldsymbol{u})\delta\varphi_0(\boldsymbol{v})} + a_{k+}(0,\boldsymbol{u})\dot{a}_{k-}(0,\boldsymbol{v})\frac{\delta^2\varphi(y)}{\delta\varphi_0(\boldsymbol{u})\delta\partial_0\varphi_0(\boldsymbol{v})}\\
&+\dot{a}_{k+}(0,\boldsymbol{u})a_{k-}(0,\boldsymbol{v})\frac{\delta^2\varphi(y)}{\delta\partial_0\varphi_0(\boldsymbol{v})\delta\varphi_0(\boldsymbol{u})} + \dot{a}_{k+}(0,\boldsymbol{u})\dot{a}_{k-}(0,\boldsymbol{v})\frac{\delta^2\varphi(y)}{\delta\partial_0\varphi_0(\boldsymbol{v})\delta\partial_0\varphi_0(\boldsymbol{u})}\Bigg).
\end{aligned} \tag{4.39}$$

Then putting equation (4.39) into equation (4.38) gives,

$$
\begin{aligned}
&\left[\int\limits_{u,v,k} [a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}]\,[a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}]\right]\varphi(x) \\
&= -i\int\limits_{u,v,k}\int\limits_{\Sigma+} d^4y G_0^R(x,y)\left[\frac{\delta^3 V}{\delta\varphi(y)^3}\left(a_{k+}(0,\boldsymbol{u})\frac{\delta\varphi(y)}{\delta\varphi_0(\boldsymbol{u})}+\dot{a}_{k+}(0,\boldsymbol{u})\frac{\delta\varphi(y)}{\delta\partial_0\varphi_0(\boldsymbol{u})}\right)\right. \\
&\times\left(a_{k-}(0,\boldsymbol{v})\frac{\delta\varphi(y)}{\delta\varphi_0(\boldsymbol{v})}+\dot{a}_{k-}(0,\boldsymbol{v})\frac{\delta\varphi(y)}{\delta\partial_0\varphi_0(\boldsymbol{v})}\right) \\
&\left.+\frac{\delta^2 V}{\delta\varphi(y)^2}\left([a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}]\,[a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}]\,\varphi(y)\right)\right] . \qquad (4.40)
\end{aligned}
$$

Recall from equation (4.34),

$$
\begin{aligned}
\mathcal{G}_t^{+-}(x,y) &= \int\limits_{u,v,k}\left([a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}]\,\varphi(x)\right)\left([a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}]\,\varphi(y)\right) \\
&\text{Using the definition of the differential operators} \\
&= \int\limits_{u,v,k}\left(a_{k+}(0,\boldsymbol{u})\frac{\delta\varphi(x)}{\delta\varphi_0(\boldsymbol{u})}+\dot{a}_{k+}(0,\boldsymbol{u})\frac{\delta\varphi(x)}{\delta\partial_0\varphi_0(\boldsymbol{u})}\right) \\
&\times\left(a_{k-}(0,\boldsymbol{v})\frac{\delta\varphi(y)}{\delta\varphi_0(\boldsymbol{v})}+\dot{a}_{k-}(0,\boldsymbol{v})\frac{\delta\varphi(y)}{\delta\partial_0\varphi_0(\boldsymbol{v})}\right) . \qquad (4.41)
\end{aligned}
$$

Thus putting the above into (4.40) gives [24],

$$
\begin{aligned}
&\left[\int\limits_{u,v,k} [a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}]\,[a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}]\right]\varphi(x) \\
&= -i\int\limits_{\Sigma+} d^4y G_0^R(x,y)\left[\frac{\delta^3 V}{\delta\varphi(y)^3}\mathcal{G}_t^{+-}(y,y)+\frac{\delta^2 V}{\delta\varphi(y)^2}\left([a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}]\,[a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}]\,\varphi(y)\right)\right] . \qquad (4.42)
\end{aligned}
$$

It is not immediately obvious what result has just been obtained. Thus one translates the above into diagrammatic notation. To do so one must re-insert the fact that this is a ϕ^4 theory.

Diagrammatic from of Equation (4.42)

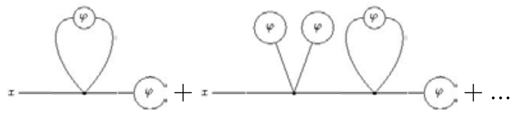
First consider this to be the ϕ^4 theory. Thus $V = \frac{g^2}{4!}\phi^4$.

The term $\mathcal{G}_t^{+-}(y,y)$ is giving a loop, this is multiplied by a term $\frac{\delta^3 V}{\delta\varphi(y)^3} = g^2\varphi(y)$. This is all multiplied by a retarded propagator.

The other term in the integral is setting up a recursive relationship whereby one recursively multiplies by $\frac{\delta^2 V}{\delta\varphi(y)^2} = \frac{g^2}{2!}\varphi^2(y)$, the classical field insert. So the recursive relationship is setting up a tree level propagator multiplying the loop correction.

The best way to see what is going on is to put this diagrammatically,

$$\left[\int\limits_{u,v,k} \left[a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}\right]\left[a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}\right]\right]\varphi(x)$$

$=$  (4.43)

Thus the operator from equation (4.36) creates a one loop correction to the classical field.

Diagrammatically $\left[\int\limits_{u,v,k} \left[a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}\right]\left[a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}\right]\right]\varphi(x)$ will be represented as

$$\left[\int\limits_{u,v,k} \left[a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}\right]\left[a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}\right]\right]\varphi(x) =$$  . (4.44)

4.4.2 One Loop Correction to $\mathcal{G}_t^{++}$

As an example one can use the $\mathcal{G}_t^{++}$ propagator although the argument presented is not unique to this propagator.

Consider $\mathcal{G}_t^{++}$, this is given by

$$\mathcal{G}_t^{++} = G_0^{++} + G_0^{+\epsilon}\frac{g^2}{2}\varphi^2 G_0^{\epsilon'+} + G_0^{+\epsilon}\frac{g^2}{2}\varphi^2 G_0^{\epsilon'\epsilon''}\frac{g^2}{2}\varphi^2 G_0^{\epsilon''+} + ... \tag{4.45}$$

where $\epsilon = +, -$, and one sums over all the possible combinations.

It is best to first consider acting with $\left[\int\limits_{u,v,k} \left[a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}\right]\left[a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}\right]\right]$ on one term in this series. From this one can see how to do the calculation. Then the result of the sequence can be presented directly.

The operator acting on the one term in the sequence is done in the same way as before, first the mathematical way is shown, the result of which is given in equation (4.51) below. This result is then interpreted diagrammatically in equation (4.53). For the result of the full sequence it is best to use diagrammatic notation directly.

The last thing presented in this section will be how a loop looks with the picture of the initial time surface in mind.

One Term in the Sequence of Equation (4.45)

So consider

$$\left[\int\limits_{u,v,k} [a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}]\,[a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}]\right] G_0^{+\epsilon}(x_4,x_3)\frac{g^2}{2}\varphi^2(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\frac{g^2}{2}\varphi^2(x_2)G_0^{\epsilon''+}(x_2,x_1)\,. \tag{4.46}$$

The term $[a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}]$ is just a differential operator that acts on φ, it won't act on a free propagator, thus the end propagators can be ignored until the end of the calculation. To shorten the expression the fact that $a_{\pm\boldsymbol{k}}(x)$ commutes with $\varphi(x)$ will be used. Also the shorthand

$$\left[\int\limits_{u,v,k} [a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}]\,[a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}]\right]\equiv\hat{\mathbb{L}} \tag{4.47}$$

with $\hat{\mathbb{L}}$ standing for the one loop operator and the shorthand

$$[a_{k+}\cdot\mathbb{T}_{\boldsymbol{u}}]\,[a_{k-}\cdot\mathbb{T}_{\boldsymbol{v}}]\equiv\hat{a}_{\pm k}\,. \tag{4.48}$$

Lastly, the $\frac{g^2}{2}$ terms will be left out for this calculation for now and put back in later. Then using the product rule and equation (4.32)

$$\begin{aligned}
&\hat{\mathbb{L}}\left[\varphi^2(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\varphi^2(x_2)\right]=\\
&\int\limits_k\Big\{2a_{k+}(x_3)a_{k-}(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\varphi^2(x_2)+2\varphi(x_3)a_{k-}(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)2\varphi(x_2)a_{k+}(x_2)\\
&+2\varphi(x_3)a_{k+}(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)2\varphi(x_2)a_{k-}(x_2)+\varphi^2(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)2a_{k+}(x_2)a_{k-}(x_2)\Big\}\\
&+\int\limits_{u,v,k}\Big\{2\varphi(x_3)\hat{a}_{\pm k}\left[\varphi(x_3)\right]G_0^{\epsilon'\epsilon''}(x_3,x_2)\varphi^2(x_2)+\varphi^2(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)2\varphi(x_2)\hat{a}_{\pm k}\left[\varphi(x_2)\right]\Big\}\,.
\end{aligned} \tag{4.49}$$

The first simplification that can be made is to utilize the occurrences of equation 4.28. There are multiplied $a_{\pm\boldsymbol{k}}$'s, with a $\boldsymbol{k}$ integral. Thus these get replaced by either $\mathcal{G}_t^{-+}$ or $\mathcal{G}_t^{+-}$ as necessary. Note that this uses up the k integral.

$$
\begin{aligned}
&\hat{\mathbb{L}}\left[\varphi^2(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\varphi^2(x_2)\right] = \\
&\Big\{2\mathcal{G}_t^{+-}(x_3,x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\varphi^2(x_2) + 4\varphi(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\mathcal{G}_t^{-+}(x_3,x_2)\varphi(x_2) \\
&+4\varphi(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\mathcal{G}_t^{+-}(x_3,x_2)\varphi(x_2) + \varphi^2(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)2\mathcal{G}_t^{+-}(x_2,x_2)\Big\} \\
&+\int\limits_{u,v,k}\Big\{2\varphi(x_3)\hat{a}_{\pm k}\left[\varphi(x_3)\right]G_0^{\epsilon'\epsilon''}(x_3,x_2)\varphi^2(x_2) + \varphi^2(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)2\varphi(x_2)\hat{a}_{\pm k}\left[\varphi(x_2)\right]\Big\} \qquad (4.50)
\end{aligned}
$$

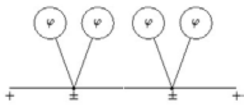
Then recognize that the last line is the sum of two terms that are found in equation (4.44). These two terms represent one loop corrections to the classical field. Thus,

$$
\begin{aligned}
&\hat{\mathbb{L}}\left[\varphi^2(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\varphi^2(x_2)\right] = \\
&\Big\{2\mathcal{G}_t^{+-}(x_3,x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\varphi^2(x_2) + 2\varphi^2(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\mathcal{G}_t^{+-}(x_2,x_2) \\
&+4\varphi(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\mathcal{G}_t^{+-}(x_3,x_2)\varphi(x_2) + 4\varphi(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\mathcal{G}_t^{-+}(x_3,x_2)\varphi(x_2)\Big\} \\
&+\Big\{2\varphi(x_3)\varphi^{1\ \mathrm{Loop}}(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\varphi^2(x_2) + 2\varphi^2(x_3)G_0^{\epsilon'\epsilon''}(x_3,x_2)\varphi(x_2)\varphi^{1\ \mathrm{Loop}}(x_2)\Big\}\ . \qquad (4.51)
\end{aligned}
$$

This is very messy to look at, for ease the above is put into diagrammatic notation. This will help to see what is going on. All the terms that were taken out of the full expression are put back in.

As a reminder on the diagrammatic notation used, a bob with φ and n legs sticking out of it represents the connected, classical n point function. So the objects with a φ and two legs sticking out from it represents the tree level propagator with φ inserts. Any free line represents the free propagator. The $+$ and $-$ is to annotate Schwinger-Keldysh propagators. Thus the object under investigation is,

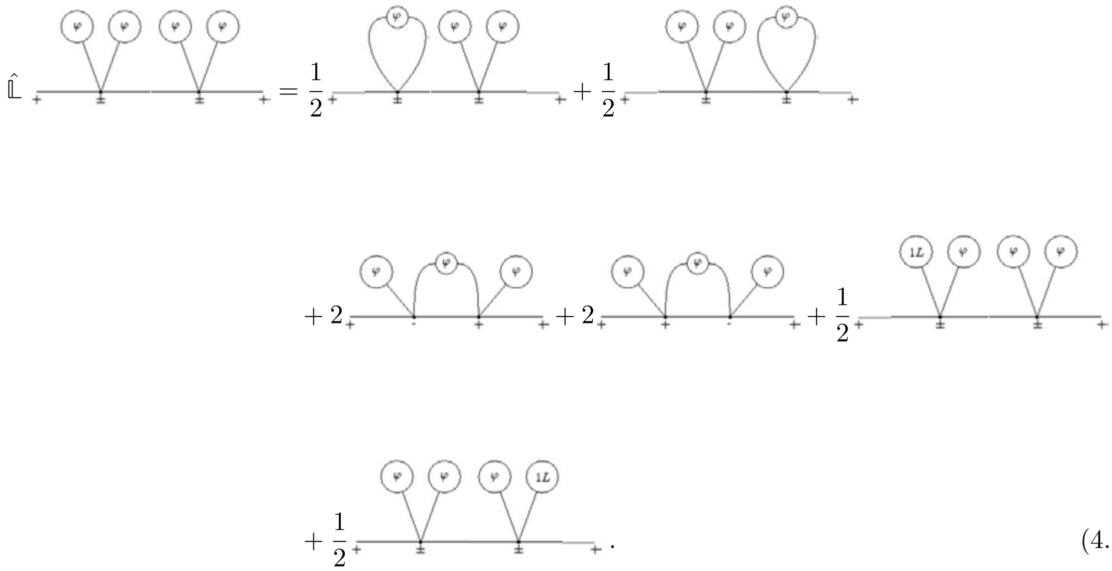
$$
G^0_{+\epsilon}(x_4,x_3)\frac{g^2}{2}\varphi^2(x_3)G^0_{\epsilon'\epsilon''}(x_3,x_2)\frac{g^2}{2}\varphi^2(x_2)G_{\epsilon''+}(x_2,x_1) =
$$



$. \qquad (4.52)$

In the above, read the vertices as a sum over $+$ and $-$ vertices.

Then in this diagramtic notation mentioned above and using notation for the 1 loop correction to the classical field introduced in equation (4.44),



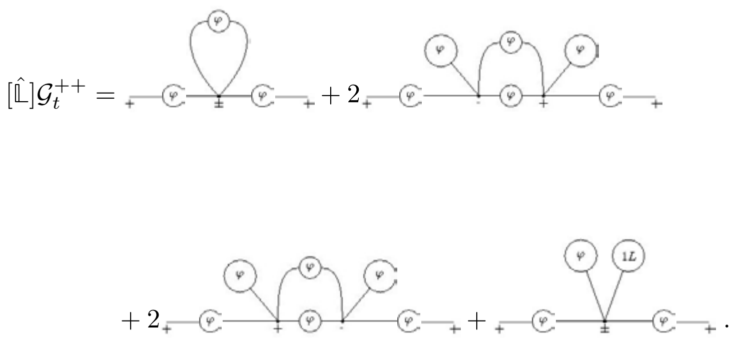
(4.53)

From the above one can interpret exactly what $\hat{\mathbb{L}}$ does. What it does is to 'amputate' two φ fields and connect them up with a $\mathcal{G}$ propagator, unless the same field is 'amputated' twice (more precisely acted upon twice by the differential operator), then this is considered the one loop correction to φ.

Now that one understands how the operators works on a term in the sequence of equation (4.45) the one loop correction to $\mathcal{G}_t^{++}$ is easily found.

The Result of $[\hat{\mathbb{L}}]\mathcal{G}_t$

Using the previous result, it can be shown that [2],



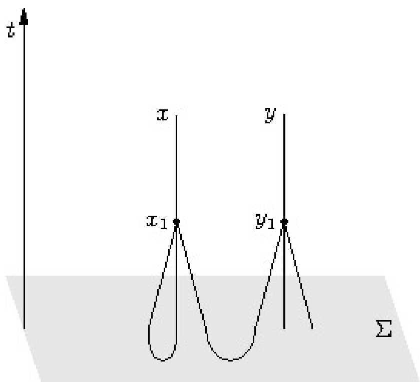
(4.54)

Thus the operator $[\hat{\mathbb{L}}]$ can generate a one loop correction to the tree level propagators. Note that the fact that this is $\mathcal{G}_t^{++}$ didn't play a role and so this operator could work on any of the Schwinger-Keldysh propagators.

Initial Time Surface

To keep touch with what loops look like in terms of the initial time surface, one of the diagrams from equation (4.53) is represented in terms of initial time diagrams.

Looking at the local loop given in this equation, in the language of the initial time surface, this is expressed as



(4.55)

where the loop occurs below the initial time surface Σ.

4.4.3 Solving the NLO Equations

One can now attempt to solve these NLO equations. For example, to solve how the propagator with one loop correction, $\hat{\mathbb{L}}\mathcal{G}_t$, evolves in time one needs to only consider the classical equation of motion with the given initial conditions. One can then act on this solution with the operator $\hat{\mathbb{L}}$ which can be done numerically by realising that the terms $[a_{\pm k} \cdot \mathbb{T}_{\boldsymbol{v}}]$ are effectively shifting the initial conditions of φ.

In doing this further problems arise, these are discussed in the next section.

4.5 Divergences

Using the one loop approximation for the propagator as above now provides the first clue that the theory contains large fluctuations: at large x^0 the oscillations of the NLO solution grow exponentially [5].

[5] actually locates where and how these divergences enter the calculations in a fully analytical manner. Since these divergences represent the dominant fluctuations in the system, these need to be incorporated and carefully resummmed in order to capture equilibrating phenomena. It is worthwhile to reproduce the argument that shows where these dominant fluctuations exist.

4.5.1 Instability analysis

In the NLO solutions, the loops are made from tree level propagators. The tree level propagators are made from the a's (see section 4.3.3) that are defined by the equation of motion given in equation (4.8). Since there are divergences in the NLO solution a good idea is to study the evolution of the a's in time since if these objects are not behaving properly, then the NLO solution won't behave properly.

To proceed one sets up an evolution equation for a in time only by using the Fourier transformation. This equation is a second order differential equation with periodic coefficients. One can then use mapping at a period to find three different types of the solutions to this equation. One of the solutions will be shown to increase exponentially with time for certain momentum modes. This is where the divergences of the a's are contained.

Time Evolution of a

To re-iterate this entire argument can be found directly in [5].

The equation of motion for a, equation (4.8), was given as

$$\left(\Box + m^2 + \frac{g^2}{2}\varphi^2\right) a = 0\ . \tag{4.56}$$

The E.O.M. for a is a differential equation in both space and time. Since only the time evolution is what is needed to be looked at, it would be nice to find a way of ignoring the spacial derivatives. This can be achieved by considering the Fourier transform of a.

The Fourier transform of $a(x)$ is,

$$a(x^0, \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} a(x^0, \boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\ . \tag{4.57}$$

It is then easily seen that $\Box a(x^0, \boldsymbol{x}) = \ddot{a}(x) + \boldsymbol{k}^2 a(x)$ and thus equation (4.56) becomes

$$\ddot{a} + \left(\boldsymbol{k}^2 + m^2 + \frac{g^2}{2}\varphi^2\right) a = 0\ . \tag{4.58}$$

From solving the equations of motion for φ it is seen that φ varies periodically in time [5]. The terms $\boldsymbol{k}$ and m are both constant with respect to time. If one now considers the term $\left(\boldsymbol{k}^2 + m^2 + \frac{g^2}{2}\varphi^2\right)$ as the coefficient of a in the differential equation, then since it is the sum of two terms constant in time and one term periodic in time, it is clear that this coefficient must be periodic in time.

Given two independent solutions to equation (4.58) a_1 and a_2, an analysis can be found by mapping at a period. This entails evolving a pair of solutions from some t_0 to some time T where T represents the period of oscillations in the oscillating coefficient.

$$\begin{pmatrix} a_1(T, \boldsymbol{k}) & a_2(T, \boldsymbol{k}) \\ \dot{a}_1(T, \boldsymbol{k}) & \dot{a}_2(T, \boldsymbol{k}) \end{pmatrix} \equiv M_{\boldsymbol{k}} \begin{pmatrix} a_1(t_0, \boldsymbol{k}) & a_2(t_0, \boldsymbol{k}) \\ \dot{a}_1(t_0, \boldsymbol{k}) & \dot{a}_2(t_0, \boldsymbol{k}) \end{pmatrix} \tag{4.59}$$

The reason this can be written using a matrix $M_{\boldsymbol{k}}$ is because equation (4.58) is linear in a [5].

After n periods nT one has

$$\begin{pmatrix} a_1(nT,\boldsymbol{k}) & a_2(nT,\boldsymbol{k}) \\ \dot{a}_1(nT,\boldsymbol{k}) & \dot{a}_2(nT,\boldsymbol{k}) \end{pmatrix} \equiv M_{\boldsymbol{k}}^n \begin{pmatrix} a_1(t_0,\boldsymbol{k}) & a_2(t_0,\boldsymbol{k}) \\ \dot{a}_1(t_0,\boldsymbol{k}) & \dot{a}_2(t_0,\boldsymbol{k}) \end{pmatrix} . \tag{4.60}$$

Large x^0 means that n is large. This shows that the asymptotic behaviour of a_1 and a_2 is driven by the eigenvalues of $M_{\boldsymbol{k}}$.

The Wronskian of the set of solutions $a_1, a_2, \dot{a}_1, \dot{a}_2$ is simply the determinant of the matrix of solutions.

Thus

$$\begin{aligned} \det\left[\begin{pmatrix} a_1(nT,\boldsymbol{k}) & a_2(nT,\boldsymbol{k}) \\ \dot{a}_1(nT,\boldsymbol{k}) & \dot{a}_2(nT,\boldsymbol{k}) \end{pmatrix}\right] &\equiv \det\left[M_{\boldsymbol{k}}^n \begin{pmatrix} a_1(t_0,\boldsymbol{k}) & a_2(t_0,\boldsymbol{k}) \\ \dot{a}_1(t_0,\boldsymbol{k}) & \dot{a}_2(t_0,\boldsymbol{k}) \end{pmatrix}\right] \\ &\quad \text{but since } \det[AB] = \det[A]\det[B] \\ &= \det\left[M_{\boldsymbol{k}}^n\right] \det\left[\begin{pmatrix} a_1(t_0,\boldsymbol{k}) & a_2(t_0,\boldsymbol{k}) \\ \dot{a}_1(t_0,\boldsymbol{k}) & \dot{a}_2(t_0,\boldsymbol{k}) \end{pmatrix}\right] . \end{aligned} \tag{4.61}$$

For the type of differential equation governing a, it is known that the Wronskian of independent solutions will be independent of time. To see this one can look up Abel's Theorem for differential equations. Thus the Wronskian at nT and at t_0 must be the same.

$$\det\left[\begin{pmatrix} a_1(nT,\boldsymbol{k}) & a_2(nT,\boldsymbol{k}) \\ \dot{a}_1(nT,\boldsymbol{k}) & \dot{a}_2(nT,\boldsymbol{k}) \end{pmatrix}\right] = \det\left[\begin{pmatrix} a_1(t_0,\boldsymbol{k}) & a_2(t_0,\boldsymbol{k}) \\ \dot{a}_1(t_0,\boldsymbol{k}) & \dot{a}_2(t_0,\boldsymbol{k}) \end{pmatrix}\right] \tag{4.62}$$

Thus $\det[M_{\boldsymbol{k}}^n] = (\det[M_{\boldsymbol{k}}])^n = 1 \Rightarrow \det[M_{\boldsymbol{k}}] = 1$.

LetΛ_1 andΛ_2 be eigenvalues of the matrix $M_{\boldsymbol{k}}$. Then since $\det[M_{\boldsymbol{k}}] = 1 \Rightarrow \Lambda_1\Lambda_2 = 1, \Lambda_1 = \Lambda_2^{-1}$. Thus the two eigenvalues are simply Λ andΛ^{-1}.

Thus the trace of the matrix can be written as

$$tr\left(M_{\boldsymbol{k}}\right) = \Lambda + \Lambda^{-1} . \tag{4.63}$$

All the dynamics of equation (4.58) can be determined by the eigenvalues Λ since, due to equation (4.59), all the time behaviour is governed by the matrix $M_{\boldsymbol{k}}$.

Let $tr\left(M_{\boldsymbol{k}}\right) = C$, then equation (4.63) becomes $\Lambda + \Lambda^{-1} = C \Rightarrow \Lambda^2 - C\Lambda + 1 = 0$

Thus

$$\Lambda = \frac{1}{2}\left(C \pm \sqrt{C^2 - 4}\right) . \tag{4.64}$$

There are three possibilities.

1) $C > 2$ or $C < -2$, then Λ is real positive. In this case the solutions of equation (4.58) diverge as can be seen by the fact that $M_{\boldsymbol{k}}$ will cause the solutions to keep growing (see equation (4.59)).

2) $C = \pm 2$. Then $\Lambda = 1$. In this case, the matrix $M_{\boldsymbol{k}}$ can be written as $P^{-1}\begin{pmatrix}1 & \alpha \\ 0 & 1\end{pmatrix} P$. In this case

$$M_{\boldsymbol{k}}^{n} = P^{-1}\begin{pmatrix}1 & n\alpha \\ 0 & 1\end{pmatrix} P\,. \tag{4.65}$$

Thus one solution is periodic and the other diverges (unless $\alpha = 0$ but that's a very specific case).

3) $-2 < C < 2$. Thus Λ becomes complex. In this case the solutions won't run away and remain stable.

It was found in [5] that for some momentum modes $\boldsymbol{k}$, $C > 2$. Thus this is where the divergences of the NLO solution are caused.

Specifically the solution of a in the region of $C > 2$, can be characterized by the Lyapunov exponent $\mu(\boldsymbol{k}, \frac{1}{2}g^2\varphi_0^2)$ where

$$\mu(\boldsymbol{k}, \frac{1}{2}g^2\varphi_0^2) \equiv \frac{1}{T}\ln Max\,\{\Lambda_{1,2}\} \tag{4.66}$$

and, asymptotically the solutions of equation (4.56) are

$$a(x^0, \mathbf{k}) \underset{x^0 \to +\infty}{\sim} e^{\mu(\boldsymbol{k}, \frac{1}{2}g^2\varphi_0^2)x^0}\,. \tag{4.67}$$

4.5.2 Analysis of Divergences

The way the above divergence lead to problems in the NLO result is seen in from equation (4.34). Here it shows that the a's are integrated over momentum. If these terms grow exponentially in time, the integrals will diverge in time.

These diverging $a_{\pm\boldsymbol{k}}$'s provide the dominant fluctuations in the system. What follows is a resummation for these terms.

4.6 Resummation

To fix the divergences found in the previous section one can introduce a resummtation of the divergences where the proposed resummation is given by [5]

$$G_{\text{Resummed}} = e^{\left[\frac{1}{2}\int d^3\boldsymbol{u} d^3\boldsymbol{v} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k}[a_{+\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{u}}][a_{-\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{v}}]\right]}\mathcal{G}_t\,. \tag{4.68}$$

Note that expanding the exponential gives

$$\mathcal{G}_t + \left[\frac{1}{2}\int d^3\boldsymbol{u}d^3\boldsymbol{v}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k}\left[a_{+\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{u}}\right]\left[a_{-\boldsymbol{k}}\cdot\mathbb{T}_{\boldsymbol{v}}\right]\right]\mathcal{G}_t + ... \tag{4.69}$$

So the first two terms give the LO and NLO terms. The rest of the terms give parts of the higher loop order contributions. At first, it seems like the procedure described above is adding an infinite tower of divergent terms since every term in the expansion contains integrals of the $a_{\pm\mathfrak{k}}$. However, this resummation scheme works.

This can be shown by analogy to a wonderful mathematical trick [5] where

$$e^{\frac{\gamma}{2}\partial_x^2}f(x) = \int\limits_{-\infty}^{\infty} dz\frac{e^{-\frac{z^2}{2\gamma}}}{\sqrt{2\pi\gamma}}f(x+z) \tag{4.70}$$

(for a proof one can expand $f(x+z)$ on the right and $e^{\frac{\gamma}{2}\partial_x^2}$ on the left).

In the above, the square of the differential operator acting on some function $f(x)$ amounts to shifting the dependency on that function by an amount z where z itself is summed over from a Gaussian distribution.

Back to equation (4.68). It is clear that $\mathcal{G}_t = \mathcal{G}_t[\varphi(x)]$. However, φ is obtained by a differential equation from its initial conditions φ_0 and $\dot{\varphi}_0$ i.e. φ is just a function that depends on φ_0 and $\dot{\varphi}_0$ or $\varphi = F[\varphi_0, \dot{\varphi}_0]$. Thus $\mathcal{G}_t[\varphi(x)] = \mathcal{G}_t[F[\varphi_0, \dot{\varphi}_0]] = \mathcal{G}_t[\varphi_0, \dot{\varphi}_0]$; $\mathcal{G}_t$ is actually a functional that depends on the initial conditions of the classical field φ_0 and $\dot{\varphi}_0$.

The exponential term of equation (4.68) contains the terms $a_{\pm\boldsymbol{k}}\cdot\mathbb{T}$ which, by definition of equation (4.12) are differential operators acting on the initial conditions of the classical field φ.

Thus equation (4.68) can be explained as follows. It is the square of a differential operator that acts on the functional dependencies of some function G_t. This is exactly the structure of the left hand side of equation (4.70). Thus [5]

$$G_{\text{Resummed}} = \int D[\alpha]D[\dot{\alpha}]Z[\alpha,\dot{\alpha}]G_t[\varphi_0+\alpha, \dot{\varphi}_0+\dot{\alpha}] \tag{4.71}$$

where the $\alpha, \dot{\alpha}$ represents shifts to the initial conditions of φ and $\dot{\varphi}_0$. The $Z[\alpha,\dot{\alpha}]$ is a Gaussian distribution of α and $\dot{\alpha}$ which needs a slightly longer discussion for its form.

From equation (4.70), it can be seen that the term (γ) multiplying the square differential operator becomes the variance term in the Gaussian. In the case of equation (4.68) the term multiplying the differential operators is (using the definition of equation (4.12)) $\frac{d^3\boldsymbol{k}}{(2\pi)^3 2k}\left\{a_{+\boldsymbol{k}}(0,\boldsymbol{u})a_{-\boldsymbol{k}}(0,\boldsymbol{v}) + \dot{a}_{+\boldsymbol{k}}(0,\boldsymbol{u})\dot{a}_{-\boldsymbol{k}}(0,\boldsymbol{v})\right\}$.

From equation (4.34), it can be seen that both $\frac{d^3\boldsymbol{k}}{(2\pi)^3 2k}a_{+\boldsymbol{k}}(0,\boldsymbol{u})a_{-\boldsymbol{k}}(0,\boldsymbol{v})$ and $\frac{d^3\boldsymbol{k}}{(2\pi)^3 2k}\dot{a}_{+\boldsymbol{k}}(0,\boldsymbol{u})\dot{a}_{-\boldsymbol{k}}(0,\boldsymbol{v})$ represent correlations at the initial time. Thus the Gaussian distribution $Z[\alpha,\dot{\alpha}]$ has a variance given by the initial correlations.

One can now numerically show that the divergences have disappeared since the above expression equilibrates. This resummation amounts to solving the LO equation for a set of initial conditions,

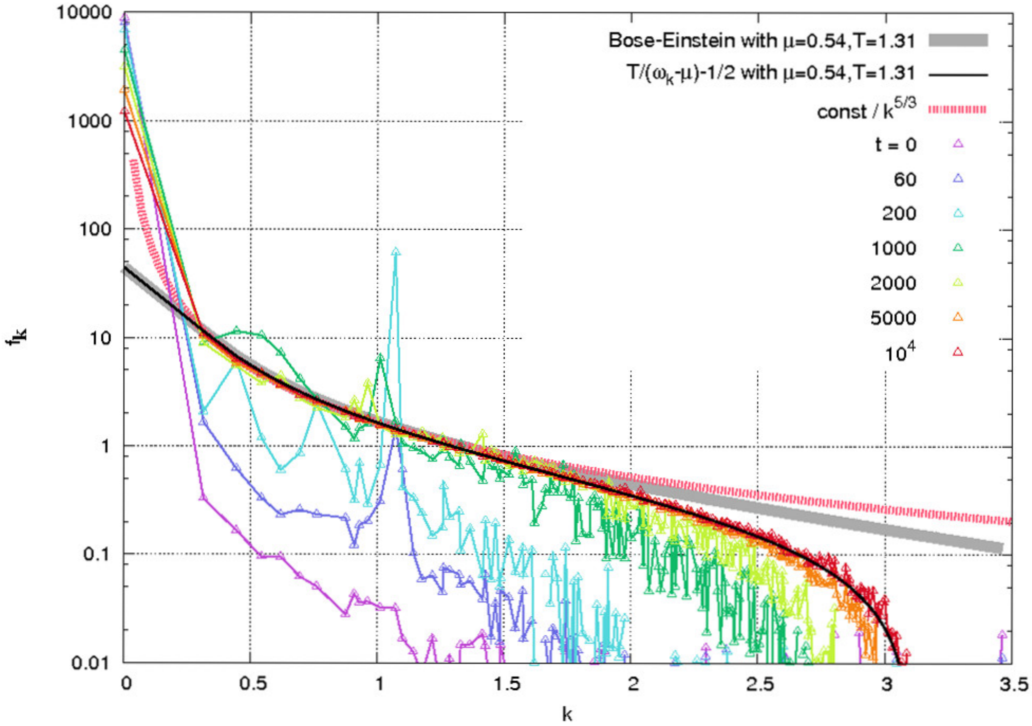


Figure 4.1: $f(k_0)$ at various times in the evolution of the system. Note that the orange triangles compared to the red triangles show the system is not changing over large time scales any more, a characteristic of equilibrium. This plot is taken out of [2], it is not calculated in this thesis

where the initial conditions are sampled from a Gaussian distribution with a variance equivalent to that of the initial fluctuations.

To show the equilibration, figure 4.1 is pulled directly from [2]. The numerical result presented here is taken as a visual emphasise that the resummation scheme does indeed reproduce something that looks like equilibration. It is not used to explain any results of this thesis and no numerical calculation was performed at any point in this thesis.

In this figure, this paper is showing numerical results of finding $f(k_0)$ (on the y-axis) from found the relationship of equation (3.147) found in section 3.3.2 . This equation is re-represented here,

$$F_\beta(k_0) = i\left(\frac{1}{2} + f(k_0)\right)\rho_\beta(k_0) \; . \tag{4.72}$$

It is pointed out that their numerical results do not find the exact Bose-Einstein distribution but this will be discussed a little further on in this chapter. The important behaviour the figure shows is that from the orange triangles, at some medium length time, to the red triangles, at some much later time, the system is not changing any more. This lack of change over large times scales is a characteristic of equilibrium. This shows that equilibration is achieved in this system when using the resummation scheme.

4.6.1 Analysis of the Resummation Scheme

In this section a brief discussion is had on why the above resummation scheme is able to cure divergences. This is followed by an indication of how one can think of the divergences, particularly their size.

How Divergences are Cured

The reason this ressumation has worked is because the terms that cause the divergences, the a's, have been moved into the initial conditions. This is seen by the fact that the exponentiation of the operators $a \cdot \mathbb{T}$, which contain the divergent a's, has been shown to be equivalent to a Gaussian sampling over the initial conditions (equating equations (4.68) and (4.71)). The initial conditions can't in themselves cause divergences and so the divergence has been cured.

Furthermore, as these divergences represent the dominant fluctuations of the system, the dominant fluctuations of the system have been taken into account in the resummation scheme.

Understanding the Fluctuations

As was stated in the introduction of this chapter, equilibration relies on the fluctuations in this system. Thus any greater understanding of the $a_{\boldsymbol{k}}$ terms one can obtain is valuable information. This resummation scheme may provide such new information, that the size of $a_{\boldsymbol{k}}$ may not grow to be parametrically larger than $\frac{1}{g}$.

Consider that the resummation scheme sums over classical objects that are created from φ. Since this resummation scheme works, this implies that the classical objects are still, parametrically, the most important objects in the system as they provide the largest contributions. Recall that $\varphi \sim O(\frac{1}{g})$ (see section 1.8.4), then since the fluctuations are corrections to this leading order, this implies that $a_{\boldsymbol{k}}$ cannot grow to a size larger than $\frac{1}{g}$.

This observation would require a more in-depth numerical analysis and may provide greater insight into the equilibration process.

4.6.2 Weaknesses of the Resummation Scheme

This resummation scheme is not devoid of issues. The first issue is that the resummation scheme leaves out a large set of diagrams in its resummation, all diagrams that contain the quantum-statistical vertex (see section 3.2.4 for a reminder). This leads to the second issue, that the resummation is unable to reproduce the correct equilibration conditions as discussed in section 3.3.2.

Resummation Scheme and Quantum-Statistical Vertices

The C-S method's G_{Resummed} can only contain classical-statistical vertices. Where these vertices are the same as those discussed in section 3.2.4. This can easily be understood through understanding how the resummation works.

The resummation works by acting on a classical object and is incapable of creating new vertices, as the operator in the exponential of equation (4.68) does not create vertices.

The classical object is made from classical field inserts that must end on a vertex with the σ part of the propagator (as discussed in section 3.2.5) and at each vertex on the classical propagator there are two classical fields inserted, thus the only possible vertex allowed in this object can be the classical-s vertex.

Then since the resummation cannot create new vertices, one can only have classical-s vertices in the resummation.

4.6.3 Equilibration in C-S Method

Remember from section 3.3 that in thermal equilibrium, a system will have the following relationship

$$F_\beta(k_0) = i\left(\frac{1}{2} + f(k_0)\right)\rho_\beta(k_0) \tag{4.73}$$

where $f(k_0) = \frac{1}{e^{\beta k_0}-1}$ is the Bose-Einstein distribution.

The Classical-Statistical method looks for this equilibrium relationship between F and ρ. It was found in [2] that the best fit for their results was given by the function $f(k_0) = -\frac{1}{2} + \frac{1}{\beta k_0}$.

This however is simply the first two terms in the Taylor expansion of $\frac{1}{e^{\beta k_0}-1}$. Thus the Classical-Statistical method is unable to reproduce the exact equilibrium distribution and is only able to produce a semi-classical approximation.

According to [28], the fact that the Classical-Statistical method doesn't incorporate all possible effects within the resummation may lead to this incorrect equilibrium distribution. Since the effects left out of the resummation are all diagrams with the quantum-statistical vertex, it seems that these vertices are important to get the tails of the Bose-Einstein distribution.

Chapter 5

Non-Equilibrium 2 P.I. Equations of Motion

The classical-statistical method of the previous chapter is just one way to discuss equilibration phenomena of out of equilibrium methods. This chapter will discuss a popular alternative to the C-S method, the 2 P.I. method.

To start the chapter, it will be shown that the 2 P.I. formalism comes out very naturally from the Schwinger-Keldysh framework. This means that the 2 P.I. formalism can work within the context of out of equilibrium field theory.

The Schwinger-Dyson (S-D) equations of motion of the one point function $\bar{\phi}$ and the connected two point function G will be derived with the 2 P.I. action based within the Schwinger-Keldysh formalism. This allows one to consider the Schwinger-Dyson (S-D) equations of motion for an out of equilibrium field theory.

The equation for the propagator will then be put into a form that will make it comparable to the Classical-Statistical method by transforming the propagator G into the F and ρ propagators of section 3.1.3. This will rely heavily on diagrammatic representation of the equations.

The consequences of these S-D equations of motion in the context of a strong source J will be investigated. It will be shown that for equilibration to occur requires the F and ρ propagators to have particular sizes in g, the coupling parameter in the theory. It has been shown numerically by [29] that equilibration is achieved and that the equilibrium distribution found in section 3.3.2 is fully reproduced by this method.

At this point one finds a serious flaw within this method. The S-D equations of motion for F and ρ contain an infinite sequence of terms that requires truncation, but the sizes of F and ρ required for equilibration to occur render the truncation invalid. This will imply that the intermediate behaviour of the system, predicted by the 2 P.I. formalism, may be subject to large uncertainties.

5.1 Using the 2 P.I. Effective Action in a Regime Experiencing Collective Effects

In this section it will be shown that the 2 P.I. action of equation (2.40), where one considers time to be given as a closed contour as was shown in figure 3.1 (see section 3.1) is equivalent to the Schwinger-Keldysh generating functional with a closed time contour as given in equation (3.71) with a Gaussian initial density matrix.

The 2 P.I. method has only two degrees of freedom, $\bar{\phi}$ and G this leads one to try parametrize the initial density matrix in $Z[J_+, J_-, \rho(t_0)]$ as a Gaussian function since a Gaussian has the capability to describe two degrees of freedom. This Gaussian function, when multiplying the rest of the terms in $Z[J_+, J_-, \rho(t_0)]$, looks like the 2 P.I. action of equation (2.40) as long as one considers time to be integrated over the whole Schwinger-Keldysh contour [7].

So to show this relationship between the 2 P.I. formalism and the Schwinger-Keldysh, one starts with $Z[J_+.J_-, \rho(t_0)]$ with a closed time contour as given in equation (3.71)

$$\begin{aligned} Z[J_+, J_-, \rho(t_0)] = &\int D[\phi^{(+)}_{t_0}(\boldsymbol{x})] D[\phi^{(-)}_{t_0}(\boldsymbol{x})] \langle \phi^{(+)}_{t_0} | \rho(t_0) | \phi^{(-)}_{t_0} \rangle \\ &\times \int D[\phi^+] \int D[\phi^-] e^{i \int\limits_{C} dx^0 \int d^3\boldsymbol{x} [\mathcal{L}[\phi(x)] + J(x)\phi(x)]} . \end{aligned} \tag{5.1}$$

The term $\langle \phi^{(+)}_{t_0} | \rho(t_0) | \phi^{(-)}_{t_0} \rangle$ is the matrix element for the initial conditions. The initial density matrix can be parametrized [7], [8] such that

$$\begin{aligned} \langle \phi^{(+)}_{t_0} | \rho(t_0) | \phi^{(-)}_{t_0} \rangle &= C e^{iF[\phi]} \\ &= C e^{i \left(F_0 + \int\limits_{C} d^4x F_1(x)\phi(x) + \frac{1}{2!} \int\limits_{C} d^4x d^4y F_2(x,y)\phi(x)\phi(y) + \dots \right)} \end{aligned} \tag{5.2}$$

with the integration $\int\limits_{C}$ meaning that the time integration runs over the complete Schwinger-Keldysh time contour. C is a normalization constant, F_0 is a scalar quantity, $F_1(x)$ is a vector type quantity, $F_2(x, y)$ is a matrix.

Assume that the density matrix is Gaussian, thus the argument of the exponential is truncated at $F_2(x, y)$. This can be inserted back into $Z[J_+, J_-, \rho]$,

$$
\begin{aligned}
Z[J_+, J_-, \rho] &= \int D[\phi_{t_0}^{(+)}(\boldsymbol{x})] D[\phi_{t_0}^{(-)}(\boldsymbol{x})] C e^{i\left(F_0 + \int\limits_C d^4x F_1(x)\phi(x) + \frac{1}{2!}\int\limits_C d^4x d^4y F_2(x,y)\phi(x)\phi(y)\right)} \\
&\quad \times \int D[\phi^+] \int D[\phi^-] e^{i\int\limits_C dx^0 \int d^3\boldsymbol{x}[\mathcal{L}[\phi(x)] + J(x)\phi(x)]} \\
&= C e^{iF_0} \int D[\phi^+] \int D[\phi^-] \\
&\quad \times e^{i\int\limits_C d^4x \mathcal{L}[\phi(x)] + i\int\limits_C d^4x[J(x)\phi(x) + F_1(x)\phi(x)] + \frac{i}{2!}\int\limits_C d^4x d^4y F_2(x,y)\phi(x)\phi(y)} \\
&= C e^{iF_0} \int D[\phi^+] \int D[\phi^-] \\
&\quad \times e^{i\int\limits_C d^4x \mathcal{L}[\phi(x)] + i\int\limits_C d^4x[J(x)\phi(x) + F_1(x)\phi(x)] + \frac{i}{2!}\int\limits_C d^4x d^4y F_2(x,y)\phi(x)\phi(y)} .
\end{aligned}
\tag{5.3}
$$

Recall from section 2.5.1 that the starting point for finding the 2 P.I. effective action was given by equation (2.40). One then can amend this equation in such a way that the time integral runs over the entire Schwinger-Keldysh contour, then equation (2.40) becomes,

$$
Z[J, R] = N \int D[\phi] e^{i\left[\int\limits_C d^4x \mathcal{L}[\phi(x)] + \int\limits_C d^4x J(x)\phi(x) + + \frac{1}{2}\int\limits_C d^4x \int\limits_C d^4y R(x,y)\phi(x)\phi(y)\right]} . \tag{5.4}
$$

Compare the above expression with equation (5.3). These look the same if one considers $J + F_1 = J$ and $F_2 = R$, they only differ by a normalization factor.

This gets one to make the claim that the non-equilibrium 2 P.I. effective action is$\Gamma[\ \bar{\phi}, G]$ of section 2.5.2, with a closed time contour [8].

Thus the 2.P.I. effective action allows one to consider the situation where there is a non-equilibrium field theory. When one considers this as a closed time contour, the equations of motion one derives from the 2 P.I. effective action describe the equations of motion in finite temperature.

To consider the equations of motion in a non-equilibrium situation, the initial set up of the problem must be that of a non-thermal density matrix $\rho(t_0)$. Thus one must discuss how the Gaussian $\rho(t_0)$ will affect the initial conditions of the evolving objects of interest, $\bar{\phi}$ and G.

5.2 Discussion on a Gaussian Initial Density Matrix

The initial density matrix is important in two ways which are actually related. The first is that, as has been previously pointed out, the structure of this object determines weather the initial ensemble is thermal or not. The second role this object plays is that, since it is setting up the initial ensemble, it encodes the initial time information of the theory. This chapter is about deriving equations of motion, these will need initial conditions. The initial conditions for the equations of motion then must be related to the Gaussian initial density matrix described in the previous section.

This section will show how the initial conditions for the degrees of freedom, namely $\bar{\phi}$ and G are contained within the Gaussian initial density matrix of the previous section. The reason that $\bar{\phi}$ and G

are the only two variables encoded in the Gaussian density matrix is that a Gaussian matrix can only describe up to two degrees of freedom and the 2 P.I. method's only degrees of freedom are $\bar{\phi}$ and G.

From [8] the most general Gaussian Density matrix can be written as

$$\begin{aligned}\left\langle \phi^{(1)}_{t_0} | \rho(t_0) | \phi^{(2)}_{t_0} \right\rangle = & \\ & \frac{1}{\sqrt{2\pi\xi^2}} exp\left\{ i\dot{\phi}\left(\phi^{(1)}_{t_0} - \phi^{(2)}_{t_0}\right) - \frac{\sigma^2+1}{8\xi^2}\left[\left(\phi^{(1)}_{t_0} - \phi\right)^2 + \left(\phi^{(1)}_{t_0} - \phi\right)^2\right]\right. \\ & + i\frac{\eta}{2\xi}\left[\left(\phi^{(1)}_{t_0} - \phi\right)^2 - \left(\phi^{(1)}_{t_0} - \phi\right)^2\right] + \frac{\sigma^2-1}{4\xi^2}\left(\phi^{(1)}_{t_0} - \phi\right)\left(\phi^{(2)}_{t_0} - \phi\right) . \end{aligned} \tag{5.5}$$

From the above matrix it is shown in appendix E.1 that [8],

$$\phi = \mathrm{Tr}\left\{\rho(t_0)\phi(t)\right\}_{|t=0} \tag{5.6}$$

$$\dot{\phi} = \mathrm{Tr}\left\{\rho(t_0)\partial_t\phi(t)\right\}_{|t=0} \tag{5.7}$$

$$\xi^2 = \mathrm{Tr}\left\{\rho(t_0)\phi(t')\phi(t)\right\}\Big|_{t=t'=0} - \phi\phi \tag{5.8}$$

$$\xi\eta = \frac{1}{2}\mathrm{Tr}\left\{\rho(t_0)\left(\partial_{t'}\phi(t')\phi(t) + \phi(t')\partial_t\phi(t)\right)\right\}\Big|_{t=t'=0} - \dot{\phi}\phi \tag{5.9}$$

$$\eta^2 + \frac{\sigma^2}{4\xi^2} = \mathrm{Tr}\left\{\rho(t_0)\partial_{t'}\phi(t')\partial_t\phi(t)\right\}\Big|_{t=t'=0} - \dot{\phi}\dot{\phi} . \tag{5.10}$$

$$\mathrm{Tr}\left\{\rho(t_0)\phi(x)...\partial_t\phi(y)...\right\} = \int d\phi^{(1)}_{t_0} d\phi^{(2)}_{t_0} \left\langle \phi^{(1)}_{t_0} | \rho(t_0) | \phi^{(2)}_{t_0} \right\rangle \left\langle \phi^{(2)}_{t_0} | \phi(x)...\partial_t\phi(y)... | \phi^{(1)}_{t_0} \right\rangle$$

and

$$\phi(t=0,\boldsymbol{x})|\phi^{(1)}(\boldsymbol{x})\rangle = \phi^{(1)}_{t_0}(\boldsymbol{x})|\phi^{(1)}_{t_0}\rangle \tag{5.11}$$

$$\phi(t'=0,\boldsymbol{x})|\phi^{(2)}(\boldsymbol{x})\rangle = \phi^{(2)}_{t_0}(\boldsymbol{x})|\phi^{(2)}_{t_0}\rangle . \tag{5.12}$$

Recall that the expectation value of an operator in a non-equilibrium field theory is given as the trace, as above. The time derivatives of the objects in equations (5.6) - (5.10) can be pulled outside the trace. Finally by noting that the full n-point function is the connected part minus the disconnected part, one can identify these five equations as the initial one and connected two point functions as they are correlations of fields with time derivatives out the front. Thus from the initial density matrix one can define the initial conditions for $\bar{\phi}$ and G.

5.3 Equations of Motion

In this section the equations of motion for G and $\bar{\phi}$ are found using the 2 P.I. formalism with a closed time contour. In both cases this is done by starting with varying the 2 P.I. action with respect to the degree of freedom needed, these were given as the stationary conditions of $\Gamma[\bar{\phi}, G]$ in section 2.5.1.

It turns out that it will be best to split the S-D equation of motion for G into the F and ρ propagators. This will expose a serious problem within the 2 P.I. formalism when describing a field in the background of strong sources.

Note that these are the only two S-D equations required to be able to describe the entire theory. This is true as all other n-point functions in the 2 P.I. formalism depend on these two degrees of freedom as was discussed in section 2.5.5.

5.3.1 S-D equation of motion for Field $\bar{\phi}$

To find the S-D equation of motion of the field, the stationary condition given by equation (2.45) of section 2.5.1 is used.

Thus [8],

$$-J(x) - \int d^4y R(x,y)\bar{\phi}(y) = \frac{\delta S[\bar{\phi}]}{\delta\bar{\phi}} - \frac{i}{2}\frac{\delta}{\delta\bar{\phi}}\mathrm{Tr}G_t^{-1}(\bar{\phi})G + \frac{\delta\Gamma_2}{\delta\bar{\phi}} \ . \tag{5.13}$$

Then using $\frac{\delta S[\bar{\phi}]}{\delta\bar{\phi}} = \left[-(\Box_x + m^2)\bar{\phi}(x) - \frac{g^2}{3!}\bar{\phi}^3(x)\right]$ and $\frac{\delta}{\delta\bar{\phi}(z)}\mathrm{Tr}G_t^{-1}(\bar{\phi})G = -ig^2\bar{\phi}(z)G(z,z)$,

$$\begin{aligned} -J(x) - \int d^4y R(x,y)\bar{\phi}(y) &= \left[-(\Box_x + m^2)\bar{\phi}(x) - \frac{g^2}{3!}\bar{\phi}^3(x)\right] \\ &\quad -\frac{1}{2}g^2\bar{\phi}(x)G(x,x) + \frac{\delta\Gamma_2}{\delta\bar{\phi}} \ . \end{aligned} \tag{5.14}$$

The $G(x,x)$ term acts like a position dependent mass shift since it is a term that acts at a point. In fact all the local terms behave just like a position dependent mass term shift. Thus all the local terms can be absorbed into a position dependent mass term such that [8],

$$M^2(x) = m^2 + \frac{g^2}{2}\bar{\phi}^2(x) + \frac{g^2}{2}G(x,x) \ . \tag{5.15}$$

Thus equation (5.14) becomes

$$\left(\Box_x + M^2(x)\right)\bar{\phi}(x) = J(x) + \int d^4y R(x,y)\bar{\phi}(y) + \frac{g^2}{3}\bar{\phi}^3(x) + \frac{\delta\Gamma_2}{\delta\bar{\phi}} \ . \tag{5.16}$$

This defines the S-D equation of motion for $\bar{\phi}$. One can see that it depends on a truncation of$\Gamma_2[\bar{\phi}, G]$.

5.3.2 S-D equation of motion for G

To find the S-D equation of motion for G, the stationary condition given by equation (2.46) of section 2.5.1 is used. In section 2.5.3 it was found that using the stationary condition one finds equation (2.63) where this is given by,

$$G^{-1}(x,z) = -G_t^{-1}(x,z) - \Sigma^R(x,z) - iR(x,z) \ . \tag{5.17}$$

From the definition of the propagator $\int d^4zG^{-1}(x,z)G(z,y) = \delta(x-y)$. Then one can multiply the above with $G(z,y)$ on the right and remembering that to multiply these objects implies an integral,

$$\begin{aligned}
\int d^4zG^{-1}(x,z)G(z,y) &= -\int d^4zG_t^{-1}(x,z)G(z,y) - \int d^4z\left[\Sigma^R(x,z)+iR(x,z)\right]G(z,y) \\
&\quad \text{Using } G_t^{-1}(x,z) = i\frac{\delta^2 S}{\delta\bar{\phi}^2} \text{ (section 2.4.4)} \\
\delta(x-y) &= i\left((\Box_x+m^2)+\frac{g^2}{2}\bar{\phi}^2(x)\right)G(x,y) - \int d^4z\left[\Sigma^R(x,z)+iR(x,z)\right]G(z,y)\,.
\end{aligned} \tag{5.18}$$

Thus [8],

$$\left((\Box_x+m^2)+\frac{g^2}{2}\bar{\phi}^2(x)\right)G(x,y)+i\int d^4z\left[\Sigma^R(x,z)+iR(x,z)\right]G(z,y) = -i\delta(x-y)\,. \tag{5.19}$$

This provides an S-D equation of motion for the full propagator.

5.4 Equations of motion for F and ρ

In this section the S-D equation of motion for G, equation (5.19), will be split up into two separate equations of motion, one for F and one for ρ. To do this split, one can exploit the useful fact that F and ρ correspond to the real and imaginary parts of G respectively (see section 3.1.3). Consider the definition of these propagators in terms of G given in equation (3.21) of section 3.1.3 where,

$$G(x,y) = F(x,y) - \frac{i}{2}\rho(x,y)\mathrm{sign}_c(x^0-y^0)\,. \tag{5.20}$$

As was discussed in section 3.1.3, F represents the real part of G whilst ρ represent the imaginary part of G. This is how one can find the equations of motion for F and ρ. One splits up equation (5.19) with F and ρ, then all the real terms are part of the S-D equation of motion for F and the terms with imaginary parts form the S-D equation of motion for ρ.

The first problem encountered is thatΣ^R needs to be split up into a real and imaginary part. This is what is done in the first part of this section. This split is done along the lines of the way G is split. In the second part one inserts the split up parts into equation (5.19). At this point the S-D equation of motion can be put into a form that is actually solvable by dealing with the time integral over the full contour of the objects found. The reason there are simplifications is that by splitting up G and Σ into F and ρ terms, one exposes the exact time dependence of G and Σ through Θ functions.

After simplifying the time integrals, the real and imaginary parts of what is left are used to find equations of motion for F and ρ. It turns out that to actually solve these equations of motion one has to truncate the Σ term. This truncation is only valid if the terms neglected in Σ remain consistently small. This is important to note as in section 5.5 it is shown that a truncation is not possible in the presence of large sources.

5.4.1 Self Energy in terms of Σ_F and Σ_ρ

Here$\Sigma\ ^R$ is split up into$\Sigma\ _F$, Σ_ρ. To do this one needs to separate off the local part of$\Sigma\ ^R$ first as only the non-local part of$\Sigma\ ^R$ is non trivial. This discussion can be found in [8].

The local part of the self energy is given as $-i\frac{g^2}{2}G(x,x)$. Looking at the definition of $\rho(x,y)$ in equation (3.18), one can see that the local $\rho(x,x)$ will disappear. Thus the local $G(x,x) = F(x,x)$.

Thus

$$\Sigma(x,y) = -i\frac{g^2}{2}F(x,x) + \bar{\Sigma}(x,y) \tag{5.21}$$

where $\bar{\Sigma}(x,y)$ is the non-local contribution to the self energy.

This non-local $\bar{\Sigma}(x,y)$ can be broken up in exactly the same manner as the propagator, thus

$$\bar{\Sigma}(x,y) = \Sigma_F(x,y) - \frac{i}{2}\Sigma_\rho \text{sign}_{\mathcal{C}}(x^0 - y^0)\ . \tag{5.22}$$

Note that the structure of$\Sigma\ _F$ and$\Sigma\ _\rho$ is discussed in section 5.4.4 below.

5.4.2 Equations of motion for F and ρ

One can now get at the equations of motion for F and ρ. Since F and ρ are the real and imaginary parts of G, one can try split up the equation of motion for G into its real and imaginary parts. This entails splitting up into F and ρ.

Start by inserting the split up$\Sigma\ ^R$ as well as the split up G into equation (5.19) to find that,

$$\begin{aligned}
-i\delta(x-y) \quad = \quad & \left((\Box_x + m^2) + \frac{g^2}{2}\bar{\phi}^2(x)\right)\left(F(x,y) - \frac{i}{2}\rho(x,y)\text{sign}_{\mathcal{C}}(x^0 - y^0)\right) \\
& + i\int d^4z \Bigg[\left(-i\frac{g^2}{2}F(x,x)\delta(x-z) + \Sigma_F(x,z) - \frac{i}{2}\Sigma_\rho(x,z)\text{sign}_{\mathcal{C}}(x^0 - z^0)\right) \\
& + iR(x,z)\Bigg]\left(F(z,y) - \frac{i}{2}\rho(z,y)\text{sign}_{\mathcal{C}}(z^0 - y^0)\right)\ .
\end{aligned} \tag{5.23}$$

To simplify one can again use the idea of using a position dependent mass term as in equation (5.15),

$$M(x) = m^2 + \frac{g^2}{2}\bar{\phi}^2(x) + \frac{g^2}{2}F(x,x) \tag{5.24}$$

then using this new mass definition and expanding further gives,

$$\begin{aligned} -i\delta(x-y) &= \left(\Box_x + M^2(x)\right)\left(F(x,y) - \frac{i}{2}\rho(x,y)\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\right) \\ &+i\int d^4z\Bigg[\left(\Sigma_F(x,z) - \frac{i}{2}\Sigma_\rho(x,z)\mathrm{sign}_{\mathcal{C}}(x^0-z^0)\right) \\ &\times\left(F(z,y) - \frac{i}{2}\rho(z,y)\mathrm{sign}_{\mathcal{C}}(z^0-y^0)\right) \\ &+ iR(x,z)\left(F(z,y) - \frac{i}{2}\rho(z,y)\mathrm{sign}_{\mathcal{C}}(z^0-y^0)\right)\Bigg] \ . \end{aligned} \tag{5.25}$$

From this expression one would like two separate equations of motion, one for F and the other for ρ, the real and imaginary parts of the above. To separate out the above expression it is best to work on equation (5.25) in two parts. First one can analyse the following,

$$\int d^4z\left(\Sigma_F(x,z) - \frac{i}{2}\Sigma_\rho(x,z)\mathrm{sign}_{\mathcal{C}}(x^0-z^0)\right)\left(F(z,y) - \frac{i}{2}\rho(z,y)\mathrm{sign}_{\mathcal{C}}(z^0-y^0)\right) \ . \tag{5.26}$$

Considering that the $\mathrm{sign}_{\mathcal{C}}$ term is built from Θ functions, one has time integration over Θ functions. This causes massive simplifications in the time integration of the above. The result is that one integrates only over past times, these are known as memory integrals [8]. The details of this calculation are tedious and are best left to appendix E.2. The end result is found in equation (E.23) [8],

$$\begin{aligned} &\int d^4z\left(\Sigma_F(x,z) - \frac{i}{2}\Sigma_\rho(x,z)\mathrm{sign}_{\mathcal{C}}(x^0-z^0)\right)\left(F(z,y) - \frac{i}{2}\rho(z,y)\mathrm{sign}_{\mathcal{C}}(z^0-y^0)\right) \\ &= -\left(i\int_{t_0}^{x^0} dz^0\int d^3\boldsymbol{z}\Sigma_\rho(x,z)F(z,y)\right) + \left(i\int_{t_0}^{y^0} dz^0\int d^3\boldsymbol{z}\Sigma_F(x,z)\rho(z,y)\right) \\ &- \left(\frac{1}{2}\int_{t_0}^{y^0} dz^0\int d^3\boldsymbol{z}\Sigma_\rho(x,z)\rho(z,y)\right) \ . \end{aligned} \tag{5.27}$$

The second thing one needs to study to split equation (5.25) into two different equations of motion, is the term

$$\Box_x\left(F(x,y) - \frac{i}{2}\rho(x,y)\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\right) \ . \tag{5.28}$$

Here one would like to know how the differential operator acts on F and ρ specifically. To do this involves finding derivatives of the Θ functions that are contained within $\mathrm{sign}_{\mathcal{C}}$. Again this entails a tedious calculation that is best left for the bulging appendices. The result from appendix E.3, equation (E.28) gives that [8],

$$\begin{aligned} \Box_x\left(F(x,y) - \frac{i}{2}\rho(x,y)\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\right) &= \Box_x F(x,y) - \frac{i}{2}\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\Box_x\rho(x,y) \\ &-i\delta(x-y) \ . \end{aligned} \tag{5.29}$$

Here the δ function comes from derivatives of the Θ function and using the equal time commutation relations contained within ρ at t_0.

Then putting together equation (5.25), equation (5.27) and equation (5.29), then splitting up the resulting equations into the real and imaginary parts gives [8];

Real:

$$\begin{aligned}\left(\Box_x + M^2(x)\right) F(x,y) &= \int_{t_0}^{x^0} dz^0 \int d^3\boldsymbol{z}\Sigma_\rho(x,z)F(z,y) - \int_{t_0}^{y^0} dz^0 \int d^3\boldsymbol{z}\Sigma_F(x,z)\rho(z,y) \\ &\quad + \int d^4z R(x,z)F(z,y) \ . \end{aligned} \tag{5.30}$$

The Imaginary part:

$$\begin{aligned}\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\left(\Box_x + M^2(x)\right)\rho(x,y) &= -\int_{t_0}^{y^0} dz^0 \int d^3\boldsymbol{z}\Sigma_\rho(x,z)\rho(z,y) \\ &\quad + \int d^4z \mathrm{sign}_{\mathcal{C}}(z^0-y^0)R(x,z)\rho(z,y) \ . \end{aligned} \tag{5.31}$$

This provides the S-D equation of motion for the F and ρ propagators.

Note that the term R is a non-physical source. Thus it is set to 0 and doesn't actually play a role in these equations of motion. From this point on it will be assumed that $R = 0$.

To actually solve these equations of motion one would need to truncateΣ_F andΣ_ρ. To truncate one needs to be able to trust the diagrams are decreasing in size. In section 5.5 it will be shown that these diagrams do not actually decrease in size, thus the 2 P.I. method has a serious flaw.

5.4.3 Initial Conditions for Equations (5.30) and (5.31)

Since the equations (5.30) and (5.31) are equations of motion, they must have initial conditions. These equations were derived from a 2 P.I. generating functional. It was stated previously that the 2 P.I. generating functional over a closed time contour is the same thing as the Schwinger-Keldysh formalism with a Gaussian initial density matrix $\rho(t_0)$. The initial density matrix contains information of the initial conditions for the theory. Thus the initial conditions of F and ρ should be related to the objects found in section 5.2.

If one considers the structure of ρ, it is seen that it is highly constrained.

From equation (3.18)

$$\rho(x,y) \equiv i\langle[\phi(x),\phi(y)]\rangle \ . \tag{5.32}$$

The ρ propagator at the initial time means that both space-time arguments are at the same time, t_0. Thus the ρ actually encodes the equal time commutation relation [8] i.e.

$$\begin{aligned}\rho(x,y)\Big|_{x^0=y^0=t_0} &= i\langle[\phi(\boldsymbol{x}),\phi(\boldsymbol{y})]\rangle\Big|_{x^0=y^0=t_0}\\ &= 0\end{aligned} \tag{5.33}$$

The first time derivative of ρ gives one of the other equal time commutation relations

$$\begin{aligned}\partial_{x^0}\rho(x,y)\Big|_{x^0=y^0=t_0} &= i\langle[\pi(\boldsymbol{x}),\phi(\boldsymbol{y})]\rangle\Big|_{x^0=y^0=t_0}\\ &= i\delta(\boldsymbol{x}-\boldsymbol{y})\ .\end{aligned} \tag{5.34}$$

These two initial conditions are fixed, they are constrained completely by the equal time commutation relations. Thus the S-D equation of motion for ρ has fixed initial conditions.

This means that the only freedom in the initial conditions for the full propagator G are completely contained in F at t_0.

This freedom of the initial conditions for the theory was shown to be equivalent to the free Gaussian parameters given by equations (5.6) - (5.10) where specifically for the connected propagator initial conditions it was given that

$$\xi^2 = \mathrm{Tr}\left\{\rho(t_0)\Phi(t)\Phi(t')\right\}\Big|_{t=t'=0} - \phi\phi \tag{5.35}$$

$$\xi\eta = \frac{1}{2}\mathrm{Tr}\left\{\rho(t_0)\left(\partial_t\Phi(t)\Phi(t') + \Phi(t)\partial_{t'}\Phi(t')\right)\right\}\Big|_{t=t'=0} - \dot{\phi}\phi \tag{5.36}$$

$$\eta^2 + \frac{\sigma^2}{4\xi^2} = \mathrm{Tr}\left\{\rho(t_0)\partial_t\Phi(t)\partial_{t'}\Phi(t')\right\}\Big|_{t=t'=0} - \dot{\phi}\dot{\phi}\ . \tag{5.37}$$

Given that F is the only place where the freedom of the initial conditions comes in this implies that [8]

$$F(x,y)\Big|_{x^0=y^0=t_0} = \xi^2 \tag{5.38}$$

$$\frac{1}{2}\left(\partial_{x^0}F(x,y) + \partial_{y^0}F(x,y)\right)\Big|_{x^0=y^0=t_0} = \xi\eta \tag{5.39}$$

$$\partial_{y^0}\partial_{x^0}F(x,y)\Big|_{x^0=y^0=t_0} = \eta^2 + \frac{\sigma^2}{4\xi^2}\ . \tag{5.40}$$

The above can be checked explicitly. As an example consider $\mathrm{Tr}\left\{\rho(t_0)\Phi(t)\Phi(t')\right\}\Big|_{t=t'=0} - \phi\phi$, this is the initial condition of G. Breaking this term up into the commutator and anti-commutator of the fields gives,

$$\mathrm{Tr}\left\{\rho(t_0)\Phi(t)\Phi(t')\right\}\Big|_{t=t'=0} = \mathrm{Tr}\left\{\rho(t_0)\frac{1}{2}\left(\{\Phi(t),\Phi(t')\} - [\Phi(t),\Phi(t')]\right)\right\}\Big|_{t=t'=0} \tag{5.41}$$

note that this is the definition of F and ρ at equal times. Then since the equal time commutation of the fields is 0, the term $[\Phi(t), \Phi(t')]$ contributes 0, this is the ρ part. Thus one finds equation (5.38).

The next two conditions can be checked in the same way, using equal time commutation relations. In this manner both equations (5.39) and (5.40) can be checked.

5.4.4 Diagrammatic form of Σ_F and Σ_ρ

There are two reasons to find the diagrammatic form ofΣ_F andΣ_ρ. The first reason is for greater understanding of the equations of motion found in equations (5.30) and (5.31). The second reason is that it will help in the following section.

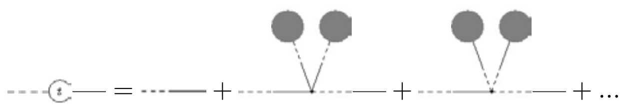
The diagrammatic structure can be obtained by looking at equation (5.30), the S-D equation of motion for F, in terms of the diagrammatic notation introduced in section 3.2.6. An outline of the exact procedure done is now presented.

What one does is to consider the S-D equations of motion of the full F propagator in terms of the notation introduced in section 3.2.6 (equation (5.45) below). Then consider multiplying both sides by the differential operator $\Box + m^2 + \frac{g^2}{2}\bar{\phi}^2$. This makes the left hand side of the equation $\left[\Box + m^2 + \frac{g^2}{2}\bar{\phi}^2\right] F$, the same as the left hand side of equation (5.30) minus the local part ofΣ^R. Then one can try to match up the right hand side of the found diagrammatic equation to equation (5.30). To be able to complete this plan, one needs to find what $\Box + m^2 + \frac{g^2}{2}\bar{\phi}^2$ acting on F_t and ρ_t is as these objects appear on the right hand side of the diagrammatic version of the S-D equation of motion for F, this is done in appendix E.4.2.

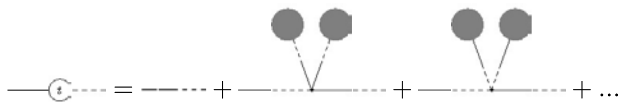
The diagrammatic structure ofΣ_F andΣ_ρ is found below in equations (5.51) and (5.50) respectively.

Diagrams

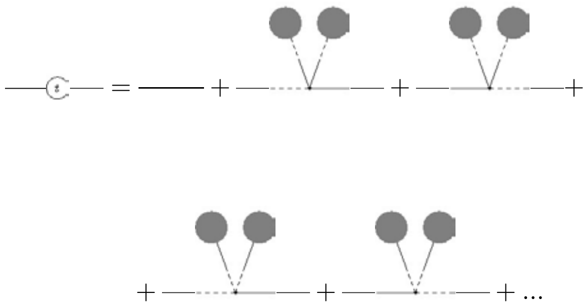
When defining composite propagators in the transformed fields σ and η, the first line in and the last line are important. In the vertices, one must sum over all possible combinations that provide the correct first line in and last line out. As an example look at the definitions of the $G_t^{\eta\sigma}$, $G_t^{\sigma\eta}$ and $G_t^{\sigma\sigma}$ propagators. Note that a line represents a free propagator in the same way as section 3.2.6. The filled in blob with the dotted end is a η field insert, the blob with a solid end is the σ field insert. Thus let,



(5.42)



(5.43)

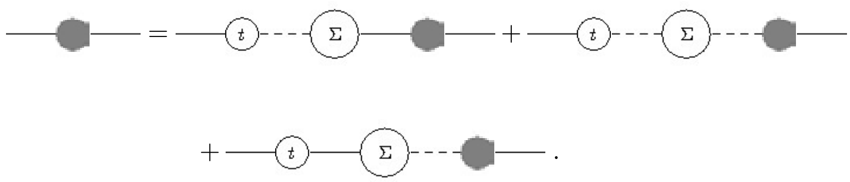


(5.44)

The left hand side diagrams are tree level diagrams, and note that, in the same way as defined in section 2.4.4, the tree here refers to the fact that there are no explicit loops. The shaded in fields inserts may contain loops.

This same requirement of the first line in and the last line out holds for the full propagator, no matter what inserts one has in the middle, as long as the fist line in and the last line out correspond to the required propagator then one has the full propagator.

This leads one to write the full F propagator with a sum of all Σ blob possibilities as [22],



(5.45)

Where the notation used is consistent with previous notation such that dark blobs with two external lines represent full propagators, where the legs to the left and right represent the type of propagator in the same way the tree propagators are defined. Thus the start and end lines are solid lines since this is an F propagator.

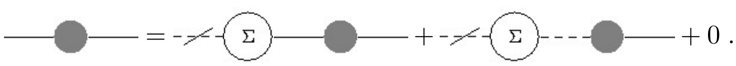
In appendix E.4.2 it is shown in equations (E.44) and (E.45) that

$$(\Box_x + m^2 + \frac{g^2}{2!}\bar{\phi}^2(x))\rho^t(x,y) = -\delta(x,y) \tag{5.46}$$

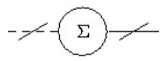
$$\tag{5.47}$$

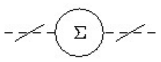
$$\left(\Box_x + m^2 + \frac{g^2}{2!}\bar{\phi}^2(x)\right) F^t(x,y) = 0 \; . \tag{5.48}$$

Note that diagrammatically, an amputated leg of a diagram is given as a line with a slash through it, then acting on both sides of equation (5.45) with $\left(\Box + m^2 + \frac{g^2}{2}\bar{\phi}^2\right)$ and using the above two expression gives

$$\left(\Box + m^2 + \frac{g^2}{2}\bar{\phi}^2\right)$$  (5.49)

Then by comparing the above to equation (5.30), where the M term defined in equation (5.15) is split up into its constituents, it is seen that

$$\Sigma_\rho - \frac{g^2}{2}F =$$  (5.50)

$$\Sigma_F =$$  . (5.51)

Again, remember that a line with a slash through it is referring to an amputated leg.

5.5 2 P.I. And Power Counting

The equations of motion described in equations (5.30) and (5.31) have a very similar form to equation (4.56) of section 4.5.1. In fact the difference is all the local and non-local self energy terms. This leads one to understand that the same instabilities that occur in section 4.5.1 must also occur in these two equations of motion if one didn't have the self energies. For these equations of motion to equilibrate then, the self energy terms must cause the growth to tail off. This can only occur if the self energy terms are at least quadratic in F and ρ and they must be of the same order in g as the instabilities.

It will turn out that there is a size of F and ρ that can lead to equilibration but, in trying to solve the equations of motion, the truncation ofΣ_F andΣ_ρ that is required is invalid unless one introduces another parameter such as N so as to perform a $\frac{1}{N}$ expansion. The truncation is invalid as there is an infinity of diagrams at each order in g contained inΣ_F andΣ_ρ when one has large sources in the system.

In this section, first the g-scalings of F and ρ are determined in the presence of strong sources. It is found that $F \sim O(g^{-2})$ and $\rho \sim O(1)$. It is then shown that this set of scalings for F and ρ should allow the equations of motion to equilibrate as the self energy terms are large enough to control the divergences of the equation. This is backed up with numerical evidence from [29].

To actually use the equations of motion one needs a truncation inΣ_F andΣ_ρ so the next step is to see if the diagrams in these two objects get parametrically smaller. To do this, one determines the relationship between the number of different propagators, vertices and field inserts in the diagrammatic notation of 3.2.6 for a general diagram (equations (5.62) and (5.63) below). Then by considering the order in g that each constituent ofΣ_F andΣ_ρ comes with, combined with relationships between the numbers of these objects, one can determine the power in g of a general diagram inΣ_F andΣ_ρ. It is found that the power in g is independent of the number of classical vertices (classical vertices as defined in section 3.2.4) in the diagram (equation (5.66) below). This invalidates any attempt to truncate without a further expansion parameter.

This implies that the process of equilibration predicted in the 2 P.I. method should is subject to large uncertainties as this method is leaving out a vast amount of information by truncating.

Finally one then can ask what happens if the powers in F and ρ are actually different to what was assumed. This is tried and it is found that any other logical choices for F and ρ result in the equations of motion in F and ρ not being able to equilibrate due to the self energy being smaller than the instabilities.

5.5.1 Powers of F and ρ in terms of g

To continue one needs to determine what the size of the propagators F and ρ are. Since this thesis is set up in a regime where the source are large, the fields are $O(g^{-1})$ as was shown at the end of section 1.7.2.

Look at the definition of the F and ρ. In equations (3.17) and (3.18) it was defined that

$$F(x,y) \equiv \frac{1}{2}\langle\{\phi(x),\phi(y)\}\rangle \tag{5.52}$$

$$\rho(x,y) \equiv i\langle[\phi(x),\phi(y)]\rangle \; . \tag{5.53}$$

Since $F \sim \phi^2$ one should start with the assumption that $F \sim 2O(g^{-1}).O(g^{-1}) \sim O(g^{-2})$.

In the previous section it was stated that the initial values of ρ are governed by the equal time commutation relations (see section 5.4.3). The equal time commutations would imply that ρ starts at the initial time with $O(1)$. Then assume ρ maintains this size throughout equilibration. This gives an estimate that $\rho \sim O(1)$.

The consequences of this size for F and ρ is now investigated.

5.5.2 Checking Equilibration Possibility

In this section it will be shown that with $F \sim O(g^{-2})$ and $\rho \sim O(1)$, the equations of motion for these objects (equations (5.30) and (5.50)) should equilibrate.

One can reproduce the equations of motion schematically from equations (5.30) and (5.31) as the details of the convolution of the functions is not important for this discussion since only the structure of these equations is being investigated.. Here the convolution will be denoted as a $\otimes$. In the following the term $M(x)$ (see equation (5.15)) of equations (5.30) and (5.31) has been split up, and $R = 0$.

$$\begin{aligned}\left(\Box_x + m^2 + \frac{g^2}{2}\bar{\phi}^2\right) F(x,y) + \frac{g^2}{2} F(x,x)F(x,y) &= \Sigma_\rho(x,z) \otimes F(z,y) \\ &\quad -\Sigma_F(x,z) \otimes \rho(z,y) \end{aligned} \tag{5.54}$$

and

$$\left(\Box_x + m^2 + \frac{g^2}{2}\bar{\phi}^2\right) \rho(x,y) + \frac{g^2}{2} F(x,x)\rho(x,y) = \Sigma_\rho(x,z) \otimes \rho(z,y) \; . \tag{5.55}$$

Consider the differential operator acting on F and ρ. Without any self energy terms these equations would be structurally the same as equation (4.56) found in section 4.5.1 where an analysis was done on the instabilities hiding in the following equation

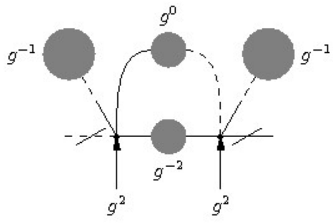
$$\left(\Box + m^2 + \frac{g^2}{2}\varphi^2\right) a = 0 \; . \tag{5.56}$$

It was found that these instabilities grew exponentially with time.

What this has to do with the current situation is that to stop runaway growth due to instabilities, the terms coming from the self energy in the equations of motion for F and ρ needs to be of the same order as the E.O.M. part. If it can be shown that these terms are much smaller, then these equations will show divergent behaviour due to the exact same instabilities of equation (4.56). By the structure of the self energy it is clear that one has terms that are at least the square of F and ρ in each E.O.M.

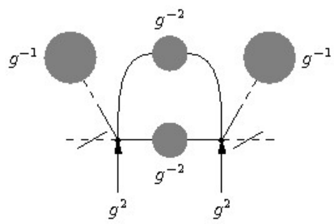
For the following remember that each coupling constant brings a factor g^2 and each field insert $\bar{\phi}$ brings $O(g^{-1})$ while from the previous section, $F \sim O(g^{-2})$ and $\rho \sim O(1)$. Each necessary piece of the self energy terms will be discussed.

First the term$\Sigma\ _\rho(x,z)$. This was shown in equation (5.50) to have a solid line and dotted line amputated as the external lines. In the next section it will be shown that a lowest order diagram in Σ_F and$\Sigma\ _\rho$ will occur when $N_{qv} = N_\eta = 0$. All one needs is at least one self energy term to be as large as the instability term as this in itself would be enough to show equilibration should occur. So one type of lowest order diagram from here is given by



$$\sim g^0 \ . \tag{5.57}$$

The term$\Sigma\ _F(x,z)$. This was shown in equation (5.51) to have a two dotted amputated lines as the external lines. A lowest order diagram from here is given by



$$\sim g^{-2} \ . \tag{5.58}$$

The term $\frac{g^2}{2}F(x,x)$ that was in $M(x)$ is $\sim O(1)$ where the g^2 is for the vertex point for a local loop at the point x and $F \sim g^{-2}$.

Finally the term $\frac{g^2}{2}\bar{\phi}^2 \sim g^2(g^{-1})^2 \sim O(1)$

So in equation (5.54),$\Sigma\ _\rho \otimes F \ \sim\ O(g^{-2})$,$\Sigma\ _F \otimes \rho \ \sim\ O(g^{-2})$, $\frac{g^2}{2}F(x,x)F9(x,y) \ \sim\ O(g^{-2})$ and $\left(\Box_x + m^2 + \frac{g^2}{2}\bar{\phi}^2\right) F \sim O(g^{-2})$. Thus

$$\underbrace{\left(\Box_x + m^2 + \frac{g^2}{2}\bar{\phi}^2\right) F(x,y)}_{O(g^{-2})} = O(g^{-2}) \ . \tag{5.59}$$

Thus the evolution equation shows that the self energy terms, the terms suppressing the growth caused by $\left(\Box_x + m^2 + \frac{g^2}{2}\bar{\phi}^2\right)$, are the same order of magnitude as the growth terms. This means that one can have equilibration for F as long as ρ is also controlled. If ρ blows up then the above expression will also diverge.

In the same way for equation (5.55),

$$\underbrace{\left(\Box_x + m^2 + \frac{g^2}{2}\bar{\phi}^2\right)\rho(x,y)}_{O(1)} = O(1)\ . \tag{5.60}$$

Thus the same argument holds for ρ. This implies that $F \sim O(g^{-2})$ and $\rho \sim O(1)$ allows the equations of motion for F and ρ to equilibrate.

This statement can be substantiated with numerical results found in [29].

Numerical Results from [29]

If one solves the 2 P.I. method equations of motion for F and ρ, it can be shown numerically that equilibration is achieved. The numerical results of [29] show that the 2 P.I. method is capable of reproducing the full quantum equilibrium limit of the full Bose-Einstein distribution for $f(k_0)$, where $f(k_0)$ is determined from the relationship (see section 3.3)

$$F_\beta(k_0) = i\left(\frac{1}{2} + f(k_0)\right)\rho_\beta(k_0)\ . \tag{5.61}$$

Figure 5.1 is a numerical result taken directly from [29]. This figure plots $f(k_0)$ on the y-axis (in the plot it is called $n(k_0)$) against k_0 (in the plot this called ω_0) for nine different times. For the purposes of this thesis, only the black dots need to be looked at as these are showing that equilibration is achieved. This is seen by the fact that over the bottom three frames, the distribution is no longer changing in time, a characteristic of equilibrium. This plot illustrates that the 2 P.I. method is capable of producing equilibration behaviour and of obtaining the correct equilibrium distribution. Again, the numerical result presented here is taken as a visual emphasise that the 2 P.I. method produces equilibration behaviour. This figure is not used to explain any results of this thesis and no numerical calculation was performed at any point in this thesis.

To be confident in this method, the self energy terms that are truncated need to be getting parametrically smaller in g. In the following section this is shown to not be the case.

5.5.3 Power Counting of Σ_F and Σ_ρ

In this section, the power counting of a general$\Sigma\ _F$ and$\Sigma\ _\rho$ diagram is done starting from $F \sim O(g^{-2})$ and $\rho \sim O(1)$.

2 P.I. Diagrams in Terms of σ and η

To continue it is best to make a table of all the important objects that will be used, although all have been defined already. This just helps clarify certain statements made later.

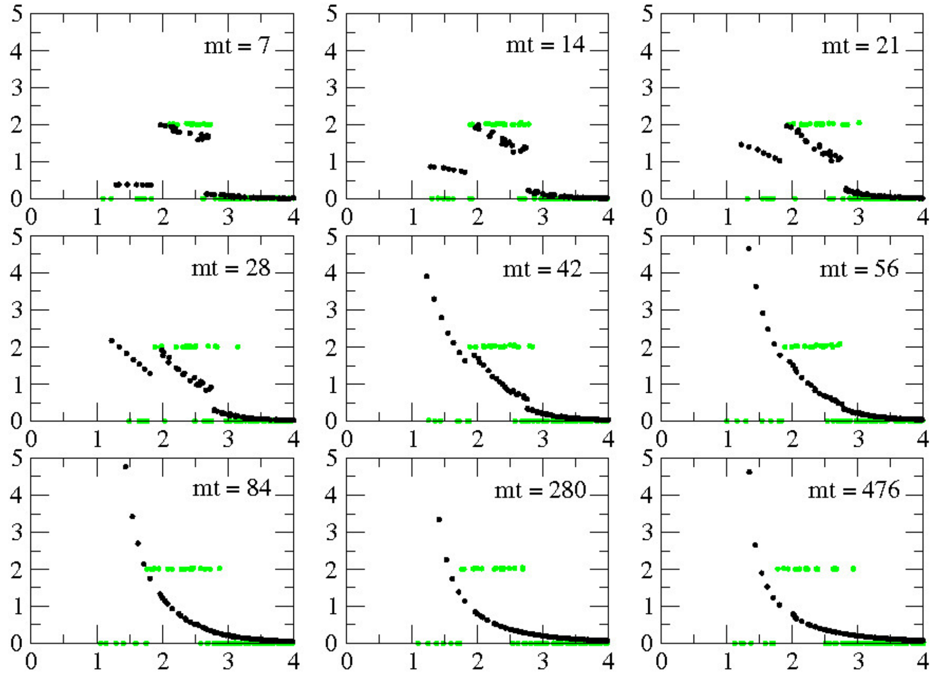


Figure 5.1: $f(k_0)$ (the plot calls it $n(k_0)$) vs. k_0 the plot calls this ω_0 at nine different times in the evolution of the system. The black dots are the ones this thesis is interested in as they show the equilibration behaviour, particularly in the last three frames. This plot is taken out of [29].

	F propagator, counted by N_F	Has two free solid ends
	ρ propagator, counted by N_ρ	Has one free solid end and one free dotted end
	σ field, counted by N_σ	Has one free solid end
	η field, counted by N_η	Has one free dotted end
	quantum-s vertex, counted by N_{qv}	Has three free dotted ends and one free solid ends
	classical-s vertex, counted by N_{clv}	Has one free dotted end and three free solid ends

Now one can define the relationships of the above objects in a general vacuum diagram by considering the amount of free spaces, provided by the vertices, which then have to be saturated by all other objects in a given diagram.

A vacuum diagram is one where all spaces on the vertices have been saturated. To explain the situation for vacuum diagrams, start by counting the 'dotted lines' counting, observing that each quantum vertex has 3 'dotted' ends, while each classical vertex has 1 dotted end. These ends can be connected up by using ρ and η insertions where the ρ provides one dotted end as does each η field. Thus the 3 dotted ends of the quantum vertex plus the 1 dotted end on the classical vertex are joined by ρ's and η's. Thus

$$N_\eta + N_\rho = N_{clv} + 3N_{qv} \; . \tag{5.62}$$

The same argument about ends can be applied for objects ending in solid lines which leads to the inclusion of N_F and N_σ and the exclusion of N_η. It is found that

$$2N_F + N_\sigma + N_\rho = 3N_{clv} + N_{qv} \ . \tag{5.63}$$

Power Counting Of Σ_F and Σ_ρ

The above only holds for the vacuum graphs. To deal withΣ_ρ andΣ_F the above needs a slight adjustment.Σ_ρ is an object with two free end where one is a solid end and the other is a dotted end as seen from equation (5.50). Thus one must subtract 1 from the right hand side of equations (5.62) and (5.63).

In the same wayΣ_F is an object where 2 dotted lines are taken from this diagram.

Now make the assumption stated at the start that $F \sim O(\frac{1}{g^2})$ and $\rho \sim O(1)$. Thus the power of a general diagram inΣ_ρ is (note the fact that the ρ doesn't contribute)

$$g^{2N_{clv}} g^{2N_{qv}} g^{-2N_F} g^{-N_\eta} g^{-N_\sigma} = g^{2(N_{clv}+N_{qv}-N_F)} g^{-N_\eta} g^{-N_\sigma} \ . \tag{5.64}$$

The above can be simplified by using equations (5.62) and (5.63).

To start one eliminates N_ρ from equations (5.62) and (5.63) which gives (note the minus 1 for each equation on the side of the vertex due to there being one less of each type of vertex).

$$\begin{aligned} 2N_F + N_\sigma - N_\eta + N_{clv} + 3N_{qv} - 1 &= 3N_{clv} + N_{qv} - 1 \\ 2N_{clv} - 2N_{qv} - 2N_F &= N_\sigma - N_\eta \\ 2(N_{clv} + N_{qv} - N_F) &= 4N_{qv} + N_\sigma - N_\eta \end{aligned} \tag{5.65}$$

This result can then be inserted into equation (5.64) such that one elimates N_{clv} and N_F in favour of N_{qv}, N_σ and N_η to obtain

$$g^{4N_{qv}+N_\sigma-N_\eta} g^{-N_\sigma} = g^{4N_{qv}-N_\eta} \ . \tag{5.66}$$

This shows that

$$\Sigma_\rho \sim g^{4N_{qv}-N_\eta} \ . \tag{5.67}$$

The same type of calculation, keeping in mind thatΣ_F as two dotted line amputated, gives that

$$\Sigma_F \sim O(g^{4N_{qv}-N_\eta-2}) \ . \tag{5.68}$$

Thus the number of classical-s vertices as well as the number of F don't affect the power counting of these diagrams.

This leads to a massive issue, diagrams with the same order of quantum-s vertices, no matter how many classical-s vertices, all have the same g scaling. They are all equally large and equally important which implies that infinitely many diagrams should be taken into consideration which, of course, isn't feasible. To be able reduce the diagrams to a finite number requires an further expansion parameter, such as $1/N$, to actually work.

Thus although the 2 P.I. method is capable of obtaining equilibrium as was shown in section 5.5.2 the above result implies that its description of equilibration behaviour is subject to very large uncertainties. This is because this method is throwing away valid information by truncating the self energy diagrams.

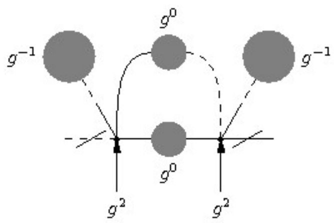
5.5.4 A different Set of Choices for F and ρ

If $F \sim O(g^{-2})$ and $\rho \sim O(1)$ then one has problems in terms of truncating. The size of these wasn't derived so maybe there is an error in the assumed sizes. One can ask what happens if $F \sim O(1)$, as this may help in the truncation. Or one could make $\rho \sim O(g^{-2})$. Problems arise in trying this too. This section will discuss what happens if both F and ρ are considered to be $O(1)$.

Again one can look to see if these choices for F and ρ can even lead to equilibration. This is done in the exact same manner as section 5.5.2.

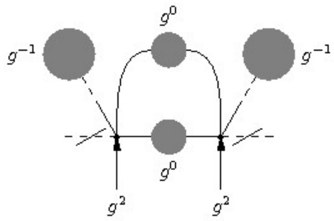
For the following remember that each coupling constant brings a factor g^2 and each field insert $\bar{\phi}$ brings $O(g^{-1})$ while each F and ρ don't contribute factors of g as it is being assumed they both $O(1)$. Each necessary piece of the self energy terms will be discussed.

The term$\Sigma_{\rho}(x, z)$. This was shown in equation (5.50) where a lowest order diagram from here is given by



$$\sim g^2 \ . \tag{5.69}$$

The term$\Sigma_{F}(x, z)$ has a lowest order diagram from here is given by



$$\sim g^2 \ . \tag{5.70}$$

The term $\frac{g^2}{2}F(x,x)$ that was in $M(x)$, $\frac{g^2}{2}F(x,x) \sim O(g^2)$.

Finally the term $\frac{g^2}{2}\bar{\phi}^2 \sim g^2(g^{-1})^2 \sim O(1)$

In equation (5.54), all the self energy terms are $O(g^2)$ due to $F \sim \rho \sim O(1)$. The evolution part of the equation is $O(1)$ i.e.

$$\underbrace{\left(\Box_x + m^2 + \frac{g^2}{2}\bar{\phi}^2\right) F(x,y)}_{O(1)} = O(g^2) \ . \tag{5.71}$$

Thus there is a problem in this evolution equation. The instabilities given by the evolution are $O(1)$ whilst the terms that should be cancelling the growth are very small compared as they $O(g^2)$ (keep in mind that g is very small by construction). This equation will not equilibrate.

In the same way for equation (5.55), all the self energy terms are $O(g^2)$. The evolution part of the equation is $O(1)$ i.e.

$$\underbrace{\left(\Box_x + m^2 + \frac{g^2}{2}\bar{\phi}^2\right) \rho(x,y)}_{O(1)} = O(g^2) \ . \tag{5.72}$$

Thus given a choice of F,ρ $\sim O(1)$, both objects would diverge in time due to the instabilities found in equation (4.56). One can show there is a similar problem with F,ρ $\sim O(g^{-2})$.

5.6 Analysis of the 2 P.I. Method

This section will briefly recap the successes and failures within the 2 P.I. method found in the previous sections.

5.6.1 Successes of the Method

In section 5.5.2 it was shown that the 2 P.I. method not only achieves equilibration, but actually is capable of producing the correct equilibration distribution of the Bose-Einstein distribution. According to [28], the reason the 2 P.I. method finds the correct equilibrium distribution is due to the fact that this method takes into account all possible effects that are included in the self energies, i.e. large classes of different types of diagrams are not excluded.

Remember that it was suggested in section 4.6.3 that the Classical-Statistical method may not achieve the correct equilibrium condition due to it not including any terms with the quantum-statistical vertex. The 2 P.I. method does contain these types of vertices, thus the fact that it can obtain the required quantum limit is consistent with the idea that both types of vertices are required to equilibrate properly.

5.6.2 Issues With The Method

To even allow one to be able to obtain equilibration in the presence of strong sources one requires that $F \sim O(g^{-2})$ and $\rho \sim O(1)$ other choices of these values lead to the S-D equations of motion for F and ρ experiencing instabilities exactly like those found in equation (4.56) of section 4.5.1. With these choices, one has an infinity of diagrams all at the same parametric order in g, in the presence of strong sources. An additional criterion is required to validly truncate this. Without this extra criterion one cannot trust the equilibration behaviour predicted by this method as it is subject to additional uncertainties, that in each case require further study to assess.

Conclusion

The goal of this thesis was to investigate and understand tools that are capable of describing equilibration phenomena in field theory. In doing this the inevitable happened, many conflicting and incomplete explanations were encountered in the literature. The most important of these concerns the nature and interpretation of the off diagonal elements of the Schwinger-Keldysh propagator matrix. During the discussion in section 3.1.6 it was shown that the off diagonal elements that are taken for granted in most of the literature, are actually a special case for an expectation value that contains a sum to all possible out states. The ground work for this interpretation was developed in [25]. The description in [25] provides the basis to conclude that in more general cases, such as out states to restricted parts of phase space, these off diagonal elements will be different to what is described in most of the literature. The discussion only took place for the free propagator matrix, an investigation into the effects of these observations on the full propagator would require further research efforts.

Within its core topic, this thesis has shown that both the classical-statistical method and the 2 P.I. method are capable of setting up equations of motion that can handle the situation of a dynamical system experiencing collective effects in the background of strong sources. Both methods are able to eventually show that equilibration occurs starting from a system far from equilibrium.

Both methods have advantages but both contain certain flaws. Both the successes and failures of the methods centre around the crucial result found in section 4.5.1. Here it was shown that equations of the form

$$\left(\Box + m^2 + \frac{g^2}{2}\varphi^2\right) a = 0 \tag{5.73}$$

have instabilities that grow with exponentially with time.

The classical-statistical method cleverly deals with this by isolating and then resumming these divergent terms such that the instabilities are removed as was shown in section 4.6. The issue is that the resummation scheme used only allows for diagrams with classical-statistical vertices and ignores all diagrams with quantum-statistical vertices, as was discussed in section 4.6.3. It is this lack of diagrams with quantum-statistical vertices that is proposed to lead to the C-S method only being capable of producing the leading order equilibration condition found in section 3.3.2 and missing the tails of the required Bose-Einstein distribution as was discussed in section 4.6.3.

In contrast the 2 P.I. method does successfully equilibrate to the full Bose-Einstein equilibrium distribution as was shown in section 5.6.1. It was suggested that it is able to reproduce this equilibration condition precisely because it contains diagrams with the quantum statistical vertices. However, this method requires an additional approximation to become practically viable as pointed out in section 5.5.3. This means the 2 P.I. method's predictions for the equilibration behaviour are subject to additional uncertainties affecting details of the equilibration process and quantities such as equilibration times.

As the flaws of the methods appear to be in some sense complimentary, a natural question to ask would be if both methods can be combined. One might imagine using the ability of the C-S method to successfully deal with the divergences due to equation 5.73 at the early times in evolution. Then once the C-S method has 'tamed' these divergences allow the 2 P.I. method to dominate the description, which could allow the system to reach the proper equilibrium condition.

It is not clear at this point if such a programme can be implemented consistently and requires a project for future research.

Appendix A

Introductory Concepts Appendix

A.1 Functional Derivatives

To define the functional derivative, the one-parameter groups of variation with compact support are required. For the functional derivative, one uses a distribution that picks a space-time point, i.e. the δ function. Define a functional as follows [12],

$$f_{i,s}(x) = f_i(x) + s_i \hat{e}_i \delta^{(4)}(x-y) \tag{A.1}$$

the i label allows for there to be components of f. The components are varied independently, so read the summation convention in the second term.

Then the functional derivative $\frac{\delta}{\delta f_j(y)}$ is defined as the differentiation with respect to s at $s=0$ i.e.

$$\frac{\delta}{\delta f_j(y)} f_i(x) \equiv \frac{d}{ds}\Big|_{s=0} f_{i,s}(x) = \delta_{ij}\delta^{(4)}(x-y) \; . \tag{A.2}$$

From the above definition, one can see that all the usual rules of differentiation hold.

A.2 Gaussian Matrix Integral

The following derivation is taken using [9]

One would like to be able to solve integrals of the form,

$$\int D[\phi] e^{-\frac{1}{2}\int dx dx' \phi(x')\left(-G_0^{-1}\right)(x',x)\phi(x)} \tag{A.3}$$

where the conventions of section 1.1 has been used for G_0^{-1}. This convention leads to the use of the double negative sign.

Start with the Gaussian integral that is well known i.e.

$$\int dx e^{-\frac{1}{2}ax^2} = \sqrt{2\pi}\frac{1}{\sqrt{a}} \ . \tag{A.4}$$

One can now generalize the above integral to n dimensions,

$$\int dx_1 ... dx_n e^{-\frac{1}{2}\boldsymbol{X}^T\boldsymbol{A}\boldsymbol{X}} \tag{A.5}$$

where $\boldsymbol{X}$ is an n-dimensional vector and $\boldsymbol{A}$ is a $n \times n$ symmetric, non-singular matrix. The solution to this integral is found by considering diagonalizing $\boldsymbol{A}$. Since $\boldsymbol{A}$ is symmetric, the diagonalizing matrices are orthogonal. Let the diagonalizing matrix be called $\boldsymbol{P}$.

Now consider changing the vector $\boldsymbol{X}$ by $\boldsymbol{P}$ i.e.

$$\boldsymbol{Y} = \boldsymbol{P}\boldsymbol{X} \ . \tag{A.6}$$

This implies that

$$dy_1....dy_n = \det[\boldsymbol{P}]dx_1...dx_n \tag{A.7}$$

where, since $\boldsymbol{P}$ is orthogonal, $\det[\boldsymbol{P}] = 1$

Then putting this together

$$\int dy_1...dy_n e^{-\frac{1}{2}\boldsymbol{Y}^T\boldsymbol{A}\boldsymbol{P}\boldsymbol{Y}} = \int dx_1...dx_n e^{-\frac{1}{2}\boldsymbol{X}^T\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{X}} \ . \tag{A.8}$$

This identifies the integral that is being investigated as an integral with a diagonal matrix in the exponential. The fact there is now a diagonal matrix can be exploited to write the exponential of the matrices as a sum of scalar terms.

$$\begin{aligned}
\int dy_1...dy_n e^{-\frac{1}{2}\boldsymbol{Y}^T\boldsymbol{A}\boldsymbol{P}\boldsymbol{Y}} &= \int dx_1...dx_n e^{-\frac{1}{2}\boldsymbol{X}^T\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{X}} \\
&= \int dx_1...dx_n e^{-\frac{1}{2}a_1x_1^2+...+a_nx_n^2} \\
&\text{Using equation (A.4)} \\
&= (2\pi)^{n/2}\left(\frac{1}{\sqrt{a_1}}...\frac{1}{\sqrt{a_n}}\right) \\
&\text{Using the determintent of a diagonal matrix} \\
&= (2\pi)^{n/2}\left(\det[A]\right)^{-1/2}
\end{aligned} \tag{A.9}$$

Functionals of one or two coordinates can be thought of as infinite dimensional vectors and matrices respectively. So the above result can be extended to infinite dimensional matrices, i.e. to functionals. Then using the path integral as defined in section 1.2,

$$\int D[\phi] e^{-\frac{1}{2}\int dx dx' \phi(x')\left(-G_0^{-1}\right)\phi(x)} = N\left(\det[G_0^{-1}]\right)^{-1/2} \tag{A.10}$$

where N is a normalization constant and using $\det[-G_0^{-1}] = C\det[G_0^{-1}]$.

One can generalize the integral of equation (A.10) to a Gaussian with a linear term. This has the following form:

$$\int D[\phi] e^{-\frac{1}{2}\int dx dx' \phi(x')\left(-G_0^{-1}\right)(x',x)\phi(x) + \int dx iJ(x)\phi(x)} . \tag{A.11}$$

For the sake of simplicity define objects, $-G_0^{-1} \equiv \bar{G}_0^{-1}$ and let $iJ \equiv \bar{J}$

Thus one looks at,

$$\int D[\phi] e^{-\frac{1}{2}\int dx dx' \phi(x')\bar{G}_0^{-1}(x',x)\phi(x) + \int dx \bar{J}(x)\phi(x)} . \tag{A.12}$$

The solution of this is started by completing the square where,

$$\begin{aligned}
&-\frac{1}{2}\int dx dx' \phi(x')\bar{G}_0^{-1}(x',x)\phi(x) + \int dx \bar{J}(x)\phi(x) = \\
&-\frac{1}{2}\int dx dx' \left[\phi(x') - \int dy G_0(x',y)\bar{J}(y)\right] \bar{G}_0^{-1}(x',x) \left[\phi(x) - \int dy' G_0(x,y')\bar{J}(y')\right] \\
&+\frac{1}{2}\int dx dx' \bar{J}(x')\bar{G}_0(x',x)\bar{J}(x) .
\end{aligned} \tag{A.13}$$

This can be checked by expanding the above out.

Then one can consider a change in integration variable such that the objects in the square brackets are re-defined:

$$\phi'(x) = \phi(x) - \int dy' \bar{G}_0(x,y')\bar{J}(y') . \tag{A.14}$$

This amounts to performing a constant shift in the integration variable of equation (A.11). Using this constant shift as re-defining the integration variable along with the completed square gives

$$\int D[\phi]e^{-\frac{1}{2}\int dxdx'\phi(x')\bar{G}_0^{-1}(x',x)\phi(x)+\int dx\bar{J}(x)\phi(x)} = e^{\frac{1}{2}\int dxdx'\bar{J}(x')\bar{G}_0(x',x)\bar{J}(x)}$$
$$\times\int D[\phi']e^{-\frac{1}{2}\int dxdx'\phi'(x')\bar{G}_0^{-1}(x',x)\phi'(x)} \ . \quad (A.15)$$

Then, using equation (A.10)

$$\int D[\phi]e^{-\frac{1}{2}\int dxdx'\phi(x')\bar{G}_0^{-1}(x',x)\phi(x)+\int dx\bar{J}(x)\phi(x)} = e^{\frac{1}{2}\int dxdx'\bar{J}(x')\bar{G}_0(x',x)\bar{J}(x)}N\left(\det[\bar{G}_0^{-1}]\right)^{-1/2} \ . \quad (A.16)$$

Since the term $N\left(\det[\bar{G}_0^{-1}]\right)^{-1/2}$ is just a number, one can consider this a normalization constant [17],

$$\int D[\phi]e^{-\frac{1}{2}\int dxdx'\phi(x')\bar{G}_0^{-1}(x',x)\phi(x)+\int dx\bar{J}(x)\phi(x)} = Ne^{\frac{1}{2}\int dxdx'\bar{J}(x')\bar{G}_0(x',x)\bar{J}(x)} \ . \quad (A.17)$$

Then, using the original terms, without bars,

$$\int D[\phi]e^{-\frac{1}{2}\int dxdx'\phi(x')\bar{G}_0^{-1}(x',x)\phi(x)+\int dx\bar{J}(x)\phi(x)} = Ne^{\frac{1}{2}\int dxdx'iJ(x')\left(-G_0(x',x)\right)iJ(x)}$$
$$= Ne^{\frac{1}{2}\int dxdx'J(x')G_0(x',x)J(x)} \ . \quad (A.18)$$

which identifies

$$\int D[\phi]e^{-\frac{1}{2}\int dxdx'\phi(x')\left(-G_0^{-1}(x',x)\right)\phi(x)+\int dxiJ(x)\phi(x)} = Ne^{\frac{1}{2}\int dxdx'J(x')G_0(x',x)J(x)} \quad (A.19)$$

A.3 2^{nd} 3^{rd} and 4^{th} Derivative of $Z_0[J]$

This appendix continues the calculation of $\left.\frac{\delta^4 Z_0}{\delta J(w)J(z)J(y)J(x)}\right|_{x=y=z=w}$ started in section 1.6. In that section, the first derivative was done, the result of which was shown to be

$$\frac{\delta}{\delta J(x)}Z_0[J] = \left[-i\int d^4x' G_0(x-x')J(x')\right]Z_0[J] \ . \quad (A.20)$$

The second derivative is as follows,

$$\begin{aligned}
\frac{\delta}{\delta J(y)J(x)}Z_0\Big|_{x=y} &= \frac{\delta}{\delta J(y)}\left(\frac{\delta}{\delta J(x)}Z_0[J]\right)\Big|_{x=y} \\
&= \frac{\delta}{\delta J(y)}\left(\int d^4x' G_0(x-x')J(x')Z_0\right)\Big|_{x=y} \\
&\quad \text{Using product rule} \\
&= \int d^4x' G_0(x-x')\delta(y-x')Z_0 + \int d^4x' G_0(x-x')J(x')\frac{\delta Z_0}{\delta J(y)}\Big|_{x=y} \\
&= G_0(x-y)Z_0 \\
&\quad +\left(\int d^4x' G_0(x-x')J(x')\right)\left(\int d^4y' G_0(y-y')J(y')\right)Z_0\Big|_{x=y} \\
&= \left(G_0(x-y) + \int d^4x' d^4y' G_0(x-x')J(x')G_0(y-y')J(y')\right)Z_0\Big|_{x=y} \\
&= \left(G_0(x-x) + \int d^4x' d^4y' G_0(x-x')J(x')G_0(x-y')J(y')\right)Z_0\ . \qquad \text{(A.21)}
\end{aligned}$$

To reduce this cumbersome calculation, new notation is introduced. Define $\int d^4x' G_0(x-x')J(x') \equiv (G_0 \otimes J)_{x'}(x)$. From the calculation to get to (A.21) from (A.20), it is seen that varying $(G_0 \otimes J)_{x'}(x)$ w.r.t $J(y)$ results in replacing $J_{x'}$ with a y. Whilst varying Z_0 w.r.t $J(y)$, in this notation, gives $(G_0 \otimes J)_{y'}(y)$.

Using this the next two derivatives can be found:

$$\begin{aligned}
\frac{\delta}{\delta J(z)J(y)J(x)}Z_0\Big|_{x=y=z} &= \frac{\delta}{\delta J(z)}\left(\frac{\delta}{\delta J(y)J(x)}Z_0[J]\right)\Big|_{x=y=z} \\
&= \frac{\delta}{\delta J(z)}\left\{\left[G_0(x-y) + (G_0 \otimes J)_{x'}(x)(G_0 \otimes J)_{y'}(y)\right]Z_0\right\}\Big|_{x=y=z} \\
&= \left[G_0(x-z)(G_0 \otimes J)_{y'}(y) + (G_0 \otimes J)_{x'}(x)G_0(y-z)\right]Z_0 + \\
&\quad \left[G_0(x-x) + (G_0 \otimes J)_{x'}(x)(G_0 \otimes J)_{y'}(y)\right]\left[(G_0 \otimes J)_{z'}(z)Z_0\right]\Big|_{x=y=z} \\
&= \Big[G_0(x-z)(G_0 \otimes J)_{y'}(y) + (G_0 \otimes J)_{x'}(x)G_0(y-z) \\
&\quad +G_0(x-x)(G_0 \otimes J)_{z'}(z) + (G_0 \otimes J)_{x'}(x)(G_0 \otimes J)_{y'}(y)(G_0 \otimes J)_{z'}(z)\Big]Z_0\Big|_{x=y=z} \\
&\quad \text{Letting } y \text{ and } z \text{ be } x \\
&= \Big[G_0(x-x)\left\{(G_0 \otimes J)_{y'}(x) + (G_0 \otimes J)_{x'}(x) + (G_0 \otimes J)_{z'}(z)\right\} - \\
&\quad (G_0 \otimes J)_{x'}(x)(G_0 \otimes J)_{y'}(x)i(G_0 \otimes J)_{z'}(x)\Big]Z_0 \qquad \text{(A.22)}
\end{aligned}$$

and then

$$
\begin{aligned}
\frac{\delta}{\delta J(w)J(z)J(y)J(x)}Z_0\Big|_{x=y=z=w} \quad &= \quad \frac{\delta}{\delta J(w)}\left(\frac{\delta}{\delta J(z)J(y)J(x)}Z_0[J]\right)\Big|_{x=y=z=w} \\
&\quad\quad \text{Let } w,\ y \text{ and } z \text{ be } x \text{ immediately} \\
&= \quad \Big[(G_0(x-x))^2 + G_0(x-x)\Big\{(G_0\otimes J)_{y'}(x)\,(G_0\otimes J)_{z'}(x) \\
&\quad\quad + (G_0\otimes J)_{x'}(x)\,(G_0\otimes J)_{z'}(x) + (G_0\otimes J)_{x'}(x)\,(G_0\otimes J)_{y'}(x)\Big\}\Big]Z_0[J] \\
&\quad\quad + \Big[G_0(x-x)\Big\{(G_0\otimes J)_{y'}(x) + (G_0\otimes J)_{x'}(x) + (G_0\otimes J)_{z'}(z)\Big\} \\
&\quad\quad + (G_0\otimes J)_{x'}(x)\,(G_0\otimes J)_{y'}(x)\,(G_0\otimes J)_{z'}(x)\Big]\left[(G_0\otimes J)_{w'}(x)Z_0\right] \ .
\end{aligned} \tag{A.23}
$$

Re-arranging the above gives [12], [13];

$$
\begin{aligned}
\frac{\delta}{\delta J(w)J(z)J(y)J(x)}Z_0\Big|_{x=y=z=w} \quad &= \quad \Big[3(G_0(x-x))^2 + 6G_0(x-x)\,(G_0\otimes J)_{y'}(x)\,(G_0\otimes J)_{z'}(x) + \\
&\quad\quad (G_0\otimes J)_{x'}(x)\,(G_0\otimes J)_{y'}(x)\,(G_0\otimes J)_{z'}(x)\,(G_0\otimes J)_{w'}(x)\Big]Z_0 \ .
\end{aligned} \tag{A.24}
$$

Appendix B

One and Two P.I. Action Appendix

B.1 Power Counting

If an expansion of a function in powers of a certain variable exists, the lowest power of that expansion can be easily found. Given a function $f(x)$ that can be expanded in powers of $\hbar$, it can be written that $f(x,\hbar) = c_0 + c_1\hbar + c_2\hbar^2 + ...$

Then the way to find the lowest power in the expansion of $f(x,\hbar)$ in terms of $\hbar$ is to do as follows. First one sets $\hbar = 0$ then $f(x,\hbar = 0) = c_0$. Thus if $f(x,\hbar = 0) = 0$ then the lowest power is higher that 0. Next check $\frac{\partial}{\partial\hbar}f(x,\hbar)\big|_{\hbar=0}$ since $\frac{\partial}{\partial\hbar}f(x,\hbar)\big|_{\hbar=0} = c_1$ if c_0 is 0. If this term is non-zero, the lowest power in $\hbar$ is 1, if its zero then its higher order in $\hbar$. One keeps taking the derivative and seeing if one gets 0 until the first non-zero value. The number of derivatives taken corresponds to the highest power in $\hbar$ in the expansion of $f(x,\hbar)$.

Thus $F[\phi,\hbar] = \ln\left[\frac{\int D[\phi]e^{g[\phi]}e^{A(\hbar)h[\phi]}}{\int D[\phi]e^{g[\phi]}}\right]$. Where $A(\hbar) = \hbar + \hbar^2 + ...$ This has an expansion in $\hbar$.

$F[\phi,\hbar = 0] = \ln\left[\frac{\int D[\phi]e^{g[\phi]}}{\int D[\phi]e^{g[\phi]}}\right] = \ln[1] = 0$. Thus the lowest power is higher than order 0 in $\hbar$.

$\frac{\partial}{\partial\hbar}F[\phi,\hbar]\big|_{\hbar=0} = \ln\left[\frac{\int D[\phi]e^{g[\phi]}(1+O(\hbar))h[\phi]e^{A(\hbar)h[\phi]}}{\int D[\phi]e^{g[\phi]}}\right]\Big|_{\hbar=0} = \ln\left[\frac{\int D[\phi]e^{g[\phi]}h[\phi]}{\int D[\phi]e^{g[\phi]}}\right] \neq 0$

Thus $F[\phi,\hbar]$ has lowest power $\hbar$ in an $\hbar$ expansion.

B.2 Eliminating R from $\Gamma[\bar{\phi}, R]$

This calculation continues from the point in section 2.5.2 where equation (2.56) was inserted into (2.54). Then using the identity $\mathrm{Tr}\ln[A] = \ln\det[A]$ leaves one with,

$$\Gamma[\bar{\phi}, G] \quad = \quad S[\bar{\phi}] + \frac{i}{2}\hbar Tr\ln\left[G^{-1} + \Sigma^R\right] - \frac{1}{2}Tr\left(iG^{-1}G + iG_t^{-1}G + i\Sigma^R G\right) + O(\hbar^2) \tag{B.1}$$

To continue one can note that $\mathrm{Tr}GG^{-1} = \mathrm{Tr}\mathbb{1}$. Then try to isolate the term $\mathrm{Tr}G^{-1}$, thus

$$
\begin{aligned}
\Gamma[\bar{\phi}, G] &= S[\bar{\phi}] + \frac{i}{2}\hbar Tr \ln\left[G^{-1}\left(1 + \frac{\Sigma^R}{G^{-1}}\right)\right] - \frac{i}{2}Tr\mathbf{1} - \frac{i}{2}Tr\left(\Sigma^R G + G_t^{-1}G\right) + O(\hbar^2) \\
&\quad \text{using } \ln[ab] = \ln[a] + \ln[b] \text{ and } \frac{1}{G^{-1}} = G \\
&= S[\bar{\phi}] + \frac{i}{2}\hbar Tr \ln\left[G^{-1}\right] + \frac{i}{2}Tr \ln\left[1 + \Sigma^R G\right] - \frac{i}{2}Tr\mathbf{1} - \frac{i}{2}Tr\left(\Sigma^R G + G_t^{-1}G\right) + O(\hbar^2) \\
&\quad \text{using } \ln[1+a] = a - \frac{1}{2}a^2 + \frac{1}{3}a^3 + \ldots \text{ when expanding around } 0 \\
&= S[\bar{\phi}] + \frac{i}{2}\hbar Tr \ln\left[G^{-1}\right] + \frac{i}{2}Tr\left[\Sigma^R G + \frac{1}{2}\Sigma^R G \Sigma^R G + \ldots\right] \\
&\qquad -\frac{i}{2}Tr\mathbf{1} - \frac{i}{2}Tr\left(\Sigma^R G\right) - \frac{i}{2}Tr\left(G_t^{-1}G\right) + O(\hbar^2)\ .
\end{aligned}
\tag{B.2}
$$

Looking at the first term in the trace , this can be seen to cancel off with the term $-\frac{i}{2}Tr\left(\Sigma^R G\right)$. Then what is left in the trace is a sequence of$\Sigma\ ^R G$ terms. Since$\Sigma\ ^R$ is the self energy it has at least one loop in it which means it must be at least $O(\hbar)$. Thus this sequence, which starts with two terms of Σ^R must be at least $O(\hbar^2)$. So [8],

$$
\Gamma[\bar{\phi}, G] = S[\bar{\phi}] + \frac{i}{2}\hbar Tr \ln\left[G^{-1}\right] - \frac{i}{2}Tr\mathbf{1} - \frac{i}{2}Tr\left(G_t^{-1}G\right) + O(\hbar^2)\ . \tag{B.3}
$$

Appendix C

Schwinger-Keldysh Appendix

C.1 Determining the Last Three Entries of $R\boldsymbol{G}R^T$

In this appendix the calculation of the entries of the transformed matrix RGR^T (section 3.2.1) in terms of G^R and G^A is continued. As a reminder, from equation (3.80)

$$R\boldsymbol{G}R^T = \begin{pmatrix} \frac{1}{4}(G^{++} + G^{-+} + G^{+-} + G^{--}) & \frac{1}{2}(G^{++} - G^{+-} + G^{-+} - G^{--}) \\ \frac{1}{2}(G^{++} + G^{+-} - G^{-+} - G^{--}) & (G^{++} - G^{+-} - G^{-+} + G^{--}) \end{pmatrix} . \tag{C.1}$$

Now to try find the remaining three entries,

$\left(R\boldsymbol{G}R^T\right)^{1,2}$: $\frac{1}{2}(G^{++} - G^{+-} + G^{-+} - G^{--})$

$$\begin{aligned} \frac{1}{2}(G^{++} - G^{+-} + G^{-+} - G^{--}) &= \frac{1}{2}\Big(\left[\Theta(x^0 - y^0) + 1 - \Theta(y^0 - x^0)\right] G^{-+}(x,y) \\ &\quad \left[\Theta(y^0 - x^0) - 1 - \Theta(x^0 - y^0)\right] G^{+-}(x,y)\Big) \\ &\quad \text{Using}\Theta(\ y^0 - x^0) + \Theta\ (x^0 - y^0) = 1 \\ &= \frac{1}{2}\Big(2\theta(x^0 - y^0)G^{-+}(x,y) - 2\theta(x^0 - y^0)G^{+-}(x,y)\Big) \\ &= \theta(x^0 - y^0)(G^{-+}(x,y) - G^{+-}(x,y)) \end{aligned} \tag{C.2}$$

Using the definition of $G^{-+}(x,y)$ and $G^{+-}(x,y)$ from equations (3.13) and (3.14)

$$\left(R\boldsymbol{G}R^T\right)^{1,2} = \theta(x^0 - y^0)\left(\langle\phi(x)\phi(y)\rangle - \langle\phi(y)\phi(x)\rangle\right) . \tag{C.3}$$

This is almost the definition of the retarded propagator $G^R(x,y)$ given in 3.22. To identify it as G^R there is a missing factor of i, thus [23]

$$\left(R\boldsymbol{G}R^T\right)^{1,2} = -iG^R(x,y) . \tag{C.4}$$

$\left(R\boldsymbol{G}R^T\right)^{2,1}$: $\frac{1}{2}(G^{++} + G^{+-} - G^{-+} - G^{--})$

This calculation is very much the same as the calculation for $\left(R\boldsymbol{G}R^T\right)^{1,2}$. Using the same ideas it is found that [23]

$$\left(R\boldsymbol{G}R^T\right)^{2,1} = -\theta(y^0 - x^0)\left(\langle\phi(x)\phi(y)\rangle - \langle\phi(y)\phi(x)\rangle\right) \ . \tag{C.5}$$

This is the definition of the advanced propagator $G^A x, y$ (given in equation (3.23)) with a missing i

Thus

$$\left(R\boldsymbol{G}R^T\right)^{2,1} = -iG^A(x,y) \ . \tag{C.6}$$

Finally

$\left(R\boldsymbol{G}R^T\right)^{2,2}$: $(G^{++} - G^{+-} - G^{-+} + G^{--})$

This calculation is very similar to that of $\left(R\boldsymbol{G}R^T\right)^{1,1}$ except that instead of adding a 1 we subtract a 1. i.e.

$$\begin{aligned}\left(R\boldsymbol{G}R^T\right)^{2,2} &= \frac{1}{4}\left((\underbrace{\theta(x^0 - y^0) + \theta(y^0 - x^0)}_{=1} - 1)(G^{-+}(x,y) + G^{+-}(x,y))\right) \\ &= 0 \end{aligned}\tag{C.7}$$

thus [23]

$$\left(R\boldsymbol{G}R^T\right)^{2,2} = 0 \ . \tag{C.8}$$

C.2 Matrix Derivative of $\mathcal{D}[J_+, J_-]$ Acting on $Z^0_{-+}[J_+, J_-]$

The object that needs to be worked out is given by

$$\int d^4x d^4y \left[(\delta[\boldsymbol{J}])^T(x)\boldsymbol{G}^0_{OD}(x,y)\left(\delta[\boldsymbol{J}]\right)(y)\right] e^{\frac{1}{2}\int d^4x' \int d^4y' \boldsymbol{J}^T(x')\boldsymbol{G}^0_D(x',y')\boldsymbol{J}(y')} \tag{C.9}$$

where,

$$(\delta[\boldsymbol{J}]) = \begin{pmatrix} \frac{\delta}{\delta J_+} \\ \frac{\delta}{\delta J_-} \end{pmatrix} \tag{C.10}$$

$$\boldsymbol{G}^0_{OD}(x,y) = \begin{pmatrix} 0 & (\overleftarrow{\Box}_x + m^2)G^0_{+-}(x,y)(\overrightarrow{\Box}_y + m^2) \\ (\overleftarrow{\Box}_x + m^2)G^0_{-+}(x,y)(\overrightarrow{\Box}_y + m^2) & 0 \end{pmatrix} \tag{C.11}$$

and $\boldsymbol{J}$ and $\boldsymbol{G}^0_D$ are given in equations (3.55) and (3.56).

The proof is started by re-writing the above expression in terms of the summation convention and labelled vector and matrix entries. Then the differentiation can be carried out, where $a, b = +, -$. Thus equation (C.9) becomes,

$$\begin{aligned}
&\int d^4x d^4y \left[(\delta[\boldsymbol{J}])^a (x) \left(\boldsymbol{G}^0_{OD}\right)^{ab} (x,y) (\delta[\boldsymbol{J}])^b (y) \right] e^{\frac{1}{2}\int d^4x' \int d^4y' \boldsymbol{J}^{a'}(x') \left(\boldsymbol{G}^0_D\right)^{a'b'}(x',y') \boldsymbol{J}^{b'}(y')} \\
&= \int d^4x d^4y \left[(\delta[\boldsymbol{J}])^a (x) \left(\boldsymbol{G}^0_{OD}\right)^{ab} (x,y) \right] \left(\frac{1}{2} \int d^4x' \int d^4y' \left\{ \delta_{a',b} \delta(x'-y) \left(\boldsymbol{G}^0_D\right)^{a'b'} (x',y') \boldsymbol{J}^{b'}(y') \right.\right. \\
&\qquad \left.\left. + \boldsymbol{J}^{a'}(x') \left(\boldsymbol{G}^0_D\right)^{a'b'} (x',y') \delta_{b,b'} \delta(y-y') \right\} \right) e^{\cdots} .
\end{aligned} \tag{C.12}$$

Then by integrating over the Dirac delta's and using the Kronecker deltas to get that

$$\begin{aligned}
&\int d^4x d^4y \left[(\delta[\boldsymbol{J}])^a (x) \left(\boldsymbol{G}^0_{OD}\right)^{ab} (x,y) (\delta[\boldsymbol{J}])^b (y) \right] e^{\frac{1}{2}\int d^4x' \int d^4y' \boldsymbol{J}^{a'}(x') \left(\boldsymbol{G}^0_D\right)^{a'b'}(x',y') \boldsymbol{J}^{b'}(y')} \\
&= \int d^4x d^4y \left[(\delta[\boldsymbol{J}])^a (x) \left(\boldsymbol{G}^0_{OD}\right)^{ab} (x,y) \right] \left(\frac{1}{2} \left\{ \int d^4y' \left(\boldsymbol{G}^0_D\right)^{bb'} (y,y') \boldsymbol{J}^{b'}(y') \right.\right. \\
&\qquad \left.\left. + \int d^4x' \boldsymbol{J}^{a'}(x') \left(\boldsymbol{G}^0_D\right)^{a'b} (x',y) \right\} \right) e^{\cdots} \\
&= \int d^4x d^4y \left(\boldsymbol{G}^0_{OD}\right)^{ab} (x,y) \left[\left(\frac{1}{2} \left\{ \int d^4y' \left(\boldsymbol{G}^0_D\right)^{bb'} (y,y') \delta_{a,b'} \delta(x-y') \right.\right.\right. \\
&\qquad \left.\left. + \int d^4x' \delta_{a,a'} \delta(x'-x) \left(\boldsymbol{G}^0_D\right)^{a'b} (x',y) \right\} \right) + \left(\frac{1}{2} \left\{ \int d^4y' \left(\boldsymbol{G}^0_D\right)^{bb'} (y,y') \boldsymbol{J}^{b'}(y') \right.\right. \\
&\qquad \left.\left. + \int d^4x' \boldsymbol{J}^{a'}(x') \left(\boldsymbol{G}^0_D\right)^{a'b} (x',y) \right\} \right) \left(\frac{1}{2} \left\{ \int d^4y'' \left(\boldsymbol{G}^0_D\right)^{ab''} (x,y'') \boldsymbol{J}^{b''}(y'') \right.\right. \\
&\qquad \left.\left.\left. + \int d^4x'' \boldsymbol{J}^{a''}(x'') \left(\boldsymbol{G}^0_D\right)^{a''a} (x'',x) \right\} \right) \right] e^{(\cdots)} .
\end{aligned} \tag{C.13}$$

To clean this up further note that $\boldsymbol{G}^0_D = \left(\boldsymbol{G}^0_D\right)^T$. This means that the order of the entry labels can be switched on this object. One can also redefine integration variables and use the fact that objects here are scalar, so everything commutes. Then using the delta functions and rearranging

$$
\begin{aligned}
&\int d^4x d^4y \left[(\delta[\boldsymbol{J}])^a (x) \left(\boldsymbol{G}^0_{OD}\right)^{ab} (x,y) (\delta[\boldsymbol{J}])^b (y) \right] e^{\frac{1}{2}\int d^4x' \int d^4y' \boldsymbol{J}^{a'}(x') \left(\boldsymbol{G}^0_D\right)^{a'b'}(x',y') \boldsymbol{J}^{b'}(y')} \\
&= \int d^4x d^4y \left(\boldsymbol{G}^0_{OD}\right)^{ab} (x,y) \left[\left(\frac{1}{2} \left\{ \left(\boldsymbol{G}^0_D\right)^{ba} (y,x) + \left(\boldsymbol{G}^0_D\right)^{ab} (x,y) \right\} \right) \right. \\
&\quad \left. + \left(\int d^4y' \int d^4y'' \boldsymbol{J}^{b'}(y') \left(\boldsymbol{G}^0_D\right)^{b'b} (y,y') \left(\boldsymbol{G}^0_D\right)^{ab''} (x,y'') \boldsymbol{J}^{b''}(y'') \right) \right] e^{(\dots)} \\
&\quad \text{Reabeling integration variables and switching labels} \\
&= \int d^4x d^4y \left(\boldsymbol{G}^0_{OD}\right)^{ab} (x,y) \left(\boldsymbol{G}^0_D\right)^{ba} (x,y) e^{(\dots)} \\
&\quad + \int d^4x d^4y d^4y' d^4y'' \boldsymbol{J}^{b''}(y'') \left(\boldsymbol{G}^0_D\right)^{b''a} (y'',x) \left(\boldsymbol{G}^0_{OD}\right)^{ab} (x,y) \left(\boldsymbol{G}^0_D\right)^{b'b} (y,y') \boldsymbol{J}^{b'}(y') e^{(\dots)} \; .
\end{aligned}
\tag{C.14}
$$

.

The term $\left(\boldsymbol{G}^0_{OD}\right)^{ab} (x,y) \left(\boldsymbol{G}^0_D\right)^{ba} (x,y)$ is 0 (See equations (3.56) and (3.59)).

The second term is found by using equations (3.56) and (3.59) with the fact that $G^0_{--}(x,y) \left(\Box_x + m^2\right) = \delta(x-y)$ and $G^0_{++}(x,y) \left(\Box_x + m^2\right) = \delta(x-y)$. Using this gives

$$
\begin{aligned}
&\int d^4x d^4y d^4y' d^4y'' \left(\boldsymbol{G}^0_D\right)^{b''a} (y'',x) \left(\boldsymbol{G}^0_{OD}\right)^{ab} (x,y) \left(\boldsymbol{G}^0_D\right)^{b'b} (y,y') \\
&= \int d^4x d^4y d^4y' d^4y'' \begin{pmatrix} 0 & \delta(y'',x) G^0_{+-}(x,y) \delta(y,y') \\ \delta(y'',x) G^0_{-+}(x,y) \delta(y,y') & 0 \end{pmatrix} \\
&\quad \text{Integrating over the } \delta \\
&= \int d^4x d^4y \begin{pmatrix} 0 & G^0_{+-}(x,y) \\ G^0_{-+}(x,y) & 0 \end{pmatrix} \; .
\end{aligned}
\tag{C.15}
$$

Inserting the above into equation (C.14) gives

$$
\begin{aligned}
&\int d^4x d^4y \left[(\delta[\boldsymbol{J}])^a (x) \left(\boldsymbol{G}^0_{OD}\right)^{ab} (x,y) (\delta[\boldsymbol{J}])^b (y) \right] e^{\frac{1}{2}\int d^4x' \int d^4y' \boldsymbol{J}^{a'}(x') \left(\boldsymbol{G}^0_D\right)^{a'b'}(x',y') \boldsymbol{J}^{b'}(y')} \\
&= \int d^4x d^4y \left(\boldsymbol{J}\right)^T (x) \begin{pmatrix} 0 & G^0_{+-}(x,y) \\ G^0_{-+}(x,y) & 0 \end{pmatrix} \mathbf{J}(y) e^{\frac{1}{2}\int d^4x' \int d^4y' \boldsymbol{J}^{a'}(x') \left(\boldsymbol{G}^0_D\right)^{a'b'}(x',y') \boldsymbol{J}^{b'}(y')} \; .
\end{aligned}
\tag{C.16}
$$

This is the result required.

C.3 Deriving the KMS Relationship

The following argument is made following [27].

Here one would like to derive the KMS relationship given by equations (3.129) and (3.130) (section 3.3.1).

The first thing to introduce is the concept of $e^{\beta \hat{H}}$ as an operator that evolves states in imaginary time

$e^{\beta\hat{H}}$ Imaginary Time Evolution

In QM the object

$$\left\langle \boldsymbol{x}'|e^{-i\hat{H}(t'-t)}|\boldsymbol{x}\right\rangle \tag{C.17}$$

is interpreted as the amplitude of finding a particle in position $\boldsymbol{x}$ at time t' after knowing it was at $\boldsymbol{x}$ at time t. This gives the interpretation of $e^{-i\hat{H}(t'-t)}$ as a time evolution operator.

In statistical physics it is know that the partition function is given by

$$\begin{aligned} Z[\beta] &= \mathrm{Tr}\left[e^{-\beta\hat{H}}\right] \\ &= \int d\boldsymbol{x} \left\langle \boldsymbol{x}|e^{-\beta\hat{H}}|\boldsymbol{x}\right\rangle \ . \end{aligned} \tag{C.18}$$

By identifying the integrand of the above with equation (C.17) a statement can be made about $e^{-\beta\hat{H}}$. Consider the integrand as an amplitude. It describes an amplitude that starts at state $|\boldsymbol{x}\rangle$ and ends at $\langle\boldsymbol{x}|$. This evolved to the new state through the use of $e^{-\beta\hat{H}}$.

It has been stated that one interprets $e^{-i\hat{H}(t'-t)}$ as an evolution in time. If time was imaginary i.e. $t \to i\tau$ and $t' \to i\tau'$, then $e^{-i\hat{H}(t'-t)} \to e^{-\hat{H}(\tau-\tau')}$. This looks the same as $e^{-\beta\hat{H}}$ where $\beta = \tau - \tau'$. This leads to the interpretation that $e^{-\beta\hat{H}}$ is an evolution operator in imaginary time.

KMS Relation

Since it has been shown that $e^{\beta\hat{H}}$ is an evolution operator in imaginary time.

$$e^{-\beta\hat{H}}\hat{\phi}(t)e^{\beta\hat{H}} = \hat{\phi}(t+i\beta) \ . \tag{C.19}$$

Then recall from equations (3.127) and (3.128) that

$$G_\beta^{-+}(t,t') = \langle\hat{\phi}(t)\hat{\phi}(t')\rangle_\beta \tag{C.20}$$

$$G_\beta^{+-}(t,t') = \langle\hat{\phi}(t')\hat{\phi}(t)\rangle_\beta = G_\beta^{-+}(t',t) \ . \tag{C.21}$$

Putting the above two facts together, [27]

$$
\begin{aligned}
G_\beta^{-+}(t,t') &= \langle \hat{\phi}(t)\hat{\phi}(t')\rangle_\beta \\
&= \mathrm{Tr}\left[e^{-\beta \hat{H}}\hat{\phi}(t)\hat{\phi}(t')\right] \\
&\text{Using cyclicity of the trace} \\
&= \mathrm{Tr}\left[\hat{\phi}(t')e^{-\beta \hat{H}}\hat{\phi}(t)\right] \\
&\text{Applying } \hat{\mathbb{1}} = e^{\beta \hat{H}}e^{-\beta \hat{H}} \\
&= \mathrm{Tr}\left[\hat{\phi}(t')e^{-\beta \hat{H}}\hat{\phi}(t)e^{\beta \hat{H}}e^{-\beta \hat{H}}\right] \\
&\text{Using equation (C.19)} \\
&= \mathrm{Tr}\left[\hat{\phi}(t')\hat{\phi}(t+i\beta)e^{-\beta \hat{H}}\right] \\
&\text{Using cyclicity of trace and equation (3.127)} \\
&= G_\beta^{-+}(t', t+i\beta) \\
&\text{Using equation (3.128)} \\
&= G_\beta^{+-}(t+i\beta, t') \; .
\end{aligned}
\tag{C.22}
$$

It is easy to show, in the same way, that [27]

$$
G_\beta^{+-}(t,t') = G_\beta^{-+}(t-i\beta, t') \; . \tag{C.23}
$$

These two results give the KMS relationship.

Appendix D

Classical-Statistical Method Appendix

D.1 Green's Solutions

This appendix deals with the Green's solutions used throughout chapter 4. The Green's solution allow one to solve an equation of motion while keeping track of the initial conditions. The proof developed here is taken from [26].

Assume the following equation was being solved for $F(x)$ where one is only interested when $x^0 > t_0$ and $F(t_0)$ and $\partial_{x^0} F(t_0)$ are known.

$$\left(\overrightarrow{\Box}_x + m^2\right) F(x) = J(x) \tag{D.1}$$

where $J(x)$ is acting as some source term.

One starts with the equation of interest at different space-time points,

$$\left(\overrightarrow{\Box}_y + m^2\right) F(y) = J(y) \; . \tag{D.2}$$

It will be shown in section E.4.1 that $\left(\Box_y + m^2\right) G_R^0(x, y) = -i\delta(x - y)$. This is equivalent to

$$G_R^0(x, y) \left(\overleftarrow{\Box}_y + m^2\right) = -i\delta(x - y) \; . \tag{D.3}$$

The arrows give the instruction for which way the operator should act, to the left or the right.

What is done now is to multiply the first, general, equation by $G_R^0(x, y)$ on the left and the second equation by $F(y)$ on the right. Then subtract the first equation from the second equation to get

$$G_R(x,y)J(y) + i\delta(x-y)F(y) = G_R^0(x,y)\left(\overrightarrow{\Box}_y + m^2\right)F(y) - G_R^0(x,y)\left(\overleftarrow{\Box}_y + m^2\right)F(y)\,. \tag{D.4}$$

The above is then integrated over y. Since the solution is only desired for $x^0 > t_0$, the y integration only happens from t_0. Integrating over the δ function gives,

$$F(x) = i\int\limits_{y^0>t_0} d^4y G_R(x,y)J(y) + i\int\limits_{y^0>t_0} d^4y\left\{G_R^0(x,y)\overleftarrow{\Box}_y F(y) - G_R^0(x,y)\overrightarrow{\Box}_y F(y)\right\}\,. \tag{D.5}$$

One can now exploit Stoke's theorem [26]. The time and spacial parts of $\Box$ will be looked at separately.

Time:

$$\begin{aligned}\int\limits_{y^0>t_0} d^4y\left\{G_R^0(x,y)\left(\overleftarrow{\partial}^2_{y^0} - \overrightarrow{\partial}^2_{y^0}\right)F(y)\right\} = \int\limits_{y^0>t_0} & d^4y\partial_{y^0}\left\{G_R^0(x,y)\overleftarrow{\partial}_{y^0}F(y)\right.\\ & \left.-G_R^0(x,y)\overrightarrow{\partial}_{y^0}F(y)\right\}\end{aligned} \tag{D.6}$$

Notice that there is a total time derivative. This means one can employ Stoke's theorem where the boundary here is the at $y^0 = t^0$. This gets to,

$$\begin{aligned}\int\limits_{y^0>t_0} d^4y\left\{G_R^0(x,y)\left(\overleftarrow{\partial}^2_{y^0} - \overrightarrow{\partial}^2_{y^0}\right)F(y)\right\} = \int\limits_{y^0=t_0} & d^3\boldsymbol{y}\left\{G_R^0(x,y)\overleftarrow{\partial}_{y^0}F(y)\right.\\ & \left.-G_R^0(x,y)\overrightarrow{\partial}_{y^0}F(y)\right\}\,.\end{aligned} \tag{D.7}$$

Note that there is no contribution from the $y^0 = +\infty$ boundary. This is due to the fact that the $\Theta(x,y)$ in $G_R^0(x,y)$ is only non-zero when $x^0 > y^0$. If $y^0 = +\infty$ then x^0 cannot be greater than y^0. Thus $\Theta(x,y)$ must be zero when $y^0 = +\infty$. This in combination with the equal time commutation relations shows why both $G_R^0(x,y)\overleftarrow{\partial}_{y^0}$ and $G_R^0(x,y)$ disappear when $y^0 = +\infty$.

Space:

$$\begin{aligned}\int\limits_{y^0>t_0} d^4y\left\{G_R^0(x,y)\left(\overleftarrow{\nabla}^2_{\boldsymbol{y}} - \overrightarrow{\nabla}^2_{\boldsymbol{y}}\right)F(y)\right\} = \int\limits_{y^0>t_0} & d^4y\nabla_{\boldsymbol{y}}\left\{G_R^0(x,y)\overleftarrow{\nabla}_{\boldsymbol{y}}F(y)\right.\\ & \left.-G_R^0(x,y)\overrightarrow{\nabla}_{\boldsymbol{y}}F(y)\right\}\end{aligned} \tag{D.8}$$

Then using Stoke's theorem to perform the spatial part of the above integral will result in an integral over time of the fields at the spatial boundary. It is reasonable to assume that the fields disappear at the spatial boundary. If this is the case, the above integral becomes a time integral over objects that are 0. Thus

$$\int\limits_{y^0>t_0} d^4y \left\{ G_R^0(x,y) \left(\overleftarrow{\nabla}_{\boldsymbol{y}}^2 - \overrightarrow{\nabla}_{\boldsymbol{y}}^2 \right) F(y) \right\} = 0 \; . \tag{D.9}$$

The above is only true if the fields go to 0 at the spatial boundary.

The above results are then inserted into equation D.5, this gives

$$F(x) = i \int\limits_{y^0>t_0} d^4y G_R(x,y) J(y) + i \int\limits_{y^0=t_0} d^3\boldsymbol{y} \left\{ G_R^0(x,y) \overleftarrow{\partial}_{y^0} F(y) - G_R^0(x,y) \overrightarrow{\partial}_{y^0} F(y) \right\} - 0 \; . \tag{D.10}$$

Thus the result can be written as [26],

$$F(x) = i \int\limits_{y^0>t_0} d^4y G_R(x,y) J(y) + i \int\limits_{y^0=t_0} d^3\boldsymbol{y} \left\{ G_R^0(x,y) \left(\overleftarrow{\partial}_{y^0} - \overrightarrow{\partial}_{y^0} \right) F(y) \right\} \; . \tag{D.11}$$

Notice that if there are trivial boundary conditions, then the above reduces to the naive solution for this type of equation.

D.2 Lipmann-Schwinger Solution

The result derived here is known as a solution to the Lippmann-Schwinger equation [16]. Where the Lippmann-Schwinger equation is given as,

$$\begin{aligned} \boldsymbol{\mathcal{G}}_t &= \boldsymbol{G_0} + \boldsymbol{G_0 \phi G_0} + \boldsymbol{G_0 \phi G_0 \phi G_0} + ... \\ &= \sum_n \boldsymbol{G_0} \left(\boldsymbol{\phi G_0} \right)^n \; . \end{aligned} \tag{D.12}$$

Where the bold font represents the Schwinger-Keldysh matrices where

$$\boldsymbol{G_0} = \begin{pmatrix} G_0^{++} & G_0^{+-} \\ G_0^{-+} & G_0^{--} \end{pmatrix} \tag{D.13}$$

and

$$\boldsymbol{\phi} = \begin{pmatrix} \frac{g^2}{2}\phi^2 & 0 \\ 0 & -\frac{g^2}{2}\phi^2 \end{pmatrix} \; . \tag{D.14}$$

To solve this above equation gives an expression for the Schwinger-Keldysh propagators in terms of the free propagators, G^R, G^A and F. It is solved by exploiting the linear transformation R of section 3.2.1.

Using the linear map R defined in equation (3.77) (see section 3.2.1) one can find that,

$$\boldsymbol{\mathcal{G}}_t^{\text{Trans}} = \sum_n \boldsymbol{G}_{\mathbf{0}}^{\text{Trans}} \left((\boldsymbol{\phi})^{\text{Trans}} \boldsymbol{G}_{\mathbf{0}}^{\text{Trans}} \right)^n \tag{D.15}$$

where,

$$\boldsymbol{G}_{\mathbf{0}}^{\text{Trans}} = \begin{pmatrix} F_0 & -iG_0^R \\ -iG_0^A & 0 \end{pmatrix} \tag{D.16}$$

and

$$\boldsymbol{\phi}^{\text{Trans}} = \begin{pmatrix} 0 & \frac{g^2}{2}\phi^2 \\ \frac{g^2}{2}\phi^2 & 0 \end{pmatrix} . \tag{D.17}$$

In this transformation the term $\left(\boldsymbol{\phi}^{\text{Trans}} \boldsymbol{G}_{\mathbf{0}}^{\text{Trans}}\right)$ can be easily diagonalized, making it easy to determine $\left(\boldsymbol{\phi}^{\text{Trans}} \boldsymbol{G}_{\mathbf{0}}^{\text{Trans}}\right)^n$.

For the sake of ease one can make the following identity,

$$\frac{g^2}{2}\phi^2 \equiv \phi_g . \tag{D.18}$$

One can work through the manipulations to find that

$$\left(\boldsymbol{\phi}^{\text{Trans}} \boldsymbol{G}_{\mathbf{0}}^{\text{Trans}}\right)^n = \begin{pmatrix} \sum_{j=0}^{n} \left(-\phi_g i G_R^0\right)^j F^0 \left(-\phi_g i G_A^0\right)^{n-j} & -iG_R^0 \left(-\phi_g i G_R^0\right)^n \\ -iG_A^0 \left(-\phi_g i G_A^0\right)^n & 0 \end{pmatrix} . \tag{D.19}$$

From equation (D.19) into (D.15) it can be seen that

$$-i\mathcal{G}_t^R = \sum_n -iG_R^0 \left(-\phi_g i G_R^0\right)^n \tag{D.20}$$

$$-i\mathcal{G}_t^A = \sum_n -iG_A^0 \left(-\phi_g i G_A^0\right)^n \tag{D.21}$$

and that,

$$\begin{aligned}
F_t &= \sum_{j=0}^{n} \left(-\phi_g i G_R^0\right)^j F^0 \left(-\phi_g i G_A^0\right)^{n-j} \\
&= \left(1 - \phi_g i G_0^R + [-\phi_g i G_0^R]^2 + ...\right) F_0 \left(1 - \phi_g i G_0^A + [-\phi_g i G_0^A]^2 + ...\right) \\
&= \left(-i G_0^R - i G_0^R[-\phi_g i G_0^R] - i G_0^R[-\phi_g i G_0^R]^2 + ...\right) \left(i G_0^R\right)^{-1} F_0 \left(i G_0^A\right)^{-1} \\
&\quad \times \left(-i G_0^A - i G_0^A[-\phi_g i G_0^A] - i G_0^A[-\phi_g i G_0^A]^2 + ...\right) \\
&= \left(-i\mathcal{G}_t^R\right) \left(i G_0^R\right)^{-1} F^0 \left(i G_0^A\right)^{-1} \left(-i\mathcal{G}_t^A\right) \\
&= \left(\mathcal{G}_t^R\right) \left(G_0^R\right)^{-1} F^0 \left(G_0^A\right)^{-1} \left(\mathcal{G}_t^A\right) .
\end{aligned} \tag{D.22}$$

Now all this is put back into the original Schwiner-Keldysh notation.

Since $R\boldsymbol{\mathcal{G}}_t R^T = \boldsymbol{\mathcal{G}}_t^{\mathrm{Trans}}$

$$\boldsymbol{\mathcal{G}}_t = R^{-1} \boldsymbol{\mathcal{G}}_t^{\mathrm{Trans}} \left(R^T\right)^{-1} \tag{D.23}$$

which means that

$$\boldsymbol{\mathcal{G}}_t = \begin{pmatrix} F_t - \frac{1}{2} i\mathcal{G}_t^R - \frac{1}{2} i\mathcal{G}_t^A & F_t - \frac{1}{2} i\mathcal{G}_t^A + \frac{1}{2} i\mathcal{G}_t^R \\ F_t - \frac{1}{2} i\mathcal{G}_t^R + \frac{1}{2} i\mathcal{G}_t^A & F_t + \frac{1}{2} i\mathcal{G}_t^A + \frac{1}{2} i\mathcal{G}_t^R \end{pmatrix} . \tag{D.24}$$

However,

$$\boldsymbol{\mathcal{G}}_t = \begin{pmatrix} \mathcal{G}_t^{++} & \mathcal{G}_t^{+-} \\ \mathcal{G}_t^{-+} & \mathcal{G}_t^{--} \end{pmatrix} \tag{D.25}$$

thus,

$$\begin{aligned}
\mathcal{G}_t^{+-} &= F_t - \frac{1}{2} i\mathcal{G}_t^A + \frac{1}{2} i\mathcal{G}_t^R \\
&\qquad \text{Using equations (3.93) and (3.94)} \\
&= F_t + \frac{1}{2}\left(\mathcal{G}_t^{++} - \mathcal{G}_t^{-+}\right) - \frac{1}{2}\left(\mathcal{G}_t^{++} - \mathcal{G}_t^{+-}\right) \\
&\qquad \text{Using equation (D.22)} \\
&= \mathcal{G}_t^R \left(G_R^0\right)^{-1} \frac{1}{2}\left(G_{+-}^0 + G_{-+}^0\right) \left(G_A^0\right)^{-1} \mathcal{G}_t^A + \frac{1}{2}\left(\mathcal{G}_t^{++} - \mathcal{G}_t^{-+}\right) - \frac{1}{2}\left(\mathcal{G}_t^{++} - \mathcal{G}_t^{+-}\right) .
\end{aligned} \tag{D.26}$$

Then multiply through by 2 and rearranging the above

$$\mathcal{G}_t^{+-} + \mathcal{G}_t^{-+} = \mathcal{G}_t^R \left(G_R^0\right)^{-1} G_{+-}^0 \left(G_A^0\right)^{-1} \mathcal{G}_t^A + \mathcal{G}_t^R \left(G_R^0\right)^{-1} G_{-+}^0 \left(G_A^0\right)^{-1} \mathcal{G}_t^A . \tag{D.27}$$

One can see that this seems to identify

$$\mathcal{G}_t^{+-} = \mathcal{G}_t^R \left(G_R^0\right)^{-1} G_{+-}^0 \left(G_A^0\right)^{-1} \mathcal{G}_t^A \ . \tag{D.28}$$

Going through the same process one finds that a consistent set of solutions are given by [16]

$$\mathcal{G}_t^{+-} = \mathcal{G}_t^R \left(G_R^0\right)^{-1} G_{+-}^0 \left(G_A^0\right)^{-1} \mathcal{G}_t^A \tag{D.29}$$
$$\mathcal{G}_t^{-+} = \mathcal{G}_t^R \left(G_R^0\right)^{-1} G_{-+}^0 \left(G_A^0\right)^{-1} \mathcal{G}_t^A \tag{D.30}$$
$$\mathcal{G}_t^{++} = \frac{1}{2}\left[\mathcal{G}_t^R \left(G_R^0\right)^{-1} F^0 \left(G_A^0\right)^{-1} \mathcal{G}_t^A + \mathcal{G}_t^R + \mathcal{G}_t^A\right] \tag{D.31}$$
$$\mathcal{G}_t^{--} = \frac{1}{2}\left[\mathcal{G}_t^R \left(G_R^0\right)^{-1} F^0 \left(G_A^0\right)^{-1} \mathcal{G}_t^A - \mathcal{G}_t^R - \mathcal{G}_t^A\right] \ . \tag{D.32}$$

By consistent what is meant is that these solutions follow the identity shown in equation (3.15) of section 3.2.1 that, $G_{++} + G_{--} = G_{-+} + G_{+-}$.

Appendix E

2 P.I. Equations of Motion Appendix

E.1 Proving equations (5.6) - (5.10)

This appendix will show that the general Gaussian density matrix of section 5.2 contains the initial correlations. Thus it will be shown that equations (5.6) - (5.10) are true. To start two properties need to be noted.

Firstly, states are normalized such that

$$\langle \phi_{t_0}^{(2)} | \phi_{t_0}^{(1)} \rangle = \delta(\phi_{t_0}^{(2)} - \phi_{t_0}^{(1)}) \; . \tag{E.1}$$

Secondly, in NRQM, the momentum operator acting in position space is a derivative. Extending this to QFT one gets that

$$\Pi(t) | \phi_{t_0}^{(1)} \rangle = -i \frac{\delta}{\delta \phi_{t_0}^{(1)}} | \phi_{t_0}^{(1)} \rangle \tag{E.2}$$

$$\langle \phi_{t_0}^{(2)} | \Pi(t') = i \frac{\delta}{\delta \phi_{t_0}^{(2)}} \langle \phi_{t_0}^{(2)} | \; . \tag{E.3}$$

Starting with proving equation (5.6) [8],

$$
\begin{aligned}
\mathrm{Tr}\left\{\rho(t_0)\phi(t)\right\}_{|t=0} &= \int d\phi_{t_0}^{(1)} d\phi_{t_0}^{(2)} \left\langle \phi_{t_0}^{(1)} | \rho(t_0) | \phi_{t_0}^{(2)} \right\rangle \left\langle \phi_{t_0}^{(2)} | \phi(t) | \phi_{t_0}^{(1)} \right\rangle \\
&\quad \text{Using equation (5.11)} \\
&= \int d\phi_{t_0}^{(1)} d\phi_{t_0}^{(2)} \left\langle \phi_{t_0}^{(1)} | \rho(t_0) | \phi_{t_0}^{(2)} \right\rangle \phi_{t_0}^{(1)} \left\langle \phi_{t_0}^{(2)} | \phi_{t_0}^{(1)} \right\rangle \\
&= \int_{-\infty}^{\infty} d\phi_{t_0}^{(1)} d\phi_{t_0}^{(2)} \left\langle \phi_{t_0}^{(1)} | \rho(t_0) | \phi_{t_0}^{(2)} \right\rangle \phi_{t_0}^{(1)} \delta(\phi_{t_0}^{(2)} - \phi_{t_0}^{(1)}) \\
&\quad \text{Using definition of } \left\langle \phi_{t_0}^{(1)} | \rho(t_0) | \phi_{t_0}^{(2)} \right\rangle \text{ (equation 5.5)} \\
&\quad \text{Integrating over the } \delta \\
&= \int_{-\infty}^{\infty} d\phi_{t_0}^{(1)} \frac{1}{\sqrt{2\pi\xi^2}} \phi_{t_0}^{(1)} \exp\left(-\frac{1}{2\xi^2}(\phi_{t_0}^{(1)} - \phi)^2\right) \\
&\quad \text{Using a constant shift of integration variable, } \phi_{t_0}^{(1)} \to \phi_{t_0}^{(1)} + \phi \\
&= \phi \, .
\end{aligned} \tag{E.4}
$$

This is as required from equation (5.6) .

Next equation (5.7) [8],

$$
\begin{aligned}
\mathrm{Tr}\left\{\rho(t_0)\partial_t\phi(t)\right\}_{|t=0} &= \mathrm{Tr}\left\{\rho(t_0)\Pi(t)\right\}_{|t=0} \\
&\quad \text{Using equation (E.2)} \\
&= \int d\phi_{t_0}^{(1)} d\phi_{t_0}^{(2)} \delta(\phi_{t_0}^{(1)} - \phi_{t_0}^{(2)}) \left(-i\frac{\delta}{\delta\phi_{t_0}^{(1)}}\right) \left\langle \phi_{t_0}^{(1)} | \rho(t_0) | \phi_{t_0}^{(2)} \right\rangle \\
&\quad \text{Differentiating } \left\langle \phi_{t_0}^{(1)} | \rho(t_0) | \phi_{t_0}^{(2)} \right\rangle \text{ (equation (5.5))} \\
&\quad \text{Then intergating over the } \delta \\
&= -\frac{1}{2\pi\xi^2} \int d\phi_{t_0}^{(1)} \exp\left(-\frac{1}{2\xi^2}(\phi_{t_0}^{(1)} - \phi)^2\right) \\
&\qquad \times i\left[i\dot{\phi} - \frac{1}{2\xi^2}(\phi_{t_0}^{(1)} - \phi) + i\frac{\eta}{\xi}(\phi_{t_0}^{(1)} - \phi)\right] \\
&\quad \text{Using a constant shift of integration variable, } \phi_{t_0}^{(1)} \to \phi_{t_0}^{(1)} + \phi \\
&= \dot{\phi} \, .
\end{aligned} \tag{E.5}
$$

This is the expression found in equation (5.7)

Onto the term $\mathrm{Tr}\left\{\rho(t_0)\phi(t)\phi(t')\right\}\Big|_{t=t'=0}$.

One can again act on the eigenstates with the operators $\phi(t)$ and $\phi(t')$. This will again leave a term that goes like $\left\langle \phi_{t_0}^{(2)} | \phi_{t_0}^{(1)} \right\rangle = \delta(\phi_{t_0}^{(1)} - \phi_{t_0}^{(2)})$. Then integrating over the δ gives,

$$\begin{aligned}
\mathrm{Tr}\left\{\rho(t_0)\phi(t')\phi(t)\right\}\Big|_{t=t'=0} &= \frac{1}{\sqrt{2\pi\xi^2}}\int\limits_{-\infty}^{\infty}\int d\phi^{(1)}_{t_0}\delta(\phi^{(2)}_{t_0}-\phi^{(1)}_{t_0})\phi^{(2)}_{t_0}\phi^{(1)}_{t_0}\left\langle\phi^{(1)}_{t_0}|\rho(t_0)|\phi^{(2)}_{t_0}\right\rangle \\
&\quad \text{Integrating over the delta and using the form of } \left\langle\phi^{(1)}_{t_0}|\rho(t_0)|\phi^{(2)}_{t_0}\right\rangle \\
&= \frac{1}{\sqrt{2\pi\xi^2}}\int\limits_{-\infty}^{\infty}\int d\phi^{(1)}_{t_0}\phi^{(1)}_{t_0}\phi^{(1)}_{t_0}\exp\left(-\frac{1}{2\xi^2}(\phi^{(1)}_{t_0}-\phi)^2\right) \\
&= \xi^2+\phi^2 \; .
\end{aligned} \tag{E.6}$$

Thus [8]

$$\xi^2 = \mathrm{Tr}\left\{\rho(t_0)\phi(t)\phi(t')\right\}\Big|_{t=t'=0} - \phi^2 \; . \tag{E.7}$$

As was given in equation (5.8)

Looking at the term $\frac{1}{2}\mathrm{Tr}\left\{\rho(t_0)\left(\partial_{t'}\phi(t')\phi(t)+\phi(t')\partial_t\phi(t)\right)\right\}\Big|_{t=t'=0}$

One can go through the now familiar process except here one needs to make use of both equation (E.2) and equation (E.3). Skipping ahead to the point after one takes the derivatives,

$$\begin{aligned}
&\frac{1}{2}\mathrm{Tr}\left\{\rho(t_0)\left(\partial_t\phi(t)\phi(t')+\phi(t)\partial_{t'}\phi(t')\right)\right\}\Big|_{t=t'=0} \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi\xi^2}}\int\limits_{-\infty}^{\infty} d\phi^{(1)}_{t_0}d\phi^{(2)}_{t_0}\delta(\phi^{(1)}_{t_0}-\phi^{(2)}_{t_0}) \\
&\quad\times\left\{\left[i\phi^{(1)}_{t_0}\left[-i\dot{\phi}-\frac{1}{2\xi^2}(\phi^{(2)}_{t_0}-\phi)-i\frac{\eta}{\xi}(\phi^{(2)}_{t_0}-\phi)\right]\right]\right. \\
&\quad+\left.\left[-i\phi^{(2)}_{t_0}\left[i\dot{\phi}-\frac{1}{2\xi^2}(\phi^{(1)}_{t_0}-\phi)+i\frac{\eta}{\xi}(\phi^{(1)}_{t_0}-\phi)\right]\right]\right\}e^{(\ldots)} \\
&\quad \text{Integrating over the } \delta \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi\xi^2}}\int\limits_{-\infty}^{\infty} d\phi^{(1)}_{t_0}\left\{2\phi^{(1)}_{t_0}\dot{\phi}+2\frac{\eta}{\xi}(\phi^{(1)}_{t_0}-\phi)\right\}\exp\left(-\frac{1}{2\xi^2}(\phi^{(1)}_{t_0}-\phi)^2\right) \\
&\quad \text{Using previous integral results} \\
&= \dot{\phi}\phi+\xi\eta.
\end{aligned} \tag{E.8}$$

Thus, as in equation (5.9) [8]

$$\xi\eta = \frac{1}{2}\mathrm{Tr}\left\{\rho(t_0)\left(\partial_t\phi(t)\phi(t')+\phi(t)\partial_{t'}\phi(t')\right)\right\}\Big|_{t=t'=0} - \dot{\phi}\phi. \tag{E.9}$$

Lastly, the term $\mathrm{Tr}\left\{\rho(t_0)\partial_{t'}\phi(t')\partial_t\phi(t)\right\}\Big|_{t=t'=0}$

This term involves a double derivative of the exponential (through equations (E.2) and (E.3)) that is $\left\langle \phi^{(1)}_{t_0} | \rho(t_0) | \phi^{(2)}_{t_0} \right\rangle$. Jumping to the point just after the derivatives have been done and then integrating over the δ,

$$
\begin{aligned}
\mathrm{Tr}\left\{\rho(t_0)\partial_{t'}\phi(t')\partial_t\phi(t)\right\}\Big|_{t=t'=0} &= \frac{1}{\sqrt{2\pi\xi^2}}\int d\phi^{(1)}_{t_0}\exp\left(-\frac{1}{2\xi^2}(\phi^{(1)}_{t_0}-\phi)^2\right) \\
&\quad\times\left[\frac{\sigma^2-1}{4\xi^2}+\left(i\dot{\phi}-\frac{1}{2\xi^2}(\phi^{(1)}_{t_0}-\phi)+\frac{i\eta}{\xi}(\phi^{(1)}_{t_0}-\phi)\right)\right. \\
&\quad\left.\times\left(-i\dot{\phi}-\frac{1}{2\xi^2}(\phi^{(1)}_{t_0}-\phi)-\frac{i\eta}{\xi}(\phi^{(1)}_{t_0}-\phi)\right)\right] \\
&\quad\text{Multiplying out and intergating} \\
&= \eta^2+\frac{\sigma^2}{4\xi^2}+\dot{\phi}^2\,. \qquad \text{(E.10)}
\end{aligned}
$$

Thus [8]

$$
\mathrm{Tr}\left\{\rho(t_0)\partial_{t'}\phi(t')\partial_t\phi(t)\right\}\Big|_{t=t'=0} = \eta^2+\frac{\sigma^2}{4\xi^2}+\dot{\phi}\dot{\phi}\,. \qquad \text{(E.11)}
$$

This is the solution as required (equation 5.10)

E.2 Determining Time Integrals of Equation (5.26)

This appendix solves the time integrals of equation (5.26) from section 5.4.2.

Expanding equation (5.26) and using the fact that $\int d^4z = \int_{\mathcal{C}} dz^0 \int d^3\boldsymbol{z}$,

$$
\begin{aligned}
\int_{\mathcal{C}} dz^0\int d^3\boldsymbol{z}\Bigg[&\underbrace{\Sigma_F(x,z)F(z,y)}_{i}-\underbrace{\frac{i}{2}\Sigma_\rho(x,z)\mathrm{sign}_{\mathcal{C}}(x^0-z^0)F(z,y)}_{ii} \\
&\underbrace{+\Sigma_F(x,z)\left(-\frac{i}{2}\rho(z,y)\mathrm{sign}_{\mathcal{C}}(z^0-y^0)\right)}_{iii} \\
&\underbrace{-\frac{i}{2}\Sigma_\rho(x,z)\mathrm{sign}_{\mathcal{C}}(x^0-z^0)\left(-\frac{i}{2}\rho(z,y)\mathrm{sign}_{\mathcal{C}}(z^0-y^0)\right)}_{iv}\Bigg]\,. \qquad \text{(E.12)}
\end{aligned}
$$

Each of the terms i, ii, iii and iv are discussed in turn. The results for each term are then inserted back into equation E.12.

<u>Term i</u>

This term is just an integration over the Schwinger-Keldysh contour. Thus integrating over all the contour means that there are contributions on the backward part of the contour that cancel those on the forward part of the contour to give 0.

Term ii

$\int_{\mathcal{C}} dz^0 \mathrm{sign}_{\mathcal{C}}(x^0 - z^0)$ needs to be considered.

Section 3.1.3 introduced the concept of the sign function. This was explained to be given by$\Theta_c(x^0 - y^0) - \Theta_c(y^0 - x^0)$. Using this,

$$\int dz^0_{\mathcal{C}} \mathrm{sign}_{\mathcal{C}}(x^0 - z^0) = \int_{\mathcal{C}} dz^0 \left(\Theta(x^0 - z^0) - \Theta(z^0 - x^0)\right) . \tag{E.13}$$

Split up the contour integral such that $\int_{\mathcal{C}} dz^0 = \int\limits_{t_0}^{x^0} dz^0 + \int\limits_{x^0}^{t_0} dz^0$.

In the first part of the contour, $\int\limits_{t_0}^{x^0} dz^0$, x^0 is always greater than z^0 and in the second part, $\int\limits_{x^0}^{t_0} dz^0$, z^0 is always greater than x^0.

Putting all this together gives

$$\begin{aligned} \int dz^0_{\mathcal{C}} \mathrm{sign}_{\mathcal{C}}(x^0 - z^0) &= \int_{\mathcal{C}} dz^0 \left(\Theta(x^0 - z^0) - \Theta(z^0 - x^0)\right) \\ &\quad \text{Splitting up the integration contour as described above} \\ &= \int\limits_{t_0}^{x^0} dz^0 \left(\Theta(x^0 - z^0) - \Theta(z^0 - x^0)\right) + \int\limits_{x^0}^{t_0} dz^0 \left(\Theta(x^0 - z^0) - \Theta(z^0 - x^0)\right) \\ &\quad \text{Using the definition of the } \Theta \text{ function and the relative size of } x^0 \text{and } y^0 \\ &= \int\limits_{t_0}^{x^0} dz^0(+1) + \int\limits_{x^0}^{t_0} dz^0(-1) \\ &= 2\int\limits_{t_0}^{x^0} dz^0 . \end{aligned} \tag{E.14}$$

Thus term ii) is $2\int\limits_{t_0}^{x^0} dz^0 \int d^3\boldsymbol{z} \frac{i}{2}\Sigma_\rho(x, z)F(z, y)$

Term iii

This term is found in the exact same way as term ii) except that

$$\int dz^0_{\mathcal{C}} \mathrm{sign}_{\mathcal{C}}(z^0 - y^0) = -2\int\limits_{t_0}^{y^0} dz^0 . \tag{E.15}$$

Thus term iii) is [8]

$$(iii) = -2\int_{t_0}^{y^0} dz^0 \int d^3\boldsymbol{z}\left(-\frac{i}{2}\Sigma_F(x,z)\rho(z,y)\right) . \tag{E.16}$$

Term iv

For iv, the term $\int_{\mathcal{C}} dz^0 \mathrm{sign}_{\mathcal{C}}(x^0 - z^0)\mathrm{sign}_{\mathcal{C}}(z^0 - y^0)$ needs to be discussed.

Again, using the definition of the sign function [8],

$$\begin{aligned}\int_{\mathcal{C}} dz^0 \mathrm{sign}_{\mathcal{C}}(x^0 - z^0)\mathrm{sign}_{\mathcal{C}}(z^0 - y^0) &= \int_{\mathcal{C}} dz^0 \left(\Theta(x^0 - z^0) - \Theta(z^0 - x^0)\right)\left(\Theta(z^0 - y^0) - \Theta(y^0 - z^0)\right) \\ &= \int_{\mathcal{C}} dz^0 \Theta(x^0 - z^0)\Theta(z^0 - y^0) - \int_{\mathcal{C}} dz^0 \Theta(x^0 - z^0)\Theta(y^0 - z^0) \\ &\quad - \int_{\mathcal{C}} dz^0 \Theta(z^0 - x^0)\Theta(z^0 - y^0) + \int_{\mathcal{C}} dz^0 \Theta(z^0 - x^0)\Theta(y^0 - z^0) \\ &. \end{aligned} \tag{E.17}$$

When there are two Θ functions multiplied together, this gives information on what the relations are between the arguments for the product to be 1. So to categorize all the Θ function products defined in equation (E.17);

$\Theta(x^0 - z^0)\Theta(z^0 - y^0)$ can only be non-zero when (by the definition of the Θ functions) $x^0 > y^0$ and $z^0 > y^0$.
$\Theta(x^0 - z^0)\Theta(y^0 - z^0)$ can only be non-zero when $x^0 > z^0$ and $y^0 > z^0$.
$\Theta(z^0 - x^0)\Theta(z^0 - y^0)$ can only be non-zero when $z^0 > x^0$ and $z^0 > y^0$.
$\Theta(z^0 - x^0)\Theta(y^0 - z^0)$ can only be non-zero when $z^0 > x^0$ and $y^0 > z^0$.

If the above conditions aren't met, the product is 0.

The best way to proceed from here is to realize that there are two different scenarios, it could be that $x^0 > y^0$ or it could be that $x^0 < y^0$.
First the situation where $x^0 > y^0$.

From here the contour integral is split up into 3 parts such that $\int_{\mathcal{C}} dz^0 = \int_{t_0}^{y^0} dz^0 + \int_{y^0}^{x^0} dz^0 + \int_{x^0}^{t_0} dz^0$. The first part is where $x^0, y^0 > z^0$ the second part is where $x^0 > z^0 > y^0$ and the third part is where $x^0, y^0 < z^0$. (Note the fact that $x^0 > y^0$ is now important).

Putting this information together with product of Θ functions requirements, from equation (E.17)

$$\begin{aligned}\int_{\mathcal{C}} dz^0 \mathrm{sign}_{\mathcal{C}}(x^0 - z^0)\mathrm{sign}_{\mathcal{C}}(z^0 - y^0) &= \int_{y^0}^{x^0} dz^0(1) - \int_{t_0}^{y^0} dz^0(1) - \int_{x^0}^{t_0} dz^0(1) + 0 \\ &= -\int_{t_0}^{y^0} dz^0(1) + \int_{y^0}^{x^0} dz^0(1) - \int_{x^0}^{t_0} dz^0(1) . \end{aligned} \tag{E.18}$$

The above can be further simplified. Notice that $\int_{\mathcal{C}} dz^0[A] = \int_{t_0}^{y^0} dz^0[A] + \int_{y^0}^{x^0} dz^0[A] + \int_{x^0}^{t_0}[A]$. The integration along the contour of a function A gives 0. Thus, by rearranging the above

$$-\int_{t_0}^{y^0} dz^0[A] - \int_{x^0}^{t_0}[A] = \int_{y^0}^{x^0} dz^0[A] \ . \tag{E.19}$$

Putting this into equation (E.18)

$$\int_{\mathcal{C}} dz^0 \text{sign}_{\mathcal{C}}(x^0 - z^0)\text{sign}_{\mathcal{C}}(z^0 - y^0) = 2\int_{y^0}^{x^0} dz^0 \ . \tag{E.20}$$

The other situation, where $x^0 < y^0$.

This will change the splitting up of the integration contour. In this scenario; $\int_{\mathcal{C}} dz^0 = \int_{t_0}^{x^0} dz^0 + \int_{x^0}^{y^0} dz^0 + \int_{y^0}^{0} dz^0$. One can then proceed as before to find,

$$\int_{\mathcal{C}} dz^0 \text{sign}_{\mathcal{C}}(x^0 - z^0)\text{sign}_{\mathcal{C}}(z^0 - y^0) = -2\int_{y^0}^{x^0} dz^0 \ . \tag{E.21}$$

The difference between equation (E.20) and equation (E.21) is the sign of the difference between x^0 and y^0. Thus it depends on the sign function $\text{sign}_{\mathcal{C}}(x^0 - y^0)$.

So term iv) is [8],

$$(iv) = 2\int_{y^0}^{x^0}\int d^3\boldsymbol{z} dz^0 \left[-\text{sign}_{\mathcal{C}}(x^0 - z^0)\frac{i}{2}\frac{i}{2}\Sigma_\rho(x,z)\rho(z,y)\right] \ . \tag{E.22}$$

Thus all the pieces of equation (E.12) have been found.

Inserting all the terms i, ii, iii, iv into equation (E.12) gives;

$$\int d^4z \left(\Sigma_F(x,z) - \frac{i}{2}\Sigma_\rho(x,z)\mathrm{sign}_{\mathcal{C}}(x^0-z^0)\right)\left(F(z,y) - \frac{i}{2}\rho(z,y)\mathrm{sign}_{\mathcal{C}}(z^0-y^0)\right)$$
$$= 0 - \left(i\int_{t_0}^{x^0} dz^0 \int d^3\boldsymbol{z}\Sigma_\rho(x,z)F(z,y)\right) + \left(i\int_{t_0}^{y^0} dz^0 \int d^3\boldsymbol{z}\Sigma_F(x,z)\rho(z,y)\right)$$
$$- \left(\frac{1}{2}\int_{t_0}^{y^0} dz^0 \int d^3\boldsymbol{z}\Sigma_\rho(x,z)\rho(z,y)\right) . \tag{E.23}$$

E.3 Looking at $\Box_x \left(F(x,y) - \frac{i}{2}\rho(x,y)\mathbf{sign}_{\mathcal{C}}(x^0-y^0)\right)$

This appendix looks at $\Box_x \left(F(x,y) - \frac{i}{2}\rho(x,y)\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\right)$ and particularly how one can consider this as the differential operator acting on the F part and the ρ part.

$$\begin{aligned}\Box_x\rho(x,y)\mathrm{sign}_{\mathcal{C}}(x^0-y^0) &= (\partial^2_{x^0} - \nabla^2_{\boldsymbol{x}})\rho(x,y)\mathrm{sign}_{\mathcal{C}}(x^0-y^0) \\ &\quad \text{Note that } \mathrm{sign}_{\mathcal{C}}(x^0-y^0) \text{ doesn't depend on } \boldsymbol{x} \\ &= \left(\partial^2_{x^0}\rho(x,y)\right)\mathrm{sign}_{\mathcal{C}}(x^0-y^0) + 2\partial_{x^0}\rho(x,y)\partial_{x^0}\mathrm{sign}_{\mathcal{C}}(x^0-y^0) + \\ &\quad \rho(x,y)\partial^2_{x^0}\mathrm{sign}_{\mathcal{C}}(x^0-y^0) - \left(\nabla^2_{\boldsymbol{x}}\rho(x,y)\right)\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\end{aligned} \tag{E.24}$$

To continue, the time derivatives of $\mathrm{sign}_{\mathcal{C}}(x^0-y^0)$ need to be discussed.

It is known that the derivative of a Θ function gives a delta function. The derivative of a δ function can be found as follows [8],

$$\begin{aligned}\int_{-\infty}^{+\infty} dx\frac{d\delta(x)}{dx}f(x) &= \delta(x)f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{+\infty} dx\delta(x)\frac{df(x)}{dx} \\ &\quad \text{The } \delta(x) \text{ disappears at the boundaries} \\ \int_{-\infty}^{+\infty} dx\frac{d\delta(x)}{dx}f(x) &= 0 + \int_{-\infty}^{+\infty} dx\left(-\delta(x)\frac{df(x)}{dx}\right) \\ &\quad \text{Equating the integrands} \\ \frac{d\delta(x)}{dx}f(x) &= -\delta(x)\frac{df(x)}{dx} \ .\end{aligned} \tag{E.25}$$

Thus

$$
\begin{aligned}
\left[\partial_{x^0}^2 \mathrm{sign}_{\mathcal{C}}(x^0-y^0)\right]\rho(x,y) &= \left[\partial_{x^0}^2\left(\Theta(x^0-y^0)-\Theta(y^0-x^0)\right)\right]\rho(x,y) \\
&\quad \text{Using the derivative of a } \Theta \text{ function is a } \delta \text{ function} \\
&= \left[\partial_{x^0}\left(\delta(x^0-y^0)-(-\delta(y^0-x^0))\right)\right]\rho(x,y) \\
&\quad \delta(x-y)=\delta(y-x) \\
&= \left[\partial_{x^0} 2\delta(x^0-y^0)\right]\rho(x,y) \\
&\quad \text{Using equation (E.25)} \\
&= -2\delta(x^0-y^0)\partial_{x^0}\rho(x,y)\ . \qquad \text{(E.26)}
\end{aligned}
$$

Putting all the above into equation (E.24)

$$
\begin{aligned}
\Box_x \rho(x,y)\mathrm{sign}_{\mathcal{C}}(x^0-y^0) &= \left(\partial_{x^0}^2\rho(x,y)\right)\mathrm{sign}_{\mathcal{C}}(x^0-y^0)+2\partial_{x^0}\rho(x,y)\left(2\delta(x^0-y^0)\right)+ \\
&\quad \partial_{x^0}\rho(x,y)\left(-2\delta(x^0-y^0)\right)-\left(\nabla_{\boldsymbol{x}}^2\rho(x,y)\right)\mathrm{sign}_{\mathcal{C}}(x^0-y^0) \\
&= \mathrm{sign}_{\mathcal{C}}(x^0-y^0)\partial_{x^0}^2\rho(x,y)-\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\nabla_{\boldsymbol{x}}^2\rho(x,y) \\
&\quad +2\partial_{x^0}\rho(x,y)\delta(x^0-y^0) \\
&\quad \rho(x,y)\delta(x^0-y^0)=\delta(\boldsymbol{x}-\boldsymbol{y}) \\
&= \mathrm{sign}_{\mathcal{C}}(x^0-y^0)\Box_x\rho(x,y)+2\delta(x-y)\ . \qquad \text{(E.27)}
\end{aligned}
$$

Thus [8],

$$
\begin{aligned}
\Box_x\left(F(x,y)-\frac{i}{2}\rho(x,y)\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\right) &= \Box_x F(x,y)-\frac{i}{2}\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\Box_x\rho(x,y) \\
&\quad -i\delta(x-y)\ . \qquad \text{(E.28)}
\end{aligned}
$$

E.4 $\Box + m^2 + \frac{g^2}{2}\bar{\phi}^2$ acting on F_t and ρ_t

This Appendix is done in two parts, first $(\Box_x + m^2)$ acting on $G_0^R(x,y)$ and $G_0^A(x,y)$ is found, then $\Box + m^2 + \frac{g^2}{2}\bar{\phi}^2$ acting on F_t and ρ_t is discussed.

E.4.1 Proving that $(\Box_x + m^2)G_0^R(x,y) = -i\delta(x-y)$

This proof follows [10].

$$
\begin{aligned}
\left(\Box_x+m^2\right)G_0^R(x,y) &= \left(\partial_{x^0}^2+\nabla_x^2+m^2\right)G_0^R(x,y) \\
&\quad \text{From equation (3.22)} \\
&= \left(\partial_{x^0}^2+\nabla_x^2+m^2\right) \\
&\qquad \times i\left[\Theta(x^0-y^0)\langle[\phi(x),\phi(y)]\rangle\right] \\
&= i\Theta(x^0-y^0)\left(\nabla_x^2+m^2\right)\langle[\phi(x),\phi(y)]\rangle \\
&\qquad + i\left(\partial_{x^0}^2\right)\Theta(x^0-y^0)\langle[\phi(x),\phi(y)]\rangle \\
&= i\Theta(x^0-y^0)\left(\nabla_x^2+m^2\right)\langle[\phi(x),\phi(y)]\rangle \\
&\qquad + i\left(\partial_{x^0}^2\Theta(x^0-y^0)\right)\langle[\phi(x),\phi(y)]\rangle \\
&\qquad + 2i\left(\partial_{x^0}\Theta(x^0-y^0)\right)\langle[\partial_{x^0}\phi(x),\phi(y)]\rangle \\
&\qquad + i\Theta(x^0-y^0)\langle\left[\partial_{x^0}^2\phi(x),\phi(y)\right]\rangle \qquad \text{(E.29)}
\end{aligned}
$$

Remember that $\partial_{x^0}\phi(x) = \pi(x)$, where $\pi(x)$ is the conjugate momenta. $\partial_{x^0}\Theta(x^0 - y^0) = \delta(x^0 - y^0)$.

By integration by parts it can be found that

$$\begin{aligned}\left(\partial^2_{x^0}\Theta(x^0 - y^0)\right)\langle[\phi(x),\phi(y)]\rangle &= \left(\partial_{x^0}\delta(x^0 - y^0)\right)\langle[\phi(x),\phi(y)]\rangle \\ &= -\delta(x^0 - y^0)\langle[\partial_{x^0}\phi(x),\phi(y)]\rangle \ .\end{aligned} \tag{E.30}$$

Putting this into equation E.29

$$\begin{aligned}\left(\Box_x + m^2\right)G_0^R(x,y) &= i\Theta(x^0 - y^0)\left(\nabla_x^2 + m^2\right)\langle[\phi(x),\phi(y)]\rangle \\ &\quad - i\delta(x^0 - y^0)\langle[\pi(x),\phi(y)]\rangle \\ &\quad + 2i\delta(x^0 - y^0)\langle[\pi(x),\phi(y)]\rangle \\ &\quad + i\Theta(x^0 - y^0)\partial^2_{x^0}\langle[\phi(x),\phi(y)]\rangle \\ &= i\Theta(x^0 - y^0)\left(\Box_x + m^2\right)\langle[\phi(x),\phi(y)]\rangle \\ &\quad + i\delta(x^0 - y^0)\langle[\pi(x),\phi(y)]\rangle \ .\end{aligned} \tag{E.31}$$

Finally, using the equal time commutation relations and that $\left(\Box_x + m^2\right)\langle[\phi(x),\phi(y)]\rangle = \left(\Box_x + m^2\right)[G^{+-}(x,y) - G^{-+}(x,y)] = 0$.

$$\begin{aligned}\left(\Box_x + m^2\right)G_0^R(x,y) &= 0 - i\delta(x^0 - y^0)\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ &= -i\delta^{(4)}(x - y)\end{aligned} \tag{E.32}$$

One can use the same argument for $G_0^A(x,y)$, except that $G_0^A(x,y) = -i\Theta(y^0 - x^0)\langle[\phi(x),\phi(y)]\rangle$. Following the same procedure will look the same except that there will be an extra $-$ due to the $\Theta(y^0 - x^0)$. Then an overall $-$ from the definition. This means one obtains the same result i.i

$$\left(\Box_x + m^2\right)G_0^A(x,y) = -i\delta^{(4)}(x - y) \ . \tag{E.33}$$

In light of these equations one then defines $(G_0^R)^{-1}$ such that [10]

$$(G_0^R)^{-1} = i(\Box + m^2) \tag{E.34}$$

and [10]

$$(G_0^A)^{-1} = i(\Box + m^2) \ . \tag{E.35}$$

E.4.2 $\Box + m^2 + \frac{g^2}{2}\bar{\phi}^2$ acting on F_t and ρ_t

Now that it has been shown that [10],

$$(\Box_x + m^2)G_0^R(x,y) = -i\delta(x,y) \quad \text{(E.36)}$$
$$(\Box_x + m^2)G_0^A(x,y) = -i\delta(x,y) \quad \text{(E.37)}$$
$$\text{(E.38)}$$

one can find $\Box + m^2 + \frac{g^2}{2}\bar{\phi}^2$ acting on F_t and ρ_t.

$\rho_0(x,y) = G_0^R(x,y) - G_0^A(x,y)$. The propagators G_0^R and G_0^A cannot both exist at the same time as they are governed by the Θ functions $\Theta(x^0 - y^0)$ and $\Theta(y^0 - x^0)$ respectively. Thus

$$(\Box_x + m^2)\rho_0(x,y) = -i\delta(x-y) \quad \text{(E.39)}$$

It is easy to find that

$$(\Box_x + m^2)F_0(x,y) = 0 \quad \text{(E.40)}$$

Since F_0 contains no Θ functions.

What is needed is $\left(\Box_x + m^2 + \frac{g^2}{2!}\bar{\phi}^2(x)\right)\rho_t(x,y)$ and $\left(\Box_x + m^2 + \frac{g^2}{2!}\bar{\phi}^2(x)\right)F_t(x,y)$

In turns out to be easier to use the alternative form of F and ρ. Namely the $G_{\sigma\sigma}$ and $G_{\sigma\eta} - G_{\eta\sigma}$ propagators respectively (see section 3.2.1).

Then the tree form of $\rho(x,y)$ is given by

$$\begin{aligned} G^t_{\sigma\eta}(x,y) - G^t_{\eta\sigma}(x,y) = \left\{G^0_{\sigma\eta}(x,y) - G^0_{\eta\sigma}(x,y)\right\} + \int d^4z G^0_{\sigma\epsilon}(x,z)\frac{g^2}{2!}\bar{\phi}^2_{\epsilon'}(x)G^t_{\epsilon''\eta}(z,y) \\ + \int d^4z' G^0_{\eta\epsilon}(x,z)\frac{g^2}{2!}\bar{\phi}^2_{\epsilon'}(x)G^t_{\epsilon''\sigma}(z,y) \end{aligned} \quad \text{(E.41)}$$

where ϵ, ϵ' and ϵ'' can be either σ or η and all the options are summed over.

The next step is to look at $(\Box_x + m^2)\rho^t(x,y)$.

Since $G_{\sigma\eta}(x,y) = -iG^R(x,y)$, $(\Box_x + m^2)G_{\sigma\eta} = -\delta(x,y)$.
Since $G_{\eta\sigma}(x,y) = -iG^A(x,y)$, $(\Box_x + m^2)G_{\eta\sigma} = -\delta(x,y)$
Since $G_{\sigma\sigma}(x,y) = F(x,y)$, $(\Box_x + m^2)G_{\sigma\sigma} = 0$

Due to the identification with G^R and G^A, the same issue with the Θ functions exists here, i.e. $G_{\sigma\eta}(x,y)$ and $G_{\eta\sigma}(x,y)$ can't exist at the same time, thus its one or the other

Then for the $G_{\sigma\eta}(x,y)$

$$
\begin{aligned}
(\Box_x + m^2)G^t_{\sigma\eta}(x,y) = (\Box_x + m^2)\Big[G^0_{\sigma\eta}(x,y) + \int d^4z G^0_{\sigma\eta}(x,z)\frac{g^2}{2!}\bar{\phi}^2_\sigma(x)G^t_{\sigma\eta}(z,y) \\
+ \int d^4z G^0_{\sigma\eta}(x,z)\frac{g^2}{2!}\bar{\phi}^2_\eta(x)G^t_{\sigma\eta}(z,y)\Big] \\
= -\delta(x,y) - \frac{g^2}{2!}\bar{\phi}^2_\sigma(x)G^t_{\sigma\eta}(x,y) - \frac{g^2}{2!}\bar{\phi}^2_\eta(x)G^t_{\sigma\eta}(x,y)\ . \qquad \text{(E.42)}
\end{aligned}
$$

Bringing objects to the other side of the equals sign

$$
(\Box_x + m^2 + \frac{g^2}{2!}\bar{\phi}^2_\sigma(x) + \frac{g^2}{2!}\bar{\phi}^2_\eta(x))G^t_{\sigma\eta}(x,y) = -\delta(x,y) \qquad \text{(E.43)}
$$

The same thing for the $G_{\sigma\eta}(x,y)$ part of ρ produces the same thing. Thus,

$$
(\Box_x + m^2 + \frac{g^2}{2!}\bar{\phi}^2(x))\rho^t(x,y) = -\delta(x,y)\ . \qquad \text{(E.44)}
$$

Once can go through the same process to find $\left(\Box_x + m^2 + \frac{g^2}{2!}\bar{\phi}^2(x)\right)F^t(x,y)$ by using $F(x,y) = G_{\sigma\sigma}(x,y)$.

Then by building in the same way one finds that

$$
\left(\Box_x + m^2 + \frac{g^2}{2!}\bar{\phi}^2(x)\right)F^t(x,y) = 0\ . \qquad \text{(E.45)}
$$

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