INTRODUCTION TO SUPERSYMMETRY

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Thesis submitted in partial fulfilment of the requirements of

Master of Science in Theoretical Physics

University of Cape Town

September 1987

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Introduction

The tendency in theoretical physics, particularly in the past few decades, has been towards unification: over the years it has emerged that increasingly many physical phenomena can be explained by a common underlying theory. Symmetry principles, both global and local, play an important role in this unification programme. Global symmetries often account for approximate regularities we observe in nature - local or gauge symmetries are understood as basic symmetries which lie at the heart of the interactions of the constituents of matter.

The use of symmetry ideas has led to enourmous progress in our understanding of nature. For example, the electroweak theory of Glashow, Salam and Weinberg[1] rests on the gauge group SU(2)⊗U(1) and unifies the electromagnetic and weak interactions. The standard model[2], which is based on the gauge group SU(3)⊗SU(2)⊗U(1), was another step forward in the quest for unification: In this theory, the strong, weak and electromagnetic interactions for three families of quarks and leptons are described in terms of one underlying interaction. The experimental verification of the standard model is, however, limited- and recent experimental results on the lifetime of the proton[3] have even reached values that indicate inconsistency of the model with observation.

If symmetries are categorised according to the degree of unification they introduce, then supersymmetry is the supreme symmetry. Haag, Sohnius and Lopuszanski[4] have shown that supersymmetry is the largest possible symmetry of the S-matrix: it unifies spacetime symmetries (the Poincaré group) with internal symmetries such as SU(3) descibing - exactly - the three colour charges of quarks, or - approximately - the multiplet structure of elementary particles consisting of up, down and strange quarks. It also - and this is the great novelty of supersymmetric theories - unifies fermions with bosons by placing them in common irreducible multiplets. Supersymmetry is thus a statistics-changing symmetry - for this reason it is also referred to as Fermi-Bose symmetry. Local supersymmetry, or supergravity is, additionally, a unification scheme for gravity and all other interactions.

Although supersymmetric models are able to unify fundamental interactions, they fail to give a realistic description of nature in the energy regions in which we are able to measure it. For example, supersymmetry predicts that for every observed fermion we should find a boson of the same mass, and vice versa; this is certainly not what we observe. So some essential ingredient is missing in these models which would justify calling them "theories" instead of "models".

The latest arrival on the market of unifying theories are superstring theories[5]. Their basic postulate is that elementary particles are one-dimensional, rather than point-like, objects, as has previously been assumed (due to an experimental limit for the size of the electron of 10^{-19}m) They have some attractive features, for example that they naturally include gravity. In fact, for a consistent formulation of quantum field theory they actually require it. For this reason, superstring theories have attracted widespread interest lately (notwithstanding the fact that it is a formidable task to find one's way through the immense variety of superstring models - and different fashions in notation - that exist today).

Superstring theories are theories containing supergravity which are formulated in larger than four-dimensional space-time. To study superstrings, it is useful to extend relativistic quantum mechanics, which is believed to correctly desribe nature in four-dimensional Minkowski space, to arbitrary space-time dimensions. In particular,for a satisfactory description of spin-½ particles in the framework of the higher-dimensional superstring theories, it is necessary to investigate the Dirac equation in higher dimensions. Secondly, a sound knowledge of the ideas of supersymmetry is required to understand the concepts of superstring theories.

My objective in this work is to draw together these two threads: the formulation of the Dirac equation in higher dimensions and the introduction of the concepts of supersymmetry. I present here ideas and results which are seldom found in the literature in easily accessible form. I do not arrive at fundamentally new results but attempt to give a review which enables the reader (and first and foremost, the writer) to cope with the technicalities of superstring theory. In particular, in chapter 1 I prove some results which are, in the superstring literature, always stated but not derived.

The thesis is organised as follows:

In chapter 1, I discuss the Dirac equation as a relativistic quantum mechanical equation describing spin-½ particles in higher-dimensional space-time. However (except for some examples), I still assume that we have just one time dimension. I consider the Clifford algebra in d-dimensional space-time, thereby investigating the representation space of the Dirac gamma-matrices and determining the size (dimension) and explicit form they should have. I also dicuss the discrete symmetries of the Dirac equation: charge conjugation, time reversal and parity or reflection symmetry. In particular, I investigate their generalisation to higher space-time dimensions. Then I take a look at the Majorana and Weyl conditions which require the particle wavefunctions to be invariant under particle ↔ antiparticle exchange and left-handed particle ↔ right-handed antiparticle exchange respectively. (That is, a Majorana spinor is an eigenspinor of the charge-conjugation symmetry and a Weyl spinor is an eigenspinor of the chirality operation. We find that to implement both conditions simultaneously, space-time has to be 2 (mod 8)- dimensional.

For an understanding of the origin of the supersymmetry algebra, a proper understanding of the Poincaré group with its irreducible representations is essential. This is given in chapter 2. The Poincaré group consists of the Lorentz group of space rotations and Lorentz boosts, to which is added the group of space-time translations. I analyse this group and find its irreducible representations.

Finally, in chapter 3, I give an introduction to supersymmetry. The concept of supersymmetry arises out of the generalisation of Lie algebras: previously, symmetries had only been described in the framework of Lie algebras; one now generalises this idea to include as possible symmetry groups also graded Lie algebras. These are algebras involving both commutation and anticommutation relations between the symmetry generators. The introduction of graded Lie structures allows the inclusion into the algebra of fermionic (anticommuting) generators. These generators are the supersymmetry generators: they produce the statistics-changing symmetry between fermions and bosons. I discuss some of the immediate consequences of th supersymmery algebra. Perhaps the most important of these is the famous "fermions = bosons" rule: this states that in any supersymmetric multiplet of particle states or fields, there are always the same number of fermions and bosons present. I calculate some supersymmetric multiplets, both in the context of single particle states and fields. Finally, I introduce the superspace-superfield approach. Superspace is a mathematical object which allows a tremendous simplification of the formulation of supersymmetry: in this picture, the usual four-dimensional coordinate space is extended to inclue a set of anticommuting coordinates. The advantage of this formulation is that now supersymmery becomes manifest: it arises naturally just like the Poincaré symmetry arises naturally in four-dimensional Minkowski space.

The notation and some conventions used are summarised in Appendix A. Each chapter, including this introduction, is concluded with a list of relevant references. References are numbered for each chapter separately.

Appendix B gives a short introduction into the methods of group classification with the aim to show how the dimensions of irreducible representations of classical groups can be derived. These methods enable us to readily arrive at the particle multiplets, given the knowledge of the spectrum generating group (the group of transformations on the symmetry generators which leave the algebra invariant).

References:

[1] See for example:

1) J. Schwinger : Ann. Phys. 2 (1975),407
2) S. Glashow : Nucl. Phys. 22 (1961), 579
3) A. Salam, J.Ward : Phys. Lett. 13 (1964), 168
4) S. Weinberg : Phys. Rev. Lett. 19 (1967),1264

[2] Some references on the standard model are [1] plus:

1) A.Salam : Elementary Particle Theory, edited by N. Swartholin, Almquist and Forlag, Stockholm (1968)
2) S. Glashow, J. Iliopoulos, L.Maiani : Phys. Rev. D2 (1970),1285
3) M.Kobayashi, M. Maskawa : Progr. Theor. Phys. 49 (1973),652

[3] R.M. Biota et al. : Phys. Rev. Lett. 51 (1983), 215

[4] R.Haag,J.Lopuszanski, M.Sohnius : All possible generators of : supersymmetries of the S-matrix, Nuclear Physics B88 (1975),257

[5] All the important papers from the early days of superstring theory are published in:

J. Schwarz (ed.) : Superstrings: the first 15 years of superstring theory, 2 volumes, World Scientific (1985)

1. The Dirac Equation in Spacetime of Arbitrary Dimension

1.1 The Clifford Algebra in d-dimensional Spacetime

Our intention is to investigate the Dirac equation in a spacetime which has an arbitrary number of dimensions instead of the familiar four of Minkowski space. In particular, we are interested to see how the consequences of the Clifford Algebra in four-dimensional spacetime change when we go to higher (lower) dimensional spaces.

We start with the Dirac equation written in covariant form without interaction term:

$$(\not{p} - m)\,\Psi = 0 \quad , \tag{1.1.1}$$

where $\not{p} = \gamma_\mu p^\mu$. (1.1.2)

The Dirac spinor Ψ is of the form

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_n \end{pmatrix} .$$

The dimension n of the Dirac matrices and hence of the Dirac spinor (which gives the degrees of freedom of the Dirac equation) depends on the dimension of the γ-matrices in (1.1.1) and is as yet unknown. The γ-matrices have to be linearly independent for the Dirac equation to make sense - this fact determines the dimension they will be required to have.

Now since

$$\not{p} = \gamma_\mu\, p^\mu, \qquad \text{where } p^\mu = i\hbar\partial^\mu = i\hbar\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right),$$

and we have to satisfy the covariant momentum-energy relation

$$m^2 = p_\mu p^\mu \tag{1.1.3}$$

or

$$m = \sqrt{p_\mu p^\mu} = \pm\gamma_\mu\, p^\mu,$$

we obtain the specifying relation for the matrices γ_μ:

$$m^2 = \gamma_\mu p^\mu\, \gamma_\nu p^\nu$$

$$= \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)p^\mu p^\nu$$ (since the expression is symmetric on interchange of p^μ and p^ν)

In order to obtain (1.1.3),

it follows that the γ-matrices must satisfy

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\,\eta_{\mu\nu}\,\mathbb{1}\,,$$

or

$$\{\gamma_\mu,\gamma_\nu\} = 2\,\eta_{\mu\nu}\,\mathbb{1}\,, \tag{1.1.4}$$

where $\eta_{\mu\nu}$ is the metric of our spacetime. If not stated otherwise, we will always use the Minkowski metric as defined in Appendix A.

To find the dimension of the Dirac matrices γ_μ given the dimension of space-time, we now explore the <u>Clifford-algebra</u> (1.1.4) in "three easy steps":[1]

i) From (1.1.4), it follows that

$$\gamma_\mu\gamma_\mu = 2\,\eta_{\mu\mu}\,\mathbb{1}$$

which implies

$$(\gamma_\mu)^2 = \begin{cases} \mathbb{1} & \mu = 0 \\ -\mathbb{1} & \mu \neq 0 \end{cases} \tag{1.1.5}$$

ii) The dimension of the metric is clearly

$$\dim(\eta_{\mu\nu}) = \dim(\text{spacetime}) = d$$

and consequently

$$\eta_{\mu\nu}\eta^{\mu\nu} = d. \tag{1.1.6}$$

iii) The Clifford algebra requires that there be d linearly independent matrices γ_μ. This enables us to find the dimension of the space that is spanned by these matrices, or, equivalently, the dimension n of the γ-matrices themselves.

Consider the matrices Γ^A, defined as follows:

$$\begin{array}{lll} \mathbb{1} & = \Gamma^0 & \\ \gamma_\mu & = \Gamma^{i_1} & i_1 = 1,2,\ldots,d \\ \gamma_\mu\gamma_\nu & = \Gamma^{i_2} & i_2 = 1,2,\ldots,\binom{d}{2} \\ \gamma_\mu\gamma_\nu\gamma_\rho & = \Gamma^{i_3} & i_3 = 1,2,\ldots,\binom{d}{3} \\ \vdots & & \\ \underbrace{\gamma_\mu\gamma_\nu\cdots\gamma_\alpha}_{\text{d matrices}} & = \Gamma^{i_d} & i_d = \binom{d}{d} = 1 \end{array}$$

where $\mu \neq \nu \neq \ldots \neq \rho = 0,1,\ldots,d-1$.

Using the Clifford-algebra of the γ_μ, (1.1.4), we can then find the properties of the Γ^A-matrices (which are not necessarily the same as the Γ-matrices of the conventional four-dimensional Dirac equation):

a) The matrices Γ^A are unimodular:

$$(\Gamma^A)^2 = \pm \mathbb{1} \qquad \text{all } \Gamma^A \qquad (1.1.7)$$

This follows at once, since $(\gamma_\mu)^2 = \pm \mathbb{1}$ and

$$\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu \qquad \nu \neq \mu \quad .$$

This holds for even and odd d.

b) If d is even, then the following <u>theorem</u> holds:

To each Γ^A except Γ^0 there corresponds at least one Γ^B such that

$$\Gamma^A\Gamma^B = -\Gamma^B\Gamma^A \qquad \text{all } \Gamma^A \neq \Gamma^0 \qquad (1.1.8)$$

This follows from the fact that for even d,

$$\{\gamma_\mu,\gamma_{d+1}\} = 0 \qquad \text{where } \gamma_{d+1} = \gamma_0\gamma_1\cdots\gamma_d \qquad (1.1.9)$$

This shows that (1.1.8) is certainly true for Γ^{i_1} because $\Gamma^{i_1} = \gamma_\mu$. To show the statement is true for Γ^{i_2}, we multiply (1.1.9) from the right by γ_ν to get

$$\{\gamma_\mu\gamma_\nu,\gamma_{d+1}\gamma_\nu\} = 0$$

Hence

$$\Gamma^{i_2} = -\gamma_{d+1}\gamma_\nu = -\gamma_0\cdots\gamma_\mu\cdots\gamma_d\,\gamma_\nu$$
$$= (-)^{d-\mu}\gamma_0\cdots\gamma_{\mu-1}\gamma_{\mu+1}\cdots\gamma_d\,\gamma_\mu\gamma_\nu$$

$$= (-)^{d-\mu}\, \gamma_0 \cdots \gamma_{\mu-1}\gamma_{\mu+1}\cdots\gamma_d\, \Gamma^{i_2} \tag{1.1.10}$$

$$\equiv -\Gamma^B\, \Gamma^{i_2} \tag{1.1.11}$$

By swopping γ-matrices in (1.1.10), we can adjust the sign such that (1.1.11) holds. Γ^B is then automatically defined. The procedure can now be repeated to show that (1.1.8) is indeed true or all Γ^A.

As an example, we find all the matrices Γ^B belonging to matrices Γ^A in four-dimensional Minkowski space:

Firstly, to Γ^{i_1} belongs Γ^{i_4} :

- $$\Gamma^{i_1}\,\Gamma^{i_4} = \gamma_\mu\, \gamma_0\gamma_1\gamma_2\gamma_3 = -\,\Gamma^{i_4}\,\Gamma^{i_1} \quad , \tag{1.1.12}$$

since we know that $\{\gamma_5,\gamma_\mu\} = 0$.

Secondly,

- $$(\gamma_\mu\gamma_\nu)(\gamma_\alpha\gamma_\beta\gamma_\delta) = -(\gamma_\alpha\gamma_\beta\gamma_\delta)(\gamma_\mu\gamma_\nu), \qquad \mu \neq \nu,\ \alpha \neq \beta \neq \delta$$

implies

$$\Gamma^{i_2}\,\Gamma^{i_3} = -\Gamma^{i_3}\,\Gamma^{i_2} \quad . \tag{1.1.13}$$

Now we can use (1.1.13) to show that to $\Gamma^A = \Gamma^{i_3}$ belongs $\Gamma^B = \Gamma^{i_2}$. Similarly from (1.1.12) we see that to $\Gamma^A = \Gamma^{i_4}$ belongs $\Gamma^A = \Gamma^{i_1}$ in the sense of (1.1.8). This concludes our example.

For <u>odd n</u>, the statement (1.1.8) is not true for all Γ^A since we now have $[\gamma_{d+1},\gamma_\mu] = 0$ for all γ_μ. So this proof does not strictly apply for the case where d is odd. This is due to the fact that for odd d, not all the matrices Γ^A are linearly independent. However as we shall see shortly, the subset of matrices $\Gamma^0, \Gamma^{i_1}, \ldots \Gamma^{i_{[d/2]}}$ forms a linearly independent set of matrices for which theorem (1.1.8) obviously holds.

c) The matrices $\Gamma^A \neq \Gamma^0$ are traceless:

From (b), we have for even d:

$$\pm \Gamma^A = - \Gamma^B \Gamma^A \Gamma^B$$

$$\Rightarrow \quad \pm \operatorname{tr} \Gamma^A = -\operatorname{tr} \Gamma^B \Gamma^A \Gamma^B$$

$$\Rightarrow \quad \pm \operatorname{tr} \Gamma^A = -\operatorname{tr} (\Gamma^B)^2 \, \Gamma^A \qquad \text{(cyclically permuted under the trace)}$$

$$= \operatorname{tr} (\Gamma^B)^2 \, \Gamma^A \qquad \text{(since } \Gamma^A \Gamma^B = -\Gamma^B \Gamma^A\text{)}$$

$$= 0 \tag{1.1.14}$$

Hence it follows that $\operatorname{tr} \Gamma^A = 0$ for all $\Gamma^A \neq \Gamma^0$ for <u>even d</u>.

Again, for <u>odd d</u>, the linearly independent <u>subset</u> of matrices excluding Γ^0 contains only traceless matrices.

d) The following <u>theorem</u> follows from our observations (a) to (c):

For a given Γ^A and a given Γ^B with $A \neq B$ there exists a $\Gamma^C \neq \Gamma^0$ such that

$$\Gamma^A \Gamma^B = \pm \Gamma^C \quad . \tag{1.1.15}$$

This is immediately evident since in any product of γ's of the form $\gamma_\alpha \gamma_\beta \ldots \gamma_\delta$, any two identical γ's will multiply out to give a factor one plus a sign. We are thus left with a product of γ's which are all different- this gives one of the Γ^A again. If $\Gamma^A = \Gamma^0$, the result is obvious: $\Gamma^B = \Gamma^C$.

e) These observations enable us to show that the set of matrices Γ^A are linearly independent for even d:

Let $\sum_{i=1}^{m} a_i \Gamma^i = 0$,

where m is the number of different matrices Γ^i formed.

Multiply by $\Gamma^A \neq \Gamma^0$:

$$\pm\, a_A \, \mathbb{1} + \sum_{i \neq A}^{m} a_i \Gamma^A \Gamma^i = 0.$$

Take the trace:

$$\pm\, n\, a_A \pm \sum_{i \neq A}^{m} a_i \operatorname{tr}(\Gamma^A \Gamma^i) = 0.$$

But we know that $\Gamma^A \Gamma^i = \pm\, \Gamma^C$ (from (d)) and tr $\Gamma^C = 0$ (from (c)). Hence

$$a_A = 0 \qquad\qquad \text{for all } a_A \neq a_0 \quad .$$

If $\Gamma^A = \Gamma^0$, then from

$$0 = \operatorname{tr}\Big(\sum_{i=1}^{m} a_i \Gamma^i \Big)$$

we get

$$0 = a_0 \operatorname{tr}(\Gamma^S)^2 + \sum_{i \neq S}^{m} a_i \operatorname{tr}(\Gamma^i)$$

$$= n\, a_0 \quad ,$$

which again gives

$$a_0 = 0.$$

Hence $a_A = 0$ for all A, and we have thus shown the linear independence of the matrices Γ^A for even d.

f) The following picture emerges:

Table 1.1: Products of γ-matrices in d spacetime dimensions

Γ^A	no. of possible different matrices
$\mathbb{1}$	$1 = \binom{d}{0}$
γ_μ	$d = \binom{d}{1}$
$\gamma_\mu\gamma_\nu$	$\binom{d}{2}$
$\vdots$	$\vdots$
$\gamma_{d+1} = \underbrace{\gamma_\mu\gamma_\nu\cdots\gamma_\omega}_{\text{d matrices}}$	$1 = \binom{d}{d}$

If d is even, then $\{\gamma_{d+1},\gamma_\mu\} = 0$ for all γ_μ since

$$\gamma_{d+1}\gamma_\mu + \gamma_\mu\gamma_{d+1} = \gamma_{d+1}\gamma_\mu + (-)^{d+1}\gamma_{d+1}\gamma_\mu = 0 \quad .$$

We thus have

$$\sum_{i=0}^{d} \binom{d}{i} = 2^d$$

linearly independent matrices.

If d is odd, then not all the matrices Γ^A are linearly independent. We have:

$$[\gamma_{d+1},\gamma_\mu] = 0 \text{ for all } \gamma_\mu.$$

Similarly,

$$\begin{aligned}
[\gamma_{d+1},\Gamma^{i_2}] &= [\gamma_{d+1},\gamma_\mu\gamma_\nu] \\
&= \gamma_{d+1}\gamma_\mu\gamma_\nu - \gamma_\mu\gamma_\nu\gamma_{d+1} \\
&= (-)^{d-1}\gamma_\mu\gamma_{d+1}\gamma_\nu - \gamma_\mu\gamma_\nu\gamma_{d+1} \\
&= \gamma_\mu[\gamma_{d+1},\gamma_\nu] \\
&= 0.
\end{aligned}$$

When continuing in this way, we find

$$\underbrace{\gamma_\mu\gamma_\nu\cdots\gamma_\omega}_{\frac{d}{2}\text{ terms}} = \pm\, \gamma_{d+1}\underbrace{\gamma_\rho\gamma_\theta\cdots\gamma_\sigma}_{\frac{d-1}{2}\text{ terms}} \quad ,$$

$$\underbrace{\gamma_\mu\gamma_\nu\cdots\gamma_\omega}_{\frac{d+1}{2}\text{ terms}} = \pm\, \gamma_{d+1}\underbrace{\gamma_\rho\gamma_\theta\cdots\gamma_\sigma}_{\frac{d-2}{2}\text{ terms}} \quad ,$$

and so forth.

We conclude that the matrices $\Gamma^{i_1},\Gamma^{i_2},\ldots,\Gamma^{i_{(d-1)/2}}$ form a complete set of $\frac{1}{2}\cdot 2^d = 2^{d-1}$ lineraly independent elements which spans the space of the γ-matrices. Since γ_{d+1} commutes with each member of this set, it follows from Schur's Lemma that

$$"\gamma_5"\ (d=\text{odd}) = \text{const.}\mathbb{1},$$

and hence γ_{d+1} is already contained in the above set. Thus for odd d, there are 2^{d-1} linearly independent matrices Γ^A.

g) In R^n (with n×n matrices) there are at most n^2 linearly independent matrices. Having found the number of possible different linearly independent matrices Γ^A, we can find the maximum size (dimension) of the Γ^A and hence also of the γ-matrices themselves. We conclude that the dimension of the γ_μ is:

$$\begin{aligned} \dim(\gamma_\mu) = 2^{[(d-1)/2]} &= 2^{d/2} \qquad ; \text{ even d} \\ &= 2^{(d-1)/2} \quad ; \text{ odd d.} \end{aligned} \tag{1.1.16}$$

1.2 Representations of the Clifford Algebra in Four-Dimensional Minkowski Space

i) The Bjorken and Drell, or Standard Representation is given by

$$\gamma_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \ , \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \ ,$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

These matrices satisfy

$$\{\gamma_\mu,\gamma_\nu\} = 2\eta_{\mu\nu}\ \mathbb{1}$$

$$(\gamma_\mu)^* = \gamma_\mu \qquad\qquad \mu \neq 2$$

$$(\gamma_2)^* = -\gamma_2 \ ,$$

$$(\gamma_i)^\dagger = -\gamma_i \qquad\qquad i = 1,2,3$$

$$(\gamma_0)^\dagger = \gamma_0$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

ii) The Weyl Representation is given by:

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad , \qquad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

Then the γ-matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\, \mathbb{1}$$

and

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

The Weyl representation is designed to have the specific form of γ_5 shown, which is particularly useful to project out left- and right-handed components of spinors.

iii) The Majorana Representation is given by:

$$\gamma_0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} , \; \gamma_1 = i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} ,$$

$$\gamma_2 = i\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} , \; \gamma_3 = i\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

The γ-matrices satyisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\, \mathbb{1}$$

and also

$$(\gamma_\mu)^* = -\gamma_\mu \quad ,$$

$$(\gamma_i)^\dagger = -\gamma_i \qquad\qquad i = 1,2,3$$

$$(\gamma_0)^\dagger = \gamma_0$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = i\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} .$$

The important aspect about the Majorana representation is that all the γ_μ are purely imaginary.

1.3 The Charge Conjugation, Parity Transformation and Time Translation Symmetry of the Dirac Equation[2]

a) Charge Conjugation:

We claim that the wavefunctions of the electron and positron both obey the Dirac equation, with the distinction that the electron has negative charge and the positron has positive charge:

$$(\not p - e\not A - m)\Psi = 0 \qquad \text{electron (E)} \tag{1.3.1}$$

$$(\not p + e\not A - m)\Psi_C = 0 \qquad \text{positron (P)} \tag{1.3.2}$$

We now require an operator C which transforms the one wavefunction into the other. To do this, we note that for the transition from (E) to (P), one has to change the relative sign of p and A. This can be achieved by complex conjugation of (E):

$$((i\hbar\partial^\mu + eA^\mu)\gamma_\mu{}^* + m)\Psi^* = 0 \tag{1.3.3}$$

To obtain (P), we now multiply this equation by a non-singular matrix C:

$$(i\hbar\partial^\mu + eA^\mu)\ C\ \gamma_\mu{}^*C^{-1} + m)\ C\Psi^* = 0 \tag{1.3.4}$$

and thus C has to satisfy

$$C\gamma_\mu^*\ C^{-1} = -\gamma_\mu \quad . \tag{1.3.5}$$

Using (1.3.5), we get from (1.3.4)

$$((i\hbar\partial^{\mu} + eA^{\mu})\gamma_{\mu} - m)\ C\ \Psi^{*} = 0,$$

hence

$$C\ \Psi^{*} = \Psi_{C}\quad .$$

If we have a Majorana representation, this gives

$$C = 1 \tag{1.3.6)(a}$$

and

$$\Psi_{C} = \Psi^{*}\quad , \tag{1.3.6)(b}$$

regardless of the dimension of space-time.

b) Parity

We want the Dirac equation to be invariant under the parity operation (which is an improper Lorentz transformation):

$$(t,\vec{x})' = (t,-\vec{x}) \tag{1.3.7)(a}$$

$$(A_0,\vec{A})' = (A_0,-\vec{A}) \tag{1.3.7)(b}$$

The corresponding transformation is

$$x^{\mu} = a^{\mu}{}_{\nu}\ x^{\nu} \tag{1.3.8}$$

with $a^{\mu}{}_{\nu} = \eta^{\mu\nu}$.

In analogy to (a) above, we must now satisfy the equation

$$a^{\mu}{}_{\nu}\gamma^{\nu} = P\, \gamma^{\nu}\, P^{-1} \tag{1.3.9}$$

for the parity operator P which gives the parity-transformed wave-function:

$$P\Psi = \Psi_P \quad . \tag{1.3.10}$$

Evaluation gives

$$P = e^{i\phi}\, \gamma^0 \quad , \; P^{-1} = e^{-i\phi}\, \gamma^0 \; . \tag{1.3.11}$$

This is valid regardless of the dimension of space-time since all that is required here is that $(\gamma^0)^2 = 1$.

If we now require that $(P)^4\Psi = \Psi$, one obtains

$$e^{i\phi} = \pm 1, \; \pm i \quad . \tag{1.3.12}$$

This result is independent of the representation used.

c) <u>Time Reversal</u>

One requires the Dirac equation to be invariant under the time reversal operation:

$$(t,\vec{x})' = (-t,\vec{x})$$

$$(A_0,\vec{A})' = (A_0,-\vec{A})$$

This transformation changes the relative sign of x^{μ} and A^{μ}, which can be achieved again by complex conjugation:

$$(\gamma^*_{\mu}\, (-i\hbar\partial^{\mu} - eA^{\mu}) - m)\, \Psi^* = 0 \tag{1.3.13}$$

In the Majorana representation, $(\gamma_\mu)^* = -\gamma_\mu$ and we can thus write

$$(\gamma_\mu(i\hbar\partial^\mu + eA^\mu) - m)\,\Psi^* = 0. \qquad (1.3.14)$$

Multiply by γ_0:

$$((-)^{\delta_{\mu 0}+1}\gamma_\mu(i\hbar\partial^\mu + eA^\mu) - m)\,\gamma_0\Psi^* = 0. \qquad (1.3.15)$$

Use the transformed coordinates:

$$x^\mu \to (-)^{\delta_{\mu 0}}\, x^\mu$$

$$A^\mu \to (-)^{\delta_{\mu 0}+1}\, A^\mu$$

This gives

$$(-\gamma_\mu(i\hbar\partial^\mu - eA^\mu) - m)\,\gamma_0\Psi^* = 0. \qquad (1.3.16)$$

For this equation to yield the Dirac equation for the transformed wavefunction Ψ_T, we need to operate on it with a matrix M such that

$$M\gamma_\mu = -\gamma_\mu M \qquad \text{all } \gamma_\mu$$

Then $\Psi_T = M\gamma_0\Psi^*$. (1.3.17)

This reasoning holds for arbitrary dimensions of space-time, provided the matrix M can be found.

For <u>even d</u>, we know that $\{\gamma_{d+1},\gamma_\mu\} = 0$ and hence clearly

$$M = \gamma_{d+1} \qquad (1.3.18)$$

For <u>odd d</u>, an M such that $\{M,\gamma_\mu\} = 0$ for all γ_μ does not exist. However, if d = odd and m = 0, the solution is M = 1 trivially.

<u>1.4 Construction of the Dirac Matrices in Arbitrary Dimensions</u>

We try to find a Majorana representation for the γ_μ in arbitrary spacetime dimensions, starting in d = 2 with the Pauli matrices. For the Majorana representation, we have to meet the requirements:

$$(\gamma_0)^2 = \mathbb{1}\ ,\ (\gamma_i)^2 = -\mathbb{1} \qquad i = 1,2,\ldots,d-1 \tag{1.4.1)(a}$$

$$(\gamma_\mu)^* = -\gamma_\mu \tag{1.4.1)(b}$$

$$\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu \qquad \mu \neq \nu \tag{1.4.1)(c}$$

We construct the representation such that we define "γ_5" = $i\gamma_0\gamma_1\cdots\gamma_d$ so that it has the meaning of the conventional four-dimensional γ_5. As before, $\gamma_{d+1} = \gamma_0\gamma_1\cdots\gamma_d$ in d-dimensional spacetime.

First, we check that for d = 1 together with our convention for the metric, there is no Majorana representation possible:

If we take the one dimension as a "time" dimension, which coincides with our definition of the Minkowski metric, then we need

$$\gamma_0^{\ 2} = 1 \tag{1.4.2}$$

$$\gamma_0^{\ *} = -\gamma_0 \tag{1.4.3}$$

But since in one spacetime dimension, the dimension of γ_0 is also 1, (1.4.2) together with (1.4.3) is not possible for any number γ_0. In d = 1, we thus cannot have a Majorana representation in <u>Minkowski space</u>.

However, if we regard the one dimension as a space dimension, we would have the metric $\eta = -1$ (i.e. not the Minkowski metric) and then we need

$$\gamma_0{}^2 = -1$$

$$\gamma_0{}^* = -\gamma_0 \qquad (1.4.4)$$

which can be satisfied by taking $\gamma_0 = i$.

• Now we consider the case <u>d = 2:</u>

The dimension of the representation is

$$\dim(\gamma_\mu) = 2^{d/2} = 2.$$

We take $\gamma_0 = \sigma_2 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -(\gamma_0)^*$,

$$\gamma_1 = i\sigma_1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -(\gamma_1)^* .$$

Then (1.4.1) has been met and

$$"\gamma_5" = i\ \gamma_0\gamma_1 = i^2\sigma_2\sigma_1 = i\sigma_3 \quad .$$

• <u>d = 3:</u>

If we now continue in this way, we must take

$$(\gamma_0)_3 = (\gamma_0)_2 = \sigma_2 \ ,$$

$$(\gamma_1)_3 = (\gamma_1)_2 = i\sigma_1 \ ,$$

$$(\gamma_2)_3 = ("\gamma_5")_2 = i\sigma_3 \ ,$$

This set satisfies (1.4.1) and

$$("\gamma_5")_3 = i\gamma_0\gamma_1\gamma_2 = -\mathbb{1}.$$

• <u>d = 4:</u>

The dimension of the representation is now

$$\dim(\gamma_\mu)_4 = 2^{4/2} = 4 \ .$$

We take
$$(\gamma_0)_4 = \begin{pmatrix} 0 & (\gamma_0)_3 \\ (\gamma_0)_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$

$$(\gamma_1)_4 = \begin{pmatrix} 0 & (\gamma_1)_3 \\ (\gamma_1)_3 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$$

$$(\gamma_2)_4 = \begin{pmatrix} 0 & (\gamma_2)_3 \\ (\gamma_2)_3 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$(\gamma_3)_4 = i \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

This is the conventionally used Majorana representation for d = 4 and gives

$$"\gamma_5" = i\gamma_0\gamma_1\gamma_2\gamma_3 = i \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \ .$$

• <u>d = 5:</u>

We continue as above and take

$$(\gamma_\alpha)_5 = (\gamma_\alpha)_4 \qquad\qquad \alpha = 0,1,2,3$$

and $(\gamma_4)_5$ must be of the form

$$(\gamma_4)_5 = iM,$$

where M is a purely real matrix which we try to determine now.

We require the validity of the Clifford algebra

$$M(\gamma_\alpha)_4 = -(\gamma_\alpha)_4 M \qquad \alpha = 0,1,2,3 \tag{1.4.5}$$

The requirement (1.4.1)(a) means

$$(iM)^2 = -\mathbb{1} \quad , \tag{1.4.6}$$

which through the reality of M implies

$$M^2 = \mathbb{1} \quad . \tag{1.4.7}$$

To find M, let

$$iM = i\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A,B,C,D purely real 2×2 matrices. Eq.(1.4.7) then is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^2+BC & AB+BD \\ CA+DC & CB+D^2 \end{pmatrix} = 1,$$

or in component matrices

$$\begin{aligned} A^2+BC &= \mathbb{1} \\ AB+BD &= 0 \\ CA+DC &= 0 \\ CB+D^2 &= \mathbb{1} \end{aligned} \tag{1.4.8}$$

We have as the first γ-matrices of the set

$$(\gamma_0)_4 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad (\gamma_1)_4 = i \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$

$$(\gamma_2)_4 = i \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad (\gamma_3)_4 = i \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \qquad (1.4.9)$$

Then (1.4.5) must hold for $(\gamma_3)_4$:

$$M(\gamma_3)_4 = -(\gamma_3)_4 M$$

This gives

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = - \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} = \begin{pmatrix} -A & B \\ -C & D \end{pmatrix}$$

In components:

$$A = 0$$
$$B = 0 \qquad (1.4.10)$$

This reduces eq. (1.4.8) to

$$BC = \mathbb{1} = CB. \qquad (1.4.11)$$

Similarly, for $(\gamma_2)_4$ from (1.4.5)

$$M(\gamma_2)_4 = -(\gamma_2)_4 M$$

gives, using (1.4.9) and (1.4.10),

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} B\sigma_3 & 0 \\ 0 & C\sigma_3 \end{pmatrix} = - \begin{pmatrix} \sigma_3 C & 0 \\ 0 & \sigma_3 B \end{pmatrix}$$

and hence

$$\begin{aligned} B\sigma_3 &= -\sigma_3 C \\ C\sigma_3 &= -\sigma_3 B. \end{aligned} \tag{1.4.12}$$

So for the matrices B,C we get from (1.4.12)

$$\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} B_1 & -B_2 \\ B_3 & -B_4 \end{pmatrix} = \begin{pmatrix} -C_1 & -C_2 \\ C_3 & C_4 \end{pmatrix} . \tag{1.4.13}$$

Similarly, from

$$M(\gamma_1)_4 = -(\gamma_1)_4 M$$

and

$$M(\gamma_0)_4 = -(\gamma_0)_4 M$$

we get

$$\begin{pmatrix} B_2 & -B_1 \\ B_4 & -B_3 \end{pmatrix} = \begin{pmatrix} C_3 & C_4 \\ -C_1 & -C_2 \end{pmatrix} \tag{1.4.14}$$

and

$$\begin{pmatrix} B_2 & B_1 \\ B_4 & B_3 \end{pmatrix} = \begin{pmatrix} -C_3 & -C_4 \\ -C_1 & -C_2 \end{pmatrix} . \tag{1.4.15}$$

From (1.4.13),(1.4.14),(1.4.15) it then follows that the components of B, C satisfy

$$B_1 = -C_1 = -C_4$$

$$B_2 = C_2 = -C_3 = C_3 = 0$$

$$B_3 = C_3 = -C_2 = C_2 = 0$$

$$B_4 = -C_4 = -C_1 = B_1 \tag{1.4.16}$$

and hence we can write down the matrices B and C in component form

$$B = \begin{pmatrix} -C_1 & 0 \\ 0 & -C_1 \end{pmatrix} \tag{1.4.17}$$

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix} . \tag{1.4.18}$$

But from (1.4.11) we also have

$$\begin{pmatrix} -C_1 & 0 \\ 0 & -C_1 \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix} = 1,$$

which gives

$$\begin{pmatrix} -C_1^2 & 0 \\ 0 & -C_1^2 \end{pmatrix} = 1$$

and hence

$$C_1^2 = -1.$$

But we have constructed M such that its components are all real - this gives a contradiction. We have thus proved that for d = 5 there does not exist a Majorana representation.

We now generalise this result to arbitrary dimensions:

Generally we want for even space-time dimensions:

$$(\gamma_i)_{d=2n} = \begin{pmatrix} 0 & (\gamma_i)_{d=2n-1} \\ (\gamma_i)_{d=2n-1} & \end{pmatrix} \quad ; \; i = 0,1,\ldots,d-2$$

$$(\gamma_{d-1})_{d=2n} = \begin{pmatrix} i\mathbb{1} & 0 \\ 0 & -i\mathbb{1} \end{pmatrix} \tag{1.4.19)(a}$$

and for odd space-time dimensions:

$$(\gamma_i)_{d=2n+1} = (\gamma_i)_{d=2n} \qquad ; i = 0,1,\ldots,d-2$$

$$(\gamma_{d-1})_{d=2n+1} = (\omega\gamma_{d+1})_{d=2n} = (\text{"}\gamma_5\text{"})_{d=2n} \,, \tag{1.4.19)(b}$$

where ω is a phase which must be adjusted.

In order to obtain a Majorana representation, we need additionally

$$(\gamma_{d-1})^*_{d=2n+1} = -(\gamma_{d-1})_{d=2n+1} \tag{1.4.20}$$

and

$$(\omega\gamma_{d+1})^2_{d=2n} = -\mathbb{1} \quad . \tag{1.4.21}$$

Multiplying out (1.4.21) gives

$$\omega^2\gamma_0\gamma_1\ldots\gamma_{d-1}\gamma_0\ldots\gamma_{d-1} \quad = -\mathbb{1}$$

$$\Rightarrow \quad \omega^2(-)^{\frac{1}{2}d(d-1)}\,(-)^{d-1} \quad = -\mathbb{1}. \tag{1.4.22}$$

This is true since

$$\gamma_0\gamma_1\ldots\gamma_{d-1}\gamma_0\ldots\gamma_{d-1} = (-)^{(d-1)+(d-2)+\ldots+1}(\gamma_0)^2(\gamma_1)^2\ldots(\gamma_{d-1})^2$$

and $\quad (\gamma_0)^2(\gamma_1)^2\ldots(\gamma_{d-1})^2 = (-)^{d-1} \quad .$

From (1.4.22) it further follows that

$$\omega^2 = (-)^{\frac{1}{2}d(d-1) - (d-1) + 1}$$
$$= (-)^{\frac{1}{2}(d-1)(d-2) + 1} \quad .$$

ω is therefore real if

$$\frac{(d-1)(d-2)}{2} + 1 = 2n,$$

i.e. if $d = \begin{cases} 4n+3 \\ 4n \end{cases}$. (1.4.23)

We thus get in general:

d	ω	γ_{d+1}	$\tilde{\gamma}_{d+1} = \omega\gamma_{d+1}$
4n	real	real	real
4n+1	imaginary	imaginary	real
4n+2	imaginary	real	imaginary
4n+3	real	imaginary	imaginary

(1.4.24)

But since we want to use

$$(\gamma_{d-1})_{d=2n+1} = (\tilde{\gamma}_{d+1})_{d=2n} \tag{1.4.25}$$

and

$$(\gamma_{d-2})_{d=2n+2} = \begin{pmatrix} 0 & (\tilde{\gamma}_{d+1})_{d=2n} \\ (\tilde{\gamma}_{d+1})_{d=2n} & 0 \end{pmatrix} , \qquad (1.4.26)$$

and for the Majorana representation we want

$$(\gamma_\mu)^* = -\gamma_\mu \qquad \text{for all } \gamma_\mu, \qquad (1.4.27)$$

there is an inconsistency whenever $(\omega\gamma_{d+1})_{d=2n} = (\tilde{\gamma}_{d+1})_{d=2n}$ is real since then eq.(1.4.27) cannot be satisfied for $(\gamma_{d-1})_{d=2n+1}$. This means effectively that in some dimensions we cannot satisfy (1.4.20) and (1.4.21) simultaneously - in this case a Majorana representation cannot exist for the next higher dimension. For example, if $\omega\gamma_{d+1}$ is real for d = 4n, then for d = 4n+1 we cannot have a Majorana representation. Thus from (1.4.23), there does not exist a Majorana representation whenever

$$d = \begin{cases} 4n+1 \\ 4n+2 \end{cases} \qquad n \neq 0 \qquad (1.4.28)$$

i.e. if

$$\begin{aligned} d = {} & 5,\ 9,\ 13,\ \ldots \\ & 6,\ 10,\ 14,\ldots \end{aligned} \qquad (1.4.29)$$

It must be stressed that this result applies to the <u>Minkowski</u> metric together with the algorithm used for obtaining the γ-matrices, eqs.(1.4.19). It will in general always be possible to find a Majorana representation for a <u>specific</u> dimension:

Let $g_{\mu\nu} = \text{diag}\{\underbrace{1,1,\ldots,1}_{s \text{ indices}};\underbrace{-1,-1,\ldots-1}_{d-s \text{ indices}}\}$

Then, for a Majorana representation, we want

$$\{\gamma_\mu,\gamma_\nu\} = 2\, g_{\mu\nu}\, \mathbb{1},$$

$$(\gamma_\mu)^2 = g_{\mu\mu}\, \mathbb{1}$$

together with

$$(\omega\gamma_{d+1})^2 = -\mathbb{1},$$

since $(\omega\gamma_{d+1})$ corresponds to the (d+1)th diagonal term of the metric in dimension d+1 which is equal to -1. Accordingly we have

$$(\omega\gamma_{d+1})^2 = \omega^2\gamma_0\gamma_1\ldots\gamma_{d-1}\gamma_0\ldots\gamma_{d-1}$$

$$= \omega^2(-)^{\frac{1}{2}d(d-1)}(\gamma_0)^2(\gamma_1)^2\ldots(\gamma_{d-1})^2$$

Therefore we must have

$$-1 = \omega^2(-)^{\frac{1}{2}d(d-1)}(-)^{d-s}$$

so that

$$\omega^2 = (-)^{\frac{1}{2}d(d-1)-(d-s)+1}$$

and

$$\omega = (\pm i)^{-\frac{1}{2}d(d-1)+(1+s)} \quad .$$

Since we now have two arbitrary parameters, ω and s, instead of one, we can adjust these always such that

$$(\omega\gamma_{d+1})^2 = -\mathbb{1},$$

and hence, given a Majorana representation in dimension d, we can find one in dimension d+1 (not necessarily with the same metric!).

We can now also see why (1.4.28) is not valid for n = 0: in this case, we have not constructed the γ-matrices from existing γ-matrices in the lower dimension d = 1.

1.5 The Majorana Condition

To investigate the Majorana condition which demands that the particle wavefunction equals the antiparticle wavefunction,

$$\Psi_C = \Psi, \tag{1.5.1}$$

we have another look at charge conjugation:

We know that for the Dirac equation to be invariant under charge conjugation, we need a matrix U such that

$$U\,\gamma_\mu^*\,U^{-1} = -\gamma_\mu \tag{1.5.2}$$

as has been discussed before.

We define now the charge-conjugate spinor Ψ_C as follows:

$$\Psi_C = U\Psi \equiv C^{-1}\gamma_0^T\,\Psi^* \equiv C^{-1}\,\bar{\Psi}^T \tag{1.5.3}$$

Eq. (1.5.2) then becomes

$$(C^{-1}\gamma_0{}^T)\gamma_\mu{}^*(C^{-1}\gamma_0{}^T)^{-1} = -\gamma_\mu$$

$$\Rightarrow \quad (C^{-1}\gamma_0{}^T\,\gamma_\mu{}^*\gamma_0{}^{-1T}C)^T = -(\gamma_\mu)^T$$

This gives

$$C^T\gamma_0{}^{-1}\gamma_\mu{}^\dagger\gamma_0 C^{-1T} = -\gamma_\mu{}^T \tag{1.5.4}$$

For simplicity, we introduce the "preferred" representation of the Clifford algebra:[3]

$$\gamma_0 = \gamma_0{}^\dagger$$

$$\gamma_i = -\gamma_i{}^\dagger \quad . \tag{1.5.5}$$

The γ_μ generate a finite group, the Clifford group in d dimensions. Since for any <u>finite</u> group, any representation is equivalent to a unitary representation, we can assume this preferred representation as follows:

$$\gamma_0^\dagger = (\gamma_0)^{-1} = \gamma_0$$

$$\gamma_i^\dagger = (\gamma_i)^{-1} = -\gamma_i \ . \qquad (1.5.6)$$

This preferred class of representations of the Dirac algebra includes, for example, the Majorana representation (where it exists). In the Majorana representation, the matrices $(\gamma_\mu)_{d=2n+2}$ take the form

$$(\gamma_0)_{d=2n+2} = \begin{bmatrix} 0 & (\gamma_0)_{d=2n} \\ (\gamma_0)_{d=2n} & 0 \end{bmatrix} \quad ; \ (\gamma_0)^2_{d=2n} = 1 \qquad (1.5.6)(a)$$

$$(\gamma_i)_{d=2n+2} = \begin{bmatrix} 0 & (\gamma_i)_{d=2n} \\ (\gamma_i)_{d=2n} & 0 \end{bmatrix} \quad ; \ (\gamma_i)^2_{d=2n} = -1 \ ; \ i < d-2 \qquad (b)$$

$$(\gamma_{d-2})_{d=2n+2} = \begin{bmatrix} 0 & "\gamma_5"_{d=2n+1} \\ "\gamma_5"_{d=2n+1} & 0 \end{bmatrix} \quad ; \ ("\gamma_5")_{d=2n+1} \propto i1 \qquad (c)$$

$$(\gamma_{d-1})_{d=2n+2} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \Rightarrow (\gamma_{d-1})^2_{d=2n+2} = -1 \qquad (d)$$

We started the construction of these matrices in d = 2, where we have

$$(\gamma_0)_2 = \sigma_2$$

$$(\gamma_0)_2^\dagger = \gamma_0$$

and hence $(\gamma_0)^\dagger_{d=2n+2} = \gamma_0$ for any d.

Similarly,

$$(\gamma_1)_{d=2} = i\sigma_1$$

$$(\gamma_1)^{\dagger}_{d=2} = -\sigma_1 \quad .$$

Also,

$$("\gamma_5")^{\dagger}_{d=2n+1} = -("\gamma_5")_{d=2n+1}$$

as is clear from (1.5.6)(c).

Finally,

$$(\gamma_{d-1})^{\dagger}_{d=2n+2} = -(\gamma_{d-1})_{d=2n+2}$$

from (1.5.6)(d).

This procedure can now be generalised to all dimensions where the Majorana representation exists.

Hence

$$(\gamma_i)^{\dagger}_{d=2n+2} = -(\gamma_i)^{\dagger}_{d=2n+2} \quad . \qquad \text{for all } i \neq 0 \quad .$$

For odd d,

$$(\gamma_\mu)_{d=2n+1} = (\gamma_\mu)_{d=2n} \qquad \text{for } \mu \neq d-1.$$

For these, we have already shown that (1.5.5) holds. Finally,

$$(\gamma_{d-1})_{d=2n+1} = (\omega\gamma_{d+1})_{d=2n} \quad ,$$

which has the form

$$(\gamma_{d-1})_{d=2n+1} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and thus clearly $(\gamma_{d-1})_{d=2n+1} = -(\gamma_{d-1})^{\dagger}_{d=2n+1}$.

This completes the proof of the fact that the Majorana representation is a preferred representation.

We show that the same holds for the Weyl representation:

For the Weyl representation, we need γ-matrices such that "γ_5" takes the form

$$"\gamma_5" = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{1.5.7}$$

Also, the Weyl representation only makes sense in even spacetime dimensions (in odd dimensions, "γ_5" $\propto 1$, i.e. not the form (1.5.7)).

Starting from the Weyl representation in d = 4 (for convenience),

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \Rightarrow \gamma_0^{\dagger} = \gamma_0$$

$$\gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \Rightarrow \gamma_i^{\dagger} = -\gamma_i$$

$$\gamma_5 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \qquad \Rightarrow \gamma_5^{\dagger} = -\gamma_5$$

we can build up Weyl representations in higher dimensions by taking

$$(\gamma_0)_{d=2n+2} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \Rightarrow (\gamma_0)^{\dagger}_{d=2n+2} = (\gamma_0)_{d=2n+2}$$

$$(\gamma_i)_{d=2n+2} = \begin{pmatrix} 0 & -(\gamma_i)_{2n} \\ (\gamma_i)_{2n} & 0 \end{pmatrix} \qquad ;i < d-2$$

with

$$(\gamma_i)_{2n} = -(\gamma_i)^{\dagger}_{2n} \quad .$$

This gives

$$(\gamma_i)^{\dagger}_{d=2n+2} = -(\gamma_i)_{d=2n+2}$$

Also,

$$(\gamma_{d-2})_{d=2n+2} = \begin{bmatrix} ("\gamma_5")_{d=2n} & 0 \\ 0 & -("\gamma_5")_{d=2n} \end{bmatrix}$$

with

$$("\gamma_5")^{\dagger}_{d=2n} = -("\gamma_5")_{d=2n}$$

and thus

$$(\gamma_{d-2})^{\dagger}_{d=2n+2} = -(\gamma_{d-2})_{d=2n+2}$$

and finally

$$(\gamma_{d-1})_{d=2n+2} = \begin{bmatrix} 0 & -(\gamma_0)_{2n} \\ (\gamma_0)_{2n} & 0 \end{bmatrix}$$

with

$$(\gamma_{d-1})^{\dagger}_{d=2n+2} = -(\gamma_{d-1})_{d=2n+2}$$

With this algorithm we get

$$("\gamma_5")_{2n+2} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

and the Weyl representation is thus included in the set of preferred representations.

We now return to equation (1.5.4):

Let $\mu = 0$: then

$$C^T\gamma_0 C^{-1T} = -\gamma_0{}^T$$

$$\Rightarrow \quad C^{-1}\gamma_0{}^T C = -\gamma_0$$

$$\Rightarrow \quad -\gamma_0{}^T = C\gamma_0 C^{-1} \quad ,$$

and if $\mu \neq 0$:

$$C^T\gamma_0{}^{-1}\gamma_i{}^\dagger\gamma_0 C^{-1T} = -\gamma_i{}^T$$

$$\Rightarrow \quad -C^T\gamma_0\gamma_i\gamma_0 C^{-1T} = -\gamma_i{}^T$$

$$\Rightarrow \quad C^T\gamma_i C^{-1T} = -\gamma_i{}^T$$

$$\Rightarrow \quad -\gamma_i{}^T = C\gamma_i C^{-1} \quad .$$

Generally we can thus say

$$-\gamma_\mu{}^T = C\gamma_\mu C^{-1} \tag{1.5.8}$$

This relation may now be explored further to determine the charge conjugation matrix C.

We take the adjoint of eq.(1.5.8):

$$-\gamma_\mu{}^T = C^{-1\dagger}\gamma_\mu C^\dagger \tag{1.5.9}$$

Using again (1.5.8), this becomes

$$C\gamma_\mu C^{-1} = C^{-1\dagger}\gamma_\mu C^\dagger$$

This gives

$$\gamma_\mu = C^{-1}C^{-1\dagger}\gamma_\mu C^\dagger C$$

$$= (C^\dagger C)^{-1}\gamma_\mu(C^\dagger C)$$

Thus

$$[\gamma_\mu, C^\dagger C] = 0 \quad . \tag{1.5.10}$$

From Schur's Lemma this means that

$$C^\dagger C = \alpha\, \mathbb{1} \qquad \text{for some constant } \alpha \tag{1.5.11}$$

and we can rescale C so that

$$C^\dagger = C^{-1} = C^* \quad . \tag{1.5.12}$$

We are now ready to investigate the symmetry of the C-matrix[3]:

Since from (1.5.8),

$$-\gamma_\mu = C^{-1T}\gamma_\mu{}^T C^T$$

$$= -C^{-1T}C\gamma_\mu C^{-1}C^T$$

$$= -(C^{-1}C^T)^T\gamma_\mu(C^{-1}C^T),$$

it follows that

$$[\gamma_\mu, C^{-1}C^T] = 0$$

and thus

$$C^{-1}C^T = p\,1 \qquad \text{(from Schur's Lemma)}$$

which gives finally

$$C^T = pC \qquad \text{with } p^2 = 1. \tag{1.5.13}$$

Now we develop the following product of matrices:

$$\begin{aligned}(C\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_r})^T &= (\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_r})^T C^T \\ &= (\gamma_{\mu_r})^T(\gamma_{\mu_{r-1}})^T\ldots(\gamma_{\mu_1})^T pC \\ &= (-)^r(C\gamma_{\mu_r}C^{-1})(C\gamma_{\mu_{r-1}}C^{-1})\ldots(C\gamma_{\mu_1}C^{-1})pC \\ &= (-)^r(C\gamma_{\mu_r}\gamma_{\mu_{r-1}}\ldots\gamma_{\mu_1}C^{-1})pC \\ &= (-)^r pC\gamma_{\mu_r}\gamma_{\mu_{r-1}}\ldots\gamma_{\mu_1} \\ &= (-)^r pC\ (-)^{\frac{1}{2}r(r-1)}\ \gamma_{\mu_1}\ldots\gamma_{\mu_r}\end{aligned}$$

Thus finally

$$(C\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_r})^T = (-)^{\frac{1}{2}r(r+1)}\ pC\ \gamma_{\mu_1}\ldots\gamma_{\mu_r}\ . \tag{1.5.14}$$

The product of matrices $C\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_r}$ is thus either symmetric $(p = 1)$ or antisymmetric $(p = -1)$. The general picture is as follows:

Table 1.3: Symmetry of the Charge Conjugation Matrix C

$C\Gamma^A$	Symmetry of $C\Gamma^A$	no. of lin. indep. matrices $C\Gamma^A$
C	$+p$	$\binom{d}{0}$
$C\gamma_\mu$	$-p$	$\binom{d}{1}$
$C\gamma_\mu\gamma_\nu$	$-p$	$\binom{d}{2}$
$C\gamma_\mu\gamma_\nu\gamma_\rho$	$+p$	$\binom{d}{3}$
$C\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$	$+p$	$\binom{d}{4}$
. . . etc.		

It can be shown easily that among a complete set of n×n matrices, there are $\frac{1}{2}n(n+1)$ symmetric and $\frac{1}{2}n(n-1)$ antisymmetric matrices. Also, the set of matrices $C\gamma_\mu$, $C\gamma_\mu\gamma_\nu$, ..., $C\gamma_\mu\gamma_\nu\ldots\gamma_\sigma$ forms a complete set of n×n matrices, where n is the dimension of the γ-matrices.This fact can be used to determine the value of p in the corresponding dimensions.

We have generally:

$$(C\gamma_{\mu_1}\ldots\gamma_{\mu_r})^T = pC\gamma_{\mu_1}\ldots\gamma_{\mu_r} \qquad r = 4n,\ 4n+3$$

$$= -pC\gamma_{\mu_1}\ldots\gamma_{\mu_r} \qquad r = 4n+1,\ 4n+2$$

$$= +pC\gamma_{\mu_1}\ldots\gamma_{\mu_r} \qquad d = \begin{cases} 8n+2 \\ 8n+3 \\ 8n+4 \end{cases} \qquad (1.5.15)(a)$$

$$= -pC\gamma_{\mu_1}\ldots\gamma_{\mu_r} \qquad d = \begin{cases} 8n+6 \\ 8n+7 \\ 8n+8 \end{cases} \qquad (1.5.15)(b)$$

$$= \text{nonexistent} \qquad d = \begin{cases} 8n+1 \\ 8n+5 \end{cases} \qquad (1.5.15)(c)$$

Comparing the number of symmetric (antisymmetric) n×n matrices in R^n with the number of symmetric (antisymmetric) matrices $C\gamma_{\mu_1}\ldots\gamma_{\mu_r}$, we find that in some dimensions there is a one-one correspondence between the two sets. Where the numbers do not agree, C cannot exist. Where they do agree, we can determine the value of p from this correspondence. For example, in d = 2, we have $\binom{2}{0}$ = 1 2×2 matrix with the symmetry +p, and $\binom{2}{1} + \binom{2}{2}$ = 3 matrices with the symmetry -p. There are a total of 4 linearly independent 2×2 matrices in R^n, of which 3 are symmetric and one is antisymmetric. We thus conclude that in this case, p = -1. Care must be taken in odd dimensions to take only the set of 2^{d-1} linearly independent matrices and not the set of 2^d possible matrices. The general result is shown in the following table:

Table 1.4: Determination of p from the Symmetry of $C\Gamma^A$

d	$\dim(\gamma^\mu)$	No. of linearly ind. $C\Gamma^A$-matrices	$\frac{1}{2}n(n+1)$ =no. of symmetric d×d matrices	$\frac{1}{2}n(n-1)$ =no. of antisymm. d×d matrices	No. of $C\Gamma^A$ matr. with symmetry +p	No. of $C\Gamma^A$ matr. with symmetry -p	p
2	2	4	3	1	1	3	-1
3	2	4	3	1	1	3	-1
4	4	16	10	6	6	10	-1
5	4	16	10	6	1	15	x
6	8	64	36	28	36	28	+1
7	8	64	36	28	36	28	+1
8	16	256	136	120	136	120	+1
9	16	256	136	120	211	45	x
10	32	1024	528	496	496	528	-1
11	32	1024	528	496	496	528	-1

From the table it becomes clear that for d = 4n+1, C cannot exist. In these dimensions, we can thus certainly not have a Majorana condition. To obtain the general result, we impose the Majorana condition and demand that it be consistent.

We had previously

$$\gamma_\mu{}^T C^T = -pC\gamma_\mu \tag{1.5.16}$$

So for p = 1, i.e. for d = 6,7,8 (mod 8)

$$\gamma_\mu{}^T C^T = -C\gamma_\mu \qquad (1.5.17)(a)$$

and for p = -1 i.e. for d = 2,3,4 (mod 8)

$$\gamma_\mu{}^T C^T = C\gamma_\mu \ . \qquad (1.5.17)(b)$$

The Majorana condition is

$$\Psi = \Psi_C = C^{-1}\bar{\Psi}^T = C^{-1}\gamma_0{}^T\Psi \ . \tag{1.5.18}$$

From (1.5.18),

$$\bar{\Psi}^T = \gamma_0{}^T\Psi_C^* = \gamma_0{}^T C^{-1*}\gamma_0{}^T\Psi$$

$$= \gamma_0{}^T C^T \gamma_0 \Psi \qquad \text{since } C^\dagger = C^{-1} = -C^*$$

$$= p\gamma_0{}^T C\gamma_0\Psi \qquad \text{since } pC = C^T$$

$$= p\gamma_0{}^T(-p\gamma_0{}^T C^T)\Psi$$

$$= -p^2 C^T\Psi = -p^3 C\Psi = -pC\Psi \ . \tag{1.5.19}$$

Now we substitute this into (1.5.18) to check if it is consistent:

$$\Psi = C^{-1}\bar{\Psi}^T = C^{-1}(-pC\Psi) = -p\Psi \quad . \qquad (1.5.20)$$

We see that for p = 1, this is inconsistent and conclude that Majorana spinors can only exist in d = 2,3,4 (mod 8).

1.6 The Weyl Condition

For the Weyl condition (which demans that left-handed and right-handed massless particles can be described by one two-component wave function), we need

$$\alpha\gamma_{d+1}\Psi = \pm\,\Psi \quad , \qquad (1.6.1)$$

where $\alpha\gamma_{d+1}$ has the meaning of γ_5 in four-dimensional spacetime.

Eq. (1.6.1) is trivially satisfied in odd dimensions since then $\gamma_{d+1}=$ const$\cdot$1. For odd dimensions, the relation is thus empty and we only have to investigate the even dimensions.

To satisfy (1.6.1), we thus need

$$(\alpha\gamma_{d+1})^2 = \mathbb{1},$$

i.e. $(\alpha^2\gamma_0\gamma_1\ldots\gamma_{d+1}\gamma_0\gamma_1\ldots\gamma_{d+1}) = \mathbb{1}$

Developing gives

$$\alpha^2(-)^{d-1+d-2+\ldots+1}\underbrace{(\gamma_0)^2(\gamma_1)^2\ldots(\gamma_{d-1})^2}_{(-)^{d-1}} = \mathbb{1}$$

$$\Rightarrow \quad \alpha^2(-)^{\frac{1}{2}d(d-1)}(-)^{d-1} = 1$$

$$\Rightarrow \quad \alpha^2(-)^{\frac{1}{2}(d-1)(d-2)} = 1$$

$$\Rightarrow \quad \alpha = (\pm i)^{\frac{1}{2}(d-1)(d-2)} \qquad (1.6.2)$$

This gives an equation for α which we can always satisfy - showing that the Weyl condition can be imposed in any even dimension.

1.7 The Majorana and Weyl Conditions imposed together

Having shown that we can impose the Majorana condition in d = 2,3,4 (mod 8) and the Weyl condition in d = 2 (mod 2) dimensions, we wish to determine in which dimensions we can impose both of them simultaneously. For this to be satisfied, the charge conjugation and helicity operation have to commute:

$$(\alpha\gamma_{d+1})C^{-1}\gamma_0{}^T\Psi^* = C^{-1}\gamma_0{}^T(\alpha\gamma_{d+1}\Psi)^* \tag{1.7.1}$$

Developing (1.7.1), we get

$$(\alpha\gamma_{d+1}\ C^{-1}\gamma_0{}^T - C^{-1}\gamma_0{}^T\alpha^*\gamma_{d+1}{}^{\dagger T})\Psi^* = 0 \tag{1.7.2}$$

Now

$$\begin{aligned}\gamma_{d+1}{}^{\dagger T} &= (\gamma_0\gamma_1\ldots\gamma_{d-1})^{\dagger T}\\ &= \{(\gamma_{d-1})^{\dagger}(\gamma_{d-2})^{\dagger}\ldots(\gamma_0)^{\dagger}\}^T\end{aligned}$$

Using the preferred representation, this becomes

$$\begin{aligned}\gamma_{d+1}{}^{\dagger T} &= (\gamma_{d-1}\gamma_{d-2}\ldots\gamma_0)^T(-)^{d-1}\\ &= (\gamma_0{}^T\gamma_1{}^T\ldots\gamma_{d-1}{}^T(-)^{d-1}\quad .\end{aligned} \tag{1.7.3}$$

Eq. (1.7.2) becomes thus

$$\begin{aligned}&\{\alpha\gamma_{d+1}\ C^{-1}\gamma_0{}^T - C^{-1}\alpha^*(-)^{d-1}\gamma_0{}^T(\gamma_0{}^T\gamma_1{}^T\ldots\gamma_{d-1}{}^T))\Psi^*\\ =\quad &\{\alpha\gamma_{d+1}\ C^{-1}\gamma_0{}^T - C^{-1}\alpha^*(-)^{d-1}(-)^{d-1}\gamma_0{}^T\gamma_1{}^T\ldots\gamma_{d-1}{}^T\gamma_0{}^T\}\Psi^*\end{aligned}$$

$$= \quad \{\alpha\gamma_{d+1}\, C^{-1}\gamma_0{}^T - \alpha^* C^{-1}\gamma_0{}^T\gamma_1{}^T \ldots \gamma_{d-1}{}^T\gamma_0{}^T\}\Psi^* . \tag{1.7.4}$$

From (1.5.8),

$C^{-1}\gamma_\mu{}^T = -\gamma_\mu{}^T C^{-1}$, so (1.7.4) becomes

$$(\alpha\gamma_{d+1}\, C^{-1}\gamma_0{}^T - \alpha^*(-)^d\gamma_0{}^T\gamma_1{}^T \ldots \gamma_{d-1}{}^T C^{-1}\gamma_0{}^T)\Psi^*$$

$$= \quad (\alpha - \alpha^*(-)^d\,)\,\gamma_{d+1}C^{-1}\gamma_0{}^T\,\Psi^*$$

$$= \quad 0 \qquad\qquad \text{if } \alpha = \alpha^*(-)^d,$$

i.e. if

$$(\pm i)^{\frac{1}{2}(d-1)(d-2)} = (\mp i)^{\frac{1}{2}(d-1)(d-2)}(-)^d$$

i.e if

$$1 = (-)^{\frac{1}{2}(d-1)(d-2)+d}$$

or if

$$2n = \frac{(d-1)(d-2)}{2} + d \quad .$$

This holds for d = 4n+2, 4n+3.

But since in odd dimensions, the Weyl condition is empty, the final result is that the Weyl and Majorana conditions can be imposed simultaneously in d = 4n+2 conditions. However, in d = 8n+6, the Majorana condition cannot be imposed, so that we are left with <u>d = 8n+2</u> in which both conditions are valid simultaneously.

We sum up the validity of the Majorana and Weyl conditions in the following

Table 1.5: Dimensions in which the Majorana and Weyl Conditions are valid

d	Weyl Cond.	Majorana Cond.	Weyl + Majorana Cond.
8n	x		
8n+1			
8n+2	x	x	x
8n+3		x	
8n+4	x	x	
8n+5			
8n+6	x		
8n+7			

Had we taken instead of the Minkowski metric a different metric, this result would obviously change. The metric $\bar{\eta}_{\mu\nu}$ = diag(+1,+1,-1,-1,...-1), for example, would give the following result:

Table 1.6: Dimensions in which the Majorana and Weyl Conditions are valid: Metric with two time dimensions

d	Weyl Cond.	Majorana Cond.	Weyl + Majorana Cond.
8n	x	x	
8n+1		x	
8n+2	x	x	x
8n+3			
8n+4	x		
8n+5			
8n+6	x		
8n+7			

In deriving the validity of the Majorana condition for this metric, the only change to the "original" derivation comes in eq.(1.5.14) which now becomes

$$(C\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_r})^T = (-)^{\frac{1}{2}(r^2-r+2)}\ pC\ \gamma_{\mu_1}\cdots\gamma_{\mu_r}\ . \qquad (1.5.14)'$$

This means that table 1.4 becomes

Table 1.7: Determination of p from the Symmetry of $C\Gamma^A$: metric with two"time" dimensions

d	$\dim(\gamma^\mu)$	No. of linearly ind.$C\Gamma^A$-matrices	$\frac{1}{2}n(n+1)$ =no. of symmetric d×d matrices	$\frac{1}{2}n(n-1)$ =no. of antisymm. d×d matrices	No. of $C\Gamma^A$ matr. with symmetry +p	No. of $C\Gamma^A$ matr. with symmetry -p	p
2	2	4	3	1	1	3	-1
3	2	4	3	1	0	4	x
4	4	16	10	6	10	6	+1
5	4	16	10	6	10	6	+1
6	8	64	36	28	36	28	+1
7	8	64	36	28	56	8	x
8	16	256	136	120	120	136	-1
9	16	256	136	120	120	136	-1
10	32	1024	528	496	496	528	-1
11	32	1024	528	496	220	804	x

From this, we can directly read off the result for the Majorana condition shown in table 1.6, using eq.(1.5.19) which is still valid.

The result for the Weyl condition clearly remains unchanged.

For the validity of the Majorana and Weyl conditions imposed together, we find that eq.(1.6.2) becomes

$$\alpha = (\pm i)^{\frac{1}{2}(d+1)(d-2)} \quad , \tag{1.6.2$'$}$$

which results in eq.(1.7.5) changing to

$$2n = \frac{(d+1)(d-2)}{2} + d,$$

and thus (1.7.6) becomes

$$d = 4n+1,\ 4n+2 \quad .$$

This concludes the proof of the result in table 1.6.

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2. The Poincaré Group

2.1 The Orthogonal Group and the Rotation Group: O(3) and SO(3)

A homogeneous linear tranformation of a vector $\vec{x}$ is written as follows:

$$\vec{x}' = R\,\vec{x} \quad , \qquad (2.1.1)(a)$$

Here R is a 3×3 matrix transforming the vector $\vec{x}$ into the new vector $\vec{x}'$. The matrix equation (2.1.1)(a) can be written in component form as

$$(x_i)' = R_{ij}x_j \quad . \qquad (2.1.1)(b)$$

We have used here Einstein's summation convention (we sum over repeated indices) - this will be assumed in what follows. Since under orthogonal transformations, the length of a vector $|\vec{x}| = x_1^2 + x_2^2 + x_3^2$ remains invariant, we must have for vectors which transform under the orthogonal matrix R

$$(x_i)'(x_i)' = x_i x_i \quad , \qquad (2.1.2)$$

that is

$$g_{ij}\ R_{ik}R_{jl}\ x_k x_l = g_{il}\ x_i x_l \quad .$$

Since x_i, x_j are arbitrary, this means

$$R_{jk}R_{jl} = \delta_{kl}$$

or in matrix notation

$$(R)^T R = \mathbb{1} \quad . \qquad (2.1.3)$$

This is the orthogonality relation specifying the orthogonal group in three dimensions, O(3).

From (2.1.3) we get

$$\det(R)^T \det(R) = 1$$

which implies

$$\det(R) = \pm 1 \quad ,$$

so the orthogonal matrices R are unimodular.

The elements R with det(R) = 1 constitute the group SO(3) - these are the rotation matrices that can be obtained from the identity by continuously changing the matrix elements.

SO(3) is <u>doubly connected</u>: For example, a rotation through an angle π has exactly the same effect as a rotation through the angle $-\pi$ (the corresponding transformation matrices are identical). So in this case, two points in the parameter space (which is composed of, for example, the angle of rotation plus two independent parameters fixing the direction of the rotation) correspond to the same element of the group. The result is that a closed path in the parameter space which passes through the parameter values $\phi = \pi$ and $\phi = -\pi$, cannot be contracted to a point. However, a path that does not pass through the values $\phi = \pm\pi$, can be contracted to a point. There are thus two distinct types of closed paths possible and we say that the group is doubly connected.

The elements of O(3) with det(R)=-1 correspond to improper rotations, i.e. rotations followed by a space inversion.

The group $J \equiv (I, I_s)$, where I_s is the inversion matrix $I_s = -I$, forms an invariant subgroup of O(3) - so does SO(3).The only element common to SO(3) and J is the identity, so one can write O(3) as the direct product of the two subgroups:

$$O(3) = SO(3) \otimes J \quad . \tag{2.1.4}$$

The relation (2.1.3) is a matrix equation containing $n^2 = 9$ components. However,the equation is symmetric in the off-diagonal terms so that $\frac{1}{2}n(n-1)$ components are dependent. This gives $n^2 - \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1) = 6$ equations in 9 parameters, leaving 3 parameters to be chosen freely; the group SO(3) thus possesses 3 generators.

From the definition of an infinitesimal generator

$$J_k = i \left. \frac{\partial R_k}{\partial \phi} \right|_{\phi=0} , \tag{2.1.5}$$

where R_k is an element of the group and ϕ represents the group parameters, together with the general expressions for the rotation matrices

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} , \tag{2.1.6)(a}$$

$$R_2 = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} , \tag{2.1.6)(b}$$

$$R_3 = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{2.1.6)(c}$$

one obtains the <u>infinitesimal generators of SO(3)</u>:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{2.1.7)(a}$$

$$J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{2.1.7)(b}$$

$$J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2.1.7)(c}$$

The generators J_i are easily seen to satisfy the commutation relations

$$[J_i,J_j] = i\varepsilon_{ijk}J_k . \tag{2.1.8}$$

Since it is impossible to find any linear combination of the generators which commutes with any particular generator of the set, the rank of the Lie algebra (i.e. the maximum number of commuting elements of the group) is 1. The rank of a Lie algebra also gives the number of Casimir operators necessary to characterise the group. The Casimir operator for the orthogonal group can be chosen as:

$$J^2 = (J_1)^2 + (J_2)^2 + (J_3)^2 \quad . \tag{2.1.9}$$

Since SO(3) is doubly connected, we must look for its <u>universal covering group</u>:

The group SU(2) is defined as the set of <u>unitary transformations</u> (with positive determinant) in a complex two-dimensional vector space:

$$(\xi^i)' = u_{ij}\xi^j \tag{2.1.10}$$

or in matrix notation

$$(\xi)' = u\xi \quad , \det(u) = 1 \quad .$$

The basis vectors of this space,

$$\xi^i = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \quad ,$$

are called <u>contravariant spinors</u> of rank 1. (They are actually vectors with respect to SU(2) and spinors with respect to SO(3)).

Correspondingly, one can define the <u>covariant spinors</u> $\eta = \xi^\dagger$ to transform according to

$$(\eta)' = \eta u^\dagger \tag{2.1.11}$$

$$\text{or} \quad (\eta_i)' = u^\dagger_{ji}\eta_j$$

$$= u^*_{ij}\eta_j \quad . \tag{2.1.12}$$

Notice that $(\xi^i)^*$ transforms like a covariant spinor: The complex conjugate of eq. (2.1.10) reads

$$(\xi^{i*})' = u_{ij}{}^*(\xi^j)^* \quad ,$$

which has the same form as (2.1.12) with

$$(\xi^i)^* \equiv \eta_i \ . \tag{2.1.13}$$

Now we can form the product of a covariant and a contravariant spinor as follows:

$$\xi^i\ \xi_j \equiv \Omega^i{}_j = \xi \otimes \xi^\dagger = \begin{pmatrix} \xi^1\xi_1 & \xi^1\xi_2 \\ \xi^2\xi_1 & \xi^2\xi_2 \end{pmatrix} \tag{2.1.14}$$

This matrix $\Omega^i{}_j$ transforms according to:

$$\begin{aligned} (\xi^i)'(\xi_j)' &= u_{ik}\ \xi^k\xi_l\ u_{lj}{}^\dagger \\ &= u_{ik}\ \Omega^k{}_l\ u_{lj}{}^\dagger \end{aligned} \tag{2.1.15}$$

This is in matrix form

$$(\Omega)' = u\ \Omega\ u^\dagger \quad . \tag{2.1.16}$$

The matrices u are unitary:

$$uu^\dagger = u^\dagger u = 1 \quad . \tag{2.1.17)(a}$$

Additionally, for SU(2) we require

$$\det(u) = 1 \tag{2.1.17)(b}$$

As a complex 2x2 matrix, u contains 8 real components. Four conditions are given by (2.1.17)(a) and one by (2.1.17)(b), thus leaving 3 parameters of the group, the same number as for SO(3).

From (2.1.17), the general form for u is:

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad , \qquad aa^* + bb^* = 1 \quad . \tag{2.1.18}$$

From this form of u we see that the spinor

$$\begin{pmatrix} (-\xi^2)^* \\ (\xi^1)^* \end{pmatrix}$$

transforms as

$$\xi = \begin{pmatrix} \xi^2 \\ \xi^1 \end{pmatrix} .$$

This is clear because ξ transfroms as

$$\begin{aligned} (\xi^1)' &= a\xi^1 + b\xi^2 \\ (\xi^2)' &= -b^*\xi^1 + a^*\xi^2 \end{aligned} \tag{2.1.19}$$

and so

$$\begin{aligned} (-\xi^2)^{*\prime} &= a(-\xi^2)^* + b(\xi^1)^* \\ (\xi^1)^{*\prime} &= -b^*(-\xi^2)^* + a^*(\xi^1)^* \quad , \end{aligned} \tag{2.1.20}$$

which is the same transformation behavior as (2.1.19).

Now we see that

$$\begin{pmatrix} (-\xi^2)^* \\ (\xi^1)^* \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (\xi^1)^* \\ (\xi^2)^* \end{pmatrix} \equiv S\xi^* \quad , \tag{2.1.21}$$

hence $\eta = \xi^{\dagger}$ transforms as

$$(S\xi)^{T} = (-\xi^{2},\xi^{1}) \quad . \tag{2.1.22}$$

So (2.1.14) becomes, using (2.1.22),

$$\xi\otimes\xi^{\dagger} = \begin{bmatrix} \xi^{1}\xi_{1} & \xi^{1}\xi_{2} \\ \xi^{2}\xi_{1} & \xi^{2}\xi_{2} \end{bmatrix} = \begin{bmatrix} -\xi^{1}\xi^{2} & \xi^{1}\xi^{2} \\ -(\xi^{2})^{2} & \xi^{1}\xi^{2} \end{bmatrix} \quad . \tag{2.1.23}$$

A <u>homomorphism between SO(3) and SU(2)</u> can now be obtained by identifying the three-dimensional vector $\vec{x}$ with $\Omega^{i}{}_{j}$ of eq. (2.1.14):

$$h \equiv -\Omega^{i}{}_{j} = \vec{\sigma}\cdot\vec{x} = \begin{pmatrix} x_3 & x_1-ix_2 \\ x_1+ix_2 & -x_3 \end{pmatrix} \tag{2.1.24}$$

where $\vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$, σ_i being the Pauli matrices.

h now transforms according to

$$(h)' = uhu^{\dagger} \tag{2.1.25}$$

and so

$$\begin{aligned} \det(h') &= \det(u)\det(h)\det(u^{\dagger}) \\ &= \det(h) \\ &= -(x_1^2 + x_2^2 + x_3^2) \\ &= -|\vec{x}|^{2} \quad , \end{aligned}$$

thus ensuring that the length of a vector remains invariant under transformations of the type (2.1.16).

Having identified $-\xi\otimes\xi^{\dagger}$ with h, we thus conclude that an SU(2) transformation on

$$\xi = \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix}$$

corresponds to an SO(3) transformation on

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

with

$$x_1 = 1/2 \{(\xi^2)^2 - (\xi^1)^2\}$$

$$x_2 = -i/2 \{(\xi^1)^2 + (\xi^2)^2\}$$

$$x_3 = \xi^1\xi^2 \quad . \tag{2.1.26}$$

Calculating explicitly the transformation (2.1.16), using (2.1.19) and (2.1.26), we obtain

$$(x_1)' = 1/2(a^2+a*^2-b^2-b*^2)x_1 - i/2(a^2 - a*^2 + b^2 - b*^2)x_2 - (a*b*+ab)x_3$$

$$(x_2)' = i/2(a^2-a*^2-b^2+b*^2)x_1 + 1/2(a^2 + a*^2 + b^2 + b*^2)x_2 - i(ab-a*b*)x_3$$

$$(x_3)' = (ab* + ba*)x_1 + i(ba*-ab*)x_2 + (|a|^2 - |b|^2)x_3 \quad . \tag{2.1.27}$$

Generally, since we can generate elements of SU(2) with the Pauli-matrices (as we shall show explicitly later), we may write a matrix u of SU(2) as

$$u = e^{-i\vec{\sigma}\cdot\vec{n}(\theta/2)} \tag{2.1.28}$$

Similarly, a matrix R of SO(3) can be written as

$$R = e^{-i\vec{J}\cdot\vec{n}\theta} \quad . \qquad (2.1.29()$$

Inserting $a = e^{-i\theta/2}$, $b = 0$ into (2.1.27), which corresponds to

$$u_z = e^{-i\sigma_z\theta/2} \quad ,$$

we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = R_z\vec{x} \quad .$$

Similarly, $a = \cos(\theta/2)$, $b = -\sin(\theta/2)$ gives

$$u_y = e^{-i\sigma_y\theta/2}$$

with

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

and $a = \cos\theta/2$, $b = -i\sin\theta/2$ gives

$$u_x = e^{-i\sigma_x\theta/2}$$

and the corresponding rotation matrix is

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad .$$

This makes explicit the correspondence between elements of the groups SO(3) and SU(2).

Since the tranformation (2.1.16) is insensitive to a change of sign in u, the two matrices u and -u of SU(2) correspond to the same element of SO(3): the simply connected SU(2) is homomorphic to the doubly connected SO(3).

2.2 The Homogeneous Lorentz Group

We define a Lorentz transformation as follows:

$$(x^\mu)' = \Lambda^\mu{}_\nu \, x^\nu \quad . \tag{2.2.1}$$

The length of a four-vector is to be conserved:

$$(x^\mu)'(x_\mu)' = x^\mu \, x_\mu \tag{2.2.2}$$

Substitution of (2.2.1) into (2.2.2) gives:

$$\Lambda^\mu{}_\nu \, x^\nu \, \Lambda^\rho{}_\sigma \, x^\rho \quad \eta_{\mu\rho} = x^\mu \, x^\rho \, \eta_{\mu\sigma}$$

so that the matrices Λ must satisfy

$$\Lambda^\mu{}_\nu \, \Lambda^\rho{}_\sigma \, \eta_{\mu\rho} = \eta_{\sigma\nu} \tag{2.2.3}$$

or in matrix notation

$$(\Lambda)^T \, \eta \, (\Lambda) = \eta \quad . \tag{2.2.4}$$

This is the <u>characteristic equation for a homogeneous Lorentz transformation</u>. We note that, since $\eta \equiv \mathrm{diag}(1,-1,-1,-1)$, eq.(2.2.4) actually characterises the group O(3,1).)

From (2.2.4) we get

$$\det(\Lambda)^T \det(\eta) \det(\Lambda) = \det(\eta),$$

implying

$$(\det(\Lambda))^2 = 1$$

or $\quad \det(\Lambda) = \pm 1 \qquad (2.2.5)$

Furthermore, if we take the (00)-component of eq.(2.2.4), we get

$$(\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2 = 1$$

Therefore

$$(\Lambda^0{}_0)^2 \geq 1$$

and

$$\Lambda^0{}_0 \geq 1 \qquad \text{or} \qquad \Lambda^0{}_0 \leq -1 \quad . \tag{2.2.6}$$

We now show that the transformations (2.2.1) form a group:

i) Let Λ_1 and Λ_2 be elements of the set L of the matrices Λ. Then we define their product to be $\Lambda_3 = \Lambda_1\Lambda_2$ in the sense of matrix multiplication and (2.2.4) becomes

$$\begin{aligned}(\Lambda_1\Lambda_2)^T \eta (\Lambda_1\Lambda_2) &= \Lambda_2{}^T\Lambda_1{}^T \eta \Lambda_1\Lambda_2 \\ &= \Lambda_3{}^T \eta \Lambda_3{}^T \\ &= \eta \quad .\end{aligned}$$

Hence eq.(2.2.4) is satisfied for Λ_3 and so Λ_3 is also a member of the group.

ii) The identity is the unit matrix which satisfies:

$$I^T \eta I = \eta$$

iii) Since $\det(\Lambda) = \pm 1 \neq 0$, we can find an inverse for each matrix Λ, which is again an element of the group:

$$(\Lambda^{-1})^T \eta \Lambda^{-1} = (\Lambda^{-1})^T \eta^{-1} \Lambda^{-1}$$
$$= (\Lambda \eta \Lambda^T)^{-1}$$
$$= \eta^{-1} = \eta \quad .$$

iv) Clearly the multiplication of matrices Λ is associative.

We call this group the <u>homogeneous Lorentz group</u> L or O(3,1).

L is <u>continuous</u> because the transformations (2.2.1) are continuous. However, L is <u>not connected</u>: From eq.(2.2.5) one sees that L is split into two disconnected parts: The transformations with $\det(\Lambda) = 1$ and those with $\det(\Lambda) = -1$. These are again each divided into two disconnected parts corresponding to $\Lambda^0{}_0 \geq 1$ and $\Lambda^0{}_0 \leq -1$ respectively. Thus L consists of 4 disconnected subsets:

<u>Table 2.1: Disconnected Subsets of the Homogeneous Lorentz Group</u>

Designation	$\det(\Lambda)$	$\Lambda^0{}_0$	Discrete transformation
$L_+^\uparrow$	+1	≥ 1	I
$L_-^\uparrow$	-1	≥ 1	$I_s = \eta$ (space inversion)
$L_-^\downarrow$	-1	≤ -1	$I_t = -\eta$ (time inversion)
$L_+^\downarrow$	+1	≤ -1	$I_{st} = -I$ (spacetime inversion)

Of these subsets only $L_+^\uparrow$ forms a subgroup since it alone contains the identity element. However, each of the other three subsets contains a discrete transformation (as listed above); so any element of L can be obtained by applying a discrete transformation together with a transformation from $L_+^\uparrow$. To find the generators of L then, it suffices to find the generators of $L_+^\uparrow$.

2.3 The Generators of the Restricted Lorentz Group $L_+^\uparrow$ (or SO(3,1))

The defining equation (2.2.4) is a 4x4 matrix equation; hence it contains 16 components. It is, however, symmetric in the components (since $\eta_{\mu\nu}$ is symmetric), so only 10 of the 16 components are independent. This leaves 6 undetermined components, i.e. the group has 6 parameters.

To obtain the 6 generators of the Lorentz group, we proceed as we did for SO(3): we write down the general form of the elements of the group:

Pure Lorentz boosts are of the form

$$L_x = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.3.1)(a}$$

$$L_y = \begin{pmatrix} \cosh\phi & 0 & -\sinh\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\phi & 0 & \cosh\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.3.1)(b}$$

$$L_z = \begin{pmatrix} \cosh\phi & 0 & 0 & -\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \tag{2.3.1)(c}$$

and space rotations are written as

$$R_i = \begin{pmatrix} 1 & 0 \\ 0 & \vec{R} \end{pmatrix} , \tag{2.3.1)(d}$$

where $\vec{R}$ is the three-dimensional space rotation matrix and has the form (2.1.6).

Now we use the definition of an infinitesimal generator, eq.(2.1.5), to obtain the infinitesimal generators of $L_+^\uparrow$:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \tag{2.3.2)(a}$$

$$J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \qquad (2.3.2)(b)$$

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (2.3.2)(c)$$

$$K_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (2.3.2)(d)$$

$$K_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (2.3.2)(e)$$

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} . \qquad (2.3.2)(f)$$

Here J_i denote rotation generators and K_i denote generators of pure Lorentz boosts. Since these 6 generators are independent and $L_+^\uparrow$ only possesses 6 infinitesimal generators, we know that we have found the complete set of generators. The generators J_i, K_i can be summarised in a tensor $M^{\mu\nu}$ of the form:

$$M^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix} . \qquad (2.3.3)$$

It is also instructive to write the generators $M^{\mu\nu}$ in operator form. This is

$$M^{\mu\nu} \equiv x^\mu p^\nu - x^\nu p^\mu \qquad (2.3.4)$$

Since $M^{\mu\nu}$ is antisymmetric, it has 6 independent components - as we have required before. The nonzero components are:

$$M^{ij} = x^i p^j - x^j p^i \qquad\qquad i,j,k = 1,2,3$$

$$= \varepsilon^{ijk} J_k \tag{2.3.5)(a}$$

and

$$M^{0i} = x^0 p^i - x^i p^0$$

$$= -x^0 p_i - x^i p_0$$

$$\equiv K_i \quad . \tag{2.3.5)(b}$$

These expressions are now the generators of the group $L_+^\uparrow$ written in operator form. We notice again that in this form, the J_i generate space rotations and the K_i pure Lorentz transformations.

A more rigid approach to arrive at the generators $M_{\mu\nu}$ is to derive them from an infinitesimal tranformation. This is done as follows:

We denote an infinitesimal Lorentz transformation by

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + i\omega_\mu{}^\nu \quad , \tag{2.3.6}$$

where $\omega_\mu{}^\nu$ is a real parameter.

Eq.(2.3.6) has to satisfy the orthogonality relation (2.2.3):

$$(\delta_\mu{}^\nu + i\omega_\mu{}^\nu)(\delta_\sigma{}^\rho + i\omega_\sigma{}^\rho)\eta_{\mu\rho} = \eta_{\sigma\nu} \tag{2.3.7}$$

To lowest order of ω, this gives

$$\eta_{\nu\sigma} + i\omega_\sigma{}^\nu + i\omega_\sigma{}^\nu = \eta_{\nu\sigma}$$

and hence $\omega_{\mu\nu}$ is antisymmetric:

$$\omega_{\mu\nu} = -\omega_{\mu\nu} \tag{2.3.8}$$

The fact that the parameters $\omega_{\mu\nu}$ are antisymmetric implies again that there are 6 independent parameters for a transformation like (2.3.6).

To find the generators, we write (2.3.6) in the form

$$\Lambda_{\mu}{}^{\nu} = \delta_{\mu}{}^{\nu} - \tfrac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})_{\mu}{}^{\nu} \quad , \tag{2.3.9}$$

where we sum over the indices α and β, and $\omega_{\alpha\beta}$ is the antisymmetric parameter of eq.(2.3.6). Comparing (2.3.6) with (2.3.9), we get a relation for the tensor $(M_{\alpha\beta})_{\mu}{}^{\nu}$:
We must have

$$\tfrac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})_{\mu}{}^{\nu} = -i\omega_{\mu}{}^{\nu} \tag{2.3.10}$$

We lower the tensor index:

$$\tfrac{1}{2}\omega_{\alpha\beta}(M^{\alpha\beta})_{\mu\sigma} = -i\omega_{\mu\sigma} \quad .$$

Since $\omega_{\mu\nu}$ is antisymmetric, the solution is

$$(M^{\alpha\beta})_{\mu\sigma} = -i(\eta^{\alpha}{}_{\mu}\eta^{\beta}{}_{\sigma} - \eta^{\alpha}{}_{\sigma}\eta^{\beta}{}_{\mu}) \tag{2.3.11}$$

and so

$$(M^{\alpha\beta})_{\mu}{}^{\nu} = -i\eta^{\nu\sigma}(\eta^{\alpha}{}_{\mu}\eta^{\beta}{}_{\sigma} - \eta^{\alpha}{}_{\sigma}\eta^{\beta}{}_{\mu}) \tag{2.3.12}$$

Eq.(2.3.12) now gives the components μ, ν of the tensor $M^{\alpha\beta}$ which contains the 6 generators of the group of transformations (2.3.6).

For example,

$$(M^{01})_{\mu\nu} = -i\eta^{\nu\sigma}(\eta^0{}_\mu\eta^1{}_\nu - \eta^1{}_\mu\eta^0{}_\nu)$$

$$= -i(\eta^0{}_\nu\eta^{1\sigma} - \eta^1{}_\mu\eta^{0\sigma})$$

$$= \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = K_1,$$

as we had previously. The other generators are obtained similarly.

Let us now work out the commutation relations between the generators of $L_+^\uparrow$.
For example,

$$[J_1,J_2] = [x^2p_3 - x^3p_2,\ x^3p_1 - x^1p_3]$$

$$= x^2\ [p_3,x^3]\ p_1 + x^1\ [x^3,\ p_3]\ p_2$$

$$= iJ_3 \quad ,$$

or in matrix form:

$$[J_1,J_2] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = iJ_3 \quad .$$

Generally one finds

$$[J_i,J_j] = i\ \varepsilon_{ijk}\ J_k \quad . \qquad (2.3.13)(a)$$

Similarly,

$$[K_i,K_j] = [x_0p^i - x^ip_0,\ x_0p^j - x^ip_0]$$

$$= -x^i\ [p_0,\ x_0]\ p^j - x^j\ [x_0,\ p_0]\ p^i$$

$$= -i\,\varepsilon_{ijk}\,J_k \quad .$$

The matrix equations yield the corresponding results. We write generally:

$$[K_i, K_j] = -i\,\varepsilon_{ijk}\,J_k \quad , \qquad (2.3.13)(b)$$

and note that the J_i are <u>Hermitean</u> while the K_i are <u>anti-Hermitean</u>.

Finally,

$$[J_i,K_j] = \varepsilon_{kli}\,[x_k,p^l - x^l p_k,\ x_0 p^j - x^j p_0]$$

$$= i\,\varepsilon_{ijk}\,(x_0 p^l - x^l p_0)$$

$$\Rightarrow \quad [J_i,K_j] = i\,\varepsilon_{ijk}\,K_k \quad . \qquad (2.3.13)(c)$$

The commutation relations (2.3.13) show that the Lie algebra of $L_+^\uparrow$ has rank 2, i.e. the maximum number of elements which commute is 2; there are thus two Casimir operators of the group.
Notice that the pure Lorentz transformations do not form a group since the generators K_i do not form a closed algebra under commutation.

A general transformation of the restricted Lorentz group can now be written as a combination of a Lorentz boost (generators K_i) and a rotation (generators J_i):

$$\Lambda(\theta\vec{n},\phi\vec{\nu}) = e^{-i\theta\,\vec{J}\cdot\vec{n}\ -\ i\phi\,\vec{K}\cdot\vec{\nu}} \qquad (2.3.14)$$

Clearly $\vec{J} = (J_1,J_2,J_3)$,

$\vec{K} = (K_1,K_2,K_3)$,

and $\vec{n}$, $\vec{\nu}$ are the unit vectors in the direction of the corresponding transformation.

Eq. (2.3.14) can be simplified as:

$$\Lambda = e^{-\frac{1}{2}\,\omega_{\mu\nu}\,M^{\mu\nu}} \quad , \tag{2.3.15}$$

where $\omega_{\mu\nu}$ is an <u>antisymmetric</u> tensor (as $M_{\mu\nu}$ is antisymmetric) and is seen to be

$$\omega_{\mu\nu} = -i \begin{pmatrix} 0 & -\phi_1 & -\phi_2 & -\phi_3 \\ \phi_1 & 0 & -\theta_3 & \theta_2 \\ \phi_2 & \theta_3 & 0 & -\theta_1 \\ \phi_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix} . \tag{2.3.16}$$

The two Casimir operators of $L_+^\uparrow$ can be written as

$$\tfrac{1}{2}\, M_{\mu\nu}M^{\mu\nu} = \vec{J}^2 - \vec{K}^2$$

and $$\frac{1}{4}\,\varepsilon^{\mu\nu\sigma\tau}\, M_{\mu\nu}\, M_{\sigma\tau} = -\vec{J}\cdot\vec{K} \quad . \tag{2.3.17}$$

Clearly these operators commute with all the generators of the group.

Alternatively, one can write the generators in Hermitean form:

$$M_i = (J_i + iK_i)$$

$$N_i = (J_i - iK_i) \quad . \tag{2.3.18}$$

Then the commutation relations are

$$[M_i, M_j] = i\varepsilon_{ijk}\, M_k$$

$$[N_i, N_j] = i\varepsilon_{ijk}\, N_k$$

$$[N_i, M_j] = 0 \quad , \qquad (2.3.13)(d)$$

i.e. M_i and N_i behave as the components of two angular momenta.

The Casimir operators now take the form:

$$M^2 = \sum_i M_i^2$$

$$N^2 = \sum_i N_i^2 \quad . \qquad (2.3.17)(a)$$

We see that now M and N each generate a group SU(2), which commute. This demonstrates that the Lorentz group is essentially SU(2)⊗SU(2), so that particle states which transform under the Lorentz group will be characterized by two angular momenta j,j′ corresponding to M and N.

2.4 The Universal Covering Group of $L_+^\uparrow$

The group $L_+^\uparrow$ is composed of the pure Lorentz transformations and the three-dimensional space rotations. Since the subset SO(3) of $L_+^\uparrow$ is doubly connected, so is $L_+^\uparrow$, and hence we need to look for the universal covering group of $L_+^\uparrow$. The procedure to follow is analogous to that used for establishing the homomorphism between SO(3) and SU(2):

We associate with each four-vector x^μ a Hermitean matrix X :

$$X \equiv \sum_\mu \sigma^\mu x^\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \quad . \qquad (2.4.1.)$$

x^μ can be written as

$$x^\mu = \tfrac{1}{2}\,\mathrm{Tr}\,(\sigma^\mu X) \quad , \qquad (2.4.2)$$

where $\sigma^\mu = (\sigma_0, \vec{\sigma}) = (1,\vec{\sigma}) = \bar{\sigma}_\mu$

and $\bar{\sigma}^{\mu} = (\sigma_0, -\vec{\sigma}) = \sigma_{\mu}$.

In this picture, the length of a four-vector is given by:

$$\det X = (x^0)^2 - (\vec{x})^2 = (x)^2 .$$

The (Lorentz) transformation of X is now performed by a complex unimodular 2x2 matrix A :

$$(X)' = A\, X\, A^{\dagger} \quad , \qquad \det A = 1. \qquad (2.4.3)$$

The length of four-vectors is preserved under the transformation (2.4.3):

$$(X')^2 = \det (X') = \det (X) = (x)^2$$

and hence A corresponds to a Lorentz transformation.

The matrices A form the group SL(2,C), which has 6 generators (between the 4 complex components, there is only one constraining complex equation: det(A)=1), like the homogeneous Lorentz group. The correspondence is given by:

$$(x^{\mu})' = \Lambda^{\mu}{}_{\nu}\, x^{\mu} \qquad \text{in L}$$

corresponds to

$$(x)' = A\, X\, A^{\dagger} \qquad \text{in SL(2,C)}$$

and
$$\begin{aligned}(x^{\mu})' &= \tfrac{1}{2} \operatorname{Tr} (\sigma^{\mu} X') \\ &= \tfrac{1}{2} \operatorname{Tr} (\sigma^{\mu} A\, X\, A^{\dagger}) \\ &= \tfrac{1}{2} \operatorname{Tr} (\sigma^{\mu} A\, \sigma^{\nu} A^{\dagger}\, x^{\nu}) \quad ,\end{aligned}$$

so that

$$\Lambda^{\mu}{}_{\nu} = \tfrac{1}{2} \operatorname{Tr} (\sigma^{\mu} A \bar{\sigma}_{\nu} A^{\dagger}) \ . \qquad (2.4.4)$$

The group SL(2,C) then forms the universal covering group of $L_{+}^{\uparrow}$, with $\Lambda^{\mu}{}_{\nu}$ defined as above.

We verify that $\Lambda(A)\ \Lambda(B) = \Lambda(AB)$: (2.4.5)

$$[\Lambda(AB)]^{\mu}_{\nu} = \tfrac{1}{2} \operatorname{Tr} (\sigma^{\mu} AB \bar{\sigma}_{\nu} (AB)^{\dagger})$$

$$= \tfrac{1}{2} \operatorname{Tr} (A^{\dagger} \sigma^{\mu} AB \bar{\sigma}_{\nu} B^{\dagger})$$

and

$$[\Lambda(A)\ \Lambda(B)]^{\mu}_{\nu} = \Lambda^{\mu}{}_{\rho} (A)\ \Lambda^{\rho}{}_{\nu}(B)$$

$$= \tfrac{1}{2} \operatorname{Tr} (\sigma^{\mu} A \bar{\sigma}_{\rho} A^{\dagger})\ \tfrac{1}{2} \operatorname{Tr} (\sigma^{\rho} B \bar{\sigma}_{\nu} B^{\dagger})$$

$$= \frac{1}{4} \operatorname{Tr} (A^{\dagger} \sigma^{\mu} A \bar{\sigma}_{\rho}) \operatorname{Tr} (\sigma^{\rho} B \bar{\sigma}_{\nu} B^{\dagger})$$

$$= \frac{1}{4} \operatorname{Tr} (A^{\dagger} \sigma^{\mu} A \sigma_{\rho}) \operatorname{Tr} (\sigma_{\rho} B \bar{\sigma}_{\nu} B^{\dagger}) \ .$$

Now

$$\operatorname{Tr} (A \sigma_{\alpha}) \operatorname{Tr} (\sigma_{\alpha} B) = 2 \operatorname{Tr} (AB)$$

so that

$$[\Lambda(A)\ \Lambda\ (B)]^{\mu}_{\nu} = \tfrac{1}{2} \operatorname{Tr} (A^{\dagger} \sigma^{\mu} AB \bar{\sigma}_{\nu} B^{\dagger}) = [\Lambda(AB)]^{\mu}_{\nu} \ ;$$

this verifies the validity of the group multiplication (2.4.5) .

By calculating $\Lambda^{\mu}{}_{\nu}(A)$ explicitly, using 2.4.4) and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \qquad ab\text{-}bc = 0 \quad , \qquad (2.4.6)$$

we can also verify (the calculation is tedious!) that

$$\det(\Lambda) = 1$$

and $\Lambda^0{}_0 \geq 1$.

Again, as we saw in the homomorphism between SO(3) and SU(2), since a sign change in A does not change the transformation (2.4.3), two matrices A and -A of SL(2,C) correspond to one matrix Λ of $L_+^\uparrow$ (SO(3,1)). The double-valued SO(3,1) is thus covered by the single-valued SL(2,C).

A matrix A of SL(2,C) can also be written in exponential form:

$$A = e^S \quad , \qquad \mathrm{Tr}\; S = 0 \ . \qquad (2.4.7)$$

There are 6 independent 2x2 traceless complex matrices, so we choose for S the 6 matrices σ_k and $i\sigma_k$.

A matrix of SL(2,C) then has the following form:

$$U = e^{-1/2\; i\theta\; \vec{\sigma}\cdot\vec{n}}$$

$$= 1 - \frac{i}{2}\;\theta\;\vec{\sigma}\cdot\vec{n} - \frac{1}{2!}\left(\frac{\theta\;\vec{\sigma}\cdot\vec{n}}{2}\right)^2 + \frac{i}{3!}\left(\frac{\theta\;\vec{\sigma}\cdot\vec{n}}{2}\right)^3 + \ldots.$$

$$= 1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 + \ldots.$$

$$- i\;\vec{\sigma}\cdot\vec{n}\left[\left(\frac{\theta}{2}\right) - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \frac{1}{5!}\left(\frac{\theta}{2}\right)^5 + \ldots.\right]$$

$$= \cos(\theta/2) - i\;\vec{\sigma}\cdot\vec{n}\;\sin(\theta/2) \qquad (2.4.8)$$

which is unitary,

or

$$H = e^{-1/2\ \phi\ \vec{\sigma}\cdot\vec{\nu}}$$

$$= 1 - \frac{1}{2}\ \theta\ \vec{\sigma}\cdot\vec{\nu} + \frac{1}{2!}\left(\frac{\phi\ \vec{\sigma}\cdot\vec{\nu}}{2}\right)^2 - \frac{1}{3!}\left(\frac{\phi\ \vec{\sigma}\cdot\vec{\nu}}{2}\right)^3 + \ldots.$$

$$= 1 + \frac{1}{2!}\left(\frac{\phi}{2}\right)^2 + \frac{1}{4!}\left(\frac{\phi}{2}\right)^4 + \ldots.$$

$$- \vec{\sigma}\cdot\vec{\nu}\left[\left(\frac{\phi}{2}\right) + \frac{1}{3!}\left(\frac{\phi}{2}\right)^3 + \frac{1}{5!}\left(\frac{\phi}{2}\right)^5 + \ldots.\right]$$

$$= \cosh(\phi/2) - \vec{\sigma}\cdot\vec{\nu}\ \sinh(\phi/2) \qquad (2.4.9)$$

which is Hermitean.

The matrices U correspond to <u>rotations</u> and the matrices H to <u>pure Lorentz boosts</u>, so that any matrix of SL(2,C) can be written as a product of H and U. The unitary matrices U form also the group SU(2) which is simply connected. Since the group of pure Lorentz transformations is also simply connected, we conclude that <u>SL(2,C) is simply connected</u> as well.

2.5 The Translation Group

A general translation can be written in the form:

$$(x^{\mu})' = x^{\mu} + a^{\mu} = f^{\mu}(x^{\nu}, a^{\alpha}) \qquad (2.5.1)$$

The definition of the inifinitesimal generators,

$$X_{\alpha} = \left.\frac{\partial f^{\mu}(x^{\nu}, a^{\beta})}{\partial a^{\alpha}}\right|_{a=0} \frac{\partial}{\partial x^{\mu}}$$

gives the generators

$$X_\alpha = \frac{\partial}{\partial x^\alpha} \quad . \tag{2.5.2}$$

In order to obtain Hermitean generators, we use

$$P_\nu = iX_\nu = i\,\frac{\partial}{\partial x^\nu} \quad . \tag{2.5.3}$$

as the generators; there are thus 4 generators for the translation group (the group properties are immediately evident). We can then, as usual, write a finite translation in the form:

$$T\,(a^\mu) = e^{-i\,a^\mu P_\mu} \tag{2.5.4}$$

2.6 The Poincaré Group (Inhomogeneous Lorentz Group)

The Poincaré group consists of Lorentz transformations and translations:

$$(x^\mu)' = \Lambda^\mu{}_\nu\, x^\nu + a^\mu \tag{2.6.1.}$$

This transformation does not leave the length of position vectors invariant, but only the distance between position vectors:

$$[\,(x_1{}^\mu)' - (x_2{}^\mu)'\,]^2 = (x_1{}^{\mu\prime} - x_2{}^{\mu\prime}\,)^2$$

The generators of the Poincaré group are $M_{\mu\nu}$, the generators of the homogeneous Lorentz transformations, and P_ν , the generators of translations, so that any Poincaré transformation can be written as a product of those two types of transformation. It is well known that the translations do not commute with the Lorentz transformations. This can easily be seen as follows:

Denote the transformation (2.6.1) by (a,Λ).Then

$$(a,I)\ (0,\Lambda) = (a,\Lambda),$$

but

$$(0,\Lambda)\ (a,I) = (\Lambda a,\Lambda) \neq (a,\Lambda) \quad .$$

The transformation (a,Λ) can also be written in matrix form

$$(a,\Lambda) = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}$$

Here Λ denotes the usual 4-dimensional Lorentz transformation matrix and a is the four-vector which gives the translation. The transformation (2.6.1) is then a 5-dimensional matrix equation:

$$(\vec{x})' = (a,\Lambda)\ \vec{x}$$

is written

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda_{(4)} & a \\ & \\ & \\ & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix}$$

or

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda x + a \\ 1 \end{pmatrix} \quad . \qquad (2.6.2)$$

In particular

$$(a,I) = \begin{pmatrix} 0_{(4)} & a \\ 0 & 1 \end{pmatrix} \quad ,$$

so a matrix representation for the generators $\hat{P}_\mu$ of the translation group is

$$\hat{P}_0 = \begin{pmatrix} 0&0&0&0&i \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} ,$$

$$\hat{P}_1 = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&i \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} ,$$

etc.

Next, we develop the commutation relations between the generators of the Poincaré group:

i) $[P_\mu, P_\nu] = 0$ (2.6.3)(i)

ii) $[M_{\mu\nu}, P_\rho] = [x_\mu P_\nu - x_\nu P_\mu, P_\rho]$

$= [x_\mu, P_\rho]\, P_\nu - [x_\nu, P_\rho]\, P_\mu$

$\Rightarrow \quad [M_{\mu\nu}, P_\rho] = i\, (\eta_{\nu\rho}\, P_\mu - \eta_{\mu\rho}\, P_\nu)$ (2.6.3)(ii)

iii) $[M_{\rho\lambda}, M_{\mu\nu}] = x_\rho\, [P_\lambda, x_\mu]\, P_\nu + x_\mu\, [x_\rho, P_\nu]\, x_\rho$

$- x_\rho\, [P_\lambda, x_\nu]\, P_\mu - x_\nu\, [x_\rho, P_\mu]\, x_\lambda$

$- x_\lambda\, [P_\rho, x_\mu]\, P_\nu - x_\mu\, [x_\lambda, P_\nu]\, x_\rho$

$+ x_\lambda\, [P_\rho, x_\nu]\, P_\mu + x_\nu\, [x_\lambda, P_\mu]\, x_\rho$

$\Rightarrow \quad [M_{\rho\lambda}, M_{\mu\nu}] = i\, (\eta_{\rho\nu} M_{\lambda\mu} + \eta_{\lambda\mu} M_{\rho\nu} - \eta_{\rho\mu} M_{\nu\lambda} - \eta_{\lambda\nu} M_{\mu\rho})$ (2.6.3)(iii)

These commutation relations suggest again that the rank of the Lie algebra is 2, so that one can find two Casimir operators.

An easy choice for a Casimir operator is $P^2 = P^\mu P_\mu$. We can see this immediately:

- $[P_\mu, P^2] = 0$

- $$[M_{\rho\nu}, P_\mu P^\mu] = P_\mu \, \eta^{\mu\sigma} \, P_i \, (\eta_{\nu\sigma} \, P_\rho - \eta_{\rho\nu} \, P_\sigma)$$
$$= 0 \quad .$$

For a second Casimir operator, one introduces the covariant spin vector:

$$W_\mu = 1/2 \; \varepsilon_{\mu\nu\sigma\tau} \; M^{\nu\sigma} \; P^\tau$$
$$= (\vec{L}\cdot\vec{P}, \; \vec{L}P_0 + \vec{K} \times \vec{P}) \qquad . \tag{2.6.4}$$

Now

$$[P_\mu, W_\nu] = 1/2 \; \varepsilon_{\nu\alpha\beta\gamma} \; [P_\mu, M^{\alpha\beta} \; P^\gamma]$$
$$= 1/2 \; \varepsilon_{\nu\alpha\beta\gamma} \; (\eta^\alpha{}_\mu \; P^\beta - \eta^\beta{}_\mu \; P^\alpha) \; P^\gamma$$
$$= i \; \varepsilon_{\nu\alpha\beta\gamma} \; P^\beta \; P^\gamma$$
$$= 0 \quad .$$

We introduce the quantity $W_\mu W^\mu = W^2$, and to find $[W^2, M_{\mu\nu}]$, we first calculate some useful commutation relations:

1) $$[x_\mu, W_\tau] = 1/2 \; \varepsilon_{\tau\alpha\beta\gamma} \; [x_\mu, M^{\alpha\beta} \; P^\gamma]$$
$$= 1/2 \; \varepsilon_{\tau\alpha\beta\gamma} \; (-\eta_\mu{}^\beta \; x^\alpha \; P^\gamma + i\eta_\mu{}^\alpha \; x^\beta \; P^\gamma - iM^{\alpha\beta} \; \eta_\mu{}^\gamma)$$
$$= -i/2 \; \varepsilon_{\tau\alpha\beta\mu} \; M^{\alpha\beta} \tag{2.6.5}$$

2) $$[M_{\mu\nu},W_\tau] = [x_\mu\, P_\nu - x_\nu\, P_\mu, W_\tau]$$

$$= -i/2\ \varepsilon_{\tau\alpha\beta\mu}\, M^{\alpha\beta}\, P_\nu + i/2\ \varepsilon_{\tau\alpha\beta\nu}\, M^{\alpha\beta}\, P_\mu$$

$$= i\ (\eta_{\nu\tau}\, W_\mu - \eta_{\mu\tau}\, W_\nu) \qquad (2.6.6)$$

Hence we have

$$[W^2, M_{\mu\nu}] = [W^\alpha W_\alpha, M_{\mu\nu}]$$

$$= [W^\alpha,\ i(g_{\mu\alpha}\, M_\nu - g_{\nu\alpha}\, M_\mu)]$$

$$= i\ (W_\mu\, M_\nu - M_\nu\, W_\mu + M_\mu\, W_\nu - M_\nu\, W_\mu)$$

$$= 0\ , \qquad \text{as required.}$$

We have thus shown that both P^2 and W^2 commute with all the generators of the group $L_+^\uparrow$ and are therefore its Casimir operators.

2.7 Generalisation to Arbitrary Dimensions

Assume now that space-time is d-dimensional. Then the defining equation for the homogeneous Lorentz group,

$$(\Lambda)^T\ \eta\ \Lambda = \eta \qquad (2.2.4)$$

is a dxd matrix equation which is symmetric, leaving d/2 (d-1) free parameters. For example, if d = 10, then the homogeneous Lorentz group has 45 generators.

We know that the (d-1)-dimensional rotation group, SO(d-1), has 1/2(d-1)(d-2) degrees of freedom, e.g.SO(3) has 3 generators, and SO(9) has 36. Also, the pure Lorentz transformations in d-dimensional space-time have d-1 generators, one for each spatial dimension. So we see again that the homogeneous Lorentz group has

$$N = 1/2\ (d-1)(d-2) + d-1 = d/2\ (d-1)$$

generators.

Similarly, the inhomogeneous Lorentz group has d additional generators for the translations. Hence the total number of generators for the Poincaré group is

$$\bar{N} = d/2\ (d-1) + d = d/2\ (d+1) \quad .$$

For example, if d = 10, then $\bar{N} = 55$.

We can easily generalise the form of the generators from d = 4 to d = n:

We have

$$M^{\mu\nu} = x^{\mu}\, P^{\nu} - x^{\nu}\, P^{\mu} \qquad \mu,\nu = 0,1,\ldots,n-1$$

$$\text{with } M^{ij} = \varepsilon^{ijk}\, L_k \qquad i,j,k = 1,2,\ldots,n-1$$

$$\text{and } M^{0i} = K_i \quad .$$

$$\text{Here } \varepsilon_{ijk} =: \begin{cases} 1 & \text{if } i,j,k = i,i+1,i+2 \\ & \text{or any even permutation thereof} \\ -1 & \text{for odd permutations of } i,i+1,i+2 \\ 0 & \text{otherwise} \end{cases}$$

Then the L_i are again the generators of rotations in (d-1)-dimensional space, and the K_i generate pure Lorentz transformations.

Finally,

$$P_{\mu} = i\frac{\partial}{\partial x^{\mu}} \quad , \qquad \mu = 0,1,\ldots,n$$

generate the translations in d-dimensional space-time.

Clearly all the commutation relations will still hold in arbitrary dimensions and hence the Casimir operators assume the same form as in 4-dimensional space-time.

2.8 Representations

2.8(a) Definition of a Group Representation[1]

Definition: If we can find a set of linear operators $T(g)$ in a linear vector space L, which correspond to the elements g of the group G in the sense that

$$T(a)T(b) = T(ab) \qquad \text{for all } a,b \in G$$

$$T(e) = 1 \quad ,$$

then this set of operators forms a representation of the group G in the space L. The dimension of the vector space is also the dimension of the representation.

Choosing a basis $\{e_1, e_2, \ldots e_n\}$ in L, one can form a matrix for each operator $T(g)$ as follows:

$$T(g)\, e_i = \sum_j T_{ji}(g)\, e_j \quad . \tag{2.8.1}$$

The transformation of an arbitrary vector x^i is then

$$(x^i)' = T_{ik}\, x^k \quad . \tag{2.8.2}$$

A representation of a group in L_n is reducible if a non-trivial subspace L_m of L_n is left invariant by the operators $T(g)$ of the representation.
For example, suppose all the matrices corresponding to $T(g)$ can be written as

$$D(g) = \begin{pmatrix} D_{11}(g) & D_{12}(g) \\ 0 & D_{22}(g) \end{pmatrix} \quad .$$

Then we can write a transformation

$$D(g)\,\vec{x} = D(g)\begin{pmatrix} x_1 \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_m \end{pmatrix}\begin{matrix} \left.\vphantom{\begin{matrix}x\\x\\x\end{matrix}}\right\} L_m \\ \left.\vphantom{\begin{matrix}x\\x\\x\end{matrix}}\right\} L_{n-m} \end{matrix} = \begin{pmatrix} D_{11}(g) & D_{12}(g) \\ & \\ 0 & D_{22}(g) \end{pmatrix}\begin{pmatrix} L_m \\ \\ L_{n-m} \end{pmatrix} .$$

It can easily be seen that in this case L_m is left invariant by the operators:

$$\begin{pmatrix} D_{11}(g) & D_{12}(g) \\ 0 & D_{22}(g) \end{pmatrix}\begin{pmatrix} L_m \\ 0 \end{pmatrix} = \begin{pmatrix} D_{11}(g)L_m \\ 0 \end{pmatrix}$$

whereas L_{n-m} is not.

The representation is <u>completely reducible</u> if also L_{n-m} is left invariant, i.e. if $D_{21}(g) = 0$:

$$D(g) = \begin{pmatrix} D_{11}(g) & 0 \\ 0 & D_{22}(g) \end{pmatrix} .$$

A representation is <u>irreducible</u> if no invariant subspace exists.

2.8(b) Tensor Representations

Consider the general form of Lorentz transformations:

$$(x^{\mu})' = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \qquad (2.2.1)$$

Here the object x^{μ} is a 4-dimensional vector on which the matrix $\Lambda^{\mu}{}_{\nu}$ acts- hence eq.(2.2.1) is a linear transformation in a 4-dimesional vector space.
A covariant vector will transform according to

$$(\psi_{\mu})' = \Lambda_{\mu}{}^{\nu}\, \psi_{\nu} \quad . \tag{2.8.3}$$

If we now generalise from the linear transformation in a 4-dimensional vector space $\mathcal{L}_4$ to a linear transformation in a 4^n -dimensional vector space $\mathcal{L}_{4^n}$, the transformation law becomes

$$(\psi_{\mu_1\mu_2\ldots\mu_n})' = \Lambda_{\mu_1}{}^{\nu_1}\, \Lambda_{\mu_2}{}^{\nu_2} \ldots \Lambda_{\mu_n}{}^{\nu_n}\, \psi_{\nu_1\ldots\,\nu_n} \tag{2.8.4}$$

Here $\mu_1,\mu_2,\ldots\mu_n$; $\nu_1,\nu_2,\ldots\nu_n$ all range from 0 to 3, and eq.(2.8.4) is the defining equation for the transformation of a tensor of rank n.
Because the identity transformation is

$$(\psi_{\mu_1\ldots\mu_n})' = \delta_{\mu_1}{}^{\nu_1} \ldots \delta_{\mu_n}{}^{\nu_n}\, \psi_{\nu_1\ldots\nu_n} \quad ,$$

we require, as usual, that

$$(\Lambda)^T\, \eta\, (\Lambda) = \eta \quad ,$$

so that there exists an isomorphism between the group of linear transformations in 4^n dimensions and SO(3,1) or the Lorentz group $L_+^{\uparrow}$. Hence one can say that the group of transformations represented by eq.(2.8.4) form a 4^n-dimensional representation of $L_+^{\uparrow}$ - these representations are called tensor representations.
The simplest example of a tensor representation is the transformation of a tensor of rank 0:

$$(\psi)' = \psi$$

This is the (trivial) scalar representation. The next higher representation would be the transformation of a tensor of rank 1, which is also called vector representation:

$$(\psi_\mu)' = \Lambda_\mu{}^\nu \psi_\nu \quad .$$

One can also think of a tensor of rank n as a 4^n-component vector in the representation space, the transformation being "summarised" into one 4×4 matrix $A_N{}^M$; in other words, the transformation

$$(\psi_{\mu_1 \dots \mu_n})' = \Lambda_{\mu_1}{}^{\nu_1} \Lambda_{\mu_2}{}^{\nu_2} \dots \Lambda_{\mu_n}{}^{\nu_n} \psi_{\nu_1 \dots \nu_n}$$

is written as

$$(\psi_N)' = A_N{}^M \psi_M \quad . \tag{2.8.5}$$

Since each matrix $\Lambda_{\mu_i}{}^{\nu_i}$ of (2.8.4) only operates om the 4-dimensional subspace of $\mathcal{L}_{4^n}$ corresponding to the index ν_i of $\psi_{\mu_1 \dots \mu_n}$, $A_N{}^M$ is the direct product of the matrices $\Lambda_{\mu_i}{}^{\nu_i}$:

$$A = \underbrace{\Lambda \otimes \Lambda \otimes \Lambda \otimes \dots\dots \otimes \Lambda}_{\text{n factors}} \quad . \tag{2.8.6}$$

This shows that all the tensor representations of the group SO(3,1) or $L_+^\uparrow$ can be built up from the vector representation.

2.8(c) The Spinor Representations of SU(2)

Consider the transformations corresponding to the group SU(2,C):

$$(\xi)' = u\, \xi \tag{2.8.7}$$

with $u\,u^\dagger = u^\dagger\,u = 1$. (2.8.8)

Because of (2.8.8), u has the form

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad , \qquad |a|^2 + |b|^2 = 1 \tag{2.8.9}$$

as we have seen before.

In component form, (2.8.7) is

$$(\xi^1)' = a\xi^1 + b\xi^2$$
$$(\xi^2)' = -b^*\xi^1 + a^*\xi^2 \quad . \tag{2.8.10}$$

Notice that the complex conjugate of eq.(2.8.10) is equivalent to (2.8.10) itself - it does not yield a different transformation behavior.

We have seen before that the group SU(2) has three parameters (as seen from eq.(2.8.9)) and is homomorphic to SO(3). The elements ξ^i of the complex linear space which transform according to (2.8.10) are called spinors with respect to the three-dimensional Euclidean space.

We now consider the linear space of monomials of degree v:

$$P_k = (\xi^1)^{v-k}\,(\xi^2)^k \quad . \tag{2.8.11}$$

Here v and k are integers such that $0 \le v \le k$.

Because for fixed v, there are v+1 different monomials of the form (2.8.11), the space is (v+1)-dimensional. A general transformation in the (v+1)-dimensional representation space will thus be represented by a (v+1)x(v+1) matrix. This can be seen by applying the transformation (2.8.10) to P_k:

$$(P_k)' = (a\xi^1 + b\xi^2)^{v-k}\,(-b^*\xi^1 + a^*\xi^2)^k \tag{2.8.12}$$

By arranging the right-hand side of eq. (2.8.12) in terms of monomials, this can be written

$$(P_k)' = \sum_{l=0}^{v} D_{kl}\ (\xi^1)^{v-l}\ (\xi^2)^l$$

$$= \sum_{l=0}^{v} D_{kl}\ P_l \quad . \qquad (2.8.13)$$

The numbers D_{kl} are the the components of the (v+1)x(v+1) matrix representing the transformations in the (v+1)-dimensional representation space - the matrix D itself is called the spinor representation of the group SU(2).
The irreducible representations of a group are usually labelled by the eigenvalues of the Casimir operators of the group.The Casimir operator of SO(3) is J^2 with the eigenvalues

$$J^2 \mid j\ m > = j\ (j+1) \mid j\ m >$$

and

$$J_3 \mid m > = m \mid m >$$

with m = -j, -j+1, ... , 0, ...,j

For a fixed j, there are therefore 2j+1 eigenvectors $\mid j\ m >$.

Correspondingly, we want a (2j+1)-dimensional representation for each fixed j - for this reason, we identify the v of the monomials above with 2j:

$$\tfrac{1}{2}\ v = j\ . \qquad (2.8.14)$$

Now j can assume any integer or half-integer value.The representations so constructed are denoted D^j and are (2j+1)x(2j+1) matrices - the elements of the corresponding (2j+1)-dimensional representation space are called spinors of rank 2j of three-dimensional real space.

Let us look at some examples:

1) We construct D^1 of SU(2):

We have j = 1,

and thus v = 2.

Hence we get the monomials

$$P_0 = (\xi^1)^2$$

$$P_1 = (\xi^1)(\xi^2)$$

$$P_2 = (\xi^2)^2 \quad .$$

We have

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

and

$$P_k' = \sum_{l=0}^{v} D_{kl} P_l \quad .$$

Now, using explicitly the transformation (2.8.10), we get

$$(P_0)' = (a\xi^1 + b\xi^2)^2$$

$$= a^2 (\xi^1)^2 + (2ab)(\xi^1)(\xi^2) + b^2 (\xi^2)^2$$

$$\equiv \sum_{l=0}^{2} D_{0l} P_l$$

This gives

$$D_{00} = a^2$$

$$D_{01} = 2ab$$

$$D_{02} = b^2 \quad .$$

With similar calculations for $(P_1)'$ and $(P_2)'$ we obtain all the elements of the matrix $D^1(u)$. The result is

$$D^1(u) = \begin{pmatrix} a^2 & 2ab & b^2 \\ -ab^* & (aa^*-bb^*) & a^*b \\ b^{*2} & -2a^*b^* & a^{*2} \end{pmatrix} .$$

Similarly, with

$$-u = \begin{pmatrix} -a & -b \\ b^* & -a^* \end{pmatrix}$$

we get the equivalent representation $D^1(-u) = D^1(u)$.

2) Naturally, $D^{\frac{1}{2}}(u)$ of SU(2) is just the self-representation:

The monomials are:

$$P_0 = \xi^1$$

$$P_1 = \xi^2 \quad ,$$

so that

$$\begin{pmatrix} P_0 \\ P_1 \end{pmatrix}' = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}' = u \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = D^{\frac{1}{2}}(u) \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} .$$

On other hand,

$$D^{\frac{1}{2}}(-u) = -u\,\xi$$

$$= -\,D^{\frac{1}{2}}(u),$$

so in this case, $D^{\frac{1}{2}}(u)$ and $D^{\frac{1}{2}}(-u)$ are not identical matrices.

From the pattern emerging, we may notice the following facts:

i) The spinor representations of SU(2) - or equivalently, SO(3) - are irreducible.

ii) There are two types of representations:

Type 1 obeys

$$D^j(u) = D^j(-u)$$

and type 2 obeys

$$D^j(u) = - D^j(-u) .$$

These are called even (type 1) and odd (type 2) representations respectively. The above examples illustrate the fact that the representations corresponding to integer j are even and those corresponding to half-integer j are odd.

Recall that each element R of SO(3) is represented by the elements u and -u of SU(2).
Now for integer j, we had

$$D^j(u) = D^j(-u),$$

so that each element R of SO(3) is represented by a matrix D^j , regardless of the sign of u. Therefore the integer j (even) representations are single-valued for SO(3). The odd representations (half-integer j) are characterised by

$$D^j(u) = - D^j(-u)$$

so that each element R of SO(3), to which correspond the elements u and -u of SU(2),is represented by two different matrices D^j and $-D^j$. Therefore the odd representations are double-valued for SO(3).

iii) The fact that there is a representation of SO(3) (SU(2)) for all values of j illustrates the fact (not shown rigorously) that all the irreducible representations of SU(2) are contained in the spinor representations.

We have already constructed the tensor representations of $L_+^\uparrow$. But since SO(3) is a subgroup of $L_+^\uparrow$, this means we have also constructed the tensor representations of SO(3). Now we have seen that all the irreducible representations of SU(2) (and therefore SO(3)) are spinor representations - so we must conclude that the tensor representations are contained in the spinor representations.
In fact, the tensor representations are single-valued for SO(3) - so we can see that the spinor representations corresponding to integer j are tensor representations. We now illustrate this with an example.

Let us consider the representation $D^1(u)$ of SU(2) with the basis $(P_0, P_1, P_2) \equiv P$. Let us transform this basis with the matrix

$$T = \begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & 2 & 0 \end{pmatrix} , \quad T^{-1} = 1/2 \begin{pmatrix} -1 & i & 0 \\ 0 & 0 & 1 \\ 1 & i & 0 \end{pmatrix}$$

with det T = -2i, det T^{-1} = i/2 ,

to obtain the transformed basis vector

$$\Psi \equiv T\,P = \begin{pmatrix} -P_0 + P_2 \\ -iP_0 - iP_2 \\ -2P_1 \end{pmatrix} \quad ..$$

Under a transformation D^1 of SU(2), P becomes

$$(P)' = D^1(u)\,P \quad .$$

Correspondingly

$$(\Psi)' = (TP)' = T(P)'$$

$$= T\,D^1\,P$$

$$= T\, D^1\, T^{-1}\, \Psi \quad . \tag{2.8.15}$$

With

$$D^1(u) = \begin{pmatrix} a^2 & 2ab & b^2 \\ -ab^* & (aa^*-bb^*) & a^*b \\ b^2 & -2a^*b^* & a^{*2} \end{pmatrix}$$

we get

$$T\, D^1(u)\, T^{-1} = \frac{1}{2} \begin{pmatrix} a^2-b^2-b^{*2}+a^{*2} & i(a^{*2}+b^{*2}-a^2-b^2) & -2(ab+a^*b^*) \\ i(a^2-b^2-a^{*2}+b^{*2}) & a^2+b^2+a^{*2}+b^{*2} & -2i(ab-a^*b^*) \\ 2(ab^*+a^*b) & 2i(a^*b-ab^*) & 2(aa^*-bb^*) \end{pmatrix} \tag{2.8.16}$$

Because of (2.8.8), $aa^* + bb^* = 1$, and after finite calculational effort we confirm that

$$\sum_i (\Psi_i')^2 = \Psi_1' + \Psi_2' + \Psi_3'$$

$$= \sum_i (\Psi_i)^2 \quad ,$$

showing that the "length" of the vector is invariant under the transformation (2.8.15).

Additionally, $\det\, (T\, D^1\, T^{-1})$

$$= \det\, (D^1)$$

$$= 1 \; , \tag{2.8.28}$$

and all the matrix elements of (2.8.16) are real.

The equations (2.8.17) and (2.8.18) show that the transformation (2.8.15) of the vector Ψ_i has all the properties of the well-known Lorentz-transformation of a vector - or, equivalently, that the transformation (2.8.15) is equivalent to the vector transformation:

$$(\Psi_i)' = \Lambda_i{}^j \Psi_j \quad . \tag{2.8.19}$$

We conclude that D^1 is identical to the vector representation.

Note that, since D^1 is the vector representation of SO(3), it is a three-dimensional representation as opposed to the four-dimensional vector representation of SO(3,1).

In an analogous manner, $D^2(u)$ of SU(2) can be shown to be identical to the tensor representation of SO(3) of rank 2, and so on.

We sumarise the observations of this section in the following

Table 2.2: Representations of SO(3) - SU(2):

$D^j(u)$ of SU(2): integer j	$D^j(u)$ of SU(2): half-integer j
even: $D^j(u) = D^j(-u)$	odd: $D^j(u) = -D^j(-u)$
single-valued for SO(3): to each R of SO(3) corresponds $D^j(u)$	double-valued for SO(3): to each R of SO(3) correspond $D^j(u), -D^j(u)$
tensor representation	genuine spinor representation

Irreducible Representations of O(3):

The irreducible representations of O(3) can be obtained from the irreducible representations of SO(3) by incorporating the inversion transformation.
Since O(3) is the direct product of SO(3) and $J \equiv (I, I_s)$:

$$O(3) = SO(3) \otimes J \quad ,$$

we can classify the irreducible representations of $O(3)$ by using the irreducible representations of $SO(3)$, $D^j(u)$.

The inversion element I_s is represented by

$$D(I_s) = \pm I \ .$$

For integer j, since $D^j(-u) = D^j(u)$, we get two possibilities for the representations $D^{(j)}$ of $O(3)$:

$$D^j(I_sR) = D(I_s)\ D^{(j)}(R) = D^j(R) \tag{2.8.20}$$

or

$$D^j(I_sR) = D(I_s)\ D^{(j)}(R) = -D^j(R) \tag{2.8.21}$$

We characterise the two different representations as follows:
If $D^{(j)}(I_sR)$ satisfies (2.8.20), it is denoted by $D^{(j+)}$, and $D^{(j)}(I_sR)$ satisfying (2.8.21) is denoted by $D^{(j-)}$. Since (2.8.20) does not reflect the inversion, it is not a faithful representation, whereas $D^{(j-)}$ obviously is faithful.

For half-integer j, for which $D^j(-u) = -D^j(u)$, there is no way to distinguish between a representation $D^j(I_sR)$ of $O(3)$ and a representation $D^j(u)$ of $SO(3)$: both can have the same value. There is thus only one double-valued representation for half-integer j:

$$\pm D^j(I_sR) = \pm\ D^{(j)}(R) \quad .$$

2.8(d) The Representations of SL(2,C)

The group SL(2,C) is the group of complex unimodular matrices in 2 dimensions:

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \qquad \det u = 1 \ , \tag{2.8.22(a)}$$

$$\text{i.e. } ad - bc = 1 \ . \tag{2.8.22(b)}$$

The transformations of SL(2,C) can be written as:

$$(\xi)' = u\,\xi \tag{2.8.23}$$

or in component form

$$(\xi^1)' = a\xi^1 + b\xi^2$$

$$(\xi^2)' = c\xi^1 + d\xi^2 \ . \tag{2.8.24}$$

The complex conjugate transformation to (2.8.24) is

$$(\xi^{\dot{1}})' = a^*\xi^{\dot{1}} + b^*\xi^{\dot{2}}$$

$$(\xi^{\dot{2}})' = c^*\xi^{\dot{1}} + d^*\xi^{\dot{2}} \tag{2.8.25}$$

with $a^*d^* - b^*c^* = 1$.

In matrix notation this is

$$(\xi^*)' = u^*\,(\xi^*) \ . \tag{2.8.26}$$

The covariant spinors are defined as follows:

$$(\eta)' = \eta\, u^{-1}$$

$$(\eta^*)' = \eta^* \, (u^*)^{-1} \quad .$$

This is so that the form of the product of spinors is invariant under complex conjugation

$$\eta \, \xi = \eta^* \, \xi^*. \tag{2.8.27}$$

In index notation, we thus have:

$$\xi^{\alpha} = u^{\alpha}{}_{\beta} \, \xi^{\beta} \tag{2.8.28)(a}$$

$$\xi^{\dot{\alpha}} = u^{*\dot{\alpha}}{}_{\dot{\beta}} \, \xi^{\dot{\beta}} \tag{2.8.28)(b}$$

$$\eta_{\alpha} = (u^{-1})_{\alpha}{}^{\beta} \, \eta_{\beta} \tag{2.8.28)(c}$$

$$\eta_{\dot{\alpha}} = (u^{-1})_{\dot{\alpha}}{}^{\dot{\beta}} \, \eta_{\dot{\beta}} \tag{2.8.28)(d}$$

The elements of the complex 2-dimensional vector space which transform according to (2.8.24) are called spinors of Minkowski space (because SL(2,C) is homomorphic to $L_+^{\uparrow}$). Because the matrices u of SL(2,C) are not unitary, the transformations (2.8.24) and (2.8.25) are not equivalent and hence constitute two different representations, acting on two different two-dimensional complex vector spaces (the basis vectors being ξ and $\dot{\xi} \equiv \xi^*$ respectively.).

We can see that the group has 6 real parameters: between the 4 complex components of u there is only one complex equation (2.8.22)(b). Because the group has two non-equivalent representations corresponding to ξ and ξ^* , we construct monomials of degree v+v = 2v:

$$P_{kk'} = (\xi^1)^{v-k} \, (\xi^{\dot{1}})^{v'-k'} \, (\xi^2)^k \, (\xi^{\dot{2}})^{k'} \tag{2.8.29}$$

with $\quad 0 \le k \le v$

and $\quad 0 \le k' \le v \quad .$

Consequently the representation space has now the dimension $(v+1)(v'+1)$.

Recall that the Casimir operators of the group $L_+^\uparrow$ are M^2 and N^2, with the eigenvalues

$$M^2 \mid j\ m\ j'\ n> = j\ (j+1) \mid j\ m\ j'\ n>$$

$$N^2 \mid j\ m\ j'\ n> = j'\ (j'+1) \mid j\ m\ j'\ n> \tag{2.8.30}$$

and

$$M_3 \mid j\ m\ j'\ n> = m \mid j\ m\ j'\ n>$$

$$N_3 \mid j\ m\ j'\ n> = n \mid j\ m\ j'\ n> \tag{2.8.31}$$

This is because M,N have the Lie algebra of angular momentum operators and can thus be interpreted as angular momenta.

Again (in analogy to the angular momentum J covered in section 8(c)), for a fixed value of j there are 2j+1 allowed values of m, and for a fixed value of j' there are 2j'+1 allowed values of n. Consequently we make the identifications

$$\tfrac{1}{2}\ v = j$$

$$\tfrac{1}{2}\ v' = j' \tag{2.8.32}$$

and call the corresponding representations $D^{jj'}$. Naturally, j and j' can now be integer of half-integer.

The elements $D^{jj'}$ are called spinors of rank $(2j+1)(2j'+1)$ and they are $(2j+1)(2j'+1)\text{x}(2j+1)(2j'+1)$ matrices - the representation space is $(2j+1)(2j'+1)$-dimensional, of course.

Let us now calculate a few examples:

a) $D^{0\frac{1}{2}}$ is based upon the elements

$$P_{00} = \xi^1$$

$$P_{10} = \xi^2 \quad .$$

The corresponding transformation is

$$\begin{pmatrix} P_{00} \\ P_{10} \end{pmatrix}' = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} P_{00} \\ P_{10} \end{pmatrix} = D^{\frac{1}{2}0}(u,u^*) \begin{pmatrix} P_{00} \\ P_{10} \end{pmatrix}$$

In components we have

$$P_{00}' = (\xi^1)' = a\xi^1 + b\xi^2 = aP_{00} + bP_{10}$$

$$P_{10}' = (\xi^2)' = c\xi^1 + d\xi^2 = cP_{00} + dP_{10} \quad .$$

Hence $D^{\frac{1}{2}0}(u,u^*) = u$.

$D^{\frac{1}{2}0}$ is thus the self-representation of SL(2,C) through the spinor ξ.

b) $D^{0\frac{1}{2}}$ is calculated similarly and one obtains

$$D^{0\frac{1}{2}}(u,u^*) = u^* \quad ,$$

so that $D^{0\frac{1}{2}}$ is the self-representation of SL(2,C), with the basis ξ^*.

c) Now we consider $D^{\frac{1}{2}\frac{1}{2}}$:

Here $j = j' = \frac{1}{2}$, giving $v = v' = 2$.

We get the monomials

$$P_{00} = \xi^1 \dot{\xi}^1$$

$$P_{01} = \xi^1 \dot{\xi}^2$$

$$P_{10} = \xi^{2} \xi^{\dot{1}}$$

$$P_{11} = \xi^{2} \xi^{\dot{2}}$$

with the corresponding transformation

$$P_{00}' = (a\xi^{1} + b\xi^{2})\ (a^{*}\xi^{\dot{1}} + b^{*}\xi^{\dot{2}})$$

$$= aa^{*}\ \xi^{1}\xi^{\dot{1}} + ab^{*}\ \xi^{1}\xi^{\dot{2}} + ba^{*}\ \xi^{2}\xi^{\dot{1}} + bb^{*}\ \xi^{2}\xi^{\dot{2}}$$

$$= aa^{*}\ P_{00} + ab^{*}\ P_{01} + ba^{*}\ P_{10} + bb^{*}\ P_{11}\ .$$

With similar calculations for P_{00}', P_{01}',P_{10}',P_{11}' we get the transformation

$$\begin{pmatrix} P_{00} \\ P_{01} \\ P_{10} \\ P_{11} \end{pmatrix}' = D^{\frac{1}{2}\frac{1}{2}}\ (u,u^{*}) \begin{pmatrix} P_{00} \\ P_{01} \\ P_{10} \\ P_{11} \end{pmatrix}$$

where

$$D^{\frac{1}{2}\frac{1}{2}}(u,u^{*}) = \begin{pmatrix} aa^* & ab^* & ba^* & bb^* \\ ac^* & ad^* & bc^* & bd^* \\ ca^* & cb^* & da^* & db^* \\ cc^* & cd^* & dc^* & dd^* \end{pmatrix} \qquad (2.8.33)$$

The unitarity relation ad-bc = 1 may be used to simplify this expression a little.

In analogy to the spinor representations of SU(2) we can note some facts straight away:

i) The group $L_+^\uparrow$ is doubly connected (as SO(3)), so there are again two kinds of irreducible representations: the double-valued and single-valued representations.

ii) Again, the single-valued representations are tensor representations and correspond to integer angular momentum: $j+j'$ = integer.
The double-valued representations are genuine spinor representations and correspond to half-integer angular momentum:
$j+j'$ = half-integer.

As an example, we show that $D^{\frac{1}{2}\frac{1}{2}}(u,u^*)$ (eq.(2.8.33)) of SL(2,C) is equivalent to the vector representation of $L_+^\uparrow$:

The representation space consists of monomials P_{00}, P_{01},P_{10},P_{11} . Again, we can transform the basis in representation space - we choose the following transformation matrix:

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \\ i & 0 & 0 & i \end{pmatrix} \tag{2.8.34}$$

$$T^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & -i \\ 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & -1 & -i \end{pmatrix}$$

which gives the new basis

$$\Psi = \frac{1}{\sqrt{2}} T\ P\ ,$$

or $\quad P = \sqrt{2}\ T^{-1}\ \Psi\ .$

More precisely:

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\ (P_{10} + P_{01}) \\ \frac{1}{2} i (P_{01} - P_{10}) \\ \frac{1}{2}\ (P_{00} - P_{11}) \\ \frac{1}{2} i (P_{00} + P_{11}) \end{pmatrix}$$

or

$$\begin{pmatrix} P_{00} \\ P_{01} \\ P_{10} \\ P_{11} \end{pmatrix} = \begin{pmatrix} \Psi_3 + i\Psi_4 \\ \Psi_1 - i\Psi_2 \\ \Psi_1 - i\Psi_2 \\ -\Psi_3 - i\Psi_4 \end{pmatrix} . \tag{2.8.35}$$

In the new basis, the transformation $D^{\frac{1}{2}\frac{1}{2}}$ of SL(2,C) is written:

$$\begin{aligned}(\Psi)' &= \frac{1}{\sqrt{2}}\, T\, P' \\ &= \frac{1}{\sqrt{2}}\, T\, D^{\frac{1}{2}\frac{1}{2}}(u,u^*)\, P \\ &= \frac{1}{\sqrt{2}}\, T\, D^{\frac{1}{2}\frac{1}{2}}(u,u^*)\, P\, \sqrt{2}\, T^{-1}\, \Psi \\ &= T\, D^{\frac{1}{2}\frac{1}{2}}(u,u^*)\, P\quad T^{-1}\, \Psi \quad .\end{aligned} \tag{2.8.36}$$

We get

$$2\, T\, D^{\frac{1}{2}\frac{1}{2}}(u,u^*)\, T^{-1} =$$

$$\begin{pmatrix} (a\bar{d}+c\bar{b}+b\bar{c}+d\bar{a}) & i(-a\bar{d}-c\bar{b}+b\bar{c}+d\bar{a}) & (a\bar{c}+c\bar{a}-b\bar{d}-d\bar{b}) & -i(a\bar{c}+c\bar{a}+b\bar{d}+d\bar{b}) \\ i(a\bar{d}+b\bar{c}-c\bar{b}-d\bar{a}) & (a\bar{d}-c\bar{b}-b\bar{c}+d\bar{a}) & i(a\bar{c}+d\bar{b}-c\bar{a}-b\bar{d}) & (a\bar{c}+b\bar{d}-c\bar{a}-d\bar{b}) \\ (a\bar{b}-c\bar{d}+b\bar{a}-d\bar{c}) & i(c\bar{d}+b\bar{a}-a\bar{b}-d\bar{c}) & (a\bar{a}-c\bar{c}-b\bar{b}+d\bar{d}) & i(c\bar{c}+d\bar{d}-a\bar{a}-b\bar{b}) \\ i(a\bar{b}+c\bar{d}+b\bar{a}+d\bar{c}) & (a\bar{b}+c\bar{d}-b\bar{a}+d\bar{c}) & i(a\bar{a}+c\bar{c}-b\bar{b}-d\bar{d}) & (a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d}) \end{pmatrix}$$

(where $\bar{a}$ means a^* etc.).

One can then confirm the following :

a) $$\sum_i (\Psi_i')^2 = \sum_i (\Psi_i)^2 \tag{2.8.37}$$

b) $\det\,(T\,D^{\frac{1}{2}\frac{1}{2}}\,T^{-1}) = \det\,D^{\frac{1}{2}\frac{1}{2}}$

$$= 1$$

$$= \det(u)\,\det(u^{*}) \quad . \qquad (2.8.38)$$

c) Ψ_1, Ψ_2, Ψ_3 are real and Ψ_4 is complex.

d) $(T\,D^{\frac{1}{2}\frac{1}{2}}\,T^{-1})_{44} = 1/2\ (\ |a|^2 + |b|^2 + |c|^2 + |d|^2)$

$$\geq 1.$$

(Proof: Suppose $|a|^2 + |b|^2 + |c|^2 + |d|^2 < 2.$

Then suppose $|b|^2 + |c|^2 \ < x$

$$\Rightarrow \qquad -\frac{x}{2} < \mathrm{Re}(bc) < \frac{x}{2}$$

Hence $\qquad |a|^2 + |d|^2 \ < x - 2 - x$

$$\Rightarrow \qquad -\frac{2-x}{2} < \mathrm{Re}(ad) < \frac{2-x}{2}$$

$$\Rightarrow \qquad \mathrm{Re}(ad-bc) \quad < 1 - \frac{x}{2} - \frac{x}{2}$$

$$= 1 \quad .$$

But we know from the unimodularity of u,u* that

$$ad - bc = 1 ,$$

which contradicts the above result. Consequently we must have that

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 \geq 2 \qquad , \text{q.e.d})$$

e) For example, $\;u = e^{-i\sigma_i/2}$

implies $D^{\frac{1}{2}\frac{1}{2}} = \begin{pmatrix} 2\sigma^i & 0 \\ 0 & 2\sigma^i \end{pmatrix}$.

Thus we see that under the transformation (2.8.36), the quantities Ψ_μ transform as four-vectors under proper Lorentz transformations, Ψ_4 being the time-component of the vector. The components of the vector Ψ_μ in terms of its spinor components are given by eq.(2.8.35).

Example:

Consider the particular Lorentz transformation

$$x_0' = x_0$$

$$x_1' = x_1 \cos\theta + x_2 \sin\theta$$

$$x_2' = -x_1 \sin\theta + x_2 \cos\theta$$

$$x_3' = x_3 \quad .$$

We use the correspondence

$$x_0 = \Psi_4$$

$$x_1 = \Psi_1$$

$$x_2 = \Psi_2$$

$$x_3 = \Psi_3$$

and use (2.8.35) to get the transformed components

$$P_{00}' = x_3' - ix_0' = x_3 - ix_0 = P_{00}$$

$$P_{01}' = x_1' - ix_2' = (x_1 - ix_2)\, e^{i\theta} = e^{i\theta} P_{01}$$

$$P_{10}' = x_1' + ix_2' = (x_1 + ix_2)\, e^{-i\theta} = e^{-i\theta} P_{10}$$

$$P_{11}' = -x_3' - ix_0' = -x_3 - ix_0 = P_{11}$$

Comparison with (2.8.33) yields

$$aa^* = 1$$

$$ad^* = e^{i\theta}$$

$$da^* = e^{-i\theta}$$

$$dd^* = 1 \;,$$

all other components being zero. This is satisfied by

$$a = \pm\, e^{i\theta/2}$$

$$d = \pm\, e^{-i\theta/2}$$

$$b,c = 0 \quad .$$

Consequently

$$u = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} = D^{\frac{1}{2}0}\,(u,u^*) \tag{2.8.39}$$

and

$$u^* = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} = D^{0\frac{1}{2}}\,(u,u^*) \quad . \tag{2.8.40}$$

Note that, if we substitute $\theta = 2\pi$ in (2.8.40), the spinor transformation reads

$$(\xi)' = u\,\xi = \begin{pmatrix} e^{i\pi} & 0 \\ 0 & e^{-i\pi} \end{pmatrix} \xi = -\xi$$

and $(\dot{\bar{\xi}})' = -\,\dot{\xi}$, i.e. spinors change sign under a rotation of 2π . This demonstrates again the fact that the spinor representation is double-valued for $L_+^\uparrow$ and hence also for SO(3): The values θ and $\theta + 2\pi$ correspond to different basis vectors ξ, but to the same rotation.

If we look at the representation $D^{\frac{1}{2}\frac{1}{2}}(u,u^*)$ of SL(2,C), eq.(2.8.33), we notice that it can be written as follows:

$$D^{\frac{1}{2}\frac{1}{2}}(u,u^*) = \begin{pmatrix} aa^* & ab^* & ba^* & bb^* \\ ac^* & ad^* & bc^* & bd^* \\ ca^* & cb^* & da^* & db^* \\ cc^* & cd^* & dc^* & dd^* \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} , \qquad (2.8.41)$$

as the direct product of two matrices is defined exactly in this way. Consequently we can write:

$$D^{jj'}(u,u^*) = D^j(u) \otimes D^{j'}(u^*) \quad .$$

However, the matrices $D^j(u)$, $D^{j'}(u^*)$ are not identical to the matrices $D^j(u)$ of SU(2) since u of SU(2) is unimodular <u>and</u> unitary, whereas u of SL(2,C) is only unimodular. However, if we restrict ourselves to the subgroup of rotations, u of SU(2) and u of SL(2,C) are identical and <u>for spatial rotations</u> we can write

$$D^{jj'} = D^j \otimes D^{j'} \quad ,$$

where D^j is now the representation of the three-dimensional rotation group.

The index j gives the spin of the particular representation D^j . Because to each D^j there corresponds a different value of the spin, the representation $D^{jj'}$ of $L_+^\uparrow$ does not possess a unique spin representation, but is built up from different spin representations D^j ,$D^{j'}$. Generally one can thus write

$$D^{jj'} = D^{j} \otimes D^{j'} \tag{2.8.42}$$

$$= D^{j+j'} \oplus D^{j+j'-1} \oplus \ldots \oplus D^{|j-j'|} \quad . \tag{2.8.43}$$

For example,

$$D^{\frac{1}{2}0} = D^{\frac{1}{2}} \oplus D^{0}$$

$$= D^{\frac{1}{2}} \quad .$$

Finally, we can obtain higher irreducible representations by direct product decomposition as follows:

$$D^{(j_1,j_2)} \otimes D^{(j_1',j_2')} = D^{(j_1+j_2',j_2+j_2')} \oplus \ldots \oplus D^{(|j_1-j_1'|,|j_2-j_2'|)} \tag{2.8.44}$$

To recapitulate:

The homogeneous Lorentz group is characterised by matrices $\Lambda^{\mu}{}_{\nu}$, to which correspond the matrices A of SL(2,C) via the homomorphism $L_{+}^{\uparrow}$ -SL(2,C). The correspondence is given by eq.(2.4.4):

$$\Lambda^{\mu}{}_{\nu} = \tfrac{1}{2} \, \mathrm{Tr} \, (\sigma^{\mu} A \, \bar{\sigma}_{\nu} \, A^{\dagger}) \quad .$$

It can be seen from the form of the equation that the four matrices A , -A, A^{*}, $-A^{*}$ all correspond to the same matrix $\Lambda^{\mu}{}_{\nu}$. We remember that the group $L_{+}^{\uparrow}$ is doubly connected so that both matrices A, -A of SL(2,C) correspond to the same element of $L_{+}^{\uparrow}$. Therefore it is clear that the two types of matrices A, A^{*} correspond to two different non-equivalent representations of $L_{+}^{\uparrow}$. It is also evident that the matrices A, A* are identical to the matrices u, u^{*} of SL(2,C), so that A,A^{*} correspond to the representations $D^{\frac{1}{2}0}(u,u^{*})$ and $D^{0\frac{1}{2}}(u,u^{*})$ of SL(2,C).

Recall from section 2.4 that a 2x2 matrix A of SL(2,C) can in general be written

$$A = e^{\frac{1}{2}i\,(i\phi\,\vec{\sigma}\cdot\vec{\nu}\,-\,\theta\,\vec{\sigma}\cdot\vec{n})} \quad . \tag{2.8.45}$$

On the other hand, we recall from section 2.3 (eq. (2.3.14)) that a general matrix $\Lambda^{\mu}{}_{\nu}$ of $L_{+}^{\uparrow}$ can be written as

$$\Lambda = e^{-i\,\theta\,\vec{J}\cdot\vec{n}\,-\,i\,\phi\,\vec{K}\cdot\vec{\nu}} \quad . \tag{2.8.46}$$

This shows that $\frac{1}{2}\vec{\sigma}$,$-\frac{1}{2}i\vec{\sigma}$ provide a 2x2 representation for the generators $\vec{L}$, $\vec{K}$ of $L_{+}^{\uparrow}$:

$$\begin{aligned} J_i &\longrightarrow \tfrac{1}{2}\,\sigma_i \\ K_i &\longrightarrow -\tfrac{1}{2}\,i\,\sigma_i \end{aligned} \tag{2.8.47}$$

Recall from (2.3.18):

$$\begin{aligned} M_i &= \tfrac{1}{2}\,(J_i + iK_i) \\ N_i &= \tfrac{1}{2}\,(J_i - iK_i) \quad , \end{aligned}$$

so the representation (2.8.47) becomes

$$\begin{aligned} M_i &\longrightarrow \tfrac{1}{2}\,\sigma_i \\ N_i &\longrightarrow 0 \quad . \end{aligned} \tag{2.8.48}$$

We have labelled the representations by the eigenvalues of the Casimir operators M^2, N^2 and their z-components M_3 ,N_3 . From (2.8.48) we see that the j-values corresponding to M_i,N_i are 1/2 and 0 respectively - hence the representation given by (2.8.45) is equivalent to $D^{\frac{1}{2}0}(u,u^*)$:

$$A = e^{-\frac{1}{2}i\,\vec{\sigma}\cdot(\vec{\theta}\,-\,i\vec{\phi})} \equiv D^{\frac{1}{2}0}(u,u^*) \quad . \tag{2.8.49}$$

The complex conjugate of (2.8.45) is

$$A^* = e^{-\frac{1}{2}i\,(-i\vec{\sigma}^*\cdot\vec{\phi} - \vec{\sigma}^*\cdot\vec{\theta})}$$

$$= e^{-\frac{1}{2}\,\vec{\phi}\cdot\vec{\sigma}^* + \frac{1}{2}i\,\vec{\theta}\cdot\vec{\sigma}^*} \quad . \tag{2.8.50}$$

This gives the representation

$$J_i \longrightarrow -\tfrac{1}{2}\,\sigma_i^{\,*}$$

$$K_i \longrightarrow -\tfrac{1}{2}\,i\,\sigma_i^{\,*} \quad . \tag{2.8.51}$$

However, we can make the unitary transformation

$$A^* \to S\,A^*\,S^{-1}$$

with $S = \sigma_2$.

Because $\sigma_2\,\sigma_i^*\,\sigma_2 = -\sigma_i$, we get the equivalent representation

$$A^* = e^{-\frac{1}{2}i\,\vec{\sigma}\cdot(\vec{\theta} + i\vec{\phi})}$$

and hence

$$J_i \longrightarrow \tfrac{1}{2}\,\sigma_i$$

$$K_i \longrightarrow \tfrac{1}{2}\,i\,\sigma_i \tag{2.8.52}$$

which corresponds to

$$M_i \longrightarrow 0$$

$$N_i \longrightarrow \tfrac{1}{2}\,\sigma_i \ ,$$

so that the representation (2.8.52) is identical to $D^{0\frac{1}{2}}(u,u^*)$. Naturally, the bases for the representations $D^{\frac{1}{2}0}$ and $D^{0\frac{1}{2}}$ are the spinors ξ and ξ^* as seen before.

Similarly, the representation $D^{\frac{1}{2}\frac{1}{2}}$ has the basis $\xi\xi^\dagger$ which transforms according to

$$(\xi\xi^\dagger) = A\,\xi\xi^\dagger\,A^\dagger \quad . \tag{2.8.53}$$

This has the same form as eq.(2.4.3) which represents the transformation of the four-vector x^μ written as a 2x2 matrix X - therefore $\xi\xi^\dagger$ can be used as the basis of the four-dimensional representation $D^{\frac{1}{2}\frac{1}{2}}$.

Note: here $\xi\xi^\dagger$ must be read as a direct product of ξ, $\xi^\dagger$:

$$\xi\xi^\dagger = \xi \otimes \xi^\dagger$$

$$= \begin{pmatrix} \xi^1\xi^{1*} & \xi^1\xi^{2*} \\ \xi^2\xi^{1*} & \xi^2\xi^{2*} \end{pmatrix} = \begin{pmatrix} \xi^1\xi^{\dot{1}} & \xi^1\xi^{\dot{2}} \\ \xi^2\xi^{\dot{1}} & \xi^2\xi^{\dot{2}} \end{pmatrix}$$

$$= \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix} \equiv X = \sum_\mu \sigma^\mu x^\mu$$

$$= \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix}$$

and we can again easily verify that

$$(x^{\mu\prime})^2 = (x^\mu)^2$$

i.e. $\det(X') = \det X$:

$$(\xi \otimes \xi^\dagger)' = \begin{pmatrix} P_{00}' & P_{01}' \\ P_{10}' & P_{11}' \end{pmatrix}$$

implies

$$\det(X') = P_{00}'\, P_{11}' - P_{01}'\, P_{10}'$$

$$= P_{00}\, P_{11} - P_{01}\, P_{10}$$

$$= \det X \quad .$$

We see this is true because

$$P_{00}'\, P_{11}' - P_{01}'\, P_{10}'$$

$$= (aa^*P_{00}+ab^*P_{01}+ba^*P_{10}+bb^*P_{11}) \times (cc^*P_{00}+cd^*P_{01}+dc^*P_{10}+dd^*P_{11})$$

$$- (ac^*P_{00}+ad^*P_{01}+bc^*P_{10}+bd^*P_{11}) \times (ca^*P_{00}+cb^*P_{01}+da^*P_{10}+db^*P_{11})$$

$$= \{\ a^*d^*(ad - bc) - b^*c^*(ad - bc)\ \}P_{00}P_{11}$$

$$+ \{\ a^*d^*(bc - ad) - b^*c^*(bc - ad)\ \}P_{01}P_{10}$$

$$= P_{00}P_{11} - P_{01}P_{10} \quad .$$

2.8(e) Representations of the Extended Lorentz Group L:

We recall the definition of covariant spinors

$$\eta' = \eta\, A^{-1}$$

i.e. $(\eta_\alpha)' = \eta_\beta\, (A^{-1})^\beta{}_\alpha$ (2.8.54)

and $(\eta^*)' = \eta^*\, A^{*-1}$

i.e. $(\eta_{\dot\alpha})' = \eta_{\dot\beta}\,(A^*)^{\dot\beta}{}_{\dot\alpha}$ (2.8.55)

as opposed to contravariant spinors

$$\xi' = A\,\xi \qquad \Rightarrow (\xi^\alpha)' = A^\alpha{}_\beta\,\xi^\beta \tag{2.8.56}$$

$$(\xi^*)' = A^*\,\xi \qquad \Rightarrow (\xi^{\dot\alpha})' = A^{*\dot\alpha}{}_{\dot\beta}\,\xi^{\dot\beta} \tag{2.8.57}$$

Now one can see that there exists a matrix C which relates A and $(A^{-1})^T$ as follows:

$$(A^{-1})^T = C\,A\,C^{-1}$$

or equivalently

$$C = A^T\,C\,A\,. \tag{2.8.58}$$

The transpose of (2.8.54) is

$$(\eta^T)' = (A^{-1})^T\,\eta^T = C\,A\,C^{-1}\,\eta^T$$

$$\Rightarrow \quad (C^{-1}\eta^T)' = A\,(C^{-1}\,\eta^T)\quad .$$

Since this is the same as eq.(2.8.56),

$$(\xi)' = A\,\xi \quad ,$$

we conclude that $C^{-1}\,\eta^T$ transforms as ξ; we can write

$$\eta^T = C\,\xi \quad ; \tag{2.8.59}$$

hence A and A^{T-1} are equivalent.

The matrix C is found to be

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\,\sigma_2 \ ,$$

and (2.8.59) shows that C transforms contravariant and covariant spinors into each other.

We find similarly under a Lorentz transformation

$$\begin{aligned} \eta^{T*} &\rightarrow (A^{*-1})^T \,\eta^{*T} \\ &= C\,A^*\,C^{-1}\,\eta^{*T} \end{aligned}$$

so that

$$(\eta^*)^T = \eta^\dagger = C\,\xi^* \tag{2.8.60}$$

and hence A^* and $A^{\dagger -1}$ are equivalent.

We notice that explicitlty,

$$C = -C^{-1} = C^* = C^T \ . \tag{2.8.61}$$

Recall that the elements of the extended Lorentz group L can be constructed by adding to the restricted Lorentz group $L_+^\uparrow$ the inversion elements I_s, I_t, I_{st}. So to obtain IR's for L, we investigate the behavior of the bases ξ, ξ^*, η, η^* of the two-dimensional IR's $D^{\frac{1}{2}0}$ and $D^{0\frac{1}{2}}$ of $L_+^\uparrow$ under these discrete transformations.

We note that the matrix X (2.4.1) transforms under the inversions as follows:

$$X \xrightarrow{I_s} X' = \begin{pmatrix} x^0+x^3 & -x^1+ix^2 \\ -x^1-ix^2 & x^0+x^3 \end{pmatrix}$$

$$= C\,X^{*}\,C^{-1} \tag{2.8.62(a)}$$

$$X \xrightarrow{I_t} X' = \begin{pmatrix} -x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & -x^0-x^3 \end{pmatrix}$$

$$= -C\,X^{*}\,C^{-1} \tag{2.8.62(b)}$$

$$X \xrightarrow{I_{st}} X' = -X \quad . \tag{2.8.62(c)}$$

From this we can choose the new bases for L, neglecting phase amgiguities:

$$\left.\begin{aligned} \xi &\xrightarrow{I_s} i\eta^{\dagger} \\ \eta^{\dagger} &\xrightarrow{I_s} i\xi \end{aligned}\right\} \tag{2.8.63}$$

This we see as follows: $\xi\xi^{\dagger}$ behaves as X under transformations of SL(2,C), so under I_s we have

$$\begin{aligned} X \xrightarrow{I_s} X' &= C\,X^{*}\,C^{-1} \\ &= C\,(\xi\xi^{\dagger})^{*}\,C^{-1} \end{aligned} \tag{2.8.64}$$

but since $\eta^{T} = C\,\xi$, we can substitute for ξ in (2.8.64):

$$\begin{aligned} X' &= C\,(C^{-1}\,\eta^{T}\,\eta^{*}\,C)^{*}\,C^{-1} \\ &= C\,C^{-1}\,\eta^{\dagger}\,\eta\,C\,C^{-1} \\ &= \eta^{\dagger}\,\eta\ . \end{aligned}$$

Now if we let

$$\eta^\dagger \xrightarrow{I_s} i\xi$$

then

$$X' = (i\xi)(i\xi)^\dagger$$
$$= \xi\xi^\dagger ,$$

so that the transformed X can again be represented by the (transformed) basis $\xi\xi^\dagger$. This confirms the choice (2.8.63) for the basis ξ.

Similarly, if we use the basis $\eta^T\eta^*$, we get under an SL(2,C) transformation:

$$X \xrightarrow{I_s} X' = C\, X^*\, C^{-1}$$
$$= C\, (\eta^T\eta^*)\, C^{-1}$$
$$= C\, \eta^\dagger\eta\, C^{-1}$$
$$= \xi^*\xi^T .$$

Let

$$\xi \xrightarrow{I_s} i\eta^\dagger ,$$

then

$$X' = (i\eta^\dagger)^*(i\eta^\dagger)^T$$
$$= \eta^T\eta^*$$

as before, so we have confirmed (2.8.63) for the basis $\eta^\dagger$.

Similarly, under the inversion I_t and I_{st}, we choose

$$\left.\begin{array}{l} \xi \xrightarrow{I_t} \eta^{\dagger} \\ \eta^{\dagger} \xrightarrow{I_t} -\xi \end{array}\right\} \tag{2.8.65}$$

and

$$\left.\begin{array}{l} \xi \xrightarrow{I_{st}} i\xi \\ \eta^{\dagger} \xrightarrow{I_{st}} -i\eta^{\dagger} \end{array}\right\} \tag{2.8.66}$$

As a special case, the IR's of the orthochronous Lorentz group $L^{\uparrow}$ (which consists of O(3) plus I_s) can be classified as follows (denote an IR of $L_+^{\uparrow}$ by $\mathcal{D}^{(j_1,j_2)}$):

a) If j_1+j_2 = integer and $j_1 \neq j_2$, then each IR of $L^{\uparrow}$ contains two IR's of $L_+^{\uparrow}$ as follows:

$$\mathcal{D}^{(j_1,j_2)} = D^{(j_1,j_2)} \oplus D^{(j_2,j_1)}$$

$$= \sum_{j=|j_1-j_2|}^{j_1+j_2} (D^{(j+)} \oplus D^{(j-)})$$

The reason for this decomposition is the following: Under space inversion, the generators J_i and K_i transform as

$$I_s L_i I_s = L_i \quad ,$$

$$I_s K_i I_s = -K_i \quad ,$$

which is clear from the operator form of the generators,

so that under space inversion

$$I_s M_i I_s = N_i$$

$$I_s N_i I_s = M_i \quad .$$

Hence the IR of $L^{\uparrow}$ contains both $D^{j_1 j_2}$ and $D^{j_2 j_1}$ of $L^{\uparrow}$. The decomposition into $D^{(j+)}$ and $D^{(j-)}$ follows from our discussion of the IR's of O(3) in section 2.8(c).

b) If j_1+j_2 = half-integer and $j_1 \neq j_2$, the IR's of $L^{\uparrow}$ contain two IR's of $L_+^{\uparrow}$ as follows:

$$\mathcal{D}^{(j_1,j_2)} = D^{(j_1,j_2)} \oplus D^{(j_2,j_1)}$$

$$= 2 \sum_{j=|j_1-j_2|}^{j_1+j_2} (D^{(j)} \quad .$$

Since we have now a genuine spinor representation there is no decomposition of the IR's into $D^{(j+)}$ and $D^{(j-)}$ of the restricted Lorentz group as in the case of the tensor representation (a) above.

2.9 Relation to Dirac Spinors

Remember that the spinors ξ form the bases to the spinor representations of SL(2,C). In particular

$$\xi' = A\ \xi \tag{2.9.1}$$

Now the covariant spinor η is given by

$$\eta^T = C\ \xi \quad . \tag{2.9.2}$$

From (2.9.1),

$$
\begin{aligned}
(C\ \xi)' &= C\ A\ \xi \\
&= C\ A\ C^{-1} C\ \xi \\
&= C\ A\ C^{-1}\ \eta^T
\end{aligned}
\tag{2.9.3}
$$

and, using (2.8.58), this becomes

$$(C\xi)' = A^{T-1}\ \eta^T$$

or , from (2.9.2),

$$(\eta^T)' = A^{T-1}\ \eta^T$$

Hence the contravariant and the covariant spinors are equivalent, i.e. they correspond to equivalent representations:

$$
\begin{aligned}
&\xi \triangleq A \\
&\eta^T \triangleq A^{T-1}\ ,
\end{aligned}
$$

so that both A and A^{T-1} correspond to the representation $D^{0\frac{1}{2}}$ of SL(2,C).

The complex conjugate A^* corresponds to the representation $D^{\frac{1}{2}0}$ of SL(2,C), as we have seen. We have

$$(\xi^*)' = A^*\ \xi^*$$

and

$$
\begin{aligned}
(C\ \xi)^{*\prime} &= C\ A^*\ C^{-1}\ C\ \xi^* \\
&= (A^{T-1})^*\ (\eta^T)^* \\
&= A^{\dagger-1}\ \eta^\dagger
\end{aligned}
$$

or $\quad \eta^{\dagger\prime} = A^{\dagger-1}\ \eta^\dagger$

so that the spinors ξ^*, $\eta^\dagger$ form the bases for the representation $D^{\frac{1}{2}0}$ of SL(2,C), which is given by A^* and $A^{\dagger -1}$.

Consider now the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\, \Psi = 0 \tag{2.9.4}$$

This we require to be Lorentz covariant:

$$(i\gamma^\mu \partial_\mu{}' - m)\, \Psi' = 0 \quad ,$$

where $\Psi' = S\,\Psi$ for some operator S,

and

$$\left(\frac{\partial}{\partial x^\mu}\right)' = \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu} \quad .$$

From (2.9.5) we get

$$S^{-1}\,(i\gamma^\mu \Lambda^\mu{}_\nu \frac{\partial}{\partial x^\nu} - m)\, S\, \Psi = 0$$

$$\Rightarrow \quad (i\, S^{-1}\, \gamma^\mu\, \Lambda^\mu{}_\nu\, S \frac{\partial}{\partial x^\nu} - m)\, \Psi = 0$$

$$\Rightarrow \quad S^{-1}\, \gamma^\mu\, \Lambda^\mu{}_\nu\, S = \gamma^\nu$$

$$\text{or} \quad S^{-1}\, \gamma^\mu\, S = \Lambda^\mu{}_\nu\, \gamma^\nu \tag{2.9.5}$$

Now S can be shown (see for example [2], p199ff) to be of the form

$$S = e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}} \tag{2.9.6}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$

and

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & -\phi_1 & -\phi_2 & -\phi_3 \\ \phi_1 & 0 & -\theta_3 & \theta_2 \\ \phi_2 & \theta_3 & 0 & -\theta_1 \\ \phi_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix} .$$

The inverse of S is

$$S^{-1} = e^{-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}$$

and

$$\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu} = \frac{i}{4}\{ -[\gamma^0,\gamma^1]\phi_1 - [\gamma^0,\gamma^2]\phi_2 - [\gamma^0,\gamma^3]\phi_3$$

$$-[\gamma^1,\gamma^2]\theta_3 + [\gamma^1,\gamma^3]\theta_2 - [\gamma^2,\gamma^3]\theta_1$$

$$= \frac{1}{2}\left\{ \sum_{i=1}^{3} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \phi_i - i \sum_{i=1}^{3} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \theta_i \right\} .$$

Here we have used the chiral representation of the Dirac algebra (see section 1.2(ii)).

Hence we can write S as follows:

$$S = \begin{pmatrix} e^{-\frac{1}{2}i\,\vec{\sigma}\cdot\vec{n}\,\theta + \vec{\sigma}\cdot\vec{\nu}\,\phi} & 0 \\ 0 & e^{-\frac{1}{2}i\,\vec{\sigma}\cdot\vec{n}\,\theta - \frac{1}{2}\vec{\sigma}\cdot\vec{\nu}\,\phi} \end{pmatrix} . \qquad (2.9.7)$$

But from section (2.4) we recall that the general form of A is

$$A = e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{n}\,\theta - \frac{1}{2}\vec{\sigma}\cdot\vec{\nu}\,\phi}$$

so that we have just shown that

$$S = \begin{pmatrix} A & 0 \\ 0 & A^{\dagger -1} \end{pmatrix} . \tag{2.9.8}$$

Consequently the Dirac spinor Ψ has the form

$$\Psi = \begin{pmatrix} \xi \\ \eta^{\dagger} \end{pmatrix} . \tag{2.9.9}$$

Then it transforms under $L_{+}^{\uparrow}$ according to:

$$(\Psi)' = S\,\Psi = \begin{pmatrix} A & 0 \\ 0 & A^{\dagger -1} \end{pmatrix} \begin{pmatrix} \xi \\ \eta^{\dagger} \end{pmatrix}$$

$$= \begin{pmatrix} A\xi \\ (\eta A^{-1})^{\dagger} \end{pmatrix}$$

as required.

We notice that the form (2.9.9) of the Dirac spinor reflects the fact that the Dirac equation describes massive spin ½ particles: The appropriate representation of $L^{\uparrow}$ in this case is

$$\mathcal{D}^{\frac{1}{2}0} = D^{\frac{1}{2}0} \oplus D^{0\frac{1}{2}} \tag{2.9.10}$$

and the Dirac spinor is actually the direct sum of the bases of $D^{\frac{1}{2}0}$ and $D^{0\frac{1}{2}}$:

$$\Psi = \begin{pmatrix} \xi \\ \eta^\dagger \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \xi^2 \\ (\eta_1)^\dagger \\ (\eta_2)^\dagger \end{pmatrix}$$

$$= \begin{pmatrix} \xi^1 \\ \xi^2 \\ 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ \eta_1^\dagger \\ \eta_2^\dagger \end{pmatrix} \quad . \tag{2.9.11}$$

We have here worked in the <u>chiral representation</u> because we can then write the Dirac spinor as follows:

$$\Psi(p) = \begin{pmatrix} \phi_R(p) \\ \phi_L(p) \end{pmatrix} \tag{2.9.12}$$

and the Dirac equation

$$(\gamma_\mu p^\mu - m)\, \Psi(p) = 0$$

becomes

$$\begin{pmatrix} -m & p_0 - \vec{\sigma}\cdot\vec{p} \\ p_0 + \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \begin{pmatrix} \phi_R(p) \\ \phi_L(p) \end{pmatrix} = 0 \tag{2.9.13}$$

For massless states we get the Weyl equations

$$(p_0 - \vec{\sigma}\cdot\vec{p})\, \phi_R(p) = 0$$

$$(p_0 + \vec{\sigma}\cdot\vec{p})\, \phi_L(p) = 0 \tag{2.9.14}$$

or, since $p_0 = |\vec{p}|$,

$$\vec{\sigma}\cdot\hat{p}\ \phi_R = \phi_R$$

$$\vec{\sigma}\cdot\hat{p}\ \phi_L = -\phi_L \quad . \tag{2.9.15}$$

Here ϕ_R and ϕ_L represent the right-handed and left-handed massless particles respectively; this means that in this form of the Dirac equation no mixing of rigt-handed and left-handed components occurs.

We see from (2.9.15) that the Weyl spinors are eigenstates of the helicity $\vec{\sigma}.\hat{p}$.

In the standard representation, we have

$$\gamma^0_{SR} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = S\ \gamma^0_{SR}\ S^{-1}$$

and we find that $S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Then the Dirac spinor in the standard representation, Ψ_{SR}, can be recovered from the spinor Ψ in the chiral representation:

$$\Psi_{SR} = S\ \Psi = S \begin{pmatrix} \phi_R(p) \\ \phi_L(p) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_R + \phi_L \\ \phi_R - \phi_L \end{pmatrix} \quad . \tag{2.9.16}$$

2.10 Representations of the Poincaré Group

A five-dimensional matrix representation of the Poincaré group has been given in section (2.6) where the Poincaré transformation (a,Λ) has been given as a five-dimensional matrix

$$(a,\Lambda) = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix} \quad .$$

The generators are then written in the form

$$K_{i_{(5)}} \longrightarrow \begin{bmatrix} K_{i_{(4)}} & 0 \\ 0 & 0 \end{bmatrix}$$

$$L_{i_{(5)}} \longrightarrow \begin{bmatrix} L_{i_{(4)}} & 0 \\ 0 & 0 \end{bmatrix}$$

where $K_{i_{(5)}}$, $L_{i_{(5)}}$ is the usual form of the generators in 4 dimensions. This representation is not unitary since the generators P_μ and K_μ are not Hermitean. In fact, since the group P is non-compact, there do not exist finite-dimensional unitary representations. For physical applications, we are interested in the (infinite-dimesional) unitary representations whose bases are the state vectors in Hilbert space. These representations then represent a symmetry operation on a physical state Ψ, $\Psi \longrightarrow \Psi'$, in the form of a unitary or antiunitary operator acting on the state vector Ψ:

$$| \Psi' > = U | \Psi > \tag{2.10.1}$$

We denote a Poincaré transformation (a,Λ) by an operator $U(a,\Lambda)$.

The group multiplication is written as

$$U(a_1,\Lambda_1)\, U(a_2,\Lambda_2) = \pm\, U(a_3,\Lambda_3) \quad , \tag{2.10.2}$$

where the sign ambiguity follows from the fact that $P_+^\uparrow$ is doubly connected.

The representations of $P_+^\uparrow$ are <u>unitary</u> since the transformations are connected to the identity, i.e. they can be obtained from the identity by continuous deformation.

The Casimir operators of the group P are

$$P^2 = P_\mu P^\mu$$

and

$$W^2 = W_\mu W^\mu \quad .$$

To obtain the unitary irreducible representations of $P_+^\uparrow$, we look for the subspaces of the Hilbert space which are invariant under transfromations of $P_+^\uparrow$.

Since P^2 is an invariant under the transformations of the restricted Poincaré group, the subspaces $| p,\sigma >$ (with the quantum numbers σ so far unspecified) corresponding to eigenvalues p^μ of P^μ with a fixed value of p^2 are invariant:

Consider a state $| p >$ with

$$P^\mu | p > = p^\mu | p > \quad . \tag{2.10.3}$$

Now we operate on $| p>$ with a Lorentz tansformation:

$$U(a,\Lambda) \; | p > = | \Lambda p > \quad . \tag{2.10.4}$$

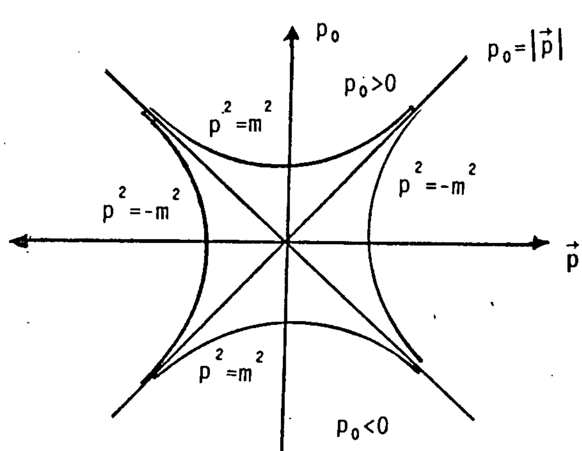
Now $P^\mu | \Lambda p > = (\Lambda p^\mu) | p >$,

but

$$(\Lambda p)^2 = (\Lambda p)^\mu (\Lambda p)_\mu = p^\mu p_\mu = p^2 \quad .$$

remains invariant. This means that in fig.1, the point representing a certain value of p moves on the same branch of the hyperboloid under transformations of $P_+^\uparrow$.

Fig 1: Hyperboloid of $p^2 = m^2$ in four-momentum space.



Furthermore, since

$$[W_\mu, W^\mu] = i\varepsilon_{\mu\nu\sigma\tau} W^\sigma P^\tau$$

and

$$[P_\mu, W_\nu] = 0,$$

we can determine the eigenvalue of W^2 and one of its components, say W_3, simultaneously.

Recall from (2.6.4) the form of W_μ :

$$W_\mu = (\vec{J}.\vec{P},\ \vec{J}P_0 + \vec{K}\times\vec{P})$$

In the rest frame of a particle, this becomes:

$$W_\mu = m(0, J_1, J_2, J_3)$$

i.e. $W^2 = -m^2\ \vec{J}^2$.

We see that this corresponds to the spin of the particle. In the rest frame, we get the eigenvalues

$$W_3 \mid p=(m,0), j_3 > = m\ j_3 \mid p=(m,0), j_3 >$$

$$W^2 \mid p=(m,0), j_3 > = -m^2\ j(j+1) \mid p=(m,0), j_3 > \quad . \qquad (2.10.5)$$

In summary: the Casimir operators J^2, W^2 of the group are associated with the invariants m^2 (mass^2) and spin. The z-component of the covariant spin, W_3 , however, is dependent on the frame of reference.

The following cases occur:

1) $p^2 = m^2 > 0$:

Because $\Lambda^0{}_0 \geq 0$ for $P_+^\uparrow$, the sign of the energy,

$$\frac{p_0}{|p_0|} ,$$

is also invariant under the transformations of $P_+^\uparrow$. Therefore to each value of p there correspond two IR's, one for a positive and one for a negative value of $\frac{p_0}{|p_0|}$.

The subgroup of $P_+^\uparrow$ that leaves a particular $\{p^\mu\}$ (for which $p^2 = m^2 > 0$) invariant (the little group of $\{p^\mu\}$) obviously has the same structure for all momenta in $\{p^\mu\}$. We choose a particular p^μ, the particle rest-frame:

$$k^\mu = (m,0,0,0) . \tag{2.10.6}$$

It is clear that the little group in this case is the rotation group SO(3) or SU(2), since this has no effect on k^μ.

Now consider an arbitrary timelike state p^μ. We can obtain this from the rest frame:

$$p^\mu = L^\mu{}_\nu(p)\ k^\nu$$

where $L^\mu{}_\nu$ denotes a Lorentz transformation,

or

$$|\ p,\sigma > = U(L(p))\ |\ k,\sigma > . \tag{2.10.7}$$

An arbitrary Lorentz transformation transforms this state into

$$U(\Lambda)\ |\ p,\sigma > = U(\Lambda)\ U(L(p))\ |\ k,\sigma >$$

$$= U(L(\Lambda p))\ U^{-1}(L(\Lambda p))\ U(\Lambda)\ U(L(p))\ |\ k,\sigma >$$

$$= U(L(p))\ U(L^{-1}(\Lambda p)\ \Lambda\ L(p))\ |\ k,\sigma > \qquad (2.10.8)$$

Now

$$L(p)\ |\ k,\sigma > = |\ p,\sigma >$$

$$\Lambda\ L(p)\ |\ k,\sigma > = |\ \Lambda\ p,\sigma >$$

$$L^{-1}(\Lambda p)\ \Lambda\ L(p)\ |\ k,\sigma > = |\ k,\sigma >$$

so that $L^{-1}(\Lambda p)\ \Lambda\ L(p)$ is a rotation.

Hence we can write the representation of $U(\Lambda)$ as follows:

$$U(\Lambda)\ |\ p,\sigma > = U(L(\Lambda p)) \sum_{\sigma'} D^{j}_{\sigma\sigma'}(R)\ |\ k,\sigma' >$$

$$= \sum_{\sigma'} D^{j}_{\sigma\sigma'}(R)\ |\ \Lambda p,\sigma' > .$$

<u>2) $p^2 = 0,\ p^\mu \neq 0$</u>:

In this case, because $m^2 = 0$, we have $W^2 = 0$ as well. Since $P_\mu P^\mu = W_\mu W^\mu = 0$, i.e.

$$W_\mu W^\mu = \tfrac{1}{2}\ \varepsilon_{\mu\nu\sigma\tau}\ M^{\nu\sigma}\ P^\tau\ P^\mu$$

$$= 0\ ,$$

we conclude that W^μ , P^μ are parallel:

$$W^\mu = \lambda\ P^\mu\ ,$$

where λ can be calculated as:

$$\lambda = \frac{W_0}{P_0} = \frac{\vec{P}.\vec{J}}{|\vec{p}|} ,$$

i.e. λ is the projection of the spin onto the direction of motion. It is called the helicity.
The two IR's can be characterised by the eigenvalues corresponding to p and λ: the basis vectors are labelled $| p,\lambda >$. For a fixed value of p, there are in general two values of λ: $\pm\lambda$, so that for each of the two values $\pm\lambda$ there exists an independent IR.

<u>3) $p^\mu = 0$:</u>

This case is identical to the homogeneous Lorentz group which has already been dicussed.

<u>4) $p^2 < 0$:</u>

This is an unphysical situation since p^μ is spacelike (imaginary). However, it could correspond to virtual particles which often have spacelike momenta.

Generally, we can write the IR's of $P_+^\uparrow$ as:

$$U(a,\Lambda) = e^{-i\, a^\mu P_\mu}\, e^{-\frac{i}{2}\, \omega_{\mu\nu}\, M^{\mu\nu}}$$

because in <u>Hilbert space</u>, the infinitesimal generators P^μ , $M_{\mu\nu}$ are represented by Hermitean operators.

This method of obtaining the irreducible representations of the Poincaré group is called the Wigner method of induced representations. This is because to arrive at the representation, one selects a fixed value of the momentum, finds a subgroup (the little group) which leaves this momentum intact and obtains a representation of this subgroup. This representation is then boosted to the required momentum. The prodedure is valid for any value of the starting momentum- this makes sense since surely the properties of a particle should not depend on the frame of reference in which we make our observations.

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3. SUPERSYMMETRY

3.1 The Supersymmetry Algebra

With the growing interest among physicists in the internal symmetries such as SU(3) etc., in the 1960's, much effort was devoted to the attempt to find a symmetry which would combine space-time symmetries (the Poincaré Group) with internal symmetries in a non-trivial way. Initial attempts all concerned symmetries (of the S-matrix, other symmetries do exist) in the framework of Lie Algebras. The most important result here is the famous no-go-theorem by Coleman and Mandula[1]:

It states that any Lie group which contains the Poincaré group P

$$[P_\mu, P_\nu] = 0 \tag{3.1.1)(a}$$

$$[P_\mu, M_{\rho\sigma}] = -i\,(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho) \tag{3.1.1)(b}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\,(g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma}) \tag{3.1.1)(c}$$

and an internal symmetry group G

$$[B_r, B_s] = i\, c_{rs}{}^t\, B_t \quad , \tag{3.1.2}$$

must be a direct product of P and G, i.e.

$$[B_r, P_\mu] = [B_r, M_{\mu\nu}] = 0 \; . \tag{3.1.3}$$

Additionally, the theorem states that the internal symmetry group must be a direct product of a compact semisimple group and U(1) groups.

In effect, this means that any group that contains the Poincaré group and internal symmetry groups in a non-trivial fashion would lead to trivial physics - it is like solving a system of three equations in two unknowns.

One important consequence of (3.1.3) is O'Raifeartaigh's theorem[2]: Since the generators of the Poincaré group commute with the generators of the internal symmetry group, so do the Casimir operators of the Poincaré group:

$$[P^2, B_r] = [W^2, B_r] = 0$$

or, in other words, the multiplets of the internal symmetry group must have the same mass and the same spin (or helicity if they are massless).

We see thus that, to obtain a symmetry that combines space-time and internal symmetries non-trivially, one has to generalise the concept of a Lie group. This was achieved by Haag, Lopuszanski and Sohnius[3] by allowing in addition to bosonic (commuting) generators, fermionic (anticommuting) generators as well. The Lie Algebra then has a graded structure: If B_r denotes bosonic (even) and F_s fermionic (odd) generators, we get:

$$
\begin{aligned}
&[B_i,B_j] = i\, c_{ij}{}^{k} B_k = -i\, c_{ji}{}^{k} B_k \\
&[F_\alpha,B_i] = s_{\alpha i}{}^{\beta} F_\beta \\
&\{F_\alpha,F_\beta\} = \gamma_{\alpha\beta}{}^{i} B_i = \gamma_{\beta\alpha}{}^{i} B_i \quad .
\end{aligned}
\tag{3.1.4}
$$

The corresponding Jacobi identities are:

$$[[B_i,B_j],B_k] + [[B_k,B_i],B_j] + [[B_j,B_k],B_i] = 0 \qquad (3.1.5)(a)$$

$$[[F_\alpha,B_i],B_j] + [[B_j,F_\alpha],B_i] + [[B_i,B_j],F_\alpha] = 0 \qquad (3.1.5)(b)$$

$$[\{F_\alpha,F_\beta\},B_i] + \{[B_i,F_\alpha],F_\beta\} - \{[F_\beta,B_i],F_\alpha\} = 0 \qquad (3.1.5)(c)$$

$$[\{F_\alpha,F_\beta\},F_\gamma] + [\{F_\gamma,F_\alpha\},F_\beta] + [\{F_\beta,F_\gamma\},F_\alpha] = 0 \qquad (3.1.5)(d)$$

The additional minus-sign in (3.1.5)(c) arises from the interchange of fermionic operators.

We call the fermionic generators $Q_{\alpha i}$, where α = 1,2 is the spinor index (in anticipation of the fact that these operators have spinor form) and i = 1,...,N accounts for the possibility that there are up to N different generators of supersymmetry.

Let us look at the identity (3.1.5)(b):

$$[[Q_{\alpha i},M_{\mu\nu}],\, M_{\rho\sigma}] + [[M_{\rho\sigma},Q_{\alpha i}],\, M_{\mu\nu}] + [[M_{\mu\nu},M_{\rho\sigma}],\, Q_{\alpha i}] = 0 \tag{3.1.6}$$

and so

$$[(s_{\mu\nu})_\alpha{}^\beta Q_{\beta i}, M_{\rho\sigma}] - [(s_{\rho\sigma})_\alpha{}^\beta Q_{\beta i}, M_{\mu\nu}]$$
$$- i[g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}, Q_{\alpha i}] = 0$$

Hence

$$[(s_{\mu\nu}),(s_{\rho\sigma})]_\alpha{}^\gamma = i(g_{\mu\sigma}(s_{\nu\rho})_\alpha{}^\gamma + g_{\nu\rho}(s_{\mu\sigma})_\alpha{}^\gamma - g_{\mu\rho}(s_{\nu\sigma})_\alpha{}^\gamma - g_{\nu\sigma}(s_{\mu\rho})_\alpha{}^\gamma) \tag{3.1.7}$$

This has the same form as eq. (3.1.1)(c), from which we conclude that the $(s_{ij})_\alpha{}^\beta$ form a representation of the Lorentz group, or, put differently, the $Q_{\alpha i}$ carry a representation of the Lorentz group through the Jacobi identity (3.1.7). We select $Q_{\alpha i}$ and $\bar{Q}^i{}_{\dot\alpha}$ to be in the lowest representations of the Lorentz group, namely in the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations respectively:

$$[Q_{\alpha i}, M_{\mu\nu}] = \tfrac{1}{2}\,(\sigma_{\mu\nu})_\alpha{}^\beta Q_{\beta i}$$

$$[\bar{Q}^i{}_{\dot\alpha}, M_{\mu\nu}] = -\tfrac{1}{2}\,\bar{Q}^i{}_{\dot\beta}(\bar\sigma_{\mu\nu})^{\dot\beta}{}_{\dot\alpha} \tag{3.1.8}$$

(see the Appendix for the definitions of $\sigma_{\mu\nu}$ and $\bar\sigma_{\mu\nu}$.).

We now recognise the spinor structure of the $Q_{\alpha i}$, $\bar{Q}^i{}_{\dot\beta}$ - they carry the spinor representation of the Lorentz group.

We have chosen $\bar{Q}^i{}_{\dot\alpha}$ such that

$$\bar{Q}^i{}_{\dot\alpha} = (Q_{\alpha i})^\dagger \quad . \tag{3.1.9}$$

This we can do since $(Q_{\alpha i})^\dagger$ and $\bar{Q}^i{}_{\dot\alpha}$ are both in the $(0,\frac{1}{2})$ representation, so we can always redefine $\bar{Q}^i{}_{\dot\alpha}$ so as to fulfill (3.1.9).

The anticommutator $\{Q,\bar{Q}\}$ must be in the $(\frac{1}{2},\frac{1}{2})$ representation of the Lorentz group if the $Q_{\alpha i}$, $\bar{Q}^i{}_{\dot\alpha}$ are as defined above. The only object in the bosonic sector which is in the $(\frac{1}{2},\frac{1}{2})$ representation, is P_μ , so that $\{Q,\bar{Q}\}$ must be proportional to P_μ:

$$\{Q_{\alpha i},\bar{Q}^j{}_{\dot\beta}\} = 2\ \delta_i{}^j(\sigma^\mu)_{\alpha\dot\beta}P_\mu \quad . \tag{3.1.10}$$

This can be seen as follows:

We consider eq. (3.1.5)(c):

$$[\ \{Q_\alpha,\bar{Q}_{\dot\beta}\},M_{\mu\nu}] + \{\ [M_{\mu\nu},Q_\alpha],\bar{Q}_{\dot\beta}] - \{\ [\bar{Q}_{\dot\beta},M_{\mu\nu}],Q_\alpha\} = 0$$

This yields, using (3.1.10) and the Poincaré algebra,

$$(\sigma^\rho)_{\alpha\dot\beta}(-ig_{\rho\mu}P_\nu+ig_{\rho\nu}P_\mu) - \tfrac{1}{2}\ (\sigma_{\mu\nu})_\alpha{}^\gamma(\sigma^\rho)_{\gamma\dot\beta}P_\rho + \tfrac{1}{2}\ (\sigma^\rho)_{\alpha\dot\gamma}(\bar\sigma_{\mu\nu})^{\dot\gamma}{}_{\dot\beta}P_\rho = 0$$

where $(\sigma^\mu)_{\alpha\dot\beta}$ are unknown structure constants which we want to determine. We get

$$-\tfrac{1}{2}\ \sigma^{\mu\nu}\sigma^\rho + \tfrac{1}{2}\ \sigma^\rho\bar\sigma^{\mu\nu} - i\ (g^{\rho\mu}\sigma^\nu - g^{\rho\nu}\sigma^\mu) = 0 \quad , \tag{3.1.11}$$

of which the only solution for σ^μ is

$$\sigma^\mu = (1,\vec\sigma) = \bar\sigma_\mu \tag{3.1.12}$$

with the Pauli matrices as defined in Appendix A. This explains the form of the structure constants $(\sigma^\mu)_{\alpha\dot\beta}$ in (3.1.10).

Since the left-hand side of eq. (3.1.10) is Hermitean, and σ_μ is also Hermitean, it follows that any factor matrix multiplying the structure constants is also Hermitean. Therefore the $Q_{\alpha i}$, $\bar{Q}^j{}_{\dot\beta}$ can always be redefined to absorb such a factor, so that we remain with the factor $\delta_i{}^j$. The factor 2 is a convenient normalisation constant. This concludes the proof of the commutation relation (3.1.10).

We now investigate the commutator $[Q_{\alpha i},P_\mu]$. Since $Q_{\alpha i}$ is in the $(\frac{1}{2},0)$ and P_μ in the $(\frac{1}{2},\frac{1}{2})$ representation, the commutator can be a combination of the $(0,\frac{1}{2})$ and $(3/2,\frac{1}{2})$ representations - however, since there is no generator in the latter representation, the most general form is:

$$[Q_{\alpha i},P_\mu] = c_{ij}(\sigma_\mu)_{\alpha\dot{\beta}}\bar{Q}^{\dot{\beta}j} \tag{3.1.13}$$

and

$$[\bar{Q}^{\dot{\alpha}i},P_\mu] = (c_{ij})^*(\sigma_\mu)^{\dot{\alpha}\beta}Q_{\beta j} \quad ,$$

so that the identity (3.1.5)(b) with $Q_{\alpha i}$, P_μ, P_ν becomes:

$$(c_{ij})(c_{jk})^*(\sigma_\mu\bar{\sigma}_\nu)_\alpha{}^\gamma - (c_{ij})(c_{jk})^*(\sigma_\nu\bar{\sigma}_\mu)_\alpha{}^\gamma = 0$$

or

$$(c_{ij})(c_{jk})^*(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)_\alpha{}^\gamma = 0 \quad .$$

Now the second term in this equation does not equal zero - hence we must have $c_{ij} = 0$ and consequently:

$$[Q_{\alpha i},P_\mu] = [\bar{Q}_{\dot{\alpha}}{}^i,P_\mu] = 0 \quad . \tag{3.1.14}$$

Finally, the supersymmetry generators can carry some representation of the (bosonic) internal symmetry group:

$$[Q_{\alpha i},B_r] = (b_r)_i{}^jQ_{\alpha j} \tag{3.1.15}$$

with $(b_r) = (b_r)^\dagger$ since the internal symmetry group is compact, and so

$$[\bar{Q}^i{}_{\dot{\alpha}},B_r] = -\bar{Q}^j{}_{\dot{\alpha}}(b_r)_j{}^i \quad . \tag{3.1.16}$$

Here the commutator only acts on the indices $i = 1,\ldots,N$ of $Q_{\alpha i}$. This reflects the fact that the space-time part of the symmetry commutes with any internal symmetry parts, as required by the Coleman-Mandula theorem. If $N = 1$, we call $B_r = R$ and so

$$[Q,R] = Q$$
$$[\bar{Q},R] = -\bar{Q} \qquad . \tag{3.1.17}$$

Now we consider the anticommutator $\{Q,Q\}$. The space-time part of this must be a sum of (0,0) (trivial) and (1,0) representations of the Lorentz group. However, $\{Q,Q\}$ commutes with P (the $(\frac{1}{2},\frac{1}{2})$ representation) and can thus not contain any term of the (1,0) representation because this would not commute with a $(\frac{1}{2},\frac{1}{2})$ term. We are thus left with the internal symmetry part of $\{Q,Q\}$:

$$\{Q_{\alpha i},Q_{\beta j}\} = 2\ \varepsilon_{\alpha\beta}Z_{ij} \qquad . \tag{3.1.18}$$

The $\varepsilon_{\alpha\beta}$ accounts for the interchange of the fermionic generators, while the Z_{ij} is a sum of the internal symmetry generators:

$$Z_{ij} = a_{ij}{}^{r}B_r \qquad . \tag{3.1.19}$$

The Z_{ij} span an invariant subalgebra of the internal symmetry algebra:

$$\begin{aligned}
[Z_{ij},B_r] &= \tfrac{1}{2}\ \varepsilon_{\alpha\beta}[\{Q_{\alpha i},Q_{\beta j}\},\ B_r] \\
&= \tfrac{1}{2}\ \varepsilon_{\alpha\beta}(\ \{Q_{\alpha i},[Q_{\beta j},B_r]\} + \{Q_{\beta j},[Q_{\alpha i},B_r]\}\ ) \\
&= \varepsilon_{\alpha\beta}(\ (b_r)_j{}^k\ \varepsilon_{\alpha\beta}Z_{ik} + \varepsilon_{\beta\alpha}\ (b_r)^k{}_i Z_{jk}\ ) \\
&= (b_r)_j{}^k Z_{ik} + (b_r)_i{}^k Z_{kj}
\end{aligned} \tag{3.1.20}$$

(since (3.1.18) implies that $Z_{ij} = -Z_{ji}$)

and

$$[Z_{ij}\ ,\ Z_{kl}] = a_{kl}{}^{r}(b_r)_i{}^{k}Z_{kj} + a_{kl}{}^{r}(b_r)_j{}^{k}Z_{ik} \quad . \tag{3.1.21}$$

Now, because $Z_{ij} = a_{ij}{}^{r}B_r$,

$$[\bar{Q}^{k\cdot}{}_{\alpha}\ ,\ Z_{ij}] = a_{ij}{}^{r}\ [\bar{Q}^{k\cdot}{}_{\alpha}\ ,\ B_r] = -a_{ij}{}^{r}\bar{Q}^{l\cdot}{}_{\alpha}\ (b_r)_l{}^{k}$$

$$= \tfrac{1}{2}\ \varepsilon_{\alpha\beta}[\bar{Q}^{k\cdot}{}_{\alpha}\ ,\ \{Q_{\alpha i},Q_{\beta j}\}\ ]$$

$$= \tfrac{1}{2}\ \varepsilon_{\alpha\beta}(\ [\{\bar{Q}^{k\cdot}{}_{\alpha},Q_{\alpha i}\}\ ,\ Q_{\beta j}] + [\{\bar{Q}^{k\cdot}{}_{\alpha},Q_{\beta j}\}\ ,\ Q_{\alpha i}\ ]\ )$$

$$= 0 \qquad\qquad \text{since } [P_{\mu},Q_{\alpha}] = 0\ .$$

and so

$$a_{ij}{}^{r}(b_r)_l{}^{k} = 0 \quad . \tag{3.1.22}$$

Hence from (3.1.21),

$$[Z_{ij},Z_{kl}] = 0$$

and $[Q,Z_{ij}] = 0$ (since $(b_r)_j{}^{i}$ is Hermitean) .

Hence the invariant algebra is Abelian. This means it contains all the generators Z_{ij} , since the Coleman-Mandula theorem restricts the internal group to be a direct product of a semisimple group and Abelian factors. We conclude that the Z_{ij} , which are referred to as central charges, commute with all other generators.

From eq. (3.1.18), we see that, because $\varepsilon_{\alpha\beta}$ is antisymmetric and the anticommutator is symmetric, that $Z_{ij} = -Z_{ji}$ and thus $a_{ij}{}^{r} = -a_{ji}{}^{r}$. If N = 1, central charges are thus excluded.

We have written the algebra in two-component notation. However, it is sometimes convenient to write it in the ordinary four-component notation, which we will add for completeness:

We can define the supersymmetry generators to be Majorana spinors[4],[5]:

$$Q_i = \begin{pmatrix} Q_{\alpha i} \\ \bar{Q}^{\dot{\alpha} i} \end{pmatrix} \quad . \tag{3.1.23}$$

This is possible because we can always redefine the $Q_{\alpha i}$ to fulfill the above relation. The conjugate spinor becomes

$$\bar{Q}_i = (Q^{\alpha}{}_i, \bar{Q}^i{}_{\dot{\alpha}}) \quad . \tag{3.1.24}$$

With

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}$$

$$\sigma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix} \quad ,$$

where $\{\gamma^{\mu},\gamma^{\nu}\} = 2\eta^{\mu\nu}\, \mathbb{1}$

and

$$\sigma^{\mu\nu} = \frac{i}{2}\, [\gamma^{\mu},\gamma^{\nu}]$$

and $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$,

we get the four-component form of the superalgebra

$$\{Q_i,\bar{Q}_j\} = 2\,(\delta_{ij}\gamma^{\mu}P_{\mu} + i\,\mathrm{Im}\, Z_{ij} + i\gamma_5 \mathrm{Re}\, Z_{ij}) \quad . \tag{3.1.25}$$

This is easily seen because

$$\{Q_i,\bar{Q}_j\} = \begin{pmatrix} \{Q_{\alpha i},Q^{\beta}{}_j\} & \{Q_{\alpha i},\bar{Q}^j{}_{\dot{\beta}}\} \\ \{\bar{Q}^{\dot{\alpha} i},Q^{\beta}{}_j\} & \{\bar{Q}^{\dot{\alpha} i},\bar{Q}^j{}_{\dot{\beta}}\} \end{pmatrix}$$

$$= \begin{pmatrix} 2\ Z_{ij} & 2\ (\sigma^{\mu})_{\alpha\dot{\beta}} P_{\mu}\ \delta_i{}^j \\ 2\ (\bar{\sigma}^{\mu})_{\alpha\dot{\beta}}\ P_{\mu}\ \delta_i{}^j & -2\ (Z_{ij})^* \end{pmatrix} .$$

We can thus summarise the algebra as follows:

Table 3.1: Summary of the Supersymmetry Algebra

2-Component notation	4-Component notation
a) $[P_{\mu}, P_{\nu}] = 0$	
b) $[P_{\mu},M_{\rho\sigma}] = -i\ (g_{\mu\rho}P_{\sigma} - g_{\mu\sigma}P_{\rho})$	
c) $[M_{\mu\nu},M_{\rho\sigma}] = -i\ (g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma})$	
d) $[B_r,B_s] = ic_{rs}{}^tB_t$	
e) $[B_r,P_{\mu}] = 0$	
f) $[B_r,M_{\mu\nu}] = 0$	
g) $[Z_{ij}$, all generators$] = 0$	

2-Component notation	4-Component notation
h) $\{Q_{\alpha i},\bar{Q}^{j}{}_{\dot\beta}\} = 2\delta_i{}^j(\sigma^\mu)_{\alpha\dot\beta}P_\mu$ i) $\{Q_{\alpha i},Q_{\beta j}\} = 2\varepsilon_{\alpha\beta}Z_{ij}$ $= 2\ \varepsilon_{\alpha\beta}a_{ij}{}^rB_r$ j) $\{\bar{Q}^{i}{}_{\dot\alpha},\bar{Q}^{j}{}_{\dot\beta}\} = -2\ \varepsilon_{\dot\alpha\dot\beta}Z^{ij}$ $= -2\ \varepsilon_{\dot\alpha\dot\beta}(Z_{ij})^*$	r) $\{Q_i,\bar{Q}_j\} = 2\ (\delta_{ij}\gamma^\mu P_\mu +$ $+ i\ \mathrm{Im}\ Z_{ij}$ $+ i\gamma_5\mathrm{Re}\ Z_{ij}\ )$
k) $[Q_{\alpha i},P_\mu] = [\bar{Q}^{i}{}_{\dot\alpha},P_\mu] = 0$ l) $[Q_{\alpha i},M_{\mu\nu}] = \frac{1}{2}\ (\sigma_{\mu\nu})_\alpha{}^\beta Q_{\beta i}$ m) $[\bar{Q}^{i}{}_{\dot\alpha},M_{\mu\nu}] = -\ \frac{1}{2}\ \bar{Q}^{i}{}_{\dot\beta}(\bar\sigma_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}$	s) $[Q_i,P_\mu] = 0$ t) $[Q_i,M_{\mu\nu}] = \frac{1}{2}\ \sigma_{\mu\nu}Q_i$
n) $[Q_{\alpha i},B_r] = (b_r)_i{}^jQ_{\alpha j}$ o) $[\bar{Q}^{i}{}_{\dot\alpha},B_r] = -\bar{Q}^{j}{}_{\dot\alpha}(b_r)_j{}^i$	
p) $[Q,R] = Q$ q) $[\bar{Q},R] = -\bar{Q}$	u) $[Q,R] = i\gamma_5Q$

(3.1.26)

3.2 Consequences of the Supersymmetry Algebra

1) Eq. (3.1.26)(k) shows that the supersymmetry generator commutes with the Hamiltonian, P_0. This means that states of non-zero energy are paired by the action of Q.

2) Eq. (3.1.26)(h) tells us that

$$H = P_0 = 1/4\ (\bar{Q}_1 Q_1 + \bar{Q}_1 Q_1 + \bar{Q}_2 Q_2 + \bar{Q}_2 Q_2) \tag{3.2.1}$$

because $(\sigma^0)_{\alpha\dot{\beta}}(\sigma_\nu)^{\alpha\dot{\beta}} = 2g^0{}_\nu = 2\delta^0{}_\nu$.

This means that $H \geq 0$, so that the vacuum energy is well defined. If supersymmetry is not broken, we have for the vacuum state

$$Q_\alpha|0> = 0$$

and thus

$$<0|\ \{Q_\alpha,\bar{Q}_{\dot{\beta}}\}\ |0> = 0 = <0|H|0> \quad , \tag{3.2.2}$$

and hence the vacuum energy is always zero!

3) Since the supersymmetry generators sit in the spin-½ representation of the Lorentz algebra (i.e. they are fermionic), their action on any state of spin j will produce states of spin j+½, j-½ - so the supersymmetry generator mixes particles with different spin - it produces a symmetry between fermions and bosons, particles that obey different statistics. We thus see that the introduction to the algebra of anticommutators lifts some of the restrictions imposed by the Coleman-Mandula theorem.

4) There are an equal number of bosonic and fermionic degrees of freedom in any representation of the supersymmetry algebra:
Let us divide the supersymmetry representation into a bosonic (B) and a fermionic (F) set. If we now operate on B with Q_α, we map B onto a

subset F' of F. Subsequent operation of $\bar{Q}_\alpha$ on F' will map F' onto some subset B' of B. If the operator P_μ is one-to-one and onto (which is usually the case), then, becaue of the algebra, so is $\{Q_\alpha,\bar{Q}_{\dot\beta}\}$, so that after two applications of the supersymmetry operation, we must be back at the original set B. This means that B and F must have the same dimensions - or, there are as many fermionic degrees of freedom as there are bosonic. Note that this applies to representations in which P_μ is a one-to-one, onto operator only.

Another way to realise the "fermions = bosons" rule is to observe that eq. (3.1.26)(k) implies that $[Q,H] = [Q,P_0] = 0$ and therefore states of nonzero energy are paired by the action of Q. Since Q is fermionic and carries spin ½, this results in the pairs containing an equal number of fermionic and bosonic states.

5) The O'Raifeartaigh theorem still holds:

$$[Q,P^2] = 0 \quad ,$$

since $[Q,P] = 0$.

Supermultiplets are thus degenerate in mass.

6) In the rest frame, $W^2 = -m^2L^2$ and thus W^2 contains the Lorentz generators $M_{\mu\nu}$ with which the supersymmetry generators do not commute, thus

$$[Q,W^2] \neq 0 \ . \qquad 3.2.3)$$

Therefore, the supersymmetry multiplets contain differnt spins but are always degenerate in mass. Since we do not observe anything like this in nature, we conclude that supersymmetry must be at least broken badly.

7) From (3.2.2), we know that the vacuum energy of a supersymmetric theory is well defined and equal to zero identically. This means that the supersymmetric vacuum state is always at the absolute minimum of the potential. Consider the following cases:[9]

Fig. 2.1: Potential $V(\phi)$ with unbroken supersymmetry and unbroken gauge symmetry

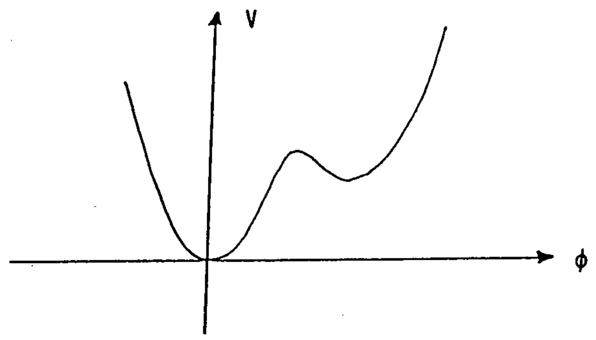


Fig. 2.2: Potential $V(\phi)$ with unbroken supersymmetry and broken gauge symmetry

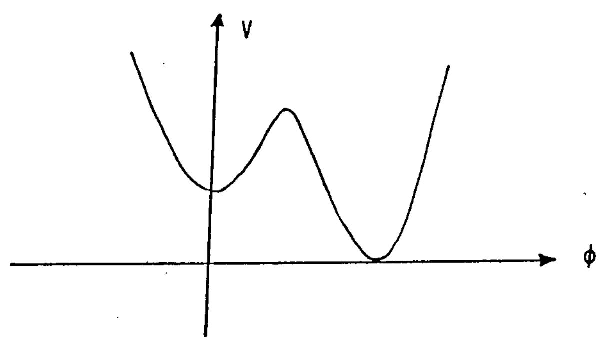


Fig. 2.3: Potential $V(\phi)$ with broken supersymmetry and unbroken gauge symmetry

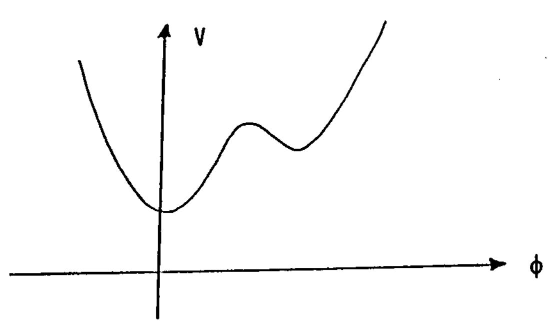
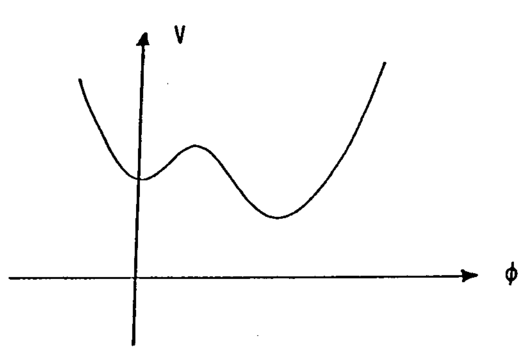


Fig. 2.4: Potential $V(\phi)$ with broken supersymmetry and broken gauge symmetry



In figures (2.1.) and (2.2), supersymmetry is unbroken and the supersymmetric vacuum state is at the minimum of the potential, i.e. $E_{vac} = 0$. If spontaneous breakdown of supersymmetry occurs, the potentials change to figures (2.3) and (2.4) respectively. The situation is not analogous to breakdown of gauge symmetries: In figures (2.2) and (2.4), the gauge symmetry will be spontaneously broken. This is the case because in these situations the (gauge) symmetric state is not a ground state.

3.3 Irreducible Representations of the Supersymmetry Algebra

To find the particle content of the supersymmetry algebra, we have to find irreducible representations of it. This can be done by the Wigner method of induced representations- so we need only find representations of the algebra for the states at rest. We consider the massive and massless cases separately.

3.3(a) Massless one-particle states: no central charges [5],[6],[7]

We select the rest-frame as follows:

$$p^{\mu} = (E,0,0,E) \tag{3.3.1}$$

For massless states, the spin and momentum operators are parallel (see chapter 2.10):

$$W_{\mu} = \lambda \, P_{\mu} \, ,$$

λ being the helicity of the particle. The space-time properties of this state are thus determined by the energy E and the helicity λ (these are sufficient to characterise the state since, as we have seen, the Poincaré group has two Casimir operators). The rest-frame algebra becomes:

$$\{Q_{\alpha i}, \bar{Q}^{j}{}_{\dot\beta}\} = 2\,\delta_i{}^j \begin{pmatrix} E+E & 0 \\ 0 & E-E \end{pmatrix}_{\alpha\dot\beta}$$

$$= 4\,\delta_i{}^j\, E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \tag{3.3.2(a)}$$

$$\{Q_{\alpha i},Q_{\beta j}\} = \{\bar{Q}^{i}{}_{\dot{\alpha}},\bar{Q}^{j}{}_{\dot{\beta}}\} = 0. \tag{3.3.2)(b}$$

or

$$\{Q_{1i},\bar{Q}^{j}{}_{\dot{1}}\} = 4\ \delta_i{}^j\ E$$

$$\{Q_{2i},\bar{Q}^{j}{}_{\dot{1}}\} = \{Q_{1i},\bar{Q}^{j}{}_{\dot{2}}\} = \{Q_{2i},\bar{Q}^{j}{}_{\dot{2}}\} = 0 \tag{3.3.3}$$

Since we require a positive metric, i.e.

$$< E,\lambda\ |\{Q,Q^\dagger\}|\ E,\lambda > = (Q|\ E,\lambda >)^2 + (Q^\dagger|\ E,\lambda >)^2 \geq 0,$$

we conclude

$$Q_{2i} = \bar{Q}^{j}{}_{\dot{2}} = 0. \tag{3.3.4}$$

We remain with a Clifford algebra

$$\{Q_{1i},\bar{Q}^{j}{}_{\dot{1}}\} = 4\ \delta_i{}^j\ E$$

with 2N elements: N of Q_{1i} and N of $\bar{Q}^{j}{}_{\dot{1}}$.

The generators which can have non-trivial action on the rest-frame states are $M_{\mu\nu}$, P_μ, B_r, $Q_1{}^i$ and $\bar{Q}^{j}{}_{\dot{1}}$. $M_{\mu\nu}$ is incorporated in the Pauli-Lubanski spin-vector and represents the helicity of the states:

$$W_\mu = \tfrac{1}{2}\ \varepsilon_{\mu\nu\sigma\tau}\ M^{\nu\sigma}\ P^\tau = \lambda\ P_\mu = \lambda\ (E,0,0,E) \tag{3.3.5}$$

In particular

$$W_0 = \tfrac{1}{2}\ (\varepsilon_{0123}M^{12}\ P^3 + \varepsilon_{0213}M^{21}\ P^3)$$

$$= M^{12}\ E = \lambda\ E$$

and hence

$$M^{12} = \lambda = J^3 .$$

Similarly,

$$W_1 = (M^{02} + M^{32})E = 0 \Rightarrow K^2 = -J^1 = 0.$$

$$W_2 = (M^{31} + M^{01})E = 0 \Rightarrow K^1 = -J^2 = 0.$$

$$W_3 = M^{12}E = \lambda\ E \Rightarrow M^{12} = \lambda = J^3 .$$

Action of B_r on the rest-frame states does not change the energy and helicity since $[Q_i, M_{\mu\nu}] = [Q_i, P_\mu] = 0$.
Also, since $[Q_i, P_\mu] = 0$, the energy-momentum content of the rest-fame states does not change upon application of Q_i, but only the helicity $W_0 = \vec{J}.\vec{P}$:

$$W_0\ Q_{\alpha i} \mid E,\lambda > = Q_{\alpha i}\ W_0 \mid E,\lambda > + [W_0,\ Q_{\alpha i}] \mid E,\lambda > \qquad (3.3.6)$$

Now $[W_0,\ Q_{\alpha i}] = [\vec{J}.\vec{P},\ Q_{\alpha i}]$

$$= \vec{P}\ [\vec{J},\ Q_{\alpha i}]$$

$$= E\ [J_3,\ Q_{\alpha i}]$$

$$= E\ [M_{12},\ Q_{\alpha i}]$$

$$= -\tfrac{1}{2}\ E\ (\sigma_{12})_\alpha{}^\beta\ Q_{\beta i}$$

$$= -\tfrac{1}{2}\ E\ (\sigma_3)_1{}^1\ Q_{1i} \qquad \text{since } Q_{2i} = 0. \qquad (3.3.7)$$

Therefore

$$W_0\, Q_{1i} \mid E,\lambda > \; = E\,(\lambda\, 1 - \tfrac{1}{2}\,(\sigma_3)\,)_1{}^1\, Q_{1i}$$

$$= E\,(\lambda - \tfrac{1}{2}\,)_1{}^1\, Q_{1i} \quad . \tag{3.3.8}$$

We see that Q_{1i} lowers the helicity of the state by ½. We have also

$$[W_0, \bar{Q}^i{}_{\dot{1}}] = \tfrac{1}{2}\, E\, (\sigma_3)_{\dot{1}}{}^{\dot{1}}\, \bar{Q}^i{}_{\dot{1}}$$

$$\Rightarrow \quad W_0\, \bar{Q}^i{}_{\dot{1}} \mid E,\lambda > \; = E\,(\lambda + \tfrac{1}{2}\,)_1{}^1\, \bar{Q}^i{}_{\dot{1}}\ , \tag{3.3.9}$$

so that $\bar{Q}^i{}_{\dot{1}}$ raises the helicity of the state by ½.

Having obtained raising and lowering operators, we can now obtain a representation by assuming a Clifford ground state $\mid E,\lambda_0 >$ as follows:

$$Q_i \mid E,\lambda_0 > \; = 0. \tag{3.3.10}$$

The higher states are the obtained by applying the raising operators $\bar{Q}^i$:

$$\bar{Q}^{i_1}\, \bar{Q}^{i_2} \,\ldots\ldots\, \bar{Q}^{i_n} \mid E,\lambda_0 > \; = \; \mid E,\ \lambda_0 + \tfrac{n}{2}\ ,\ i_1, i_2, \ldots, i_n > \quad . \tag{3.3.11}$$

Since $\{\bar{Q},\bar{Q}\} = 0$, these states must be totally antisymmetric in the indices i_α, and therefore the sequence stops after each of the operators has been applied exactly once, giving the top helicity $\lambda_0 + \frac{N}{2}$. Because of the antisymmetry of the Q's, there are $\binom{N}{n}$ states of a given helicity $\lambda_0 + \frac{n}{2}$. For example, if $N = 4$ and $\lambda_0 = -1$, we get

n	0	1	2	3	4
λ	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
multiplicity = $\binom{N}{n}$	1	4	6	4	1

However, if N = 1, $\lambda_0 = 0$, we get

n	0	1
λ	0	$\frac{1}{2}$
mult.	1	1

For Lorentz-covariant theories, we require the spectrum to be PCT-invariant, meaning that for each state with helicity λ there should be a state with helicity $-\lambda$. Therefore we must add to the above spectrum the PCT-conjugate spectrum and obtain for N = 1, $\lambda_0 = 0$ the spectrum:

λ	$-\frac{1}{2}$	0	$\frac{1}{2}$
mult.	1	2	1

which means that we now have two spin-0 scalar fields (one scalar, one pseudoscalar) and one spin-$\frac{1}{2}$ Majorana field.

Generally we obtain the following multiplicity tables:

Table 3.1: Multiplicities for massless supersymmetric spectra without central charges : N = 1:

					CPT- invariant			
$\lambda \downarrow$ $\lambda_0 \rightarrow$	1/2	1	3/2	2	1/2	1	3/2	2
2				1				1
3/2			1	1			1	1
1		1	1			1	1	
1/2	1	1			1	1		
0	1				2			

Table 3.2: Multiplicities for massless supersymmetric spectra without central charges: N = 2:

					CPT- invariant			
$\lambda \downarrow$ \ $\lambda_0 \rightarrow$	1/2	1	3/2	2	1/2	1	3/2	2
2				1				1
3/2			1	2			1	2
1		1	2	1		1	2	1
1/2	1	2	1		1	2	1	
0	2	1			2	2		

Table 3.3: Multiplicities for massless supersymmetric spectra without central charges : N = 4:

						CPT- invariant	
$\lambda \downarrow$ \ $\lambda_0 \rightarrow$	0	1/2	1	3/2	2	2	3/2
2					1	1	
3/2				1	4	4	1
1			1	4	6	6	4
1/2		1	4	6	4	4	7
0	1	4	6	4	1	2	8
-1/2	4	6	4	1		4	7
-1	6	4	1			6	4
-3/2	4	1				4	1
-2	1					1	

Table 3.4: Multiplicities for massless supersymmetric spectra without central charges : N = 8:

$\lambda_0 \downarrow$ \ $\lambda \rightarrow$	2	3/2	1	1/2	0	-1/2	-1	-3/2	-2
2	1	8	28	56	70	56	28	8	1

From these tables we see that for N > 4, the PCT- self-conjugate multiplet contains helicities greater than 1 - which is not desirable for renormalisable theories. Also, for N > 8, the PCT- self-conjugate multiplet contains helicities greater than 2 - which is not allowed for consistent theories of gravity. Hence the number of supersymmetry generators is restricted to $N \leq 4$ for renormalisable theories of supersymmetry and $N \leq 8$ for supergravity theories.

The total number of states in a multiplet is

$$\sum_{n=0}^{N} \binom{N}{n} = 2^N .$$

We can obtain the fermions = bosons rule from the binomial expansion of the coefficients:

We expand the following:

$$(1 - 1)^N = 0$$

This gives for even N,

$$\sum_{m=0}^{N/2} \binom{N}{2m} - \sum_{m=0}^{N/2-1} \binom{N}{2m+1} = 0 \tag{3.3.12}$$

and for odd N,

$$\sum_{m=0}^{N/2-1} \binom{N}{2m} - \sum_{m=0}^{N/2-\frac{1}{2}} \binom{N}{2m+1} = 0. \tag{3.3.13}$$

Now depending on the value of λ_0, in both of the expansions above, one term will represent the total number of states with half-integer helicities (fermions) and the other term the total number of of states with integer helicity (bosons). The expansions above thus proves that there are an equal number of bosonic and fermionic states. Also, since the total number of states equals 2^N, there are thus 2^{N-1} bosonic and 2^{N-1} fermionic states.

The fact that we get exactly $\begin{pmatrix} N \\ n \end{pmatrix}$ states of a given helicity $\lambda_0 + \frac{n}{2}$ by applying n raising operators to the ground staste, points to the fact that the massless supersymmetric spectrum forms the totally antisymmetric supersymmetric irreducible representation of U(N): The dimension of the representation is thus given by considering the totally antisymmetric Young tableau for U(N):

n rows

We say that the massless spectrum is generated by U(N) - this does not necessarily mean that U(N) is an actual symmetry of the underlying field theory - it just means that the elements of the <u>algebra</u> are invariant under U(N) transformations and that U(N) is the largest invariance group of the algebra.

<u>3.3(b) Massless Representations with Central Charges:</u>

Since the central charges commute with all generators, we can diagonalise them to a convenient form. Thus we can bring them to the form of an antisymmetric N×N matrix with complex entries Z_{ij}. Now there exists a theorem [15] which states that an antisymmetric complex matrix can be transformed by a unitary transformation into a form

$$Z_{ij} = (U\ Z\ U^T)_{ij} = \bar{Z}_{ij}$$

$$= \begin{bmatrix} 0 & D_{(N/2)} \\ -D_{(N/2)} & 0 \end{bmatrix} \qquad \text{for even N, and}$$

$$= \begin{bmatrix} 0 & D_{(N-1)/2} & 0 \\ -D_{(N-1)/2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{for odd N} \tag{3.3.14}$$

where $D_{N/2}$, $D_{(N-1)/2}$ are diagonal matrices with non-negative eigenvalues $z_1, z_2, \ldots, z_{[N/2]}$.
Obviously, the commutation relation (3.1.26)(h) remains unchanged, so we redefine the supercharges

$$Q_{\alpha i} \to U_i^{\ j}\ Q_{\alpha j}$$

$$\bar{Q}^j{}_{\dot\alpha} \to (\bar{Q}^i{}_{\dot\alpha})\ (U^{-1})_i^{\ j}\ . \tag{3.3.15}$$

Corresponding to the form of (3.3.14) of the central charges, we can now split the entire algebra into two parts, i.e. we treat the parts i = 1, 2, ...,[N/2] and i = [N/2]+1, ..., N separately (If N is odd, the last index N stands separately and is not affected). This can be done by doubling the index i:

$$i \to a,r : \quad a = 1,2 \quad r = 1,2, \ldots [N/2]. \tag{3.3.16}$$

We can then rewrite the algebra in terms of these generators:

$$\{Q_{\alpha i},\bar{Q}^j{}_{\dot\beta}\} \to \{Q_{\alpha ar},\bar{Q}^{bs}{}_{\dot\beta}\}$$

$$= 2\ \delta_a^{\ b}\ \delta_r^{\ s}\ (\sigma^\mu)_{\alpha\dot\beta}\ P_\mu \tag{3.3.17)(a}$$

$$\{Q_{\alpha i}, Q_{\beta j}\} \rightarrow \{Q_{\alpha ar}, Q_{\beta bs}\}$$

$$= 2\ \varepsilon_{\alpha\beta}\ \varepsilon_{ab}\ \delta_{rs}\ z_r \tag{3.3.17)(b}$$

$$\{\bar{Q}^{ar}{}_{\dot{\alpha}}, \bar{Q}^{bs}{}_{\dot{\beta}}\} = -2\ \varepsilon_{\dot{\alpha}\dot{\beta}}\ \varepsilon^{ab}\ \delta^{rs}\ z_r \quad . \tag{3.3.17)(c}$$

For odd N, we have also

$$\{Q_{\alpha N}, \bar{Q}^{i}{}_{\dot{\beta}}\} = 2\ \delta_N{}^i\ (\sigma^\mu)_{\alpha\dot{\beta}}\ P_\mu \ . \tag{3.3.17)(d}$$

The derivation of $Q_{2i} = \bar{Q}^j{}_{\dot{2}} = 0$ is not affected by the presence of central charges, so this result still stands.

Equation (3.3.17) has the form

$$\left\{ \begin{pmatrix} Q_{11r} \\ Q_{21r} \\ Q_{12r} \\ Q_{22r} \end{pmatrix}, \begin{pmatrix} Q_{11s} \\ Q_{21s} \\ Q_{12s} \\ Q_{22s} \end{pmatrix} \right\} = \left\{ \begin{pmatrix} Q_{11r} \\ 0 \\ Q_{12r} \\ 0 \end{pmatrix}, \begin{pmatrix} Q_{11s} \\ 0 \\ Q_{12s} \\ 0 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} \{Q_{11r}, Q_{11s}\} & 0 & \{Q_{11r}, Q_{11s}\} & 0 \\ 0 & 0 & 0 & 0 \\ \{Q_{12r}, Q_{11s}\} & 0 & \{Q_{12r}, Q_{11s}\} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= 2 \begin{pmatrix} 0 & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & 0 \end{pmatrix} \delta_{rs}\ z_r \tag{3.3.18}$$

From this form of the algebra it is evident that $z_r = 0$ for all r.

For odd N, there is the extra relation $\{Q_{\alpha N}, Q_{\beta i}\} = 0$, which does not place any restriction on the central charges.

Hence we see that massless particles represent central charges trivially.

3.3(c) Massive Representations : No Central Charges[8]:

We look again at the rest frame in which

$$P_\mu = (m,0,0,0) \tag{3.3.19}$$

and the supersymmetry algebra becomes:

$$\{Q_{\alpha i},\bar{Q}^j{}_{\dot\beta}\} = 2m\,\delta_i{}^j\,\delta_{\alpha\dot\beta}$$
$$\{Q_{\alpha i},Q_{\beta j}\} = 0$$
$$\{\bar{Q}^i{}_{\dot\alpha},\bar{Q}^j{}_{\dot\beta}\} = 0. \tag{3.3.20}$$

In this case, the algebra obtained has 4N elements- or 2N fermionic degrees of freedom (meaning that there are 2N raising operators). Again, we can define a Clifford ground state such that

$$Q_{\alpha i}\,|m,s_0,s_3\rangle = 0. \tag{3.3.21}$$

However, in the massive case the ground state is characterised by the value of s_0 as well as s_3, so there are several states which satisfy the ground state condition; we have a ground state multiplet with

$$s_3 = -s_0,-s_0+1,\ \ldots\ ,s_0. \tag{3.3.22}$$

Since the supersymmetry operators carry half-integer spin, application of Q or $\bar{Q}$ onto a state with spin s will produce a linear combination of states with spins $s+\frac{1}{2}$ and $s-\frac{1}{2}$. The spectrum is built up from the ground state by applying the raising operators $\bar{Q}$ succesively. Because these states must be totally symmetric under interchange of index pairs, the highest spin occurs when each of the N $\bar{Q}^i{}_{\dot 1}$ or the N $\bar{Q}^i{}_{\dot 2}$ have been applied exactly once, giving a spin of $s + \frac{N}{2}$. Mixing raising operators $\bar{Q}^i{}_{\dot 1}$ and N $\bar{Q}^i{}_{\dot 2}$ always results in spins lower than $s + \frac{N}{2}$ since any product $\bar{Q}^i{}_{\dot 1}\ \bar{Q}^i{}_{\dot 2}$ carries spin zero and so does not increase the spin of the original state. Therefore, if all 2N raising oprators have been applied once, we obtain again a state of spin s_0.

In analogy to the massless case, where the dimension of a representation was 2^N, it is 2^{2N} for the massive case since there are now 2N raising operators:

$$\sum_{n=0}^{2N} \binom{2N}{n} = 2^{2N} .$$

Again, this representation splits into two irreducible parts of dimension 2^{2N-1}, one for bosons and fermions each.

To generate the spectrum of the massive states, we must find the invariance group (the group of transformations of the Q, $\bar{Q}$ between each other that leaves the algebra invariant) of the supersymmetry algebra (3.3.20). In order to make the structure apparent, we write the generators in the form

$$Q_{\alpha I} \equiv \begin{pmatrix} Q_{\alpha i} \\ \bar{Q}^{\alpha i} \end{pmatrix} . \qquad \text{with } I \equiv i. \tag{3.3.23}$$

The algebra (3.3.20) then becomes

$$\{Q_{\alpha I}, Q_{\beta J}\} = \varepsilon_{\alpha\beta} \begin{pmatrix} 0 & 2m \\ -2m & 0 \end{pmatrix}_{IJ} \tag{3.3.24}$$

$$= 2m\, \varepsilon_{\alpha\beta} \in_{IJ},$$

where now I,J = 1,2, ..., 2N.

This form of the algebra makes apparent its SU(2)⊗USp(2N) structure:

Since Q_α and $\bar{Q}^\alpha$ transform in the same way under Lorentz transformations, the form (3.3.23) of the generators is justified and gives a unique transformation behavior for the generator $Q_{\alpha I}$.
The largest invariance group of (3.3.24) is clearly SO(4N). This is apparent when one realises that the expression $\varepsilon_{\alpha\beta} \in_{IJ}$ is actually a 4N×4N matrix, and that the commutator on the right hand side of eq. (3.3.24) can be written as $Q_{\alpha I}\, Q_{\beta J}{}^T + Q_{\beta J}\, Q_{\alpha I}{}^T$ which is clearly invariant under a transformation

$$Q_{\alpha I} \rightarrow Q_{\alpha I}\, O, \tag{3.3.25}$$

where O is an orthogonal matrix.

The form (3.3.24) of the algebra further exhibits an explicit $SU(2)\otimes USp(2N) \supset SO(4N)$ invariance. The SU(2) invariance is due to the Lorentz indices α, β: both $Q_{\alpha i}$ and $\bar{Q}^{\alpha i}$ are invariant under SU(2) transformations as is clear from their transformation behavior under Lorentz transformations.

The USp(2N) invariance is due to the I- indices:
We have

$$(Q_{\alpha I})^{\dagger T} = \begin{pmatrix} \bar{Q}_{\alpha}{}^{i} \\ Q^{\alpha}{}_{i} \end{pmatrix} = \varepsilon_{\alpha\beta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} Q_{\beta i} \\ \bar{Q}^{\beta i} \end{pmatrix}$$

$$\equiv \varepsilon_{\alpha\beta} \in_{IJ} Q_{\beta J} \quad . \tag{3.3.26}$$

For this to be invariant under a transformation

$$Q_{\alpha I} \longrightarrow S\, Q_{\alpha I}, \tag{3.3.27}$$

we must have

$$(SQ)^{\dagger T} = \varepsilon \in (SQ)$$

or

$$S^{\dagger T}\, Q^{*} = \varepsilon \in SQ$$

and hence

$$Q^{*} = \varepsilon\, (S^{\dagger T-1} \in S)\, Q \quad .$$

If S is unitary, then

$$S^{-1} = S^{\dagger}, \quad \text{i.e. } S^{\dagger -1} = S$$

and the condition for (3.3.26) to be invariant is thus

$$S^{-1} = S^{\dagger} , \tag{3.3.28)(a}$$

$$S^{T} \in S = \in \ . \tag{3.3.28)(b}$$

Eq. (3.3.28)(a) gives the unitarity, while eq. (3.3.28)(b) is the group structure of the symplectic group. The matrix $\in$ is 2N-dimensional, so the invariance group is USp(2N) (see Appendix B for a discussion of the properties of the symplectic group Sp(2n)).

The massive supersymmetric spectrum is now obtained from the following <u>theorem:</u>[5]

If the Clifford vacuum is a scalar unde SU(2) (and the internal symmetry group which is left out of the discussion here altogether) then the content of the massive irreducible representation is the following:

$$2^{2N} = [\tfrac{N}{2}\ ,(0)] + [\tfrac{N-1}{2}\ ,\ (1)] + \ldots + [0,\ (N)], \tag{3.3.29}$$

where the first entry in the bracket denotes the spin of the respective representation and the second entry (k) denotes the kth-fold antisymmetric irreducible representation of USp(2n) belonging to this spin (see Appendix B for a method to calculate the dimension of the irreducible representations of USp(2n)).

Consider for example the case N = 2, $s_0 = 0$:

The calssifying group is USp(4)⊗SU(2), and we have one spin-1 (SU(2) triplet), four spin-½ (SU(2) doublet) and five spin-0 (SU(2) singlet) states, so that the total number of states is indeed $1\times3 + 4\times2 + 5\times1 = 2^4$, as required.

If the Clifford vacuum is not a singlet under SU(2), i.e. if the vacuum has spin, then the spectrum is obtained by taking the tensor product between the vacuum and the representation given above[8]. The easiest way to work out this spectrum is to rely on the fact that we must have 2^{2N-1} fermionic and 2^{2N-1} bosonic states. So to the (shifted) spectrum generated by USp(2N) for the degenerate vacuum, we have to add enough terms so that we end up with the required amount of fermionic and bosonic states.

We summarise these results as follows:

Table 3.5: Multiplicities for massive supersymmetric spectra without central charges

spin ↓ \ N →	1				2			3		4
2				1			1		1	1
3/2			1	2		1	4	1	6	8
1		1	2	1	1	4	5+1	6	14+1	27
1/2	1	2	1		4	5+1	4	14	14+6	48
0	2	1			5	4	1	14	14	42

Recall that for a field theory to be renormalisable, the particles must have spin $\leq \frac{1}{2}$. This shows that for renormalisable coupling to massive matter, N = 1.

3.3(d) Massive Representations with Central Charges:

The supersymmetry algebra which includes central charges is given be eq. (3.3.17). To make the structure more evident, we write the generators as

$$A^{\pm}_{\alpha r} = \tfrac{1}{2}\,(Q_{\alpha 1 r} \pm \bar{Q}^{\dot{\alpha} 2 r}), \tag{3.3.30(a)}$$

$$(A^{\pm}_{\alpha r})^{\dagger} = \tfrac{1}{2}\,(Q^{\cdot 1r}_{\alpha} \pm \bar{Q}_{\dot{\alpha} 2 r}) \tag{3.3.30(b)}$$

and thus the algebra becomes

$$\{A^{\pm}_{\alpha r}, A^{\pm}_{\beta s}\} = \{A^{\pm}_{\alpha r}, A^{\mp}_{\beta s}\} = \{A^{\pm}_{\alpha r}, (A^{\mp}_{\beta s})^{\dagger}\} = 0 \tag{3.3.31(a)}$$

$$\{A^{\pm}_{\alpha r}, (A^{\pm}_{\beta s})^{\dagger}\} = \delta_{\alpha\beta}\, \delta_{rs}\, (m \pm z_r) \quad . \tag{3.3.31(b)}$$

Since we must have for each Q individually

$$\{Q,Q^{\dagger}\} \geq 0$$

and hence also

$$\{A^{\pm}_{\alpha r},(A^{\pm}_{\beta s})^{\dagger}\} \geq 0, \tag{3.3.32}$$

we must have

$$m \geq z_r \quad ; \; r = 1,2, \ldots, [\tfrac{N}{2}] \quad . \tag{3.3.33}$$

If we have $z_1 = z_2 = \ldots z_q = m$, then the first 2q operators $A^{-}_{\alpha r}$, which satisfy this constraint, are represented trivially. The result is to effectively reduce the number of supersymmetry generators, so that we are left with (N-q)- extended supersymmetry. Clearly the invariance group in that case is SU(2)⊗USp(2N-2q). However, these states now appear twice: Under a PCT transformation we have

$$Q_{\alpha ar} \xrightarrow{PCT} iQ^{*}_{\alpha ar},$$

with the effect that

$$z_r \xrightarrow{PCT} -z_r \quad .$$

This means that the massive multiplet with central charges has twice the number of degrees of freedom of the represenation of the Clifford algebra without central charges. The spectrum thus looks as follows:

Table 3.6: Supersymmetric spectrum for massive particles with central charges

spin ↓ \ N →	N=1 one Z			N=4 one Z		N=4 two Z		
2								
3/2			2x1		2x1			2x1
1		2x1	2x2	2x1	2x4		2x1	2x2
1/2	2x1	2x2	2x1	2x4	2x(5+1)	2x1	2x2	2x1
0	2x2	2x1		2x5	2x4	2x2	2x1	

3.4 The N = 1 Chiral Multiplet

To formulate a supersymmetric field theory one must find a representation of the supersymmetry algebra on fields. The construction of such a representation is analogous to that of representations on particle states discussed above. The difference is that for a representation on fields, the supersymmetry operators must act on multiplets of fields instead of single-particle states.

We thus need a method to enforce the supersymmetry algebra on fields. But since, quite generally,

$$[\Phi,P_\mu] = [\Phi,i\partial_\mu] = i\partial_\mu\Phi, \qquad (3.4.1)$$

and the anticommutator of the supersymmetry generators is an expression in P_μ, we know the action of the such generators on fields and can hence construct the multiplets. As an example, we discuss here the construction of the chiral multiplet.

For the construction of representations on particle states (section 3.3), we started from a Clifford ground state $Q|\Phi\rangle = 0$. Analogously, for the chiral representation we start with a scalar field A on which we impose the "ground state" constraint

$$[A,\bar{Q}_{\dot\alpha}] = 0 \quad . \qquad (3.4.2)$$

We then define the field into which A transforms as

$$[A,Q_\alpha] = 2i\Psi_\alpha \quad . \tag{3.4.3}$$

Since A is a scalar field, it is bosonic, so that Ψ_α must be fermionic. The fields into which Ψ_α transforms are then defined as

$$\{\Psi_\alpha,Q_\beta\} = -iF_{\alpha\beta} \tag{3.4.4}$$

$$\{\Psi_\alpha,\bar{Q}_{\dot\beta}\} = \chi_{\alpha\dot\beta} \quad . \tag{3.4.5}$$

$F_{\alpha\beta}$ and $\chi_{\alpha\dot\beta}$ are again bosonic. We see that tensor fields are transformed into spinor fields and vice versa.

The Jacobi identity (3.1.5)(c) now gives for A:

$$[\{Q_\alpha,\bar{Q}_{\dot\beta}\},A] + \{[A,Q_\alpha],\bar{Q}_{\dot\beta}\} - \{[\bar{Q}_{\dot\beta},A],Q_\alpha\} = 0 \quad ,$$

or because of (3.4.1) and the constraint (3.4.2),

$$\{[A,Q_\alpha],\bar{Q}_{\dot\beta}\} = 2i\, \sigma^\mu{}_{\alpha\dot\beta}\, \partial_\mu A \quad . \tag{3.4.6}$$

We see that for this to be nontrivial, A has to be complex. Otherwise A = constant and we would have a trivial representation.

Further, (3.4.6) yields

$$2i\, \chi_{\alpha\dot\beta} = 2i\, \sigma^\mu{}_{\alpha\dot\beta}\, \partial_\mu A \quad . \tag{3.4.7}$$

The identity

$$[\{Q_\alpha,Q_\beta\},A] + \{[A,Q_\alpha],Q_\beta\} - \{[Q_\beta,A],Q_\alpha\} = 0$$

yields

$$F_{\alpha\beta} = -F_{\beta\alpha} \quad ,$$

and hence $F_{\alpha\beta}$ is complex and

$$F_{\alpha\beta} = F\varepsilon_{\alpha\beta} \quad . \tag{3.4.8}$$

We now proceed for F as we did for A, i.e. we define the fields into which F transforms:

$$[F,Q_{\alpha}] = 2\lambda_{\alpha} \tag{3.4.9}$$

$$[F,\bar{Q}_{\dot{\alpha}}] = 2\bar{\Lambda}_{\dot{\alpha}} \quad . \tag{3.4.10}$$

The graded Jacobi identities involving Ψ_{α} are

$$[\{\Psi_{\alpha},Q_{\beta}\},\bar{Q}_{\dot{\beta}}] + [\{\Psi_{\alpha},\bar{Q}_{\dot{\beta}}\},Q_{\beta}] - [\Psi_{\alpha},\{Q_{\beta},\bar{Q}_{\dot{\beta}}\}] = 0 \quad , \tag{3.4.11}$$

$$[\{\Psi_{\alpha},Q_{\beta}\},Q_{\gamma}] + [\{\Psi_{\alpha},Q_{\gamma}\}Q_{\beta}] - [\Psi_{\alpha},\{Q_{\beta},Q_{\gamma}\}] = 0 \quad . \tag{3.4.12}$$

From (3.4.11), we get, using (3.4.4) and (3.4.5),

$$-2i\varepsilon_{\alpha\beta}\bar{\Lambda}_{\dot{\beta}} = 2i(\sigma^{\mu}{}_{\beta\dot{\beta}}\,\partial_{\mu}\Psi_{\alpha} - \sigma^{\mu}{}_{\alpha\dot{\beta}}\,\partial_{\mu}\Psi_{\beta}) \quad ,$$

so that $\bar{\Lambda}_{\dot{\beta}}$ is a function of Ψ:

$$\bar{\Lambda}_{\dot{\beta}} = 2\sigma^{\mu}{}_{\alpha\dot{\beta}}\,\partial_{\mu}\Psi^{\alpha} \tag{3.4.13}$$

Similarly, from (3.4.12) we get

$$\varepsilon_{\alpha\beta}\lambda_{\gamma} + \varepsilon_{\alpha\gamma}\lambda_{\beta} = 0$$

with the solution

$$\lambda_\gamma = 0 \quad . \tag{3.4.14}$$

Thus any further application of the supersymmetry generators just brings us back to one of the fields F,A,Ψ. These fields are the <u>component</u> fields of the multiplet.

To show that the algebra is closed, we have to investigate the remaining Jacobi identities. These are:

- $\{[F,Q],\bar{Q}\} + \{[F,\bar{Q}],Q\} = [F,\{Q,\bar{Q}\}]$

This implies

$$[F,\{Q,\bar{Q}\}] = 2i\ \sigma^\mu\partial_\mu[\Psi,Q]$$

$$= 2i\ \sigma^\mu\partial_\mu F \quad , \tag{3.4.15}$$

which is consistent with the algebra.
Secondly,

- $\{[F,Q],Q\} + \{[f,Q],Q\} = [F,\{Q,Q\}]$

implies

$$[F,\{Q,Q\}] = 0 \tag{3.4.16}$$

and similarly

- $[F,\{\bar{Q},\bar{Q}\}] = 0 \quad (3.4.17)$
- $[\Psi,\{\bar{Q},\bar{Q}\}] = 0 \quad . \quad (3.4.18)$

Thus we have proved that the algebra is closed and contains only the elements Ψ, F and A which form the so-called <u>chiral or scalar multiplet</u>

$$\Phi = (A;\Psi;F) \quad . \tag{3.4.19}$$

The name <u>scalar multiplet</u> arises because the lowest component field A is a scalar field. We shall see later why the name <u>chiral multiplet</u> is appropriate.

Because the generators Q have the mass dimension ½ (this follows from the algebra), fields of dimension l transform into fields of dimension l+½ or derivatives of lower dimension. Thus if A has mass dimension 1, then Ψ has dimension 3/2 and F has dimension 2. Also, we can define the mass dimension of a multiplet as being that of the field with the lowest mass dimension (the ground state field A in the chiral multiplet). Ψ has then the mass dimension 1.

We summarise the algebra:

$$[A, Q_\alpha] = 2i\Psi_\alpha \qquad (3.4.20)(a)$$

$$[A, \bar{Q}_{\dot{\alpha}}] = 0 \qquad (3.4.20)(b)$$

$$\{\Psi_\alpha, \bar{Q}_{\dot{\beta}}\} = \sigma^{\mu}{}_{\alpha\dot{\beta}}\, \partial_\mu A \qquad (3.4.20)(c)$$

$$\{\Psi_\alpha, Q_\beta\} = -i\varepsilon_{\alpha\beta} F \qquad (3.4.20)(d)$$

$$[F, Q_\alpha] = 0 \qquad (3.4.20)(e)$$

$$[F, \bar{Q}_{\dot{\alpha}}] = 2\, \partial_\mu \Psi^\beta\, \sigma^{\mu}{}_{\beta\dot{\alpha}} \qquad (3.4.20)(f)$$

Because of the graded nature of the supersymmetry algebra, the notation becomes sometimes clumsy. A simplification is introduced when we use <u>anticommuting spinor parameters ζ^α</u> which are defined as being fermionic, i.e. they anticommute with fermionic objects and commute with bosonic objects:

$$\{\zeta^\alpha, \zeta^\beta\} = \{\zeta^\alpha, Q_\beta\} = \{\zeta^\alpha, \bar{Q}_{\dot{\beta}}\} = [P_\mu, \zeta^\alpha] = \ldots = 0 \qquad (3.4.21)$$

In terms of these, the supersymmetry algebra without central charges reduces to the very simple form:

$$[\zeta Q, \bar{Q}\,\bar{\zeta}] = 2\,\zeta\,\sigma_\mu\,\bar{\zeta}\,P^\mu \qquad (3.4.22)(a)$$

$$[\zeta Q, \zeta Q] = [\bar{\zeta}\,\bar{Q}, \bar{\zeta}\,\bar{Q}] = 0 \quad . \qquad (3.4.22)(b)$$

The corresponding algebra on the component fields, eq.(3.4.20), becomes

$$[A,\zeta Q] = 2i\zeta\Psi \tag{3.4.23)(a}$$

$$[\Psi,\zeta Q] = i\zeta^{\beta}\varepsilon_{\alpha\beta}F = -i\zeta F \tag{3.4.23)(b}$$

$$[\Psi,\bar{Q}\zeta] = \sigma^{\mu}\,\partial_{\mu}\,\bar{\zeta} \tag{3.4.23)(c}$$

$$[F,\bar{Q}\,\bar{\zeta}] = 2\,\partial_{\mu}\Psi\,\sigma^{\mu}\,\bar{\zeta} \tag{3.4.23)(d}$$

We use the anticommuting spinor parameters to define the infinitesimal supersymmetry variation of a field as:

$$\delta\Phi = -i[\Phi,\zeta Q + \bar{\zeta}\,\bar{Q}] \quad . \tag{3.4.24}$$

This definition is a straightforward analogy to the usual definition of an operator acting on a field, such as for example eq.(3.4.1).

Now, using (3.3.24) and the algebra (3.4.23), the infinitesimal transformations of the component fields are

$$\delta A = 2\zeta\Psi \tag{3.4.25)(a}$$

$$\delta\Psi = -\zeta F - i\partial_{\mu}A\,\sigma^{\mu}\,\bar{\zeta} \tag{3.4.25)(b}$$

$$\delta F = -2i\,\partial_{\mu}\Psi\,\sigma^{\mu}\,\bar{\zeta} \tag{3.4.25)(c}$$

The anti-chiral multiplet can be obtained from Φ by Hermitean conjugation since the constraint

$$[A,\bar{Q}_{\dot{\alpha}}] = 0$$

implies

$$[A^{\dagger},Q_{\alpha}] = 0 \ , \tag{3.4.26}$$

and (3.4.26) gives then the groundstate constraint for the anti-chiral multiplet. Its component fields tranform as

$$\delta A^{\dagger} = 2\bar{\Psi}\,\bar{\zeta} \tag{3.4.27)(a}$$

$$\delta\Psi = -F^{\dagger}\,\bar{\zeta} + i\zeta\,\sigma^{\mu}\partial_{\mu}A^{\dagger} \tag{3.4.27)(b}$$

$$\delta F^{\dagger} = 2i\,\zeta\,\sigma^{\mu}\partial_{\mu}\bar{\Psi} \quad . \tag{3.4.27)(c}$$

We have constructed the chiral multiplet by starting from a ground state field and then applying the supersymmetry transformations successively, until no new fields were formed in this way. Another method to construct a multiplet is to assume a set of fields $\Phi = (A,\Psi,\ldots)$, each of which satisfy an as yet unknown transformation law like (3.3.24). We start out by defining the transformation of the ground state field A. One could then demand that the commutators of the transformations close in the following sense:

$$[\delta_{\zeta},\delta_{\eta}]\Phi = 2i\,(\zeta\sigma^{\mu}\,\bar{\eta} - \eta\,\sigma^{\mu}\,\bar{\zeta})\,\partial_{\mu}\Phi \tag{3.4.28}$$

Matching components then gives us the transformation laws of the fields of the multiplet. Eq.(3.4.28) is derived from the supersymmetry algebra, eq.(3.4.22).

Using this, we get

$$\begin{aligned}[\delta_{\zeta},\delta_{\eta}] &= [-i(\eta Q + \bar{Q}\,\bar{\eta}),\ -i(\zeta Q + \bar{Q}\,\bar{\zeta})] \\ &= 2i\,(\zeta\sigma^{\mu}\,\bar{\eta} - \eta\,\sigma^{\mu}\,\bar{\zeta})\,\partial_{\mu}\end{aligned}$$

as required.

The chiral multiplet has 8 degrees of freedom: Two each for the complex scalar fields A and F and four for the complex spinor field Ψ. Once again, we see that the numbers of bosonic and fermionic degrees of freedom are equal. Since any multiplet must contain at least one spinor (because of the spinorial character of Q), with four degrees of freedom, 8 is the minimum number of degrees of freedom for a multiplet of fields and the chiral (anti-chiral) multiplet is thus irreducible.

The character of the various component fields (scalar, pseudocsalar, etc.) becomes more evident when we write the transformation in four-component notation:
We define a Majorana spinor ζ^{i} from the spinor $\zeta^{\alpha i}$ as follows:

$$\zeta^{\alpha i} Q_{\alpha i} + \bar{Q}^{i}{}_{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}}{}_{i} \equiv \bar{\zeta} Q \quad . \tag{3.4.29}$$

We then have the infinitesimal supersymmetry variation in four-spinor form:

$$\delta\Phi = -i[\Phi, \bar{\zeta}Q] \quad , \tag{3.4.30}$$

and with the four-component from of the anticommutator, $\{Q_i, \bar{Q}_j\}$, eq. (3.1.26)(r), we get

$$\begin{aligned}[\delta_\eta, \delta_\zeta] &= [-i\bar{\zeta}Q, -i\bar{\eta}Q] \\ &= -\bar{\eta}\ \{Q,Q\}\bar{\zeta} \\ &= \bar{\eta}\ \{Q,\bar{Q}\}\zeta \\ &= 2i\bar{\eta}\ \gamma^{\mu}\ \zeta\ \partial_{\mu}\Phi + 2i\ \bar{\eta}^{i}[\Phi,\ \mathrm{Im}Z_{ij} + \gamma_5 \mathrm{Re}\ Z_{ij}]\zeta^{j}\end{aligned} \tag{3.4.31}$$

If we now split the multiplet Φ into its real and imaginary parts as follows

$$\Phi = (A;\Psi;F) \equiv (A,B;\Psi;F,G) \quad , \tag{3.4.32}$$

where Ψ is now

$$\Psi = \begin{pmatrix} \Psi_\alpha \\ \bar{\Psi}_{\dot{\alpha}} \end{pmatrix} \quad , \tag{3.4.33}$$

and use the chiral representation of the Dirac algebra,

$$\gamma_\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \quad , \quad \gamma_5 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad ,$$

then it is possible to write the two-component spinors Ψ_α in terms of the four-component spinors Ψ as follows:

$$\Psi_\alpha = \tfrac{1}{2}(\mathbb{1} + i\gamma_5)\Psi \tag{3.4.34)(a}$$

$$\bar{\Psi}_{\dot\alpha} = \tfrac{1}{2}(\mathbb{1} - i\gamma_5)\Psi \quad . \tag{3.4.34)(b}$$

The infinitesimal transformation (3.4.25)(a) can then be written as

$$\delta(A + iB) = \zeta\ (\mathbb{1} + i\gamma_5)\Psi \quad ,$$

and (3.4.25)(d) becomes

$$\delta(A^\dagger - iB^\dagger) = (\mathbb{1} - i\gamma_5)\Psi\ \bar{\zeta} \quad ,$$

where it must be stressed that here ζ, $\bar{\zeta}$ are the two-component spinor parameters.
Using the definition (3.4.34) for the four-component spinor parameters, this then becomes

$$\delta A = \bar{\zeta}\ \Psi \tag{3.4.35)(a}$$

$$\delta B = \bar{\zeta}\ \gamma_5\ \Psi \quad , \tag{3.4.35)(b}$$

where now

$$A = \begin{pmatrix} A \\ A^\dagger \end{pmatrix} \quad , \quad B = \begin{pmatrix} B \\ B^\dagger \end{pmatrix} ,$$

and $\bar{\zeta}$ is the four-component spinor parameter.

The transformations on Ψ, $\bar{\Psi}$ and F, $F^{\dagger}$ of the chiral multiplet can be summarised similarly. The overall result is

$$\delta A = \bar{\zeta}\ \Psi \qquad (3.4.35)(a)$$

$$\delta B = \bar{\zeta}\ \gamma_5\ \Psi \qquad (3.4.35)(b)$$

$$\delta\Psi = -(F + \gamma_5 G)\zeta\ - i\gamma^{\mu}\partial_{\mu}(A + \gamma_5 B)\zeta \qquad (3.4.35)(c)$$

$$\delta F = i\bar{\zeta}\ \gamma^{\mu}\ \partial_{\mu}\Psi \qquad (3.4.35)(d)$$

$$\delta G = i\bar{\zeta}\ \gamma_5\ \gamma^{\mu}\ \partial_{\mu}\Psi \quad . \qquad (3.4.35)(e)$$

These transformations now describe a chiral multiplet which contains both the chiral and the anti-chiral multiplets described above, but here all the fields are real. Hermitean conjugation thus does not yield a new multiplet. The form (3.4.35) of the multiplet exhibits the parity character of the fields: A and F are scalar and B and G are pseudoscalar fields.

To construct an <u>invariant action</u> which describes the field content of the chiral multiplet, it is necessary to find combinations of fields which transform into a divergence. Then the integral of such a combination L will be invariant by Gauss' theorem. The general procedure is thus to write down a combination of fields L and determine the tranformation properties of the unknown fields by demanding that L transforms like a divergence. Thus we form the product

$$L = Af + Bg + \bar{\Psi}\chi + Fa + Gb \qquad (3.4.36)$$

of our chiral multiplet $\Phi = (A,B;\Psi;F,G)$ and its <u>contragradient</u> multiplet

$$\Phi' = (a,b;\chi;f,g) \quad . \qquad (3.4.37)$$

L transforms as

$$\delta L = A\delta f + B\delta g + \bar{\Psi}\delta\chi + F\delta a + G\delta b + f\bar{\zeta}\Psi + g\bar{\zeta}\gamma_5\Psi - \bar{\zeta}(F + \gamma_5 G)\chi$$

$$+ i\bar{\zeta}\,\not\partial(A + \gamma_5 B)\chi + i\bar{\zeta}\not\partial\,\Psi\,a + i\bar{\zeta}\,\gamma_5\not\partial\,\Psi\,b \qquad (3.4.38)$$

The only terms in this expression which contain derivatives are the last three - in order to obtain a divergence, these must be partially integrated to give

$$i\bar{\zeta}\,\gamma_\mu\partial^\mu(A + \gamma_5 B)\chi + i\bar{\zeta}\,\gamma_\mu\partial^\mu\,\Psi\,a + i\bar{\zeta}\,\gamma_5\,\gamma_\mu\partial^\mu\,\Psi\,b$$

$$= \partial^\mu[\,i\bar{\zeta}(A+\gamma_5 B)\gamma_\mu\chi + i\bar{\zeta}(a-\gamma_5 b)\gamma_\mu\Psi\,] - i\bar{\zeta}(A+\gamma_5 B)\not\partial\chi - i\bar{\zeta}\not\partial(a-\gamma_5 b)\Psi$$

Comparing terms in A, B, etc. gives the resulting transformation laws for Φ':

$$\delta f = i\bar{\zeta}\,\not\partial\chi \qquad (3.4.39)(a)$$

$$\delta g = i\bar{\zeta}\,\gamma_5\,\not\partial\chi \qquad (3.4.39)(b)$$

$$\delta\chi = -(f+\gamma_5 g)\zeta - i\not\partial(a+\gamma_5 b)\zeta \qquad (3.4.39)(c)$$

$$\delta a = \bar{\zeta}\chi \qquad (3.4.39)(d)$$

$$\delta b = \bar{\zeta}\gamma_5\chi\quad, \qquad (3.4.39)(e)$$

which are again the transformation laws for the component fields of a chiral multiplet.

This example shows that for the chiral multiplet, an invariant Lagrangian can be constructed by forming the product of two chiral multiplets Φ, Φ' in such a way that each term of the product has the same mass dimension. For this reason, we have in our example formed the products Af, Fa, etc. which all have the dimension 3 if A,B,a,b have mass dimension 1. The chiral multiplet is said to be self-contragradient because the multiplet contragradient to it is again a chiral multiplet.

3.5 N = 1 Multiplets in General [6]

The preceding chapter gives a detailed discussion of one kind of multiplet of the N = 1 supersymmetry algebra, the chiral multiplet. A different multiplet is obtained when we do not use the chirality constraint, eq.(3.4.2), but start out on a general ground state field C with no constraints imposed on it. The procedure is similar to the construction of the chiral multiplet, but we obtain more component fields. The resulting multiplet is called the general multiplet V and has the real field content

$$V = (C;\chi;M;N;A_\mu;\lambda;D) \quad , \tag{3.5.1}$$

where C,N,D are pseudoscalar, M is a scalar, A_μ is a vector and χ and λ are spinor fields.

The general multiplet (also called vector multiplet because its lowest component transforms as a vector) is not irreducible like the chiral multiplet, but partly reducible.

A third possibility of a multiplet is the kinetic multiplet $T\Phi$, which is obtained from the chiral multiplet by taking the "d'Alembertian" of some of the component fields:

$$T\Phi = (F,G;i\partial\!\!\!/\Psi;-\Box A,-\Box B) \tag{3.5.2}$$

and we notice that

$$T^2\Phi = -\Box\Phi \quad . \tag{3.5.3}$$

The kinetic multiplet is useful for the construction of Lagrangians because generally we require a Lagrangian to contain derivatives of fields as well as fields themselves.

It is also possible to form products of multiplets which give rise to new multiplets of the algebra. This is done by taking combinations of products of component fields of certain multiplets as the component fields of the product multiplets. For example, the product $\Phi_3 = \Phi_1 \cdot \Phi_2$ of two chiral multiplets is formed by using the two chirality constraints for the (complex) multiplets Φ_1 and Φ_2, $[A_1,\bar{Q}] = 0$ and $[A_2,\bar{Q}] = 0$, as a basis for the chirality constraint for the product multiplet: This means that the chirality constraint for Φ_3 is $[A_1A_2,\bar{Q}] = 0$. This automatically results in an assignment of the field combinations for the (real) product multiplet:

$$\Phi_1\cdot\Phi_2 = (A_1A_2-B_1B_2,\ B_1A_2+A_1B_2;\ (A_1-\gamma_5B_1)\Psi_2+(A_2-\gamma_5B_2)\Psi_1;$$

$$F_1A_2 + A_1F_2 + B_1G_2 + G_1B_2 + \bar{\Psi}_1\Psi_2;$$

$$G_1A_2 + A_1G_2 - B_1F_2 - F_1B_2 - \bar{\Psi}_1\gamma_5\Psi_2) \quad . \qquad (3.5.4)$$

It is clear that the product multplet is again a chiral multiplet.

There are other products of multiplets possible, but these will not be of interest here. Obviously, in any form of product between multiplets, the multiplets from which we form products must always have the same number of component fields, so that the procedure of forming products be well defined.

These definitions of the multiplets allow us to state the form of the Lagrangian of the Wess-Zumino model of simple (N = 1) supersymmetry:

$$L = (\tfrac{1}{2}\Phi\cdot T\Phi - \tfrac{m}{2}\Phi\cdot\Phi - \tfrac{g}{3}\Phi\cdot\Phi\cdot\Phi)_F \qquad (3.5.5.)$$

where $(\ldots)_F$ means taking the F-componet of the density in brackets. Taking the highest component of a density as constructed in (3.4.36) makes sense since the highest term of any supersymmetric multiplet (in the case of the chiral multiplet, this is the F-term) transforms into a total divergence. This can be explicitly seen in the case of the chiral multiplet from its transformation law (3.4.25). A space-time integral of such a total derivative is then an invariant quantity under supersymmetry transformations.

In components, (3.5.5) becomes

$$\tfrac{1}{2}\Phi\cdot T\Phi = \tfrac{1}{2}(\partial_\mu A\,\partial^\mu A + \partial_\mu B\partial^\mu B + i\bar{\Psi}\partial\!\!\!/\Psi + F^2 + G^2) + 4\text{-div}$$

$$\equiv L_0, \qquad (3.5.6)(a)$$

$$-\tfrac{1}{2}m\Phi\cdot\Phi = -m(AF + BG + \tfrac{1}{2}\bar{\Psi}\Psi) \equiv L_m \qquad (3.5.6)(b)$$

$$-\tfrac{g}{3}\,\Phi\cdot\Phi\cdot\Phi = -g\,\left((A^2-B^2)F + 2ABG + \bar{\Psi}(A-\gamma_5 B)\Psi \right) \equiv L_g \quad . \qquad (3.5.6)(c)$$

The component form of the Lagrangian makes it easy to derive the equations of motion for the component fields of the multiplets involved. These can be seen to be

$$T\Phi = m\Phi + g\,\Phi\cdot\Phi \quad , \qquad (3.5.7)$$

or in component form

$$F = mA + g(A^2 - B^2) \qquad (3.5.8)(a)$$

$$G = mB + 2gAB \qquad (3.5.8)(b)$$

$$i\partial\!\!\!/\Psi = m\Psi + 2g(A-\gamma_5 B)\Psi \qquad (3.5.8)(c)$$

$$-\Box A = mF + 2g(AF + BG + \tfrac{1}{2}\bar{\Psi}\Psi) \qquad (3.5.8)(d)$$

$$-\Box B = mG + 2g\,(AG - BF - \tfrac{1}{2}\bar{\Psi}\gamma_5\Psi) \qquad (3.5.8)(e)$$

The three terms L_0, L_m, L_g of the Wess-Zumino Lagrangian transform as densities under the supersymmetry transformations (3.4.35). Since the transformations (3.4.35) contain no information of the Lagrangian (the equations of motion are not manifest), we say the Lagrangian (3.5.5) exhibits off-shell supersymmetry.

Looking again at the equations of motion (3.5.8) of the Wess-Zumino Lagrangian, we notice that whereas the equations for F,G are purely algebraic, Ψ, A and B satisfy genuine wave equations. Therefore we can eliminate F and G from the Lagrangian by using their equations of motion. This results in an on-shell Lagrangian (which now explicitly contains only the fields A, B and Ψ) and the on-shell equations of motion

$$(\Box+m^2)\, A = -mg(3A^2 + B^2) - 2g^2A(A^2 + B^2) - g\bar{\Psi}\Psi \qquad (3.5.9)(a)$$

$$(\Box+m^2)\, B = -2mgAB - 2g^2B(A^2 + B^2) + g\bar{\Psi}\gamma_5\Psi \qquad (3.5.9)(b)$$

$$(i\partial\!\!\!/ - m)\Psi = 2g(A - \gamma_5 B)\Psi \qquad (3.5.9)(c)$$

Obviously, the elimination of F and G from the Lagrangian makes it necessary to eliminate them from the transformation laws as well. We get the on-shell transformations

$$\delta A = \bar{\zeta}\Psi \qquad (3.5.10)(a)$$

$$\delta B = \bar{\zeta}\, \gamma_5\, \Psi \qquad (3.5.10)(b)$$

$$\delta\Psi = -\{i\partial\!\!\!/ + m + g(A\!\!\!/ + \gamma_5 B)\}(A + \gamma_5 B)\zeta \quad , \qquad (3.5.10)(c)$$

which are the transformations (3.4.37) with F and G replaced by their equations of motion.

Now the on-shell Lagrangian still transforms as a density under the on-shell transformation laws. However, these now depend on the model (i.e. on the components m, g), and the fields must always be taken on-shell, i.e. subject to their equations of motion. This is called <u>on-shell supersymmetry</u>. The fields F and G which are eliminated from the Lagrangian because they do not contribute to the interaction, are called auxiliary fields.

<u>3.6 Superspace</u>[9],[10],[11]

We have now constructed supersymmetric fields by extending the ordinary spacetime (Poincaré) symmetries: To the generators of the spacetime symmetries, we have adjoined N spinorial generators Q which generate the supersymmetry. The realisation of supersymmetry came from the fact that the anticommutator of the two spinorial generators Q (where Q is a four-component spinor) is proportional to the translation operator, thereby producing the symmetry between fermions and bosons. Ordinary fields are functions of spacetime coordintes and hence the supersymmetry algebra can be enforced on fields to obtain field multiplets which turn out to contain an equal number of fermionic and bosonic component fields[13],[14]. This procedure is cumbersome since we have to calculate each of the component fields individually.

A different and more compact approach to studying supersymmetry is the superspace-superfield approach. This method is due to A.Salam and J.Strathdee[12]. In the superspace approach, ordinary space-time is extended to include extra coordinates, which anticommute. The anticommuting coordinates take the form of N two-component Weyl spinors, or 2N anticommuting spinor parameters. If we now define spinor fields as functions over this space, then a big advantage of this approach emerges: one can expand a superfield in a Taylor series with respect to the anticommuting coordinates; this series will terminate because any product of two identical anticommuting parameters vanishes. The individual terms in the series give the component fields of the ordinary space-time approach. In superspace, the supersymmetry transformations are just given as rotations and translations in spacetime and anticommuting coordinates, i.e. supersymmetry is manifest.

Let us now write down a general tramsformation on superspace. As usual, we write a finite transformation in terms of an exponential involving the generators of the symmetry. The supersymmetry transformation is thus generally written as

$$S(x,\theta,\bar{\theta}) = e^{i(-x \cdot P + \theta Q + \bar{Q}\,\bar{\theta})} \qquad (3.6.1)$$

To find the multiplication law for two transformations of this kind, we use the Hausdorf formula:

$$e^{A}\, e^{B} = e^{A+B+\frac{1}{2}[A,B]+\text{higher terms}} \qquad (3.6.2)$$

The higher terms vanish because of the anticommuting nature of θ and $\bar{\theta}$. We use the superalgebra in commutator form, eq.(3.4.22), to obtain

$$\begin{aligned} & S(0,\alpha,\bar{\alpha})\; S(x^{\mu},\theta,\bar{\theta}) \\ &= e^{i(\alpha\, Q + \bar{Q}\,\bar{\alpha})}\; e^{i(-x \cdot P + \theta\, Q + \bar{Q}\,\bar{\theta})} \\ &= e^{i[(\alpha+\theta)Q+\bar{Q}(\bar{\alpha}+\bar{\theta})-x \cdot P]}\; e^{\frac{1}{2}([i\alpha Q,i\bar{Q}\,\bar{\theta}] + [i\bar{Q}\,\bar{\alpha},i\theta Q])} \\ &= S(-x^{\mu} - i\alpha\sigma_{\mu}\bar{\theta} + i\theta\sigma_{\mu}\bar{\alpha},\ \theta+\alpha,\ \bar{\theta} + \bar{\alpha}) \end{aligned} \qquad (3.6.3)$$

In complete analogy, the finite supersymmetry transformation of a superfield $\Phi(x,\theta,\bar{\theta})$ becomes

$$S(x,\alpha,\bar{\alpha})\ \Phi(0,\theta,\bar{\theta}) = \Phi(x - i\alpha\ \sigma_\mu\bar{\theta} + i\theta\sigma_\mu\bar{\alpha},\ \theta + \alpha,\ \bar{\theta} + \bar{\alpha}) \tag{3.6.4}$$

We can use the form (3.6.4) of a finite supersymmetry transformation to obtain a differential operator form for the generators Q and $\bar{Q}$. Consider for example the transformation

$$S(0,\alpha,0)\ \Phi(0,0,\bar{\theta}) = \Phi(-i\alpha\ \sigma_\mu\bar{\theta},\alpha,\bar{\theta}) \quad .$$

Using the explicit form (3.6.1), this becomes

$$e^{-i\alpha Q}\ \Phi(0,0,\bar{\theta}) = \Phi(-i\alpha\ \sigma_\mu\bar{\theta},\alpha,\bar{\theta})$$

We expand this around α, keeping only α-terms of order 1:

$$(1 - i\alpha\ Q)\ \Phi(0,0,\bar{\theta}) = \Phi(-i\alpha\ \sigma_\mu\bar{\theta},\alpha,\bar{\theta})$$

and hence

$$-i\ Q = \frac{1}{\alpha}\left[\ \Phi(-i\alpha\ \sigma_\mu\bar{\theta},\alpha,\bar{\theta}) - \Phi(-i\alpha\ \sigma_\mu\bar{\theta},0,\bar{\theta})\ \right]$$

$$- i\sigma_\mu\ \bar{\theta}\left(\frac{\Phi(-i\alpha\sigma_\mu\bar{\theta},0,\bar{\theta}) - \Phi(0,0,\bar{\theta})}{-i\alpha\sigma_\mu\bar{\theta}}\right)$$

Taking the limit as $\alpha \longrightarrow 0$, we get Q as a differential operator:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\ \sigma^\mu{}_{\alpha\dot{\beta}}\ \bar{\theta}^{\dot{\beta}}\ \partial_\mu \tag{3.6.5)(a}$$

If we use other special transformations, we get the same form for Q_α; so (3.6.5)(a) is the most general form of Q_α. $\bar{Q}_{\dot{\alpha}}$ can similarly be found to be:

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\ \theta^{\beta}\ \sigma^{\mu}{}_{\beta\dot{\alpha}}\ \partial_{\mu} \qquad (3.6.5)(b)$$

Finally, we write an infinitesimal supersymmetry variation of a superfield $\Phi(x,\theta,\bar{\theta})$ in the obvious form:

$$\begin{aligned} \delta_S\Phi(x,\theta,\bar{\theta}) &= \alpha\ Q + \bar{Q}\ \bar{\alpha}\ \Phi(x,\theta,\bar{\theta}) \\ &= (\alpha\frac{\partial}{\partial\theta} + \bar{\alpha}\frac{\partial}{\partial\bar{\theta}} - i(\alpha\sigma_{\mu}\ \bar{\theta} - \theta\ \sigma_{\mu}\ \bar{\alpha})\partial^{\mu}\ )\ \Phi \qquad (3.6.6) \end{aligned}$$

The covariant derivative of a superfield is an operator which anticommutes with the infinitesimal variation:

$$D_{\alpha}\ \delta_S\Phi = -\delta_S D_{\alpha}\Phi \qquad (3.6.7)$$

D_{α} and $\bar{D}_{\dot{\alpha}}$ are easily found to be

$$D_{\alpha} = \partial_{\alpha} + i\ \sigma^{\mu}{}_{\alpha\dot{\beta}}\ \bar{\theta}^{\dot{\beta}}\ \partial_{\mu} \qquad (3.6.8)(a)$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\ \theta^{\beta}\ \sigma^{\mu}{}_{\beta\dot{\alpha}}\ \partial_{\mu}\quad . \qquad (3.6.8)(a)$$

Instead of the parametrisation (3.6.6) of the superfield, we can arbitrarily shift the whole parametrisation of the superspace. Two useful examples of such a reparametrisation are the so-called left and right representations of the superfield given by:

$$\begin{aligned} \Phi(x,\theta,\bar{\theta}) &= \Phi_L(x_{\mu} + i\theta\sigma_{\mu}\ \bar{\theta},\theta,\bar{\theta}) \\ &= \Phi_R(x_{\mu} - i\theta\sigma_{\mu}\ \bar{\theta},\theta,\bar{\theta}) \qquad (3.6.9) \end{aligned}$$

If we now define the corresponding supersymmetry transformation as

$$S_L = e^{i(\theta Q\ -\ x.P)}\ e^{i\bar{Q}\ \bar{\theta}}$$

$$S_R = e^{i(\bar{Q}\,\bar{\theta}\; - \; x.P)}\, e^{i\theta\, Q} \;, \tag{3.6.10}$$

the infinitesimal supersymmetry variation becomes

$$\delta\Phi_L = (\alpha \frac{\partial}{\partial\theta} + \bar{\alpha} \frac{\partial}{\partial\bar{\theta}} + 2i\; \theta\; \sigma_\mu\; \bar{\alpha}\partial^\mu\;)\;\Phi \tag{3.6.11(a)}$$

$$\delta\Phi_R = (\alpha \frac{\partial}{\partial\theta} + \bar{\alpha} \frac{\partial}{\partial\bar{\theta}} - 2i\; \alpha\; \sigma_\mu\; \bar{\theta}\; \partial^\mu\;)\;\Phi \tag{3.6.11(b)}$$

and the three differently parametrised fields Φ, Φ_L, Φ_R transfrom identically under supersymmetry transformations in the corresponding parametrisation. The covariant derivatives get reparametrised as well:

$$D_L = \frac{\partial}{\partial\theta} + 2i\sigma_\mu\; \bar{\theta}\; \partial^\mu \tag{3.6.12(a)}$$

$$\bar{D}_L = - \frac{\partial}{\partial\bar{\theta}} \tag{3.6.12(b)}$$

$$D_R = \frac{\partial}{\partial\theta} \tag{3.6.12(c)}$$

$$\bar{D}_R = - \frac{\partial}{\partial\bar{\theta}} - 2i\; \theta\; \sigma_\mu\; \partial^\mu \tag{3.6.12(d)}$$

The left-right parametrisation makes the derivation of the scalar or chiral superfield particularly simple. The so-called left-handed chiral superfield $\Phi(x,\theta,\bar{\theta})$ arises out of the covariant condition $\bar{D}\Phi = 0$. In the left representation, this is $\bar{D}_L\Phi_L = -\frac{\partial}{\partial\bar{\theta}}\,\Phi_L = 0$, and hence Φ_L is a function of x and θ only. The Taylor series is then:

$$\Phi_L(x,\theta) = A(x) + \theta^\alpha\; \Psi_\alpha(x) + \theta^\alpha\theta^\beta\varepsilon_{\alpha\beta}F(x) \quad , \tag{3.6.13}$$

and the infinitesimal variation becomes

$$\delta\Phi_L(x,\theta) = \delta A + \theta\delta\Psi + \theta\theta\delta F \quad . \tag{3.6.14}$$

We can compute (3.6.14) explicitly using (3.6.11)(a). Comparing coefficients of θ in the result and in (3.6.14) gives us the transformation behavior of the multiplet: it is exactly the transformation behavior of eq.(3.4.25), so we have recovered the chiral multiplet. A similar treatment for $D\Phi = 0$ gives then the anti-chiral or right-handed chiral multiplet.

Also, if we have no covariant condition for the field, then the general expansion is

$$V(x,\theta,\bar{\theta}) = C(x) + \theta\Phi(x) + \bar{\theta}\,\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\,\bar{\theta}\,N(x) + \theta\bar{\theta}\,V(x)$$
$$+ \theta\theta\bar{\theta}\,\bar{\lambda}(x) + \bar{\theta}\,\bar{\theta}\,\bar{\theta}\,\Psi(x) + \theta\,\theta\,\bar{\theta}\,\bar{\theta}\,D(x) \qquad (3.6.15)$$

The vector superfield V has to satisfy the reality condition $V = V^{\dagger}$, which somewhat reduces the degrees of freedom of (3.6.15). To obtain the vector multiplet as derived in the component field approach, component fields have to be redefined and rearranged. The result is[7]:

$$V(x,\theta,\bar{\theta}) = C(x) + i\theta\chi(x) - i\overline{\theta\chi}(x) + \frac{i}{2}\,\theta\theta(M(x) + iN(x))$$
$$- \frac{i}{2}\,\bar{\theta}\,\bar{\theta}\,(M(x) - iN(x)) - \theta\sigma^{\mu}\bar{\theta}\,A_{\mu} + i\,\theta\theta\bar{\theta}(\bar{\lambda}(x) +$$
$$+\frac{i}{2}\,\bar{\sigma}^{\mu}\partial_{\mu}\chi(x)) - i\,\bar{\theta}\,\bar{\theta}\,\bar{\theta}\,(\lambda(x) + \frac{i}{2}\,\sigma^{\mu}\partial_{\mu}\,\bar{\chi}(x))$$
$$+ \frac{1}{2}\,\theta\,\theta\,\bar{\theta}\,\bar{\theta}\,(D(x) + \frac{1}{2}\,\Box C(x) \qquad (3.6.16)$$

Now all the component fields are real (because the of reality condition).

A general supersymmetric Lagrangian can now be built by using the two kinds of superfields, scalar and vector superfields, as well as products of these fields. Any product of (N = 1) superfields will be again one of the two types as is clear from the fact that a Taylor expansion of a field which exhibits N = 1 supersymmetry cannot yield a higher field than a vector field. The general supersymmetric Lagrange density is thus of the form

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_D,$$

where $\mathcal{L}_F$ denotes the F-component of a scalar density and $\mathcal{L}_D$ the D-component of a vector density respectively.
Integration in superspace is defined through the relations

$$\int d\theta = 0$$

and

$$\int d\theta \cdot \theta = 1 \quad .$$

An invariant Lagrangian can then be obtained as follows:

$$L = \int d^4x \; (\int d^2\theta \; d^2\bar{\theta} \; \mathcal{L}_D + \int d^2\theta \; \mathcal{L}_F + \text{Hermitean conjugate}).$$

3.7 Concluding Remarks

At this point the basic framework for supersymmetric field theories has been laid. Now there is room to build different supersymmetric models and compare their predictions with the requirements of ordinary gauge field theories (if not with nature!), like the absence of quadratic divergences, finiteness, etc.

The simplest supersymmetric model is the N = 1 Wess-Zumino model, the Lagrangian of which we have shown and which is very well understood. It is renormalisable and essentially free of divergences. Extended (N > 1) supersymmetric models are less well understood- in particular, the superspace approach is less well developed for theories of extended supersymmetry. The most popoular theory of extended supersymmetry is the N = 4 supersymmetric Yang-Mills theory which is finite.

The process by which one attempts to render supersymmetry natural (i.e. break the degeneracy in mass of the supermultiplets which have equal bosonic and fermionic content, contrary to observations), spontaneous supersymmetry breaking, in particular, is dependent on the model, i.e. on the Lagrangian we choose for a particular model. Supersymmetry is spontaneously broken if the auxiliary fields (the fields of highest mass dimension in the multiplet, which transform into a total divergence under supersymmetry transformations) have nonvanishing vacuum expectation values. The fermionic partner of the auxiliary field receives a vanishing expectation value- it is the massless Goldstone <u>fermion</u> of broken supersymmetry. In each model, the supersymmetry breaking mechanism is different, depending on the field conntent of the corresponding Lagrangian.

The discussion of <u>global</u> supersymmetry in the superspace approach lends itself to generalisation to <u>local</u> supersymmetry: Global supersymmetry was formulated in superspace with a spinor parameter θ which transformed as a two-spinor and was independent of space-time coordinates. In local supersymmetry , this parameter becomes dependent on space-time; this means that the algebra (3.4.22) is space-time dependent: the product of two supersymmetry transformations now gives a space-time translation that differs from point to point. The fact that this is equivalent to a general coordinate transformation leads to gravity appearing in supersymmetric theories: the birth of supergravity.

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Acknowledgements

I am indebted to and would therefore like to thank:

- The CSIR for financial assistance
- My supervisor, Professor Raoul Viollier, for motivation, his willingness to help in any kind of situation, his patience and for financial support (which includes support to attend the "Spring School on Superstrings" held at Trieste, Italy in 1987)
- Professor Johann Rafelski for always encouraging exitement in physics
- Dr. Petr Zimak, Dr. Rudolph Tegen, Andrew Stoddart and Detlof von Oertzen for helpful discussions and hints
- Allard Schnabel for efficiently eliminating any problems in connection with computing and word-processing facilities
- The whole theory group for a stimulating atmosphere
- Nan Warner for typing parts of the manuscript
- Detlof von Oertzen for moral support

Appendix A: Notation and Conventions

The Minkowski metric of the signature

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{A.1}$$

was used throughout, so that

$$p^{\mu} = i\hbar \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \tag{A.2}$$

$$p_{\mu} = i\hbar \left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \tag{A.3}$$

$$x^{\mu} = (t, \vec{x}), \tag{A.4}$$

$$x_{\mu} = (t, -\vec{x}) \tag{A.5}$$

and $[p^{\mu}, x^{\mu}] = i\eta^{\mu\nu}$. (A.6)

The Pauli matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.7)(a}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{A.7)(b}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.7)(c}$$

$$\sigma_0 = 1 \tag{A.7)(d}$$

and the spacetime index on the Pauli matrices is defined as

$$\sigma^{\mu} = \bar{\sigma}_{\mu} = (1, \vec{\sigma}) \tag{A.8}$$

$$\sigma_{\mu} = \bar{\sigma}^{\mu} = (1, -\vec{\sigma}) \quad . \tag{A.9}$$

Representations of the Lorentz group:

Recall from eq. (2.3.18) the representation of the Lorentz group:

$$M_i = \tfrac{1}{2}(J_i + iK_i) = \tfrac{1}{2}(\varepsilon_{kji}M_{kj} + iM_{0i}) \tag{A.10}$$

$$N_i = \tfrac{1}{2}(J_i - iK_i) = \tfrac{1}{2}(\varepsilon_{kji}M_{kj} - iM_{0i}) \quad . \tag{A.11}$$

The ($\frac{1}{2}$0) representation corresponds to

$$r(\vec{M}) = \tfrac{1}{2}\vec{s} \tag{A.12}$$

$$r(\vec{N}) = 0, \tag{A.13}$$

so that

$$r(M_{0i}) = -\tfrac{1}{2}i\sigma^i \equiv \tfrac{1}{2}\sigma^{0i} \tag{A.14}$$

$$r(\varepsilon_{kji}M_{kj}) = \tfrac{1}{2}\sigma^i \equiv \tfrac{1}{2}\sigma^{kj} \quad . \tag{A.15}$$

Note that

$$\tfrac{1}{2}[\tfrac{1}{2}i(\sigma^k\bar{\sigma}^j - \sigma^j\bar{\sigma}^k)] = \frac{i}{4}\,(-2\sigma^k\sigma^j)$$

$$= \tfrac{1}{2}\varepsilon_{kji}\sigma^i \tag{A.16}$$

and

$$\frac{i}{4}(\sigma^0\bar{\sigma}^i - \sigma^i\bar{\sigma}^0) = -\frac{i}{2}\,\sigma^0\sigma^i$$

$$= -\frac{i}{2}\,\sigma^i, \tag{A.17}$$

so that we can write generally

$$\sigma^{\mu\nu} = \frac{i}{2}\,(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu) \quad . \tag{A.18}$$

Similarly, the $(0\frac{1}{2})$ representation is

$$r(\vec{J}) = 0 \tag{A.19}$$

$$r(\vec{K}) = \tfrac{1}{2}\vec{\sigma} \quad . \tag{A.20}$$

Then

$$r(M_{0i}) = \tfrac{1}{2} i\sigma^{i} \equiv \tfrac{1}{2}\bar{\sigma}^{0i} \tag{A.21}$$

$$r(\varepsilon_{ijk} M_{kj}) = \tfrac{1}{2}\sigma^{i} \equiv \tfrac{1}{2}\bar{\sigma}^{kj} \tag{A.22}$$

and we see that we can write

$$\bar{\sigma}^{\mu\nu} = \frac{i}{2}\,(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu}) \quad . \tag{A.23}$$

Spinor notation:

Two-spinor indices are raised and lowered by the Levi-Civita tensor

$$\varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}} = \varepsilon^{\alpha\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^{2} \tag{A.24}$$

in the following sense

$$\Psi_{\alpha} = \varepsilon_{\alpha\beta}\, \Psi^{\beta} \tag{A.25}$$

$$\bar{\Psi}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\, \bar{\Psi}^{\dot{\beta}} \tag{A.26}$$

$$\Psi^{\alpha} = \Psi_{\beta}\, \varepsilon^{\beta\alpha} \tag{A.27}$$

$$\bar{\Psi}^{\dot{\alpha}} = \bar{\Psi}_{\dot{\beta}}\, \varepsilon^{\dot{\beta}\dot{\alpha}} \quad . \tag{A.28}$$

Contraction of spinor indices is defined such that

$$\Psi^2 = \Psi^\alpha \Psi_\alpha = -\Psi_\alpha \Psi^\alpha = \varepsilon_{\alpha\beta} \Psi^\alpha \Psi^\beta \tag{A.29}$$

$$\bar{\Psi}^2 = \bar{\Psi}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}} = -\varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\Psi}^{\dot{\alpha}} \bar{\Psi}^{\dot{\beta}} \tag{A.30}$$

$$\zeta\Psi = \zeta^\alpha \Psi_\alpha = -\zeta_\alpha \Psi^\alpha \tag{A.31}$$

$$\bar{\Psi}\,\bar{\zeta} = \bar{\Psi}_{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} = -\bar{\Psi}^{\dot{\alpha}} \bar{\zeta}_{\dot{\alpha}} \tag{A.32}$$

The complex conjugate spinor is

$$\bar{\Psi}^{\dot{\alpha}} = (\Psi^\alpha)^* \tag{A.33}$$

and

$$(\zeta\,\Psi)^* = (\zeta^\alpha \Psi_\alpha)^* = (\Psi_\alpha)^* (\zeta^\alpha)^* = \bar{\Psi}^{\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} = \bar{\Psi}\,\bar{\zeta} \tag{A.34}$$

APPENDIX B: Overview over some Classical Groups

B.1 Decription and Order of Classical Groups:

1) General linear group GL(n,C) : The most general group of matrices possible is GL(n,C), the general linear group of complex invertible matrices. Since there are no restrictions on the group parameters, the order of the group is

$$r_{GL(n,C)} = 2n^2 .$$

The general linear group of real matrices obviously has order

$$r_{GL(n,R)} = n^2 .$$

2) The special linear group SL(n,C) contains those matrices of GL(n,C) which have a determinant of +1. This condition gives two constraints for the parameters, so

$$r_{SL(n,C)} = 2(n^2 - 1)$$

$$r_{SL(n,R)} = n^2 - 1 .$$

We have

$$GL(n,C) \supset GL(n,R) \supset SL(n,R)$$

$$GL(n,C) \supset SL(n,C) \supset SL(n,R)$$

3) The Unitary group U(n) consists of all complex n×n matrices which satisfy

$$u\, u^\dagger = 1 = u^\dagger\, u .$$

From the constraining equation for the parameters,

$$\sum_t a_{it}\, a^*_{tj} = \delta_{ij} ,$$

we see we have n^2 constraints for $2n^2$ parameters, hence

$$r_{U(n)} = n^2 .$$

The constraining equation also says that

$$|a_{ij}|^2 < 1,$$

so that the parameter domain is bounded - so U(n) is compact.

Obviously, $GL(n,C) \supset U(n)$.

The pseudo-unitary group U(p,q) leaves intact the form

$$-u_1 u_1^* - u_2 u_2^* - \ldots - u_p u_p^* + u_{p+1} u_{p+1}^* + \ldots u_{p+q} u_{p+q}^* ,$$

or

$$u\ g\ u^\dagger = g.$$

Since this does not introduce any extra constraints on the parameters as compared to the restriction on U(p+q) (only the signature of the matrices is changed), we get

$$r_{U(p,q)} = n^2 .$$

Of course

$$GL(p+q,C) \supset U(p,q) .$$

4) Special unitary groups SU(n) are unitary matrices with determinant of +1, hence

$$r_{SU(n)} = n^2 - 1.$$

It is clear that

$$SU(n) = U(n) \cap SL(n,C).$$

5) **The Orthogonal group O(n,C)** consists of all complex orthogonal matrices A with

$$A^T A = 1.$$

From this condition it follows that det A = ± 1, thus splitting the group into two disconnected pieces. Since the constraint equation is symmetric, we get

$$r_{O(n,C)} = 2n^2 - n(n+1) = n(n-1).$$

and

$$r_{O(n,R)} = n/2\ (n-1).$$

Also, O(n,C) and its subgroups are compact and

$$SO(n,C) = SL(n,C) \cap O(n,C).$$

6) **The symplectic group Sp(2n,C)** is the group of transformations which leave the skew-symmetric form

$$\{\vec{x},\vec{y}\} = x_i\, g_{ik}\, y_k \tag{B.1}$$

invariant, where $g_{ik} = -g_{ki}$. The skew symmetry of the matrix means that

$$g^T = -g$$

and so in n dimensions, taking the determinant on both sides, we get

$$\det g = (-1)^n \det g.$$

From this we see that the symplectic group can only be defined for even dimensions 2n.
The condition that (B.1) be invariant under a transformation a means

$$a^T g\, a = g. \tag{B.2}$$

We can select a symplectic coordinate basis in R_{2n} such that

$$\vec{x} = \sum_{i=1}^{n} x_i\, e_i + x_{i'}\, e_{i'}\ .$$

with $\{e_i, e_j\} = \{e_{i'}, e_{j'}\} = 0,$

$$\{e_i, e_{j'}\} = 1.$$

Then the form (B.1) becomes $\{\vec{x},\vec{y}\} = \epsilon_{ij}\, x_i\, y_j$

with

$$\epsilon_{ij} = \begin{cases} 1 & i = \alpha,\ j = \alpha' \\ -1 & i = \alpha',\ j = \alpha \\ 0 & \text{otherwise} \end{cases}$$

The form of the matrix $\epsilon = \epsilon_{ij}$ then depends on the order in which we write the indices:

$$\epsilon = \begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix}$$

order of indices: $1, 1', 2, 2', \ldots, n, n'$

$$\text{or} \quad \left(\begin{array}{cccccc|cccccc} & & & & & & 1 & & & & & \\ & & & & & & & 1 & & & & \\ & & & & & & & & 1 & & & \\ & & & & & & & & & \cdot & & \\ & & & & & & & & & & \cdot & \\ & & & & & & & & & & & 1 \\ \hline -1 & & & & & & & & & & & \\ & -1 & & & & & & & & & & \\ & & -1 & & & & & & & & & \\ & & & \cdot & & & & & & & & \\ & & & & \cdot & & & & & & & \\ & & & & & -1 & & & & & & \end{array}\right)$$

order of indices: 1,2, ... ,n,1′,2′,...,n′

$$\text{or} \quad \left(\begin{array}{cccc|cccc} & & & & & & & 1 \\ & & & & & & \cdot & \\ & & & & & \cdot & & \\ & & & & 1 & & & \\ \hline & & & -1 & & & & \\ & & \cdot & & & & & \\ & \cdot & & & & & & \\ -1 & & & & & & & \end{array}\right)$$

order of indices:1,2,...,n,n′,...,1′

The constraining equation (B.2) becomes in the symplectic basis

$$a^T \in a = \in$$

and is antisymmetric in the components; so we are left with

$$r_{Sp(2n,C)} = 2n(2n+1),$$

$$r_{Sp(2n,R)} = n(2n+1).$$

The unitary symplectic group is

$$USp(2n,C) = U(2n) \cap Sp(2n,C).$$

The unitarity condition gives another $4n^2$ constraints- however, one can split U into a symmetric and an antisymmetric part , the symmetric part giving n(2n+1), the antisymmetric part giving n(2n-1) constraints. The n(2n-1) antisymmetric constraints are contained in the symplectic part- leaving n(2n+1) extra constraints, and so

$$r_{USp(2n,C)} = n(2n+1)$$

$$r_{USp(2n,R)} = n/2\ (2n+1)\ .$$

B.2 Classification of Classical Groups: Dynkin Diagrams

The following analysis applies to semisimple Lie algebras. Such an algebra is of rank l if l of its generators commute. If we now call the l commuting generators H_i (i = 1,2, ..., l) and the remaining N-l generators E_α, (α = N-l, ...,N), then we can write the algebra in a suitably normalised form (called the Cartan-Weyl basis) as

$$[H_i,H_k] = 0 \qquad i,k = 1,2, \ldots, l \qquad \text{(B.3)(a)}$$

$$[H_i,E_\alpha] = \alpha_i E_\alpha \qquad \text{(B.3)(b)}$$

$$[E_\alpha,E_\beta] = N_{\alpha\beta}\ E_{\alpha+\beta} \qquad \alpha+\beta \neq 0 \qquad \text{(B.3)(c)}$$

$$[E_\alpha,E_{-\alpha}] = \sum_i \alpha_i\ H_i \qquad \text{(B.3)(d)}$$

The α_i are called <u>roots</u> of the semisimple Lie algebra. We can interpret them as the covariant components of a vector (the radix vector) of an l-dimensional subspace of the N-dimensional representation space. To obtain a classification scheme, we still need the following:

1) <u>Definition:</u>

The scalar product of radix vectors is defined as:

$$(\alpha,\beta) = \sum_i \alpha_i \beta_i \ .$$

2) Lemma 1:

If α is a root of (B.3), then $-\alpha$ is also a root.

3) Lemma 2:

If α, β are roots of (B.3), then $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

4) Lemma 3:

If α, β are roots of (B.3), then $\beta - \frac{2\alpha\ (\alpha,\beta)}{(\alpha,\alpha)}$ is also a root.

5) The angle between two roots is, according to (1),

$$\cos\phi = \frac{(\alpha,\beta)}{\sqrt{(\alpha,\alpha)(\beta,\beta)}}$$

and can thus be

$$\phi = 0°,\ 30°,\ 45°,\ 60°,\ 90°.$$

The ratio of lengths of roots is given by

$$k = \sqrt{\frac{(\alpha,\alpha)}{(\beta,\beta)}}$$

and we have the possibilities:

ϕ	k^2
30°	3
45°	2
60°	1
90°	undetermined

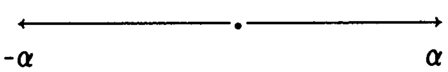
(B.4)

These statements can be deduced from the three Lemmas above, whose proof is far too laborious to give here.

As an example, we draw the vector diagrams of a few algebras:

1) l = 1:

From (B.3)(a) and Lemma 1, we see that there are only two roots ,α and $-\alpha$. Hence ϕ = 0°, and the diagram is



2) l = 2:

According to the different choices of angle, we get different vector diagrams. We illustrate the case of ϕ = 30°:

Eq. (B.3)(b) gives us two roots. We choose $\alpha = (1,0)$ and suppose $\beta>\alpha$. Then from (B.4) we see that $(\beta,\beta) = 3$ and thus $\beta = (\frac{3}{2},\frac{\sqrt{3}}{2})$. From Lemma 3, $\beta - \frac{2\alpha\ (\alpha,\beta)}{(\alpha,\alpha)} = \beta - \alpha$, and hence also $\alpha - \beta$, are also roots of the algebra. Continuing in this way, we obtain a set of 12 vectors, so the vector diagram looks as follows:

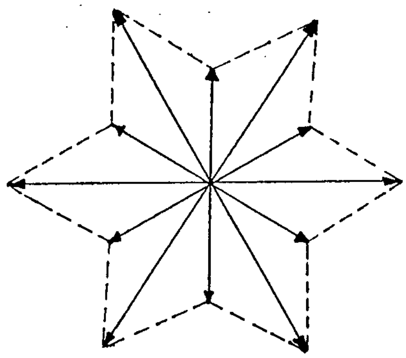
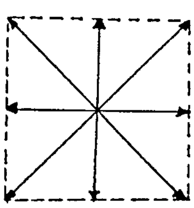
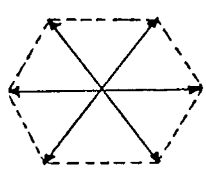


Figure B.1: Root vector diagram corresponding to $l=2$, $\phi=30°$.

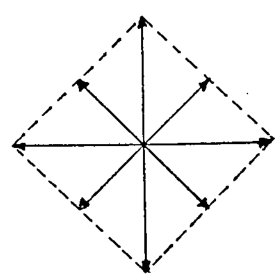
The other root vector diagrams corresponding to $\phi=45°$, $60°$, $90°$ are:



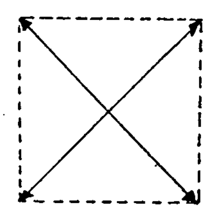
$\phi=45°$



$\phi=60°$



$\phi=90°$



$\phi=90°$

For each of the angles we obtain a different type of diagram which corresponds to a different Lie algebra. The more commonly known of these diagrams are sometimes labelled A_l, B_l, C_l, D_l - this labelling is due to Cartan and classifies the classical Lie algebras of rank l - there are also some special Lie algebras labelled E_l, F_l etc., but we will not consider these here.

If $l\geq 2$, these vector diagrams cannot be drawn anymore since of course the dimensions of the vector spaces get larger. However, if we denote by $\vec{e}_i$ the basis vectors of the l-dimensional space (for example, in 2 dimensions, $\vec{e}_1 = (1,0)$, $\vec{e}_2 = (0,1)$), then we can generalise the procedure illustrated above and write down the root vectors for the four different types of diagram:

Table B.5: Roots of Classical Groups

type	order	root vectors	corresponding Lie algebra
B_l	$l(2l+1)$ (including l null vectors)	$\pm\vec{e}_i, \pm\vec{e}_i \pm\vec{e}_j$ $i,j = 1, \ldots, l$	$SO(2l+1)$
C_l	$l(2l+1)$ (including l null vectors)	$\pm 2\vec{e}_i, \pm\vec{e}_i \pm\vec{e}_j$ $i,j = 1, \ldots, l$	$Sp(2l)$
D_l	$l(2l-1)$	$\pm\vec{e}_i \pm\vec{e}_j$ $i,j = 1, \ldots, l$	$SO(2l)$
A_l	$l(l+2)$ (including l null vectors)	$\vec{e}_i - \vec{e}_j = \vec{e}_k$ $i,j = 1, \ldots, l+1$ project e_k onto an l-dimens. subspace	$SU(l+1)$

The correspondence between the types of algebra A_l, B_l, etc. and the groups $SO(2l+1)$, $Sp(2l)$ etc. is obtained by comparing the orders of the groups, taking into account in each case the rank of the algebra.

To obtain a simple way to illustrate these higher dimensional root diagrams, we rely on the fact that we can classify the algebra with only l roots. We define a <u>positive</u> root as a root whose first non-zero component (in any arbitrary basis) is positive. Further, a <u>simple</u> root is one which cannot be expressed as a sum of positive roots of the algebra.

For example, consider B_2:

According to (B.5), the root vectors are

$$(1,0),\ (-1,0),\ (0,1),\ (0,-1),\ (1,1),\ (1,-1),\ (-1,1),\ (-1,-1),$$

The positive roots are

(1,0), (0,1), (1,1), (1,-1)

and the simple roots are

(0,1) and (1,-1),

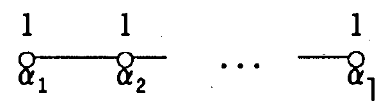
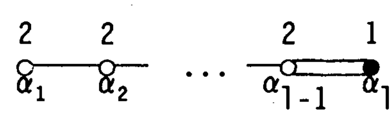
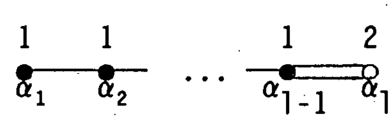
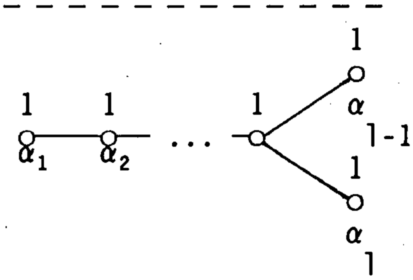
as can easily be checked. There are exactly 1 simple roots for a semi-simple Lie algebra of rank 1.

For the angles between simple roots there are only the following possibilities:

ϕ	$\frac{(\beta,\beta)}{(\alpha,\alpha)}$
120°	1
135°	2
150°	3
90°	undetermined

<u>Dynkin's rule</u> now gives a simple method to draw higher dimensional root diagrams: each root is represented by a small circle- the circle representing the roots with the the shortest length are filled in. Since there are at most two different lengths of roots for semisimple Lie algebras, this does not mean any loss of information. The circles are connected with one, two or three lines according to whether the angle between the roots is 120°, 135°, or 150°. If the angle is 90°, the circles are not connected. We can then extend table (B.5) to include the Dynkin diagram for each kind of algebra:

Table B.6: Dynkin diagrams for classical groups
(in this table the numbers above the circles correspond to the scalar products (α_i,α_i))

type	order	group	Dynkin diagram
A_l	$l(l+2)$	$SU(l+1)$	
B_l	$l(2l+1)$, $l \geq 2$	$SO(2l+1)$	
C_l	$l(2l+1)$, $l \geq 3$	$Sp(2l)$	
D_l	$l(2l-1)$, $l \geq 4$	$SO(2l)$	

We illustrate this procedure for B_2:

The simple roots are:

$$\alpha = (0,1)$$

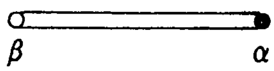
$$\beta = (1,-1)$$

Now $(\alpha,\alpha) = 1$ and $(\beta,\beta) = 2$, so

$$\cos\phi = \frac{(\alpha,\beta)}{\sqrt{(\alpha,\alpha)(\beta,\beta)}} = -\frac{1}{\sqrt{2}}$$

and hence $\phi = 135°$

and the Dynkin diagram is



.

We now look at C_3:

The root vectors are:

(2,0,0) , (0,2,0) , (0,0,2) , (-2,0,0) , (0,-2,0) ,(0,0,-2),
(1,1,0) ,(1,0,1) , (0,1,1) , (-1,1,0) ,(-1,0,1) , (0,-1,1),
(1,-1,0) , (1,0,-1),(0,1,-1) ,(-1,-1,0) , (-1,0,-1) , (0,-1,-1),

the simple roots are

$$\alpha_3 = (0,0,2),\ \alpha_1 = (1,-1,0),\ \alpha_2 = (0,1,-1)$$

so

$$\cos \phi_{\alpha_1,\alpha_2} = -1/2 \Rightarrow \phi_{\alpha_1,\alpha_2} = 120°.$$

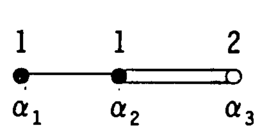
and $\dfrac{(\alpha_1,\alpha_1)}{(\alpha_2,\alpha_2)} = 1.$

$$\cos \phi_{\alpha_2,\alpha_3} = \frac{-1}{\sqrt{2}} \Rightarrow \phi_{\alpha_2,\alpha_3} = 135°.$$

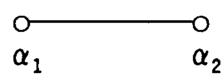
and $\dfrac{(\alpha_3,\alpha_3)}{(\alpha_2,\alpha_2)} = 2.$

$\cos \phi_{\alpha_1,\alpha_3} = 0.$

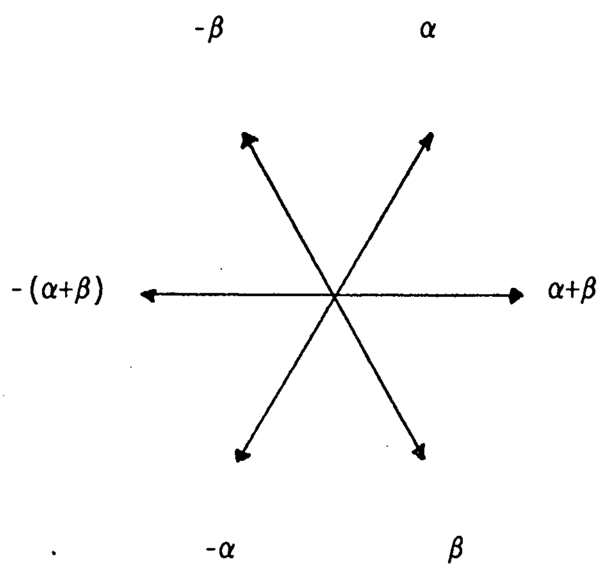
Thus the Dynkin diagram is:



It is now easy to deduce from any Dynkin diagram the Lie algebra of the correponding group. We illustrate this with the diagram for $SU(3) \approx SU(2+1)$. The Dynkin diagram is



from which we see that there are two simple roots of equal length. Because $(\alpha,\beta) = -1$, $\alpha - 2\beta \frac{(\alpha,\beta)}{(\beta,\beta)} = \alpha+\beta$ - therefore the positive roots are α, β, $\alpha+\beta$. Also, from lemma 1, $-\alpha$, $-\beta$, $-(\alpha+\beta)$ are roots. The root diagram is thus



There are no more roots since the order of $SU(3)$ is $l(l+2)=8$, which includes $l=2$ null vectors.

Normalising the positive roots such that

$$\sum_{\alpha_i} \alpha_i \alpha_j = \delta_{ij} ,$$

i.e. $\sum_{\alpha} \alpha_i \alpha_i = n$ $\qquad$ n = no. of components in α $= l = 2$ here

we obtain

$$\alpha_1 = \frac{1}{2\sqrt{3}}(1,\sqrt{3})$$

$$\alpha_2 = \frac{1}{2\sqrt{3}}(1,-\sqrt{3})$$

$$\alpha_3 = \alpha_1 + \alpha_2 = \frac{1}{\sqrt{3}}(1,0)$$

For the Lie algebra we still need to know the normalisation factor $N_{\alpha,\beta}$. We use the so-called Cartan-Weyl normalisation:

Corresponding to a row of roots containing β:

$$\beta+i\alpha,\ \beta+(j-1)\alpha,\ \ldots,\ \beta,\ \ldots,\ \beta-k\alpha,$$

the normalisation constants are given by

$$N_{\alpha\beta}\, N_{-\alpha,\alpha+\beta} = \tfrac{1}{2}\, j(k+1)\ (\alpha,\alpha)$$

with the phase convention

$$N_{\alpha\beta} = -N_{\beta\alpha} = -N_{-\alpha,-\beta} = N_{-\beta,-\alpha}\ .$$

In our case, the row containing β is

$$\beta,\ \beta+\alpha\ ,$$

so $j=1$ and $k=0$.

We thus get

$$N_{\alpha\beta}\, N_{-\alpha,\alpha+\beta} = 1/6\ ,$$

so that we have

$$N_{\alpha\beta} = N_{-\alpha,\alpha+\beta} = 1/\sqrt{6} \quad .$$

Consequently the Lie algebra is

$$[H_1, E_{\pm\alpha}] = \pm \frac{1}{2\sqrt{3}} E_{\pm\alpha} \qquad [H_2, E_{\pm\alpha}] = \pm \tfrac{1}{2} E_{\pm\alpha}$$

$$[H_1, E_{\pm\beta}] = \pm \frac{1}{2\sqrt{3}} E_{\pm\beta} \qquad [H_2, E_{\pm\alpha}] = \mp \tfrac{1}{2} E_{\pm\beta}$$

$$[H_1, E_{\pm(\alpha+\beta)}] = \pm \frac{1}{\sqrt{3}} E_{\pm(\alpha+\beta)} \qquad [H_2, E_{\pm(\alpha+\beta)}] = 0$$

$$[E_\alpha, E_{-\alpha}] = \frac{1}{2\sqrt{3}} H_1 + \tfrac{1}{2} H_2 \qquad [E_\beta, E_{-\beta}] = \frac{1}{2\sqrt{3}} H_1 - \tfrac{1}{2} H_2$$

$$[E_{\alpha+\beta}, E_{-(\alpha+\beta)}] = \frac{1}{\sqrt{3}} H_1 \qquad [E_\alpha, E_\beta] = \frac{1}{\sqrt{6}} E_{\alpha+\beta}$$

$$[E_\alpha, E_{\alpha+\beta}] = 0 \qquad [E_\beta, E_{\alpha+\beta}] = 0$$

$$[E_\alpha, E_{-(\alpha+\beta)}] = -\frac{1}{\sqrt{3}} E_{-\beta} \qquad [E_\beta, E_{-(\alpha+\beta)}] = \frac{1}{\sqrt{3}} E_{-\alpha}$$

$$[H_i, H_j] = 0 \tag{B.7}$$

B.3 Construction of Irreducible Representatations of Classical Lie algebras:

We start by defining the Cartan matrix A_{ij}:

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad .$$

The Cartan matrices for our four types of algebra are thus

$$A(A_l) = \begin{pmatrix} 2 & -1 & 0 & . & . & . & 0 & 0 \\ -1 & 2 & -1 & . & . & . & 0 & 0 \\ 0 & -1 & 2 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & 0 & 2 & -1 \\ . & . & . & . & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$A(B_l, C_l) = \begin{pmatrix} 2 & -1 & 0 & . & . & . & 0 & 0 \\ -1 & 2 & -1 & . & . & . & 0 & 0 \\ 0 & -1 & 2 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & 0 & 2 & -2 \\ . & . & . & . & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$A(D_l) = \begin{pmatrix} 2 & -1 & 0 & . & . & . & 0 & 0 \\ -1 & 2 & -1 & . & . & . & 0 & 0 \\ 0 & -1 & 2 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & 2 & -1 & -1 \\ . & . & . & . & 0 & -1 & 2 & 0 \\ . & . & . & . & 0 & -1 & 0 & 2 \end{pmatrix}$$

The Cartan matrix gives a nice book-keeping device to check the structure of the roots of the algebra: it gives the scalar products between all the simple roots.

To find an irreducible representation of an algebra, we need to find a method of distinguishing and consistently labelling various representations. In order to achieve this, we introduce the concept of the <u>weight</u> of a state:

A Lie algebra of rank l has l commuting operators H_l to which we can construct a simultaneous eigenfunction such that

$$H_{\alpha_i} |u\rangle = \Lambda_i |u\rangle$$

Then the l eigenvectors Λ_i , just like the l simple root vectors of the algebra, form a vector in an l-dimensional space, referred to as the weight vector of the vector $|u\rangle$. Also, we see that if to $H_\alpha |u\rangle$ belongs the weight Λ, then to the vector $E_\beta |u\rangle$ belongs the weight $(\Lambda+\beta)$:

$$H_{\alpha_i} E_\beta |u\rangle = ([H_{\alpha_i}, E_\beta] + E_\beta H_{\alpha_i}) |u\rangle$$

$$= (\beta_i + \Lambda_i) E_\beta |u\rangle \quad .$$

In a straightforward analogy to root vectors, a weight is said to be positive if its first nonzero component is positive, and the highest weight is the weight whose first nonvanishing component is greater than that of all the other weights in the particular representation. The representation space R_ϕ of a representation ϕ is then subdivided into a number of different <u>layers</u>, all of which have different weights, i.e. the representation space R_ϕ is a direct procuct of weight subspaces R_ϕ^Λ corresponding to differnt weights Λ.

To build up weight patterns, we need the following:

<u>Definition:</u>

If Λ is the greatest weight and M any other weight in the representation, then

$$\delta(\Lambda) = 2 \sum_{\alpha \text{ simple}} \Lambda_\alpha$$

and the quantitiy

$$\gamma(M) = \tfrac{1}{2} (\delta(\Lambda) - \delta(M))$$

counts the number of simple roots that one has to subtract from Λ to arrive at M. By subtracting simple roots from Λ one by one, we arrive at the different layers of a representation:

$$\Delta_\phi = \Delta_\phi^0 \ \Delta_\phi^1 \ \ \ldots . \ \Delta_\phi^T \quad .$$

The representation Δ_ϕ is built up from layers Δ_ϕ^k for which $\gamma(M) = k$, and T is the number of layers minus one. T is called the <u>height</u> of the representation and is given by theorem 5.
A certain layer may contain more than one weight vector: The layer k may have a multiplicity $S_k(\phi)$ greater than one. Obviously, the dimension of the representation is the sum of the multiplicities of the different layers:

$$S_0(\phi) + S_1(\phi) + \ldots + S_T(\phi) = N(\phi).$$

The number of layers can also be calculated directly:

Since, if $\Lambda \in \Delta_\phi^0$ and $\Lambda' \in \Delta_\phi^T$, then

$$\delta(\Lambda) + \delta(\Lambda') = 0,$$

and we have

$$\delta(\Lambda) - \delta(\Lambda') = 2T(\phi),$$

as can be seen from the definitions above.

<u>Theorem 1:</u>

A sequence of weights $M_1, M_2, \ldots, M_k$ with

$$M_2 - M_1 = M_3 - M_4 = \ldots = M_k - M_{k-1} = \alpha$$

and $M_1 - \alpha$, $M_k + \alpha \in \Delta_\phi$, is called an α series of weights. Arranging the weights in the order

$$\Lambda - r\alpha, \ldots, \Lambda - q\alpha,$$

we have

$$\frac{2(\Lambda,\alpha)}{(\alpha,\alpha)} = r - q.$$

This theorem gives us a method to determine whether a given vector is a weight or not.

Theorem 2:

For any weight Λ and root α, $\frac{2\alpha(\Lambda,\alpha)}{(\alpha,\alpha)}$ is an integer and $\Lambda - \frac{2\alpha(\Lambda,\alpha)}{(\alpha,\alpha)}$ is a weight.

Theorem 3:

The multiplicity of the highest weight of a representation is 1: the highest weight is simple.

Theorem 4:

For a weight to be the highest weight of a representation, all the numbers

$$\Lambda_\alpha = \frac{2(\Lambda,\alpha)}{(\alpha,\alpha)}$$

must be non-negative.

According to this theorem, we can label irreducible representations by labelling the circles of the Dynkin diagrams with the numbers Λ_α.

Theorem 5:

If Λ is the greatest weight of an irreducible representation, then

$$T(\phi) = \sum_{\alpha \text{ simple}} r_\alpha \Lambda_\alpha$$

where r takes the following values:

Table B.8: Values for r to use in Theorem 5

A_n	B_n	C_n	D_n
n	$n(n+1)/2$	n^2	$n(n-1)/2$ $n(n-1)/2$
$(n-1)2$	$(n-1)(n+2)$	$(n-1)(n+1)$	$(n-2)(n+1)$
$(n-2)3$	$(n-2)(n+3)$	$(n-2)(n+2)$	$(n-3)(n+2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(n-k-1)k$	$(n-k+1)(n+k)$	$(n-k+1)(n+k-1)$	$(n-k+1)(n+k-2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(n-1)2$	$(2n-1)/2$	$(2n-2)2$	$(2n-3)2$
n	$2n$	$2n-1$	$(2n-3)2$

We can now calculate all the weights in a given represenation if we are given the greatest weight. Starting from Δ_ϕ^0 , we can find the elements of Δ_ϕ^1 by finding all the simple roots for which $\Lambda - \alpha \in \Delta_\phi$. From theorem 1, we can see that $\Lambda - \alpha \in \Delta_\phi$ if

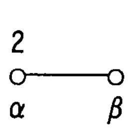
$$\Lambda_\alpha + q \geq 0 \tag{B.9}$$

where $\Lambda - k\alpha \in \Delta_\phi$ for $q \geq -k$

and $\qquad \in \Delta_\phi$ for $q = -k-1$

Doing this for all lower layers, we find the complete irreducible representation.

We now calculate an example: we want to establish the weight pattern for the Dynkin diagram



The 2 above the circle corresponding to the simple root α is now the number

$$\Lambda_\alpha = \frac{2(\Lambda,\alpha)}{(\alpha,\alpha)}$$

designating the highest weight of the irreducible representation.

To find the greatest weight, we put

$$\Lambda = a\alpha + b\beta$$

and calculate

$$\Lambda_\alpha = 2 = \frac{2(a\alpha + b\beta,\alpha)}{(\alpha,\alpha)} .$$

Since $(\alpha,\alpha) = (\beta,\beta) = 1$

and $(\alpha,\beta) = -\frac{1}{2}$,

we get

$$\Lambda_\alpha = 2 = 2a - b.$$

Also

$$\Lambda_\beta = 0 = -a + 2b,$$

and so

$$a = \frac{4}{3}$$

and $b = \frac{2}{3}$

and the greatest weight is

$$\Lambda = \frac{4\alpha + 2\beta}{3} .$$

We now calculate $T(\phi)$ using table (B.8) and theorem 5:

$$T(\phi) = 2\Lambda_\alpha + 2\Lambda_\beta = 4,$$

so there are 5 layers.

Now we investigate whether $\Lambda - \alpha$ is a weight.

From (B.9) and theorem 1, we have

$$\begin{aligned} \Lambda - \alpha &\in \Delta_\phi \quad \text{for } q \geq -1, \\ &\in \Delta_\phi \quad \text{for } q = -2; \end{aligned}$$

but $\Lambda_\alpha = 2$, so $\Lambda_\alpha + q = 1 > 0$

and $\Lambda - \alpha = \frac{\alpha + 2\beta}{3} \in \Delta_\phi$.

Similarly, we check if $\Lambda - \beta \in \Delta_\phi$:

$$\begin{aligned} \Lambda - \beta &\in \Delta_\phi \qquad \text{for } q \geq -1 \\ &\in \Delta_\phi \qquad \text{for } q = -2; \end{aligned}$$

with $\Lambda_\beta = 0$ it follows

$$\Lambda_\beta - 1 = -1 \geq 0$$

and so $\Lambda - \beta$ is not a root.

By continuing in this way and making use of theorem 3, we get the weight pattern

$$\frac{4\alpha + 2\beta}{3}$$

$$\frac{2\alpha + 2\beta}{3}$$

$$\frac{-(2\alpha - 2\beta)}{3} \qquad \frac{2\alpha + 2\beta}{3}$$

$$\frac{-(2\alpha + 2\beta)}{3}$$

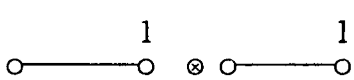
$$\frac{-(2\alpha + 4\beta)}{3}$$

B.4 Kronecker Products

From the simple irreducible representations constructed in the last chapter one can now build higher irreducible representations by forming direct products (Kronecker products) of representations: As we have seen, the direct product of two representations is found by multiplying each "member" of the first representation with each member of the second representation. The dimension of the product space will then be the product of the dimensions of the parent spaces.

Consider as an example the direct product of two

representations of SU(3):



$$= \begin{array}{ccc} \Lambda_1 \bullet & & \bullet \Lambda_1 \\ \Lambda_2 \bullet & \otimes & \bullet \Lambda_2 \\ \Lambda_3 \bullet & & \bullet \Lambda_3 \end{array}$$

with $\Lambda_1 = \dfrac{\alpha + 2\beta}{3}$,

$\Lambda_2 = \dfrac{\alpha - \beta}{3}$,

$\Lambda_3 = \dfrac{2\alpha - \beta}{3}$

This is

$$
= \begin{array}{ccccc}
 & & \cdot\ \Lambda_1 + \Lambda_1 & & \\
 & \Lambda_1 + \Lambda_2\ \cdot & & \cdot\ \Lambda_2 + \Lambda_1 & \\
\Lambda_3 + \Lambda_1\ \cdot & & \Lambda_2 + \Lambda_2\ \cdot & & \cdot\ \Lambda_1 + \Lambda_3 \\
 & \Lambda_3 + \Lambda_2\ \cdot & & \cdot\ \Lambda_2 + \Lambda_3 & \\
 & & \cdot\ \Lambda_3 + \Lambda_3 & &
\end{array}
$$

$$
= \circ\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\circ} \;\oplus\; \overset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\circ \;.
$$

This example illustrates the following facts:

1) Consider an n-dimensional representation being spanned by n basis vectors ξ_1, ξ_2, $\ldots,\xi_n$. The basis is called a canonical basis if the basis vectors are arranged in order of decreasing weight, i.e. if the weight Λ_i belongs to the basis vector ξ_i then

$$\Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_n \;.$$

Now if we form the Kronecker product of the two representations, then we get n^2 basis vectors which can be divided into a symmetric and an antisymmetric set of basis vectors. In our example, n = 3, so our basis is (ξ_1,ξ_2,ξ_3). The basis for the symmetric part of the Kronecker product is then

$$(\xi_1^2,\xi_2^2,\xi_3^2,\xi_1\xi_2 + \xi_2\xi_1,\xi_1\xi_3 + \xi_3\xi_1,\xi_2\xi_3 + \xi_3\xi_2)$$

and that of the antisymmetric part is

$$(\xi_1\xi_2 - \xi_2\xi_1, \xi_1\xi_3 - \xi_3\xi_1, \xi_2\xi_3 - \xi_3\xi_2).$$

2) Let $(\xi_1, \xi_2, \ldots, \xi_n)$ be a canonical basis of the original representation ϕ. Then $\phi^{\{k\}}$ denotes the symmetric part of the kth Kronecker product and $\phi^{\{1^k\}}$ denotes the antisymmetric terms.

If ξ_i belongs to the weight Λ_i then the system of weights for the symmetric part of $\phi^{\{k\}}$ is

$$\Lambda_{i_1} + \Lambda_{i_2} + \ldots + \Lambda_{i_k} \quad (i_k \geq i_{k-1} \geq \ldots \geq i_1)$$

with greatest weight $k\Lambda_1$,

and the system of weights for the antisymmetric part of $\phi^{\{1^k\}}$ is

$$\Lambda_{i_1} + \Lambda_{i_2} + \ldots + \Lambda_{i_k} \quad (i_k > i_{k-1} > \ldots > i_1)$$

with greatest weight

$$\Lambda_1 + \Lambda_2 + \ldots + \Lambda_k.$$

In our example,

$$\left(\circ\!\!-\!\!-\!\!-\!\!\overset{1}{\circ}\right)^{\{k\}} =$$

$\cdot\ \Lambda_1 + \Lambda_1$

$\cdot\ \Lambda_1 + \Lambda_2$

$\Lambda_2 + \Lambda_2\ \cdot \qquad \cdot\ \Lambda_1 + \Lambda_3$

$\cdot\ \Lambda_2 + \Lambda_3$

$\cdot\ \Lambda_3 + \Lambda_3$

and $\overset{\quad\quad 1}{(\circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ)}^{\{1^k\}} =$

$$\begin{array}{l} \bullet \quad \Lambda_1 + \Lambda_1 \\ \bullet \quad \Lambda_1 + \Lambda_2 \\ \bullet \quad \Lambda_2 + \Lambda_3 \quad . \end{array}$$

3) The dimension of the symmetric and antisymmetric parts of ϕ^k are

$$N(\phi^{\{k\}}) = \binom{n+k-1}{k}$$

and

$$N(\phi^{\{1^k\}}) = \binom{n}{k} \ .$$

In our example,

$$N(\phi^{\{k\}}) = \binom{3+2-1}{2} = 5$$

and

$$N(\phi^{\{1^k\}}) = \binom{3}{2} = 3.$$

4) A <u>basic</u> representation is a representation that cannot be built up from Kronecker products. This means that its greatest weight cannot be a sum of weights of lower representations - so it must be of such a form that the greatest weight in the Dynkin diagram is labelled with a 1, all the other Λ_α being zero. There are thus l basic representations for any classical Lie group. An <u>elementary representation</u> is one which has its greatest weight at an end point of its Dynkin diagram.
Our example now also illustrates that an arbitrary basic represenation may be constructed from any elementary representation by the process of antisymmetrisation. For example,

$$(\overset{1}{\circ}\!-\!\!-\!\circ\!-\!\!-\!\circ)^{\{1^2\}} = (\circ\!-\!\!-\!\overset{1}{\circ}\!-\!\!-\!\circ)$$

and

$$(\overset{1}{\circ}\!-\!\!-\!\circ\!-\!\!-\!\circ)^{\{1^3\}} = (\circ\!-\!\!-\!\circ\!-\!\!-\!\overset{1}{\circ}) \quad .$$

So to obtain an elementary representation, we pick the basic representationwith the greatest weight and antisymmetrise it. We denote this operation by $\phi^{\{1^k\}}$.

The observation (4) leads to the following <u>theorem</u> by Dynkin:

The roots of the groups A_l, B_l, C_l, D_l can be expressed in terms of the weights of the <u>elementary</u> representations as follows:

<u>Table B.10: Roots of Semisimple algebras in terms of weights of elementary representations</u>

Group	roots	p,q for roots	p,q for simple roots
A_l:	$\alpha = \{\lambda_p - \lambda_q\}$,	$p \neq q; p,q = 1,2,\ldots,l+1$	$p > q$
B_l:	$\alpha = \{\lambda_p, \pm\lambda_p \pm \lambda_q\}$	$p \neq q; p,q = 1,2,\ldots,l$	$p > q$
C_l:	$\alpha = \{\pm 2\lambda_p, \pm\lambda_p \pm \lambda_q\}$	$p \neq q; p,q = 1,2,\ldots,l$	$p > q$
D_l:	$\alpha = \{\pm\lambda_p \pm \lambda_q\}$	$p \neq q; p,q = 1,2,\ldots,l$	$p > q$

so that the roots are:

Table B.11: Simple roots of Lie groups in terms of Weights

group: A_l	B_l	C_l	D_l
$\alpha_1 = \lambda_1 - \lambda_2$	$\alpha_1 = \lambda_1 - \lambda_2$	$\alpha_1 = \lambda_1 - \lambda_1$	$\alpha_1 = \lambda_1 - \lambda_2$
$\alpha_2 = \lambda_2 - \lambda_3$	$\alpha_2 = \lambda_2 - \lambda_3$	$\alpha_2 = \lambda_2 - \lambda_3$	$\alpha_2 = \lambda_2 - \lambda_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\alpha_{n-1}=\lambda_{n-1}-\lambda_n$	$\alpha_{n-1}=\lambda_{n-1}-\lambda_n$	$\alpha_{n-1}=\lambda_{n-1}-\lambda_n$	$\alpha_{n-2}=\lambda_{n-2}-\lambda_{n-1}$
$\alpha_n = \lambda_n-\lambda_{n+1}$	$\alpha_n = \lambda_n$	$\alpha_n = 2\lambda_n$	$\alpha_n = \lambda_{n-1}+\lambda_n$ $\alpha_{n-1} = \lambda_{n-1}-\lambda_n$

We can write the greatest weight Λ in the form

$$\Lambda = \sum_{i=1}^{l} l_i \lambda_i \qquad \text{or } \Lambda = \sum_{i=1}^{l+1} l_i \lambda_i \quad \text{for } A_l$$

with

$$\Lambda_{\alpha_k} = \frac{2(\Lambda,\alpha_k)}{(\alpha_k,\alpha_k)} \equiv a_k .$$

Using table (B.11), this gives

group	values of l_i
A_l	$l_k = l_{l+1} + \sum_{i=k}^{l} a_i$ $l_{l+1} = \frac{-1}{l+1} \sum_{i=1}^{l} i\, a_i$
B_l	$l_k = \frac{a_l}{2} + \sum_{i=k}^{l-1} a_i$
C_l	$l_k = \sum_{i=k}^{l} a_i$
D_l	$l_k = \frac{a_{l-1} - a_l}{2} + \sum_{i=k}^{l-2} a_i$

B.5 Dimensions of Irreducible Representations

Now, having expressed the simple roots in terms of the weights of elementary representations, we can use the properties of the Cartan matrix to find the scalar products of weights (λ_i,λ_j). We find

Table B.12: Scalar products of weights of elementary representations

group	scalar product of weights	value of K
A_l	$(\lambda_i,\lambda_i) = lK;\quad i=1,2,\ldots,l+1$ $(\lambda_i,\lambda_j) = -K;\quad i\neq j$	$\frac{1}{2(l+1)^2}$
B_l	$(\lambda_i,\lambda_i) = K;\quad i=1,2,\ldots,l$ $(\lambda_i,\lambda_j) = 0\ ;\quad i\neq j$	$\frac{1}{2(2l-1)^2}$
C_l	$(\lambda_i,\lambda_i) = K;\quad i=1,2,\ldots,l$ $(\lambda_i,\lambda_j) = 0\ ;\quad i\neq j$	$\frac{1}{4(l+1)^2}$
D_l	$(\lambda_i,\lambda_i) = K;\quad i=1,2,\ldots,l$ $(\lambda_i,\lambda_j) = 0\ ;\quad i\neq j$	$\frac{1}{4(l-1)^2}$

The constant K is determined by the requirement that the roots form an orthonormal set, i.e.

$$\sum_\alpha (\alpha_i,\alpha_j) = \delta_{ij}$$

so that

$$\sum_\alpha (\alpha,\alpha) = 1.$$

Finally, we are ready to work out the dimension of any irreducible representation. The derivation is laborious, so we state the formula which is due to Weyl:

$$N(\phi) = \prod_{\alpha \text{ pos.}} \frac{(\Lambda+g,\alpha)}{(g,\alpha)}$$

$$\text{with } g = \tfrac{1}{2} \sum_{\alpha \text{ pos.}} \alpha \tag{B.13}$$

By writing g as

$$g = \Sigma\, g_i \lambda_i$$

and using table (B.11) for the expansion (B.13), we obtain the values for g_i:

Table B.14: Values for g_i

group	g_i
A_l	$\frac{l}{2} - i + 1$
B_l	$l + i + \frac{1}{2}$
C_l	$l - i + 1$
D_l	$l - i$

For example, for A_2 we have the roots α_1, α_2, $\alpha_3 = \alpha_1 + \alpha_2$, so

$$g = \tfrac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3) = (\alpha_1 + \alpha_2) = (\lambda_1 - \lambda_2 + \lambda_2 - \lambda_3) = \lambda_1 - \lambda_3$$

$$= \Sigma\, g_i \lambda_i,$$

so that

$g_1 = 1, g_2 = 0, g_3 = -1$, which agrees with table (B.14).

We now work out $N(\phi)$:

For A_1, we have

$$\pi_{\alpha\ \text{pos.}} (g,\alpha) = \pi_{\alpha\ \text{pos.}} (\Sigma\, g_i\, \alpha_i, \alpha)$$

$$= \pi_{\alpha\ \text{pos.}} \Sigma\, \lambda_i(\eta_i, \lambda_p - \lambda_q), \quad p>q \quad \text{because of table B.10}$$

$$= nK\, \pi_{\alpha\ \text{pos.}} (g_p - g_q) \quad , \ p>q \qquad \text{because of table B.12 .}$$

Also,

$$\pi_{\alpha\ \text{pos.}} (\Lambda + g,\alpha) = \pi_{\alpha\ \text{pos.}} (\Sigma\, l_i\lambda_i + \Sigma\, g_i\lambda_i, \alpha)$$

$$= \pi_{\alpha\ \text{pos}} \Sigma\, m_i\, (\lambda_i, \alpha)$$

$$= nK\, \pi_{\alpha\ \text{pos}} (m_p - m_q); \quad p>q \ ,$$

so that

$$N(\phi) = \pi_{p>q} \left(\frac{m_p - m_q}{g_p - g_q} \right) \quad \text{for } A_1.$$

The general result is

Table B.15: Dimensions of Irreducible Representations

group	$N(\phi)$
A_1	$\prod_{p>q} \left(\frac{m_p - m_q}{g_p - g_q} \right)$
B_1, C_1	$\prod_{p>q} \left(\frac{m_p}{g_p} \right) \prod_{p>q} \left(\frac{m_p - m_q}{g_p - g_q} \right) \prod_{p>q} \left(\frac{m_p + m_q}{g_p + g_q} \right)$
D_1	$\prod_{p>q} \left(\frac{m_p - m_q}{g_p - g_q} \right) \prod_{p>q} \left(\frac{m_p + m_q}{g_p + g_q} \right)$

Examples are:

group	$N(\phi)$	first terms
$A_1=SU(2)$	$l_1 - l_2 + 1$	1,2,3
$B_2=SO(5)$	$\frac{(l_1+l_2+2)(l_1-l_2+1)(2l_1+3)(2l_2+1)}{6}$	1,5,10,14
$C_2=Sp(4)$	$\frac{(l_1+2)(l_2+1)(l_1-l_2+1)(l_1+l_2+3)}{6}$	1,4,5,10,14
$A_2=SU(3)$	$\frac{(l_1-l_2+1)(l_2-l_3+1)(l_1-l_3+2)}{2}$	1,3,6
$C_3=Sp(6)$		1,6,14,14,21
$C_4=Sp(8)$		1,8,27,48,42

The sole purpose of this appendix was to show how the dimensions of irreducible representations can be derived. These are important as the spectra of supersymmetric theories are generated by various groups. It was therefore attempted to show a calculating method- most proofs of theorems have been omitted. These can be found in the original papers or books by Weyl, Cartan, Dynkin etc., for example:

E.B.Dynkin: Am. Math.Soc. Transl. **6**,308 (1962)

E.B.Dynkin: Am. Math.Soc. Transl. **6**,111 (1965)

E.B.Dynkin: Am. Math.Soc. Transl. **6**,245 (1965)

H.Weyl: Math. Zeitsch. **23**,271 (1925) and **24**, 328,377,789 (1926)

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[2] W.Greiner, B.Müller : Quantenmechanik 2: Symmetrien, Harri Deutsch Publ. (1985)

[3] B.G.Wybourne : Classical Groups for Physicists, John Wiley (1974)

[4] E.Stiefel, A.Faessler : Gruppentheoretische Methoden und ihre Anwendung, Teubner, 1979

[5] J.P.Elliott, P.G.Dawber : Symmetry in Physics, Macmillan Press, 1979

Extensive tables of irreducible representations of classical groups can be found in:

[6] B.G.Wybourne : Symmetry Principles and Atomic Spectroscopy, Wiley-Interscience (1970)

[7] R.Slansky : Group Theory for Unified Model Building, Physics Report 79 No.1, 1-128 (1981)